ADVANCES IN APPLIED MATHEMATICS 1, 22–36 (1980)

A Novel Approach to the Solution of Boundary-Layer Problems

CARL M. BENDER,* FRED COOPER, AND G. S. GURALNIK†

Theoretical Division, Los Alamos Scientific Laboratory, Los Alamos, New Mexico 87545

ERIC MJOLSNESS

Department of Physics, Washington University, St. Louis, Missouri 63130

HARVEY A. ROSE

Controlled Thermonuclear Research Division, Los Alamos Scientific Laboratory, Los Alamos, New Mexico 87545

AND

DAVID H. SHARP

Theoretical Division, Los Alamos Scientific Laboratory, Los Alamos, New Mexico 87545

By replacing a differential equation boundary-layer problem by its discrete lattice equivalent we are able to treat the resulting equation as a regular perturbation problem. We obtain the solution on the lattice as a regular perturbation series in inverse powers of the lattice spacing. To obtain the answer to the continuum problem we extrapolate the solution to the lattice problem to zero lattice spacing. This extrapolation, which is a Padé-like procedure, yields good numerical results for a wide range of problems.

I. INTRODUCTION

Recently we have been conducting research on the approximate solution of quantum field theories on a lattice and the problem of taking the continuum limit of such theories [1–5]. We have come to realize that some

*Permanent address: Department of Physics, Washington University, St. Louis, Missouri 63130.

†Permanent address: Department of Physics, Brown University, Providence, Rhode Island 02912.

0196-8858/80/010022-15$05.00/0

of the methods we have been using are applicable in a wide area of linear and nonlinear problems having boundary-layer-like features. The purpose of this paper is to illustrate and examine these techniques by solving several standard boundary-layer examples.

The method we use here consists of three stages. First, we convert the differential equation to a difference equation on a discrete lattice in the coordinate possessing boundary-layer structure. In many cases this discretization converts what is typically a singular perturbation problem in the continuum into a regular perturbation problem on the lattice. The second stage consists of solving the lattice equations in the form of regular perturbation series in powers of $1/a$, where a is the lattice spacing. This is a somewhat odd approach because we are ultimately interested in recovering the continuum limit $a \to 0$ and in this limit every term in the perturbation series becomes infinite. This is the way in which the singular nature of the original continuum perturbation problem manifests itself. The third stage consists of a Padé-like extrapolation procedure for obtaining a finite continuum limit of the lattice perturbation series.

Before proceeding further we describe in detail the extrapolation procedure we use in this paper. The essential problem we must deal with is how to extrapolate a perturbation series of the form

$$Q(\epsilon) = \epsilon^{\alpha} \sum_{n=0}^{\infty} a_n \epsilon^n, \qquad \alpha \neq 0, \tag{1.1}$$

which has been derived assuming that ϵ is small, to a finite value at $\epsilon = \infty$. Let us truncate this series after the ϵ^N term. For small ϵ, we can raise the expression

$$\epsilon^{\alpha} \sum_{n=0}^{N} a_n \epsilon^n$$

to the N/α power and write the result in the form

$$\epsilon^N \Big/ \sum_{n=0}^{N} b_n \epsilon^n,$$

where the coefficients b_n are uniquely determined by a_n, $n = 0, 1, \ldots, N$. Now we take the limit $\epsilon \to \infty$ and obtain $1/b_N$. We define

$$Q_N = \left(\frac{1}{b_N}\right)^{\alpha/N} \tag{1.2}$$

as the Nth approximant to the limiting value $Q(\infty)$. In many cases we find that the sequence of approximants Q_N rapidly converges to $Q(\infty)$ [6].

The following problem illustrates this extrapolation technique.

