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A New Basis for Singularly Perturbed Problems: A Representation Theorem*

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1. INTRODUCTION

In this work we solve a problem in analysis which has been stimulated by the theory of inviscid flow in fluid dynamics and by problems in singular perturbations. It is a well-known fact that the functions ϵ^i, $i = 0, 1, \ldots$ are "too narrow a family of functions to describe the asymptotic expansion of a function $y(t, \epsilon)$ at $\epsilon = 0$." The most trivial initial-value problem,

$$\epsilon y' + y = 0, \qquad y(0) = 1, \qquad 0 < t, \qquad \epsilon > 0,$$

has the solution $y(t, \epsilon)$

$$y(t, \epsilon) = \exp - t\epsilon^{-1}.$$

Any attempt to find an asymptotic expansion

$$y(t, \epsilon) \sim \sum_{\nu=0}^{\infty} y_\nu(t)\epsilon^\nu$$

with $y_\nu(t)$, $\nu = 0, 1, \ldots$ continuous functions of t for $0 \le t \le 1$ fails to preserve all values of $y(t, \epsilon)$ for $\epsilon > 0$ and ϵ small. This manifests itself in the fact that for all $\nu = 0, 1, \ldots, y_\nu(t) \equiv 0$, and one of the explanations that accounts for this phenomenon is that $y_\nu(t)$ solves "differential equations" (actually in this case algebraic ones) of "lower order." Therefore, not all the boundary conditions of the "higher-order equation," $\epsilon y' + y = 0$ could be taken care of by the $y_\nu(t)$, $\nu = 0, 1, \ldots$. If one tries a generalized asymptotic representation

$$y(t, \epsilon) \sim \sum_{\nu=0}^{\infty} y_\nu(t)\phi_\nu(\epsilon)$$

*This paper is dedicated to W. R. Wasow on the occasion of his seventieth birthday for his many contributions to the field of singular perturbations.

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such that

$$\phi_\nu(\epsilon) = o(\phi_{\nu+1}(\epsilon)), \qquad \nu = 1, \ldots$$
$$\phi_0(\epsilon) \equiv 1,$$

(see [7, p. 25]), then still no remedy is obtained. This state of affairs stimulated the idea that in order to construct at a first stage the limit

$$\lim_{\epsilon \to 0^+} y(t, \epsilon) = y(t, 0^+)$$

one should start from a "different basis." If we let $y(t, \epsilon)$ denote the solution of the somehow vague singularly perturbed differential equation $L_\epsilon y = 0$ with the somehow vague boundary conditions $By = 0$, we would like to *construct a pointwise* representation of $y(t, \epsilon)$ such that *one* and *the same* representation will yield the values of $y(t, \epsilon)$ as well as the values of $y(t, 0^+)$. From all the optional techniques to solve this problem we chose one which was most promising for solving nonlinear singularly perturbed problems. The price paid for choosing this method, which we call the "discrete power series method," is an extra assumption. It says that $y(t, \epsilon)$ is a holomorphic function of ϵ, for ϵ in some open domain D_ϵ such that ϵ, $0 < \epsilon < \epsilon_0$, belongs to D_ϵ and therefore $\epsilon = 0$ is the closure of D_ϵ. For many practical problems this assumption is reasonable. For instance, the Burger equation (solved explicitly by Hopf and Cole; see [5] and [2], respectively),

$$u_t + uu_x = \epsilon u_{xx}, \qquad u(x, 0) = f(x),$$

possesses holomorphic solutions for all $\epsilon \neq 0$. For $\epsilon \to 0^+$ there exists a limit. (See [9, Chap. 3].) It is an obvious expectation to produce $y(t, 0^+)$ from the values of $y(t, \epsilon)$ for ϵ bounded *away* from $\epsilon = 0$ by virtue of the holomorphic property of $y(t, \epsilon)$. We would like to stress that from the point of view of applied math and physics, this technique has the ability to *construct* $y(t, \epsilon)$ for $\epsilon \to 0^+$, even in the absence of a rigorous asymptotic analysis.

In the language of fluid dynamics the technique says that we are going to calculate the inviscid flow from the knowledge of the flow with high coefficients of viscosity. In cases where a procedure for the construction of a solution to an inviscid flow problem is not available, a good guess of the proper domain D_ϵ mentioned above may lead to the desired result. We now detach ourselves from the "small parameter ϵ" and formulate the problems we will solve.

Let

$$\sum_{n=0}^{\infty} y_n(t) u^n \tag{1.1.1}$$

be a power series convergent in the disk D,

$$D = \{u \mid |u| < 1\}. \tag{1.1.2}$$

The coefficients $y_n(t)$ are functions (scalar or vectorial) of the variable $t \in J$ where J is an interval

$$J = \{t \mid 0 \le t \le 1\} \tag{1.1.3}$$

and J_∞ is the interval

$$J_\infty = \{t \mid 0 \le t < \infty\}. \tag{1.1.4}$$

The series (1.1.1) represents an analytic function $y(t, u)$ of the variable u on

$$I = \{u \mid -1 < u < 1\}. \tag{1.1.5}$$

1. Find in terms of $y_n(t)$ a new representation of $y(t, u)$ which is valid for $u \in I$ and u close to 1 and such that if

$$\lim_{u \to 1^-} y(t, u) = y(t, 1), \tag{1.1.6}$$

then the new representation will yield the value $y(t, 1)$.

Moreover, let the analytic function $y(t, u)$ have an asymptotic expansion

$$y(t, u) \sim \sum_{\nu=0}^{\infty} q_\nu(t)(1 - u)^\nu \tag{1.1.7}$$

for certain values of $t \in J$.

2. Find in terms of $y_n(t)$, $n = 0, 1, \ldots$ the coefficients $q_\nu(t)$, $\nu = 0, 1, \ldots$ of the asymptotic expansion. Another important problem that is worth mentioning is how to find the asymptotic expansion for $t \to \infty$.

In order to solve these problems we turn to an old idea, namely, the expansion of $y(t, u)$ in terms of orthogonal polynomials on I with a certain weight $\omega(u)$. Since we need a pointwise representation of $y(t, u)$ on $\bar{I}$ ($\bar{I}$ being the closure of I) rather than an $L^2_\omega(I)$ expansion, we will have to appeal to summalility matrix methods. The combination of

$$\omega(u) = (1 - u^2)^{1/2} \tag{1.1.8}$$

which induces Tshebysheff's polynomials of the second kind and Cesaro's summability method—$(C, 2)$ (see [4, p. 96]), turns out to do a good job.

In order to achieve this goal we use the triangular scheme shown in Fig. 1.1 (where $a_i(t)$ are the Fourier coefficients of $y(t, u)$).

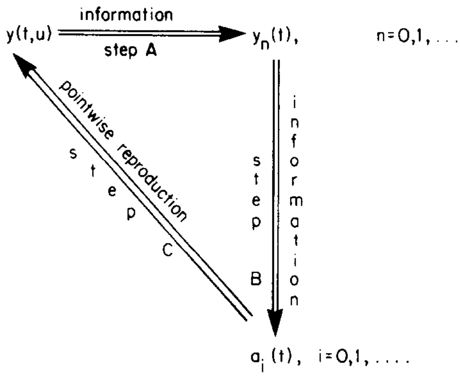

FIGURE 1

In step A we mimic well-known formulas (for the sake of self-containment) in order to obtain information on $y_n(t)$, $n = 0, 1, \ldots$ from the behavior of $y(t, u)$. Step A is the subject of Section 2.

In step B, we assume some rate of growth on $y_n(t)$, $n = 0, 1, \ldots$ in order to express $a_i(t)$, $i = 0, 1, \ldots$ in terms of $y_n(t)$, $n = 0, 1, \ldots$. This is accomplished in Section 3.

In step C we reproduce pointwise the function $y(t, u)$ for u close to $u = 1$ by $y_n(t)$, $n = 0, 1, \ldots$ via $a_i(t)$, $i = 0, 1, \ldots$. This is the topic of Sections 4 and 5. The main result of this work is a representation theorem. It is formulated in Section 6 with an application.

It is worth noting that there are techniques provided by summability theory (e.g., see [4, p. 187]) which take a divergent series $\Sigma_{n=0}^{\infty} y_n$ into the "correct value" on condition that $u = 1$ is in "the polygon of summability."

This means that $u = 1$ is a point where the function (1.1.1) is holomorphic. The result to be demonstrated in this work provides a transformation which takes $\Sigma_0^{\infty} y_n$ into the "correct value" even if $u = 1$ is a *singular point of* (1.1.1), for example, an *essential singularity*. This is the case in many flow problems and in singular perturbations.

We remark that for the sake of clarity we end each proof of a lemma or a theorem by E.O.P.

2. Estimation Lemmas for $y_n(t)$

ASSUMPTION 1.2.1. The function $y(t, u)$ is holomorphic in D for each $t \in J$ (or $t \in J_\infty$).

ASSUMPTION 1.2.2. We denote $y(t, u) \in H^p$, $p > 1$ if

$$\sup_{0<r<1} \left(\int_0^{2\pi} |y(t, re^{i\theta})|^p d\theta \right)^{1/p} < \infty.$$

LEMMA 1.2.1. *Let assumptions* 1.2.1 *and* 1.2.2 *hold. Then for some* $M(t)$

$$|y_n(t)| \le M(t), \qquad n = 0, 1, \ldots . \tag{1.2.1}$$

If we have

$$|y(t, u)| \le m(t) \tag{1.2.2}$$

for $|u| < 1$, *then*

$$|y_n(t)| \le m(t). \tag{1.2.3}$$

Proof. By Cauchy's formula

$$y_n(t) = \frac{1}{2\pi r^n}\int_0^{2\pi} y(t, re^{i\theta})e^{-in\theta}d\theta. \tag{1.2.4}$$

Using Holder's inequality one obtains

$$|y_n(t)| \le \frac{1}{2\pi r^n}\left(\int_0^{2\pi}|y(t, re^{i\theta})|^p d\theta\right)^{1/p}\left(\int_0^{2\pi}1^q d\theta\right)^{1/q} \le \frac{(2\pi)^{1/q}}{2\pi r^n}K_p(t), \tag{1.2.5}$$

where

$$K_p(t) := \sup_{0<r<1}\left(\int_0^{2\pi}|y(t, re^{i\theta})|^p d\theta\right)^{1/p} \tag{1.2.6}$$

$$1/p + 1/q = 1, \qquad p > 1. \tag{1.2.7}$$

Letting $r \to 1^-$ in (1.2.5) yields the desired result with

$$M(t) = \frac{(2\pi)^{1/q}}{2\pi}K_p(t). \tag{1.2.8}$$

Moreover, if $K_p(t)$ is bounded for $t \in J$, we let

$$\sup_t K_p(t) = K_p, \qquad \sup_t M(t) = M, \tag{1.2.9}$$

and obtain

$$|y_n(t)| \le M. \tag{1.2.10}$$

It is quite obvious that if

$$|y(t, re^{i\theta})| \le m(t), \tag{1.2.11}$$

for $0 \le r < 1$, $0 \le \theta < 2\pi$, we obtain by Cauchy's formula

$$|y_n(t)| \le m(t). \tag{1.2.12}$$

E.O.P.

The conditions of Lemma 1.2.1 are sufficient to produce (1.2.1) and (1.2.3) but are not necessary.

The function $y = (1 + u^{2j})^{-1}$ with $j > 0$ has the expansion

$$y = \sum_{k=0}^{\infty} (-1)^k u^{2jk} \tag{1.2.13}$$

with

$$|y_n| \le 1 \tag{1.2.14}$$

but does not satisfy any condition of Lemma 1.2.1.

