CHAPTER

12

Deferred Annuity

INTRODUCTION

We have so far discussed about annuities in which periodic payments are made at the end (or beginning) of every payment period, the first being paid at the end (or the beginning) of the first period. But there are some obligations for which the first payment may not be paid at the end of the period. For example, consider an education loan. Such loans are provided to students for the purpose of higher education. Suppose that a bank provides an education loan of Rs. 1,00,000 to a student and the loan is to be repaid in 20 equal annual instalments and the borrower is expected to repay the loan after the expiry of a certain number of years (say 5 years). The first instalment is paid at the end of 6^{th} year and the last is paid at the end of 25^{th} year. Such kind of annuities are called deferred annuities.

Thus, the deferred annuity is defined as an ordinary annuity in which the first payment is postponed until the expiry of certain time period which is equivalent to certain number of payment intervals.

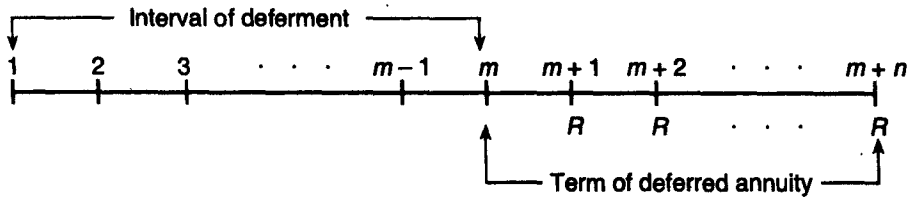
If the term of an ordinary annuity of 'n' periods begins after the lapse of 'm' periods, then the annuity is said to be an annuity of 'n' periods, deferred 'm' periods.

In deferred annuity, the first payment is made at the end of $(m + 1)^{th}$ period. But the term of deferred annuity begins at the beginning of the $(m + 1)^{th}$ payment period, *i.e.,* immediately after the expiry of m periods.

The length of time between now and the beginning of the term of the deferred annuity is called the *interval of deferment.*

It should be noted that the interval of deferment ends are period before the first payment is due.

This has been illustrated in the following figure :



Remark : Deferred annuity can be a due deferred annuity. If the deferment period of an annuity due is equal to m periods, then the first payment is made at the beginning of $m + 1$ periods.

AMOUNT OF A DEFERRED ANNUITY

The amount of a deferred annuity is the value of the annuity at the end of the term. Since the amount is the sum of future values of every periodic payment at the end of the term, the amount of a deferred annuity does not depend on the deferment interval. Therefore, this amount is obviously the amount of an ordinary annuity.

Thus, if a deferred annuity consists of n periodic payments of Rs. R each at the rate of i per period and the annuity is deferred for m periods, then the amount A of this annuity is the amount of an ordinary annuity consisting of n periodic payments of R each at the rate i per period.

The amount of an deferred annuity is thus given by

$$A = R\left[\frac{(1+i)^n - 1}{i}\right]$$

or

$$A = R\ s_{\overline{n}|i}$$

Example 1 : *An annuity consists of 12 semi-annual payments of Rs. 2,000 each the first being payable at the end of 5 years. Find the amount of this annuity, if money is worth 6% converted semi-annually.*

***Solution* :** The first payment is made after $1\frac{1}{2}$ year, *i.e.,* after 3 payments.

$\therefore$ It is a deferred annuity.

Amount of a deferred annuity is given by

$$A = R\left[\frac{(1+i)^n - 1}{i}\right]$$

Here periodic payment R = Rs. 2,000

Rate of interest $r = 6\% = 0.06$

Interest is compounded semi-annually.

$$\therefore \quad i = \frac{r}{2} = \frac{0.06}{2} = 0.03$$

Given that $n = 12$

$$A = 2{,}000\left[\frac{(1.03)^{12}-1}{0.03}\right]$$

$$A = 2{,}000\left[\frac{1.42576088-1}{0.03}\right]$$

$$A = 2{,}000\left[\frac{0.42576088}{0.03}\right]$$

$$A = 2{,}000\ [14.192029] = 28{,}384.06$$

$\therefore$ The amount is Rs. 28384.06

Example 2 : *Find the amount of a sequence of semi-annual payments of Rs. 800 each, the first being made at the end of $6\frac{1}{2}$ years and the last at the end of 14^{th} year. Find the amount of this annuity, if money is worth 8% converted semi-annually.*

Solution **:** Since the periodic payment is payable after 6 years, it is a deferred annuity.

Deferment interval is 6 years.

$\therefore$ Term of the annuity is 8 years.

Since the payments are made semi-annually, $n = 2 \times 8 = 16$

Rate of interest $r = 8\% = 0.08$

Interest is compounded semi-annually

$$\therefore \quad i = \frac{r}{2} = \frac{0.08}{2} = 0.04$$

Periodic payment R = Rs. 800

Amount of a deferred annuity is given by

$$A = R\left[\frac{(1+i)^n - 1}{i}\right]$$

Here $$A = 800\left[\frac{(1.04)^{16}-1}{0.04}\right]$$

$$A = 800\left[\frac{1.87298125-1}{0.04}\right]$$

$$A = 800\ [21.82453] = 17{,}459.66$$

$\therefore$ The amount is Rs. 17,459.66.

