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LOCALLY PURE TOPOLOGICAL ABELIAN GROUPS: ELEMENTARY INVARIANTS*

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0. Introduction

In the present article we will give a classification in terms of numerical invariants of certain saturated topological abelian groups, or equivalently of the corresponding complete theories. It is intended to provide a topological analog of results of Eklof and Fisher in the discrete (i.e. non-topological) case [2]. Let us first review the background.

0.1. *Model theory of abelian groups*

Traditional model theory deals with first-order theories of algebraic systems. A basic result in the model theory of abelian groups, obtained by Szmielew [13] in 1955, is the decidability of the full theory of abelian groups. Szmielew uses the method of elimination of quantifiers, which typically produces the sharpest results.

More abstract model theoretic methods can be used to obtain Szmielew's results. In the process the results lose some of their effectivity, but gain in algebraic content. Eklof and Fisher [2] reworked Szmielew's results in terms of a detailed analysis of saturated abelian groups. They were able to give a complete classification of somewhat (i.e. ω_1-) saturated abelian groups. As it turned out, the algebraic tools needed for this are all to be found in Kaplansky's monograph [7].

0.2. *Model theory of topological abelian groups: negative results*

More recently a topological model theory has been developed which deals with the first-order theories of topological algebraic systems. It was not clear initially what one should mean by 'first-order logic' in a topological context, but a

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convincing candidate emerged in the form of the logic L^t (t for 'topological') introduced by McKee [10] and developed in [3, 5, 14]. This logic will be presented in Section 1 below.

When we began to look systematically at the possibility of extending the results of Szmielew and Eklof–Fisher to a topological context it became clear that there are substantial obstructions to such a program, as manifested for example by:

Fact A [1]. *The theory of torsionfree Hausdorff topological abelian groups is undecidable.*

The proof of this fact led us to examine the class of locally pure topological abelian groups. For our present purposes these may be taken to be the topological abelian groups possessing a neighborhood basis of pure open subgroups at the identity (alternatively one may consider more generally groups which are elementarily equivalent to such a group). A second negative result should be mentioned:

Fact B [1]. *The theory of locally pure Hausdorff topological abelian groups is undecidable.*

The analysis of topological abelian groups which are both torsionfree and locally pure turned out be be more fruitful.

0.3. The present paper

Main Theorem. *The isomorphism types of saturated, torsionfree, locally pure topological abelian groups can be classified in terms of simple numerical invariants. In fact all such groups are of the form*

$$(*) \qquad \textit{discrete} \oplus \textit{Trivial} \oplus \exp(A, B, \mu)$$

where a trivial group is one with no proper open subset, and $\exp(A, B, \mu)$ *is a group which is described explicitly in Definition* 2.11 *below, in terms of a pair of* (*discrete*) *torsionfree abelian groups* $B \subseteq A$ *and a cardinal* μ.

At the request of the referee we have rewritten the paper to bring out more clearly the algebraic content of the analysis, since the saturation hypothesis is used in a limited number of ways. Algebraists unfamiliar with saturation may think of it as a completeness or compactness condition analogous to algebraic compactness.

Notice that we do not classify ω_1-saturated groups of the stated type, but only the fully saturated ones. We do not see how one could analyze the topological structure under weaker hypotheses.

One technical point which should be emphasized is the connection of the work reported here with the work of Kokorin and Kozlow [8]. Our numerical invariants will be just the Szmielew invariants of the first two factors in (*) together with the

Kokorin–Kozlow invariants for the pair (A, B). There is, however, a 'missing link'. As it stands, the work of Kokorin and Kozlow does not look much like what we do here, but in [12] their results are reworked in the manner of [2] and in the process the relationship to the present paper becomes evident.

Returning to the question of decidability, which guided us in our initial investigations, we derive as an immediate corollary to the main theorem:

Corollary. *The L^t-theory of torsionfree, locally pure topological abelian groups is decidable.*

Ziegler has pointed out that this corollary can be obtained quite rapidly from Gurevich's remarkable decidability result for ordered abelian groups [6]. On the other hand a 'saturated models version' of Gurevich's result has been sought for some time without success.

The paper is structured as follows: After reviewing the logic L^t and discussing the key notion of local purity in Section 1 below, we will describe in Section 2 our set of elementary invariants associated with topological abelian groups. The Main Theorem will be proved in Section 3.

In particular the first two sections are devoted entirely to preliminaries, culminating in the introduction of our 'standard invariants' in Definition 2.28. Lemma 2.29 states that these are indeed L^t elementary invariants, and Theorem 2.30 states that they are a complete set of elementary invariants, which is the model theoretic form of our main result. In Section 2 we give only the trivial part of the proof of this theorem, namely the passage from the structural form of the main theorem to its model-theoretic form.

1. Preliminaries: local purity

1.1. The logic L^t

We will present the first-order topological logic L^t in a form specifically adapted to the discussion of first-order properties of topological groups.

Definition 1.1. Let LG be the usual first-order language of group theory (written additively) and let LG^{II} be the extension of LG to the following weak second-order logic:

(1) *Syntax:* Conventional second-order logic with second-order variables $X, Y, \ldots,$ second-order constants, and the binary relation symbol $\in$. The class of formulas is closed under second-order quantification.

(2) *Semantics:* a structure for LG^{II} is a group G with a family $\mathcal{B}$ of subsets of G.

(3) *Interpretation:* $\in$ represents membership. Second-order variables range over $\mathscr{B}$.

The logic L^t is obtained as a sublogic of LG^{II}.

Definition 1.2. (1) An occurrence of a second-order variable X in an LG^{II} formula φ is said to be *positive* (resp. *negative*) if it is governed by an even (resp. odd) number of negation symbols, (in this connection we take the propositional connectives to be $\neg$, $\vee$, & only; thus $\varphi \to \psi$ abbreviates $\neg\varphi \vee \psi$).
(2) L^t is a sublogic of LG^{II} with the same semantics but a restricted syntax: a formula φ of LG^{II} belongs to L^t iff for each subformula $\exists X\,\psi$ (resp. $\forall X\,\psi$) of φ all occurrences of X in ψ are negative (resp. positive).

Example. The following three sentences of L^t:
(1) $\forall X\,\forall Y\,\exists Z\,\forall x\,(x \in Z \to x \in X\ \&\ x \in Y)$,
(2) $\forall X(0 \in X)$,
(3) $\forall X\,\exists Y\,\forall x, y\,(x \in Y\ \&\ y \in Y \to x - y \in X)$
assert that the family $\mathscr{B}$ constitutes a neighborhood basis at 0 for a topology τ on G such that (G, τ) is a topological group.

Notation. Elementary equivalence with respect to the logic L^t is denoted: $\equiv_t$.

The following are easily verified [3, 5, 11].

Fact 1.3. *If $\mathscr{B}_1$, $\mathscr{B}_2$ are neighborhood bases for the same topology on the topological group G, then:*

$$(G, \mathscr{B}_1) \equiv_t (G, \mathscr{B}_2).$$

Fact 1.4. *The logic L^t satisfies the Compactness Theorem.*

Definition 1.5. Let $(G, \mathscr{B})$ be a structure for L^t. A *type* Σ over $(G, \mathscr{B})$ is a set of formulas of L^t involving a fixed finite set of first- and second-order variables $x_1, \ldots, x_n$, $X_1, \ldots, X_m$ such that
(1) all constants occurring in formulas in Σ denote elements of G or sets in $\mathscr{B}$;
(2) the variables X_i occur only negatively in formulas of Σ;
(3) Σ is finitely satisfiable in $(G, \mathscr{B})$ (using elements $a_1, \ldots, a_n$ in G and sets $A_1, \ldots, A_m$ in $\mathscr{B}$).

Remark. It is easy to dispense with condition (2) above, but including this restriction yields a smoother general theory.

Definition 1.6. (1) For μ an infinite cardinal $(G, \mathscr{B})$ is *μ-saturated* iff each type Σ over $(G, \mathscr{B})$ which involves fewer than μ constants in $(G, \mathscr{B})$ is realized in $(G, \mathscr{B})$.

(2) $(G, \mathcal{B})$ is *saturated* iff $(G, \mathcal{B})$ is μ-saturated for $\mu = \mathrm{card}(G \cup \mathcal{B})$.

(3) G is *μ-saturated* (resp. *saturated*) iff there is a neighborhood basis $\mathcal{B}$ for G such that $(G, \mathcal{B})$ is μ-saturated (resp. saturated).

Remark. The types referred to in clause (1) above may be taken to involve a single free variable.

Fact 1.7. *Let μ be a regular cardinal. Then each L^t-structure $(G, \mathcal{B})$ is L^t-elementarily equivalent to a μ^+-saturated structure of cardinality 2^μ.*

Fact 1.8. *If G, G' are saturated topological groups of the same cardinality and $G \equiv_t G'$, then G is topologically isomorphic to G'.*

Fact 1.9. *Suppose that H is an L^t-definable subgroup of the μ-saturated topological group G. Then H (resp. G/H if H is normal) is μ-saturated in the induced topology.*

1.2. Saturation and local purity

Recall that a topological space is called a *P-space* if any countable intersection of open sets is open (equivalently, any countable intersection of neighborhoods of p is a neighborhood of p, for each point p). If a topological group is a P-space, we will call it a *P-topological group.*

We deal with topological groups as pointed topological spaces equipped with a distinguished basis for the neighborhoods at the identity. In this context, if $\mathcal{B}$ is the distinguished basis, then let $G_\delta(\mathcal{B})$ denote the collection of all countable intersections of members of $\mathcal{B}$. In particular, if the space is a P-space, then the elements of $G_\delta(\mathcal{B})$ are neighborhoods of the base point.

Lemma 1.10. *If G is μ-saturated, then the intersection of fewer than μ open sets is open. In particular if μ is uncountable, then G is a P-topological group.*

Proof. Fix a neighborhood basis $\mathcal{B}$ for G at 0 such that $(G, \mathcal{B})$ is μ-saturated. Let $\{W_\alpha : \alpha < \lambda\}$ be a family of open sets and $\lambda < \mu$. For any $x \in W = \bigcap\{W_\alpha : \alpha < \lambda\}$ there are $U_\alpha \in \mathcal{B}$ such that $x + U_\alpha \subseteq W_\alpha$. The set of L^t-formulas

$$\{\text{“}X \subseteq U_\alpha\text{”}: \alpha < \lambda\}$$

with X a second-order variable is a finitely satisfiable type in L^t over $(G, \mathcal{B})$. Letting $U \in \mathcal{B}$ realize this type yields $x + U \subseteq W_\alpha$ for all $\alpha < \lambda$. Thus W is open.

Lemma 1.11. *If G is a P-topological group, then there is a neighborhood basis for G at 0 consisting exclusively of open subgroups of G. More precisely, if $\mathcal{B}$ is any basis for G at 0, then $G_\delta(\mathcal{B})$ contains such a basis.*

Proof. For any neighborhood A of 0 choose sets B_n in $\mathcal{B}$ such that:

$$B_0 \subseteq A, \qquad B_{n+1} - B_{n+1} \subseteq B_n \quad \text{for each } n.$$

Let $A' = \bigcap\{B_n : n \in \omega\}$. Then A' is in $G_\delta(\mathcal{B})$, so A' is a neighborhood of 0. A' is a subgroup by construction and hence A' is also open.

Corollary. *Any topological group is elementarily equivalent to a topological group having a neighborhood basis of open subgroups at the origin.*

Definition 1.12. Let G be a topological abelian group.

(1) G is *locally pure* iff G has a neighborhood basis of pure open subgroups at 0.

(2) G is of *locally pure type* iff G satisfies the following axiom for all primes p:
(LP-p) $\forall X\, \exists Y\, \forall x\, (px \in Y \rightarrow \exists\, z \in X\, (pz = px))$.

