Annals of Pure and Applied Logic 160 (2009) 238–254

Contents lists available at ScienceDirect

Annals of Pure and Applied Logic

journal homepage: www.elsevier.com/locate/apal

Constructive equivalence relations on computable probability measures

Laurent Bienvenu [a], Wolfgang Merkle [b,*]

[a] Laboratoire d'Informatique Fondamentale, CNRS & Université de Provence, Marseille, France
[b] Institut für Informatik, Ruprecht-Karls-Universität Heidelberg, Germany

ARTICLE INFO

Article history:
Available online 8 February 2009

AMS subject classifications:
03D55
03D80
03D99

Keywords:
Algorithmic randomness
Computable analysis
Probability measures
Kolmogorov complexity

ABSTRACT

A central object of study in the field of algorithmic randomness are notions of randomness for sequences, i.e., infinite sequences of zeros and ones. These notions are usually defined with respect to the uniform measure on the set of all sequences, but extend canonically to other computable probability measures. This way each notion of randomness induces an equivalence relation on the computable probability measures where two measures are equivalent if they have the same set of random sequences.

In what follows, we study the equivalence relations induced by Martin-Löf randomness, computable randomness, Schnorr randomness and Kurtz randomness, together with the relations of equivalence and consistency from probability theory. We show that all these relations coincide when restricted to the class of computable strongly positive generalized Bernoulli measures. For the case of arbitrary computable measures, we obtain a complete and somewhat surprising picture of the implications between these relations that hold in general.

1. Introduction and overview

A central object of study in the field of algorithmic randomness are notions of randomness for sequences, i.e., infinite sequence of zeros and ones. Since early work by von Mises around 1920, several notions of randomness have been developed. The most satisfactory concept is probably Martin-Löf randomness, which was introduced by Martin-Löf in 1966 [6], but other concepts introduced later have received considerable attention too, for example, computable randomness, Schnorr randomness, and Kurtz randomness. These concepts are usually defined with respect to the uniform measure on the set of all sequences, and in this setting the relations between the different notions have been studied intensively. We refer the reader to Downey and Hirschfeldt [2] for an excellent comprehensive and detailed survey on algorithmic randomness and the various notions of randomness for sequences.

The mentioned notions of randomness extend canonically to other computable probability measures. This way every randomness notion induces an equivalence relation on the class of computable measures where two measure are equivalent if the corresponding sets of random sequences are the same. In what follows, we focus on the question of which implications hold between these equivalence relations, i.e., we ask for example whether the coincidence of Schnorr randomness for two given measures implies that also the two corresponding notions of Kurtz randomness are the same.

First, in Section 3, we consider the restricted case of computable strongly positive generalized Bernoulli measures and we extend a well-known result of Vovk [16] on Martin-Löf randomness and a partial result of Muchnik, Semenov, and Uspensky [9] to all four mentioned notions of randomness. We also show that, like in the result of Vovk on Martin-Löf randomness, for both computable randomness and Schnorr randomness, there is a dichotomy in the sense that for any two

* Corresponding author.
E-mail addresses: laurent.bienvenu@lif.univ-mrs.fr (L. Bienvenu), merkle@math.uni-heidelberg.de (W. Merkle).

0168-0072/$ – see front matter © 2009 Elsevier B.V. All rights reserved.
doi:10.1016/j.apal.2009.01.002

strongly positive generalized Bernoulli measures the respective sets of random sequences either coincide or are disjoint, whereas for Kurtz randomness only a slightly weaker result is true.

The case of arbitrary computable measures is considered in Section 4, and there we obtain a complete picture of the implications and nonimplications that hold between the equivalence relations induced by Martin-Löf randomness, computable randomness, Schnorr randomness, and Kurtz randomness, as well as the notions of equivalence and consistency from probability theory. This picture is somewhat surprising because it does not reflect at all the implications between the underlying notions of randomness.

In what follows we deal only with computable measures because there is no completely natural extension of the standard notions of randomness to noncomputable measures. More precisely, when working with noncomputable measures one faces the following dilemma. If the computation model used in the definition of the randomness notion may access the measure as an oracle, then measures that are numerically very close may have vastly different computational power. But otherwise, when information about the measure is lacking, there may be sequences that are random in the formal sense but from an intuitive point of view may be considered as being highly nonrandom with respect to the measure under consideration.

2. Notation and concepts

2.1. Probability measures on Cantor space

In the following, the terms word and sequence refer to finite and infinite, respectively, binary sequences, unless explicitly stated otherwise. The set of all words and the set of all sequences are denoted by 2^* and by 2^ω, where the latter is also referred to as Cantor space. We write $w = w(0) \dots w(n-1)$ for a word w of length n and similarly $A = A(0)A(1) \dots$ for a sequence A. We write $w \upharpoonright n$ and $A \upharpoonright n$ for the finite word consisting of the first n bits of a word w or a sequence A. The empty word is denoted by ϵ, e.g., $A \upharpoonright 0 = \epsilon$ for any sequence A. The prefix relation on $2^\omega \cup 2^*$ is denoted by $\sqsubseteq$. Furthermore, for a subset $\mathcal{X}$ of Cantor space, let $\overline{\mathcal{X}}$ denote the relative complement of $\mathcal{X}$ in 2^ω.

The basic open sets, i.e., the subsets of Cantor space of the form

$$[u] = \{X \in 2^\omega : u \sqsubseteq X\}$$

where u is a word form a basis for the standard topology on Cantor space, which can be characterized equivalently as the product topology of the discrete topology on the set $\{0, 1\}$. For a set of words U, we write $[U]$ for the open set $\bigcup_{u \in U} [u]$.

By the following celebrated result due to Caratheodory, a probability measure on Cantor space is already determined by its restriction to the set of basic open sets.

Theorem 1 (*Caratheodory's Extension Theorem*). *Let m be a function defined on the basic open sets of Cantor space that takes real values in the interval* $[0, 1]$ *such that* $m([\epsilon]) = 1$ *and for any word u it holds that*

$$m([u]) = m([u0]) + m([u1]).$$

Then there exists a unique probability measure μ on Cantor space that extends m and is defined on the σ-algebra induced by the basic open sets.

For convenience, in what follows the term measure will refer to probability measure on Cantor space, unless explicitly specified otherwise. By the extension theorem, we can identify a measure μ with its restriction to the basic open sets, and accordingly we write $\mu(u)$ for $\mu([u])$. The canonical measure on 2^ω is the uniform measure λ, also known as Lebesgue measure, defined by $\lambda(u) = 2^{-|u|}$ for all words u.

Caratheodory's extension theorem allows us to give a simple definition of computable measure. Before, we recall the notion of a computable real-valued function on words.

Definition 2. A function $f : 2^* \to \mathbb{R}$ is *computable* if there is a computable function $\varphi : 2^* \times \mathbb{N} \to \mathbb{Q}$ such that for all u and n, $|\varphi(u, n) - f(u)| \leq 2^{-n}$.

Definition 3. A measure μ is *computable* if the function $u \mapsto \mu(u)$ is computable.

For further use, recall from classical probability theory the two following fundamental relations between measures .

Definition 4. Let μ and ν be two probability measures on the same set.

The measures μ and ν are *equivalent*, $\mu \sim \nu$ for short, if they have the same null sets. The measures μ and ν are *consistent* if there is no set which has measure 1 with respect to one of the measures and has measure 0 with respect to the other one.

Equivalence of measures is indeed an equivalence relation on the class of all measures on Cantor space, whereas consistency is reflexive and symmetric but in general not transitive.

2.2. Algorithmic random sequences

With a measure understood, by definition a subset of Cantor space is null if it is contained in open sets of arbitrarily small nonzero measure. By a result of Ville [15], null sets of Cantor space can be characterized equivalently by the existence of a betting strategy that succeeds on every sequence in the set. Both characterizations of the notion of null set can be made effective, and depending on the chosen computation model, this way one obtains various notions of an effective null set. Unlike for the classical notion, in the effective setting there are singleton sets of the form $\{A\}$ that are not null, in which case the sequence A is called random.

In the remainder of this section, we review some basic notions of algorithmic random sequence. We first review some standard notions of random sequence and their definitions in terms of betting strategies, more precisely, in terms of the equivalent notion of martingale; then we state some standard characterizations of these notions in terms of tests, that is, in terms of sequences of open sets that have measures tending to 0. We refer the reader to the monograph by Downey and Hirschfeldt [2] for a more detailed account of random sequences and for proofs of the results reviewed in this section.

A betting strategy can be identified with a player who successively bets money on the next bit of an infinite binary sequence, never betting more than the current capital. The pay-off of this gamble is fair where, of course, the meaning of fairness depends on the underlying measure. Such betting strategies can be represented by their capital functions, which are called martingales

Definition 5. Let μ be a measure. A *μ-martingale* is a function d from the set of words to the nonnegative reals such that for all words u it holds that

$$d(u)\mu(u) = d(u0)\mu(u0) + d(u1)\mu(u1). \tag{1}$$

A martingale d is said to be *normed* if $d(\epsilon) = 1$. A μ-martingale d *succeeds* on a sequence A, if it holds that

$$\limsup_{n\to\infty} d(A \upharpoonright n) = +\infty.$$

Definition 6. Let μ be a measure. We say that μ is *nowhere vanishing* if $\mu(u) > 0$ for every word u (otherwise the measure μ is said to be *vanishing*).

For any given nowhere vanishing measure μ and with a suitable formalization of betting strategy understood, there is a one-to-one correspondence between μ-martingales and pairs of a betting strategy and a value of the initial capital. Furthermore, the following folklore result asserts an exact correspondence between μ-martingales and measures.

Lemma 7. *For every nowhere vanishing measure μ, the normed μ-martingales are exactly the functions of the form ξ/μ, where ξ is a measure. For every computable nowhere vanishing measure μ, the computable normed μ-martingales are exactly the functions of the form ξ/μ where ξ is a computable measure.*

Remark 8. Observe that given two vanishing computable measures ξ and μ and with some convention for the value of fractions with zero denominator understood, in general the quotient ξ/μ is not a computable function.

For a corresponding counterexample, assume that the value of 0/0 has been set to some not necessarily finite value, and let c be a natural number that differs from this value by at least 1. Then let measures ξ and μ be defined as follows. For all $k \geq 0$, let $z_k = 0^k 1$, let

$$\xi(z_k) = \mu(z_k) = \frac{1}{2^{k+1}},$$

and let ξ and μ be uniformly distributed on extensions of z_k10 and z_k11 in the sense that for all words u, it holds that $\mu(z_k1u0) = \mu(z_k1u1)$, and similarly for ξ. Furthermore, let $\xi(z_k0) = \mu(z_k0) = 0$ in case k is not in the halting problem, and, otherwise, in case k is enumerated first after s steps of some fixed enumeration of the halting problem, let $\xi(z_k0) = c/s$ and $\mu(z_k0) = 1/s$, fixing $\xi(z_k1)$ and $\mu(z_k1)$ accordingly. By construction, the measures ξ and μ are computable, however the quotient ξ/μ is not. For a proof of the latter, observe that approximating the value of the quotient at place z_k0 with error at most 1/3 suffices to decide whether k is in the halting problem.

The fairness condition (1) implies the following basic result by Ville [15], which we state for further use.

Theorem 9 (*Ville*). *Let μ be a measure and let d be a μ-martingale. For all reals $k > 0$, we have*

$$\mu\left\{A \in 2^\omega : \sup_n d(A \upharpoonright n) \geq k\, d(\epsilon)\right\} \leq \frac{1}{k}.$$

Definition 10. A function $f : 2^* \to \mathbb{R}$ is *left-computable* if there is a computable function $\varphi : 2^* \times \mathbb{N} \to \mathbb{Q}$ such that for any word u the values $\varphi(u, n)$ converge nondecreasingly to $f(u)$, i.e., it holds that

$$\lim_{n\to\infty} \varphi(u, n) = f(u) \quad \text{and} \quad \varphi(u, n) \leq \varphi(u, n+1) \quad \text{for all } n.$$

Definition 11. An *order* is a function $g : \mathbb{N} \to \mathbb{N}$ that is nondecreasing and unbounded.

