Géza Schay

A Concise Introduction to Linear Algebra

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Preface

This book is designed for a one-semester, post-calculus linear algebra course, primarily intended for mathematics, physics, and computer science majors. While basic calculus is a prerequisite for such a course, very little of it is used in the book. Certainly, multivariable calculus is not required. Vectors are treated fully in Chapter 1, but for classes familiar with them, this chapter may be skipped or just reviewed briefly. Complex numbers, series, and exponentials are presented briefly in an appendix, but they are needed only in Section 7.4, which may not be covered in some courses.

The selection of topics conforms to a large extent to the recommendations of the Linear Algebra Curriculum Study Group.[1] The main differences are that the book begins with a chapter on Euclidean vector geometry, mostly in three dimensions; determinants are treated more fully and are placed just before eigenvalues, which is where they are needed; the LU factorization is relegated to Chapter 8 on numerical methods; and the facts about linear transformations are collected in one chapter and are treated in more detail.

This book is considerably shorter than the 400 to 800 pages of most introductory linear algebra books, which are more suitable for two- or three-semester courses.

While many applications are presented, they are mostly taken from physics, and several new ones have been added in the second edition. However, these examples give only a glimpse of how the subject is used in other fields, and further details are left to texts in those fields. There is, though, a section on computer graphics and a chapter on numerical methods. Also, most sections contain MATLAB® exercises. On the other hand, we hope that the student's interest will be aroused not only by the possible applications, but also by the geometrical background and the beautiful structure of linear algebra. Nevertheless, for readers especially interested in applications, a list of the ones discussed follows this preface.

The more difficult exercises and theorems are marked by an asterisk. Some exercises are used to develop new topics, whose inclusion in the main text would have disrupted the flow of ideas. The symbols ■ and ⧫ are used to indicate the end of proofs and examples, respectively.

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[1] David Carlson, Charles R. Johnson, David C. Lay, A. Duane Porter. The Linear Algebra Curriculum Study Group Recommendations for the First Course in Linear Algebra. *College Mathematics Journal*, 24:1 (1993) 41–46.

In this second edition, in response to the concerns of some users of the first edition, many of the earlier proofs and explanations have been expanded and a few new ones added. Also, exercises involving laborious computations have been replaced by simpler ones, and some new ones have been added.

Foreword to Instructors

- The brevity mentioned above makes the book easier to use. Important points are not drowned in a sea of detail, and instructors and students do not have to search for what to keep and what to omit. In a minimal course, however, the following sections may be omitted entirely: Section 4.3 on computer graphics, Section 5.1 on orthogonal projections and least squares, Section 6.2 on cofactor expansions of determinants, Cramer's rule, etc., Section 6.3 on the cross product, Sections 7.3 and 7.4 on principal axes and complex matrices, and Chapter 8 on numerical methods. Theorem 3.4.8 (The Exchange Theorem) may also be omitted, since an alternative direct proof of the dimension theorem is provided in the new edition.
- The geometric content is heavily emphasized. In fact, as mentioned above, the book begins with a chapter on Euclidean vector geometry, mostly in three dimensions. Most other similar textbooks start with the solution of linear systems. We believe that this early introduction of the geometrical background helps students to visualize the concepts of linear algebra and provides easy concrete examples. Additionally, many students in this course, e.g., computer science majors, are not required to take multivariable calculus, and do not see this important material anywhere else.
- In the first chapter, the equations of planes are given in both parametric and nonparametric form, in contrast to most calculus books, which present only the nonparametric form. Many examples and exercises illustrate the transition from one form to the other. However, we avoid using the cross product at this stage, because it is only available in $\mathbb{R}^3$. We use the method of solving simultaneous equations to obtain a normal vector to a plane, and this topic is revisited as an example to Gaussian elimination. On the other hand, Section 6.3 is devoted to the cross product as an illustration of the use of determinants, and it is only at that point that it is used to obtain a normal vector to a plane.
- The "back and forth" process between parametric and nonparametric equations for lines and planes lays the groundwork for the same transition between describing a subspace of $\mathbb{R}^n$ as a set of linear combinations or as the solution set of a homogeneous system of linear equations, that is, as the column space of a matrix or the null space of another matrix. Another generalization of this issue is finding orthogonal complements of subspaces of $\mathbb{R}^n$ given in either form.
- Many books use the notation $\|\mathbf{p}\|$ for the length of a vector $\mathbf{p}$ in $\mathbb{R}^n$, but we prefer $|\mathbf{p}|$, because in $\mathbb{R}^1$ length is the absolute value, and there is no

reason to change notation for higher dimensions, just as there was none in using + for addition of both scalars and of vectors. The notation $\|\mathbf{p}\|$ is left for other norms.

- Important concepts are presented as definitions and theorems. Students are advised to memorize them. It is not enough just to understand the material; the main concepts must be remembered well to be able to build on them.
- Except for the Spectral Theorem in the complex case and theorems from other fields of mathematics, all theorems are proved. It is thus left to the instructor to adjust the level of the course from the computational to the fairly theoretical by omitting as many or as few proofs as desired.
- Great care has been taken to motivate every new concept, even those that many books do not, such as dot product, matrix operations, linear independence (not just in two or three dimensions), determinants, eigenvalues, and eigenvectors.
- The letter symbols are selected to reflect the connections between related quantities, a principle often ignored in other linear algebra books. Vectors and their components, matrices and their column and row vectors and entries are denoted by the same letters with different fonts, like $\mathbf{v}$, v_i and A, $\mathbf{a}_i$, $\mathbf{a}^j$, a_{ij}. The main exception is the unit matrix, which is, bowing to tradition, denoted by I, its columns by $\mathbf{e}_i$, and its entries by δ_{ij}.
- Only standard notation is used, so that students who study further, will have no difficulty in reading applied or more advanced texts. Nonstandard notation, such as the use of a list in parentheses for column vectors and in brackets for row vectors, or $\overrightarrow{\mathbf{a}}_i$ or $\mathbf{A}_i$ for a row vector of a matrix, found in some other introductory linear algebra books, is avoided. We use $\mathbf{a}_i$ for the column vectors of a matrix A and $\mathbf{a}^i$ for its row vectors. This is standard notation in more advanced books. (See, e.g., *Introduction to Linear and Nonlinear Programming* by David G. Luenberger, Addison-Wesley, 1973.) We also use $\mathbf{x}_A = (x_{A1}, x_{A2}, \ldots, x_{An})^T$ for the coordinate vector of a vector $\mathbf{x}$ relative to an ordered basis or basis matrix A. (Compare this, e.g., with the notation $[\mathbf{x}]_{\mathcal{B}} = (c_1, c_2, \ldots c_n)^T$ of *Linear Algebra and Its Applications* by David Lay, Addison-Wesley, 1993, where the brackets on the left are superfluous, the coordinates of $\mathbf{x}$ are denoted by the unrelated letter c, and the basis $\mathcal{B}$ is not indicated on the right, not to mention that we need an ordered basis or basis matrix here.) Our notation makes the notoriously messy topic of change of basis much simpler.
- Similarity of matrices is introduced in the context of changing bases.
- Most introductory linear algebra books introduce determinants by unmotivated formulas. This book introduces them by three simple properties, expanding on the approach in Strang.[2]

[2] Gilbert Strang, *Linear Algebra and its Applications*, 3rd ed. Harcourt Brace, San Diego, 1988.

- MATLAB exercises at the end of most sections reinforce and expand the linear algebra material. They also provide some introduction to MATLAB, but should be used in conjunction with a MATLAB manual.
- The appendix on implication and equivalence introduces the student in an informal way to certain crucial elements of proofs, and is highly recommended reading for most.
- All displayed equations are numbered, and in the new edition, mnemonic headings are appended to all definitions, theorems, figures, and examples. These numbers and headings should make references to these items easier and make their connections more transparent.

Foreword to Students

Linear algebra is probably your first mathematics course in which the theory is just as important as the computations. To study from this book you have to carefully read the text with paper and pencil in hand.

The book starts out gently, with analytic geometry, but soon the algebra takes over and the subject becomes more abstract, which may cause some difficulty for some of you.

Studying this kind of mathematics involves three interwoven steps:

1. You must understand the material.
2. You must learn the concepts thoroughly so that you remember them and can apply them knowledgeably.
3. You must practice it, doing exercises.

Each of these steps is necessary and supports the others.

In many other subjects, understanding is not a problem, and so many students believe that once they pass that hurdle, they have done enough. Not true: If you understand something in class, that does not mean you will know it the next day. You must study after every class and make sure that you are able to explain the material in your own words so that you do not forget it. If you don't, then you have to start over again on your own, with the class attendance wasted. You will need to study several hours after every class. This is especially important, because most concepts are built upon each other. For instance, vectors, introduced in Section 1.1, are used throughout the book; matrices introduced in Section 2.2 are used throughout the rest of the book, and so on.

On the other hand, you cannot do mathematics by rote memorization without understanding, because the subject is generally too complicated for that. Also, doing that would defeat the whole purpose of studying mathematics, which is the comprehension of its logic and the ability to use it in applications—not just in those that were presented, but in other similar (or even somewhat different) applications.

Working out solutions to the exercises reinforces both the learning and the understanding of the material and is often also useful in its own right, because many exercises involve important applications of the theory.

In studying linear algebra, you have to thoroughly understand and remember the definitions first, since everything else is built on them. If you don't remember a definition, you cannot possibly understand the theory that depends on it and the exercises that make use of it.

