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Mass Renormalization for the $\lambda\phi^4$ Euclidean Lattice Field

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The inverse correlation length or physical mass of the $\lambda\phi_n^4 + \sigma\phi_n^2$ Euclidean lattice field is shown to be a continuous-increasing function of σ in the single-phase region. By a suitable choice of σ, the inverse correlation length can be set equal to any strictly positive value.

I. INTRODUCTION

The $\lambda\phi_n^4 + \sigma\phi_n^2$ Euclidean lattice field is a Markov random field on $\mathbb{Z}^n$. A random field ϕ on $\mathbb{Z}^n$ is determined by a probability measure on the set of functions $\phi : \mathbb{Z}^n \to \mathbb{R}$. The probability measure associated with a Markov random field has the property that for any $x \in \mathbb{Z}^n$ Borel set $B \in \mathbb{R}$, and finite $\Lambda \subseteq \mathbb{Z}^n$ containing Λ_x, the $2n$ nearest neighbors of x,

$$P(\phi(x) \in B | \phi(y), y \in \Lambda) = P(\phi(x) \in B | \phi(y), y \in \Lambda_x).$$

In the field theory literature this is referred to as the local Markov property [4]. In the Markov random field we consider, the associated measure is also stationary, or invariant under lattice translations and rotations.

When $n = 1$, a time-reversible stationary Markov process on $\mathbb{R}$ is typically characterized by a self-adjoint second-order ordinary differential operator, the infinitesimal generator of the process [10]. These operators have purely discrete spectra with unique lowest eigenstates. The lowest eigenstate determines the equilibrium measure of the process. Its uniqueness assures us that there is a unique stationary Markov process associated with a given infinitesimal generator.

The Markov random fields we consider are also intimately related to their infinitesimal generator, called the physical Hamiltonian, which is

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formally an elliptic partial differential operator in an infinite number of variables. When $n \geq 2$, these operators may have a degenerate lowest eigenvalue, and therefore, there may be more than one stationary Markov random field associated with a given Hamiltonian. This is the phenomenon of multiple phases.

Nondegeneracy of the lowest eigenvalue is equivalent to the ergodicity of our field under lattice translations [4]. In this case

$$\lim_{\Lambda \to \mathbf{Z}^n} \frac{1}{|\Lambda|} \sum_{x \in \Lambda} f(\phi(x)) = \text{constant a.e.},$$

while in the degenerate case, when the field is not ergodic, the above limit will in general be a nonconstant random variable, the value of which depends on the phase we are observing. If the field is nonergodic, it can always be decomposed into ergodic pure phases; see, for example, [4].

In the Markov random fields considered here, there is a simple criteria for ergodicity: the vanishing of the long-range order.

Explicitly, let $E_\sigma(\cdot)$ denote expectations for the Markov random field associated with $\lambda\phi_n^4 + \sigma\phi_n^2$, emphasizing the σ dependence. $E_\sigma(\cdot)$ is obtained as a limit of expectations $E_{\sigma, L}(\cdot)$ for the theory with half-Dirichlet boundary conditions in the region L. For details consult [1].

The long-range order $\mathcal{L}(\sigma) \geq 0$ is defined by

$$\mathcal{L}(\sigma)^2 = \lim_{t \to \infty} E_\sigma(\phi(0, \ldots, 0)\phi(t, 0, \ldots, 0)).$$

If $n \geq 2$, the set of σ with $\mathcal{L}(\sigma) = 0$, is a proper right half-line, since $\mathcal{L}(\sigma)$ is decreasing [2], and $\mathcal{L}(\sigma) \neq 0$ for σ sufficiently negative [3]. The right half-line, where $\mathcal{L}(\sigma) = 0$, which is the region of ergodicity [1], is therefore called the single-phase region.

