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Are Adobe Walls Optimal Phase-Shift Filters?*

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The adobe house construction gives an automatic, air conditioning effect because the rooms tend to be cool at midday and warm at night. Presumably this is brought about by the walls acting as a heat filter so that there is nearly a 12-hour phase lag. This raises the question of how to optimize the adobe phenomenon by a suitable design of the walls. In this study it is supposed possible to make the walls of layered construction with layers having different thermal resistivity. Such a layered wall can be modeled electrically as a ladder filter of capacitors and resistors. The input to the ladder is a sinusoidal voltage. Then the following question arises: If the filter capacitors have given values, how should the resistors be chosen so that the output voltage has a given phase lag but least attenuation? It is found possible to answer this question by use of a special variational principle. Applying this analysis to building construction shows how to maximize oscillation of interior temperature with a phase lag of a prescribed number of hours.

1. INTRODUCTION

The daily variation in outdoor temperature causes a related variation of indoor temperature in an unheated building. However, the indoor temperature is not directly proportional to the outdoor temperature because the walls of the building act as a filter. Thus the high and low temperatures outside cause high and low temperatures inside but there is a time lag. The amount of the lag depends on the nature of the outer walls. Moreover, the magnitude of the oscillation of temperature inside is not as great as the magnitude of the oscillation outside. This attenuation of oscillation is also dependent on the nature of the walls.

The time lag of peak temperatures can be of practical value in hot climates. Thus, if by a suitable wall design the lag can be made to be around 12 hours, then the interior of the building will be coolest during the heat of the day. Presumably the adobe houses of the American Southwest made use of this natural air conditioning method.

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To enhance the cooling effect the wall design should not only give a desired lag but should also produce the least attenuation in temperature oscillation. It is this optimization question which prompted our study. We are now led to the conclusion that the wall should not be homogeneous. One feasible design is for the wall to be composite, made of several layers having different thermal resistivities. We give an algorithm which prescribes these resistivities.

To treat this optimization question we find it advantageous to use an electrical model of heat flow. Thus a nonhomogeneous wall is modeled as an electrical filter of the ladder type composed of resistors and capacitors. The capacitors all share a common ground terminal. The input to the filter is a sinusoidal voltage. The output is the voltage across the last capacitor. The output is also a sinusoid but suffers both attenuation in amplitude and lag in phase. Thus our optimization question can be stated as follows: Given the values of the capacitors and a desired phase lag what should the values of the resistors be in order to minimize the attenuation? A variational analysis leads to the following theorem. If a filter is optimal, then at each resistor,

$$\text{phase}\, y + \text{phase}\, y^* = \text{constant}, \qquad (*)$$

where y is the current through the resistor and y^* is the current through the resistor in a reverse state. A reverse state corresponds to shorting the input end of the filter and energizing the output end. We term relation (*) the *equiphase condition*. It serves as a basis for an algorithm to construct filters giving minimum attenuation.

The algorithm for filter construction is easy to implement on a computer. Moreover, we are thereby enabled to give an empirical formula expressing the attenuation of an optimum filter in terms of the phase shift and the capacity ratio. The capacity ratio is approximately the ratio of the mass of the walls to the mass of the interior of the building. We may picture the wall mass as a "thermal flywheel" and the interior mass as the "load" to be driven through a suitable resistive coupling.

For the problems posed, a layered wall offers considerable practical advantage over a uniform wall. Thus, in a typical situation, a uniform wall would give over twice the attenuation of a layered wall. Moreover, the uniform wall would need to be more massive to achieve the desired phase lag.

2. Flow of Heat through a Wall

Let the temperature of a building wall be $U(x, t)$ where x is the distance measured from the outside ($x = 0$) toward the inside ($x = L$). The time is denoted by t. Let the heat current density be $Y(x, t)$. Then the temperature and current in the wall satisfy the system of differential equations:

$$\frac{\partial U}{\partial x} = -rY, \qquad \frac{\partial Y}{\partial x} = -c\frac{\partial U}{\partial t}. \tag{a}$$

Here r is the thermal resistivity of the wall and c is the heat capacity per unit volume of the wall. It is often the case that walls are of layered construction. In that case r and c would be functions of x.