Next in importance come the theorems, lemmas (minor or auxiliary theorems), and corollaries. These are usually preceded by introductory examples and followed by further examples that illuminate various aspects and applications of the theorems. You must study these examples together with the theorems and their proofs. It is permissible to read everything just superficially at first, to get a basic understanding, but after that, you must study it again in detail. When studying a theorem, isolate the conditions or hypotheses which make it tick. Try to see where these conditions are used in the proof, and what would happen if a condition were changed or omitted. After pinpointing the conditions, do the same for the conclusions, and last, try to follow the steps of the proof. This is where the paper and pencil come in: Write these steps down. Close the book and write down the conditions, the conclusions, or the whole statement that you are studying. Try to fill in steps that are just briefly indicated in the proofs. If the proof has a reference to some earlier material, be sure to look it up and explain to yourself how it is used. The same advice applies to the follow-up examples as well: make sure you see where the conditions of the theorem are used and why they are necessary, and follow the computations on paper.

There is an appendix on implication and equivalence, which introduces in an informal way certain crucial elements of proofs. It is highly recommended reading for all those who have not seen this material before.

Finally, after you have gone through the steps listed above, you will be ready to tackle exercises. The odd-numbered ones have solutions available in a Students' Solution Manual on the book's webpage. Do those exercises first; they are usually similar to examples in the text. Don't look at the solution before making a really serious attempt to solve a problem on your own. If a problem looks too difficult at first, then look at a similar example in the text or go back and review the definition or theorem that the problem is intended to illustrate. A problem that you have solved stays much better in your mind than one that you have merely read, and its structure becomes much clearer. But, of course, once you have solved a problem, there is no harm in looking up the solution. You may even learn a different way of solving it, or find an error in your solution (or perhaps in the solution manual).

If you follow the advice above, you will probably find linear algebra to be a very interesting and enjoyable subject, but if you don't, then it may become an unpleasant chore.

Acknowledgments

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The second edition owes many improvements to suggestions of Stephen Parrott, John Lutts, and an anonymous reviewer, which are gratefully acknowledged. Finally, I am indebted to my colleague Dennis Wortman for thoroughly checking the latest version of the manuscript and suggesting many changes, corrections, and additions of every possible type, from commas to clarifications.

Géza Schay

List of Applications

1. Center of mass. Exercises 1.1.7, 1.1.8.
2. Work as a dot product. Example 1.2.4.
3. Equations of lines and planes. Section 1.3.
4. An electrical network, Kirchhof's laws. Example 2.3.2.
5. A connection matrix for an airline. Example 2.4.8.
6. Population changes. Example 2.4.9.
7. The structure of the system expressing Kirchhof's laws. Example 3.5.7.
8. Hermite polynomials. Example 3.6.4.
9. Legendre polynomials. Exercise 3.6.11.
10. Computer graphics. Section 4.3.
11. Orthogonal projections and least-squares approximations. Section 5.1.
12. Coriolis force. Example 6.3.5.
13. Lorentz force. Example 6.3.6.
14. Systems of difference and differential equations. Section 7.2.
15. Population growth. Example 7.2.1.
16. An electric circuit with resistor, condenser, and coil. Example 7.2.2.
17. A predator-prey population model. Exercise 7.2.11.
18. Conic sections and quadric surfaces. Section 7.3.

1. *Analytic Geometry of Euclidean Spaces*

1.1 Vectors

We begin by describing some geometrical concepts. This approach may seem strange in a book on algebra, but the influence of geometry is fundamental to our subject, since the underlying geometrical ideas provide motivation, examples, and applications for the algebraic constructions. In fact, the adjective "linear" in this book's title means "pertaining to lines" (which in mathematics usually mean straight lines), and indicates the geometric origins of linear algebra.

In physics, several important notions such as displacement, velocity, and force possess not just a magnitude but a direction as well. These are typical of a large class of quantities called *vectors*, which can be depicted by arrows showing the desired directions and representing the vectors' magnitudes by their lengths. In geometry, we can use them to locate points and also, as we shall see later, to write equations of lines and planes. Let us look at a few such examples before stating formal definitions.

Example 1.1.1. (Position Vectors). Either in the plane or in three-dimensional space, consider a fixed point O and other points P, Q, R, and draw correspondingly labeled arrows $\mathbf{p}$, $\mathbf{q}$, $\mathbf{r}$ from O to the other points (see Figure 1.1). These arrows are called the *position vectors* or *radius vectors*

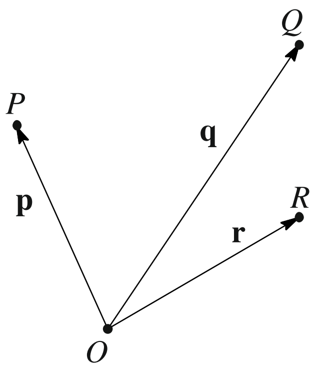

Fig. 1.1. Position vectors

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of P, Q, R relative to the point O, which is usually regarded as the origin of a coordinate system. Such vectors are also sometimes called bound vectors, for they are bound to the origin, in contrast to free vectors to be introduced shortly. The position vector of the point O is a special vector $\mathbf{0}$, called the zero or null vector, whose length is 0, and whose direction is undefined. ♦

Whereas in print, vectors are generally denoted by lowercase boldface letters such as $\mathbf{p}$, $\mathbf{q}$, $\mathbf{r}$, or by symbols like $\overrightarrow{OP}$, $\overrightarrow{OQ}$, in handwriting, boldface would be difficult and so $\underline{p}, \underline{q}$ or $\overrightarrow{p}$, $\overrightarrow{q}$, etc. are used instead.

Since position vectors and points are in one-to-one correspondence, you may wonder why we need position vectors at all. The answer is that various arithmetic operations that would make no sense with points can be performed with vectors, and will lend themselves to all kinds of useful constructions. Such operations are also essential for the vectors of physics.

Example 1.1.2. (Adding Forces). If, in Figure 1.2, $\mathbf{p}$ and $\mathbf{q}$ represent two *forces* acting simultaneously on a point mass at O, then the single force represented by $\mathbf{r}$, to be defined as $\mathbf{p} + \mathbf{q}$, would have the same effect. ♦

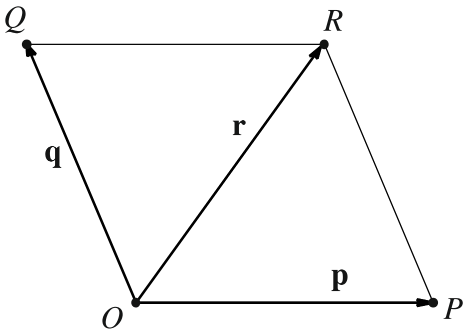

Fig. 1.2. $\mathbf{r} = \mathbf{p} + \mathbf{q}$

Example 1.1.3. (AddingDisplacements). If, in Figure 1.2, $\mathbf{p}$ and $\mathbf{q}$ represent simultaneous *displacements*, then $\mathbf{r}$ represents their combined effect. This happens, for example, if a person on a boat at O walks to Q while the point O of the boat moves, together with the boat, to P (and the point Q of the boat to R). Then, as seen from the shore, the person ends up at R. ♦

The last two examples illustrate how addition of such vectors is defined. Given any pair $\mathbf{p}$ and $\mathbf{q}$ as in Figure 1.2, the corresponding points O, P, Q determine a parallelogram $OPRQ$, and the sum $\mathbf{p} + \mathbf{q}$ is defined as the diagonal vector $\mathbf{r} = \overrightarrow{OR}$. This is called the *parallelogram law* of vector addition.

A second operation we consider is multiplication of vectors by scalars. (In this context real numbers are usually called scalars, since they can be pictured on a scale, unlike vectors.) Let c be any scalar and $\mathbf{p}$ any vector, as in the previous examples. The vector $c\mathbf{p}$ is defined as the vector whose length is $|c|$ times the length of $\mathbf{p}$ and whose direction is the same as that of $\mathbf{p}$ if $c > 0$, and opposite if $c < 0$. If $c = 0$, then $c\mathbf{p}$ is the zero vector. Two examples of this type of multiplication are shown in Figure 1.3.

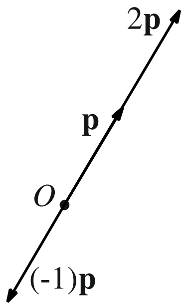

Fig. 1.3. Scalar multiples of a vector

The discussion has been somewhat informal so far, because we have not really specified very precisely the sets of vectors under consideration. It is best to remedy this omission by introducing a coordinate system into the picture.

If we consider the position vector $\mathbf{p}$ of a point P in a plane (see Figure 1.4) and introduce a Cartesian coordinate system, then we can represent the vector $\mathbf{p}$, as well as the point P, by the ordered pair (p_1, p_2) of coordinates, and write $\mathbf{p} = (p_1, p_2)$. For this representation to be of any use, we recast the parallelogram law and the multiplication of vectors by scalars in terms of the coordinates, as follows.

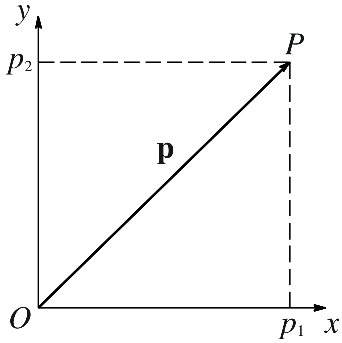

Fig. 1.4. The coordinates of a point P are the components of its position vector $\mathbf{p}$

For any two vectors $\mathbf{p} = (p_1, p_2)$ and $\mathbf{q} = (q_1, q_2)$, Figure 1.5 illustrates that if $\mathbf{r} = \mathbf{p} + \mathbf{q}$ is the diagonal of the parallelogram spanned by $\mathbf{p}$ and $\mathbf{q}$,

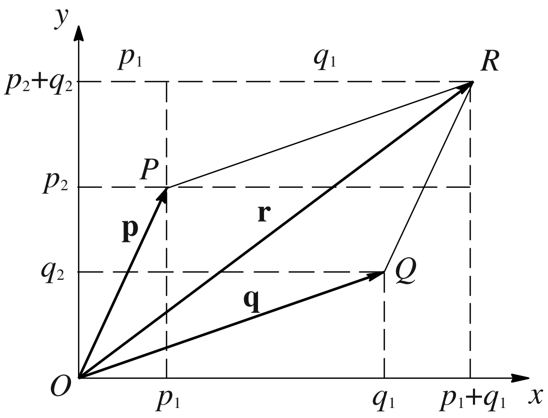

Fig. 1.5. The parallelogram law in terms of coordinates

then $\mathbf{p} + \mathbf{q} = (p_1 + q_1, p_2 + q_2)$ must hold; that is, we must simply add the corresponding coordinates. Similarly, we must have $c\mathbf{p} = (cp_1, cp_2)$ for every scalar c.