In the single-phase region, the gap m between the first two eigenvalues is called the inverse correlation length, since events which are separated by distances greater than m^{-1} are "essentially" uncorrelated. Explicitly, if $\mathcal{O}_j^{\pm}$ denotes the algebra generated by $\phi(x)$, $x_1 \geq j$ or $x_1 \leq j$, respectively,

$$|E_\sigma(fg) - E_\sigma(f)E_\sigma(g)| \leq \exp(-m|j-k|)\|f\|_2\|g\|_2$$

for any $f \in \mathcal{O}_j^-$, $g \in \mathcal{O}_k^+$; $j \leq k$. In analogy with continuum field theory, the inverse correlation length is also called the physical mass. Let us define

$$m(\sigma) = \lim_{t \to \infty} -\log(E_\sigma(\phi(0, \ldots, 0)\phi(t, 0, \ldots, 0)))/t.$$

In the single-phase region $m(\sigma)$ is the physical mass [1]. $m(\sigma)$ is always monotone increasing. The critical point σ_c is defined as $\sigma_c = \sup\{\sigma | m(\sigma) = 0\}$. Since $\mathcal{L}(\sigma) \neq 0$ implies $m(\sigma) = 0$, we see that the region $\{\sigma | \sigma > \sigma_c\}$ is in the single-phase region.

Our aim in this paper is to show that $m(\sigma)$ depends continuously on σ, and in particular that $m(\sigma)$ approaches zero as $\sigma\downarrow\sigma_c$. Since it is known that $m(\sigma)\uparrow\infty$ as $\sigma\uparrow\infty$ [9], we see that for any $m > 0$, one can choose $\sigma > \sigma_c$ such that the physical mass of the $\lambda\phi_n^4 + \sigma\phi_n^2$ Markov random field is equal to m. This is called mass renormalization.

Our proof of the mass renormalizability of the $\lambda\phi_4^4$ lattice field theory completes the first step in a program proposed by Glimm and Jaffe [6] for the construction of a continuum $\lambda\phi_4^4$ field theory. In this approach, one lets the lattice spacing tend to zero, while holding the mass fixed. Hence the importance of knowing that a lattice theory with a given mass exists.

Glimm and Jaffe were the first to obtain results on the continuity of the physical mass [7]. Using the Lebowitz inequality, they were able to show that the physical mass is Lipschitz continuous away from its critical point, in $\lambda\phi_2^4$ continuum theories. In Section V we present a proof, along similar lines, of the Lipschitz continuity of the physical mass away from its critical point in $\lambda\phi_n^4$ lattice theories. The fact that we are interested in a lattice theory, where we do not have Euclidean invariance at our disposal, necessitates substantial modifications of the Glimm–Jaffe proof. The proof will also yield

$$m(\sigma) \leq c|\sigma - \sigma_c|^{1/n},$$

near the critical point.

Baker [8] has made similar use of the Lebowitz inequality to show continuity of a pseudomass and indicated how this might be used to show continuity of the physical mass at the critical point. This is carried out in detail in Sections II, III, and IV. We use a pseudomass which differs from Baker's and provides us with a simpler route to continuity.

For the convenience of the reader we briefly recall certain results used in this paper. For details see [1].

Let A, B be monomials in the field ϕ. That is, let $A = \prod_{x\in K}\phi(x)^{j_x}$, $K \subseteq \mathbb{Z}^n, j_x \in N$. The GKS inequalities (for Griffiths, Kelly, and Sherman) state that

$$\begin{aligned} E_{\sigma,L}(A) &\geq 0, \\ E_{\sigma,L}(AB) &\geq E_{\sigma,L}(A)E_{\sigma,L}(B). \end{aligned} \tag{1}$$

This implies that

$$E_{\sigma,L}(A) \leq E_{\sigma,L'}(A); \qquad L \subseteq L'$$

and allows us to define

$$E_\sigma(A) = \lim_{L\uparrow\mathbb{Z}^n} E_{\sigma,L}(A).$$

The Lebowitz inequality is

$$E_{\sigma,L}(\phi(x)\phi(y)\phi(z)\phi(w)) - E_{\sigma,L}(\phi(x)\phi(y))E_{\sigma,L}(\phi(z)\phi(w)) \\ \leq E_{\sigma,L}(\phi(x)\phi(z))E_{\sigma,L}(\phi(y)\phi(w)) + E_{\sigma,L}(\phi(x)\phi(w))E_{\sigma,L}(\phi(y)\phi(z)).$$