LEMMA 1.2.2. *Assume k a nonnegative integer and*

$$y_k(\theta) := \lim_{r \to 1^-} \frac{\partial^k y(t, u)}{\partial u^k} \qquad \textit{with} \qquad u = re^{i\theta} \tag{1.2.15}$$

to be an integrable function of θ for $0 < \theta < \pi$ for t fixed. Then, for any $\delta > 0$ sufficiently small, there exists $n_0(t)$ such that for $n > n_0(t)$,

$$|y_n(t)| \le \delta n^{-k}. \tag{1.2.16}$$

If, in addition, $\partial^k y(t, u)/\partial u^k$ is Lipschitzian of order α, $0 < \alpha \le 1$, namely, that uniformly for $0 \le \theta \le \pi$, $u = re^{i\theta}$,

$$\left| \frac{\partial^k y(t, u + h)}{\partial u^k} - \frac{\partial^k y(t, u)}{\partial u^k} \right| \le |h|^\alpha m(t), \tag{1.2.17}$$

then there exists $n_0(t)$ such that for $n > n_0(t)$,

$$|y_n(t)| \le \tilde{m}(t) n^{-k+\alpha} \tag{1.2.18}$$

for some $\tilde{m}(t)$.

Proof. We have (1.2.16) by the Riemann–Lebesgue method (see [8, p. 403]), and (1.2.17) follows by [8, p. 426]. E.O.P.

It is possible to derive Lemmas 1.2.1 and 1.2.2 such that the bounds obtained on $y_n(t)$ will hold uniformly for $t \in \tilde{J}$, where $\tilde{J}$ is some subset of J.

The modifications needed then are as follows. In Lemma 1.2.1, if we assume that Assumptions 1.2.1 and 1.2.2 hold uniformly for $t \in \tilde{J}$, then $M(t)$ in (1.2.1) will be bounded by some M independent of t. If we assume in (1.2.2) that $m(t)$ is independent of t, then the same bound will appear in (1.2.3).

Similarly in Lemma 1.2.2, if (1.2.15) holds uniformly for $t \in \tilde{J}$, then $n_0(t)$ accompanied by (1.2.16) will be the same for all $t \in \tilde{J}$. If (1.2.16) holds uniformly for $t \in \tilde{J}$, then an $\tilde{m}(t)$ in (1.2.17) independent of t could be found.

3. The Fourier Coefficients

LEMMA 1.3.1. *Let $a_i(t)$ be the Fourier coefficients of the expansion of $y(t, u)$ in terms of Tshebysheff's polynomials of the second kind.* (*See* [6, *p.* 114].)

Consider

$$\phi_{i\nu} := \int_{-1}^{1} (1 - u^2)^{1/2} u^\nu \frac{\sin(i + 1) \arc \cos u}{(1 - u^2)^{1/2}} du,$$

$$i = 0, 1, \ldots, \quad \nu = 0, 1, \ldots . \tag{1.3.1}$$

Let

$$\sum_{\nu=0}^{\infty} \phi_{i\nu} y_\nu(t) \tag{1.3.2}$$

be convergent for $i = 0, 1, \ldots$. Then

$$a_{2l}(t) = \left(\sum_{s=0}^{\infty} \phi_{2l, 2s} y_{2s}(t) \right) \sqrt{2/\pi}, \qquad l = 0, 1, \ldots, \tag{1.3.3}$$

$$a_{2l+1}(t) = \left(\sum_{s=0}^{\infty} \phi_{2l+1, 2s+1} y_{2s+1}(t) \right) \sqrt{2/\pi}, \qquad l = 0, 1, \ldots . \tag{1.3.4}$$

Proof. By use of the substitution

$$u = \cos \theta, \qquad -\sin \theta \, d\theta = du, \tag{1.3.5}$$

one obtains

$$\phi_{i\nu} = \int_0^\pi \cos^\nu \theta \sin(i + 1) \theta \sin \theta \, d\theta. \tag{1.3.6}$$

By the well-known orthogonality property of the sequence $\sin k\theta$, $k = 1, 2, \ldots$, we notice immediately that

$$\phi_{00} = \pi/2, \phi_{10} = 0, \ldots, \phi_{i0} = 0, \qquad i = 1, 2, \ldots . \tag{1.3.7}$$

In order to obtain the general result we rewrite (1.3.6) in the form

$$\phi_{i\nu} = \tfrac{1}{2}\int_0^{\pi} \cos^{\nu}\theta\left[\cos i\theta - \cos(i+2)\theta\right], \tag{1.3.8}$$

and use [3, p. 374, formula 17] to define $\psi_{i\nu}$ by

$$\psi_{i\nu} = \int_0^{\pi} \cos^{\nu}\theta \cos i\theta \, d\theta = \left[1 + (-1)^{\nu+i}\right]\int_0^{\pi/2} \cos^{\nu}\theta \cos i\theta \, d\theta,$$

$$i = 0, 1, \ldots, \quad \nu = 0, 1, \ldots . \tag{1.3.9}$$

We obtain from (1.3.9) that

$$\psi_{i\nu} = 0 \tag{1.3.10}$$

for

$$\nu + i = 2k + 1, \qquad k = 0, 1, \ldots . \tag{1.3.11}$$

By [3, p. 374, formula 17] we also have

$$\psi_{i\nu} = \left[1 + (-1)^{\nu+i}\right]$$

$$\times \begin{cases} s\dfrac{\nu!}{(i-\nu)(i-\nu+2)\cdots(i+\nu)} & \text{for } \nu < i \\ \dfrac{\pi}{2^{\nu+1}}\dbinom{\nu}{(\nu-i)/2} & \text{for } i \le \nu \text{ and } \nu - i = 2k \\ \dfrac{\nu!}{(2k+1)!!\,(2i+2k+1)!!} & \text{for } i < \nu \text{ and } \nu - i = 2k+1 \end{cases}$$

$$k = 0, 1, \ldots \tag{1.3.12}$$

and in case $\nu < i$ above

$$\begin{aligned} s &= 0 && \text{for } i - \nu = 2k, \\ &= 1 && \text{for } i - \nu = 4k + 1, \\ &= -1 && \text{for } i - \nu = 4k - 1. \end{aligned} \tag{1.3.13}$$

We denote

$$(2k+1)!! = 1 \cdot 3 \cdot 5 \cdots (2k-1)(2k+1). \tag{1.3.14}$$

A second conclusion, that

$$\psi_{i\nu} = 0 \tag{1.3.15}$$

for $\nu < i$, also immediately follows from (1.3.12), (1.3.13). This is so since $(\nu + i)$ is an odd (even) number iff $|\nu - i|$ is odd (even). Since (1.3.12) implies

$$1 + (-1)^{\nu+i} = 0$$

for $(\nu + i)$ odd, it is enough to consider the case $(\nu + i)$ even for $\nu < i$. By (1.3.13) we find $s = 0$ when $(\nu + i)$ is even and (1.3.15) follows.

Thus let $(\nu + i)$ be an even integer. Then

$$\psi_{i\nu} = \frac{\pi}{2^\nu}\binom{\nu}{(\nu - i)/2} \qquad \text{for } \nu \geq i \text{ and } (\nu + i) \text{ is even.} \quad (1.3.16)$$

In all other cases $\psi_{i\nu} = 0$. Therefore, by (1.3.8) we have for $(\nu + i)$ even

$$\phi_{i\nu} = \psi_{i\nu} - \psi_{i+2,\nu} = \frac{\pi}{2^\nu}\left[\binom{\nu}{(\nu - i)/2} - \binom{\nu}{(\nu - i - 2)/2}\right]. \tag{1.3.17}$$

In particular, we have

$$\phi_{ii} = \pi/2^i \tag{1.3.18}$$

$$\phi_{i,i+1} = 0 \tag{1.3.19}$$

$$\begin{aligned}\phi_{i,\nu} &= \frac{\pi}{2^\nu}\left[\frac{\nu!}{\left(\frac{\nu+i}{2}\right)!\left(\frac{\nu-i}{2}\right)!} - \frac{\nu!}{\left(\frac{\nu-i-2}{2}\right)!\left(\frac{\nu+i+2}{2}\right)!}\right] \\ &= \frac{\pi}{2^\nu}\frac{\nu!\,(i+1)}{\left(\frac{\nu-i-2}{2}\right)!\left(\frac{\nu+i}{2}\right)!}\cdot\frac{4}{(\nu-i)(\nu+i+2)}\end{aligned}$$

$$\text{for } (i + 2) \leq \nu,\ (\nu + i) \text{ even,} \quad (1.3.20)$$

$$= \frac{\pi}{2^\nu}\binom{\nu}{(\nu - i)/2}\frac{2(i+1)}{(\nu + i + 2)}. \tag{1.3.21}$$

If we adopt the convention that $\binom{\nu}{k} = 0$ for $k = -1, -2, \ldots,$ we obtain

$$\phi_{i\nu} = \frac{\pi}{2^{\nu-1}}\binom{\nu}{(\nu - i)/2}\frac{(i+1)}{(\nu + i + 2)}, \qquad \nu + i = 0, 2, 4, \ldots . \tag{1.3.22}$$

Let us write down the form of the upper triangular infinite matrix $\phi_{i\nu}$:

$$(\phi_{i\nu}) = \frac{\pi}{2}\begin{bmatrix} 1 & 0 & 1/2^2 & 0 & \cdots \\ 0 & 1/2 & 0 & 1/2^2 & \cdots \\ 0 & 0 & 1/2^2 & 0 & \cdots \\ 0 & 0 & 0 & 1/2^3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

$i = 0, 1, \ldots,\ \nu = 0, 1, \ldots$.

By (1.3.4),

$$a_i(t) = \sqrt{\frac{2}{\pi}} \sum_{\nu=0}^{\infty} \phi_{i\nu} y_\nu(t). \tag{1.3.23}$$

Let $i = 2l$, $l = 0, 1, \ldots$. Then, since we just showed that $\phi_{2l,\nu} = 0$ for ν odd, one deduces (1.3.3) and

$$a_{2l}(t) = \sqrt{\frac{\pi}{2}} \sum_{k=l}^{\infty} \binom{2k}{k-l} \frac{(2l+1)}{(l+k+1)2^{2k}} y_{2k}(t). \tag{1.3.24}$$

Let $i = 2l + 1$, $l = 0, 1, \ldots$. Then, since $\phi_{2l+1,\nu} = 0$ for ν even, we deduce (1.3.4) and

$$a_{2l+1}(t) = \sqrt{\frac{\pi}{2}} \sum_{k=l}^{\infty} \binom{2k+1}{k-l} \frac{(l+1)}{(k+l+2)2^{2k}} y_{2k+1}(t). \tag{1.3.25}$$

E.O.P.

LEMMA 1.3.2. *Let $y(t, u)$, a function of $u \in I$ and of the parameter $t \in J$, be defined by the absolutely converging series,*

$$\sum_{n=0}^{\infty} y_n(t) u^n = y(t, u), \qquad u \in I. \tag{1.3.26}$$

Denote by $a_i(t)$, $i = 0, 1, \ldots$ its Fourier coefficients. Let t be such that

$$|y(t, u)| \le m(t) < \infty \tag{1.3.27}$$

for $u \in I$, for some $m(t)$. If

$$X_i = \sqrt{\frac{2}{\pi}} \sum_{\nu=0}^{\infty} \phi_{i\nu} y_\nu(t) \qquad i = 0, 1, \ldots \tag{1.3.28}$$

converge, then

$$a_i(t) = X_i, \qquad i = 0, 1, \ldots .$$

Proof. Consider the function

$$y(t, xu) = \sum_{n=0}^{\infty} y_n(t) u^n x^n \tag{1.3.29}$$

of the parameter x, $0 < x < 1$. It is easily verified that also

$$|y(t, xu)| \le m(t) < \infty \tag{1.3.30}$$

for $0 < x < 1$, $u \in I$. Moreover,

$$\lim_{x \to 1^-} y(t, xu) = y(t, u) \tag{1.3.31}$$

for $u \in I$.

By the Lebesgue-dominated convergence theorem [8, p. 345],

$$\lim_{x \to 1^-} \sqrt{\frac{2}{\pi}} \int_{-1}^{1} y(t, xu)(1 - u^2)^{1/2} \frac{\sin(i + 1) \arccos u}{(1 - u^2)^{1/2}} du = a_i(t). \tag{1.3.32}$$

But for $0 < x < 1$ we may write by Abel's theorem [8, p. 9]

$$\lim_{x \to 1^-} \sqrt{\frac{2}{\pi}} \sum_{\nu=0}^{\infty} x^\nu y_\nu(t) \int_{-1}^{1} (1 - u^2)^{1/2} \frac{\sin(i + 1) \arccos u}{(1 - u^2)^{1/2}} du = \sqrt{\frac{2}{\pi}} \sum_{\nu=0}^{\infty} y_\nu(t) \phi_{i\nu} \tag{1.3.33}$$

and the result follows. E.O.P.

