CHAPTER

1

Permutation and Combination

INTRODUCTION

It is often necessary to calculate the number of different ways in which something can be done or can happen. In this chapter, we shall study some techniques of answering some counting problems without actually counting or listing all of them. These techniques are useful in determining the number of different ways of arranging or selecting different items.

FUNDAMENTAL PRINCIPLE OF COUNTING

PRINCIPLE I : ADDITIVE PRINCIPLE

This principle states that : Let A and B be two events that cannot occur simultaneously. Then if A can occur in m ways and B can occur in n ways, it follows that the number of ways in which either A or B can occur in $m + n$.

This principle can be generated, for more than two events, as follows :

"If one event can happen in m_1 ways, following which another event can happen in m_2 ways, following which another event can happen in m_3 ways and so on that all events, in succession can happen in $m_1 + m_2 + m_3 + \ldots\ldots$ different ways.

For example, in a class room there are 20 boys and 15 girls. If the teacher wants to select either a boy or a girl to represent the class, the number of ways in which the teacher can perform the selection is 20 + 15, *i.e.*, 35 ways.

PRINCIPLE II : MULTIPLICATION PRINCIPLE

This principle states that : If an event can happen in 'm' different ways following which another event can happen in 'n' different ways, then both event in succession can happen in exactly 'mn' different ways.

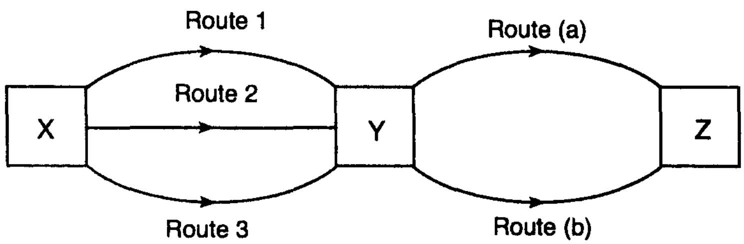
This principle can also be generalised, for even more than two operations, as follows :

If an event can happen in 'm_1' different ways, following which another event can happen in 'm_2' different ways, following which another event can happen in 'm_3' different ways and so on, then all event in succession can happen in exactly $m_1 \times m_2 \times m_3 \times \ldots\ldots$ different ways.

For example, in a class room there are 20 boys and 15 girls. If the teacher wants to select one boy and one girl to represent the class, the number of ways in which the teacher can perform the selection is 20×15, *i.e.*, 300 ways.

Consider the following illustration :

There are 3 stations X, Y and Z and 3 routes to go from X to Y and 2 routes from X to Z. This is shown in the following figure :



A person can go from station X to Y in three different ways and from Y to Z in 2 different ways.

By the fundamental principle of counting, the number of ways for a person to go from X to Z is $3 \times 2 = 6$, *i.e.*, $(1, a)$, $(1, b)$, $(2, a)$, $(2, b)$, $(3, a)$ and $(3, b)$.

Example 1 : *There are eight doors in the college hall. In how many ways can a student enter the college hall through one door and come out through a different door ?*

Solution **:** The student can enter the hall in 8 ways and corresponding to each way of entering, there are only 7 ways of coming out, since he has to come out through a different door.

Hence the required number of ways $= 8 \times 7 = 56$.

Example 2 : *Given 7 flags of different colours, how many different signals can be generated if a signal requires the use of two flags, one below the other ?*

Solution **:** The upper flag can be any one of the 7 flags and the lower flag can be any one of the remaining 6 flags.

$\therefore$ Required no. of signals which can be generated $= 7 \times 6 = 42$.

Example 3 : *A customer forgets a four-digit code for an Automatic Teller Machine (ATM) in a bank. However, he remembers that this code consists of digits 3, 5, 6 and 9. Find the largest possible number of trials necessary to obtain the correct code.*

Solution **:** The digits are 3, 5, 6 and 9.

No. of ways of choosing first digit $= 4$

No. of ways of choosing second digit = 3

No. of ways of choosing third digit = 2

No. of ways of choosing fourth digit = 1

∴ Total number of ways of choosing four digits = $4 \times 3 \times 2 \times 1 = 24$

∴ In all there could be 24 codes and customer would have to try 24 trials to obtain the correct code.

Example 4 : *A coin is tossed three times and the outcomes are recorded. How many possible outcomes are there ?*

Solution : In a single toss of a coin, there are 2 possible outcomes. Either 'Head' of 'tail' can appear on the upper-most face. Therefore, when a coin is tossed three time, number of possible outcomes are $2 \times 2 \times 2 = 2^3 = 8$.

The possible outcome are

HHH, HHT, HTH, HTT, THH, THT, TTH, TTT

Example 5 : *How many 3-digit numbers can be formed from the digits 1, 2, 3, 4 and 5 ,if*

(*a*) *repetition of digits is allowed ?* (*b*) *repetition of digits is not allowed ?*

Solution : Total number of digits = 5

(*a*) *When repetition of digits is allowed* :

Unit's place can be filled by any one of the five digits

∴ No. of ways to fill units place = 5

Since the repetition is allowed, the ten's place can also be filled by any one of the five digits.

∴ No. of ways to fill the 10's place = 5

The 100's place can also be filled by any one of the five digits.

∴ No. of ways to fill the 100's place = 5

This is shown as follows :

100's	10's	1's
5	5	5

∴ The required number of 3-digit numbers = $5 \times 5 \times 5 = 125$

(*b*) *When repetition is not allowed* :

Unit's place can be filled by any one of 5 digits

∴ No. of ways to fill units place = 5

Since repetition is not allowed, there are only 4 choices

∴ No. of ways to fill 10's place = 4

Now only 3 digits are left

∴ No. of ways to fill 100's place = 3

This is shown as follows :

100's	10's	1's
3	4	5

∴ The required number of 3-digit numbers = 5 × 4 × 3 = 60

Example 6 : *How many 3-digit even numbers can be formed using the digits 1, 2, 3, 4, and 5, if*

(*a*) *repetition of digits is allowed ?* (*b*) *repetition of digits is not allowed ?*

***Solution* :**

Total number of digits = 5 (*i.e.* 1, 2, 3, 4, 5)

Total number of even digits = 2 (*i.e.*, 2, and 4)

(*a*) *When the repetition is allowed* :

100's	10's	1's
5	5	2

No. of ways to fill unit's place = 2

No. of ways to fill 10's place = 5

No. of ways to fill 100's place = 5

∴ Required number of 3-digit even number = 2 × 5 × 5 = 50

(*b*) *When the repetition is not allowed* :

100's	10's	1's
3	4	2

No. of ways to fill unit's place = 2

No. of ways to fill 10's place = 4

No. of ways to fill 100's place = 3

∴ required number of 3-digit even numbers = 2 × 4 × 3 = 24

Example 7 : *How many 3-digit numbers can be formed from the digits 0, 1, 2, 3, and 4, if*

(*a*) *repetition is allowed ?* (*b*) *repetition is not allowed ?*

***Solution* :**

Total number of digits = 5

(*a*) *When repetition is allowed* :

100's	10's	1's
4	5	5

'0' cannot be placed in 100's place while forming 3-digit number

∴ No. of ways to fill 100's place = 4

No. of ways to fill 10's place = 5

No. of was to fill 1's place = 5

∴ required number of 3-digit numbers = 4 × 5 × 5 = 100

(b) *When repetition is not allowed* :

100's	10's	1's
4	4	3

No. of ways to fill 100's place = 4

Now 4 digits are left

∴ No. of ways to fill 10's place = 4

Now 3 digits are left

∴ No. of ways to fill 1's place = 3

∴ required number of 3-digit number = 4 × 4 × 3 = 48

Example 8 : *How many different numbers below 1000 can be formed form the digits 3, 4, 6, 7 and 8, if* :

(a) *no digit is repeated ?* (b) *digits can be repeated ?*

Solution : The numbers less than 1000 can be :

(i) One-digit numbers (ii) Two-digit numbers (iii) Three-digit numbers.

(a) *No digit is repeated* :

(i) One digited numbers are 3, 4, 6, 7 and 8. These are 5 numbers.

(ii) No. of ways of filling unit's place = 5

No. of ways of filling ten's place = 4

∴ No. of 2 digited numbers = 5 × 4 = 20.

(iii) No. of ways of filling unit's place = 5

No. of ways of filling ten's place = 4

No. of ways of filling hundred's place = 3

∴ No. of 3 digited numbers = 5 × 4 × 3 = 60

∴ Total numbers = 5 + 20 + 60 = 85.