The temperature and current are determined by solving the system (a) subject to boundary conditions at $x = 0$ and $x = L$. If the building were a hollow shell, then the boundary conditions on the inside surface ($x = L$) would be $Y = 0$. However, the usual situation is for the interior of the building to have a total heat capacity comparable to the total heat capacity of the wall. Let the *load capacitance* C be defined as the total heat capacity of the interior divided by the area of the wall. Let V be the temperature of the interior material. Then the following boundary condition is assumed at $x = L$:

$$Y(L_3t) = C\frac{dV(t)}{dt} = \frac{U(L, t) - V(t)}{R_L}, \tag{b}$$

where the constant R_L is the coupling resistance between the wall and the interior material. It is known that R_L is not zero because the air layer at the surface of the wall acts to resist heat transfer. Thus R_L is the "skin resistance" of unit area of the wall.

There is also a skin resistance R_o at the outer surface of the wall. Thus the following boundary condition is assumed at $x = 0$:

$$R_o Y(0, t) = E(t) - U(0, t). \tag{c}$$

Here E is the ambient temperature outside the building.

Attention is to be directed to the steady situation when the ambient temperature has a daily sinusoidal variation about a mean value. It is convenient to choose the temperature scale so that the mean temperature is zero. Then using the conventional complex representation of sinusoids we write

$$E(t) = e_1 e^{i\omega t},$$

where ω is the angular frequency and e_1 is the constant amplitude of oscillation. Let

$$U(x, t) = u(x)e^{i\omega t}, \qquad Y(x, t) = y(x)e^{i\omega t}.$$

Then the heat equations (a) become

$$\frac{du}{dx} = -ry, \qquad \frac{dy}{dx} = -i\omega cu. \tag{a'}$$

Let $V(t) = ve^{i\omega t}$. Then the boundary conditions (b) and (c) become

$$V + Ry(L) = u(L) = (1 + i\omega RC)v, \tag{b'}$$

$$u(0) + R_0 y(0) = e_1. \tag{c'}$$

Let the *attenuation factor* f be defined as

$$f = |e_1/v|.$$

Let the *capacity ratio* g be defined as

$$g = \int_0^L c(x)\, dx/C.$$

Since heat capacity is approximately proportional to weight the total heat capacity of the wall and the total heat capacity of the interior are not readily varied. Consequently the capacity ratio should not be regarded as a design variable. However, there is very little restriction on resistance. Thus the basic design question can be phased as follows.

PROBLEM 1. *Let the heat capacity function $c(x)$ of the wall and the load capacitance C of the interior of a building be given. Then seek the resistance function $r(x)$ of the wall and the skin resistances R_0 and R_L of the wall so as to minimize the attenuation of interior temperature oscillation subject to a prescribed phase lag.*

To make the problem more precise it is reasonable to restrict $r(x)$ and $c(x)$ to the class of piecewise continuous functions with positive lower bounds. Also it must be recognized that there may be no minimum but only an infimum.

INVARIANCE LEMMA. *The infimum value of the attenuation in Problem 1 depends only on the capacity ratio g and not on the choice of the heat capacity function $c(x)$.*

Proof. We show that if given a system with a certain attenuation, then the same attenuation can be achieved by a standard system. The standard system has a wall of unit thickness ($L = 1$) and a unit heat capacity function ($c(x) = 1$). The load capacitance of the interior in the standard system is taken to be $1/g$.

Let $K = \int_0^L c(z)dz$ and let $\varphi(x) = K^{-1}\int_0^x c(z)dz$. Then $\varphi(x)$ is a smooth increasing function and $\varphi(L) = 1$. In the differential equations (a′) make the change of variables

$$\bar{x} = \varphi(x), \qquad \bar{u} = u, \qquad \bar{y} = y/K.$$

Then clearly we obtain a standard system

$$\frac{d\bar{u}}{d\bar{x}} = -\bar{r}\bar{y}, \qquad \frac{d\bar{y}}{d\bar{x}} = -i\omega\bar{u},$$

where $\bar{r} = K^2 r/c$. Since the quotient of admissible functions is admissible, we see that $\bar{r}$ is piecewise continuous.

Let $\bar{R}_o = R_o/K$; this satisfies the boundary condition (c′) at $x = 0$. Let $\bar{R}_1 = R_L Cg$; this satisfies the boundary condition (b′) at $x = 1$. Then the standard system, so chosen, has the same value of e_1 and v as the given system. Thus the two systems have the same attenuation. This is seen to complete the proof of the lemma.

The differential equations (a) for temperature and heat current in a wall can be interpreted as the equations for voltage and electric current on a suitable electrical transmission line. Thus Problem 1 has a direct electrical analog. As is well known, a transmission line can be approximated by a network of lumped elements, the so-called ladder filter. In this way Problem 1 can be discretized.