Example 3 : *At 6 months interval Mr. A deposits Rs. 100 in a savings bank account which credit interest at the rate of 8% per annum compounded semi-annually. The first deposit was made when A's son was 6 years old and the last deposit was made when his son was 21 years old. The money remained in the account and was presented to the son on his 25^{th} birthday. How much did he receive ?*

Solution **:** Since first deposit is made at the end of 6 years, it is a deferred annuity.

Periodic payment R = Rs. 100

Payment are made on semi-annual basis.

Interval of deferment = $5\frac{1}{2}$ years

$\therefore$ $m = 5\frac{1}{2} \times 2 = 11$

Last payment was made at the end of 21 years.

$\therefore$ Term of this annuity = $15\frac{1}{2}$ years

$\therefore$ $n = 15\frac{1}{2} \times 2 = 31$

Rate of interest $r = 8\% = 0.08$

Interest is compounded semi-annually.

$\therefore$ $i = \frac{r}{2} = \frac{0.08}{2} = 0.04$

Amount at the end of 21st year is the amount of this deferred annuity of Rs. 100.

Amount of deferred annuity is given by :

$A = R \; s_{\overline{n}|i}$

$\therefore$ $A = 100 \; s_{\overline{31}|0.04}$

$A = 100 \; (59.3283353) = 5{,}932.83$

$\therefore$ Amount after 21 years is Rs. 5,932.83

This amount remained in account for 4 more years.

Now $n = 4 \times 2 = 8$

Amount is given by

$A = P(1 + i)^n$

Here P = Rs. 5932.83 and $i = 0.04$

$\therefore$ $A = 5{,}932.83 \; (1.04)^8$

$A = 5{,}932.83 \; (1.36856905) = 8{,}119.49$

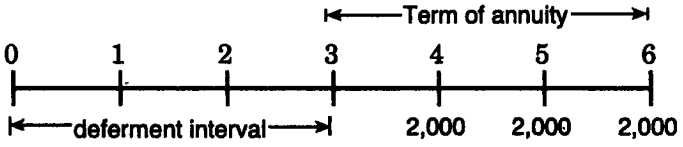
$\therefore$ Amount at the end of 25 years is Rs. 8,119.49.

PRESENT VALUE OF DEFERRED ANNUITY

The present value of a deferred annuity for 'n' periods, deferred 'm' periods, is the value of the annuity at the beginning of the interval of deferment. In other words, the present value of a deferred annuity is the sum of present values of each instalment at the beginning of the deferment interval. We have seen that the amount of an ordinary annuity and the amount of a deferred annuity are the same. But present value of deferred annuity is not the same as the value of an ordinary annuity.

For example, suppose that a person borrows a certain sum from a bank now at 6% effective rate of interest and agrees to pay Rs. 2,000 at the end of every year; the first instalment is to be paid at the end of 4th year.

This is shown in the following figure :



Present value of first instalment of Rs. 2,000

$= \text{Rs. } 2,000\ (1.06)^{-(3+1)} = \text{Rs. } 2,000\ (1.06)^{-4} = \text{Rs. } 1,584.19$

Present value of second instalment of Rs. 2,000

$= \text{Rs. } 2,000\ (1.06)^{-(3+2)} = \text{Rs. } 2,000\ (1.06)^{-5} = \text{Rs. } 1,494.52$

Present value of third instalment of Rs. 2,000

$= \text{Rs. } 2,000\ (1.06)^{-(3+3)} = \text{Rs. } 2,000\ (1.06)^{-6} = \text{Rs. } 1,409.92$

$\therefore$ The present value of this annuity is

$P = \text{Rs. } (1,584.19 + 1,494.52 + 1,409.92) = \text{Rs. } 4,488.63$

Thus, we say that a loan of Rs. 4,488.63 at a rate of 6% per annum compounded annually is repaid by three annual instalments of Rs. 2,000 (recall that in case of ordinary annuity of Rs. 2,000 at 6% effective for 3 years is Rs. 5,346.02).

Now we derive the formula for the present value of a deferred annuity.

Let R be the periodic payment, i the rate of interest per period, n the number of payment and m the deferment interval.

Since the first payment of R is made at the end of $(m + 1)$ periods, the present value of this R is $R\ (1 + i)^{-(m+1)}$.

The second payment of R is made at the end of $(m + 2)$ period. So the present value of second payment is $R\ (1 + i)^{-(m+2)}$ and so on.