Definition 12. Let μ be a computable probability measure.

A sequence A is *μ-Martin-Löf random* if no left-computable μ-martingale succeeds on A.

A sequence A is *μ-computably random* if no computable μ-martingale succeeds on A.

A sequence A is *μ-Schnorr random* if there is no computable μ-martingale and no computable order h such that for infinitely many n it holds that $h(n) \leq d(A \upharpoonright n)$.

A sequence A is *μ-Kurtz random* if there is no computable μ-martingale and no computable order h such that for all n it holds that $h(n) \leq d(A \upharpoonright n)$.

The subsets of Cantor space of all μ-Martin-Löf random, all μ-computably random, all μ-Schnorr random, and all μ-Kurtz random sequences are denoted by $\mu\mathbf{MLR}$, $\mu\mathbf{CR}$, $\mu\mathbf{SR}$, and $\mu\mathbf{KR}$, respectively.

Martin-Löf randomness has been introduced by Martin-Löf [6] and is meant to capture all effective statistical tests. Schnorr [11,12] introduced computable randomness and Schnorr randomness as weaker, but in some sense more effective notions of randomness. Kurtz randomness was introduced by Kurtz [4] and is also known as weak randomness; the characterization in terms of martingales used above to define the concept is due to Wang [17].

Remark 13. For any measure μ, the notion of μ-Martin-Löf randomness and of μ-computable randomness remain the same if we require in the definition of success of a martingale d on a sequence A not just $\limsup_n d(A \upharpoonright n) = +\infty$ but that the martingale succeeds in the stronger sense that $\lim_n d(A \upharpoonright n) = +\infty$.

For Martin-Löf randomness, this follows from the standard proof of the first assertion in Theorem 16 where for a given μ-Martin-Löf test one constructs a left-computable μ-martingale that succeeds in the mentioned stronger sense on all sequences that are covered by this test. For computable randomness, by the standard technique of putting away one unit of capital whenever a certain threshold is reached, any computable μ-martingale can be transformed into another computable μ-martingale, which is of a special form known as savings martingale, that succeeds in the stronger sense exactly on the sequences on which the first martingale succeeds.

The definitions of Schnorr randomness and Kurtz randomness can be equivalently reformulated as according to Proposition 14, as can be verified by applying the savings technique discussed in Remark 13 (see Downey and Hirschfeldt [2] for a full proof of this fact).

Proposition 14. *A sequence A is μ-Schnorr random if and only if there exists no computable μ-martingale d and computable function $f : \mathbb{N} \to \mathbb{N}$ such that $d(A \upharpoonright f(n)) \geq n$ holds for infinitely many n.*

A sequence A is μ-Kurtz random if and only if there exists no computable μ-martingale d and computable function $f : \mathbb{N} \to \mathbb{N}$ such that $d(A \upharpoonright f(n)) \geq n$ holds for all n.

Definition 15. A subset $\mathcal{V}$ of Cantor space is said to be *effectively open*, if there exists a computably enumerable set A of words such that $\mathcal{V}$ is equal to $\bigcup_{u \in A} [u]$. A sequence $\{\mathcal{V}_n\}_{n\in\mathbb{N}}$ of subsets of Cantor space is said to be *uniformly effectively open* if there exists a computable function $(n, k) \mapsto u_{n,k}$ from pairs of natural numbers to words such that for all n, the set $\mathcal{V}_n$ is equal to $\bigcup_{k\in\mathbb{N}} [u_{n,k}]$.

A μ-Martin-Löf test is a uniformly effectively open sequence $\{\mathcal{V}_n\}_{n\in\mathbb{N}}$ such that for all n, we have $\mu(\mathcal{V}_n) \leq 1/2^n$.

A sequence A *passes* a μ-Martin-Löf test $\{\mathcal{V}_n\}_{n\in\mathbb{N}}$ if $A \notin \bigcap_n \mathcal{V}_n$. A μ-Martin-Löf test *covers* a sequence if the sequence does not pass the test, and the test *covers* a subset of Cantor space if it covers every sequence in the set.

Theorem 16. *A sequence is μ-*Martin-Löf random *if and only if the sequence passes all μ-Martin-Löf tests.*

*A sequence is μ-*Schnorr random *if and only if the sequence passes all μ-Martin-Löf tests $\{\mathcal{V}_n\}_{n\in\mathbb{N}}$ such that $\mu(\mathcal{V}_n) = 1/2^n$ for all n.*

*A sequence is μ-*Kurtz random *if and only if the sequence passes all μ-Martin-Löf tests $\{\mathcal{V}_n\}_{n\in\mathbb{N}}$ such that $\mathcal{V}_n = [U_n]$ for a finite set U_n where a canonical index of U_n can be computed from n.*

Remark 17. The concepts of Martin-Löf randomness and Schnorr randomness remain the same if one uses in their definitions $\varepsilon(n)$ in place of 2^{-n} where $\varepsilon : \mathbb{N} \to \mathbb{Q}$ is a computable function that tends to 0 as n goes to infinity. More precisely, the concepts remain the same if we define a Martin-Löf test to be a uniformly effectively open sequence $\{\mathcal{V}_n\}_{n\in\mathbb{N}}$ where $\mu(V_n) \leq \varepsilon(n)$ for any such function $\varepsilon(n)$, and likewise, one requires in the definition of Schnorr randomness $\mu(V_n) = \varepsilon(n)$.

Another characterization of Martin-Löf randomness can be obtained by a variant of Martin-Löf tests known as Solovay tests [2]; in a similar fashion, Downey and Griffiths [1] gave the following useful characterization of Schnorr randomness.

Proposition 18. *A sequence A is μ-Schnorr random if and only if for any uniformly effectively open sequence of sets $\{\mathcal{V}_n\}_{n\in\mathbb{N}}$ such that $\sum_{n\in\mathbb{N}} \mu(\mathcal{V}_n)$ is finite and equal to a computable real number, the sequence A belongs to only finitely many sets $\mathcal{V}_n$.*

It is straightforward to see that a sequence A is covered by a finite μ-Martin-Löf test of the type used in Theorem 16 for characterizing Kurtz randomness if and only if it belongs to the complement of an effectively open set of measure 1. Accordingly, one obtains the following characterization of Kurtz randomness due to Kurtz [4].

Theorem 19. *A sequence is μ-Kurtz random if and only if it belongs to all effectively open sets $\mathcal{U}$ of μ-measure 1.*

3. Generalized Bernoulli measures

Before we turn to arbitrary computable measures in Section 4, we consider in this section the restricted case of generalized Bernoulli measures. The latter are measures on Cantor space that are obtained by choosing the bits of a sequence A by independent tosses of biased coins such that the probability of $A(i)$ to be 1 is p_i, that is, the probabilities for a bit to be 0 or 1 depend on the position i of the bit, and only on i.

Definition 20. Let $\{p_i\}_{i\in\mathbb{N}}$ be a sequence of real numbers such that $p_i \in [0, 1]$ for all i. The *generalized Bernoulli measure* μ of parameter $\{p_i\}_{i\in\mathbb{N}}$ is defined, for all words u by

$$\mu([u]) = \prod_{i<|u|,\ u(i)=0} (1 - p_i) \prod_{i<|u|,\ u(i)=1} p_i.$$

Furthermore, if there exists a constant $\varepsilon > 0$ such that for all i the real p_i is contained in the closed interval $[\varepsilon, 1 - \varepsilon]$, then the measure μ is said to be *strongly positive*.

Definition 21. The generalized Bernoulli measure μ of parameter $\{p_i\}_{i\in\mathbb{N}}$ is computable if and only if $\{p_i\}_{i\in\mathbb{N}}$ is a computable sequence of real numbers in the sense that the real-valued function $i \mapsto p_i$ is computable.

Remark 22. For generalized Bernoulli measure the Kolmogorov 0-1-law holds, and accordingly the concepts of equivalence and consistency are the same.

Recall that the Kolmogorov 0-1-law holds for some measure if any set that is closed under finite variation has either measure 0 or measure 1, where a set $\mathcal{X}$ is closed under finite variation if it contains for all n and all words u of length n the set

$$\mathcal{X}_u = \{Y \in 2^\omega : Y = u(0)\dots u(n-1)X(n)X(n+1)\dots \text{ for some } X \in \mathcal{X}\}.$$

For generalized Bernoulli measures, one argues as in the proof of the Lebesgue density theorem that given any set $\mathcal{X}$ of nonzero measure and any real number $\delta < 1$, there is a basic open set such that the relative measure of $\mathcal{X}$ in this basic open sets exceeds δ. Consequently, if the set $\mathcal{X}$ is closed under finite variation, then the measure of $\mathcal{X}$ exceeds any $\delta < 1$, hence must be equal to 1.

Now, by definition, for any pair of measures equivalence implies consistency. If, on the other hand, two generalized Bernoulli measures μ and ν are not equivalent, say, there is a set $\mathcal{X}$ such that $\mu(\mathcal{X}) > 0$ and $\nu(\mathcal{X}) = 0$, then if we let $\mathcal{X}'$ be the closure of $\mathcal{X}$ under finite variation, we have $\mu(\mathcal{X}') = 1$ and $\nu(\mathcal{X}') = 0$, where the latter follows because $\mathcal{X}'$ is a countable union of sets of the form $\mathcal{X}_u$, where all such sets have ν-measure 0.

Generalized Bernoulli measures are rather simple generalizations of the uniform measure, even so they have interesting applications in the field of algorithmic randomness [7,13,14]. For example, they have been used by Shen [13] to separate Martin-Löf randomness and Kolmogorov–Loveland stochasticity (for a definition of the latter, see for example [8]), the coincidence of which was left as an open question by Kolmogorov.

In 1948, Kakutani [3] characterized equivalence of strongly positive generalized Bernoulli measures in terms of the differences of corresponding elements in the parameters.

Theorem 23 (*Kakutani*). *Let μ and ν be two strongly positive generalized Bernoulli measures of parameter $\{p_i\}_{i\in\mathbb{N}}$ and $\{q_i\}_{i\in\mathbb{N}}$, respectively.*

(a) *If $\sum_i (p_i - q_i)^2 < +\infty$, then μ and ν are equivalent.*
(b) *If $\sum_i (p_i - q_i)^2 = +\infty$, then μ and ν are inconsistent.*

Vovk [16] proved an analogue of Theorem 23 in terms of Martin-Löf randomness.

Theorem 24 (*Vovk*). *Let μ and ν be computable strongly positive generalized Bernoulli measures of parameter $\{p_i\}_{i\in\mathbb{N}}$ and $\{q_i\}_{i\in\mathbb{N}}$, respectively.*

(a) *If $\sum_i (p_i - q_i)^2 < +\infty$, then $\mu\mathbf{MLR} = \nu\mathbf{MLR}$.*
(b) *If $\sum_i (p_i - q_i)^2 = +\infty$, then $\mu\mathbf{MLR} \cap \nu\mathbf{MLR} = \emptyset$.*

Note that implication (b) in Theorem 24 strengthens the corresponding implication in Kakutani's theorem because if the sets $\mu\mathbf{MLR}$ and $\mu\mathbf{MLR}$ are disjoint, then each of the sets witnesses that the measures μ and ν are inconsistent. In fact, implication (b) in Theorem 24, hence also the argument just given, remains valid with computably random sequences in place of the Martin-Löf random ones, as has been demonstrated by Muchnik et al. [9].

Theorem 25 (*Muchnik, Semenov, and Uspensky*). *Let μ and ν be two computable strongly positive generalized Bernoulli measures, respectively of parameter $\{p_i\}_{i\in\mathbb{N}}$ and $\{q_i\}_{i\in\mathbb{N}}$. If $\sum_i (p_i - q_i)^2 = +\infty$, then $\mu\mathbf{CR} \cap \nu\mathbf{CR} = \emptyset$ (and a fortiori, $\mu\mathbf{MLR} \cap \nu\mathbf{MLR} = \emptyset$).*

In view of Theorems 24 and 25, it is suggesting to ask whether Theorem 24 remains true if one replaces **MLR** by **CR**, by **SR**, or by **KR**. Theorem 31 answers this question in the affirmative for the cases of computable randomness and Schnorr randomness, whereas for Kurtz randomness implication (a) in Theorem 24 is true but implication (b) is false in general. Before we state and demonstrate Theorem 31 and the related Theorem 32, which are the main results of this section, we introduce some notation and derive a technical lemma.

