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# On the Operations of Pair Production, Transmutations, and Generalized Random Walk*

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## I

The aim of this paper is twofold. I shall discuss here some of the abstractions in mathematical schemata suggested by the problems of the physical world, and their autonomous development, which in turn leads to our formulation of physical theories; and I shall present some general considerations involving branching processes, including the interactions between pairs (and more general $k$-tuples of such), and problems involving random walks of elements which interact among themselves to produce more elements of this sort.

The dichotomy between two points of view, one considering the mathematics and logic itself as a primary basis in the Kantian sense, and the other complementary view considering our ideas as formed by the actions of the external world, is very old. The idea of the number system, the integers, the rational numbers, the continuum of real and complex numbers, developing in the more abstruse elements of more general algebras, may be not only stimulated but forced by the nature and the property of the physical worlds. The same could be asserted for the ideas of geometry as creations of our mind; or else these may be regarded as a result of the experience of our senses and experimentation.

Modern theoretical physics has become increasingly abstract in its basic formulations. Dirac and some other physicists express the view that the criterion of whether a theory may ultimately turn out to be true [sic!] depends on the beauty of the principles, and the mathematical formulation of the basic laws. So, for example, Heisenberg, in a speech given a few years ago, contended that the principles of symmetry in the mathematical patterns of physical theories, together with their simplicity, will guide us to

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new and more comprehensive physical insights. But Heisenberg himself, and certainly Einstein, were not in favor of too much complication or sophistication in the mathematical concepts to be used. Whether the degree of complexity is largely a question of time and habit is very hard to say.

I intend to indicate several areas where new mathematical questions might be stimulated by the present indications of the behavior of physical realities in the world of the very small, the subatomic regions, and in the large, with perhaps the intervention of *actual* infinities which could play a role in the foundations of physical theories.

Geometry, which in its origins could be considered a physical reality, developed the ideas of the Euclidean spaces to ever more intricate mathematical systems. If we consider for the moment only the topology of space, at present almost exclusively assumed to be locally Euclidean in most physical ideas, one may already ask for justifications. Einstein's program of geometrizing all fundamental physics is based on not only the locally Euclidean topological character, but on differentiable if not analytic metrics on it. Differentiability of course implies smoothness and linearity in the small. This is certainly adequate for phenomena involving gravitational forces alone. The world of atoms and the nuclei show an increasingly stronger and violent local behavior, the electromagnetic forces being more powerful than the gravitational ones by a factor of $10^{40}$, and still more in the nuclear and subnuclear dimensions. An attempt to build a geometry to describe the physical phenomena in such regions—philosophically and epistemologically one of the most wonderful ideas promulgated by Einstein—cannot proceed through classical mathematical analysis. At distances of, say, $10^{-8}$cm, not to mention $10^{-11}$ or $10^{-13}$, when the phenomena of nuclear interactions take place, some other possible metric geometry or perhaps even before that, the topology itself of space–time, may be different. As Hardy has said, mathematicians, who are, among other things, makers of patterns, have provided other ones which may suit the new physical insights. The topology may be non-Euclidean locally—it could be "granular"—which does not necessarily mean finite. There need not be a minimal length which is not consistent with the invariants under a Lorentz group of transformations, but there could be arbitrarily small distances, actual or "potential," an infinity of such, not necessarily leading to a Euclidean continuum. In fact, in Everett and Ulam [1], we have discussed the analogs of the Lorentz group defined for transformations of a space topologically equivalent to a Cantor discontinuum and algebraically defined as a product of a three-variable $p$-iadic number system ("space") and a one-dimensional $p$-iadic continuum ("time") where the transformations preserve the "light cones," that is, sets whose three-dimensional $p$-iadic norms are equal to the fourth-time-norm. This turns out to be a group different from the Lorentz group.

Of course one should not take literally this particular construction as physically meaningful, but one could imagine space–time structures of this sort corresponding to subatomic dimensions. Indeed, for distances of the order $10^{-6}$ or $10^{-7}$ cm, the world seems to be very well described locally by the Euclidean metric. If one wants to have changes in the much smaller dimensions, one should therefore do it so that the transition is not too abrupt. As Everett once said in a discussion of this matter, one would have to do it in such a way that the "seams would not show" when you sew together the geometry in the exceedingly small and the more macroscopic dimensions.

