Recent Development in the Theory of Connections and Holonomy Groups

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Introduction

The purpose of the present article is to give an exposition of some of the developments in contemporary differential geometry, such as the theo-

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ry of connections in fiber bundles, the theory of holonomy groups of linear and Riemannian connections, the results on transformation groups of a geometric object, and differential geometry of homogeneous spaces.

We wish to explain what we mean by contemporary differential geometry as contrasted to classical differential geometry. Classical differential geometry, such as we understand as a title for a course, is a study of curves and surfaces in 3-dimensional Euclidean space whose principal method is differential calculus. A generalization of intrinsic geometry of surfaces is, of course, Riemannian geometry, which dates from the time of Riemann himself. Not only in this branch of differential geometry, but also in other branches such as affine, projective, or conformal differential geometry, there is such a vast amount of literature that we would not venture to give a historic account of the development of differential geometry. But one thing seems certain. That is that the work of Elie Cartan [1-4] on connections, holonomy groups, and homogeneous spaces, is the source of all that is interesting in contemporary differential geometry. These theories are not only of their own interest but also serve as the best foundation of all work in differential geometry.

Contemporary differential geometry is the study of a geometric object given on a differentiable manifold. We shall again not try to define "geometric objects." It is sufficient to understand, for example, a connection, a linear connection, a Riemannian metric, or a Kählerian metric, and so on. Given such a structure Γ on a differentiable manifold M, the first object is to study the properties of Γ. Then, there are problems such as the study of the group of automorphisms $A(\Gamma)$ of the structure Γ and its relation to properties of Γ, or the study of relationship between properties of Γ and the topological properties of the manifold M.

It is only after the concept of a differentiable manifold was introduced that the work of Elie Cartan has been fully clarified, understood, and developed. As for the so-called global theory of Lie groups, the book of C. Chevalley [1] appeared in 1946. The concept of a connection in a fiber bundle was first defined rigorously by Ehresmann [1] in 1950. It was then possible to develop the theory of connections in a form which is completely intelligible to mathematicians of today. Functions, mappings, vector fields, and so on, which come up in the study, will have a definite domain of definition, thus eliminating ambiguities which existed even in the statements of theorems, as we often find in books and papers before this period. The emphasis of "global" aspects of a geometric structure has been apparent in connection with the topological structure, once the notion of the differentiable manifold took the place of an ambiguous term like "space of $x^1, \ldots, x^n$." But even in the study of local aspects, many things have remained unclarified until very recently. In fact, one of the interesting problems in differential geometry is to study the variation of local phenomena from point to point and relate them to global phenomena. We shall discuss a few problems of this nature in this article.

Thus, if the concept of a differentiable manifold has given a topological background for contemporary differential geometry and opened a new field "differential geometry and topology," there are still two more characteristic features of contemporary differential geometry. One is the emphasis of the algebraic point of view. For example, tangent vectors are treated not just individually, but are considered as elements of a vector space which they form. Instead of defining them as objects with components with respect to a coordinate system, they are better defined as derivations of the algebra of differentiable functions into the real number field. Similarly, covariant differentiation associated with a linear connection is understood not as a mere rule of computation but as a derivation of the algebra of tensor fields into the tensor algebra over the tangent space at each point of the manifold. This approach not only simplifies and clarifies many definitions and proofs, but it also makes it possible to distinguish algebraic aspects of tensor calculus from analytical aspects. The algebraic method becomes indispensable when we deal with holonomy groups or transformation groups. Elementary knowledge of Lie theory simplifies and unifies many results which were individually verified by tedious computation before. More profound knowledge on Lie groups and Lie algebras is needed for the deeper study of holonomy groups and transformation groups.

Now the last point we wish to mention is the question related to analytical assumptions for a manifold and its geometric structure in question. In classical literature, for example, analyticity of a Riemannian metric was often assumed without explicit mention. We now make clear what degree of differentiability we assume in the beginning of any theory. The most frequent one is that of class C^∞. Here, an important problem is to distinguish results which are valid under differentiability assumptions and those which are valid only under analyticity assumptions. We shall mention a few examples of this nature in our article.

What we mean by contemporary differential geometry should now be clear. In this article, we shall give a brief survey of the foundation of the theory of connections and holonomy groups and its applications which have been developed since about 1950. The selection of material to be included was made so as to give a few central ideas about the problems and methods in this branch of differential geometry rather than to give a complete account of the development. It goes without saying that there are many more important contributions to contemporary differential geometry that are not mentioned at all in this article.

The references at the end are listed by authors. They have been kept to the minimum since more of the related references are easily found in the papers and books on our list. For a detailed account of the theory of connections and holonomy groups, see Ehresmann [1], Chern [1], Lichnerowicz [3], Nomizu [5], and Kobayashi [5]. A survey of recent results is found in Lichnerowicz [2, 4] and Nomizu [6]. For applications of the concept of a connection to cohomology theory, which we did not mention in this article, see H. Cartan [1] and Chern [1].

Preliminaries

The basic concepts on differentiable manifolds, Lie groups and fiber bundles, are now standard (see Chevalley [1], Steenrod [1], Nomizu [5]). We shall give here a brief résumé of definitions and notations which we use in this article.

By differentiability, we always mean that of class C^∞. Let M be a differentiable manifold of dimension n. The tangent space at u, denoted by $T_u(M)$, is an n-dimensional real vector space consisting of all tangent vectors X at u. A tangent vector X at u is a linear mapping of the algebra of all differentiable functions on M into the real number field R such that $X(fg) = (Xf)\,g(u) + f(u)\,Xg$ for arbitrary differentiable functions f and g. A vector field X on M is an assignment of a tangent vector X_u to each point $u \in M$. We consider differentiable vector fields, that is, vector fields X such that the function Xf defined by $(Xf)_u = X_u f$, $u \in M$, is differentiable whenever f is so. For two differentiable vector fields, X and Y, the bracket $[X, Y]$ is a vector field defined by $[X, Y]f = X(Yf) - Y(Xf)$, where f is an arbitrary differentiable function. We have the Jacobi identity: $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$. The set of all differentiable vector fields on M forms a Lie algebra over R (of infinite dimensions).

Over each tangent space $T_u(M)$, we consider the tensor algebra $T(u)$ which is the direct sum of tensor spaces $T^r_s(u)$ of type (r, s) (r: contravariant degree, s: covariant degree). A tensor field K of type (r, s) is an assignment of an element $K_u \in T^r_s(u)$ to each point u, and differentiability can be defined suitably. In particular, a tensor field K of type (1,1) can be considered as a field of linear transformations of each tangent space $T_u(M)$. A tensor field K of type $(1, s)$ can be considered as a field of multilinear mappings of $T_u(M) \times \ldots \times T_u(M)$ (s-times product) into $T_u(M)$, denoted by $K(X_1, \ldots, X_s) \in T_u(M)$, where $X_1, \ldots, K_s \in T_u(M)$.

By an r-dimensional distribution Δ on M, we mean a field of r-dimensional subspace Δ_u of $T_u(M)$, $u \in M$. It is called differentiable, if for each point u, there is a neighborhood U of u and r differentiable vector fields $X_1, \ldots, X_r$ on U which span the subspace Δ_v at each point $v \in U$. It is called involutive, if, for vector fields X and Y which belong to Δ (that is, X_u, $Y_u \in \Delta_u$ for each u), $[X, Y]$ belongs to Δ. The Frobenius theorem in a global formulation says that for any differentiable and involutive distribution, there is a unique maximal integral manifold through each point.

A differential form of degree r is a skew-symmetric covariant tensor field of degree r. For a 1-form ω, we have the formula

$$2(d\omega)(X, Y) = X.\omega(Y) - Y.\omega(X) - \omega([X, Y])$$

for arbitrary vector fields X and Y, where $X.\ \omega(Y)$ is the function obtained by applying the vector field X to the function $\omega(Y)$. If V is a finite-dimensional vector space, a V-valued differential form ω of degree r on M is defined as a skew-symmetric multilinear mapping, at each $u \in M$, of $T_u(M) \times \ldots \times T_u(M)$ (r-times) into V. When a basis $\{e_1, \ldots, e_k\}$ of V is chosen, ω is of the form $\omega = \sum_{i=1}^k \omega^i e_i$, where $\omega^1, \ldots, \omega^k$ are usual r-forms on M. The exterior derivative is defined as $d\omega = \sum_{i=1}^k (d\omega^i) e_i$.

A Lie group G is a group which is at the same time a differentiable manifold such that the group operations are differentiable. The connected component G^0 of the identity e is an open (and hence closed) subgroup of G. We usually impose the condition that G satisfies the second axiom of countability, which is equivalent to the condition that the coset space G/G^0 is at most countable. It is known that a Lie group admits a structure of real analytic manifold such that the group operations are analytic.

Given a Lie group G, the Lie algebra $\mathfrak{g}$ of G is defined as the set of all left invariant (or right invariant) vector fields on G, where the bracket $[X, Y]$ is naturally defined. As a vector space, $\mathfrak{g}$ is isomorphic with the tangent space $T_e(G)$ since a left invariant vector field is completely determined by its value at e. The Maurer-Cartan form θ on G is defined as a $\mathfrak{g}$-valued 1-form on G such that $\theta_a(X) = L_a^{-1}. X$ for any tangent vector X at $a \in G$, where L_a denotes the differential of the left translation L_a by a.

A Lie subgroup G' of a Lie group G is defined as a subgroup of G which is at the same time a submanifold of G, G' itself a Lie group with respect to its differentiable structure. We do not assume G' to be closed in G. There is a natural 1-1 correspondence between the set of connected Lie subgroups of G and the set of subalgebras of $\mathfrak{g}$. In particular, every element $A \in \mathfrak{g}$ generates a 1-parameter subgroup denoted by $\exp tA$. A theorem of Yamabe-Kuranishi says that an arcwise connected subgroup of a Lie group is a Lie subgroup.

A Lie group G is said to be a Lie transformation group on a differentiable manifold M if the following conditions are satisfied:

(a) For each $a \in G$, there corresponds a differentiable transformation of M, denoted by $x \to ax$; $x \in M$.

(b) If $a, b \in G$, then $a(bx) = (ab)x$ for every $x \in M$.

(c) The mapping $(a, x) \in G \times M \to ax \in M$ is differentiable.

We say that G acts effectively on M if $ax = x$ for every $x \in M$ implies that $a = e$. G acts freely on M if $ax = x$ for some $x \in M$ implies that $a = e$; in other words, no other element than e has a fixed point in M. A 1-parameter group of transformations on M is the real additive group R acting on M as a Lie transformation group. When G is a Lie transformation group on M, we have a natural homomorphism of the Lie algebra $\mathfrak{g}$ (of right invariant vector fields) into the Lie algebra of all differentiable vector fields on M: for any $X \in \mathfrak{g}$, let the corresponding vector field X^* be defined by

$$X_u^* f = \lim_{t \to 0} \frac{1}{t} [f(\exp (tA)\, u) - f(u)]$$

at each $u \in M$ and for any differentiable function f on M. In case G is effective, this homomorphism is an isomorphism. In case G acts freely, X^* has no zero point for any nonzero element $X \in \mathfrak{g}$.

Let M be a differentiable manifold and G a Lie group. A differentiable manifold P is called a principal fiber bundle over the base space M with structure group G if the following conditions are satisfied:

(a) G acts on P differentiably and freely; we denote the action by $(x, a) \in P \times G \to xa \in P$.

(b) M is the quotient space of P by the equivalence relation induced by G and the canonical projection $\pi: P \to M$ is differentiable.

(c) P is locally trivial, that is, every point $u \in M$ has a neighborhood U such that $\pi^{-1}(U)$ has a diffeomorphism $x \in \pi^{-1}(U) \to (\pi(x), \phi(x)) \in U \times G$ such that $\phi(xa) = \phi(x)$ a for every $a \in G$.

For each $u \in M$, $\pi^{-1}(u)$ is a closed submanifold, called the fiber over u. For any $x \in P$, the fiber through x, which is $\pi^{-1}(\pi(x))$, is diffeomorphic with the space G. For each $x \in P$, $A \in \mathfrak{g} \to A_x^*$ is an isomorphism of the vector space $\mathfrak{g}$ onto the tangent space at x to the fiber through x. For each $A \in \mathfrak{g}$, the fundamental vector field A^* is defined as the image of the natural homomorphism of $\mathfrak{g}$ into the Lie algebra of all vector fields on P.

Let $P(M, G)$ be a principal fiber bundle over M with structure group G. If F is a differentiable manifold on which G acts differentiably: $(a, \xi) \in G \times F \to a\xi \in F$, we can construct an associated fiber bundle $E(M, F\, G, P)$ over M with standard fiber F and structure group G. An element $x \in P$ can be regarded as a diffeomorphism of F onto the fiber of E over $u = \pi(x)$. The most important example of this situation in differential geometry is P = bundle of frames over a differentiable manifold, with structure group $GL(n, R)$, E = tangent bundle whose standard fiber is an n-dimensional real vector space with a fixed basis $(e_1, ..., e_n)$.

For a fiber bundle $E(M, F, G, P)$, a cross section is a differentiable mapping f of M into E such that $\pi . f$ = identity. A cross section always exists if M satisfies the second axiom of countability, and if the standard fiber F is diffeomorphic with Euclidean space R^k.

In the text, we shall assume that a given manifold M is connected and satisfies the second axiom of countability.

1. Theory of Connections

1.1. Connection in a Principal Fiber Bundle. Let $P(M, G)$ be a principal fiber bundle over the base manifold M with structure group G. At each point x of P, let P_x be the tangent space at x of P and G_x the subspace of P_x tangent to the fiber through x. A *connection* Γ on P is an assignment of a subspace Q_x of P_x to each point x of P which satisfies the following conditions:

(a) $P_x = G_x + Q_x$ (direct sum).

(b) For every $a \in G$ and $x \in P$, Q_{xa} is the image of Q_x by R_a.

(c) Q_x depends differentiably on x.