In this paper we will not analyze Problem 1 further by differential equation methods (we hope to do so at a later time). Instead we will jump to the discretized version which we term Problem 2. The discrete model has the following properties:

(i) The model has direct physical interpretation.

(ii) The skin resistances R_o and R_L enter the problem formulation automatically as r_1 and r_n in Fig. 1.

(iii) Problem 2 is compact in nature and so has a minimum, not merely an infimum.

(iv) A difference equation analysis leads directly to an optimization algorithm.

(v) The algorithm is quite easy to evaluate with a computer.

(vi) A theorem for a discrete system often has a limiting version for the limiting continuous system. Thus we expect the equiphase theorem to hold also for Problem 1.

(vii) The invariance lemma just proved shows that it is necessary to consider only the simple case in which the capacity function is unity. In the discrete model this suggests that all the capacitors except the last be taken with value unity.

In the formulation of Problem 1 it is assumed that it is possible to vary the skin resistances. This might be achieved practically by adding a thin layer of very resistive material to the wall surface.

3. Characterizing Optimum Phase-shift Filters

The problem of the adobe wall leads us to an electrical model of the wall. This model is an RC ladder filter of the form shown in Fig. 1. The filter is driven by a generator having sinusoidal voltage e_1 at unit angular frequency. The n meshes of the filter have capacitors with positive values, $c_1, c_2, \ldots, c_n$. The capacitors are joined by resistors having nonnegative values $r_1, r_2, \ldots, r_n$. The currents through the resistors have complex values $y_1, y_2, \ldots, y_n$. Let v_n be the voltage at the last capacitor.

We regard e_1 as the input voltage and v_n as the output voltage. Let ψ denote the voltage ratio of input to output. Thus

$$\psi = \frac{e_1}{v_n} = fe^{i\theta},$$

where $f > 0$ and $0 \leq \theta < 2\pi$. We term f the *attenuation factor* and θ the *phase lag*. The discrete analog of Problem 1 may now be phrased as follows.

Problem 2. *Given the capacitance values $c_1, c_2, \ldots, c_n$ and an angle β in the range $0 < \beta < 2\pi$, seek the resistance values $r_1, r_2, \ldots, r_n$ so as to minimize the attenuation factor f subject to the constraint $\theta \geq \beta$ on the phase lag.*

To treat this problem we express Kirchhoff's laws for the network of Fig. 1 as follows:

$$\begin{aligned} e_1 &= (r_1 + s_1)y_1 - s_1 y_2 \\ e_2 &= -s_1 y_1 + (r_2 + s_2 + s_1)y_2 - s_2 y_3 \\ &\vdots \\ e_n &= -s_{n-1} y_{n-1} + (r_n + s_n + s_{n-1})y_n. \end{aligned} \tag{1}$$

Here $s_k = 1/ic_k$ is the impedance of the kth capacitor. Here e_k is the voltage of a generator in series with the kth resistor. Of course in Fig. 1, $e_k = 0$ if $k > 1$. Equations (1) are seen to be the well-known network mesh equations.

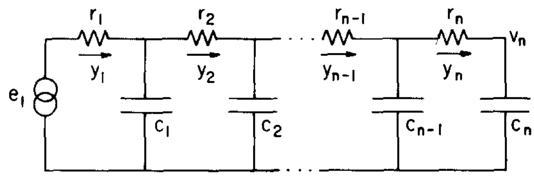

FIG. 1. The ladder filter of n meshes.

The mesh equations may be expressed in abbreviated notation as

$$e_j = \sum_{k=1}^{n} Z_{jk} y_k \qquad j = 1, \ldots, n, \tag{2}$$

where Z_{jk} is seen to be a symmetric matrix. Since all $C_k > 0$ it can be shown that Z_{jk} is nonsingular.

RECIPROCITY LEMMA. *If* (e_j', y_j') *and* (e_j^*, y_j^*) *are two solutions of the network equations, then*

$$\sum_{j=1}^{n} e_j' y_j^* = \sum_{j=1}^{n} e_j^* y_j'. \tag{3}$$

Proof. This is an obvious consequence of relation (2) because $Z_{jk} = Z_{kj}$.