In light of the above discussion we now make this formal definition.

Definition 1.1.1. ***(Two-Dimensional Euclidean Vector Space).*** *The set of all ordered pairs of real numbers, together with the two algebraic operations defined below, is called the two-dimensional Euclidean vector space* $\mathbb{R}^2$.

The elements of $\mathbb{R}^2$ *are called two-dimensional vectors (or coordinate vectors) and we define the operations of vector addition and multiplication of a vector by a scalar by*

$$(p_1, p_2) + (q_1, q_2) = (p_1 + q_1, p_2 + q_2), \tag{1.1}$$

and

$$c(p_1, p_2) = (cp_1, cp_2) \tag{1.2}$$

for every (p_1, p_2), (q_1, q_2) *and any scalar* c.

The scalars p_1 *and* p_2 *are called the components of the vector* $\mathbf{p} = (p_1, p_2)$. *Furthermore, two vectors are said to be equal if and only if their corresponding components are equal.*

Example 1.1.4. (A Parallelogram). In Figure 1.6, let the points $P = (1, 5)$ and $Q = (3, 1)$ be given. Then the corresponding coordinate vectors are $\mathbf{p} = (1, 5)$ and $\mathbf{q} = (3, 1)$, and the position vector of the point R that makes $OQRP$ into a parallelogram is $\mathbf{r} = \mathbf{p} + \mathbf{q} = (1 + 3, 5 + 1) = (4, 6)$. The midpoint M of the parallelogram has the position vector $\frac{1}{2}\mathbf{r} = (2, 3)$. ♦

The following simple properties follow from Definition 1.1.1 and the algebraic properties of real numbers. They will be used in the definition of a general vector space in Chapter 3.

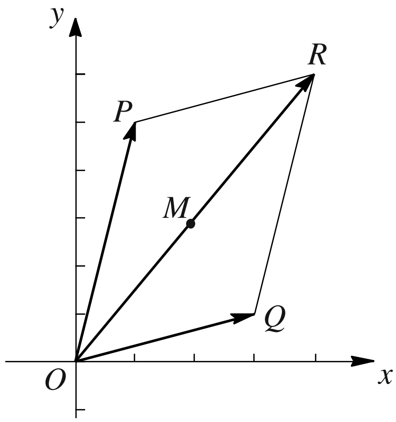

Fig. 1.6. The midpoint of a parallelogram in terms of the position vectors of the vertices

Theorem 1.1.1. ***(Basic Properties of Vectors in*** $\mathbb{R}^2$***).*** *For all vectors* $\mathbf{p}, \mathbf{q}, \mathbf{r}$ *in* $\mathbb{R}^2$ *and all scalars* a, b *we have:*
1. $\mathbf{p} + \mathbf{q} = \mathbf{q} + \mathbf{p}$ *(commutativity of addition),*
2. $(\mathbf{p} + \mathbf{q}) + \mathbf{r} = \mathbf{p} + (\mathbf{q} + \mathbf{r})$ *(associativity of addition),*
3. There is a vector $\mathbf{0}$ *such that* $\mathbf{p} + \mathbf{0} = \mathbf{p}$ *for all vectors* $\mathbf{p}$ *(existence of zero vector),*
4. For each vector $\mathbf{p}$ *there is a vector* $-\mathbf{p}$ *such that* $\mathbf{p} + (-\mathbf{p}) = \mathbf{0}$ *(existence of additive inverse),*
5. $1\mathbf{p} = \mathbf{p}$ *(rule of multiplication by 1),*
6. $a(b\mathbf{p}) = (ab)\mathbf{p}$ *(associativity of multiplication by scalars)*[1]*,*
7. $(a + b)\mathbf{p} = a\mathbf{p} + b\mathbf{p}$ *(first distributive law),*
8. $a(\mathbf{p} + \mathbf{q}) = a\mathbf{p} + a\mathbf{q}$ *(second distributive law).*

Proof. 1. $\mathbf{p} + \mathbf{q} = (p_1 + q_1, p_2 + q_2) = (q_1 + p_1, q_2 + p_2) = \mathbf{q} + \mathbf{p}$.

2. $(\mathbf{p} + \mathbf{q}) + \mathbf{r} = (p_1 + q_1, p_2 + q_2) + (r_1, r_2) = (p_1 + q_1 + r_1, p_2 + q_2 + r_2) = (p_1, p_2) + (q_1 + r_1, q_2 + r_2) = \mathbf{p} + (\mathbf{q} + \mathbf{r})$.

3. Defining $\mathbf{0} = (0, 0)$ we have $\mathbf{p} + \mathbf{0} = (p_1 + 0, p_2 + 0) = (p_1, p_2) = \mathbf{p}$.

4. Defining $-\mathbf{p} = (-p_1, -p_2)$ we have $\mathbf{p} + (-\mathbf{p}) = (p_1 + (-p_1), p_2 + (-p_2)) = (0, 0) = \mathbf{0}$.

We leave the rest to the reader. ■

Let us remark that Properties 2, 6, 7, and 8 can be extended to several vectors and scalars much as for numbers, and we shall use such extensions without further ado.

Subtraction of vectors can be defined just as for numbers.

[1] "Associativity" is nonstandard here; there is no commonly used name for this property.

Definition 1.1.2. ***(Subtraction in*** $\mathbb{R}^2$***).*** *For every* $\mathbf{p} = (p_1, p_2)$, $\mathbf{q} = (q_1, q_2) \in \mathbb{R}^2$, *we define*

$$\mathbf{p} - \mathbf{q} = \mathbf{p} + (-\mathbf{q}). \tag{1.3}$$

The definitions lead at once to the following alternative expressions for the negatives of vectors and for their subtraction in terms of components.

Theorem 1.1.2. ***(Negative and Subtraction in*** $\mathbb{R}^2$ ***in Terms of Components).*** *For every* $\mathbf{p} = (p_1, p_2)$, $\mathbf{q} = (q_1, q_2) \in \mathbb{R}^2$,

$$-\mathbf{p} = (-1)\,(p_1, p_2) \tag{1.4}$$

and

$$\mathbf{p} - \mathbf{q} = (p_1 - q_1, p_2 - q_2). \tag{1.5}$$

Example 1.1.5. (A Subtraction in $\mathbb{R}^2$***).*** Let $\mathbf{p} = (1, -3)$ and $\mathbf{q} = (-4, 5)$. Then $-\mathbf{p} = (-1)(1, -3) = (-1, 3)$, $-\mathbf{q} = (-1)(-4, 5) = (4, -5)$, and $\mathbf{p} - \mathbf{q} = (1, -3) + (4, -5) = (5, -8)$. ♦

We have the following list of further properties of vectors.

Theorem 1.1.3. ***(Properties of Vectors in*** $\mathbb{R}^2$ ***Involving 0 and Subtraction).*** *For all vectors* $\mathbf{p}, \mathbf{q}, \mathbf{x}$ *in* $\mathbb{R}^2$ *and all scalars c and d we have*
1. $0\mathbf{p} = \mathbf{0}$,
2. $c\,\mathbf{0} = \mathbf{0}$,
3. $\mathbf{p} + \mathbf{x} = \mathbf{q}$ *if and only if* $\mathbf{x} = \mathbf{q} - \mathbf{p}$,
4. If $c\mathbf{p} = \mathbf{0}$ *then either* $c = 0$ *or* $\mathbf{p} = \mathbf{0}$ *or both,*
5. $(-c)\mathbf{p} = c(-\mathbf{p}) = -(c\mathbf{p})$,
6. $c(\mathbf{p} - \mathbf{q}) = c\mathbf{p} - c\mathbf{q}$,
7. $(c - d)\mathbf{p} = c\mathbf{p} - d\mathbf{p}$.

Proof. We prove only Property 3. There are two statements here: the "if" and the "only if" part. Writing $\mathbf{p} = (p_1, p_2)$, $\mathbf{q} = (q_1, q_2)$, and $\mathbf{x} = (x_1, x_2)$, if $\mathbf{x} = \mathbf{q} - \mathbf{p}$, then we have

$$\mathbf{x} = (x_1, x_2) = (q_1 - p_1, q_2 - p_2) \tag{1.6}$$

and so

$$\mathbf{p} + \mathbf{x} = (p_1 + (q_1 - p_1), p_2 + (q_2 - p_2)) = (q_1, q_2) = \mathbf{q} \tag{1.7}$$

must hold.