Finally, the Osterwalder–Schrader reconstruction theorem provides us with a Hilbert space $\langle \cdot, \cdot \rangle_\sigma$, vectors $u(x_1, \ldots, x_{n-1})$ indexed by $\mathbb{Z}^{n-1}$, and a contraction semigroup, which can be shown to be of the form e^{-mH}, $H \geq 0$, such that

$$E_\sigma(\phi(0, \ldots, 0)\phi(x_1, \ldots, x_n)) = \langle u(0, \ldots, 0), e^{-|x_1|H}u(x_2, \ldots, x_n)\rangle_\sigma. \tag{2}$$

II. Definition and Properties of $M(\sigma)$

In this section we define the pseudomass $M(\sigma)$. $M(\sigma)$ will be shown to be continuous in Section IV.

Fix $\sigma_0 < \sigma_c$. Throughout this paper it will be understood that $\sigma \geq \sigma_0$. The GKS inequalities (1) imply [2] that

$$0 < E_\sigma(\phi^2(0)) \leq E_{\sigma_0}(\phi^2(0)) \doteq A^2/2 < \infty.$$

Set $\psi(r) = \phi(r)/A$, and note that

$$E_{\sigma,L}(\psi^2(r)) \leq E_\sigma(\psi^2(r)) \leq \tfrac{1}{2}, \tag{3}$$

and

$$m(\sigma) = \lim_{t\to\infty} -\log[E_\sigma(\psi(0, \ldots, 0)\psi(t, 0, \ldots, 0))]/t. \tag{4}$$

$M(\sigma)$ will be defined as $\lim_{L\uparrow\mathbb{Z}^n} M(L, \sigma)$, where

$$M(L, \sigma) \\ = \sup\left\{ m \middle| E_{\sigma,L}(\psi(r)\psi(s)) \leq \frac{\exp(-m|r-s|)}{1 + (m|r-s|)^{n-1/2}} \quad r, s \in L, r \neq s \right\}. \tag{5}$$

To see that these quantities are well defined, note that the function $f(x, m) = m + \log(1 + (mx)^{n-1/2})/x$, $m \geq 0$ for fixed $x > 0$, is a strictly increasing function of m, with $f(x, 0) = 0$, $f(x, m)\uparrow\infty$, as $m\uparrow\infty$.

Given a region L, and $r, s \in L$, by (3), $-\log E_{\sigma,L}(\psi(r)\psi(s)) > 0$. If

$r \neq s$, the equation for m,

$$f(|r - s|, m) = -\log[E_{\sigma, L}(\psi(r)\psi(s))]/|r - s|, \tag{6}$$

has a unique solution which we denote by $M(L, \sigma, r, s)$. Set $M(L, \sigma) = \inf\{M(L, \sigma, r, s)|r, s \in L, r \neq s\}$. Since $f(|r - s|, \cdot)$ is increasing,

$$f(|r - s|, M(L, \sigma)) \leq -\log[E_{\sigma, L}(\psi(r)\psi(s))]/|r - s|, \qquad r, s \in L, r \neq s, \tag{7}$$

with equality for a least one pair $r, s \in L$. This shows the equivalence of the present definition of $M(L, \sigma)$ with (5).

Note that one expects $(n - 1)/2$ as the exponent in the denominator in (5). Our choice is motivated only by a desire to simplify our proofs.

If $L' \supseteq L$, and $r, s \in L$, $-\log E_{\sigma, L'}(\psi(r)\psi(s)) \leq -\log E_{\sigma, L}(\psi(r)\psi(s))$, by GKS. The monotonicity of $f(|r - s|, \cdot)$ shows $M(L', \sigma, r, s) \leq M(L, \sigma, r, s)$; hence,

$$M(L', \sigma) \leq M(L, \sigma). \tag{8}$$

Thus, $M(\sigma) = \lim_{L\uparrow \mathbf{Z}^n} M(L, \sigma)$ is well defined and

$$0 \leq M(L, \sigma)\downarrow M(\sigma). \tag{9}$$

III. A Comparison of $m(\sigma)$ and $M(\sigma)$

THEOREM 1.

