New Journal of Physics
The open access journal at the forefront of physics

Deutsche Physikalische Gesellschaft DPG IOP Institute of Physics

Toric codes and quantum doubles from two-body Hamiltonians

To cite this article: Courtney G Brell *et al* 2011 *New J. Phys.* **13** 053039

View the article online for updates and enhancements.

Related content

- Topological color codes and two-body quantum lattice Hamiltonians
  M Kargarian, H Bombin and M A Martin-Delgado
- Perturbative 2-body parent Hamiltonians for projected entangled pair states
  Courtney G Brell, Stephen D Bartlett and Andrew C Doherty
- Qudit surface codes and gauge theory with finite cyclic groups
  Stephen S Bullock and Gavin K Brennen

Recent citations

- Anyons are not energy eigenspaces of quantum double Hamiltonians
  Anna Kómár and Olivier Landon-Cardinal
- Limits on the storage of quantum information in a volume of space
  Steven T. Flammia *et al*
- Non-commuting two-local Hamiltonians for quantum error suppression
  Zhang Jiang and Eleanor G. Rieffel

This content was downloaded from IP address 117.211.180.14 on 17/01/2018 at 05:56

New Journal of Physics
The open-access journal for physics

Toric codes and quantum doubles from two-body Hamiltonians

Courtney G Brell[1,5], Steven T Flammia[2,3], Stephen D Bartlett[1] and Andrew C Doherty[1,4]

[1] Centre for Engineered Quantum Systems, School of Physics, The University of Sydney, Sydney, Australia
[2] Perimeter Institute for Theoretical Physics, Waterloo, Canada
[3] California Institute of Technology, Institute for Quantum Information, Pasadena, CA, USA
[4] School of Mathematics and Physics, The University of Queensland, St Lucia, Australia
E-mail: cbrell@physics.usyd.edu.au

New Journal of Physics **13** (2011) 053039 (47pp)
Received 18 December 2010
Published 20 May 2011
Online at http://www.njp.org/
doi:10.1088/1367-2630/13/5/053039

Abstract. We present here a procedure to obtain the Hamiltonians of the toric code and Kitaev quantum double models as the low-energy limits of entirely two-body Hamiltonians. Our construction makes use of a new type of perturbation gadget based on error-detecting subsystem codes. The procedure is motivated by a projected entangled pair states (PEPS) description of the target models, and reproduces the target models' behavior using only couplings that are natural in terms of the original Hamiltonians. This allows our construction to capture the symmetries of the target models.

[5] Author to whom any correspondence should be addressed.

© IOP Publishing Ltd and Deutsche Physikalische Gesellschaft

Contents

1. Introduction **2**
2. Results and methods **3**
2.1. The quantum double models . . . 3
2.2. Methods . . . 5
2.3. Discussion . . . 8
3. Example: the toric code **10**
3.1. Projected entangled pair states (PEPS) representation of the toric code . . . 10
3.2. Solving the code gadget Hamiltonians . . . 11
3.3. Perturbation analysis . . . 13
4. Review of quantum double models **15**
4.1. Simplifications in the case of cyclic groups . . . 17
5. Our construction for the cyclic quantum double models **18**
5.1. Code gadgets on lattice edges . . . 18
5.2. Coupling the code gadgets . . . 20
6. Our construction for general quantum double models **23**
6.1. Code gadget operators . . . 23
6.2. Code gadget Hamiltonian . . . 25
6.3. Coupling the code gadgets . . . 26
Acknowledgments **33**
Appendix A. Perturbation theory **33**
Appendix B. Extension to arbitrary graph **34**
Appendix C. Proof of theorem 1 **35**
Appendix D. Proof of theorem 3 **36**
Appendix E. Error operations in general quantum double models **41**
References **45**

1. Introduction

There has been a surge of interest recently in spin lattice models exhibiting topological order in their ground states, because of their strange and interesting properties. From the perspective of condensed matter physics and quantum many-body theory, these models have recently led to major advances in the understanding of the nature of quantum phase transitions and topological order in two-dimensional (2D) systems. The topological properties of these models are also of interest in quantum computing and quantum error correction. A system with topological order can possess intrinsic error correction or protection capabilities. These are exploited for quantum data storage [1–3] or quantum information processing [4–6] with high error thresholds. The encoded logical operations in topological models are associated with nontrivial homology cycles on a lattice of spins. A lattice that has a nontrivial topology (such as a torus or punctured disc) can encode quantum information into its ground states that is robust to small local perturbations of the Hamiltonian.

Many models that are relatively simple (from a theoretical point of view) contain topologically ordered ground states. The toric code and its generalization to the quantum

double models [7] constitute a significant class of exactly solvable models containing a range of different topological orders. Quantum double models of non-Abelian groups can support non-Abelian anyons, quasiparticles whose exchange statistics transcend the traditional Bose–Einstein/Fermi–Dirac dichotomy that is ubiquitous in three dimensions. Braiding such quasiparticles can be used in universal quantum computation [8–11].

However, these models (and other more general topological models [12]) consist of many-body interactions that are quite challenging to implement experimentally, as they usually involve interactions between four or more bodies. By contrast, most natural couplings are only two-body. It is therefore of interest to find systems with only two-body interactions that can realize topologically ordered phases.

An example of a two-body system with a topologically ordered ground state is the Kitaev honeycomb model [13]. This well-studied model is being pursued experimentally in a number of systems, but unfortunately it cannot be used for universal quantum computation.

Aside from explicitly finding two-body models that reproduce a particular desired type of topological order (certainly challenging), one can use perturbative techniques to reproduce an existing many-body model as the low-energy effective behavior of a two-body system. The perturbative gadgets approach [14–17] is the standard tool for achieving this, but it has a number of drawbacks. By tailoring the perturbation gadgets to specific classes of models, one might hope to obtain a simpler construction circumventing many of these difficulties.

Here we present a new type of perturbation gadget that works by encoding the logical qudits of the target models in quantum error-detecting codes. This allows us to reproduce the properties of topological models as the low-energy effective Hamiltonians of two-body systems. Here we concentrate specifically on the quantum double models, but we anticipate that a similar mechanism could be tailored to other classes of models (e.g. string net models [12]). Our construction is natural, in the sense that all the interactions of our system are very closely related to the interactions of the target model, and because of this, an extensive number of symmetries of the target model are preserved exactly from the level of the physical lattice.

Unlike Kitaev's honeycomb model, our constructions are not exactly solvable. However, our results are a significant extension of Kitaev's method in that they can yield any type of topological order (i.e. different types of anyons) within the class spanned by the quantum double models, including those that are universal for quantum computation.

2. Results and methods

In this section, we give an overview of our results and the methods used to obtain them. To avoid obscuring the essence of our work with unnecessary technical details, we will use the toric code model as a concrete example in many places. However, we stress that our results immediately carry over to all of the cyclic ($\mathbb{Z}_d$) quantum double models, and extend to general non-Abelian quantum double models with just minor adjustments.

2.1. The quantum double models

The quantum double models [7] are a class of spin-lattice models that exhibit topological order. They can be used as topological quantum error-correcting codes based on the algebra of the Drinfeld double $\mathcal{D}(G)$ of a group G. The simplest member, corresponding to the group $\mathbb{Z}_2$, is the well-studied toric code model.

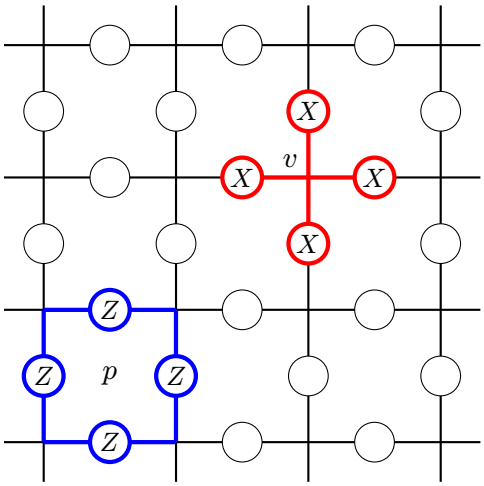


Figure 1. The toric code defined on a square lattice with qubits on the edges. Each colored region represents one of the two types of terms in the Hamiltonian. The star terms (in red) act around a vertex v with a Pauli X on each qubit and the plaquette terms (in blue) act on the qubits around the boundary of p with a Pauli Z operator.

For simplicity, we define the quantum double models on a square lattice (although they can be defined on any oriented graph) with qudits (d-level quantum systems) on the edges, as in figure 1. The lattice can be embedded into any 2D orientable surface, such as a torus. The Hamiltonian for the model takes the form

$$H_{\mathrm{QD}} = -\sum_{v} A(v) - \sum_{p} B(p), \tag{1}$$

where v denotes a vertex of the lattice and p denotes a plaquette. In the simple case of the toric code, the vertex and plaquette operators are defined by

$$A(v) \equiv \bigotimes_{e\in+(v)} X_e \quad \text{and} \quad B(p) \equiv \bigotimes_{e\in\square(p)} Z_e, \tag{2}$$

where the form of the star of a vertex $+(v)$ and the boundary of a plaquette $\square(p)$ can be seen in figure 1, and X_e and Z_e are Pauli matrices acting on the qubit located on the edge e. Since we are working on a square lattice, each of these terms are clearly four-body. For more complicated quantum double models, the $A(v)$ and $B(p)$ operators take slightly different forms, but will always consist of operators acting on the star of a vertex or the boundary of a plaquette. The exact details of the quantum double construction can be found in section 4.

When the surface in which the lattice is embedded has genus g, the ground space of the toric code is 4^g-fold degenerate and thus can encode $2g$ qubits. This particular encoding has generated much interest because it is in some ways naturally robust to local errors [1, 7, 18, 19]. The dimension of the codespace for a non-Abelian quantum double model is more complicated, but on a torus the degeneracy of the ground space is equal to the total number of particle types [20, 21].

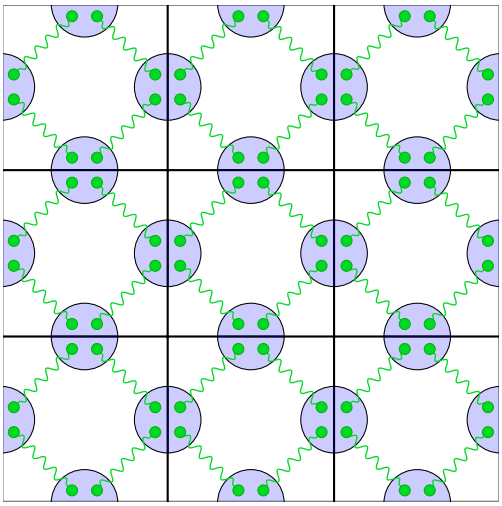


Figure 2. PEPS description on a square lattice. Each qubit (for the toric code, or qudit in general) on an edge in the original model is replaced by four qubits (or qudits in general). Qubits connected by a wavy line are in a maximally entangled state. Each blue circle represents a projection down to a single encoded qubit. The quantum states in the support of these projectors are encoded qubits, entangled with each other. The global state space contains exactly the states in the ground space of the original toric code model. With toroidal boundary conditions, the ground space is fourfold degenerate. Our construction proceeds by simulating these local projections and the entangling interactions with two-body Hamiltonians.

2.2. Methods

2.2.1. Overview of our construction. Our main result is the construction of a wholly two-body Hamiltonian that reproduces the quantum double Hamiltonian of (1) as its low-energy limit. Our procedure uses only qudits of the same dimension as in the target model. Furthermore, the couplings that we use have the same form as in the original model. As an example, note how each $A(v)$ and $B(p)$ term from (2) consists of the tensor product of four Pauli X terms or four Pauli Z terms, respectively. Our construction for the toric code will involve products of only two Pauli X terms or Z terms.

Each qudit on the edges of the original model is encoded in four physical qudits, shown in figure 2. The two-body interactions among these four qudits will give us an effective single-qudit degree of freedom in the low-energy limit. This gives a fourfold increase in the number of qudits required to construct our model as compared with the target model. We will couple neighboring encoded qudits perturbatively, and the perturbation expansion will yield the desired Hamiltonian at fourth order. This order is related to the coordination number of the underlying lattice and the number of edges bordering each plaquette, both of which are four for a square lattice. In contrast, on a honeycomb lattice terms will arise at third and sixth order in perturbation theory (although there would still only be a fourfold increase in the number of physical qudits required).

We achieve these results with a blend of techniques from condensed matter physics and quantum information theory. Before sketching how we use the techniques, let us briefly describe each technique.

2.2.2. Projected entangled pair states (PEPS). Our construction is inspired by PEPS, a class of quantum states particularly well suited for describing the ground states of interacting quantum many-body systems [22–27]. Indeed, for the case of 1D systems this is provably the case [28]. The basic idea of these states is to use virtual pairs of entangled systems to simulate correlations. For every coupling between neighboring systems on a lattice, a maximally entangled state (of some chosen dimension) is introduced between the systems. These virtual entangled pairs are then projected down to a 'physical' subspace with a dimension equal to that of the spins in the original model. An illustrative construction is depicted in figure 2 for the special case of a square lattice.

One would expect the kinds of models we are studying to have efficient PEPS representations of their ground spaces because of the facts that they obey an area law, possess a spectral gap above the ground state and contain finite correlations. In fact, the PEPS representation of the toric code ground space has been studied by Verstraete *et al* [29], and a PEPS representation of general quantum double models is also known [30]. PEPS descriptions have also been developed for all of the string-net ground spaces [31, 32], and the symmetries of these PEPS descriptions have been explored [30]. The quantum double models on trivalent lattices can be mapped to string net models [33], and generalizations of the quantum double models are also being interpreted as extended string net models [34]. In our construction we have an implicit PEPS representation for the quantum double models that is presumably equivalent to those already known (this is provably the case for our construction of the toric code).

2.2.3. Perturbation gadgets. The next technique that we use is that of *perturbation gadgets* [14–17]. Perturbation gadgets are a method for systematically reducing the complexity of a many-body coupling between a large number of quantum systems. The gadgets generally consist of introducing some ancilla qudits that act as *conductors*, in the sense of the conductor of an orchestra. By having a strong coupling to the conductor, n separate primary qudits can synchronize their behavior in a way that mimics an n-body coupling at low energies, but by using couplings having only a fraction of the 'body-ness'. By recursively applying these general constructions, one can arrive at a strictly two-body Hamiltonian for an arbitrary n-body coupling. The cost is that this coupling only occurs at a higher order in perturbation theory.

Our construction is a new variant on the perturbative gadgets approach. The concept is very similar, but instead of beginning from the original lattice and adding ancilla qubits to break up many-body interactions, we begin by encoding the qudits of the original lattice into a four-qudit system. These systems are then coupled via relatively weak two-body interactions, which enables us to treat the entire model perturbatively and show that it reproduces the target model in the low-energy limit. We will refer to these four-qudit encoded systems as 'code gadgets'. Apart from the intrinsic interest of a new approach, we also manage to bypass a number of pitfalls that a naive application of perturbation gadgets can encounter. In particular, the resource cost of perturbative gadget schemes scales poorly with the system complexity, and a naive application of the technique can lead to the energy gap scaling with the system size or the fidelity of the topologically ordered states [17, 35]. While this can be avoided, it remains a problem when applying the method in general. Additionally, while the couplings can be reduced to only two-body, the nature of these two-body couplings is in general vastly different from that of the couplings of the original model. It is possible that by taking advantage of structure a much simpler construction could be devised that is specifically tailored to the model, using couplings that exploit this structure. This is the approach taken by Koenig [36], who showed that a simple 'clock' gadget could reduce the complexity of the quantum double models to only three-body

terms. Our construction also follows a similar strategy in some sense, but uses perturbative couplings that are direct analogues of the original models' terms on simple surfaces.

The types of gadgets that we use are adapted for use with states that are ground states of local Hamiltonians and have a simple PEPS description [37]. The virtual entangled pairs of the PEPS description are promoted into real physical systems with a coupling such that their ground state is a maximally entangled state. Then the PEPS projection can be done by using strong interactions between the systems within a code gadget that energetically favors the subspace defined by the PEPS projection. When this coupling within a site is much stronger than the entangling coupling between sites, the resulting Hamiltonian at low energy approximates the desired many-body Hamiltonian, with the same ground space up to perturbative corrections. This technique was originally used [37, 38] to find two-body Hamiltonians whose ground state encodes the cluster state [39], a state that is universal for measurement-based quantum computing. Our technique is very similar, in that the target models are reproduced in an encoded manner, i.e. the four-qudit code gadgets of our construction serve as the logical spins of the target model.

2.2.4. Subsystem quantum error-detecting codes. Finally, we make use of quantum error-detecting codes [40–42] to ensure that all the undesirable terms in the perturbative expansion do not couple to the low-energy sector of our model. Recall that a quantum error-detecting code consists of a subspace of a larger Hilbert space, which is protected from a set of errors on the large Hilbert space. This protected subspace, the codespace, is used to store encoded logical information. The mapping between the codespace and the physical Hilbert space defines the encoded logical operators. Detectable errors move the system out of the codespace and so can be detected by a suitable measurement.

Specifically, we use a particular type of quantum code known as a subsystem code [43, 44]. Compared to stabilizer codes, subsystem codes are no more powerful in terms of the number of errors they can detect; rather, their power lies in a simplification of the recovery operations required for fixing the errors. The physical Hilbert space is partitioned into distinct subsystems: the logical subsystem and a gauge subsystem. As in a stabilizer code [42], stabilizer operators are defined such that the logical codespace is in the mutual +1 eigenspace of these operators. If an error moves the system out of this +1 eigenspace, it can be detected by the measurement of the stabilizer and must actively be corrected. In contrast, the logical state is taken to be invariant under a transformation on the gauge system and so any errors that occur on this gauge space are passively avoided.

The code we use in our gadget construction is designed so that any operations contributing to unwanted terms in the perturbative expansion will be detected as errors, and map the ground space to another energy eigenspace with higher energy. In this way, the code ensures that the low-energy behavior of our system will remain error-free, in the sense that only desirable terms (i.e. those of the target model) will remain. Errors that move the system out of the codespace need not explicitly be corrected, because they will be suppressed heavily by an energy penalty for doing so. Note that the 'errors' that occur in the gadget code are unrelated to errors appearing in the target model we are trying to replicate. Instead, they mix the protected low-energy sector with higher-energy (unprotected) sectors, thus preventing our system from mimicking the target model perfectly.

Our models are similar in many respects to the topological subsystem codes of Bombin [45]. These models, based on the color codes [46, 47] and their qudit generalizations [48], yield a subsystem code using local two-body gauge generators. We note that the

construction of [45] requires a three-colorable lattice, whereas our method applies to any lattice for which the desired model possesses a PEPS description. Additionally, Bombin's models possess an exactly degenerate ground space, while our models' ground spaces are only approximately degenerate (as in the Kitaev honeycomb model [13]), but where the degeneracy is only broken at very high order (roughly the linear size of the system). One consequence of this exact degeneracy in the Bombin model is that it is straightforward to define quantum error correction in the ground space [49], where perfect recovery operations will yield perfect recovery from correctable errors. By contrast, in our model even perfect recovery operations will yield some small error due to the splitting of the ground space. While the small error in recovery seems unavoidable in our model, it might still be the case that our model yields higher thresholds due to the substantially simpler stabilizer measurements that are required. It is an open problem to define error correction and exactly quantify the errors incurred by the splitting for our model. Aside from both of our constructions being viewed as a generalization of the honeycomb model, we are not aware of any deeper connection between them.