Conversely, the "only if" part of Property 3 is equivalent to saying that if $\mathbf{p} + \mathbf{x} = \mathbf{q}$, then $\mathbf{x} = \mathbf{q} - \mathbf{p}$. (See Appendix 1.) So, to prove this part, assume $\mathbf{p} + \mathbf{x} = \mathbf{q}$. Then we can write this equation in components as

$$(p_1 + x_1, p_2 + x_2) = (q_1, q_2) \tag{1.8}$$

and, because the equality of two vectors means that the corresponding components must be equal, we have

$$p_1 + x_1 = q_1 \tag{1.9}$$

and

$$p_2 + x_2 = q_2. \tag{1.10}$$

Solving these equations for x_1 and x_2 and combining them into a vector, we get

$$\mathbf{x} = (x_1, x_2) = (q_1 - p_1, q_2 - p_2) = \mathbf{q} - \mathbf{p}. \tag{1.11}$$

■

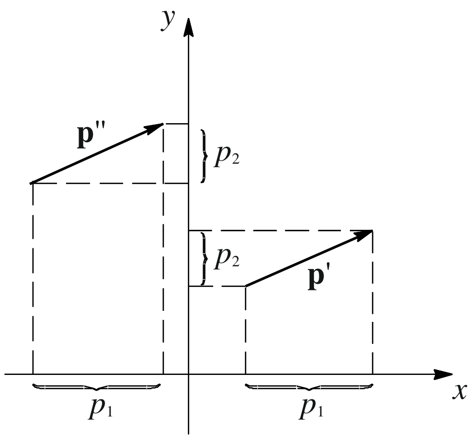

Fig. 1.7. Representative arrows of a vector in $\mathbb{R}^2$

There is an additional, important way of associating arrows with ordered pairs of coordinates. If we draw an arrow $\mathbf{p}'$ anywhere in a coordinate system (see Figure 1.7), not necessarily at the origin, then we can still project it perpendicularly onto the axes and consider the signed lengths p_1, p_2 of the projections to be the components of a vector in $\mathbb{R}^2$. Of course, any other arrow, such as $\mathbf{p}''$, obtained from $\mathbf{p}'$ by a parallel shift, will produce the same p_1, p_2 values. Thus for a given vector $(p_1, p_2) \in \mathbb{R}^2$ there corresponds a class $\mathbf{p}$ of infinitely many arrows parallel to each other and equal in length,[2] all having the same signed scalar projections p_1 and p_2. The arrows like $\mathbf{p}'$ and $\mathbf{p}''$ are

[2] Equivalence class is a standard term used for sets whose members constitute all objects equivalent to each other under a certain type of relation called an equivalence relation.

equivalent representatives of the class $\mathbf{p}$. Such classes of equivalent arrows are called *free vectors*, since the arrows can be shifted freely. We usually identify the free vector $\mathbf{p}$ with the vector $(p_1, p_2) \in \mathbb{R}^2$, that is, we write $\mathbf{p} = (p_1, p_2)$. This should not lead to confusion, just as referring to a point as (x, y) instead of a point P with coordinates (x, y) does not.

A free vector can be represented by any one of its arrows; that is, the whole class $\mathbf{p}$ is known if any member of $\mathbf{p}$ is known. Unfortunately, many people confuse the class $\mathbf{p}$ with the individual arrows, and call $\mathbf{p}'$ and $\mathbf{p}''$ equal *vectors*, rather than just equivalent representative *arrows* of the vector $(p_1, p_2) \in \mathbb{R}^2$ or of the free vector $\mathbf{p}$.

Why do we use free vectors at all? There are at least three reasons. First, they arise rather naturally as representations of coordinate vectors, as we have just seen. Second, in physical applications some vector quantities are not bound to any fixed point. (For example, the velocity vector of a non-rotating object can reasonably be drawn at any point of the object.) Third, the pictures of many constructions become simpler and less cluttered if we use well-positioned representative arrows, rather that just vectors at O. Many examples of this usage will follow, but here we just look at a variation of the addition of vectors in terms of free vectors. In Figure 1.8, let the arrows marked $\mathbf{p}$ and $\mathbf{q}$ represent the free vectors $\mathbf{p}$ and $\mathbf{q}$ (we shall abbreviate this statement from now on as customary to "let $\mathbf{p}$ and $\mathbf{q}$ be two vectors as shown"). Then, obviously, their sum is represented by the arrow $\mathbf{p} + \mathbf{q}$. This is sometimes called the *triangle law* of vector addition. If the arrows represent displacements, then it is the natural description of their sum, that is, of one displacement followed by another.

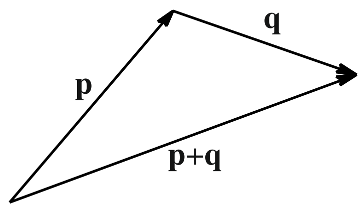

Fig. 1.8. The triangle law of vector addition

Now let us turn to the addition of several, say four, vectors $\mathbf{p}$, $\mathbf{q}$, $\mathbf{r}$, $\mathbf{s}$ as given in Figure 1.9. By repeated application of the triangle law we get the sum as shown. (Because of the associativity of vector addition, just as with numbers, we do not need parentheses in the sum.) Contrast the simplicity of this construction with the mess we would get if all vectors were drawn at O.

Since we have $(\mathbf{p} + \mathbf{q}) - \mathbf{p} = \mathbf{q}$, we can relabel Figure 1.8 to illustrate the subtraction of vectors, by writing $\mathbf{r}$ for $\mathbf{p} + \mathbf{q}$ and $\mathbf{r} - \mathbf{p}$ for $\mathbf{q}$ as in Figure 1.10. The triangle law applied to Figure 1.10 shows that $\mathbf{p} + (\mathbf{r} - \mathbf{p}) = \mathbf{r}$, as it should be, and that $\overrightarrow{PR} = \mathbf{r} - \mathbf{p}$. This construction is especially useful for

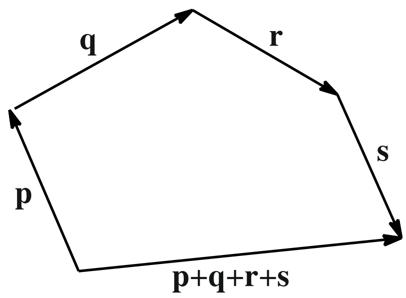

Fig. 1.9. Addition of several vectors by the triangle law

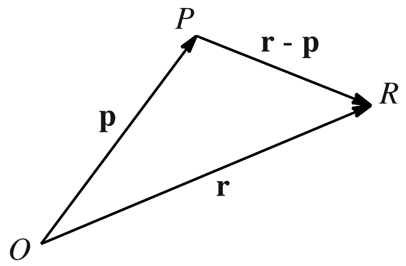

Fig. 1.10. Subtraction of vectors

obtaining the coordinate vectors of arrows joining given points as in the following example.

Example 1.1.6. (A Vector with Given Endpoints). Given two points $P = (1, 2)$ and $R = (3, 6)$ in the plane, find the coordinate vector of $\overrightarrow{PR}$.

We can write the position vectors of the given points as $\mathbf{p} = (1, 2)$ and $\mathbf{r} = (3, 6)$, and so $\overrightarrow{PR} = \mathbf{r} - \mathbf{p} = (3 - 1, 6 - 2) = (2, 4)$. ♦

Example 1.1.7. (Finding Various Points of a Parallelogram). Given three points $A = (4, 3)$, $B = (-1, 4)$, and $C = (0, -2)$ in the plane (see Figure 1.11), find the coordinates of the point D that makes $ABDC$ a parallelogram, and those of the midpoint M of the parallelogram.

The position vectors of the given points are $\mathbf{a} = (4, 3)$, $\mathbf{b} = (-1, 4)$, and $\mathbf{c} = (0, -2)$. Then, finding first $\overrightarrow{AB} = \mathbf{b} - \mathbf{a} = (-5, 1)$ and $\overrightarrow{AC} = \mathbf{c} - \mathbf{a} = (-4, -5)$, we can use them to find $\overrightarrow{AD} = (\mathbf{b} - \mathbf{a}) + (\mathbf{c} - \mathbf{a}) = (-9, -4)$. Now $\mathbf{d} = \mathbf{a} + \overrightarrow{AD} = (-5, -1)$, and this ordered pair also gives the coordinates of D. The position vector $\mathbf{m}$ of the midpoint M can be obtained as $\mathbf{m} = \mathbf{a} + \frac{1}{2}\overrightarrow{AD} = (4, 3) + \frac{1}{2}(-9, -4) = (-\frac{1}{2}, 1)$. ♦

We now define the three-dimensional Euclidean vector space $\mathbb{R}^3$ of coordinate vectors.

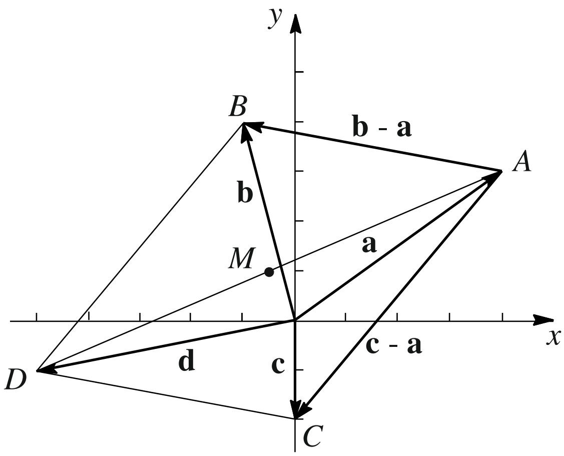

Fig. 1.11. Computing the coordinates of vertex D and midpoint M of a parallelogram, given three vertices A, B, C

Definition 1.1.3. ***(Three-Dimensional Euclidean Vector Space).*** *The vector space* $\mathbb{R}^3$ *is the set of all ordered triples* (p_1, p_2, p_3) *of real numbers with the operations defined componentwise just as in* $\mathbb{R}^2$*: For all* (p_1, p_2, p_3), (q_1, q_2, q_3) *and every scalar* c,

$$(p_1, p_2, p_3) + (q_1, q_2, q_3) = (p_1 + q_1, p_2 + q_2, p_3 + q_3), \tag{1.12}$$

and

$$c(p_1, p_2, p_3) = (cp_1, cp_2, cp_3). \tag{1.13}$$

Again, $\mathbb{R}^3$ *is called the three-dimensional Euclidean vector space and its elements are called three-dimensional vectors (or coordinate vectors). The scalars* p_1, p_2, p_3 *are called the components of the vector* $\mathbf{p} = (p_1, p_2, p_3)$ *and two vectors are said to be equal if and only if their corresponding components are equal.*

Just as in two dimensions, if we introduce a Cartesian coordinate system with the origin at O, then every $\mathbf{p} \in \mathbb{R}^3$ can be regarded as the position vector of the corresponding point P. (See Figure 1.12). Thus we identify the arrow $\mathbf{p}$ with the coordinate vector (p_1, p_2, p_3).