$$M(\sigma) \leq m(\sigma) \leq c(n)M(\sigma). \tag{10}$$

Proof. To establish the left-hand inequality, fix $r = (t, 0, \ldots, 0)$. For any region $L \supseteq \{(0, \ldots, 0), (t, 0, \ldots, 0)\}$, by (7) and (9),

$$\begin{aligned} &-\log E_{\sigma, L}(\psi(0, \ldots, 0)\psi(t, 0, \ldots, 0))/t \\ &\qquad \geq M(\sigma) + \log(1 + (M(\sigma)t)^{n-1/2})/t. \end{aligned} \tag{11}$$

Since the right-hand side of (11) is independent of L, we have

$$\begin{aligned} &-\log E_{\sigma}(\psi(0, \ldots, 0)\psi(t, 0, \ldots, 0))/t \\ &\qquad \geq M(\sigma) + \log(1 + (M(\sigma)t)^{n-1/2})/t \end{aligned}$$

and the left-hand inequality of (10) follows on letting $t \to \infty$. Using (2) and (3), in the case $|x_1| = \max|x_i| \geq C(n)|x|$, where $x = (x_1, \ldots, x_n)$,

$$E_\sigma(\psi(0, \ldots, 0)\psi(x_1, \ldots, x_n)) = \langle u(0, \ldots, 0), e^{-|x_1|H}u(x_2, \ldots, x_n)\rangle_\sigma$$
$$\leq \exp(-m(\sigma)|x_1|) \leq \exp(-cm(\sigma)|x|) \tag{12}$$

or

$$-\log E_\sigma(\psi(0)\psi(x))/|x| \geq cm(\sigma). \tag{13}$$

To prove the right-hand inequality of (10) fix L and $r, s \in L$ with $M(L, \sigma, r, s) = M(L, \sigma)$; then by (13)

$$cm(\sigma) \leq -\log E_\sigma(\psi(r)\psi(s))/|r-s| \leq -\log E_{\sigma, L}(\psi(r)\psi(s))/|r-s|$$
$$\leq M(L, \sigma) + \log\left(1 + (M(L, \sigma)|r-s|)^{n-1/2}\right)/|r-s|$$
$$\leq c(n)M(L, \sigma) \tag{14}$$

because $\log(1 + x^{n-1/2}) \leq d(n)x$, $x > 0$. Since the left-hand side of (14) is independent of L, the proof of (10) follows on letting $L \uparrow \mathbb{Z}^n$.

IV. The Continuity of $M(\sigma)$

In this section we present a streamlined version of Baker's proof [8] of the continuity of $M(\sigma)$.

Theorem 2. *For any $\sigma_1 < \infty$, there is a $c(\sigma_1) < \infty$ such that*

$$0 \leq M(\sigma')^{2n} - M(\sigma)^{2n} \leq c(\sigma_1)(\sigma' - \sigma); \; \sigma_0 \leq \sigma \leq \sigma' \leq \sigma_1. \tag{15}$$

Since $M(\sigma_c) = 0$ by (10), setting $\sigma = \sigma_c$ in (15) and using (10), we have

Corollary 3.

$$m(\sigma) \leq c(\sigma_1)|\sigma - \sigma_c|^{1/2n}, \qquad \sigma_0 \leq \sigma \leq \sigma_1. \tag{16}$$

This exhibits the continuity of $m(\sigma)$ at σ_c.

Proof. By (9), it suffices to prove (15) with $M(\sigma)$ replaced by $M(L, \sigma)$, for some $c(\sigma_1)$ independent of L. But $M(L, \sigma)$, being the minimum of a finite number of analytic functions $M(L, \sigma, r, s)$, possesses a right-hand derivative $D^+M(L, \sigma)$, and it suffices to show

$$D^+M(L, \sigma) \leq c(\sigma_1)/M(L, \sigma)^{2n-1}, \qquad \sigma_0 \leq \sigma \leq \sigma_1 \tag{17}$$

for some $c(\sigma_1)$ independent of L. (Note that $M(L, \sigma) > 0$.) Furthermore, for each σ there will always be some $r, s \in L$, $r \neq s$ with