2.3. Discussion

Our results allow us to replicate in the low-energy limit the Hamiltonians of certain topological models, but a more ambitious goal is to reproduce the topological order in the ground state wavefunctions of these models, as well as in their low-lying excited states. To demonstrate that our constructions reproduce *all* the topological properties of the original models, we would need to show additional properties. First, we would need to show that these topological orders are stable under the kinds of perturbative corrections that our procedure introduces. There are two separate types of corrections that could threaten the stability of the topological properties of our models. The first are the very high-order corrections where perturbation terms can form nontrivial homology cycles on the surface of the model. These will occur at order $2L$ in perturbation theory (with L being the smallest linear dimension of the surface) and split the ground space degeneracy of the encoded model. They can be heavily suppressed by increasing L or by increasing the bare energy gap of the system. The second kinds of corrections to consider are those which leave the protected ground space of our code gadgets. These allow transitions into higher energy subspaces where our encodings fail. These kinds of corrections are suppressed energetically by the energy gap. Thus, we would also need to demonstrate the stability of the energy gap in the thermodynamic limit. It would also be quite interesting to compute the topological entanglement entropy in the ground state [50–53]. Like the toric code model in two dimensions [1, 54, 55], the topological order of Kitaev's non-Abelian quantum double model is not expected to persist at finite temperature. We expect that our models will have similar behavior to the original quantum double model at finite temperature, and in particular that they will suffer from the same 'thermal fragility' of the topological order.

Even more ambitiously, one might hope to show that our models remain gapped even in the presence of arbitrary local perturbations, with only very small splitting of the ground state degeneracy and the degeneracy of the excited states. For topological models whose Hamiltonians consist of the sum of commuting projectors, just such a result was shown by Bravyi *et al* [18, 19]. Unfortunately, their techniques cannot be directly applied to our models since our Hamiltonians are not sums of commuting projectors. In fact, our models do not have frustration-free ground states either (meaning that the ground states are not minimum-energy eigenstates of each term separately in the Hamiltonian) and hence other results on frustration-free systems also do not apply.

As one might expect from a perturbative construction, our effective Hamiltonian can be thought of as the target Hamiltonian plus perturbative corrections. Generically in this type of construction, the symmetries of the target model will be recovered approximately, up to these corrections. In our model, the encoded string operators of the target models will not commute with the perturbative terms in the Hamiltonian, and the corresponding ground space degeneracy will also be split. This splitting will be exponentially suppressed by the size of the lattice (as noted above), and so these symmetries will be recovered approximately as one might expect. In the quantum double models, there are also an extensive number of vertex and plaquette operators that commute with the Hamiltonian and form a quantum double algebra. It is possible to construct encoded counterparts of these operators on our model, that also commute with our full Hamiltonian and form an equivalent algebra. In contrast to a generic perturbative construction, this large symmetry group is reproduced exactly by our model. This symmetry group severely constrains the arbitrarily high-order terms arising from the perturbation expansion, and prevents undesirable terms from appearing. In fact, higher order terms in our effective Hamiltonian will act on the logical codespace as products of (commuting) lower-order terms until the perturbative order is sufficiently high to form nontrivial loops over the lattice and break the ground space degeneracy.

As well as capturing symmetries of the target models, our construction is also natural in the sense that it is built from miniature quantum double models overlaid on each other. In the case of Abelian quantum double models (including the toric code), there is an exact correspondence between our constructions and quantum double models on simple surfaces. For these models, the ground space of our code gadget is chosen to coincide with a subspace of the quantum double ground space on a four-qudit torus (see section 6.2). That is, our codespace is stabilized by the same operators as the ground space of the quantum double Hamiltonian on this torus. These stabilizers would normally be four-body, so in order for our codespace to obtain this property from a two-body Hamiltonian we must sacrifice the degeneracy of one of the two qudits a torus can typically encode. In addition, the perturbative bond terms we introduce can be interpreted as quantum double models on a two-qudit sphere (see section 6.3). All the terms present in our model reflect the construction of the quantum double models on small surfaces. In non-Abelian models, the correspondence in the Hamiltonian is not as precise due to the distinction between left and right regular representations. However, it will still be true that the ground spaces of our code gadgets will correspond to a subspace of the quantum double ground space on a four-qudit torus, and the ground space of the perturbative bond terms will correspond to the ground space of the relevant quantum double model on a two-qudit sphere.

Our work is inspired by the construction of Bartlett and Rudolph [37], who used encoded qubits to reproduce the cluster state as the ground state of a two-body Hamiltonian. As with the model we present, the encoding is closely related to the PEPS description of the target state. Our work generalizes this type of construction by using a subsystem code (as opposed to a subspace code). In order to reproduce the PEPS space as the ground space of a two-body Hamiltonian, we have had to sacrifice the extra gauge degrees of freedom in our model. This would not have been possible if we had used a subspace code, where these gauge degrees of freedom are not available.

The procedure we present reproduces the quantum double Hamiltonian in the perturbative coupling limit. If we consider the opposite (strong coupling) limit in our model, the system will act as a disconnected set of maximally entangled pairs. In this limit, the lattice can be thought of as a set of disconnected quantum double models on spheres (up to the caveats noted above). There must be some phase transition(s) between these states, and the respective topologies in the two limits are suggestive of the kind of behavior studied by Gils *et al* [56]. The phase transition

would be between different topologies in the sense that the quantum double model would act on the entire lattice together in one limit, and at some critical coupling strength break down to act on disconnected portions of it.

A robust topologically ordered system would also have anyonic low-energy excitations which, ideally, would be similar to those of the desired quantum double model. One would need to consider the effect of the perturbative corrections in our model on these anyons and any other excitations, as for example in [57–59].

It would also be interesting to explicitly and rigorously show that cooling our Hamiltonians by coupling to a local bath can bring them to the ground space quickly. (Cooling to a particular ground *state* of the degenerate ground space is more difficult, but also quite interesting.) Because of the frustration in the model, it is not immediately obvious that this can be achieved efficiently, i.e. in an amount of time polynomial in the size of the system. But because the low-energy effective theory is frustration-free, it certainly seems plausible that the cooling can be done efficiently.

It is noteworthy that the codes we utilize in our construction for general quantum double models are examples of extensions of the stabilizer formalism to non-Abelian groups. This has not been closely studied previously, and in that sense the quantum codes we use may be of interest in their own right.

We believe that the kind of approach we employ here to reproduce topological orders may be more generally applicable to other systems with efficient PEPS representations. Some work toward extending this treatment to the class of string net models [12] supports this belief, with some caveats, and will be presented in a future publication.

3. Example: the toric code

As a simple and illustrative example of our scheme, we now demonstrate how to construct a two-body Hamiltonian for which the low-energy behavior reproduces the standard toric code Hamiltonian on a square lattice. This simple example possesses all of the key features of our general construction for the quantum double models.

3.1. Projected entangled pair states (PEPS) representation of the toric code

We begin our construction by replacing each qubit on the edges of the toric code lattice with four qubits, as in figure 2. We use the term *bond* to refer to the wavy lines in figure 2 that connect the maximally entangled pairs of physical qubits. By contrast, we use the word *edge* to denote the edges of the original lattice. In the PEPS description, the projection operators acting on each edge will entangle these four qubits into a single qubit. We achieve this projection in the ground space of a Hamiltonian defined on each edge e.

$$H(e) = -\left(\begin{array}{c|c} X & I \\ X & I \end{array}\right)_e - \left(\begin{array}{c|c} I & X \\ I & X \end{array}\right)_e - \left(\begin{array}{c|c} Z & Z \\ I & I \end{array}\right)_e - \left(\begin{array}{c|c} I & I \\ Z & Z \end{array}\right)_e . \tag{3}$$

This notation is a convenient visual shorthand for the tensor product of the operators acting on the given physical qubits. For each edge, the collection of four-qubit forms our ‘code gadget’. The Hamiltonian contains only two-body terms acting within the gadget itself.

Instead of the explicit projection mechanism of the PEPS scheme to reduce the Hilbert space, our model simply suppresses by energy penalty states that lie outside the desired PEPS projection. It can be shown that the projectors to the ground space of our edge Hamiltonian are equivalent to the projectors in [29] since they are equal modulo the gauge freedom in choosing the PEPS description; however, explicitly demonstrating this equivalence is tedious (although straightforward) and so we omit this.

Next, we introduce entanglement across the bonds by coupling sites on different edges. Thus for each bond b, define the perturbation term

$$V(b) = -\, X \overset{b}{\sim\!\sim} X \;-\; Z \overset{b}{\sim\!\sim} Z\,, \tag{4}$$

chosen because it possesses a maximally entangled state as its ground state[6]. Our unperturbed Hamiltonian is summed over all edges e:

$$H_0 = \sum_e H(e), \tag{5}$$

and our perturbation term is summed over all bonds b:

$$V = \sum_b V(b). \tag{6}$$

For those readers familiar with PEPS, it may seem counterintuitive to treat the bond term as small compared to the code gadget Hamiltonian (which is simulating the PEPS projection). We will see that this is in fact the correct approach to recover the target model. We introduce a coupling strength λ, which is a small parameter compared to the strength of the main terms in our Hamiltonian (which we have taken to have unit norm). The full Hamiltonian describing our lattice is then given by

$$H = H_0 + \lambda V. \tag{7}$$

Now we need to compute the perturbative low-energy effective Hamiltonian to leading nontrivial order in λ. We will find the exact ground space of $H(e)$ in the next section and then show that the perturbations λV will generate operators that reproduce an encoded toric code Hamiltonian (1) and (2) at fourth order in λ.

3.2. Solving the code gadget Hamiltonians

We must first demonstrate that $H(e)$ of an edge e has a 2D degenerate ground space. We will show that this ground space is in fact the codespace of a subsystem quantum error-detecting code, which will greatly assist the perturbative analysis in the next section. Our analysis of this Hamiltonian for the toric code construction follows Bacon [61]. Because we are always working on a particular arbitrary edge (within a particular code gadget), we will suppress the label e in this section.

[6] An alternative choice would be $V(b) = -\, Y \overset{b}{\sim\!\sim} Y$. This choice also approximately realizes the toric code Hamiltonian in the small λ limit. The resulting Hamiltonian (7) is precisely equivalent to Kitaev's exactly solvable honeycomb model on a mosaic tiling [60]. We do not consider this possibility further since it is unclear how to generalize it to more complicated quantum double models. It is interesting to note, however, that this alternative choice results in a model with topological order in the large λ limit, in contrast to our models that become valence bond solids.

The code gadget Hamiltonian H (3) possesses a number of constants of motion. That is, we can define operators that commute with each other, and with the Hamiltonian. In fact, some of these operators are the *stabilizers* of a quantum code, and so we label them S. They form a commutative group and are generated by

$$S_X \equiv \begin{array}{|c|c|}\hline X & X \\ \hline X & X \\ \hline \end{array} \quad \text{and} \quad S_Z \equiv \begin{array}{|c|c|}\hline Z & Z \\ \hline Z & Z \\ \hline \end{array}. \tag{8}$$

We can also define other joint operators to complete the algebra of our code gadget. We will call these operators *gauge* operators and *logical* operators, and denote them with appropriate subscripts.

$$X_{\mathsf{G}} \equiv \begin{array}{|c|c|}\hline I & X \\ \hline I & X \\ \hline \end{array}, \quad Z_{\mathsf{G}} \equiv \begin{array}{|c|c|}\hline Z & Z \\ \hline I & I \\ \hline \end{array} \quad \text{and} \quad X_{\mathsf{L}} \equiv \begin{array}{|c|c|}\hline X & X \\ \hline I & I \\ \hline \end{array}, \quad Z_{\mathsf{L}} \equiv \begin{array}{|c|c|}\hline I & Z \\ \hline I & Z \\ \hline \end{array}. \tag{9}$$

We see immediately that these operators encode two orthogonal copies of the Pauli algebra, and so they define two encoded qubits. In terms of these new operators, we can rewrite the gadget Hamiltonian

$$H(e) = -X_{\mathsf{G}}(1 + S_X) - Z_{\mathsf{G}}(1 + S_Z). \tag{10}$$

The protected subspace of our code, also corresponding to the ground space of the Hamiltonian, is a subspace of the +1 eigenspace of the stabilizers S_X and S_Z. Any single-qubit error will anticommute with at least one of these and so could be detected (although not unambiguously) by measurement of the stabilizers. This means that any single-qubit error will necessarily move the system out of its ground space, and will be suppressed by the energy penalty for doing so.

We can easily check that the logical operators X_{L} and Z_{L} also commute with H. However, neither X_{G} nor Z_{G} commutes with H. Given all these facts, H decomposes into a direct sum of four copies of $\mathsf{L} \otimes \mathsf{G}$, each labeled by the pairs of eigenvalues $(\pm 1, \pm 1)$ of S_X and S_Z, and furthermore the energies in the logical space are degenerate. Thus, we only have to solve the Hamiltonian on the gauge subspace of each stabilizer eigenvalue to find the ground space.

It turns out that the ground space is contained in the $(+1, +1)$ block, and it is exactly twofold degenerate. In appendix D we provide a proof of this statement and its generalization to all quantum double models. It can be shown that within this codespace the encoded computational basis states take the (unnormalized) form:

$$|0_{\mathsf{L}}\rangle = \left(1 + \sqrt{2}\right) \left[\begin{array}{|c|c|}\hline |0\rangle & |0\rangle \\ \hline |0\rangle & |0\rangle \\ \hline \end{array} + \begin{array}{|c|c|}\hline |1\rangle & |1\rangle \\ \hline |1\rangle & |1\rangle \\ \hline \end{array} \right] + \begin{array}{|c|c|}\hline |0\rangle & |1\rangle \\ \hline |0\rangle & |1\rangle \\ \hline \end{array} + \begin{array}{|c|c|}\hline |1\rangle & |0\rangle \\ \hline |1\rangle & |0\rangle \\ \hline \end{array} \tag{11}$$

$$|1_{\mathsf{L}}\rangle = \left(1 + \sqrt{2}\right) \left[\begin{array}{|c|c|}\hline |1\rangle & |1\rangle \\ \hline |0\rangle & |0\rangle \\ \hline \end{array} + \begin{array}{|c|c|}\hline |0\rangle & |0\rangle \\ \hline |1\rangle & |1\rangle \\ \hline \end{array} \right] + \begin{array}{|c|c|}\hline |1\rangle & |0\rangle \\ \hline |0\rangle & |1\rangle \\ \hline \end{array} + \begin{array}{|c|c|}\hline |0\rangle & |1\rangle \\ \hline |1\rangle & |0\rangle \\ \hline \end{array}. \tag{12}$$

Now that we have determined that the ground space encodes a qubit in an error-detecting code, we can perform perturbative analysis to compute the low-energy effective Hamiltonian.

3.3. Perturbation analysis

We now introduce the perturbative coupling of (4) between our encoded qubits on the lattice. We use the Green's function perturbation method, following Kitaev [13] (see also [62]), to calculate the leading nontrivial order in the effective Hamiltonian, defined as $H_{\text{eff}} = E_0 + \Sigma(E_0)$, with the unperturbed ground state energy of the lattice E_0, and Σ is the self-energy. In this, we have approximated the self-energy as being independent of E for $E \approx E_0$. More details of our perturbation formalism are given in appendix A.

The key to our analysis is the fact that quantum codes map detectable errors to orthogonal states. In terms of the gadget Hamiltonians $H(e)$, this means that any single-qubit Pauli operator anti-commutes with either (or both) S_X or S_Z, and hence maps ground states to orthogonal states, since these states must lie in some block of $H(e)$ other than the $(+1, +1)$ block.

The consequence is that the perturbation analysis greatly simplifies. It immediately gives the result that all odd-order perturbation terms will vanish, as they will necessarily leave two code gadgets in excited states. The terms at second order only contribute an energy shift, since those that do not vanish act twice on the same qubits; hence they act proportionally to the identity. The first nontrivial terms appear at fourth order, and we can write the fourth-order effective Hamiltonian as follows:

$$H_{\text{eff}}^{(4)} = \lambda^4 \Upsilon V (G_0(E_0) V)^3 \Upsilon, \tag{13}$$

where Υ is the projector to the ground space of the unperturbed system and $G_0(E) = (E - H_0)^{-1}(1 - \Upsilon)$ is the Green's function (resolvent) projected to vanish on ground states. The Υ will project the stabilizer and gauge degrees of freedom down to a single state, but will act identically on the logical degree of freedom.

The nontrivial fourth-order terms arise by constructing joint operators around a plaquette or around a vertex which leave all the code gadgets in the ground space. By expanding (13) and ignoring constant energy shifts, we can express the effective Hamiltonian to fourth order as

$$H_{\text{eff}} \propto -\lambda^4 \sum_v \Upsilon \hat{A}(v) \Upsilon - \lambda^4 \sum_p \Upsilon \hat{B}(p) \Upsilon, \tag{14}$$

where v and p sum over all vertices and plaquettes, respectively. The operators $\hat{A}(v)$ and $\hat{B}(p)$ are each a sum of two terms,

$$\hat{A}(v) = \hat{A}(X, v) + \hat{A}(Z, v) \quad \text{and} \quad \hat{B}(p) = \hat{B}(X, p) + \hat{B}(Z, p), \tag{15}$$

where these terms are quite cumbersome to express algebraically, so we define them pictorially in figure 3. Basically, these operators act on those pairs of qubits in an edge that are closest to the center of a given plaquette or vertex.

We can derive from figure 3 that $\hat{A}(X, v)$ acts as a tensor product of logical X_{L} operators on each of the edges surrounding v. This is not immediately obvious; we must use the fact that in the ground space $X_{\text{L}} = S_X X_{\text{L}}$ to exchange the action of the logical operators between pairs of qubits at a particular edge. Similarly, $\hat{B}(Z, p)$ acts as a tensor product of logical Z_{L} operators around the plaquette p. The other operators $\hat{A}(Z, v)$ and $\hat{B}(X, p)$ act as gauge operators in a similar fashion. When these encoded gauge operators are mapped back to the ground space by Υ, they contribute only a constant energy shift, which we can ignore.

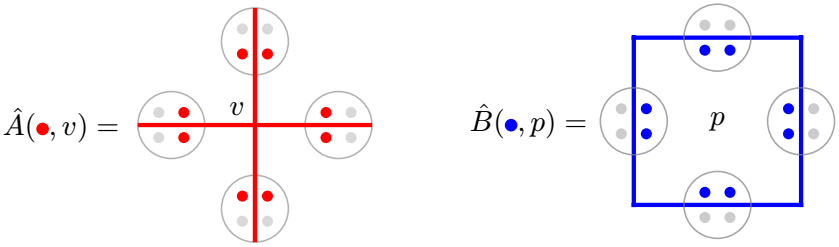


Figure 3. Encoded operators. For each operator, the nontrivial operator acts on the colored qubits. For example, $\hat{A}(X, v)$ is a tensor product of X operators on each of the colored qubits surrounding the vertex v.

We can think of these $\hat{A}$ and $\hat{B}$ operators as acting equivalently to some logical operators within the ground space. If we define

$$\hat{A}_{\mathsf{L}}(v) \equiv \bigotimes_{e \in +(v)} X^e_{\mathsf{L}} \quad \text{and} \quad \hat{B}_{\mathsf{L}}(p) \equiv \bigotimes_{e \in \square(p)} Z^e_{\mathsf{L}}, \tag{16}$$

then we can see that within the codespace, they are equivalent (up to muliplicative and additive constants) to $\hat{A}$ and $\hat{B}$, respectively. That is,

$$\Upsilon \hat{A}_{\mathsf{L}}(v) \Upsilon \propto \Upsilon \hat{A}(v) \Upsilon - \text{const} \quad \text{and} \quad \Upsilon \hat{B}_{\mathsf{L}}(p) \Upsilon \propto \Upsilon \hat{B}(p) \Upsilon - \text{const}. \tag{17}$$

We can think of the logical Hamiltonian acting within the codespace as being comprised of these operators, such that

$$H_{\mathsf{L}} = -\lambda^4 \sum_v \hat{A}_{\mathsf{L}}(v) - \lambda^4 \sum_p \hat{B}_{\mathsf{L}}(p). \tag{18}$$

When restricted to the codespace, this is exactly the effective Hamiltonian we previously derived (again, up to multiplicative constants and energy shifts), so that

$$H_{\text{eff}} \propto \Upsilon H_{\mathsf{L}} \Upsilon + \text{const}. \tag{19}$$

Noting that the logical operators $\hat{A}_{\mathsf{L}}$ and $\hat{B}_{\mathsf{L}}$ act on the logical state exactly like the toric code vertex and plaquette terms, we can see that on the logical space, our effective Hamiltonian is the toric code Hamiltonian of (2) up to constants, as claimed.