Given a solution (e_j, y_j) of the mesh equations, consider a variation in the solution due to variations $\delta r_1, \ldots, \delta r_n$ in $r_1, \ldots, r_n$. The e_j and c_j are held constant so there will be variations $\delta y_1, \ldots, \delta y_n$. Then taking the variation of the mesh equation (2) gives

$$0 = \sum_{k=1}^{n} Z_{jk} \delta y_k + y_j \delta r_j \qquad j = 1, \ldots, n.$$

It is useful to write this in the form

$$e_j' = \sum_{k=1}^{n} Z_{jk} y_k' \qquad j = 1, \ldots, n, \tag{4}$$

where $y_j' = \delta y_j$ and $e_j' = -y_j \delta r_j$. We term (e_j', y_j') a variational solution to the mesh equations. The existence of a unique variational solution results from the fact that Z_{jk} can be shown to be nonsingular.

VARIATIONAL LEMMA. *Let δy_n be the variation in the current y_n due to variations $\delta r_1, \ldots, \delta r_n$ in the resistors. If capacitors and generators are unvaried,*

$$\delta y_n = -(e_n^*)^{-1} \sum_{j=1}^{n} y_j y_j^* \delta r_j, \tag{5}$$

where (e_j^*, y_j^*) is a solution of the network equations such that $e_n^* \neq 0$, $e_j^* = 0$ if $j \neq n$.

Proof. We term (e_j^*, y_j^*) a *reverse solution,* because input and output of the filter have been reversed, as shown in Fig. 2. Substitute a reverse solution (e^*, y^*) and a variational solution into the reciprocity equation (3). The left side of (3) is evaluated as

$$\sum_1^n e_j^* y_j' = e_n^* \delta y_n.$$

The right side of (3) is evaluated as

$$\sum_1^n e_j' y_j^* = -\sum_1^n y_j y_j^* \delta r_j.$$

Equating the left side and the right side gives

$$e_n^* \delta y_n = -\sum_1^n y_j y_j^* \delta r_j.$$

This proves the variational lemma.

The input impedance Z of the filter is defined as

$$Z = e_1 / y_1.$$

Then we see that

$$Z y_1 y_1^* = e_1 y_1^* = e_n^* y_n, \tag{6}$$

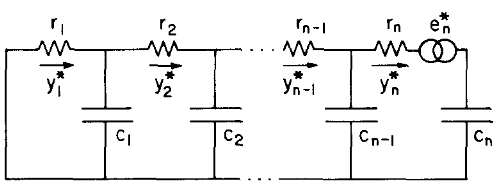

FIG. 2. The reversed filter.

where the second equation comes from the reciprocity lemma. Now note that $v_n = s_n y_n$, so

$$-\frac{\delta y_n}{y_n} = -\frac{\delta v_n}{v_n} = \frac{\delta\psi}{\psi} = \frac{\delta f}{f} + i\delta\theta. \tag{7}$$

Substituting (6) and (7) in the variational expression (5) gives another formulation.

$$\frac{\delta f}{f} + i\delta\theta = \frac{1}{Z}\sum_{j=1}^{n}\frac{y_j y_j^*}{y_1 y_1^*}\,\delta r_j. \tag{5a}$$

IMPEDANCE LEMMA. *The impedance Z of an RC ladder filter has a negative imaginary part and a positive real part. (It is supposed that at least one resistance is positive.)*

Proof. A filter of one mesh has impedance $r + s = r - i/c$ and so satisfies the lemma. For n meshes the series and parallel formulas give

$$Z = r_1 + \left(s_1^{-1} + Z_1^{-1}\right)^{-1},$$

where Z_1 is the impedance of the filter with the first mesh removed. This is seen to furnish an inductive proof of the lemma.

LOCAL DECAY LEMMA. *In the RC ladder filter with positive resistors the resistor currents (and the capacitor voltages) decay both in amplitude and phase from mesh to mesh. However the phase decay is less than 90 degrees.*

Proof. By Kirchhoff's current law

$$y_j = y^c + y_{j+1} \qquad j = 1, \ldots, n-1 \tag{8}$$

where y^c is the current through the capacitance c_j. The section of the filter following c_j is also an RC ladder with impedance Z_j. By the impedance lemma $Z_j = a - ib$ with $a > 0$, $b > 0$. Let v_j be the voltage across capacitance c_j; then Kirchhoff's voltage law gives

$$v_j = s_j y^c = Z_j y_{j+1}.$$

Eliminating y^c between this and (8) gives

$$y_j/y_{j+1} = Z_j/s_j + 1 = 1 + c_j b + i c_j a.$$

Then y_j/y_{j+1} has absolute value greater than 1 and phase between 0 and 90 degrees. This proves the statement about resistor currents. The statement about capacitor voltages follows by a similar argument.

COROLLARY 1. *The phase lag of an RC ladder filter with n meshes is less than n × 90 degrees. The attenuation factor is greater than* 1.