$$M(L, \sigma) = M(L, \sigma, r, s) \tag{18}$$

such that

$$D^+M(L, \sigma) = \frac{d}{d\sigma} M(L, \sigma, r, s).$$

Differentiating (6) and using the Lebowitz inequality [1, p. 326],

$$\begin{aligned}
&\frac{dM(L, \sigma, r, s)}{d\sigma}\left[1 + \frac{(M(L, \sigma)|r - s|)^{n-3/2}(n - 1/2)}{1 + (M(L, \sigma)|r - s|)^{n-1/2}}\right] \\
&\quad = \sum_{t \in L} \frac{E_{\sigma, L}(\psi(r)\psi(s)\phi^2(t)) - E_{\sigma, L}(\psi(r)\psi(s))E_{\sigma, L}(\phi^2(t))}{|r - s|E_{\sigma, L}(\psi(r)\psi(s))} \\
&\quad \leq c \sum_{t \in L} \frac{E_{\sigma, L}(\psi(r)\psi(t))E_{\sigma, L}(\psi(s)\psi(t))}{|r - s|E_{\sigma, L}(\psi(r)\psi(s))}.
\end{aligned} \tag{19}$$

Note that since $\psi = \phi/A$ where $A^2 = 2E_{\sigma_0}(\phi^2(0))$, our constant c depends on the lattice spacing.

If $t = r$ or s, use (3) to bound the summand in (19). Otherwise, use (5) to bound the numerator and (18) for the denominator to obtain

$$\begin{aligned}
D^+M(L, \sigma) &\leq c\frac{2}{|r - s|} + \sum_{t \neq r, s} \frac{\exp(M(L, \sigma)(|r - s| - |t - s| - |t - r|))}{|r - s|} \\
&\quad \times \frac{1 + (|r - s|M(L, \sigma))^{n-1/2}}{[1 + M(L, \sigma)|t - r|^{n-1/2}][1 + M(L, \sigma)|t - s|^{n-1/2}]} \\
&\leq \frac{c(\sigma_1)}{M(L, \sigma)^{2n-1}}\left[1 + \sum_{t \neq r, s} \frac{\exp(M(L, \sigma)(|r - s| - |t - s| - |t - r|))}{|r - s|^{3/2-n}|t - r|^{n-1/2}|t - s|^{n-1/2}}\right],
\end{aligned} \tag{20}$$

where use has been made of the fact that $|r - s| \geq 1$, and $M(L, \sigma)$ is uniformly bounded in L and $\sigma_0 \leq \sigma \leq \sigma_1$, by monotonicity.

It is easily seen that the sum in (20) is bounded in terms of integrals of the form

$$\begin{aligned}
&\int \frac{|a|^{n-3/2} d^n t}{|t|^{n-1/2}|t - a|^{n-1/2}} \\
&\leq \int \frac{d^n t}{|t|^{n-1/2}|t - a/|a|\,|^{n-1/2}} \leq c \text{ independent of } a; \qquad |a| \geq 1.
\end{aligned}$$

V. Lipschitz Continuity of $m(\sigma)$

In this section we modify the methods of Glimm and Jaffe [7] to prove

Theorem 4.

$$|m(\sigma')^n - m(\sigma)^n| \leq c|\sigma' - \sigma| \tag{21}$$

for σ, σ' in any finite interval, with c depending on the interval.

In particular we have

Corollary 5.

$$m(\sigma) \leq k|\sigma - \sigma_c|^{1/n} \tag{22}$$

in a neighborhood of the critical point.

Proof of Theorem 4. Since our proof of Lipschitz continuity in the lattice $\lambda\phi_n^4$ theory differs significantly from the Glimm–Jaffe proof for the continuum $\lambda\phi_2^4$ theory, all details of the proof will be spelled out.

