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# LOCAL DEFINABILITY THEORY

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The main results of the theory of definability in pure logic can be grouped, roughly, into two classes: those of *local* and those of *global* nature.

As an example of the first, we can mention Scott's definability theorem [22]. In admittedly vague terms it states that a relation "determined" by the relations of a countable structure can be defined in the structure by an infinitary formula of the language $L_{\omega_1\omega}$ (see [9] for references on infinitary languages). The other class is exemplified by Svenonius' theorem [25]: assume that in every model $(\mathfrak{A}, P)$ of a complete first-order theory $T(P)$, $P$ is "determined" by the relations of $\mathfrak{A}$. Then in every model $(\mathfrak{A}, P)$ of $T(P)$, $P$ is "uniformly" definable by a first order formula (i.e., the definition does not depend on the model).

For first order logic, we can usually obtain a global result by "globalizing" local results for saturated or special models. The

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compactness theorem does the job (see for instance [2], [10], [11], [18]). Similarly, we can easily "localize" global results.

For the infinitary language $L_{\omega_1\omega}$, on the other hand, we have a split: there are local results such as Scott's theorem just mentioned and some global results (i.e., the analogue of Beth's theorem in [17]). However, in general, we cannot obtain one from the other.

In this paper we shall study further this "local theory of definition". By this term we shall mean (as a first approximation) the theory of Galois connection between groups of permutations of a set $X$ and objects constructed from $X$ (i.e., relations, sets of relations, etc.) which are "determined" (in a sense to be made precise later) by these groups. Similar programs have been considered earlier by M.Krasner [12] and J.Sebastião e Silva [24].

In § 1 we study groups of permutations of a set $X$ acting on relations (and sets of relations of $X$). By introducing a natural topology, a form of Baire category theorem can be proved for certain groups (Theorem 1.1.17 and 1.1.18). Several consequences are pointed out.

In § 2 we consider the problem of introducing algebraic structures on the objects correlated with the groups to complete our Galois theory. Only the case when the objects are sets of relations has been considered. We make our objects into polyadic algebras and Galois correspondences are set up between certain polyadic algebras and some groups of permutations (Theorems 2.3.12 and 2.3.14). This solves a problem raised by A.Daigneault. Some local results about $L_{\mu^+\mu}$ and $L_{\omega_1\omega}$ are also obtained.

The main result in § 3 is an improvement of the Chang-Makkai theorem [2,18] and proves a conjecture of M.Makkai. The proof makes use of certain results of § 1, "specialized" to special structures.

A much weaker version of § 1 appeared in our dissertation [20] as section 2 of chapter 3 and the main result of § 3 in [21].

We are highly indebted to our thesis advisor Professor W.Craig. He has emphasized the possibility of "non-linguistic" approaches to logic and the present paper is written in this spirit. We would also like to thank Professor M.Makkai. He suggested that Baire category techniques used by us in [20] could be applied to problems of definability. The possibility of interpreting some of our results in the context of a Galois theory was pointed out by Professor A.Daigneault, who referred us to the previous works of Krasner and Sebastião e Silva. We express him our thanks. Finally, we would also like to thank Mr. Vincent Papillon who took notes during a course we taught at Montreal including some of these subjects and made some valuable comments.

Our notation and terminology are taken for the most part from [19] and follow the recommendations of [0]. We shall not attempt the (often) hopeless task of making a list of all the notations to be used. ("Do not scratch if it doesn't itch!") We just notice that *structure* will be understood in the sense of relational structure with finitary or infinitary relations, possibly with distinguished elements.

We shall freely indulge in confusions of use and mention.

## § 1. Definability without language

### 1.1. *Topological groups*

Throughout this section, we let $A$ be an infinite set and we let $A!$ be the group of permutations of $A$.

1.1.1. *Definition*

(i) For every partial mapping $f$ from $A$ into $A$, we let

$$[f] = \{\pi \in A! : \pi \supseteq f\} .$$

(ii) For every infinite cardinal $\mu$ we define a class $\mathcal{B}_\mu$ of subsets of $A!$ as follows:

$Q \in \mathcal{B}_\mu$ iff $Q = [f]$ for some partial mapping from $A$ into itself such that $\overline{\overline{\mathrm{dom}(f)}} < \mu$.

1.1.2. For every infinite cardinal $\mu$, $\mathcal{B}_\mu$ is a basis for a topology on $A!$.

*Proof:* Obviously $A! = \bigcup \mathcal{B}_\mu$. Let $[f]$, $[g] \in \mathcal{B}_\mu$ and let $\pi \in [f] \cap [g]$. Hence $\pi \supseteq f$ and $\pi \supseteq g$. Let $h = \pi | \mathrm{dom}(f) \cup \mathrm{dom}(g)$. Since $\overline{\overline{\mathrm{dom}(f)}} < \mu$ and $\overline{\overline{\mathrm{dom}(g)}} < \mu$, thus $\overline{\overline{\mathrm{dom}(h)}} < \mu$. Hence $\pi \in [h] \in \mathcal{B}_\mu$ and $[h] \subseteq [f] \cap [g]$.

1.1.3. *Definition.* The topology defined by $\mathcal{B}_\mu$ on $A!$ is called the *$\mu$-topology.*

1.1.4. For every infinite cardinal $\mu$, $A!$ provided with the $\mu$-topology is a Hausdorff topological group.

*Proof:* Let $\pi, \pi' \in A!$ be such that $\pi \neq \pi'$. Hence $\pi(a) \neq \pi'(a)$ for some $a \in A$. Therefore $\pi \in [\pi | \{a\}]$, $\pi' \in [\pi' | \{a\}]$ and $[\pi | \{a\}] \cap [\pi' | \{a\}] = 0$. To finish our proof, we check that $A!$ is a topological group. Let $I: A! \to A!$ and $C: A! \times A! \to A!$ be defined by $I(\pi) = \pi^{-1}$ and $C(\pi_1, \pi_2) = \pi_1 \circ \pi_2$, respectively. If $\pi \in I^{-1}[f]$, then $\pi \in [\pi \mid \text{range}(f)] \subseteq I^{-1}[f]$, i.e., $f$ is continuous. Similarly, if $(\pi_1, \pi_2) \in C^{-1}[f]$, then

$$(\pi_1, \pi_2) \in [\pi_1 \mid \text{dom}(\pi_2 \circ f)] \times [\pi_2 \mid \text{dom}(f)] \subseteq C^{-1}[f] ,$$

i.e., $C$ is also continuous.

Henceforth, we shall tacitly use the fact that a subgroup of a topological group is itself a topological group (with the induced topology).

1.1.5. *Definition.* Let $\mu$ be an infinite cardinal and let $M \subseteq A!$ be the topological space with the induced $\mu$-topology.

(i) $N$ is *$\mu$-meager* in $M$ iff $N = \bigcup\{N_\xi : \xi \in \mu\}$, for some sequence $\langle N_\xi : \xi \in \mu\rangle$ of nowhere dense subsets of $M$.
(ii) $N$ is *co-$\mu$-meager* in $M$ if $M \setminus N$ is $\mu$-meager in $M$.
(iii) $M$ is a *$\mu$-Baire space* iff 0 is the only open $\mu$-meager subset of $M$.
(iv) $M$ is a *$\mu$-Baire group* iff $M$ is a topological group which is also a $\mu$-Baire space.

The assumption that a group is $\mu$-Baire has several interesting consequences as the rest of 1.1 shows.

1.1.6. Let $\mu$ be an infinite cardinal and let $G \subseteq A!$ be a $\mu$-Baire group. Then either $\overline{\overline{G}} > \mu$ or $G \cap [\text{id}_X] = \{\text{id}_A\}$, for some $X \subseteq A$ such that $\overline{\overline{X}} < \mu$.

*Proof:* Assume that $\overline{\overline{G}} \leqslant \mu$. Then $\{\text{id}_A\}$ is not nowhere dense in $G$. In fact, assume the contrary. Then for each $\pi \in G$, $\{\pi\}$ is nowhere dense in $G$ by the homogeneity of $G$ (since every topological group is homogeneous). Therefore $G = \bigcup\{\{\pi\} : \pi \in G\}$ is $\mu$-meager in $G$, a contradiction. Hence $G \cap [\text{id}_X] \subseteq \{\text{id}_A\}$ for some subset $X \subseteq A$ such that $\overline{\overline{X}} < \mu$.

1.1.7. *Definition.* Let $G \subseteq A!$ be a group, $\pi \in G$, $\mu$ a cardinal, $\xi \in \mu$, $s \in {}^{\xi}A$ and $P \subseteq {}^{\xi}A$.

(i) The *image* of $s$ (respectively of $P$) under $\pi$ is defined as $\pi^*s = \langle \pi(s(\eta)) \colon \eta \in \xi \rangle$ (respectively as $\pi^*P = \{\pi^*s \colon s \in P\}$).
(ii) The *orbit* of $s$ (respectively of $P$) under $G$ is defined as $O^G(s) = \{\pi^*s \colon \pi \in G\}$ (respectively as $O^G(P) = \{\pi^*P \colon \pi \in G\}$).
(iii) The *group of stability* of $s$ (respectively of $P$) relative to $G$ is defined as $G(s) = \{\pi \in G \colon \pi^*s = s\}$ (respectively as $G(P) = \{\pi \in G \colon \pi^*P = P\}$).

1.1.8. Assume that $\mu$ is an infinite cardinal, $\xi \in \mu$, $P \subseteq {}^{\xi}A$ and $s \in {}^{\xi}A$. Then

(i) $G(s)$ is both open and closed in $G$ (provided with the $\mu$-topology).
(ii) $G(P)$ is closed in $G$ (provided with the $\mu$-topology).

*Proof:* (i) follows from $G(s) = G \cap [\mathrm{id}_{\mathrm{rg}(s)}]$.
(ii) Let $\pi \in \overline{G(P)}$ ($\overline{G(P)}$ denotes the topological closure of $G(P)$) and $s \in P$. Since $\overline{\overline{\mathrm{rg}(s)}} < \mu$ and $\pi \in [\pi | \mathrm{rg}(s)] \in \mathcal{B}_{\mu}$, $[\pi | \mathrm{rg}(s)] \cap G(P) \neq 0$. Let $\sigma \in [\pi | \mathrm{rg}(s)] \cap G(P)$. Then $\sigma | \mathrm{rg}(s) = \pi | \mathrm{rg}(s)$ and $\sigma^*s \in P$, since $\sigma^*P = P$. Hence $\pi^*s \in P$ and this shows that $\pi^*P = P$, i.e., $\pi \in G(P)$.

In terms of these notions we can introduce a *topological* measure for the dependence of a relation on a group.

1.1.9. *Definition.* Let $G \subseteq A!$ be a group, $\mu$ an infinite cardinal, $\xi \in \mu$ and $P \subseteq {}^{\xi}A$.

(i) $P$ is *$\mu$-determined* by $G$ (written $P$ is $G-\mu-$det) iff $G(P) = G$.
(ii) $P$ is *$\mu$-weakly-determined* by $G$ (written $P$ is $G-\mu-$w.det) iff $G(P)$ is open in $G$ (provided with the $\mu$-topology) †.

† Conditions (i) and (ii) are the "non-linguistic" counterparts of "definable" and "definable with parameters". See 1.1.13, 1.1.14, 2.2.1, 2.2.2 and 3.1.6.

(iii) $P$ is *$\mu$-free* over $G$ (written $P$ is $G$-$\mu$-free) iff $G(P)$ is nowhere dense in $G$ (provided with the $\mu$-topology).

1.1.10. Assume that $\mu$ is an infinite cardinal, $G \subseteq A!$, $\xi \in \mu$ and $P \subseteq {}^{\xi}A$. Then $P$ is $G$-$\mu$-w.det iff $P$ is $G \cap [\mathrm{id}_X]$-$\mu$-det, for some $X \subseteq A$ such that $\overline{\overline{X}} < \mu$.

**Proof**: Assume that $P$ is $G$-$\mu$-w.det. Since $G(P)$ is open in $G$ (1.1.9 (ii)) and $\mathrm{id}_A \in G(P)$, then $G \cap [\mathrm{id}_X] \subseteq G(P)$ for some $X \subseteq A$ such that $X < \mu$. Clearly $P$ is $G \cap [\mathrm{id}_X]$-$\mu$-det. Conversely, suppose that $P$ is $G \cap [\mathrm{id}_X]$-$\mu$-det for some $X \subseteq A$ of power less than $\mu$. Then $G \cap [\mathrm{id}_X] \subseteq G(P)$ and this shows that $\mathrm{id}_A$ is an interior point of $G(P)$. By the homogeneity of a topological group, every point of $G(P)$ is interior, i.e., $G(P)$ is open in $G$.

Instead of "open" and "nowhere dense" in 1.1.9 we could have used "not $\mu$-meager" and "$\mu$-meager" respectively, at least from $\mu$-Baire groups. In fact,

1.1.11. Assume that $\mu$ is an infinite cardinal $G \subseteq A!$ a $\mu$-Baire group, $\xi \in \mu$ and $P \subseteq {}^{\xi}A$. Then

(i) $P$ is $G$-$\mu$-det iff $G(P)$ is co-$\mu$-meager in $G$;
(ii) $P$ is $G$-$\mu$-w.det iff $G(P)$ is not $\mu$-meager in $G$;
(iii) $P$ is $G$-$\mu$-free iff $G(P)$ is $\mu$-meager in $G$.

**Proof**: (i) Assume that $G(P)$ is co-$\mu$-meager in $G$. Since $G(P)$ is closed in $G$ (1.1.8 (ii)), $G \sim G(P)$ is open and $\mu$-meager in $G$. Then $G \sim G(P) = 0$, i.e., $P$ is $G$-$\mu$-det. The other implication is trivial.

(ii) Assume that $P$ is $G$-$\mu$-w.det. Then $G(P)$ is open in $G$ (1.1.9 (ii)) and since $G(P) \neq 0$, then $G(P)$ is not $\mu$-meager in $G$. Assume, on the other hand, that $G(P)$ is not $\mu$-meager in $G$. Hence $G(P)$ is not nowhere dense in $G$ and this implies that $\overline{G(P)}$ has an interior point. By 1.1.8 (ii), $\overline{G(P)} = G(P)$ and by the homogeneity of a topological group, $G(P)$ is open in $G$.

(iii) Assume that $G(P)$ is $\mu$-meager in $G$. Hence $G(P)$ doesn't contain any non-empty open subsets of $G$. Since $\overline{G(P)} = G(P)$ (by 1.1.8 (ii)), this shows that $G(P)$ is nowhere dense in $G$.

Besides the *topological* measure, we have an obvious *set-theoretical* measure for the dependence of a relation $P$ on a group $G$: the cardinality of $O^G(P)$.