AN ELEMENTARY EXAMPLE. Consider the transcendental equation

$$\ln x + \frac{1}{1-x} = 0. \tag{1.3}$$

To find the root of this equation between 0 and 1 we introduce a perturbation parameter ϵ:

$$\ln x + \frac{1}{1+\epsilon}\left(\frac{\epsilon}{1-x} + 1\right) = 0. \tag{1.4}$$

Note that we recover the original equation (1.3) in the limit $\epsilon \to \infty$. However, it is easiest to solve for $x(\epsilon)$ as a power series in ϵ because the unperturbed problem for $\epsilon = 0$,

$$\ln x + 1 = 0,$$

is trivially solvable. We find the following perturbation series for $x(\epsilon)$:

$$\begin{aligned} x(\epsilon) = \frac{1}{e}(1 &- 0.58198\epsilon + 1.28714\epsilon^2 - 3.16768\epsilon^3 \\ &+ 8.52949\epsilon^4 - 24.64515\epsilon^5 + 75.21698\epsilon^6 \\ &- 239.05397\epsilon^7 + 783.87521\epsilon^8 \\ &- 2633.66832\epsilon^9 + 9021.28163\epsilon^{10} - \cdots). \end{aligned} \tag{1.5}$$

The series in (1.5) is not of the form in (1.1), but it can easily be cast into that form by taking the logarithm of $ex(\epsilon)$:

$$\begin{aligned} -\ln[ex(\omega)] = \epsilon(0.58198 &+ 1.11779\epsilon + 2.48430\epsilon^2 - 6.26489\epsilon^3 \\ &+ 17.40965\epsilon^4 - 51.89367\epsilon^5 + 162.61270\epsilon^6 \\ &- 528.40893\epsilon^7 + 1764.43470\epsilon^8 \\ &- 6017.07240\epsilon^9 + \cdots). \end{aligned} \tag{1.6}$$

The next part of the extrapolation procedure consists of raising the series in parentheses in (1.6) to the powers $-1, -2, -3, -4, \cdots$. It is remarkable that while the coefficients in all of these series ultimately alternate in sign, the first N coefficients in the series raised to the $-N$ power are all positive. This result ensures that the extrapolants Q_N are always real. The extrapolants, when exponentiated, give the following rapidly convergent sequence for x, the root of (1.3):

$$\begin{aligned}
x_1 &= 0.271713639,\\
x_2 &= 0.255145710,\\
x_3 &= 0.260300667,\\
x_4 &= 0.258935592,\\
x_5 &= 0.259336423,\\
x_6 &= 0.259219343,\\
x_7 &= 0.259254556,\\
x_8 &= 0.259243826,\\
x_9 &= 0.259247147,\\
x_{10} &= 0.259246107,\\
x_{11} &= 0.259246436,\\
x_{12} &= 0.259246331,\\
x_{13} &= 0.259246365,\\
x_{14} &= 0.259246353.
\end{aligned}$$

x_{14} is accurate to 1 part in 10^9.

In Sections II through V we show how to use these extrapolation techniques to solve the following singular perturbation problems:

(i) Given

$$m\frac{d^2y}{dt^2} + \beta\frac{dy}{dt} + ky = I_0\delta(t) \qquad \text{(damped linear oscillator)}$$

with $y(0-) = y'(0-) = 0$, find $y'(0+)$;

(ii) given

$$\epsilon^2\frac{d^2y}{dx^2} + y - y^3 = 0 \qquad \text{(``kink'' equation)}$$

with $y(0) = 0$ and $y(+\infty) = 1$, find $y'(0)$;

(iii) given

$$2\epsilon y'''(x) + y(x)y''(x) = 0 \qquad \text{(Blasius equation)}$$

with $y(0) = y'(0) = 0$ and $y'(+\infty) = 1$, find $y''(0)$;

(iv) given

$$U_t = \nu U_{xx} + \delta(x)\delta(t) \qquad \text{(Green's function for diffusion equation)}$$

find $u(0, t)$.