(b) *Digits can be repeated* :

(i) One digited numbers = 5

(ii) No. of ways of filling unit's place = 5

No. of ways of filling ten's place = 5 (∵ Repetition is allowed)

∴ No. of 2 digited numbers = 5 × 5 = 25

(iii) No. of ways of filling unit's place = 5

No. of ways of filling ten's place = 5

No. of ways of filling hundred's place = 5

∴ No. of 3 digited numbers = 5 × 5 × 5 = 125

∴ Total numbers = 5 + 25 + 125 = 155.

Example 9 : *Ten people compete in a race. In now many ways the first three prices can be distributed ?*

Solution : First price goes to any one of the 10 people

Second price goes to any one of the remaining 9 people

Third price goes to any of the remaining 8 people

$\therefore$ No. of ways to distribute the prices = $10 \times 9 \times 8 = 720$.

Example 10 : *How many automobile license plates can be made, if the inscription on each contains two different letters followed by three different digits ?*

Solution : Out of 5 inscriptions, first 2 are of different letters (out of 26) and the other 3 are of different digits (out of 10).

No. of ways of 1^{st} inscription = 26

No. of ways of 2^{nd} inscription = 25

No. of ways of 3^{rd} inscription = 10

No. of ways of 4^{th} inscription = 9

No. of ways of 5^{th} inscription = 8

$\therefore$ Total number of license plates = $26 \times 25 \times 10 \times 9 \times 8$ = 4,68,000.

EXERCISES

1. A hall has 3 entrances and 4 exits. In how many ways can a man enter and exit from the hall ? **[Ans.** 12]
2. In a railway compartment, 6 seats are vacant on a bench. In how many ways can 3 passengers sit on them ? **[Ans.** 120]
3. If there are 20 steamers playing between places *A* and *B*, in how many ways could the round trip from *A* be made if the return was made on (*a*) the same steamer, (*b*) a different steamer. **[Ans.** (*a*) 20, (*b*) 380]
4. There are 5 routes between city *X* and city *Y*. In how many different ways can a man go from city *X* to city *Y* and return, if for return journey :
   (*a*) any of the routes is taken (*b*) the same routes is taken
   (*c*) the same route is not taken. **[Ans.** (*a*) 25, (*b*) 5, (*c*) 20]
5. Four coins are tossed simultaneously. In how many ways can they fall ? **[Ans.** 32]
6. In how many ways can 4 students draw water from 4 taps, if no tap remain unused ? **[Ans.** 24]
7. It has been decided that the flag of a newly formed forum will be in the form of three blocks one below the other, each coloured different. If there are five different colours on the whole to choose from, how many such designs are possible ? **[Ans.** 60]
8. Given 6 flags of different colours, how many different signals can be generated, if a signal requires the use of two flags, one below the other ? **[Ans.** 30]

9. For a set of 6 true or false questions no student has written all correct answers and no two students have given the same sequence of answer. What is the maximum number of students in the class for this to be possible ? [**Ans.** 63]

10. A team consists of 6 boys and 4 girls and the other has 5 boys and 3 girls. How many single matches can be arranged between the two terms, when a boy plays against a boys and a girls plays against a girl ? [**Ans.** 42]

11. A class consists of 20 girls and 15 boys. In how many ways can a president, vice president, treasurer and secretary be chosen if the treasurer must be a girls, the secretary must be a boy and a student may not hold more than one office ?

[**Ans.** 31680]

12. How many four digit numbers can be formed out of the digits 1, 2, 3, 4, 5, and 6, when no digit is repeated in the same number ? [**Ans.** 360]

13. How many three digit numbers can be formed without using the digits, 1, 2, 3, 4 ?

[**Ans.** 180]

14. From the numbers 1, 2, 3, 4, 5, and 6, how many 3-digit odd numbers can be formed when (*a*) the repetition of the digits is allowed, (*b*) the repetition of the digits is not allowed. [**Ans.** (*a*) 180, (*b*) 60]

15. How many words (with or without meaning) of four distinct letters of the English alphabet are there ? [**Ans.** 358800]

16. Find the number of possible even numbers which have three digits ? [**Ans.** 450]

17. How many numbers can be formed using the digits 1, 2, 3, and 9, if repetition of digits is not allowed ? [**Ans.** 64]

18. How many 6-digit telephone number can be constructed if each number starts with 32 and no digit appears more than once ? [**Ans.** 1680]

19. How many 2-digit even numbers can be formed from the digits 1, 2, 3, 4 and 5, if (*a*) the repetition of digits is allowed (*b*) the repetition of digits is not allowed.

[**Ans.** (*a*) 10, (*b*) 8]

20. How many 3-letter code words are possible using the first 10 letters of English alphabet, if (*a*) no letter can be repeated ? (*b*) letters are repeated ?

[**Ans.** (*a*) 720, (*b*) 1000]

21. How many numbers are there between 100 and 1000 in which all the digits are distinct ? [**Ans.** 648]

22. How many number are there between 100 and 1000 such that every digit is either 2 or 9 ? [**Ans.** 8]

23. How many numbers are there between 100 and 1000 which have exactly one of their digits as 7 ? [**Ans.** 225]

24. Twelve students compete in a race. In how many was can the first three prices be distributed ? [**Ans.** 1320]

25. In how many ways can 4 different prizes be awarded among 6 contestants, so that a contestant may receive : (*a*) at most one prize, (*b*) any number of prizes.

[**Ans.** (*a*) 360, (*b*) 1296]

26. A number of lock on a suitcase has 3 wheels each labelled with ten digits from 0 to 9. If opening of the lock is a particular sequence of three digits with no repeats, how many such sequences will be possible ? **[Ans.** 720]

27. The licence plates for vehicles registered in Delhi consists of 3 letters from English alphabet followed by 1, 2, 3 or 4 digits. The letter on the extreme left has be '*D*'. For the 1-digit number plates the number '0' is not allowed. For others, the digits and the letters of course can repeat but the number should be significant. Determine the possible number of licence plates. **[Ans.** 6759324]

28. A code word is to consist of two distinct English alphabet followed by two distinct numbers from 1 to 9. For example GH 79 is a code world. How many such code words are possible ? **[Ans.** 46800]

FACTORIAL NOTATION

There are some occasions when we wish to consider the product of first n natural numbers. The continued product of first n natural numbers (beginning with 1 and ending with n) is called n-factorial or factorial n and is denoted by $n\,!$ or $\underline{|n}$.

Thus,

$$n\,! = 1\,.\,2.\,3\,.\,...\,(n-1)\,.\,n$$

For example,

$$5\,! = 1 \times 2 \times 3 \times 4 \times 5 = 120$$

$$3\,! = 1 \times 2 \times 3 = 6$$

Remarks :

(*a*) $0\,! = 1$ (*b*) $1\,! = 1$ (*c*) $n\,! = (n-1)\,! \times n$

for example $7\,! = 7 \times 6\,!$

Example 11 : *Evaluate the following* :

(*a*) $\frac{9\,!}{7\,!}$, (*b*) $\frac{12\,!}{7\,!\,5\,!}$, (*c*) $\frac{6\,!}{2 \times 4\,!}$

***Solution* :**

(*a*) $\frac{9\,!}{7\,!} = \frac{9 \times 8 \times 7\,!}{7\,!} = 9 \times 8 = 72$

(*b*) $\frac{12\,!}{7\,!\,5\,!} = \frac{12 \times 11 \times 10 \times 8 \times 9 \times 7\,!}{7\,! \times 1 \times 2 \times 3 \times 4 \times 5} = 792$

(*c*) $\frac{6\,!}{2 \times 4\,!} = \frac{6 \times 5 \times 4\,!}{2 \times 4\,!} = 15$

Example 12 : *Which of the following statements are correct*

(*a*) $2\,! + 3\,! = 5\,!$ (*b*) $(4\,!)(2\,!) = 8\,!$

(*c*) $5\,(4\,!) = (5 \times 4)\,!$ (*d*) $\frac{6\,!}{2\,!} = 3\,!$

Solution :

(*a*) $2! + 3! = (2 \times 1) + (3 \times 2 \times 1) = 2 + 6 = 8$

$5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$

$\therefore\ 2! + 3! \neq 5!$

(*b*) $(4!)(2!) = (4 \times 3 \times 2 \times 1) \times (2 \times 1) = 24 \times 2 = 48$

$8! = 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 1 = 40320$

$\therefore\ (4!)(2!) \neq 8!$

(*c*) $5 \times (4!) = 5! \neq 20!$

$\therefore\ 5(4!) \neq 20!$

(*d*) $\frac{6!}{2!} = \frac{6 \times 5 \times 4 \times 3 \times 2!}{2!} = 360$

$3! = 3 \times 2 \times 1 = 2$

$\therefore\ \frac{6!}{2!} \neq 3!$

PERMUTATION

An arrangement in a definite order of a number of things taking some or all of them at a time is called a **permutation.** The total number of permutations of n distinct things taking r $(1 \leq r \leq n)$ at a time is denoted by ^nP_r, or by $P(n, r)$.