In three-dimensional space we can again represent coordinate vectors also by arrows drawn anywhere, not just at the origin, and we define free vectors much as in the plane.

About Figure 1.12, let us remark that the x-axis is meant to be interpreted as pointing out of the paper towards the reader. (This sense is not obvious: if you stare at the picture hard, you may see it as pointing into the paper.) In three dimensions, two kinds of coordinate systems are possible: the kind

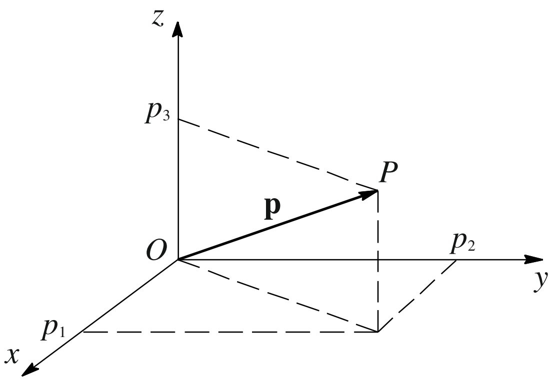

Fig. 1.12. Coordinates in $\mathbb{R}^3$

pictured here and its mirror image. The one shown is called a right-handed coordinate system, since the x, y, z axes point like the thumb, index, and middle finger of the right hand, respectively. (See Figure 1.13.) By convention, left-handed coordinate systems are rarely used.

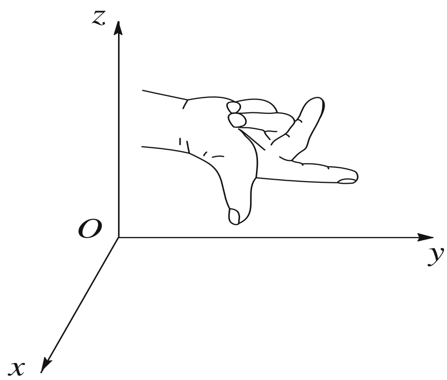

Fig. 1.13. The right hand gives the orientation of the right-handed coordinate system

Although there is no way of picturing it when $n > 3$, the n-dimensional vector space $\mathbb{R}^n$ is defined algebraically as follows.

Definition 1.1.4. ***(n-Dimensional Euclidean Vector Space).*** *For every positive integer* n, $\mathbb{R}^n$ *is the vector space of ordered* n*-tuples* $(x_1, x_2, \ldots, x_n)$ *of real numbers, with the basic operations defined by*

$$(x_1, x_2, \ldots, x_n) + (y_1, y_2, \ldots, y_n) = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n) \quad (1.14)$$

and

$$c(x_1, x_2, \ldots, x_n) = (cx_1, cx_2, \ldots, cx_n) \tag{1.15}$$

for all vectors $(x_1, x_2, \ldots, x_n)$ *and* $(y_1, y_2, \ldots, y_n)$ *and every scalar* c.

Again, $\mathbb{R}^n$ *is called the n-dimensional Euclidean vector space, and its elements are called n-dimensional vectors (or coordinate vectors). The scalars* $x_1, x_2, \ldots, x_n$ *are called the components of the vector* $\mathbf{x} = (x_1, x_2, \ldots, x_n)$*, and two vectors are said to be equal if and only if their corresponding components are equal.*

Such coordinate vectors with $n > 3$ arise in many applications. For instance, in physics the configuration space of n point-like particles is defined as the $3n$-dimensional vector space $\mathbb{R}^{3n}$ whose vectors are made up of the particles' coordinates, and the phase space as the $6n$-dimensional vector space $\mathbb{R}^{6n}$ whose vectors' components are the coordinates and the momentum components of the particles. Similarly, in theoretical economics the prices and quantities of n commodities are frequently represented by n-dimensional vectors.

We can define negatives and subtraction of vectors as before.

Definition 1.1.5. ***(Negative and Subtraction in*** $\mathbb{R}^n$***).*** *For all* $\mathbf{p}, \mathbf{q} \in \mathbb{R}^n$ *we define*

$$-\mathbf{p} = (-\mathbf{1})\mathbf{p} \tag{1.16}$$

and

$$\mathbf{p} - \mathbf{q} = \mathbf{p} + (-\mathbf{q}). \tag{1.17}$$

Additionally, we adopt the following notational conventions:

$$\mathbf{p}c = c\mathbf{p} \text{ and } \frac{\mathbf{p}}{c} = \frac{1}{c}\mathbf{p} \tag{1.18}$$

for all vectors and scalars (except for $c = 0$ in the latter, of course).

If $n = 1$, then $\mathbb{R}^1$ denotes the one-dimensional vector space formed by the set $\mathbb{R}$ of real numbers itself, with ordinary addition and multiplication serving as the vector operations. Although $\mathbb{R}$ has more structure than that of a vector space, it is still customary to write $\mathbb{R}$ not just for the field of real numbers but for the vector space $\mathbb{R}^1$ as well.

The theorems we stated for vectors in the plane also remain valid for vectors in $\mathbb{R}^n$, for every positive integer n, with the obvious change to n components where needed. Also, in $\mathbb{R}^n$, just as in $\mathbb{R}^2$, Properties 2, 6, 7, and 8 of Theorem 1.1.1 can be extended to several vectors and scalars.

Before closing this section, let us consider a three-dimensional example.

Example 1.1.8. (Midpoint of a Line Segment in $\mathbb{R}^3$***).*** Given the points $P = (1, 2, 3)$ and $Q = (-1, 6, 5)$, find the midpoint M of the line segment PQ.

Just as in two dimensions, we can write $\overrightarrow{PQ} = \mathbf{q} - \mathbf{p} = (-2, 4, 2)$ and the position vector $\mathbf{m}$ of the point M as

$$\mathbf{m} = \mathbf{p} + \frac{1}{2}\overrightarrow{PQ} = (1, 2, 3) + \frac{1}{2}(-2, 4, 2) = (0, 4, 4). \tag{1.19}$$

♦

Notice that in the example above we could also have written

$$\mathbf{m} = \mathbf{p} + \frac{1}{2}(\mathbf{q} - \mathbf{p}) = \mathbf{p} + \frac{1}{2}\mathbf{q} - \frac{1}{2}\mathbf{p} = \frac{1}{2}(\mathbf{p} + \mathbf{q}), \tag{1.20}$$

which gives a general formula for the midpoint of a line segment.

In later chapters we shall discuss many further examples of vector spaces. Most of them do not resemble sets of directed line segments at all, but their vector space structure allows us to study their common properties together. The set of all polynomials in one variable, the set of all polynomials of degree less than some arbitrary number, the set of all functions continuous on a given interval, the set of all solutions of certain differential equations, etc. are all vector spaces, just to mention a few.

Exercises

Exercise 1.1.1. Referring to Figure 1.2, find expressions in terms of $\mathbf{p}$ and $\mathbf{q}$ of the free vectors $\overrightarrow{PR}$, $\overrightarrow{PQ}, \overrightarrow{QP}$ and of the vectors $\overrightarrow{QC}$, $\overrightarrow{PC}$, and $\overrightarrow{OC}$, where C denotes the center of the parallelogram.

Exercise 1.1.2. If in Figure 1.5 $\mathbf{p} = (1, 3)$ and $\mathbf{q} = (4, 2)$, then what are $\mathbf{r}$, $\mathbf{p} - \mathbf{q}$, and $\mathbf{q} - \mathbf{p}$ in terms of components?

Exercise 1.1.3. Let $\mathbf{p} = (2, 3, -1)$ and $\mathbf{q} = (1, 2, 2)$ be two vectors in $\mathbb{R}^3$. Find $\mathbf{p} + \mathbf{q}$, and draw all three vectors from the origin in the xyz coordinate system to illustrate the parallelogram law in three dimensions.

Exercise 1.1.4. In $\mathbb{R}^3$ the unit cube is defined as the cube with vertices $(0, 0, 0), (0, 0, 1), (0, 1, 0), (1, 0, 0), (0, 1, 1), (1, 0, 1), (1, 1, 0), (1, 1, 1)$. Find the position vectors (in coordinate form) of the midpoints of the edges, the centers of the faces, and the center of the whole cube.

Exercise 1.1.5. Draw a diagram to illustrate the second distributive law of vectors (Property 8 of Theorem 1.1.1) with $a = 2$ and $\mathbf{p}, \mathbf{q}$ any vectors in $\mathbb{R}^2$.

***Exercise 1.1.6.** Prove the last four parts of Theorem 1.1.3.

Exercise 1.1.7. Given n point masses m_i, $i = 1, 2, \ldots, n$, at the points with position vectors $\mathbf{r}_i$ in either two or three dimensions, their center of mass is defined as the point with position vector $\mathbf{r} = \frac{1}{M}\sum_{i=1}^{n} m_i\mathbf{r}_i$, where

$M = \sum_{i=1}^{n} m_i$ is the total mass. (If the masses are equal, their center of mass is called their centroid.) If three mass points are given with $m_1 = 2$, $m_2 = 3$, $m_3 = 5$, $\mathbf{r}_1 = (2, -1, 4)$, $\mathbf{r}_2 = (1, 5, -6)$, and $\mathbf{r}_3 = (-2, -5, 4)$, then find $\mathbf{r}$.