The second statement is not true for general RC electrical filters. There are simple counterexamples with an attenuation factor less than 1. However, such electrical networks have no thermal counterpart.

THEOREM 1. *Suppose that the attenuation of an RC ladder filter is a relative minimum subject to a phase lag constraint. Then for* $k = 1, \ldots, n$,

$$\text{phase}\, y_k + \text{phase}\, y_k^* = \text{constant}. \tag{9}$$

Here $y_1, \ldots, y_n$ *are resistor currents of the filter and* $y_1^*, \ldots, y_n^*$ *are resistor currents of the reversed filter.*

Proof. It is easy to show that no mesh current can vanish. Thus the phases of y_k and y_k^* in (9) are well defined. We term relation (9) the *equiphase condition*.

At first suppose that all the resistance values are positive. Then if only r_j and r_{j+1} are varied, formula (5a) becomes

$$\frac{\delta f}{f} + i\delta\theta = \frac{1}{Z}\left(\frac{y_j y_j^*}{y_1 y_1^*}\right)(\delta r_j + \varphi \delta r_{j+1}), \tag{5b}$$

where $\varphi = y_{j+1} y_{j+1}^* / y_j y_j^*$. The variations δr_j and δr_{j+1} can be arbitrary real numbers, so if φ were not real they could be chosen so that $\delta f < 0$ and $\delta\theta > 0$. This means that f does not have a relative minimum. The contradiction shows that φ is real.

It follows from the local decay lemma that y_{j+1}/y_j is in the fourth quadrant. The corresponding statement for the reversed filter is that y_{j+1}^*/y_j^* is in the first quadrant. Then since $\varphi = (y_{j+1}/y_j)(y_{j+1}^*/y_j^*)$, it follows that φ is in the right half-plane. However, it was shown above that φ is real. The conclusion is that φ is positive. In other words,

$$\text{phase}\, y_j + \text{phase}\, y_j^* = \text{phase}\, y_{j+1} + \text{phase}\, y_{j+1}^*$$

for $j = 1, \ldots, n - 1$. This proves (9) in the case that all resistance values are positive.

THEOREM 2. *Suppose that the attenuation of an RC ladder filter is a relative minimum subject to a phase lag constraint. Then all resistances have positive values.*

Proof. At first suppose that only $r_1 = 0$. Then in the variational formula (5b) take $j = 1$ so

$$\frac{\delta f}{f} + i\delta\theta = \frac{1}{Z}(\delta r_1 + \varphi \delta r_2). \tag{5c}$$

Since $r_1 = 0$ the network equations (1) give for the first mesh $0 = s_1 y_1^* - s_1 y_2^*$. Thus $y_1^* = y_2^*$ and

$$\varphi = \frac{y_2 y_2^*}{y_1 y_1^*} = \frac{y_2}{y_1} = g - ih.$$

Here, by virtue of the local decay lemma, $g > 0$ and $h > 0$. Then by the impedance lemma $Z^{-1} = G + iH$ with $G > 0$ and $H > 0$. Then take $\delta r_1 = 1$ and $\delta r_2 = -1/g$ so

$$\frac{\delta f}{f} + i\delta\theta = -\frac{Hh}{g} + i\frac{Gh}{g}.$$

This shows that $\delta f < 0$ and $\delta\theta > 0$, which contradicts the assumption that the attenuation is a relative minimum. Hence $r_1 > 0$.

Next suppose that only $r_2 = 0$ at the minimum. Then c_1 and c_2 are connected in parallel, giving a joint capacitance $c_1' = c_1 + c_2$. This joining gives a filter with all positive resistances and Theorem 1 applies. Now it is observed that the network equations (1) give for the second mesh

$$0 = -s_1 y_1 + (s_1 + s_2)y_2 - s_2 y_3.$$

In other words

$$y_2 = \mu y_1 + \nu y_3$$

where $\mu + \nu = 1$, $\mu > 0$, and $\nu > 0$. Likewise

$$y_2^* = \mu y_1^* + \nu y_3^*.$$

Multiplying these last two equations gives

$$\varphi = \frac{y_2 y_2^*}{y_1 y_1^*} = \mu^2 + \nu^2 \frac{y_3 y_3^*}{y_1 y_1^*} + \mu\nu\left(\frac{y_3^*}{y_1^*} + \frac{y_3}{y_1}\right).$$