Definition 26. Let d be a μ-martingale for some measure μ. For any word u, let

$$\mathrm{stake}(d,u) = d(u) - \min(d(u0), d(u1)) \quad \text{and} \quad \mathrm{ratio}(d,u) = \frac{\mathrm{stake}(d,u)}{d(u)}$$

where we set $\mathrm{ratio}(d,u) = 0$ in case $d(u) = 0$. Furthermore, let

$$\mathrm{guess}(d,u) = \begin{cases} 0 & \text{if } d(u0) \geq d(u1), \\ 1 & \text{if } d(u0) < d(u1). \end{cases}$$

Suppose a martingale d, which we identify with the corresponding betting strategy, has already scanned a prefix u of an infinite sequence and now bets on the first bit not in u. In this situation, the martingale bets in favor of $\mathrm{guess}(d,u)$, moreover, the martingale bets an absolute amount of capital of $\mathrm{stake}(d,u)$, which is a fraction of $\mathrm{ratio}(d,u)$ of its current capital.

Definition 27. Let d be a μ-martingale and d' be a ν-martingale for some measures μ and ν. Then d' is a *stake-martingale* of d, if for all words u it holds that

$$\mathrm{guess}(d',u) = \mathrm{guess}(d,u) \quad \text{and} \quad \mathrm{stake}(d',u) = \mathrm{ratio}(d,u),$$

except that $\mathrm{stake}(d',u) = 0$ whenever $\mathrm{ratio}(d,v)$ exceeds $d'(v)$ for some prefix v of u. In the latter situation the stake martingale d' is said to be *broke at* u, and d' *goes broke on* a sequence, if d' is broke at some prefix of the sequence.

Observe that for given measures μ and ν and a μ-martingale d, for any given initial capital there is a unique ν-martingale d' that is a stake martingale with respect to d.

Next we observe that λ-martingales d and d' where d' is a stake martingale of d succeed on the same sequences, except for the sequences on which d succeeds and d' goes broke.

Proposition 28. *Let d and d' be λ-martingales where d is normed and d' is a stake martingale of d, and let A be sequence on which d' is not going broke. Then for all n, we have $\ln d(A \upharpoonright n) \leq d'(A \upharpoonright n)$, in particular, if d succeeds on A then d' succeeds on A.*

Proof. By elementary curve sketching, one infers easily that for any real number $t > -1$ we have

$$\ln(1+t) \leq t. \tag{2}$$

Setting x_k equal to either $\mathrm{ratio}(d, A \upharpoonright k)$ or $-\mathrm{ratio}(d, A \upharpoonright k)$ depending on whether $\mathrm{guess}(d, A \upharpoonright k)$ agrees or disagrees with the correct value $A(k)$, we obtain

$$\ln d(a \upharpoonright n) = \ln \prod_{k=0}^{n} (1 + x_k) = \sum_{k=0}^{n} \ln(1 + x_k) \leq \sum_{k=0}^{n} x_k = d'(A \upharpoonright n). \quad \square$$

Proposition 28 and its proof essentially extend to μ-martingales for strongly positive generalized Bernoulli measures μ different from the uniform measure and even to mixed cases of a μ-martingale d and a ν-martingale d', provided that μ and ν are sufficiently close. The proof of this extension uses a variant of (2) where the difference between $\ln(1+t)$ and t is quantified by a term $c_m t^2$ for some constant c_m.

Lemma 29. *For any natural number m there is a constant c_m such that for all real numbers t in the half-open interval $(-1, m]$ it holds that*

$$\ln(1+t) \leq t - c_m t^2. \tag{3}$$

Proof. If we set $c_m = 1/m'$ for any natural number $m' \geq 2(m+1)$, then (3) is true for any t in the interval $(-1, m]$, simply because the functions

$$t \mapsto \ln(1+t) \quad \text{and} \quad t \mapsto t - c_m t^2$$

attain the same value for $t = 0$, while the former function grows faster in the interval $(-1, 0)$ and grows more slowly in the interval $(0, m)$ than the latter function. $\square$

Proposition 30. *Let μ and ν be two computable strongly positive generalized Bernoulli measures of parameter $\{p_i\}_{i\in\mathbb{N}}$ and $\{q_i\}_{i\in\mathbb{N}}$, respectively, such that $\sum_i (p_i - q_i)^2 < +\infty$. Let d be a computable normed μ-martingale and let A be a sequence such that $\lim d(A \upharpoonright n) = +\infty$. Then there exists a (not necessarily normed) computable ν-martingale d' such that for all n it holds that*

$$\ln d(A \upharpoonright n) \leq d'(A \upharpoonright n) + O(1).$$

Proof. The ν-martingale d' will be equal to a stake martingale of d, i.e., we have

$$\text{guess}(d', u) = \text{guess}(d, u) \quad \text{and} \quad \text{stake}(d', u) = \text{ratio}(d, u).$$

For the moment, assume that d' may incur debts, i.e., d' never goes broke but bets an amount of $\text{ratio}(d, u)$ on any word u, where d' then may attain negative values. We will argue later that when first multiplying the stakes of d' with some appropriate constant and then converting back d' to a standard stake martingale that stops betting when it is about to go broke, we obtain a martingale as desired.

For ease of notation, set $\rho_n = \text{ratio}(d, A \restriction n)$. For each n, there are then the three following cases.

	$d(A \restriction n+1)/d(A \restriction n)$	$d'(A \restriction n+1) - d'(A \restriction n)$
$\text{guess}(d, A \restriction n) \neq A_n$	$1 - \rho_n$	$-\rho_n$
$\text{guess}(d, A \restriction n) = A_n = 0$	$1 + \rho_n \frac{p_n}{1-p_n}$	$\rho_n \frac{q_n}{1-q_n}$
$\text{guess}(d, A \restriction n) = A_n = 1$	$1 + \rho_n \frac{1-p_n}{p_n}$	$\rho_n \frac{1-q_n}{q_n}$

By setting x_n equal to $-\rho_n$, or $\rho_n \frac{p_n}{1-p_n}$, or $\rho_n \frac{1-p_n}{p_n}$ in the three different cases, respectively, the entries in the table above can be rewritten as follows.

	$d(A \restriction n+1)/d(A \restriction n)$	$d'(A \restriction n+1) - d'(A \restriction n)$
$\text{guess}(d, A \restriction n) \neq A_n$	$1 + x_n$	x_n
$\text{guess}(d, A \restriction n) = A_n = 0$	$1 + x_n$	$x_n\left(1 + \frac{q_n - p_n}{p_n(1-q_n)}\right)$
$\text{guess}(d, A \restriction n) = A_n = 1$	$1 + x_n$	$x_n\left(1 + \frac{p_n - q_n}{q_n(1-p_n)}\right)$

By induction it follows for all n, that

$$d(A \restriction n) = \prod_{k=0}^{n-1}(1 + x_k).$$

By strong positivity, choose $\varepsilon > 0$ such that all p_i and q_i are contained in the interval $[\varepsilon, 1-\varepsilon]$ and set $m = \lceil \varepsilon^{-1} \rceil$. Then by definition all x_n are in the interval $[-1, m]$, hence by Lemma 29 there is a positive constant c_m such that for all k, we have

$$\ln\, d(A \restriction n) = \ln \prod_{k=0}^{n-1}(1 + x_k) = \sum_{k=0}^{n-1} \ln(1 + x_k) \leq \sum_{k=0}^{n-1} x_k - \sum_{k=0}^{n-1} c_m x_k^2. \tag{4}$$

Concerning the martingale d', for all n and for all three cases discussed above, we have

$$x_n - m^2 |x_n||p_n - q_n| \leq d'(A \restriction n+1) - d'(A \restriction n),$$

hence, by induction, it follows for all n that

$$\sum_{k=0}^{n-1} x_k - \sum_{k=0}^{n-1} m^2 |x_k||p_k - q_k| \leq d'(A \restriction n). \tag{5}$$

Next set $c = \sqrt{\sum_{i=0}^{\infty}(p_i - q_i)^2}$ and choose a constant c' such that for all $t \geq 0$,

$$m^2 c \sqrt{t} \leq c_m t + c'.$$

Then we obtain by the Cauchy–Schwarz inequality

$$\begin{aligned} \sum_{k=0}^{n-1} m^2 |x_k||p_k - q_k| &\leq m^2 \sqrt{\sum_{k=0}^{n-1}(p_k - q_k)^2} \sqrt{\sum_{k=0}^{n-1} x_k^2} \\ &= m^2 c \sqrt{\sum_{k=0}^{n-1} x_k^2} \leq c_m \left(\sum_{k=0}^{n-1} x_k^2\right) + c'. \end{aligned}$$

Together with (4) and (5), this yields

$$\ln\, d(A \restriction n) \leq d'(A \restriction n) + c'.$$

Recall that up to now, we have assumed that d' is a normed ν-martingale of a special type that is allowed to incur debts. Furthermore, by assumption, we have

$$\lim_n d(A \restriction n) = +\infty, \quad \text{hence } \lim_n d'(A \restriction n) = +\infty,$$

and thus there is a natural number m such that $-(m-1) < d'(A \upharpoonright n)$ for all n. Consequently, if we multiply the stakes of d' by $1/m$, we obtain a new ν-martingale of special type that never goes broke on A. If we transform this ν-martingale back to a stake martingale, i.e., one that stops betting when being about to go broke, we obtain a ν-martingale as desired. □

Theorem 31. *Let μ and ν be computable strongly positive generalized Bernoulli measures of parameter $\{p_i\}_{i\in\mathbb{N}}$ and $\{q_i\}_{i\in\mathbb{N}}$, respectively. If $\sum_i (p_i - q_i)^2 < +\infty$ holds, then $\mu\mathbf{CR} = \nu\mathbf{CR}$, $\mu\mathbf{SR} = \nu\mathbf{SR}$, and $\mu\mathbf{KR} = \nu\mathbf{KR}$.*

Proof. Suppose that $A \notin \mu\mathbf{CR}$, i.e. by Remark 13 there exists a computable normed μ-martingale d such that $\lim_n d(A \upharpoonright n) = +\infty$. By Proposition 30 there exists a computable ν-martingale d' such that $d'(A \upharpoonright n) \geq \ln d(A \upharpoonright n)$ for all n. This implies in particular that $\lim_n d'(A \upharpoonright n) = +\infty$, i.e. $A \notin \nu\mathbf{CR}$. Suppose now that $A \notin \mu\mathbf{SR}$. Then d can be chosen such that in addition $d(A \upharpoonright n) \geq g(n)$ for some computable order g and infinitely many n, hence $d'(A \upharpoonright n) \geq \ln g(n)$ for infinitely many n's. Since $\ln g(n)$ is an order, this proves $A \notin \nu\mathbf{SR}$. A similar proof can be given for Kurtz randomness with "all n" in place of "infinitely many n's". We thus have proved $\nu\mathbf{CR} \subseteq \mu\mathbf{CR}$, $\nu\mathbf{SR} \subseteq \mu\mathbf{SR}$, $\nu\mathbf{KR} \subseteq \mu\mathbf{KR}$ and the theorem follows by symmetry. □

Theorem 31 can be strengthened to a full dichotomy in the style of Vovk's theorem, and this is stated in Theorem 32. In the latter theorem, the equivalence of the assertions (i) through (iv) in (a) and in (b) hold by the theorems of Kakutani and of Vovk and by Remark 22, and the implications (v) → (iv) in (a) and (i) → (v) in (b) have been demonstrated by Muchnik, Semenov, and Uspensky [9].