Let

$$F_t(D) = \sum_{\substack{|x_i| \leq D^3 \\ i \neq 1}} \psi(t, x_2, \ldots, x_n).$$

By (12) and translation invariance

$$D^{3(n-1)} E_\sigma(\psi(0, \ldots, 0)\psi(D, 0, \ldots, 0)) \\ \leq E_\sigma(F_0(D)F_D(D)) \leq D^{6(n-1)}\exp(-m(\sigma)D)$$

so that

$$m(\sigma) = \lim_{D\to\infty} -\log E_\sigma(F_0(D)F_D(D))/D.$$

Our main task will be to show that for any $\sigma_1 < \infty$ and $\epsilon > 0$, there exists c independent of ϵ with

$$0 \leq \frac{d}{d\sigma}\left(-\log\left[E_\sigma(F_0(D)F_D(D))\right]/D\right) \\ \leq \left(c/m(\sigma)^{n-1}\right) + O(D^{-1/2}); \sigma_1 \geq \sigma \geq \sigma_c + \epsilon, \tag{23}$$

where the term $O(D^{-1/2})$ may depend on ϵ, but is independent of σ in the interval $\sigma_1 \geq \sigma \geq \sigma_c + \epsilon$. For assume (23) is proven. From the monotonicity of $m(\sigma)$, we have, after integrating (23) and letting $D \to \infty$,

$$0 \leq m(\sigma') - m(\sigma) \leq c(\sigma' - \sigma)/m(\sigma_c + \epsilon)^{n-1}; \qquad \sigma_1 \geq \sigma' \geq \sigma \geq \sigma_c + \epsilon.$$

This shows that $m(\sigma)$ is absolutely continuous in $\sigma_1 \geq \sigma \geq \sigma_c + \epsilon$, and consequently, has a derivative a.e. and is the integral of its derivative. Since we have, similarly, for $h \geq 0$,

$$0 \leq m(\sigma + h) - m(\sigma) \leq ch/m(\sigma)^{n-1}; \qquad \sigma_1 \geq \sigma \geq \sigma_c + \epsilon$$

we see that

$$\frac{d}{d\sigma} m(\sigma)^n = n\frac{dm(\sigma)}{d\sigma} \cdot m^{n-1}(\sigma) \leq nc,$$

and therefore,

$$m^n(\sigma') - m^n(\sigma) \leq nc(\sigma' - \sigma); \qquad \sigma_1 \geq \sigma' \geq \sigma \geq \sigma_c + \epsilon,$$

with c independent of ϵ. Since we have shown in Section IV that $m(\sigma)$ is continuous at σ_c, (21) and (22) follow on letting $\epsilon \to 0$.

Let us now prove (23). Differentiating and using the Lebowitz and GKS inequalities [1] we have

$$\begin{aligned} 0 &\leq \frac{d}{d\sigma}\left[-\log E_\sigma(F_0(D)F_D(D))/D \right] \\ &= \sum_t \frac{E_\sigma\big(F_0(D)F_D(D)\phi^2(t)\big) - E_\sigma(F_0(D)F_D(D))E_\sigma(\phi^2(t))}{DE_\sigma(F_0(D)F_D(D))} \\ &\leq c\sum_t \frac{E_\sigma(F_0(D)\psi(t))E_\sigma(\psi(t)F_D(D))}{DE_\sigma(F_0(D)F_D(D))}. \end{aligned} \tag{24}$$

Let us first show that for large D we need only consider the sum in (24) over t in the region $\mathrm{I} = \{0 \leq t_1 \leq D, |t_i| \leq D^3\}$.

Consider the region $\mathrm{II} = \{D \leq t_1 \leq D + D^{1/2}, |t_i| \leq D^3\}$. Since H is positive $E_\sigma(F_0(D)F_s(D))$ is decreasing in s. Therefore,

$$\begin{aligned} &\sum_{t\in \mathrm{II}} E_\sigma(\psi(t)F_0(D))E_\sigma(F_D(D)\psi(t)) \\ &\quad\leq \left[\sum_{D\leq t_1 \leq D + D^{1/2}} E_\sigma\big(F_0(D)F_{t_1}(D)\big) \right] \sup_{t\in \mathrm{II}} E_\sigma(\psi(t)F_D(D)) \\ &\quad\leq D^{1/2}E_\sigma(F_0(D)F_D(D)) \sum_{x\in Z^{n-1}} \exp(-Cm(\sigma)|x|), \quad [\text{by } (12)] \\ &\quad\leq D^{1/2}E_\sigma(F_0(D)F_D(D))k/m(\sigma)^{n-1}. \end{aligned}$$

Hence, the summation over region II contributes

$$O(D^{-1/2})/m(\sigma_c + \epsilon)^{n-1}$$

to (23).