II. Damped Linear Oscillator

We are interested in obtaining the solution to

$$m\frac{d^2y}{dt^2} + \beta\frac{dy}{dt} + ky = I_0\delta(t) \tag{2.1}$$

with $y(0-) = y'(0-) = 0$ and $m > 0$ small. For small m the solution $y(t)$ exhibits a boundary layer at $t = 0$ of thickness m/β. The exact solution to (2.1) satisfies

$$y'(0+) = I_0/m. \tag{2.2}$$

Our objective here is to reproduce the result in (2.2) by expanding (2.1) on the lattice in powers of $\epsilon = m/(\beta a)$, where a is the lattice spacing. On the lattice we replace

$$\begin{aligned} y(t) &\to y_n, \\ y'(t) &\to (y_{n+1} - y_n)/a, \\ y''(t) &\to (y_{n+1} - 2y_n + y_{n-1})/a^2, \\ \delta(t) &\to \delta_{n,0}/a. \end{aligned} \tag{2.3}$$

Thus, the lattice version of (2.1) is

$$\epsilon(y_{n+1} - 2y_n + y_{n-1}) + y_{n+1} - y_n + kay_n/\beta = I_0\delta_{n,0}/\beta. \tag{2.4}$$

When $\epsilon = 0$ the boundary condition on y_n is

$$y_0 = 0.$$

With this boundary condition the solution to (2.4) with $\epsilon = 0$ is

$$\begin{aligned} y_n^{(0)} &= 0, & n &= 0, \\ &= \left(1 - \frac{ka}{\beta}\right)^{n-1}\frac{I_0}{\beta}, & n &> 0. \end{aligned} \tag{2.5}$$

In the continuum limit $na = t$, $n \to \infty$, $a \to 0$, $y_n^{(0)}$ in (2.5) approaches the continuum outer solution; that is, the solution to (2.1) with $m = 0$:

$$y(t) = \frac{I_0}{\beta}e^{-kt/\beta}. \tag{2.6}$$

Now we look for a solution to (2.4) as a regular perturbation series in powers of ϵ:

$$y_n = y_n^{(0)} + \epsilon y_n^{(1)} + \epsilon^2 y_n^{(2)} + \cdots. \tag{2.7}$$

Substituting (2.7) into (2.4) and matching like powers of ϵ gives the following recursion relation for the perturbation coefficients $y_n^{(m)}$:

$$\begin{aligned} y_1^{(m)} &= I_0/\beta \qquad m = 0, \\ &= -y_1^{(m-1)} \qquad m \neq 0, \\ y_{n+1}^{(m)} &= \left(1 - \frac{ka}{\beta}\right) y_n^{(m)} + 2y_n^{(m-1)} - y_{n+1}^{(m-1)} - y_{n-1}^{(m-1)}, \end{aligned} \tag{2.8}$$

where we have used $y_0 = y_{-1} = 0$ as boundary conditions for the full second-order difference equation (2.4).

The solution to (2.8) at the $n = 1$ lattice point is the series

$$y_1 = \frac{ID}{\beta}(1 - \epsilon + \epsilon^2 - \epsilon^3 + \cdots). \tag{2.9}$$

Thus, we calculate from (2.3) that

$$\begin{aligned} y'(0) &= \lim_{a \to 0} \frac{y_1 - y_0}{a} \\ &= \lim_{\epsilon \to \infty} \frac{\epsilon I_0}{m}(1 - \epsilon + \epsilon^2 - \epsilon^3 \cdots). \end{aligned}$$

Using the formula for the Nth approximants Q_N in (1.2) we find that all of the b_N in (1.2) for the series

$$\epsilon(1 - \epsilon + \epsilon^2 - \epsilon^3 \cdots)$$

are 1. Hence,

$$Q_N = I_0/m, \qquad \text{all } N.$$

Thus, the difference equation technique is exact to all orders!

