CHAPTER 10

Immediate Annuity

INTRODUCTION TO ANNUITY

In our daily life, we see a lost of transactions taking place, like sales or purchase of goods, renting or hiring, borrowing or lending of money and etc. Some of these transactions involving money are sometimes done by a single payment at a future date. But not many people are in a position to deposit a large sum of money at one time in an account. They make the payments in instalments over a period of time. For example, a customer, instead of paying the entire price of Rs. 20,000 for a refrigerator, many pay a part of it, say Rs. 5,000 initially and remaining in equal monthly instalments. These instalment are so determined that they compensate the seller for his waiting time and are based on compound interest rate.

If a depositor makes equal deposits at regular intervals, he is contributing to an annuity. So by an annuity, we mean a sequence of equal periodic payments made at equal intervals of time. The payments may be generally made weekly, monthly, quarterly, half-yearly or yearly. For example, premium for a life insurance policy may be made in equal instalments periodically, say semi-annually. The equal monthly payment that a retired person receives in the form of pension is another example of annuity. Even bank loans are repaid in equal instalments, say monthly, over a period of time. All these transactions have a common thing equal payments at equal intervals of time. In this chapter, we discuss various kinds of annuities and study in detail the present value and future value concepts of immediate annuity.

ANNUITY : DEFINITION

An annuity is defined as follows :

"An annuity is a sequence of equal payments made at equal intervals of time."

The term annuity is derived from the Latin word 'annum'. Although this word indicates that payments must be annual, its meaning in practice, has been broadened to include any periodic payment of a fixed amount made at a regular interval which may be more or less than one year. The person obligated to make such payments is called *annuitator* where as the person entitled to receive such payments is called annuitant.

SOME BASIC TERMS

(*i*) Period Payments

The size of each payment of an annuity is called the periodic payment or periodic rent of the annuity.

For example, if a sequence of payments of Rs. 2,000 is made at the beginning of every month, then we say that it is annuity with periodic payment of Rs. 2,000.

(*ii*) Payment Date

The date on/by which a periodic payment is to be made is called payment date.

For example,. if a bank asks a borrower to pay the monthly instalment against a personal loan on 5th of every month this date will be the payment date.

(*iii*) Payment Period

The period of time elapsing between two successive payment dates of an annuity is called its payment period or payment interval or rent period.

For example, if a sequence of payments is to be made on 1st January every year, then it is an annuity with payment period of one year. The payment period of an annuity can be a year, an half-year, a quarter, a month, a week and etc.

(*iv*) Term

The total time from the beginning of the first payment period to the end of the last payment period is called the term of an annuity.

For example, if a series of payments was made on 1st January every year from 1990 to 1999, it is an annuity with a 10-year term and payment period of one year.

It should be noted that the term of an annuity can also be a certain number of months. For example, if a loan is to be repaid in 6 equal monthly instalments, then it is an annuity with 6-months term and payment period of one month.

(*v*) Amount or Future Value

It is the sum of all the payments made and the interest earned on them at the end of the term of the annuity.

In other words, the amount of an annuity is the sum of the terminal or future values of each of the periodic payments. Here, the term terminal value means the sum of the payments made and the interest earned on that payment.

(*vi*) Present Value or Capital Value

The present value or capital value of an annuity is the sum of the present values of all the payments.

In other words, the present value of an annuity is the amount of money that must be invested now to purchase the payments due in future.

TYPES OF ANNUITIES

There are many types of annuities which are classified based on different factors.

(*i*) Annuities based on their Term

Following are the types of annuities which are classified based on the term of the annuity :

(*a*) Annuity Certain

(*b*) Contingent Annuity

(*c*) Perpetual Annuity

(*a*) *Annuity Certain*

An annuity payable for a certain fixed term is called Annuity certain. This kind of annuity begins and ends on certain fixed dates. The term of an annuity is fixed and so its payments extend over a fixed period of time.

The payment for hike-purchase of durables like Television, Refrigerator, Car, etc. are usually annuity certain, because the buyer knows the specified dates on which the instalments are to be made and the number of instalments to be made. Repayment of bank loan and Bank recurring deposits are also annuity certain.

(*b*) *Contingent Annuity*

Contingent annuities one whose payments continue for a period of time which depends on the happening of an event the date of which cannot be accurately foretold. If the contingency did not happen at all, then the periodic payments should be paid till the end of designated term.

For example, payment of life insurance premium is contingent annuity. Whenever the insured person dies, the payment of premium is stopped.

(*c*) *Perpetual Annuity*

A perpetual annuity, simply perpetuity is an annuity whose payments continue forever (infinite number of years). In perpetuity, the beginning date is known but the terminal date is not known, because the term of this annuity is infinite.

For example, in endowment funds of trusts the principal amount is not touched and the interest earned is used for welfare activities. Since the principal is not touched, a certain sum is received periodically forever.

(*ii*) Annuities based on Payment Date

Following are the types of annuities classified based on the payment interval and time of payment :

(*a*) Ordinary or Immediate Annuity

(*b*) Annuity Due

(*c*) Deferred Annuity.

(a) *Immediate Annuity*

An ordinary annuity or immediate annuity is an annuity in which every periodic payment is made at the end of the corresponding payment period.

Even though, the first payment is made at the end of the first payment period, the term of an ordinary annuity begins with the beginning of the first payment period and ends on the day of the last payment.

For example, repayment of a bank loan is an ordinary annuity.

It should be noted that an ordinary annuity is an annuity certain but the converse may not be true.

(b) *Annuity Due*

If each payment of annuity is made at the beginning of the corresponding payment period is called annuity due.

Here, even though the last payment is made at the beginning of the last payment period, the term of an annuity due ends at the end of the last payment period. The term of an annuity due begins on the day, the first payment is made.

For example, a saving scheme of five years in which equal deposits are made at the beginning of each year is an annuity due. In this annuity every payment is an instalment, the first payment earns interest for five years, the second for four years and the last for one year.

Payment of premium for a life insurance policy is the most precise example for annuity due. In life insurance, the first premium is to be made at the commencement of the policy.

(c) *Deferred Annuity*

Deferred stands for postponed. A deferred annuity is an annuity in which the first payment is postponed for a period of time which is equivalent to a certain number of payment periods.

In deferred annuity, the term begins after the expiry of this postponed period. This period is called period of deferment. It is understood from the above discussion that in deferred annuity the term is known but postponed for a certain number of intervals.

A deferred annuity can either be a deferred ordinary annuity or a deferred annuity due. But it is generally an ordinary annuity.

For example, when a house building loan is given by an employer to an employee, generally the repayment in equal instalments by the employee does not start thereof but may begin after some time, say five years after the loan is sanctioned. The interval of five years is called the deferred period.

(*iii*) Annuities based on Payment Period and Conversion Period

Annuities based on payment period and conversion period are classified as follows :

(*a*) Simple Annuity

(*b*) Complex Annuity

(*a*) *Simple Annuity*

An annuity in which the payment period coincides with the interest conversion period is called simple annuity.

For example, consider an annuity of Rs. 1000 payable at the end of every quarter, interest being calculated at 8% per annum compounded quarterly. Here payment is made at the end of every quarter and interest is also compounded on quarterly basis. This kind of annuity is called simple annuity.

(*b*) *Complex Annuity*

An annuity in which payment period differs from the interest conversion period is called a complex annuity.

For example, suppose that a man deposits Rs. 500 at the end of every month in an account and the bank calculates interest at 6% per annum compounded half-yearly. This kind of annuity is called complex annuity. In this case, payment period is one month whereas the interest conversion period is six months. In other words, six periodic payments are made in one interest conversion period.

(*iv*) Annuities based on the Period Payment

Annuities based on the amount of periodic payment are classified as follows :

(*a*) Uniform Annuity

(*b*) Variable Annuity

(*a*) *Uniform Annuity*

If the periodic payment are all equal through out the term of the annuity, then the annuity is called uniform annuity. It is also known as fixed annuity.

Annuities are generally uniform, since equal payment are made at equal time interval. For example, repayment of a personal loan is a uniform annuity. The debtor is to repay the loan in equal instalments.

(*b*) *Variable Annuity*

It the periodic payment of an annuity changes every year or every payment period, then the annuity is called variable annuity.

Penson plan is an example of variable annuity. Certain additional features may also be added to the plan. The annuity may increase as a result of vested bonuses, if the plan is a participating one.

CHARACTERISTICS OF AN ANNUITY

- Annuity is a recurring payment of fixed amount, *i.e.*, a periodic payment.
- Annuity is payable at an equal interval of time, either annually, semi-annually, quarterly or the like.
- Annuity is paid against a lump sum received at present or a compound amount receivable after a certain period.

- Annuity is paid either to discharge an existing obligation instalmentwise or to create a fund to be accumulated in future.
- If the annuity is paid against a lump sum received at present, then every payment will contain interest for the outstanding principal and the sum of all instalments (periodic payments) will be more than the present value. The difference is the total interest paid.
- If the compound amount (Future value) is receivable after a certain term, then every periodic payment will bring interest and the sum of the instalments will be less than the future value. The difference is the total interest received.
- Annuity is paid either at the beginning or at the end of each period.
- In annuity, every payment is a case of compound interest. To find the future value (or present value) of an annuity, future or present value of every periodic payment is calculated on the basis of compound interest and the sum is found out.
- Annuity is paid on the basis of a contract by one party called annuitator to another party called annuitant.