Lemma 1.13. *Let G be an ω_1-saturated topological abelian group. Then G is locally pure iff G is of locally pure type.*

Proof. In the nontrivial direction, assume that G is of locally pure type. Fix a basis $\mathcal{B}$ of neighborhoods of 0 for G such that $(G, \mathcal{B})$ is ω_1-saturated. Let U be an arbitrary neighborhood of 0. Since G is a P-topological group we can apply the definition of 'locally pure type' to construct a sequence of open subgroups B_n of G contained in U such that:

$$\text{For all primes } p: \quad B_{n+1} \cap pG \subseteq pB_n, \tag{1}$$

$$B_n \text{ is in } G_\delta(\mathcal{B}). \tag{2}$$

Let $B = \bigcap\{B_n : n \in \omega\}$. Then B is an open subgroup of G contained in U. It remains to be seen that B is pure in G. Fix $b \in B \cap pG$ and consider the L^t-type

$$\{px = b\} \cup \{\text{'}x \in B_n\text{'} : n \in \omega\}.$$

To make this be a type over $(G, \mathcal{B})$ we have to read '$x \in B_n$' as the set $\{x \in B_{n,k} : k \in \omega\}$ where $B_{n,k} \in \mathcal{B}$ and $B_n = \bigcap\{B_{n,k} : k \in \omega\}$. If $a \in G$ is an element realizing the above type, then $pa = b$ and $a \in B$, as desired.

Remarks. (1) If G is torsionfree the final argument is superfluous, so it is then enough to assume that G is a P-topological group.

(2) The pure open subgroups B arising in the above proof are in $G_\delta(\mathcal{B})$.

Corollary. *Every group of locally pure type is elementarily equivalent to a locally pure group. Hence the theory of the class of locally pure groups is axiomatized by the axioms for topological groups together with the axioms* (LP-p) *above for each prime p.*

2. Numerical invariants

2.1. Szmielew invariants

We review the definition and basic properties of the Szmielew invariants using the notation of [2] with minor alterations.

Notation 2.1.

$C_{p,n}$ = the cyclic group of order p^n:
Z_p = the group of rational p-adic integers (i.e. rational numbers with denominators prime to p under addition):
C_{p^∞} = the Prüfer p-group, realizable e.g. as the additive group Q/Z_p;
Q = the additive group of rationals.

Notation 2.2. Let A be an abelian group, μ a cardinal, m, n integers.

(1) $A^{(\mu)}$ is the direct sum of μ copies of A. A^μ is the direct product of μ copies of A.

(2) $A[n] = \{a \in A : na = 0\}$, $mA[n] = (mA)[n]$.

(3) If $pA = \{0\}$ for some prime p, then $\dim A$ denotes the dimension of A as a vector space over the Galois field F_p.

Definition 2.3. (Szmielew invariants). Let G be an abelian group.

$$\alpha_{p,n}(G) = \begin{cases} \dim(p^{n-1}G[p]/p^nG[p]) & \text{if this is finite,} \\ \infty & \text{otherwise;} \end{cases}$$

$$\beta_p(G) = \begin{cases} \lim_n \dim(p^nG/p^{n+1}G) & \text{if this is finite,} \\ \infty & \text{otherwise;} \end{cases}$$

$$\gamma_p(G) = \begin{cases} \lim_n \dim(p^nG[p]) & \text{if this is finite,} \\ \infty & \text{otherwise;} \end{cases}$$

$$\delta(G) = \begin{cases} 0 & \text{if } G \text{ is of bounded exponent,} \\ \infty & \text{otherwise.} \end{cases}$$

Thus for any invariant $\iota = \alpha_{p,n}, \beta_p, \gamma_p, \delta$, $\iota(G)$ is a natural number or the symbol ∞. In the torsionfree case $\alpha_{p,n} = \gamma_p = 0$, $\delta = \infty$ (for G nontrivial) and for all $n \geqslant 0$

$$\beta_p(G) = \dim(p^nG/p^{n+1}G).$$

The following topological terminology is useful in the discussion of (non-topological) abelian groups. A variant of it will be introduced below in the context of topological groups and will be a basic tool in our analysis.

Definition 2.4. Let G be an abelian group.

(1) For p a prime, the p-adic topology on G is determined by the neighborhood basis:

$$\{p^n G\colon n \geqslant 0\}.$$

(2) The Z-topology on G is the join of the p-adic topologies. A basis for the Z-topology is given by:

$$\{nG\colon n \geqslant 1\}.$$

(3) If G is Hausdorff in the Z-topology (i.e. G contains no infinitely divisible element), then $\bar{G}$ denotes the completion of G in the Z-topology. (The p-adic topologies, and hence the Z-topology, are associated with an obvious choice of pseudometrics). In particular:

(4) $\bar{Z}_p$ is the completion of Z_p, namely the additive group of all p-adic integers.

We can now state the main technical result of [2].

Fact 2.5. *Let μ be an uncountable cardinal. If G is a μ-saturated abelian group, then:*

$$G = \left(\prod_p \left(\sum_n [C_{p,n}^{(\alpha^1_{p,n})} \oplus Z_p^{(\beta^1_p)}]\right)^{-}\right) \oplus \sum_p C_p^{(\gamma^1_p)} \oplus Q^{(\delta^1)}$$

where for ι any invariant:

$$\iota^1 \begin{cases} = \iota(G) & \text{if this is finite,} \\ \geqslant \mu & \text{if } \iota(G) = \infty. \end{cases}$$

Remark. In particular every μ-saturated abelian group for uncountable μ is Z-complete.

We mention a useful additivity property:

Fact 2.6. *If B is a pure subgroup of the abelian group A and ι is one of the Szmielew invariants, then:*

$$\iota(A) = \iota(B) + \iota(A/B)$$

with the usual rules governing the symbol ∞.

This follows from the following two facts:

Fact 2.7. *If B is a Z-complete pure subgroup of the abelian group A, then B is a direct summand of A.* ([4, *Theorem* 39.1].)

Fact 2.8. *For each Szmielew invariant ι and each integer n there is a set $s(\iota, n)$ of first-order sentences such that an abelian group G satisfies $s(\iota, n)$ iff $\iota(G) > n$.*

We will need a more precise statement:

Supplement to Fact 2.8. *If we adjoin new predicates $D_n(x)$ to the language of abelian groups and add corresponding axioms*

$$\text{“}\forall x(D_n(x) \leftrightarrow n \text{ divides } x)\text{”}$$

to the axioms for abelian groups, then the sets $s(\iota, n)$ described in Fact 2.8 may be taken to be existential.

Fact 2.9. *If $A, H \subseteq G$ are abelian groups with H Z-complete, $A \cap H = \{0\}$ and $A \oplus H$ pure in G, then there is a subgroup K of G such that*

$$A \subseteq K \quad \text{and} \quad G = K \oplus H.$$

Proof. Apply Fact 2.7 to G/A and HA/A.

2.2. Local divisibility

To obtain additional numerical invariants for topological abelian groups we introduce a further notion which plays a fundamental role in our analysis.

Definition 2. Let G be a topological abelian group, $g \in G$, p a prime and $n \in \omega$

(1) *n locally divides* g iff G satisfies $\forall X \, \exists x \, (g\text{-}nx \in X)$,

(2) $G_{p,n} = \{x \in G: p^n \text{ locally divides } x\}$,

(3) $G_{p,\infty} = \bigcap\{G_{p,n}: n \in \omega\}$.

Notice that $G_{p,n}$, $G_{p,\infty}$ are subgroups of G. Indeed $G_{p,n}$ is the closure in G of p^nG.

It is useful to consider these notions in conjunction with the following example.

Definition 2.11. Let μ be a cardinal and let A, B be abelian groups with $B \subseteq A$.

(1) A^μ is the direct product of μ copies of A equipped with the topology determined by the neighborhood basis

$$\{U_\alpha: \alpha < \mu\}$$

where $U_\alpha = \{x \in A^\mu: x_i = 0 \text{ for all } i < \alpha\}$.

(2) $\exp(A, B, \mu) = \{x \in A^\mu: \text{ for some } b \in B,\ \alpha < \mu \text{ we have } \forall i(\alpha < i < \mu \rightarrow x_i = b)\}$. Give $\exp(A, B, \mu)$ the topology induced by A^μ.

(3) $\exp(A, \mu) = \exp(A, \{0\}, \mu)$.

If B is identified with the group of B-valued constant functions in A^μ, then

$$\exp(A, B, \mu) = \exp(A, \mu) \oplus B.$$

Remark. If $G = \exp(A, B, \mu)$ and $G_0 = \exp(A, \mu)$, then for any prime p and $n \in \omega$:

$$G_{p,n} = p^nG_0 \oplus (B \cap p^nA),$$

$$G_{p,\infty} = p^\infty G_0 \oplus (B \cap p^\infty A) \quad \text{where } p^\infty H = \bigcap\{p^nH: n \in \omega\}.$$

The following example is particularly incisive. In fact our main technical result (Theorem 3.19) states that essentially only examples of the following type need be considered.

Example 2.12. Set

$$A = Q^{(\delta)} \oplus \prod_p (\bar{A}_p \oplus A_{p,\infty})$$

where:

(1) $A_p = \sum \{A_{p,n} : 0 \leq n < \infty\}$;
(2) $A_{p,0} = Z_p^{(\beta_{p,0})} \oplus Z_p^{(\beta'_{p,0})}$;
(3) $A_{p,n} = Z_p^{(\beta_{p,n})}$ for $0 < n < \infty$;
(4) $A_{p,\infty}$ = the divisible hull of $B_{p,\infty}$.

Set

$$B = D \oplus \prod_p (B_p^* \oplus B_{p,\infty})$$

where
(5) D is a divisible subgroup of $Q^{(\delta)}$;
(6) $B_p = \sum \{B_{p,n} : 0 \leq n < \infty\}$;
(7) B_p^* = completion of B_p in the topology induced on B_p by the p-adic topology on A;
(8) $B_{p,0} = Z_p^{(\beta_{p,0})} \subseteq A_{p,0}$;
(9) $B_{p,n} = p^n A_{p,n}$ for $0 < n < \infty$;
(10) $B_{p,\infty} = (Z_p^{(\beta_{p,\infty})})^- \subseteq A_{p,\infty}$.
(Here $^-$ denotes the completion of $Z_p^{(\beta_{p,\infty})}$ in its own Z-adic topology, in particular $B_{p,\infty}$ is Hausdorff with respect to this topology.)

(Up to elementary equivalence all pairs of torsionfree abelian groups are of this form [8].)

Here δ and the betas are freely chosen cardinals.

Example 2.12. (continued). Let $G = \exp(A, B, \mu)$ and $G_0 = \exp(A, \mu)$ for some cardinal μ. Then for any prime:

(11)
$$\begin{aligned} G/G_{p,1} &\cong G_0/pG_0 \oplus B/(B \cap pA) \\ &\cong \exp(A/pA, \mu) \oplus B_{p,0}/pB_{p,0} \\ &\cong \exp(C_p^{(\varepsilon)}) \oplus C_p^{(\beta_{p,0})}, \quad \text{with } \varepsilon = \beta'_{p,0} + \sum_{n<\infty} \beta_{p,n}; \end{aligned}$$

(12)
$$\begin{aligned} G_{p,n}/(pG_{p,n-1} + G_{p,n+1}) &\cong (B \cap p^n A)/(p(B \cap p^{n-1}A) + (B \cap p^{n+1}A)) \\ &\cong B_{p,n}/pB_{p,n} \cong C_p^{(\beta_{p,n})}; \end{aligned}$$

(13)
$$\begin{aligned} G_{p,\infty}/pG_{p,\infty} &\cong (B \cap p^\infty A)/p(B \cap p^\infty A) \\ &\cong B_{p,\infty}/pB_{p,\infty} \cong C_p^{(\beta_{p,\infty})}. \end{aligned}$$

This example suggests the introduction of the following numerical invariants:

Definition 2.13. If G is a topological abelian group set:
(1) $\beta_{p,0}(G) = \dim G/G_{p,1}$;
(2) $\beta_{p,n}(G) = \dim(G_{p,n}/(pG_{p,n-1} + G_{p,n+1})$ for $0 < n < \infty$;
(3) $\beta_{p,\infty}(G) = \inf_n \dim(G_{p,n}/(G_{p,n} \cap pG)$.