The higher-order terms in the expansion for the self-energy (see appendix A) will generally act on the logical space-like products of the terms appearing in (18). Moreover, for low energies all these terms will be negative, since the perturbation term and the Green's function will both be non-positive and so each term in the expansion will be negative. All these terms in the self-energy expansion will commute, and so the ground space should remain the +1 eigenspace of the terms in (18), as desired. There will be some corrections to the excited spectrum of the effective Hamiltonian due to these higher-order corrections and due to the energy dependence of the self-energy, as discussed in appendix A. But as we are mainly interested in the topologically ordered ground space, this does not concern us especially. At very high order it will be possible to construct terms that run all the way around the torus. These errors will corrupt the logical state. If the linear size of the torus is N, these terms will appear at $2N$th order, and will be suppressed by a factor of λ^{2N}.

This result allows us to take a system of only two-body couplings and in the low-energy limit reproduce the Hamiltonian of the toric code. This means we might expect to be able to

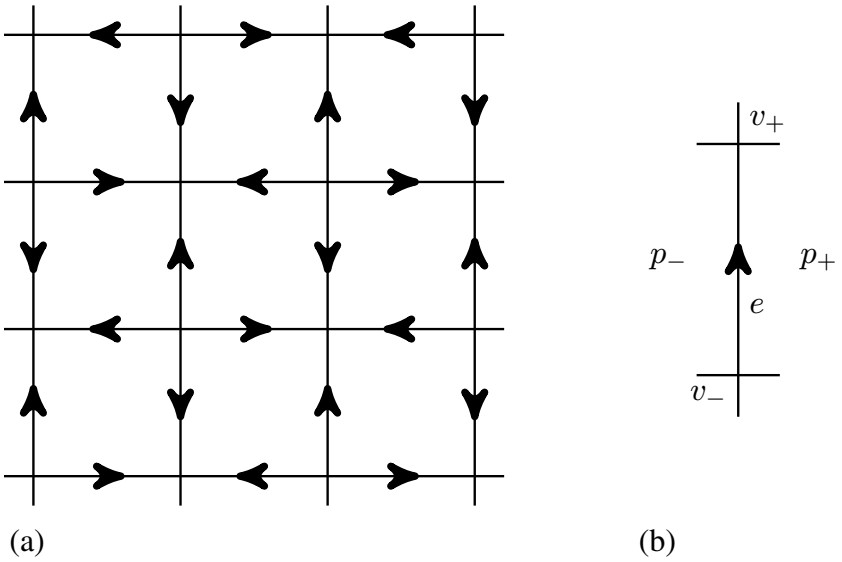


Figure 4. (a) A directed square lattice, and the orientation of + and − vertices and plaquettes relative to the edge direction. Each vertex consists either of all 'inward' edges or all 'outward' edges. Plaquettes consist of alternately directed edges as you traverse their boundary. (b) If an edge spans the pair of vertices (v_+, v_-), then the edge is oriented toward v_+. The plaquettes are labeled with signs, where p_+ is on the right of the edge, following the given orientation.

use the topological properties of the toric code to protect quantum information without the requirement for experimentally problematic many-body couplings. A similar result of obtaining the toric code in a limit was observed in Kitaev's honeycomb model [13]. In contrast to our construction, this honeycomb model is exactly solvable. Although we can only solve our model perturbatively, we can generalize it relatively easily to more complicated quantum double models (and lattices other than square), as will be seen in the following sections.

4. Review of quantum double models

The quantum double models consist of coupled finite-dimensional quantum systems on the edges of a lattice, and their ground states exhibit topological order [7]. In this section, we will define the quantum double Hamiltonian that will become the target model of the perturbative two-body systems that we will work with in the subsequent sections.

As with the toric code, we will work with a square lattice for concreteness. The lattice can be embedded into any orientable two-dimensional surface. With each edge e of the lattice we will associate an orientation, as in figure 4. Although this orientation could be arbitrary, we chose the orientation in figure 4 because it has the convenient feature that each vertex can be labeled with either a '+' or a '−' sign. In fact, any bipartite lattice can be partitioned in such a way.

We associate each quantum double model with a finite group G, and a local Hilbert space for each edge $\mathbb{C}^{|G|}$, with $|G|$ being the order of the group. On each edge e of the lattice, there exists a natural orthonormal basis $\{|g\rangle_e,\ g \in G\}$ for these degrees of freedom. The total Hilbert space is then the tensor product of the local Hilbert spaces over all the edges.

We now define a number of operators that act on these edge degrees of freedom. For each edge e, define operators associated with left (+) and right (−) group multiplication and group projectors as follows:

$$L_+^g(e) \equiv \sum_h |gh\rangle\langle h|_e, \qquad L_-^g(e) \equiv \sum_h |hg^{-1}\rangle\langle h|_e, \tag{20}$$

$$T_+^g(e) \equiv |g\rangle\langle g|_e, \qquad T_-^g(e) \equiv |g^{-1}\rangle\langle g^{-1}|_e. \tag{21}$$

The operators on a single edge form an algebra defined by commutation relations

$$L_\pm^g T_\pm^h = T_\pm^{gh} L_\pm^g, \qquad L_\pm^g T_\mp^h = T_\mp^{hg^{-1}} L_\pm^g. \tag{22}$$

Clearly, operators acting on different edges commute. We also associate a sign $\pm$ with each vertex and plaquette relative to their incident edge. This is illustrated in figure 4. If an edge spans the vertices (v_-, v_+), then the arrow along the edge points away from v_- and toward v_+. Plaquettes to the left of an edge when looking along its direction are labelled p_-, while those to the right are labeled p_+.

It is convenient to associate a particular operator with a (plaquette, edge) pair or a (vertex, edge) pair. That is, depending on the sign of the vertex or plaquette under consideration at the time ($v_\pm$ or $p_\pm$) with respect to the edge under consideration, the sign of the group multiplication and projection operators can be inferred. Explicitly, we define

$$L^g(e, v_\pm) \equiv L_\pm^g(e), \qquad T^g(e, p_\pm) \equiv T_\pm^g(e). \tag{23}$$

The L operators play the role of a generalized Pauli X operator in the group element basis, insofar as they move a particular state $|g\rangle$ through the space of elements. To make the analogy to the toric code more apparent, we can also construct some operators that act as generalized Pauli Zs for an arbitrary group algebra. Let π be a unitary irreducible representation of G. Then we can define a type of Fourier transform by

$$Z_\pm^{\pi_{ij}} \equiv \sum_g [\pi(g)]_{ij} T_\pm^g, \tag{24}$$

where $[\pi(g)]_{ij}$ is the (i, j)th element of the representation matrix for group element g in representation π. Equivalently, we can invert this expression to obtain

$$T_\pm^g = \frac{1}{|G|} \sum_\pi d_\pi \sum_{ij} [\pi(g)]_{ij} Z_\pm^{\pi_{ij}}, \tag{25}$$

where the sum is over the complete set of unitarily inequivalent irreps of G and d_π is the dimension of the irrep π. Although the quantum doubles are typically defined in terms of T operators, the algebra of these generalized Z operators gives the most convenient form for a particular calculation later on.

Given these preliminary operators, for each vertex v we can define operators

$$A^g(v) \equiv \bigotimes_{e \in +(v)} L^g(e, v), \tag{26}$$

where $+(v)$ is the set of edges incident on the vertex v (recall figure 1). We can average these operators over the group to give the projector

$$A(v) \equiv \frac{1}{|G|} \sum_g A^g(v). \tag{27}$$

Given a plaquette p and a fiducial edge on its boundary that we label e_1, we can define an operator

$$B^g(p) \equiv \sum_{g_k \cdots g_1 = g} \bigotimes_{e_i \in \square(p)} T^{g_i}(e_i, p), \tag{28}$$

where e_i are the boundary edges taken as the plaquette is traversed clockwise starting with e_1, and there are k total edges on the boundary of p. These operators are all orthogonal projectors, but note that this definition depends on the choice of the fiducial edge e_1. However, if we consider the operator

$$B(p) \equiv B^1(p), \tag{29}$$

where 1 is the identity element of the group G, then it is easy to see that B no longer depends on the choice of this fiducial edge.

For all p and all v, the $B(p)$ operators and the $A(v)$ commute pairwise amongst themselves and with each other. Finally, the following Hamiltonian defines the quantum double model,

$$H = -\sum_v A(v) - \sum_p B(p), \tag{30}$$

in a manner that directly generalizes the toric code.

4.1. Simplifications in the case of cyclic groups

It will be instructive to treat the cyclic groups $\mathbb{Z}_d$ before moving on to the general case. For that reason, we revisit the above discussion specialized to this setting. Because each of the $|G|$ representations of the (Abelian) groups $\mathbb{Z}_d$ is 1D, we can relate the L and Z operators by a simple discrete Fourier transform for these groups. In cyclic groups (with $d = |G|$), the group multiplication operation is addition (modulo d), and so the left and right multiplication operations are equivalent. Following this, the convention for the identity element in cyclic groups is 0 as opposed to 1 for general groups. With no need for unique left and right multiplication operators, we define the cyclic L operator by

$$L \equiv \sum_{h=0}^{d-1} |h+1\rangle\langle h|. \tag{31}$$

The addition within the ket is performed modulo d. Note that the group action of any other element can be achieved in these models by taking powers of this (unitary) operator. We also define a set of group projection operators

$$T^g \equiv |g\rangle\langle g|. \tag{32}$$

Following the general case, we also define a generalized Pauli Z operator. As representations of the cyclic groups are 1D, we can define a primitive Z corresponding to a representation ω:

$$Z \equiv \sum_{h=0}^{d-1} \omega^h T^h, \tag{33}$$

where ω is a primitive dth root. Other representations of the group correspond to powers of ω, so to obtain the operators corresponding to these representations we need only take powers of this Z operator. Thus we regard the powers of Z as being labeled by representations of $\mathbb{Z}_d$.

If we take a discrete Fourier transform on the basis $|g\rangle$, we obtain

$$|\gamma\rangle = \frac{1}{\sqrt{d}} \sum_j \omega^{\gamma j} |j\rangle, \tag{34}$$

where we label the Fourier basis with Greek letters (corresponding to irreps). This transformation diagonalizes the L operators:

$$L = \sum_\gamma \omega^{-\gamma} |\gamma\rangle\langle\gamma|, \tag{35}$$

and so we can see that the Z and L operators are simply a basis change away from each other in these cyclic models, as was the case with the Pauli matrices for the toric code. As they are both unitary, we can also say that

$$L^\dagger = L^{-1} = L^{d-1}, \tag{36}$$

$$Z^\dagger = Z^{-1} = Z^{d-1}. \tag{37}$$

In terms of the definition of the quantum double model for cyclic groups, the only changes we need to make to become consistent with this simplified set of operators is to slightly redefine the associations of L and T operators with $\pm$ vertices and plaquettes, i.e.

$$L^g(e, v_\pm) \equiv L^{\pm g}(e), \qquad T^g(e, p_\pm) \equiv T^{\pm g}(e). \tag{38}$$

With this in mind, the quantum double Hamiltonian is defined exactly as in the general case.

5. Our construction for the cyclic quantum double models

In this section, we will show how our construction on the toric code generalizes naturally to the quantum doubles of cyclic groups. The toric code model corresponds to the quantum double of the group $\mathbb{Z}_2$; here we extend this treatment to the quantum double of $G = \mathbb{Z}_d$, where $|G| = d$ is the order of the group. This analysis could be extended to general Abelian groups. However, for simplicity, and because the fully general case is considered in the next section, we restrict our attention to cyclic groups in this section.

In order to reproduce the cyclic quantum double models, two features must be added to the simple toric code construction. To begin with, qubits at each site must be replaced by d-dimensional qudits, with appropriate generalized Pauli operators defined on them, as introduced in section 4. The group multiplication operator L plays the role of the X operator in the toric code, and the newly generalized Z operator plays the role of the Pauli Z. These operators obey the commutation relation

$$Z^a L^b = \omega^{ab} L^b Z^a. \tag{39}$$

The other feature we will add to our construction at this juncture is the notion of directed edges, as discussed in section 4. As in the toric code ($\mathbb{Z}_2$) case, we will now proceed with our construction explicitly on the square lattice.

5.1. Code gadgets on lattice edges

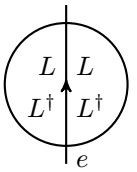
We use a very similar construction to the toric code to encode our qudits for cyclic quantum double models. Each logical qudit is encoded using a subsystem code constructed from four

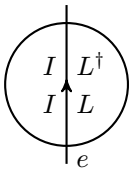
physical qudits (figure 2 shows the scheme). The d^4-dimensional space of these edges is partitioned as

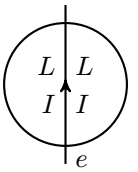
$$\mathcal{H}(e) = \bigoplus_S \mathcal{H}_{\mathsf{L}} \otimes \mathcal{H}_{\mathsf{G}}, \tag{40}$$

where the direct sum is over eigenvalues of stabilizers S. Our codespace is in the +1 eigenspace of two stabilizer operators S_L and S_Z defined below. The remaining d-dimensional degrees of freedom are encoded as qudits, one of which (the gauge qudit $\mathcal{H}_{\mathsf{G}}$) we fix in a single state in the codespace. The other qudit is used as our logical space $\mathcal{H}_{\mathsf{L}}$. If an error occurs, it will flip a stabilizer operator, move the code gadget out of the codespace and incur an energy penalty.

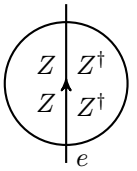
Physically, we provide the codespace with these properties as the ground space of a two-body Hamiltonian. Before we write it, we will first introduce the gauge, logical and stabilizer joint operators as we did for the toric code.

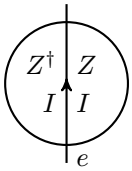
$S_L \equiv$ 

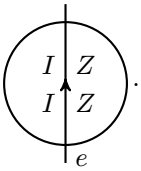
, $L_{\mathsf{G}} \equiv$ 

, $L_{\mathsf{L}} \equiv$ 

,

$S_Z \equiv$ 

, $Z_{\mathsf{G}} \equiv$ 

, $Z_{\mathsf{L}} \equiv$ 

. (41)

To avoid confusion as much as possible, we will distinguish typographically G for gauge and G for group and similarly L for logical and L for group multiplication. It is simple to verify that the operators defining each separate degree of freedom commute with each other (i.e. stabilizers commute with gauge and logical operators, and gauge operators commute with logical operators). It can also be seen that the logical operators satisfy the desired algebra of the cyclic quantum double models:

$$Z^a_{\mathsf{L}} L^b_{\mathsf{L}} = \omega^{ab} L^b_{\mathsf{L}} Z^a_{\mathsf{L}} \tag{42}$$

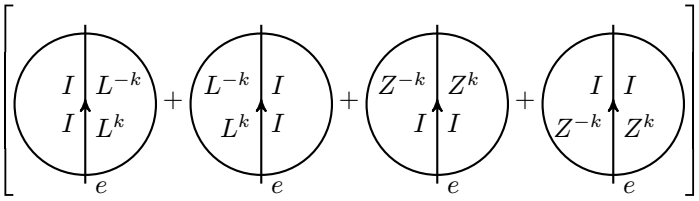
and the gauge operators satisfy an equivalent algebra:

$$Z^a_{\mathsf{G}} L^{-b}_{\mathsf{G}} = \omega^{ab} L^{-b}_{\mathsf{G}} Z^a_{\mathsf{G}}. \tag{43}$$

We define the Hamiltonian on a single code gadget (associated with edge e) as

$$H(e) = -\frac{1}{d} \sum_k \left[L^k_{\mathsf{G}} + (L^k_{\mathsf{G}} S^k_L)^\dagger + Z^k_{\mathsf{G}} + (Z^k_{\mathsf{G}} S^k_Z)^\dagger \right]_e . \tag{44}$$

This equation can be represented diagrammatically as

$H(e) = -\frac{1}{d} \sum_k$ 

. (45)

In this form, it is easy to see that each term in the Hamiltonian acts only on two qudits.

Multiplication by the stabilizers (as in $L_{\mathsf{G}}S_L$) effectively moves a gauge operator from two qudits onto the opposite two. In the $d = 2$ case, this Hamiltonian does not directly reduce to the one quoted for the toric code earlier (equation (10)) because of the inclusion of the identity ($k = 0$) term. However, this term only induces a constant energy shift and so can be disregarded for our purposes.

We now turn to the properties of the ground space of this Hamiltonian. It is clear that this Hamiltonian commutes with the logical operators, but it is less obvious whether its ground space possesses the other properties we require. In appendix C, we prove the following theorem.

Theorem 1. *The Hamiltonian (44) has a d-fold degenerate ground space that is in the common* +1 *eigenspace of both* S_L *and* S_Z.

This result, combined with the fact that our logical operators commute with the Hamiltonian, gives us a ground space to use as an encoded logical codespace $\mathcal{H}_{\mathsf{L}}$.

5.2. Coupling the code gadgets

The lattice is connected exactly as is the case for the toric code (figure 2), with qudits from neighboring edges linked via an entangling bond. We have an unperturbed Hamiltonian for each edge qudit given as in the previous section,

$$H_0 = \sum_e H(e), \tag{46}$$

where the index e denotes a particular edge qudit. We then introduce the bond term

$$V = \sum_b V(b) \tag{47}$$

$$= -\sum_b \sum_{k=0}^{d-1} \left[L^k \overset{b}{\sim\!\sim} L^k + Z^k \overset{b}{\sim\!\sim} Z^{-k} \right] \tag{48}$$

coupling the physical qudits connected by bond b. The ground state of this bond term is a maximally entangled state of dimension d between the two qudits.

We are interested in reproducing the quantum double Hamiltonian in an encoded form, so to concisely state our objective, we define the encoded A and B operators,

$$\hat{A}(v) \equiv \frac{1}{|G|} \sum_g \bigotimes_{e \in +(v)} L_{\mathsf{L}}^g(e, v), \tag{49}$$

$$\hat{B}(p) \equiv \sum_{g_k \ldots g_1 = 0} \bigotimes_{e_i \in \square(p)} T_{\mathsf{L}}^{g_i}(e_i, p), \tag{50}$$

with being L_{L} defined in (41) and $T_{\mathsf{L}} = \frac{1}{d} \sum_k \omega^k Z_{\mathsf{L}}^k$ is the encoded group projection operator. We can then state the main result of this section as theorem 2.

Theorem 2. *The Hamiltonian* $H = H_0 + \lambda V$ *with* H_0 *and* V *defined as in (46) and (47) on a square lattice has a low-energy behaviour described by an effective Hamiltonian of the form*

$$H_{\text{eff}} = c_I I - \left(c_A \lambda^4\right) \sum_v \hat{A}(v) - \left(c_B \lambda^4\right) \sum_p \hat{B}(p) + \mathcal{O}(\lambda^5) \tag{51}$$

for some constants c independent of λ, *where N is the number of sites on the lattice.*

The consequence of this theorem is our system's low-energy effective Hamiltonian replicating the low-energy sector of the quantum double model (for cyclic groups at this stage).