Thus,

Present value of 1st payment, $P_1 = R\ (1 + i)^{-(m+1)}$

Present value of 2nd payment, $P_2 = R\ (1 + i)^{-(m+2)}$

Present value of 3rd payment, $P_3 = R\ (1 + i)^{-(m+3)}$

$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$

Present value of n^{th} payment, $p_n = R\ (1 + i)^{-(m+n)}$

Present value of this deferred annuity is given by

$$P = P_1 + P_2 + P_3 + \ldots\ldots + P_n$$

$$\Rightarrow \quad P = R\ (1 + i)^{-(m+1)} + R\ (1 + i)^{-(m+2)} + \ldots\ldots + R\ (1 + i)^{-(m+n)}$$

$$\Rightarrow \quad P = R\ (1 + i)^{-m}\ (1 + i)^{-1} + R\ (1 + i)^{-m}\ (1 + i)^{-2} + \ldots\ldots + R\ (1 + i)^{-m}\ (1 + i)^{-n}$$

$$\Rightarrow \quad P = R\ (1 + i)^{-m}\ [(1 + i)^{-1} + R\ (1 + i)^{-2} + \ldots\ldots + (1 + i)^{-n}]$$

$$\Rightarrow \quad P = R\ (1 + i)^{-m}\ (1 + i)^{-1} \left[\frac{(1 + i)^{-n} - 1}{(1 + i)^{-1} - 1}\right] \quad \because\ a + ar + ar^2 + \ldots ar^{n-1} = a \cdot \frac{r^n - 1}{r - 1}$$

$$\Rightarrow \quad P = R(1+i)^{-m} \cdot \frac{1}{(1+i)} \left[\frac{(1+i)^{-n}-1}{(1+i)^{-1}-1}\right]$$

$$\Rightarrow \quad P = R(1+i)^{-m} \left[\frac{(1+i)^{-n}-1}{1-(1+i)}\right]$$

$$\Rightarrow \quad P = R(1+i)^{-m} \left[\frac{(1+i)^{-n}-1}{-i}\right]$$

$$\Rightarrow \quad P = R(1+i)^{-m} \left[\frac{1-(1+i)^{-n}}{i}\right]$$

$$\Rightarrow \quad P = \frac{R}{i}[(1+i)^{-m} - (1+i)^{-(m+n)}]$$

Thus, the present value of a deferred annuity is given by

$$\boxed{P = \frac{R}{i}[(1+i)^{-m} - (1+i)^{-(m+n)}]}$$

Remark 1 : When the periodic payment is Re. 1, then the present value of a deferred annuity is given by $\frac{(1+i)^{-m} - (1+i)^{-(m+n)}}{i}$. This is denoted by $a^{(m)}_{\overline{n}|i}$.

Therefore, we express the present value as follows :

$$P = R\ a^{(m)}_{\overline{n}|i}.$$

Remark 2 : We have derived that

$$a^{(m)}_{\overline{n}|i} = \frac{(1+i)^{-m} - (1+i)^{-(m+n)}}{i}$$

$$\Rightarrow \quad a^{(m)}_{\overline{n}|i} = \frac{1-(1+i)^{-(m+n)} - (1-(1+i)^{-m})}{i}$$

$$\Rightarrow \quad a^{(m)}_{\overline{n}|i} = \frac{1-(1+i)^{-(m+n)}}{i} - \frac{1-(1+i)^{-m}}{i}$$

$$\Rightarrow \quad a^{(m)}_{\overline{n}|i} = a_{\overline{m+n}|i} - a_{\overline{m}|i}$$

$\therefore$ The present value of a deferred annuity can also be expressed as

$$\boxed{P = R\left[a_{\overline{m+n}|i} - a_{\overline{m}|i}\right]}$$

CONTINUOUS COMPOUNDING

We have derived that

$$P = R\left[a_{\overline{m+n}|i} - a_{\overline{m}|i}\right]$$

i.e. $$P = R\ a_{\overline{m+n}|i} - R\ a_{\overline{m}|i}$$

Thus, present value of a deferred annuity of Rs. R, payable at a rate of i per period for n periods, deferred m periods can be expressed as the present value of an ordinary annuity of

Rs. R at a rate of i per period for $(m + n)$ periods minus the present value of an ordinary annuity of Rs. R at a rate of i per periods for m periods.

We know that the present value of R of an ordinary annuity for n $(= t)$ years at the rate of r per annum compounded continuously is given by

$$P = \int_0^{t(=n)} Re^{-rt}\, dt$$

$\therefore$ The preset value of a deferred annuity is give by

$$P = \int_0^{t(=m+n)} Re^{-rt}\, dt - \int_0^{t(=m)} Re^{-rt}\, dt$$

$$\Rightarrow \quad P = \int_{t=m}^{t=m+n} Re^{-rt}\, dt \qquad \because \int_0^{m+n} f(x)\, dx = \int_0^{m} f(x)dx + \int_m^{m+n} f(x)dx$$

$$\Rightarrow \quad P = \frac{R}{-r}\left[e^{-rt}\right]_m^{m+n} \qquad \Rightarrow \int_0^{m+n} f(x)\, dx - \int_0^{m} f(x)dx = \int_m^{m+n} f(x)dx$$

$$\Rightarrow \quad P = \frac{-R}{r}\left[e^{-(m+n)r} - e^{-mr}\right]$$

$$\Rightarrow \quad P = \frac{R}{r}\left[e^{-mr} - e^{-(m+n)r}\right]$$

Thus, the present value of a deferred annuity of Rs. R to begin at the end of m periods and to continue for n periods at a rate of i per period is given by

$$\boxed{P = \frac{R}{r}\left[e^{-mr} - e^{-(m+n)r}\right]}$$

Example 4 : *Find the present value of 20 semi-anual payments of Rs. 500 each, the first being made at the end of $6\frac{1}{2}$ years, if the money is worth 10% compounded semi-annually.*