LEMMA 1.3.3. *Let*

$$|y_\nu(t)| \le m(t) \nu^\alpha \tag{1.3.34}$$

with

$$\alpha < 0.5. \tag{1.3.35}$$

Then

$$\sum_{\nu=0}^{\infty} \phi_{i\nu} y_\nu(t) \tag{1.3.36}$$

are absolutely convergent for $i = 0, 1, \ldots$.

Let us find out the asymptotic behavior of $\phi_{2l,2s}$, $\phi_{2l+1,2s+1}$ by use of Stirling's formula [8, p. 58, formula (3)].

$$\begin{aligned}
\ln\frac{2\phi_{i\nu}}{\pi} &= -(\nu-1)\ln 2 + \ln \nu! - \ln\frac{(\nu-i)}{2}! - \ln\left(\frac{\nu+i}{2}\right)! \\
&\qquad + \ln(i+1) - \ln(\nu+i+2) \\
&= -(\nu-1)\ln 2 + \ln(i+1) - \ln(\nu+i+2) \\
&\quad + \left(\nu+1-\frac{1}{2}\right)\ln(\nu+1) - (\nu+1) + \ln\sqrt{2\pi} \\
&\quad - \left(\frac{\nu-i}{2}+1-\frac{1}{2}\right)\ln\left(\frac{\nu-1}{2}+1\right) + \left(\frac{\nu-i}{2}+1\right) - \ln\sqrt{2\pi} \\
&\quad - \left(\frac{\nu+i}{2}+1-\frac{1}{2}\right)\ln\left(\frac{\nu+i}{2}+1\right) + \left(\frac{\nu+i}{2}+1\right) - \ln\sqrt{2\pi} + o(1).
\end{aligned} \tag{1.3.37}$$

(The $o(1)$ symbol is with respect to $\nu \to \infty$.)

It follows by letting

$$\frac{\nu-i}{2} = k, \qquad k = 0, 1, \ldots \tag{1.3.38}$$

that

$$\ln\frac{2\phi_{i,2k+i}}{\pi} = 1 - \ln\sqrt{2\pi} + o(1) + \ln(i+1) + h(i,k) \tag{1.3.39}$$

with

$$\begin{aligned}
h(i,k) &= \left(2k+i+\tfrac{1}{2}\right)\ln(2k+i+1) - \left(k+\tfrac{1}{2}\right)\ln(k+1) \\
&\quad - \left(k+i+\tfrac{1}{2}\right)\ln\left(k+i+\tfrac{1}{2}\right) \\
&\quad - (2k+i-1)\ln 2 - \ln 2(k+i+1).
\end{aligned} \tag{1.3.40}$$

(The $o(1)$ should be interpreted now with respect to $k \to \infty$.)

We may rewrite $h(i,k)$ as follows:

$$\begin{aligned}
h(i,k) &= \left(2k+i+\tfrac{1}{2}\right)\ln(k+((i+1)/2)) - \left(k+\tfrac{1}{2}\right)\ln(k+1) \\
&\quad - \left(k+i+\tfrac{1}{2}\right)\ln\left(k+i+\tfrac{1}{2}\right) \\
&\quad + \tfrac{1}{2}\ln 2 - \ln(k+i+1) \\
&= \left[2k+i+\tfrac{1}{2}\right) - \left(k+\tfrac{1}{2}\right) - \left(k+i+\tfrac{1}{2}\right)\Big]\ln k + \tfrac{1}{2}\ln 2 \\
&\quad - \ln(k+i+1) + f(i,k) \\
&= -\ln\sqrt{k}\,(k+i+1) + \tfrac{1}{2}\ln 2 + f(i,k)
\end{aligned} \tag{1.3.41}$$

with

$$f(i,k) = \left(2k + i + \frac{1}{2}\right)\ln\left(1 + \frac{i+1}{2k}\right) - \left(k + \frac{1}{2}\right)\ln\left(1 + \frac{1}{k}\right)$$
$$- \left(k + i + \frac{1}{2}\right)\ln\left(1 + \frac{i+\frac{1}{2}}{k}\right). \tag{1.3.42}$$

Thus, we have so far

$$\ln\frac{2\phi_{i,2k+i}}{\pi} = 1 - \ln\sqrt{\pi} + \ln(i+1) - \ln\sqrt{k}\,(k+i+1)$$
$$+ f(i,k) + o(1). \tag{1.3.43}$$

We use Cauchy's theorem to obtain for $x > 0$,

$$\frac{(\alpha + \beta x)\ln(1+\theta x)}{x} = \beta\ln(1 + \theta x^*) + \frac{(\alpha + \beta x^*)}{(1 + \theta x^*)}$$

for some x^*, $0 < x^* < x$. If we let $x = k^{-1}$, then

$$\left(2k + i + \frac{1}{2}\right)\ln\left(1 + \frac{i+1}{2k}\right) = \left(i + \frac{1}{2}\right)\ln\left(1 + \frac{i+1}{2}x_1^*\right)$$
$$+ \frac{2 + \left(i + \frac{1}{2}\right)x_1^*}{1 + ((i+1)/2)x_1^*} \tag{1.3.44}$$

for $0 < x_1^* < k^{-1}$,

$$\left(k + \frac{1}{2}\right)\ln\left(1 + \frac{1}{k}\right) = \frac{1}{2}\ln(1 + x_2^*) + \frac{1 + \frac{1}{2}x_2^*}{1 + x_2^*} \tag{1.3.45}$$

for $0 < x_2^* < k^{-1}$, and

$$\left(k + i + \frac{1}{2}\right)\ln\left(1 + \frac{1+\frac{1}{2}}{k}\right) = \left(i + \frac{1}{2}\right)\ln\left(1 + \left(i + \frac{1}{2}\right)x_3^*\right)$$
$$+ \frac{1 + \left(i + \frac{1}{2}\right)x_3^*}{1 + \left(i + \frac{1}{2}\right)x_3^*} \tag{1.3.46}$$

for $0 < x_3^* < k^{-1}$.

From (1.3.44), (1.3.45), and (1.3.46) it turns out that for i *fixed* and $k \to +\infty$, $f(i,k) \to 0$. By elaborating a little bit on (1.3.44), (1.3.45), and (1.3.46), one can estimate the rate of convergence of $f(i,k)$ to 0.

This will show that for fixed i,

$$f(i,k) = O(k^{-1}). \tag{1.3.47}$$

We return now to (1.3.43) to obtain

$$\phi_{i,2k+i} = \sqrt{\pi}\,\frac{(i+1)e}{\sqrt{k}\,(k+1)}\left[1 + o(1)\right]. \tag{1.3.48}$$

Therefore, (1.3.34) and (1.3.35) imply

$$\left|\sum_{k=0}^{\infty} \phi_{i,2k+i}\, y_{2k+i}(t)\right| \le m(t)\sqrt{\pi}\, e(i+1) \sum_{k=0}^{\infty} \frac{(2k+i)^{\alpha}}{\sqrt{k}\,(k+1)}. \tag{1.3.49}$$

The series on the right of (1.3.49) are absolutely convergent and the result follows. E.O.P.

The fact is that

$$\lim_{\nu\to\infty} a_\nu(t) = 0 \tag{1.3.50}$$

follows immediately by application of the Riemann–Lebesgue theorem. The proof that under certain circumstances (1.3.50) holds uniformly in $t \in J$ needs a bit of explanation. This is offered in the following lemma.

LEMMA 1.3.4. *Let* $|y(t, u)| \le M$ *for* $u \in I$ *and* $t \in J$,

$$\int_{-1}^{1} y(t, u)\omega(u) P_\nu(u)\, du = a_\nu(t) \tag{1.3.51}$$

where $\omega(u) \in C(I)$ *is a nonnegative function and* $P_\nu(u)$, $u = 0, 1, \ldots$ *are the orthogonal polynomials corresponding to* $\omega(u)$ *on* I.

Let $r_n(t)$ *be the notation of*

$$r_n(t) = \sum_{\nu=n}^{\infty} a_\nu^2(t). \tag{1.3.52}$$

Then

$$\lim_{\nu\to\infty} a_\nu(t) = 0 \tag{1.3.53}$$

uniformly in $t \in J$.

Proof. We first assume that we have already proved that $a_\nu(t)$, $\nu = 0, 1, \ldots$ are continuous functions of t for $t \in \bar{J}$. Also,

$$\int_{-1}^{1} y^2(t, u)\omega(u)\, du = \sum_{\nu=0}^{\infty} a_\nu^2(t) \tag{1.3.54}$$

is a continuous function of t for $t \in \bar{J}$.

Therefore, the functions $r_n(t)$, $n = 0, 1, \ldots$ are all continuous functions of t in $t \in \bar{J}$, and therefore, each $r_n(t)$ attains its maximum in $\bar{J}$ at a point t_n.

In fact, we will prove that uniformly for $t \in \bar{J}$

$$\lim_{n\to\infty} r_n(t) = 0. \tag{1.3.55}$$

Assume, on the contrary, that for any $\theta > 0$, θ arbitrarily small, there are $t \in \bar{J}$

$$r_n(t) < \theta$$

for n sufficiently large. This implies that there exists an infinite subsequence of n_i and an infinite sequence of $t_i \in \bar{J}$ such that

$$r_{n_i}(t_i) \geq \theta > 0. \tag{1.3.56}$$

We are going to show that this leads to a contradiction. Let

$$M_{n_i} = \max_t r_{n_i}(t) = r_{n_i}(t_{n_i}) \geq r_{n_i}(t_i) \geq \theta > 0 \tag{1.3.57}$$

for some t_{n_i}, $t_{n_i} \in \bar{J}$. Without loss of generality, assume that t_{n_i}, $i = 0, 1, \ldots$ is an infinite sequence such that

$$\lim_{i\to\infty} t_{n_i} = \hat{t} \tag{1.3.58}$$

$$\lim_{i\to\infty} M_{n_i} = \hat{M} = \limsup M_n. \tag{1.3.59}$$

For any $\delta(n) > 0$, $n = 0, 1, \ldots$ there exists a $\mu(n) > 0$ s.t.

$$|t - \hat{t}| < \mu(n) \tag{1.3.60}$$

implies

$$|r_n(t) - r_n(\hat{t})| < \delta(n), \qquad n = 0, 1, \ldots \tag{1.3.61}$$

Pick up a sequence of $t_{k(n)}$ s.t.

$$|t_{k(n)} - \hat{t}| < \mu(n) \tag{1.3.62}$$

implies

$$-\delta(n) < \sum_{\nu=k(n)}^{\infty} a_\nu^2(t_{k(n)}) - \sum_{\nu=k(n)}^{\infty} a_\nu^2(\hat{t}) < \delta(n) \tag{1.3.63}$$

or

$$\sum_{k(n)}^{\infty} a_\nu^2(\hat{t}) - \delta(n) < \sum_{k(n)}^{\infty} a_\nu^2(t_{k(n)}) \leq \sum_{k(n)}^{\infty} a_\nu^2(\hat{t}) + \delta(n). \tag{1.3.64}$$

Then

$$\sum_{k(n)}^{\infty} a_\nu^2(\hat{t}) - \delta(n) \le M_{k(n)} \le \sum_{k(n)}^{\infty} a_\nu^2(\hat{t}) + \delta(n). \tag{1.3.65}$$

But by the convergence of $\Sigma_0^\infty a_\nu^2(\hat{t})$, it turns out that

$$\lim_{n\to\infty} \sum_{k(n)}^{\infty} a_\nu^2(\hat{t}) = 0. \tag{1.3.66}$$

Obviously, we may choose $k(n) \to \infty$ for $n \to \infty$. We may also choose a sequence $\delta(n) \to 0$. Then, we find by (1.3.65) that

$$\lim_{n\to\infty} M_{k(n)} = 0. \tag{1.3.67}$$

This contradicts (1.3.56) with $\theta > 0$ and the result follows. E.O.P.