Proof of theorem 2. We again follow the perturbative analysis as described in appendix A. As such, this will require evaluating terms in the perturbative expansion of the self-energy at order n:

$$\Sigma^{(n)}(E_0) = \lambda^n \Upsilon V (G_0(E_0)V)^{(n-1)} \Upsilon, \tag{52}$$

with G_0 being the Green's function for the system (vanishing on ground states) and Υ is the projector to the mutual ground space of each of the code gadgets.

Before we begin presenting our proof in earnest, it is useful to comment on the kinds of terms that will be preserved and those that will vanish in the ground space. All the operators defined for cyclic groups have some commutation relation $MN = \alpha NM$ for some complex α. It is then a simple result to show that for our set of stabilizers any operator that does not commute with each stabilizer will necessarily excite a ground state to an orthogonal state. This means that an operator M with $\alpha \neq 1$ for N any stabilizer will become a detectable error on the gadget's quantum code and will take the gadget to an orthogonal subspace. This implies that it will vanish in our perturbative treatment, as terms arising in our effective Hamiltonian are restricted to the ground space. In this way we need only consider error-free terms (i.e. terms that commute with all stabilizers) in our effective Hamiltonian.

From this discussion, we immediately see that first order terms will vanish, as they will necessarily leave two code gadgets in excited states. The only non-vanishing second order terms will be proportional to identity. In contrast to the toric code presented earlier, there will be non-vanishing third order terms due to the inclusion of an identity component in the perturbation term (48). However, these terms will be proportional to the second order terms, and so will be trivial. At fourth order we find the nontrivial vertex and plaquette terms that survive. To write each of these terms more explicitly, we must now distinguish between inwards-directed vertices (v_+) and outward-directed vertices (v_-), as well as the two kinds of plaquettes. We will label the plaquettes as left (p_l) and right (p_r), depending on the orientation of the top edge. In this scheme, the effective Hamiltonian will take the form

$$H_{\text{eff}}^{(4)} = \text{const} - \sum_{k=0}^{d-1} \left[\sum_{v_+} H_{v_+}^k + \sum_{v_-} H_{v_-}^k + \sum_{p_l} H_{p_l}^k + \sum_{p_r} H_{p_r}^k \right]. \tag{53}$$

We can write each of these parts individually using the notation of figure 5:

$$H_{v_+}^k = \kappa_v^L \lambda^4 \Upsilon \hat{A}(L^k, L^k, v)\Upsilon + \kappa_v^Z \lambda^4 \Upsilon \hat{A}(Z^k, Z^{-k}, v)\Upsilon, \tag{54}$$

$$H_{v_-}^k = \kappa_v^L \lambda^4 \Upsilon \hat{A}(L^k, L^k, v)\Upsilon + \kappa_v^Z \lambda^4 \Upsilon \hat{A}(Z^k, Z^{-k}, v)\Upsilon, \tag{55}$$

$$H_{p_l}^k = \kappa_p^L \lambda^4 \Upsilon \hat{B}(L^k, L^{-k}, p)\Upsilon + \kappa_p^Z \lambda^4 \Upsilon \hat{B}(Z^k, Z^k, p)\Upsilon, \tag{56}$$

$$H_{p_r}^k = \kappa_p^L \lambda^4 \Upsilon \hat{B}(L^k, L^{-k}, p)\Upsilon + \kappa_p^Z \lambda^4 \Upsilon \hat{B}(Z^k, Z^k, p)\Upsilon. \tag{57}$$

The κ are constants that take into account the sum of products of the Green's functions in the perturbation. They can be calculated for a given d once the spectrum of H_0 has been found. They must be nonzero for each of these terms because they do not return to the ground state before the end of the perturbation and so the Green's function will never vanish.

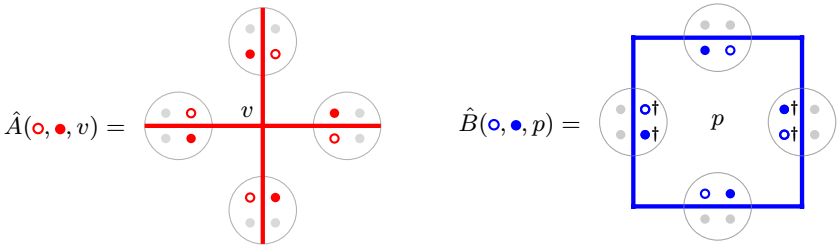


Figure 5. Physical operators for a cyclic quantum double model. Qudits at locations denoted by open circles will be acted upon by the same single-qudit operator (and similarly for those represented by full circles). The adjoint (†) of a given operator is applied to the qudits so labeled. Note the similarity of this diagram to figure 3; however, in contrast to the toric code case, the operators acting on adjacent qubits are now different.

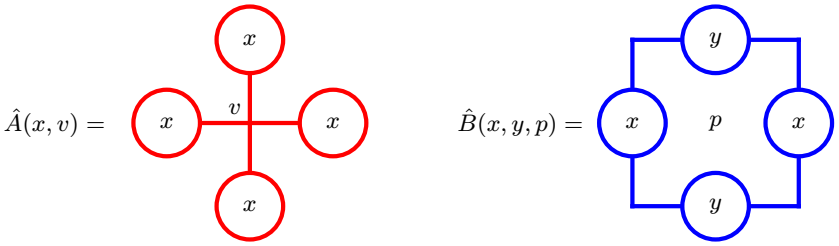


Figure 6. Encoded operators for a cyclic quantum double model. Here each of x or y represents a four-qudit logical or gauge operator. We use an overloaded notation here for $\hat{A}$ and $\hat{B}$ such that when their arguments are encoded four-qudit operators they take this form, whereas when the arguments are single-qudit operators, they take the form of figure 5.

In terms of encoded logical, gauge and stabilizer operators, we can use a similar notation (as seen in figure 6) to write these terms:

$$H^k_{v_+} = \kappa^L_v \lambda^4 \Upsilon \hat{A}(L^k_{\mathsf{L}} S^{-k}_L, v)\Upsilon + \kappa^Z_v \lambda^4 \Upsilon \hat{A}(Z^{-k}_{\mathsf{G}} S^{-k}_Z, v)\Upsilon, \tag{58}$$

$$H^k_{v_-} = \kappa^L_v \lambda^4 \Upsilon \hat{A}(L^k_{\mathsf{L}}, v)\Upsilon + \kappa^Z_v \lambda^4 \Upsilon \hat{A}(Z^{-k}_{\mathsf{G}}, v)\Upsilon, \tag{59}$$

$$H^k_{p_l} = \kappa^L_p \lambda^4 \Upsilon \hat{B}(L^k_{\mathsf{G}}, L^{-k}_{\mathsf{G}} S^{-k}_L, p)\Upsilon + \kappa^Z_p \lambda^4 \Upsilon \hat{B}(Z^{-k}_{\mathsf{L}} S^{-k}_Z, Z^k_{\mathsf{L}}, p)\Upsilon, \tag{60}$$

$$H^k_{p_r} = \kappa^L_p \lambda^4 \Upsilon \hat{B}(L^k_{\mathsf{G}} S^k_L, L^{-k}_{\mathsf{G}}, p)\Upsilon + \kappa^Z_p \lambda^4 \Upsilon \hat{B}(Z^{-k}_{\mathsf{L}}, Z^k_{\mathsf{L}} S^k_Z, p)\Upsilon. \tag{61}$$

We now evaluate the Υ projectors. Stabilizers S_Z & $S_L \to I$ in the ground space by theorem 1, and each gauge operator will simply evaluate as a constant (expectation value) in the ground space. We can again disregard these constant terms as irrelevant energy shifts. This leaves us with

$$H^k_{v_+} = \kappa^L_v \lambda^4 \Upsilon \hat{A}(L^k_{\mathsf{L}}, v)\Upsilon, \tag{62}$$

$$H^k_{v_-} = \kappa^L_v \lambda^4 \Upsilon \hat{A}(L^k_{\mathsf{L}}, v)\Upsilon, \tag{63}$$

$$H^k_{p_l} = \kappa^Z_p \lambda^4 \Upsilon \hat{B}(Z_{\mathsf{L}}^{-k}, Z_{\mathsf{L}}^k, p)\Upsilon, \tag{64}$$

$$H^k_{p_r} = \kappa^Z_p \lambda^4 \Upsilon \hat{B}(Z_{\mathsf{L}}^{-k}, Z_{\mathsf{L}}^k, p)\Upsilon. \tag{65}$$

By substituting the definitions of the Z operators in terms of T projectors, and using the orthogonality of the characters ω, it is simple to verify that these B terms are identical to the B terms of (29). Although the $\pm$ signs associated with the vertices (and edges in general) in these definitions may not immediately seem consistent with the quantum double Hamiltonian presented earlier (30) for cyclic groups, we have the freedom to rearrange the sums over k in the Hamiltonian. When we consider that we can take $k \to -k$ whenever we like, it becomes clear that these terms are indeed identical to those appearing in the quantum double Hamiltonian. This gives the result that the effective Hamiltonian will take form (51). □

At all orders $< 2L$ (L is the smallest linear dimension of the surface into which the model is embedded) the terms in the self-energy expansion (see appendix A) will act on the logical state like products of the fourth-order terms. These terms all commute and will not map the ground space of the quantum double model out of the +1 eigenspace of the encoded vertex and plaquette terms (62)–(65).

6. Our construction for general quantum double models

General quantum double models can have a much more complicated algebra than the simple cyclic ones studied previously (see section 4). However, this class includes non-Abelian models that are able to perform universal quantum computation, and so they are of key interest in this study.

6.1. Code gadget operators

As in the previously studied $\mathbb{Z}_d$ quantum double models, we encode each logical qudit in a code gadget consisting of four strongly coupled physical qudits (the scheme is again as in figure 2). For a given group G, the code gadget on each edge then has a $|G|$-fold degenerate ground space, which we use as a logical qudit. The difference in this scheme is that the generalizations of the gauge operators and stabilizers will not commute in general and so these degrees of freedom are not separable. This is contrary to the normal use of the terms 'stabilizer' and 'gauge', but we will abuse the terminology and continue to use these terms in analogy to their cyclic counterparts. The operators we define here directly generalize those used in the previous sections. The differences arise from the non-commutativity of the group multiplication operations and the fact that the irreducible representations of these general groups can be multidimensional (as opposed to the cyclic groups, which have 1D irreps). With this in mind, we can define logical and 'gauge' operators on the code gadget as follows:

$$\begin{aligned} L^g_{\mathsf{L}+} &\equiv \left(\begin{array}{c|c} L^g_+ & L^g_+ \\ I & I \end{array}\right), \qquad T^g_{\mathsf{L}+} \equiv \sum_{g_2 g_3 = g} \left(\begin{array}{c|c} I & T^{g_2}_+ \\ I & T^{g_3}_+ \end{array}\right), \\ L^g_{\mathsf{G}-} &\equiv \left(\begin{array}{c|c} I & L^g_- \\ I & L^g_+ \end{array}\right), \qquad T^g_{\mathsf{G}+} \equiv \sum_{g_1 g_2 = g} \left(\begin{array}{c|c} T^{g_1}_- & T^{g_2}_+ \\ I & I \end{array}\right). \end{aligned} \tag{66}$$

We can also define operators to describe the 'stabilizer' degrees of freedom:

$$\begin{aligned} S_L^g &\equiv \left(\begin{array}{c|c} L_-^{g^{-1}} & L_-^{g^{-1}} \\ L_+^{g^{-1}} & L_+^{g^{-1}} \end{array}\right), & S_L &\equiv \frac{1}{|G|}\sum_g S_L^g, \\ S_T^g &\equiv \sum_{g_1 g_2 g_3 g_4 = g} \left(\begin{array}{c|c} T_-^{g_1} & T_+^{g_2} \\ T_-^{g_4} & T_+^{g_3} \end{array}\right), & S_T &\equiv S_T^1, \end{aligned} \tag{67}$$

where the identity group element is denoted by 1. These operators are clearly defined in a very natural way with respect to the quantum double algebra. In these models, the only operators we strictly require to stabilize (act identically within) the ground space of our edge qudits are the projectors S_L and S_T. In spite of this, we will extend our notational abuse and continue to refer to S_L^g and S_T^g as 'stabilizer operators'.

The operators we have defined now constitute a minimum operator basis for the degrees of freedom of our encoded qudit. We can also define additional operators:

$$\begin{aligned} L_{\mathsf{L}-}^g &\equiv \left(\begin{array}{c|c} I & I \\ L_-^g & L_-^g \end{array}\right), & T_{\mathsf{L}-}^g &\equiv \sum_{g_4 g_1 = g} \left(\begin{array}{c|c} T_-^{g_1} & I \\ T_-^{g_4} & I \end{array}\right), \\ L_{\mathsf{G}+}^g &\equiv \left(\begin{array}{c|c} L_-^g & I \\ L_+^g & I \end{array}\right), & T_{\mathsf{G}-}^g &\equiv \sum_{g_3 g_4 = g} \left(\begin{array}{c|c} I & I \\ T_-^{g_4} & T_+^{g_3} \end{array}\right). \end{aligned} \tag{68}$$

The + and − subscripts on the joint operators here refer to whether these operators possess the algebra of left or right multiplication (or projection) operators, i.e.

$$L_\pm^g T_\pm^h = T_\pm^{gh} L_\pm^g, \tag{69}$$

$$L_\pm^g T_\mp^h = T_\mp^{hg^{-1}} L_\pm^g. \tag{70}$$

The stabilizer operators themselves satisfy the quantum double algebra

$$S_L^g S_T^h = S_T^{g^{-1}hg} S_L^g. \tag{71}$$

The logical operators commute with both the gauge and stabilizer operators, but in this scheme the gauge and stabilizer operators do not commute with each other.

We can also derive the Z operators corresponding to our T operators (see section 4):

$$\begin{aligned} Z_{\mathsf{L}+}^{\pi_{ij}} &= \sum_m \left(\begin{array}{c|c} I & Z_+^{\pi_{im}} \\ I & Z_+^{\pi_{mj}} \end{array}\right), & Z_{\mathsf{G}+}^{\pi_{ij}} &= \sum_m \left(\begin{array}{c|c} Z_-^{\pi_{im}} & Z_+^{\pi_{mj}} \\ I & I \end{array}\right), \\ Z_{\mathsf{L}-}^{\pi_{ij}} &= \sum_m \left(\begin{array}{c|c} Z_-^{\pi_{mj}} & I \\ Z_-^{\pi_{im}} & I \end{array}\right), & Z_{\mathsf{G}-}^{\pi_{ij}} &= \sum_m \left(\begin{array}{c|c} I & I \\ Z_-^{\pi_{mj}} & Z_+^{\pi_{im}} \end{array}\right). \end{aligned} \tag{72}$$

By construction, these operators satisfy the relevant algebra for Z operators.

We have now defined operators on the code gadget corresponding to both the left regular representation and the right regular representation of the group G for the encoded gauge and logical qudits. These definitions are redundant, in that the gauge or logical state can be uniquely defined through the action of only one L operator and only one T operator (or equivalently Z operator). This gives us some freedom in the definition of the logical state. We will choose to define it through the action of $L_{\mathsf{L}-}^g$ and $T_{\mathsf{L}+}^g$. The action of $L_{\mathsf{L}+}^g$ and $T_{\mathsf{L}-}^g$ will then be poorly

defined in general with respect to the logical state, but we will choose the codespace such that they act appropriately within it.

In order to understand how these operators act on the logical state, it is useful at this point to explicitly identify the encoding scheme of our code gadget. We can label the state of each physical qudit within a given code gadget as follows:

$$|h_a, h_b, k_{\mathsf{G}}, k_{\mathsf{L}}\rangle = \left(\begin{array}{c|c} |h_a\rangle & |k_{\mathsf{G}} h_a\rangle \\ |h_b^{-1} h_a^{-1} k_{\mathsf{L}}\rangle & |h_a^{-1} k_{\mathsf{G}}^{-1} k_{\mathsf{L}}\rangle \end{array}\right). \tag{73}$$

Here the logical and gauge states of the system are labelled by k_{L} and k_{G}, respectively, and will transform under the action of the logical or gauge operators. The remaining labels h define how the stabilizers S_{L} and S_{G} act on the system. We can directly see the action of our encoded operators on these states. For example,

$$L^g_{\mathsf{L}-}|h_a, h_b, k_{\mathsf{G}}, k_{\mathsf{L}}\rangle = |h_a, h_b, k_{\mathsf{G}}, k_{\mathsf{L}} g^{-1}\rangle, \tag{74}$$

$$T^g_{\mathsf{L}+}|h_a, h_b, k_{\mathsf{G}}, k_{\mathsf{L}}\rangle = \delta_{g k_{\mathsf{L}}}|h_a, h_b, k_{\mathsf{G}}, k_{\mathsf{L}}\rangle. \tag{75}$$

We chose to define the logical state k_{L} by the action of these two operators and as such they act on k_{L} as might be expected. The stabilizer operators act as

$$S^g_L\,|h_a, h_b, k_{\mathsf{G}}, k_{\mathsf{L}}\rangle = \left|h_a g, g^{-1} h_b g, k_{\mathsf{G}}, k_{\mathsf{L}}\right\rangle, \tag{76}$$

$$S^g_T\,|h_a, h_b, k_{\mathsf{G}}, k_{\mathsf{L}}\rangle = \delta_{g h_b}\,|h_a, h_b, k_{\mathsf{G}}, k_{\mathsf{L}}\rangle\,. \tag{77}$$

The action of the remaining encoded operators is not so simple. In general, these will mix the logical or gauge states with the h_a or h_b. However, we will construct the codespace such that within it logical operators will act only on the logical state, and gauge operators similarly act appropriately.

6.2. Code gadget Hamiltonian

Now that we have defined a set of operators we consider the Hamiltonian of a code gadget. We require that this Hamiltonian consist only of two-body terms and possess a ground space that can be used as a codespace for our logical qudit. For this to happen, the ground space must be stabilized by S_L and S_T, and be exactly $|G|$-fold degenerate. The significance of these two stabilizers is that they are terms in the quantum double Hamiltonian defined on a small four-qudit torus (as depicted in figure 7). In that sense, we are creating a code that is very similar to a miniature quantum double model. Of course, our code gadget will consist of only two-body terms, and to achieve this we must sacrifice the gauge degrees of freedom in our model.

To this end, we generalize the Hamiltonian we used for cyclic groups previously to give

$$H(e) = -\frac{J_L}{|G|}\sum_g \left[L^g_{\mathsf{G}+} + L^g_{\mathsf{G}-}\right] - \frac{J_Z}{|G|}\sum_{\pi,i} d_\pi \left[Z^{\pi_{ii}}_{\mathsf{G}+} + Z^{\pi_{ii}}_{\mathsf{G}-}\right] \tag{78}$$

$$= -\frac{J_L}{|G|}\sum_g \left[L^g_{\mathsf{G}+} + L^g_{\mathsf{G}-}\right] - J_Z\left[T^1_{\mathsf{G}+} + T^1_{\mathsf{G}-}\right]. \tag{79}$$

These two forms can be seen to be equivalent using the definitions of the Z_{G} operators (72). In the latter form, it is clear that our unperturbed Hamiltonian is very closely related to the quantum double Hamiltonian (30). It is also clear that the logical operators will commute

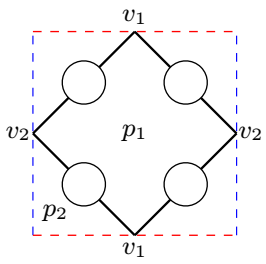


Figure 7. The four-qudit torus is constructed by identifying the opposite vertices of a square. The dashed horizontal (red) lines are identified, and the dashed vertical (blue) lines are identified. There are two equivalent plaquette stabilizers corresponding to p_1 and p_2 and two (generally inequivalent) vertex stabilizers corresponding to v_1 and v_2 in this quantum double model. Our stabilizers S_L and S_T are obtained by choosing one of each. For Abelian models, the two vertex stabilizers are also equivalent, and so S_L and S_T generate the full stabilizer group of the quantum double model on this surface for those groups.

with the Hamiltonian (as each term is a gauge operator) and so we can expect at least the $|G|$-fold degeneracy of our logical qubit. The following theorem encapsulates the remaining requirements of our codespace.