Our next theorem states that, under some conditions on $\mu$, these measures coincide. We first state two lemmas:

1.1.12. Assume that $\mu$ is an infinite cardinal, $G \subseteq A!$ is a group, $\xi \in \mu$, $P, Q \subseteq {}^{\xi}A$. If $P$ is $G$-$\mu$-free, then $G(P, Q) = \{\pi \in G \colon \pi^* P = Q\}$ is nowhere dense in $G$.

**Proof**: For each $\pi \in G$, the translations $f_\pi$ defined by $f_\pi(\sigma) = \pi \circ \sigma$, for all $\sigma \in G$, are homeomorphisms of the space $G$. Assume $G(P, Q) \neq 0$ (otherwise $G(P, Q)$ is clearly nowhere dense in $G$) and let $\pi \in G(P, Q)$ and let $\pi \in G(P, Q)$. Since $f_{\pi-1}\ G(P, Q) = G(P)$ and $G(P)$ is nowhere dense in $G$ it is easily checked that $G(P, Q)$ is nowhere dense in $G$.

1.1.13. Assume that $\mu$ is an infinite cardinal, $G \subseteq A!$ is $\mu$-Baire group, $\xi \in \mu$ and $P \subseteq {}^{\xi}A$. If $P$ is $G$-$\mu$-free, then $\overline{\overline{O^G(P)}} > \mu$.

**Proof**: Assume that $\overline{\overline{O^G(P)}} \leq \mu$. Hence $G = \bigcup \{G(P, Q) \colon Q \in O^G(P)\}$ is $\mu$-meager in $G$, since $G(P, Q)$ is nowhere dense, for each $Q \in O^G(P)$ (1.1.12). This contradicts our supposition that $G$ is a $\mu$-Baire group.

1.1.14. **Theorem**: *Assume that $\mu$ is an infinite regular cardinal such that $\mu = 2^{\underline{\mu}} = \Sigma < 2^{\lambda} \colon \lambda \in \mu >$, $G \subseteq A!$ a $\mu$-Baire group, $\xi \in \mu$ and $P \subseteq {}^{\xi}A$. Then*

(i) $P$ is $G$-$\mu$-det iff $\overline{\overline{O^G(P)}} = 1$;
(ii) $P$ is $G$-$\mu$-w.det iff $\overline{\overline{O^G(P)}} \leq \mu$;
(iii) $P$ is $G$-$\mu$-free iff $\overline{\overline{O^G(P)}} > \mu$.

**Proof**: (i) is obvious and (ii) follows from (iii). Assume that $P$ is not $G$-$\mu$-free. Hence (by 1.1.11), $P$ is $G$-$\mu$-w.det. By 1.1.10, $P$ is $G \cap [\mathrm{id}_X]$-$\mu$-det, for some $X \subseteq A$ such that $\overline{\overline{X}} < \mu$. Hence $\overline{\overline{O^{G}(P)}} \leq \mu^{\overline{\overline{X}}} \leq \mu^{\mu} = \mu$.

1.1.15. Assume that $\mu$ is an infinite cardinal, $G \subseteq A!$ a $\mu$-Baire group and $\xi \in \mu$. If $X$ is a set of $G$-$\mu$-free relations on $A$ of rank $\xi$ and $\overline{\overline{X}} \leq \mu$, then there is some $\pi \in G$ such that $\pi^*X \cap X = 0$.

**Proof**: Since $G(P, Q)$ is nowhere dense in $G$, for every $P, Q \in X$ (1.1.12), $\bigcup \{G(P, Q)\colon P, Q \in X\}$ is $\mu$-meager in $G$ (because $\overline{\overline{X}} < \mu$). Therefore there is some $\pi \in G \sim \bigcup\{G(P, Q)\colon P, Q \in X\}$, i.e., $\pi \notin G(P, Q)$ for every $P, Q \in X$. Clearly $\pi^*X = \{\pi^*P\colon P \in X\}$ is disjoint from $X$.

We shall now give sufficient conditions for a group $G \subseteq A!$ to be $\mu$-Baire.

1.1.16. **Definition**. Let $\kappa$ and $\mu$ be infinite cardinals and let $G \subseteq A!$ be a group. $G$ is $\mathfrak{B}_\kappa$-*$\mu$-compact* iff for every family $\{Q_i\colon i \in I\}$ of elements of $\mathfrak{B}_\kappa$ (the basis of the $\kappa$-topology) such that $\overline{\overline{I}} < \mu$, if the intersection of every subfamily of $\{Q_i \cap G\colon i \in I\}$ of power less than $\kappa$ is non-empty, then $\bigcup\{Q_i \cap G\colon i \in I\} \neq 0$ †.

1.1.17. **Theorem**. *Assume that $\overline{\overline{A}} = \mu$ is an infinite regular cardinal and $G \subseteq A!$ is a $\mathfrak{B}_{\aleph_0}$-$\mu$-compact group which is closed in the $\aleph_0$-topology. Then $G$ is a $\mu$-Baire group.*

1.1.18. **Theorem**. *Assume that $\mu$ is an infinite regular cardinal, $\mu^+$ the successor of $\mu$ and $G \subseteq A!$ is a $\mathfrak{B}_{\aleph_0}$-$\mu^+$-compact group which is closed in the $\aleph_0$-topology. Then $G$ is a $\mu$-Baire group.*

† We owe this formulation of $\mathfrak{B}_\kappa$-$\mu$-compactness to I.Fleischer, who simplified a previous (equivalent) definition.

We postpone the proofs of these theorems until the end of this section.

As it will be shown in §2 (2.2.1 and 2.2.2) and §3 (3.1.6), 1.1.13 and 1.1.14 can be interpreted as very general local versions of the Chang-Makkai theorem [2, 18]. To make them more presentable to the logical community and tie these and some other results of this section to definability theorems, we "realize" some subgroups of $A!$ as groups of automorphisms of relational structures with domain $A$.

## 1.2. *Groups and relational structures*

In the rest of this section, we consider structures with domain $A$.

1.2.1. **Definition.** Let $\mu$ be a cardinal. A structure $\mathfrak{A}$ is called *$\mu$-homogeneous* iff any isomorphism between two substructures of $\mathfrak{A}$ of power less than $\mu$ can be extended to an automorphism of tt.

Although this notion, due to B.Jónsson [6], seems to be too restricted, we shall see that for some purposes it imposes no real restrictions on a structure, i.e. under some conditions any structure $\mathfrak{A}$ can be "homogenized" by adding relations, without changing Aut($\mathfrak{A}$), the group of automorphisms of the original $\mathfrak{A}$ (1.2.4).

1.2.2. Assume that $\mu$ is an infinite cardinal, $\mathfrak{A}$ a relational structure having relations of rank less than $\mu$ and $G = \mathrm{Aut}(\mathfrak{A})$. Then $G$ is closed in the $\mu$-topology.

**Proof:** Let $\mathfrak{A} = \langle A, R_i \rangle_{i \in I}$. Then clearly $G = \bigcap \{G(R_i) \colon i \in I\}$ and hence $G$ is closed by 1.1.9 (ii).

1.2.3. Assume that $\mu$ is an infinite cardinal and $G \subseteq A!$ a group.

Then $\operatorname{Aut}[\langle A, O^G(s)\rangle_{s \in {}^{\mu}A}) = \bar{G}$, i.e., the closure of $G$ in the $\mu$-topology.

**Proof**: Let $\mathfrak{A} = \langle A, O^G(s)\rangle_{s \in {}^{\mu}A}$. Clearly $G \subseteq \operatorname{Aut}(\mathfrak{A})$ and this implies (by 1.2.2) that $\bar{G} \subseteq \operatorname{Aut}(\mathfrak{A})$. Assume that $\pi \in \operatorname{Aut}(\mathfrak{A})$. Let $Q \in \mathfrak{B}_\mu$ such that $\pi \in Q$. By the definition of $\mathfrak{B}_\mu$, $Q = [\pi | X]$ for some subset $X \subseteq A$ such that $\overline{\overline{X}} < \mu$. Let $s \in {}^{\mu}A$ be such that $X = \operatorname{range}(s)$. Since $s \in O^G(s)$ and $\pi \in \operatorname{Aut}(\mathfrak{A})$, then $\pi^* s \in O^G(s)$. Therefore there is some $\sigma \in G$ such that $\pi^* s = \sigma^* s$, i.e., $\sigma \in [\pi | X] \cap G$. Hence $Q \cap G \neq 0$ and thos shows that $\pi \in \bar{G}$.

1.2.4. Assume that $\mu$ is an infinite cardinal, $\mathfrak{A}$ a relational structure having relations of rank less than $\mu$ and $G = \operatorname{Aut}(\mathfrak{A})$. Then $\mathfrak{A}^{\mu} = (\mathfrak{A}, O^G(s))_{s \in {}^{\mu}A}$ is $\mu$-homogeneous and $\operatorname{Aut}(\mathfrak{A}^{\mu}) = G$.

**Proof**: By 1.2.2 and 1.2.3, $G = \operatorname{Aut}(\mathfrak{A}^{\mu})$.

Let $f\colon \mathfrak{A}^{\mu} | B \to \mathfrak{A}^{\mu} | f^*B$ be an isomorphism between the two substructures $\mathfrak{A}^{\mu} | B$ and $\mathfrak{A}^{\mu} | f^*B$ of domain $B$ and $f^*B$ respectively and such that $\overline{\overline{B}} < \mu$. Let $s$ be a sequence in ${}^{\mu}A$ such that $\operatorname{range}(s) = B$. Since $s \in O^G(s) \cap {}^{\operatorname{dom}(s)}B$ and $f$ is an isomorphism, $f^*s \in O^G(s) \cap {}^{\operatorname{dom}(s)}f^*B$. Hence, there is some $\pi \in G$ such that $\pi^* s = f^* s$. Clearly $\pi \supseteq f$ and $\pi \in G = \operatorname{Aut}(\mathfrak{A}^{\mu})$.

The last three lemmas have the immediate corollary:

1.2.5. **Theorem**. *Assume that $\mu$ is an infinite cardinal and $G \subseteq A!$ is a group. Then the following are equivalent:*

(i) $G$ is closed in the $\mu$-topology;
(ii) $G = \operatorname{Aut}(\mathfrak{A})$, for some structure $\mathfrak{A}$ having relations of rank less than $\mu$;
(iii) $G = \operatorname{Aut}(\mathfrak{A})$, for some $\mu$-homogeneous structure $\mathfrak{A}$ having relations of rank less than $\mu$;
(iv) $G = \operatorname{Aut}(\langle A, O^G(s)\rangle_{s \in {}^{\mu}A})$.

This theorem (minus the clause mentioning homogeneity) was independently obtained by B.Jónsson [8]. It should be noted, however, that his result has already appeared in print.

1.2.6. Assume that $\kappa$ and $\mu$ are infinite cardinals such that for every cardinal $\nu < \mu$, $\nu^{\underset{\smile}{\kappa}} < \mu$ and $\kappa$ is regular. Assume that $G \subseteq A!$ is a group. Then the following are equivalent:

(i) $\langle A, O^G(s)\rangle_{s \in {}^{\underset{\smile}{\kappa}}A}$ is $\mu$-homogeneous;
(ii) For every family $\{Q_i : i \in I\}$ of elements of $\mathfrak{B}_\kappa$ such that $\overline{\overline{I}} < \mu$, if the intersection of every subfamily of $\{Q_i \cap G : i \in I\}$ of power less than $\kappa$ is non-empty, then $\bigcap\{Q_i \cap \overline{G} : i \in I\} \neq 0$ ($\overline{G}$ denotes the closure of $G$ in the $\kappa$-topology).

**Proof**: Assume (ii). Let $\mathfrak{A} = \langle A, O^G(s)\rangle_{s \in {}^{\underset{\smile}{\kappa}}A}$ and let $f\colon \mathfrak{A}|B \to \mathfrak{A}|C$ be an isomorphism such that $B = \mathrm{dom}(f)$ has power less than $\mu$. Let $F \subseteq B$ be such that $\overline{\overline{F}} < \kappa$. We can find a sequence $s \in {}^{\underset{\smile}{\kappa}}A$ such that $F = \mathrm{range}(s)$. Since $s \in O^G(s) \cap {}^{\xi}B$, for some $\xi \in \kappa$ and $f$ is an isomorphism, $f^*s \in O^G(s)$, i.e., there is some $\sigma \in G$ such that $\sigma|F = f|F$. This implies that $\sigma \in [f|F] \cap G$. We have shown that $[f|F \cap G \neq 0$ for every $F \subseteq B$ such that $\overline{\overline{F}} < \kappa$. Let $\mathcal{F}$ be a subfamily of $\{[f|F] \cap G : F \subseteq B$ and $\overline{\overline{F}} < \kappa\}$ of power less than $\kappa$, i.e., $\mathcal{F} = \{[f|F_i] \cap G : F_i \subseteq B, \overline{\overline{F_i}} < \kappa$ and $i \in I\}$ for some $I$ such that $\overline{\overline{I}} < \kappa$. This implies that $\bigcap \mathcal{F} = [f|\bigcup\{F_i : i \in I\}] \cap G \neq 0$, since $\overline{\overline{\bigcup[F_i : i \in I]}} < \kappa$ by the regularity of $\kappa$. Since $F_0 = \{[f|F] : F \subseteq B$ and $\overline{\overline{F}} < \kappa\}$ has power less than $\mu$ (in virtue of the hypothesis on $\kappa$ and $\mu$), $\bigcap\{[f|F] \cap \overline{G} : F \subseteq B$ and $\overline{\overline{F}} < \kappa\} \neq 0$, i.e., there is some $\sigma \in \overline{G}$ such that $\sigma|\mathrm{dom}(f) = f$. Since $\overline{G} = \mathrm{Aut}(\mathfrak{A})$ (by 1.2.3) this implies, in turn, that $\mathfrak{A}$ is $\mu$-homogeneous.

Assume now that $\mathfrak{A} = \langle A, O^G(s)\rangle_{s \in {}^{\underset{\smile}{\kappa}}A}$ is $\mu$-homogeneous. Let $\{Q_i : i \in I\}$ be a family statisfying the hypotheses of (ii). By 1.1.1 (ii), for every $i \in I$ there is a partial function $f_i$ from $A$ into $A$ such that $Q_i = [f_i]$ and $\overline{\overline{\mathrm{dom}(f_i)}} < \kappa$. Let $B = \bigcup\{\mathrm{dom}(f_i) : i \in I\}$. Clearly $\overline{\overline{B}} < \mu$. We define a function $f\colon B \to A$ as follows: $f(b) = f_i(b)$ if there is some $i \in I$ such that $b \in \mathrm{dom}(f_i)$. We first show that $f$ is well-defined. Assume that $b \in \mathrm{dom}(f_i) \cap \mathrm{dom}(f_j)$ for some $i, j \in I$.