Solution **:** Since the first payment is made at the end of $6\frac{1}{2}$ years, it is a deferred annuity

Periodic payment R = Rs. 500

Deferment period = 6 years

Payment are made semi-annually

$\therefore \quad m = 2 \times 6 = 12$

No. of payment periods, $n = 20$

Rate of interest, $r = 10\% = 0.10$

Interest compounded semi-annually

$$\therefore \quad i = \frac{r}{2} = \frac{0.10}{2} = 0.05$$

The present value of a deferred annuity is given by

$$P = \frac{R}{i}[(1+i)^{-m} - (1+i)^{-(m+n)}]$$

$$P = \frac{500}{0.05}[(1.05)^{-12} - (1.05)^{-32}]$$

$$P = 10{,}000\ [0.55683742 - 0.20986617]$$

$$P = 10{,}000\ [0.34697125) = 3{,}469.71$$

The present value is Rs. 3,469.71

Example 5 : *Mr. X borrows a certain sum of money at 8% per annum compound interest and agrees to pay both principal and interest in 10 equal yearly instalments of Rs. 1,200 each. If the first instalment is to be paid at the end of 5 years from the date of borrowing and the other yearly instalments are paid regularly at the end of the subsequent years, find the sum borrowed by him.*

Solution **:** Since the first instalment is paid at the end of 5th year, it is a deferred annuity.

Payments are made annually

$\therefore$ Deferment interval $m = 4$ (years)

Periodic payment R = Rs. 1,200

No. of periods $n = 10$

Rate of interest $i = r = 8\% = 0.08$

Present value of a deferred annuity is given by

$$P = \frac{R}{i}[(1+i)^{-m} - (1+i)^{-(m+n)}]$$

$$\therefore\ P = \frac{1{,}200}{0.08}[(0.08)^{-4} - (1.08)^{-14}]$$

$$P = 15{,}000\ [0.73502985 - 0.34046104]$$

$$P = 15{,}000\ [0.394556881] = \text{Rs. } 5{,}918.53$$

$\therefore$ The amount borrowed is Rs. 5,918.53.

Alternative Method :

Present value of a deferred annuity is given by

$$P = R\ [a_{\overline{m+n}|i} - a_{\overline{m}|i}]$$

$$\therefore\ P = 1{,}200\ [a_{\overline{14}|0.08} - a_{\overline{4}|0.08}]$$

$$P = 1{,}200\ [8.24423698 - 3.31212684]$$

$$P = 1{,}200\ (4.93211) = 5{,}918.53$$

$\therefore$ The amount borrowed is 5,918.53.

Example 6 : *Find the present value of an annuity consisting of 60 monthly payments of Rs. 150 each, the first being made at the end of 3 years, if the money is worth 12% per annum compounded monthly.*

Solution : Since the first payment is made at the end of 3 years, it is a deferred annuity.

Monthly instalment R = Rs. 150

No. of instalments $n = 60$

Deferment period = 2 years and 11 months

$\therefore \quad m = 2 \times 12 + 11 = 35$

Rate of interest $r = 12\% = 0.12$

Interest is compounded monthly.

$$\therefore \quad i = \frac{r}{12} = \frac{0.12}{12} = 0.01$$

The present value of deferred annuity is given by

$$P = \frac{R}{i}[(1+i)^{-m} - (1+i)^{-(m+n)}]$$

$$\therefore \quad P = \frac{150}{0.01}[(1.01)^{-35} - (1.01)^{-95}]$$

$$P = 15{,}000\left[\frac{1}{(1.01)^{35}} - \frac{1}{(1.01)^{95}}\right]$$

Let $x = (1.01)^{35}$

$\log x = 35 \times 0.0043 = 0.1505$

x = antilog (0.1505)

$x = 1.415$

Let $y = (1.01)^{95}$

$\log y = 95 \log 1.01$

$= 95 \times 0.0043 = 0.4085$

y = antilog (0.4085)

$y = 2.562$

$$\therefore \quad P = 15{,}000\left[\frac{1}{1.415} - \frac{1}{2.562}\right]$$

$P = 15{,}000\,[0.706714 - 0.39320]$

$P = 15{,}000\,(0.316394) = 4745.91$

$\therefore$ The present value is Rs. 4,745.91.

Example 7 : *X buys a machine for which he agrees to make quarterly payments of Rs. 1,800, the first is being made at the end of $3\frac{1}{2}$ years and the last at the end of 8 years. Find the purchasing price of the machine, if the money is worth 9% per annum compounded quarterly.*

Solution : Since the payments are make at the end of $3\frac{1}{2}$ years. It is a deferred annuity.

Quarterly payment R = Rs. 1,800

Period of deferment $= 3\frac{1}{4}$ years

$\therefore \quad m = 3 \times 4 + 1 = 13$

Term of the annuity is $4\frac{3}{4}$ years

$\therefore \quad n = 4 \times 4 + 3 = 19$

Rate of interest $r = 9\% = 0.09$

Interest is compounded quarterly.

$$\therefore \quad i = \frac{r}{4} = \frac{0.09}{4} = 0.0225$$

Purchasing price of the machine is the present value of the deferred annuity of Rs. 1,800.