Theorem 3. *The Hamiltonian (78) has a $|G|$-fold degenerate ground space that is in the common $+1$ eigenspace of S_L and S_T.*

We provide a proof of this theorem in appendix D. This theorem places restrictions on the form of the ground space which ensure that undesirable terms in the effective Hamiltonian will couple to higher-energy sectors and vanish. Now that we have a suitable codespace for our logical qudit, we proceed to reproducing the target model.

6.3. Coupling the code gadgets

We place our code gadgets on the edges of the lattice and couple them using the same geometric scheme as the toric code and cyclic group cases (illustrated in figure 2), where each physical qudit is perturbatively coupled to a physical qudit from a neighboring code gadget. Our uncoupled (unperturbed) Hamiltonian for each edge of the lattice is as given in (78).

$$H_0 = \sum_e H(e) \tag{80}$$

$$= \sum_e \left[-\frac{J_L}{|G|} \sum_g \left[L^g_{\mathsf{G}+} + L^g_{\mathsf{G}-} \right] - \frac{J_Z}{|G|} \sum_{\pi,i} d_\pi \left[Z^{\pi_{ii}}_{\mathsf{G}+} + Z^{\pi_{ii}}_{\mathsf{G}-} \right] \right]_e . \tag{81}$$

As before, the subscript e refers to a particular edge (a particular code gadget). The perturbation term is generalized straightforwardly from the cyclic case to take the form

$$V = \sum_b V(b) \tag{82}$$

$$= -\sum_b \left[\sum_k L^k \overset{b}{\sim\!\sim} L^k + \sum_{\pi,k,m} d_\pi \, Z^{\pi_{km}} \overset{b}{\sim\!\sim} Z^{\pi_{mk}} \right] , \tag{83}$$

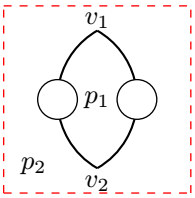


Figure 8. The two-qudit sphere is constructed by identifying every point on the boundary of a plane. This boundary is represented by dashed red lines. The solid lines correspond to edges on the lattice, with a qudit located on each edge. The loop formed by these edges is homologically equivalent to an equator around the sphere. There exists a plaquette stabilizer for each plaquette p_1 and p_2 and similarly a vertex stabilizer for each vertex v_1 and v_2 in this quantum double model. The terms in our bond Hamiltonian are obtained by choosing one plaquette and one vertex stabilizer from these. For Abelian models the two vertex stabilizers are also equivalent, as are the two plaquette stabilizers, and so the bond Hamiltonian consists of a full generating set of the stabilizer group of the quantum double model on this surface for these groups.

where the L or Z operator associated with a particular qudit has definite $\pm$ subscript depending on its location as follows:

$$L: \begin{pmatrix} + & + \\ - & - \end{pmatrix} \qquad Z: \begin{pmatrix} - & + \\ - & + \end{pmatrix}. \tag{84}$$

For example, an L operator in the top right-hand corner of a code gadget (with the edge taken running up the page) will take the form L_+. This is motivated by the definitions of $v_\pm$ and $p_\pm$ of the quantum double model in section 4. As in the $\mathbb{Z}_d$ models, it is clear that the coupling terms are very closely related to the quantum double Hamiltonian. In fact, these bond terms could be considered a quantum double Hamiltonian on a small (two-qudit) sphere, with the corresponding nondegenerate ground state (see figure 8).

Again we must define the encoded A and B operators we will reproduce in our perturbative expansion (with 1 being the identity group element for general groups):

$$\hat{A}(v) \equiv \frac{1}{|G|} \sum_g \bigotimes_{e \in +(v)} L_{\mathrm{L}}^g(e, v), \tag{85}$$

$$\hat{B}(p) \equiv \sum_{g_k \cdots g_1 = 1} \bigotimes_{e_i \in \square(p)} T_{\mathrm{L}}^{g_i}(e_i, p). \tag{86}$$

The main result of this section is theorem 4.

Theorem 4. *The Hamiltonian $H = H_0 + \lambda V$ with H_0 and V defined as in (80) and (82) on a square lattice has a low energy behaviour described by an effective Hamiltonian of the form*

$$H_{\mathrm{eff}} = c_I I - \left(c_A \lambda^4\right) \sum_v \hat{A}(v) - \left(c_B \lambda^4\right) \sum_p \hat{B}(p) + \mathcal{O}(\lambda^5) \tag{87}$$

for some constants c independent of λ and N, where N is the number of sites on the lattice. The encoded low energy behavior of the system to this order is described by the quantum double Hamiltonian (30) up to additive and multiplicative constants.

This theorem implies that our effective Hamiltonian correctly reproduces the low-energy sector of the quantum double Hamiltonian for any group. Before we prove it, we make some comments on the operators that will arise in the perturbative treatment. For cyclic models, we were able to make some general arguments to exclude all unwanted operators from arising in the effective Hamiltonian. Unfortunately, no equivalent general argument has been found in the case of the general quantum double models. For this reason, we have explicitly shown this property for the relevant operators in appendix E. The results are an intuitive generalization of the cyclic case, in that the only nonvanishing operators will be those contributing encoded operators that act on the gauge or logical subspaces. We use these results now to calculate the nonvanishing terms in the effective Hamiltonian.

Proof of theorem 4. We follow the same perturbative treatment as used previously (see appendix A). The results of appendix E effectively show that no second order terms are able to survive except those proportional to identity (exactly as in the cyclic case). Similarly, all first order, and all non-trivial third order, terms will vanish. It is clear then that no nontrivial operators will appear below fourth order. The effective Hamiltonian at order 4 is then given by

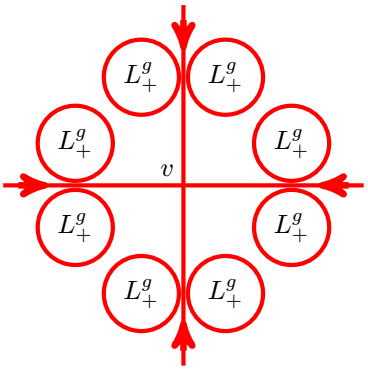
$$H_{\text{eff}}^{(4)} = \lambda^n \Upsilon V (G_0(E_0)V)^{(n-1)} \Upsilon, \tag{88}$$

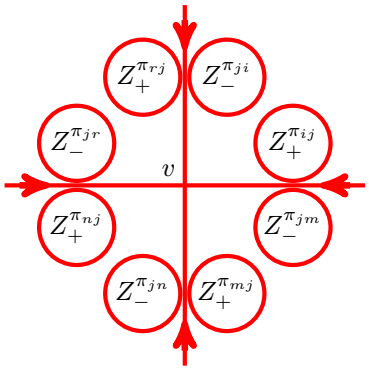
with G_0 being the Green's function vanishing on ground states (taken at the unperturbed ground state energy E_0) and Υ again the projector to the ground space of the code gadgets. At this order, we will find nontrivial terms around plaquettes and vertices consisting of products of Z or L operators. As in the $\mathbb{Z}_d$ case, it is useful to distinguish the two different kinds of vertices and the two different kinds of plaquettes here (see figure 4). We have defined the lattice such that vertices can either consist of all inwardly directed edges (v_+) or all outwardly directed edges (v_-). Plaquettes can have their top edge directed either left (p_l) or right (p_r).

Disregarding constant energy shifts, we can then separate terms in the Hamiltonian by vertex or plaquette type:

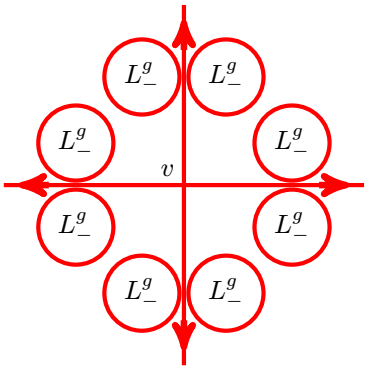
$$H_{\text{eff}}^{(4)} = -\Upsilon \left[\sum_{v_+} H_{v_+} + \sum_{v_-} H_{v_-} + \sum_{p_l} H_{p_l} + \sum_{p_r} H_{p_r} \right] \Upsilon. \tag{89}$$

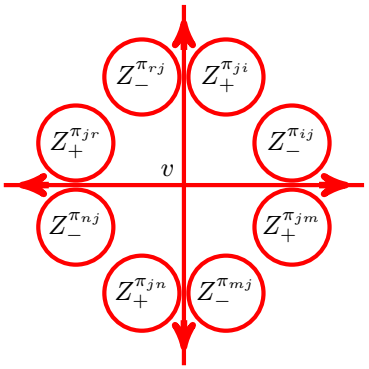
As we did for the cyclic groups, we will write these operators pictorially to make the physical location of a particular operator obvious. For simplicity, we have neglected to draw those qudits on the outer edge of the vertex/plaquette under consideration, where these terms will act trivially. With this in mind, individual terms in the effective Hamiltonian can be written as

$H_{v_+} = \kappa_v^L \lambda^4 \sum_g$ 

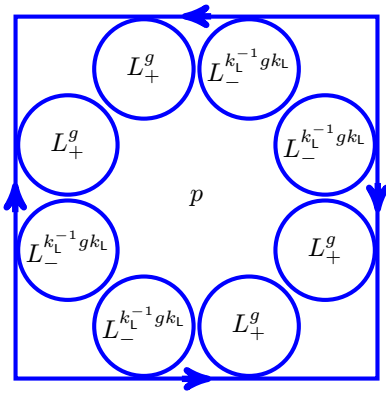
$+\kappa_v^Z d_\pi^4 \lambda^4 \sum_{\pi,i,r,m,n} \sum_j$ 

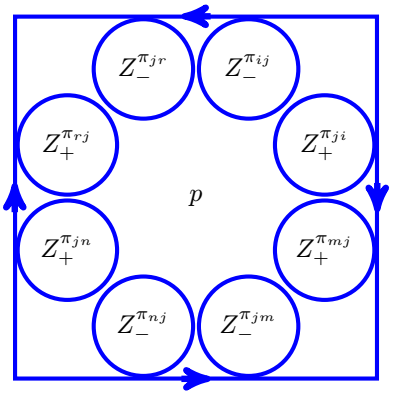
(90)

$$H_{v_-} = \kappa_v^L \lambda^4 \sum_g$$ 

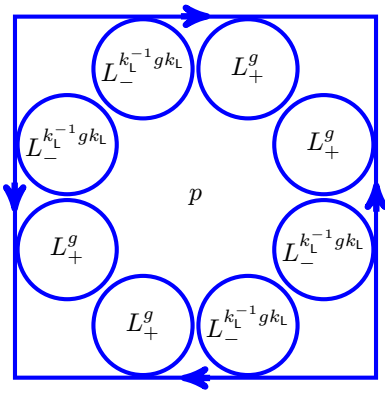
$$+\kappa_v^Z d_\pi^4 \lambda^4 \sum_{\pi,i,r,m,n} \sum_j$$ 

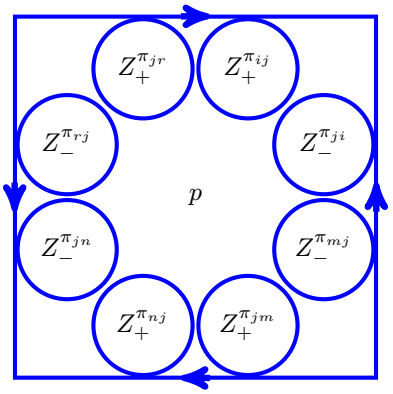
(91)

$$H_{p_l} = \kappa_p^L \lambda^4 \sum_g$$ 

$$+\kappa_p^Z d_\pi^4 \lambda^4 \sum_{\pi,i,r,m,n} \sum_j$$ 

(92)

$$H_{p_r} = \kappa_p^L \lambda^4 \sum_g$$ 

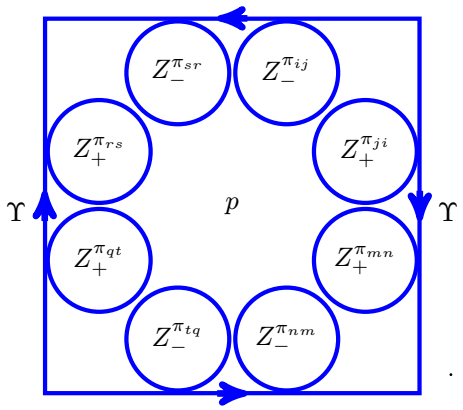
$$+\kappa_p^Z d_\pi^4 \lambda^4 \sum_{\pi,i,r,m,n} \sum_j$$ 

.

(93)

Here the κs are proportionality constants arising from the pertubation treatment. For a specific model, they can readily be calculated. It should be reasonably clear that no other nontrivial fourth-order terms (or lower) than those shown above will survive (see appendix E). The higher-order terms will act in the logical space as products of the fourth order terms, until the order is sufficiently high to form noncontractible loops over the lattice.

The most illustrative of the fourth order terms are the plaquette Z terms. As such, we will present a brief demonstration of how these operators arise. On our plaquette, at the fourth order the most general Z operator to be constructed has the form



Here we have already used the fact that Z operators with different representations will move out of the ground space and so we have only used one representation π. From equations (E.12) and (E.14), we find the condition $j = s = n = q$ for nonvanishing terms. From this, we immediately obtain the term appearing above in the effective Hamiltonian for this plaquette. Similar considerations give the other plaquette and vertex terms.

Given the operators (90)–(93) in their current form, it is difficult to immediately see how many of them act on the logical state. In order to study this, we must revisit the encoding scheme defined in (73) and the ground space studied in appendix D. We will study the action of each operator on the basis

$$|\sigma_{mn}, h, k_{\mathsf{G}}, k_{\mathsf{L}}\rangle \equiv \frac{\sqrt{d_\sigma}}{\sqrt{|G|}} \sum_{g_\sigma \in \mathcal{G}} [\sigma(g_\sigma)]_{mn} |g_\sigma, g_\sigma^{-1} h g_\sigma, k_{\mathsf{G}}, k_{\mathsf{L}}\rangle, \tag{94}$$

particularly in the ground space where $\sigma_{mn} = I_{11}$ and $h = 1$. From this, we can examine the effect on the logical state k_{L} in the codespace. The operators in question act on these states as follows:

$$\begin{pmatrix} L_+^g & I \\ L_-^{k_{\mathsf{L}}^{-1} g k_{\mathsf{L}}} & I \end{pmatrix} |I_{11}, 1, k_{\mathsf{G}}, k_{\mathsf{L}}\rangle = \frac{1}{\sqrt{|G|}} \sum_{g_\sigma} |g g_\sigma, 1, k_{\mathsf{G}} g^{-1}, k_{\mathsf{L}}\rangle \tag{95}$$

$$= \frac{1}{\sqrt{|G|}} \sum_{\tilde{g}_\sigma} |\tilde{g}_\sigma, 1, k_{\mathsf{G}} g^{-1}, k_{\mathsf{L}}\rangle \tag{96}$$

$$= |I_{11}, 1, k_{\mathsf{G}} g^{-1}, k_{\mathsf{L}}\rangle \tag{97}$$

and

$$\begin{pmatrix} I & L_+^g \\ I & L_-^{k_{\mathsf{L}}^{-1} g k_{\mathsf{L}}} \end{pmatrix} |I_{11}, 1, k_{\mathsf{G}}, k_{\mathsf{L}}\rangle = \frac{1}{\sqrt{|G|}} \sum_{g_\sigma} |g_\sigma, 1, g k_{\mathsf{G}}, k_{\mathsf{L}}\rangle \tag{98}$$

$$= |I_{11}, 1, g k_{\mathsf{G}}, k_{\mathsf{L}}\rangle. \tag{99}$$

In the codespace, these operators have the effective action equivalent to $L^g_{\mathsf{G}-}$ or $L^g_{\mathsf{G}+}$, respectively. They differ from the previously defined versions, but the key is that they act only on the gauge state in this subspace. We will denote these operators as $L^g_{\mathsf{G}'-}$ and $L^g_{\mathsf{G}'+}$. In a similar way, we can look at the action of the Z-like operators:

$$\sum_m \left(\begin{array}{c|c} Z^{\pi_{mj}}_- & Z^{\pi_{im}}_+ \\ I & I \end{array} \right) |I_{11}, 1, k_{\mathsf{G}}, k_{\mathsf{L}}\rangle = [\pi(k_{\mathsf{G}})]_{ij} |I_{11}, 1, k_{\mathsf{G}}, k_{\mathsf{L}}\rangle. \tag{100}$$

This operator acts quite similarly (identical up to conjugacy) to $Z^{\pi_{ij}}_{\mathsf{G}+}$. In the ground space, we will be able to ignore it as a constant expectation value (as the gauge state will be fixed). As such, we will denote this operator $Z^{\pi_{ij}}_{\mathsf{G}'+}$. The second of the Z operators is a little more complicated. To this end, we must consider the exact form of the ground states:

$$|\psi_0\rangle = |I_{11}, 1, 1, k_{\mathsf{L}}\rangle + |I_{11}, 1, I_{11}, k_{\mathsf{L}}\rangle. \tag{101}$$

This allows us to see the action of these operators.

$$\sum_m \left(\begin{array}{c|c} I & I \\ Z^{\pi_{im}}_- & Z^{\pi_{mj}}_+ \end{array} \right) |I_{11}, 1, 1, k_{\mathsf{L}}\rangle \quad = [\pi(k_{\mathsf{L}}^{-1} k_{\mathsf{L}})]_{ij} |I_{11}, 1, 1, k_{\mathsf{L}}\rangle \tag{102}$$

$$= \delta_{ij} |I_{11}, 1, 1, k_{\mathsf{L}}\rangle, \tag{103}$$

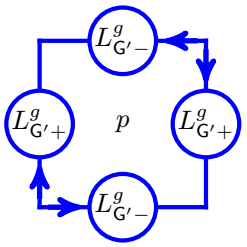
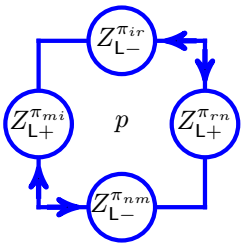
$$\sum_m \left(\begin{array}{c|c} I & I \\ Z^{\pi_{im}}_- & Z^{\pi_{mj}}_+ \end{array} \right) |I_{11}, 1, I_{11}, k_{\mathsf{L}}\rangle = \frac{1}{\sqrt{|G|}} \sum_{g_\pi} [\pi(k_{\mathsf{L}}^{-1} g_\pi^{-1} k_{\mathsf{L}})]_{ij} |I_{11}, 1, g_\pi, k_{\mathsf{L}}\rangle. \tag{104}$$

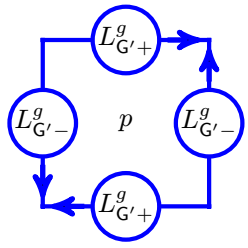
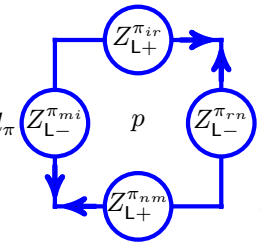
On the ground space, this acts not dissimilarly to $Z^{\pi_{ij}}_{\mathsf{G}-}$. The gauge state becomes mixed up somewhat, but because of the particular definite gauge state we have in the codespace, an operator of this form can be evaluated as a constant. This is a result of the fact that the overlap of this state (104) with the ground space is independent of k_{L} (as can be easily verified). With this in mind, we will write this operator as $Z^{\pi_{ij}}_{\mathsf{G}'-}$ from now on.