Since $[f_i] \cap [f_j] \cap G \neq 0$ by hypothesis, then there is some $\sigma \in G$ such that

$$\sigma | \mathrm{dom}\,(f_i) = f_i \qquad \text{and} \qquad \sigma | \mathrm{dom}\,(f_j) = f_j\,.$$

This implies that $f_i(b) = \sigma(b) = f_j(b)$, i.e., $f$ is well-defined. A similar argument shows that $f$ is $1-1$. We now show that $f\colon \mathfrak{A}|B \to \mathfrak{A}|f^*B$ is an isomorphism. Let $s \in O^G(t) \cap {}^{\xi}B$ for some $\xi \in \kappa$ and let $F =$ range$(s)$. Hence $F \subseteq \bigcup\{\mathrm{dom}\,(f_j)\colon j \in J\}$ for some $J \subseteq I$ of power $\bar{\bar{J}} < \kappa$. By the hypothesis on $\{Q_i\colon i \in I\}$, $\bigcap\{[f_j] \cap G\colon j \in J\} \neq 0$, i.e., there is some $\sigma \in G$ such that $\sigma | \mathrm{dom}\,(f_j) = f_j$ for all $j \in J$. This implies that $f^*s = \sigma^*s \in O^G(t)$. By the $\mu$-homogeneity of $\mathfrak{A}$, $f$ can be extended to some $\sigma \in \mathrm{Aut}\,(\mathfrak{A}) = \bar{G}$ (by 1.2.3). Clearly $\sigma \in \bigcap\{Q_i \cap \bar{G}\colon i \in I\}$.

1.2.7. **Theorem.** *Assume that $\kappa$ and $\mu$ are infinite cardinals such that for every cardinal $\nu < \mu$, $\nu^{\underline{\kappa}} < \mu$ and $\kappa$ is regular. Assume that $G \subseteq A!$ is a group. Then the following are equivalent:*

(i) $G$ is $\mathfrak{B}_\kappa$-$\mu$-compact and closed in the $\kappa$-topology;
(ii) $G = \mathrm{Aut}\,(\langle A, O^G(s)\rangle_{s \in {}^{\underline{\kappa}}A})$ and $\langle A, O^G(s)\rangle_{s \in {}^{\underline{\kappa}}A}$ is $\mu$-homogeneous;
(iii) $G = \mathrm{Aut}\,(\mathfrak{A})$, for some $\mu$-homogeneous structure $\mathfrak{A}$ having relations of rank less than $\kappa$.

**Proof**: By 1.2.5 and 1.2.6 we only need to show that (iii) implies (ii).

Assume that $\mathfrak{A} = \langle A, R_i\rangle_{i \in I}$ is a $\mu$-homogeneous structure having relations of rank less than $\kappa$. Let $G = \mathrm{Aut}\,(\mathfrak{A})$ and $X = \langle A, O^G(s)\rangle_{s \in {}^{\underline{\kappa}}A}$. Assume that $f\colon X|B \to X|C$ is an isomorphism such that $\mathrm{dom}\,(f) = B$ has power less than $\kappa$. We shall show that $f$ is an isomorphism of $\mathfrak{A}|B$ onto $\mathfrak{A}|C$. In fact, let $s \in R_i \cap {}^{\xi}B$ for some $\xi \in \kappa$, Since $s \in O^G(s) \cap {}^{\xi}B$, $f^*s \in O^G(s)$, i.e., there is some $\sigma \in G$ such that $f^*s = \sigma^*s$. But $\sigma^*s \in R_i$ and this completes the proof of our claim. Since $\mathfrak{A}$ is $\mu$-homogeneous, there is some $\pi \in G = \mathrm{Aut}\,(\mathfrak{A})$ which extends $f$.

1.2.8. **Theorem.** *Assume that $\mu$ is an infinite regular cardinal and $\mathfrak{A}$ a $\mu$-homogeneous structure of power $\mu$ having only finitary relations. Then* Aut$(\mathfrak{A})$ *is a $\mu$-Baire group.*

**Proof**: Immediate from 1.2.7 and 1.1.17.

1.2.9. **Theorem.** *Assume that $\mu$ is an infinite regular cardinal, $\mu^+$ the successor of $\mu$ and $\mathfrak{A}$ is a $\mu$-homogenous structure having only finitary relations. Then* Aut$(\mathfrak{A})$ *is a $\mu$-Baire group.*

**Proof**: Immediate from 1.2.7 and 1.1.18.

**Proof of Theorem** 1.1.17: Let $G \subseteq A!$ be a group satisfying the hypotheses of 1.1.17. By 1.2.7 there is some $\mu$-homogeneous structure $\mathfrak{A}$ with domain $A$ having finitary relations only such that $G = \text{Aut}(\mathfrak{A})$.

Let $\langle a_\xi : \xi \in \mu \rangle$ be a list of all the elements of $A$. Assume that $O$ is a non-empty open subset of $G$ and $\langle N_\xi : \xi \in \mu \rangle$ is a sequence of nowhere dense subsets of $G$.

We shall build a chain $\langle f_\xi : \xi \in \mu \rangle$ of partial functions from $A$ into itself such that $f = \bigcup\{f_\xi : \xi \in \mu\} \in O \sim \bigcup\{N_\xi : \xi \in \mu\}$.

Since $O \sim \overline{N_0}$ is a non-empty open subset of $G$, there is some partial mapping $f_0$ from $A$ into itself such that $\overline{\overline{\text{dom}(f_0)}} < \mu$ and $0 \neq [f_0] \subseteq O \sim \overline{N_0}$.

Let us suppose that $\xi \in \mu$ and $f_\xi$ is defined in such a way that $\overline{\overline{\text{dom}(f_\xi)}} < \mu, f_\xi \supseteq f_\eta$ for all $\eta \in \xi$ and $[f_\xi] \neq 0$. Then $[f_\xi] \sim \overline{N_\xi}$ is a non-empty open subset of $G$. Hence there is some partial mapping $g$ from $A$ into itself such that $\overline{\overline{\text{dom}(g)}} < \mu$ and $0 \neq [g] \subseteq [f_\xi] \sim \overline{N_\xi}$.

Let $\pi \in [g]$ and let $X = \text{dom}(g) \cup \text{dom}(f_\xi) \cup [a_\eta : \eta < \xi] \cup [\pi^{-1}(a_\eta) : \eta \in \xi]$.

Let us define $f_{\xi+1} = \pi | X$. It is easily checked that $\overline{\overline{\text{dom}(f_{\xi+1})}} < \mu$, $f_{\xi+1} \supseteq f_\eta$ for all $\eta \leq \xi$ and $[f_{\xi+1}] \neq 0$.

If $\lambda \in \mu$ is a limit ordinal, we define $f_\lambda = \bigcup[f_\xi : \xi \in \lambda]$. Again $\overline{\overline{\text{dom}(f_\lambda)}} < \mu$ (by the regularity of $\mu$) and $f_\lambda \supseteq f_\eta$ for all $\eta \in \lambda$. Furthermore $[f_\lambda] \neq 0$, since $f_\lambda$ is a partial isomorphism of $\mathfrak{A}$ into itself (the union of a chain of partial isomorphisms of $\mathfrak{A}$ into itself is again a partial isomorphism, since $\mathfrak{A}$ has only finitary relations) and $\mathfrak{A}$ is $\mu$-homogeneous. Finally, let $f = \bigcup\{f_\xi : \xi \in \mu\}$. It is easily checked that $f \in \mathrm{Aut}(\mathfrak{A})$ and $f \in O$, since $f \in [f_0] \subseteq O$. Furthermore, $f \in [f_{\xi+1}]$ for all $\xi \in \mu$ and this implies that $f \notin \overline{N_\xi}$, for all $\xi \in \mu$. A fortiori, $f \notin \bigcup\{N_\xi : \xi \in \mu\}$.

**Proof of Theorem 1.1.18**: Let $G \subseteq A!$ be a group satisfying the hypothesis of 1.1.18. By 1.2.7, there is some $\mu^+$-homogeneous structure $\mathfrak{A}$ with domain $A$ having finitary relations only such that $G = \mathrm{Aut}(\mathfrak{A})$.

Let $\langle a_\xi : \xi \in \mu' \rangle$ be a list of all the elements of $A$ (by allowing repetitions, we may assume that $\mu' \geq \mu$). Assume that $O$ is a non-empty open subset of $G$ and $\langle N_\xi : \xi \in \mu] \rangle$ is a sequence of nowhere dense subsets of $G$.

Exactly as before (1.1.17) we build a chain $\langle f_\xi : \xi \in \mu] \rangle$ of partial functions and we define $f = \bigcup [f_\xi : \xi \in \mu]$. As before, we can check that $f$ is a partial isomorphism of $\mathfrak{A}$ into itself. By the $\mu^+$-homogeneity of $\mathfrak{A}$, $[f] \neq 0$. Let $\pi \in [f]$. It is easily checked that $\pi \in O \sim \bigcup [N_\xi : \xi \in \mu]$.

1.2.10. Assume that $\overline{\overline{A}} = \aleph_0$ and $\mathfrak{A}$ is a countable structure having only finitary relations. Then $\mathrm{Aut}(\mathfrak{A})$ is a $\aleph_0$-Baire group.

**Proof**: Immediate from 1.2.5 and 1.2.8.

It should be pointed out that we have developed the topological notions here introduced just for the purposes at hand, without attempting an exhaustive study. We have disregarded, for instance, the fact that the groups of permutations act on topological spaces (even metrizable in some cases) and not just on sets, i.e., we are dealing with transformation groups.

## §2. Local definability and Galois connections

### 2.1. *Orbital structures*

We shall see that the key results to establish our Galois connection described in the introduction will be 1.2.3 and Scott's definability theorem. However, since the latter is not true for uncountable structures *, we first study those structures for which it holds.

2.1.1. **Definition.** Let $\kappa$ and $\mu$ be infinite cardinals, let $\mathfrak{A}$ be a structure of power $\mu$ and let $G = \mathrm{Aut}(\mathfrak{A})$.

(i) $\mathfrak{A}$ *is orbital* iff $\mathfrak{A}$ has at most $\mu$ relations of rank $< \mu$ and every orbit $O^G(s)$ (for each $s \in {}^{\underset{\smile}{\mu}}A$) is definable from the relations of $\mathfrak{A}$ by a formula of $L_{\mu^+\mu}$ **;
(ii) $\mathfrak{A}$ is *$\kappa$-orbital* iff $\mathfrak{A}$ is orbital having relations of rank less than $\kappa$ and $\mathfrak{A}^{\underset{\smile}{\kappa}} = (\mathfrak{A}, O^G(s))_{s \in {}^{\underset{\smile}{\kappa}}A}$ is $\mu$-homogeneous.

2.1.2. Assume that $\kappa$ and $\mu$ are infinite cardinals such that $\mu = \mu^{\underset{\smile}{\kappa}} = \sup\{\mu^\lambda : \lambda \in \kappa\}$ and $\mathfrak{A}$ is a structure of power $\mu$ having at most $\mu$ relations of rank less than $\kappa$. Then the following are equivalent:

(i) $\mathfrak{A}$ is $\kappa$-orbital;
(ii) for all $\xi \in \kappa$ and all $P \subseteq {}^{\xi}A$, if $P$ is $\mathrm{Aut}(\mathfrak{A})$-$\mu$-det, then $P$ is definable in $\mathfrak{A}$ by a formula of $L_{\mu^+\mu}$.

For the proof, we need the following simple lemma:

2.1.3. Assume that $\mu$ is an infinite cardinal, $\mathfrak{A}$ is a $\mu$-homogeneous structure of power $\mu$ having at most $\mu$ relations of rank less than $\mu$, $\xi \in \mu$ and $P \subseteq {}^{\xi}A$. Then $P$ is $\mathrm{Aut}(\mathfrak{A})$-$\mu$-det iff $P$ is definable from the relations of $\mathfrak{A}$ by a quantifier-free formula of $L_{\mu^+\mu}$.

* See footnote on page 000.
** We owe this terminology to V.Papillon.

**Proof**: For each $s \in {}^{\xi}A$, let $\Delta(\mathfrak{A} \upharpoonright \text{range}(s))$ be the diagram of the structure $\mathfrak{A} \upharpoonright \text{range}(s)$, i.e., the set of all atomic sentences or its negations (in a language having individual constants for the elements of range($s$)) which are true in $\mathfrak{A}$. If $v \in {}^{\xi}\text{Var}$ is a sequence of variables, let

$$\psi_s(v) = \text{Sub} \bigwedge \Delta(\mathfrak{A} \upharpoonright \text{range}(s)) \binom{s}{v} ,$$

i.e., the result of replacing in the (infinite) conjunction of all the sentences of $\Delta(\mathfrak{A} \upharpoonright \text{range}(s))$, every name $s_\eta$ by the variable $v_\eta$ ($\eta \in \xi$). It is easily checked that

$$(\mathfrak{A}, P) \models \forall v (Pv \leftrightarrow \bigvee_{s \in P} \psi_s(v)) .$$

Furthermore $\bigvee_{s \in P} \psi_s(v)$ is a quantifier-free formula of $L_{\mu^+\mu}$.

**Proof of 2.1.2**: Assume that $\mathfrak{A}$ is $\kappa$-orbital, $G = \text{Aut}(\mathfrak{A})$, $\xi \in \kappa$ and $P \subseteq {}^{\xi}A$ is $G$-$\mu$-det. Then $\mathfrak{A}' = (A, O^G(s))_{s \in {}^{\xi}A}$ is $\mu$-homogeneous having $\mu$ relations (since $\mu^{\xi} = \mu$) and $\text{Aut}(\mathfrak{A}') = G$ (1.2.4 and 1.2.5). By 2.1.3, $P$ is definable in $\mathfrak{A}'$ from the orbits $\{O^G(s): s \in {}^{\xi}A\}$ by a quantifier-free formula $\Phi(v)$ of $L_{\mu^+\mu}$. Since $\mathfrak{A}$ is orbital, for each $s \in {}^{\xi}A$, $O^G(s)$ is definable in $\mathfrak{A}$ by some formula $\Phi_s$ of $L_{\mu^+\mu}$. Let $\Phi^*(v) = \text{Sub}\, \Phi(v) \binom{O^G(s)}{\Phi_s}_{s \in {}^{\mu}A}$. It is easily checked that $(\mathfrak{A}, P) \models \forall v\, (Pv \leftrightarrow \Phi^*(v))$.