DIFFERENCE BETWEEN COMPOUND INTEREST AND ANNUITY

- In case of compound interest, a lump sum (principal) is paid only once. But incase of annuity, a series of equal periodic payments are payable (or receivable) at equal interval of time.
- The purpose of investment under compound interest system is to crate a heavy fund at the end of a certain term to meet an obligation. But the purpose of investment under annuity system is to create a heavy fund through periodic contribution of small amounts to meet an obligation.
- Compound interest system operates without any element of annuity system, but annuity system cannot operate with out the element of compound interest system.
- Annuities can be classified into different types based on the payment date, payment interval and etc. But there is no such classification in case of compound interest.
- In case of compound interest system, the rate of interest may vary from time to time. But in case of annuity system, interest is generally compounded at an agreed fixed rate of interest.
- In case of multiple investments under compound interest, the investments need not to made at equal time intervals. But instalments should be made at equal time intervals in the annuity system.

Note : In this chapter, we will study the concept of ordinary annuity in detail and succeeding chapters are devoted for annuity due, differed annuity and perpetuity.

ORDINARY (OR IMMEDIATE) ANNUITY

As we have discussed earlier in this chapter, in an ordinary annuity, the periodic payments are made at the end of each payment period. We consider ordinary annuities that are certain and simple with equal periodic payments. In this chapter, we study the annuities that are subject to the following conditions :

(*a*) The periodic payments are made at equal time interval.

(*b*) The payment period coincides with the interest conversion period.

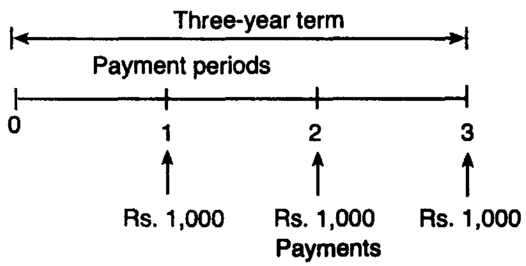
(*c*) The periodic payments are made at the end of every payment period.

Remark : If the payment date (*i.e.* beginning or end of the payment period) is not mentioned, it should be then assumed that payments are made at the end of the payment periods and the annuity is ordinary annuity.

AMOUNT OF AN ORDINARY ANNUITY

The amount or future value of an annuity is the sum of all the payments made and the interest earned on them at the end of the term of the annuity. In other words, future value of an ordinary annuity is the sum of the compound amounts of all the payments accumulated at the end of the term.

For example, consider an ordinary annuity of Rs. 1,000 payable at the end of every year for 3 years at 8% effective rate of interest.



The payment of Rs. 1,000 made at the end of 1st year will earn interest for 2 years and this amount will be accumulated to 1,000 $(1.08)^2$, *i.e.,* Rs. 1,166.40.

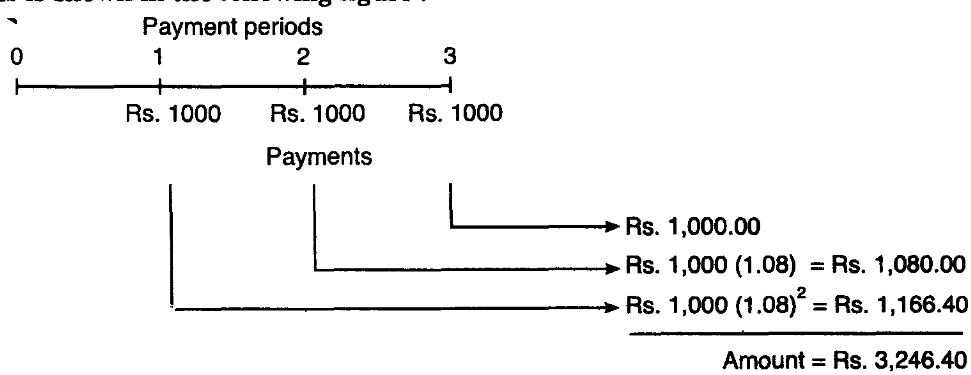
The second payment of Rs. 1,000 made at the end of 2nd year will earn interest for 1 year, since this amount remains in the account for one completed year. At the end of the term of 3 years, this payment will be accumulated to Rs. 1,000 (1.08), *i.e.,* Rs. 1,080.00.

The last payment, *i.e.,* the third payment will earn no interest, because as soon as the payment is made, the term ends. Therefore, the amount of this payment will remain same, *i.e.,* Rs. 1,000.

$\therefore$ The amount of this annuity

A = 1,166.40 + 1,080 + 1,000 = Rs. 3,246.40

This is shown in the following figure :



Now we try to derive the formula for future value of an ordinary annuity in a general problem.

Let Rs. R be the size of the periodic payment of an ordinary annuity.

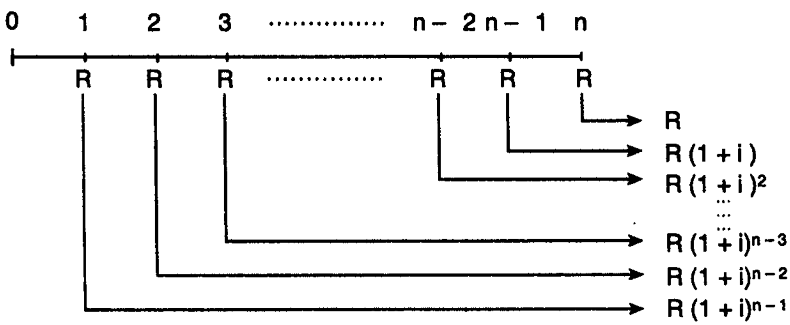
Let r be the annual rate of interest and m be the number of conversion periods per year.

$\therefore$ The rate of interest per conversion period is given by

$i = r/m$

Let 'n' be the total number of conversion periods.

The first payment Rs. R, made at the end of the first period, earns interest for $(n - 1)$ periods, hence its compound amount would be $R\,(1 + i)^{n-1}$. Similarly, the second payment of Rs. R earns interest for $(n - 2)$ years, the third for $(n - 3)$ years and so on. The last payment of Rs. R made at the end of the term does not earn any interest and its value is simply Rs. R. This is illustrated in the following figure :



Amount of 1[st] instalment, $A_1 = R\,(1 + i)^{n-1}$

Amount of 2[nd] instalment, $A_2 = R\,(1 + i)^{n-2}$

$\vdots$

Amount of $(n - 2)^{nd}$ instalment $A_{n-2} = R\,(1 + i)^2$

Amount of $(n - 1)^{st}$ instalment $A_{n-1} = R\,(1 + i)$

Amount of n^{th} instalment $A_n = R$

The Amount of this annuity is given by

$$A = A_1 + A_2 + A_3 + \ldots\ldots A_{n-2} + A_{n-1} + A_n$$

$$\Rightarrow \quad A = R(1 + i)^{n-1} + R\,(1 + i)^{n-2} + \ldots\ldots + R\,(1 + i)^2 + R\,(1 + i)^1 + R$$

$$\Rightarrow \quad A = R + R\,(1 + i) + R\,(1 + i)^2 + \ldots\ldots + R\,(1 + i)^{n-2} + R\,(1 + i)^{n-1}$$

$$\Rightarrow \quad A = R\,[1 + (1 + i) + (1 + i)^2 + \ldots\ldots + (1 + i)^{n-2} + (1 + i)^{n-1}]$$

$$\Rightarrow \quad A = R\left\{\frac{(1 + i)^{n-1}}{(1 + i) - 1}\right\} \qquad \because 1 + r + r^2 \ldots r^{n-1} = \frac{r^n - 1}{r - 1}$$

$$\Rightarrow \quad A = R\left[\frac{(1 + i)^n - 1}{i}\right]$$

Thus, amount of an ordinary annuity of Rs. R payable at the end of every period for 'n' periods at a rate of i per period is given by

$$\boxed{A = R\left[\frac{(1+i)^n - 1}{i}\right]}$$

Remark : When the periodic payment is Re. 1, then amount of annuity is given by $\frac{(1+i)^n - 1}{i}$. This quantity is denoted by $s_{\overline{n}|i}$.

$$\therefore \quad S = R.\, s_{\overline{n}|i} \text{ where } s_{\overline{n}|i} = \frac{(1+i)^n - 1}{i}$$

$s_{\overline{n}|i}$ is read as "s sub n at the rate i".

The values of $s_{\overline{n}|i}$ for various n and i are given in table III.

CONTINUOUS COMPOUNDING

The amount A of an ordinary annuity in which Rs. R are paid each year for t (= n) years at the rate of interest of r per annum compounded continuously is given by

$$A = \int_0^{n(=t)} R\, e^{rt}\, dt$$

$$\Rightarrow \qquad \boxed{A = \frac{R}{r}(e^{rt} - 1)}$$

TOTAL INTEREST EARNED

A is the amount at the end of the term and Rs. R is the periodic payment.

If there are n payments, then the total payments paid = nR.

We know that the amount is the sum of total payments and the interest accumulated on every payment.