The number of permutations of n things taking r at a time is given by

$\therefore\quad {}^nP_r = n(n-1)(n-2) \ldots\ldots r$ factors, $1 \leq r \leq n$.

Alternatively $\boxed{{}^nP_r = \frac{n!}{(n-r)!}}$, $1 \leq r \leq n$.

Remark 1. The number of permutations of 'n' different thinks taking at a time is equal to n !

$${}^nP_n = \frac{n!}{(n-n)!} = \frac{n!}{0!} = n!$$

Remark 2. It should be noted carefully that in ^nP_r, one count only those permutation in which repetition of things is not allowed.

Remark 3. The number of all permutation of 'n' different objects taking r at a time, when a particular object is always included in each arrangement is

$$r \cdot {}^{n-1}P_{r-1}$$

Remark 4. The number of all permutations of 'n' different objects taking r at a time, when a particular object is always excluded in each arrangement is

$${}^{n-1}P_r$$

Example 13 : *Evaluate the following :*

(a) $^7P_3$, *(b)* $^6P_4$

Solution :

(a) $$^7P_3 = \frac{7!}{(7-3)!} = \frac{7!}{4!} = \frac{7 \times 6 \times 5 \times 4!}{4!} = 210$$

Alternatively, $^7P_3 = 7 \times 6 \times 5 = 210$

(b) $$^6P_4 = \frac{6!}{(6-4)!} = \frac{6!}{2!} = \frac{6 \times 5 \times 4 \times 3 \times 2!}{2!} = 360$$

Alternatively $^6P_4 = 6 \times 5 \times 4 \times 3 = 360$

Example 14 : *If 5P (5, r) = 2P (6, r – 1), find r.*

Solution : Given that

$$^5P_r = 2 \,.\, ^6P_{r-1}$$

$$\Rightarrow \quad \frac{5!}{(5-r)!} = 2 \cdot \frac{6!}{6-(r-1)!}$$

$$\Rightarrow \quad \frac{5!}{(5-r)!} = 2 \cdot \frac{6!}{(7-r)!}$$

$$\Rightarrow \quad \frac{5!}{(5-r)!} = 2 \cdot \frac{6 \times 5!}{(7-r)(6-r)(5-r)!}$$

$$\Rightarrow \quad 1 = \frac{2 \times 6}{(7-r)(6-r)}$$

$$\Rightarrow \quad (7-r)(6-r) = 12$$

$$\Rightarrow \quad (7-r)(6-r) = 4 \times 3$$

$$\Rightarrow \quad (7-r)(6-r) = (7-3)(6-3)$$

$$\therefore \quad r = 3$$

Example 15 : *If 16 $^nP_3$ = 13 . $^{n+1}P_3$, find n.*

Solution :

$$16 \,.\, ^nP_3 = 13 \,.\, ^{n+1}P_3$$

$$\Rightarrow \quad 16 \cdot \frac{n!}{(n-3)!} = 13 \cdot \frac{(n+1)!}{(n+1-3)!}$$

$$\Rightarrow \quad 16 \cdot \frac{n!}{(n-3)!} = 13 \cdot \frac{(n+1)!}{(n-2)!}$$

$$\Rightarrow \quad 16 \cdot \frac{n!}{(n-3)!} = 13 \cdot \frac{(n+1)\, n!}{(n-2)(n-3)!}$$

$$\Rightarrow \quad 16 = \frac{13(n+1)}{(n-2)}$$

$$\Rightarrow \quad 16(n-2) = 13(n+1)$$

$$\Rightarrow \quad 3n = 45 \Rightarrow n = 15$$

$$\therefore \quad n = 15$$

Example 16 : *Prove that* ${}^nP_r = {}^{n-1}P_r + r \cdot {}^{n-1}P_{r-1}$

Solution :

$$\text{RHS} = {}^{n-1}P_r + r \cdot {}^{n-1}P_{r-1}$$

$$= \frac{(n-1)!}{(n-1-r)!} + r\frac{(n-1)!}{(n-1-(r-1))!}$$

$$= \frac{(n-1)!}{(n-r-1)!} + r\frac{(n-1)!}{(n-r)!}$$

$$= (n-1)!\left[\frac{1}{(n-r-1)!} + \frac{r}{(n-r)!}\right]$$

$$= (n-1)!\left[\frac{(n-r)}{(n-r)(n-r-1)!} + \frac{r}{(n-r)!}\right]$$

$$= (n-1)!\left[\frac{(n-r)}{(n-r)!} + \frac{r}{(n-r)!}\right] = (n-1)!\left[\frac{n-r+r}{(n-r)!}\right]$$

$$= (n-1)!\frac{n}{(n-r)!} = \frac{n(n-1)!}{(n-r)!}$$

$$= \frac{n!}{(n-r)!} = {}^nP_r = \text{LHS}.$$

Hence proved.

Example 17 : *In how many ways can five children stand in a queue ?*

***Solution* :** Number of ways in which 5 children stand in queue

= Number of ways in which 5 children can be arranged among themselves

$= {}^5P_5 = 5\,!$ $\qquad [\because\ {}^nP_n = n\,!]$

$= 5 \times 4 \times 3 \times 2 \times 1 = 120.$

∴ The required number of ways is 120.

Example 18 : *Seven persons are participating in a race. In how many can the first three prizes be own ?*

***Solution* :** Total number of participant $n = 7$

No. of prizes $r = 3$

∴ The required number of ways $= {}^7P_3 = 7 \times 6 \times 5 = 210$

∴ The required no. of ways = 210.

Example 19 : *(a) It is required to seat 5 men and 4 women in a row so that the women occupy the even places. How many such arrangements are possible ?*

(b) If it is required to seat 5 men and 2 women in a row so that the women occupy the even places, how many such arrangements are possible ?

Solution :

(*a*) Total number of position = 9

No. of even positions = 4

and No. of women = 4

Given that the women occupy even places.

∴ No. of ways to arrange the women = $^4P_4 = 4! = 4 \times 3 \times 2 \times 1 = 24$

No. of remaining position = 5

No. of persons to be occupied = 5 (men)

∴ No. of ways to occupy the remaining positions = $^5P_5 = 5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$

∴ No. of ways to occupy all 9 position = $24 \times 120 = 2880$

∴ Required number of arrangements = 2880

(*b*) Total number of positions = 7

No. of even positions = 3 (2^{nd}, 4^{th}, and 6^{th})

No. of women = 2

Given that women occupy the even places.

∴ No. of ways to arrange the women = $^3P_2 = 3 \times 2 = 6$

No. of remaining position = 5

No. of remaining persons = 5 (men)

∴ No. of ways to occupy the remaining positions

$= {}^5P_5 = 5! = 5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$

∴ The number of ways to occupy all 7 positions = $4 \times 120 = 720$

∴ Required number of arrangement of 720

Example 20 : *In how many ways can 6 exam papers can be arranged so that the best and the worst papers are never placed together ?*

Solution :

Total number of papers = 6

No. of best papers = 4

No. of worst papers = 2

First we arrange the best papers.

No. of ways to arrange the best papers = $^4P_4 = 4! = 24$

There are 5 places for the worst papers as shown in the following figure.

× B × B × *B* × B ×

∴ No. of ways in which the worst papers can be arranged = $^5P_2 = 5 \times 4 = 20$

∴ No. of arrangements of all 6 papers = $24 \times 20 = 480$

∴ The required number of arrangements = 480.

Alternative Method :

No. of ways in which all 6 papers can be arranged with out any restriction = 6!

Now, consider the two worst papers as one units.

∴ No. of units to be arranged = 5 (*i.e.* 4 + 1)

No. of ways to arrange this 5 units = 5 !

In each such arrangements, the worst papers can be arranged among themselves in 2 ! ways.

∴ The number of ways in which the worst papers are placed together is 5 ! × 2 !

The number of arrangement in which the worst papers never place together

= Total no. of possible arrangement – The no. of arrangement in which they are placed together.

$= 6! - 5! \times 2!$

$= 6 \times 5! - 5! \times 2$ [∵ $2! = 2$]

$= 5! [6 - 2] = 5! \times 4 = 120 \times 4 = 480$

∴ The required number of arrangement = 480

Example 21 : *A family of 4 brothers and 3 sisters is to be arranged for a photograph in one row. In how many ways can they be seated if*

(a) all sisters sit together ? *(b) no two sisters sit together ?*

Solution :

Given, No. of brothers = 4

No. of sisters = 3

(*a*) We take 3 sisters as one unit

∴ No. of units to be arranged = 4 + 1 = 5

∴ No. of ways to arrange all units = 5 !