Present value of deferred annuity is given by

$$P = \frac{R}{i}[(1+i)^{-m} - (1+i)^{-(m+n)}]$$

$$\therefore \quad P = \frac{1,800}{0.0225}[(1.0225)^{-13} - (1.0225)^{-32}]$$

$$P = 80,000\,[0.74881905 - 0.49065233]$$

$$P = 80,000\,(0.25816672) = \text{Rs. } 20,653.34$$

$\therefore$ Price of the machine is Rs. 20,653.34.

Example 8 : *A man borrows Rs. 80,000 at compound interest of 5% per annum and agrees to repay the money in 15 equal yearly instalments. What should be the annual instalment if the first payment is paid at the end of 5 years ?*

Solution : Since the first payment is made at the end of 5 years, it is a deferred annuity.

Deferment interval = 4 years

$\therefore \quad m = 4$

Let R be the annual instalment.

Amount borrowed P = Rs. 80,000

No. of instalments $n = 15$

Rate of interest $r = i = 0.05$

Present value of a deferred annuity is given by

$$P = R\,[a_{\overline{m+n}|i} - a_{\overline{m}|i}]$$

$$\therefore \quad 80,000 = R\,[a_{\overline{19}|0.05} - a_{\overline{4}|0.05}]$$

$$\Rightarrow \quad 80,000 = R\,[12.08532086 - 3.54595050]$$

$$\Rightarrow \quad 80,000 = R\,(8.53937036)$$

$$R = \frac{80,000}{8.53937036} = 9,368.37$$

$\therefore$ The annual instalment is Rs. 9,368.37.

Example 9 : *An annuity consists of 6 annual payment of Rs 2,400 each, the first being made at the end of 5 years. If money is worth 6% per annum compounded continuously, what is the present value of this annuity ?*

Solution : Since first payment is made at the end of 5 years, it is a deferred annuity.

Deferment interval	$m = 4$
No. of payments	$n = 6$
Periodic payment	R = Rs. 2,400
Rate of interest	$r = 6\% = 0.06$

Interest is compounded continuously.

Present value of deferred annuity in case of continuous compounding is given by

$$P = \int_{m}^{m+n} Re^{-rt}\, dt$$

i.e. $$P = \frac{R}{r}[e^{-mr} - e^{-(m+n)r}]$$

$\therefore$ $$P = \frac{2,400}{0.06}[e^{-0.24} - e^{-0.60}]$$

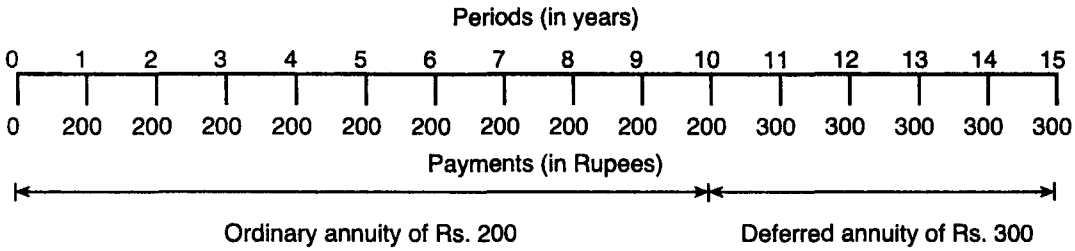
$$P = 40,000\,[0.78663 - 0.54881]$$

$$P = 40,000\,(0.23782) = 9,512.80$$

$\therefore$ The present value is Rs. 9,512.80

Example 10. *An annuity payable for 15 years certain, the first payment falling due at the end of 1st year. The annuity is payable at the rate of Rs. 200 per annum during the first 10 years and the rate of Rs. 300 per annum during the remaining 5 years. Calculate the present value of the annuity on the basis of interest at 7% per annum.*

Solution : Periodic payment are shown in the following figure :



Consider the first 10 payments of Rs. 200 :

It is an ordinary annuity of Rs. 200 for 10 years.

Period payment R = Rs. 200

No. of periods $n = 10$

Rate of interest $i = r = 7\% = 0.07$

Present value of an ordinary annuity is given by

$$P = R\, a_{\overline{n}|i}$$

$\therefore$ $$P = 200\, a_{\overline{10}|0.07}$$

$$P = 200\,(7.02358154) = \text{Rs. } 1,404.72 \qquad ...(1)$$

Consider the last 5 payments of Rs. 300 :

It is a deferred annuity of Rs. 300, the first payment being due at the end of 11 years.