It is possible to find estimates on the *rate of convergence* of $a_\nu(t)$ to zero uniformly for t by imposing more conditions on $y(t, u)$.

Let us prove the following lemma.

LEMMA 1.3.5. *Let $y(t, u)$ be such that*

$$|y(t, -1)| \le K, \tag{1.3.68}$$

where K is a constant. Let $P_\nu(u)$ in (1.3.51) *correspond to*

$$\omega(u) = \sqrt{1 - u^2}\ . \tag{1.3.69}$$

Let

$$\int_0^\pi \left(\frac{\partial}{\partial\theta} [y(t, \cos\theta) \sin\theta] \right)^2 d\theta \le M, \tag{1.3.70}$$

where M is a constant. Then, there exists a constant $\beta > 0$ s.t.

$$|a_\nu(t)| \le \beta/(\nu + 1) \tag{1.3.71}$$

$\nu = 0, 1, \ldots,\ t \in \bar{J}$.

Proof. By applying the transformation

$$u = \cos\theta \tag{1.3.72}$$

to (1.3.51), one finds that

$$a_\nu(t) = \sqrt{\frac{2}{\pi}} \int_0^\pi F(t, \theta) \sin(\nu + 1)\theta \, d\theta, \tag{1.3.73}$$

where

$$F(t, \theta) := y(t, \cos\theta)\sin\theta. \tag{1.3.74}$$

By integration by parts one finds

$$\sqrt{\frac{\pi}{2}}\, a_\nu(t) = \left[\frac{F(t, 0^+) - F(t, \pi)(-1)^{\nu+1}}{(\nu+1)}\right] + \frac{L(t)}{\nu+1}, \tag{1.3.75}$$

where

$$L(t) = \int_0^\pi F_\theta(t, \theta)\cos(\nu+1)\theta\, d\theta. \tag{1.3.76}$$

But Schwarz's inequality yields

$$|L(t)| \le \left(\int_0^\pi F_\theta^2(t, \theta)\, d\theta\right)^{1/2} \sqrt{\pi/2}\,. \tag{1.3.77}$$

By (1.3.77) we get

$$|L(t)| \le \sqrt{M}\,\sqrt{\pi/2}\,. \tag{1.3.78}$$

Also,

$$F(t, \pi) = 0. \tag{1.3.79}$$

In order to estimate $F(t, 0^+)$ we use the identity

$$F(t, \theta) = \int_\pi^\theta F_\theta(t, s)\, ds. \tag{1.3.80}$$

Since $F_\theta(t, s)$ belongs to $L^2(0, \pi)$ it also belongs to $L^1(0, \pi)$ [8, p. 381]. Therefore,

$$|F(t, 0^+)| \le \sqrt{\pi M}\,. \tag{1.3.81}$$

Combining (1.3.78) with (1.3.79) and (1.3.81) in formula (1.3.75) yields (1.3.71) with

$$\beta = (1 + \sqrt{2}\,)\sqrt{M}\,. \tag{1.3.82}$$

E.O.P.

Since we want to use the results of this chapter in singularly perturbed problems, it is instructive to notice that $y(t, u)$, the solution of

$$(1 - u)y' + y = 0, \qquad y(0) = 1,$$

satisfies the conditions of the above lemma.

4. Reproducing the Limit Value

Let T be a regular matrix summability method. See [4, p. 43]. Let (C, k), (H, k), $k = 1, 2, \ldots$ denote the Cesaro and Holder regular methods [4, pp. 94–97], respectively. We adopt the following conventions.

If

$$\sum_{i=0}^{\infty} a_i \tag{1.4.1}$$

is an infinite series and A_k, $k = 0, 1, \ldots$ is the sequence of its partial sums

$$A_k := \sum_{i=0}^{k} a_i, \tag{1.4.2}$$

then

$$[(H, 1)A_k]_n := (n+1)^{-1}\left(\sum_{k=0}^{n} A_k\right), \tag{1.4.3}$$

$$[(H, 2)A_k]_N = (N+1)^{-1} \sum_{n=0}^{N} [(H, 1)A_k]_n. \tag{1.4.4}$$

Definition 1.4.1. We say that the sequence A_k is summable $(H, 2)$ to the sum A if

$$\lim_{N\to\infty} [(H, 2)A_k]_N = A \tag{1.4.5}$$

and we denote

$$(H, 2)\left(\sum_{i=0}^{\infty} a_i\right) = A. \tag{1.4.6}$$

We denote by

$$A_n^1 = \sum_{k=0}^{n} A_k \tag{1.4.7}$$

$$A_N^2 = \sum_{k=0}^{N} A_k^1 \tag{1.4.8}$$

$$[(C, 2)A_k]_N = \frac{2A_N^2}{(N+2)(N+1)}. \tag{1.4.9}$$

DEFINITION 1.4.2. We say that the sequence A_k is summable $(C, 2)$ to the limit A if

$$\lim_{N\to\infty} \left[(C, 2)A_k\right]_N = A \tag{1.4.10}$$

and we write in this case

$$(C, 2)\left(\sum_{i=0}^{\infty} a_i\right) = A. \tag{1.4.11}$$

In what follows, we also use the notation

$$q_{1,\alpha}(\alpha, t) := \frac{\partial}{\partial\alpha} q_1(\alpha, t) \tag{1.4.12}$$

$$\phi_\theta(t, \theta) := \frac{\partial}{\partial\theta}\phi(t, \theta) \tag{1.4.13}$$

etc.

LEMMA 1.4.1. *Let $y(t, \cos\theta)$ be an integrable function of θ for $0 \le \theta \le \pi$. Let*

$$\sqrt{\frac{2}{\pi}} \sum_{i=0}^{\infty} a_i(t) \frac{\sin(i+1)\text{arc}\cos u}{\sqrt{1-u^2}} = S_\infty(t, u) \tag{1.4.14}$$

be the Fourier expansion of $y(t, u)$ in terms of Tshebysheff's polynomials of the second kind on I. Let

$$\sqrt{\frac{2}{\pi}} \sum_{i=0}^{\infty} (i+1)a_i(t) = S_\infty(t, 1) \tag{1.4.15}$$

be the expansion of (1.4.14) *at $u = 1$. Let $y(t, u)$ be a continuous function of u for $u = 1$ s.t. for some $t \in \bar{J}$,*

$$\lim_{u\to 1^-} y(t, u) = y(t, 1). \tag{1.4.16}$$

Then,

$$(C, 2)\left(\sum_{i=0}^{\infty} (i+1)a_i(t)\right) = y(t, 1)\sqrt{\pi/2}\,. \tag{1.4.17}$$

More precisely, for ϵ arbitrarily small, $\epsilon > 0$, there exists $N_0(t)$ s.t.

$$\left|\left[(C, 2)\sum_{i=0}^{n} (i+1)a_i(t)\right]_N - \sqrt{(\pi/2)}\, y(t, 1)\right| \le \epsilon \tag{1.4.18}$$

if $N \ge N_0(t)$.

Proof. We have by [6, p. 68] with

$$S_n(t,u) = \sum_{i=0}^{n} a_i(t)\omega_i(u) \tag{1.4.19}$$

that

$$S_n(t,u) - y(t,u) = \left(\sqrt{\lambda_{n+1}}\right)\int_{-1}^{1}\sqrt{1-\eta^2}\left[\frac{y(\eta)-y(u)}{\eta-u}\right] \times\left[\omega_{n+1}(\eta)\omega_n(u) - \omega_{n+1}(u)\omega_n(\eta)\right] d\eta, \tag{1.4.20}$$

where [6, p. 115]

$$\omega_n(\eta) = \sqrt{\frac{2}{\pi}}\;\frac{\sin(n+1)\arccos\eta}{\sin\eta}, \qquad n = 0, 1, \ldots \tag{1.4.21}$$

are Tshebysheff's polynomials of the second kind.

Using the substitution

$$\arccos\eta = \theta, \qquad 0 \le \theta \le \pi, \tag{1.4.22}$$

one obtains

$$d\theta = -d\eta\Big/\sqrt{1-\eta^2}\,. \tag{1.4.23}$$

Also, let

$$u = \cos\psi, \qquad 0 \le \psi \le \pi. \tag{1.4.24}$$

Also, by [6, p. 107] it follows that

$$\lambda_{n+1} = \tfrac{1}{4}. \tag{1.4.25}$$

It is easily verified that

$$(\pi/4)(S_n(t,u) - y(t,u)) = \int_0^{\pi} R(\theta,\psi)A_n\, d\theta, \tag{1.4.26}$$

where

$$R(\theta,\psi) = \frac{\sin\theta}{\sin\psi}\left[\frac{y(t,\cos\theta) - y(t,\cos\psi)}{\cos\theta - \cos\psi}\right] \tag{1.4.27}$$

$$A_n = \sin(n+2)\theta\,\sin(n+1)\psi - \sin(n+2)\psi\,\sin(n+1)\theta. \tag{1.4.28}$$

Using the trigonometric identities,

$$2 \sin \alpha \sin \beta = \cos(\alpha - \beta) - \cos(\alpha + \beta), \tag{1.4.29}$$

$$2 \cos \alpha \sin \beta = \sin(\beta - \alpha) + \sin(\beta + \alpha), \tag{1.4.30}$$

one obtains

$$\begin{aligned} A_n = \tfrac{1}{2}[&\cos(n+1)(\theta - \psi) - \cos(n+1)(\theta + \psi)](\cos\theta - \cos\psi) \\ &+ \sin(n+1)(\theta + \psi)[\sin\theta - \sin\psi] + \sin(n+1)(\psi - \theta) \\ &\times (\sin\theta - \sin\psi)]. \end{aligned} \tag{1.4.31}$$

Let us focus on the integrand $R(\theta, \psi)A_n$ as a function of both variables in the rectangle B,

$$B = \{\langle\theta, \psi\rangle | 0 \le \theta \le \pi, 0 \le \psi \le \pi\}.$$

Let us write

$$R(\theta, \psi)A_n = [K_1(\theta, \psi) + K_2(\theta_2, \psi)][y(t, \cos\theta) - y(t, \cos\psi)], \tag{1.4.32}$$

where

$$K_1(\theta, \psi) = \frac{1}{2}\frac{\sin\theta}{\sin\psi}[\cos(n+1)(\theta - \psi) - \cos(n+1)(\theta + \psi)] \tag{1.4.33}$$

$$K_2(\theta, \psi) = \frac{1}{2}\frac{\sin\theta}{\sin\psi}[\sin(n+1)(\theta + \psi) + \sin(n+1)(\psi - \theta)] \times \left(-\cot\frac{\theta + \psi}{2}\right). \tag{1.4.34}$$

It is easily verified that

$$\frac{\sin\theta - \sin\psi}{\cos\theta - \cos\psi} = -\frac{2\sin\dfrac{\theta - \psi}{2}\cos\dfrac{\theta + \psi}{2}}{2\sin\dfrac{\theta - \psi}{2}\sin\dfrac{\theta + \psi}{2}} = -\cot\frac{\theta + \psi}{2}. \tag{1.4.35}$$

Using trigonometric identities, one obtains

$$K_1(\theta, \psi) = \sin\theta(\sin(n+1)\theta)\frac{\sin(n+1)\psi}{\sin\psi} \tag{1.4.36}$$

$$K_2(\theta, \psi) = -\sin\theta\left(\cot\frac{\theta + \psi}{2}\right)(\cos(n+1)\theta)\left(\frac{\sin(n+1)\psi}{\sin\psi}\right). \tag{1.4.37}$$

Consider the coefficient $K_1(\theta, \psi)$ in (1.4.36). Obviously $K_1(\theta, \psi)$ is a continuous function of θ, ψ for

$$-\infty < \theta < +\infty, \qquad -\infty < \psi < +\infty.$$

Moreover,

$$\lim_{\theta \to j\pi} K_1(\theta, \psi) = 0, \qquad j = 0, \pm 1, \pm 2, \ldots$$

uniformly for $-\infty < \psi < +\infty$ since

$$K_1(\theta, \psi) = O((\theta - j\pi)^2(n + 1)) \tag{1.4.38}$$

with respect to $\theta \to j\pi$.