No. of ways to arrange all sisters among themselves = 3 !

∴ The number of ways in which all sisters sit together

$= 5! \times 3! = 120 \times 6 = 720$

∴ The required no. of arrangements = 720.

(*b*) First we arrange 4 brothers

No. of ways to arrange the brothers = 4 !

× *B* × B × B × B ×

There are 5 places in which 3 sisters can be seated.

∴ No. of ways to arrange the sisters = $^5P_3$

∴ No. of ways in which all can be seated

$= 5! \times {}^5P_3 = 60 \times 24 = 1440$

∴ The required no. of arrangements = 1440.

Example 22 : *A team has 7 players. In how many ways they can be seated in a row, if the captain sits in the middle seat ?*

Solution : Since the captain is fixed in the middle seat, the remaining six players can be seated in the remaining six seats in 6 ! was, *i.e.,* 720 ways.

Example 23 : *In how many ways can 5 English 4 Tamil and 2 Hindi books can be arranged if the books of each different language are kept together ?*

Solution : We take 5 English books as one unit; 4 Tamil books as one unit; and 2 Hindi books as one unit.

No. of ways to arrange 3 units = $^3P_3 = 3\,!$

Now we arrange each language books among themselves.

No. of ways to arrange English books $= {}^5P_5 = 5\,!$

No. of ways to arrange Tamil books $= {}^4P_4 = 4\,!$

No. of ways to arrange Hindi books $= {}^2P_2 = 2\,!$

$\therefore$ No. of ways to arrange all books $= 3\,! \times 5\,! \times 4\,! \times 2\,! = 6 \times 120 \times 24 \times 2 = 34560$

$\therefore$ The required no. of arrangements = 34560.

Example 24 : *How many four digit numbers are there with distinct digits ?*

Solution : There are 10 digits (0, 1, 2, ..., 9).

The 1000's place cannot be filled with 0.

$\therefore$ No. of ways to fill the 1000's place = 9

The remaining 3 places can be filled with any three of the remaining 9 digits.

$\therefore$ No. of ways to fill the remaining places $= {}^9P_3$

$\therefore$ No. of four digits numbers with distinct digits $= 9 \times {}^9P_3 = 9 \times 9 \times 8 \times 7 = 4536$

$\therefore$ Required no of four digit numbers = 4536.

Example 25 : *How many 3-letter words can be made using the letters of the word "DELHI".*

Solution : No. of letters in the word 'DELHI' = 5

$\therefore$ No. of 3-letter words $= {}^5P_3 = 5 \times 4 \times 3 = 60$

$\therefore$ Required no. of words is 60.

Example 26 : *How many 3-letter words can be made using the letters of the word "ORIGINAL", if*

(a) 'N' is always included in all words ?

(b) 'N' is always excluded in all words ?

(c) the word starts with 'N' ?

Solution : No. of letters in the word 'ORIGINAL' is 8 and all letters are distinct.

(*a*) No. of ways to arrange '*N*' = 3

After arranging '*N*' two places and 7 letters are left.

$\therefore$ No. of ways to arrange the remaining 2 places $= {}^7P_2$

∴ No. of words in which N is always included = $3 \times {}^7P_2 = 3 \times 7 \times 6 = 126$

∴ Required number of words is 126.

(b) If 'N' is excluded, No. of ways in which the remaining 7 letters can be arranged taking three at a time = ${}^7P_3 = 7 \times 6 \times 5 = 210$

∴ Required number of words = 210

(c) If the 3-letter word starts with 'N',

No. of ways to arrange 'N' = 1

No. of ways to fill the remaining 2 places = ${}^7P_2$

∴ No. of 3-letter words starting with 'N' = $1 \times {}^7P_2 = 1 \times 7 \times 6 = 42$

∴ Required number of words is 42.

Example 27 : *In how many ways can the letters of the word 'STRANGE' be arranged so that*

(a) the vowels are never separated

(b) the vowels never come together, and

(c) the vowels occupy only the odd places.

***Solution* :**

No. of letters of the word 'STRANGE' = 7

No. of vowels in the word = 2 (A and E)

∴ No. of consonants in the word = 5

(a) We take the two vowels as one unit

∴ No. of units to be arranged = 6 (5 + 1)

No. of ways to arrange the 6 units = ${}^6P_6 = 6\,!$

No. of ways to arrange the vowels among themselves = ${}^2P_2 = 2\,!$

N o. of arrangements in which vowels are never separated = $6\,!\ \ 2\,! = 720 \times 2 = 1440$

∴ The required number of arrangements = 1440

(b) First we arrange the consonants

No. of ways to arrange the consonants = ${}^5P_5 = 5\,!$

$\times\, C \times C \times C \times \mathrm{C} \times C \,\times$

There are 6 places in which vowels can be arranged

∴ No. of ways to arrange the vowels = ${}^6P_2$

∴ No. of arrangements in which the vowels never come together

$= 5\,! \times {}^6P_2 = 120 \times 6 \times 5 = 3600$

∴ The required number of arrangements is 3600.

(c)

O	E	O	E	O	E	O

No. of odd places = 4

The two vowels are to occupy any two of these 4 odd places.

$\therefore$ No. of ways to arrange the vowels = $^4P_2$

Now the remaining 5 places can be filled with the 5 consonants

$\therefore$ No. of ways to arrange the consonants = 5 !

$\therefore$ The no. of arrangements in which the vowels occupy odd places

$^4P_2 \times 5\,! = 4 \times 3 \times 120 = 1440$

$\therefore$ The required number of arrangements is 1440.

EXERCISES

1. Evaluate

(a) $^4P_3$ (b) $^6P_2$ (c) $^{20}P_4$ (d) $^9P_9$

[**Ans.** (a) 24, (b) 30, (c) 116280, (d) 362880]

2. Show that $^{10}P_3 = {}^9P_3 + 3\,{}^9P_2$

3. (a) Find n, if $2\,{}^nP_3 = {}^{n+1}P_3$ and $n > 2$. [**Ans.** $n = 5$]

(b) Find n, if $^nP_6 = 3\,{}^nP_5$. [**Ans.** $n = 8$]

4. (a) Find r, if $^{15}P_{r-1} : {}^{16}P_{r-2} = 3 : 4$ [**Ans.** 14]

(b) Find r, if $^5P_r = {}^6P_{r-1}$ [**Ans.** 4]

5. (a) Show that $^nP_n = 2.\,{}^nP_{n-2}$

(b) Show that $^nP_r = n.\,{}^{n-1}P_{r-1}$

6. Eight children are to be seated on a bench. In how many ways can the children be seated ? [**Ans.** 40320]

7. From among the 20 teachers of a college, one director and one dean are to be appointed. In how many ways can this be done ? [**Ans.** 380]

8. From a pool of 12 candidates, in how many ways can we select president, vice president, secretary and a treasurer, if each of the 12 candidates can hold any office ? [**Ans.** 11880]

9. Six candidates are called for interview to fill four posts in an office. Assuming that each candidate is fit for each post, determine the number of ways in which the four posts can be filled ? [**Ans.** 360]

10. In how many ways can 6 boys and 5 girls be arranged for a group photograph, if the girls are to sit on chairs in a row and the boys are to stand in a row behind them ? [**Ans.** 86400]

11. When a group photograph is taken, all the 5 teachers should be in the first row and all the 8 students should be in the second row. If the two corners of the second row are reserved for the two talent students, interchangeable only between them, and if the middle seat of the front row is reserved for the principal, how many arrangements are possible ? [**Ans.** 34560]

12. In how many ways can 5 boys and 3 girls be arranged in a row so that no two girls may sit together ? [**Ans.** 14400]

13. There are 8 students appearing in an examination of which 3 have to appear in a Mathematics paper and the remaining 5 in different subjects. In how many ways can they be made to sit in a row, if no two candidates in Mathematics sit next to each other. [**Ans.** 14400]

14. Six men and four women are to sit in a row so that the women occupy the even places. Find the number of all possible arrangements. [**Ans.** 86400]

15. How many signals can be given with 6 flags of different colours such that :

    (*a*) exactly three flags can be used for a signal ?

    (*b*) at most three flags are to be used for a signal ?

    (*c*) at least three flags are to be used for a signal ?

    (*d*) any number of flags may be used for a signal ?

    [**Ans.** (*a*) 120, (*b*) 156, (*c*) 1920, (*d*) 1956]

16. In how many ways can 9 examination papers be arranged so that the best and the worst papers never come together ? [**Ans.** 282240]

17. A family of 5 brothers and 3 sisters is to be arranged for a photograph in one row. In how many ways can they be seated, if

    (*a*) all sisters sit together ?