The last condition means the following. Given an arbitrary vector field X on P, we have at each point $x \in P$ a unique decomposition $X_x = Y_x + Z_x$, with $Y_x \in G_x$ and $Z_x \in Q_x$. Thus, we get a vector field Y: $x \to Y_x$, called the vertical component of X, and a vector field Z: $x \to Z_x$, called the horizontal component of X. Condition (c) means that if X is differentiable, so is Y (and hence Z). A tangent vector at x is called *horizontal* if it is in Q_x. A differentiable curve is called *horizontal* if its tangent vectors are all horizontal.

Given a connection Γ in P, we define a 1-form ω on P with values in the Lie algebra $\mathfrak{g}$ of G as follows. We know that the subspace G_x is the set of all vectors of the form A^*_x, $A \in \mathfrak{g}$, where A^* is the fundamental vector field corresponding to A. We define ω_x as a linear mapping of P_x into $\mathfrak{g}$ which maps $(A^*)_x$ upon $A \in \mathfrak{g}$, and which maps every $Z \in Q_x$ into 0. As a consequence of condition (b) the *connection form* ω has the property that

$$R^*_a\, \omega = \mathrm{ad}(a^{-1})\, \omega$$

for $a \in G$. We shall say that ω is of type ad(G).

A connection Γ in P enables us to define the notion of *parallel displacement*. Given any piecewise differentiable curve u_t, $0 \leqq t \leqq 1$, in M, we shall obtain an isomorphism of the fiber $\pi^{-1}(u_0)$ upon the fiber $\pi^{-1}(u_1)$, namely, a diffeomorphism which commutes with the action of the structure group. This is done by taking a horizontal curve x_t starting from each point $x \in \pi^{-1}(u_0)$ which projects on the given curve u_t; the end point of such a curve will be the image of x by the parallel displacement along u_t. Intuitively speaking, suppose a given curve is an integral curve of a vector field X on M. We may get a horizontal vector field X^* on P which projects on X. The integral curve of X^* through x then gives a horizontal curve which projects upon u_t. The local existence of parallel displacement follows from this argument. More precisely, we can prove the following:

Proposition. *For any piecewise differentiable curve* $\tau : u_t$, $0 \leqq t \leqq 1$, *in* M, *and for any point* x *in* $\pi^{-1}(u_0)$, *there is a unique horizontal curve* x_t *such that* $x_0 = x$, *and* $\pi(x_t) = u_t$.

The parallel displacement along τ will be denoted by the same letter τ. Now the natural thing is to consider the effects of parallel displacement along all closed curves at a point u. Let u be an arbitrary point of M and let us fix an arbitrary point x over u. For any closed curve τ at u, the parallel displacement τ is an isomorphism of the fiber $\pi^{-1}(u)$ onto itself such that $\tau(ya) = \tau(y)\, a$ for every point y and $a \in G$. Thus,

τ is completely determined by a unique element $a \in G$ such that $\tau(x) = xa$, since then for any point $y \in \pi^{-1}(u)$ we have $y = xb$ for some $b \in G$ and $\tau(y) = xab$. If τ and μ are two closed curves at u, the parallel displacement along the composite curve $\tau \cdot \mu$ is simply the product of the parallel displacements along μ and τ. In fact, if $\tau(x) = xa$, and $\mu(x) = xb$, then $(\tau \cdot \mu)(x) = \tau(xb) = \tau(x)\, b = xab$. The set of parallel displacements along all closed curves τ at u forms a group of transformations of the fiber over u, which is isomorphic with a subgroup Φ_x of G consisting of all $a \in G$ such that $\tau(x) = xa$ for some closed curve τ at u. We call Φ_x the *holonomy group* of the given connection at x. If we change a reference point x either in the same fiber or elsewhere, we get essentially isomorphic groups.

The first step in the study of the holonomy group is to show that it is a Lie group. Once this is done, we can talk about its Lie algebra which will be closely related to the invariants which can be defined infinitesimally from a given connection.

From the construction of parallel displacement along a curve it follows that if τ_s is a family of curves from u_0 to u_1 depending differentiably on the parameter s, then the parallel displacement also depends differentiably on s. We define the restricted holonomy group Φ_x^0 (with reference point x) as the subgroup of the holonomy group Φ_x consisting of parallel displacements along all closed curves which are homotopic to zero. Then Φ_x^0 is isomorphic with an arcwise connected subgroup of the structure group G. By a theorem of Kuranish-Yamabe, it follows that Φ_x^0 is a connected Lie group, which is an invariant subgroup of Φ_x. Now consider the fundamental group $\pi_1(M)$ and a natural homomorphism of $\pi_1(M)$ onto Φ_x/Φ_x^0. Since $\pi_1(M)$ is countable when M satisfies the second axiom of countability, Φ_x/Φ_x^0 is also countable. We can therefore make Φ_x into a differentiable manifold such that Φ_x^0 is an open submanifold. It is in this manner that we regard Φ_x as a Lie group.

1.2. Curvature Form. Let ω be the connection form of a given connection Γ in $P(M, G)$. We define the *curvature form* Ω of Γ in the following way. The exterior differential $d\omega$ is a 2-form on P with values in $\mathfrak{g}$. We define

$$\Omega(X, Y) = (d\omega)(hX, hY)$$

for any vector fields X and Y on P, where h denotes the horizontal component. It is clear from this definition that Ω is again a 2-form on P with values in $\mathfrak{g}$ and that $\Omega_x(X, Y) = 0$ if X_x or Y_x is vertical.

From the fact that exterior differentiation d and the operation of taking horizontal components commute with the action of $a \in G$ on P, it follows that Ω satisfies, just like the connection form itself, the condition

$$R_a^* \Omega = \mathrm{ad}(a^{-1})\, \Omega$$

for any $a \in G$.

The curvature form Ω of a connection corresponds to the curvature tensor of Riemannian geometry. In fact, we shall later see that in the case of a linear connection, the classical curvature tensor can be derived from the curvature form. We have two important identities concerning the curvature form of an arbitrary connection.

The first on is the following *structure equation*:

$$\Omega(X, Y) = (d\omega)(X, Y) + (\tfrac{1}{2})\,[\omega(X), \omega(Y)]$$

which is valid for all vector fields X and Y on P. The proof is given by checking the equation in the following three cases: (1) X and Y are horizontal; (2) X is horizontal and Y is vertical; (3) X and Y are vertical. In particular, when X and Y are horizontal, the equation reduces to the definition of Ω since $\omega(X) = \omega(Y) = 0$. But we have also $2(d\omega)(X, Y) = X.\omega(Y) - Y.\omega(X) - \omega([X, Y]) = -\omega([X, Y])$ so that $\omega([X, Y]) = -2\Omega(X, Y)$. If the vertical component of $[X, Y]$ at $x \in P$ is equal to A_x^*, where $A \in \mathfrak{g}$, then $A = -2\Omega_x(X, Y)$. This fact plays an important role later.

Another identity on the curvature form is that $d\Omega(X, Y, Z) = 0$ if X, Y, and Z are horizontal. This equation is a generalization of Bianchi's identity in Riemannian geometry.

1.3. Homomorphism of Connections. Let $Q(M, H)$ be a principal fiber bundle over M with structure group H. A differentiable mapping f of Q into $P = P(M, G)$ is called a *homomorphism* if $f(xa) = f(x)\phi(a)$ for all $x \in Q$ and $a \in H$, where ϕ is a certain differentiable homomorphism of the Lie group H into G, and if the induced mapping of the base space M is a diffeomorphism of M onto itself. This concept includes the following special cases:

(a) $Q = P$ and ϕ is the identity automorphism of $H = G$; f is then called an *automorphism* of the principal fiber bundle $P(M, G)$.

(b) H is a Lie subgroup of G, ϕ is the injection of H into G, and the induced mapping f of the base is the identity transformation. In this case, we say that f is an injection, and that $Q(M, H)$ is a subbundle of

$P(M, G)$. We shall also say that the structure group G of a given bundle $P(M, G)$ is ***reducible*** to a Lie subgroup H if there is a principal bundle $Q(M, H)$ with an injection f into P.

Let f be a homomorphism of $Q(M, H)$ into $P(M, G)$ corresponding to a homomorphism $\phi : H \to G$. Given a connection Γ_Q in Q, we can obtain a connection Γ_P in P in such a way that f maps the horizontal subspace at each $x \in P$ upon the horizontal subspace at $f(x) \in P$. It is sufficient to define the horizontal subspace at each $f(x)$, $x \in P$, to be the image of the horizontal subspace at x by f and translate it to any other point of the same fiber by the action of the structure group G. In general, when a homomorphism f maps a connection Γ_Q in $Q(M, H)$ into a connection Γ_P in $P(M, G)$ in this manner, we say that f preserves the given connections. It is almost clear that such a mapping f maps horizontal curves in Q into horizontal curves in P and induces a natural homomorphism of the holonomy group of Γ_Q with reference point x into the holonomy group of Γ_P with reference point $f(x)$.

This fairly formal argument is very useful in two ways. The first application is, of course, to the construction of a connection in P from a connection in a smaller bundle Q. The second is the converse. Given a connection Γ_P in a bundle $P(M, G)$, we say that Γ_P can be reduced to a connection Γ_Q if there is a reduced bundle $Q(M, H)$ with an injection f into P and a connection Γ_Q in Q which is mapped upon Γ_P by f. In this case, properties of Γ_P can be studied from those of Γ_Q.

As a general theorem, we have the following.

Reduction theorem. *Let $P(M, G)$ be a principal fiber bundle with a connection Γ. Let Φ be the holonomy group of with reference point x in P. Then, the structure group G is reducible to Φ, and the connection Γ is reducible to a connection in the reduced bundle $Q(M, \Phi)$ whose holonomy group is exactly Φ.*

To get an intuitive picture of this theorem, let Q be the set of points in P which can be joined to x by a horizontal curve. For every $a \in \Phi$, the point xa is, of course, in Q. More generally, if $y \in Q$, then $ya \in Q$ for every $a \in \Phi$. In fact, denoting by $y \sim z$ the fact that two points y and z can be joined by a horizontal curve, we see that $y \sim x$, $x \sim xa$ so that $ya \sim x$ because $y \sim z$ implies that $yb \sim zb$ for every $b \in G$. This means that the group Φ leaves the set Q invariant. We can prove rigorously that Q is a principal bundle over M with structure group Φ. From the definition of Q, it is clear that at any point y of Q the horizontal curves starting from y are all contained in Q and hence the horizontal subspace at y is tangent to Q. This means that Q has a natural connection related to Γ_P as desired.

Finally, in case f is an automorphism of $P(M, G)$ which preserves a given connection Γ, we say that f is an automorphism of the connection Γ. For a general connection, the group of all automorphisms is not a Lie group; that is, there are always many more automorphisms than can be controlled by a finite number of parameters.

1.4. Holonomy Theorem. We are now in a position to state the following theorem, originally due to E. Cartan [1] and rigorously proved by Ambrose-Singer [1].

THEOREM. *Let Γ be a connection in $P(M, G)$, and let Φ be the holonomy group of Γ at $x \in P$. The holonomy algebra (Lie algebra of Φ) is the subalgebra of $\mathfrak{g}$ (Lie algebra of G) which is generated by all elements of the form $\Omega_y(X, Y)$, where y is an arbitrary point of P which can be joined to x by a horizontal curve, and X and Y are arbitrary horizontal vectors at y.*

To prove this theorem, let Q be the reduced bundle consisting of all points y which can be joined to x by a horizontal curve. We know that Q has a natural connection Γ_Q induced from that of P. Since the curvature form of Γ_Q is simply the restriction of that of Γ, it is sufficient to prove the theorem for Γ_Q. Thus in the original bundle $P(M, G)$, we may assume that every point y of P can be joined to x by a horizontal curve. Now let $\mathfrak{g}'$ be the subalgebra generated by all elements of the form $\Omega_y(X, Y)$ as stated in the theorem. We show that $\mathfrak{g}' = \mathfrak{g}$. At each point y, let Δ_y be the subspace of P_y spanned by the horizontal subspace Q_y and the subspace $\mathfrak{g}'_y = \{A^*_y \mid A \in \mathfrak{g}'\}$. Admitting that the distribution Δ is differentiable and involutive, we immediately get the theorem because the maximal integral manifold $P(x)$ of Δ through x contains all horizontal curves starting from x and thus coincides with P. This implies that $\dim \mathfrak{g}' = \dim \mathfrak{g}$, and hence, $\mathfrak{g}' = \mathfrak{g}$. Thus, it remains to verify that Δ is differentiable and integrable. The first condition is easy to verify. To prove the second, let X and Y be vector fields which belong to the distribution Δ and we show that $[X, Y]$ belongs to Δ. The essential case is the case where X and Y are both horizontal. If we recall the remark we made about the structure equation, we see that the vertical component of $[X, Y]_y$ is of the form A^*_y, where $A = -2\Omega_y(X, Y)$ belongs to $\mathfrak{g}'$. Thus, the vertical component as well as the horizontal component of $[X, Y]$ belong to Δ. We have thus proved the theorem.

In the above theorem, we make the following observation. If we take the smallest linear subspace $\mathfrak{m}$ of $\mathfrak{g}$ containing all elements $\Omega_v(X, Y)$, then it is a subalgebra and hence, by the theorem, coincides with the Lie algebra of Φ. In fact, using $R_a^*\Omega = \mathrm{ad}(a^{-1})\,\Omega$, we see that the linear subspace $\mathfrak{m}$ is invariant by ad (Φ). Since exp X, $X \in \mathfrak{m}$, is contained in Φ by the theorem, we see that $[X, \mathfrak{m}] \subset \mathfrak{m}$ for every $X \in \mathfrak{m}$; that is, $\mathfrak{m}$ itself is closed with respect to the bracket operation.

The theorem in this section will serve as a basic lemma in a more profound study of the holonomy group which we shall take up later.