Theorem 32. *Let μ and ν be computable strongly positive generalized Bernoulli measures of parameter $\{p_i\}_{i\in\mathbb{N}}$ and $\{q_i\}_{i\in\mathbb{N}}$, respectively.*

(a) *The following are equivalent.*
- (i) $\sum_i (p_i - q_i)^2 < +\infty$,
- (ii) *μ and ν are consistent*,
- (iii) *μ and ν are equivalent*,
- (iv) $\mu\mathbf{MLR} = \nu\mathbf{MLR}$,
- (v) $\mu\mathbf{CR} = \nu\mathbf{CR}$,
- (vi) $\mu\mathbf{SR} = \nu\mathbf{SR}$,
- (vii) $\mu\mathbf{KR} = \nu\mathbf{KR}$.

(b) *The following are equivalent.*
- (i) $\sum_i (p_i - q_i)^2 = +\infty$,
- (ii) *μ and ν are inconsistent*,
- (iii) *μ and ν are not equivalent*,
- (iv) $\mu\mathbf{MLR} \cap \nu\mathbf{MLR} = \emptyset$,
- (v) $\mu\mathbf{CR} \cap \nu\mathbf{CR} = \emptyset$,
- (vi) $\mu\mathbf{SR} \cap \nu\mathbf{SR} = \emptyset$.
- (vii) $\mu\mathbf{SR} \cap \nu\mathbf{KR} = \mu\mathbf{KR} \cap \nu\mathbf{SR} = \emptyset$.

Proof. For part (a) and for part (b), assertions (i) through (iv) are pairwise equivalent as an immediate consequence of Kakutani's and Vovk's theorem and by Remark 22.

For (a), the fact that assertion (i) implies assertions (iv) through (vii) is just Theorem 31, and by Theorem 38, which will be shown in the next section, even for arbitrary measures μ and ν, each of the assertions (iv) through (vii) implies consistency.

For (b), the implications (vii) → (vi), (vi) → (v), and (v) → (iv) are immediate by the way the involved sets are nested, while even for arbitrary measures μ and ν, inconsistency implies (vii), as stated in Proposition 40 in the next section. □

Remark 33. There are computable strongly positive generalized Bernoulli measures μ and ν that are inconsistent but where $\mu\mathbf{KR}$ and $\nu\mathbf{KR}$ have a nonempty intersection. Accordingly, assertion (vii) in part (b) of Theorem 32 cannot be replaced by $\mu\mathbf{KR} \cap \nu\mathbf{KR} = \emptyset$.

For a proof, consider the generalized Bernoulli measures μ and ν where the probability for a single bit to be 1 is 1/2 and 1/3, respectively. Then μ and ν are inconsistent by Theorem 32, however, one can construct a sequence R that is Kurtz random with respect to both measures by a noneffective finite-extension construction that diagonalizes against all pairs of a computable martingale d and computable order h. More precisely, all such pairs (d, h) are considered in some order and for the pair currently considered one extends the already constructed prefix of R to a word w such that $d(w) < h(|w|)$. Observe that by a slight variation of this construction the set R can be chosen to be computable in the halting problem.

As a slight digression from our investigation of effective random sequences, we derive next some results on effective null sets. Recall that notions of effective null sets can be introduced similarly to the introduction of various notions of random sequence in terms of martingales in Definition 12. Given a computable measure μ, for example, a set is a *μ-computably null set* if there is a computable μ-martingale that succeeds on every sequence in the set, a set is a *μ-Schnorr null set* if there is a computable μ-martingale d and a computable order h such that for every sequence A in the set there are infinitely many n such that $d(A \upharpoonright n)$ exceeds $h(n)$, and a set is a *μ-Kurtz null set* if there is a computable μ-martingale d and a computable order h such that for every sequence A in the set $d(A \upharpoonright n)$ exceeds $h(n)$ for all n.

Recall further that similarly to the characterizations of effective random sequences in terms of tests in Theorem 16, notions of effective μ-null sets can be characterized by requiring that there is an appropriate μ-test $(\mathcal{V}_n)_{n\in\mathbb{N}}$ such that the set under consideration is contained in the intersection of the sets $\mathcal{V}_n$. For example, a set is a μ-Schnorr null set if and only if there is a μ-Schnorr test $(\mathcal{V}_n)_{n\in\mathbb{N}}$ such that the set is contained in $\cap_{n\in\mathbb{N}}\mathcal{V}_n$.

For the various notions of effective random sequence, a sequence A is random with respect to a given computable measure if and only if the singleton set $\{A\}$ is a null set of the corresponding type. In particular, if for two given computable measures the notions of effective null set are the same, this implies that also the corresponding notions of random sequence are the same. In the case of computable strongly positive generalized Bernoulli measures this implication is in fact an equivalence, as is immediate by Theorem 32 and the following theorem.

Theorem 34. *Let μ and ν be computable strongly positive generalized Bernoulli measures. Then the following assertions are equivalent.*

(a) *The concepts of Martin-Löf null set with respect to μ and to ν coincide.*
(b) *The concepts of computable null set with respect to μ and to ν coincide.*
(c) *The concepts of Schnorr null set with respect to μ and to ν coincide.*
(d) *The concepts of Kurtz null set with respect to μ and to ν coincide.*
(e) *The measures μ and ν are equivalent.*

Proof. First, observe that by the discussion preceding Theorem 34 each of the assertions (a) through (d) implies that the corresponding notions of random sequence are the same, and hence by Theorem 32 implies assertion (e). So it remains to show that assertion (e) implies the other four assertions.

In the case of assertion (d), in fact this implication holds for arbitrary computable measures μ and ν because with respect to any computable measure a set is Kurtz null if and only if the set is contained in a null set that is the complement of an effectively open set. Furthermore, the implication from (e) to (a) is immediate by Kakutani's and Vovk's theorem, and because due to the existence of maximum Martin-Löf null classes any two computable measures have the same Martin-Löf null classes if and only if the two measures have the same Martin-Löf random sequences.

That assertion (e) implies assertions (b) and (c) can be demonstrated by a slight extension of the proof of Theorem 31. We will give the somewhat more complicated proof for assertion (c) and omit the very similar considerations for assertion (b).

Given a computable normed μ-martingale d and a computable order h, we argue that there is a computable ν-martingale d'' and a computable order h'' such that for all sequences A for which there are infinitely many n such that $d(A \upharpoonright n)$ exceeds $h(n)$, there are also infinitely many n where $d''(A \upharpoonright n)$ exceeds $h''(n)$. This then shows that every Schnorr null set with respect to μ is a Schnorr null set with respect to ν, and by symmetry assertion (c) follows.

Recall the construction of the martingale d' in the proof of Theorem 31, where d' was allowed to incur debts, and the transformation of d' into a normed stake martingale that is not allowed to incur debts and where the stakes of d' are multiplied with some appropriate factor. Let d'_m be equal to a normed stake martingale constructed this way while using the factor $1/m$. Furthermore, let

$$d'' = \sum_{m=1}^{\infty} \frac{1}{2^m} d'_m.$$

Then d'' is a normed martingale, which is computable because in order to compute $d''(u)$ up to an error of $1/2^e$, by construction it suffices to evaluate the infinite sum in the definition of d'' up to the index $k = |u| + e$. Furthermore, we infer similarly to the proof of Theorem 31 that for any sequence A such that $d(A \upharpoonright n)$ exceeds $h(n)$ for infinitely many n, for any sufficiently large number m, the martingale d'_m does not go broke on A and that we have for all n,

$$\ln d(A \upharpoonright n) \leq m + m\, d'_m(A \upharpoonright n),$$

hence $d''(A \upharpoonright n)$ exceeds $h''(n) = \ln \ln h(n)$ for infinitely many n. $\square$

Remark 35. By Ville's characterization [15] of null sets as the sets on which some martingale succeeds, the argument in the proof of Proposition 30 extends to a proof of implication (a) in Kakutani's theorem in essentially the same way as with the corresponding implications in Theorem 34. However, implication (a) in Vovk's theorem does not follow this way because it is open whether a stake martingale of a left-computable martingale is again left-computable.

We conclude the discussion of generalized Bernoulli measures by arguing that in the theorems of this section the hypothesis of strong positivity is necessary, where we use a straightforward effective version of the Borel–Cantelli lemma.

Proposition 36. *Let $\{p_i\}_{i\in\mathbb{N}}$ be a computable sequence taking its values in $(0, 1)$ and converging to 0. Let μ be the generalized Bernoulli measure of parameter $\{p_i\}_{i\in\mathbb{N}}$.*

(a) *If $\sum_i p_i < +\infty$, then $\mu\mathbf{MLR} = \mu\mathbf{CR} = \{0, 1\}^*0^\omega$.*
(b) *If $\sum_i p_i = +\infty$, then $\mu\mathbf{CR} \cap \{0, 1\}^*0^\omega = \emptyset$ (which implies a fortiori that $\mu\mathbf{MLR} \cap \{0, 1\}^*0^\omega = \emptyset$).*

Proof. (a) Suppose $\sum_i p_i < +\infty$. Let A be any sequence in $\{0, 1\}^*0^\omega$. If we choose m such that for all $n > m$, $A_n = 0$ and set $u = A \upharpoonright m$, we have

$$\mu(A) = \mu(u) \prod_{i=m+1}^{\infty} (1 - p_i),$$

where the first factor $\mu(u)$ is nonzero because the p_i differ from 0 and 1, and the second factor is nonzero, too, since $\sum_i p_i < +\infty$. This means that every singleton in $\{0, 1\}^*0^\omega$ has positive measure, hence is necessarily μ-Martin-Löf random, i.e., it holds that $\{0, 1\}^*0^\omega \subseteq \mu\mathbf{MLR}$. The inclusion $\mu\mathbf{MLR} \subseteq \mu\mathbf{CR}$ being immediate, it remains to show that $\mu\mathbf{CR} \subseteq \{0, 1\}^*0^\omega$.

Let B be an infinite sequence which is not in $\{0,1\}^*0^\omega$. Let d be the (computable) μ-martingale which at the i-th move bets the fraction p_i of its capital on the value of $B(i)$ to be 1. One has for all n

$$d(B \upharpoonright n) = \prod_{i<n,\ B(i)=0} (1-p_i) \prod_{i<n,\ B(i)=1} (2-p_i).$$

Again, the first product converges to some positive number since $\sum_i p_i < +\infty$, whereas the second product tends to infinity, since $B(i) = 1$ for infinitely many i's. This means that $B \notin \mu\mathbf{CR}$.

Conversely, suppose that $\sum_i p_i = +\infty$. Let A be an element of $\{0,1\}^*0^\omega$. Let m be such that for all $i > m$: $A_i = 0$. Let d be the (computable) μ-martingale which at the i-th move bets nothing if $i \le m$ and bets all its capital on the value of A_i to be 0 if $i > m$. We get:

$$d(A \upharpoonright n) = \prod_{m<i<n} \left(1 + \frac{p_i}{1-p_i}\right) \ge \prod_{m<i<n} \left(1 + p_i\right),$$

which tends to infinity because of $\sum_i p_i = +\infty$. This proves $A \notin \mu\mathbf{CR}$. $\square$

Corollary 37. *There are computable generalized Bernoulli measures μ and ν of parameter $\{p_i\}_{i\in\mathbb{N}}$ and $\{q_i\}_{i\in\mathbb{N}}$, respectively, such that*

$$\sum_i (p_i - q_i)^2 < +\infty, \quad \textit{but} \quad \mu\mathbf{MLR} \ne \nu\mathbf{MLR} \quad \textit{and} \quad \mu\mathbf{CR} \ne \nu\mathbf{CR},$$

where indeed $\mu\mathbf{CR} \cap \nu\mathbf{CR} = \emptyset$, and hence also $\mu\mathbf{MLR} \cap \nu\mathbf{MLR} = \emptyset$.

Proof. The corollary is immediate by applying Proposition 36 to the measures with parameters given by $p_i = \frac{1}{i+1}$ and $q_i = \frac{1}{(i+1)^2}$. $\square$

4. Computable measures

We now move on to the general case of arbitrary computable measures. The Theorem 38 provides a complete picture of how the equivalence relations induced on the class of computable measures by the concepts of Martin-Löf randomness, computable randomness, Schnorr randomness and Kurtz randomness compare to each other. The implication structure between these relations as stated in Theorem 38 is surprising in so far as it does not reflect the implications that hold between the underlying notions of randomness.