$\therefore$ The total interest earned is given by

$$\boxed{I = A - nR}$$

Example 1 : *Find the amount of an ordinary annuity of Rs. 5,000 payable at the end of each year for 3 years at 8% per annum compounded annually.*

Solution **:** Amount A of an ordinary annuity of Rs. R per period for n periods at the rate of i per period is given by

$$A = R\left(\frac{(1+i)^n - 1}{i}\right)$$

Here, periodic payment R = Rs. 5,000

rate of interest $r = 8\% = 0.08$

Interest is compounded annually

$\therefore \quad i = r = 0.08$ and $n = t = 3$ years

$$\therefore \quad A = 5{,}000\left(\frac{(1.08)^3 - 1}{0.08}\right)$$

$$A = 5{,}000\left(\frac{1.259712 - 1}{0.08}\right)$$

$$A = 5{,}000\left(\frac{0.259712}{0.08}\right) = 5{,}000\ (3.2464)$$

$$A = 16{,}232$$

$\therefore$ The amount is Rs. 16,232

Alternatively,

$$A = R\ s_{\overline{n}|i}$$

Here $A = 5{,}000\ S_{\overline{3}|0.08}$

$$A = 5{,}000\ (3.2464000) = 16{,}232$$

$\therefore$ The amount is Rs. 16,232

Example 2 : *Find the amount of an ordinary annuity if payment of Rs. 500 is made at the end of every quarter for 10 years at the rate of 8% per annum compounded quarterly.*

Solution : The amount A of a ordinary annuity of Rs. R per period for n periods at the rate of i per period is given by

$$A = R\left[\frac{(1+i)^n - 1}{i}\right]$$

Here Periodic payment R = Rs. 500

Rate of interest $r = 8\% = 0.08$

Interest is compounded quarterly

$$\therefore \ i = \frac{r}{4} = \frac{0.08}{4} = 0.02 \qquad \text{and} \qquad n = 4 \times 10 = 40$$

$$\therefore \quad A = 500\left[\frac{(1.02)^{40} - 1}{0.02}\right]$$

Let $x = (1.02)^{40}$

then $\log x = 40 \log 1.02$

$$\log x = 40 \times 0.0086 = 0.344$$

$$x = \text{antilog}\ (0.344) = 2.2080$$

$$\therefore \ A = 500\left[\frac{2.2081 - 1}{0.02}\right]$$

$$A = 500\left[\frac{1.2080}{0.02}\right] = 500\ (60.4)$$

$$A = 30{,}200$$

$\therefore$ The amount is Rs. 30,200.

Example 3: *A person decides to put aside Rs. 100 at the end of every month in a money market fund that pay interest at the rate of 8% compounded monthly. After making 12 deposits, how much money does he have ?*

Solution : Since the payments are made at the end of every month, it is an ordinary annuity.

The amount of an ordinary annuity of Rs. R per period for n periods at a rate of i per period is given by

$$A = R\left[\frac{(1+i)^n - 1}{i}\right]$$

Here, R = Rs. 100

Rate of interest $r = 8\% = 0.08$

Interest is compounded monthly.

$$\therefore \quad i = \frac{r}{2} = \frac{0.08}{12} = 0.00667 \text{ and } n = 12$$

$$n = 12$$

$$\therefore \quad A = 100\left[\frac{(1.00667)^{12} - 1}{0.00667}\right]$$

$$A = 100\left[\frac{1.0830425 - 1}{0.00667}\right]$$

$$A = 100\left[\frac{0.0830425}{0.00667}\right] = 100\,(12.4501)$$

$$A = 1245.01$$

$\therefore$ The amount is Rs. 1,245.

Example 4: *At 6 months interval, Mr. X deposited Rs. 100 in a savings A/c which carries interest at 10% p.a. compounded semi-annually. The first deposit was made when his son was 6 months old and last deposit was made when his son was 8 years old. The money remained in the account and was presented to his son when he was 10 years old. What much did he received ?*

Solution. Rs. 100 was deposited at 6 months interval for 8 years at 10%

i.e. R = Rs. 100

Rate of interest $r = 10\% = 0.10$

Interest is compounded semi-annually.

$$i = \frac{r}{2} = 0.05 \quad \text{and} \quad n = 8 \times 2 = 16$$

The amount of an ordinary annuity is given by

$$A = R\left[\frac{(1+i)^n - 1}{i}\right]$$

$\therefore$ Amount at the end of 8 years is

$$A = 100\left[\frac{(1.05)^{16}-1}{0.05}\right]$$

Let $x = (1.06)^{16}$

$$\log x = 16 \log 1.05$$

$$= 16 \times 0.0212 = 0.339$$

$$x = \text{antilog}\,(0.339) = 2.183$$

$$A = 100\left[\frac{2.183-1}{0.05}\right]$$

$$= 100\left[\frac{1.183}{0.05}\right]$$

$$= 100\,(23.66) = 2{,}366$$

This will remain in the account for 2 more years.

$\therefore$ $n = 2 \times 2 = 4$ and $i = 0.05$

In case of compound interest, the amount is given by

$$A = P\,(1+i)^n$$

$\therefore$ Amount at the of 10th year is :

$$A = 2{,}366\,(1.05)^4$$

$$= 2{,}366 \times 1.2155$$

$$= 2{,}875$$

$\therefore$ The amount is Rs. 2,875.

Example 5: *If a bank pays 6% interest compounded quarterly, what equal deposits have to be made at the end of each quarter for 3 years to have Rs. 1,500 at the end of 3 years ?*

Solution : Given that Amount A = Rs. 1,500.

Since the payment are made at the end of quarter, it is the case of ordinary annuity.

Rate of interest $r = 6\% = 0.06$

$$i = \frac{0.06}{4} = 0.015 \quad \text{and} \quad n = 3 \times 4 = 12$$

Let Rs. R be the equal payment.

The amount of an ordinary annuity is given by

$$A = R\left[\frac{(1+i)^n - 1}{i}\right]$$

$$1{,}500 = R\left[\frac{(1.015)^{12}-1}{0.015}\right]$$

$$1{,}500 = R\left(\frac{1.1956-1}{0.015}\right)$$

$$1{,}500 = R \times 13.04$$

$$R = \frac{1{,}500}{13.04} = 115$$

$\therefore$ The quarterly payment is Rs. 115.

Example 6: *A company set aside a sum of Rs. 4,500 at the end of each year for 7 years to pay off a debenture issue of Rs. 40,000. If the fund accumulates at 9% compounded annually, find the surplus after full redemption of the debenture issue.*

Solution : Here, periodic payment R = Rs. 4,500.

Interest is compounded annually.

$\therefore$ $i = r = 9\% = 0.09$ and $n = t = 7$.

Since the sum R is set aside at the end of every year, it is an ordinary annuity.

The amount of an ordinary annuity is given by

$$A = R\left[\frac{(1+i)^n - 1}{i}\right]$$

$$\therefore \quad A = 4{,}500\left[\frac{(1.09)^7 - 1}{0.09}\right]$$

$$\Rightarrow \quad A = 4{,}500\left[\frac{1.828039 - 1}{0.09}\right]$$

$$\Rightarrow \quad A = 4{,}500\left[\frac{0.828039}{0.09}\right] = 41{,}401.96$$

$\therefore$ Amount Accumulated = Rs. 41,401.96

Value of debenture = Rs. 40,000

$\therefore$ The surplus = Rs. (41,401.96 – 40,000) = Rs. 1,401.96

$\therefore$ The surplus is Rs. 1,401.96

Example 7 : *Mr. X deposits Rs. 5,000 at the end of every 6 months into his saving bank account. The bank calculates interest at a rate of 11% per annum compounded semi-annually.*

(*a*) *What account will be accumulated at the end of 12 years ?*

(*b*) *What is the amount he will earn as interest ?*

Solution :

(*a*) Since the deposit is made at the end of every 6 months, it is an ordinary annuity.

Given that, R = Rs. 500

Rate of interest, $r = 11\% = 0.11$

Interest compounded semi-annually.

$$\therefore \quad i = \frac{r}{2} = \frac{0.11}{2} = 0.055 \text{ and } n = 2 \times 12 = 24$$

Amount of an ordinary annuity is given by

$$A = R\left[\frac{(1+i)^n - 1}{i}\right]$$

$$\therefore \quad A = 5{,}000\left[\frac{(1.055)^{24} - 1}{0.055}\right]$$

$$A = 5{,}000\left[\frac{3.61459 - 1}{0.055}\right]$$

$$A = 5{,}000\left[\frac{2.61459}{0.55}\right] = 5{,}000\,[47.538]$$

$$A = 2{,}37{,}690$$

$\therefore$ The amount after 12 years is Rs. 2,37,690

(*b*) The total interest earned is given by

$$I = A - nR$$

$$\therefore \quad I = \text{Rs. } (2{,}37{,}690 - 24 \times 5{,}000)$$

$$I = (2{,}37{,}690 - 1{,}20{,}000) = \text{Rs. } 1{,}17{,}690$$

$\therefore$ The interest earned is Rs. 1,17,690.

Example 8 : *Find the number of years for which an annuity of Rs. 1,500, payable per annum accumulates to Rs. 30,000 at the rate of 9% effective.*

Solution : Since the payment date is not mentioned, it is assumed to be an ordinary annuity.

Periodic Payment, R = Rs. 1,500

Accumulate amount, A = Rs. 30,000

Rate of interest $i = r$ = 9% effective = 0.09

The problem is to find 'n'

The amount of an ordinary annuity is given by

$$A = R\left[\frac{(1+i)^n - 1}{i}\right]$$

$$\therefore \quad 30{,}000 = 1{,}500\left[\frac{(1.09)^n - 1}{0.09}\right]$$

$$\Rightarrow \quad (1.09)^n - 1 = \frac{30{,}000 \times 0.09}{1{,}500} = 1.8$$

$$\Rightarrow \quad (1.09)^n = 2.8$$

Taking log on both sides, we get

$$n \log 1.09 = \log 2.8$$

$$n = \frac{\log (2.08)}{\log (1.09)} = \frac{0.4472}{0.374} = 11.957$$

$\therefore$ The number of years is approximately 12 years.

Example 9 : *A deposits annually Rs. 200 15th for 10 years, the first deposit being made one year from now; and after 10 years the annual deposit is enhanced to Rs. 300 per annum. Immediately after depositing the 15th payment he closes his account. What is the amount payable to him, if the interest is allowed at 6% effective ?*

Solution : Since the deposits are made at the end of every year, it is an ordinary annually.

This problem contain two annuities : An annuity of Rs. 200 for 10 years and thereafter an annuity of Rs. 300 for 5 years.

Period in years

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	200	200	200	200	200	200	200	200	200	200	300	300	300	300	300

Payments (in Rupees)

Consider the annuity of Rs. 200 for 10 years :

Periodic Payment $\quad R = \text{Rs. } 200$

Rate of interest $\quad r = 6\% = 0.06$

Interest is compounded annually.