Let us show that $K_2(\theta, \psi)$ is a bounded function of θ, ψ for $0 \leq \theta \leq \pi$, $0 \leq \psi \leq \pi$. It suffices to consider

$$K_3(\theta, \psi) = \frac{\sin \theta}{\sin((\theta + \psi)/2)}. \tag{1.4.39}$$

The only problematic points of possible nonboundedness in the rectangle B may occur at

$$(\theta + \psi)/2 = j\pi, \qquad j = 0, 1, \ldots . \tag{1.4.40}$$

But for $j = 0$ we need consider only θ, ψ small and positive. Obviously,

$$K_3(\theta, \psi) = \frac{\sin \theta}{\theta} \frac{\theta + \psi}{\sin((\theta + \psi)/2)} \frac{\theta}{\theta + \psi}. \tag{1.4.41}$$

For

$$0 < \theta < \pi/2, \qquad 0 < \psi < \pi/2 \tag{1.4.42}$$

$$\theta/ (\theta + \psi) \leq 1. \tag{1.4.43}$$

A similar discussion for $j = 1$ leads to consideration of values of θ, ψ close to π. Since

$$K_3(\theta, \psi) = \frac{\sin(\pi - \theta)}{\sin\left(\pi - \dfrac{\theta + \psi}{2}\right)},$$

it suffices to consider

$$\frac{\pi - \theta}{(\pi - \theta) + (\pi - \psi)}, \tag{1.4.44}$$

which satisfies

$$\frac{\pi - \theta}{(\pi - \theta) + (\pi - \psi)} \leq 1 \tag{1.4.45}$$

for

$$0 < \pi - \theta < \pi/2, \qquad 0 < \pi - \psi < \pi/2. \tag{1.4.46}$$

Moreover, for any $0 < \delta < \pi$, one finds that $K_3(\theta, \psi)$ is a continuous function of θ, ψ for

$$\delta \leq \theta \leq \pi - \delta, \qquad \delta \leq \psi \leq \pi - \delta, \tag{1.4.47}$$

Since we are interested in particular in the evaluation of (1.4.26) for $u = 1$ or $\psi = 0$, we find that by (1.4.36) and (1.4.37),

$$a_k(\theta) := K_1(\theta, 0) = k(\sin k\theta) \sin \theta \tag{1.4.48}$$

$$b_k(\theta) := K_2(\theta, 0) = -2k\left(\cos k\frac{\theta}{2}\right) \cos^2\frac{\theta}{2}. \tag{1.4.49}$$

Let us compute the arithmetic means of a_k, b_k by use of the identities from [3, pp. 30–31, 24–25]:

$$\sum_{k=0}^{n} \sin k\theta = \frac{1}{2}\left[\cos(2n+1)\frac{\theta}{2} - \cos\frac{\theta}{2}\right] \csc\frac{\theta}{2}, \tag{1.4.50}$$

$$\sum_{k=0}^{n} k \sin k\theta = \frac{1}{4}\sin(n+1)\theta \csc^2\frac{\theta}{2} - \frac{1}{2}(n+1) \cos(2n+1)\frac{\theta}{2}\csc\frac{\theta}{2}, \tag{1.4.51}$$

$$\sum_{k=0}^{n} \cos k\theta = \frac{1}{2}\sin(2n+1)\frac{\theta}{2}\csc\frac{\theta}{2} + \frac{1}{2}, \tag{1.4.52}$$

$$\sum_{k=0}^{n} k \cos k\theta = \frac{(n+1)}{2}\sin(2n+1)\frac{\theta}{2}\csc\frac{\theta}{2} - \frac{1}{4}(1 - \cos(n+1)\theta) \csc^2\frac{\theta}{2}. \tag{1.4.53}$$

The arithmetic means of the sequence $a_k(\theta)$, $k = 1, 2, \ldots$ is found by (1.4.48),

$$\begin{aligned} H_n^1(\theta) &:= (n+1)^{-1} \sum_{k=0}^{n} a_k(\theta) \\ &= \left[\frac{1}{4}(n+1)^{-1} \sin(n+1)\theta \csc^2\frac{\theta}{2} - \frac{1}{2}\cos(2n+1)\frac{\theta}{2}\csc\frac{\theta}{2}\right] \sin\theta. \end{aligned} \tag{1.4.54}$$

It is easily verified that $H_n^1(\theta)$ is bounded for $0 \le \theta \le \pi$, $n = 1, 2, \ldots$. Denote this bound by M_1 so that

$$|H_n^1(\theta)| \le M_1 = \left(\frac{\pi}{2} - 1\right). \tag{1.4.55}$$

Consider

$$I_1(\delta) = \int_0^\delta H_n^1(\theta)[y(t, \cos\theta) - y(t, 1)]\, d\theta. \tag{1.4.56}$$

It is easily deduced that

$$|I_1(\delta)| \le \delta\left(\frac{\pi}{2} - 1\right)\|y(t, \cos\theta) - y(t, 1)\|_{E_\delta}. \tag{1.4.57}$$

We denote by $|\ |$ the absolute value and by $\|\ \|_{E_\delta}$ we denote

$$\|\ \|_{E_\delta} = \sup_{E_\delta} |\ |. \tag{1.4.58}$$

Let

$$I_1(\pi) = \int_\delta^\pi H_n^1(\theta)[y(t, \cos\theta) - y(t, 1)]\, d\theta. \tag{1.4.59}$$

Then by the Lebesgue–Riemann theorem we know there exists an absolutely continuous function $\phi(t, \theta)$ on $0 \le \theta < \pi$ such that

$$|y(t, \cos\theta) - y(t, 1) - \phi(t, \theta)| \le \mu, \tag{1.4.60}$$

where μ is arbitrarily small, and therefore,

$$I_1(\pi) = K_1(\pi) + K_2(\pi), \tag{1.4.61}$$

where

$$K_1(\pi) = \int_\delta^\pi H_n^1(\theta)[y(t, \cos\theta) - y(t, 1) - \phi(t, \theta)]\, d\theta, \tag{1.4.62}$$

and

$$K_2(\pi) = \int_\delta^\pi H_n^1(\theta)\phi(t, \theta)\, d\theta. \tag{1.4.63}$$

Obviously,

$$|K_1(\pi)| \le \mu(\pi - \delta)\left(\frac{\pi}{2} - 1\right) \tag{1.4.64}$$

and

$$|K_2(\pi)| = \left(\int_\delta^\pi H_n^1(\theta)\, d\theta\right)\phi(t, \pi) - \int_\delta^\pi \left(\int_\delta^\theta H_n^1(\eta)\, d\eta\right)\phi_\theta(t, \theta)\, d\theta. \tag{1.4.65}$$

By straightforward integration we have

$$\left|\int_\delta^\pi H_n^1(\theta)\, d\theta\right| \leq (\csc^2 \delta/2)(n+1)^{-1}\alpha, \tag{1.4.66}$$

where α is some positive constant. This implies that there exists $\hat{\alpha} > 0$ such that

$$|K_2(\pi)| \leq \hat{\alpha}(n+1)^{-1} \csc^2 \delta/2. \tag{1.4.67}$$

It turns out that

$$|I_1(\pi)| \leq \mu(\pi - \delta)\left(\frac{\pi}{2} - 1\right) + \hat{\alpha}(n+1)^{-1} \csc^2 \delta/2. \tag{1.4.68}$$

Let us consider now the arithmetic means of $b_k(\theta)$. By (1.4.53), one obtains

$$\begin{aligned} B_n^1 &= \left(\sum_{k=0}^n b_k\right)(n+1)^{-1} \\ &= \left(\sin(2n+1)\frac{\theta}{2}\csc\frac{\theta}{2} - \frac{1}{(n+1)}\sin^2(n+1)\frac{\theta}{2}\csc^2\frac{\theta}{2}\right)\left(-\cos^2\frac{\theta}{2}\right). \end{aligned} \tag{1.4.69}$$

One sets

$$B_n^1 = (c_n + e_n)\left(-\cos^2\frac{\theta}{2}\right), \tag{1.4.70}$$

where

$$c_n = -\sin(2n+1)\frac{\theta}{2}\csc\frac{\theta}{2}, \tag{1.4.71}$$

$$e_n = \frac{1}{(n+1)}\sin^2(n+1)\frac{\theta}{2}\csc^2\frac{\theta}{2}. \tag{1.4.72}$$

Let us estimate

$$E_n := E_n(\delta) + E_n(\pi) \tag{1.4.73}$$

with

$$E_n(\delta) = \int_0^\delta e_n(\theta) \cos^2\frac{\theta}{2}\left[y(t, \cos\theta) - y(t, 1)\right] d\theta \tag{1.4.74}$$

$$E_n(\theta) = \int_\delta^\pi e_n(\theta) \cos^2\frac{\theta}{2}\left[y(t, \cos\theta) - y(t, 1)\right] d\theta. \tag{1.4.75}$$

This integral involves a nonnegative kernel which simplifies the task of finding the estimates:

$$E_n(\pi) = \frac{1}{(n+1)}\int_0^\pi \sin^2(n+1)\frac{\theta}{2}\csc^2\frac{\theta}{2}\cos^2\frac{\theta}{2}(y(t, \cos\theta) - y(t, 1))\, d\theta. \tag{1.4.76}$$

Let

$$\Delta E_n(\delta) := \frac{1}{(n+1)}\int_0^\delta \sin^2(n+1)\frac{\theta}{2}\cos^2\frac{\theta}{2}\left[\csc^2\frac{\theta}{2} - \left(\frac{1}{2}\theta^2\right)^{-1}\right] \times \left[y(t, \cos\theta) - y(t, 1)\right] d\theta. \tag{1.4.77}$$

Since

$$g(\theta) = \csc^2\frac{\theta}{2} - \left(\frac{\theta}{2}\right)^{-2} \tag{1.4.78}$$

is a continuous function for $0 \le \theta \le \pi$, one obtains

$$|\Delta E_n(\delta)| \le \frac{1}{(n+1)}\delta \|y(t, \cos\theta) - y(t, 1)\|_{E_\delta}(\|g(\theta)\|_{E_\delta}). \tag{1.4.79}$$

Therefore, the estimation of $E_n(\delta)$, can be done via the estimation of (1.4.79) and

$$\hat{E}_n(\delta) := \frac{1}{(n+1)}\int_0^\delta \left(\frac{\theta}{2}\right)^{-2} \sin^2(n+1)\frac{\theta}{2}\cos^2\frac{\theta}{2}(y(t, \cos\theta) - y(t, 1))\, d\theta, \tag{1.4.80}$$

$$\hat{E}_n(\delta) \le \|y(t, \cos\theta) - y(t, 1)\|_{E_\delta} q_n(\delta), \tag{1.4.81}$$

$$q_n(\delta) = \frac{1}{(n+1)}\int_0^\delta \left(\frac{\theta}{2}\right)^{-2} \sin^2(n+1)\frac{\theta}{2}\cos^2\frac{\theta}{2}\, d\theta. \tag{1.4.82}$$

Performing the transformation

$$(n+1)\frac{\theta}{2} = \gamma, \tag{1.4.83}$$

$$d\theta = \frac{2}{n+1}d\gamma, \tag{1.4.84}$$

we have

$$q_n(\delta) = 2\int_0^{(n+1)\frac{\delta}{2}} \gamma^{-2} \sin^2 \gamma \cos^2 \frac{\gamma}{(n+1)}\, d\gamma, \tag{1.4.85}$$

or

$$q_n(\delta) \le q, \tag{1.4.86}$$

with

$$q := 2\int_0^{\infty} \gamma^{-2} \sin^2 \gamma \, d\gamma. \tag{1.4.87}$$

So, we are left with the evaluation,

$$C_n^1 = \int_0^{\pi} c_n(\theta) \cos^2 \frac{\theta}{2} [y(t, \cos\theta) - y(t, 1)]\, d\theta. \tag{1.4.88}$$

We apply to the sequence C_n^1, the arithmetic means and thus obtain by [3, p. 30, formula 1.342, No. 3],

$$\begin{aligned} C_n^2 &= (n+1)^{-1} \sum_{k=0}^{n} C_k^1 \\ &= \frac{1}{(n+1)} \int_0^{\pi} \sin^2(n+1)\frac{\theta}{2} \csc^2 \frac{\theta}{2} \cos^2 \frac{\theta}{2} (y(t, \cos\theta) - y(t, 1))\, d\theta. \end{aligned} \tag{1.4.89}$$

Comparison of (1.4.89) with the formula (1.4.73) for E_n shows that

$$E_n = C_n^2. \tag{1.4.90}$$

Let us show that given $\epsilon > 0$, ϵ arbitrarily small, $N_0(t)$ can be found s.t. for $N > N_0(t)$ and (1.4.18) holds.