2.1.4. Assume that $\kappa$ and $\mu$ are infinite cardinals, $\kappa \leq \mu$. $\mathfrak{A}$ is a $\mu$-homogeneous structure of power $\mu$ having at most $\mu$ relations of rank less than $\kappa$. Then $\mathfrak{A}$ is $\kappa$-orbital.

**Proof**: Immediate from 2.1.3.

2.1.5. **Remark**: (i) In view of 2.1.2, Scott's definability theorem can be stated as follows:

Let $\mathfrak{A}$ be a countable structure having countably many finitary relations. Then $\mathfrak{A}$ is orbital.

(ii) Assume that $\mu$ is an infinite cardinal and $\mathfrak{A}$ is a structure of power $\mu$ having finitary relations. Then $\mathfrak{A}$ is homogeneous (of degree $\mu$) in the sense of [19] if the orbits $O^G(s)$ (for $G = \mathrm{Aut}(\mathfrak{A})$ and $s \in {}^{\underline{\mu}}A$) are definable in $\mathfrak{A}$ by sets of first-order formulas.

Another important local theorem for $L_{\omega_1\omega}$ is Scott's isomorphism theorem [22]. This theorem cannot be generalized to uncountable structures * and now we turn our attention to those structures for which it holds.

2.1.6. Assume that $\mu$ is an infinite cardinal and $(\mathfrak{A}, s)$ is definable, up to isomorphism, by a sentence of $L_{\mu^+\mu}$ among the structures of power $\mu$, for all $s \in {}^{\underline{\mu}}A$. Then $\mathfrak{A}$ is orbital.

**Proof**: Let $\xi \in \mu$, $s \in {}^{\xi}A$ and $G = \mathrm{Aut}(\mathfrak{A})$. By hypothesis, there is a sentence $\Phi_s(s_\eta)_{\eta \in \xi}$ which defines $(\mathfrak{A}, s)$ (up to isomorphism) among the structures of power $\mu$. It is easily checked that $\mathrm{Sub}\ \Phi_s \begin{pmatrix} s_\eta \\ v_\eta \end{pmatrix}_{\eta \in \xi}$ defines $O^G(s)$, i.e., $\mathfrak{A}$ is orbital (since $\mathfrak{A}$ must have at most $\mu$ relations).

We don't know whether the converse of 2.1.6 holds, although it seems unlikely that it does.

However, we have been able to establish a partial converse (for $\aleph_0$-orbital structures). We first need the following lemma:

* A counter-example (for regular uncountable cardinals) has been constructed by M. Morley; D.Kueker (whom we owe this information) has employed this counter-example to show that Scott's definability theorem fails (for regular uncountable cardinals). In our terminology, there are structures of regular uncountable powers which are not orbital.

2.1.7. Assume that $\kappa$ and $\mu$ are infinite cardinals such that $\mu = \mu^{\underline{\kappa}}$ and $\mathfrak{A}$ a structure of power $\mu$. For every $\xi \in \kappa$ and every $s \in {}^{\xi}A$, there is a formula $\Phi_s(v)$ of $L_{\mu^+\mu}$ such that

(i) $(\mathfrak{A}, s) \models \phi_s(s)$;
(ii) $(\mathfrak{A}, t) \models \phi_s(t)$ implies $(\mathfrak{A}, s) \equiv_{L_{\mu^+\mu}} (\mathfrak{A}, t)$, i.e., $(\mathfrak{A}, s)$ and $(\mathfrak{A}, t)$ have the same true $L_{\mu^+\mu}$ sentences.

**Proof**: Let $\xi \in \mu$ and $s \in {}^{\xi}A$.

Let $\langle t_\eta : \eta \in \mu\rangle$ be a list of all sequences $t$ in ${}^{\xi}A$ such that $(\mathfrak{A}, s) \not\equiv (\mathfrak{A}, t)$ (we use our hypothesis that $\mu^{\underline{\kappa}} = \mu$). Therefore, for each $\eta \in \mu$, there is some $\phi_\eta(v)$ in $L_{\mu^+\mu}$ such that

$$(\mathfrak{A}, s) \models \phi_\eta(s) \quad \text{and} \quad (\mathfrak{A}, t) \models \neg\, \phi_\eta(t_\eta)\,.$$

Let us define $\phi_s(v) = \bigwedge\{\phi_\eta(v): \eta \in \mu\}$. Then $\phi_s(v)$ is a formula of $L_{\mu^+\mu}$ and it is easy to see that (i) and (ii) are satisfied.

2.1.8. **Theorem.** *Assume that* $\mu$ is an infinite regular cardinal such that $\mu = 2^{\underline{\mu}}$ and $\mathfrak{A}$ a $\mu$-homogeneous structure of power $\mu$ having at most $\mu$ finitary relations. Then $\mathfrak{A}$ *is definable (up to isomorphism) by a sentence of* $L_{\mu^+\mu}$, *among structures of power* $\mu$.

**Proof**: The proof, as in the case of our previous lemma, is a modification of the original proof of Scott [23] and so we sketch it only.

Let $\psi$ be the conjunction of the following sentences:

(i) $\bigwedge\limits_{a \in A} \exists v_0\ \phi_{\langle a\rangle}(v_0)\,,$

(ii) $\bigwedge\limits_{\xi \in \mu}\ \bigwedge\limits_{s \in {}^{\xi}A} \forall v\,(\phi_s(v) \to \bigwedge\limits_{a \in A} \exists w\ \phi_{s\langle a\rangle}(v, w))\,,$

(iii) $\bigwedge\limits_{\xi \in \mu}\ \bigwedge\limits_{s \in {}^{\xi}A} \forall v\,(\phi_s(v) \to \forall w \bigvee\limits_{a \in A} \phi_{s\langle a\rangle}(v, w))\,,$

(iv) $$\bigwedge_{\substack{\delta\in\mu\\ \delta \text{ limit}}} \bigwedge_{s\in {}^{\delta}A} \forall v \left(\left[\bigwedge_{\xi\in\delta} (\phi_{s|\xi}(v|\xi)\right] \to \phi_s(v)\right),$$

(v) $$\bigwedge_{\xi\in\mu} \bigwedge_{s\in{}^{\xi}A} \forall v\, (\phi_s(v) \to \mathrm{Sub} \bigwedge \Delta(\mathfrak{A}\restriction\mathrm{range}(s)) \binom{s}{v},$$

where $\phi_s$ is obtained by 2.1.7. (Sub $\bigwedge(\mathfrak{A}|\mathrm{range}(s)) \binom{s}{v}$ is defined in the proof of 2.1.3).

Clearly $\psi$ is a sentence of $L_{\mu^+\mu}$.

We now show that $\mathfrak{A} \models \psi$. The clauses (i), (ii), (iii) and (v) follow the original proof of Scott. To show (iv), let $\delta \in \mu$ be a limit ordinal and let $s, t \in {}^{\delta}A$ be such that $(\mathfrak{A}, t) \models \phi_{s|\xi}(t|\xi)$ for all $\xi \in \delta$. Define $f$: range($s$) → range($t$) as follows $f(s_\xi) = t_\xi$ for $\xi \in \delta$. By ($v$) and the fact that all the relations of $\mathfrak{A}$ are finitary, $f$: $\mathfrak{A}|\mathrm{range}(s) \simeq \mathfrak{A}|\mathrm{range}(t)$ is a partial isomorphism of $\mathfrak{A}$ into itself such that $\overline{\overline{\mathrm{dom}\,(f)}} < \mu$. Since $\mathfrak{A}$ is $\mu$-homogeneous, there is some $\pi \in \mathrm{Aut}\,(\mathfrak{A})$ such that $\pi \supseteq f$. From $(\mathfrak{A}, s) \models \phi_s(s)$ we obtain $(\mathfrak{A}, \pi^*s) \models \phi_s(\pi^*s)$, i.e., $(\mathfrak{A}, t) \models \phi_s(t)$.

Assume now that $\mathfrak{B} \models \psi$ and $\overline{\overline{\mathfrak{B}}} = \mu$. To show that $\mathfrak{B} \simeq \mathfrak{A}$, we employ the usual Cantor type argument. The only novelty (with respect to Scott's proof) is the appearance of limit ordinals which are handled by (iv).

2.1.9. **Theorem**. *Assume that $\mu$ is an infinite regular cardinal such that $\mu = 2^{\underline{\mu}}$ and $\mathfrak{A}$ is an $\aleph_0$-orbital structure of power $\mu$. Then $\mathfrak{A}$ is definable (up to isomorphism) by a sentence of $L_{\mu^+\mu}$ among structures of power $\mu$.*

**Proof**: By 2.1.8, $\mathfrak{A}^{\underline{\omega}}$ is definable (up to isomorphism) by a sentence of $L_{\mu^+\mu}$ among the structures of power $\mu$. Since $\mathfrak{A}$ is orbital, for each $s \in {}^{\underline{\omega}}A$ $O^G(s)$ is definable in $\mathfrak{A}$ by some formula $\phi_s(v)$ of

$L_{\mu+\mu}$. It is easily checked that $\psi^* = \mathrm{Sub}\begin{pmatrix} O^G(s) \\ \phi_s \end{pmatrix}_{s \in {}^{\omega}A}$ defines $\mathfrak{A}$ (up to isomorphism) among structures of power $\mu$.

From our proof, it is clear that there is connection between orbital and homogeneous structures. To make it explicit, we define

2.1.10. **Definition.** Let $\kappa$ and $\mu$ be infinite cardinals and let $\mathfrak{A}$ be a structure power $\mu$. $\mathfrak{A}^{\underline{d\kappa}} = (\mathfrak{A}, \langle \Phi^{\mathfrak{A}}: \Phi$ is a formula of $L_{\mu+\mu}$ having a set of free variables of cardinality less than $\kappa\rangle)$.

2.1.11. Assume that $\kappa$ and $\mu$ are infinite cardinals such that $\mu = \mu^{\underline{\kappa}}$ and $\mathfrak{A}$ is a structure of power $\mu$ having at most $\mu$ relations of rank less than $\kappa$. Then the following are equivalent:

(i) $\mathfrak{A}$ is $\kappa$-orbital;
(ii) $\mathfrak{A}^{\underline{d\kappa}}$ is $\mu$-homogeneous.

**Proof.** Assume that $\mathfrak{A}$ is $\kappa$-orbital. Let $\xi \in \kappa$, $s \in {}^{\xi}A$, $G = \mathrm{Aut}(\mathfrak{A})$ and let $\Phi_s$ be a formula in $L_{\mu+\mu}$ which defines $O^G(s)$, i.e., $\Phi_s^{\mathfrak{A}} = O^G(s)$. Then $(\mathfrak{A}, \Phi_s^{\mathfrak{A}})_{s \in {}^{\underline{\kappa}}A}$ is $\mu$-homogeneous by 1.2.4. A fortiori, $\mathfrak{A}^{\underline{d\kappa}}$ is $\mu$-homogeneous. Assume that $\mathfrak{A}^{\underline{d\kappa}}$ is $\mu$-homogeneous. Let $\xi \in \kappa$, $s \in {}^{\xi}A$ and let $\Phi_s$ be the formula of $L_{\mu+\mu}$ given by 2.1.7. Assume that $(\mathfrak{A}, t) \models \Phi_s(t)$. By 2.1.7 (ii), $(\mathfrak{A}, s) \equiv_{L_{\mu+\mu}} (\mathfrak{A}, t)$. Let us define $f$: range$(s) \to$ range$(t)$ as follows $f(s_\eta) = t_\eta$ for all $\eta \in \xi$. It is easy to see that $\mathfrak{A}^{\underline{d\kappa}} | \mathrm{range}(s) \simeq_f \mathfrak{A}^{\underline{d\kappa}} | \mathrm{range}(t)$. Since $\overline{\overline{\mathrm{range}(s)}} < \mu$ and $\mathfrak{A}^{\underline{d\kappa}}$ is $\mu$-homogeneous, there is some $\pi \in \mathrm{Aut}(\mathfrak{A}^{\underline{d\kappa}})$ such that $\pi \supseteq f$. Hence $(\mathfrak{A}, s) \simeq_\pi (\mathfrak{A}, t)$ and this shows that $\Phi_s$ defines $O^G(s)$.

The following result tells us that if the group of automorphism of a structure has a "sufficient degree of compactness", all these notions of orbital structures coincide and Scott's definability theorem holds iff Scott's isomorphism theorem does.

2.1.12. Assume that $\mu$ is an infinite cardinal and $\mathfrak{A}$ is a structure of power $\mu$ having at most $\mu$ finitary relations. Assume, furthermore, that $\mathrm{Aut}(\mathfrak{A})$ is $\mathfrak{B}_{\aleph_0}$-$\mu$-compact. Then the following are equivalent:

(i) $\mathfrak{A}$ is orbital;
(ii) $\mathfrak{A}$ is $\aleph_0$-orbital.

Furthermore, if $\mu = \mu^{\underline{\mu}}$ the following condition is equivalent to (i) and (ii):

(iii) $(\mathfrak{A}, s)$ is definable (up to isomorphism) by an $L_{\mu^+\mu}$ sentence among structures of power $\mu$, for every $s \in {}^{\xi}A$ and every $\xi \in \mu$.

**Proof**: Assume that $\mathfrak{A}$ is orbital. By 1.2.5 and 1.2.7 (ii), $\langle A, O^G(s)\rangle_{s \in \underline{{}^{\omega}A}}$ is $\mu$-homogeneous (letting $G = \mathrm{Aut}(\mathfrak{A})$). A fortiori, $\mathfrak{A}^{\underline{\omega}}$ is $\mu$-homogeneous and this shows (ii). Under the assumption that $\mu = \mu^{\underline{\mu}}$ 2.1.9 shows that (ii) implies (iii), since $(\mathfrak{A}, s)$ is easily seen to be $\aleph_0$-orbital. Finally (iii) implies (i) is a consequence of 2.1.6.

2.1.13. Assume that $\mu$ is an infinite cardinal such that $\mu = \mu^{\underline{\mu}}$ and $\mathfrak{A} = \langle A, R_i\rangle_{i\in I}$ is a structure of power $\mu$ having at most $\mu$ finitary relations. Assume, furthermore, that $\mathrm{Aut}(\mathfrak{A})$ is $\mathfrak{B}_{\aleph_0}$-$\mu$-compact. Then there exists $\phi(R_i, S_j)_{i\in I,\, j\in J}$ in $L_{\mu^+\mu}$ such that $J < \mu$, each $S_j$ is a new finitary relation symbol, $\models \exists \langle S_j\rangle_{j\in J}\, \phi \leftrightarrow \exists! \langle S_j\rangle_{j\in J}\, \phi$ and for all $\mathfrak{B}$ of power $\mu$, $\mathfrak{B} \models \exists \langle S_j\rangle_{j\in J}\, \phi$ iff $\mathfrak{B} \simeq \mathfrak{A}$.