Given these definitions, we can now write the terms in the effective Hamiltonian in a more succinct format:

$$H_{v_+} = \kappa_v^L \lambda^4 \sum_g \; L^g_{\mathsf{L}+} \; L^g_{\mathsf{L}+} \; v \; L^g_{\mathsf{L}+} \; L^g_{\mathsf{L}+} + \kappa_v^Z \lambda^4 \sum_{\pi,i,r,m,n} d_\pi \; Z^{\pi_{nr}}_{\mathsf{G}'+} \; Z^{\pi_{ri}}_{\mathsf{G}'+} \; v \; Z^{\pi_{im}}_{\mathsf{G}'+} \; Z^{\pi_{mn}}_{\mathsf{G}'+} \tag{105}$$

$$H_{v_-} = \kappa_v^L \lambda^4 \sum_g \; L^g_{\mathsf{L}-} \; L^g_{\mathsf{L}-} \; v \; L^g_{\mathsf{L}-} \; L^g_{\mathsf{L}-} + \kappa_v^Z \lambda^4 \sum_{\pi,i,r,m,n} d_\pi \; Z^{\pi_{nr}}_{\mathsf{G}'-} \; Z^{\pi_{ri}}_{\mathsf{G}'-} \; v \; Z^{\pi_{im}}_{\mathsf{G}'-} \; Z^{\pi_{mn}}_{\mathsf{G}'-} \tag{106}$$

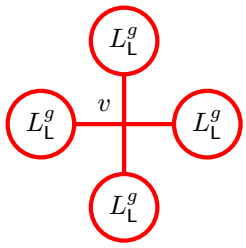
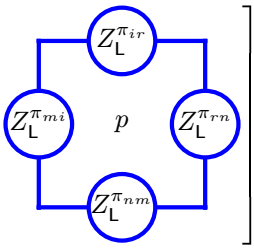
$$H_{p_l} = \kappa_p^L \lambda^4 \sum_g$$  $$+ \kappa_p^Z \lambda^4 \sum_{\pi,i,r,m,n} d_\pi$$  (107)

$$H_{p_r} = \kappa_p^L \lambda^4 \sum_g$$  $$+ \kappa_p^Z \lambda^4 \sum_{\pi,i,r,m,n} d_\pi$$  . (108)

We have been able to treat each code gadget in the Z terms separately, even though in the previous definitions they shared a common index (j). The reason why we are able to separate them in this way is because in the ground space for any given j we can use (E.12)–(E.18) to rewrite each term as an average over all values of j. This allows us to rewrite each term as a summation over different indices, which gives us their quoted form. The factor of d_π appearing in the terms (105)–(108) then comes from the remaining sum over j. As a simplified explicit example, consider

$$\Upsilon \sum_j \begin{pmatrix} Z_-^{\pi_{jm}} & Z_+^{\pi_{ij}} \\ I & I \end{pmatrix} \otimes \begin{pmatrix} Z_-^{\pi_{jp}} & Z_+^{\pi_{mj}} \\ I & I \end{pmatrix} \Upsilon = \Upsilon \sum_j \sum_{j'} \frac{1}{d_\pi} \begin{pmatrix} Z_-^{\pi_{j'm}} & Z_+^{\pi_{ij'}} \\ I & I \end{pmatrix} \otimes \begin{pmatrix} Z_-^{\pi_{jp}} & Z_+^{\pi_{mj}} \\ I & I \end{pmatrix} \Upsilon. \quad (109)$$

We can disregard all the terms in the Hamiltonian that do not act on the logical subspace (as they will only introduce some constant energy shift for our purposes). We now also drop the $\pm$ subscript on logical operators, with the observation that they are consistent with the edge orientation conventions introduced in section 4. With this in mind, our effective Hamiltonian reduces to

$$H_{\text{eff}} = -\lambda^4 \Upsilon \Bigg[\sum_v \kappa_v^L \sum_g$$  $$+ \sum_p \kappa_p^Z \sum_{\pi,i,r,m,n} d_\pi$$  $$\Bigg] \Upsilon ,$$ (110)

with each operator defined as in equations (105–108), and the $\pm$ of the logical operators defined by the edge orientation. Using the definitions of the Z operators in terms of T operators, and the orthogonality of group characters, it is not difficult to show that these operators are indeed equivalent to those of the target quantum double Hamiltonian. This gives our Hamiltonian the form (87), as claimed. □

As in the Abelian case, all orders $< 2L$ (L being the smallest linear dimension of the surface into which the model is embedded) of the self-energy expansion (see appendix A) will act on the logical state like products of the fourth order terms. These terms all commute and

will not map the ground space of the quantum double model out of the +1 eigenspace of the encoded vertex and plaquette terms (62)–(65). Beyond this order perturbative corrections to the self-energy will be able to form homologically nontrivial loops on the surface. These will be heavily suppressed for large L and large energy gap.

Acknowledgments

We thank Miguel Aguado, Sergey Bravyi, Robert Koenig, Spyridon Michalakis, David Poulin and Guifre Vidal for interesting and fruitful discussions. SDB and ACD acknowledge support from the ARC via the Centre of Excellence in Engineered Quantum Systems (EQuS), project number CE110001013. Research at Perimeter Institute is supported by the Government of Canada through Industry Canada and by the Province of Ontario through the Ministry of Research and Innovation. This work was supported in part by NSA/ARO under grant no. W911NF-09-1-0442 and by NSF under grant no. PHY-0803371.

Appendix A. Perturbation theory

We will now give a brief introduction to the formalism we use to perform perturbation calculations, such as to introduce lattice couplings between edge qudits. We follow the resolvent or Green's function approach in [13] and, in general, we are only interested in the leading nonconstant order in the effective Hamiltonian. Given the Hamiltonian

$$H = H_0 + \lambda V, \tag{A.1}$$

where H_0 has a subspace of degenerate eigenvectors with energy E_0, let Υ be the projector onto the eigenspace of the eigenvalue E_0 of H_0. In our case we are interested in the situation where E_0 is the ground state energy of H_0. In degenerate perturbation theory, one generally aims to find an effective Hamiltonian H_{eff} that acts on the subspace given by Υ and that has the same eigenvalues as H, in other words, an effective Hamiltonian that describes how the perturbation term V acts within the ground space of the unperturbed Hamiltonian. We use the Green's function formalism of [13] for this calculation and find the self-energy $\Sigma(E)$, which is given by the perturbation expansion:

$$\Sigma(E) = \Upsilon \left(\lambda V + \lambda^2 V G_0(E) V + \lambda^3 V G_0(E) V G_0(E) V + \cdots\right) \Upsilon. \tag{A.2}$$

At the lowest nontrivial order of perturbation theory, we have

$$H_{\text{eff}} \simeq E_0 + \Sigma(E_0). \tag{A.3}$$

The unperturbed Green's function for excited states of H_0 is denoted by $G_0(E) = (E - H_0)^{-1}(1 - \Upsilon)$, such that this function vanishes in the ground space. At higher orders in perturbation theory, one would need to take into account the E dependence of the self-energy around $E \approx E_0$ in order to find the effective Hamiltonian; however, as we are interested only in the lowest nontrivial order in perturbation theory throughout this paper, we do not do so.

We can expand the self-energy order by order as follows (neglecting constant energy shifts):

$$\Sigma(E_0) = \sum_n \Sigma^{(n)}(E_0), \tag{A.4}$$

$$\Sigma^{(n)}(E_0) = \lambda^n \Upsilon V (G_0(E_0) V)^{(n-1)} \Upsilon. \tag{A.5}$$

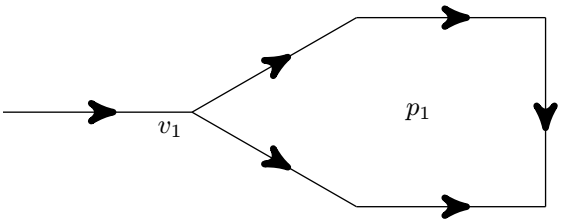


Figure B.1. A section of a simple directed graph.

The Green's function G_0 is always evaluated at the unperturbed ground state energy E_0, and will be nonpositive.

The product of operators obtained between Υs will vanish unless it remains within (or at least overlaps in some nontrivial way) the ground space of every code gadget on the lattice. It will be simple to eliminate many operators that will vanish or act trivially on the ground space. At the nth order, the self-energy will consist of a sum of terms

$$\Sigma^{(n)}(E_0) = \sum_j \lambda^n \Upsilon \kappa_j A_j \Upsilon, \tag{A.6}$$

with κ_j being constants. They take into account the Green's function terms appearing in the perturbative analysis. Summed over each possible ordering of the V terms that will give the same A_j, κ_j is the product of the $(n-1)$ Green's functions appearing in the nth-order terms. In our case, this sum will run over the $n!$ ways of ordering the perturbations, which multiply together to give the operator A_j.

Appendix B. Extension to arbitrary graph

It is not difficult to extend our treatment from the explicit square lattice to an arbitrary directed graph. As each edge is associated with two plaquettes on any nonintersecting 2D graph (neglecting boundaries), each edge qudit is associated with exactly four nearest neighbors regardless of the form of the lattice. This allows us to retain our previous definitions for edge qubit gauge and logical operators as in section 6.1.

If we apply a perturbation to the uncoupled edge qudits exactly as before, we will again see plaquette and vertex terms arising in the effective Hamiltonian. Of course, they may not arise at the same order in this treatment (e.g. for a hexagonal lattice, plaquettes arise at sixth and vertices arise at third order, and on a general graph, plaquette boundaries and vertex stars will not be uniform in size). It is possible that this may have some undesirable effect on excited states of the effective Hamiltonian, but the higher-order terms that will be able to survive the perturbation will act in the logical space as products of existing (commuting) terms until the perturbative order is high enough to form nontrivial homology cycles over the lattice. Below this order, the ground space of the effective Hamiltonian will remain within the ground space of the encoded quantum double Hamiltonian.

We can see, in figure B.1, part of a simple directed graph. If we consider the perturbation term exactly as in the general group square lattice treatment (82), the vertex term will take the form (up to additive and multiplicative constants)

$$H(v) \sim \Upsilon \sum_g \prod_{e \to v} L^g_{\mathsf{L}+} \prod_{e \leftarrow v} L^g_{\mathsf{L}-} \Upsilon + \Upsilon \sum \prod_{e \to v} Z_{\mathsf{G}'+} \prod_{e \leftarrow v} Z_{\mathsf{G}'-} \Upsilon, \tag{B.1}$$

where $e \leftarrow v$ denotes those edges that run out of the vertex v, and $e \rightarrow v$ denotes those that run toward the vertex. The exact form of the Z term is not difficult to calculate but is only sketched in this notation for clarity. In any case, the vertex Z term will evaluate as a constant in the ground space and so can be disregarded for our purposes. This leaves the products of the terms for inwards-directed edges and outwards-directed edges. If we again look at the definitions of the quantum double model in section 4, we can see that this will give a vertex term consistent with the target Hamiltonian.

Similarly, as we traverse a plaquette, two kinds of terms will arise:

$$H(v) \sim \Upsilon \sum_g \prod_{e \rightarrow p} L^g_{\mathsf{G}-} \prod_{e \leftarrow p} L^g_{\mathsf{G}+} \Upsilon + \Upsilon \sum \prod_{e \rightarrow v} Z_{\mathsf{L}+} \prod_{e \leftarrow v} Z_{\mathsf{L}-} \Upsilon. \tag{B.2}$$

Here the notation $e \rightarrow p$ is taken to mean that the edge is on the left of the plaquette when looking along e and similarly $e \leftarrow p$ has the edge on the right of p. The indices of the Z terms are suppressed here for clarity, but they can easily be restored by comparing with (110) and can be verified to be consistent with the quantum double Hamiltonian. Of course, the gauge L term will be evaluated as a constant and disregarded. This will leave us with only the desired logical operators on both plaquettes and vertices, and we will have successfully reproduced the quantum double Hamiltonian (with the caveat that the terms may arise at different order, and with different coefficients).

Appendix C. Proof of theorem 1

Proof. We will label eigenstates of Z by group elements $|h\rangle$, and the corresponding eigenstates of L by (unitary, irreducible) representations $|\sigma\rangle = \frac{1}{\sqrt{d}} \sum_h \omega^{\sigma h} |h\rangle$. Note that for these cyclic groups, the representations are all 1D. We begin by defining projectors

$$P_i^k = \frac{1}{d} \sum_l \omega^{kl} S_i^l, \tag{C.1}$$

with inverses

$$S_i^k = \sum_l \omega^{-kl} P_i^l, \tag{C.2}$$

where $i \in \{L, Z\}$. We can then write our Hamiltonian as

$$H = -\frac{1}{d} \sum_k \left[L^k_{\mathsf{G}} \left(1 + \sum_h \omega^{-kh} P^h_L \right) + Z^k_{\mathsf{G}} \left(1 + \sum_\sigma \omega^{-k\sigma} P^\sigma_Z \right) \right]. \tag{C.3}$$

If we block diagonalize with respect to both h and σ, our proof will amount to showing that the unique ground state is in the $h = 0$ and $\sigma = 0$ block. In group theoretic terms, this corresponds to h the identity group element and σ the trivial representation (which we will denote now as $\sigma = I$). Our Hamiltonian within the (h, σ) block takes the form

$$H^{h,\sigma} = -\frac{1}{d} \sum_k \left[L^k_{\mathsf{G}} (1 + \omega^{-kh}) + Z^k_{\mathsf{G}} (1 + \omega^{-k\sigma}) \right]. \tag{C.4}$$

We can rewrite the $\sum_k L_{\mathsf{G}}^k$ and $\sum_k Z_{\mathsf{G}}^k$ operators as projectors on the gauge subspace onto group elements or representations:

$$\sum_k \omega^{-kh} L_{\mathsf{G}}^k = d|h\rangle\langle h| \quad \text{and} \quad \sum_k \omega^{-k\sigma} Z_{\mathsf{G}}^k = d|\sigma\rangle\langle\sigma|, \tag{C.5}$$

with $|\sigma\rangle$ being a representation state and $|h\rangle$ a group element state. The representation states are discrete Fourier transforms of the element states. Then we have

$$H^{h,\sigma} = -\big[|0\rangle\langle 0| + |h\rangle\langle h| + |I\rangle\langle I| + |\sigma\rangle\langle\sigma|\big]. \tag{C.6}$$

Here $|0\rangle$ is the group identity element and $|I\rangle$ corresponds to the trivial representation of the group. Since $H^{h,\sigma}$ is a negative sum of projectors, all its eigenvalues are nonpositive. We can then write a triangle inequality for the magnitude of least eigenvalue by taking the operator norm,

$$\|H^{h,\sigma}\| \leqslant \||0\rangle\langle 0| + |I\rangle\langle I|\| + \||h\rangle\langle h| + |\sigma\rangle\langle\sigma|\|. \tag{C.7}$$

We know that this inequality is saturated only if $(|0\rangle\langle 0| + |I\rangle\langle I|)$ is parallel to $(|h\rangle\langle h| + |\sigma\rangle\langle\sigma|)$ in their largest eigenspace. But unless $(h, \sigma) = (0, I)$, these vectors are never parallel. Then we have

$$\|H^{0,I}\| = 2\||0\rangle\langle 0| + |I\rangle\langle I|\| > \|H^{h,\sigma}\| \quad \text{when } (h, \sigma) \neq (0, I). \tag{C.8}$$

We can now say that the ground space must have $h = 0$ and $\sigma = I$. Furthermore, when this is the case, we see from (C.6) that the only nonvanishing eigenvectors of this block are linear combinations of $|0\rangle$ and $|I\rangle$. We find by inspection that the two independent eigenvectors are

$$|\psi_\pm\rangle = |0\rangle \pm |I\rangle, \tag{C.9}$$

with eigenvalues

$$\lambda_\pm = -2\left[1 \pm \frac{1}{\sqrt{d}}\right]. \tag{C.10}$$

This gives for the unique ground state of the gauge qudit

$$|\psi_0^{\mathsf{G}}\rangle = |0\rangle + |I\rangle. \tag{C.11}$$

Given that the stabilizer and gauge degrees of freedom have a unique ground state and the fact that the Hamiltonian commutes with the logical operators, we can also say that the Hamiltonian (44) has a d-fold degenerate ground space encoding the logical state. □

Appendix D. Proof of theorem 3

Proof. We wish to prove that our Hamiltonian (78) is stabilized as claimed. The key to proving this theorem is choosing a suitable basis. We have previously defined the states as in the physical basis of (73). Here, we will need to look at some more sophisticated bases that make the actions of our encoded operators even more transparent. □

D.1. Alternative bases

Consider first the gauge representation basis:

$$|\sigma_{mn}, h, \pi_{ij}, k_{\mathsf{L}}\rangle \equiv \frac{\sqrt{d_\pi d_\sigma}}{|G|} \sum_{g_\pi, g_\sigma \in G} [\pi(g_\pi g_\sigma)]_{ij} [\sigma(g_\sigma)]_{mn} |g_\sigma, g_\sigma^{-1} h g_\sigma, g_\pi, k_{\mathsf{L}}\rangle. \tag{D.1}$$

Here, π and σ are (unitary irreducible) representations of dimension d_π and d_σ, respectively. The π_{ij} variable defines the gauge state, which is why we call this the gauge representation basis. It is clear that there are a total of $|G|^4$ basis states, as needed, and orthonormality can be easily verified. We would like to know how the $\sum_g L^g_{\mathsf{G}\pm}$ projectors in the Hamiltonian (78) will act on this basis.