Lemma 2.14. (i) *For any locally pure torsion free group*

$$G_{p,\infty} \cap pG = pG_{p,\infty}$$

(ii) *For any ω_1-saturated topological abelian group*

$$G_{p,\infty} \cap pG = pG_{p,\infty}$$

and

$$\beta_{p,\infty}(G) = \dim G_{p,\infty}/(G_{p,\infty} \cap pG).$$

It will be important that the second part of (ii) *can be stretched a little to yield:*
(iii) *If $(G, \mathscr{B})$ is a μ^+-saturated topological abelian group, H a pure open subgroup which is the intersection of μ elements of $\mathscr{B}$, then*

$$\beta_{p,\infty}(H) = \dim H_{p,\infty}/(H_{p,\infty} \cap pH).$$

Proof. Easy. □

The remainder of this subsection is devoted to establishing general properties of these invariants. We begin with a definability lemma.

Lemma 2.15. *Let p be a prime, $k \in \omega$, $0 \leq n < \infty$, then*
(1) *If $\iota = \beta_{p,n}$ then there is an L^ι-sentence $s(\iota, k)$ such that for all topological abelian groups G:*

$$G \text{ satisfies } s(\iota, k) \quad \text{iff } \beta_{p,n}(G) > k.$$

(2) *If $\iota = \beta_{p,\infty}$, then there is a set $s(\iota, k)$ of L^ι-sentences such that for all topological abelian groups G:*

$$G \text{ satisfies } s(\iota, k) \quad \text{iff } \beta_{p,\infty}(G) > k.$$

Proof. Obvious. □

Lemma 2.16. *Let H be a pure open subgroup of the topological group G. Then for all n, $0 < n \leq \infty$:*

$$\beta_{p,n}(G) = \beta_{p,n}(H).$$

Proof. If H is a direct factor of G, then the claim may be verified by a simple and explicit calculation.

To prove the lemma in general, consider an ω_1-saturated elementary extension (G', H') of the pair (G, H). Then H' will be a direct summand of G' by Fact 2.7 and we will have using Lemma 2.15:

$$\beta_{p,n}(G) = \beta_{p,n}(G') = \beta_{p,n}(H') = \beta_{p,n}(H). \quad \square$$

The invariant $\beta_{p,0}$ requires a different analysis, as one would expect in view of the discrepancy between Example 2.12 (11) and Definition 2.13 (1).

Lemma 2.17. *If H is a pure open subgroup of the torsionfree topological abelian group G, then:*

$$\beta_{p,0}(G) = \beta_p(G/H) + \beta_{p,0}(H).$$

Proof. Arguing as in the first part of the preceding proof, we may suppose that there is a direct decomposition $G = G' \oplus H$, where G' carries the discrete topology and H carries the induced topology. Then:

$$G_{p,1} \cong pG' \oplus H_{p,1}$$
$$G/G_{p,1} \cong G'/pG' \oplus H/H_{p,1}$$

and since G' is torsionfree:

$$\beta_p(G/H) = \beta_p(G') = \dim(G'/pG').$$

The result follows. $\square$

Remark. If $G = \exp(A, B, \mu)$ where $B \subseteq A$ are as in Example 2.12, then we can retrieve the cardinals $\beta_{p,n}$ $0 < n \leq \infty$ used in the construction of the pair (A, B) in an L^t-definable way from G. The same is not true for $\beta_{p,0}$ or $\beta'_{p,0}$ and the L^t-theory of G also does not give information on the divisible group D. This is the reason for requiring (1) and (2) in Theorem 3.19 below.

2.3. Tight groups and kernels

In this subsection we will show how to split off the uninteresting portions of a saturated torsionfree locally pure topological abelian group.

Definition 2.18. Let G be a topological abelian group and let ι be one of the Szmielew invariants or $\beta_{p,0}$ for some prime p.

(1) $\iota_*(g) = \inf\{\iota(H)$: H a pure open subgroup of $G\}$,

(2) $\iota^*(G) = \sup\{\iota(G/H)$: H a pure open subgroup of $G\}$.

Thus $\iota_*(G)$, $\iota^*(G)$ are natural numbers or the symbol ∞.

Lemma 2.19. *If ι is a Szmielew invariant or $\beta_{p,0}$ for some prime p and if n is an integer, then there is a set $s_*(\iota, n)$ of L^t-sentences such that for all locally pure topological abelian groups G:*

$$G \text{ satisfies } s_*(\iota, n) \quad \text{iff } \iota_*(G) > n.$$

Proof. Suppose first that ι is a Szmielew invariant and let $s(\iota, n)$ be the set of sentences described in Fact 2.8 and its supplement, formulated as existential sentences in an expanded language with divisibility predicates D_n. If $\varphi = \exists \boldsymbol{x}\, \varphi_0(\boldsymbol{x})$ is such a sentence and φ_0^1 is the translation of φ_0 back into the ordinary language of abelian groups, set

$$\varphi_* = \forall X\, \exists \boldsymbol{x} \in X\, \varphi_0^1(\boldsymbol{x})$$

and take

$$s_*(\iota, n) = \{\varphi_*: \varphi \in s(\iota, n)\}.$$

Since the restriction of a divisibility predicate D_n from G to a pure subgroup H yields the divisibility predicate on H, the set $s_*(\iota, n)$ has the desired meaning.

Consider now the case $\iota = \beta_{p,0}$. Examining the definition of $\beta_{p,0}$, we see that the condition '$\beta_{p,0}(G) > n$' can be expressed by a sentence of the form:

$$\varphi(\iota, n) = \exists \boldsymbol{x}\, \exists Y\, \forall\, y \in Y\, \varphi_0(\boldsymbol{x}, y)$$

where φ_0 is quantifier-free in the language containing divisibility predicates. Set

$$s_*(\iota, n) = \{\text{`}\forall X\, \exists \boldsymbol{x} \in X\, \exists Y\, \forall\, y \in Y\, \varphi_0^1(\boldsymbol{x}, y)\text{'}\}.$$

It is again easy to see that this has the intended meaning. □

Lemma 2.20. *For each prime p and $n \in \omega$ there is an L^t-sentence $\varphi^*(p, n)$ such that any locally pure torsionfree topological abelian group G satisfies $\varphi^*(p, n)$ just in case*

$$\beta_p^*(G) > n.$$

Proof. As we now consider only torsionfree groups $s(\beta_p, n)$ may be taken to contain only a single sentence φ. Let X be a second-order variable which may be thought of intuitively as representing an open subgroup, and let φ/X be a natural formalization of $G/X \models \varphi$. Inspection of φ will reveal that X only occurs negatively in φ/X, and hence

$$\exists X\, \varphi/X$$

is an L^t-sentence, which we will denote $\varphi^*(p, n)$. Clearly this sentence has the intended meaning. □

Definition 2.21. The topological abelian group G is *tight* if for ι any Szmielew invariant or $\iota = \beta_{p,0}$, we have for all n:

$$G \text{ satisfies } s_*(\iota, n) \quad \text{iff } \iota(G) \geq n.$$

Lemma 2.22. *If G is a locally pure P-topological abelian group, then there is a pure open subgroup H of G so that:*

(1) *H is tight,*
(2) *$\iota(H) = \iota_*(G)$ for all Szmielew invariants ι.*
(3) *$\beta_{p,0}(H) = \beta_{p,0,*}(G)$ for all primes p,*
(4) *$\iota(G/H) = \iota^*(G)$ for all Szmielew invariants ι.*

Furthermore, if $\mathscr{B}$ is any basis for G at 0, then:

(5) *H can be taken to be in $G_\delta(\mathscr{B})$.*

Proof. We remark that (1) follows from (2) and (3), since $\iota_*(H) = \iota_*(G)$. Now for $H_1 \subseteq H_2$ pure open subgroups of G we can derive from Fact 2.6 and Lemma 2.17 for ι a Szmielew invariant and p a prime:

(i) $\iota(H_1) \leqslant \iota(H_2)$;
(ii) $\iota(G/H_1) \geqslant \iota(G/H_2)$;
(iii) $\beta_{p,0}(H_1) \leqslant \beta_{p,0}(H_2)$.

Consider now a Szmielew invariant ι. If $\iota_*(G) = \infty$ set $H_*(\iota) = G$ and otherwise choose a pure open subgroup $H_*(\iota)$ of G so that

$$\iota(H_*(\iota)) = \iota_*(G).$$

Similarly, if $\iota^*(G)$ is finite, fix a pure open subgroup $H^*(\iota)$ such that

$$\iota(G/H^*(\iota)) = \iota^*(G),$$

while if $\iota^*(g) = \infty$ choose for each k a pure open subgroup H_k such that

$$\iota(G/H_k) \geqslant k$$

and let

$$H^*(\iota) = \bigcap\{H_k : k \in \omega\}.$$

In the same way associate to each prime p a pure open subgroup $H(p)$ satisfying;

$$\beta_{p,0}(H(p)) = \beta_{p,0,*}(G).$$

Finally take

$$H \subseteq \bigcap_\iota [H_*(\iota) \cap H^*(\iota)] \cap \bigcap_p H(p)$$

a pure open subgroup of G, and use (i)–(iii) to verify that H is a suitable subgroup. □

Remark 2.23. For groups G satisfying the assumptions of Lemma 2.21 we have for all Szmielew invariants ι:

$$\iota_*(G) + \iota^*(G) = \iota(G).$$

Indeed, let H be the pure open subgroup constructed in the above lemma. by Lemmas 2.19 and 2.20 we may assume that the pair (G, H) is ω_1-saturated. Now Fact 2.6 may be applied.

Lemma 2.24. *If G is a locally pure topological abelian group, then for any prime p:*

$$\beta_{p,0,*}(G)=0 \quad or \quad \beta_{p,0,*}(G)=\infty.$$

Proof. Suppose that $\beta_{p,0,*}(G)<\infty$ and that $H\subseteq G$ is a pure open subgroup such that $\beta_{p,0}(H)=n<\infty$. Choose representatives $a_1,\ldots,a_k$ for all non-zero elements of $H/H_{p,1}$ and choose a pure open subgroup $H_1\subseteq H$ such that for all $1\leqslant i\leqslant k$: $(a_i+H_1)\cap pH=\emptyset$.

For any $h\in H_1$, $h\notin H_{p,1}$ there is an element $a=a_i$ with $a-h\in H_{p,1}$, and thus there is $h'\in H_1$ such that $a-h-h'\in pH$, so $(a+H_1)\cap pH\neq\emptyset$, a contradiction. Thus $H_1\subseteq H_{p,1}$ and therefore

$$\beta_{p,0}(H_1)=0$$

and $\beta_{p,0,*}(G)=0$ as claimed. □

Lemma 2.25. *If G is a topological abelian group such that $\beta_{p,0}(G)=0$, then $\beta_{p,n}(G)=0$ for all $n<\infty$.*

Proof. If on the contrary $\beta_{p,n}(G)>0$ and $x\in G_{p,n}$ but $x\notin G_{p,n+1}$, then choose a neighborhood X for which

$$(x+X)\cap p^{n+1}G=\emptyset.$$

Choose a neighborhood Y for which $Y-Y\subseteq X$ and choose $y\in Y$, $g\in G$ with

$$x-y=p^n g.$$

Now since $\beta_{p,0}(G)=0$ we have $g\in G_{p,1}$, and we obtain easily

$$(x-y+Y)\cap p^{n+1}G\neq\emptyset$$

yielding a contradiction. □

Remark 2.26. For any topological abelian group G

(1) $\beta_p(G)=\infty$ if $\beta_{p,0}(G)=\infty$;
(2) $\beta_p(G)=\beta_{p,\infty}(G)$ if $\beta_{p,0}(G)=0$;
(3) $\beta^*_p(G)=\infty$ if $\beta_{p,0,*}(G)=\infty$.

If G is a tight, locally pure P-topological group, then also

(4) $\beta^*_p(G)=0$ if $\beta_{p,0}(G)=0$.

We will now consider a notion which is in some vague sense dual to the notion of a tight subgroup.

Definition 2.27. The *kernel* N of a topological group G is the intersection of all neighborhoods at 0.

Notice that N is a closed, L^t-definable, normal subgroup of G. By Fact 1.9 N and G/N inherit whatever degree of saturation G possesses.

At this point we can give a precise formulation of our classification of saturated torsionfree topological abelian groups of locally pure type.