**Proof**: Let $G = \mathrm{Aut}(\mathfrak{A})$. By 2.1.12, $\mathfrak{A}^{\underline{\omega}} = (\mathfrak{A}, O^G(s))_{s\in \underline{{}^{\omega}A}}$ is $\mu$-homogeneous. By 2.1.8, $\mathfrak{A}^{\underline{\omega}}$ is definable (up to isomorphism) by a sentence $\Phi_0(R_i, O^G(s))_{i\in I,\, s\in \underline{{}^{\omega}A}}$ in $L_{\mu^+\mu}$ among structures of power $\mu$. Clearly $\mathfrak{A} \models \exists \langle S_s\rangle_{s\in \underline{{}^{\omega}A}}\, \phi$, where

$$\phi = \mathrm{Sub}\, \phi_0 \binom{O^G(s)}{S_s}_{s\in \underline{{}^{\omega}A}}$$

Let $\mathfrak{B}$ be a structure of power $\mu$ such that $\mathfrak{B} \models \exists \langle S_s\rangle_{s\in \underline{{}^{\omega}A}}\, \phi$. Hence there is some family $\langle S_s : s \in \underline{{}^{\omega}A}\rangle$ of relations on $B$ such that $(\mathfrak{B}, S_s)_{s\in \underline{{}^{\omega}A}} \models \phi$. Therefore $(\mathfrak{B}, S_s)_{s\in \underline{{}^{\omega}A}} \simeq (\mathfrak{A}, O^G(s))_{s\in \underline{{}^{\omega}A}}$ and this clearly implies that $\mathfrak{B} \simeq \mathfrak{A}$.

To finish the proof, we shall show that $\phi(R_i, S_s)_{s \in {}^{\omega}A} \wedge \phi(R_i, S'_s)_{s \in {}^{\omega}A} \to S_s = S'_s$ is logically valid, for every $s \in {}^{\omega}A$. By (a form of) the Löwenheim-Skolem theorem for $L_{\mu^+\mu}$ (see for instance [9], 10.3.5), it is enough to show that this sentence is valid in every structure of power $\mu$.

Let $(\mathfrak{B}, S_s, S'_s)_{s \in {}^{\omega}A} \models \phi(R_i, S_s)_{s \in {}^{\omega}A} \wedge \phi(R_i, S'_s)_{s \in {}^{\omega}A}$ and $\bar{\bar{\mathfrak{B}}} = \mu$. Hence $(\mathfrak{B}, S_s)_{s \in {}^{\omega}A} \simeq_f (\mathfrak{A}, O^G(s))_{s \in {}^{\omega}A}$ for some $f$ and $(\mathfrak{B}, S'_s)_{s \in {}^{\omega}A} \simeq_f (\mathfrak{A}, f^*S'_s)_{s \in {}^{\omega}A}$.

This implies that $(\mathfrak{A}, O^G(s))_{s \in {}^{\omega}A} \simeq_f (\mathfrak{A}, f^*S'_s)_{s \in {}^{\omega}A}$ for some $\pi \in G$ which, in turn, implies that $O^G(s) = f^*S'_s$, for every $s \in {}^{\omega}A$. From this we conclude that $S_s = S'_s$ for every $s \in {}^{\omega}A$.

We do not know whether the hypothesis of 2.1.13 implies that $\mathfrak{A}$ is definable by an $L_{\mu^+\mu}$ sentence (equivalent, whether being orbital can be deleted from the definition of an $\aleph_0$-orbital structure).

## 2.2. *Local theory for* $L_{\mu^+\mu}$

We have seen that Scott's isomorphism theorem holds for $\aleph_0$-orbital structures (2.1.9). We shall presently show that these structures play (with respect to $L_{\mu^+\mu}$) a role similar to that of countable models with respect to $L_{\omega_1\omega}$.

2.2.1. **Theorem.** *Assume that $\mu$ is an infinite cardinal such that $\mu = \mu^{\underline{\mu}}$ and $\mathfrak{A}$ is an $\aleph_0$-orbital structure of power $\mu$. Let $\xi \in \mu$, $G =$ Aut $(\mathfrak{A})$ and $P \subseteq {}^{\xi}A$. Then the following are equivalent:*

(i) $P$ is $G$-$\mu$-w.det;
(ii) $G(P)$ is open in $G$ (provided with the $\mu$-topology);
(iii) $\overline{\overline{O^G(P)}} \leq \mu$;
(iv) There is a formula $\Phi(v, w)$ of $L_{\mu^+\mu}$ such that $v \in {}^{\xi}$Var and $w \in {}^{\underline{\mu}}$Var and

$$(\mathfrak{A}, P) \models \exists w\, \forall v\, (Pv \leftrightarrow \Phi(v, w))\,.$$

**Proof**: From 1.2.4, $G = \mathrm{Aut}(\mathfrak{A}^{\omega})$. The equivalence of (i), (ii) and (iii) now follows from 1.2.8 and 1.1.14. It is obvious that (iv) implies (iii). We now show that (i) implies (iv). Assume that $P$ is $G$-$\mu$-w.det. Hence $P$ is $G \cap [\mathrm{id}_X]$-$\mu$-det, for some $X \subseteq A$ such that $\overline{\overline{X}} < \mu$. Let $s$ be a sequence which well-orders $X$. Since $\mathfrak{A}$ is orbital, $(\mathfrak{A}, s)$ is easily seen to be orbital. Furthermore $P$ is $\mathrm{Aut}(\mathfrak{A}, s)$-$\mu$-det and hence definable by a formula $\Phi(v, s)$ of $L_{\mu^+\mu}$ (2.1.2). This implies (iv) and our proof is complete.

For the particular case $\mu = \aleph_0$, we have the following corollary:

2.2.2. **Theorem**. *Assume that* $\mathfrak{A}$ *is a countable structure with countably many finitary relations,* $n \in \omega$ *and* $P \subseteq {}^nA$. *Let* $G = \mathrm{Aut}(\mathfrak{A})$. *Then the following are equivalent:*

(i) $P$ is $G$-$\aleph_0$-w.det;
(ii) $G(P)$ is open in $G$ (provided with the $\aleph_0$-topology);
(iii) $\overline{\overline{O^G(P)}} \leq \aleph_0$;
(iv) $\overline{\overline{O^G(P)}} < 2^{\aleph_0}$;
(v) there is a formula $\Phi(v_0, ..., v_{n-1}, w_0, ..., w_{m-1})$ of $L_{\omega_1\omega}$ such that

$$(\mathfrak{A}, P) \models \exists w_0, ..., w_{m-1} \quad \forall v_0, ..., v_{n-1} \quad (Pv_0, ..., v_{n-1} \leftrightarrow \Phi).$$

**Proof**: The equivalence of (i), (ii), (iii) and (v) follows from 1.2.4, 2.2.1 and Scott's definability theorem (2.1.5 (i)). To show that (iv) implies (iii), we notice that $O^G(P)$ is an analytic set in the space ${}^{n\omega}2$ (with the product topology) and hence its cardinality is either finite, countable or $2^{\aleph_0}$ (see Kuratowski [16]).

Theorem 2.2.2 was found independently by D.Kueker [14] (although without clause (2)) and by us in our dissertation [20]. His proof is much different from ours. Several of Kueker's results can be set in the present context. We just give one example.

2.2.3. **Theorem.** *Assume that $\mu$ is an infinite cardinal such that $\mu = \mu^{\underline{\mu}}$ and tt is an $\aleph_0$-orbital structure of power $\mu$. Then the following are equivalent:*

(i) $\overline{\overline{G}} \leq \mu$;

(ii) There are formulas $\Phi$, $\langle \psi_\xi : \xi \in \mu \rangle$ of $L_{\mu^+\mu}$ such that

$$\mathfrak{A} \models \exists v\, \Phi \quad \text{and}$$

$$\mathfrak{A} \models \forall v\, \forall w_0\, (\Phi \rightarrow \bigvee_{\xi \in \mu} \forall z_0\, (z_0 = w_0 \leftrightarrow \psi_\xi))\ ;$$

(iii) There is a formula $\Phi$ in $L_{\mu^+\mu}$ such that for all $P \subseteq A$ there are formulas $\langle \psi_\xi : \xi \in \mu \rangle$ of $L_{\mu^+\mu}$ satisfying

$$\mathfrak{A} \models \exists v\, \Phi \quad \text{and}$$

$$\mathfrak{A} \models \forall v\, (\Phi \rightarrow \bigvee_{\xi \in \mu} \forall w_0\, (Pw_0 \leftrightarrow \psi_\xi))\ .$$

**Proof**: From 1.2.4, $G = \mathrm{Aut}(\mathfrak{A}^{\cong})$. It follows from 1.2.8 that $G$ is a $\mu$-Baire group. From 1.1.6, $G \cap [\mathrm{id}_X] = \{\mathrm{id}_A\}$, for some $X \subseteq A$ such that $\overline{\overline{X}} < \mu$. The rest of the proof follows Kueker [14] (see also [15]) and is omitted.

2.2.4. **Remark.** It has been noted by several people (Barwise, Makkai, Weinstein, etc.) that several "local" preservation theorems established by Keisler [10, 11] for saturated structures and first-order language hold for countable structures and $L_{\omega_1\omega}$. For the sake of completeness, we give two examples.

2.2.5. Assume that $\mathfrak{A}$ and $\mathfrak{B}$ are countable structures having finitary relations only such that every existential $L_{\omega_1\omega}$ sentence true in $\mathfrak{A}$ is also true in $\mathfrak{B}$. Then $\mathfrak{A}$ is embeddable into $\mathfrak{B}$.

2.2.6. Assume that $\mathfrak{A}$ and $\mathfrak{B}$ are countable structures having finitary relations only such that every positive $L_{\omega_1\omega}$ sentence true in $\mathfrak{A}$ is true in $\mathfrak{B}$. Then $\mathfrak{B}$ is an homomorphic image of $\mathfrak{A}$.

We shall sketch the proof of 2.2.6.

We write $\mathfrak{A} \underset{\text{pos}}{\Rightarrow} \mathfrak{B}$ to mean that every positive $L_{\omega_1\omega}$ sentence true in $\mathfrak{A}$ is true in $\mathfrak{B}$. We proceed by induction. Assume that

$$(\mathfrak{A}, a_0, ..., a_n) \underset{\text{pos}}{\Rightarrow} (\mathfrak{B}, b_0, ..., b_n) \text{ for some } n \in \omega .$$

If $n$ is even, let $a_{n+1}$ be the first element of $A$ (in some well-ordering). Let $\langle b'_k : k \in \omega \rangle$ be a list of all the elements $b'_k$ of $B$ such that

$$(\mathfrak{A}, a_0, ..., a_n, a_{n+1}) \underset{\text{pos}}{\Rightarrow} (\mathfrak{B}, b_0, ..., b_n, b'_k) .$$

For each $k \in \omega$, let $\phi_k(a_0, ..., a_n, v_0)$ be a positive formula of $L_{\omega_1\omega}$ such that

$$(\mathfrak{A}, a_0, ..., a_n, a_{n+1}) \models \phi_k(a_0, ..., a_n, a_{n+1})$$

and

$$(\mathfrak{B}, b_0, ..., b_n, b'_k) \models \neg \phi_k(b_0, ..., b_n, b'_k) .$$

Define

$$\phi_{\langle a_0, ..., a_n, a_{n+1}\rangle}(a_0, ..., a_n, v_0) = \bigwedge_{k \in \omega} , \quad \phi_k(a_0, ..., a_n, v_0).$$

Since

$$\exists v_0 \; \phi_{\langle a_0, ..., a_n, a_{n+1}\rangle}(a_0, ..., a_n, v_0)$$

is positive and true in $(\mathfrak{A}, a_0, ..., a_n)$,

$$(\mathfrak{B}, b_0, ..., b_1) \models \exists v_0 \; , \quad \phi_{\langle a_0, ..., a_n, a_{n+1}\rangle}(b_0, ..., b_n, v_0) .$$

Let $b_{n+1}$ be the first element of $B$ (in a suitable well-ordering) which satisfies

$$\phi_{\langle a_0, \ldots, a_n, a_{n+1}\rangle}(b_0, \ldots, b_n, v_0) .$$

Clearly

$$(\mathfrak{A}, a_0, \ldots, a_{n+1}) \underset{\text{pos}}{\Rightarrow} (\mathfrak{B}, b_0, \ldots, b_{n+1}) .$$

If $n$ is odd, we start with an element of $B$, etc. The proof of 2.2.5 is similar, but simpler and is omitted.

## 2.3. *Galois connections for orbital algebras*

The connection between groups and structures given by 1.2.5 and 1.2.7 suggests the possibility of a Galois theory between these objects. However, since a structure is not determined by its group of automorphisms, we consider (instead of a structure) the set of relations "determined" by that structure. The only trouble is that we cannot talk about the group of automorphisms of a set of relations. Our remedy is to make these sets into polyadic algebras.

2.3.1. **Definition**. Let $\kappa$ and $\mu$ be infinite cardinals, and let $X$ be a set of power $\mu$. Furthermore, let $\mathfrak{A}$ be a structure of domain $X$. We define $[\mathfrak{A}]^{\kappa}_{\mu} = \{R\colon R \subseteq {}^{\xi}X$ for some $\xi \in \kappa$ and $R$ is definable from the relations of $\mathfrak{A}$ by a formula of $L_{\mu^+\mu}\}$.

2.3.2. Assume that $\kappa$ and $\mu$ are infinite cardinals such that $\mu = \mu^{\underline{\kappa}} = \sup\{\mu^{\lambda}\colon \lambda \in \kappa\}$ and $\mathfrak{A}$ is a $\kappa$-orbital structure with domain $X$. Then $[\mathfrak{A}]^{\kappa}_{\mu} = \{R\colon R \subseteq {}^{\xi}X$ for some $\xi \in \kappa$ and $R$ is $(\mathfrak{A})$-$\mu$-det$\}$.

**Proof**: Immediate from 2.1.2.

As a corollary we obtain

2.3.3. Assume that $\kappa$ and $\mu$ are infinite cardinals such that $\mu = \mu^{\underline{\kappa}} = \sup [\mu^\lambda : \lambda \in \kappa]$, $\mathfrak{A}$ and $\mathfrak{B}$ are $\kappa$-orbital structures with domain $X$. Then $[\mathfrak{A}]^{\underline{\kappa}} = [\mathfrak{B}]^{\underline{\kappa}}$ iff $\mathrm{Aut}(\mathfrak{A}) = \mathrm{Aut}(\mathfrak{B})$.

2.3.4. **Definition.** Let $\kappa$ and $\mu$ be infinite cardinals $\kappa \leq \mu$ and $\xi \in \kappa$.