$\therefore \quad i = r = 0.06 \quad \text{and} \quad n = t = 10$

The amount of an ordinary annuity is given by

$$A = R\, s_{\overline{n}|i}$$

$$\therefore \quad A = 200\, s_{\overline{10}|0.06}$$

$$\therefore \quad A = 200\,(13.1807949) = 2{,}636.16$$

Since the annuity ends at the end of 15th year, this amount will remain in the account for 5 more years, earning compound interest.

$\therefore$ The amount accumulate at the end of 15th year

$$= 2{,}636.16\,(1.06)^5 \qquad \because A = P\,(1+i)^n$$

$$= 2{,}636.16\,(1.338226)$$

$$= \text{Rs. } 3{,}527.78 \qquad \text{...(1)}$$

Now consider the annuity of Rs. 300 for 5 years starting form 11th year :

$$R = \text{Rs. } 300 \text{ and } n = 5$$

Amount at the end of 15th year is

$$= 300\, s_{\overline{5}|0.06}$$

$$= 300\,(5.6370930)$$

$$= \text{Rs. } 1{,}691.13 \qquad \text{...(2)}$$

The required amount after 15 year is

$$(1) + (2) = \text{Rs. } 3{,}527.78 + \text{Rs. } 1{,}691.13$$

$$= \text{Rs. } 5{,}218.91$$

$\therefore$ The total amount is Rs. 5,218.91

Example 10 : *An account fetches interest at 5% per annum compounded continuously. An individual deposits Rs. 1,000 each year in the account. Find how much will be in the account after 5 years.*

Solution : Given that R = Rs. 1,000

Rate of interest $r = 5\% = 0.05$

No. of years $t = 5$

$\therefore \quad rt = 0.05 \times 5 = 0.25$

Interest is compounded continuously.

The amount of annuity is given by

$$A = \frac{R}{r}[e^{rt} - 1]$$

$$\therefore \quad A = \frac{1{,}000}{0.05}[e^{0.25} - 1]$$

$$A = 20{,}000\,[1.284 - 1]$$

$$A = 20{,}000\,(0.284) = 5{,}680$$

$\therefore$ The amount is Rs. 5,680.

Example 11 : *Find*

(*a*) *How much should be deposited in a bank each year in order to accumulate Rs. 50,000 in 6 years, if the interest is calculated at a rate of 6% per annum compounded continuously ?*

(*b*) *How long it will take for an annuity of Rs. 5,000 to amount to Rs. 91,091 at a rate of 11% per annum compounded continuously ?*

Solution **:**

(*a*) Let Rs. *R* be deposited each year.

Accumulated amount A = Rs. 50,000

Rate of interest $r = 6\% = 0.06$

Number of years $t = 6$ years

$\therefore \; rt = 0.06 \times 6 = 0.36$

Amount of an a annuity is given by

$$A = \frac{R}{r}[e^{rt} - 1]$$

$$50{,}000 = \frac{R}{0.06}[e^{036} - 1]$$

$$\Rightarrow \quad 3{,}000 = R\,(e^{0.36} - 1)$$

Let $x = e^{0.36}$

$$\log x = 0.36 \log e$$

$$\log x = 0.36 \times 0.4343 = 0.156348$$
$$x = \text{antilog}\,(0.1563) = 1.4332$$
$$\therefore \quad 3{,}000 = \text{Rs.}\,(1.4332 - 1)$$
$$\Rightarrow \quad 3{,}000 = \text{R}\,(0.4332) \Rightarrow \text{R} = 6{,}925.21$$

$\therefore$ The periodic deposit = Rs. 6,925.21

(*b*) Let '*t*" be the number of years

Given that R = Rs. 5,000 and A = Rs. 91,091

Rate of interest $r = 11\% = 0.,011$

Interest is compounded continuously.

Amount of the annuity is given by

$$\text{A} = \frac{\text{R}}{r}[e^{rt} - 1]$$
$$\therefore \quad 91{,}091 = \frac{5{,}000}{0.11}[e^{011t} - 1]$$
$$\Rightarrow \quad e^{0.11t} - 1 = 2.004002$$
$$\Rightarrow \quad e^{0.11t} = 3.004002$$

Taking long on both sides we get

$$0.11t \log e = \log 3.004002$$
$$t = \frac{\log 3.004002}{0.11 \log e}$$
$$t = \frac{0.47769}{0.11 \times 0.4343}$$
$$t = 10$$

$\therefore$ The required number of years is 10 years.

EXERCISES

1. Find the amount of an ordinary annuity of Rs. 1,000 payable at the end of each year for 3 years at a rate of 10% per annum compounded annually. **[Ans.** Rs. 3,310]
2. Find the amount of an ordinary annuity of Rs. 3,000 at the rate of 5% per annum compounded annually for 8 years. **[Ans.** Rs. 28,628]
3. Find the amount of the following annuities :
   (*a*) Rs. 200 per year for 5 years at 8% per year compounded annually.
   (*b*) Rs. 500 payable at the end of each year for 14 years at 5% effective rate of interest. **[Ans.** (*a*) Rs. 1,173; (*b*) Rs. 9,810
4. Find the amount of an annuity consisting of payments of Rs. 600 at the end of every three months for 3 years at the rate of 8% compounded quarterly. **[Ans.** Rs. 8,047.25]
5. Find the amount of an ordinary annuity, if payment of Rs. 200 is made at the end of every month for 3 years at the rate 18% compounded monthly. **[Ans.** Rs. 9,023.10]

6. For the purpose of his daughter's marriage, a man deposits Rs. 3,000 at the end of each 6 month period in a fund paying 8% per year compounded semi-annually. Find the amount accumulated at the end of 18 years. [**Ans.** Rs. 2,32,795]
7. Find the amount of an ordinary annuity of 12 monthly payments of Rs. 1,000 that earn interest at 12% per year compounded monthly. [**Ans.** Rs. 12,682.50]
8. A man deposited Rs. 500 at the end of each year for 5 years. He made his first payment at the end of 1990 and the last payment at the end of 1994. How much should have been there on 31^{st} December 1994, if the rate of interest was 10% compounded annually ? [**Ans.** Rs. 30,525.50]
9. A man plans to deposit a sum of Rs. 750 in a savings account at the end of this month and the same amount at the end of each following months. To what sum will the investment grow at the end of 5 years, if the rate of interest is 5% per annum compounded monthly ? [**Ans.** Rs 51,004.56]
10. Find the amount of the following ordinary annuities :

    (*a*) Rs. 1,000 a year for 5 years at 7% per year compounded annually.

    (*b*) Rs. 500 per quarter for 10 years at 8% per year compounded quarterly.

    (*c*) Rs. 4,000 each six months for 15 years at 5% per year compounded semi-annually.

    (*d*) Rs. 40 each month for five years at 6% per year compounded monthly.

    [**Ans.** (*a*) Rs. 5,750.74; (*b*) Rs. 30,200.99; (*c*) Rs. 1,75,610.81; (*d*) Rs. 2,790.80]
11. A person wishes to deposit Rs. 2,500 per year in a savings account which earns interest of 10% per year compounded annually. Assume the first deposit is made at the end of this current year and addition deposits at the end of each following year.

    (*a*) To what sum will the investment grow at the time of the 20^{th} deposit ?

    (*b*) How much interest will be earned ? [**Ans.** (*a*) Rs. 1,43,187.50; (*b*) Rs. 93,187.50]
12. A company sets aside a sum of Rs. 5,000 annually for 10 years to payoff a debenture issue of Rs. 60,000. If the fund accumulates at 5% per annum compounded annually, find the surplus after full redemption of the debenture issue. [**Ans.** Rs. 2,889.46]
13. A company set aside a sum of Rs. 4,000 at the end of each year for 7 years to payoff a debenture issue of Rs. 35,000. If the surplus after full redemption of the debenture issue. [**Ans.** Rs. 2,948.68]
14. An annuity consists of equal payments at the end of each month for 2 years is to be purchased for Rs. 2,000. If the interest rate is 6% per annum compounded monthly, how much is each payment ? [**Ans.** Rs. 78.64]
15. For the purpose of education of his child, Ram want to accumulate Rs. 50,000 by making equal payments at the end of each quarter for next 5 years. What will be the size or these investments, if money is worth 6% converted quarterly ?

    [**Ans.** Rs. 2,162.29]
16. Mr. Shyam deposits an amount of Rs. 2,000 at the end of every 3 months into his savings bank account. The bank calculates interest at the rate of 7% per annum compounded semi-annually.

(*a*) What amount will be accumulated at the end of 10 years in his account ?

(*b*) What is the total interest earned from these deposits ?

[**Ans.** (*a*) Rs. 1,14,468.27; (*b*) Rs. 34,468.27]

17. How much money must be deposited at the end of each quarter if the objective is to accumulate Rs. 5,00,000 after 5 years ? Assume interest is earned at the rate of 10 per cent per year compounded quarterly. How much interest will be earned ?

[**Ans.** Rs. 19,573.56; Rs. 1,08,528.08]

18. A company must accumulate Rs. 12,000 at the end of 10 years to replace certain component of its machine. What sum must the company invest at the end of each year, assuming interest is earned at the rate of 4% per year compounded annually ?

[**Ans.** Rs. 1,002.08]

19. If a bank pays interest at the rate of 5% per annum compounded annually, what equal deposits have to be made at the end of ever year to have Rs. 1,00,000 at the end of 10 years. How much interest will be earned on these deposits ?