    (*b*) no two sisters sit together ? [**Ans.** (*a*) 4320, (*b*) 14400]

18. How many 6 digit telephone numbers can be constructed if each number starts with 35 and no digit appears more than once ? [**Ans.** 1680]

19. How many different numbers between 100 and 1000 can be formed using the digits 0, 1, 2, 3, 4, 5, and 6 assuming that, in any number, the digits are not repeated. How many of these will be divisible by 5 ? [**Ans.** 180, 55

20. Find the number of 4-digit numbers that can be formed using the digits 1, 2, 3, 4, 5 if no digit is used more than once in a number. How many of these numbers will be even ? [**Ans.** 120, 48]

21. How many numbers of six digits can be formed from the digits 1, 2, 3, 4, 5, and 6 (no digit being repeated). How many of these are divisible by 5 ? [**Ans.** 720, 120]

22. How many numbers greater than 4000 can be formed with the digits 2, 3, 4, 5, and 6 when no digit is repeated ? [**Ans.** 192]

23. How many numbers lying between 2000 and 4000 can be formed with the digits 1, 2, 3, 4, 5, 6 ? [**Ans.** 120]

24. Ritu wants to arrange 3 English, 2 Hindi and 4 French books on a shelf. If the books on the same subject are different, determine :

    (*a*) the number of possible arrangements.

    (*b*) the number of possible arrangements, if all the books on a subject are to be together. [**Ans.** (*a*) 362880, (*b*) 1728]

25. Find the number of ways in which 8 different books can be arranged on a shelf so that 2 particular books are

    (*a*) always together. (*b*) never together. [**Ans.** (*a*) 10080, (*b*) 30240]

26. Find the number of permutations of 8 things taking 5 at a time in which 2 particular things are always

(*a*) included. (*b*) excluded. [**Ans.** (*a*) 2400, (*b*) 720]

27. The letters of the word TUESDAY are arranged in a line, each arrangement ending with letter S. How many different arrangements are possible ? How many of them start with letter *D* ? [**Ans.** 720, 120]

28. In how many ways can the letters of the word 'FRACTION' be arranged so that no two vowels are together ? [**Ans.** 14400]

29. How many words can be formed from the letters of the word 'DAUGHTER' so that

(*a*) The vowels always come placed together ?

(*b*) The vowels are never placed together ? [**Ans.** (*a*) 4320, (*b*) 36000]

30. How many different words can be formed of the letters of the word "COMBINE" so that :

(*a*) vowels always remain together ? (*b*) no two vowels are together ?

(*c*) vowels may occupy odd places ? [**Ans.** (*a*) 720, (*b*) 1440, (*c*) 576]

31. In how many ways can the letters of the word 'TOWER' be arranged so that the letters '*O*' and 'E' occupy only even places ? [**Ans.** 12]

32. How many words can be formed out of the letters of the word 'PECULIAR' beginning into '*P*' and ending with '*R*' ? How many of them will have '*P*' and '*R*' at end places ?

[**Ans.** 720, 1440]

33. How many permutations can be made out of the letters of the word 'TRIANGLE' ? How many of these :

(*a*) begin with *T* ? (*b*) end with *E* ?

(*c*) begin with *T* and end with *E* ? (*c*) *T* and *E* occupy the end places ?

[**Ans.** 40320, (*a*) 5040, (*b*) 5040, (*c*) 720, (*d*) 1440]

34. How many different words can be formed of the letters of the word 'MALENKOV' so that :

(*a*) the first letter is a vowel ? (*b*) no two vowels are together ?

(*c*) vowels may occupy odd places ? (*d*) vowels being always together ?

[**Ans.** (*a*) 15120, (*b*) 120, (*c*) 2880, (*d*) 4320]

35. How many different words containing all the letters of the word 'LOGARITHM' can be formed ? How many of these :

(*a*) begin with '*L*' ? (*b*) end with '*M*' ?

(*c*) begin with '*L*' and end with '*M*' ? (*d*) have '*L*' and '*M*' in end places ?

(*e*) begin with '*LOG*' ? (*f*) end with '*THM*' ?

(*g*) have vowels together ? (*h*) have no vowels together ?

(*i*) have vowels in the end places ?

(*j*) have vowels and consonants in their relative position ?

(*k*) have vowels in odd places ? (*l*) have vowels in even places ?

(*m*) have vowels in 2^{nd}, 3^{rd} and 4^{th} places ?

[**Ans.** 362880, (*a*) 40320, (*b*) 40320, (*c*) 5040, (*d*) 10080, (*e*) 720, (*f*) 720, (*g*) 30240, (*h*) 151200, (*i*) 8640, (*j*) 4320, (*k*) 43200, (*l*) 17280, (*m*) 4320]

PERMUTATIONS WHEN ALL OBJECTS ARE NOT DISTINCT

In problems of counting, sometimes repetitions are allowed in the arrangements of objects or distinctions between some of the objects are ignored.

The number of permutations of n things taken all at a time, when p things are of one kind, q of second kind, r of third kind and so on is given by

$$\frac{n!}{p!\,q!\,r!\,......}$$

For example consider the word 'SEE'.

We take first E as E_1 and second E as E_2

The word has 3 letters, and the number of permutations in which 3 letters can be arranged is 3 !, *i.e.*, 6 [SE_1E_2, SE_2E_1, E_1SE_2, E_1E_2S, E_2E_1S, E_2SE_1]

Clearly,

$$\left.\begin{matrix} SE_1E_2 \\ SE_2E_1 \end{matrix}\right\} \text{ is SEE}$$

$$\left.\begin{matrix} E_1SE_2 \\ E_2SE_1 \end{matrix}\right\} \text{ is ESE}$$

and $$\left.\begin{matrix} E_1E_2S \\ E_2E_1S \end{matrix}\right\} \text{ is EES}$$

Thus, no. of ways in which the letters of the word 'SEE' can be arranged is 3, *i.e.*, $\frac{3!}{2!}$.

Example 28 : *How many different arrangements can be made by using 4 red and 3 black identical balls ?*

Solution

No. of red balls = 4

No. of black balls = 3

Total no. of balls = 7

No. of ways in which they can be arranged $= \frac{7!}{4!\,3!} = \frac{5040}{24 \times 6} = 35$

$\therefore$ The required no. of permutations is 35.

Example 29 : *In how many ways can the letters of the following words can be arranged ?*
(a) APPLE *(b) ASSASSINATION* *(c) PERMUTATION*

Solution :

(*a*) In the word APPLE, there are 5 letters in which P occurs two times.

∴ The no. of permutations of this word $= \frac{5!}{2!} = \frac{120}{2} = 60$

∴ The Required number of permutations is 60.

(*b*) In the word ASSASSINATION, there are 13 letters in which *A* occurs 3 times, S occurs 4 times, I occurs 2 times and N occurs 2 times.

∴ No. of ways the letters of this word can be arranged $= \frac{13!}{3!\,4!\,2!\,2!} = 10810800$

∴ The required number of permutations = 10810800

(*c*) In the word PERMUTATION

There are 11 letters in which T occurs two times

∴ The number of ways the letters of this word can be arranged $= \frac{11!}{2!} = 19958400$

∴ The required no. of permutation is 19958400.

Example 30 : *In how many of the distinct permutations of the letters in the word 'MATHEMATICS' do the vowels occur together ?*

Solution **:** In the word 'MATHEMATICS', there are 11 letters in which 4 letters (A, E, A, I) are vowels and remaining seven are consonants.

Taking 4 vowels as one unit, we have 8 (*i.e.* 7 + 1) units to be arranged in which 'M' occurs twice and 'T' occurs twice.

∴ No. of ways in which the 8 units can be arranged $= \frac{8!}{2!\,2!} = 10080$

In the single unit of 4 vowels, 'A' occurs twice

∴ No. of ways in which the vowels can be arranged among themselves $= \frac{4!}{2!} = 12$

Total number of arranges in which the vowels occur together = 10080 × 12 = 120960

∴ The required number of permutations is 120960.

Example 31 : *In how many of the distinct permutations of the letters in the word "MISSISSIPPI", do the four I's not come together ?*

Solution **:** In the word MISSISSIPPI, there are 11 letters in which *I* occurs four times, S occurs four times and P occurs twice.

Total number of permutations $= \frac{11!}{(4!)\,(4!)\,(2!)} = 34640.$

Now we find the permutations in which 4 I's are together. For this we treat 4 I's as one unit. Therefore, we have to find the arrangements of 8 units (I I I I), M, S, S, S, S, P, P

Here *S* occurs 4 times and P occurs 2 times

No. of permutations $= \frac{8!}{4!\,2!} = 840$

$\therefore$ No. of permutations in which all I's do not come together = 34650 – 840 = 333810.