III. Static "Kink" Solution to Nonlinear Classical Wave Equation

A static (time-independent) solution to the two-dimensional nonlinear wave equation,

$$\epsilon^2(u_{xx} - u_{tt}) + u - u^3 = 0, \tag{3.1}$$

is the so-called kink solution:

$$u(x) = \tanh\left(\frac{x}{\sqrt{2}\,\epsilon}\right). \tag{3.2}$$

The kink solution satisfies the boundary conditions $u(0) = 0$ and $u(+\infty) = 1$. Observe that for small ϵ there is a boundary layer of thickness ϵ at $x = 0$ in which the solution $u(x)$ rapidly rises from 0 to 1. To the right of this boundary layer, u is nearly constant and equal to its outer value at $x = \infty$.

OBserve from (3.2) that

$$u'(0) = \frac{1}{\sqrt{2}\,\epsilon}. \tag{3.3}$$

Once $u(0)$ and $u'(0)$ are known, one can immediately reconstruct the full Taylor expansion of $u(x)$ at the origin from the static limit of (3.1):

$$\epsilon^2 u_{xx} + u - u^3 = 0. \tag{3.4}$$

We now show how to solve the singular boundary-value problem (3.4) as a regular perturbation problem on the lattice. Replacing

$$\begin{aligned} x &\to na, \\ u(x) &\to u_n, \\ u''(x) &\to (u_{n+1} - 2u_n + u_{n-1})/a^2, \end{aligned}$$

gives the difference equation

$$\delta(u_{n+1} - 2u_n + u_{n-1}) + u_n - u_n^3 = 0, \tag{3.5}$$

where

$$\delta = \epsilon^2/a^2 \tag{3.6}$$

is a small parameter because on the lattice a is held fixed and ϵ is small.

We seek a solution odd under reflection $n \to -n$ having a regular expansion of the form

$$u_n = u_n^{(0)} + \delta u_n^{(1)} + \delta^2 u_n^{(2)} + \delta^3 u_n^{(3)} + \cdots. \tag{3.7}$$

The solution $u_n^{(0)}$ to the unperturbed equation (3.5) with $\delta = 0$ is

$$\begin{aligned} u_n^{(0)} &= 1 \qquad n \geq 1, \\ &= 0 \qquad n = 0, \\ &= -1 \qquad n \leq -1. \end{aligned} \tag{3.8 }$$

This solution satisfies the same boundary conditions as the exact kink solution to the original differential equation (3.4).

Substituting (3.7) in (3.5) and comparing coefficients of like powers of δ gives the following recursion relation for the coefficients $u_n^{(k)}$:

$$u_n^{(k)} = \tfrac{1}{2}u_{n+1}^{(k-1)} - u_n^{(k-1)} + \tfrac{1}{2}u_{n-1}^{(k-1)} - \tfrac{1}{2}\sum_{l=1}^{k-1} u_n^{(k-l)}\left[u_n^{(l)} + \sum_0^l u_n^{(p)}u_n^{(l-p)}\right]. \tag{3.9}$$

This recursion relation must be solved subject to the boundary condition

$$u_0^{(k)} = 0 \qquad k = 0, 1, 2, 3, \ldots . \tag{3.10}$$

Upon solving (3.9) we observe that the boundary structure develops as the order of perturbation theory in powers of δ increases. In particular, in kth order $u_n^{(k)}$ is nonzero for $1 \le n \le k$ and $u_n^{(k)} = 0$ for $n > k$, $k \ge 1$. Thus the matrix $u_n^{(k)}$ is triangular for $k \ge 1$. Here are the first few entries:

$$\begin{aligned}
u_0 &= 0,\\
u_1 &= 1 - \tfrac{1}{2}\delta + \tfrac{1}{8}\delta^2 + \tfrac{11}{128}\delta^4 + \cdots,\\
u_2 &= 1 \qquad - \tfrac{1}{4}\delta^2 + \tfrac{5}{16}\delta^3 - \tfrac{15}{32}\delta^4 + \cdots,\\
u_3 &= 1 \qquad - \tfrac{1}{8}\delta^3 + \tfrac{9}{32}\delta^4 + \cdots,\\
u_4 &= 1 \qquad - \tfrac{1}{16}\delta^4 + \cdots .
\end{aligned} \tag{3.11}$$