(i) If $R \subseteq {}^\xi X$, we let $\hat{R}: {}^\mu X \to 2$ be defined as follows:

$$\hat{R}(f) = 1 \text{ iff } f | \xi \in R \text{ for all } f \in {}^\mu X\ ;$$

(ii) If $R \subseteq {}^\mu X$, we let $\hat{R}$ be the characteristic function, i.e.,

$$\hat{R}(f) = 1 \text{ iff } f \in R, \text{ for all } f \in {}^\mu X\ ;$$

(iii) $C_X^\kappa = \{\hat{R}: R \subseteq {}^\xi X \text{ for some } \xi \in \kappa\}$;
(iv) $C_X^\mu = \{\hat{R}: R \subseteq {}^\mu X\}$.

The sets $C_X^\kappa$ and $C_X^\mu$ have natural structures of polyadic algebras with equality (see [4] and [6] for references on polyadic algebras). We call $C_X^\mu$ the *full polyadic algebra* and $C_X^\kappa$, the *full polyadic algebra with elements of support less than* $\kappa$.

(v) If $\mathfrak{A}$ is any structure with domain $X$, we let $\mathcal{R}^{\underline{\kappa}}(\mathfrak{A}) = [\hat{R}: R \in \{\mathfrak{A}]^{\underline{\kappa}}\}$.

The next lemma is obvious.

2.3.5. Assume that $\mathfrak{A}$ is a structure with domain $X$ and relations of rank at most $\mu$. Then $\mathcal{R}^{\underline{\kappa}}(\mathfrak{A})$ is a (polyadic) subalgebra of $C_X^\mu$ (with the polyadic operations induced by $C_X^\mu$).

We now define the Galois group of an algebra.

2.3.6. **Definition.** (i) If $A \subseteq C_X^\kappa$ is a (polyadic) algebra, we let $\mathrm{gg}(A)$, the *Galois group* of $A$, be the group of automorphism of $C_X^\kappa$ which leave $A$ pointwise fixed;

(ii) If $G \subseteq \mathrm{Aut}(C_X^{\kappa})$ is a group, we let ff$(G)$, the *fixed algebra* of $G$, be the algebra of all the elements of $C_X^{\kappa}$ which are left fixed by $G$.

2.3.7. Assume that $\mathfrak{A}$ is a structure with domain $X$. Then $\mathrm{Aut}(\mathfrak{A})$ is naturally isomorphic to gg$(\mathcal{R}^{\underline{\kappa}}(\mathfrak{A}))$, for every infinite cardinal $\kappa$.

**Proof**: Immediate consequence of Theorem 1.4 of [4].

In view of 2.3.7 we identify gg$(\mathcal{R}^{\underline{\kappa}}(\mathfrak{A}))$ with $\mathrm{Aut}(\mathfrak{A})$. Under this identification, gg becomes a map from certain algebras into subgroups of permutations of $X!$.

2.3.8. **Definition**. Let $A \subseteq C_X^{\kappa}$ be a (polyadic) algebra. We call $A$ *$\kappa$-orbital* iff $A = \mathcal{R}^{\underline{\kappa}}(\mathfrak{A})$ for some $\kappa$-orbital structure $\mathfrak{A}$ with domain $X$.

2.3.9. Assume that $\kappa$ and $\mu$ are infinite cardinals such that $\mu = \mu^{\underline{\kappa}}$ and $A \subseteq C_X^{\kappa}$ is a (polyadic) algebra. Then the following are equivalent:

(i) $A$ is $\kappa$-orbital;
(ii) $A$ is $\mathcal{R}^{\underline{\kappa}}(\mathfrak{A})$, for some $\mu$-homogeneous structure having at most $\mu$ relations of rank less than $\kappa$;
(iii) There is a $G \subseteq X!$ which is closed in the $\kappa$-topology such that $A = \mathcal{R}^{\underline{\kappa}}(\langle X, O^G(s)\rangle_{s \in {}^{\underline{\mu}}X})$.

**Proof**: (i) ⇒ (ii): Assume that $A$ is $\kappa$-orbital. Hence $A = \mathcal{R}^{\underline{\kappa}}(\mathfrak{A})$ for some $\kappa$-orbital structure $\mathfrak{A}$ with domain $X$. By 2.1.11, $\mathfrak{A}^{\underline{\mathrm{d}\kappa}}$ is $\mu$-homogeneous. Clearly $\mathcal{R}^{\underline{\kappa}}(\mathfrak{A}^{\underline{\mathrm{d}\kappa}}) = \mathcal{R}^{\underline{\kappa}}(\mathfrak{A}) = A$ and $\mathfrak{A}^{\underline{\mathrm{d}\kappa}}$ has $\mu$ relations of rank less than $\kappa$.

(ii) ⇒ (iii): Assume that $A = \mathcal{R}^{\underline{\kappa}}(\mathfrak{A})$ for some $\mu$-homogeneous structure having at most $\mu$ relations of rank less than $\kappa$. Let $G = \mathrm{Aut}(\mathfrak{A})$ and $\mathfrak{A}' = \langle X, O^G(s)\rangle_{s \in {}^{\underline{\kappa}}X}$. Then $G = \mathrm{Aut}(\mathfrak{A}')$ and $G$ is closed in the $\kappa$-topology. Furthermore $\mathfrak{A}$ and $\mathfrak{A}'$ are $\mu$-homogeneous (1.2.4). This implies that $\mathfrak{A}$ and $\mathfrak{A}'$ are $\kappa$-orbital (2.1.4). By 2.3.3, $[\mathfrak{A}]^{\underline{\kappa}} = [\mathfrak{A}']^{\underline{\kappa}}$, i.e., $\mathcal{R}^{\underline{\kappa}}(\mathrm{tt}') = \mathcal{R}^{\underline{\kappa}}(\mathfrak{A}) = A$.

(iii) ⇒ (i): Immediate from 2.1.4.

We can ask whether there is an "intrinsic" characterization of orbital algebras. We shall answer this question only for the case $\mu = \aleph_0$ (2.3.11). We need some definitions.

2.3.10. **Definition.** Let $\mu$ be an infinite cardinal and let $X$ be a set of power $\mu$.

(i) If $B$ is a subset of $C_X^{\mu}$, we let $B^{\mu}$ be the smallest (polyadic) subalgebra of $C_X^{\mu}$ which is $\mu$-complete (in the boolean sense);
(ii) If $B$ is a subset of $C_X^{\mu}$, we let $B_{\underline{\omega}}^{\underline{\mu}} = B^{\mu} \cap C_{\overline{X}}^{\underline{\mu}}$.

2.3.11. Assume that $\overline{\overline{X}} = \aleph_0$ and $A \subseteq C_X^{\underline{\omega}}$ is a (polyadic) algebra. The the following are equivalent:

(i) $A$ is orbital;
(ii) $A = (B^{\omega})^{\underline{\omega}}$ for some countable subset $B$ of $A$.

**Proof**: Assume that $A$ is orbital. Hence $A = \mathcal{R}^{\underline{\omega}}(\mathfrak{A})$, for some orbital structure $\mathfrak{A} = \langle X, R_n \rangle_{n \in \omega}$. We let $B = \{\hat{R}_n : n \in \omega\}$. It is easily (but teadiously) shown that $B^{\omega} = \{\hat{R}$: $R$ is finitary or infinitary relation on $X$ definable in $\langle X, R_n \rangle_{n \in \omega}$ by a formula of $L_{\omega_1 \omega}\}$. This clearly implies that $(B^{\omega})^{\underline{\omega}} = \mathcal{R}^{\underline{\omega}}(\mathfrak{A}) = A$. Assume now that $A = (B^{\omega})^{\underline{\omega}}$, for some countable subset $B$ of $A$. Since $A$ is locally finite, we can write $B = \{\hat{R}_n : n \in \omega\}$, for some sequence $\langle R_n : n \in \omega \rangle$ of finitary relations on $X$. By Scott's definability theorem (in the version of 2.1.5 (i)), $\mathfrak{A} = \langle X, R_n \rangle_{n \in \omega}$ is orbital. Again we can check that $\mathcal{R}^{\underline{\omega}}(\mathfrak{A}) = (B^{\omega})^{\underline{\omega}} = A$. We omit details.

2.3.12. **Theorem.** *Assume that $\mu$ is an infinite cardinal such that $\mu = \mu^{\underline{\mu}}$ and $X$ is a set of power $\mu$. Then* gg *is an anti-isomorphism between the lattice of the orbital (polyadic) subalgebras of $C_X^{\underline{\mu}}$ and the lattice of the subgroup of $X!$ which are closed in the $\mu$-topology.*

**Proof**: Assume that $A$ is an orbital subalgebra of $C_X^\mu$. Hence $A = \mathcal{R}^\mu(\mathfrak{A})$, for some orbital structure $\mathfrak{A}$ with domain $X$. In view of our identification (2.3.7) $\mathrm{gg}(A) = \mathrm{Aut}(\mathfrak{A})$ and this implies that $\mathrm{gg}(A)$ is closed in the $\mu$-topology (1.2.2). Let $\mathrm{gg}(A) = \mathrm{gg}(B)$ for some orbital algebras $A, B$. Then $A = \mathcal{R}^\mu(\mathfrak{A})$ and $B = \mathcal{R}^\mu(\mathfrak{B})$ for some orbital structures $\mathfrak{A}$ and $\mathfrak{B}$ with domain $X$. But $\mathrm{Aut}(\mathfrak{A}) = \mathrm{gg}(A) = \mathrm{gg}(B) = \mathrm{Aut}(\mathfrak{B})$ and hence $A = B$ by 2.3.3. It remains to be checked that gg is onto. Let $G \subsetneq X!$ be a group closed in the $\mu$-topology. By 1.2.5, $G = \mathrm{Aut}(\mathfrak{A})$ for some $\mu$-homogeneous structure tt having at most $\mu$ relations of rank less than $\mu$. By 2.1.4, $\mathfrak{A}$ is orbital. Let $A = \mathcal{R}^\mu(\mathfrak{A})$. Then $A$ is orbital and $\mathrm{gg}(A) = \mathrm{Aut}(\mathfrak{A}) = G$. The rest of the conclusion is trivially checked.

2.3.13. **Remark.** Theorem 2.3.12 extends a result of Daigneault [4], who had obtained a Galois connection of this type for finite $X$. It is closely related to one of the main theorems of the "Abstract Galois theory" of M.Krasner [12, 13]. However, Krasner obtains a Galois connection between all the subgroups of $X!$ and all the complete (in the boolean sense) polyadic subalgebras of $C_X^\mu$, i.e., he considers algebras with elements of infinite support. For logical purposes, at least, it seems more natural to consider locally finite algebras only and A.Daigneault had raised the problem (independently from us) of characterizing the algebras and the groups in this case. 2.3.12 solves this problem for countable $X$ (letting $\mu = \aleph_0$) and 2.3.14 in the general case (letting $\kappa = \aleph_0$). An independent solution for countable $X$ has been found by K.R.Driessel [27].

2.3.14. **Theorem.** *Assume that $\mu$ and $\kappa$ are infinite cardinals such that for every cardinal $\nu < \kappa$, $\nu^{\underline{\kappa}} < \mu$, $\kappa$ is regular and $X$ is a set of power $\mu$. Then* gg *is an anti-isomorphism between the set of the $\kappa$-orbital subalgebras of $C_X^\kappa$ ordered by inclusion and the set of the $\mathcal{B}_\kappa$-$\mu$-compact subgroups of $X!$ which are closed in the $\kappa$-topology (ordered by inclusion).*

**Proof**: Let $A \subseteq C_X^\kappa$ be a $\kappa$-orbital algebra and let $G = gg(A)$. Hence $A = \mathcal{R}\underline{\kappa}(\mathfrak{A})$ for some $\kappa$-orbital structure $\mathfrak{A}$. In virtue of our identification, $G = \mathrm{Aut}(\mathfrak{A})$. By 1.2.4, $G = \mathrm{Aut}(\mathfrak{A}\underline{\kappa})$. Then $G$ is $\mathcal{B}_\kappa$-$\mu$-compact and closed in the $\kappa$-topology (1.2.7).

Assume that $G \subseteq X!$ is a $\mathcal{B}_\kappa$-$\mu$-compact group which is closed in the $\kappa$-topology. By 1.2.7, $G = \mathrm{Aut}(\mathfrak{A})$ for some $\mu$-homogeneous structure $\mathfrak{A}$ having relations of rank less than $\kappa$. We let $A = \mathcal{R}\underline{\kappa}(\mathfrak{A})$. By 2.1.4, $A$ is $\kappa$-orbital and $gg(A) = \mathrm{Aut}(\mathfrak{A}) = G$ (by our identification).

Another of the main theorems of Krasner theory can be proved for $\aleph_0$-orbital algebras, i.e.

2.3.15. **Theorem**. *Assume that $\mu$ is an infinite cardinal such that $\mu = \mu^{\underline{\mu}}$ and $X$ a set of power $\mu$. Then any isomorphism between $\aleph_0$-orbital (polyadic) subalgebras of $C_X^\omega$ can be extended to an automorphism of $C_X^\omega$.*

**Proof**: Let $f: A \to B$ be a (polyadic) isomorphism between $\aleph_0$-orbital algebras $A, B$. Let $\mathfrak{A} = \langle X, R_i \rangle_{i \in I}$ be an $\aleph_0$-orbital structure such that $A = \mathcal{R}\underline{\omega}(\mathfrak{A})$ and let $\mathfrak{B} = \langle X, fR_i \rangle_{i \in I}$. Since $f$ is a polyadic isomorphism, $B = \mathcal{R}\underline{\omega}(\mathfrak{B})$ and $\mathfrak{A} \equiv_{L_{\mu^+\mu}} \mathfrak{B}$. By 2.1.9, $\mathfrak{A} \simeq_\pi \mathfrak{B}$ for some $\pi \in X! = \mathrm{Aut}(C_X^\omega)$. Clearly $\pi \supseteq f$.

2.3.16. **Remark**. (i) The idea of considering topological groups in infinite field extensions goes back to Dedekind, but it was first developed by Krull (see [1] for references);

(ii) We do not know whether 2.3.15 holds for orbital algebras, although it seems unlikely that it does;

(iii) In view of 2.3.11, Scott's definability theorem can be "interpreted" as establishing the connection of 2.3.12 (for $\mu = \aleph_0$). Whether Scott's isomorphism theorem can be similarly "interpreted" is not known. In fact, the whole subject of "higher order" Galois connections is wide open (see some remarks in [24] and [27]).

## §3. Weak definability for models of prescribed cardinality

### 3.1. *Saturated and special structures*

We now specialize some of our results of § 1 to saturated and special structures (for references, see [3], [19]).