[**Ans.** Rs. 7,950.46; Rs. 20,495.40]

20. To save for his son's education, Mr. Singh deposited Rs. 1,000 at the end of each 6 months period into a savings account paying 4% interest compounded semi-annually. The first deposit was made when his son was 6 months old and the last deposit was made when his son was 21 years old. The money was kept in the account and was presented to his son on his 25th birth day. How much did he receive ?

[**Ans.** Rs. 75,996.43]

21. An amount of Rs. 1,500 is deposited in a savings bank account at the end of every 3 months for a period of 8 years and no payment is made there after. But the accounts will be closed at the end of 10th year. Assuming that the bank calculates interest at a rate of 6% per annum compounded quarterly, calculate the amount which will be received at the time of closure of the account ? [**Ans.** Rs. 68,752.58]

22. What amount should be set aside at the end of each year to accumulate Rs. 1,48,970 at the end of 8 years at 5% effective. [Given $(1.05)^8 = 1.468$] [**Ans.** Rs. 15,915.60]

23. A deposits annually Rs. 200 for 10 years, the first deposit being made one year from now. After 10 years the annual deposit is enhanced to Rs. 300 per annum,. At the end of 20th year, he closes his account. What is the amount payable to him at the end of 20th year, if the interest is allowed at 9% effective ? [**Ans.** Rs. 8,659.19]

24. A sum of Rs. 500 is deposited in a bank at the end of every 6 months. What is the amount to the credit of the depositor at the end of 10 years, if interest is credited to the account at the rate of 6% for the first five years and 8% thereafter ?

[**Ans.** 14,487.72]

25. A person deposits Rs. 2,187 at the end of ever 3 months in a fund created to fulfil his obligation of Rs. 50,000 which is due after a certain number of years. If the money is worth 6% per annum compounded quarterly, find the number of years.

[**Ans.** 5 years]

26. The amount of an annuity of Rs. 500 payable at the end of each year for 12 years is Rs. 8,433. If the interest is compounded at 6% effective, find the number of years.

[**Ans.** 10 years]

27. Mr. X has decided to invest Rs. 500 at the end of each year. He did so for 7 years. Due to some unavoidable situations, he could not make the payments for the next 4 years. He again invested Rs. 500 per annum for the next 4 years beginning from the end of the 12^{th} year. Find the amount to his credit at the end of the 15^{th} year assuming interest at effective rate of 9% per annum. [**Ans.** Rs. 11,452.75]

28. An annuity is payable for 15 years certain, the first payment falling due at the end of first year. The annuity is payable at the rate of Rs. 5,000 per annum during the first 10 years and Rs. 3,000 per annum during the remaining 5 years. Calculate the future value of the annuity on the basis of interest at 4% per annum.

[**Ans.** Rs. 89,285.29]

29. If Rs. 500 is deposited each year in a savings bank account paying 5.5% per annum compounded continuously, how much will be in the account after 4 years ?

[**Ans.** Rs. 2,237.27]

30. A bank pays interest at the rate of 8% per annum compounded continuously. Find how much should be deposited in the bank each year in order to accumulate Rs. 10,000 in 5 years. [**Ans.** 1,626.67]

31. An account fetches interest at 10% per annum compounded continuously. A man deposits Rs. 600 at the end of each year in his account. If he wishes to accumulate a sum of Rs. 4,932, how many consecutive deposits he has to make in his account ?

[**Ans.** 6]

32. Find the amount of an annuity of Rs. 1,200 at the end of every year for 3 years at a rate of 5% per annum compounded continuously. [**Ans.** Rs. 3,883.35]

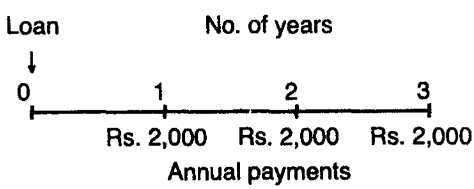
33. How much need to be saved each year in a savings account paying 6% per annum compounded continuously in order to accumulate Rs. 6,000 in three years ?

[**Ans.** Rs. 1,825.56]

PRESENT VALUE OF AN ORDINARY ANNUITY

We have so far studied annuity problems in which payments were made with the objective of obtaining a certain lump sum at a future date. In this section, we shall study problems in which a lump sum is invested at year zero at a certain rate of compound interest so that a series of equal payments can be obtained over some future period of time. The lump sum investment is generally known as the present value of the ordinary annuity. The present value of an ordinary annuity is the sum of the present values of each instalment.

For example, suppose that a person purchases a personal loan from a bank at present at 6% per annum compounded annually and agrees to pay Rs. 2,000 at the end of every year for 3 years. This is shown in the following figure :



The present value of the first instalment of Rs. 2,000 will be Rs. 2,000 $(1.06)^{-1}$, *i.e.*, Rs. 1,886.79.

The present value of the second instalment of Rs. 2,000 will be Rs. 2,000 $(1.06)^2$, *i.e.*, Rs. 1,779.99.

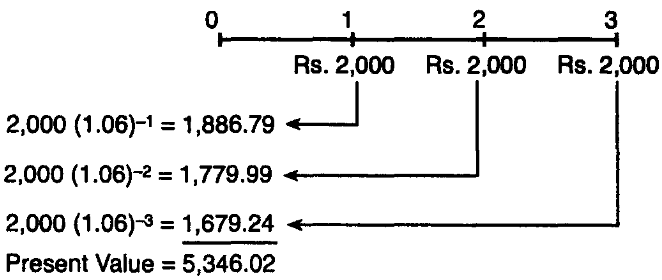
The present value of the third (final) instalment of Rs. 2,000 will be Rs. 2,000 $(1.06)^{-3}$, *i.e.*, Rs. 1,679.24.

$\therefore$ The Present Value of this annuities

$$P = \text{Rs. } 1{,}886.79 + \text{Rs. } 1{,}779.99 + \text{Rs. } 1{,}679.24 = \text{Rs. } 5{,}346.02$$

$$P = \text{Rs. } 5{,}346 \text{ (approximately)}$$

This is shown in the following figure :



Thus, we say that a loan of Rs. 5,346 at a rate of 6% per annum compounded annually is repaid by three annual instalments of Rs. 2,000.

The benefits are receivable in the form of either money such as loan, etc. or asset such as house, etc. at the beginning of the annuity. And this is repaid through a certain number of equal instalments. Therefore every intalment contains some certain amount of interest.

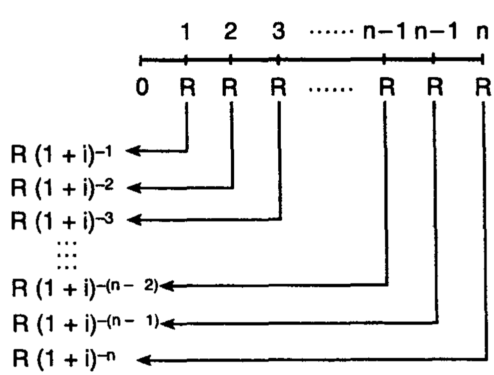
Therefore, the present value of an annuity will be always less than the sum of the all periodic payments.

Now we derive the formula for the present value of an ordinary annuity.

Consider an annuity of n payments of Rs. R each, where the interest rate is 'i' per period.

The first payment of Rs. R is made at the end of first period. So the present value of this Rs. R is calculated for one period. Its present value is $R(1 + i)^{-1}$.

The second payment of Rs. R is made after 2 payment periods and the present value of this Rs. R is $R\,(1 + i)^{-2}$ and so on. This is illustrated in the following figure :



Thus,

The present value of 1st instalment, $P_1 = R\,(1+i)^{-1}$

The present value of 2nd instalment, $P_2 = R\,(1+i)^{-2}$

⋮ ⋮ ⋮ ⋮ ⋮ ⋮ ⋮

The present value of $(n-1)$st instalment, $P_{n-1} = R(1+i)^{-(n-1)}$

The present value of nth instalment, $P_n = R(1+i)^{n}$

The present value of this ordinary annuity in given by

$$P = P_1 + P_2 + P_3 + \ldots\ldots P_{n-1} + P_n$$

$$\Rightarrow \quad P = R\,(1+i)^{-1} + R\,(1+i)^{-2} + \ldots\ldots + R\,(1+i)^{-(n-1)} + R\,(1+i)^{-n}$$

$$\Rightarrow \quad P = R\,[(1+i)^{-1} + (1+i)^{-2} + \ldots\ldots + (1+i)^{-(n-1)} + (1+i)^{-n}]$$

$$\Rightarrow \quad P = R\,(1+i)^{-1}\,[1 + (1+i)^{-1} + \ldots + (1+i)^{-(n-2)} + (1+i)^{-(n-1)}]$$

$$P = R\,(1+i)^{-1}\left[\frac{1-(1+i)^{-n}}{1+(1+i)^{-1}}\right] \qquad \because 1 + r + r^2 \ldots\ldots r^{n-1} = \frac{1-r^n}{1-r}$$

$$P = \frac{R}{1+i}\left[\frac{1-(1+i)^{-n}}{1-(1+i)^{-1}}\right] \qquad \text{where } r < 1$$

$$P = R\left[\frac{1-(1+i)^{-n}}{(1+i)-1}\right]$$

$$P = R\left[\frac{1-(1+i)^{-n}}{i}\right]$$

Thus, the present value P of an ordinary annuity of Rs. R per payment period for n periods at the rate of i per period is given by

$$\boxed{P = R\left[\frac{1-(1+i)^{-n}}{i}\right]}$$

Remark : When the periodic payment is Re. 1, then the present value of an ordinary annuity is given by $\frac{1-(1+i)^{-n}}{i}$. This quantity is denoted by $a_{\overline{n}|i}$

$$\therefore \quad P = R\,a_{\overline{n}|i} \text{ where } a_{\overline{n}|i} = \frac{1-(1+i)^{-n}}{i}$$

The values of $a_{\overline{n}|i}$ for various n and i are given in the table IV.