It is a peculiarity of our method that to any finite order in perturbation theory the thickness of the boundry layer, which is na, vanishes in the limit of zero lattice spacing $a \to 0$. Nevertheless, we can easily determine $u'(0)$ [as well as all higher derivatives of $u(x)$] from the lattice series (3.11). Using the definition

$$u'(0) \equiv \frac{u_1 - u_0}{a},$$

we obtain

$$\begin{aligned}
u'(0) &= \frac{1}{a}\left(1 - \frac{\delta}{2} + \frac{\delta^2}{8} + \frac{11}{128}\delta^4 - \frac{23}{128}\delta^5 \cdots\right)\\
&= \frac{\sqrt{\delta}}{\epsilon}\left(1 - \frac{\delta}{2} + \frac{\delta^2}{8} + \frac{11}{128}\delta^4 - \frac{23}{128}\delta^5 \cdots\right),
\end{aligned} \tag{3.12}$$

where we have used (3.6) to eliminate a. The series in (3.12) should reproduce the result in (3.3).

We have obtained 50 terms in the series in (3.12) using the MACSYMA computer program at MIT and have used the extrapolation scheme described in (1.2) to obtain the extrapolants Q_N to the series in (3.12).

(Although this series alternates in sign, the extrapolants, as in the example discussed in Section I, are all postive.) These extrapolants are to be compared with the exact coefficient $1/\sqrt{2}$ of $1/\epsilon$. We find that the extrapolants rapidly approach $1/\sqrt{2} = 0.70711\ldots$:

$$
\begin{aligned}
Q_1 &= 1.0,\\
Q_2 &= 0.84090,\\
Q_3 &= 0.78193,\\
Q_4 &= 0.75724,\\
Q_5 &= 0.74076,\\
Q_6 &= 0.73121,\\
Q_7 &= 0.72393,\\
Q_8 &= 0.71905,\\
Q_9 &= 0.71515,\\
Q_{10} &= 0.71231.
\end{aligned}
$$

The extrapolants continue decreasing until they undershoot the exact value $0.70711\ldots$. They continue decreasing until they reach a minimum in 24th order:

$$Q_{24} = 0.70198.$$

The relative error between this value and the exact answer is less than 1%. Then the extrapolants gradually rise until they recross the value $0.7071\ldots$ at 41st order and continue rising. It appears to us that the extrapolants will continue to rise from here on. This suggests to us that for this problem, unlike the example in Section II, our approximation method is asymptotic in nature. Like the Stirling series for the Gamma function and other asymptotic series, early terms in the series comprise a good approximation to the answer until some optimal order is reached. Afterward, the direct approximants from these series diverge. We have not found a summability method for improving the results of our lattice calculations.

IV. Blasius Equation

The Blasius equation,

$$2\epsilon y'''(x) + y(x)y''(x) = 0, \qquad y(0) = y'(0) = 0, \qquad y'(+\infty) = 1, \tag{4.1}$$

arises in the boundary-layer description of fluid flow across a flat plate.

The exact solution to this problem exhibits boundary-layer structure at the origin $x = 0$. The outer solution is $y = x$. In the boundary layer the derivative of y, which is the fluid velocity parallel to the plate divided by the fluid velocity at ∞, rapidly changes from 0 to 1 as x increases over a narrow range. A quantity of physical interest is $y''(0)$, which apart from dimensional parameters determines the stress on the plate.

To solve (4.1) on a lattice we make the following substitutions:

$$\begin{aligned} x &\to na, \\ y(x) &\to y_n, \\ y''(x) &\to (y_{n+1} - 2y_n + y_{n-1})/a^2, \\ y'''(x) &\to (y_{n+1} - 3y_n + 3y_{n-1} - y_{n-2})/a^3. \end{aligned} \tag{4.2}$$

This choice of differences is consistent with the following boundary conditions:

$$\begin{aligned} y(0) = y'(0) = 0 &\to y_0 = y_{-1} = 0, \\ y'(+\infty) = 1 &\to y_n \sim na(n \to \infty). \end{aligned} \tag{4.3}$$

These choices of boundary conditions give a function y_n which is symmetric when n is reflected about $-\frac{1}{2}$.