Definition 2.28. Let G be a topological abelian group with kernel N. We associate to G the following family $\Gamma(G)$ of *standard invariants of* G:

(1) $\iota(N)$ for each Szmielew invariant ι;
(2) $\iota^*(G/N)$ for each Szmielew invariant ι;
(3) $\beta_{p,0,*}(G/N)$ for each prime p;
(4) $\beta_{p,n}(G/N)$ for each prime p and $0<n\leqslant\infty$;
(5) $\delta_*(G/N)$ (this simply signals whether or not G/N is discrete).

Lemma 2.29. *Any two L^t-equivalent torsionfree topological abelian groups of locally pure type have the same standard invariants.*

Proof. Combine Fact 2.8, Lemma 2.15, Lemma 2.19 and Lemma 2.20. □

Our main result is that the converse of this lemma is also true. As we will now show, the converse of Lemma 2.29 is a consequence of Theorem 3.20 and Lemma 3.21 below.

Theorem 2.30. *If G_1, G_2 are locally pure, torsionfree, topological abelian groups having the same standard invariants, then*

$$G_1 \equiv_t G_2.$$

Proof. We will make use of Theorem 3.20 and Lemma 3.21 stated and proved in Section 3 below. In particular let μ be as stated in Theorem 3.20 and assume $(G_1, \mathscr{B}_1), (G_2, \mathscr{B}_2)$ are μ-saturated with $\text{card}(G_i) = \text{card}(\mathscr{B}_i) = \mu$ for $i = 1, 2$. The kernels N_i of G_i are pure in G_i since they are intersections of pure torsionfree subgroups. By Fact 2.7 there are algebraic isomorphisms:

$$f_i : G_i \rightarrow G_i/N_i \oplus N_i$$

such that f_i is the identity on N_i. Giving N_i the trivial topology and G_i/N_i the quotient topology, we find that f_i is a topological isomorphism.

The topological groups $\bar{G}_i = G_i/N_i$ are again μ-saturated with saturation bases $\bar{\mathscr{B}}_i$ of the form $\{X/N_i : X \in \mathscr{B}_i\}$, card $(\bar{G}i) = \text{card}(\bar{\mathscr{B}}_i) = \mu$. Notice that the groups $\bar{G}_i$ are Hausdorff. If the $\bar{G}_i$ are discrete, then $\beta_{p,0}(\bar{G}_i) = \beta_p(\bar{G}_i)$. Thus $\bar{G}_1 \cong \bar{G}_2$. If both $\bar{G}_i$ are not discrete, then choose, using Lemma 2.22, tight pure open subgroups $\bar{H}_i \subseteq \bar{G}_i$, satisfying:

(1) $\beta_p(\bar{G}_i/\bar{H}_i) = \beta_p(\bar{G}_i)$ for all p,
(2) $\bar{H}_i$ is the intersection of countably many sets in $\bar{\mathscr{B}}_i$. Then again using

saturation and Fact 2.7 we find algebraic isomorphisms

$$g_i : \bar{G}_i \to (\bar{G}_i/\bar{H}_i) \oplus \bar{H}_i$$

such that g_i is the identity on $\bar{H}_i$, and g_i becomes a topological isomorphism if we give $\bar{H}_i$ the induced topology and take $\bar{G}_i/\bar{H}_i$ to be discrete.

Now Lemma 2.16, tightness, and our hypothesis yield for $0 \leqslant n \leqslant \infty$:

$$\beta_{p,n}(\bar{H}_1) = \beta_{p,n}(\bar{H}_2).$$

Theorem 3.20 and Lemma 3.21, then yield a topological isomorphism:

$$\bar{H}_1 \simeq_t \bar{H}_2.$$

By construction the groups $\bar{G}_1/\bar{H}_1$ and $\bar{G}_2/\bar{H}_2$ have the same Szmielew invariants, and the saturation of $\bar{G}_1, \bar{G}_2$ may be combined with [2] to yield an algebraic isomorphism between these groups, which of course is a topological isomorphism since they carry the discrete topology. Similarly $N_1 \simeq_t N_2$. Combining these three topological isomorphisms yields the desired result:

$$G_1 \simeq_t G_2. \quad \square$$

3. Structural analysis of locally groups

3.1. Preliminaries

In this subsection we will establish notation used in the structural analysis of locally pure groups, particularly saturated ones. For the convenience of the reader the more important items of notation together with the main structural relationships between them will be collected in a table at the end of Section 3.

Definition 3.1. Let G be a topological abelian group.

(1) The *local p-adic topology* on G is determined by the neighborhood basis: $\{G_{p,n} : n \geqslant 0\}$.

(2) The *local Z-topology* on G is the join of the local p-adic topologies on G.

(3) If G is Hausdorff in the local Z-topology, then $\bar{\bar{G}}$ denotes the completion of G in the local Z-topology.

The notation $\bar{\bar{G}}$ is usually applied in contexts in which the local Z-topology coincides with one of the local p-adic topologies. As stated previously, a *single* bar denotes the Z-completion of a Z-Hausdorff group. When G is not Hausdorff we do not attempt to define either $\bar{G}$ or $\bar{\bar{G}}$, but we may in any case say that G is *complete* iff every Cauchy sequence converges to at least one limit.

Remark 3.2. If G is an ω_1-saturated topological abelian group, then G is both Z-complete and locally Z-complete (cf. Lemma 1.2 of [2]).

We recall also the following general fact concerning Z-complete abelian groups.

Fact 3.3. *If G is Z-complete, then for each prime p and each $g \in G$ there is an element g_p in G such that:*
(1) $g - g_p \in p^\infty G$;
(2) g_p *is p'-divisible.*

Proof. See e.g. [2, Lemma 1.3] and [4, Theorem 39.1]. □

(**Notation 3.4.** For a prime p we call an element g of an abelian group G *p'-divisible* if g is divisible by every n relatively prime to p. G is p'-divisible if every $g \in G$ is.)

Proposition 3.5. *Let G be a Z-complete abelian group. Define:*

$$D \text{ is the kernel of } G \text{ in the } Z\text{-topology}, \tag{1}$$

Then:

$$D \text{ is the maximal divisible subgroup of } G. \tag{2}$$

Choose an arbitrary complement R to D:

$$G = R \oplus D. \tag{3}$$

Define:

$$R_p = \{r \in R\colon\ r \text{ is } p'\text{-divisible}\}.$$

Then:

$$R = \prod_p R_p \tag{4}$$

$$R \text{ and all the } R_p \text{ are } Z\text{-complete.} \tag{5}$$

Notes on the proof. (2) is easy and depends directly on Z-completeness. Fact 3.3 is used to prove (4). (2) is used to get (3). For (5): any direct factor of a Z-complete group is Z-complete.

Convention. All of the notation established in the above proposition is fixed for the remainder of this paper.

Lemma 3.6. *Let G be a Z-complete torsionfree topological abelian group, then $R_p \cap G_{p,\infty}$ is a direct summand of R_p.*

Proof. By torsionfreeness $G_{p,\infty}$ is a pure subgroup of G and therefore $R_p \cap G_{p,\infty}$ is a pure subgroup of G and a fortiori of R_p. Z-completeness of $R_p \cap G_{p,\infty}$ follows easily from the Z-completeness of R_p and we may then appeal to Fact 2.7. □

Remark. Consider $R_p = H$ as a topological subgroup of G. It might be tempting to conjecture that $H \cap G_{p,n} = H_{p,n}$ for all $n \leqslant \omega$, but this is false. Consider $G = \exp(Q, Z_p, \omega)$. The maximal divisible subgroup D of G is identified as $D = \exp(Q, \omega)$. Let H be the subgroup of G consisting of all constant functions with value in Z_p. Then $G = D \oplus H$. We get $G_{p,n} = G$ for all $0 \leqslant n \leqslant \infty$. But the topology induced on H by the topology of G is discrete; thus $H_{p,n} = p^n H$ $0 \leqslant n < \infty$, $H_{p,\infty} = \{0\}$.

Notation 3.7. (1) $R(p, \infty) = R_p \cap G_{p,\infty}$;

(2) $R_p = R(p, \infty) \oplus R'_p$;

(3) $R(p, n) = R'_p \cap G_{p,n}$ for $0 \leqslant n < \infty$.

Thus we have in particular $R(p, 0) = R'_p$.

Lemma 3.8. *If G is a Z-complete, locally Z-complete torsionfree topological abelian group, then R'_p is Z-complete and locally Z-complete (with respect to the local Z-topology on G).*

Proof. As a direct summand of a Z-complete group R'_p is Z-complete. For the local Z-completeness notice that we have a decomposition $G = R'_p \oplus R^*_p$ taking

$$R^*_p = D \oplus \left(\prod_{q \neq p} R_q \right) \oplus R(p, \infty).$$

We have to check that any limit in G of a local Z-Cauchy sequence in R'_p projects to a limit in R'_p and this is clear since $R^*_p \subseteq G_{p,\infty}$. □

For the remainder of this subsection G is a Z-complete, locally Z-complete torsionfree topological abelian group.

Lemma 3.9. (1) $G_{p,n} = R(p, n) \oplus G_{p,\infty}$ *for* $0 \leqslant n < \infty$;

(2) $G_{p,\infty} = R(p, \infty) \oplus \prod_{q \neq p} R_q$.

Proof. (1) $R(p, n) \cap G'_{p,\infty} = \{0\}$ follows right from the definition. For $g \in G_{p,n}$ write $g = g' + g^*$ with $g' \in R'_p$ and $g^* \in R_p$, then $g' \in R(p, n)$ and $g^* \in G_{p,\infty}$.

(2) Follows easily from $\prod_{q \neq p} R_q \subseteq G_{p,\infty}$. □

Lemma 3.10. (1) $R(p, 0)/R(p, 1) = G/G_{p,1}$;

(2) *for* $0 < n < \infty$

$$R(p, n)/(pR(p, n-1) + R(p, n+1)) = G_{p,n}/(pG_{p,n-1} + G_{p,n+1});$$

(3) $R(p, \infty)/pR(p, \infty) = G_{p,\infty}/pG_{p,\infty}$.

Proof. (1) The natural injective homomorphism from $R(p, 0)/R(p, 1)$ into $G/G_{p,1}$ is surjective by Lemma 3.9(1).

(2) Similarly the natural homomorphism

$$\tau : R(p, n)/(pR(p, n-1)+R(p, n+1)) \rightarrow G_{p,n}/(pG_{p,n-1}+G_{p,n+1})$$

is surjective by Lemma 3.9(1). To see that it is injective fix $r \in R(p, n) \cap (pG_{p,n-1}+G_{p,n+1})$ and suppose $r = pg_1 + g_2$ with $g_1 \in G_{p,n-1}$ and $g_2 \in G_{p,n+1}$. Applying the projection from G to R'_p it follows that $r \in pR(p, n-1)+R(p, n+1)$.

(3) The natural homomorphism

$$\sigma : R(p, \infty)/pR(p, \infty) \rightarrow G_{p,\infty}/pG_{p,\infty}$$

is surjective by Lemma 3.9(2). $R(p, \infty)$ is a pure subgroup of G as the intersection of two torsionfree pure subgroups. Hence $R(p, \infty) \cap pG_{p,\infty} = pR(p, \infty)$ and σ is injective. □

Notation 3.11.

(1) $V(p, 0) = R(p, 0)/R(p, 1)$;

$V(p, n) = R(p, n)/(pR(p, n-1)+R(p, n+1))$ for $0<n<\infty$.

These are vector spaces over the Galois field F_p.

(2) $X(p, n)$ = a set of representatives in $R(p, n)$ for a basis of $V(p, n)$. Since for $0 \leq n \leq \infty$ the groups $R(p, n)$ are p'-divisible and torsionfree we can regard them as Z_p-modules. Hence we may define:

(3) $R(p, n)^0$ = the Z_p-span of $X(p, n)$ in $R(p, n)$ for $0 \leq n < \infty$;

(4) $R(p, \infty)^0$=some p-basic submodule of $R(p, \infty)$;

(5) $X(p, \infty) = Z_p$-basis for $R(p, \infty)^0$.

Thus we have $R(p, \infty) = [R(p, \infty)^0]^-$.

(6) $B_p = (\sum \{R(p, n)^0 : 0<n<\infty\})^=$;

(7) $B = \prod_p B_p$;

(8) $B_\infty = \prod_p R(p, \infty)$.