Exercise 1.1.8. Show that the definition of the center of mass, given in Exercise 1.1.7, does not depend on the choice of the point O. That is, if the origin of a new coordinate system is denoted O' and the position vectors relative to it $\mathbf{r}'_i$, then $\mathbf{r}' = \frac{1}{M}\sum_{i=1}^{n} m_i \mathbf{r}'_i$ gives the position vector of the center of mass in the new system. (*Hint*: $\mathbf{r}'_i = \overrightarrow{O'O} + \mathbf{r}_i$.)

Exercise 1.1.9. Let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be the position vectors of the vertices of a triangle. The point given by $\mathbf{p} = \frac{1}{3}(\mathbf{a} + \mathbf{b} + \mathbf{c})$ is the triangle's centroid. Show that it lies one third of the way from the midpoint of any side to the opposite vertex on the line joining these points. (Such a line is called a median of the triangle.) Draw an illustration.

Exercise 1.1.10. Let $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ be the position vectors of the vertices of a tetrahedron. The point given by $\mathbf{p} = \frac{1}{4}(\mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d})$ is the tetrahedron's centroid. Show that it lies one fourth of the way from the centroid of the vertices of any face to the opposite vertex on the line joining these points. (Such a line is called a median of the tetrahedron.) Draw an illustration. Notice that the centroid divides each median in the ratio 1 to 3 in contrast to the ratio 1 to 2 for triangles. Now one vertex balances three vertices instead of two.

Exercise 1.1.11. Show that for any tetrahedron the halfway point M on the line joining the midpoints of opposite edges is the tetrahedron's centroid (defined in Exercise 1.1.10). Thus all three such lines meet in the centroid.

MATLAB Exercises

In MATLAB, vectors mean coordinate vectors of any dimension and are denoted by names of up to 19 characters. Extra characters beyond 19 are ignored. There are several ways of entering them. In the following very simple exercises we explore these ways and various arithmetic operations with vectors. (In the exercises we use boldface characters for vectors and keywords; in entering MATLAB expressions you have to ignore the boldface.)

Exercise 1.1.12. Enter $\mathbf{u} = [1 - 3\ 6\ 0]$. (Be sure to leave spaces between the numbers.) Enter $\mathbf{v} = [1, -2, 2, 5]$. Describe and explain the results of the commands:

a. $\mathbf{u} + \mathbf{v}$,
b. $2\mathbf{u}$,
c. $2 * \mathbf{u}$,

d. $\mathbf{s} = \mathbf{u}/2$,
e. $\mathbf{u}/\mathbf{v}$,
f. $\mathbf{u}./\mathbf{v}$,
g. $\mathbf{u}(2)$,
h. $\mathbf{t} = \mathbf{u}/\mathbf{u}(2)$,
i. **format rat; s**,
j. $\mathbf{t}$,
k. **format short; s**,
l. $\mathbf{s} = 3 : 8$,
m. $\mathbf{t} = 1 : 0.2 : 2.8$,
n. **length**$(\mathbf{t})$.

Exercise 1.1.13. Let $\mathbf{r}_1 = (23, -31, 0)$, $\mathbf{r}_2 = (35, 14, -72)$, $\mathbf{r}_3 = (52, -25, 44)$, and $\mathbf{r}_4 = (12, 52, 24)$ be the position vectors of the vertices of a tetrahedron. Use MATLAB to compute the coordinates of its centroid. (See Exercise 1.1.7.)

Exercise 1.1.14. Redo the computations of Exercise 1.1.7 with MATLAB.

1.2 Length and Dot Product of Vectors in $\mathbb{R}^n$

The vectors of two and three dimensions have an additional property not covered by the definitions of Section 1.1, namely length, which we want to discuss now.

If we depict a vector $(x, y) \in \mathbb{R}^2$ by an arrow $\mathbf{p}$ anywhere in the xy system, then the Theorem of Pythagoras tells us that the length of $\mathbf{p}$, denoted by $|\mathbf{p}|$, is given by $|\mathbf{p}| = \sqrt{x^2 + y^2}$, and so we define $|(x, y)| = \sqrt{x^2 + y^2}$.

In $\mathbb{R}^3$ we can deduce a similar formula as follows. For $\mathbf{p} = (x, y, z)$, as shown in Figure 1.14, two applications of the Theorem of Pythagoras give $d^2 = x^2 + y^2$ and $|\mathbf{p}|^2 = d^2 + z^2$, and so $|\mathbf{p}|^2 = x^2 + y^2 + z^2$. Thus we define $|(x, y, z)|^2 = x^2 + y^2 + z^2$.

These formulas suggest the following generalization for every positive integer n.

Definition 1.2.1. ***(Length).*** *For all vectors in $\mathbb{R}^n$, we define the length of $(x_1, x_2, \ldots, x_n)$ by*

$$|(x_1, x_2, \ldots, x_n)| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}. \tag{1.21}$$

This length has some basic properties, summarized below.

Theorem 1.2.1. ***(Properties of Length).*** *For all vectors $\mathbf{p}, \mathbf{q}$ in $\mathbb{R}^n$ and all scalars c we have*

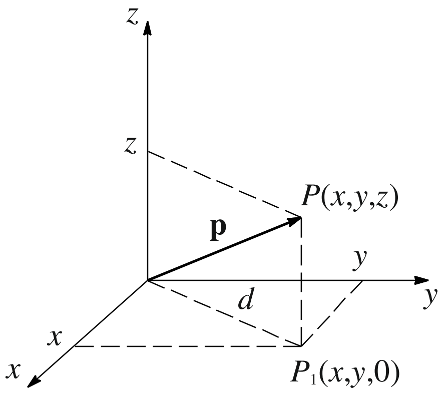

Fig. 1.14. The length of a vector in $\mathbb{R}^3$

1. $|\mathbf{p}| \geq 0$, *with equality holding only for* $\mathbf{p} = \mathbf{0}$.
2. $|c\mathbf{p}| = |c||\mathbf{p}|$, *and*
3. $|\mathbf{p} + \mathbf{q}| \leq |\mathbf{p}| + |\mathbf{q}|$. *(triangle inequality).*

The proofs of the first two parts are straightforward and are therefore omitted, and a proof of the third part (whose name is explained by Figure 1.8) is left to Exercise 1.2.13.

The three properties above hold for many functions of vectors, not just for their length, and every function satisfying them is called a *norm* on $\mathbb{R}^n$.

The length $|\mathbf{p}|$ can be used to associate with every nonzero vector $\mathbf{p} \in \mathbb{R}^n$ a vector of length 1, called a *unit vector*, pointing in the same direction, namely $\mathbf{u}_p = \frac{\mathbf{p}}{|\mathbf{p}|}$. That $|\mathbf{u}_p| = 1$ can be seen from Part 2 of Theorem 1.2.1 by substituting $c = \frac{1}{|\mathbf{p}|}$ in it.

Definition 1.2.2. ***(Distance).*** *The distance between two points* P, Q *with position vectors* $\mathbf{p}, \mathbf{q} \in \mathbb{R}^n$ *is defined as the length of* $\overrightarrow{QP} = \mathbf{p} - \mathbf{q}$, *that is, as* $|\mathbf{p} - \mathbf{q}|$.

If we want to define multiplication of vectors in $\mathbb{R}^n$, the most natural idea is to multiply them componentwise. However, another procedure makes a more useful definition, as suggested by the following applications.

Suppose we have n commodities with unit prices $(p_1, p_2, \ldots, p_n)$ and we want to buy the quantities $(q_1, q_2, \ldots, q_n)$ of each. The total amount we have to pay is then given by $p_1q_1 + p_2q_2 + \cdots + p_nq_n$. In probability theory, the same expression gives the expected value of a random variable, with the q_i denoting the possible values and the p_i their probabilities. In physics, the x-coordinate of the center of mass of point-masses p_i having x-coordinates q_i is given by the same formula divided by the total mass. Later we shall see that the formula in physics that gives the work done by a force moving an object also reduces to the same kind of expression, and in geometry too we shall put it to good use in several ways. Consequently, the following definition has been adopted.

Definition 1.2.3. ***(Scalar Product or Dot Product).*** *For all vectors* $\mathbf{p} = (p_1, p_2, \ldots, p_n)$ *and* $\mathbf{q} = (q_1, q_2, \ldots, q_n)$ *in* $\mathbb{R}^n$*, their scalar or dot product is*

$$\mathbf{p} \cdot \mathbf{q} = p_1 q_1 + p_2 q_2 + \cdots + p_n q_n. \tag{1.22}$$

Example 1.2.1. (A Dot Product). Let $\mathbf{p} = (1, 2, 3)$ and $\mathbf{q} = (1, 4, -3)$. Then

$$\mathbf{p} \cdot \mathbf{q} = 1 \cdot 1 + 2 \cdot 4 + 3 \cdot (-3) = 0. \tag{1.23}$$

♦

In addition to illustrating the computation of such products, this example shows that the scalar product of two vectors can well be zero even if the factors are not. This will turn out to be a very important and useful property.

As we can see, the scalar product results in a scalar, which explains its first name, as opposed to other products to be defined later. As for the second name, we generally denote this product by a dot.

The scalar product has some simple properties.

Theorem 1.2.2. ***(Properties of the Dot Product).*** *For all vectors* $\mathbf{p}, \mathbf{q}$ *and* $\mathbf{r}$ *in* $\mathbb{R}^n$ *for any* n *and for every scalar* c *we have*

1. $\mathbf{p} \cdot \mathbf{q} = \mathbf{q} \cdot \mathbf{p}$,
2. $\mathbf{p} \cdot (\mathbf{q} + \mathbf{r}) = \mathbf{p} \cdot \mathbf{q} + \mathbf{p} \cdot \mathbf{r}$,
3. $c(\mathbf{p} \cdot \mathbf{q}) = (c\mathbf{p}) \cdot \mathbf{q} = \mathbf{p} \cdot (c\mathbf{q})$, *and*
4. $\mathbf{p} \cdot \mathbf{p} = |\mathbf{p}|^2$.