In connection with Theorem 38, note that the fact that $\mu\mathbf{CR} = \nu\mathbf{CR}$ implies $\mu\mathbf{MLR} = \nu\mathbf{MLR}$ is due to Muchnik, Semenov, and Uspensky [9]. The other assertions of Theorem 38 are immediate from the results proven in the remainder of this section.

Theorem 38. *For all computable probability measures μ and ν, the following implications hold. Except for the transitive closure of the implications shown, no other implication is true in general.*

$\mu\mathbf{CR} = \nu\mathbf{CR}$

↓

$\mu\mathbf{MLR} = \nu\mathbf{MLR}$ ↘ ↙ $\mu\mathbf{SR} = \nu\mathbf{SR}$

μ, ν equivalent

↓

$\mu\mathbf{KR} = \nu\mathbf{KR}$

↓

μ, ν consistent

Proposition 39. *Let μ and ν be two computable measures.*

(a) *If $\mu\mathbf{MLR} = \nu\mathbf{MLR}$, then μ and ν are equivalent.*
(b) *If $\mu\mathbf{SR} = \nu\mathbf{SR}$, then μ and ν are equivalent.*

Proof. We prove the assertions (a) and (b) simultaneously. Suppose that μ and ν are not equivalent, where by symmetry we may assume that there exists a set $\mathcal{X}$ such that $\mu(\mathcal{X}) = 0$ and $\nu(\mathcal{X}) > 0$. Let q be a rational number such that $\nu(\mathcal{X}) > q > 0$. By definition of measure, $\mu(\mathcal{X})$ is equal to the infimum of $\mu(\mathcal{W})$ over all open sets $\mathcal{W}$ that contain $\mathcal{X}$. Hence, for all $k \in \mathbb{N}$, there exists an open set $\mathcal{W}$ such that $\mu(\mathcal{W}) < 2^{-k}$ and where $\mathcal{W}$ contains $\mathcal{X}$, hence $q < \nu(\mathcal{W})$. The basic open sets $[u]$ form a basis for the topology on Cantor space, thus the open set $\mathcal{W}$ can be written as the union of open sets $[w_i]$ over a finite or infinite prefix-free set of words $\{w_1, w_2, \ldots\}$. Since measures are monotonic and continuous, there exists a constant $N > 0$ such that for $U = \{w_1, \ldots, w_N\}$ we have

$$\mu(U) < \frac{1}{2^k} \quad \text{and} \quad q \le \nu(U).$$

Hence, for any k, there is a finite prefix-free set of words U_k such that we have $\mu(U_k) \leq 2^{-k}$ and $q \leq \nu(U_k)$, and hence for all n, if we let

$$\mathcal{V}_n = \bigcup_{k>n} [U_k], \quad \text{we have } \mu(\mathcal{V}_n) \leq \sum_{k>n} \mu([U_k]) \leq \sum_{k>n} 2^{-k} \leq 2^{-n}.$$

Next observe that the sets U_k can be chosen such that from k one can compute a canonical index for U_k, e.g., by running through all finite sets of words until one is found that satisfies the required properties. This way, the sequence $\{\mathcal{V}_n\}_{n\in\mathbb{N}}$ is uniformly effectively open and the mapping $n \mapsto \mu(\mathcal{V}_n)$ is computable. Indeed, for all k, $\mu(U_k)$ is bounded by 2^{-k}, hence, to compute $\mu(\bigcup_{k>n}[U_k])$ up to precision 2^{-s} it suffices to compute the μ-measure of the effectively given finite set $\bigcup_{n<k\leq s}[U_k]$.

By the preceding discussion, the sequence $\{\mathcal{V}_n\}_{n\in\mathbb{N}}$ is a μ-Schnorr test, hence the set $\mathcal{G} = \bigcap_n \mathcal{V}_n$ contains no μ-Schnorr random sequence. On the other hand, we have $\nu(\mathcal{G}) > 0$ because the sequence $\{\mathcal{V}_n\}_{n\in\mathbb{N}}$ is nested and for all n, we have $\nu(\mathcal{V}_n) \geq q$ because $\mathcal{V}_n$ contains $[U_{n+1}]$. In summary, $\mathcal{G}$ contains some ν-Martin-Löf random sequence and a fortiori some ν-Schnorr random sequence but contains no μ-Schnorr sequence and a fortiori no μ-Martin-Löf random sequence, hence it follows that $\mu\mathbf{MLR} \neq \nu\mathbf{MLR}$ and $\mu\mathbf{SR} \neq \nu\mathbf{SR}$. □

Proposition 40. *Let μ and ν be two computable measures. If μ and ν are inconsistent, then we have*

$$\mu\mathbf{SR} \cap \nu\mathbf{KR} = \mu\mathbf{KR} \cap \nu\mathbf{SR} = \mu\mathbf{SR} \cap \nu\mathbf{SR} = \emptyset.$$

Proof. For any set that has μ-measure 1 and ν-measure 0, the complement of the set has μ-measure 0 and ν-measure 1, hence we can choose a set $\mathcal{X}$ of the latter type for any computable measures μ and ν that are inconsistent. By an argument similar to the one used in the proof of Proposition 39, we infer that for every k there is a finite set of words U_k such that

$$\mu\,(U_k) < \frac{1}{2^k} \quad \text{and} \quad 1 - 1/k \leq \nu\,(U_k)$$

where a canonical index for U_k can be computed from k. By setting $\mathcal{V}_n$ as before equal to the union of the open sets $[U_{n+1}], [U_{n+2}], \ldots$, we obtain a μ-Schnorr test $\{\mathcal{V}_n\}_{n\in\mathbb{N}}$, where every component $\mathcal{V}_n$ is an open set of ν-measure 1 and thus contains all ν-Kurtz random sequences. Consequently, the set $\bigcap_n \mathcal{V}_n$ does not contain any μ-Schnorr random sequence, but contains all the ν-Kurtz random sequences, hence $\mu\mathbf{SR}\cap\nu\mathbf{KR} = \emptyset$. The remaining assertions of the proposition are immediate by symmetry and by the inclusion $\mathbf{SR} \subseteq \mathbf{KR}$. □

From Proposition 40, the following corollary is immediate because for any computable measure μ, the μ-Schnorr random sequences form a nonempty subset of $\mu\mathbf{KR}$.

Corollary 41. *Let μ and ν be two computable measures. If $\mu\mathbf{KR} = \nu\mathbf{KR}$, then μ and ν are consistent.*

Proposition 42. *Let μ and ν be two computable measures. If μ and ν are equivalent, then $\mu\mathbf{KR} = \nu\mathbf{KR}$.*

Proof. The proof is straightforward. Suppose there is a sequence A that is, say, ν-Kurtz random and not μ-Kurtz random. This means that there exists an effectively open set $\mathcal{U}$ of μ-measure 1 such that $A \notin \mathcal{U}$. Since A is ν-Kurtz random, $\mathcal{U}$ has necessarily ν-measure less than 1. Hence, $\mathcal{U}$ witnesses that μ and ν are not equivalent. □

We have proven all the implications of Theorem 38. We now turn to the more delicate task of showing that all other implications between the equivalence relations we study do not hold. When constructing corresponding counter-examples, the sets introduced in the following definition will play a crucial role.

Definition 43. Let μ and ν be computable nowhere vanishing measures, and $k \in \mathbb{R}^+$. We define:

$$\mathcal{L}^k_{\mu/\nu} = \left\{X \in 2^\omega :\ \sup_n \frac{\mu(X \upharpoonright n)}{\nu(X \upharpoonright n)} \geq k\right\} \quad \text{and} \quad \mathcal{L}^\infty_{\mu/\nu} = \bigcap_{k\in\mathbb{N}} \mathcal{L}^k_{\mu/\nu}.$$

Proposition 44. *Let μ and ν be nowhere vanishing computable measures. Then for every $k \in \mathbb{R}^+$, it holds that*

$$\mu(\mathcal{L}^k_{\nu/\mu}) \leq 1/k \quad \textit{and} \quad \mu(\mathcal{L}^\infty_{\nu/\mu}) = 0.$$

Moreover, we have $\mathcal{L}^\infty_{\nu/\mu} \cap \mu\mathbf{CR} = \emptyset$, and thus a fortiori $\mathcal{L}^\infty_{\nu/\mu} \cap \mu\mathbf{MLR} = \emptyset$.

Proof. Observe that ν/μ is a μ-martingale according to Lemma 7, and that this martingale is computable and succeeds exactly on the sequences in $\mathcal{L}^\infty_{\nu/\mu}$. The assertions on $\mathcal{L}^\infty_{\nu/\mu}$ are then immediate. The inequality $\mu(\mathcal{L}^k_{\nu/\mu}) \leq 1/k$ holds by Theorem 9. □

The equivalence relations we study are indeed closely related to the sets of the form $\mathcal{L}^\infty_{\mu/\nu}$.

Proposition 45. *For every pair μ and ν of computable nowhere vanishing measures the following equivalences hold:*

(a) *μ and ν are equivalent if and only if $\mu(\mathcal{L}^\infty_{\mu/\nu}) = \nu(\mathcal{L}^\infty_{\nu/\mu}) = 0$,*
(b) *$\mu\mathbf{MLR} = \nu\mathbf{MLR}$ if and only if $\mathcal{L}^\infty_{\mu/\nu} \cap \mu\mathbf{MLR} = \mathcal{L}^\infty_{\nu/\mu} \cap \nu\mathbf{MLR} = \emptyset$,*
(c) *$\mu\mathbf{CR} = \nu\mathbf{CR}$ if and only if $\mathcal{L}^\infty_{\mu/\nu} \cap \mu\mathbf{CR} = \mathcal{L}^\infty_{\nu/\mu} \cap \nu\mathbf{CR} = \emptyset$.*

Proof. For all three equivalences the "only if" direction is immediate from Proposition 44. Let us now prove the "if" directions. By symmetry for assertion (a) it suffices to demonstrate that every set that is ν-null is also μ-null, and similarly for the two other assertions it suffices to demonstrate $\mu\mathbf{MLR} \subseteq \nu\mathbf{MLR}$ and $\mu\mathbf{CR} \subseteq \nu\mathbf{CR}$, respectively.

Note that from the definition of $\mathcal{L}^k_{\mu/\nu}$ it is immediate that for every open set $\mathcal{U}$ and all nowhere vanishing measures μ and ν it holds that

$$\mu(\mathcal{U} \cap \overline{\mathcal{L}^k_{\mu/\nu}}) \leq k\,\nu(\mathcal{U}). \tag{6}$$

(a) Suppose $\mu(\mathcal{L}^\infty_{\mu/\nu}) = \nu(\mathcal{L}^\infty_{\nu/\mu}) = 0$, and let $\mathcal{X}$ be a set such that $\nu(\mathcal{X}) = 0$. For given $k \in \mathbb{N}$, let $\mathcal{U}$ be an open set that contains $\mathcal{X}$ such that $\nu(\mathcal{U}) \leq 1/k^2$. Then we have

$$\begin{aligned}\mu(\mathcal{X}) &\leq \mu(\mathcal{U})\\ &\leq \mu(\mathcal{U} \cap \mathcal{L}^k_{\mu/\nu}) + \mu(\mathcal{U} \cap \overline{\mathcal{L}^k_{\mu/\nu}})\\ &\leq \mu(\mathcal{L}^k_{\mu/\nu}) + k\,\nu(\mathcal{U})\\ &\leq \mu(\mathcal{L}^k_{\mu/\nu}) + 1/k.\end{aligned}$$

This is true for all k, and by assumption $\mu(\mathcal{L}^\infty_{\mu/\nu}) = 0$, where the latter is equivalent to $\mu(\mathcal{L}^k_{\mu/\nu})$ tending to 0 when k goes to infinity, hence $\mu(\mathcal{X}) = 0$.