It can be easily verified that these operators act in the following way on the physical basis:

$$L^{\tilde{g}}_{\mathsf{G}+} |h_a, h_b, k_{\mathsf{G}}, k_{\mathsf{L}}\rangle = |h_a \tilde{g}^{-1}, \tilde{g} h_b \tilde{g}^{-1}, k_{\mathsf{G}} h_a \tilde{g} h_a^{-1}, k_{\mathsf{L}}\rangle, \tag{D.2}$$

$$L^{\tilde{g}}_{\mathsf{G}-} |h_a, h_b, k_{\mathsf{G}}, k_{\mathsf{L}}\rangle = |h_a, h_b, k_{\mathsf{G}} h_a \tilde{g}^{-1} h_a^{-1}, k_{\mathsf{L}}\rangle. \tag{D.3}$$

This gives for the action of the L projectors in our new basis

$$\frac{1}{|G|} \sum_{\tilde{g}} L^{\tilde{g}}_{\mathsf{G}+} |\sigma_{mn}, h, \pi_{ij}, k_{\mathsf{L}}\rangle$$

$$= \frac{\sqrt{d_\pi d_\sigma}}{|G|^2} \sum_{g_\pi, g_\sigma, \tilde{g}} [\pi(g_\pi g_\sigma)]_{ij} [\sigma(g_\sigma)]_{mn} |g_\sigma \tilde{g}^{-1}, \tilde{g} g_\sigma^{-1} h g_\sigma \tilde{g}^{-1}, g_\pi g_\sigma \tilde{g} g_\sigma^{-1}, k_{\mathsf{L}}\rangle. \tag{D.4}$$

Introducing the substitutions

$$g'_\sigma = g_\sigma \tilde{g}^{-1}, \tag{D.5}$$

$$g'_\pi = g_\pi g_\sigma \tilde{g} g_\sigma^{-1}, \tag{D.6}$$

we find that

$$\frac{1}{|G|} \sum_{\tilde{g}} L^{\tilde{g}}_{\mathsf{G}+} |\sigma_{mn}, h, \pi_{ij}, k_{\mathsf{L}}\rangle$$

$$= \frac{\sqrt{d_\pi d_\sigma}}{|G|^2} \sum_{g'_\pi, g'_\sigma, \tilde{g}} \sum_s [\pi(g'_\pi g'_\sigma)]_{ij} [\sigma(g'_\sigma)]_{ms} [\sigma(\tilde{g})]_{sn} |g'_\sigma, g'^{-1}_\sigma h g'_\sigma, g'_\pi, k_{\mathsf{L}}\rangle. \tag{D.7}$$

The grand orthogonality theorem (E.10) can then be used over $\tilde{g}$ to give

$$\frac{1}{|G|} \sum_{\tilde{g}} L^{\tilde{g}}_{\mathsf{G}+} |\sigma_{mn}, h, \pi_{ij}, k_{\mathsf{L}}\rangle = \delta_{\sigma I} \frac{\sqrt{d_\pi d_\sigma}}{|G|} \sum_{g'_\pi, g'_\sigma} [\pi(g'_\pi g'_\sigma)]_{ij} [\sigma(g'_\sigma)]_{11} |g'_\sigma, g'^{-1}_\sigma h g'_\sigma, g'_\pi, k_{\mathsf{L}}\rangle \tag{D.8}$$

$$= \delta_{\sigma I} |\sigma_{mn}, h, \pi_{ij}, k_{\mathsf{L}}\rangle, \tag{D.9}$$

where I is the trivial irrep of the group G. We can perform a similar treatment for the other L projector, giving

$$\frac{1}{|G|} \sum_{\tilde{g}} L^{\tilde{g}}_{\mathsf{G}-} |\sigma_{mn}, h, \pi_{ij}, k_{\mathsf{L}}\rangle$$

$$= \frac{\sqrt{d_\pi d_\sigma}}{|G|^2} \sum_{g'_\pi, g_\sigma, \tilde{g}} \sum_s [\pi(g'_\pi g_\sigma)]_{is} [\pi(\tilde{g}^{-1})]_{sj} [\sigma(g_\sigma)]_{mn} |g_\sigma, g_\sigma^{-1} h g_\sigma, g'_\pi, k_{\mathsf{L}}\rangle \tag{D.10}$$

with substitutions

$$g'_\pi = g_\pi g_\sigma \tilde{g} g_\sigma^{-1}. \tag{D.11}$$

Again, this can be simplified through orthogonality to give

$$\frac{1}{|G|}\sum_{\tilde{g}} L^{\tilde{g}}_{\mathsf{G}-}|\sigma_{mn}, h, \pi_{ij}, k_\mathsf{L}\rangle = \delta_{\pi I}\frac{\sqrt{d_\pi d_\sigma}}{|G|}\sum_{g'_\pi, g_\sigma}[\pi(g'_\pi g_\sigma)]_{11}[\sigma(g_\sigma)]_{mn}|g_\sigma, g_\sigma^{-1}hg_\sigma, g'_\pi, k_\mathsf{L}\rangle \tag{D.12}$$

$$= \delta_{\pi I}|\sigma_{mn}, h, \pi_{ij}, k_\mathsf{L}\rangle. \tag{D.13}$$

To treat the $T_{G\pm}$ operators analogously, we need to look at another basis in which the gauge state takes a particular group element as opposed to an element of a representation. We define this gauge element basis as

$$|\sigma_{mn}, h, k_\mathsf{G}, k_\mathsf{L}\rangle \equiv \frac{\sqrt{d_\sigma}}{\sqrt{|G|}}\sum_{g_\sigma\in\mathcal{G}}[\sigma(g_\sigma)]_{mn}|g_\sigma, g_\sigma^{-1}hg_\sigma, k_\mathsf{G}, k_\mathsf{L}\rangle, \tag{D.14}$$

i.e. instead of using a representation for the gauge degree of freedom, here a single group element is used. In terms of the physical basis, the $T^1_{G\pm}$ projectors act in the following way:

$$T^1_{\mathsf{G}+}|h_a, h_b, k_\mathsf{G}, k_\mathsf{L}\rangle = \delta_{1k_\mathsf{G}}|h_a, h_b, k_\mathsf{G}, k_\mathsf{L}\rangle, \tag{D.15}$$

$$T^1_{\mathsf{G}-}|h_a, h_b, k_\mathsf{G}, k_\mathsf{L}\rangle = \delta_{(h_a h_b h_a^{-1})k_\mathsf{G}}|h_a, h_b, k_\mathsf{G}, k_\mathsf{L}\rangle, \tag{D.16}$$

where 1 is the identity group element. We are now in a position to demonstrate that the T operators explicitly project to single states in the gauge element basis:

$$T^1_{\mathsf{G}+}|\sigma_{mn}, h, k_\mathsf{G}, k_\mathsf{L}\rangle = \delta_{1k_\mathsf{G}}\frac{\sqrt{d_\sigma}}{\sqrt{|G|}}\sum_{g_\sigma}[\sigma(g_\sigma)]_{mn}|k_\mathsf{G}, \sigma_{mn}, h, k_\mathsf{L}\rangle \tag{D.17}$$

$$= \delta_{ek_\mathsf{G}}|R_{mn}, h, k_\mathsf{G}, k_\mathsf{L}\rangle, \tag{D.18}$$

$$T^1_{\mathsf{G}-}|\sigma_{mn}, h, k_\mathsf{G}, k_\mathsf{L}\rangle = \delta_{(g_\sigma g_\sigma^{-1}hg_\sigma g_\sigma^{-1})k_\mathsf{G}}\frac{\sqrt{d_\sigma}}{\sqrt{|G|}}\sum_{g_\sigma}[\sigma(g_\sigma)]_{mn}|g_\sigma, g_\sigma^{-1}hg_\sigma, k_\mathsf{G}, k_\mathsf{L}\rangle \tag{D.19}$$

$$= \delta_{hk_\mathsf{G}}|\sigma_{mn}, h, k_\mathsf{G}, k_\mathsf{L}\rangle. \tag{D.20}$$

D.2. Solving the Hamiltonian

The Hamiltonian of interest is given by

$$H = -\sum_g L^g_{\mathsf{G}+} - \sum_g L^g_{\mathsf{G}-} - T^1_{\mathsf{G}+} - T^1_{\mathsf{G}-}. \tag{D.21}$$

Given that we now know explicitly to what states each of these terms project, we can write instead

$$H = -\sum_{\pi_{ij},h,k_\mathsf{L}}|I_{11}, h, \pi_{ij}, k_\mathsf{L}\rangle\langle I_{11}, h, \pi_{ij}, k_\mathsf{L}| - \sum_{\sigma_{mn},h,k_\mathsf{L}}|\sigma_{mn}, h, I_{11}, k_\mathsf{L}\rangle\langle\sigma_{mn}, h, I_{11}, k_\mathsf{L}|$$
$$-\sum_{\sigma_{mn},h,k_\mathsf{L}}|\sigma_{mn}, h, 1, k_\mathsf{L}\rangle\langle\sigma_{mn}, h, 1, k_\mathsf{L}| - \sum_{\sigma_{mn},h,k_\mathsf{L}}|\sigma_{mn}, h, h, k_\mathsf{L}\rangle\langle\sigma_{mn}, h, h, k_\mathsf{L}|. \tag{D.22}$$

Each of these terms are pairwise orthogonal for different values of h or k_{L}. We can then immediately block diagonalize the Hamiltonian by these two variables. As these labels will not participate overtly in the calculation, we will suppress them henceforth for clarity. We can also block diagonalize by a representation and one of its indices as follows:

$$H_{\sigma,n}^{h,k_{\mathrm{L}}} = -\sum_m \Big[|I_{11}, \sigma_{mn}\rangle\langle I_{11}, \sigma_{mn}| + |\sigma_{mn}, I_{11}\rangle\langle \sigma_{mn}, I_{11}| + |\sigma_{mn}, 1\rangle\langle \sigma_{mn}, 1| + |\sigma_{mn}, h\rangle\langle \sigma_{mn}, h| \Big]. \tag{D.23}$$

That the projectors in each block are orthogonal to those in different blocks can be verified by examining the inner products presented in section D.3.

Looking at only one block of the Hamiltonian, we can split the terms into two vectors:

$$\begin{aligned} -H_{\sigma,n}^{h,k_{\mathrm{L}}} = &\Big[\sum_m |I_{11}, \sigma_{mn}\rangle\langle I_{11}, \sigma_{mn}| + \sum_m |\sigma_{mn}, 1\rangle\langle \sigma_{mn}, 1| \Big] \\ &+ \Big[\sum_m |\sigma_{mn}, I_{11}\rangle\langle \sigma_{mn}, I_{11}| + \sum_m |\sigma_{mn}, h\rangle\langle \sigma_{mn}, h| \Big]. \end{aligned} \tag{D.24}$$

We can bound the length of each of these vectors individually by considering them as sub-Hamiltonians:

$$-H_A = \sum_m |I_{11}, \sigma_{mn}\rangle\langle I_{11}, \sigma_{mn}| + \sum_m |\sigma_{mn}, 1\rangle\langle \sigma_{mn}, 1|, \tag{D.25}$$

$$-H_B = \sum_m |\sigma_{mn}, I_{11}\rangle\langle \sigma_{mn}, I_{11}| + \sum_m |\sigma_{mn}, h\rangle\langle \sigma_{mn}, h|. \tag{D.26}$$

Each of these sub-Hamiltonians can be further block diagonalized by m:

$$-H_A^m = |I_{11}, \sigma_{mn}\rangle\langle I_{11}, \sigma_{mn}| + |\sigma_{mn}, 1\rangle\langle \sigma_{mn}, 1|, \tag{D.27}$$

$$-H_B^m = |\sigma_{mn}, I_{11}\rangle\langle \sigma_{mn}, I_{11}| + |\sigma_{mn}, h\rangle\langle \sigma_{mn}, h|. \tag{D.28}$$

We can solve these sub-Hamiltonians in each block by calculating the overlap of the two projectors (these calculations are shown in section D.3):

$$\langle I_{11}, \sigma_{mn}|\sigma_{mn}, 1\rangle = \frac{1}{\sqrt{|G|}}, \tag{D.29}$$

$$\langle \sigma_{mn}, I_{11}|\sigma_{mn}, h\rangle = \frac{1}{\sqrt{|G|}}. \tag{D.30}$$

It can be shown that the eigenvalues of these sub-Hamiltonians will then be

$$\lambda_m = -1 \pm |C|, \tag{D.31}$$

where C is the inner product calculated above. As such, we can say that the norms of the sub-Hamiltonians are equal, independent of m block, and have the value $1 + \frac{1}{\sqrt{|G|}}$. Importantly, their length is independent of the choice of σ_{mn}, π_{ij}, h and k_{L}.

In terms of the full Hamiltonian and block structure, we have two vectors whose lengths are constant for a given group. We can then bound the eigenvalues of the full Hamiltonian as

follows. We can use a triangle inequality argument to give

$$\|H_{\sigma,n}^{h,k_{\mathsf{L}}}\| \leqslant \Big\| \sum_m |I_{11}, \sigma_{mn}\rangle\langle I_{11}, \sigma_{mn}| + \sum_m |\sigma_{mn}, 1\rangle\langle\sigma_{mn}, 1| \Big\| + \Big\| \sum_m |\sigma_{mn}, I_{11}\rangle\langle\sigma_{mn}, I_{11}| + \sum_m |\sigma_{mn}, h\rangle\langle\sigma_{mn}, h| \Big\|. \tag{D.32}$$

We know that each individual vector has a constant length. We also know that this inequality will only be saturated when the two vectors are parallel (which will only happen when they are equal). This is also when $\sigma = I$, $n = 1$ and $h = e$. As desired, there is no dependence on k_{L}.

Since the eigenvectors of H are nonpositive, the largest magnitude eigenvalue will correspond to the ground space. The triangle inequality argument amounts to showing that the ground state must rest in the block defined by $H_{I,1}^{1,k_{\mathsf{L}}}$. By inspection, there will be two independent eigenvectors in this block for each value of k_{L} (we return the h index, but the k_{L} index remains suppressed here)

$$|\psi_\pm\rangle = |I_{11}, 1, 1\rangle \pm |I_{11}, 1, I_{11}\rangle \tag{D.33}$$

with eigenvalues

$$\lambda_\pm = -2\left(1 \pm \frac{1}{\sqrt{|G|}}\right).$$

Clearly $|\psi_+\rangle$ defines a $|G|$-fold degenerate ground space of our system. It is simple to verify that the stabilizers have the desired relations with this state

$$\sum_g S_L^g |\psi_+\rangle = |\psi_+\rangle, \tag{D.34}$$

$$S_T^1 |\psi_+\rangle = |\psi_+\rangle \tag{D.35}$$

and thus the Hamiltonian (78) behaves as claimed. □

D.3. Useful inner products

Here we wish to calculate inner products between states in the gauge representation basis and the gauge element basis. In general, we have

$$\langle\sigma_{mn}, h, \pi_{ij}, k_{\mathsf{L}}|\sigma'_{m'n'}, h', k_{\mathsf{G}}, k'_{\mathsf{L}}\rangle = \frac{\sqrt{d_\pi d_{\sigma'} d_\sigma}}{|G|^{3/2}} \sum_{g_\pi, g_\sigma, g_{\sigma'}} [\pi(g_\pi g_\sigma)]^*_{ij} [\sigma(g_\sigma)]^*_{mn} [\sigma'(g_{\sigma'})]_{m'n'} \times \langle g_\sigma, g_\sigma^{-1} h g_\sigma, g_\pi, k_{\mathsf{L}}|g_{\sigma'}, g_{\sigma'}^{-1} h' g_{\sigma'}, k_{\mathsf{G}}, k'_{\mathsf{L}}\rangle \tag{D.36}$$

$$= \frac{\sqrt{d_\pi d_{\sigma'} d_\sigma}}{|G|^{3/2}} \sum_{g_\pi, g_\sigma, g_{\sigma'}} [\pi(g_\pi g_\sigma)]^*_{ij} [\sigma(g_\sigma)]^*_{mn} \sigma'(g_{\sigma'})]_{m'n'} \delta_{g_\sigma g_{\sigma'}} \delta_{hh'} \delta_{g_\pi k_{\mathsf{G}}} \delta_{k_{\mathsf{L}} k'_{\mathsf{L}}} \tag{D.37}$$

$$= \frac{\sqrt{d_\pi d_{\sigma'} d_\sigma}}{|G|^{3/2}} \delta_{hh'} \delta_{k_{\mathsf{L}} k'_{\mathsf{L}}} \sum_{g_\sigma} [\pi(k_{\mathsf{G}} g_\sigma)]^*_{ij} [\sigma(g_\sigma)]^*_{mn} [\sigma'(g_\sigma)]_{m'n'}. \tag{D.38}$$

In particular, the special cases we are interested can be simplified. With I the trivial representation, we have

$$\langle \sigma_{mn}, h, I_{11}, k_{\mathsf{L}} | \sigma'_{m'n'}, h', k_{\mathsf{G}}, k'_{\mathsf{L}} \rangle = \frac{\sqrt{d_{\sigma'} d_{\sigma}}}{|G|^{3/2}} \delta_{hh'} \delta_{k_{\mathsf{L}} k'_{\mathsf{L}}} \sum_{g_\sigma} [\sigma(g_\sigma)]^*_{mn} [\sigma'(g_\sigma)]_{m'n'} \tag{D.39}$$

$$= \frac{1}{|G|^{1/2}} \delta_{hh'} \delta_{k_{\mathsf{L}} k'_{\mathsf{L}}} \delta_{\sigma\sigma'} \delta_{mm'} \delta_{nn'} \qquad \forall k_{\mathsf{G}} \tag{D.40}$$

and

$$\langle I_{11}, h, \pi_{ij}, k_{\mathsf{L}} | \sigma'_{m'n'}, h', k_{\mathsf{G}}, k'_{\mathsf{L}} \rangle = \frac{\sqrt{d_{\pi} d_{\sigma'}}}{|G|^{3/2}} \delta_{hh'} \delta_{k_{\mathsf{L}} k'_{\mathsf{L}}} \sum_{g_\sigma} [\pi(k_{\mathsf{G}} g_\sigma)]^*_{ij} [\sigma'(n_R)]_{m'n'} \tag{D.41}$$

$$= \frac{\sqrt{d_{\pi} d_{\sigma'}}}{|G|^{3/2}} \delta_{hh'} \delta_{k_{\mathsf{L}} k'_{\mathsf{L}}} \sum_{g_\sigma} \sum_{s} [\pi(k_{\mathsf{G}})]^*_{is} [\pi(g_\sigma)]^*_{sj} [\sigma'(n_R)]_{m'n'} \tag{D.42}$$

$$= \frac{1}{|G|^{1/2}} \delta_{hh'} \delta_{k_{\mathsf{L}} k'_{\mathsf{L}}} \delta_{\pi\sigma'} \delta_{jn'} [\pi(k_{\mathsf{G}})]^*_{im'}, \tag{D.43}$$

particularly,

$$\langle I_{11}, h, \pi_{ij}, k_{\mathsf{L}} | \sigma'_{m'n'}, h', 1, k'_{\mathsf{L}} \rangle = \frac{1}{|G|^{1/2}} \delta_{hh'} \delta_{k_{\mathsf{L}} k'_{\mathsf{L}}} \delta_{GR'} \delta_{im'} \delta_{jn'} \qquad \forall k_{\mathsf{G}}. \tag{D.44}$$

The orthonormality of the gauge representation basis is also useful when written in the form

$$\langle I_{11}, h, \pi_{ij}, k_{\mathsf{L}} | \sigma_{mn}, h', I_{11}, k'_{\mathsf{L}} \rangle = \delta_{\pi\sigma} \delta_{\sigma I} \delta_{i1} \delta_{nj} \delta_{m1} \delta_{n1} \delta_{hh'} \delta_{k_{\mathsf{L}} k'_{\mathsf{L}}}. \tag{D.45}$$

Appendix E. Error operations in general quantum double models

In the case of cyclic quantum double models, we were able to provide a simple general argument as to why many of the terms in the effective Hamiltonian vanished. In the more general case, this kind of simple argument is no longer applicable, and so we are forced to take an exhaustive survey of the terms that may arise in the effective Hamiltonian. We aim to show that any undesired terms will vanish in the ground space.