Remark 3.12. $G_{p,1} \subseteq pG + (B_p \oplus R(p, \infty))$. This follows immediately from Lemma 3.9(1) and the definition of $R(p, 1)^0$. Notice that $V(p, 1) \cong R(p, 1)/pR'_p$.

For maximal efficiency we will now prove one technical result from which the remaining assertions in this subsection will follow:

Theorem 3.13. *Let* $x = \sum \{x_m : 0 \leq m < \infty\}$ *with* $x_m \in R(p, n)^0$ *and all but finitely many of the* x_m *equal* 0. *Then for each n:*

(1) $x \in p^n G$ *iff for all* m $x_m \in p^m R(p, m)^0$,

(2) $x \in G_{p,n}$ *iff for all* $m < n$ $x_m \in p^{n-m} R(p, m)^0$.

Proof. We treat both cases simultaneously, but in dealing with (2) we may suppose without loss of generality that $x_m = 0$ for $m > n$. The implications from right to left are trivial. Going the other way, we proceed by induction on n, the case $n = 0$ being trivial. Consider therefore the passage from n to $n+1$. If each x_m is p-divisible, we conclude at once by induction. Every $x_m \neq 0$ may be represented as

$$x_m = p^{i(m)} y_m$$

with

$$y_m \in R(p, m) \setminus (pR(p, m-1) + R(p, m+1)) \quad \text{for } 0 < m < \infty$$

and

$$y_0 \in R_p \setminus R(p, 1).$$

Indeed, set $x_m = \sum \{z(x) \cdot x \colon x \in X(p, m)\}$ for certain $z(x) \in Z_p$ and $0 = z(x)$ for almost all x. Take $i(m)$ to be the largest i for which p_i divides all $z(x)$ in Z_p. Take $y_m = p^{-i} x_m$.

If not all x_m are p-divisible, let m_0 be the least index such that x_{m_0} is not p-divisible. Thus $x_{m_0} \neq 0$ and $i(m_0) = 0$, and $m_0 \leq n+1$ in the second case. Now $x \in p^{n+1}G$ (resp. $x \in G_{p,n+1}$) yields:

$$y_{m_0} \in R(p, 1) \qquad \text{if } m_0 = 0;$$

$$y_{m_0} \in pR(p, m_0 - 1) + R(p, m_0 + 1) \quad \text{if } 0 < m_0 < \infty$$

contradicting our choice for y_m.

Corollary 3.14. *The sum* $\sum \{R(p, n)^0 \colon 0 \leq n < \infty\}$ *is direct.*

Proof. If $\sum \{x_m \colon 0 \leq m < \infty\} = 0$ with $x_m \in R(p, m)^0$, then Theorem 3.13(1) shows $x_m \in R(p, m) \cap p^\infty G = \{0\}$. □

Corollary 3.15. *For* $0 \leq m < \infty$ *the p-adic topology and the local p-adic topology on G induce the same topology on* $R(p, m)^0$.

Proof. By Theorem 3.13 $R(p, m)^0 \cap p^n G = R(p, m)^0 \cap G_{p,m+n}$. □

Lemma 3.16. $R'_p = (\bigoplus \{R(p, n)^0 \colon 0 \leq n < \infty\})^=$.

Proof. The inclusion $\supseteq$ follows from Lemma 3.8. So it remains to show that $\sum \{R(p, n)^0 \colon 0 \leq n < \infty\}$ is locally p-adically dense in R'_p. It suffices to show:

(m) For $m < \infty$ and $r \in R(p, m)$ there is an element $b \in \sum \{R(p, n)^0 \colon 0 \leq n < \infty\}$ such that

$$r \in b + R(p, m+1).$$

This is done by induction on m. The case $m=0$ is easy, and for $m>0$ there is an element $a \in R(p, m)^0$ so that

$$r-a \in pR(p, m-1)+R(p, m+1)$$

so that for some $r' \in R(p, m-1)$ we have

$$r \in a+pr'+R(p, m+1).$$

By induction hypothesis write $r' \in b'+R(p, m)$ with $b' \in \sum\{R(p, n)^0 \colon 0 \leqslant n<\infty\}$ and taking $b=a+pb'$ we conclude

$$r \in b+R(p, m+1). \quad \square$$

Lemma 3.17. *For all* $k \geqslant 0$:

$$(\sum\{R(p, n)^0 \colon k \leqslant n<\infty)^{=}=\prod\{[R(p, n)^0]^{-} \colon k \leqslant n<\infty\}.$$

Proof. Set $E=\sum\{R(p, n)^0 \colon k \leqslant n<\infty\}$. By Corollary 3.14 there are canonical projections

$$\pi_n \colon E \rightarrow [R(p, n)^0] \quad \text{for } k \leqslant n<\infty$$

which are continuous in view of Theorem 3.13(2). Hence we obtain continuous extensions:

$$\bar{\pi}_n \colon E^{=} \rightarrow [R(p, n)^0]^{-}$$

which can be combined into a map

$$\pi \colon E^{=} \rightarrow \prod\{[R(p, n)^0]^{-} \colon k \leqslant n<\infty\}$$

which is again continuous. It remains to be seen that π is an isomorphism.

Injectivity. If $b \in E^{=}$ and $\pi(b)=0$ let $b=\lim_i b_i$ in the local p-topology, $b_i \in E$. Then for each $m \geqslant k$ $\lim_i \pi_m(b_i)=0$ in the p-adic topology on $R(p, m)^0$. Then for m fixed and n large:

$$\pi_l(b_n) \in p^m R(p, l) \quad \text{for} \quad 0 \leqslant l \leqslant m.$$

So by Theorem 3.13(2) $b_n \in R(p, m)$. Thus $b \in \bigcap\{R(p, m) \colon m \geqslant k\} \cap R_p'=\{0\}$.

Surjectivity. If for $i \geqslant k$ $a_i \in R(p, i)^0$ and $b_n=\sum\{a_i \colon k \leqslant i \leqslant n\}$, then $\{b_n\}_{n \in \omega}$ is a Cauchy-sequence in the local p-adic topology, hence converges to some $b \in E^{=}$ and it is easily checked that for all $i \geqslant k$ $\pi_i(b)=a_i$. $\square$

Corollary 3.18. *B is Z-complete.*

Proof. By Lemma 3.17 $R_p'=[R(p, 0)^0] \oplus B_p$. Thus B_p is Z-complete as a direct summand of the Z-complete group R_p'. Now Z-completeness of B follows. $\square$

3.2. The structure theorem

We can now state a structure theorem for the appropriate class of saturated topological abelian groups. The proof of this result will occupy this and the next two subsections.

Theorem 3.19. *Let μ be an uncountable regular cardinal such that $\lambda^\omega < \mu$ for all $\lambda < \mu$. Let $(G, \mathcal{B})$ be a saturated torsionfree, tight Hausdorff topological abelian group of locally pure type of cardinality μ with* card$(\mathcal{B}) = \mu$, *then*

$$G = \exp(A, C, \mu)$$

where A, C have the form described in Example 2.12 and satisfy in addition the following restrictions:

(1) $D = \{0\}$;
(2) $\beta_{p,0} = 0$, $\beta'_{p,0} = \beta_{p,0}(G)$;
(3) $\delta = \infty$.

We will in fact prove a little more. In the following version of Theorem 3.19 we isolate exactly what consequences of saturation we will use:

Theorem 3.20. *Let μ be an uncountable regular cardinal such that $\lambda^\omega < \mu$ for all $\lambda < \mu$. Let G be a torsionfree, tight, Hausdorff, topological abelian group of locally pure type with* card$(G) = \mu$ *such that*

(1) *G is Z-complete.*
(2) *G is locally Z-complete.*
(3) *There is a neighborhood basis at 0 $\{H^0_\alpha: \alpha < \mu\}$ with the properties*
  (3.1) *H^0_α is a pure open subgroup of G.*
  (3.2) *H^0_α is Z-complete.*
  (3.3) *For $\alpha < \beta$ $H^0_\beta \subseteq H^0_\alpha$.*
  (3.4) *For limit δ: $H^0_\delta = \bigcap\{H^0_\alpha: \alpha < \delta\}$.*
  (3.5) *For all $\alpha < \mu$: $\delta(H^0_\alpha/H^0_{\alpha+1}) = \mu$ and $\beta_p(H^0_\alpha/H^0_{\alpha+1}) = \beta_{p,0}(G)$ for each p.*
  (3.6) *G is μ-pseudocomplete relative to the chain $\{H^0_\alpha: \alpha < \mu\}$, i.e. for each $\beta < \mu$, if $\{g_i: i < \beta\}$ is a sequence of elements of G such that $g_j \in g_i + H_{i+1}$ for $j > i$ then there is an element $g \in G$ with $g \in g_i + H_{i+1}$ for all $i < \beta$.*
(4) *For all primes p:*

$$\beta_{p,\infty}(G) = \dim G_{p,\infty}/(G_{p,\infty} \cap pG).$$

Then the conclusions of Theorem 3.19 hold true.

Of course we have to convince our readers that every group satisfying the hypotheses of Theorem 3.19 also satisfies the hypotheses of Theorem 3.20. We will do better:

Lemma 3.21. *Let μ be an uncountable cardinal and $(G^*, \mathcal{B})$ a saturated torsion-free Hausdorff topological abelian group of locally pure type, $G^* =$* card$(\mathcal{B}) = \mu$, *and G a tight, pure open subgroup of G^* which is the intersection of less than μ elements in $\mathcal{B}$.*

Then G satisfies the hypotheses of Theorem 3.20.

Proof. (1), (2), and (4) pose no problem (cf. Remark 3.2 and Lemma 2.14(iii)). The groups H^0_α will be constructed inductively on α and every H^0_α will be the intersection of less than μ many sets in $\mathscr{B}$. This will automatically make Remark 3.2 and Lemma 3.6 true. At limit stages δ we take $H^0_\alpha = \bigcap\{H^0_\alpha: \alpha<\delta\}$. Now consider the step from α to $\alpha+1$. For every p we will construct a pure open subgroup $K_p \in G_\delta(\mathscr{B})$ such that

$$\beta_p(H^0_\alpha/K_p) = \beta_{p,0}(G).$$

By Lemma 2.24, the fact that G is tight and the intersection of less than μ many elements in $\mathscr{B}$ and μ-saturation of $(G^*, \mathscr{B})$ there are only two possibilities:

$$\beta_{p,0}(G)=0 \quad \text{or} \quad \beta_{p,0}(G)=\mu.$$

If $\beta_{p,0}(G)=0$, then let K_p be any pure open subgroup in $G_\delta(\mathscr{B})$. Otherwise we have by tightness and the Remark 2.26 for every pure open subgroup K' of G $\beta_p(K') \geq \omega_0$. This allows to choose a sequence $(U_n)_{n\in\omega}$ of sets from $\mathscr{B}$ such that

(1) $U_n \subseteq H^0_\alpha$ for all n,

(2) $\bigcap\{U_n: n\in\omega\}$ is a pure open subgroup of G,

(3) for all $\gamma<\mu$ the types

$$\{\{x_\nu \in U_n: \nu<\gamma, n\in\omega\} \cup \{\textstyle\sum\{\lambda_\nu x_\nu: \nu\in J\} \neq 0: J \text{ a finite subset of } \gamma \text{ and } \lambda: J \to C_{p,1} \text{ not constant} = 0\}$$

are finitely satisfiable.

Now we take $K_p = \bigcap\{U_n: n\in\omega\}$. Analogously we construct a pure open subgroup $K_0 \in G_\delta(\mathscr{B})$ such that

$$\delta(H^0_\alpha/K_0) = \mu.$$

Finally $H^0_{\alpha+1} = K_0 \cap \bigcap\{K_p: p \text{ a prime}\}$. □

Our immediate goal is to recast Theorem 3.20 in a more explicit form, see Theorem 3.23 below. The latter will eventually be reduced to Theorem 3.26 of Section 3.3.

Definition 3.22. Let G be a locally pure topological abelian group, μ a cardinal.

(1) A *fundamental chain* for G of length μ is a family $\{H_\alpha: \alpha<\mu\}$ of pure open subgroups of G such that:

(1.1) $\{H_\alpha: \alpha<\mu\}$ is a neighborhood basis for G.

(1.2) $H_0 = G$.