But what about the nature of the topology itself? Is the topology or the metric more primary or fundamental? Mathematicians in general would say that it is the former.

Assuming now, for the time being, a Euclidean topology, locally or even in the large, we might ask the question: What kind of metric consistent with this topology is the most "natural"?

I would like to mention here a problem concerning this question: In a special case, let us suppose that our space is Euclidean—one, two, three, or more dimensions. Let us consider all possible metrics which give this topology, but would allow the greatest possible freedom of transformations or motions preserving the given metric—the isometric transformations of the space onto itself. These form a group. For the Euclidean metric, these are translations, rotations, and inversions. Given some other metric, given still the Euclidean topology, consider the group of isometric transformations preserving it. The problem which I formulated some two years ago is: Do there exist metrics for which the group of their isometries would be maximal in the sense that for any other metric, the corresponding group, considered as an *abstract group* would be a proper subgroup of it, and in addition not isomorphic to the whole group? I do not know the answer to this question, in this generality, even for a small number of dimensions. E. Ihrig has informed me in a letter that he can show, under the assumption that the metrics are analytically "decent," that the Euclidean metric is maximal in this sense or in that respect "natural." Of course, the problem has sense on any topological space, and in particular one can pose it for the Cantor discontinuum. Perhaps for this space there is no "natural" metric in the above sense. There are curious metrics on it; for example, if we consider the example of Antoine, who imbedded the Cantor set in three-dimensional space in such a way that a closed curve linking it cannot be contracted to a point outside.

One might say that questions of this sort could have been considered a long time ago, even by physicists, viz., the discussion by Poincaré in one of his general books under the heading, "Why is Space Three-Dimensional?".

Similar questions should be asked about space–time. There one needs a more general setup since the idea of distance used is not metric in the sense mathematicians employ, since it need not be positive-definite. On the whole there is very little in the literature dealing with a possible generalization of the Lorentz space definition to more general spaces, not merely the spaces endowed with Riemannian metrics—in analogy to the generalizations of the Euclidean and classical spaces to more abstract, general metric ones. As an aside I should mention that there are of course other mathematical ways to "characterize" the Euclidean metric by general properties. For example, if a convex body has all its plane sections isometric to each other, it must be a sphere. If all its plane sections through a certain point are equivalent by affine transformations, it is an ellipsoid. This was proved in three dimensions [2]. One might ask analogous questions in Hilbert space, etc.

The relations between metrics and linearity is evident through a theorem ascertaining that all isometric transformations of a Banach vector space are linear [3]. This relation is one indication of the role of linearity in problems of mathematical physics. Many of the most important equations of mathematical physics are linear. (I remember a remark made by Fermi in a conversation: "I do not believe that it says in the Bible that all the laws of physics should be linear!") I might mention in this connection some problems about nonlinear transformations or functional equations, some of which are quite recent.

A number of studies were stimulated by a numerical work using early electronic computers by Fermi, Pasta, and myself [4] on the behavior of a vibrating string in which a small nonlinear term was added to Hooke's law (e.g., a quadratic one). The behavior of the vibrations presented a rather unexpected periodicity instead of a gradually increasing complication and "ergodicity." The high modes were not activated, but instead the first few modes played a sort of game of musical chairs among themselves. After some thousand or so would be linear variation periods the string would come quite close to its original starting single sinusoidal shape. The problem was considered on a discrete system of points by iteration in time of a nonlinear recurrence.

Much subsequent work was and is being done both in the differential or difference equation formulation. The phenomena of solitons are increasingly being studied with some results leading to possible applications in describing phenomena involved in nuclear and elementary particle physics.

Parallel to these studies an investigation is proceeding on a rather broad front of the mathematics of iteration of transformations which are nonlinear but still rather simple algebraically, for example, quadratic and broken linear ones acting on the Euclidean spaces or manifolds (the circumference of a circle, the sphere, etc.). In Stein and Ulam [5], we have undertaken a systematic investigation of iterates of certain quadratic (or cubic) transformations in a plane—three or more dimensions. A wealth of rather strange phenomena was discovered through numerical work. The iterates show in some cases convergence to an invariant system of points and in some cases are finite and periodic in the limit on these points; in some other cases, in three or more dimensions, the iterates of starting points converge to rather strange or "pathological"-looking sets which may be continuous curves or Cantor discontinua.