This theorem can be proved simply by writing the expressions in terms of components and using the definitions; the proof is left to the reader.

Let us mention that every product that produces a scalar and satisfies the first three properties of Theorem 1.2.2 is called *an inner product.*[3] Given an inner product, the fourth property in Theorem 1.22 can then be used to define a corresponding norm.

Let us return to the case of zero products and prove the following fact.

Theorem 1.2.3. ***(Orthogonality in*** $\mathbb{R}^2$ ***and*** $\mathbb{R}^3$***).*** *Let* $\mathbf{p}$ *and* $\mathbf{q}$ *denote arbitrary nonzero vectors in* $\mathbb{R}^2$ *or* $\mathbb{R}^3$*. Then* $\mathbf{p}$ *and* $\mathbf{q}$ *are orthogonal*[4] *to each other if and only if* $\mathbf{p} \cdot \mathbf{q} = 0$.

[3] Hermann Grassmann (1809–1877), who invented the dot product (together with most of vector algebra), gave this name to it to distinguish it from his exterior product, which we shall not discuss, and from the outer product, to be defined in Section 2.2. He was led to these names by geometrical considerations.

[4] "Orthogonal" is synonymous with "perpendicular" but is, for some reason, preferred in linear algebra.

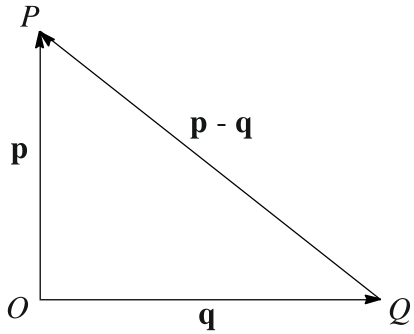

Fig. 1.15. A right triangle in terms of vectors

Proof. Assume first $\mathbf{p} \perp \mathbf{q}$. Then by the Theorem of Pythagoras

$$|\mathbf{p}-\mathbf{q}|^2 = |\mathbf{p}|^2 + |\mathbf{q}|^2 \tag{1.24}$$

(see Figure 1.15). Using Part 4 of Theorem 1.2.2 we can rewrite this expression as

$$(\mathbf{p}-\mathbf{q}) \cdot (\mathbf{p}-\mathbf{q}) = \mathbf{p} \cdot \mathbf{p} + \mathbf{q} \cdot \mathbf{q} \tag{1.25}$$

and as

$$\mathbf{p} \cdot \mathbf{p} - 2\mathbf{p} \cdot \mathbf{q} + \mathbf{q} \cdot \mathbf{q} = \mathbf{p} \cdot \mathbf{p} + \mathbf{q} \cdot \mathbf{q}, \tag{1.26}$$

from which $\mathbf{p} \cdot \mathbf{q} = 0$ follows at once.

Conversely, if $\mathbf{p} \cdot \mathbf{q} = 0$, then Equation 1.26 holds, which implies Equation 1.24. But the converse of the Pythagorean Theorem says that Equation 1.24 implies that the OPQ triangle is a right triangle with $\mathbf{p} \perp \mathbf{q}$, provided $\mathbf{p} \neq \mathbf{0}$ and $\mathbf{q} \neq \mathbf{0}$. ∎

By convention, it is generally agreed to call the zero vector orthogonal to every vector. Then we can conclude $\mathbf{p} \perp \mathbf{q}$ if and only if $\mathbf{p} \cdot \mathbf{q} = 0$, whether $\mathbf{p}$ or $\mathbf{q}$ is $\mathbf{0}$ or not.

Definition 1.2.4. ***(Orthogonality in*** $\mathbb{R}^n$***).*** *In* $\mathbb{R}^n$, *for every* $n > 3$, *we define two vectors* $\mathbf{p}$ *and* $\mathbf{q}$ *to be orthogonal to each other if* $\mathbf{p} \cdot \mathbf{q} = 0$.

With this definition, the proof of the theorem above can be reinterpreted for $\mathbb{R}^n$, for any $n > 0$, and shows that the Theorem of Pythagoras and its converse hold there as well.

Theorem 1.2.4. ***(Theorem of Pythagoras in*** $\mathbb{R}^n$***).*** *Let* $\mathbf{p}$ *and* $\mathbf{q}$ *denote arbitrary nonzero vectors in* $\mathbb{R}^n$ *with* $n > 0$. *Then* $\mathbf{p}$ *and* $\mathbf{q}$ *are orthogonal to each other if and only if*

$$|\mathbf{p}-\mathbf{q}|^2 = |\mathbf{p}|^2 + |\mathbf{q}|^2. \tag{1.27}$$

Next, we want to define angles in $\mathbb{R}^n$ for $n > 3$. Consider two arbitrary *nonzero* vectors $\mathbf{p}$ and $\mathbf{q}$ in $\mathbb{R}^n$, drawn from the point O, and decompose $\mathbf{p}$ as

$$\mathbf{p} = \mathbf{p}_1 + \mathbf{p}_2 \tag{1.28}$$

into two components, parallel and orthogonal to $\mathbf{q}$, respectively. We can obtain this decomposition by writing

$$\mathbf{p}_1 = c\mathbf{u}_q, \tag{1.29}$$

where $\mathbf{u}_q = \mathbf{q}/|\mathbf{q}|$ is the unit vector in the direction of $\mathbf{q}$, and determining c so that $\mathbf{p}_2 = \mathbf{p} - \mathbf{p}_1 = \mathbf{p} - c\mathbf{u}_q$ is orthogonal to $\mathbf{p}_1$ or equivalently to $\mathbf{u}_q$. To this end, we set

$$\mathbf{p}_2 \cdot \mathbf{u}_q = \mathbf{p} \cdot \mathbf{u}_q - c\mathbf{u}_q \cdot \mathbf{u}_q = \mathbf{p} \cdot \mathbf{u}_q - c = 0, \tag{1.30}$$

from which we see that

$$c = \mathbf{p} \cdot \mathbf{u}_q = \frac{\mathbf{p} \cdot \mathbf{q}}{|\mathbf{q}|}. \tag{1.31}$$

Now, once $\mathbf{p}$ is decomposed into a sum of two orthogonal vectors as in Equation 1.28, then the Theorem of Pythagoras shows that

$$|\mathbf{p}|^2 = |\mathbf{p}_1|^2 + |\mathbf{p}_2|^2. \tag{1.32}$$

Hence,

$$|\mathbf{p}_1|^2 \leq |\mathbf{p}|^2, \tag{1.33}$$

and from here, by Part 1 of Theorem 1.2.1,

$$|\mathbf{p}_1| \leq |\mathbf{p}|. \tag{1.34}$$

The discussion above was valid in $\mathbb{R}^n$, for any $n \geq 2$. However, in the special case of $\mathbb{R}^2$, we can visualize these vectors as follows. If $c > 0$, then in the POP_1 triangle the cosine of the angle θ between $\mathbf{p}$ and $\mathbf{p}_1$ or, equivalently between the given nonzero vectors $\mathbf{p}$ and $\mathbf{q}$, is given by the ratio $|\mathbf{p}_1| / |\mathbf{p}| = c/|\mathbf{p}|$. (See Figure 1.16.) However, if $c < 0$, then the angle θ between $\mathbf{p}$ and $\mathbf{q}$ is the supplement of the angle at O in the POP_1 triangle, and so $\cos\theta = -|\mathbf{p}_1| / |\mathbf{p}|$, which is again $c/|\mathbf{p}|$. (See Figure 1.17.) If $c = 0$, then $\theta = \pi/2$ and $\mathbf{p}_1 = \mathbf{0}$.

In $\mathbb{R}^n$, for any $n \geq 2$, Equation 1.29 shows that $|\mathbf{p}_1| = |c|$, and so, using Equation 1.31 and inequality 1.34, we have

$$\left|\frac{c}{|\mathbf{p}|}\right| = \frac{|\mathbf{p} \cdot \mathbf{q}|}{|\mathbf{p}|\,|\mathbf{q}|} \leq 1, \tag{1.35}$$

for any nonzero vectors $\mathbf{p}$ and $\mathbf{q}$. Thus, there exists a unique angle $\theta \in [0, \pi]$ whose cosine is $c/|\mathbf{p}|$, and so, analogously to the case in $\mathbb{R}^2$, we make the following definition.

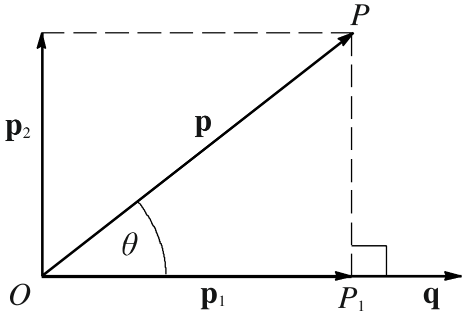

Fig. 1.16. Projection of a vector **p** onto a vector **q** when $c > 0$

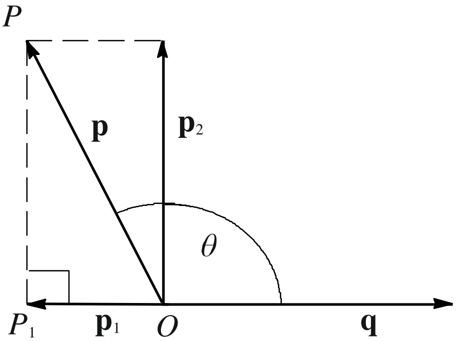

Fig. 1.17. Projection of a vector **p** onto a vector **q** when $c < 0$

Definition 1.2.5. ***(Angle in*** $\mathbb{R}^n$***).*** *In* $\mathbb{R}^n$, *for* $n > 2$, *we define the angle between two nonzero vectors* $\mathbf{p}$ *and* $\mathbf{q}$ *as*

$$\theta = \arccos \frac{\mathbf{p} \cdot \mathbf{q}}{|\mathbf{p}|\,|\mathbf{q}|}, \tag{1.36}$$

and if $\mathbf{p} = \mathbf{0}$ *or* $\mathbf{q} = \mathbf{0}$ *or both, then we set* $\theta = \pi/2$.