(1.3) $H_\alpha \subseteq H_\beta$ for $\beta \leq \alpha$.

(1.4) $H_\delta = \bigcap\{H_\alpha: \alpha<\delta\}$ for limit ordinals δ.

(1.5) G is μ-pseudocomplete relative to the chain $\{H_\alpha: \alpha<\mu\}$.

(2) A fundamental chain $\{H_\alpha: \alpha<\mu\}$ for G is *complemented* if there is a family $\{A_{\alpha,\beta}: \alpha<\beta<\mu\}$ of subgroups of G such that:

(2.1) $H_\alpha = A_{\alpha,\beta} \oplus H_\beta$ for $\alpha<\beta<\mu$;

(2.2) $A_{\alpha,\gamma} = A_{\alpha,\beta} \oplus A_{\beta,\gamma}$ for $\alpha<\beta<\gamma<\mu$.

Given such a complemented fundamental chain for G we adopt the following:

Notation. (1) $A_\alpha = A_{\alpha,\alpha+1}$;

(2) $\pi_\alpha : G \to H_\alpha$ is the projection associated to the decomposition $G = A_{0,\alpha} \oplus H_\alpha$;

(3) $\pi'_\alpha = \pi_\alpha - \pi_{\alpha+1}$.

Examples. In the notation of Definition 2.11, if $G = \exp(A, B, \mu)$, then $\{U_\alpha \cap G: \alpha < \mu\}$ is a fundamental chain for G with complements:

$$A_{\alpha,\beta} = \prod \{A_i: \alpha \leq i \leq \beta\}$$

where A_i is the ith copy of A.

Theorem 3.23. *Under the hypotheses of Theorem* 3.20 *there exist*:

(i) *a fundamental chain* $\{H_\alpha: \alpha < \mu\}$ *for* G *with complements* $A_{\alpha,\beta}$:

(ii) *abelian groups* $C \subseteq A$

(iii) *isomorphisms* $f_\alpha : A_\alpha \to A$

with the following properties:

(1)
$$A = Q^{(\mu)} \oplus \prod_p \bar{A}_{p,0} \oplus \prod_p \left(\left(\sum_{0<n<\infty} A_{p,n}\right)^- \oplus A_{p,\infty}\right)$$

(1.1) $A_{p,n} = Z_p^{(\beta_{p,n}(G))}$ *for* $0 \leq n < \infty$;

(1.2) $A_{p,\infty} =$ *the divisible hull of* $C^*_{p,\infty}$.

(2)
$$C = \prod_p \left(\sum_{0<n<\infty} C_{p,n}\right)^- \oplus C^*_{p,\infty}$$

(2.1) $C_{p,n} = p^n A_{p,n}$ $0 < n < \infty$;

(2.2) $C_{p,\infty} = Z_p^{(\beta_{p,\infty}(G))}$.

(*Here* $^-$ *denotes completion with respect to the* Z-*adic topology on* A *and* $C^*_{p,\infty}$ *denotes the completion of* $C_{p,\infty}$ *in its own* p-*adic topology*.)

(3) *For any* $g \in G$ *the function:*

$$\hat{g}(\alpha) = f_\alpha(\pi'_\alpha(g))$$

is eventually constant with a value in C *and every element of* C *is the eventual value of such a function* $\hat{g}$.

Now we show that this result implies Theorem 3.20.

Proof of Theorem 3.20. We use the notation of Theorem 3.23. Set

$$H = \prod \{A_\alpha: \alpha < \mu\}$$

equipped with the topology defined by the neighborhood basis consisting of the subgroups U_α defined by

$$U_\alpha = \{h \in H: h_\beta = 0 \quad \text{for } \beta < \alpha\}.$$

Define a map $\tilde{}: G \to H$ by:

$$(\tilde{g})_\alpha = \pi'_\alpha(g) \quad \text{for } g \in G.$$

We show:

(α) For $g \in G$: $g \in H_\alpha$ iff $\tilde{g} \in U_\alpha$.

The inclusion $\tilde{H}_\alpha \subseteq U_\alpha$ is clear, and the reverse is proved by induction on α. The main point is that if $\alpha = \beta + 1$ and $\tilde{g} \in U_\alpha$, then also $\tilde{g} \in U_\beta$ and thus $g \in H_\beta$ by the induction hypothesis. Then the condition $\pi'_\beta(g) = 0$ yields $\pi_\alpha(g) = \pi_\beta(g) = g$, so $g \in H_\alpha$ as claimed.

As a particular consequence of (α), if $\tilde{g} = 0$, then g belongs to each H_α and hence $g = 0$. It follows using (α) that G is topologically isomorphic with the topological subgroup $\tilde{G}$ of H.

We combine the isomorphisms $f_\alpha : A_\alpha \to A$ to get a topological isomorphism

$$f : H \to A.$$

Condition (3) of Theorem 3.23 implies that:

$$f[\tilde{G}] \subseteq \exp(A, C, \mu)$$

and to complete the proof it suffices to show that this inclusion can be improved to equality.

Let $k_\alpha = \{h \in H$: for all $\beta \geqslant \alpha$ $h_\beta = 0\}$. In view of condition (3) of Theorem 3.23 it suffices to show that $K_\alpha \subseteq \tilde{G}$. Indeed we claim:

(K) $K_\alpha = \tilde{A}_{0,\alpha}$.

The inclusion $\tilde{A}_{0,\alpha} \subseteq K_\alpha$ is clear and the reverse is proved by induction on α. In this connection we have to deal primarily with the case of a limit ordinal α.

Suppose then that $h \in K_\alpha$ and by the induction hypothesis choose $g_i \in A_{0,i}$ for $i < \alpha$ satisfying:

$$(\tilde{g}_i)_j = \begin{cases} h_j & \text{for } j \leqslant i, \\ 0 & \text{for } j > i. \end{cases}$$

Applying clause (1.5) of Definition 3.22 yields an element x of G satisfying:

$$x - g_i \in H_{i+1} \quad \text{for } i < \alpha.$$

Let g be the projection of x on $A_{0,\alpha}$. Then for $i \geqslant \alpha$ $\pi'_i(g) = 0$ and for $i < \alpha$:

$$\pi'_i(g) = \pi'_i(x) = \pi'_i(g_i) = h_i.$$

Thus the proof is complete. □

3.3. Theorem 3.26

This subsection will be devoted to a further reformulation of Theorem 3.20 which will be stated as Theorem 3.26 below. The notation associated with the preliminary analysis given in Section 3.1 will now begin to play a role.

Definition 3.24. A subgroup I of

$$\prod_p \left[\left(\sum_{0<n<\infty} R(p,n)^0\right)^{=} \oplus R(p,\infty)\right] = B \oplus B_\infty$$

is *rectangular*, if

(1) $I = \prod_p [(\sum_{0<n<\infty} (I \cap R(p,n)^0))^{-} \oplus (I \cap R(p,\infty)];$

(2) $I^0_{p,n} = I \cap R(p,n)^0 = Z_p$-span of $I \cap X(p,n)$ for $0<n<\infty$;

(3) $I_{p,\infty} = I \cap R(p,\infty) = (I \cap X(p,\infty))^{-}$.

Notation 3.25. Fix a family $\{I_\alpha : \alpha<\mu\}$ of rectangular subgroups such that

(1) $I_\alpha \subseteq I_\beta$ for $\alpha<\beta<\mu$;

(2) $\bigcup \{I_\alpha : \alpha<\mu) = \prod_p ((\sum_{0<n<\infty} R(p,n)^0)^{-} \oplus R(p,\infty);$

(3) $\operatorname{card}(I_\alpha)<\mu$ for all $\alpha<\mu$.

Our condition on μ that $\lambda^\omega<\mu$ for $\lambda<\mu$, is imposed in order to ensure the existence of such a family of rectangular subgroups.

Theorem 3.26. *If G is as in Theorem 3.20, then there is a fundamental chain $\{H_\alpha : \alpha<\mu\}$ for G with complements $\{A_{\alpha,\beta} : \alpha<\beta<\mu)$ so that:*

(1) *$\beta_p(A_\alpha) = \beta_{p,0}(G)$ for all primes p, $\delta(A_\alpha) = \mu$.*

(2) *Setting $R_\alpha(p,n) = \pi'_\alpha[I_\alpha \cap R(p,n)^0]$ for $0<n<\infty$ and $R_\alpha(p,\infty) = \pi'_\alpha[I_\alpha \cap R(p,\infty)]$*

(2.1) *π'_α is injective on I_α;*

(2.2) *$R_\alpha(p,n) \subseteq p^n A$ for $0<n\leqslant\infty$;*

(2.3) *$[(\sum_{0<n<\infty} p^{-n} R(p,n))^{-} \oplus p^{-\infty} R(p,\infty)]$ is a pure subgroup of A_α for each prime p (here $p^{-\infty} R_\alpha(p,\infty)$ is the divisible hull of $R_\alpha(p,\infty)$ and $^{-}$ denotes the completion in the Z-adic topology on A_α).*

(3) *For $g \in G$ and $\alpha<\mu$: if $\pi_\alpha(g)$ is divisible, then for some $\beta<\mu$ $\pi_\beta(g)=0$.*

(4) *For $g \in G$ and $\alpha<\mu$: if $\pi_\alpha(g)$ is p'-divisible, then for some β $\pi_\beta(g) \in G_{p,1}$.*

Reduction of 3.23 to 3.26. Let $\{H_\alpha : \beta<\mu\}$ be a fundamental chain for G with complements $\{A_{\alpha,\beta} : \alpha<\beta<\mu\}$ as supplied by Theorem 3.26. We must produce groups A, C and isomorphisms $f_\alpha : A_\alpha \to A$.

Set:
$$\tilde{R}_\alpha(p) = \left(\sum_{0<n<\infty} p^{-n} R_\alpha(p,n)\right)^{-};$$

$$\tilde{R}_\alpha = \prod_p \tilde{R}_\alpha(p);$$

$$\tilde{R}_\alpha(p,\infty) = \text{divisible hull of } R_\alpha(p,\infty);$$

$$\tilde{R}_{\alpha,\infty} = \prod_p \tilde{R}_\alpha(p,\infty).$$

It follows from clause (2.3) that $\sum\{\tilde{R}_\alpha(p) : p \text{ prime}\}$ has intersection $\{0\}$ with the maximal divisible subgroup D_α of A_α. Thus we find a complementary summand

A_α^r such that

$$A_\alpha = D_\alpha \oplus A_\alpha^r \quad \text{and} \quad \tilde{R}_\alpha(p) \subseteq A_\alpha^r \quad \text{for all } p$$

Since A_p is Z-complete as a direct summand of the Z-complete group G we get

$$A_\alpha^r = \prod_p A_{\alpha,p} \quad \text{with } A_{\alpha,p} = \{a \in A_\alpha^r : a \text{ is } p\text{'divisible}\}.$$

Since $\tilde{R}_\alpha(p)$ is p'-divisible we get $\tilde{R}_\alpha(p) \subseteq A_{\alpha,p}$ and thus $\tilde{R}_\alpha$ may be embedded into A_α^r. Since for all p card$(\tilde{R}_\alpha(p,\infty)) < \mu$ we may also embed $\prod_p \tilde{R}_\alpha(p,\infty)$ into D_α. Set:

$$C_\alpha(p) = \Big(\sum_{0<n<\infty} R_\alpha(p,n)\Big)^-$$

(where $^-$ denotes completion in the p-adic topology of A_α)

$$C_\alpha = \prod_p C_\alpha(p), \qquad C_{\alpha,\infty} = \prod_p R_\alpha(p,\infty).$$

Since $\tilde{R}_\alpha \oplus \tilde{R}_{\alpha,\infty}$ is a pure Z-complete subgroup of A_α we have a decomposition:

$$A_\alpha = A'_\alpha \oplus \tilde{R}_\alpha \oplus \tilde{R}_{\alpha,\infty}.$$

Since card$(\tilde{R}_\alpha \oplus \tilde{R}_{\alpha,\infty}) < \mu$ therefore A'_α has by clause 3.26(1) the same invariants as A_α and hence is isomorphic with A_α.