CONTINUOUS COMPOUNDING

The present value P of an ordinary annuity in which Rs. R are paid for each year for n (= t) years at the rate of r per annum compounded continuously is given by

$$P = \int_0^{n\,(=t)} R\,e^{-rt}\,dt$$

$$P = R \int_0^{n\,(=t)} e^{-rt}\,dt$$

$$P = R \cdot \left.\frac{e^{-rt}}{-r}\right]_0^t$$

$$P = \frac{R}{-r}[e^{-rt} - e^0] = \frac{R}{-r}[e^{-rt} - 1]$$

$$\Rightarrow \qquad P = \frac{R}{r}[1 - e^{-rt}]$$

Thus, the present value in this case is given by

$$\boxed{P = \frac{R}{r}[1 - e^{-rt}]}$$

TOTAL INTEREST PAID

Let P be the present value of an ordinary annuity and Rs. R, the periodic payment.

If there are 'n' payments, then the total payments made = nR.

We know that every periodic payment contains interest and the present value is the total of the instalments paid minus the total interest

i.e. $\quad P = nR - I$

$$\boxed{I = nR - P}$$

Example 12 : *Find the present value of an annuity of Rs. 2,000 payable at the end of each year for 10 years, if money is worth 4% effective.*

Solution **:** Present value of an ordinary annuity is given by

$$P = R\left[\frac{1-(1+i)^{-n}}{i}\right]$$

Given that R = Rs. 2,000

Interest is compounded annually

$\because \quad i = r = 4\% = 0.04$ and $n = 10$

$$\therefore \quad P = 2{,}000\left[\frac{1-(1.04)^{-10}}{0.04}\right]$$

$$P = 2{,}000\left(\frac{1-0.67556417}{0.04}\right)$$

$$P = 2{,}000\left(\frac{0.32443583}{0.04}\right) = 2{,}000\ (8.110896)$$

$P = 16221.79$

$\therefore$ The present value is Rs. 16,221.79.

Example 13 : *An equipment is purchased on an instalment basis such that Rs. 500 is to be paid at the end of every month for five years. If the money is worth 6% per annum compounded monthly, find the purchasing price of the equipment.*

Solution : The purchasing price of the equipment is the present value of the annuity of Rs. 500.

Since the instalment is paid at the end of every month, it is an ordinary annuity.

Here R = Rs. 500

Rate of interest $r = 6\% = 0.06$

Interest is compounded monthly.

$$\therefore \quad i = \frac{r}{12} = \frac{0.06}{12} = 0.005 \quad \text{and} \quad n = 5 \times 12 = 60$$

Present value of an ordinary annuity is given by

$$P = R\left[\frac{1-(1+i)^{-n}}{i}\right]$$

$$\therefore \quad P = 500\left[\frac{1-(1.005)^{-60}}{0.005}\right]$$

$$P = 1{,}00{,}000\,[1-(1.005)^{-60}]$$

$$\text{Let} \quad x = (1.005)^{-60} = \frac{1}{(1.005)^{60}} = \frac{1}{y} \text{ (say)}$$

$$\log y = 60 \log (1.005)$$

$$\log y = 60 \times 0.00212 = 0.1272$$

$$\therefore \quad y = \text{antilog } 0.1272 = 1.3403$$

$$\therefore \quad x = \frac{1}{1.303} = 0.746102$$

$$\therefore \quad P = 1{,}00{,}000\,(1-0.746102)$$

$$P = 1{,}00{,}000\,(0.253898) = \text{Rs. } 25{,}389.80$$

$\therefore$ The purchasing price is Rs. 25,389.80

Example 14 : *A person buys a house for which he agrees to pay Rs. 5,00,000 now and Rs. 5,000 at the end of each month for 6 years. If the money is worth 12% compounded monthly, what is the cash price of the house ?*

Solution : The cash price of the house = Down payment + Present value of the annuity of Rs. 5,000. ...(1)

Cash down = Rs. 5,00,000

Since the instalments are paid at the end of every month, it is an ordinary annuity.

The present value of an ordinary annuity is given by

$$P = R\left[\frac{1-(1+i)^{-n}}{i}\right]$$

Here Instalment amount R = Rs. 5,000

Rate of interest $r = 12\% = 0.12$

Interest is compounded monthly.

$$\therefore \quad i = \frac{r}{12} = \frac{0.12}{12} = 0.01 \quad \text{and} \quad n = 6 \times 12 = 72$$

$$\therefore \quad P = 5{,}000\left[\frac{1-(1.01)^{-72}}{0.01}\right]$$

$$P = 5{,}000\left[\frac{1-0.48849606}{0.01}\right] = 5{,}000\left[\frac{0.51150394}{0.01}\right]$$

$$P = 5{,}000\,[51.150394] = \text{Rs. } 2{,}55{,}751.97$$

$\therefore$ (1) $\Rightarrow$ The cash price of the house = 5,00,000 + 2,55,751.97 = Rs. 7,55,751.97

$\therefore$ Price of the house is Rs. 7,55,751.97

Example 15 : *A person purchases a car on instalment basis such that instalments of Rs. 10,000 each payable at the end of every year for 20 years and a final payment of Rs. 50,000 one year later. If the rate of interest is 9% per annum compounded annually, find the cash down price of the car.*

Solution :

$$\text{Cash price of the car} = \left\{\begin{array}{c}\text{Present value of the} \\ \text{annuity of Rs. 10,000}\end{array}\right\} + \left\{\begin{array}{l}\text{Present value of the} \\ \text{final payment of Rs. 50,000}\end{array}\right\}$$

The present value of an ordinary annuity is given by

$$P = R\left[\frac{1-(1+i)^{-n}}{i}\right]$$

Here periodic payment, R = Rs. 10,000

Rate of interest $r = 9\% = 0.09$

Interest is compounded annually

$\therefore \quad i = r = 0.09 \quad$ and $\quad n = 20.$

$$\therefore \quad P = 10{,}000\left[\frac{1-(1.09)^{-20}}{0.09}\right]$$

$$P = 10{,}000\left[\frac{1-0.17843085}{0.09}\right]$$

$$P = 10{,}000\left[\frac{0.82156915}{0.09}\right] = 10{,}000\,(9.128546)$$

$$P = \text{Rs. } 91{,}285.46 \qquad \ldots(1)$$

Final payment of Rs. 50,000 is paid at the end of 21^{st} year.

Present value of this final payment

$$= 50,000\,(1.09)^{-21} \qquad \because\; P = A(1+i)^{-n}$$

$$= 50,000\,(0.163698) = \text{Rs. } 8,184.90 \qquad \ldots(2)$$

Cash price of the car

$$(1) + (2) = \text{Rs. } 91,285.46 + \text{Rs. } 8,184.90$$

$$= \text{Rs. } 99,470.36$$

$\therefore$ Price of the car is Rs. 99,470.36.

Example 16 : *A loan of Rs. 30,000 is to be repaid with interest at 6% per annum compounded half yearly in equal instalments payable at the end of every 6 months for 10 years. What is the size of each instalment ? What amount will be paid as total interest for this loan ?*

Solution :

Let R be the size of each instalment.

Present Values or loan amount = Rs. 30,000

Rate of interest r = 6% = 0.06

Interest is compounded semi-annually.

$$\therefore \qquad i = \frac{r}{2} = \frac{0.06}{2} = 0.03 \quad \text{and } n = 2 \times 10 = 20$$

Present value of an ordinary annuity is given by

$$P = R\left[\frac{1-(1+i)^{-n}}{i}\right]$$

$$\therefore \qquad 30,000 = R\left[\frac{1-(1.03)^{-20}}{0.03}\right]$$

$$\Rightarrow \qquad 30,000 = R\left[\frac{1-0.55367575}{0.03}\right]$$

$$\Rightarrow \qquad R = \frac{30,000 \times 0.03}{0.44632425}$$

$$R = 2,016.47$$

$\therefore$ The size of each instalment is Rs. 2,016.47

The total interest paid is given by

$$I = nR - P$$

$$I = (20 \times 2016.47) - 30,000$$

$$I = 40329.40 - 30,000 = 10,329.40$$

$\therefore$ The total interest paid is Rs. 10,329.40

Example 17 : *What will be the present value of a continuous income stream of Rs. 3,500 per annum for four years if it is discounted continuously at the rate of 5% per year ?*

Solution : The present value of an ordinary annuity when interest is compounded continuously is given by

$$P = \frac{R}{r}[1 - e^{-rt}]$$

Here periodic payment R = Rs. 3,500

Rate of interest $r = 5\% = 0.05$

No. of years $t = 4$. $\therefore$ $rt = 0.05 \times 4 = 0.2$

$$\therefore \quad P = \frac{3,500}{0.05}[1 - e^{-0.2}]$$

$$P = 70,000\,[1 - 0.81873]$$

$$P = 70,000\,[0.18127] = 12,688.90$$

$\therefore$ The present value is Rs. 12,688.90.

Example 18. *A person deposits his whole fortune of Rs. 20,000 in a bank and settles to withdraw Rs. 1,800 per year for his personal expenses. He begins to withdraw from the end of the first year and goes on spending at this rate. In how many years the entire amount will be ruined, if the interest is reckoned at 5% per annum (a) compounded annually ? (b) compounded continuously ?*

Solution : Since the amount is withdrawn at the end of every year, it is an ordinary annuity.