As an example of such quadratic transformations I might mention two to give an idea of the forms which have been investigated on a great number of such examples.

I. There are three types of particles. We assume that

type 2 with type 2 produce type 1,
type 3 and type 3 produce type 1,
type 1 and type 2 produce type 1,
type 1 and type 3 produce type 2,
type 2 and type 3 produce type 2,
type 1 and type 1 produce type 3.

If we start with a very large number of particles of this type, their *fractions* of the total being denoted by $x_1, x_2, x_3$, respectively $(x_1 + x_2 + x_3 = 1)$, then in the next generation the fractions will become

$$x_1' = x_2^2 + x_3^2 + 2x_1x_2,$$
$$x_2' = 2x_1x_3 + 2x_2x_3,$$
$$x_3' = x_1^2.$$

II. With an analogous notation for four variables ("colors"), the transformation could be, for example,

$$x_1' = x_2^2 + 2x_1x_3 + 2x_2x_3 + x_4^2,$$
$$x_2' = x_1^2 + 2x_2x_4,$$
$$x_3' = 2x_3x_4 + 2x_1x_2,$$
$$x_4' = x_3^2 + 2x_1x_4.$$

Often the behavior and morphology of the iterates limiting configurations depend very sensitively on the value of the numerical parameters in the transformation. Already in problems in one dimension the existence of a finite periodic limiting set changes abruptly to very chaotic and quasi-ergodic behavior. A number of papers on results on the "chaos" of such sorts has been written during the last few years.

One interpretation of quadratic transformations of this sort might be the following one: If we imagine a great number of particles, each of one of the several, say three, kinds of "colors," we may consider random pairings between such with the subsequent production of particles whose color depending on the colors of the parents is prescribed by a given rule. In this way we obtain a new generation of particles and the process continues by repetition or iteration. This latter refers to the transformation describing the *ratios* of the number of proportion of particles of each kind as the number of generations grows indefinitely. The numbers in each generation are given by the variables obtained by iteration from the initial ones given by the above type of recurrence relations.

## II

By an elementary branching process we mean a probabilistic schema of the following sort: Starting with one, or a number of elements, we assume probabilities, for each of these elements to produce, in one unit of time $0, 1, 2, \ldots,$ or $n$ new elements. We assume the probability for that to be given: $p_0, p_1, \ldots, p_n$. In the simplest case we assume that these are constant in time and the process repeats. So, starting with one particle we obtain a tree which graphically represents a history of such a process. As an interpretation we may take the problem of neutron multiplication—starting with one, we may obtain more of these which in turn produce still more, etc. The mathematics of such a discrete branching process is conveniently formulated by the use of generating functions:

$$f(x) = p_0 + p_1 x + p_2 x^2 + \cdots + p_n x^n + \cdots .$$

The generating function for the number of particles in the second generation will be

$$g(x) = f(f(x)).$$

In general, the generating function for the number of elements in the $k$th generation is given by $f^n(x) = f(f^{n-1}(x))$.

If the elements or particles are of different kinds, say two, "red" and "black," we have a more general generating *transformation* in two variables.

$$T: \begin{cases} x' = f_1(x, y) = \sum_{i,j=0}^{\infty} p_{i,j} x^i y^j, \\ y' = f_2(x, y) = \sum_{i,j=0}^{\infty} r_{i,j} x^i y^j. \end{cases}$$

The coefficients $p_{i,j}$ denote the probability that the "$x$" (black) particles will produce $i$ black ones and $j$ (red) ones. Correspondingly, the $r_{i,j}$ is the probability of the production of $i$ black and $j$ red ones by a "$y$" red particle. The iteration of this transformation gives the probabilities of numbers of red and black particles in the subsequent generation.

One of the important theorems concerns the *ratio* of the number of the red to the black particles. Assuming that the system is supercritical, that is, that if it does not die out with probability 1, the theorem asserts that with probability 1 the ratio will tend to the value given by the eigenvector of the linear transformation defined by the *expected values* of the production of "$x$" and "$y$" by black and red particles and the expected values from the production by red ones. This linear transformation in our example in two dimensions has the characteristic vector given by the Frobenius–Perron theorem. Everett and I have proved the above "strong ratio theorem" for this type of transformation in $n$ dimensions [6].