Periodic payment R = Rs. 300

Determent interval $m = 10$

Payment periods = $n = 5$

Rate of interest $i = r = 7\% = 0.07$

Present value of a deferred annuity is given by

$$P = R\,[a_{\overline{m+n}|i} - a_{\overline{m}|i}]$$

$$\therefore \quad P = 300\,[a_{\overline{15}|0.07} - a_{\overline{10}|0.07}]$$

$$P = 300\,(9.10791401 - 7.02358154)$$

$$P = 300\,(2.08433247) = \text{Rs. } 625.30 \qquad \ldots(2)$$

$\therefore$ Present value of the desired 15 payments is given by

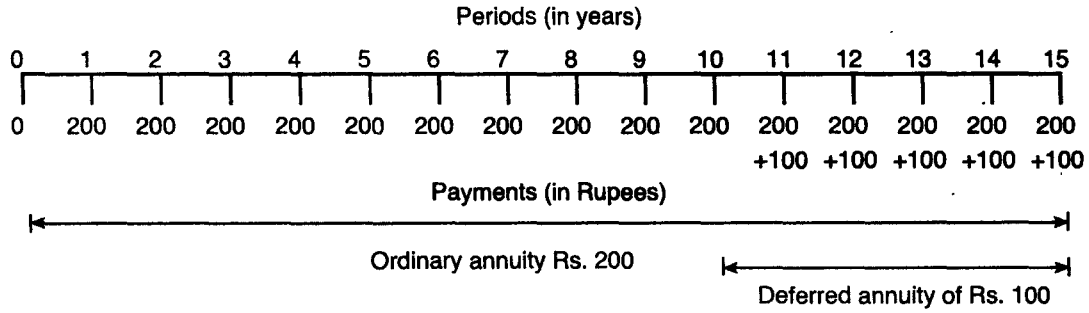
(1) + (2) = Rs. 1,404.72 + Rs. 625.30

= Rs. 2,030.02

$\therefore$ The required present value is approximately Rs. 2,030.

Alternative method :

The given annuity can be alternatively expressed in the following manner :



Present value of the given annuity can be expressed as the sum of the present value of an ordinary annuity of Rs. 200 for 15 years and the present value of a deferred annuity of Rs. 100 for 5 years, deferred 10 years.

Consider the ordinary annuity of Rs. 200 for 15 years :

Periodic payment R = Rs. 200

No. of periods $n = 15$

Rate of interest $r = i = 0.07$

$\therefore$ The present value of this annuity is given by

$$P = 200\,a_{\overline{15}|0.07}$$

$$P = 200\,(9.10791401)$$

$$P = \text{Rs. } 1,821.58 \qquad \ldots(3)$$

Consider the deferred annuity of Rs. 100 :

Periodic payment R = Rs. 100

Deferment interval $m = 10$

No. of payment $n = 5$

Present value of this annuity is given by

$$P = 100\,[a_{\overline{15}|\,0.07} - a_{\overline{10}|\,0.07}]$$

$$P = 100\,[9.10791401 - 7.02358154]$$

$$P = 100\,[2.08433247]$$

$$P = \text{Rs. } 208.43 \qquad \ldots(4)$$

Present value of the desired 15 payment is given by

(3) + (4) = Rs. 1,821.58 + Rs. 208.43

= Rs. 2,030.01

$\therefore$ The required present value is Rs. 2,030.01

Example 11 : *A is entitled to the following benefits :*

(a) Five annual payment at the rate of Rs. 200 per annum, the first being due at the end of one year from now.

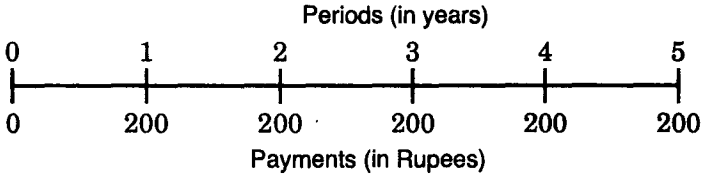
(b) Thereafter 6 annual payments at the rate of Rs. 300 per annum, the first of these being due at the end of 11 years from now.

(c) An additional lump sum of Rs. 2,000 at the end of 10 years.

Find the present value of his benefits at the rate of interest of 6% per annum. Compounded annually.

Solution : The required present value is the sum of present values of the three different benefits.

(a) In case of 5 annual payments of Rs. 200 :



Since the payments are made at the end of every year for 5 years, it is an ordinary annuity.

Periodic payment R = Rs. 200

No. of payments $n = 5$

Rate of interest $i = r = 6\% = 0.06$

Present value of an ordinary annuity is given by

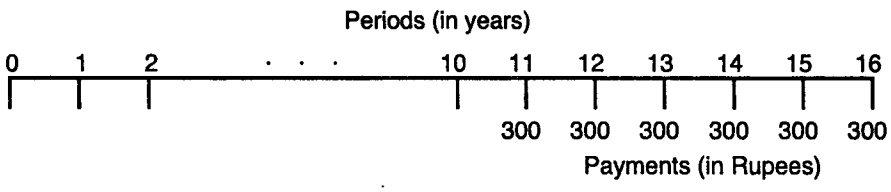
$$P = R\,a_{\overline{n}|\,i}$$

$\therefore$ $$P = 200\,a_{\overline{5}|\,0.06}$$

$$P = 200\,(4.21236379)$$

$$P = \text{Rs. } 842.47 \qquad \ldots(1)$$

(*b*) *In case of 6 annual payments of Rs. 300* :



Since the first payment is due at the end of 11 years, it is a deferred annuity.

Periodic payment R = Rs. 300

Deferment interval $m = 10$

Payment periods $n = 6$

Present value of a deferred annuity is given by

$$P = R\,[a_{\overline{m+n}|i} - a_{\overline{m}|i}]$$

$$\therefore \quad P = 300\,[a_{\overline{16}|0.06} - a_{\overline{10}|0.06}]$$

$$P = 300\,[10.10589527 - 7.36008705]$$

$$P = 300\,(2.74580822)$$

$$P = \text{Rs. } 823.74 \qquad \ldots(2)$$

(*c*) In case of lump sum of Rs. 2,000 at the end of 10 years.