Now define isomorphisms $f_{\alpha,\beta} : A_\alpha \to A_\beta$ for $\alpha < \beta < \mu$ and subgroups $S_\alpha \subseteq A_\alpha$ for $\alpha < \mu$ satisfying:

(1) $(\{A_\alpha\}, \{f_{\alpha,\beta}\})$ is a directed system.

(2) S_α is a direct summand of A'_α having the same Szmielew invariants.

(3) $f_{\alpha,\beta}[S_\alpha] \subseteq S_\beta$.

(4) If $x \in X(p,n) \cap I_\alpha$ with $0 < n \leq \infty$, then $f_{\alpha,\beta}(\pi'_\alpha(x)) = \pi'_\beta(x)$.

To see that such a collection of maps and subgroups can be constructed, proceed by induction on β. If we have constructed $f_{\alpha,\beta}$ for $\alpha < \beta$ and S_α for $\alpha \leq \beta$, then it is easy to choose $S_{\beta+1}$ and $f_{\beta,\beta+1}$ suitably, bearing in mind the structure of A_β and $A_{\beta+1}$ and making use of clause 3.26(2.1) in connection with (4) above.

To carry through the induction at a limit ordinal δ, suppose that $f_{\alpha,\beta}$ and S_α have been constructed for $\alpha < \beta < \delta$. Then

$$(\{A_\alpha : \alpha < \delta\}, \{f_{\alpha,\beta} : \alpha < \beta < \delta\})$$

is a directed system whose limit $(L, \{g_\alpha\})$ is isomorphic to each of the A_α. The limit of the groups $C_\alpha \oplus C_{\alpha,\infty}$ is a subgroup of L which may be identified with the subgroup $\bigcup \{I_\alpha : \alpha < \delta\}$ of I_δ.

Choose a subgroup S_δ of A'_δ and an isomorphism $g : L \to A_\delta$ carrying $\varinjlim S_\alpha$ into a subgroup of S_δ and extending the natural map from $\varinjlim(C_\alpha \oplus C_{\alpha,\infty})$ into $C_\delta \oplus C_{\delta,\infty}$. Take $f_{\alpha,\delta} = g \cdot g_\alpha$. This completes our description of the inductive construction.

Now let $(A, \{f_\alpha : \alpha < \mu\})$ be the direct limit of the system so constructed. Let C be the limit of $(\{C_\alpha \oplus C_{\alpha,\infty}\}, \{f_{\alpha,\beta}\})$ as a subgroup of A. As $\mu > \omega$ is regular, the group $\tilde{C} = \varinjlim(\tilde{R}_\alpha \oplus \tilde{R}_{\alpha,\infty})$ is a Z-complete pure subgroup of A, hence has a complement $A' = A/\tilde{C}$. Since furthermore $\varinjlim S_\alpha$ is isomorphic with a pure subgroup of $A/\tilde{C}$ it is clear that A' has the same invariants as $\lim S_\alpha$. Thus:

$$\delta(A') = \delta(G), \qquad \beta_p(A') = \beta_{p,0}(G).$$

Thus clauses (1) and (2) of Theorem 2.23 are satisfied.

We conclude with the verification of clause (3) of Theorem 3.23. By our construction, for each $b \in B \oplus B_\infty$ $f_\alpha(\pi'_\alpha(b))$ is eventually constant. It remains to be seen that for every g in G $f_\alpha(\pi'_\alpha(g))$ is eventually constant with value in C. To this end it will suffice to show that for some α:

$$\pi_\alpha(g) \in \pi_\alpha[B \oplus B_\infty].$$

In view of clause (3) of Theorem 3.26 we may suppose that $g \in R$. For each prime p let g_p be the projection of g on R_p. It will suffice to prove the following for each prime p:

(p) There is an $\alpha < \mu$ and $b_p \in B \oplus B_\infty$ such that:

(*) $(g_p - b_p) \in p^\infty G$.

Indeed, granted (p) for all p, then for $\alpha < \mu$ large there are $b_p \in B \oplus B_\infty$ so that (*) holds for all p. There is then by Z-completeness of $B \oplus B_\infty$ some $b \in B \oplus B_\infty$ so that

$$b - b_p \in p^\infty G \quad \text{for all } p$$

and therefore:

$$\pi_\alpha(g - b) \text{ is divisible.}$$

Then in view of clause (3) of Theorem 3.26 $\pi_\beta(g - b) = 0$ for large β and our claim follows.

Thus we need only prove (p). Let p be fixed. We claim that for each n:

(n) There is an $\alpha < \mu$ and $b_n \in B \oplus B_\infty$ so that $\pi_\alpha(g_p - b_n) \in p^n G$.

This then easily yields (p) by a limit argument.

We prove (n) by induction, starting at $n = 0$. Suppose then that b_n is given, and let us find b_{n+1}. Set $\pi_\alpha(g - b_n) = p^n g'$. It will then suffice to find $b \in B \oplus B_\infty$ for which $\pi_\beta(g' - b) \in pG$ for some β.

By clause (4) of Theorem 3.26 we may suppose that $g' \in G_{p,1}$. Then Remark 3.12 completes the proof.

3.4. Proof of Theorem 3.26

Let us formulate the result to be proved a little more precisely. Fix an enumeration $\{g^\alpha : \alpha < \mu\}$ of G.

Reformulation 3.27. Under the hypotheses of Theorem 3.26 we seek a complemented fundamental chain $\{H_\alpha : \alpha < \mu\}$ satisfying:

(1) & (2) as given in Theorem 3.26.

(3′) For all $\alpha < \mu$: if $i = i(\alpha)$ is the least index such that $\pi_\alpha(g^i)$ is divisible and nonzero, then $\pi_{\alpha+1}(g^i) = 0$.

(4′) For all $\alpha < \mu$ and all primes p: if $i = i(\alpha, p)$ is the least index such that $\pi_\alpha(g^i)$ is p'-divisible but not in $G_{p,1}$, then $\pi_{\alpha+1}(g^i) \in G_{p,1}$.

(5) $H_{\alpha+1} \subseteq H^0_\alpha$.

(6) For all $\alpha < \mu$ there is some γ such that $H_\alpha = H^0_\gamma$.

Clearly we will then have Theorem 3.26.

The construction of such a fundamental chain is carried out inductively. The case $\alpha = 0$ is trivial and we will now check that the case $\alpha = \delta$ a limit ordinal is straightforward.

Construction at limit steps 3.28. Assume therefore that H_β, $A_{\beta,\gamma}$ for $\beta < \gamma < \delta$ have been constructed. Then it is only required to find H_δ, $A_{\beta,\delta}$ satisfying the requirements for a complemented fundamental chain, as the other conditions are vacuous. Set

$$H_\delta = \bigcap \{H_\beta : \beta < \delta\}, \qquad A = \sum \{A_{0,\beta} : \beta < \delta\}.$$

We have $H_\delta = H^0_\gamma$ for some $\gamma < \mu$ by assumption (3.4) of Theorem 3.20 and $H_\beta \subseteq H^0_\beta$ for all $\beta < \delta$ certainly implies $H_\delta \subseteq H^0_\delta$. Obviously $A \cap H_\delta = \{0\}$ and it is easily seen that $A \oplus H_\delta$ is pure in G. As $H_\delta = H^0_\gamma$ is Z-complete there is by Fact 2.9 a complement $A_{0,\delta}$ to H_δ in G containing A. Define the remaining complements by:

$$A_{\beta,\delta} = A_{0,\delta} \cap H_\beta \quad \text{for } \beta < \delta.$$

It is not difficult to check:

(1) $H_\beta = A_{\beta,\delta} \oplus H_\delta$,

(2) $A_{\beta,\delta} = A_{\beta,\gamma} \oplus A_{\gamma,\delta}$.

However, the treatment of our induction at successor stages is more demanding. It is necessary to construct $H_{\alpha+1}$, A_α satisfying the stipulations above. The remaining complements may then be defined by

$$A_{\beta,\alpha+1} = A_{\beta,\alpha} \oplus A_\alpha.$$

The necessary construction will be described in the remainder of this subsection. It is accordingly assumed that H_i has been constructed for $i \leqslant \alpha$ and $A_{i,j}$ has been constructed for $i < j \leqslant \alpha$, satisfying the relevant stipulations. Hence π_i has been determined for $i \leqslant \alpha$.

Lemma 3.29. (1) *If $x \in B \oplus B_\infty$ and $\pi_\alpha(x) \in pG$, then $x \in pG$.*

(2) *π_α induces an isomorphism of $B \oplus B_\infty$ onto $\pi_\alpha[B \oplus B_\infty]$.*

(3) *$\pi_\alpha[B \oplus B_\infty]$ is a pure, Z-complete subgroup of H_α.*

Proof. (1) $B\oplus B_\infty \subseteq G_{p,1}$. Fix h in H_α so that $x-h\in pG$. Applying the projection onto $A_{0,\alpha}$ yields: $x-\pi_a(x)\in pG$ and the claim follows.

(2) The claim is that the restriction of π_α to $B\oplus B_\infty$ is injective. This will follow from (1) using the relation: $(B\oplus B_\infty)\cap\bigcap_p p^\infty G=\{0\}$.

(3) $(B\oplus B_\infty)$ is a pure, Z-complete subgroup of G.

Now using Fact 2.7 and Lemma 3.29(3) decompose H_α as follows:

$$H_\alpha = H'_\alpha \oplus \pi_\alpha[B\oplus B_\infty].$$

Furthermore decompose H'_α in the usual fashion:

$$H'_\alpha = D_\alpha \oplus R_\alpha, \qquad R_\alpha = \prod_p R_{\alpha,p}$$

with D_α divisible, R_α reduced and $R_{\alpha,p}$ p'-divisible. □

As these decompositions are not at all canonical they need not be compatible with the original decomposition of G, but we have some useful information:

Lemma 3.30. (1) $R_{\alpha,p}\cap G_{p,1}=pR_{\alpha,p}$.

(2) $\beta_p(R_{\alpha,p})=\beta_{p,0}(G)$.

Proof. (1) If $r\in R_{\alpha,p}\cap G_{p,1}$, then by Remark 3.12 we find $b\in B\oplus B_\infty$ such that $r-b\in pG$. Projecting into H_α:

$$r-\pi_\alpha(b)\in pH_\alpha.$$

$R_{\alpha,p}\oplus\pi_\alpha[B\oplus B_\infty]$ is a direct summand of H_α, so $r\in pR_{\alpha,p}$.

(2) In view of (1), there is a canonical monomorphism:

$$\tau: R_{\alpha,p}/pR_{\alpha,p}\to H_\alpha/(H_\alpha)_{p,1}$$

which is clearly surjective. Hence:

$$\beta_p(R_{\alpha,p})=\beta_{p,0}(H_\alpha)=\beta_{p,0}(G)$$

since G is tight. □

Returning to our construction, in connection with conditions (3′), (4′) of reformulation 3.27 we establish the following notation:

$$g_0=\pi_\alpha(g^{i(\alpha)}) \text{ if this is divisible and nonzero;}$$

r_p = the projection of $\pi_\alpha(g^{i(\alpha,p)})$ on $R_{\alpha,p}$ if the latter element is p'-divisible, but not in $G_{p,1}$.

In the event that some of these elements fail to exist, the corresponding parts of the following construction may be omitted.

Fixing a Z_p-basis $X_{\alpha,p,0}$ for a dense free submodule of $R_{\alpha,p}$ for each p, we may define rectangularity for subgroups of $R_\alpha\oplus\pi_a[B\oplus B_\infty]$ in the obvious way. Let I

be a rectangular subgroup of $R_\alpha \oplus \pi_\alpha[B \oplus B_\infty]$ such that

(1) $\mathrm{card}(I) < \mu$;

(2) $I \cap \pi_\alpha[B \oplus B_\infty] = \pi_\alpha[I_\alpha]$;

(3) $r_p \in I$ for all p.

Taking into account the properties of μ such an I certainly exists.

We are now ready to choose $H_{\alpha+1}$, but we first isolate a preparatory lemma.

Lemma 3.31. *If I, D_0 are subgroups of G of cardinality less than μ, then there is $\beta < \mu$ such that:*

(1) $D_0 \cap H^0_\beta = \{0\}$,

(2) $I \cap (H^0_\beta + p^n G) \subseteq G_{p,n}$ *for all p and* $0 < n \leqslant \infty$.