The theorem can be formulated in a concise way using the idea of measure in the space of all trees corresponding to such multicolored branching processes [7].

A branching process of the above type represents a "mitosis" process where one particle by itself may produce 0, 1, . . . particles of various types. A more general theory, with more and wider applications in several schemata of physical processes involves a production and transmutation of elements which combine in pairs to produce other particles. In a biological formulation it would correspond to sexual reproduction. A graphical representation of such a schema would not involve a tree but rather a number of trees whose segments combine in pairs (say at random) and each initiates 0, 1, 2, . . . new particles with various probability. These then combine again to produce the next generation, etc.

We shall illustrate now how such processes involving transmutations and "pair production" may serve as a stage for a number of fundamental physical processes. We shall try to indicate how the various "colors" may correspond not only to the various *kinds* of particles but to positions, momenta, or points in space–time of the particles, much as in the branching process discussed above, the colors could be interpreted as corresponding to location, velocities, and directions of the neutrons in an active assembly.

In the linear problems one can have of course, as a special case, such interpretations:

Consider the time-independent Schrödinger equation,

$$\Delta\psi + [E - V(x)]\psi = 0.$$

$V(x)$ is a given potential; $E$ is the eigenvalue. We may introduce a new function

$$U = e^{-Et}\psi(x, y),$$

where $t$ is a parameter, not necessarily interpretable as time. The new function will obey the equation

$$\frac{\partial U}{\partial t} = \Delta U - V \cdot U.$$

This corresponds to a process of diffusion where the Laplacian describes the random walk and $V(x)$ the potential, that is, a multiplication at a place $x$. Numerically, we may try to solve this equation by discrete steps in $t$ or by iteration. From the Frobenius–Perron theorem we know that asymptotically for large values of $t$ a "steady state" in the distribution of the function $\psi(x)$ will be established, that is, the ratios of the population in various positions will tend to fixed values, the population itself everywhere changing exponentially. In this fashion one may establish a numerical way to get an idea of the function $\psi(x)$, at least for the largest value $E$. As a matter of fact Fermi and I tried such computations for some special forms of $V(x)$.

How does one obtain the remaining eigenvalue and eigenfunction? For the moment suppose we want to consider the second one. One way to do it is to consider a population of "particles" of two colors, black and red. These will diffuse in $x$ in time $t$ through random walk and multiply in each step according to the values $V(x)$, but encountering each other they will annihilate each other. In this fashion we will approach for large $t$ a distribution corresponding to the second largest eigenvalue. (Actually this bookkeeping of black and red is used in financial accounts.)

Here we get the first example of what will be further elaborated for processes involving random walks by multicolored elements which produce other elements: If the production is by the particles themselves according to a prescribed $V$, we have a model of the linear Schrödinger equation. Suppose now that in addition to the process of annihilation which we described above, we have production by a pair of elements which "collide." This will then be an approximate formulation of a nonlinear equation of Schrödinger type but with a potential replaced now by the action of densities of the particles of different colors. For example, we could get a process which, in the limit, is describable by a nonlinear system of partial differential equations with a potential being replaced or complemented by the nonlinear, say quadratic or cubic, terms.

We may imagine further that the "particles" performing the random walk are elements of a more general algebraic nature. The colors may correspond to, not as in the first example, values 1, $-1$, which upon contact annihilate each other or produce 0, but they could be complex numbers, the colors corresponding to 1, $-1$, $i$, $-i$, and the rule of production could be quite general. In fact, one may imagine that the elements performing the random walk are matrices.

In such more general cases we have a process of "pair trees," an example of which was mentioned above.

A great deal of mathematical work is required to prove the analogs of what is known for the linear case of the branching processes. A conjectural theorem would assert the existence of ergodic limits, that is, the first means in time of the ratios of the density of population of various colors. More precisely one would like to know whether given a cone of directions issuing from the origin, the given quadratic transformation iterates fall into this cone in a sequence of times in such a way that the first mean of sojourn in this cone exists. Moreover, the above, which could be true for almost every (in the sense of measure) starting point in space, has a sojourn limit whose values, given the cone, may assume only a finite number of values depending on the initial point.