1.5. Existence of Connections. We have not said a word about the existence of connections in a principal fiber bundle yet. *A principal fiber bundle $P(M, G)$ always admits a connection*, provided it satisfies the second axiom of countability as we always assume in the whole theory. In case the structure group G is connected, one can prove the existence of a connection as follows. By a theorem in fiber bundles, the structure group G can be reduced to any one of its maximal compact subgroups, that is, there is a bundle $Q(M, H)$ with compact structure group H which admits an injection into $P(M, G)$. In the principal bundle Q, we take an arbitrary Riemannian metric, and by the standard averaging process, we can obtain a Riemannian metric g invariant by the action of H. At each point x of Q, we take the horizontal subspace to be the orthogonal complement of the tangent subspace to the fiber with respect to g. It is easy to see that we actually get a connection in Q. By a previous result, this connection can be extended to a connection in P.

This proof assumes the existence of a Riemannian metric on any differentiable manifold (satisfying the second axiom of countability), which in turn can be proved either by applying the existence theorem of a cross section or, more directly, by using a partition of unity (see Section ?.1). Similar methods will prove the existence of a connection in an arbitrary principal fiber bundle in the following form of extension theorem:

THEOREM. *Let $P(M, G)$ be a principal fiber bundle. Let U and V be open sets in M such that $\bar{U} \subset V$. For any connection Γ' in $\pi^{-1}(V)$, there is a connection in P which coincides with Γ' in $\pi^{-1}(U)$.*

An interesting result follows from this extension theorem and the holonomy theorem. It is essentially the converse to the reduction theorem, and it shows us that the holonomy group can be quite arbitrary.

THEOREM. (Nomizu [3]). *Let $P(M, G)$ be a principal fiber bundle, and assume that the structure group can be reduced to a connected Lie subgroup H of G. Then, there exists a connection in P whose restricted holonomy group is exactly H.*

In particular, let M be an arbitrary manifold and G an arbitrary connected Lie group. In the product bundle $P = M \times G$, there is a connection whose holonomy group is precisely G, showing that an *arbitrary connected Lie group can be realized as the holonomy group of a certain connection* in a bundle over an arbitrary manifold. For example, an *arbitrary connected Lie subgroup G of $GL(n, R)$ can be realized as the holonomy group of a linear connection on n-dimensional Cartesian space R^n* (J. Hano and H. Ozeki [1]).

1.6. Local and Infinitesimal Holonomy Groups. Generalizing the work of Nijenhuis [1] on the holonomy groups of linear connections, Ozeki [1] gave a systematic study of local and infinitesimal holonomy groups of an arbitrary connection. We shall now give a brief sketch of the main results.

Let $P(M, G)$ be a principal fiber bundle with a connection Γ. For each $x \in P$, we define the local holonomy group at x in the following way. Let $\Phi(U, x)$ be the holonomy group of Γ restricted to the bundle $\pi^{-1}(U)$, where U is an arbitrary connected open neighborhood of $u = \pi(x)$. We define the local holonomy group $\Phi^*(x)$ as the intersection of all $\Phi(U, x)$ (regarded as subgroups of the structure group G) where U is an arbitrary open neighborhood of $u = \pi(x)$. If U_k, $k = 1, 2, \ldots$, form a complete system of neighborhoods of u which are connected, simply connected, and open, then we have a decreasing sequence of connected Lie subgroups $\Phi(U_k, x)$ of G, By the dimension argument, we see that after some positive integer, all $\Phi(U_k, x)$ must be the same, and thus $\Phi^*(x) = \Phi(U_k, x)$ for almost all k, showing that the local holonomy group $\Phi^*(x)$ is a connected Lie subgroup of the holonomy group $\Phi(x)$. If we choose xa, $a \in G$, instead of x as a reference point, we easily see that $\Phi^*(xa) = \mathrm{ad}(a^{-1})\, \Phi^*(x)$ just like $\Phi(xa) = \mathrm{ad}(a^{-1})\, \Phi(x)$ for the holonomy groups. Thus dim $\Phi^*(x)$ is constant on each fiber, and we define $r^*(u) = \dim \Phi^*(x)$ with $\pi(x) = u$. We can easily verify that $r^*(u)$ is an upper semicontinuous function on M; that is, the set of points u with $r^*(u) \leqq \alpha$ is open for any integer α. We have then:

THEOREM 1. *The restricted holonomy group $\Phi^0(x)$ is generated by all $\Phi^*(y)$, where y is an arbitrary point P which can be joined to x by a horizontal curve.*

THEOREM 2. *If dim $\Phi^*(x)$ is constant on P, then $\Phi^0(x)$ coincides with $\Phi^*(x)$, where x is an arbitrary point of P.*

Now we define the infinitesimal holonomy group. At each $x \in P$, let $\mathfrak{m}_0(x)$ be the subspace of the Lie algebra $\mathfrak{g}$ of G which is spanned by all $\Omega_x(X, Y)$, where X and Y are arbitrary horizontal vectors at x. By induction, we define $\mathfrak{m}_k(x)$ to be the subspace of $\mathfrak{g}$ spanned by $\mathfrak{m}_{k-1}(x)$ and by the elements of $\mathfrak{g}$ of the form

$$(V_k)_x V_{k-1} \cdot \dots \cdot V_1 \cdot \Omega(X, Y)$$

where $X, Y, V_1, \dots, V_k$ are arbitrary horizontal vector fields defined in a neighborhood of x. Set $\mathfrak{m}'(x)$ to be the union of all $\mathfrak{m}_k(x)$, $k = 0, 1, 2, \dots$. We can prove that $\mathfrak{m}'(x)$ is a subalgebra of $\mathfrak{g}$, and that it is contained in the Lie algebra of the local holonomy group $\Phi^*(x)$. We define the infinitesimal holonomy group $\Phi'(x)$ to be the Lie subgroup of G which corresponds to the subalgebra $\mathfrak{m}'(x)$. Again, it is not difficult to see that $\Phi'(xa) = \mathrm{ad}(a^{-1})\, \Phi'(x)$, which is essentially a consequence of the fact that the curvature form is of type ad(G). Thus we may define a function on M by $r'(u) = \dim \Phi'(x)$ with $\pi(x) = u$. Contrary to the function $r^*(u)$, $r'(u)$ is lower semicontinuous; that is, the set of points u of M such that $r'(u) \geqq \alpha$ is open for any integer α. We have then:

THEOREM 3. *The restricted holonomy group $\Phi^0(x)$ is generated by all $\Phi'(y)$, where y is an arbitrary point of P which can be joined to x by a horizontal curve.*

THEOREM 4. *If dim $\Phi'(x)$ is constant on P, then $\Phi'(x) = \Phi^*(x) = \Phi^0(x)$ for an arbitrary point x of M.*

THEOREM 5. *In case the bundle $P(M, G)$ and its connection are real analytic,* dim *$\Phi'(x)$ is constant on P.*

In the nonanalytic case, it is easy to give an example that dim $\Phi'(x)$ is not constant on P. We can give a little more precise analysis of this situation. At any rate, this is one of the examples which show an essential distinction between the analytic assumptions and differentiable assumptions as mentioned in the introduction.

1.7. Invariant Connection. Let $P(M, G)$ be a principal fiber bundle, and let K be a Lie transformation group consisting of automorphisms of the bundle P. We ask whether there exists a connection on P which is invariant by every element of K.

We first consider the case of a 1-parameter group ϕ_t of automorphisms of P which leaves a connection Γ invariant. Let X be the vector field induced by ϕ_t on P. Let x be an arbitrary but fixed point of P and consider the orbit $x_t = \phi_t(x)$. Let $u_t = \pi(x_t)$ in M and denote by τ_t the parallel displacement along the curve from u_0 to u_t. Then, for each t, $\tau_t^{-1} x_t$ being a point in the same fiber as $x_0 = x$, we have $\pi_t^{-1} x_t = x_t s_t^{-1}$ for some $s_t \in G$. We thus obtain a curve s_t in G with $s_0 = e$. The first observation is that s_t is a 1-parameter subgroup of G, as follows from the following calculation. Let y_t be the horizontal curve which starts at x_0 and which projects on the curve u_t. Then we have $y_t = x_t s_t$. The tangent vector $\mathbf{y}_t$ of this curve being of the form $\mathbf{y}_t = x_t \mathbf{s}_t + \mathbf{x}_t s_t$, we have for the connection form ω

$$0 = \omega(\mathbf{y}_t) = s_t^{-1} \mathbf{s}_t + \mathrm{ad}(s_t^{-1})\, \omega(\mathbf{x}_t)$$

and hence

$$\mathbf{s}_t s_t^{-1} = -\, \omega(\mathbf{x}_t)$$

The connection form ω being invariant by ϕ_t, we have

$$\omega(\mathbf{x}_t) = \omega(\phi_t \mathbf{x}_0) = \omega(\mathbf{x}_0) = \omega_x(X)$$

Thus $\mathbf{s}_t s_t^{-1} = -A$, where $A = \omega_x(X) \in \mathfrak{g}$, which shows that $s_t = \exp(-tA)$; that is, s_t^{-1} is the 1-parameter subgroup $\exp(tA)$.

Now assume that a connected Lie group K acts as a group of automorphisms on the bundle P, and assume that K is fiber-transitive; that is, for any two fibers of P, there is an element $k \in K$ which maps one into the other. For an arbitrary fixed point x_0 of P, let $u_0 = \pi(x_0)$ as before. By assumption, for any point u of M, there is an element $k \in K$ which, in the base M, maps u_0 into u. Thus, the base space M is acted upon by K transitively: $M = K/H$, where H denotes the isotropy group at u_0. Since H maps the fiber through x_0 into itself, we can define a mapping ψ of H into the structure group G by setting $h(x_0) = x_0\psi(h)$. The mapping ψ is a homomorphism of H into G. We denote by the same letter ψ the corresponding homomorphism of the Lie algebra $\mathfrak{h}$ into $\mathfrak{g}$.

Under these assumptions, we have the following theorem.

THEOREM. (Wang [1]) *There is a one-one correspondence between the set of K-invariant connections on P and the set of linear mappings Ψ of the Lie algebra $\mathfrak{k}$ of K into $\mathfrak{g}$ which satisfies the following two conditions:* (1) *Ψ coincides with ψ on $\mathfrak{h}$*; (2) *$\Psi[ad(h)] = ad[\psi(h)]\, \Psi$ for every $h \in H$.*

In fact, for any K-invariant connection, we set $\Psi(X) = \omega_{x_0}(X^*)$ for each $X \in k$, where X^* is the vector field on P induced by the 1-parameter group $\phi_t = \exp(tX)$ acting on P. The element $A = \Psi(X)$ has the geometric meaning which we have explained. It is easy to prove that the mapping Ψ has properties (1) and (2). Conversely, given such a mapping Ψ of $\mathfrak{k}$ into $\mathfrak{g}$, we can define the horizontal subspace at x_0 to be the set of tangent vectors of the form $(X^*)_{x_0} - (\Psi(X))^*_{x_0}$, where X runs through $\mathfrak{k}$, and $(\Psi(X))^*$ is the fundamental vector field corresponding to $\Psi(X) \in \mathfrak{g}$. Using (1) and (2), we can show easily that by translating this horizontal subspace at x_0 to any point of P by the actions of K and the structure group G, we get a K-invariant connection on P.

In particular, in the case where $M = K/H$ is reductive, let $\mathfrak{k} = \mathfrak{m} + \mathfrak{h}$, with $\text{ad}(H)\, \mathfrak{m} = \mathfrak{m}$. Then, the mapping Ψ, which is 0 on $\mathfrak{m}$, and coincides with ψ on $\mathfrak{h}$, gives rise to an invariant connection, which we call the canonical connection on P (corresponding to the choice of $\mathfrak{m}$). This connection has a remarkable property. Let X be an arbitrary element in the subspace $\mathfrak{m}$ and let $\phi_t = \exp(tX)$. By the geometric interpretation of $\Psi(X) = 0$, it follows that the 1-parameter subgroup s_t consists of the identity element so that the orbit $x_t = \phi_t(x_0)$ itself is a horizontal curve. This means that *the parallel displacement along the curve $u_t = \pi(x_t)$ of the fiber over u_0 coincides with the 1-parameter group of transformations ϕ_t.* This is a generalization of a basic property of the canonical linear connection on a symmetric homogeneous space which we shall discuss later.

2. Linear Connections

2.1. Basic Concepts. Let M be a differentiable manifold. The notion of a linear connection will enable us to define, for each piecewise differentiable curve τ from a point u to another point v, a well determined linear isomorphism of the tangent space $T_u(M)$ onto $T_v(M)$, called the parallel displacement along τ. We define a *linear connection* on M as connection in the principal fiber bundle $L(M)$, called the *bundle of linear frames*, which we explain: At each point u of M, a frame is an ordered basis $x = (X_1, ..., X_n)$, $n = \dim M$, of the tangent space $T_u(M)$.

Let $L(M)$ be the set of all frames at all points of M. If U is a coordinate neighborhood in M with local coordinates $(u^1, ..., u^n)$, then an arbitrary frame at any point of U can be expressed as

$$X_i = \sum_{j=1}^{n} a_i^j \left(\frac{\partial}{\partial u^j}\right), \qquad \det(a_i^j) \neq 0,$$

$i = 1, 2, ..., n$. It is possible to introduce a differentiable structure in $L(M)$ in such a way that (u^i, a_i^j), $i, j = 1, ..., n$, form local coordinates in the set $\pi^{-1}(U)$ of $L(M)$, where π is the projection of $L(M)$ onto M defined by $\pi(x) = u$ for a frame x at u. We can let the general linear group $GL(n, R)$ act on $L(M)$ by

$$x \cdot a = \left(\sum_{j=1}^{n} a_1^j X_j, ..., \sum_{j=1}^{n} a_n^j X_j\right)$$

$x = (X_1, ..., X_n)$ and $a = (a_i^j) \in GL(n, R)$ and show that $L(M)$ forms for a principal fiber bundle over M with structure group $GL(n, R)$.