Let $y_3/y_1 = De^{-i\gamma}$ and $y_3^*/y_1^* = D^* e^{i\gamma}$ so

$$\varphi = \mu^2 + \nu^2 DD^* + \mu\nu(D^* + D)\cos\gamma + i\mu\nu(D^* - D)\sin\gamma.$$

It follows from the local decay lemma that $D^* > 1 > D$ and $0 < \gamma < 90°$. Hence $\varphi = g + ih$ where $g > 0$ and $h > 0$. Substitution in (5c) gives

$$\frac{\delta f}{f} + i\delta\theta = \frac{1}{Z}\left[\delta r_1 + (g + ih)\delta r_2\right].$$

If we take $\delta r_2 = 1$ and $\delta r_1 = -g$ it is seen that $\delta f < 0$ and $\delta\theta > 0$, a contradiction. Hence $r_2 > 0$.

Next suppose only $r_3 = 0$; then the variational formula (5b) for $j = 2$ can be written as

$$\frac{\delta f}{f} + i\delta\theta = \frac{1}{Z}\left(\delta r_2 + \frac{y_3 y_3^*}{y_2 y_2^*}\,\delta r_3\right)\left(\frac{y_2 y_2^*}{y_1 y_1^*}\right).$$

Since the last factor is positive we can use a similar argument to show that $\delta f < 0$ and $\delta\theta > 0$. Thus contradiction shows that $r_3 > 0$.

This process can be repeated up to $j = n - 2$. Finally we come to case $j = n - 1$ when it is supposed that only $r_n = 0$. Then the variational formula (5b) for $j = n - 1$ can be written as

$$\frac{\delta f}{f} + i\delta\theta = \frac{1}{Z}(\delta r_{n-1} + \varphi\delta r_n)\left(\frac{y_{n-1} y_{n-1}^*}{y_1 y_1^*}\right).$$

However, the network equations (1) give $y_{n-1}/y_n = (c_{n-1}/c_n + 1) = A > 0$ and so

$$\varphi = \frac{y_n y_n^*}{y_{n-1} y_{n-1}^*} = \frac{y_n^*}{A y_{n-1}^*} = g + ih.$$

By the local decay lemma $g > 0$ and $h > 0$. Repeating previous arguments we are again led to a contradiction. This shows that $r_n > 0$.

Next use induction and suppose that it has been shown that $k - 1$ resistors cannot be zero at the minimum. Then the case of k resistors being zero can be reduced to the case $k - 1$ by following the same arguments given above. This is seen to complete the proof of Theorem 2 and Theorem 1 as well.

COROLLARY 2. *Let Z be the impedance of an equiphase filter; then*

$$\text{phase } Z = \text{phase } Z^*,$$

where Z^* is the impedance of the reversed filter.

Proof. The reciprocity lemma gives

$$e_1 y_1^* = e_n^* y_n.$$

But $e_1 = Z y_1$ and $e_n^* = Z^* y_n^*$, so

$$Z y_1 y_1^* = Z^* y_n y_n^*.$$

By the equiphase property $y_n y_n^*/y_1 y_1^*$ is a positive number, so the proof is complete.

It is clear, by a compactness argument, that if the phase constraint $\theta \geq \beta$ can be satisfied, then there are resistance values $r_1, r_2, \ldots, r_n$ which give a minimum of attenuation. We now assume that the variation of the resistance values with β is smooth.

COROLLARY 3. *The rate of change of the minimum attenuation f with the phase lag θ is*

$$\frac{d \ln f}{d\theta} = -\frac{\operatorname{Re} Z}{\operatorname{Im} Z}$$

where Z is the impedance.

Proof. By the assumption of smoothness the variational formula (5a) can be written as

$$Z\left(\frac{d \ln f}{d\theta} + i\right) = \sum_{j=1}^{n} \frac{y_j y_j^*}{y_1 y_1^*} \frac{dr_j}{d\theta}.$$

Here the summation on the right side has only real terms. Thus the imaginary part of the left side is zero. This is seen to prove the corollary.

COROLLARY 4. *The minimum attenuation decreases as the number of meshes increases, provided the capacity ratio is held constant.*

Proof. Consider an optimum filter with n meshes and introduce a new mesh by replacing one of the capacitors (not the last) by two capacitors in parallel, each with half the capacitance. Of course this does not change the capacity ratio. This filter with $n + 1$ meshes is not optimal because according to Theorem 2 there should be a positive resistance between the two half capacitors. This proves the corollary.