A number of numerical experiments made at the Los Alamos Scientific Laboratory seem to support this conjecture. Actually, in the cases examined, such pair production processes led to spatial configurations with the points of a given color congregating in definite regions.

As an example of such transformations I shall write the following:

$$\begin{aligned}
&x \,\&\, x \to x \text{ and } x\\
&y \,\&\, y \to x \text{ and } z\\
&z \,\&\, z \to y \text{ and } y\\
&x \,\&\, y \to x \text{ and } y \text{ and } z\\
&x \,\&\, z \to y \text{ and } z\\
&y \,\&\, z \to z
\end{aligned}$$

$$\begin{aligned}
&x \,\&\, x \to 0 \text{ (nothing)}\\
&y \,\&\, y \to y \text{ and } z\\
&z \,\&\, z \to x \text{ and } z\\
&x \,\&\, y \to x \text{ and } y \text{ and } z\\
&x \,\&\, z \to x \text{ and } z\\
&y \,\&\, z \to y \text{ and } y
\end{aligned}$$

The symbol & means the confluence of particles of the type $x, y$ or $x, z$, etc.; $x$ & $x$ means two particles of type $x$ combining.

If you assume a very large number of particles of each type paired by the symbol & at random and producing the next generation, we may consider the fractions of the total number of the particles of each type denoted by the symbols $\bar{x}, \bar{y}, \bar{z}$, etc. The transformation governing the change of these variables may be given by a general quadratic form,

$\bar{x}_i = \sum_j \alpha_{ij} \bar{x}_{i1}, \bar{x}_j$, plus linear terms corresponding to production by transmutation of single particles.

The example of the quadratic transformation, studied previously with P. Stein, describes the case where pairs of elements produce a single element or, say two elements of the same color, given a pair of parents. In the general case, which may serve as a model for a number of physical situations symbolically, the colors of the offspring from a pair of parents may differ. Needless to say, one really could study a more general case where more than two offspring are produced. As we shall intimate later, this would be the case to model multiple production of "photons, electrons, neutrinos" from a single collision between two particles of a given kind.

In addition to theorems which would throw light on the relative proportion of elements of each color one would like to have an algorithm permitting estimation of behavior in time (i.e., as a function of the index of iteration), the growth, decay, periodicity, or quasi-periodicity of this number.

As we shall see later, however, in a very general formulation of a pair production process in which we consider the positions in space and in physical time as "colors" themselves, the morphology of spatial and time configurations becomes describable by the frequencies of occupation of these in the sequence of iterations themselves.

As one of the very simplest possible examples, let us consider a problem of particles of two colors, red and black, distributed on a division of the circumference of a circle; let us say each is distributed uniformly on 100 points as a subdivision of a circle. The game was played with the following rules: Each of the particles, red or black, moves with equal probability clockwise or counterclockwise to the next point. The red particles produce, by themselves, on the average 1.10 particles of the same type; the black ones only 1.05. (This is obtained by assuming that each particle reproduces itself with probability 9/10 and produces two with probability 1/10, if it is red, itself with probability 0.95 and two like itself with probability 0.05 if it is black. This process of multiplication and random walking is repeated cycle by cycle. We assume after the initial start that the red and black particles occupying the same position annihilate each other. Under a number of initial distributions the repetition of the process may lead to a distribution of the red elements on one-half of the circumference, the black ones occupying the remaining positions. The problem may lead to a "critical" system if the number of positions on the circle and the probabilities of multiplication ("potential") are in a certain relation to each other.

The Schrödinger type of equation governing this would have the diffusion term depending on the number of positions given on the circumference of the circle. The "potential" is the probability of multiplication of each type by itself and the "critical" size of the population, that is, the case where neither population grows or diminishes under the iterations $x$ corresponds to the case where the eigenvalue $E = 1$. Since we have both positive and negative "particles," we really have a solution for the second eigenfunction and its eigenvalue. The "critical" system would correspond to the case where $E = 1$ when the number of positions, i.e., the length of the step and of the diffusion, is related in a certain way to the "multiplication" $V$.