Lump sum A = Rs. 2,000

No. of conversion periods $n = 10$

Present value of a lump sum is given by

$$P = A\,(1 + i)^{-n}$$

$$\therefore \quad P = 2{,}000\,(1.06)^{-10}$$

$$P = 2{,}000\,(0.05583948)$$

$$P = \text{Rs. } 1{,}116.79 \qquad \ldots(3)$$

The present value of the benefits is given by

(1) + (2) + (3) = Rs. 842.47 + Rs. 823.74 + Rs. 1,116.79

= Rs. 2,783.

$\therefore$ The required present value is Rs. 2,783.

Example 13 : *Establish the relation between the present value of an ordinary annuity of Re. 1 for n periods at the rate of i per period and the preset value of a deferred annuity of Re. 1 for n periods, deferred m periods at the same rate per period.*

Solution **:** The present value of an ordinary annuity of Re. 1 for n period at the rate of i per period is given by

$$a_{\overline{n}|i} = \frac{1 - (1 + i)^{-n}}{i} \qquad \ldots(1)$$

The present value of a deferred annuity of Re. 1 for n periods, deferred m periods at the rate of i per period is given by

$$a^{(m)}_{\overline{n}|i} = \frac{(1+i)^{-m} - (1+i)^{-(m+n)}}{i} \qquad \ldots(2)$$

$$\Rightarrow \quad a^{(m)}_{\overline{n}|i} = \frac{(1+i)^{-m} - (1+i)^{-m}(1+i)^{-n}}{i}$$

$$\Rightarrow \quad a^{(m)}_{\overline{n}|i} = (1+i)^{-m}\left[\frac{1-(1+i)^{-n}}{i}\right]$$

$$\Rightarrow \quad a^{(m)}_{\overline{n}|i} = (1+i)^{-m}\, a_{\overline{n}|i}$$

EXERCISES

1. Find the amount of an annuity consisting of 6 annual payments of Rs. 1,200 each, the first being made at the end of 5 years, if the money is worth 6% effective.

[**Ans.** Rs. 8,730.38]

2. Find the amount of an annuity consisting of 20 semi-annual payment of Rs. 500 each, the first being made at the end of $4\frac{1}{2}$ years, if the money is worth 5% per annum compounded semi-anually. [**Ans.** Rs. 12,772.28]

3. Find the amount of a sequence of annual payments of Rs. 8,000 each, the first being made at the end of 5 years and the last at the end of 10 years, if the money is 5% effective. [**Ans.** Rs. 54,415.30]

4. An annuity consists of 12 semi-annual payments of Rs. 2,000 each, the first being payable at the end of 2 years. Find the amount of this annuity, if money is worth 6% compounded semi-anually. [**Ans.** Rs. 28,384.06]

5. Find the amount of a sequence of quarterly payments of Rs. 3,000 each, the first being made at the end of 4 years and the last at the end of 7 years, if the interest is calculated at 10% per annum compounded quarterly. [**Ans.** Rs. 45,421.32]

6. A sequence of payments of Rs. 200 are made on monthly basis, the first being made at the end of 3 years and the last at the end of 5 years. Find the amount of this annuity, if interest is calculated at 12% per annum compounded monthly.

[**Ans.** 5,648.64]

7. X deposited Rs. 500 at the end of every three months in a savings account which credits interest at the rate of 7% per annum compounded quarterly. The first deposit was made when X's son was 5 years old and the last deposit was made when his son was 20 years old. The money remained in the account and was presented to the son on his 25th birthday. How much did he received ? [**Ans.** Rs. 76,049.30]

8. Mr. A wishes to provide for his son's education by investing a certain sum of money three months after his son's 12th birthday, and an equal sum quarterly thereafter. If this investment yields interest at the rate of 6% compounded quarterly, find the quarterly investment, if Rs. 40,000 is to be available for the son on his 17th birthday.

[**Ans.** Rs. 1,729.83]

9. Find the present value of a sequence of annual payments of Rs. 4,000 each, the first being made at the end of 5 years and the last at the end of 10 years, if money is worth 5% effective. **Ans.** Rs. 16,708.65]

10. Find the present value of a sequence of annual payments of Rs. 25,000 each, the first being made at the end of 7 years and the last at the end of 16 years, if money is worth 7% effective. Also find the amount of this annuity.