Although the following notion will play an auxiliary role only 3.1.2 may have some independent interest.

3.1.1. **Definition.** Let $\mathfrak{A}$ be a relational structure of similitary type $\rho$.

(i) $\mathfrak{A}$ is *$\mu$-saturated* iff whenever $X \subseteq |\mathfrak{A}|$ is of power less than $\mu$, $\mathfrak{B} = (\mathfrak{A}, x)_{x \in X}$ and $\Sigma$ is a set of formulas with one free variable (in the language of type $\rho$ enriched with names for the elements of $X$), if every finite subset of $\Sigma$ is satisfiable in $\mathfrak{B}$, then so is $\Sigma$.

(ii) $\mathfrak{A}$ is *$\mu$-typical* iff every reduct $\mathfrak{A} \upharpoonright J$ of $\mathfrak{A}$ such that $\overline{\overline{J}} < \overline{\overline{\mathfrak{A}}}$ is $\mu$-saturated *.

3.1.2. Assume that $\mu$ is an uncountable regular cardinal and $\mathfrak{B}$ a structure of power at most $2^{\underline{\mu}}$ having at most $\mu$ relations. Then there is some $\mu$-typical structure $\mathfrak{A}$ of power at most $2^{\underline{\mu}}$ such that $\mathfrak{B} \succcurlyeq \mathfrak{A}$.

**Proof**: Since the method of proof is well-known, we just sketch the proof. Let $\mathfrak{B} = \langle B, S_i \rangle_{i \in I}$. By hypothesis on $\overline{\overline{I}}$ we can write $I = \bigcup\, [J_\xi : \xi \in \mu]$ so that $\overline{\overline{J}}_\xi < \mu$ for all $\xi \in \mu$. We may assume that $J_\lambda = \bigcup\, [J_\xi : \xi \in \lambda]$ if $\lambda$ is a limit ordinal.

We build a sequence $\langle \mathfrak{A}_\xi : \xi \in \mu \rangle$ of structures in such a way that the following conditions are satisfied for all $\xi \in \mu$:

* For the particular case that $|\mathfrak{A}| = \mu = \mu^{\underline{\mu}}$, $\mathfrak{A}$ is $\mu$-typical iff $\mathfrak{A}^* = (\mathfrak{A}, \Phi^{\mathfrak{A}} : \Phi$ is a formula of $L_{\omega\omega}$) is $(\mathfrak{F}(L_{\mu+\mu}), M)$-typical in the sense of [20], where $M = \{\mathfrak{B} : \mathfrak{B} \subseteq \mathfrak{C}^*$, for some $\mathfrak{C} \equiv \mathfrak{A}\}$. This formulation of the notion of a $\mu$-saturated structure is due to H.J. Keisler (see [19] for references).

$(\mathrm{i})_\xi$ $\mathfrak{B} \upharpoonright J_\xi \preccurlyeq \mathfrak{A}_\xi$ ;
$(\mathrm{ii})_\xi$ $\mathfrak{A}_\eta \preccurlyeq \mathfrak{A}_\xi \upharpoonright J_\eta$ for all $\eta \in \xi$ ;
$(\mathrm{iii})_\xi$ $\mathfrak{A}_\xi$ is $\mu$-saturated of power at most $2^{\underline{\mu}}$.

Assume that $\mathfrak{A}_\xi$ has been defined. Let $\Sigma = \Delta^c(\mathfrak{B} \upharpoonright J_\xi) \cup \Delta^c(\mathfrak{A}_\xi)$ where $\Delta^c$ stands for "complete diagram" (i.e., $\Delta^c(\mathfrak{C})$ is the set of sentences true in $\mathfrak{C}$ in a language which has names for all the elements of the domain of $\mathfrak{C}$). It is easily checked that $\Sigma$ is consistent. By the theorem of existence of saturated structures, we can get $\mathfrak{A}_{\xi+1}$ which satisfies $(\mathrm{i})_{\xi+1}$, $(\mathrm{ii})_{\xi+1}$, $(\mathrm{iii})_{\xi+1}$. At limit ordinals and at the end of our process, we take unions (although our structures have different "similarity type", it is clear what we mean by unions). Let $\mathfrak{A} = \bigcup \{\mathfrak{A}_\xi : \xi \in \mu\}$. Using the fact that a reduct of a saturated structure is saturated, $\mathfrak{A}$ is easily seen to be $\mu$-typical.

We shall derive two corollaries of this result:

3.1.3. Assume that $\mu$ is an infinite regular cardinal such that $\mu = 2^{\underline{\mu}}$ and $(\mathfrak{A}, P)$ is a $\mu$-saturated structure of power $\mu$. Let $G = \mathrm{Aut}(\mathfrak{A})$. If $\overline{\overline{O^G(P)}} < \mu$, then $O^G(P)$ is finite.

**Proof**: We may assume that $\mu > \aleph_0$, otherwise 3.1.3 is trivially true.

Assume that $O^G(P)$ is infinite and consider $\Sigma = \{\mathrm{Sub}\ \sigma \left(\frac{P}{\underline{P}_\xi}\right) :$ $\xi \in \mu$ and $\sigma$ true in $(\mathfrak{A}, P)\} \cup [\underline{P}_\xi \neq \underline{P}_\eta : \xi \neq \eta] \cup \mathrm{Th}(\mathfrak{A}, P)$ as a set of sentences in a language having the type of $(\mathfrak{A}, P)$, enriched with new symbols $\underline{P}_\xi$ of the same rank as $P$. Using the assumption that $O^G(P)$ is infinite, we can show that every finite subset of $\Sigma$ is consistent, i.e., $\Sigma$ is consistent. By 3.1.2, there is a $\mu$-typical $(\mathfrak{A}', P', P'_\xi)_{\xi \in \mu} \in \mathrm{Mod}(\Sigma)$ of power $\mu^{\underline{\mu}} = \mu$. Since $(\mathfrak{A}', P')$ is $\mu$-saturated (by the definition of a $\mu$-typical structure) and $(\mathfrak{A}', P' \equiv (\mathfrak{A}, P)$ then $(\mathfrak{A}', P') \simeq (\mathfrak{A}, P)$. Let $G' = \mathrm{Aut}(\mathfrak{A}')$. We now show that $\overline{\overline{O^{G'}(P')}} \geq \mu$. In fact $(\mathfrak{A}', P'_\xi) \equiv (\mathfrak{A}', P')$ for all $\xi \in \mu$. Furthermore $(\mathfrak{A}', P_\xi)$ is $\mu$-saturated (by the same argument as above) for all $\xi \in \mu$. Hence $(\mathfrak{A}', P'_\xi) \simeq (\mathfrak{A}', P')$, for all $\xi \in \mu$. This implies that $[P'_\xi : \xi \in \mu] \subseteq O^{G'}(P')$, i.e., $\overline{\overline{O^{G'}(P')}} \geq \mu$, since the $P'_\xi$ are different. This is a contradiction, since $\overline{\overline{O^G(P)}} < \mu$.

To formulate our next corollary, we consider a first-order language $L$ of similarity type $\rho$ and we let $L(\underline{P})$ be the language obtained from $L$ by adding a new relation symbol $\underline{P}$.

3.1.4. Assume that $T$ is a theory in the language $L(\underline{P})$. Then the following are equivalent:

(i) For every $(\mathfrak{A}, P) \in \operatorname{Mod} T$,

$$|\{P' : (\mathfrak{A}, P') \simeq (\mathfrak{A}, P)\}| < \aleph_0 ;$$

(ii) For every infinite $(\mathfrak{A}, P) \in \operatorname{Mod} T$,

$$|\{P' : (\mathfrak{A}, P') \simeq (\mathfrak{A}, P)\}| < \overline{\overline{\mathfrak{A}}} ;$$

(iii) There is an infinite cardinal $\mu$ such that for every $(\mathfrak{A}, P) \in \operatorname{Mod} T$,

$$|\{P' : (\mathfrak{A}, P') \simeq (\mathfrak{A}, P)\}| < \mu .$$

**Proof**: (ii) ⇒ (i): Assume that $(\mathfrak{A}, P)$ is a model of $T$ such that $|\{P' : (\mathfrak{A}, P') \simeq (\mathfrak{A}, P)\}| \geq \aleph_0$. For simplicity, assume that there is an infinite regular cardinal $\mu$ such that $\mu = 2^{\underline{\mu}}$ and $\mu > \max(\overline{\overline{\rho}}, \aleph_0)$ (see remark 3.1.5 (ii)). Consider, as before, $\Sigma = \{\operatorname{Sub} \sigma (P/P_\xi) : \xi \in \mu \text{ and } \sigma \text{ true in } (\mathfrak{A}, P)\} \cup \{P_\xi \neq P_\eta : \xi \neq \eta\} \cup \operatorname{Th}(\mathfrak{A}, P)$.

Exactly as before we can obtain $(\mathfrak{A}', P') \in \operatorname{Mod} T$ of power $\mu$ such that $|O^{G'}(P')| \geq \mu$, contradicting (ii). The implication (iii) ⇒ (i) is similar but simpler. To show the other implications we just notice that if $T$ has only finite models, (i), (ii) and (iii) are trivially satisfied.

3.1.5. **Remark.** (i) Instead of (i) in 3.1.4 we could have considered

(i)′: For every structure $\mathfrak{A}$ of type $\rho$, $|[P: (\mathfrak{A}, P) \in \text{Mod } T]| < \aleph_0$. Similarly, we could have considered (ii)′ and (iii)′ (obtained from (ii) and (iii) in a similar manner). It is easy to see, however, that these are all equivalent, i.e., (i) $\Leftrightarrow$ (i)′ $\Leftrightarrow$ (ii) $\Leftrightarrow$ (ii)′ $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iii)′. There is a syntactical condition equivalent to (i) in Kueker [14], which simplifies an earlier (unpublished) condition by W. Craig.

(ii) We could have considered special structures (in the sense of [3]) instead of saturated structures in Definition 3.1.1. In this way, we could have obtained the existence of $\mu$-typical (in this new sense) structures of power $\mu$ for cardinals $\mu$ of the form $\beth_\delta$ where $\delta$ is a limit ordinal (see [3], [19] for references). 3.1.3 can now be proved for special structures of one of these powers. This also allows us to eliminate the assumption of the existence of a regular cardinal $\mu$ such that $\mu = 2^{\underline{\mu}}$ in 3.14.

We now use 1.1.13 to derive the local version of the Chang-Makkai theorem in the version of Makkai [18].

3.1.6. Assume that $\mu$ is an infinite regular cardinal and $(\mathfrak{A}, P)$ is a $\mu$-saturated structure of power $\mu$. Let $G = \text{Aut}(\mathfrak{A})$. Then

(i) $|O^G(P)| = 1$ iff $P$ is definable in $\mathfrak{A}$ by a first-order formula from the relations of $\mathfrak{A}$.

(ii) $|O^G(P)| \leq \mu$ iff $P$ is definable in $\mathfrak{A}$ by a first-order formula from the relations of $\mathfrak{A}$ and finitely many individuals from $|\mathfrak{A}|$.

**Proof**: (i) can be found in [18].

(ii) Let $\mathfrak{A}^* = (\mathfrak{A}, \Phi^{\mathfrak{A}})_{\Phi \in F}$, where $F$ is the set of first-order formulas of the language of the same similarity type of $\mathfrak{A}$. Since $\mathfrak{A}$ is also $\mu$-saturated, $\mathfrak{A}^*$ is $\mu$-homogeneous. Furthermore, $\text{Aut}(\mathfrak{A}^*) = G$ as is easily checked. By 1.2.8, $G$ is a $\mu$-Baire group. Assume that $|O^G(P)| \leq \mu$. By 1.1.3, $P$ is not $G$-$\mu$-free, i.e., $P$ is $G$-$\mu$-w.det. By 1.1.10, $P$ is $G \cap [\text{id}_X]$-$\mu$-det for some subset $X \subseteq A$ such that

$\overline{\overline{X}} < \mu$. Since $(\mathfrak{A}, x)_{x \in X}$ is also $\mu$-saturated, we can conclude (ii) from (i).

3.1.7. Assume that $\kappa$ and $\mu$ are infinite cardinals such that $\mathrm{cf}(\kappa) \geq \mu^+$, $\mu$ is regular and $\langle P_\xi : \xi \in \mu \rangle$ is a sequence of relations on $A$ such that $(\mathfrak{A}, P_\xi)$ is a special structure of power $\kappa$, for each $\xi \in \mu$. If no $P_\xi$ is definable in $\mathfrak{A}$ by a first-order formula from the relations of $\mathfrak{A}$ and finitely many individuals of $A$, then there is an automorphism $\pi \in \mathrm{Aut}(\mathfrak{A})$ such that $\pi P_\xi \neq P_\eta$ for all $\xi, \eta \in \mu$.

**Proof**: Let $G = \mathrm{Aut}(\mathfrak{A})$ and $X = \{P_\xi : \xi \in \mu\}$. Assume that some $P_\xi$ is $G$-$\mu$-w.det. By 1.1.10, $P_\xi$ is $G \cap [\mathrm{id}_X]$-$\mu$-det, for some $Y \subsetneq A$ such that $\overline{\overline{Y}} < \mu$. This implies that $P_\xi$ is $\mathrm{Aut}((\mathfrak{A}, y)_{y \in Y})$-$\mu$-det. Since $(\mathfrak{A}, y)_{y \in Y}$ is also special, an argument similar to the proof of 3.1.6 (i) shows that $P_\xi$ is definable in this structure, i.e., $P_\xi$ is definable from the relations of $\mathfrak{A}$ and finitely many individuals of $A$, a contradiction. Therefore each $P_\xi$ is $G$-$\mu$-free. Let $\mathfrak{A}^* = (\mathfrak{A}, \Phi^{\mathfrak{A}})_{\Phi \in F}$ as in the proof of 3.1.6. Hence $G = \mathrm{Aut}(\mathfrak{A}^*)$ and 1.2.8 implies that $G$ is a $\mu$-Baire group. By 1.1.15, there is some $\pi \in G$ such that $\pi^* X \cap X = 0$, i.e., $\pi P_\xi \neq P_\eta$ for all $\xi, \eta \in \mu$.

To formulate our next result, we consider a first-order language $L$ and we let $L(\{\underline{P}_i : i \in I\})$ be the language obtained from $L$ by adding a set of new relation symbols $\{\underline{P}_i : i \in I\}$, all having the same rank $n$.