By (1.4.54), (1.4.56), (1.4.59), (1.4.69)–(1.4.78), (1.4.80), and (1.4.88)–(1.4.90) we have

$$\begin{aligned} r_N &:= [(H, 2)(S_n(t, u) - y(t, u))]_N \\ &= \frac{4}{\pi} \left[[(H, 1)(I_1(\delta) + I_1(\pi))]_N + [(H, 1)E_n]_N - C_N^2 \right]. \end{aligned} \tag{1.4.91}$$

Let $\rho > 0$ be a small number. Since (1.4.16) holds, there exists a δ s.t. for $0 \le \theta \le \delta < \pi$,

$$|y(t, \cos\theta) - y(t, 1)| < \rho. \tag{1.4.92}$$

This implies

$$\|y(t, \cos\theta) - y(t, 1)\|_{E_\delta} \le \rho, \tag{1.4.93}$$

and therefore,

$$|I_1(\delta)| \le \delta\rho\left(\frac{\pi}{2} - 1\right). \tag{1.4.94}$$

We choose in (1.4.68), $\mu = \rho$ and after δ is fixed we find $n_1(t)$ large enough that for $n \ge n_1(t)$

$$1/(n+1) < \rho. \tag{1.4.95}$$

We deduce from (1.4.68) and (1.4.95) that

$$|I_1(\pi)| \le \rho\left((\pi - \delta)\left(\frac{\pi}{2} - 1\right) + \hat{\alpha}\csc^2\delta/2\right) \tag{1.4.96}$$

for $n > n_1(t)$. By what we have just said (1.4.95) implies that for $n > n_1(t)$,

$$|\Delta E_n(\delta)| \le \delta\rho^2\|g(\theta)\|_{E_\delta}, \tag{1.4.97}$$

$$|\hat{E}_n(\delta)| \le \rho q, \tag{1.4.98}$$

and

$$|E_n(\pi)| \le \rho(\csc^2\delta/2)\beta. \tag{1.4.99}$$

($E_n(\pi)$ is estimated in a manner similar to the way $I_1(\pi)$ was estimated with β some positive constant.)

We notice by the definition of the $(H, 1)$, $(C, 1)$ methods that if a sequence A_n, $n = 0, 1, \ldots$ satisfies

$$|A_n| \le \alpha, \tag{1.4.100}$$

then also

$$|[(H, 1)A_n]_n| \le \alpha. \tag{1.4.101}$$

Now combine (1.4.94) with (1.4.96) and (1.4.97)–(1.4.99) to obtain

$$|I_1| \le \rho\left[\pi\left(\frac{\pi}{2} - 1\right) + 2\csc^2\delta/2\right], \tag{1.4.102}$$

$$|E_n| \le \rho\left[q + (\csc^2\delta/2)\beta + \delta\rho\|g(\theta)\|_E\right]. \tag{1.4.103}$$

However, by virtue of (1.4.100), (1.4.101) and (1.4.102), (1.4.103), we obtain

$$|r_N| \le \frac{4}{\pi}\rho K \tag{1.4.104}$$

with

$$K := \left[\pi\left(\frac{\pi}{2} - 1\right) + 2q + (\hat{\alpha} + 2\beta)\csc^2\delta/2 + 2\delta\rho\|g(\theta)\|_E\right]. \tag{1.4.105}$$

Choose now

$$\rho = \frac{\pi}{4}\epsilon K^{-1} \tag{1.4.106}$$

and the result follows. E.O.P.

Remark. If (1.4.16) holds uniformly with respect to $t \in J_1$, $J_1 \subset J$, then one can show that $N_0(t)$ associated with (1.4.18) can be chosen independent of t, $t \in J_1$.

COROLLARY. *The conclusions of Lemma* 1.4.1 *hold with* $(H, 1)$, $(H, 2)$ *replaced, respectively, by* $(C, 1)$, $(C, 2)$.

Proof. This is a result of the equivalence of the (H, k) and the (C, k) methods. See [4, p. 103].

Remark. The singular point $u = -1$ or $\psi = \pi$ can be treated similarly by considering the function $\tilde{y}(t, u)$

$$\tilde{y}(t, u) := y(t, -u) \tag{1.4.107}$$

at $u = 1$.

5. Reproducing the Function

LEMMA 1.5.1. *Let $y(t, u)$ be continuous at $u = u_0 \in I$.*

Let $y(t, \cos\theta)$ be an integrable function of θ for $0 \le \theta \le \pi$.

Let (1.4.14) *be its Fourier expansion in Tshebysheff's polynomials of the second kind. Then given ϵ arbitrarily small, there exists $n_0(t)$ s.t.*

$$\left|\sqrt{\frac{2}{\pi}}\sum_{i=0}^{k-1} a_i(t)\frac{\sin(i+1)\arccos u}{\sqrt{1-u^2}} - y(t, u)\right| < \epsilon \tag{1.5.1}$$

for $k > k_0(t)$ and $u = u_0$.

Proof. As we did before, we proceed to find estimates. We rewrite the formula (1.4.14) as follows:

$$\int_0^\pi K_1(\theta, \psi)[y(t, \cos\theta) - y(t, \cos\psi)]\,d\theta = J_{1k} - J_{2k} \tag{1.5.2}$$

with

$$J_{1k} = \int_0^{\pi} V_{1k}\, d\theta \tag{1.5.3}$$

$$J_{2k} = \int_0^{\pi} V_{2k}\, d\theta \tag{1.5.4}$$

$$V_{1k} = \frac{1}{2}\frac{\sin\theta}{\sin\psi}\cos k(\theta-\psi)[y(t,\cos\theta) - y(t,\cos\psi)] \tag{1.5.5}$$

$$V_{2k} = \frac{1}{2}\frac{\sin\theta}{\sin\psi}\cos k(\theta+\psi)[y(t,\cos\theta) - y(t,\cos\psi)]. \tag{1.5.6}$$

Using the transformation in J_{1k},

$$\theta - \psi = \alpha, \tag{1.5.7}$$

and using the transformation in J_2,

$$\theta + \psi = \alpha, \tag{1.5.8}$$

one finds that

$$J_{1k} = \frac{1}{2}\int_{-\psi}^{\pi-\psi} \frac{\sin(\alpha+\psi)}{\sin\psi}\cos k\alpha[y(t,\cos(\alpha+\psi)) - y(t,\cos\psi)]\, d\alpha \tag{1.5.9}$$

$$J_{2k} = \frac{1}{2}\int_{\psi}^{\pi+\psi} \frac{\sin(\alpha-\psi)}{\sin\psi}\cos k\alpha[y(t,\cos(\alpha-\psi)) - y(t,\cos\psi)]\, d\alpha. \tag{1.5.10}$$

We set

$$J_{1k} - J_{2k} = J_{3k} + J_{4k} + J_{5k} + J_{6k} \tag{1.5.11}$$

with

$$J_{3k} = \frac{1}{2}\int_{\psi}^{\pi-\psi} (\cos k\alpha) y(t,\cos\psi)\left(\frac{\sin(\alpha-\psi) - \sin(\alpha+\psi)}{\sin\psi}\right) d\alpha \tag{1.5.12}$$

$$J_{4k} = \frac{1}{2}\int_{\psi}^{\pi-\psi} \cos k\alpha \times\left[\frac{\sin(\alpha+\psi)y(t,\cos(\alpha+\psi)) - \sin(\alpha-\psi)y(t,\cos(\alpha-\psi))}{\sin\psi}\right] d\alpha \tag{1.5.13}$$

$$J_{5k} = \frac{1}{2}\int_{-\psi}^{\psi} \frac{\sin(\alpha+\psi)}{\sin\psi}\cos k\alpha[y(t,\cos(\alpha+\psi)) - y(t,\cos\psi))]\, d\alpha \tag{1.5.14}$$

$$J_{6k} = -\frac{1}{2}\int_{\pi-\psi}^{\pi+\psi} \frac{\sin(\alpha-\psi)}{\sin\psi}\cos k\alpha[y(t,\cos(\alpha-\psi)) - y(t,\cos\psi)]\, d\alpha. \tag{1.5.15}$$

Letting in (1.4.16),

$$\alpha = \pi + u, \tag{1.5.16}$$

we obtain

$$J_{6k} = \frac{(-1)^k}{2} \int_{-\psi}^{\psi} \frac{\sin(\alpha - \psi)}{\sin \psi} \cos k\alpha [y(t, -\cos(\alpha - \psi)) - y(t, \cos \psi)]\, d\alpha. \tag{1.5.17}$$

We substitute in J_{3k} the identities

$$\sin(\alpha - \psi) - \sin(\alpha + \psi) = -2 \sin \psi \cos \alpha \tag{1.5.18}$$

and

$$2 \cos k\alpha \cos \alpha = \cos(k+1)\alpha + \cos(k-1)\alpha. \tag{1.5.19}$$

We obtain

$$\begin{aligned} J_{3k} &= \frac{-1}{2} \int_{\psi}^{\pi-\psi} y(t, \cos \psi)[\cos(k+1)\alpha + \cos(k-1)\alpha]\, d\alpha \\ &= -\frac{1}{2} y(t, \cos \psi) \left[\frac{\sin(k+1)(\pi - \psi) - \sin(k+1)\psi}{k+1} \right. \\ &\qquad \left. + \frac{\sin(k-1)(\pi - \psi) - \sin(k-1)\psi}{(k-1)} \right] \\ &= \frac{1}{2} y(t, \cos \psi) \left[\frac{\sin 2(k+1)\psi}{k+1} + \frac{\sin 2(k-1)\psi}{(k-1)} \right]. \end{aligned} \tag{1.5.20}$$

We now have a set of identities which will be used for future estimations of the contributions of $K_1(\theta, \psi)$ to the value of (1.4.26).

Let us turn now to the corresponding identities which will help us estimate the contributions of $K_2(\theta, \psi)$ in the evaluation of (1.4.26).

The procedure is very much the same.

The main change in the identities corresponding to $K_2(\theta, \psi)$ will essentially be due to the replacement of $\sin \theta$ in V_{1k}, V_{2k} by

$$-\sin \theta \cot \frac{\theta + \psi}{2} \tag{1.5.21}$$

as a result of the additional factor $\cot((\theta + \psi)/2)$.

A slight change in form is also due to the fact that we will deal with $\sin k\alpha$ rather than with $\cos k\alpha$.