Several problems were run on a computer where the multiplication depends on the presence of the particle of the opposite color. This corresponds to the system of nonlinear Schrödinger-type equations.

Our experiments performed on the computer in Los Alamos with R. Schrandt were for the case of the circumference of a circle. One may of course study such systems on the surface of a sphere, on the entire space, or on various manifolds.

A simple problem was studied computationally by imagining a number of particles whose colors may be interpreted as the energies of the particles. A great number, e.g., 100 or more, are assumed to start with, and the values of the colors are interpreted numerically from 1 to 100 with, say, a uniform initial distribution. We then assume that the particles are paired at random and each pair produces two new ones with the values of their energies adding up to the original sum of the energies of their parents. Otherwise the two energies ("colors") can be assumed, for example, to be distributed at random by, for instance, taking the first one at random uniformly from 1 to the value of the sum of the parents and taking for the second the complement to the sum which the parents have in their collision. Obtaining a new set of these energies we repeat the process and iterate it a great number of times.

As expected, of course, the new distribution of energy tends to an exponential (Boltzmann type of distribution).

The recipe for defining the energy of the two colliding elements could be different: One can take a number $\alpha$ from 0 to 1 uniformly and give the energies to the progeny by putting 1 to $\sin^2\alpha$ of the total energy of the parents and $\cos^2\alpha$ to the second offspring. When this recipe is applied and the process is iterated again, a limiting distribution seems to have been approached, but this time of the shape of the Maxwell type.

A third recipe could be to take two numbers $\alpha$, $\beta$ uniformly on $(0, 1)$ and then form two numbers $\alpha' = \alpha/(\alpha + \beta)$ and $\beta' = \beta/(\alpha + \beta)$. Again there seems to be a convergence to a Maxwell-type distribution after many iterations, but shifted with respect to the one in our second exercise.

More generally, if the recipe involves a distribution of the energies after collision given by a function $f(\alpha)$ ($\alpha$ is the fraction of the total energy), the fraction of the total energy of the parents to be given to the first particle is

$f(\alpha)$ and $1 - f(\alpha)$ to the second, then the iteration of the process may presumably, at least for certain $f(x)$, yield a convergence to a function $g(x)$ for the distribution of the energies. In a certain sense $g(x)$ would be a sort of "collision transform" of $f(x)$.

A more interesting exercise would involve a schematized treatment by our pairing process to imitate the conservation of both the energy and the momenta, all treated as discrete colors. We inted to pursue such model calculations in the future.

Conceptually such extremely simple and schematic models as the above are first steps toward a more fundamental and physically interesting one. In the problems of statistical mechanics the interaction between particles of various types is compounded by an iteration process if one assumes sufficiently short times between steps. The two famous papers by Fermi [8], one dealing with the theory of radiation (i.e., emission and absorption of photons interacting with atoms, and of course both with each other), the second on the theory of $\beta$ decay and interactions between nuclei and electrons and neutrinos, are monumental examples of a study of specific, complicated physical systems requiring a very great number of variables if one wanted to use the terminology of colors, also for positions and momenta.

If one wanted to take physical interpretation of the mathematics of transmuting and pair production systems somewhat more seriously, one could imagine that the entities like the $\psi$ functions correspond to assemblies of an enormous number of virtual or fictitious particles transmuting according to certain algebraic rules somewhat like above. Even considering the substrata for the $\psi$'s, that is, the space or the space–time as being "manufactured" by such processes. For example, a branching process with transmutations, i.e., a system of many-colored elements, can serve to define a discrete metric in the following way:

Suppose, in a simple case, we have two variables, $x$ and $y$, which we interpret as steps on a grid in the plane in two directions; each step is taken with equal probability $= \frac{1}{2}$, in either direction and in a plus or in a minus unit of distance. Starting with a great number of particles at the origin, we may consider the probability of occupation of a point with integer-valued coordinates, $(x, y)$. We can define a distance from the origin to a point as a function of the inverse of the probability of getting there or else a function of the inverse of the density of the occupation at this point. Analogously, of course, for a distance between any two points, $(x_1, y_1)$ and $(x_2, y_2)$. This kind of distance on a discrete graph or, as we may call it—a tree—of the transmutations of the colors corresponding to the two variables and the absolute value of the number of steps may serve, in the limit of the numbers tending to infinity, as an approximation to, say, the Euclidean metric in a space of the two variables, $(x, y)$. Similarly, of course, in three or more dimensions. In this way, one may consider the metric to be of combinatorial or probabilistic origin. The non-Euclidean character would come through a "bias" in the manufacture of the "space symbols."