We briefly indicate how the parallel displacement of tangent spaces is obtained from a connection in $L(M)$. Let τ be a curve from u to v in M. For an arbitrary point $x = (X_1, ..., X_n)$ in $L(M)$ over u, we can take a unique horizontal curve τ^* in $L(M)$ which starts at x and which projects on τ. The end point of τ^* is a certain frame $y = (Y_1, ..., Y_n)$ at v. Let τ be the linear mapping of $T_u(M)$ onto $T_v(M)$ which maps X_i upon Y_i for all $i = 1, 2, ..., n$. This mapping is independent of the choice of a frame x at u because if we take xa instead of x, then the end point of the horizontal curve will change into ya, and the linear mapping determined by the frames xa and ya is the same as τ.

Once the parallel displacement of tangent vectors is defined, we define a *geodesic* as a curve u_t in M whose tangent vectors $\mathbf{u}_t$ are parallel along the curve. We can also define the parallel displacement of tensors of various types. Let τ be the parallel displacement of $T_u(M)$ onto $T_v(M)$. We can extend τ to a linear isomorphism of the tensor algebra over $T_u(M)$ onto the tensor algebra over $T_v(M)$ (by using the inverse of the transpose of τ for the covariant tensors). We denote by the same letter τ this isomorphism of the tensor algebras.

We now introduce the concept of *covariant differentiation*. Let X be a vector field defined on M or in a neighborhood of a point u. Let K be an arbitrary tensor field, say, of type (r, s), defined in a neighborhood of u. Let u_t be the integral curve of X with origin u. Denoting by τ_t the parallel displacement (of vectors and tensors) along the curve u_t

from $u_0 = u$ to u_t, $\tau_t^{-1}(K)_{u_t}$ is a tensor of type (r, s) at u. We define

$$(\nabla_X K)_u = \lim_{t \to 0} (1/t) [\tau_t^{-1} (K)_{u_t} - (K)_u]$$

which is again a tensor of type (r, s) at u. This tensor is called the covariant derivative of the tensor field K with respect to X at u. $(\nabla_X K)_u$ can be defined essentially in the same way in case K is defined only along the integral curve of X through u. It is also to be noted that $(\nabla_X K)_u$ depends only on the value of X at u, so that $\nabla_X K$ makes sense if $X \in T_u(M)$, and if K is defined in a neighborhood of u.

If K is a tensor field of type (r, s), then the covariant derivative $\nabla_X K$ is of the same type. K and $\nabla_X K$ can be regarded as a linear mapping of $T_u(M) \times \ldots \times T_u(M)$ (s-times product) into the vector space of contravariant tensors of degree r at u. We define the *covariant differential* ∇K of the tensor field K as a tensor field of type $(r, s + 1)$ which, as a linear mapping of $T_u(M) \times \ldots \times T_u(M)$ [$(s + 1)$-times product] into the vector space of contravariant tensors of degree r, maps $(X_1, X_2, \ldots, X_{r+1})$ upon $(\nabla_{X_1} K)(X_2, \ldots, X_r)$.

For a tangent vector X at a point u, covariant differentiation ∇_X is a derivation of the algebra of tensor fields on M into the tensor algebra at u, namely, a linear mapping which satisfies the following conditions:

(a) It preserves the type.

(b) $\nabla_X(K \otimes L) = (\nabla_X K) \otimes L + K \otimes (\nabla_X L)$, where $\otimes$ denotes the tensor product.

(c) $\nabla_X(CK) = C(\nabla_X K)$ for any contraction C on a fixed pair of upper and lower indices. Let us recall that a linear mapping of a vector space V into itself can be extended to a derivation of the tensor algebra over V. Applying this, we see that a linear endomorphism A of $T_u(M)$ can be extended to a derivation of the tensor algebra at u, or, it can be regarded as a derivation of the algebra of tensor fields into the tensor algebra at u.

We have the following theorem, due to Kostant [1].

THEOREM. *Let M be a differentiable manifold with a linear connection. Every derivation of the algebra of tensor fields on M into the tensor algebra at a point $u \in M$ is of the form $\nabla_X + A$, where X is a certain tangent vector at u, and A is the derivation which is given by a certain linear endomorphism of the tangent space $T_u(M)$.*

2.2. Curvature and Torsion. Let $L(M)$ be the bundle of linear frames over a differentiable manifold M of dimension n. If V_n denotes a vector space with a fixed basis $(e_1, ..., e_n)$, every element $x \in L(M)$ may be regarded as a linear isomorphism of V_n onto the tangent space $T_u(M)$, where $u = \pi(x)$, which maps each e_i upon X_i if x is a frame $(X_1, ..., X_n)$ at u. The structure group $GL(n, R)$ and its Lie algebra $\mathfrak{gl}(n, R)$ act linearly on V_n; if $x \in L(M)$, and $a \in GL(n, R)$, then $x . (a\xi) = (xa) . \xi$ for every $\xi \in V_n$.

Assume now that there is a linear connection Γ on M. The curvature form Ω of the corresponding connection in $L(M)$ is a $\mathfrak{gl}(n, R)$-valued 2-form on $L(M)$ of type ad$[GL(n, R)]$. We can derive the classical curvature tensor field R of the linear connection Γ as follows. For each $u \in M$ and $X, Y \in T_u(M)$, we define $R(X, Y)$ to be the linear endomorphism of $T_u(M)$ given as the composite $x . \Omega_x(X^*, Y^*) . x^{-1}$, where x is an arbitrary point of $L(M)$ with $\pi(x) = u$ and X^*, Y^* are horizontal vectors at x with $\pi(X^*) = X$ and $\pi(Y^*) = Y$. It is easy to verify that the linear endomorphism $x . \Omega_x(X^*, Y^*) . x^{-1}$ is independent of the choice of x with $\pi(x) = u$. Now R is the tensor field of type (1,3) which associates to a triple (X, Y, Z) of tangent vectors at u the vector $R(X, Y) . Z$. Since $\Omega(Y^*, X^*) = - \Omega(X^*, Y^*)$, it follows that $R(Y, X) = R(X, Y)$.

A special feature of the linear connection compared with a connection in an arbitrary fiber bundle is that there is another natural form, called the torsion form, which will give the classical torsion tensor field on M. First, we define a V_n-valued 1-form θ on $L(M)$ independently of any connection. At each $x \in L(M)$, we set $\theta(X) = x^{-1}. \pi(X)$ for an arbitrary tangent vector X at x. The form θ satisfies $R_a^*\theta = a^{-1}\theta$ for every $a \in GL(n, R)$. When a linear connection is given, its *torsion form* is defined by $\Theta_x(X, Y) = (d\theta)_x (hX, hY)$ for any tangent vectors X and Y at x. The torsion tensor field T of the linear connection is derived as follows. For each u, $T_u(X, Y) = x. \Theta_x(X^*, Y^*)$, where x, X^*, and Y^* have the same meanings as in the case of the curvature tensor. The torsion tensor field is the tensor field of type (1,2) which associates to a pair of tangent vectors X, Y at u the tangent vector $T(X, Y)$. We have $T(Y, X) = - T(X, Y)$.

The torsion and curvature tensor fields are the basic invariants of a given linear connection. In many problems concerning a linear connection, however, the successive covariant differentials $\nabla T, \nabla^2 T, ..., \nabla^k T, ..., \nabla R, \nabla^2 R, ..., \nabla^k R, ...$ also come in. When the torsion tensor T is identically zero (as in the case of a Riemannian connection which we discuss later), the curvature tensor R and its successive covariant differentials determine the geometric properties of the given linear connection, at least locally. We shall see later how these tensors determine the holonomy group and the automorphism group of a given linear connection.

2.3. Holonomy Groups. Given a linear connection Γ on M, the holonomy group of the connection in $L(M)$ is defined as a subgroup of $GL(n, R)$ whenever a reference point x is fixed. We may also consider the holonomy group of Γ as a group of linear transformations of $T_u(M)$ when a reference point u is fixed in M. In fact, given a closed curve τ at u, the parallel displacement along τ is a linear transformation of $T_u(M)$, and the totality of these linear transformations for all closed curves forms the holonomy group. The restricted holonomy group is the subgroup consisting of parallel displacements along all closed curves which are homotopic to zero.

We can define, as a special case of the general theory we treated in Section 1, the local holonomy group and the infinitesimal holonomy group at u. The former is defined by taking closed curves at u which stay in a sufficiently small neighborhood of u. A rigorous definition is obtained easily. The infinitesimal holonomy group for a linear connection, on the other hand, can be defined using the curvature tensor and all its successive covariant differentials [that is, without going to the bundle $L(M)$] in the following way. At each u, consider linear endomorphisms of $T_u(M)$ of the form $R(X, Y)$, $(\nabla_Z R)(X, Y)$, $(\nabla_W \nabla_Z R)(X, Y)$, ... (all covariant derivatives), where $X, Y, Z, W, \dots$ are arbitrary tangent vectors at u. All these linear endomorphisms span a subalbegra $\mathfrak{m}'$ of the Lie algebra consisting of all linear endomorphisms of $T_u(M)$. The Lie subgroup generated by $\mathfrak{m}'$ is the infinitesimal holonomy group at u (regarded as a subgroup of the group of all linear transformations of $T_u(M)$). The results in Section 1.6 are valid, of course, in this case. The main result is that if the infinitesimal holonomy group has the same dimension at every point u of M (which is the case when the manifold M and its linear connection are analytic), then the restricted holonomy group is equal to the infinitesimal holonomy group at every point. This means that in the analytic case, the curvature tensor and all its successive covariant differentials at a single point determine the restricted holonomy group completely.

2.4. Invariant Linear Connections on a Homogeneous Space. Let M be a homogeneous space G/H of a connected Lie group G over a closed subgroup H. The existence and properties of invariant linear connections on G/H (invariant by the natural action of G) were studied by Nomizu [1]. A generalization of this study has since been made by Wang [1]. Here we shall derive the linear case from the general results which we discussed in Section 1.7.

Let o be the origin of G/H, namely, the point represented by the coset H. Let us assume that G/H is reductive; that is, the Lie algebra $\mathfrak{g}$ is the direct sum of the subalgebra $\mathfrak{h}$ corresponding to H and a subspace $\mathfrak{m}$ such that $\mathrm{ad}(H)\,\mathfrak{m} = \mathfrak{m}$. By considering the bundle G over G/H with structure group H acting on G to the right, we may regard $\mathfrak{m}$ as the horizontal subspace at e of a G-invariant connection on the bundle G (G acting on itself to the left). Since the projection of G onto G/H maps $\mathfrak{m}$ isomorphically onto $T_0(G/H)$, we may identify $\mathfrak{m}$ with $T_0(G/H)$. Every element $h \in H$ leaves o invariant and induces a linear transformation $\tilde{h}$ of $T_0(G/H)$. The set of all these linear transformations forms what is called the linear isotropy group $\tilde{H}$ at o. The structure group $GL(n, R)$ of the bundle of linear frames $L(M)$ can be regarded as the group of all linear transformations of $T_0(G/H)$ (by fixing a certain basis in it). The mapping $h \in H \rightarrow \tilde{h} \in GL(n, R)$ is essentially the homomorphism ψ in the notation of Section 1.7. In terms of $\mathfrak{m}$, the mapping $\psi(h)$ is the restriction of $\mathrm{ad}(h)$ in $\mathfrak{g}$ to the subspace $\mathfrak{m}$ because for any $X \in \mathfrak{m}$, $\pi(hX) = \pi((\mathrm{ad}\, h)\, X) \,.\, h = \pi(\mathrm{ad}\,(h)\, X)$ so that $\tilde{h}X = \mathrm{ad}(h)\, X$ by identifying $\mathfrak{m}$ and $T_0(G/H)$. The Lie algebra homomorphism ψ of $\mathfrak{h}$ into $\mathfrak{gl}(n, R)$ is then given by $\psi(X) = \mathrm{ad}(X)$ in $\mathfrak{m}$.

Now according to a general result, there is a 1-1 correspondence between the set of G-invariant linear connections on G/H and the set of linear mappings Ψ of $\mathfrak{g}$ into the vector space of all linear endomorphisms of $\mathfrak{m}$ such that $\Psi(X) = \mathrm{ad}(X)$ for $X \in h$ and $\Psi(\mathrm{ad}(h)\, X) = \mathrm{ad}(h)\Psi(X)\,\mathrm{ad}(h)^{-1}$ for every $X \in \mathfrak{m}$ and $h \in H$. Thus, it is essentially the mapping Ψ of $\mathfrak{m}$ into the vector space of all linear endomorphisms of $\mathfrak{m}$ which determines a G-invariant linear connection on G/H.

When a G-invariant linear connection Γ is given on G/H, all the tensor fields on G/H which naturally arise from it are invariant by G and thus determined by their values at the origin o. There is a 1-1 correspondence between the G-invariant tensor fields on G/H and the set of tensors over the vector space $\mathfrak{m}$ which are invariant by $\mathrm{ad}(H)$ acting on $\mathfrak{m}$. In particular, the torsion tensor T and the curvature tensor R of Γ can be expressed as follows:

$$T(X, Y) = \Psi(X)Y - \Psi(Y)X - [X, Y]_m$$
$$R(X, Y) = [\Psi(X), \Psi(Y)] - \Psi([X, Y]_m) - \mathrm{ad}([X, Y]_h)$$

where $X, Y \in \mathfrak{m}$ and $[X, Y]_m$ (resp. $[X, Y]_h$) denotes the $\mathfrak{m}$-component (resp. $\mathfrak{h}$-component) of the element $[X, Y]$ in $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$. Similarly, the covariant derivatives of these tensor fields can be expressed algebraically in terms of $\mathfrak{m}$. For example,

$$(\nabla_Z R)(X, Y) = [\Psi(Z), R(X, Y)] - R(\Psi(Z)X, Y) - R(X, \Psi(Z)Y)$$

for all $X, Y, Z \in \mathfrak{m}$, where the first term of the right hand side is the bracket of two linear endomorphims of $\mathfrak{m}$.

We note that a homogeneous space G/H is a real analytic manifold, and that any G-invariant linear connection on G/H is real analytic. Thus, we can apply the results on the infinitesimal holonomy groups to the determination of the holonomy group of a G-invariant linear connection on G/H. The general result in this case appears to be the following.