We let

$$\int_0^{\pi} K_2(\theta, \psi)[y(t, \cos\theta) - y(t, \cos\psi)]\, d\theta = L_{1k} - L_{2k}, \quad (1.5.22)$$

with

$$L_{1k} = \int_0^{\pi} W_{1k}\, d\theta, \quad (1.5.23)$$

$$L_{2k} = \int_0^{\pi} W_{2k}\, d\theta, \quad (1.5.24)$$

$$W_{1k} = \frac{1}{2}\frac{\sin\theta}{\sin\psi}\cot\frac{\theta+\psi}{2}\sin k(\theta-\psi)[y(t,\cos\theta) - y(t,\cos\psi)], \quad (1.5.25)$$

$$W_{2k} = \frac{1}{2}\frac{\sin\theta}{\sin\psi}\cot\frac{\theta+\psi}{2}\sin k(\theta+\psi)[y(t,\cos\theta) - y(t,\cos\psi)]\, d\alpha. \quad (1.5.26)$$

We use in L_{1k}, L_{2k} the transformations (1.5.7), (1.5.8) to obtain

$$L_{1k} = \frac{1}{2}\int_{-\psi}^{\pi-\psi} \frac{\sin(\alpha+\psi)\cot((\alpha/2)+\psi)}{\sin\psi} \times \sin k\alpha[y(t, \cos(\alpha+\psi)) - y(t,\cos\psi)]\, d\alpha, \quad (1.5.27)$$

$$L_{2k} = \frac{1}{2}\int_{\psi}^{\pi+\psi} \frac{\sin(\alpha-\psi)\cot(\alpha/2)\sin k\alpha}{\sin\psi} \times [y(t,\cos(\alpha-\psi)) - y(t,\cos\psi)]\, d\alpha. \quad (1.5.28)$$

We let

$$L_{1k} - L_{2k} = L_{3k} + L_{4k} + L_{5k} + L_{6k}, \quad (1.5.29)$$

with

$$L_{3k} = \frac{1}{2}\int_{\psi}^{\pi-\psi}\left[\frac{(\sin k\alpha)y(t,\cos\psi)(\sin(\alpha-\psi)\cot(\alpha/2) - \sin(\alpha+\psi)\cot((\alpha/2)+\psi))}{\sin\psi}\right] d\alpha, \quad (1.5.30)$$

$$L_{4k} = \frac{1}{2}\int_{\psi}^{\pi-\psi}\frac{\sin k\alpha[\sin(\alpha+\psi)\cot((\alpha/2)+\psi)y(t,\cos(\alpha+\psi) - \sin(\alpha-\psi)\cot(\alpha/2)y(t,\cos(\alpha-\psi)]}{\sin\psi}\, d\alpha, \quad (1.5.31)$$

$$L_{5k} = \frac{1}{2}\int_{-\psi}^{\psi}\frac{\sin(\alpha+\psi)\cot((\alpha/2)+\psi)}{\sin\psi} \times \sin k\alpha[y(t,\cos(\alpha+\psi)) - y(t,\cos\psi)]\, d\alpha, \quad (1.5.32)$$

$$L_{6k} = -\frac{1}{2}\int_{\pi-\psi}^{\pi+\psi}\frac{\sin(\alpha-\psi)\cot(\alpha/2)\sin k\alpha}{\sin\psi} \times [y(t,\cos(\alpha-\psi)) - y(t,\cos\psi)]\, d\alpha. \quad (1.5.33)$$

We substitute (1.5.16) in (1.5.33) to obtain

$$L_{6k} = \frac{(-1)^k}{2} \int_{-\psi}^{\psi} \frac{\sin(\alpha - \psi) \tan(\alpha/2) \sin k\alpha}{\sin \psi}$$

$$\times \left[y(t, -\cos(\alpha - \psi)) - y(t, \cos \psi) \right] d\alpha. \tag{1.5.34}$$

We are ready now to proceed with the estimations.

By (1.5.12) we obtain

$$|J_{3k}| \le \frac{1}{2} |y(t, \cos \psi)| \frac{k}{k^2 - 1}, \qquad k = 1, 2, \ldots . \tag{1.5.35}$$

Let $q(\alpha, \psi)$ be an absolutely continuous function of α, $0 \le \alpha \le \pi$ s.t.

$$|\sin(\alpha + \psi) y(\cos(\alpha + \psi)) - \sin(\alpha - \psi) y(t, \cos(\alpha - \psi)) - q(\alpha, \psi)| \le \mu_4; \tag{1.5.36}$$

then,

$$|J_{4k}| \le \frac{\mu_4}{2} \frac{|\pi - 2\psi|}{\sin \psi} + k^{-1} \frac{[2\tilde{q} + |\pi - 2\psi| \tilde{q}_\alpha]}{\sin \psi}. \tag{1.5.37}$$

The inequality (1.5.37) was deduced by integration by parts

$$\frac{1}{2} \int_{\psi}^{\pi - \psi} \cos k\alpha q(\alpha, \psi)\, d\alpha$$

$$= k^{-1} \Big[\sin k(\pi - \psi) q(\pi - \psi, \psi) - (\sin k\psi) q(\psi, \psi)$$

$$- \int_{\psi}^{\pi - \psi} (\sin k\alpha) q_\alpha(\alpha, \psi)\, d\alpha \Big] \tag{1.5.38}$$

and by estimation with the constants $\tilde{q}$, $\tilde{q}_\alpha$ which satisfy

$$|q(\alpha, \psi)| \le \tilde{q}, \qquad 0 \le \alpha \le \pi, \tag{1.5.39}$$

$$|q_\alpha(\alpha, \psi)| \le \tilde{q}_\alpha, \qquad 0 \le \alpha \le \pi. \tag{1.5.40}$$

Let $q_1(\alpha, \psi)$ be an absolutely continuous function of α for $-\psi \le \alpha \le \psi$ s.t.

$$|y(t, \cos(\alpha + \psi)) - y(t, \cos \psi) - q_1(\alpha, \psi)| \le \mu_5, \tag{1.5.41}$$

then it is possible to find the estimation

$$|J_{5k}| = \mu_5 \frac{\psi}{\sin\psi} + k^{-1}\left[\frac{|\sin 2\psi|}{2\sin\psi} + \frac{\psi}{\sin\psi}(\tilde{q}_1 + \tilde{q}_{1,\alpha})\right] \quad (1.5.42)$$

$$\frac{1}{2}\int_{-\psi}^{\psi} \frac{\sin(\alpha+\psi)}{\sin\psi}(\cos k\alpha) q_1(\alpha,\psi)\,d\alpha$$

$$= \frac{k^{-1}}{2}\frac{\sin 2\psi}{\sin\psi}\cos k\psi - \frac{k^{-1}}{2}$$

$$\times \int_{-\psi}^{\psi} \frac{\sin k\alpha}{\sin\psi}\left[\cos(\alpha+\psi)q_1 + \sin(\alpha+\psi)q_{1,\alpha}(\alpha,\psi)\right]d\alpha. \quad (1.5.43)$$

Similarly, let $q_2(\alpha, \psi)$ be an absolutely continuous function of α for $-\psi \le \alpha \le \psi$ s.t.

$$|y(t, -\cos(\alpha - \psi)) - y(t, \cos\psi) - q_2(\alpha,\psi)| \le \mu_6. \quad (1.5.44)$$

Then

$$|J_{6k}| \le \mu_6 \frac{\psi}{\sin\psi} + k^{-1}\left[\frac{|\sin 2\psi|}{2\sin\psi} + \frac{\psi}{\sin\psi}(\tilde{q}_2 + \tilde{q}_{2,\alpha})\right] \quad (1.5.45)$$

with

$$|q_2(\alpha,\psi)| \le \tilde{q}_2, \quad (1.5.46)$$

$$|(q_2)_\alpha(\alpha,\psi)| \le \tilde{q}_{2,\alpha}, \quad (1.5.47)$$

($(q_1)_\alpha$, $(q_2)_\alpha$ means differentiation w.r.t. α.

We now turn to the estimation of L_{jk}, $j = 3, 4, 5, 6$. Let $p_3(\alpha, \psi)$ be an absolutely continuous function such that

$$\left|\sin(\alpha-\psi)\cot\frac{\alpha}{2} - \sin(\alpha+\psi)\cot\left(\frac{\alpha}{2}+\psi\right) - p_3(\alpha,\psi)\right| \le \gamma_3 \quad (1.5.48)$$

for $\psi \le \alpha \le \pi - \psi$. Then,

$$|L_{3k}| \le \gamma_3 \frac{|\pi - 2\psi||y(t,\cos\psi)|}{2\sin\psi} + k^{-1}\left[\frac{|y(t,\cos\psi)|}{\sin\psi}\tilde{p}_{3,\alpha}\frac{|\pi-2\psi|}{2}\right], \quad (1.5.49)$$

where $\tilde{p}_{3,\alpha}$ is a constant such that

$$|p_{3,\alpha}(\alpha,\psi)| \le \tilde{p}_{3,\alpha}. \quad (1.5.50)$$

This inequality can be improved since it holds uniformly for $0 < \psi < \pi$.

Let $p_4(\alpha, \psi)$ be an absolutely continuous function of α, for $\psi \leq \alpha \leq \pi - \psi$ s.t.

$$\left|\sin(\alpha + \psi) \cot\left(\frac{\alpha}{2} + \psi\right)y(t, \cos(\alpha + \psi)) \right.$$
$$\left. - \sin(\alpha - \psi) \cot\frac{\alpha}{2} y(t, \cos(\alpha - \psi) - p_4(\alpha, \psi)\right| \leq \gamma_4. \tag{1.5.51}$$

Then

$$|L_{4k}| \leq \frac{\gamma_4|\pi - 2\psi|}{2 \sin \psi} + k^{-1}\frac{|\pi - 2\psi|}{2 \sin \psi}\tilde{p}_{4, \alpha}, \tag{1.5.52}$$

where $\tilde{p}_{4, \alpha}$ is a constant s.t.

$$|p_{4, \alpha}(\alpha, \psi)| \leq \tilde{p}_{4, \alpha}, \tag{1.5.53}$$

for $\psi \leq \alpha \leq \pi - \psi$.

We would like to mention that according to inequalities (1.4.43), (1.4.45) we can stipulate that in the corresponding domains of integration, we have for suitable constants

$$\left|\sin(\alpha - \psi) \cot\frac{\alpha}{2}\right| \leq C_1, \tag{1.5.54}$$

$$\left|\sin(\alpha + \psi) \cot\left(\frac{\alpha}{2} + \psi\right)\right| \leq C_2, \tag{1.5.55}$$

uniformly for $0 < \psi < \pi$.

Let $p_5(\alpha, \psi)$, $p_6(\alpha, \psi)$ be absolutely continuous functions of α for $-\psi \leq \alpha \leq \psi$ s.t.

$$\left|\sin(\alpha + \psi) \cot\left(\frac{\alpha}{2} + \psi\right)y(t, \cos(\alpha + \psi)) - y(t, \cos \psi) - p_5(\alpha, \psi))\right| \leq \gamma_5, \tag{1.5.56}$$

$$\left|\left(\sin(\alpha - \psi) \tan\frac{\alpha}{2}\right)y(t, -\cos(\alpha - \psi)) - y(t, \cos \psi) - p_6(\alpha, \psi))\right| \leq \gamma_6. \tag{1.5.57}$$

Then

$$|L_{5k}| \leq \frac{\gamma_5\psi}{\sin \psi} + k^{-1}\frac{\psi}{\sin \psi}\tilde{p}_{5, \alpha}, \tag{1.5.58}$$

$$|L_{6k}| \leq \frac{\gamma_6\psi}{\sin \psi} + k^{-1}\frac{\psi}{\sin \psi}\tilde{p}_{6, \alpha}, \tag{1.5.59}$$

with $\tilde{p}_{5, \alpha}$, $\tilde{p}_{6, \alpha}$ constants s.t. for $-\psi \leq \alpha \leq \psi$

$$|p_{5, \alpha}(\alpha, \psi)| \leq \tilde{p}_{5, \alpha}, \tag{1.5.60}$$

$$|p_{6, \alpha}(\alpha, \psi)| \leq \tilde{p}_{6, \alpha}. \tag{1.5.61}$$

Let $\epsilon > 0$ be arbitrarily small. Obviously,

$$\left| \sum_{i=0}^{k-1} a_i(t) \frac{\sin(i+1) \arccos u}{\sqrt{1-u^2}} - \sqrt{\frac{\pi}{2}}\, y(t,u) \right| \leq b \qquad (1.5.62)$$

with

$$b = \sum_{j=3}^{6} (|J_{jk}| + |L_{jk}|). \qquad (1.5.63)$$

We let in (1.5.36), (1.5.41), (1.5.44), (1.5.48), (1.5.51), (1.5.56), (1.5.57),

$$\rho = \mu_4 = \mu_5 = \mu_6 = \gamma_3 = \gamma_4 = \gamma_5 = \gamma_6. \qquad (1.5.64)$$

This is possible since μ_j, $j = 4, 5, 6$ and γ_j, $j = 3, 4, 5, 6$ could be chosen arbitrarily small.