One could even try to define nonpositive definite metrics by using, instead of $+1$ and $-1$ as steps, imaginary units.

In order to obtain, heuristically, a feeling for the behavior of such systems, a number of numerical problems were run with Myron Stein and Robert Schrandt on the Los Alamos Laboratory computer. (The work with Paul Stein referred to earlier was now paralleled by using broken linear transformations instead of quadratic ones.) The following two could serve as examples:

$$\begin{aligned} x' &= (2x + 3y + z) \quad \text{mod } 1 \\ y' &= (x + y + 2z) \quad \text{mod } 1 \\ z' &= 1 - x' - y' \end{aligned}$$

with the new variables normalized to have sum = 1

$$\begin{aligned} x' &= 2.5x - y - z - 5w \\ y' &= 3x + y + 2z + 2w \\ z' &= 4x + 3y + z + w \\ w' &= 2x + 3y + 2z + 2w \end{aligned}$$

again normalized to have the new variables have the sum = 1

Again the behavior of the iterates resembles in its morphology, that exhibited by the quadratic and cubic transformations. Perhaps combinatorially such transformations, the graphs of which are broken lines or polyhedra, are simpler to discuss theoretically.

A simple class of production of elements by pairs of elements is the group operation. One could imagine that our "particles" are elements of a group. Combining pairs of such particles, one may, for example, define as their products two elements: If $x, y$ are two elements of a group which were associated, their offspring are the two elements $x \cdot y$ and $y \cdot x$ which in a noncommutative group will be, in general, different. Assume now that we make a pairing of all the elements of a group at random and by the rule above produce a new set of elements of the group. (This may in general lead to a smaller set.)

One such problem involved starting with the group of the 120 elements of all the permutations of five letters. Many computations were run showing that in the majority of cases the number of elements after the iterations oscillated somewhat, but remained in the neighborhood of about half of the total. (It was never quite the normal subgroup of even permutations!) In order to obtain statistics of the behavior under repeated iterations one had to run a number of cases since the pairing of elements in each stage was made at random.

A problem corresponding to the above but starting with the semigroup of all transformations of a set on itself rather than the one-to-one permutations led, after a number of iterations as expected, to a radical diminution of the number of transformations remaining.

Similar problems were considered for some few finite groups of matrices under composition.

Another pastime was to start with a number of elements, each an integer from 1 to $p - 1$ where $p$ is a prime. The rule of composition was to assign to two integers $a$, $b$ two new ones, for example, $a'$, $b'$ where $a' = a \cdot b \bmod p$ and $b' = a + b \bmod p$. This led, of course, after a number of iterations, to a much smaller number of elements which transform into themselves periodically under iteration.

Some other calculations run on computers involve rules of generation of progeny leading to sometimes three or sometimes just one or zero new elements. The results of our exploratory numerical work will be published in new Los Alamos Reports by R. Schrandt and myself, and by Myron Stein and myself.

Perhaps one may consider mathematical exercises of this type merely as a preparation for some rules of composition which have more tangible physical interpretation. Purely schematically some of the concepts of modern physics, e.g., the consideration of elementary particles as being built themselves from combinations of others—quarks or partons—are of this sort. It might be, of course, that the elements whose transmutations and combinations form new systems are themselves representable as collections resulting from combinations of a perhaps very great number of still other ones of a previous stage. This process could even go back through an infinity of such stages.

In a sense processes of construction and growth appearing in biological situations have features common to the ones we have described. In a series of papers written with Schrandt [9, 10; see also 11], we have studied a construction of "cells" growing on a rectangular or cubic grid in space produced by single or pairs of adjacent cells with rules involving changes of kind ("color") and some expressing the properties of "contact inhibition." In the terminology of this paper the occupation of new positions in the next generation is also a change of "color". The organization of cellular automata involves inter alia also such processes.

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