Proof. Let $D_0 \setminus \{0\} = \{d_j : j < \gamma\}$ and $I = \{i_j : j < \gamma'\}$ with $\gamma, \gamma' < \mu$. For each p and $0 < n < \infty$ if $i_j \notin G_{p,n}$, then there is some $\beta(j, p, n) < \mu$ such that $i_j \notin H_{\beta(j,p,n)} + p^n G$. Set $\beta = \sup\{\beta(j, p, n) : p, j, n\}$. Since μ is uncountable and regular we have $\beta < \mu$ and H^0_β satisfies (2) for $0 < n < \infty$. The case $n = \infty$ follows easily from this. Since G is Hausdorff and μ regular, we can easily shrink H^0_β to satisfy (1). □

Now the desired group $H_{\alpha+1}$ may be selected as the group H of the following proposition.

Proposition 3.32. *There is some $\beta < \mu$ such that for $H = H^0_\beta$:*

(1) $H \subset H_\alpha$;

(2) $\langle I, g_0 \rangle \cap H = \{0\}$;

(3) $I \cap (H + p^n G)\ G_{p,n}$ *for all p and* $0 < n \leqslant \infty$;

(4) $\beta_p(H_\alpha / H) = \beta_{p,0}(G)$ *for all p;*

(5) $\delta(H_\alpha / H) = \delta(G)$.

Proof. Easy consequence of Lemma 3.31 and assumption (3.5) of Theorem 3.20. Our final objective is to construct a suitable complementary summand A_α to $H = H_{\alpha+1}$ in H_α. This requires considerable care.

We will first construct a homomorphism $h : I \to H$ in such a way that we can subsequently make $\pi_{\alpha+1}$ coincide on I with h. The construction of h proceeds in two steps.

Notation. $I_\infty = \prod_p (I \cap \pi_\alpha[B_{p,\infty}])$. Thus by rectangularity

$$I_\infty = I \cap \prod_p \pi_\alpha[B_{p,\infty}] = I \cap \pi_\alpha[B_\infty].$$

Lemma 3.33. *There is a divisible subgroup $D' \subseteq H_\alpha$ and a homomorphism:*

$$h_\infty : I_\infty \to H$$

such that:

(1) $g_0 \in D'$;

(2) $D' \cap H = \{0\}$;

(3) $x - h_\infty(x) \in D'$ *for* $x \in I_\infty$.

Proof. Set $X = I \cap \pi_\alpha[X(p, \infty)]$ where $X(p, \infty)$ is our fixed basis for $R(p, \infty)$. For each $x \in X$ and each $n \in \omega$ there is because of $X \subseteq G_{p,\infty}$ an element $h(n, x) \in H$ such that $x - h(n, x) \in p^n G$. W.l.o.g. we may choose $h(n, x)$ to be p'-divisible. Then $\{h(n, x)\}_{n \in \omega}$ is a Z-Cauchy-sequence in H and since H is Z-complete $\lim_n h(n, x) = h_x \in H$ exists, where we may again assume w.l.o.g. that h_x is p'-divisible. Thus for all $x \in X$

$$x - h_x \text{ is divisible.}$$

W.l.o.g. let h_x be in some fixed reduced part H^r of H.

Since H is Z-complete, H^r is Z-complete and Z-Hausdorff, thus the map

$$x \to h_x$$

for $x \in \bigcup \{I \cap \pi_\alpha[X(p, \infty)]: p \text{ prime}\}$ extends to a homomorphism

$$h_\infty: I_\infty \to H.$$

Let D' be the divisible hull of $\langle \{a - h_\infty(a): a \in I_\infty\} \cup \{g_0\} \rangle$ in G. D' is a divisible group and by Proposition 3.32(2) $D' \cap H = \{0\}$. □

The next problem is to extend h_∞ to a homomorphism from I to H. More precisely:

Lemma 3.34. *There is a homomorphism*

$$h: I \to H$$

extending h_∞ *such that*

(1) *If* $x \in I \cap \pi_\alpha[B \oplus B_\infty] \cap G_{p,n}$, $0 < n < \infty$, *then* $x' \in p^n G$;

(2) $h(x) = 0$ *for* $x \in I \cap R_\alpha$.

Here we are introducing the abbreviation $x' = x - h(x)$.

Proof. As I is rectangular we decompose it as follows:

$$I = I_1 \oplus I_2 \oplus I_\infty,$$

$$I_1 = I \cap R_\alpha, \qquad I_2 = I \cap \pi_\alpha[B], \qquad I_\infty = I \cap \pi_\alpha[B_\infty] \quad \text{(as above)}.$$

We shall construct homomorphisms

$$h_1: I_1 \to H, \qquad h_2: I_2 \to H, \qquad h_\infty: I_\infty \to H$$

so that the desired homomorphism will be:

$$h = h_1 \oplus h_2 \oplus h_\infty: I \to H.$$

h_∞ has been constructed in Lemma 3.33 and we set $h_1 = 0$ in accordance with (2). By rectangularity and simple properties of the projection π_α we have:

(A) $$I_2 = I \cap \pi_\alpha[B] = \prod_p (I \cap \pi_\alpha[B_p]).$$

(B) $$I \cap \pi_\alpha[B_p] = I \cap \pi_\alpha\left[\left(\sum_{0<n<\infty} R(p,n)^0\right)^=\right]$$
$$= I \cap \left(\sum_{0<n<\infty} \pi_\alpha[R(p,n)^0]\right)^=$$
$$= \left(\sum_{0<n<\infty} \pi_\alpha[I_\alpha \cap R(p,n)^0]\right)^= .$$

(C) $$\pi_\alpha[I_\alpha \cap R(p,n)^0] = Z_p \cdot \pi_\alpha[I_\alpha \cap X(p,n)].$$

Since $X(p,n) \subseteq G_{p,n}$ we find for each $x \in I_\alpha \cap X(p,n)$ some element $h_x \in H$ such that

(D) $$x - h_x \in p^n G.$$

W.l.o.g. we may choose $h_x \in H^r$ and h_x p'-divisible. Thus it is possible to extend

$$x \to h_x$$

to a homomorphism

$$h_{p,n} : \pi_\alpha[I_\alpha \cap R(p,n)^0] \to H.$$

By Corollary 3.13 and Lemma 3.29(2) we may combine $h_{p,n}$ for $0<n<\infty$ to obtain a homorphism

$$h_p' : \pi_\alpha\left[\sum_{0<n<\infty} (I_\alpha \cap R(p,n)^0)\right] \to H.$$

By (D) and Lemma 3.13(2) we have:

(E) $$\text{for all } x \in \operatorname{dom}(h_p') \cap G_{p,n}: \quad x - h_p'(x) \in p^n G.$$

By Lemma 3.13 h_p' is continuous from the local Z-topology to the Z-topology. Since we have arranged $\operatorname{Im}(h_p') \subseteq H^r$ and H^r is Z-complete and Z-Hausdorff we can extend h_p' to h_p^*:

$$h_p^* : I \cap \pi_\alpha[B_p] \to H.$$

Finally again using Z-completeness of H we set:

$$h_2 = \prod_p h_p : I_2 \to H$$

It is easily checked that (E) implies (1). □

The construction of A_α depends on two lemmas.

Lemma 3.35. *If A, H are pure subgroups of G such that*
(1) $A \cap (H+pG) \subseteq pG$ *for all p and*
(2) $A \cap H$ *is Z-Hausdorff,*
then $A \cap H = \{0\}$ and $A \oplus H$ is pure in G.

Proof. For $a \in A \cap H$ it follows by repeated application of (1) that for all primes p: $a \in p^{\infty}G$. Hence by (2) $a=0$. Now if $a \in A$, $h \in H$ and $a+h \in pG$, then $a \in A \cap (H+pG) \subseteq pG$, so $a \in pA$. Hence $h \in pH$ and $a+h \in p(A \oplus H)$. □

Notation. In the notation of Lemma 3.34 set

$$I' = \{x' : x \in I \oplus \pi_\alpha[B \oplus B_\infty]\}.$$

Let A be the divisible hull in G of $D' + I'$.

Lemma 3.36. (1) $A \cap H = \{0\}$.
(2) *$A \oplus H$ is pure in G.*

Proof. We verify the two conditions of Lemma 3.35.

Condition 1: If $a \in A \cap (H+pG)$, then by definition there are $m>0$, $d \in D'$ and $x \in I \cap \pi_\alpha[B \oplus B_\infty]$ with

$$ma = x' + d.$$

W.l.o.g. $d=0$. Let $m = p^k m_0$ with $(p, m_0) = 1$. Then

$$x' \in m(H+pG) \subseteq H + p^{k+1}G.$$

Thus by Proposition 3.32(3):

$$x \in I \cap \pi_\alpha[B \oplus B_\infty] \cap (p^{k+1}G + H) \subseteq G_{p,k+1}.$$

Now Lemma 3.34(1) yields:

$$x' \in p^{k+1}G.$$

Thus $m_0 a \in pG$ and $a \in pG$ follows.

Condition 2. If $a \in A \cap H \cap \bigcap \{nG : n>0\}$ write again $ma = x' + d$ as above. This implies

$$x \in I \cap \pi_\alpha[B \oplus B_\infty] \cap (H + p^{\infty}G).$$

Thus by Proposition 3.32(3)

$$x \in G_{p,\infty}.$$

Now Lemma 3.33(3) implies $x' \in D'$ and therefore $ma = x' + d \in D' \cap H$, which gives by Lemma 3.33(2) $a=0$.

Applying Fact 2.9 and the previous two lemmas we find a group $A_\alpha = A_{\alpha,\alpha+1}$ satisfying:

$$H_\alpha = A_\alpha \oplus H \quad \text{and} \quad A \subseteq A_\alpha.$$

Notice that h and $\pi_{\alpha+1}$ coincide on I. □

Lemma 3.37. *For* $b \in I \cap \pi_\alpha[B \oplus B_\infty]$ *and all primes* p:

$$p^n \textit{ divides } \pi'_\alpha(b) \textit{ in } A_\alpha \textit{ iff } b \in G_{p,n}.$$

Proof. If $b \in I \cap \pi_\alpha[B \oplus B_\infty] \cap G_{p,n}$, then by 3.34(1)

$$\pi'_\alpha(b) \in p^n G \cap A_\alpha = p^n A_\alpha.$$

If conversely $\pi'_\alpha(b) \in p^n G$, then by Proposition 3.32(3) $b \in G_{p,n}$ as claimed. □

We claim now that the requirements of Theorem 3.26 as reformulated in 3.27 are satisfied:
3.26(1) follows from 3.32(4)(5).
3.26(2.1) follows from 3.32(2).
3.26(2.2) follows from 3.33(3) and 3.34(1).
3.26(2.3) follows from 3.37 and the construction of h_2 in 3.34.
3.26(3′) follows from $g_0 \in A_\alpha$.
3.26(4′) follows from 3.34(2), since this implies $\pi_{\alpha+1}(g^{i(\alpha,p)}) \in \pi_{\alpha+1}[B \oplus B_\infty] \subseteq G_{p,1}$.

Index of notation and structural relationships

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$X(p,n)$ = set of representatives in $R(p,n)$ for a basis of $V(p,n)$ ($0 \leqslant n < \infty$)	3.11(2)
$R(p,n)^0 = Z_p$-span of $X(p,n)$ ($0 \leqslant n < \infty$)	3.11(3)
$R(p,\infty)^0$ = basic submodule of $R(p,\infty)$	3.11(4)
$X(p,\infty) = Z_p$-basis for $R(p,\infty)^0$	3.11(5)
$B_p = (\sum \{R(p,n)^0: 0 < n < \infty\})^=$	3.11(6)
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$B_\infty = \prod_p R(p,\infty)$	3.11(8)
$(R(p,n)^0)^- = (R(p,n)^0)^=$ ($0 \leqslant n < \infty$)	3.15

$R_p' = (\bigoplus \{R(p,n)^0 : 0 \leq n < \infty\})^=$	3.16
$= \prod_p \{[R(p,n)^0]^- : 0 \leq n < \infty\}$	3.17
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$I \cap \pi_\alpha[B \oplus B_\infty] = \pi_\alpha[I_\alpha]$	p. 80

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