It is convenient to introduce a scaled dependent variable

$$f_n = y_n/a$$

so that the boundary conditions in (4.3) become independent of a. The function f_n satisfies the equation

$$\begin{aligned} 2\delta(f_{n+1} - 3f_n + 3f_{n-1} - f_{n-2}) \\ + f_n(f_{n+1} - 2f_n + f_{n-1}) = 0, \end{aligned} \tag{4.4}$$

where

$$\delta = \epsilon/a^2. \tag{4.5}$$

We expand f_n as a series in powers of δ:

$$f_n = f_n^{(0)} + f_n^{(1)}\delta + f_n^{(2)}\delta^2 + \cdots \tag{4.6}$$

and substitute (4.6) into (4.4). Matching powers of δ gives for $k \geq 1$

$$\begin{aligned} \sum_{j=0}^{k} f_n^{(k-j)}\left[f_{n+1}^{(j)} - 1 f_n^{(j)} + f_{n-1}^{(j)} \right] \\ + 2\left[f_{n+1}^{(k-1)} - 3f_n^{(k-1)} + 3f_{n-1}^{(k-1)} - f_{n-2}^{(k-1)} \right] = 0. \end{aligned} \tag{4.7}$$

To zeroth order in δ the solution to (4.4) is

$$\begin{aligned} f_n^{(0)} &= n, \qquad n \ge 0, \\ f_{-1}^{(0)} &= 0, \\ f_{-n-1} &= f_n, \qquad n \ge 0. \end{aligned} \tag{4.8}$$

As in the "kink" problem in Section III, the boundary-layer structure propagates outward away from the origin as the order is increased. In the kth order the boundary-layer disturbance reaches the kth lattice point. To the right of this lattice point the solution $f_n^{(k)}$ ($n \ge k$) remains undisturbed in the sense that

$$f_k^{(k)} = f_{k+1}^{(k)} = f_{k+2}^{(k)} = \cdots . \tag{4.9}$$

As in the "kink" problem, $f_n^{(k)}$ is not constant in a triangular matrix. The first few solutions for f_n illustrate this triangular nature of the matrix:

$$\begin{aligned} f_0 &= 0, \\ f_1 &= 1 - 2\delta + 2\delta^2 + \tfrac{8}{3}\delta^3 - 6\delta^4 - \tfrac{184}{15}\delta^5 + \cdots , \\ f_2 &= 2 - 2\delta \qquad + \tfrac{16}{3}\delta^3 - 4\delta^4 - \tfrac{208}{15}\delta^5 + \cdots , \\ f_3 &= 3 - 2\delta \qquad + 4\delta^3 - 2\delta^4 - \tfrac{64}{5}\delta^5 + \cdots , \\ f_4 &= 4 - 2\delta \qquad + 4\delta^3 - \tfrac{8}{3}\delta^4 - \tfrac{176}{15}\delta^5 + \cdots , \\ f_5 &= 5 - 2\delta \qquad + 4\delta^3 - \tfrac{8}{3}\delta^4 - 12\delta^5 + \cdots . \end{aligned}$$

To compute $y''(0)$ we evaluate

$$\begin{aligned} \frac{y_1 - 2y_0 + y_{-1}}{a^2} &= \frac{f_1}{a} \\ &= \frac{\sqrt{\delta}}{\sqrt{\epsilon}}\left(1 - 2\delta + 2\delta^2 + \frac{8}{3}\delta^3 - 6\delta^4 \cdots\right), \end{aligned} \tag{4.10}$$

where we have used (4.5). The exact value for $y''(0)$ obtained numerically is [8]

$$y''(0) = \frac{1}{\sqrt{\epsilon}}(0.33206)\cdots . \tag{4.11}$$

We have calculated 38 terms in the series in (4.10). The extrapolants Q_N so far are monotonically decreasing. The first five are

$$Q_1 = 0.5/\sqrt{\epsilon}\,,$$
$$Q_2 = 0.4204/\sqrt{\epsilon}\,,$$
$$Q_3 = 0.3948/\sqrt{\epsilon}\,,$$
$$Q_4 = 0.3819/\sqrt{\epsilon}\,,$$
$$Q_5 = 0.3742/\sqrt{\epsilon}\,.$$