THEOREM. (Nomizu [2]) *The holonomy algebra of a G-invariant linear connection Γ on G/H determined by Ψ is equal to the smallest Lie algebra $\mathfrak{h}^*$ of endomorphisms of $\mathfrak{m}$ such that* (1) $R(X, Y) \in \mathfrak{h}^*$ *for all* $X, Y \in \mathfrak{m}$, and (2) $[\Psi(X), \mathfrak{h}^*] \subset \mathfrak{h}^*$ *for every* $X \in \mathfrak{m}$.

The proof is purely algebraic and consists of showing that h^* coincides with the Lie algebra of the infinitesimal holonomy group which is spanned by $R(X, Y)$, $(\nabla_Z R)(X, Y)$, $(\nabla_W \nabla_Z R)(X, Y)$, ...

2.5. Symmetric Homogeneous Spaces. An interesting class of homogeneous spaces is the following. Let G be a connected Lie group, and let σ be an automorphism of G of period 2. Let H_σ be the set of all elements x of G such that $x^\sigma = x$. H_σ is a closed subgroup. Now a homogeneous space G/H is called a *symmetric homogeneous space* (for σ) if the closed subgroup H lies between H and its identity component.

Let G/H be a symmetric homogeneous space for the involutive automorphism σ. Since σ leaves every element of H invariant, it induces an involutive diffeomorphism σ_0 of G/H onto itself, which has the origin o as an isolated fixed point. In the Lie algebra $\mathfrak{g}$ of G, let $\mathfrak{m}$ be the set of elements $X \in \mathfrak{g}$ such that $X^\sigma = -X$, and let $\mathfrak{h}$ be the set of elements $X \in \mathfrak{g}$ such that $X^\sigma = X$, where σ again denotes the automorphism of

$\mathfrak{g}$ induced by the group automorphism σ. We see that $\mathfrak{g}$ is the direct sum of the subspace $\mathfrak{m}$ and the subalgebra $\mathfrak{h}$. By assumption on H, it follows that $\mathfrak{h}$ is nothing but the Lie subalgebra corresponding to H. For a symmetric homogeneous space G/H, we use this natural decomposition $\mathfrak{g} = \mathfrak{m} + h$, where $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$.

Let us consider an invariant linear connection on symmetric G/H which is given by Ψ such that $\Psi(X) = 0$ for every $X \in \mathfrak{m}$. This is a special case of what we called the canonical connection in Section 1.7. By the formulas in the preceding section, we see that the torsion tensor T is identically zero, and that the curvature tensor R is given by

$$R(X, Y) = -\,ad([X, Y])$$

for $X, Y \in \mathfrak{m}$, where $[X, Y] \in \mathfrak{h}$. *The covariant differential ∇R of the curvature tensor is equal to zero.* As a consequence of the theorem in Section 2.4, the holonomy algebra is generated by $\mathrm{ad}([X, Y])$ for all $X, Y \in \mathfrak{m}$. If we denote by $\mathfrak{h}_1$ the linear subspace of $\mathfrak{h}$ spanned by all $[X, Y]$, $X, Y \in \mathfrak{m}$, we easily see that $\mathfrak{h}_1$ is a subalgebra, in fact, an ideal, of $\mathfrak{h}$.

THEOREM 1. *The holonomy algebra of the canonical connection on symmetric G/H is an ideal of $ad(\mathfrak{h})$ acting on $\mathfrak{m}$. In other words, the restricted holonomy group is contained in the isotropy group H.*

We have also:

THEOREM 2. *The canonical connection is invariant by the symmetry σ_0 and by the symmetry around any point of G/H.*

Some of the remarkable properties of the canonical connection on symmetric G/H are shared by certain connections on a reductive homogeneous space G/H. Let us fix a decomposition $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$ where $\mathrm{ad}(H)\,\mathfrak{m} = \mathfrak{m}$. *The canonical connection of the first kind* on G/H is the invariant linear connection determined by Ψ such that

$$\Psi(X)\,Y = (\tfrac{1}{2})\,[X, Y]_{\mathfrak{m}}, \qquad X, Y \in \mathfrak{m}$$

The torsion tensor of this connection is zero, while the expression of the curvature torsion is rather complicated in terms of the bracket operation in $\mathfrak{g}$. For each $X \in \mathfrak{m}$, the 1-parameter subgroup $\exp(tX)$ projects on a geodesic in G/H. We have also the *canonical connection of the second kind* which is the invariant linear connection determined by the mapping Ψ such that $\Psi(X) = 0$ for every $X \in \mathfrak{m}$. The torsion tensor T is given by $T(X, Y) = -[X, Y]_{\mathfrak{m}}$, while the curvature tensor R is given by $R(X, Y) = -\text{ad}([X, Y]_{\mathfrak{h}})$. The restricted holonomy group is contained in the linear isotropy group just as in the case of the canonical connection of symmetric G/H. In the case of a symmetric homogeneous space, the canonical connections of the first kind and of the second kind coincide because then $[X, Y]_{\mathfrak{m}} = 0$ for $X, Y \in \mathfrak{m}$. This explains why the canonical connection on symmetric G/H behaves so nicely.

THEOREM 3. *For the canonical connection of the second kind on reductive G/H, the covariant differentials of the torsion tensor and the curvature tensor are both zero: $\nabla T = 0$ and $\nabla R = 0$.*

Conversely, the property $\nabla T = 0$ and $\nabla R = 0$ characterizes a reductive homogeneous space (with canonical connection of the second kind) at least locally.

THEOREM 4. (Nomizu [1]) *Let M be a differentiable manifold with a linear connection such that $\nabla T = 0$ and $\nabla R = 0$. Then M is locally isomorphic with a certain reductive homogeneous space with canonical connection of the second kind.*

THEOREM 5. (Kobayashi [1]) *If M is simply connected and complete, moreover, then M is isomorphic with a reductive homogeneous space G/H with canonical connection of the second kind.*

This last theorem is proved by using an interpretation of the conditions $\nabla T = 0$ and $\nabla R = 0$ in terms of the bundle of frames over M (also see Nomizu [5, pp. 69-73]). Another characterization of such spaces was given by Kostant [3].

A supplementary result is the following. *Let G/H be a homogeneous space with an invariant linear connection. If the restricted holonomy group is irreducible and contained in the linear isotropy group H, then $\nabla T = 0$ and $\nabla R = 0$* (Nomizu [4]).

3. Riemannian Connections

3.1. Riemannian Metrics and Orthogonal Bundles. Let M be a differentiable manifold of dimension n. By a Riemannian metric on M, we mean a positive definite symmetric covariant tensor field of degree 2 on M. Thus, a Riemannian metric g defines, at each point u of M, a positive definite inner product in the tangent space $T_u(M)$ which we shall denote by $g_u(X, Y)$, where $X, Y \in T_u(M)$.

Let g be a Riemannian metric on M. An orthogonal frame at $u \in M$ is an orthonormal basis $(X_1, ..., X_n)$ of $T_u(M)$ with respect to the inner product determined by g. The set of all orthonormal frames at all points of M forms a subset $O(M)$ of the bundle of frames $L(M)$. We can make $O(M)$ into a principal fiber bundle over M with orthogonal group $O(n)$ as structure group, just in the same way as we made $L(M)$ into a principal fiber bundle with general linear group $GL(n, R)$ as structure group. Indeed, $O(M)$ is then a subbundle of $L(M)$. This means that a given Riemannian metric g on M gives a reduction of $L(M)$ to an orthogonal bundle with structure group $O(n)$.

Conversely, assume that we are given a reduction of $L(M)$ to an orthogonal bundle $O(M)$. We regard, as before, each point x of $L(M)$ as a linear isomorphism of the vector space V_n with fixed basis $(e_1, ..., e_n)$ onto $T_u(M)$ with $\pi(x) = u$. In V_n, we consider the inner product which makes $(e_1, ..., e_n)$ an orthonormal basis so that $(e_i, e_j) = \delta_{ij}$ (Kronecker's delta). Now for each point $u \in M$, we take an arbitrary point $x \in O(M)$ over u and define $g_u(X, Y) = (x^{-1}X, x^{-1}Y)$. This value is independent of the choice of $x \in O(M)$ over u, because if we choose $y \in O(M)$ over u, then $y = xa$ for some $a \in O(n)$, and hence $(y^{-1}X, y^{-1}Y) = (a^{-1}x^{-1}X, a^{-1}x^{-1}Y) = (x^{-1}X, x^{-1}Y)$. Thus, by making use of the given orthogonal bundle $O(M)$, we can define an inner product in each tangent space $T_u(M)$ and get a Riemannian metric on M.

The above argument shows that a choice of a Riemannian metric on M corresponds to a choice of a reduction of the structure group $GL(n, R)$ to $O(n)$. Assuming that M satisfies the second axiom of countability, such a reduction is always possible because the factor space $GL(n, R)/O(n)$ is diffeomorphic with Euclidean space of suitable dimensions. This gives an existence proof for Riemannian metrics. Another proof can be provided by using a partition of unity. Let $\{U_k\}$ be a countable locally finite covering of M by open sets diffeomorphic with a cube in R^n, and let $\{\phi_k\}$ be a partition of unity subordinated to $\{U_k\}$, namely, a family of nonnegative differentiable functions such that each ϕ_k has a compact support contained in some U_j and that $\Sigma_k \phi_k = 1$ on M. We may choose any Riemannian metric g_k in each U_k and set $g = \Sigma_k \phi_k g_k$. At each $u \in M$, and for any $X, Y \in T_u(M)$, $g(X, Y) = \Sigma_k \phi_k(u)\, g_k(X, Y)$ is well defined as a finite sum and gives an inner product in $T_u(M)$, thus defining a Riemannian metric g on M.

We shall mention two examples.

(a) Let M be a Riemannian manifold, namely, a differentiable manifold with a fixed choice of Riemannian metric g. Let N be a differentiable manifold immersed in M; that is, a differentiable manifold which admits a differentiable mapping f into M which is regular at every point of N (when f is 1-1, N is an imbedded submanifold). For each $u \in N$ we define the inner product in $T_u(N)$ by setting $h_u(X, Y) = g_u(fX, fY)$ for $X, Y \in T_u(N)$. The Riemannian metric h on N so obtained is called the induced metric. In particular, any immersed manifold in Euclidean space has a naturally induced metric. In case of a 2-dimensional surface immersed in R^3, it is the metric considered in classical surface theory.

(b) Let G/H be a homogeneous space of a connected Lie group over a compact subgroup H. At the origin o represented by the coset H, the tangent space $T_o(G/H)$ is acted upon by the linear isotropy group $\tilde{H}$. Since H is compact, we can choose an inner product g_o in $T_o(G/H)$ which is invariant by $\tilde{H}$. For an arbitrary point u of G/H, we choose $a \in G$ with $u = a.o$ and define $g_u(X, Y) = g_o(a^{-1}X, a^{-1}Y)$ for $X, Y \in T_u(M)$. This value is independent of the choice of $a \in G$ with $a.o = u$. The Riemannian metric g on G/H so obtained is clearly invariant by every $a \in G$. With a fixed choice of G-invariant Riemannian metric, G/H will be called a Riemannian homogeneous space. In particular, if G is a compact Lie group, it may be looked upon as a homogeneous space of the direct product $G \times G$ over the diagonal subgroup $H = \{(a, a) \mid a \in G\}$ in such a manner that $(a, b) \in G \times G$ acts on G by $x \to axb^{-1}$ for every $x \in G$. An invariant metric on $(G \times G)/H$ is then a Riemannian metric on G which is invariant by left as well as right translations. In the case of the compact semisimple Lie group G, there is a particular important choice of bi-invariant Riemannian metric on it. Let $\mathfrak{g}$ be the Lie algebra of G as identified with the tangent space $T_e(G)$ at the identity element. Let $\phi(X, Y)$ be the Killing form on $\mathfrak{g}$ defined by $\phi(X, Y) = \text{trace}\ [\text{ad}(X)\,\text{ad}(Y)]$ for $X, Y \in \mathfrak{g}$. Since G is compact and semisimple, we know that ϕ is a negative definite symmetric bilinear form on $\mathfrak{g} \times \mathfrak{g}$. By choosing $g_e = -\phi$, and translating the inner product g_e to other points of G by left translations, we get a Riemannian metric on G. This metric is invariant by right translations too, because the Killing form ϕ and hence g_e is invariant by $\text{ad}(G)$ acting on $\mathfrak{g}$.

Homogeneous spaces and, in particular, compact Lie groups provided with suitable invariant Riemannian metrics are important objects of study in differential geometry in connection with the theory of Lie groups.

Let M be a differentiable manifold and g a fixed Riemannian metric on M. An arbitrary connection in the orthogonal bundle $O(M)$ associated to g is called a *metric connection* on M. The geometric significance is that such a connection can be considered as a connection in the bundle of frames $L(M)$ (hence as a linear connection on M) and the parallel displacement of tangent vectors with respect to that connection is an isometric mapping between tangent spaces each provided with the inner product determined by g. This last property is equivalent to the condition $\nabla g = 0$, that is, the covariant derivative of the metric tensor g is equal to 0.

The choice of a metric connection for a given Riemannian metric is by no means unique. The first important theorem in Riemannian geometry is the following.

THEOREM. *Given a Riemannian metric g, there is a unique metric connection whose torsion tensor is zero.*

The unique connection is called the *Riemannian connection* (or *Levi-Civita connection*) associated with the given Riemannian metric g. Unfortunately, the known proofs of this basic theorem all involve a certain amount of calculations, in terms of coordinates or otherwise. It will be interesting, at least esthetically, to find an intrinsic proof which accounts for the condition that the torsion is zero. Here we shall indicate how the Riemannian connection can be obtained in terms of covariant differentiation. Given g, we define for any vector fields X and Y on M the operation of covariant differentiation $\nabla_X Y$ by the following formula:

$$2g(\nabla_X Y, Z) = X \cdot g(Y, Z) + Y \cdot g(X, Z) - Z \cdot g(X, Y) \\ + g([X, Y], Z) + g([Z, X], Y) + g(X, [Z, X])$$

where Z is an arbitrary vector field on M. We can show that $\nabla_X Y$ determined in this way is the covariant differentiation with respect to a unique Riemannian connection.