3.1.8. Assume that $T$ is a complete theory in the language $L(\{\underline{P}_i : i \in I\})$. Assume that for every $(\mathfrak{A}, P_i)_{i \in I} \in \mathrm{Mod}\, T$ and every $\pi \in \mathrm{Aut}(\mathfrak{A})$, $\pi P_i = P_j$ for some $i, j \in I$. Then there is some $i \in I$ and some formula $\theta(x_1, \ldots, x_n, v_1, \ldots, v_k)$ of the language $L$ such that

$$T \vdash \exists v_1, \ldots, v_k, \quad \forall x_1, \ldots, x_n ,$$

$$(P_i x_i, \ldots, x_n \leftrightarrow \theta(x_1, \ldots, x_n, v_1, \ldots, v_k)) .$$

**Proof**: (in sketch). If $T$ has finite models, the conclusion of 3.18 is automatically satisfied. If $T$ has infinite models, we take special structures and we use 3.1.7.

### 3.2. *Chang-Makkai theorem for prescribed cardinalities*

We now state the main result of section 3. We assume that $L$ is a first order language of similar type $\rho$ and $L(\underline{P})$ is the language obtained from $L$ by adding a new $n$-ary relation symbol $\underline{P}$.

3.2.1. **Theorem**. *Assume that $T$ is a theory in a first-order language $L(P)$. Let $\kappa = \bar{\bar{\rho}} \cdot \aleph_0$ and let $\mu = 2^{\underline{\kappa}}$. Then the following are equivalent:*

(i) For every infinite structure $\mathfrak{A}$ of type $\rho$ and power $\mu$,

$$|\{P: (\mathfrak{A}, P) \in \mathrm{Mod}\, T\}| < 2^{\mu} .$$

(ii) For every infinite $(\mathfrak{A}, P) \in \mathrm{Mod}\, T$ of power $\mu$,

$$|\{P': (\mathfrak{A}, P') \simeq (\mathfrak{A}, P)\}| < 2^{\mu} .$$

(iii) There are formulas $\theta_i(x_1, ..., x_n, v_1, ..., v_k)$ $i = 1, ..., n$ such that

$$T \vdash \bigvee_{1 \leq i \leq n} \exists v_1, ..., v_k , \forall x_1, ..., x_n ,$$

$$(\underline{P}x_1, ..., x_n \leftrightarrow \theta_i(x_1, ..., x_n, v_1, ..., v_k)) .$$

**Proof**: We shall build a "tree" of structures. By compactness we may assume that $T$ is complete. Furthermore, if $T$ has only finite models, (i), (ii) and (iii) are trivially satisfied. Hence we may assume that $T$ has only infinite models.

Assume that (iii) does not hold.

We shall build a sequence of structures $\langle(\mathfrak{A}_\xi, P_\alpha^{(\xi)}, f_{\alpha\beta}^{(\xi)})_{\alpha,\beta\in {}^\xi 2}:$ $\xi \in \mu\rangle$ such that the following conditions are satisfied for all $\xi \in \mu$:

(i)$_\xi$ $\overline{\overline{\mathfrak{A}}}_\xi \leq \mu$ ;

(ii)$_\xi$ $(\mathfrak{A}_\eta, P_{\alpha|\eta}^{(\eta)}) \preccurlyeq (\mathfrak{A}_\xi, P_\alpha^{(\xi)})$ for all $\eta \in \xi$ and all $\alpha \in {}^\xi 2$ ;

(iii)$_\xi$ $f_{\alpha\beta}^{(\xi)} \mid A_\eta = f_{\alpha|\eta\ \beta|\eta}^{(\eta)}$, for all $\eta \in \xi$ and all $\alpha, \beta \in {}^\xi 2$ ;

(iv)$_\xi$ $f_{\alpha\beta}^{(\xi)}\ P_\alpha^{(\xi)} = P_\beta^{(\xi)}$, for all $\alpha, \beta \in {}^\xi 2$ ;

(v)$_\xi$ $f_{\alpha\beta}^{(\xi)} \in \mathrm{Aut}(\mathfrak{A}_\xi)$, for all $\alpha, \beta \in {}^\xi 2$ ;

(vi)$_\xi$ $P_\alpha^{(\xi)} \neq P_\beta^{(\xi)}$, for all $\alpha, \beta \in {}^\xi 2$ such that $\alpha \neq \beta$ .

For $\xi = 0$, we let $(\mathfrak{A}_0, P_0, f_{00}) = (\mathfrak{A}_0, P_0, \mathrm{id}_{A_0})$ be any structure of power $\mu$ such that $(\mathfrak{A}_0, P_0) \in \mathrm{Mod}\ T$ (this structure can be obtained by the Löwenheim-Skolem theorem).

Assume that we have defined $(\mathfrak{A}_\xi, P_\alpha^{(\xi)}, f_{\alpha\beta}^{(\xi)})_{\alpha,\beta\in {}^\xi 2}$. Let $L' = L(\{\underline{P}_\alpha: \alpha \in {}^\xi 2\} \cup \{\underline{f}_{\alpha\beta}: \alpha, \beta \in {}^\xi 2\} \cup \{\underline{a}: a \in A_\xi\} \cup \{\underline{\pi}\})$ be the language obtained from $L$ be adding new relations symbols $\underline{P}_\alpha, \underline{f}_{\alpha\beta}$, $\underline{a}, \underline{\pi}$ for $\alpha, \beta \in {}^\xi 2$ such that each $\underline{P}_\alpha$ is $n$-ary each $\underline{f}_{\alpha\beta}$ is binary and $\underline{\pi}$ is also binary.

Let $\Sigma_{\xi+1}$ be the union of the following three sets of sentences of $L'$:

(1) $\mathrm{Th}(\mathfrak{A}_\xi, P_\alpha^{(\xi)}, f_{\alpha\beta}^{(\xi)}, a)_{a\in A_\xi,\ \alpha,\beta\in {}^\xi 2}$ ;

(2) The set if sentences asserting that $\pi$ is an automorphism of the type $\rho$, which extends $\mathrm{id}_{A_\xi}$;

(3) The set of sentences asserting that $\underline{\pi}\underline{P}_\alpha \neq \underline{P}_\beta$ for all $\alpha, \beta \in {}^\xi 2$.

It is clear that $|\Sigma_{\xi+1}| \leq \mu$. Let $(\mathfrak{B}, Q_\alpha, g_{\alpha\beta})_{\alpha,\beta\in {}^\xi 2} \succcurlyeq (\mathfrak{A}_\xi, P_\alpha^{(\xi)}, f_{\alpha\beta}^{(\xi)})_{\alpha,\beta\in {}^\xi 2}$ be a special structure such that $\mathrm{cf}(\overline{\overline{\mathfrak{B}}}) > \mu^{++}$. By 3.1.7 applied to $\kappa = \overline{\overline{\mathfrak{B}}}$, $\mu^+$ and $(\mathfrak{B}, a)_{a\in A_\xi}$ which is also special, there is some $\pi \in \mathrm{Aut}(\mathfrak{B}, a)_{a\in A_\xi}$ such that $\pi Q_\alpha \neq Q_\beta$ for $\alpha, \beta \in {}^\xi 2$. (Clearly, no $Q_\alpha$ is definable from parameters in $(\mathfrak{B}, a)_{a\in A_\xi}$ since we are assuming that 3.2.1 (iii) does not hold.) Therefore $(\mathfrak{B}, Q_\alpha, g_{\alpha\beta}, \pi, a)_{a\in A_\xi,\ \alpha,\beta\in {}^\xi 2} \in \mathrm{Mod}\ \Sigma_{\xi+1}$. By the

Löwenheim-Skolem theorem we can obtain a structure

$$(\mathfrak{A}_{\xi+1}, P_\alpha^{(\xi+1)}, f_{\alpha\beta}^{(\xi+1)}, \pi, a)_{a\in A_\xi,\, \alpha,\beta\in {}^\xi 2} \in \operatorname{Mod} \Sigma_{\xi+1}$$

such that $|\mathfrak{A}_{\xi+1}| \supseteq |\mathfrak{A}_\xi|$ and $\bar{\bar{\mathfrak{A}}}_{\xi+1} \leq \mu$. We define

$$P_{\alpha^\frown\langle 0\rangle}^{(\xi+1)} = P_\alpha^{(\xi+1)}, \quad P_{\alpha^\frown\langle 1\rangle}^{(\xi+1)} = P_\alpha^{(\xi+1)},$$

$$f_{\alpha^\frown\langle 0\rangle,\, \beta^\frown\langle 0\rangle}^{(\xi+1)} = f_{\alpha\beta}^{(\xi+1)}, \quad f_{\alpha^\frown\langle 0\rangle,\, \beta^\frown\langle 0\rangle}^{(\xi+1)} = \pi \circ f_{\alpha\beta}^{(\xi+1)},$$

$$f_{\alpha^\frown\langle 1\rangle,\, \beta^\frown\langle 1\rangle}^{(\xi+1)} = \pi \circ f_{\alpha\beta}^{(\xi+1)} \circ \pi^{-1}, \quad f_{\alpha^\frown\langle 0\rangle,\, \alpha^\frown\langle 1\rangle}^{(\xi+1)} = \pi .$$

By a straightforward computation, we can verify that conditions $(\mathrm{i})_{\xi+1} - (\mathrm{vi})_{\xi+1}$ are satisfied. In fact, $(\mathrm{i})_{\xi+1}$ is obvious. By construction

$$(\mathfrak{A}_\xi, P_\alpha^{(\xi)}) \preccurlyeq (\mathfrak{A}_{\xi+1}, P_{\alpha^\frown\langle 0\rangle}^{(\xi+1)})$$

for all $\alpha \in {}^\xi 2$ and we now check that

$$(\mathfrak{A}_\xi, P_\alpha^{(\xi)} \preccurlyeq (\mathfrak{A}_{\xi+1}, P_{\alpha^\frown\langle 1\rangle}^{(\xi+1)})$$

for all $\alpha \in {}^\xi 2$. Let

$$(\mathfrak{A}_{\xi+1}, P_{\alpha^\frown\langle 1\rangle}^{(\xi+1)}) \models \phi[a_1, ..., a_m]$$

for some $a_1, ..., a_m \in A_\xi$. Then

$$(\mathfrak{A}_{\xi+1}, \pi^{-1} P_{\alpha^\frown\langle 1\rangle}^{(\xi+1)}) \models \phi[\pi^{-1} a_1, ..., \pi^{-1} a_m] .$$

Since

$$\pi | A_\xi = \mathrm{id}_{A_\xi}, \quad (\mathfrak{A}_{\xi+1}, P_{\alpha^\frown\langle 0\rangle}^{(\xi+1)}) \models \phi[a_1, ..., a_m] .$$

By the construction this implies

$$(\mathfrak{A}_\xi, P_\alpha^{(\xi)}) \models \phi[a_1, ..., a_m]$$

and this completes $(\text{ii})_{\xi+1}$. The rest of the conditions can be easily verified. We just notice that for $(\text{iii})_{\xi+1}$ we use the fact that $\pi|A_\xi = \text{id}_{A_\xi}$; in $(\text{iv})_{\xi+1}$ we use the fact that the composition of two automorphisms is an automorphism and in $(\text{vi})_\xi$ we notice that

$$(\mathfrak{A}_{\xi+1}, P_{\alpha^\frown\langle 0\rangle}^{(\xi+1)}, P_{\beta^\frown\langle 0\rangle}^{(\xi+1)}) \equiv_{\alpha,\beta\in{}^\xi 2} (\mathfrak{A}_\xi, P_\alpha^{(\xi)}, P_\beta^{(\xi)})_{\alpha,\beta\in{}^\xi 2}\ .$$

Assume that $\lambda \in \mu$ is a limit ordinal and that we have defined the structures up to $\lambda$. We define

$$\mathfrak{A}_\lambda = \bigcup \{\mathfrak{A}_\xi : \xi \in \lambda\}\ ;$$

$$P_\alpha^{(\lambda)} = \bigcup \{P_{\alpha|\xi}^{(\xi)} : \xi \in \lambda\}\ , \text{ for all } \alpha \in {}^\lambda 2\ ;$$

$$f_{\alpha\beta}^{(\lambda)} = \bigcup \{f_{\alpha|\xi,\,\beta|\xi}^{(\xi)} : \xi \in \lambda\}\ , \text{ for all } \alpha, \beta \in {}^\lambda 2\ .$$

(We notice that these definitions make sense because of $(\text{ii})_\xi$ and $(\text{iii})_\xi$).

The verification of $(\text{i})_\lambda - (\text{vi})_\lambda$ is simpler than in the previous case and is omitted. We just notice that $(\text{ii})_\lambda$ and $(\text{v})_\lambda$ use Tarski's union theorem [26].

Finally, let

$$\mathfrak{A} = \bigcup \{\mathfrak{A}_\xi : \xi \in \mu\}\ ;$$

$$P_\alpha = \bigcup \{P_{\alpha|\xi}^{(\xi)} : \xi \in \mu\}\ , \text{ for all } \alpha \in {}^\mu 2\ ;$$

$$f_{\alpha\beta} = \bigcup \{f_{\alpha|\xi,\,\beta|\xi}^{(\xi)} : \xi \in \mu\}\ , \text{ for all } \alpha, \beta \in {}^\mu 2\ ;$$

and $* = \langle 0, 0, 0, ... \rangle$.

Then $(\mathfrak{A}, P_*) \in \text{Mod } T$. (By Tarski's union theorem and $\overline{\overline{\mathfrak{A}}} = \mu$.) Furthermore if $G = \text{Aut}(\mathfrak{A})$, then $|O^G(P_*)| = 2^\mu$, since for each $\alpha \in {}^\mu 2$, $P_\alpha \in O^G(P_*)$ and $P_\alpha \neq P_\beta$ whenever $\alpha \neq \beta$, as it is easily verified. It follows that (ii) does not hold. The other implications are obvious.

3.2.2. **Remark.** For the particular case $\kappa = \aleph_0$ (and hence $\mu = \aleph_0$), this theorem was conjectured by M.Makkai (private communication). He had proved a special case, namely the case that Mod $(T)$ is a $UC_\Delta$ class (though allowing operation symbols in $\rho$). His (unpublished) proof made use of a partition theorem of [5]. Recently, D.Kueker has kindly informed us that several years ago C.C. Chang had proved (but not published) the equivalence of (iii) and the following condition (ii)′ (for the special case $\kappa = \aleph_0$):

(ii)′ For every infinite countable $(\mathfrak{A}, P) \in \text{Mod } T$,

$$|\{P' : (\mathfrak{A}, P') \simeq (\mathfrak{A}, P)\}| < \aleph_1 .$$

His proof is quite different from ours and used Vaught's two cardinal theorem [19]. After seeing our abstract [21], Chang pointed out that we can obtain 3.2.1 (for the particular case $\kappa = \aleph_0$ again) by combining his result with Theorem 2.2.2. His method, however, does not seem to yield 3.2.1 in the general case.

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