Next, let III $= \{D + D^{1/2} \le t_1, |t_i| \le D^3, i \ne 1\}$. As before

$$\sum_{t \in \mathrm{III}} E_\sigma(F_0(D)\psi(t))E_\sigma(\psi(t)F_D(D))$$

$$= \sum_{t_1 \ge D + D^{1/2}} \sum_{\substack{|t_i| \le D^3 \\ i \ne 1}} E_\sigma(F_0(D)\psi(t_1, \mathrm{t}))E_\sigma(\psi(t_1, \mathrm{t})F_D(D))$$

$$\le E_\sigma(F_0(D)F_D(D)) \sum_{t_1 \ge D + D^{1/2}} \sup_{\substack{|t_i| \le D^3 \\ i \ne 1}} E_\sigma(\psi(t_1, \mathrm{t})F_D(D))$$

$$\le E_\sigma(F_0(D)F_D(D)) \sum_{t_1 \ge D + D^{1/2}} D^{3(n-1)}\exp(-m(\sigma)(t_1 - D)) \quad [\text{by } (12)]$$

$$\le E_\sigma(F_0(D)F_D(D))D^{3(n-1)}\exp(-m(\sigma)D^{1/2})/m(\sigma)$$

$$\le E_\sigma(F_0(D)F_D(D))D^{3(n-1)}\exp(-m(\sigma_c + \epsilon)D^{1/2})/m(\sigma_c + \epsilon)$$

in $\sigma \ge \sigma_c + \epsilon$, so that the summation over III contributes only negligibly to (23) for large D.

Consider next the region IV $= \{0 \le t_1 \le D^2, D^3 \le t_2 \le D^3 + D^2, |t_i| \le D^3, i \ne 1, 2\}$. Let $A_t = \{z | z_1 = t, z_2 \ge D^3 - D^2, |z_i| \le D^3\}$ and $B = \{z | z_1 = 0, z_2 \le D^3 - D^2, |z_i| \le D^3\}$ and note that

$$\sum_{t \in \mathrm{IV}} E_\sigma(F_0(D)\psi(t))E_\sigma(\psi(t)F_D(D))$$

$$\le \sum_{t \in \mathrm{IV},\, x \in A_0,\, y \in A_D} E_\sigma(\psi(x)\psi(t))E_\sigma(\psi(t)\psi(y))$$

$$+2 \sum_{t \in \mathrm{IV},\, x \in B} E_\sigma(\psi(x)\psi(t))E_\sigma(\psi(t)F_D(D)).$$

If we reflect the points x, y, t in the hyperplane $z_2 = D^3 - D^2$, we see that the first sum is bounded by the sum in (24) over regions I, II, and III. The second sum, using the fact that t is at least a distance D^2 from x is bounded by

$$cD^{6(n-1)+3(n-2)+4}\exp(-m(\sigma)D^2) \le cD^{10n}\exp(-m(\sigma_c + \epsilon)D^2).$$

Again, by monotonicity of the two-point function in σ

$$E_\sigma(F_0(D)F_D(D)) \ge E_{\sigma_1}(F_0(D)F_D(D)) \ge O(\exp(-2m(\sigma_1)D))$$

for D large enough. Hence the sum over $t \in$ IV, $x \in B$ contributes negligibly to (23).

Similarly, we may easily control the sum over all t with $\mathrm{dist}(t, I) \le D^2$.

Finally, by (12)

$$\sum_{t\,:\,\mathrm{dist}(t,\,I)\geq D^2} E_\sigma(F_0(D)\psi(t))E_\sigma(\psi(t)F_D(D))$$
$$\leq cD^{6(n-1)}\sum_{|x|\geq D^2}\exp(-cm(\sigma)|x|)$$
$$\leq cD^{8(n-1)}\exp(-cm(\sigma)D^2)/m(\sigma)^n$$
$$\leq cD^{8(n-1)}\exp\big(-cm(\sigma_c+\epsilon)D^2\big)/m(\sigma_c+\epsilon)^n$$

as before we see that this sum contributes only negligibly to (23).