Let n be the number of years

Here present value, P = Rs. 20,000

Periodic payment R = Rs. 1,800

Rate of interest $r = 5\% = 0.05$

(*a*) *In case of interest compounded annually* :

$$i = r = 0.05$$

Present value of an ordinary annuity is given by

$$P = R\left[\frac{1-(1+i)^{-n}}{i}\right]$$

$$\therefore \quad 20,000 = 1,800\left[\frac{1-(1.05)^{-n}}{0.05}\right]$$

$$\Rightarrow \quad 1-(1.05)^{n} = \frac{20,000 \times 0.05}{1,800} = 0.5556.$$

$$\Rightarrow \quad (1.05)^{-n} = 1 - 0.5556 = 0.4444$$

$$\Rightarrow \quad (1.05)^{n} = \frac{1}{0.4444} = 2.2502$$

Taking log on both sides, we get

$$n \log (1.05) = \log 2.2502$$

$$n = \frac{\log 2.2502}{\log 1.05}$$

$$n = \frac{0.35218}{0.02119} = 16.62$$

$\therefore$ The required number of years is 16.62 years.

(b) *In case of interest compounded continuously* :

Present value of an ordinary annuity is given by

$$P = \frac{R}{r}[1 - e^{-rt}]$$

$$\therefore \quad 20{,}000 = \frac{1{,}800}{0.05}[1 - e^{-0.056}]$$

$$\Rightarrow 1 - e^{-0.05t} = \frac{20{,}000 \times 0.05}{1{,}800} = 0.5556$$

$$\Rightarrow e^{-0.05t} = 1 - 0.5556 = 0.4444$$

$$\Rightarrow e^{0.05t} = \frac{1}{0.4444} = 2.2502$$

Taking log on both sides,

$$0.05t \log e = \log 2.2502$$

$$\therefore \quad t = \frac{\log 2.2502}{0.05 \times \log e}$$

$$t = \frac{0.35218}{0.05 \times 0.4343} = 16.22$$

$\therefore$ The required number of years is 16.22 years.

Example 19 : *What should be the quarterly sales volume of a company if it desires to earn an 8% annual return convertible quarterly on its investment of Rs. 1,50,000 ? Quarterly costs are Rs. 2,000. The investment will have ten years life with no scrap value.*

Solution : Let Rs. x be the quarterly sales volume.

Quarterly cost = Rs. 2,000

$\therefore$ Quarterly return, R = Rs. $(x - 2{,}000)$

The present value or capital value = Rs. 1,50,000

Rate of interest $r = 8\% = 0.08$

No. of years $t = 10$

Interest is compounded quarterly,

$$\therefore \quad i = \frac{r}{4} = \frac{0.08}{4} = 0.02 \quad \text{and} \quad n = 4 \times 10 = 40$$

The present value of an ordinary annuity is given by

$$P = R\left[\frac{1 - (1 + i)^{-n}}{i}\right]$$

$$\therefore \quad 1{,}50{,}000 = (x - 2{,}000)\left[\frac{1-(1.02)^{-40}}{0.02}\right]$$

$$\Rightarrow \quad 1{,}50{,}000 = (x - 2{,}000)\left[\frac{1-0.45289042}{0.02}\right]$$

$$\Rightarrow \quad 1{,}50{,}000 = (x - 2{,}000)\,(27.355479)$$

$$\Rightarrow \quad x - 2{,}000 = \frac{1{,}50{,}000}{27.355479} = 5{,}483.36$$

$$\therefore \quad x = 7{,}483.36$$

∴ The sales volume is Rs. 7,483.36

Example 20 : *A series of 8 annual sums of money is payable, the first payment taking place at the end of one year from now. The first five payments are Rs. 300 each and the last three payments are Rs. 200 each. Find the present value of these payments, if money is worth 8% per annum compounded annually.*

Solution **:** We express the payment as a series of uniform payments. For this purpose, consider the first five payments of Rs. 300 each as made of two parts (Rs. 200 + Rs. 100)

(Period in years)

0	1	2	3	4	5	6	7	8
	200	200	200	200	200	200	200	200
	+100	+100	+100	+100	+100			

(Payments in Rupees)

Thus, we get two series of uniform payments : One Rs. 200 payable for 8 years and another Rs. 100 payable for 5 years.

∴ The required present value = Present value of Rs. 200 payable for 8 years + Present value of Rs. 100 payable for 5 years.

Since the payments are made at the end of every year, it is an ordinary annuity.

The present value of an ordinary annuity is given by

$$P = R\, a_{\overline{n}|i}$$

Here rate of interest $i = r = 8\% = 0.08$.

In case of Rs. 200 payable for 8 years.

$$R = \text{Rs. } 200 \text{ and } n = 8 \text{ years}$$

$$\therefore \quad P_1 = 200\, a_{\overline{8}|0.08}$$

$$P_1 = 200\,(5.74663894) = 1{,}149.33$$

$$P_1 = \text{Rs. } 1{,}149.33 \qquad \ldots(1)$$

In case of Rs. 100 payable for 5 years

$$R = \text{Rs. } 100 \quad \text{and} \quad n = 5$$

$$\therefore \quad P_2 = 100\, a_{\overline{5}|0.08}$$

$$P_2 = 100\,(3.99271004) = 399.27$$

$$\therefore \quad P_2 = \text{Rs. } 399.27 \qquad \ldots(2)$$

The required present value is :

$$(1) + (2) = 1{,}149.33 + 399.27 = 1{,}548.60$$

$\therefore$ The present value is Rs. 1,548.60

Example 21 : *A person purchased a house paying Rs. 5,00,000 cash down and promising to pay Rs. 15,000 every month for the next 5 years. The seller charges interest at 9% per annum compounded quarterly.*

(a) What was the cash price of the house ?

(b) If he missed the first 15 payments, what must he pay at the time of 16^{th} payment to bring him up to date ?

(c) After making 22 payments, he wishes to discharge his remaining indebtedness by a single payment at the time when 23^{rd} regular payment is due. What single payment must he pay ?

(d) If he missed the first 20 payments, what must he may when the 21^{st} is due to discharge his entire indeptness ?

Solution :

Here R = Rs. 15,000, $r = 9\% = 0.09$

Interest is compounded monthly.

$$\therefore \quad i = \frac{r}{12} = \frac{0.09}{12} = 0.0075 \quad \text{and} \quad n = 12 \times 5 = 60$$

(*a*) Cash price = down payment + present value of the annuity of Rs. 15,000

$= 5{,}00{,}000 + 15{,}000\, a_{\overline{60}|0.0075}$ $\qquad \because r = 0.09$

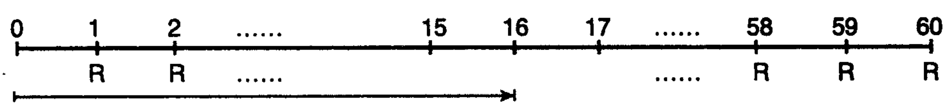
$= 5{,}00{,}000 + 15{,}000\,(48.17337352)$ $\qquad i = r/12 = 0.0075$

$= 5{,}00{,}000 + 722600.60$ $\qquad n = 12 \times 5 = 60$

$= 12{,}22{,}600.60$

$\therefore$ Cash price of the house is Rs. 12,22,600.60

(*b*)



If he missed the first 15 payment,

Amount payable at the 16^{th} payment = Future value of the annuity of Rs. 15,000 for 16 periods

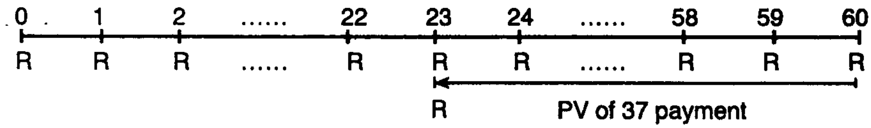
$= 15{,}000\, s_{\overline{16}|0.0075}$

$= 15{,}000\,(16.93228183)$

$= 2{,}53{,}984.23$

$\therefore$ The amount payable is Rs. 2,53,984.23

(*c*)



After making 22 payment,

Amount payable at the time of 23rd payment

= 23rd payment + Present value of remaining 37 (*i.e.* 60 – 23) payments.

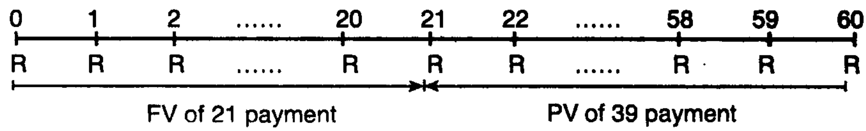
$= 15{,}000 + 15{,}000\ a_{\overline{37}|0.0075}$

$= 15{,}000 + 4{,}83{,}078.99$

$= 4{,}98{,}078.99$

∴ The amount payable is Rs. 4,98,078.99

(*d*)



If he missed the first 20 payments, to discharge his indebtness, amount payable at the time of 21st payment

= Future value of first 21 payment + Present value of remaining 39 payments

$= 15{,}000\ s_{\overline{21}|0.0075} + 15{,}000\ a_{\overline{39}|0.0075}$

$= 15{,}000\ [\, s_{\overline{21}|0.0075} + a_{\overline{39}|0.0075}\,]$

$= 15{,}000\ [22.65240312 + 33.70529048]$

$= 15{,}000\ [56.38769360]$

$= 8{,}45{,}365.41$

The amount payable is Rs. 8,45,365.41

Example 23 : *Prove that* $i = \dfrac{1}{a_{\overline{n}|i}} - \dfrac{1}{s_{\overline{n}|i}}$

***Solution* :**

$$\text{RHS} = \frac{1}{a_{\overline{n}|i}} - \frac{1}{s_{\overline{n}|i}}$$

$$= \frac{1}{\left(\dfrac{1-(i+i)^{-n}}{i}\right)} - \frac{1}{\dfrac{(1+i)^n - 1}{i}}$$

$$= \frac{i}{1-(1+i)^{-n}} - \frac{i}{(1+i)^n - 1}$$

$$= \frac{i}{1 - \dfrac{1}{(1+i)^n}} - \frac{i}{(1+i)^n - 1}$$

$$= \frac{i(1+i)^n}{(1+i)^n - 1} - \frac{i}{(1+i)^n - 1}$$

$$= \frac{i}{(1+i)^n - 1}((1+i)^n - 1)$$

$$= i = \text{RHS.}$$

Hence proved.