(b) Suppose $\mathcal{L}^\infty_{\mu/\nu} \cap \mu\mathbf{MLR} = \mathcal{L}^\infty_{\nu/\mu} \cap \nu\mathbf{MLR} = \emptyset$. Let $A \notin \nu\mathbf{MLR}$. In case A is a member of $\mathcal{L}^\infty_{\mu/\nu}$, by assumption $A \notin \mu\mathbf{MLR}$ holds and we are done. Otherwise, there is a nonzero natural number k such that $A \notin \mathcal{L}^k_{\mu/\nu}$. Now fix a ν-Martin-Löf test $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$ where $A \in \bigcap_n \mathcal{U}_n$ and a computable function $(n, i) \mapsto u_{n,i}$ such that for all n, the set $\mathcal{U}_n$ is the disjoint union of the basic open sets $[u_{n,1}], [u_{n,2}], \ldots$. For all n, let

$$\mathcal{V}_n = \bigcup\{[u_{n,i}] \colon i \in \mathbb{N} \text{ and } \mu(u_{n,i}) < k\,\nu(u_{n,i})\}.$$

Then $\{\mathcal{V}_n\}_{n\in\mathbb{N}}$ is a uniformly effectively sequence of open sets which, by definition, satisfies $\mu(\mathcal{V}_n) < k\,\nu(\mathcal{U}_n)$ for all n, hence $\{\mathcal{V}_n\}_{n\in\mathbb{N}}$ is a μ-Martin-Löf test by Remark 17. But $A \notin \mathcal{L}^k_{\mu/\nu}$, hence for all n, $\mu(A \restriction n) < k\,\nu(A \restriction n)$ hence $A \in \mathcal{V}_n$ for all n and consequently $A \in \bigcap_n \mathcal{V}_n$ and $A \notin \mu\mathbf{MLR}$.

(c) Suppose $\mathcal{L}^\infty_{\mu/\nu} \cap \mu\mathbf{CR} = \mathcal{L}^\infty_{\nu/\mu} \cap \nu\mathbf{CR} = \emptyset$. Let $A \notin \nu\mathbf{CR}$, i.e., there is a ν-martingale that succeeds on A and thus, by Lemma 7, there exists a computable measure ξ such that

$$\limsup_{n\to\infty} \frac{\xi(A \restriction n)}{\nu(A \restriction n)} = +\infty. \tag{7}$$

In case A is a member of $\mathcal{L}^\infty_{\mu/\nu}$, by assumption $A \notin \mu\mathbf{CR}$ holds and we are done. Otherwise, the quotients $\mu(A \restriction n)/\nu(A \restriction n)$ are bounded from above and (7) remains valid with ν replaced by μ, which by Lemma 7 implies $A \notin \mu\mathbf{CR}$. □

In what follows, our main tool for the construction of counter-examples will be Proposition 45 and its variant Proposition 47. The latter is used in constructions of a measure μ where we only define the values of $\mu(u)$ for words u whose length is a power of 3. First notice that if a function $u \mapsto m(u)$ is computable when restricted to the words whose length is a power of 3, and if the condition

$$m(u) = \sum_{\{w\colon |w|=3|u|\}} m(w)$$

is satisfied for all such words, then m canonically extends to a computable measure μ by setting $\mu(u) = m(u0) + m(u1)$ inductively in decreasing order of length for all words u such that $3^s < |u| < 3^{s+1}$. Similarly, if a function $u \mapsto d(u)$ is computable when restricted to the words whose length is a power of 3, and if the condition

$$2^{2|u|}d(u) = \sum_{\{w\colon u\sqsubseteq w \text{ and } |w|=3|u|\}} d(w)$$

is satisfied for all such words, then d canonically extends to a computable λ-martingale.

That said, we need to make sure that things still work if we restrict our attention to words whose length is a power of 3. But this is quite naturally the case, as the cylinders $[w]$ generated by such words form a basis for the topology of 2^ω. For example, if $\mathcal{U}$ is an effectively open set, one can transform any enumeration of $\mathcal{U}$ into an enumeration that uses only such cylinders: instead of enumerating a cylinder $[u]$ where $3^s < |u| \leq 3^{s+1}$, just enumerate all cylinders $[w]$ such that $u \sqsubseteq w$ and $|w| = 3^{s+1}$. Based on this observation, we introduce the following definition.

Definition 46. Let μ and ν be nowhere vanishing computable measures. For every $k \in \mathbb{R}^+$, we set

$$\widehat{\mathcal{L}}^k_{\mu/\nu} = \left\{A \in 2^\omega \colon \sup_n \frac{\mu(A \restriction 3^n)}{\nu(A \restriction 3^n)} \geq k\right\} \quad \text{and} \quad \widehat{\mathcal{L}}^\infty_{\mu/\nu} = \bigcap_{k\in\mathbb{N}} \widehat{\mathcal{L}}^k_{\mu/\nu}.$$

Then we get the desired variant of Proposition 45.

Proposition 47. *For every pair μ and ν of nowhere vanishing computable measures the following equivalences hold,*

(a) *μ and ν are equivalent if and only if $\mu(\widehat{\mathcal{L}}^\infty_{\mu/\nu}) = \nu(\widehat{\mathcal{L}}^\infty_{\nu/\mu}) = 0$,*
(b) *$\mu\mathbf{MLR} = \nu\mathbf{MLR}$ if and only if $\widehat{\mathcal{L}}^\infty_{\mu/\nu} \cap \mu\mathbf{MLR} = \widehat{\mathcal{L}}^\infty_{\nu/\mu} \cap \nu\mathbf{MLR} = \emptyset$,*
(c) *$\mu\mathbf{CR} = \nu\mathbf{CR}$ if and only if $\widehat{\mathcal{L}}^\infty_{\mu/\nu} \cap \mu\mathbf{CR} = \widehat{\mathcal{L}}^\infty_{\nu/\mu} \cap \nu\mathbf{CR} = \emptyset$.*

Similarly, one obtains the following characterizations of Schnorr randomness and Kurtz randomness, which can be verified by using savings martingales as discussed in Remark 13.

Proposition 48. *Let μ be a computable measure. A sequence A is μ-Schnorr random if and only if there exists no computable μ-martingale d and computable order g such that $d(A \upharpoonright 3^n) \geq g(n)$ for infinitely many n.*
A sequence A is μ-Kurtz random if and only if there exists no computable μ-martingale d and computable order g such that $d(A \upharpoonright 3^n) \geq g(n)$ for all n.

As further steps towards the construction of counter-examples, we show the following proposition and state a remark.

Proposition 49. *Let $A \in \lambda\mathbf{SR}$, and suppose that A is Δ^0_2 (i.e. computable in the halting problem). There exists a nowhere vanishing computable measure μ such that $A \notin \mu\mathbf{SR}$ and*

$$\widehat{\mathcal{L}}^\infty_{\mu/\lambda} = \emptyset \quad \text{and} \quad \widehat{\mathcal{L}}^\infty_{\lambda/\mu} = \{A\}.$$

Proof. We will in fact construct a computable λ-martingale d such that $d(A \upharpoonright 3^n)$ tends to 0 as n tends to infinity, in such a way that $d(A \upharpoonright 3^n) \leq 1/n$ for infinitely many n and if $B \neq A$, $d(B \upharpoonright 3^n)$ will be eventually constant. Then, setting $\mu(u) = \lambda(u)d(u)$ for all words u, μ will be as wanted. By the above discussion, we will only define $d(u)$ for those words u whose length is a power of 3, which we do inductively.

Since A is Δ^0_2, it is the pointwise limit of a sequence of words $\{w_s\}_{s\in\mathbb{N}}$. We can moreover assume that $\lim_{s\to+\infty} |w_s| = +\infty$, that $|w_s| \leq 3^s$ for all s, and that w_s is a prefix of A for infinitely many s.

Let $E_k = \{u1^{k|u|} : u \in 2^*\}$. Notice that every λ-Schnorr random sequence has only finitely many prefixes in E_2 and has no prefix in E_k for all k that are sufficiently large. For the sake of simplicity, we assume that A has no prefix in the set $E = E_2$, which is indeed true for some finite variant of A, and leave the virtually identical argument for larger values of k to the reader.

The martingale d is defined inductively as follows, where it is immediate from the construction that d is computable. Initially, we let $d(\epsilon) = d(0) = d(1) = 1$. Then, supposing that $d(u)$ has already been defined for all u of length 3^s, define d on words of length 3^{s+1} as follows. Let u be word of length 3^s. For each extension u' of u that has length 3^{s+1}, define $d(u')$ as follows:

- if u is not an extension of w_s, set $d(u') = d(u)$,
- if u is an extension of w_s and u' is not in E, i.e., $u' \neq u1^{2|u|}$, set $d(u') = \frac{d(u)}{s+1}$.

Finally, set $d(u1^{2|u|})$ in such a way that the average of the values $d(u')$ over all words u' that extend u and have length 3^{s+1} is equal to $d(u)$.

We turn to the verification.

Claim 1. *The function $n \mapsto d(B \upharpoonright 3^n)$ is eventually constant for all $B \neq A$.*

Proof. The words w_s converge pointwise to the sequence A, hence for any sequence $B \neq A$ there exists s_0 such that for all $s > s_0$, the word w_s is not a prefix of B, and thus, by construction of d, for all $s > s_0$, $d(B \upharpoonright 3^s) = d(B \upharpoonright 3^{s_0})$.

Claim 2. *The function $n \mapsto d(A \upharpoonright 3^n)$ t tends to 0 and $d(A \upharpoonright 3^n) \leq 1/n$ for infinitely many n.*

Proof. The claim is a direct consequence of the definition of d. Since A has no prefix in E, by construction of d, one has for all s either $d(A \upharpoonright 3^{s+1}) = d(A \upharpoonright 3^s)$ or $d(A \upharpoonright 3^{s+1}) = d(A \upharpoonright 3^s)/(s+1)$. Hence $s \mapsto d(A \upharpoonright 3^s)$ is non-increasing and is smaller than $1/s$ for all s such that w_s is a prefix of A, which happens infinitely often.

Let us now consider $\mu = \lambda\, d$. By the above discussion, $\widehat{\mathcal{L}}^\infty_{\mu/\lambda} = \emptyset$, $\widehat{\mathcal{L}}^\infty_{\lambda/\mu} = \{A\}$. Moreover, if we consider the μ-martingale $d' = \frac{\lambda}{\mu}$ we see that for infinitely many s, $d'(A \upharpoonright 3^s) \geq s$. Hence, by Proposition 48, $A \notin \mu\mathbf{SR}$. □

Remark 50. There are Δ^0_2 sequences in $\lambda\mathbf{CR} \setminus \lambda\mathbf{MLR}$ and in $\lambda\mathbf{SR} \setminus \lambda\mathbf{CR}$.

For a proof it suffices to observe that the standard constructions of sequences in the two sets under consideration can be performed effectively if the halting problem is given as an oracle. Alternatively, one can use the stronger result by Nies, Stephan, and Terwijn [10] that a Turing degree is high if and only if it contains a sequence in $\lambda\mathbf{CR} \setminus \lambda\mathbf{MLR}$ if and only if it contains a sequence in $\lambda\mathbf{SR} \setminus \lambda\mathbf{CR}$.