The required number of permutations is 33810.

Example 32 : *How many different words can be formed with the letters of the word HARYANA ? How many of these :*

(a) have H and N together ?

(b) begin with H and end with N ?

(c) have vowels together ?

***Solution* :**

In the word HARYANA, there are 7 letters in which A occurs thrice.

$\therefore$ No. of arrangements $= \frac{7\,!}{3\,!} = 840$

Required number of arrangements is 840.

(*a*) Taking H and N as one unit, we have 6 (5 + 1) units to be arranged in which A is repeated thrice.

No. of ways to arrange the 6 units $= \frac{6\,!}{3\,!} = \frac{720}{6} = 120$

No. of ways to arrange H and N among themselves = 2 ! = 2

$\therefore$ No. of arrangements in which H and N are together = 120 × 2 = 240

$\therefore$ Required number of arrangements = 240

(*b*) No. of ways to arrange H and N = 1 ! × 1 ! = 1

No. of ways to arrange remaining 5 letters $= \frac{5\,!}{3\,!} = 20$

$\therefore$ The required number of arrangements = 1 × 20 = 20

(*c*) Taking 3 vowels together as one unit, we have 5 (4 + 1) units.

No. of ways to arrange the 5 units = 5 ! = 120

No. of ways to arrange the vowels among themselves $= \frac{3\,!}{3\,!} = 1$

$\therefore$ Total no. of arrangements = 120 × 1 = 120

$\therefore$ The Required number of arrangements = 120

Example 33 : *How many (6-digit) numbers greater than 4,00,000 can be formed using the digits 0, 3, 3, 4, 4, 5.*

***Solution* :** There are 6 digits (0, 3, 3, 4, 4, 5).

We have to use all 6 digits to form the number.

The number greater than 400000 will have either 4 or 5 at the extreme left.

If 4 occupies the extreme left, the remaining 5 place can be filled with 0, 3, 3, 4, 5

Here 3 occurs twice

$\therefore$ No. of such arrangements $= \frac{5\,!}{2\,!} = \frac{120}{2} = 60$

If 5 occupies the extreme left, the remaining 5 places can be filled with 0, 3, 3, 4, 5.

Here 3 repeats twice and 4 repeats twice.

$\therefore$ No. of such arrangement $= \frac{5!}{2!\,2!} = \frac{120}{2 \times 2} = 30$

$\therefore$ Total number of arrangements = 60 + 30 = 90

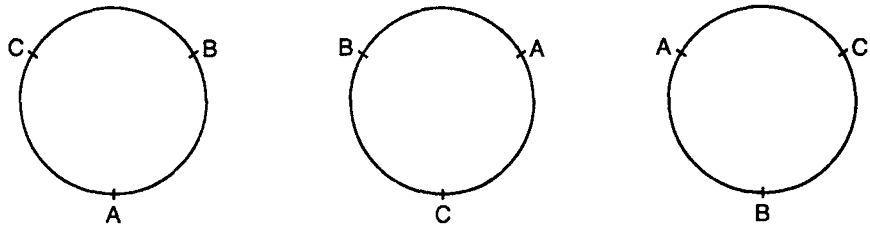
$\therefore$ Required number of 6-digit numbers is 90.

CIRCULAR PERMUTATION

The *circular permutations* are permutations of certain things in the form of a circle.

For example, *ABC*, *BCA*, and *CAB* are three different linear permutations, but round a circle these three different arrangements give only *one* circular permutation *ABC* read in the anticlockwise direction. In circular permutations, there is neither a beginning nor an end.

It is illustrated in the following figure :



All the three figures represents only one arrangement in which *B* is in the right side of *A* and *C* is in the left side of A.

Let x be the required number of circular permutations. To each one of these x circular permutations, there corresponds n linear permutations starting from each one of n things in the circular permutations and read in the anticlockwise direction.

$\therefore$ All circular permutations give rise to $x.n$ linear permutations.

$$\therefore \quad x \cdot n = n!$$

$$\therefore \quad x = \frac{n!}{n} = (n-1)!$$

Thus, the number of circular permutations of n different things is given by $(n-1)!$

In circular permutations, the permutations are always read in anti-clockwise direction.

Example 34 : *In how many ways can 7 persons sit in a round table ?*

Solution **:** The number of circular permutations of n different objects is given by $(n-1)!$

$\therefore$ No. of ways in which 7 persons can sit in a round table is $(7-1)!$ *i.e.,* $6!$ *i.e.* 720

Example 35 : *Four men and four women sit around a round table. In how many ways they can be seated so that*

(a) no two women sit together ? *(b) all women sit together ?*

Solution **:** No. of men = 4

No. of women = 4

(*a*) First four men can be seated around the round table.

∴ No. of ways to arrange the men = (4 – 1) ! = 3 !

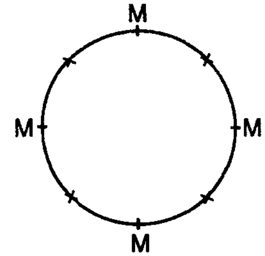
There are 4 places in between the men.

∴ No. of ways to arrange the women = $^4P_4$ = 4 !

∴ No. of arrangement in which no women sit together

= 3 ! × 4 ! = 6 × 24 = 1444

∴ Required number of arrangements is 144.



(*b*) Taking women as one unit, there are 5 (= 4 + 1) units to be arranged.

No. of ways in which 5 units can be arranged in circular form = (5 – 1) ! = 4 !

No. of ways to arrange the women among themselves = 4 !

∴ No. of arrangements in which all women sit together = 4 ! × 4 ! = 24 × 24 = 576

∴ Required number of arrangements is 576.

Example 36 : *There are 5 men and 4 women to sit around a round table. In how many ways can they sit so that no two women sit together*

Solution

First, we arrange 5 men around the round table.

∴ No. of ways to arrange the men = (5 – 1) ! = 4 !

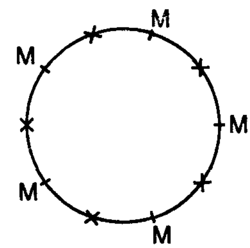
There are 5 places in between the men.

∴ No. of ways to arrange the women = $^5P_4$

∴ No. of ways in which no two women sit together = 4 ! × $^5P_4$

= 24 × 5 × 4 × 3 × 2 = 2880

∴ Required number of arrangement is 2880.



Example 37 : *Three boys and three girls are to be seated around a table in a circle. Among them, the boy X does not want any girl neighbour and the girl Y does not want any boy neighbour. How many such arrangements are possible ?*

Solution : No. of boys = 3

No. of girls = 3

Seating arrangement is shown in the figure.

Boy *X* (B_1) will have boys as neighbours

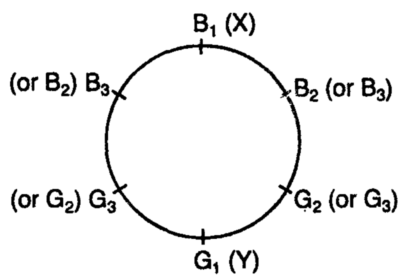
∴ No. of ways to arrange the remaining two boys = 2!

Girl *Y* (G_1) will have girls as neigbours

∴ No. of ways to arrange the remaining two girls = 2 !

∴ Total no. of arrangements = 2 ! × 2 ! = 2 × 2 = 4

∴ Required number of arrangements is 4.



EXERCISES

1. There are 5 red, 4 white and 3 blue marbles in a bag. These are drawn one by one and arranged in a row. Assuming that all the 12 marbles are drawn, determine the number of different arrangements. **[Ans. 27700]**

2. In how many ways can 4 red, 3 yellow and 2 green discs be arranged in a row, if the discs of the same colour are indistinguishable ? [**Ans.** 1260]

3. In how many different ways, can the letters of the following words be arranged ?

   (*a*) COMMERCE . (*b*) ALLAHABAD [**Ans.** (*a*) 5040, (*b*) 7560]

4. In how many different ways can the letters of the following words be arranged :

   (*a*) MOON (*b*) NOON [**Ans.** (*a*) 12, (*b*) 6]

5. Find the number of different words beginning with *P* which can be formed by using the letters of the word 'PERMUTATION'. [**Ans.** 1814400]

6. How many different signals can be transmitted by arranging 3 red, 2 yellow and 2 green flags on a pole ? (Assume that all the 7 flags are used to transit a signal).

   [**Ans.** 210]

7. In how many ways can 5 flags, in which 3 are red, one is white and one is blue, be arranged one below the other, if flags of one-colour are not distinguishable ? [**Ans.** 20]

8. In how many ways can the letters of the word 'MANWINDER' be arranged so that

   (*a*) *A* and *D* may always be together ? (*b*) *A* and *D* may never be together ?