EXERCISES

1. Find the present value of each of the following ordinary annuities :
   (a) Rs. 5,000 a year for 10 years at 7% per year compounded annually.
   [**Ans.** Rs. 35,117.91]
   (b) Rs. 1,000 per quarter for 5 years at 8% per year compounded quarterly.
   [**Ans.** 16,351.43]
   (c) Rs. 2,000 every six months for 12 years at 5% per year compounded semi-annually. [**Ans.** Rs. 35,769.97]
2. An equipment is purchased on an instalment basis such that Rs. 1,200 is to paid at the end of every month for eight years. If the money is worth 9% per annum compounded monthly, find the purchasing price. [**Ans.** Rs. 81,910.13]
3. If money is worth 6% compounded once in two months, find the present value of an annuity whose annual payment is Rs. 1,800 which is payable once in two months for 5 years. [**Ans.** Rs. 52,117.57]
4. Find the present value of an annuity of Rs. 3,000 at the end of each year for 15 years, if money is worth 5% effective. [**Ans.** Rs. 31,138.97]
5. Find the present value of the following annuities :
   (a) An annuity of Rs. 1,500 payable at the end of each year for 9 years at 7% effective. [**Ans.** Rs. 9,772.85]
   (b) An annuity of Rs. 2,500 payable at the end of every 6 months for 5 years at 6% per annum converted semi-annually. [**Ans.** Rs. 21,325.51]
6. Find the sum of money received by a pensoner at age 58, if he wants to commute his annual pension of Rs. 1;200 for a present payment, when compound interest is reckoned at 4% effective and the expectation of his life is assessed at 10 years only.
   [**Ans.** Rs. 9,717]
7. Calculate the present value of an annuity of Rs. 5,000 per annum for 12 years, the interest being charged at 4% per annum compounded annually. [**Ans.** Rs. 46,925.37]
8. A fixed royalty of Rs. 1,000 per year is granted to an author by the publisher of a book for 20 years. The right of receiving the royalty is sold after 12 years elapsed. Find the nearest rupee the price at which it is sold, assuming money is worth 12% per annum compounded annually. [**Ans.** Rs. 4,967.64]
9. A man buys car on instalment basis such that he pays Rs. 50,000 on signing of the contract and remaining in 4 equal instalments of Rs. 20,000; the first is being paid at

the end of first year and so on for each year. If the rate of interest is 8% effective, find the cash price of the car. [**Ans.** Rs. 1,16,242.54]

10. Mr. X wants to purchase a house which will cost him Rs. 6,00,000 at the time of his retirement, which is due after 16 years. How much should he deposit at the end of each year in a bank paying interest at rate of 15% per annum compounded annually in order to accumulate the above said amount ? [**Ans.** Rs. 10,768.12]

11. A man purchased a piece of land paying Rs. 50,000 in cash and the balance in 20 instalments of Rs. 8,000 each at the end of each year. If the interest was reckoned at 16%, how much he should have paid if he had purchased it cash down ?
[**Ans.** Rs. 97,430]

12. A machine is purchased for Rs. 30,000 down and payments of Rs. 2,500 at the end of every 6 months for 6 years. If the interest is at 8% compounded semi-annually, find the corresponding cash price of the machine. [**Ans.** Rs. 53,462.68]

13. *XYZ* Ltd. purchases a machine on instalment basis making 24 monthly payments of Rs. 6,000 each and a final payment of Rs. 15,000 one month later. If the rate of interest is 9% compounded monthly, find the cash price of the machine.
[**Ans.** Rs. 1,43,779.04]

14. A loan of Rs. 3,000 is to be repaid with interest at 6% per annum by means of an immediate annuity for 10 years. Find the level payment. What is the amount paid as total interest ? [**Ans.** Rs. 407.60, Rs. 1,076]

15. A man buys a car for Rs. 80,000 paying a down payment of Rs. 20,000. He agrees to pay the remaining amount in 10 equal instalments, the first is to be paid one year after the date of purchase. Calculate the amount of each instalment, compound interest being calculated at the rate of 5% per annum. [**Ans.** Rs. 7,770.27]

16. A person borrows Rs. 25,000 on the understanding that it is to be paid back in 8 equal instalments at intervals of 6 months, the first payment to be made six months after the money was borrowed. Calculate the value of each instalment, if money is worth 6% per annum compounded semi-annually. [**Ans.** Rs. 3,561.41]

17. If the present value and amount of an ordinary annuity of Re. 1 per annum for n years are Rs. 8.1109 and Rs. 12.0061 respectively, find the rate of interest and the value of n without consulting the compound interest table.
[**Ans.** i = 4% and n = 10 years]

18. Mr. X sells his old car for Rs. 1,00,000 to buy a new model costing Rs. 2,50,000. He pays Rs. x cash down and remaining by payments of Rs. 7,000 at the end of each month for 18 months. If the rate of interest is 7% compounded monthly, find x.
[**Ans.** Rs. 30,719.07]

19. Find the present value of an annuity of Rs. 100 per annum, assumed to be payable for 10 years, at the rate of 4% per annum compounded continuously.
[**Ans.** Rs. 824.19]

20. A man buys a car worth Rs. 2,00,000 paying 40% cash and the balance to be paid in 20 equal annual instalments. Calculate the value of each instalment if the money is worth 7.5% per annum compounded continuously. [**Ans.** Rs. 11,584]

21. According to an investment proposal, an initial investment of Rs. 40,000 is expected to yield a uniform income of Rs. 5,000 per annum for a certain number of years. What is the expected payback period, that is after what time, the initial investment will be recovered, if the money is worth 6% per annum (*a*) compound annually and (*b*) compounded continuously? [**Ans.** 11.23 years, (*b*) 10.22 years.]

22. A man has been accumulating a fund at 3% effective which will provide him with an income of Rs. 20,000 per year for 15 years, the first payment on his 60th birthday. If he now wishes to reduce the number of payment to 10, what should he receive annually? [**Ans.** Rs. 27,989.80]

23. What amount should be set aside at end of every year to replace the loan of Rs. 11,310 in two years at the rate of 6% per annum compounded continuously? [**Ans.** Rs. 6,000]

24. Rakesh borrowed Rs. 80,000 from a bank to help finance the purchase of a house. The bank charges interest at a rate of 9% per year on the unpaid balance, with interest computation made at the end of each month. He has agreed to repay the loan in equal monthly instalments over 30 years. How much should each payment be, if the loan is to be amortised at the end of the term? [**Ans.** Rs. 653.45]

25. A company borrows a loan of Rs. 4,00,950 on condition to repay it with compound interest at 6% per annum by annual instalments of Rs. 1,50,000 each. In how many years will the debt be paid off? [**Ans.** 3 years]

26. Find the capital value of uniform income stream of Rs. R per year for m years, reckoning interest continuously at 100 r% per year, if income is forever. [**Ans.** R/r]

27. According to an investment proposal an initial investment of Rs. 50,000 is expected to yield a uniform income stream of Rs. 5,000 per annum. If money is worth 5% per annum compounded continuously, what is the expected pay-back period, *i.e.,* after what time, the initial investment will be recovered? [**Ans.** 13.87 years]

28. A person has a right to receive Rs. 350 per annum for 8 years and Rs. 250 per annum for next 6 years. The first annual sum is due at the end of first year. What is the present value of his right, assuming interest rate of 5% per annum. [**Ans.** Rs. 3,120.98]

29. Show that $\dfrac{a_{\overline{6n}|i}}{a_{\overline{6n}|i}} = 1 + V^{3n}$ where $V = (1 + i)^{-1}$

30. X purchased a machine paying Rs. 50,000 down and promising to pay Rs. 2,000 every 3 months for 10 years. Seller charges interest at 8% compounded quarterly.

(*a*) What was the cash price of the machine?

(*b*) If X missed the first 10 payments, what must he pay at the time of 11th payment to bring him up to date.

(*c*) After making 8 payment, X wishes to discharge his entire remaining indebtedness by a single payment at the time when 9th regular payment was due. What must he pay in addition to the regular payment then due?

(*d*) If X missed the first 10 payments, what must he pay when the 11^{th} payment is due to discharge his entire indebtedness ?

[**Ans.** (*a*) 1,04,711; (*b*) 24,337.40; (*c*) 45,875.40; (*d*) 68,026.20]

31. A purchased a television paying Rs. 5,000 down and promising to pay Rs. 200 every three months for next 4 years. The seller charges interest at 8% per annum compounded quarterly.

(*a*) What is the cash price of television ?

(*b*) If *A* missed the first eight payments, what must he pay at the time the fourth is due to bring him upto date ? [**Ans.** (*a*) Rs. 7,715.54; (*b*) Rs. 1,950.92]

32. Shares in a mining company are expected to produce dividends of Rs. 0, 30, 24, 16, and 8 in the present and in the four following years and to be worth nothing thereafter. If the interest is added once yearly at 5%, find the present or capital value of the holdings. [**Ans.** Rs. 70.74]

33. A company purchased a machine paying Rs. 25,000 down payment and promising to pay Rs. 3,000 every month for next 2 years. The seller charges interest @ 12% per annum compounded quarterly. After making 10 payments, if the company wishes to discharge the remaining in debtness by a single payment at the time when the 11^{th} payment is due, what must the company pay ? [**Ans.** Rs. 49,853.37]

34. Establish the following relation : $s_{\overline{n}|i} = (1 + i)^n \, a_{\overline{n}|i}$.