Therefore,

$$b \leq \rho\alpha_1 + k^{-1}\alpha_2,$$

where

$$\alpha_1 = \frac{|\pi - 2\psi| + 4\psi + \frac{1}{2}|\pi - 2\psi||y(t, \cos\psi)|}{\sin\psi} \qquad (1.5.65)$$

$$\begin{aligned} \alpha_2 = {} & \frac{2\tilde{q} + |\pi - 2\psi|\tilde{q}_\alpha}{\sin\psi} + |y(t,\cos\psi)| + \frac{|\sin 2\psi|}{\sin\psi} + \frac{\psi}{\sin\psi}(\tilde{q}_1 + \tilde{q}_{1,\alpha}) \\ & + \frac{\psi}{\sin\psi}(\tilde{q}_2 + \tilde{q}_{2,\alpha}) + \frac{|y(t,\cos\psi)|\tilde{p}_{3,\alpha}|\pi - 2\psi|}{2\sin\psi} \\ & + \frac{|\pi - 2\psi|}{2\sin\psi}\tilde{p}_{4,\alpha} + \frac{\psi}{\sin\psi}(\tilde{p}_{5,\alpha} + \tilde{p}_{6,\alpha}). \end{aligned} \qquad (1.5.66)$$

Since α_1, α_2 are bounded, we choose $k_0(t)$ s.t.

$$1/k_0(t) \leq \rho. \qquad (1.5.67)$$

Then for $k > k_0(t)$,

$$b \leq \rho\left(\alpha_1 + \frac{k_0(t)}{k}\alpha_2\right) \leq \rho(\alpha_1 + \alpha_2). \qquad (1.5.68)$$

We choose

$$\rho = \epsilon / (\alpha_1 + \alpha_2) \qquad (1.5.69)$$

and the result follows. E.O.P

Remark. One can obtain uniform bounds with respect to t, $t \in J_1$, $J_1 \subset J$, if proper assumptions are made on the continuity of $y(t, u)$ at $u = u_0$.

6. A Representation Theorem

We can now formulate a representation theorem.

(Representation) Theorem 1.6.1. (i) *Let* $y(t, \epsilon)$ *be a holomorphic function of* ϵ *for* $\operatorname{Re} \epsilon > 0$, *and* $t \in J$.

(ii) *Let*

$$y(t, \epsilon(u)) = \sum_{\nu=0}^{\infty} y_\nu(t) u^\nu \tag{1.6.1}$$

be the power series expansion of $y(t, u)$ *in the disk* D *where*

$$u = (\alpha - \epsilon)/(\alpha + \epsilon). \tag{1.6.2}$$

(*The transformation* (1.6.2) *takes* $\operatorname{Re} \epsilon > 0$ *onto* D. *See* [8, *p.* 193].)

(iii) *Assume*

$$|y_\nu(t)| = O(\nu^\alpha) \tag{1.6.3}$$

with $\alpha < 0.5$.

(iv) *Assume* $y(t, \epsilon(\cos\theta))$ *to be an integrable function of* θ *on* $0 \leq \theta \leq \pi$. *Then*

(I) *if* $y(t, \epsilon)$ *is continuous at* $\epsilon = \epsilon_0$ *where* ϵ_0 *is any point on* $0 \leq \epsilon < \infty$ *including* $\epsilon_0 = 0$ (*continuity at* $\epsilon_0 = 0$ *means that there exists* $y(t, 0^+)$), *we have*

$$y(t, \epsilon) = y_0(t) + \Delta, \tag{1.6.4}$$

where

$$\Delta := \frac{2}{\pi}(C, 2) \sum_{i=0}^{\infty} \frac{\alpha + \epsilon}{2\sqrt{\alpha\epsilon}} \sin\left[(i + 1) \operatorname{arc}\cos\frac{\alpha - \epsilon}{\alpha + \epsilon}\right]\left(\sum_{\nu=1}^{\infty} \phi_{i\nu} y_\nu(t)\right), \tag{1.6.5}$$

$\phi_{i\nu}$ *are given by* (1.3.6) *and* $\sum_{\nu=0}^{\infty} \phi_{i\nu} y_\nu(t)$, $i = 0, 1, \ldots$ *are absolutely converging series*;

(II) *in addition, if there exists*

$$y(\infty, \epsilon) := \lim_{t\to\infty} y(t, \epsilon) \tag{1.6.6}$$

for Re $\epsilon > 0$, *then there exists*

$$y_\nu(\infty) := \lim_{t\to\infty} y_\nu(t), \qquad \nu = 0, 1, \ldots; \tag{1.6.7}$$

(III) *if*

$$y_\nu(\infty) = O(\nu^\alpha) \tag{1.6.8}$$

with $\alpha < 0.5$, *then* (1.6.5) *holds for all* ϵ *with* $t = \infty$ *in* (1.6.4).

Proof. Since $y(t, \epsilon(\cos\theta))$ is an integrable function of θ, all the Fourier coefficients of its expansion in terms of Tshebysheff's polynomials of the second kind exist. Moreover, since (1.6.3) holds, $a_i(t)$, $i = 0, 1, \ldots$ are given thanks to Lemma 1.3.3 by the absolutely converging series (1.3.3)–(1.3.4).

If $\epsilon = \epsilon_0 > 0$ is a continuity point of $y(t, \epsilon)$ corresponding to $u = u_0$, then by Lemma 1.5.1, the series in (1.4.14) converges.

Since $(C, 2)$ is a regular summability method applied to a converging series, the same limit is obtained. In case $\epsilon = \epsilon_0 = 0$ we use Lemma 1.4.1 and the result follows by extracting in each $a_i(t)$ the term with $\phi_{i0} y_0(t)$ and noticing that $\phi_{i0} = 0$ for $i = 1, 2, \ldots$.

Assume now that (1.6.6) holds. By Cauchy's formula (1.6.7) also holds. Moreover, for all $|u| < 1$,

$$y(\infty, \epsilon(u)) = \sum_{\nu=0}^{\infty} y_\nu(\infty) u^\nu \tag{1.6.9}$$

is a holomorphic function in $|u| < 1$.

We apply the previous argument and the result follows. E.O.P.

A Worked Example. Consider the most elementary singularly perturbed problem

$$\epsilon y' + y = 0 \qquad y(0) = 1, \qquad 0 \le t < \infty, \epsilon > 0. \tag{1.6.10}$$

Its solution is

$$y = \exp - t\epsilon^{-1},$$

which is a bounded function for all $t \in J_\infty$ and Re $\epsilon > 0$. Also, y is a holomorphic function for Re $\epsilon > 0$.

We take Re $\epsilon > 0$ into $|u| < 1$ by

$$\epsilon = \alpha \frac{1-u}{1+u}, \qquad \alpha > 0 \tag{1.6.11}$$

and expand y in a power series of u.

Carrying out the expansion in (1.6.10) yields the recursive set of differential equations

$$\alpha y_0' + y_0 = 0, \qquad y_0(0) = 1, \tag{1.6.12}$$

$$\alpha y_\nu' + y_\nu = a y_{\nu-1}' - y_{\nu-1}, \qquad y_\nu(0) = 0, \qquad \nu = 1, 2, \ldots . \tag{1.6.13}$$

By induction, one obtains

$$\alpha y_\nu' + y_\nu = -2(y_0 + y_1 + \cdots y_{\nu-1}), \qquad \nu = 1, 2, \ldots . \tag{1.6.14}$$

We set

$$y_\nu = y_0 P_\nu, \qquad \nu = 1, 2, \ldots$$

to obtain the set of equations

$$P_\nu' = -2\alpha^{-1}(P_0 + P_1 + \cdots P_{\nu-1}). \tag{1.6.15}$$

We prove now by induction that

$$P_\nu(t) = \left(z\int_0^t\right)\left[1 + z\int_0^t\right]^{\nu-1} P_0, \tag{1.6.16}$$

where $(z\int_0^t)$ is an operator which has to be interpreted as

$$\left(z\int_0^t\right)^j P_0 = z^j \int_0^t \frac{(t-s)^{j-1}}{(j-1)!} P_0(s)\, ds. \tag{1.6.17}$$

We take $P_0 \equiv 1$. For $\nu = 1$ we have from (1.6.13),

$$\alpha P_1' = -2P_0$$

so that

$$P_1 = -z\int_0^t P_0(s)\, ds \tag{1.6.18}$$

in accordance with (1.6.16).

Assume (1.6.16) to hold for $1, \ldots, \nu$. We substitute in (1.6.15) for $\nu + 1$ to obtain

$$P_{\nu+1}' = z\left[P_0(t) + \sum_{k=1}^{\nu} z\int_0^t \left[1 + z\int_0^t\right]^{k-1} P_0(s)\right]$$

$$= z\left[P_0(t) + \left[z\int_0^t \frac{\left[1 + z\int_0^t\right]^\nu - 1}{\left[1 + z\int_0^t\right] - 1}\right] P_0(s)\, ds = z\left[1 + z\int_0^t\right]^\nu P_0.\right. \tag{1.6.19}$$

Integration of (1.6.19) yields the desired result.

Since $P_0 \equiv 1$, all $P_\nu(t)$ turn out to be polynomials. So it turns out that

$$P_\nu(t) = z\sum_{j=0}^{\nu-1}\binom{\nu-1}{j}z^j\int_0^t \frac{(t-s)^j}{j!}\,ds = \sum_{j=0}^{\nu-1}\binom{\nu-1}{j}\frac{(zt)^{j+1}}{(j+1)!}. \tag{1.6.20}$$

Formula (1.6.20) can be verified rigorously by some other method. We have

$$\lim_{\epsilon\to 0} y(t,\epsilon) = 1 \qquad \text{if } t = 0,$$
$$= 0 \qquad \text{if } t > 0, \tag{1.6.21}$$

and

$$\lim_{t\to\infty} y(t,\epsilon) = 0 \qquad \text{for } \operatorname{Re}\epsilon > 0. \tag{1.6.22}$$

We recall the coefficients $\phi_{i\nu}$ given by (1.3.8) and more explicitly by (1.3.9)–(1.3.22).

Since

$$|y(t,\epsilon)| \le 1 \qquad \text{for } t \in J_\infty,\ \operatorname{Re}\epsilon > 0, \tag{1.6.23}$$

by virtue of Lemma 1.3.3 all conditions of Representation Theorem 1.6.1 hold, (with $\alpha = 0$ in (1.6.3)); therefore,

$$\exp - t\epsilon^{-1} = \exp - t\alpha^{-1} + \Delta \tag{1.6.24}$$

with

$$\Delta := \frac{2}{\pi}(C,2)\sum_{i=0}^{\infty}\frac{(\alpha+\epsilon)}{2\sqrt{\alpha\epsilon}}\sin\left[(i+1)\arccos\frac{\alpha-\epsilon}{\alpha+\epsilon}\right]$$
$$\times\left(\sum_{\nu=1}^{\infty}\phi_{i\nu}\exp - t\alpha^{-1}\sum_{j=0}^{\nu-1}\binom{\nu-1}{j}\frac{(-2\alpha^{-1}t)^{j+1}}{(j+1)!}\right). \tag{1.6.25}$$

In particular,

$$\lim_{\epsilon\to 0^+}\exp - t\epsilon^{-1} = \exp - t\alpha^{-1} + \Delta_0 \tag{1.6.26}$$

with

$$\Delta_0 = \frac{2}{\pi}(C,2)\sum_{i=0}^{\infty}(i+1)$$
$$\times\left[\sum_{\nu=1}^{\infty}\phi_{i\nu}\exp - t\alpha^{-1}\left(\sum_{j=0}^{\nu-1}\binom{\nu-1}{j}\frac{(-2\alpha^{-1}t)^{j+1}}{(j+1)!}\right)\right]. \tag{1.6.27}$$

Of course, for $t = 0$, Δ_0 becomes 0, and for $t > 0$, there is convergence which cannot be uniformly valid for all t $0 < t \leq M$. Since

$$\lim_{t\to+\infty} y_0(t)P_\nu(t) = 0, \qquad \nu = 0, 1, \ldots,$$

we get from the representation theorem the value of

$$\lim_{t\to\infty} y(t, \epsilon) = 0, \qquad \operatorname{Re}\epsilon > 0. \tag{1.6.28}$$

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