Once the Riemannian connection is introduced, we can talk about its holonomy group. In the orthogonal bundle $O(M)$ with structure group $O(n)$, the holonomy group of the Riemannian connection is obtained as a subgroup of $O(n)$. In terms of the parallel displacement of tangent vectors, the holonomy group with a reference point $u \in M$

consists of orthogonal transformations of $T_u(M)$ provided with the inner product g_u. Orthogonality allows us to study the holonomy group much more closely than the case of a general linear connection as we shall see later.

3.2. Basic Concepts. On a differentiable manifold M with a linear connection, it is known that we can choose the so-called normal coordinates. For any point $u \in M$, there is a neighborhood U of u with local coordinates $(u^1, ..., u^n)$ with origin u such that every geodesic in U issuing from u can be expressed by a system of linear equations $u^i = a^i t$, $i = 1, 2, ..., n$, where $(a^1, ..., a^n)$ are a set of constants which determine the direction of the geodesic, and t is the parameter. This fact remains valid in a Riemannian manifold. A Riemannian manifold has a stronger property of having convex neighborhoods in the following precise sense.

THEOREM. *Eeach point u of a Riemannian manifold M has a neighborhood W such that* (1) *any two points v and w can be joined by a geodesic C which is the shortest among all geodesics in M joining v and w*; (2) *such a geodesic C is unique and lies in W*; (3) *C is a unique geodesic lying in W which joins v and w.*

Stated and proved in this precise form, Theorem 1 has many applications. For example, we can prove the existence of a simple covering of a differentiable manifold. For any open covering $\{U_i\}$ of an arbitrary differentiable manifold M, there is an open covering $\{V_i\}$ such that (a) the closure of V_i is compact and contained in some U_j; (b) $\{V_i\}$ is locally finite (that is, every compact subset of M meets only a finite number of V_i's; (c) any nonvoid intersection of a finite number of V_i's is diffeomorphic with an open cell in R^n. The proof consists of using any Riemannian metric on M and choosing V_i's satisfying (a) and (b) by standard technique with the requirement that each of them is convex. If each V_i is chosen to be convex in the sense of the preceding theorem, then any nonvoid intersection is still convex in the sense that any two points can be joined by a unique shortest geodesic in M which lies in the intersection. This implies that the intersection is diffeomorphic with an open cell. The existence of a simple covering was utilized by A. Weil [1] in his proof of the de Rham theorem on cohomology of differential forms.

Another important concept on a Riemannian manifold is that of distance. For any two points u and v of M, let $d(u, v)$ be the infinimum of the length of all curves joining u and v; the length of a curve $u(t)$, $a \leqq t \leqq b$, being defined by $\int_a^b g(\mathbf{u}_t, \mathbf{u}_t)^{1/2}\, dt$ where $\mathbf{u}_t$ denote the tangent vectors of the curve. We can prove that $d(u, v)$ satisfies the axioms of distance and that it gives the same topology on M as the original manifold topology.

The natural metric d on a Riemannian manifold M has close relationship with differential geometric properties of M. First of all, *the Riemannian metric g itself can be canonically reconstructed from the metric d*, as was shown by Palais [2]. In particular, any homeomorphism of M onto itself which preserves the metric d turns out to be a diffeomorphism which preserves the Riemannian metric g (result originally due to Myers and Steenrod [1]). The central idea for the proof is the following formula which shows that the Riemannian metric can be locally approximated by the usual Euclidean metric. Let o be an arbitrary point of M and let $x(s)$ and $y(s)$ be two geodesics in the directions of unit vectors X and Y, respectively, where s is the arc length on each geodesic. Let $L(s)$ be the length of the unique geodesic joining the two points $x(s)$ and $y(s)$ for sufficiently small s. Then the angle θ between X and Y is given by

$$\sin \frac{\theta}{2} = \lim_{s \to 0} \frac{L(s)}{2s}$$

When s is sufficiently small so that $x(s)$ and $y(s)$ stay in a convex neighborhood of o, we have of course $d(o, x(s)) = d(o, y(s)) = s$ and $L(s) = d(x(s), y(s))$. The above formula expresses the angle θ in terms of the metric d. Once the angle θ is obtained, the inner product $g(X, Y)$ is equal to $\cos \theta$.

A similar result is this: A continuous curve $u(t)$ in a metric space is called a segment if

$$d(u(t_1), u(t_2)) + d(u(t_2), u(t_3)) = d(u(t_1), u(t_3))$$

for any values $t_1 \leqq t_2 \leqq t_3$ of the parameter. In a Riemannian manifold M, a segment is a geodesic.

In order to deal with global properties of M, the most important concept is that of completeness. A Riemannian manifold M is called *complete* if the metric d is complete; that is, if every Cauchy sequence with respect to d has a limit point. *This condition is equivalent to either of the following two conditions (Hopf-Rinow)*: (1) *every geodesic of M*

can be extended indefinitely with respect to its arc-length; (2) *every bounded set (with respect to d) is relatively compact.*

If M is complete, any two points u, v of M can be joined by a geodesic whose length is equal to d(u, v). (For the proof, see de Rham [1], Appendix.)

Another useful property of a complete Riemannian manifold is that it cannot be imbedded as an open submanifold of another Riemannian manifold.

A Riemannian homogeneous space $M = G/H$ (H compact) is always complete, because the metric d is also G-invariant, and hence, M is uniformly locally compact; that is, there exists $\varepsilon > 0$ such that the ε-neighborhood of every point of M is relatively compact.

Another condition which is often imposed on a Riemannian manifold M is that of simply-connectedness. When M is not simply connected, we take the simply connected covering manifold $\tilde{M}$ of M and introduce a natural Riemannian metric on $\tilde{M}$ by using the projection $\pi: \tilde{M} \to M$, which is an immersion mapping. The Riemannian properties of $\tilde{M}$ and M are locally the same so that, for example, a geodesic in $\tilde{M}$ projects on a geodesic of M and, conversely, a geodesic of M can be lifted to a geodesic in $\tilde{M}$. The restricted holonomy group of M is the same as the holonomy group of $\tilde{M}$.

3.3. Holonomy Groups. We now study the holonomy groups of a Riemannian manifold M in detail. Let Φ be the holonomy group of M with reference point $u \in M$ which is a group of orthogonal transformations of $T_u(M)$. Suppose that $T_u(M)$ admits a nontrivial subspace T'_u which is invariant by Φ. For any point $v \in M$, let τ be an arbitrary curve which joins u and v. The parallel displacement of T'_u along τ gives rise to a subspace T'_v of $T_v(M)$. Since T'_u is invariant by Φ, the subspace T'_v is determined independently of the choice of τ. We thus get a distribution T' which assigns to each point $v \in M$ the subspace T'_v. The distribution T' is differentiable and involutive. In order to prove that it is involutive, we have to show that if vector fields X and Y belong to T', so does $[X, Y]$. Since the torsion tensor $T[X, Y] = \nabla_X Y - \nabla_Y X - [X, Y]$ is zero, it is sufficient to show that $\nabla_X Y$ (as well as $\nabla_Y X$) belongs to T'. Now if Y belongs to T', then $\nabla_X Y$ belongs to T' for an arbitrary vector field X. In fact, $(\nabla_X Y)_v$ is equal to $\lim_{t \to 0} t^{-1}(\tau_t^{-1} \cdot Y_{v_t} - Y_v)$, where v_t is the integral curve of X with origin v, and τ_t is the parallel displacement along the curve v_t from v to v_t. Since Y_{v_t} belongs to T', so does $\tau_t^{-1} \cdot Y_{v_t}$, because T' is obtained by parallel displacement from T'_u. Thus $(\nabla_X Y)_v$ belongs to T'_v.

Let M' be the maximal integral manifold of the distribution T' through u. M' has an induced Riemannian metric. It turns out that M' is a totally geodesic submanifold of M, that is, every geodesic in M' is a geodesic in M. From this it follows that if M is complete so is M'.

Now going back to Φ acting on $T_u(M)$, let T''_u be the orthogonal complement of T'_u. Then $T_u(M)$ is the direct sum of two subspaces T'_u and T''_u which are both invariant by Φ. From the subspace T''_u we obtain a distribution T'' and the maximal integral manifold M'' of T'' through u. From the consideration of these two mutually complementary distributions, it follows that there are neighborhoods V, V' and V'' of u in M, M' and M'' respectively, such that V is the direct product of V' and V''. There exists a system of local coordinates $(u^1, ..., u^p, u^{p+1}, ..., u^n)$ in V such that $(u^1, ..., u^p)$ are coordinates in V' and $(u^{p+1}, ..., u^n)$ are coordinates in V''. After this purely topological argument, we can show that the Riemannian metric in V is the direct product of the metrics of V' and V'', in terms of coordinates, $g(\partial/\partial u^i, \partial/\partial u^j)$, $i, j = 1, 2, ..., p$ (resp. $i, j = p + 1, ..., n$) are independent of $u^{p+1}, ..., u^n$ (resp. $u^1, ..., u^p$). It follows that the holonomy group of V is the direct product of the holonomy groups of V' and V''.

Thus we have obtained a *local decomposition* of M when the holonomy group is not irreducible.

We now consider a most natural decomposition of $T_u(M)$. Let T_0 be the set of tangent vectors at u which are fixed by every element of Φ. Then $T_u(M)$ is the direct sum of T_0 and its orthogonal complement T'. We can obtain a decomposition of $T_u(M)$ into the direct sum $T_0 + T_1 + ... + T_k$ of mutually orthogonal subspaces with irreducible $T_i (1 \leqq i \leqq k)$.

Assuming that M is simply connected, we can prove the following. According to a decomposition $T_u(M) = T_0 + T_1 + ... + T_k$, the holonomy group Φ is the direct product $\Phi_0 \times \Phi_1 \times ... \times \Phi_k$, where Φ_0 is the identity on $T_u(M)$, and each $\Phi_i (1 \leqq i \leqq k)$ acts trivially on T_j for $j \neq i$ and irreducibly on T_i. The proof makes use of the local decomposition of M corresponding to the decomposition of $T_u(M)$ and the following factorization lemma, a convenient form of the principle of monodromy.

Factorization Lemma. (Lichnerowicz [2]) *In a topological space M, let U_x be a neighborhood of a point $x \in M$ chosen once for all for each $x \in M$. Every closed curve which is homotopic to zero can be transformed, by substituting a curve of the form $\tau^{-1} \cdot \tau$ (τ^{-1} is the curve obtained by reversing the orientation of the curve τ, and the dot means the product of* *curves, namely, the succession of two curves) by a trivial curve, or vice versa, into a product of curves each of the form* $\tau \cdot \mu \cdot \tau^{-1}$, *where* τ *is a curve from a point, say,* x, *to another, say,* y, *and* μ *is a closed curve at* y *which is contained in* U_y. As consequences of these considerations, we have the following results.

THEOREM 1. (Borel-Lichnerowicz [1]) *The restricted holonomy group of a Riemannian manifold is a closed subgroup of* $SO(n)$.

THEOREM 2. (de Rham [1]) *The decomposition* $T_u(M) = T_0 + T_1 + ... T_k$ *with properties mentioned above is unique up to the order.*

The local decomposition of M based on $T_u(M) = T_0 + T_1 + ... + T_k$ can be made into a global decomposition when M is simply connected and complete. We have, namely, the following important theorem:

THEOREM 3. (de Rham [1]) *A simply connected and complete Riemannian manifold* M *is the direct product* $M_0 \times M_1 \times ... \times M_k$, *where* M_0 *is a Euclidean space, and each* M_i, $1 \leqq i \leqq k$, *is an irreducible Riemannian manifold.*

Many problems concerning a complete Riemannian manifold can be reduced to the case of an irreducible Riemannian manifold by first going to the universal covering and then using the above theorem. Irreducible Riemannian manifolds thus play the role of simple Lie groups, while the Euclidean spaces correspond to abelian Lie groups. The analogy, however, is not quite complete. Berger [1, 2] studied the classification of irreducible Riemannian manifolds.

3.4. Induced Connections. Among other general results on Riemannian connections and holonomy groups, we shall state a result on the submanifolds imbedded in a Euclidean space.

Let R^{n+k} be a Euclidean space of $(n + k)$-dimensions. Let $M_{n,k}$ be the Grassmann manifold of n-planes in R^{n+k}, namely, the set of all n-planes through the origin of R^{n+k}. If R^n is a fixed n-plane and R^k is its orthogonal complement, then we may identify $M_{n,k}$ with the homogeneous space $O(n + k)/O(n) \times O(k)$, where $O(n)$ is the orthogonal subgroup of $O(n + k)$ which leaves R^k pointwise fixed, and $O(k)$ is the orthogonal subgroup which leaves R^n pointwise fixed. Let $P_{n,k} = O(n + k)/O(k)$ be the Stiefel manifold. It is a principal fiber bundle over $M_{n,k}$ with structure group $O(n)$ in the natural fashion.