In order to undertake this study, it is useful to note the following relations:

$$Z^{\pi_{ij}}_{\pm} L^{g}_{\pm} = \sum_{k} [\pi(g)]_{ik} L^{g}_{\pm} Z^{\pi_{kj}}_{\pm}, \tag{E.1}$$

$$Z^{\pi_{ij}}_{\pm} L^{g}_{\mp} = \sum_{k} L^{g}_{\mp} Z^{\pi_{ik}}_{\pm} [\pi(g^{-1})]_{kj}. \tag{E.2}$$

The kinds of operators we consider will take the form

$$\begin{aligned}
z_a &= \left(\begin{array}{c|c} Z^{\pi_{ij}}_{-} & Z^{\sigma_{kl}}_{+} \\ I & I \end{array}\right), \quad z_b = \left(\begin{array}{c|c} I & I \\ Z^{\pi_{ij}}_{-} & Z^{\sigma_{kl}}_{+} \end{array}\right), \quad z_c = \left(\begin{array}{c|c} I & Z^{\pi_{ij}}_{+} \\ I & Z^{\sigma_{kl}}_{+} \end{array}\right), \quad z_d = \left(\begin{array}{c|c} Z^{\pi_{ij}}_{-} & I \\ Z^{\sigma_{kl}}_{-} & I \end{array}\right), \\
l_a &= \left(\begin{array}{c|c} L^{g}_{+} & L^{g'}_{+} \\ I & I \end{array}\right), \quad l_b = \left(\begin{array}{c|c} I & I \\ L^{g}_{-} & L^{g'}_{-} \end{array}\right), \quad l_c = \left(\begin{array}{c|c} I & L^{g}_{+} \\ I & L^{g'}_{-} \end{array}\right), \quad l_d = \left(\begin{array}{c|c} L^{g}_{+} & I \\ L^{g'}_{-} & I \end{array}\right).
\end{aligned} \tag{E.3}$$

We will refer to these as 'error terms' or 'error operators'. For each error term $\hat{E}$, we will calculate $\Upsilon\hat{E}\Upsilon$, where Υ is the projector on to the ground state. Note that $\Upsilon = S\Upsilon$ for either of the stabilizers S_T or S_L. First consider the error term z_c:

$$\Upsilon z_c \Upsilon = \Upsilon z_c S_L \Upsilon \tag{E.4}$$

$$= \frac{1}{|G|}\sum_g \Upsilon z_c S_L^g \Upsilon \tag{E.5}$$

$$= \frac{1}{|G|}\sum_g \Upsilon \left(\begin{array}{c|c} L_-^{g^{-1}} & Z_+^{\pi_{ij}} L_-^{g^{-1}} \\ L_+^{g^{-1}} & Z_+^{\sigma_{kl}} L_+^{g^{-1}} \end{array}\right) \Upsilon \tag{E.6}$$

$$= \frac{1}{|G|}\sum_g \Upsilon (S_L^{g^{-1}})^\dagger \sum_{m,n} [\pi(g)]_{mj}[\sigma(g^{-1})]_{kn} \left(\begin{array}{c|c} I & Z_+^{\pi_{im}} \\ I & Z_+^{\sigma_{nl}} \end{array}\right) \Upsilon. \tag{E.7}$$

Now, because the ground state is stabilized by the projector S_L, it is easy to check that it must also be stabilized by each of S_L^g individually. This allows us to write

$$\Upsilon z_c \Upsilon = \frac{1}{|G|}\Upsilon \sum_{m,n,g} [\pi(g)]_{mj}[\sigma(g^{-1})]_{kn} \left(\begin{array}{c|c} I & Z_+^{\pi_{im}} \\ I & Z_+^{\sigma_{nl}} \end{array}\right) \Upsilon, \tag{E.8}$$

$$= \frac{1}{|G|}\Upsilon \sum_{m,n,g} [\pi(g)]_{mj}[\sigma(g)]^*_{nk} \left(\begin{array}{c|c} I & Z_+^{\pi_{im}} \\ I & Z_+^{\sigma_{nl}} \end{array}\right) \Upsilon, \tag{E.9}$$

where we have used the unitarity of the representation σ in the last line. Now we make use of the grand orthogonality theorem for unitary representations:

$$\sum_g [\pi(g)]^*_{ij}[\sigma(g)]_{kl} = \frac{|G|}{d_\pi}\delta_{\pi\sigma}\delta_{ik}\delta_{jl} \tag{E.10}$$

and we can write

$$\Upsilon z_c \Upsilon = \Upsilon \sum_{m,n} \frac{1}{d_\pi}\delta_{\pi\sigma}\delta_{mn}\delta_{jk} \left(\begin{array}{c|c} I & Z_+^{\pi_{im}} \\ I & Z_+^{\sigma_{nl}} \end{array}\right) \Upsilon \tag{E.11}$$

$$= \Upsilon \sum_{m} \frac{1}{d_\pi}\delta_{\pi\sigma}\delta_{jk} \left(\begin{array}{c|c} I & Z_+^{\pi_{im}} \\ I & Z_+^{\sigma_{ml}} \end{array}\right) \Upsilon. \tag{E.12}$$

This gives us the result that all errors of this type will vanish from the effective Hamiltonian unless they satisfy the conditions $\pi = \sigma$ and $j = k$. It also allows us to rewrite an operator of the form z_c satisfying these conditions as an average over the shared index in the ground state. This will become important when we performed the perturbation calculations.

We can find analogous results for the other z-error terms,

$$\Upsilon z_d \Upsilon = \Upsilon \sum_{m,n,g} [\pi(g^{-1})]_{im}[\sigma(g)]_{nl} \left(\begin{array}{c|c} Z_-^{\pi_{mj}} & I \\ Z_-^{\sigma_{kn}} & I \end{array}\right) \Upsilon \tag{E.13}$$

$$= \Upsilon \sum_m \frac{1}{d_\pi} \delta_{\pi\sigma}\delta_{il} \left(\begin{array}{c|c} Z_-^{\pi_{mj}} & I \\ Z_-^{\sigma_{km}} & I \end{array} \right) \Upsilon, \tag{E.14}$$

$$\Upsilon z_a \Upsilon = \Upsilon \sum_{m,n,g} [\pi(g^{-1})]_{im} [\sigma(g)]_{nl} \left(\begin{array}{c|c} Z_-^{\pi_{mj}} & Z_+^{\sigma_{kn}} \\ I & I \end{array} \right) \Upsilon \tag{E.15}$$

$$= \Upsilon \sum_m \frac{1}{d_\pi} \delta_{\pi\sigma}\delta_{il} \left(\begin{array}{c|c} Z_-^{\pi_{mj}} & Z_+^{\sigma_{km}} \\ I & I \end{array} \right) \Upsilon, \tag{E.16}$$

$$\Upsilon z_b \Upsilon = \Upsilon \sum_{m,n,g} [\pi(g)]_{mj} [\sigma(g^{-1})]_{kn} \left(\begin{array}{c|c} I & I \\ Z_-^{\pi_{im}} & Z_+^{\sigma_{nl}} \end{array} \right) \Upsilon \tag{E.17}$$

$$= \Upsilon \sum_m \frac{1}{d_\pi} \delta_{\pi\sigma}\delta_{jk} \left(\begin{array}{c|c} I & I \\ Z_-^{\pi_{im}} & Z_+^{\sigma_{ml}} \end{array} \right) \Upsilon. \tag{E.18}$$

The l error terms require a slightly different treatment. Consider first l_a:

$$\Upsilon l_a \Upsilon = \Upsilon l_a S_T^1 \Upsilon \tag{E.19}$$

$$= \Upsilon \sum_{g_1 g_2 g_3 g_4 = 1} \left(\begin{array}{c|c} L_+^g T_-^{g_1} & L_+^{g'} T_+^{g_2} \\ T_-^{g_4} & T_+^{g_3} \end{array} \right) \Upsilon \tag{E.20}$$

$$= \Upsilon \sum_{g_1 g_2 g_3 g_4 = 1} \left(\begin{array}{c|c} T_-^{g_1 g^{-1}} & T_+^{g' g_2} \\ T_-^{g_4} & T_+^{g_3} \end{array} \right) l_a \Upsilon \tag{E.21}$$

$$= \Upsilon \sum_{g_1 g g'^{-1} g_2 g_3 g_4 = 1} \left(\begin{array}{c|c} T_-^{g_1} & T_+^{g_2} \\ T_-^{g_4} & T_+^{g_3} \end{array} \right) l_a \Upsilon. \tag{E.22}$$

The first operator is now orthogonal to the projector S_T^1 unless $g' g^{-1} = 1$. Because the ground space is stabilized by S_T^1, we can say that this vanishes unless $g = g'$. This gives

$$\Upsilon l_a \Upsilon = \Upsilon \delta_{gg'} \left(\begin{array}{c|c} L_+^g & L_+^{g'} \\ I & I \end{array} \right) \Upsilon. \tag{E.23}$$

i.e. error terms of the form l_a will vanish unless they obey $g = g'$. The procedure for l_b proceeds similarly to yield

$$\Upsilon l_b \Upsilon = \Upsilon \sum_{g_1 g_2 g_3 g' g^{-1} g_4 = 1} \left(\begin{array}{c|c} T_-^{g_1} & T_+^{g_2} \\ T_-^{g_4} & T_+^{g_3} \end{array} \right) l_b \Upsilon \tag{E.24}$$

$$= \Upsilon \delta_{gg'} \left(\begin{array}{c|c} I & I \\ L_-^g & L_-^{g'} \end{array} \right) \Upsilon. \tag{E.25}$$

The error operators l_c and l_d are more complicated:

$$\Upsilon l_c \Upsilon = \Upsilon l_c \sum_{g_1 g_2 g_3 g_4 = 1} \left(\begin{array}{c|c} T_-^{g_1} & T_+^{g g_2} \\ T_-^{g_4} & T_+^{g_3 g'^{-1}} \end{array} \right) \Upsilon. \tag{E.26}$$

We can imagine the second operator acting on the state $|h_a, h_b, k_{\mathsf{G}}, k_{\mathsf{L}}\rangle$. This leads to this operator vanishing unless

$$h_a^{-1} g^{-1} k_{\mathsf{L}} g' k_{\mathsf{L}}^{-1} h_a h_b = 1.$$

In the ground subspace, we know $h_b = 1$; this implies that

$$g = k_{\mathsf{L}} g' k_{\mathsf{L}}^{-1}, \tag{E.27}$$

i.e. the nonvanishing operators of the form l_c depend explicitly on the logical state. A similar result is obtained for the error term l_d:

$$\Upsilon l_d \Upsilon = \Upsilon l_d \sum_{g_1 g_2 g_3 g_4 = 1} \left(\begin{array}{c|c} T_-^{g_1 g^{-1}} & T_+^{g_2} \\ T_-^{g' g_4} & T_+^{g_3} \end{array} \right) \Upsilon. \tag{E.28}$$

The second operator vanishes unless

$$h_a^{-1} g k_{\mathsf{L}} g'^{-1} k_{\mathsf{L}}^{-1} h_a h_b = 1. \tag{E.29}$$

In the codespace this is equivalent to

$$g = k_{\mathsf{L}} g' k_{\mathsf{L}}^{-1}. \tag{E.30}$$

There are two remaining kinds of terms that will arise in our perturbation treatment. They may take a diagonal form, e.g.

$$\hat{E} \sim \left(\begin{array}{c|c} \hat{A} & I \\ I & \hat{B} \end{array} \right),$$

in which case they will not contribute to low-order terms (and will have little impact on the form of higher-order terms). A similar treatment to appendix E and equations (95)–(104) reveals that these operators act as both gauge and logical operators, just as in the cyclic case.

Alternatively, error operators may consist of mixed L and Z operators, e.g.

$$\hat{E} \sim \left(\begin{array}{c|c} Z_-^{\pi_{ij}} & L_+^{g} \\ I & I \end{array} \right).$$

It is a simple calculation to show that these kinds of operators will vanish except when $\pi = I$ (the trivial representation) and $g = 1$. That is, the only nonvanishing operator of this form is the identity operator. These results show that the encoding we use will indeed prevent undesirable excitations from being permitted, leaving only the terms from the target Hamiltonian to arise in our perturbative treatment.

References

[1] Dennis E, Kitaev A, Landahl A and Preskill J 2002 Topological quantum memory *J. Math. Phys.* **43** 4452–505

[2] Wang C, Harrington J and Preskill J 2003 Confinement–Higgs transition in a disordered gauge theory and the accuracy threshold for quantum memory *Ann. Phys.* **303** 31–58

[3] Duclos-Cianci G and Poulin D 2010 Fast decoders for topological quantum codes *Phys. Rev. Lett.* **104** 050504

[4] Raussendorf R and Harrington J 2007 Fault-tolerant quantum computation with high threshold in two dimensions *Phys. Rev. Lett.* **98** 190504

[5] Raussendorf R, Harrington J and Goyal K 2006 A fault-tolerant one-way quantum computer *Ann. Phys.* **321** 2242–70

[6] Raussendorf R, Harrington J and Goyal K 2007 Topological fault-tolerance in cluster state quantum computation *New J. Phys.* **9** 199

[7] Kitaev A Y 2003 Fault-tolerant quantum computation by anyons *Ann. Phys.* **303** 2–30

[8] Mochon C 2003 Anyons from nonsolvable finite groups are sufficient for universal quantum computation *Phys. Rev.* A **67** 022315

[9] Mochon C 2004 Anyon computers with smaller groups *Phys. Rev.* A **69** 032306

[10] Preskill J 1998 *Fault-Tolerant Quantum Computation* (Singapore: World Scientific)

[11] Ogburn W R and Preskill J 1999 Topological quantum computation ed C Williams *Quantum Computing and Quantum Communications* (*Lecture Notes in Computer Science* vol 1509) (Berlin: Springer) pp 341–56

[12] Levin M A and Wen X-G 2005 String-net condensation: a physical mechanism for topological phases *Phys. Rev.* B **71** 045110

[13] Kitaev A 2006 Anyons in an exactly solved model and beyond *Ann. Phys.* **321** 2–111 (January Special Issue)

[14] Kempe J, Kitaev A and Regev O 2006 The complexity of the local Hamiltonian problem *SIAM J. Comput.* **35** 1070–97

[15] Bravyi S, DiVincenzo D P, Loss D and Terhal B M 2008 Quantum simulation of many-body Hamiltonians using perturbation theory with bounded-strength interactions *Phy. Rev. Lett.* **101** 070503

[16] Jordan S P and Farhi E 2008 Perturbative gadgets at arbitrary orders *Phys. Rev.* A **77** 062329

[17] Oliveira R and Terhal B M 2008 The complexity of quantum spin systems on a two-dimensional square lattice *Quantum Inf. Comput.* **8** 0900–24

[18] Bravyi S, Hastings M B and Michalakis S 2010 Topological quantum order: stability under local perturbations *J. Math. Phys.* **51** 093512

[19] Bravyi S and Hastings M B 2010 A short proof of stability of topological order under local perturbations arXiv:1001.4363

[20] Nayak C, Simon S H, Stern A, Freedman M and Das Sarma S 2008 Non-abelian anyons and topological quantum computation *Rev. Mod. Phys.* **80** 1083–159

[21] Aguado M 2011 From entanglement renormalisation to the disentanglement of quantum double models arXiv:1101.0527

[22] Affleck I, Kennedy T, Lieb E H and Tasaki H 1988 Valence bond ground states in isotropic quantum antiferromagnets *Comm. Math. Phys.* **115** 477–528

[23] Fannes M, Nachtergaele B and Werner R F 1992 Finitely correlated states on quantum spin chains *Commun. Math. Phys.* **144** 443

[24] Hastings M B 2006 Solving gapped Hamiltonians locally *Phys. Rev.* B **73** 085115

[25] Pérez-García D, Verstraete F, Wolf M M and Cirac J I 2007 Matrix product state representations *Quantum Inf. Comput.* **7** 401

[26] Verstraete F, Wolf M, Pérez-García D and Cirac J I 2006 Projected entangled states: properties and applications *Int. J. Mod. Phys.* B **20** 5142–53

[27] Vidal G 2003 Efficient classical simulation of slightly entangled quantum computations *Phys. Rev. Lett.* **91** 147902

[28] Hastings M B 2007 An area law for one-dimensional quantum systems *J. Stat. Mech.* **8** P08024

[29] Verstraete F, Wolf M M, Pérez-García D and Cirac J I 2006 Criticality, the area law, and the computational power of projected entangled pair states *Phys. Rev. Lett.* **96** 220601
[30] Schuch N, Cirac I and Pérez-García D 2010 Peps as ground states: degeneracy and topology *Ann. Phys.* **325** 2153–92
[31] Gu Z-C, Levin M, Swingle B and Wen X-G 2009 Tensor-product representations for string-net condensed states *Phys. Rev.* B **79** 085118
[32] Buerschaper O, Aguado M and Vidal G 2009 Explicit tensor network representation for the ground states of string-net models *Phys. Rev.* B **79** 085119
[33] Buerschaper O and Aguado M 2009 Mapping Kitaev's quantum double lattice models to Levin and Wen's string-net models *Phys. Rev.* B **80** 155136
[34] Buerschaper O, Christandl M, Kong L and Aguado M 2010 Electric–magnetic duality and topological order on the lattice arXiv:1006.5823
[35] Van den Nest M, Luttmer K, Dür W and Briegel H J 2008 Graph states as ground states of many-body spin-1/2 Hamiltonians *Phys. Rev.* A **77** 012301
[36] Koenig R 2010 Simplifying quantum double Hamiltonians using perturbative gadgets *Quantum Inf. Comput.* **10** 292–324
[37] Bartlett S D and Rudolph T 2006 Simple nearest-neighbor two-body Hamiltonian system for which the ground state is a universal resource for quantum computation *Phys. Rev.* A **74** 040302
[38] Griffin T and Bartlett S D 2008 Spin lattices with two-body Hamiltonians for which the ground state encodes a cluster state *Phys. Rev.* A **78** 062306
[39] Raussendorf R and Briegel H J 2001 A one-way quantum computer *Phys. Rev. Lett.* **86** 5188–91
[40] Shor P W 1995 Scheme for reducing decoherence in quantum computer memory *Phys. Rev.* A **52** R2493–6
[41] Steane A M 1996 Error correcting codes in quantum theory *Phys. Rev. Lett.* **77** 793–7
[42] Gottesman D 1997 Stabilizer codes and quantum error correction *PhD Thesis* California University of Technology arXiv:quant-ph/9705052
[43] Kribs D, Laflamme R and Poulin D 2005 Unified and generalized approach to quantum error correction *Phys. Rev. Lett.* **94** 180501
[44] Kribs D W, Laflamme R, Poulin D and Lesosky M Operator quantum error correction *Quantum Inf. Comput.* **6** 383–99
[45] Bombin H 2010 Topological subsystem codes *Phys. Rev.* A **81** 032301
[46] Bombin H, Kargarian M and Martin-Delgado M A 2009 Quantum 2-body Hamiltonian for topological color codes *Fortschr. Phys.* **57** 1103
[47] Kargarian M, Bombin H and Martin-Delgado M A 2010 Topological color codes and two-body quantum lattice Hamiltonians *New J. Phys.* **12** 025018
[48] Sarvepalli P 2010 Topological color codes over higher alphabet *IEEE Information Theory Workshop (Dublin, August 30–September 3)* pp 1–5
[49] Suchara M, Bravyi S and Terhal B M 2011 Constructions and noise threshold of topological subsystem codes *J. Phys. A: Math. Theor.* **44** 155301
[50] Hamma A, Ionicioiu R and Zanardi P 2005 Bipartite entanglement and entropic boundary law in lattice spin systems *Phys. Rev.* A **71** 022315
[51] Kitaev A and Preskill J 2006 Topological entanglement entropy *Phys. Rev. Lett.* **96** 110404
[52] Levin M and Wen X-G 2006 Detecting topological order in a ground state wave function *Phys. Rev. Lett.* **96** 110405
[53] Flammia S T, Hamma A, Hughes T L and Wen X-G 2009 Topological entanglement rényi entropy and reduced density matrix structure *Phys. Rev. Lett.* **103** 261601
[54] Nussinov Z and Ortiz G 2008 Autocorrelations and thermal fragility of anyonic loops in topologically quantum ordered systems *Phys. Rev.* B **77** 064302
[55] Castelnovo C and Chamon C 2007 Entanglement and topological entropy of the toric code at finite temperature *Phys. Rev.* B **76** 184442

[56] Gils C, Trebst S, Kitaev A, Ludwig A W W, Troyer M and Wang Z 2009 Topology-driven quantum phase transitions in time-reversal-invariant anyonic quantum liquids *Natl. Phys.* **5** 834–9
[57] Dusuel S, Schmidt K P and Vidal J J 2008 Creation and manipulation of anyons in the Kitaev model *Phys. Rev. Lett.* **100** 177204
[58] Schmidt K, Dusuel S and Vidal J 2008 Emergent fermions and anyons in the Kitaev model *Phys. Rev. Lett.* **100** 057208
[59] Vidal J, Schmidt K and Dusuel S 2008 Perturbative approach to an exactly solved problem: Kitaev honeycomb model *Phys. Rev.* B **78** 245121
[60] Yang S, Zhou D L and Sun C P 2007 Mosaic spin models with topological order *Phys. Rev.* B **76** 180404
[61] Bacon D 2003 Decoherence, control, and symmetry in quantum computers *PhD Thesis* University of California, Berkeley
[62] Bergman D L, Shindou R, Fiete G A and Balents L 2007 Degenerate perturbation theory of quantum fluctuations in a pyrochlore antiferromagnet *Phys. Rev.* B **75** 094403