Two other properties are worth noting: First observe that at the minimum the phase constraint is tight. This can be seen from the variational formula (5a). Second note that it is possible to have relative minima which are not absolute minima. This could happen, for example, if the successive phase shifts total $2\pi + \beta$ instead of β.

4. CONSTRUCTION OF AN EQUIPHASE FILTER

An algorithm is now to be given to form a filter which satisfies the equiphase condition (9) of Theorem 1. The capacitor values $c_1, c_2, \ldots, c_{n-1}$ are any given positive numbers. However, the terminating capacitance c_n is not arbitrary but is fixed by the algorithm. On the other hand, the algorithm has two arbitrary parameters. They could be used to specify the input impedance.

There is no loss of generality in taking the following normalization conditions

$$y_1 = 1, \qquad y_1^* = 1. \tag{10}$$

Since $e_1^* = 0$, the first mesh equation gives

$$y_2^* = 1 + ir_1c_1. \tag{11}$$

Now we introduce two arbitrary positive parameters Q and P and define

$$y_2 = Q(1 - iP). \tag{12}$$

Thus

$$y_2y_2^* = Q(1 + r_1c_1P) + i(r_1c_1 - P).$$

But $y_1y_1^* = 1$, so to satisfy the equiphase condition we must set

$$P = r_1c_1. \tag{13}$$

The input impedance is $Z = e_1/y_1$, and the first mesh equation gives

$$Z = \left[(1 + Q)P - i(1 - Q)\right]/c_1. \tag{14}$$

As a convenient notation let

$$p_j = r_jc_j, \qquad h_j = c_{j-1}/c_j \qquad \text{for } j = 2, \ldots, n - 1. \tag{15}$$

Then the jth mesh equation for $2 \leq j \leq n - 1$ takes on the form

$$y_{j+1} = (ip_j + h_j + 1)y_j - h_jy_{j-1} \tag{16}$$

and

$$y_{j+1}^* = (ip_j + h_j + 1)y_j^* - h_jy_{j-1}^*. \tag{17}$$

Multiply equations (16) and (17) to obtain the positive quantity $y_{j+1}y_{j+1}^*$. Thus

$$y_{j+1}y_{j+1}^* = \left\{-p_j^2 + (h_j + 1)^2 + i2p_j(h_j + 1)\right\}y_jy_j^* + h_j^2y_{j-1}y_{j-1}^*$$
$$- \left(ip_jh_j + h_j^2 + h_j\right)\left(y_jy_{j-1}^* + y_{j-1}y_j^*\right).$$

Divide this relation by $y_jy_j^*$ and let

$$x_j = y_{j-1}/y_j \qquad x_j^* = y_{j-1}^*/y_j^* \tag{18}$$

so

$$1/x_{j+1}x_{j+1}^* = -p_j^2 + (h_j + 1)^2 + i2p_j(h_j + 1) + h_j^2x_jx_j^*$$
$$- \left(ip_jh_j + h_j^2 + h_j\right)\left(x_j + x_j^*\right).$$

By the equiphase condition, $x_j x_j^*$ is positive, so taking the imaginary part of this last equation gives

$$0 = \mathrm{Im}\left[-\left(h_j^2 + h_j\right)(x_j + x_j^*)\right] + \mathrm{Re}\left[2p_j(h_j + 1) - p_j h_j(x_j + x_j^*)\right].$$

Solving for p_j gives

$$p_j = \frac{\left(h_j^2 + h_j\right)\mathrm{Im}(x_j + x_j^*)}{2(h_j + 1) - h_j \mathrm{Re}(x_j + x_j^*)}. \tag{19}$$

It is now seen that relations (16), (17), and (19) are recursion formulas which can be used to determine $y_3, y_4, \ldots, y_n$ and $y_3^*, y_4^*, \ldots, y_n^*$ and $p_2, p_3, \ldots, p_{n-1}$.

It remains to consider the last mesh. The currents y_{n-1} and y_n have been determined but not r_n and c_n. To determine these we write the last equation as

$$-\frac{y_{n+1}}{ic_{n-1}} + \left(r_n + \frac{1}{ic_n} + \frac{1}{ic_{n-1}}\right)y_n = 0. \tag{20}$$

Thus

$$x_n = \frac{y_{n-1}}{y_n} = ir_n c_{n-1} + \frac{c_{n-1}}{c_n} + 1$$

so

$$r_n = (\mathrm{Im}\ x_n)/c_{n-1} \tag{21}$$

and

$$c_n = c_{n-1}/\left(\mathrm{Re}\ x_n - 1\right). \tag{22}$$

It is now seen that all elements of the filter have been determined.