We first introduce a connection in the bundle $P_{n,k}$ over $M_{n,k}$. Denoting the Lie algebra of $O(n)$ by $\mathfrak{o}(n)$, we have

$$\mathfrak{o}(n+k) = \mathfrak{o}(n) + \mathfrak{o}(k) + \mathfrak{m}_{n,k}$$

where $\mathfrak{o}(n+k)$ consists of all skew-symmetric matrices of degree $n+k$, $\mathfrak{o}(n)$ consists of all matrices of the form

$$\left\| \begin{matrix} A & 0 \\ 0 & 0 \end{matrix} \right\|$$

with skew-symmetric A of degree n, $\mathfrak{o}(k)$ consists of all matrices of the form

$$\left\| \begin{matrix} 0 & 0 \\ 0 & B \end{matrix} \right\|$$

with skew-symmetric B of degree k, and $\mathfrak{m}_{n,k}$ consists of all matrices of the form

$$\left\| \begin{matrix} 0 & C \\ {}^tC & 0 \end{matrix} \right\|$$

where C is an arbitrary $k \times n$ matrix. The above decomposition is a direct sum of vector subspaces such that $\mathfrak{m}_{n,k}$ is invariant by $ad(O(n) \times O(k))$ acting on $\mathfrak{o}(n+k)$. On the group $O(n+k)$, let θ be the left invariant Maurer-Cartan form with values in $\mathfrak{o}(n+k)$. Let ω be the $\mathfrak{o}(n)$-component of θ with respect to the above decomposition. Since $O(n)$ and $O(k)$ commute elementwise as subgroups of $O(n+k)$, it follows that ω induces an $\mathfrak{o}(n)$-valued form on $P_{n,k}$ in the natural fashion. This form, still denoted by ω, satisfies the conditions of a connection form, and thus defines a connection in the bundle $P_{n,k}$ over $M_{n,k}$. The connection so obtained is called the *canonical connection in* $P_{n,k}$.

Now let us consider an imbedded submanifold M of dimension n in R^{n+k}. We define a mapping f of M into $M_{n,k}$ as follows. For each point $u \in M$, $f(u)$ is a point of $M_{n,k}$ represented by the n-plane through the origin which is parallel to the tangent plane of M at u in R^{n+k}. Let $O(M)$ be the bundle of orthogonal frames over M. We define a mapping $\tilde{f}$ of $O(M)$ into $P_{n,k}$ as follows. For each $x \in O(M)$ which is an orthogonal frame at $u \in M$, we take the orthogonal n-frame at the origin of R^{n+k} and add k more vectors to obtain an orthogonal $(n+k)$-frame. The matrix represented by this $(n+k)$-frame is an element of $O(n+k)$. The point of $P_{n,k} = O(n+k)/O(k)$ represented by this orthogonal matrix depends only on x and not on the way of adding k more vectors in the above process. In this way, we obtain a mapping $\tilde{f}$ of $O(M)$ into $P_{n,k}$. It can be verified that $\tilde{f}$ is a bundle mapping of $O(M)$ over M into $P_{n,k}$ over $M_{n,k}$ which induces the mapping f of the base M into $M_{n,k}$.

The main result here is the following. By the bundle mapping $\tilde{f}$ of $O(M)$ into $P_{n,k}$, we get the $\mathfrak{o}(n)$-valued form $\tilde{f}^*.\omega$ from the canonical connection form on $O(M)$. *The form $\tilde{f}^*.\omega$ coincides with the connection form of the Riemannian connection of the embedded submanifold M* (Kobayashi [2]).

If M is an oriented n-dimensional submanifold in R^{n+k}, let $SO(M)$ be the bundle of oriented orthogonal frames over M with structure group $SO(n)$. Instead of $P_{n,k}$ over $M_{n,k}$, we consider $P'_{n,k} = SO(n + k)/SO(k)$ which is a bundle over $M = M_{n,k} = SO(n + k)/SO(k) \times (SO(n)$ with structure group $SO(n)$. By defining the canonical connection ω on $P_{n,k}$ and the mappings f, $\tilde{f}$, we have the same result as above. In particular, let M be an oriented n-dimensional hypersurface in R^{n+1}. Then $P_{n,1} = SO(n + 1)$, $Mn, 1 = SO(n + 1)/SO(n) = S^n$, and the mapping f of M into S^n is nothing but the spherical map of Gauss. The above result then gives an interpretation of the classical Levi-Civita connection.

More detailed study of the questions related to the mapping $M \to M_{n,k}$ are found in Kobayashi [2, 3, 4], and Chern [2]. In the case of a submanifold M imbedded in a Riemannian manifold N, a similar result was given by S. Takizawa [1]. These are all closely related to the study of characteristic classes.

3.5. Killing Vector Fields. Let M be a Riemannian manifold. A diffeomorphism θ of M which preserves the Riemannian metric is called an isometry. Let X be a vector field defined on M. For each point $u \in M$, there exists a neighborhood V of u, a positive number $\varepsilon > 0$, and a family of transformations ϕ_t, $|t| < \varepsilon$, such that:

(a) For each t with $|t| < \varepsilon$, ϕ_t is a diffeomorphism of V onto an open set $\phi_t(V)$ of M.

(b) The mapping $(t, v) \to \phi_t(V)$ is differentiable.

(c) If $|t|$, $|s|$, $|t + s| < \varepsilon$, and if $\phi_t(v) \in V$, $v \in V$, then $\phi_s \cdot \phi_t(v) = \phi_{s+t}(v)$; and which is related to X by

(d) $X_v f = \lim_{t \to 0} (1/t) [f(\phi_t(v)) - f(v)]$ for every $v \in V$ and for every differentiable function f.

The family ϕ_t is called a local group of local transformations generated by X in a neighborhood of the point u. In particular, if $\phi_t(v)$ is defined for all real numbers t and for all points v of M, we say that ϕ_t is the (global) 1-parameter group of transformations of M generated by X and that X is globally integrable.

A vector field X is called a *Killing vector field* if a local 1-parameter group of local transformations ϕ_t generated by X in a neighborhood of each point consists of local isometries, that is, if each ϕ_t preserves the Riemannian metric. We have the following:

THEOREM. (Kobayashi [5]) *If M is a complete Riemannian manifold, every Killing vector field is globally integrable.*

For any arbitrary vector field X on M, we define a tensor field A_X of type (1,1), that is, a field of linear endomorphisms of tangent spaces, by $A_X(Y) = -\nabla_Y X$ for every tangent vector Y. Since the torsion tensor $T(X, Y) = \nabla_X X - \nabla_Y X - [X, Y]$ is zero, we have

$$L_X = \nabla_X + A_X$$

where L_X is the Lie differentiation with respect to X which is considered as a derivation of the algebra of tensor fields together with ∇_X and A_X. The vector field X is Killing if and only if $L_X g = 0$, where g is the Riemannian metric. Rewriting this condition, we have:

(1) *X is Killing if and only if A_X is a skew-symmetric endomorphism at each point.*

The following two lemmas can be proved with a little amount of tensor calculus.

(2) *If X and Y are Killing vector fields, then*

$$A_{[X,Y]} = [A_X, A_Y] - R(X, Y).$$

(3) *If X is a Killing vector field, then*

$$\nabla_V X = -A_X \cdot V \qquad \text{and} \qquad \nabla_V(A_X) = R(X, V)$$

for an arbitrary vector V (the first is nothing but the definition of A_X; see Kostant [1]).

Let us denote by $\mathfrak{k}(M)$ the set of all Killing vector fields X defined on M. From (1), it follows that $\mathfrak{k}(M)$ forms a vector space over the real number field R. From (2), it follows that $\mathfrak{k}(M)$ is closed with respect to the bracket operation $[X, Y]$ so that $\mathfrak{k}(M)$ forms a Lie algebra over R.

Assuming that M is connected, we prove that $\mathfrak{k}(M)$ is finite-dimensional. Let o be an arbitrary point of M. For any curve v_t from o to an arbitrary point $v \in M$, let V_t be the family of tangent vectors. If we denote by X_t and A_t the restrictions of the vector field X and the associated tensor field A_x to the curve v_t, respectively, then X_t and A_t satisfy the system of differential equations

$$\nabla_{V_t} X_t = - A_t \cdot V_t$$

$$\nabla_{V_t}(A_t) = R(X_t, V_t)$$

Since the system admits a unique solution (X_t, A_t) for any initial condition at o, we see that the mapping $X \in \mathfrak{k}(M) \rightarrow (X_o, (A_X)_o)$ is 1-1. In fact, this is a linear isomorphism of $\mathfrak{k}(M)$ into the vector space $\mathfrak{g}_0(M) = T_o(M) + E_o(M)$, where $E_o(M)$ is the vector space consisting of all skew-symmetric endomorphisms of the tangent space $T_o(M)$. Since $\dim \mathfrak{g}_o(M) = n + n(n-1)/2$, we see that $\dim \mathfrak{k}(M) \leqq n + n(n-1)/2$.

Thus, we have shown that $\mathfrak{k}(M)$ forms a finite-dimensional Lie algebra over the real number field. It is this fact that lies in the background of the theorem that the group $I(M)$ of all isometries of M can be made into a Lie group (Myers-Steenrod [1]). When M is complete, the Lie algebra of $I(M)$ is precisely $\mathfrak{k}(M)$. When M is not complete, the Lie algebra of $I(M)$ is the subalgebra of $\mathfrak{k}(M)$ consisting of all globally integrable vector fields in $\mathfrak{k}(M)$. These results follow from a general theory of Lie transformation groups as developed by Palais [1]. We shall not go into this theory but shall indicate how $\mathfrak{k}(M)$ is determined from the curvature tensor R and its covariant differentials $\nabla^m R$.

At each point u, we have already considered the vector space $\mathfrak{g}_u = T_u(M) + E_u(M)$. In $\mathfrak{g}_u$, we define the subspace $\mathfrak{k}(u)$ consisting of all pairs $(X, A) \in \mathfrak{g}_u$ satisfying the conditions $(\nabla_X + A) . (\nabla^m R) = 0$ for all $m = 0, 1 \ldots$, where ∇_X, A, and their sum, are considered as derivations of the algebra of tensor fields into the tensor algebra at u.

On the other hand, let $\mathfrak{k}^*(u)$ be the set of all germs of Killing vector fields defined in a neighborhood of u. Every element of $\mathfrak{k}^*(u)$ is a Killing vector field defined in some neighborhood of u, and two such Killing vector fields are defined to be the same element of $\mathfrak{k}^*(u)$ if they coincide with each other in a sufficiently small neighborhood of u. As before, $\mathfrak{k}^*(u)$ is a finite-dimensional Lie algebra.

Let X be a Killing vector field defined in a neighborhood of u. Since $L_X \cdot g = 0$, it follows that $L_X \cdot (\nabla^m R) = 0$ for all $m = 0, 1, \ldots$.

Thus $(X_u, (A_X)_u)$ belongs to $\mathfrak{k}(u)$. We thus obtain a natural linear mapping of $\mathfrak{k}^*(u)$ into $\mathfrak{k}(u)$. On the other hand, by restriction of the domain of definition, we have a linear mapping of $\mathfrak{k}(M)$ into $\mathfrak{k}^*(u)$ for every $u \in M$.

The determination of $k(M)$ from the curvature tensor and all its covariant differentials is carried out in two steps. First, under what conditions is the mapping $\mathfrak{k}^*(u) \to \mathfrak{k}(u)$ *onto*? Second, under what conditions is the mapping $\mathfrak{k}(M) \to \mathfrak{k}^*(u)$ *onto*? The second is a topological question of extending a local Killing vector field to a global Killing vector field where the principle of monodromy plays a role. The first is concerned with the determination of a local Killing vector field from an algebraic tensor structure at one point, and illustrates the significance of tensor analysis in differential geometry. The analogy of these two questions to the study of local and infinitesimal holonomy groups is apparent. We now state the main results (Nomizu [7]).

A point $u \in M$ is called $\mathfrak{k}$-regular (resp. $\mathfrak{k}^*$-regular) if dim $\mathfrak{k}(v)$ [resp. dim $\mathfrak{k}^*(v)$] is constant in a neighborhood of u.

THEOREM 1. *If M and its Riemannian metric are analytic, then every point is $\mathfrak{k}$-regular.*

THEOREM 2. *A $\mathfrak{k}$-regular point is $\mathfrak{k}^*$-regular.*

THEOREM 3. *If u is $\mathfrak{k}$-regular, the mapping $\mathfrak{k}^*(u) \to \mathfrak{k}(u)$ is onto.*

THEOREM 4. *If M is simply connected, and if every point is $\mathfrak{k}^*$-regular, the mapping $\mathfrak{k}(M) \to \mathfrak{k}^*(u)$ is onto for every u.*

Thus, if an analytic Riemannian manifold M is simply connected, $\mathfrak{k}(M)$ is completely determined by the curvature tensor and all its covariant differentials at one point. If, moreover, M is complete, the group $I(M)$ is completely determined by the same invariants. This corresponds to the fact that a simply connected and complete analytic Riemannian manifold is determined from any small piece.

A classical result is that $\mathfrak{k}(u) = \mathfrak{g}(u)$ for every u if and only if M is of constant curvature. The various possibilities for the dimension of $I(M)$ have been studied by various authors (see Yano [1]).

We now come to another type of problem concerning Killing vector fields. In Section 2, we indicated the relationship between the holonomy group and the isotropy group of a homogeneous space G/H with the canonical connection of the second kind. On a Riemannian manifold M, the following question has been studied. Let X be a Killing vector field and A_X the associated tensor field of type (1,1). At an arbitrary point u of M, does the endomorphism A_X belong to the holonomy algebra at u? In general, A_X belongs to the normalizor of the holonomy algebra. Concerning this problem, we have:

THEOREM 5. (Lichnerowicz [1]) *If a Riemannian manifold M is irreducible, and the Ricci tensor is not zero, then A_X belongs to the holonomy algebra at every point.*

THEOREM 6. (Kostant [1]) *If a Riemannian manifold M is compact, the same conclusion holds as in Theorem 5.*

The proof of Theorem 5 depends on the algebraic argument on the irreducibility combined with the consideration of a Kählerian connection which M may admit (we shall discuss this question in Section 4). The proof of Theorem 6 utilizes the Green-Stokes formula in the form $\int_M \operatorname{div} Y \, dv = 0$ for an arbitrary vector field Y on a compact Riemannian manifold M with volume element dv. The Ricci tensor is closely related to Killing vector fields as is suggested by Theorem 5 and by the following theorem:

THEOREM 7. (Bochner [1]) *If the Ricci tensor of a compact Riemannian manifold M is negative definitive, M admits no nontrivial Killing vector field.*

The basic formula for this theorem is

$$\operatorname{div}(A_X \cdot X) = -S(X, X) - \operatorname{trace}(A_X)^2$$

where X is a Killing vector field and S is the Ricci tensor.