As N increases, Q_N becomes very flat:

$$Q_{25} = 0.3502/\sqrt{\epsilon}\,,$$
$$Q_{26} = 0.3500/\sqrt{\epsilon}\,,$$
$$Q_{37} = 0.3485/\sqrt{\epsilon}\,,$$
$$Q_{38} = 0.3484/\sqrt{\epsilon}\,.$$

The relative error between the exact answer in (4.11) and Q_{38} is about 5%. We will not discuss here the many ways to extrapolate Q_N to its limiting value Q_∞.

V. Green's Function for Diffusion Equation

In this section we consider the diffusion equation with a point source in the space and time variables

$$u_t = \nu u_{xx} + \delta(x)\delta(t), \tag{5.1}$$

subject to the initial condition

$$u(x, t) = 0 \qquad \text{for } t < 0. \tag{5.2}$$

The exact solution to (5.1)–(5.2) is the Green's function:

$$u(x, t) = \frac{\theta(t)}{\sqrt{4\pi t\nu}} \exp\left(-\frac{x^2}{4\pi\nu t}\right). \tag{5.3}$$

Observe that for small ν and fixed t, $u(x, t)$ has a boundary layer structure at $x = 0$ of thickness $\sqrt{\nu t}$.

In what follows we present a method for calculating $u(0, t)$. [This method can be easily adapted to calculate all the spatial derivatives of $u(x, t)$ at $x = 0$.] From (5.3) the exact value of $(u(x, t)$ at $x = 0$ is

$$u(0, t) = \frac{\theta(t)}{\sqrt{4\pi t\nu}} \doteqdot 0.282095 \frac{\theta(t)}{\sqrt{\nu t}}. \tag{5.4}$$

On the lattice we discretize in the spatial variable. Thus, (5.1) becomes

$$\frac{\partial u_n}{\partial t} = \epsilon(u_{n+1} - 2u_n + u_{n-1}) + \delta(t)\delta_{n,0/a}, \tag{5.5}$$

where

$$\epsilon = \nu/a^2. \tag{5.6}$$

To solve (5.5) we substitute

$$u_n(t) = \sum_{k=0}^{\infty} \epsilon^k u_n^{(k)}(t) \tag{5.7}$$

into (5.5). This gives the partial difference–differential equation

$$\partial u_n^{(k)}/\partial t = u_{n+1}^{(k-1)} - 2u_n^{(k-1)} + u_{n-1}^{(k-1)} \qquad (k \geq 1) \tag{5.8}$$

and

$$\partial u_n^{(0)}/\partial t = \delta(t)\delta_{n,0}/a. \tag{5.9}$$

The solution to (5.9) is

$$u_n^{(0)}(t) = \theta(t)\delta_{n,0}/a. \tag{5.10}$$

To solve (5.8) we set

$$u_n^{(k)}(t) = \theta(t)t^k w_n^{(k)}/a, \tag{5.11}$$

where $w_n^{(k)}$ is independent of t. This gives the simpler recursion relation

$$w_n^{(k)} = \frac{1}{k} w_{n+1}^{(k-1)} - 2w_n^{(k-1)} + w_{n-1}^{(k-1)}. \tag{5.12}$$