The complex attenuation factor ψ is given as

$$\begin{aligned}\psi = e_1/v_n &= \{r_1 y_1 + (y_1 - y_2)/ic_1\}/\{y_n/ic_n\}\\ &= (c_n/c_1)(y_1 - y_2 + ir_1 c_1 y_1)/y_n.\end{aligned}$$

Substituting (11), (12), and (22) gives

$$\psi = \frac{c_{n-1}}{c_1}\frac{\{1 - Q + iP(1 + Q)\}}{(\mathrm{Re}\ x_n - 1)y_n}. \tag{23}$$

This determines the real attenuation $f = |\psi|$ and the phase lag θ = phase ψ.

Another quantity of physical significance is the *capacity ratio* g defined as

$$g = (c_1 + c_2 + \ldots + c_{n-1})/c_n. \tag{24}$$

The last capacitance c_n is termed the *load capacitance*.

The algorithm just defined leads to a ladder filter of the form shown in Fig. 1 provided that the choice of the $n + 1$ positive numbers c_1, $c_2, \ldots, c_{n-1}$ and P and Q leads to positive values in the expression for p_2, $p_3, \ldots, p_{n-1}$ and r_n and c_n. This is an equiphase filter because it is seen that the choice of p_j by expression (19) for $j = 2, 3, \ldots, n - 1$ causes $y_{j+1}y^*_{j+1}$ to be positive.

It is easy to see that if the filter is terminated at the jth mesh rather than at the nth mesh then again an equiphase filter results, provided the terminating r_j and c_j are suitably redefined. We will let these new values be denoted by capital letters R_j and C_j and term them *skin resistance* and *load capacitance*. Thus

$$R_j = \frac{1}{c_{j-1}} \operatorname{Im} x_j \tag{21a}$$

$$C_j = c_{j-1}/(\operatorname{Re} x_j - 1). \tag{22a}$$

5. Methods of Calculation

The algorithm just described is very easy to implement with a computer. It is first necessary to select the capacitances $c_1, \ldots, c_{n-1}$. The simple model in which each $c_j = 1$ reveals much of the general behavior. Then we select Q and P to satisfy $Q^2 + Q^2P^2 < 1$. For $j = 2, \ldots,$ let $x_j = y_{j-1}/y_j$ and make the following recursive calculations:

$$r_j = 2[\operatorname{Im}(x_j + x_j^*)]/[4 - \operatorname{Re}(x_j + x_j^*)],$$
$$y_{j+1} = (2 + ir_j)y_j - y_{j-1},$$
$$y^*_{j+1} = (2 + ir_j)y_j^* - y^*_{j-1}.$$

Then for each value of j we may tabulate:

$R_j = \operatorname{Im} x_j$	skin resistance,
$C_j = [\operatorname{Re} x_j - 1]^{-1}$	load capacitance,
$\psi_j = [1 - Q + iP(1 + Q)]C_j y_j^{-1}$	Complex attenuation,
$g_j = (j - 1)/C_j$	capacity ratio.

Thus for each value of j an equiphase filter results. We have made a number of calculations of this kind and also in the case where the capacitances change in a geometric progression.

For the purpose of studying the adobe wall problem it is of main interest to have phase lags in the range of 90 to 180° and to have capacity ratios of at least unity. It is not difficult to select Q and P and j to obtain data in the above ranges. For j about 7 or greater the results do not depend appreciably on j.

An important parameter in the calculations is denoted by λ and is defined in terms of the impedance as

$$\lambda = -\frac{\operatorname{Im} Z}{\operatorname{Re} Z} = \frac{1 - Q}{(1 + Q)P}. \tag{25}$$

Note that λ appears in Corollary 3. Our numerical results show that the minimum attenuation f, the phase lag θ, and the impedance Z are related by the following approximate formula:

$$\frac{1}{\ln f} = \frac{2\lambda}{\theta} - \frac{1}{8} \qquad (\theta \text{ in radians}). \tag{26}$$

The accuracy is highest when f is small.

It is of especial importance to know how f and θ depend on the capacity ratio g. Our numerical results show that f, θ, and g are related by the following approximate formula:

$$f = \frac{e^{\theta/45}}{3g^{1/2}} + 1 \qquad (\theta \text{ in degrees}). \tag{27}$$

This is accurate within 10% for $1 \leq g \leq 10$ and $\theta \geq 90°$.

Formulas (26) and (27) should be sufficiently accurate for design purposes. In a sequel to this paper we shall give practical considerations and numerical data.

References

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