   [**Ans.** (*a*) 40320, (*b*) 141120]

9. How many words can be formed with the letters of the word 'PARALLEL' so that all L's do not come together ? [**Ans.** 3000]

10. How many different words can be formed with the letters of the word CAPTAIN ? In how many of these *C* and *T* are never together ? [**Ans.** 2520, 1800]

11. If the different permutations of all the letters of the word EXAMINATION are listed as in a dictionary, how many items are there in this list before the first word starting with E ? [**Ans.** 907200]

12. Find the number of different words which can be formed by using all the letters of the words INSTITUTION. In how many of them :

    (*a*) the three *T*s are together ? (*b*) the first two letters are the two *N*s ?

    [**Ans.** 554400, (*a*) 30240, (*b*) 10080]

13. Find the number of arrangements which can be made from the letters of the word ALGEBRA without altering the positions of the vowels and consonants. [**Ans.** 72]

14. How many words can be formed by arranging the letters of the word 'UNIVERSITY' so that the vowels remain together ? [**Ans.** 60480]

15. Find the number of arrangements that can be made out of the letters of the word COMBINATION. In how many of these, vowels occur together ?

    [**Ans.** 4989600, 75600]

16. How many six digited numbers can be formed by using the digits 1, 1, 1, 2, 2, 3 ? How many of these are greater than 3,00,000 ? [**Ans.** 60, 10]

17. How many number can be formed with the digits 1, 2, 3, 4, 3, 2, 1 so that the odd digits always occupy the odd places. [**Ans.** 18]

18. How many numbers greater than a million can be formed by using the digits 4, 6, 0, 6, 7, 4, 6 ? [**Ans.** 360]

19. How many numbers greater than 50,000 can be formed by using the digits 3, 5, 6, 6, 7, 9 ? [**Ans.** 48]
20. Four persons, A, B, C and D are to be seated at a circular table. In how many ways can they be seated ? [**Ans.** 6]
21. In how many ways can 8 persons form a ring ? [**Ans.** 5040]
22. In how many ways can 7 people be arranged at a round table so that 2 particular persons may be together ? [**Ans.** 240]
23. In how many ways can 5 persons A, B, C, D and E sit around a circular table, if
    (*a*) B and D sit next to each other ? (*b*) A and D do not sit next to each other ?
    [**Ans.** (*a*) 12, (*b*) 12]
24. In how many ways can 6 boys be arranged at a round table so that two particular boys may be together ? [**Ans.** 240]
25. There are six gentlemen and four ladies to dine at a round table. In how many ways can they sit among themselves so that no two ladies are together ? [**Ans.** 43200]
26. In how many ways can 5 gentlemen and 5 ladies be seated at a round table so that no two gentlemen are together ? [**Ans.** 2880]
27. The chief ministers of 11 states of India meet to discuss the current issues. In how many ways can they seat themselves at a round table so that the chief ministers of states X and Y sit together ? [**Ans.** 725760]
28. In how many ways 11 members of a committee sit at a round table so that the secretary and the joint secretary are always the neighbours of the president ?
    [**Ans.** 80640]

COMBINATION

A selection (group) of a number of things taking some or all of them at a time is called a *combination*. The selections are different from permutations in the sense that in a permutation, the order of things is taken into consideration whereas in case of selection, the order of things is immaterial and we consider only the things which are occurring in a selection. For example, *ab* and *ba* are two distinct permutations but same selection.

The total number of combinations of n distinct things taking $r (1 \leq r \leq n)$ at a time is denoted by ^nC_r or by $C(n, r)$.

Consider the following example :

The permutation of 3 things x, y, z taking 2 at a time are :

$xy,\ yx,\ xz,\ zx,\ yz,\ zy$

$\therefore \quad {}^3P_2 = 6$

But the combination of 3 thing x, y, z taking 2 at a time are :

$xy \qquad yz \qquad xy$

$\therefore \quad {}^3C_2 = 3$

xy and yx represents the same combination and two different permutations.

Thus permutations are arrangements in definite order, whereas combinations are 'selections' in which order of objects does not matter.

The number of combinations of n things taking r at a time is equal to

$${}^nC_r = \frac{n!}{r!\,(n-r)!},\ 1 \le r \le n.$$

Let the required number of combinations of n things taken r at a time be x.

Take one of these combinations. It contains r things which can be arranged among themselves in $r!$ ways. Thus, one combination gives rise to $r!$ permutations.

Total number of permutations of n things taken r at a time = ${}^nP_r$.

$$\therefore \quad x\,r! = {}^nP_r \quad \text{or} \quad x = \frac{{}^nP_r}{r!}$$

$$\therefore \quad {}^nC_r = \frac{{}^nP_r}{r!} = \frac{1}{r!}\cdot\frac{n!}{(n-r)!} \qquad \left[\because\ {}^nP_r = \frac{n!}{(n-r)!}\right]$$

Hence $\boxed{{}^nC_r = \frac{n!}{r!\,(n-r)!}}$

Remark 1 : ${}^nC_n = 1$

Proof : $${}^nC_n = \frac{n!}{n!\,(n-n)!} = \frac{n!}{n!\,0!} = \frac{n!}{n!} = 1$$

Remark 2 : ${}^nC_r = {}^nC_{n-r}$

Proof : $${}^nC_{n-r} = \frac{n!}{(n-r)!\,(n-(n-r))!} = \frac{n!}{(n-r)!\,r!} = {}^nC_r$$

Remark 3 : ${}^nC_0 = 1$

Proof : $${}^nC_0 = \frac{n!}{0!\,(n-0)!} = \frac{n!}{1.n!} = 1$$

Remark 4 : $${}^nC_r = \frac{n\,(n-1)\,(n-2)\ \ldots\ldots\ r \text{ terms}}{1.2\,3\ \ldots\ldots\ r.},\ \textit{i.e. } {}^nC_r = \frac{{}^nP_r}{r!}$$

Remark 5. The number of combinations of n different objects taken r at a time in such a way that a particular object is always included is ${}^{n-1}C_{r-1}$.

Remark 6. The number of combinations of n different objects taken r at a time ion such a way that a particular object is always excluded is ${}^{n-1}C_r$

Example 38 : *Evaluate the following :*

(a) ${}^7C_3$ *(b)* ${}^{25}C_4$ *(c)* ${}^{42}C_{39}$

Solution :

(a) $${}^7C_3 = \frac{7\times 6\times 5}{1\times 2\times 3} = 35$$

(b) $${}^{25}C_4 = \frac{25\times 24\times 23\times 22}{1\times 2\times 3\times 4} = 12650$$

(c) $${}^{42}C_{39} = {}^{42}C_{42-39} = {}^{42}C_3 = \frac{42\times 41\times 40}{1\times 2\times 3} = 11480$$

Example 39 : *Prove that if n and r natural members such that $1 \le r \le n$, then*

$${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$$

Solution :

$$\text{LHS} = {}^nC_r + {}^nC_{r-1} = \frac{n!}{r!\,(n-r)!} + \frac{n!}{(r-1)!\,(n-r+1)!}$$

$$= \frac{n!}{(r-1)!\,(n-r)!} \times \left[\frac{1}{r} + \frac{1}{n-r+1}\right]$$

$$= \frac{n!}{(r-1)!\,(n-r)!}\left[\frac{n-r+1+r}{r\,(n-r+1)}\right]$$

$$= \frac{n!}{(r-1)!\,(n-r)!}\left[\frac{n+1}{r\,(n-r+1)}\right]$$

$$= \frac{(n+1)\,n!}{r(r-1)!\,(n-r+1)(n-r)!}$$

$$= \frac{(n+1)!}{r!\,(n-r+1)!} = {}^{n+1}C_r = \text{RHS}$$

$\therefore \quad {}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r.$

Example 40 : *In how many ways can a committee be selected from 15 persons, if committee is to have*

(a) 3 members ? *(b) 13 members ?*

Solution

(*a*) Total no. of persons = 15

No. of ways of selecting 3 people $= {}^{15}C_3 = \frac{15 \times 14 \times 13}{1 \times 2 \times 3} = 455$

$\therefore$ Required number of ways is 455

(*b*) No. of ways of selecting 13 people ${}^{15}C_{13} = {}^{15}C_{15\ 13} = {}^{15}C_2 = \frac{15 \times 14}{1 \times 2} = 105$

$\therefore$ Required number of ways is 105.