This equation has an exact closed-form solution in terms of binomial coefficients in the form of a triangular matrix. We are interested in the point at $x = 0$; the relevant result is

$$w_0^{(k)} = (-1)^k (2k)!\,(k!)^{-3}. \tag{5.13}$$

Combining (5.13) with (5.7) and (5.11) gives

$$u_0(t) = \frac{1}{a}\theta(t)\sum_{k=0}^{\infty}\frac{(-\epsilon t)^k(2k)!}{(k!)^3}. \tag{5.14}$$

Before discussing the extrapolation of truncations of this series for $a \to 0$ and $\epsilon \to \infty$, we observe that this series can be summed exactly in closed form. (The sum is formed by taking the Laplace transform in the t variable term by term, summing the resulting binomial expansion, and then taking the inverse Laplace transform.) We find that

$$u_0^{(t)} = \theta(t)\sqrt{\epsilon/\nu}\, I_0(2t\epsilon)e^{-2t\epsilon}. \tag{5.15}$$

This can immediately be extrapolated to its continuum value by fixing t and taking $\epsilon \to \infty$. Here we use the asymptotic behavior

$$I_0(z) \sim \frac{1}{\sqrt{2\pi z}}e^z(z \to +\infty)$$

to obtain the exact answer in (5.5).

However, since in general lattice series of the form in (5.14) cannot be summed in closed form we carry out the extrapolation procedure for truncations of the series in (5.14) after the ϵ^N term. The first few extrapolants Q_N divided by $\theta(t)/\sqrt{\nu t}$ are

$$\begin{aligned} Q_1 &= 0.5,\\ Q_2 &= 0.435,\\ Q_3 &= 0.408,\\ Q_4 &= 0.393,\\ Q_5 &= 0.384. \end{aligned}$$

After this value, the extrapolants become very flat with increasing N:

$$\begin{aligned} Q_{10} &= 0.362,\\ Q_{15} &= 0.354,\\ Q_{20} &= 0.349,\\ Q_{25} &= 0.346,\\ Q_{30} &= 0.344,\\ Q_{35} &= 0.343,\\ Q_{40} &= 0.342. \end{aligned}$$

Thus, we have obtained the answer in (5.5) up to a relative error of about 18%. Once again there is a smooth and monotonic approach toward the correct answer. However, in this case it would clearly be desirable to have a more rapidly converging extrapolation technique.

ACKNOWLEDGMENTS

We are indebted to the Laboratory for Computer Science at MIT for allowing us the use of MACSYMA to perform algebraic manipulation. We are grateful to the Department of Energy for partial financial support. One of us, G. S. G., would also like to thank the Brown University Materials Research Laboratory for partial support under its NSF grant.

REFERENCES

1. C. M. BENDER, F. COOPER, G. S. GURALNIK, AND D. H. SHARP, *Phys. Rev. D* **19** (1979), 1865.
2. C. M. BENDER, F. COOPER, G. S. GURALNIK, R. ROSKIES, D. H. SHARP, AND M. L. SILVERSTEIN, *Phys. Rev. D* **20** (1979), 1374.
3. C. M. BENDER, F. COOPER, G. S. GURALNIK, R. ROSKIES, AND D. H. SHARP, *Phys. Rev. Lett.* **43** (1979), 537.
4. C. M. BENDER, F. COOPER, G. S. GURALNIK, H. ROSE, AND D. H. SHARP, Strong-coupling expansion for classical statistical dynamics, *J. Stat. Phys.*, in press.
5. C. M. BENDER, G. S. GURALNIK, AND M. L. SILVERSTEIN, WKB approximation for quantum theory on a lattice, *Phys. Rev. D* **20** (1979), 2583.
6. There is a way to improve the extrapolation scheme discussed here which seems to avoid the nonuniform convergence that results from interchanging two limits (order to perturbation theory $\rightarrow \infty$ and $\epsilon \rightarrow \infty$). See Ref. [3].
7. J. D. COLE, "Perturbation Methods in Applied Mathematics," Blaisdell, Waltham, Mass., 1968.
8. H. SCHLICHTING, "Boundary-Layer Theory," p. 129, McGraw-Hill, New York, 1968.