Proposition 51. (a) *There exists a computable measure μ that is equivalent to λ and nonetheless satisfies $\lambda\mathbf{MLR} \neq \mu\mathbf{MLR}$, $\lambda\mathbf{CR} \neq \mu\mathbf{CR}$, $\lambda\mathbf{SR} \neq \mu\mathbf{SR}$.*
(b) *There exists a computable measure μ such that $\lambda\mathbf{MLR} = \mu\mathbf{MLR}$ and $\lambda\mathbf{CR} \neq \mu\mathbf{CR}$.*
(c) *There exists a computable measure μ such that $\lambda\mathbf{CR} = \mu\mathbf{CR}$ and $\lambda\mathbf{SR} \neq \mu\mathbf{SR}$.*

Proof. (a) Let A be a Δ_2^0 member of $\lambda\mathbf{MLR}$ (such as Chaitin's constant Ω). Let μ be, by Proposition 49, a (nowhere vanishing) computable measure such that $\widehat{\mathcal{L}}^{\infty}_{\mu/\lambda} = \emptyset$, $\widehat{\mathcal{L}}^{\infty}_{\lambda/\mu} = \{A\}$ and $A \notin \mu\mathbf{SR}$. Since $\lambda(\{A\}) = 0$, by Proposition 47, the measures λ and μ are equivalent. Moreover, since $A \in \lambda\mathbf{MLR} \subset \lambda\mathbf{CR} \subset \lambda\mathbf{SR}$, and $A \notin \mu\mathbf{SR}$, it follows that $\lambda\mathbf{MLR} \neq \mu\mathbf{MLR}$, $\lambda\mathbf{CR} \neq \mu\mathbf{CR}$, $\lambda\mathbf{SR} \neq \mu\mathbf{SR}$.
(b) By Remark 50, let B be a Δ_2^0 sequence such that $B \in \lambda\mathbf{CR} \setminus \lambda\mathbf{MLR}$. By Proposition 49, there exists a (nowhere vanishing) computable measure μ such that $\widehat{\mathcal{L}}^{\infty}_{\mu/\lambda} = \emptyset$, $\widehat{\mathcal{L}}^{\infty}_{\lambda/\mu} = \{B\}$ and $B \notin \mu\mathbf{SR}$. By Proposition 47, we have $\lambda\mathbf{MLR} = \mu\mathbf{MLR}$ (since $B \notin \lambda\mathbf{MLR}$) and $\lambda\mathbf{CR} \neq \mu\mathbf{CR}$ (since $B \in \lambda\mathbf{CR} \setminus \mu\mathbf{CR}$).
(c) By Remark 50, let C be a Δ_2^0 sequence such that $C \in \lambda\mathbf{SR} \setminus \lambda\mathbf{CR}$. By Proposition 49, there exists a (nowhere vanishing) computable measure μ such that $\widehat{\mathcal{L}}^{\infty}_{\mu/\lambda} = \emptyset$, $\widehat{\mathcal{L}}^{\infty}_{\lambda/\mu} = \{C\}$ and $C \notin \mu\mathbf{SR}$. By Proposition 47, we have $\lambda\mathbf{CR} = \mu\mathbf{CR}$ (since $C \notin \lambda\mathbf{CR}$) and $\lambda\mathbf{SR} \neq \mu\mathbf{SR}$ (since $C \in \lambda\mathbf{SR} \setminus \mu\mathbf{SR}$). □

The following lemma will be used in the proof of Proposition 53.

Lemma 52. *Let μ and ν be two nowhere vanishing computable measures and $A \in 2^{\omega}$. If $A \in \nu\mathbf{SR} \setminus \mu\mathbf{SR}$, then there exists a computable order g such that $\frac{\nu(A\restriction 3^n)}{\mu(A\restriction 3^n)} \geq g(n)$ holds infinitely often.*

Proof. Let $A \in \nu\mathbf{SR} \setminus \mu\mathbf{SR}$. By Lemma 7 and Proposition 48, there exists a computable measure ξ and a computable order h such that $\frac{\xi(A\restriction 3^n)}{\mu(A\restriction 3^n)} \geq h(n)$ for infinitely many n. Then $g = \lfloor\sqrt{h}\rfloor$ is a computable order and since $A \in \nu\mathbf{SR}$, for almost all n it holds that $\frac{\xi(A\restriction 3^n)}{\nu(A\restriction 3^n)} \leq g(n)$. Hence, for infinitely many n

$$\frac{\nu(A \restriction 3^n)}{\mu(A \restriction 3^n)} \geq \frac{\xi(A \restriction 3^n)}{\mu(A \restriction 3^n)} \frac{\nu(A \restriction n)}{\xi(A \restriction 3^n)} \geq \frac{h(n)}{g(n)} \geq g(n). \quad \square$$

Proposition 53. *There exists a computable measure μ such that $\lambda\mathbf{SR} = \mu\mathbf{SR}$, $\lambda\mathbf{CR} \neq \mu\mathbf{CR}$ and $\lambda\mathbf{MLR} \neq \mu\mathbf{MLR}$.*

Proof. Let Ω be Chaitin's constant, which is in $\lambda\mathbf{MLR}$. We will construct, in a very similar way as for Proposition 49, a computable λ-martingale d and a corresponding computable measure $\mu = \lambda d$ such that

$$\widehat{\mathcal{L}}^{\infty}_{\mu/\lambda} = \emptyset \quad \text{and} \quad \widehat{\mathcal{L}}^{\infty}_{\lambda/\mu} = \{\Omega\},$$

where now we want Ω to be μ-Schnorr random. Again we will construct d such that we have $\lim_n d(\Omega \restriction 3^n) = 0$, whereas for any sequence $B \neq \Omega$, the values $d(B \restriction 3^n)$ will be eventually equal to a nonzero constant. In addition, we will ensure that $d(\Omega \restriction 3^n)$ decreases so slowly that the values $\frac{\lambda(\Omega\restriction 3^n)}{\mu(\Omega\restriction 3^n)}$ tend to infinity more slowly than any computable order, hence Ω will be μ-Schnorr random by Lemma 52.

Since Ω is a left-computable sequence, let $\{w_s\}_{s\in\mathbb{N}}$ be a sequence of words such that Ω is the pointwise limit of this sequence, $\lim_{s\to+\infty} |w_s| = +\infty$, $|w_s| \leq 3^s$ for all s, w_s is a prefix of Ω for infinitely many s, and if w_s is a prefix of Ω, $w_s \sqsubseteq w_t$ for all $t > s$. Up to replacing the sequence Ω by one of its finite variants, we can assume that Ω has no prefix in the set E_2 as defined in the proof of Proposition 49.

Let $F : \mathbb{N} \to \mathbb{N}$ be such that $F(0) = 0$, and for all $s > 0$, if $w_s \sqsubseteq \Omega$, then $F(s+1) = |w_s|$ and, otherwise, $F(s+1) = F(s)$. By definition of the w_s, for all s, the initial segment of w_s which coincides with Ω has at least length $F(s)$.

Claim. *F tends to infinity slower than any computable order.*

Proof. Let g be a computable order, and suppose that $g(n) \leq F(n)$ for infinitely many n. Up to taking g even more slow-growing, we can assume that $g(0) = 0$ and $g(n+1) \leq g(n) + 1$ for all n. In other words, we can suppose that the range of g is $\mathbb{N}$. Then, for all $i \in \mathbb{N}$, let n_i be the largest integer such that $g(n_i) = i$. Since $g(n) \leq F(n)$ for infinitely many n, it follows that $i = g(n_i) \leq F(n_i)$ for infinitely many i. Since $w_n \restriction F(n) = \Omega \restriction F(n)$ for all n, it follows that for infinitely many i, $\Omega \in [w_{n_i} \restriction i]$. And since

$$\sum_i \lambda\left([w_{n_i} \restriction i]\right) = \sum_i 2^{-i} = 2,$$

we can apply Proposition 18, from which we get that Ω is not Schnorr random, a contradiction.

We now construct inductively a λ-martingale d such that for all s it holds that $d(\Omega \restriction 3^s) = F(s)^{-1}$, whereas for any sequence $B \neq \Omega$, the values $d(B \restriction n)$ are eventually constant. Initially, we let $d(\epsilon) = d(0) = d(1) = 1$. Supposing that $d(u)$ has already been defined for a word u of length 3^s, define $d(u')$ for every extension u' of u of length 3^{s+1} as follows:

- if u is not an extension of w_s, set $d(u') = d(u)$
- if u is an extension of w_s and u' is not in E_2 (i.e. $u' \neq u1^{2|u|}$) set $d(u') = \frac{1}{|w_s|}$.

Finally, set $d(u1^{2|u|})$ in such a way that the average of the values $d(u')$ over all words u' that extend u and have length 3^{s+1} is equal to $d(u)$.

Set $\mu = \lambda\, d$. It remains to show that μ is as wanted. First, we see that, for the same reason as in the proof of Proposition 49, $s \mapsto d(B \upharpoonright 3^s)$ is eventually constant for $B \neq \Omega$. Second, we see that, since Ω has no prefix in E_2 and by definition of F, for all s, $d(\Omega \upharpoonright 3^{s+1}) = F(s+1)^{-1}$.

To complete the proof, notice that the μ-martingale d succeeds on Ω hence we have $\Omega \notin \mu\mathbf{CR}$ (a fortiori $\Omega \notin \mu\mathbf{MLR}$). It follows that $\lambda\mathbf{MLR} \neq \mu\mathbf{MLR}$ and $\mu\mathbf{CR} \neq \mu\mathbf{CR}$. However, we have $\lambda\mathbf{SR} = \mu\mathbf{SR}$. Indeed by the previous lemma, $\mu\mathbf{SR} \setminus \lambda\mathbf{SR} = \emptyset$ since $\widehat{\mathcal{L}}^{\infty}_{\mu/\lambda} = \emptyset$, and $\lambda\mathbf{SR} \setminus \mu\mathbf{SR} = \emptyset$ since $\widehat{\mathcal{L}}^{\infty}_{\lambda/\mu} = \{\Omega\}$ and $\frac{\lambda(\Omega \upharpoonright 3^n)}{\mu(\Omega \upharpoonright 3^n)} = F(n)$, with $F(n) = o(g(n))$ for every computable order g.

Proposition 54. *There exists a computable measure μ such that μ and λ are consistent and $\lambda\mathbf{KR} \neq \mu\mathbf{KR}$.*

Proof. Let δ be the measure such that $\delta(\{0^\omega\}) = 1$, which is clearly computable, and set $\mu = \delta/2 + \lambda/2$. For any $\mathcal{X} \subseteq 2^\omega$, if $0^\omega \in \mathcal{X}$, then $\mu(\mathcal{X}) = 1/2 + \lambda(\mathcal{X})/2$, and if $0^\omega \notin \mathcal{X}$, then $\mu(\mathcal{X}) = \lambda(\mathcal{X})/2$. In both cases, it is impossible that one of the values $\lambda(\mathcal{X})$ and $\mu(\mathcal{X})$ is equal to 1 and the other is equal to 0, hence λ and μ are consistent. On the other hand, $0^\omega \notin \lambda\mathbf{KR}$ but $0^\omega \in \mu\mathbf{KR}$. □

The proof of our next and last counter-example uses the notion of λ-2-ML randomness, i.e. Martin-Löf randomness relativized to the halting problem $\emptyset'$. The following theorem due to Nies, Stephan and Terwijn [10] gives a nice characterization of λ-2-ML randomness in terms of plain Kolmogorov complexity, which we denote by C. For an extensive survey of Kolmogorov complexity, we refer to Li and Vitanyi [5].

Theorem 55. *A sequence A is λ-2-ML random if and only if there exists a constant c such that $\mathrm{C}(A \upharpoonright n) \geq n - c$ for infinitely many n. Moreover, there exists a computable upper-bound C^* of C such that this characterization of λ-2-ML randomness remains valid with C^* in place of C.*

Proposition 56. *There exists a (nowhere vanishing) computable probability measure μ such that $\mu\mathbf{KR} = \lambda\mathbf{KR}$ and μ is not equivalent to λ.*

Proof. We will construct a computable measure μ such that λ and μ have the same Kurtz random sequences and yet are not equivalent. As in the proof of Proposition 53, the construction will be done by constructing a λ-martingale d and setting $\mu = d\,\lambda$. And here again, we will only define d on words the length of which is a power of 3, the values on the other words being implicitly defined.

Let C^* be a computable function that characterizes λ-2-ML randomness according to Theorem 55. Here we can assume that C^* has in addition the property that for some constant c_1 and for all words u and v, it holds that

$$\mathrm{C}^*(u1^{2|u|}v) \leq 2|u| + |v| + c_1. \tag{8}$$

In order to see this, let c_1 be a constant such that (8) holds for all u and v with C^* replaced by C. For every word w, set

$$\mathrm{C}^{**}(w) = \min\{\mathrm{C}^*(w), |w| - i_w + c_1\}$$

where i_w is the largest length i such that w has a prefix of the form $u1^{2|u|}$ with $|u| = i$. Then C^{**} is computable and satisfies (8), and since we have $C \leq \mathrm{C}^{**} \leq \mathrm{C}^*$, Theorem 55 remains true with C^{**} in place of C^*. In summary, we can assume that C^* satisfies (8) because otherwise we may simply replace C^* by C^{**}.