3.6. Riemannian Symmetric Spaces. Let G/H be a symmetric homogeneous space. If the subgroup H is compact, G/H admits a G-invariant Riemannian metric, and any such metric gives rise to the canonical connection on G/H which we explained in Section 2.5. These Riemannian symmetric homogeneous spaces were first studied by E. Cartan [2]. As consequences of a more general theory we developed in Section 2, we have the following results for such spaces G/H. *The symmetry around each point of G/H is an isometry. The covariant derivatives of an arbitrary G-invariant tensor field are zero. In particular,*

$\nabla R = 0$ and every invariant differential form on G/H is closed. The restricted holonomy group is contained in the linear isotropy group.

A Riemannian manifold M is called *locally symmetric* if $\nabla R = 0$. This condition is equivalent to the following. For each u in M, there is an involutive isometry of a certain neighborhood U of u onto itself which admits u as an isolated fixed point. M is called *globally symmetric* if, for each u, M admits an involutive isometry which has u as an isolated fixed point. A complete globally symmetric Riemannian space can be expressed as a symmetric homogeneous space G/H. *If a locally symmetric Riemannian space M is simply connected and complete, M is globally symmetric.*

The inclusion of the holonomy group in the linear isotropy group characterizes a locally symmetric Riemannian space. Let M be a Riemannian manifold. For each $u \in M$, let H_u be the set of all isometries of M which fix u.

THEOREM. (Nomizu [4]) *If M is complete, and if the holonomy group at $u \in M$ is contained in the linear isotropy group H_u for every u, then M is locally symmetric.*

Riemannian homogeneous spaces which are not symmetric have been studied by Nomizu [2] and Kostant [2]. Among them, an important class consists of Riemannian homogeneous spaces G/H whose Riemannian metric gives rise to the canonical connection of the first kind with respect to a certain decomposition $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$, $\mathrm{ad}(H)\,\mathfrak{m} = \mathfrak{m}$, of the Lie algebra $\mathfrak{g}$ of G. If G is compact, G/H always admits this kind of metric. A symmetric G/H, of course, has this property. For this class of Riemannian homogeneous spaces G/H, we can prove that if G is simple, then the restricted holonomy group is irreducible (see Nomizu [6]).

4. Kählerian Connections

4.1. Complex Structure on a Real Vector Space. Let V be a finite-dimensional vector space over the field of real numbers R. A *complex structure* I on V is a linear transformation of V such that $I^2 = -E$, where E denotes the identity transformation. When a complex structure I is given, V can be made into a vector space over the field of complex numbers C by defining scalar multiplication by:

$$(a + b\sqrt{-1})\,x = ax + bIx, \qquad x \in V,\ a, b \in R$$

V admits a basis $\{e_1, ..., e_n, Ie_1, ..., Ie_n\}$ so that $\dim V = 2n$. In the vector space V considered over C, the elements $e_1, ..., e_n$ form a basis. We shall denote by V_{2n} the original real vector space with a complex structure I and by $V_n(C)$ the complex vector space obtained from V_{2n} by the above process.

Conversely, if $V(C)$ is a complex vector space, the set $V(C)$ can be regarded as a real vector space where we can define a complex structure I by $Ix = \sqrt{-1}\, x$, $x \in V$.

Let V_{2n} be a real vector space with complex structure I. An inner product on V_{2n} will be called *hermitian* if $(Ix, Iy) = (x, y)$ for all $x, y \in V_{2n}$. In the complex vector space $V_n(C)$, we define $(x, y)^* = (x, y) - \sqrt{-1}\,(Ix, y)$. We have easily $(x, y)^* = \overline{(y, x)^*}$ and $(\alpha x, y)^* = \alpha(x, y)^*$ for $\alpha \in C$. Thus $(x, y)^*$ is a hermitian inner product in the complex vector space $V_n(C)$.

A linear transformation A of V_{2n} which commutes with I can be considered as a linear transformation of $V_n(C)$, since

$$f[(a + b\sqrt{-1}\, x] = f(ax + bIx) = af(x) + bIf(x) = (a + b\sqrt{-1})f(x)$$

for $x \in V_n(C)$ and $a, b \in R$. When V_{2n} admits a hermitian inner product, any orthogonal transformation of V_{2n} can be considered as a unitary transformation of $V_n(C)$ with respect to the hermitian inner product $(\ ,\ )^*$.

We can treat the concept of a quaternion structure on a real vector space V in a similar fashion. It is defined as a pair of linear transformations (I, J) such that $I^2 = -E$, $J^2 = -E$ and $JI = -IJ$. In this case, $K = IJ$ is again a linear transformation with $K^2 = -E$, and I, J, and K behaving just like quaternion units i, j, and k with respect to multiplication. If V admits a quaternion structure (I, J, K), $\dim V = 4n$ and $V_{4n} \equiv V$ can be made into a vector space $V_n(Q)$ over the field of quaternions Q in such a way that

$$(a + bi + cj + dk)\, x = ax + bIx + cJx + dKx$$

for $x \in V_n(Q) = V$ and $a, b, c, d \in R$. A linear transformation of V_{4n} which commutes with I, J (and hence with K) can be considered as a linear transformation of $V_n(Q)$. An inner product on V_{4n} which is invariant by I, J gives rise to a symplectic inner product on $V_n(Q)$ in the natural fashion.

4.2. Almost Complex Structure on a Differentiable Manifold. Let M be a differentiable manifold. An almost complex structure on M is a tensor field I of type (1,1) such that $I^2 = -E$. At each point of M, the endomorphism I_x defines a complex structure in the tangent space $T_x(M)$. Thus, if M admits an almost complex structure I, dim $M = 2n$. A complex manifold M, when considered as a real analytic manifold, admits of course a natural almost complex structure I in the following way. For any point x of M, let $(z^1, ..., z^n)$ be complex coordinates with origin x. The complex analytic homeomorphism $(z^1, ..., z^n) \to (\sqrt{-1}\, z^1, ..., \sqrt{-1}\, z^n)$ of a neighborhood of x onto itself fixes the point x and induces a linear transformation I_x of the tangent space $T_x(M)$, considered as the tangent space of a real analytic manifold M. The linear transformation I_x is independent of the choice of complex coordinates $(z^1, ..., z^n)$.

Given an almost complex structure I on a differentiable manifold M, we define a tensor field J of type (1,2) by

$$J(X, Y) = I[X, Y] - [IX, Y] - [X, IY] - I[IX, IY]$$

where X and Y are arbitrary vector fields on M. J is just like the torsion tensor of a linear connection and is skew-symmetric in X and Y. The following theorem has been known for a real analytic manifold and an analytic almost complex structure for many years (see, for example, Fröhlicher [1]), but has been recently proved under differentiability assumptions (Newlander-Nirenberg [1]).

THEOREM. *An almost complex structure I is integrable (that is, it arises from a complex structure on M) if and only if the associated tensor J is identically zero.*

From the point of view of differential geometry, we first consider a linear connection which has a particular property with respect to a given almost complex structure I on M. A linear connection Γ will be called *almost Kählerian* if the covariant differential of the tensor field I is equal to zero. This condition is clearly equivalent to the condition that the parallel displacement along an arbitrary curve in M is compatible with the complex structures of the tangent spaces at both ends of the curve. In particular, every transformation belonging to the holonomy group with reference point x commutes with I_x so that the holonomy group can be regarded as a subgroup of $GL(n, C)$, where dim $M = 2n$. On an almost complex manifold M, an almost Kählerian linear connection always exists. In fact, from the point of view of fiber bundles, the structure $GL(2n, R)$ of the bundle of frames over M can be reduced to the subgroup $GL(n, C)$ [naturally imbedded in $GL(2n, R)$] and an arbitrary connection in the reduced bundle determines an almost Kählerian linear connection on M.

The torsion tensor of any such linear connection has a close relation with the tensor J. In fact, we have:

THEOREM. *If an almost complex manifold M admits an almost Kählerian linear connection whose torsion tensor is zero, then M is a complex manifold.*

Conversely, on a complex manifold, there is always a linear connection with zero torsion such that $\nabla I = 0$. For the proof of these results, see, for example, Fröhlicher [1].

Particularly important are complex manifolds which admit a Riemannian metric whose Riemannian connection satisfies $\nabla I = 0$. More precisely, let M be a complex manifold. Regarded as a real analytic manifold with complex structure I, M will be denoted by M_{2n}. A Riemannian metric g on M_{2n} is called a *Kählerian metric* if it is hermitian: $g(IX, IY) = g(X, Y)$ for arbitrary vectors X and Y, and if the Riemannian connection satisfies $\nabla I = 0$. The pair (I, g) is called a *Kählerian structure on* M_{2n}.

On a Kählerian manifold M_{2n}, we define a 2-form F by $F(X, Y) = g(IX, Y)$ for arbitrary tangent vectors X and Y. From the conditions $\nabla I = 0$ and $\nabla g = 0$, it follows that $\nabla F = 0$ and hence $dF = 0$, since dF is equal to the alternation of the covariant differential ∇F. In case M_{2n} admits a Riemannian metric g satisfying $g(IX, IY) = g(X, Y)$ with respect to the complex structure I, we can still define the 2-form F. It can be proved that if $dF = 0$, then g is a Kählerian metric (see Fröhlicher [1]). The condition $dF = 0$ is sometimes taken as the definition of a Kählerian metric. At any rate, F with $\nabla F = 0$ is a harmonic form and, in case M is compact, defines a nontrivial 2-dimensional cohomology class. Likewise, F^k is a harmonic form and gives a nontrivial cohomology class of dimension $2k$, $1 \leqq k \leqq n - 1$. For the cohomology of Kählerian manifolds, see Lichnerowicz [3] and Weil [2].

4.3. Holonomy Groups and Ricci Curvature. Let (I, g) be a Kählerian structure on M_{2n}. For an arbitrary point u in M_{2n}, every element of the holonomy group Φ_u leaves g_u invariant and commutes with the complex structure I_u so that it can be considered as a subgroup of the unitary group $U(n)$. Infinitesimally, the endomorphisms of the form $R(X, Y)$, $(\nabla_Z R)(X, Y)$, ... (all successive covariant derivatives) commute with I_u and are skew-symmetric with respect to g_u. The Ricci tensor S can be given the following expression

$$S(X, Y) = -\tfrac{1}{2}\operatorname{trace}(I \cdot R(X, Y))$$

where X and Y are arbitrary tangent vectors. Using this formula we can prove:

THEOREM. (Lichnerowicz [1]) *The Ricci tensor is not identically zero if and only if the restricted holonomy group has a nondiscrete center.*

In other words, the Ricci tensor is identically zero if and only if the restricted holonomy group is contained in the special unitary group $SU(n)$.

As an application of this theorem, we obtain the following result which gives Theorem 5, Section 3.5. *Let M be a Riemannian manifold. If the restricted holonomy group Φ^0 is irreducible and the Ricci tensor is not zero, then the identity component $N^0(\Phi^0)$ of the normalizor $N(\Phi^0)$ of Φ^0 in $O(n)$ coincides with Φ^0.*

In fact, consider the set A of all endomorphisms of $T_u(M)$ which commute with every element of Φ^0. Since Φ^0 is irreducible, A turns out to be a finite-dimensional division algebra over the real number field. By a well known theorem in algebra, A is isomorphic with either R or C or Q. In case A is isomorphic with R, it follows easily that the center of Φ^0 is, at most, one-dimensional. In case A is isomorphic with C or Q, A contains an endomorphism I_u of $T_u(M)$ such that $I_u^2 = -E$ (E: identity). By going to the universal covering if necessary, we can define a complex structure I on M by parallel displacement of I_u, and prove that (I, g) is a Kählerian structure on M. Now then, by the above theorem, Φ^0 has a nondiscrete center. Since Φ^0 can be regarded as irreducible as a subgroup of $U(n)$, the center must be of dimension 1. We have thereby proved that, at any rate, the center of Φ^0 is of dimension $\leqq 1$. Since Φ^0 is a compact connected Lie group with, at most, a one-dimensional center, the connected component of the group of automorphisms coincides with the group of inner automorphisms. This implies that $N^0(\Phi^0) = \Phi^0$.

For a Kählerian structure (I, g) on M, we can ask the following question. Let $I(M)$ be the group of all isometries of M with respect to g, and $K(M)$ the group of all transformations of M which preserve the complex structure (namely, the group of all complex analytic transformations). What is the relationship between $I(M)$ and $K(M)$? This problem has been treated by several authors (see, for example, Kobayashi-Nomizu [1], Y. Matsushima [2], and the references there).

4.4. Examples. (a) Let N be a complex submanifold imbedded in a Kählerian manifold M. The induced metric of N is then Kählerian. The complex projective space $P_n(C)$ admits the so-called Fubini-Study metric which is Kählerian. Thus any nonsingular algebraic variety in $P_n(C)$ admits a natural Kählerian structure. Conversely, a compact Kählerian manifold which satisfies a certain condition (the condition of the Hodge metric that the form F represents a rational cohomology class) is known to be an algebraic variety (a result of Kodaira [1]).

(b) A bounded domain D in the space C^n of several complex variables admits the so-called Bergman metric (Bergman [1]), which is Kählerian. This metric is invariant by every complex analytic transformation of the domain D. Differential geometric properties such as the completeness condition of the Bergman metric, can be studied in relation to function-theoretic properties of the domain (see Kobayashi [6] and the references there).

(c) Kählerian homogeneous spaces, namely, homogeneous spaces G/H which admit a G-invariant Kählerian structure, are particularly interesting and important. The problem of E. Cartan [4] whether a homogeneous bounded domain is symmetric or not has been answered affirmatively by Hano [1] in the case in which it admits a transitive unimodular group of complex analytic transformations, but negatively by Pyateckii and Sapiro [1] in the general case. For this subject, see E. Cartan [4], Lichnerowicz [1], Hano-Matsushima [1]. Matsushima [1], Hano [1], and the references there.

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