Example 41 : *Find the number of ways of selecting 9 balls from 6 red balls, 5 white balls, 5 blue balls, if each selection consists of 3 balls of each colour.*

Solution :

No. of red balls = 6

No. of white balls = 5

No. of blue balls = 5

No. of ways to select 3 red balls $= {}^6C_3$

No. of ways to select 3 white balls $= {}^5C_3$

No. of ways to select 3 blue balls $= {}^5C_3$

$\therefore$ No. of ways to select 9 balls $= {}^6C_3 \times {}^5C_3 \times {}^5C_3 = 20 \times 10 \times 10 = 2000$

$\therefore$ Required number of ways is 2000.

Example 42 : *A committee of 5 members is to be formed out of 6 men and 4 ladies. In how many ways can this be done if*

(a) *exactly two women are included ?* (b) *at least two women are included ?*

(c) *at most two women are included ?*

Solution :

Total No. of men $= 6$

Total No. of women $= 4$

(a) If exactly two women are to be selected, then only three men can be selected.

$\therefore$ No. of ways to select 2 women $= {}^4C_2$

No. of ways to select 3 men $= {}^6C_3$

$\therefore$ No. of ways to select the committee of 5 $= {}^4C_2 \times {}^6C_3 = 6 \times 20 = 120$

$\therefore$ Required number of ways is 120.

(b) If at least two women are to be selected, the no. of women may be 2, 3, or 4.

The committee may consist as follows :

Women	*Men*
2	3
3	2
4	1

$\therefore$ No. of ways to select the committee of 5 members

$= ({}^4C_2 \times {}^6C_3) + ({}^4C_3 \times {}^6C_2) + ({}^4C_4 \times {}^6C_1)$

$= (6 \times 20) + (4 \times 15) + (1 \times 1) = 120 + 60 + 1 = 181$

$\therefore$ Required number of ways is 181.

(c) If at most two women are to be selected, then number of women may be 0, 1 or 2

Selection in made as follows :

Women	*Men*
0	5
1	4
2	3

$\therefore$ No. of ways to select the committee of 5 members

$= ({}^4C_0 \times {}^6C_5) + ({}^4C_1 \times {}^6C_4) + ({}^4C_2 \times {}^6C_3)$

$= (1 \times 6) + (4 \times 15) + (6 \times 20) = 6 + 60 + 120 = 186$

$\therefore$ Required number of ways is 186.

Example 43 : *In how many ways can a student choose a programme of 5 courses if 9 courses are available and 2 courses are compulsory for every student ?*

Solution :

No. of courses available = 9

No. of courses in the programme = 5

Since two papers are compulsory, these papers have to be always included. Remaining 3 papers are to selected from 7 courses.

$\therefore$ No. of ways to select 5 courses = $^{9-2}C_{5-2} = {}^7C_3 = \frac{7 \times 6 \times 5}{1 \times 2 \times 3} = 35$

$\therefore$ Required number of ways is 35

Example 44 : *In how many ways can we select a cricket eleven from 15 players in which only 5 players can bowl, if the team must include exactly four bowlers ?*

Solution :

No. of players = 15

No. of bower = 5

The term contains 11 players in which number of bowlers are 4.

$\therefore$ No. of ways of selecting 4 bowlers = $^5C_4$

The remaining 7 player are selected from 10 players in $^{10}C_7$ ways

$\therefore$ No. of ways in which the team is selected = $^5C_4 \times {}^{10}C_7 = 5 \times 120 = 600$

$\therefore$ Required number of ways is 600.

Example 45 : *A team of 8 players is to be selected from a group of 12 players and one of the player is then to be selected as captain and another as vice-captain. In how many ways can the team be selected ?*

Solution :

No. of ways in which 8 players can be selected from 12 players is $^{12}C_8$.

In selection of captain and vice-captain, the order of selection does matter.

$\therefore$ No. of ways in which the captain and vice-captain are selected from 8 players in $^8P_2$

$\therefore$ No. of ways in which the team is selected = $^{12}C_8 \times {}^8P_2 = 495 \times 56 = 27720$

$\therefore$ Required number of ways is 27720.

Example 46 : The English alphabet has 5 vowels and 21 consonants. How many words with two different vowels and two different consonants can be formed from the alphabet ?

Solution : To form four-letter words, first we have to select 2 vowels and 2 consonants.

$\therefore$ No. of ways of selecting 2 vowels = $^5C_2$

No. of ways of selecting 2 consonants = $^{21}C_2$

After selecting 2 vowels and 2 consonants, we arrange the four letters to form 4-letter words.

$\therefore$ No. of ways to arrange the 4 letters = ${}^4P_4 = 4\,!$

$\therefore$ No. of 4-letter words = ${}^5P_2 \times {}^{21}C_2 \times 4\,! = 10 \times 210 \times 24 = 50400$

$\therefore$ Required numbers of words is 50400

EXERCISES

1. Evaluate :

   (a) ${}^9C_4$ (b) ${}^{15}C_{14}$ (c) ${}^{20}C_{20}$ (d) ${}^{24}C_3$

   [**Ans.** (a) 126, (b) 15, (c) 1, (d) 3036]

2. Find n if

   (a) ${}^nC_8 = {}^nC_6$ (b) $12\ {}^nC_2 : {}^{2n}C_3$ [**Ans.** (a) 14, (b) 5]

3. Find n, if

   (a) ${}^{2n}C_3 : {}^7C_2 = 12 : 1$ (b) ${}^{n+2}C_8 : {}^{n-2}P_4 = 57 : 16$ [**Ans.** (a) 5, (b) 19]

4. Show that :

   (a) $r\,.\,{}^nC_r = n\,.\,{}^{n-1}C_{r-1}$ (b) ${}^nC_r + {}^nC_{r+1} = {}^{n+1}C_{r+1}$ (c) ${}^nC_r \times {}^rC_s = {}^nC_s \times {}^{n-s}C_{r-s}$

5. In how many ways can a team of 3 boys and 3 girls be selected from 5 boys and 4 girls ? [**Ans.** 40]

6. How many selection of 4 books can be made from 8 different books ? [**Ans.** 70]

7. How many different committees can be formed consisting of 4 men and 3 women out of 7 men and 5 women ? [**Ans.** 350]

8. From a class of 25 students, 10 are to be chosen for an excursion party. There are three students who decide that either all of them will join or none of them will join. In how many ways can the 10 students be chosen ? [**Ans.** 651244]

9. A boy has 3 library tickets and 8 books of his interest are in the library. Of these books, he does not want to borrow Business Mathematics, unless Financial Maths is also borrowed. In how many ways can he choose the three books to be borrowed ?

   [**Ans.** 41]

10. In how many ways can 5 members forming a committee out of 10 be selected so that :

    (a) two particular members must be included ?

    (b) two particular members must not be included ? [**Ans.** (a) 56, (b) 56]

11. In how many ways can a football team of 11 players be selected from 15 players ? How many of these will

    (a) include one particular player ?

    (b) exclude one particular player ? [**Ans.** 1365, (a) 1001, (b) 364]

12. A committee of 7 is to be formed from 9 boys and 5 girls. In how many ways can this be done when the committee contains

    (a) exactly 3 girls ? (b) at least 3 girls ? [**Ans.** 1260, (b) 1716]

13. The question paper on Mathematics and Statistics contains 10 questions divided into two groups of 5 questions each. In how many ways can an examinee select 6 questions taking at least two questions from each group ? [**Ans.** 200]

14. From 4 officers and 8 jawans, in how many ways can 6 be chosen (*a*) to include exactly one officer ? (*b*) to include at least one officer ? [**Ans.** (*a*) 224, (*b*) 896]

15. Out of 6 boys and 4 girls, a committee of 6 is to be formed. In how many ways can this be done if the committee contains :

    (*a*) exactly 2 girls ? (*b*) at least 2 girls ? (*c*) at most 2 girls ?

    [**Ans.** (*a*) 90, (*b*) 185, (*c*) 115]

16. How many committees of 5 members each can be formed with 8 officials and 4 non-official members in the following cases :

    (*a*) each consists of 3 officials and 2 non-official members ?

    (*b*) each consists of at least two non-official members ?

    (*c*) a particular official members is never included ?

    (*d*) a particular non-official member is always included ?

    [**Ans.** (*a*) 336, (*b*) 456, (*c*) 462, (*d*) 330]

17. Out of 7 consonants and 4 vowels, how many words can be made each containing 3 consonants and 2 vowels ? [**Ans.** 25200]

18. How many words containing 4 consonants and 3 vowels can be formed from 6 consonants and 5 vowels ? [**Ans.** 756000]

19. Among 20 members of a cricket club, there are two wicket-keepers and 5 bowlers. In how many ways can eleven players be chosen so as to include only one of the wicket-keepers and atleast three bowlers ? [**Ans.** 54054]

20. A box contains 2 different white balls, 3 different black balls and 4 different red balls. In how many ways can 3 balls be drawn from the box if at least one black ball is to be included in the draw. [**Ans.** 64]