The set of λ-2-ML random sequences has λ-measure 1 and, by choice of C^* according toTheorem 55, this set is equal to the nested countable union

$$\bigcup_{c \in \mathbb{N}} \{A\colon \exists^\infty n\ \mathrm{C}^*(A \upharpoonright n) \geq n - c\}.$$

Thus, there exists some $c_0 \in \mathbb{N}$ such that the set

$$\mathcal{R} = \{A\colon \exists^\infty n\ \mathrm{C}^*(A \upharpoonright n) \geq n - c_0\}$$

has nonzero λ-measure.

For any sequence X, define the function h_X by

$$h_X(s) = \#\left\{t < s\colon \exists n \in (3^t, 3^{t+1}]\ \mathrm{C}^*(X \upharpoonright n) \geq n - c_0\right\}.$$

Then h_A is eventually constant for all $A \notin \mathcal{R}$, whereas h_A is an order for all $A \in \mathcal{R}$. Furthermore, in the latter case there is no computable order g where $g \leq h_A$, as can be shown by the following line of argument due to Nies, Stephan and Terwijn [10]. Suppose that there were such a computable order g. Then A belongs to the set of all sequences X such that $g(s) \leq h_X(s)$ for all s, which is a Π^0_1 class because the values of h_X can be effectively approximated from above given oracle access to X. Thus, A belongs to a Π^0_1 class which is a subclass of $\mathcal{R}$ and hence contains only λ-2-ML random sequences. This is a contradiction since by the Kreisel Basis Theorem, every nonempty Π^0_1 class contains a Δ^0_2 sequence.

The martingale d is defined inductively, where it is immediate from the construction that d is computable. Initially, set $d(\epsilon) = d(0) = d(1) = 1$. Supposing that $d(u)$ has already be defined for all u of length 3^s, define d on words of length 3^{s+1} as follows. Let u be a word of length 3^s. For each extension w of u that has length 3^{s+1} and differs from $u1^{2|u|} = u1^{(3^{s+1}-3^s)}$,

- set $d(w) = d(u)/2$ in case there is $n \in (3^s, 3^{s+1}]$ where $C^*(w \upharpoonright n) \geq n - c_0$,
- set $d(w) = d(u)$, otherwise.

Finally, set $d(u1^{2|u|})$ in such a way that the average of the values $d(u')$ over all words u' that extend u and have length 3^{s+1} is equal to $d(u \upharpoonright 3^s)$.

When analyzing the behavior of d on a sequence A, we distinguish the following three cases. In connection with the discussion of these cases, recall that $A \in \mathcal{R}$ holds if and only if there are infinitely many n such that $C^*(A \upharpoonright n) \geq n - c_0$. First, in case $A \in \mathcal{R}$, the sequence A is in particular λ-2-ML random and thus has only finitely many prefixes of the form $u1^{2|u|}$, hence up to a fixed positive multiplicative constant we have $d(A \upharpoonright 3^s) = 2^{-h_A(s)}$ for all s. Second, if $A \notin \mathcal{R}$ and A has only finitely many prefixes of the form $u1^{2|u|}$, then $d(A \upharpoonright 3^{s+1})$ is equal to $d(A \upharpoonright 3^s)$ for almost all s. Third, if $A \notin \mathcal{R}$ and A has infinitely many prefixes of the form $u1^{2|u|}$, then A has prefixes of the form $u1^{2|u|}$ for arbitrarily long u. But for all sufficiently long u and all v, we have

$$C^*(u1^{2|u|}v) \leq 2|u| + |v| + c_1 \leq |u1^{2|u|}v| - c_0,$$

hence for all sufficiently large s and for $u = A \upharpoonright 3^s$, we have $d(w) = d(u)$ for all extensions w of u of length 3^{s+1} that differ from $u1^{2|u|}$, where then also $d(u1^{2|u|}) = d(u)$ follows. Notice that in all the above cases $n \mapsto d(A \upharpoonright n)$ is bounded from above by a constant.

Set $\mu = d\lambda$. Then μ is obviously computable since d is computable. Let us prove that μ also has the other required properties.

Claim 1. *The measures μ and λ are not equivalent.*

Proof. We have $\lambda(\mathcal{R}) > 0$ by choice of $\mathcal{R}$. On the other hand, $d' = 1/d = \lambda/\mu$ is a μ-martingale by Proposition 7, where for every sequence $A \in \mathcal{R}$ and for all s, up to a positive multiplicative constant, $d'(A \upharpoonright 3^s) = 2^{h_A(s)}$. Since h_A is an order for all $A \in \mathcal{R}$, this proves that d' succeeds on all $A \in \mathcal{R}$. Hence, $\mathcal{R} \cap \mu\mathbf{CR} = \emptyset$ and thus $\mu(\mathcal{R}) = 0$.

Claim 2. *It holds that $\mu\mathbf{KR} \subseteq \lambda\mathbf{KR}$.*

Proof. Let $A \notin \lambda\mathbf{KR}$. Then there exists a computable λ-martingale d_0 and a computable order g such that $d_0(A \upharpoonright n) \geq g(n)$ for all n. Moreover, the λ-martingale $d = \mu/\lambda$ is bounded from above on any sequence, hence we can fix a constant $r > 0$ such that $d(A \upharpoonright n) \leq r$ for all n. Now in order to see that $A \notin \mu\mathbf{KR}$, it suffices to observe that for the μ-martingale $d_1 = d_0 \frac{\lambda}{\mu}$ it holds for all n that

$$d_1(A \upharpoonright n) = d_0(A \upharpoonright n)\frac{\lambda(A \upharpoonright n)}{\mu(A \upharpoonright n)} = \frac{d_0(A \upharpoonright n)}{d(A \upharpoonright n)} \geq \frac{g(n)}{r}.$$

Claim 3. *It holds that $\lambda\mathbf{KR} \subseteq \mu\mathbf{KR}$.*

Proof. Given any sequence $A \notin \mu\mathbf{KR}$, fix a computable μ-martingale d_2 and a computable order f such that $d_2(A \upharpoonright n) \geq f(n)$ for all n and define a computable λ-martingale d_3 by

$$d_3 = d_2\frac{\mu}{\lambda}, \quad \text{that is,} \quad d_3 = d_2 d = \frac{d_2}{d'}.$$

We distinguish two cases. In case $A \notin \mathcal{R}$, we have already seen that almost all values $d(A \upharpoonright 3^n)$ are equal to a constant $1/r' > 0$, hence $d_3(A \upharpoonright 3^n) \geq \frac{g(n)}{r}$ for almost all n and consequently $A \notin \lambda\mathbf{KR}$ according to Proposition 48. In the second case, i.e., $A \in \mathcal{R}$, A is in particular λ-2-ML random, hence the λ-martingale d_3 is bounded on A from above, say by a constant $r'' > 0$, and we obtain for all n

$$d'(A \upharpoonright n) = d_2(A \upharpoonright n)\frac{d'(A \upharpoonright n)}{d_2(A \upharpoonright n)} \geq \frac{f(n)}{r''}.$$

Recall that $d'(A \upharpoonright 3^s) = 2^{h_A(s)}$ up to a positive multiplicative constant. It follows for some constant r''' and for all s that

$$2^{h_A(s)} \geq \frac{f(3^s)}{r'''}$$

and hence

$$h_A(s) \geq \log\left(\frac{f(3^s)}{r''}\right),$$

which contradicts the fact that h_A cannot be bounded from below by a computable order. Hence, the second case will not occur, and this finishes the proof of Claim 3. □

The last counter-example we gave showed that two measures which have the same Kurtz random sequences need not be equivalent. This has consequences for the possible effectivizations of the following classical result.

Proposition 57. *Let μ and ν be two probability measures on the Cantor space. Then the following assertions are equivalent.*

(i) *The measures μ and ν are equivalent.*
(ii) *The measures μ and ν have the same closed null sets.*
(iii) *The measures μ and ν have the same G_δ null sets.*

It is suggesting to ask whether an effectivized version of Proposition 57 is true, where attention is restricted to computable measures and in place of closed sets and G_δ sets one considers Π^0_1 sets and Π^0_2 sets, respectively. As a corollary to the results already shown, we obtain that for the effective version the equivalence (i) $\leftrightarrow$ (iii) remains valid whereas (i) $\leftrightarrow$ (ii) is false in general.

Proposition 58. *Two computable probability measures μ and ν on the Cantor space are equivalent if and only if they have the same Π^0_2 null sets.*

There are computable probability measures μ and ν on the Cantor space that have the same Π^0_1 null sets but are not equivalent.

Proof. Concerning the first assertion, it suffices to observe that in the proof of Proposition 39 we have shown that if two computable probability measures μ and ν are not equivalent, then there exists a Π^0_2 set that is a null set with respect to exactly one of the two measures. Concerning the second assertion, we have shown in the proof of Proposition 55 that there are two computable probability measures that have the same Kurtz random sequences but are not equivalent. But having the same Kurtz random sequences and having the same Π^0_1 null sets is the same because if there is a Π^0_1 set that is null, say, for μ, but has positive measure for ν, then this set cannot contain any Kurtz random sequence with respect to μ but has a nonempty intersection with the class of Kurtz random sequences with respect to ν, which has ν measure 1. □

Acknowledgements

The authors are grateful to Alexander Shen for helpful discussions and to an anonymous referee for numerous comments and suggestions. The second author has been partially supported by the German Science Foundation (DFG) under grant ME 1806/3-1.

References

[1] R. Downey, E. Griffiths, On Schnorr randomness, Journal of Symbolic Logic 69 (2) (2004) 533–554.
[2] R. Downey, D. Hirschfeldt, Algorithmic randomness and complexity, Manuscript, 645 pages, 2008.
[3] S. Kakutani, On equivalence of infinite product measures, Annals of Mathematics 49 (1948) 214–224.
[4] S. Kurtz, Randomness and genericity in the degrees of unsolvability, Ph.D. Thesis, University of Illinois at Urbana-Champaign, Department of Mathematics, 1981.
[5] M. Li, P. Vitanyi, An Introduction to Kolmogorov Complexity and its Applications, second edition, Springer, New York, 1997.
[6] P. Martin-Löf, The definition of random sequences, Information and Control 9 (6) (1966) 602–619.
[7] W. Merkle, The Kolmogorov–Loveland stochastic sequences are not closed under selecting subsequences, Journal of Symbolic Logic 68 (2003) 1362–1376.
[8] W. Merkle, J. Miller, A. Nies, J. Reimann, F. Stephan, Kolmogorov–Loveland randomness and stochasticity, Annals of Pure and Applied Logic 138 (1–3) (2006) 183–210.
[9] An.A. Muchnik, A.L. Semenov, V.A. Uspensky, Mathematical metaphysics of randomness, Theoretical Computer Science 207 (2) (1998) 263–317.
[10] A. Nies, F. Stephan, S. Terwijn, Randomness relativization and Turing degrees, Journal of Symbolic Logic 70 (2) (2005) 515–535.
[11] C.P. Schnorr, A unified approach to the definition of random sequences, Mathematical Systems Theory 5 (1971) 246–258.
[12] C.P. Schnorr, Zufälligkeit und Wahrscheinlichkeit, in: Lecture Notes in Mathematics, vol. 218, Springer-Verlag, 1971.
[13] A.Kh. Shen, On relations between different algorithmic definitions of randomness, Soviet Mathematics Doklady 38 (1988) 316–319.
[14] M. van Lambalgen, Random sequences, Doctoral Dissertation, Department of Mathematics and Computer Science, University of Amsterdam, 1987.
[15] J. Ville, Étude critique de la notion de collectif, Gauthiers-Villars, Paris, 1939.
[16] V.G. Vovk, On a randomness criterion, Soviet Mathematics Doklady 35 (1987) 656–660.
[17] Y. Wang, Randomness and complexity, Doctoral Dissertation, Universität Heidelberg, Mathematische Fakultät, 1996.