It only remains to study the sum in (24) over region I. We have

$$\sum_{t\in \mathrm{I}}\frac{E_\sigma(F_0(D)\psi(t))E_\sigma(\psi(t)F_D(D))}{DE_\sigma(F_0(D)F_D(D))}$$
$$\leq \sup_{0\leq t_1\leq D}\sum_{\substack{|t_i|\leq D^3\\ i\neq 1}}\frac{E_\sigma(F_0(D)\psi(t_1,\mathbf{t}))E_\sigma(\psi(t_1,\mathbf{t})F_D(D))}{E_\sigma(F_0(D)F_D(D))}. \tag{25}$$

Translation invariance shows that for any $t=(t_1,s)\in \mathrm{I}$

$$D^{3(n-1)}E_\sigma(\psi(t_1,\mathbf{s})F_D(D))\leq \sum_{\substack{|t_i|\leq 3D^3\\ i\neq 1}}E_\sigma(\psi(t_1,\mathbf{t})F_D(D))$$
$$=\sum_{a;\,a_i=0,\,2D^3,\,-2D^3}E_\sigma\Bigg[\sum_{\substack{|t_i|\leq D^3\\ i\neq 1}}\psi(t_1,a+t)F_D(D)\Bigg]$$
$$\leq 3^{n-1}E_\sigma\big(F_{t_1}(D)F_D(D)\big),$$

where the last inequality follows using the transfer matrix $e^{-H(D-t_1)}$ and the Schwarz inequality. Therefore,

$$\sum_{t\in \mathrm{I}}\frac{E_\sigma(F_0(D)\psi(t))E_\sigma(\psi(t)F_D(D))}{DE_\sigma(F_0(D)F_D(D))}$$
$$\leq c\sup_{0\leq t_1\leq D}\frac{E_\sigma\big(F_0(D)F_{t_1}(D)\big)E_\sigma\big(F_{t_1}(D)F_D(D)\big)}{D^{3(n-1)}E_\sigma(F_0(D)F_D(D))}. \tag{26}$$

The Schwarz inequality and the positivity of H imply $\langle u, Hu\rangle_\sigma \leq \|u\|\,\|Hu\|$. Therefore, with $u = \sum_{|x_i|\leq D^3} u(x)$,

$$\frac{d^2}{ds^2}\left[E_\sigma(F_0(D)F_s(D))E_\sigma(F_s(D)F_D(D))\right]$$

$$= \frac{d^2}{ds^2}\left(\langle u, e^{-sH}u\rangle_\sigma \langle u, e^{-(D-s)H}u\rangle_\sigma\right)$$

$$= \langle H^2u, e^{-sH}u\rangle_\sigma \langle u, e^{-(D-s)H}u\rangle_\sigma + \langle u, e^{-sH}u\rangle_\sigma \langle H^2u, e^{-(D-s)H}u\rangle_\sigma$$

$$-2\langle Hu, e^{-sH}u\rangle_\sigma \langle Hu, e^{-(D-s)H}u\rangle_\sigma$$

$$\geq \Big[\langle Hu, e^{-sH}Hu\rangle_\sigma^{1/2} \langle u, e^{-(D-s)H}u\rangle_\sigma^{1/2}$$

$$- \langle u, e^{-sH}u\rangle_\sigma^{1/2} \langle H, ue^{-(D-s)H}Hu\rangle_\sigma^{1/2}\Big]^2 \geq 0.$$

Therefore, the sup in (26) must occur at either $t = 0$ or $t = D$. Then by (12)

$$\sum_{t\in I} \frac{E_\sigma(F_0(D)\psi(t))E_\sigma(\psi(t)F_D(D))}{DE_\sigma(F_0(D)F_D(D))}$$

$$\leq cE_\sigma(F_0(D)F_0(D))/D^{3(n-1)}$$

$$\leq c \sum_{x\in Z^{n-1}} E_\sigma(\psi(0, \ldots, 0)\psi(0, x))$$

$$\leq c \sum_{x\in Z^{n-1}} \exp(-cm(\sigma)|x|)$$

$$\leq c/m(\sigma)^{n-1}.$$

This completes the proof of (23) and with it the proof of Lipschitz continuity (21) and (22).

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