[**Ans.** Rs. 1,24,917.08, Rs. 3,45,411.20]

11. Mr. X buys a piece of land for which he agrees to make 10 annual payments of Rs. 1,50,000 each, first being made at the end of 3 years. find the equivalent cash price of this property, if money is worth 5% effective. [**Ans.** Rs. 10,50,576.24]

12. A man borrows a certain sum of money at 10% per annum compound interest and agrees to pay both the principal and interest in 20 equal semi-annual instalments of Rs. 8,000 each. If the first instalments is to be paid at the end of 5 years from the date of borrowing and the other instalments are paid regularly at the end of subsequent 6 months, find the sum borrowed by him. [**Ans.** Rs. 99,861.48]

13. Find the present value of an annuity consisting of 52 monthly payments of Rs. 250 each, the first being made at the end of 2 years if money is worth 5% compounded monthly. [**Ans.** Rs. 10,602.33]

14. What is the present values of an annuity of Rs. 400 payable at the end of every month, if the first payment is made at the end of $3\frac{1}{2}$ years and last at the end of 7 years and money is worth 12% per annum compounded monthly. [**Ans.** Rs. 9,259.51]

15. A company buys a machine for Rs. 2,30,000 and agrees to make 10 annual payments, the fist being made at the end of 3 years. find the annual payment, if the money is worth $6\frac{1}{2}$% effective. [**Ans.** Rs. 36,288.48]

16. A man borrowed Rs. 50,000 at compound interest of 7% per annum and has to repay the money in 20 equal yearly instalments. What should be the instalment of repayment has to start 3 years hence ? [**Ans.** Rs. 5,403.52]

17. A house is sold for Rs. 75,000 down and 15 semi-annual payments of Rs. 6,000 each, the first due 4 years hence. If the money is worth 8% compounded semi-anually, what is the cash price of the house ? [**Ans.** Rs. 1,25,694.36]

18. A borrows Rs. 1,75,000 from a bank which computes interest at a rate of 6% per annum compounded quarterly. He agrees to repay the loan in 30 equal quarterly instalments, the first being made at the end of $3^1/_2$ years and the other instalments at the end of subsequent quarter. Find the amount of monthly instalment.

[**Ans.** Rs. 8,842.98]

19. A person purchased a house on 1st Jan. 1983 for which he agreed pay Rs. 1,50,000 down and the remaining amount in monthly instalments of Rs. 2,750. The first payment was made on 30th Sept. 1985 and last on 31st Dec. 1990. If the interest was calculated at 9% per annum compounded monthly, what was the cash price of the house ? [**Ans.** 2,59,732.67]

20. If Rs. 8,000 is to be paid at the end of every year, the first payment is to be made 5 years hence and the interest rate is 5% effective, how many payments will be required to discharge a debt of Rs. 54,415 ? [**Ans.** 10]

21. *A* had decided to invest Rs. 500 at the end of each year. He did so far 7 years. Then there was a gap of 4 years. He could again invest Rs. 500 per annum for the next 4 years beginning from the end of the 12^{th} year. Find the amount to his credit at the end of the 15^{th} year assuming interest at effective rate of 9% per annum.

[**Ans.** Rs. 11,452.75]

22. A man took a loan which was to be repaid by monthly instalment of Rs. 2,000 for 5 years. He paid the instalments up to the end of two years. Then he left for USA on a foreign assignment and returned back after one year. He did not pay any instalment in the third year and wants to start the payment from the beginning of the fourth year. Find how much he should pay each month so that his loan will be cleared by the end of five years as agreed upon, the rate of interest being 12% compounded monthly. [**Ans.** Rs. 3,050.42]

23. A person has a right to receive Rs. 350 per annum for 8 years and Rs. 250 per annum for the next 6 years. The first annual sum is due at the end of 6 years from now. What is the present value of his rights, assuming interest rate of 5% effective ?

[**Ans.** Rs. 2,445.36]

24. Calculate the present value of a deferred annuity payable for 10 years certain, the first payment falling due at the end of 6 years from the present time. The annuity is payable at the rate of Rs. 100 per annum for the first 5 years and 200 per annum thereafter. Interest is calculated at a rate of 5% effective. [**Ans.** Rs. 870.82]

25. An annuity is payable for 15 years certain, the first payment falling due at the end of first year. The annuity is payable at the rate of Rs. 500 per annum during the first 10 years and Rs. 300 per annum during the remaining 5 years. Calculate the present value of the annuity on the basis of interest at 4% effective. [**Ans.** Rs. 4,957.70]

26. *A* is entitled to Rs. 200 per annum for 4 years, the first being payable after one year from now. For the next 5 years he is entitled to Rs. 150 per annum. Find the present value of these payments at 7% per annum. Find also the accumulated value of the annuity at the end of 9 years. [**Ans.** Rs. 1,146.64, Rs. 2,108.06]

27. Establish the relation : $a^{(m)}_{\overline{n}|i} = a_{\overline{m+n}|i} - a_{\overline{m}|i}$.

28. An annuity consists of 15 equal annual payments of Rs. 7,500, the first being made at the end of 6 years. Find the present value of this annuity, if interest is compounded continuously at 6% per annum. [**Ans.** Rs. 54,953.75]

29. Mr. *X* borrowed Rs. 10,000 from a bank and agreed to repay the debt in 10 equal annual instalment, the first being made after 5 year from the date of borrowing. If the interest is calculated at a rate of 7% per annum compounded continuously, what would have been the size of the annual instalment ? [**Ans.** Rs. 1,839.83]

30. Find the present value of a sequence of annual payments of 2,500 each, the first being made at the end of 7 years and the last at the end of 16 years, if the money is worth 5% per annum compounded continuously. Also find the amount of this annuity. [**Ans.** Rs. 14,574.50, Rs. 32,435]