Note that we can have $\theta = \pi/2$ for nonzero $\mathbf{p}$ and $\mathbf{q}$, if $\mathbf{p} \cdot \mathbf{q} = 0$, in which case $\mathbf{p}$ and $\mathbf{q}$ are orthogonal to each other.

In $\mathbb{R}^2$, the following theorem is proved by the discussion above, and in $\mathbb{R}^n$, it is just a reformulation of Definition 1.2.5.

Theorem 1.2.5. ***(The Dot Product in Terms of Lengths and Angle).*** *Let* $\mathbf{p}$ *and* $\mathbf{q}$ *be any vectors in* $\mathbb{R}^n$ *for any* $n \geq 2$. *Then*

$$\mathbf{p} \cdot \mathbf{q} = |\mathbf{p}||\mathbf{q}| \cos\theta, \tag{1.37}$$

where $\theta \in [0, \pi]$ *is the angle between* $\mathbf{p}$ *and* $\mathbf{q}$.

The discussion leading to Theorem 1.2.5 has an important by-product, which is worth stating separately, in part as a definition and in part as a corollary.

Definition 1.2.6. ***(Orthogonal Projection).*** *Let* $\mathbf{p}$ *and* $\mathbf{q} \neq \mathbf{0}$ *be any vectors in* $\mathbb{R}^n$ *for any* $n \geq 2$. *Then the vector* $\mathbf{p}_1$ *is called the orthogonal projection of* $\mathbf{p}$ *onto the line of* $\mathbf{q}$, *and is also denoted by* $proj_{\mathbf{q}}(\mathbf{p})$.

Corollary 1.2.1. ***(Computing the Orthogonal Projection).*** *In* $\mathbb{R}^n$ *for any* $n \geq 2$, *the orthogonal projection of* $\mathbf{p}$ *onto* $\mathbf{q} \neq \mathbf{0}$ *is given by*

$$proj_{\mathbf{q}}(\mathbf{p}) = |\mathbf{p}| \cos\theta \frac{\mathbf{q}}{|\mathbf{q}|} = \frac{\mathbf{p} \cdot \mathbf{q}}{|\mathbf{q}|^2}\mathbf{q} \tag{1.38}$$

and its length is given by

$$|proj_{\mathbf{q}}(\mathbf{p})| = |\mathbf{p}||\cos\theta| = \frac{|\mathbf{p} \cdot \mathbf{q}|}{|\mathbf{q}|}. \tag{1.39}$$

One of the uses of Theorem 1.2.5 is that of computing the angle between two vectors given in coordinate form.

Example 1.2.2. (An Angle in $\mathbb{R}^3$***).*** Let $\mathbf{p} = (1,2,3)$ and $\mathbf{q} = (1,-2,2)$. Then

$$\cos\theta = \frac{\mathbf{p} \cdot \mathbf{q}}{|\mathbf{p}||\mathbf{q}|} = \frac{1 \cdot 1 + 2 \cdot (-2) + 3 \cdot 2}{\sqrt{1^2 + 2^2 + 3^2} \cdot \sqrt{1^2 + (-2)^2 + 2^2}} \approx 0.2673$$

and $\theta \approx 74.5\,°$. ♦

Example 1.2.3. (A Projection in $\mathbb{R}^3$***).*** Let us consider the same vectors $\mathbf{p}$, $\mathbf{q}$ as in the previous example, and decompose $\mathbf{p}$ into the sum of two vectors $\mathbf{p}_1$ and $\mathbf{p}_2$, parallel and orthogonal to $\mathbf{q}$ respectively, as in the proof of Theorem 1.2.5. Then, from Corollary 1.2.1,

$$proj_{\mathbf{q}}(\mathbf{p}) = \frac{\mathbf{p} \cdot \mathbf{q}}{|\mathbf{q}|^2}\mathbf{q} = \frac{1 \cdot 1 + 2 \cdot (-2) + 3 \cdot 2}{1^2 + (-2)^2 + 2^2}(1,-2,2) = \frac{1}{3}(1,-2,2) \tag{1.40}$$

and

$$\mathbf{p}_2 = \mathbf{p} - proj_{\mathbf{q}}(\mathbf{p}) = (1,2,3) - \frac{1}{3}(1,-2,2) = \frac{1}{3}(2,8,7). \tag{1.41}$$

We can easily check that $\mathbf{p}_2$ is orthogonal to $\mathbf{q}$ by computing their dot product:

$$\mathbf{p}_2 \cdot \mathbf{q} = \frac{1}{3}(2,8,7) \cdot (1,-2,2) = \frac{1}{3}(2 - 16 + 14) = 0. \tag{1.42}$$

♦

The following important result follows immediately from inequality 1.35 and Theorem 1.2.5.

Theorem 1.2.6. ***(Cauchy's Inequality).***[5] *For all vectors* $\mathbf{p}, \mathbf{q}$ *in* $\mathbb{R}^n$, *for any* $n \geq 2$,

$$|\mathbf{p} \cdot \mathbf{q}| \leq |\mathbf{p}||\mathbf{q}|, \tag{1.43}$$

with equality holding if and only if $\mathbf{p}$ *is parallel to* $\mathbf{q}$.

Proof. If $\mathbf{p}$ and $\mathbf{q}$ are nonzero vectors in $\mathbb{R}^n$, then inequality 1.43 is just a rearrangement of Inequality 1.35, and if $\mathbf{p} = \mathbf{0}$ or $\mathbf{q} = \mathbf{0}$ or both, then the result is trivially true with $0 = 0$. (The vector $\mathbf{0}$ is considered to be also parallel and not just orthogonal to any vector.) Thus all that remains to show is that equality holds for nonzero vectors if and only if $\mathbf{p}$ is parallel to $\mathbf{q}$. Now if $\mathbf{p}$ and $\mathbf{q}$ are nonzero and $|\mathbf{p} \cdot \mathbf{q}| = |\mathbf{p}||\mathbf{q}|$, then, by Theorem 1.2.5, $|\cos\theta| = 1$, and so $\theta = 0$ or $\theta = \pi$, and in both cases $\mathbf{p}$ is parallel to $\mathbf{q}$. Conversely, if $\mathbf{p}$ and $\mathbf{q}$ are nonzero and $\mathbf{p}$ is parallel to $\mathbf{q}$, then $\mathbf{p} = a\mathbf{q}$, for some scalar a, and so $|\mathbf{p} \cdot \mathbf{q}| = |a|\,|\mathbf{q} \cdot \mathbf{q}| = |a|\,|\mathbf{q}|^2$ and $|\mathbf{p}||\mathbf{q}| = |a\mathbf{q}||\mathbf{q}| = |a|\,|\mathbf{q}|^2$, that is, the two sides in Cauchy's inequality are equal. ■

(For an alternate proof, see Exercise 1.2.12.)

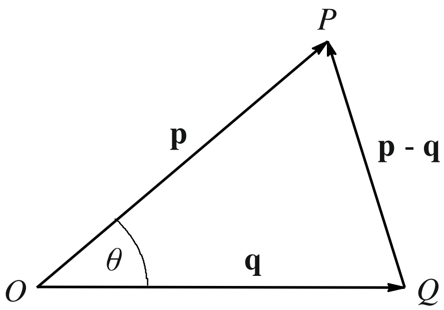

Fig. 1.18. An arbitrary triangle to illustrate the relationship between the law of cosines and the dot product

Also, given the properties of the dot product expressed in Theorem 1.2.2, in $\mathbb{R}^2$ Theorem 1.2.5 is equivalent to the law of cosines: In Figure 1.18 we have $(\mathbf{p} - \mathbf{q}) \cdot (\mathbf{p} - \mathbf{q}) = \mathbf{p} \cdot \mathbf{p} + \mathbf{q} \cdot \mathbf{q} - 2\mathbf{p} \cdot \mathbf{q}$. By using Theorem 1.2.5 this equation can be rewritten as $|\mathbf{p} - \mathbf{q}|^2 = |\mathbf{p}|^2 + |\mathbf{q}|^2 - 2|\mathbf{p}||\mathbf{q}|\cos\theta$, and this is exactly the law of cosines for the triangle OPQ in $\mathbb{R}^2$.

[5] Named after Augustin Louis Cauchy (1789–1857), and also named sometimes after V. I. Bunyakovsky (1804–1889) and H. A. Schwarz (l843–1921), who generalized it to integrals.

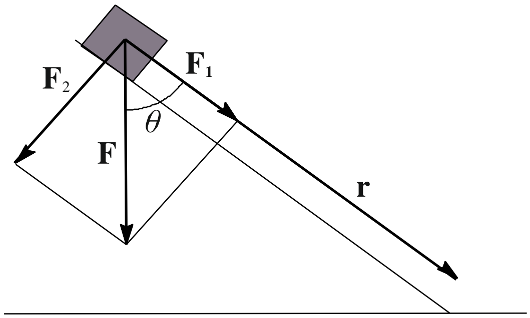

Fig. 1.19. Decomposition of a force $\mathbf{F}$ into the sum of two forces: $\mathbf{F}_1$ along a given $\mathbf{r}$ and $\mathbf{F}_2$ orthogonal to $\mathbf{r}$

Example 1.2.4. (Work as a Dot Product). In first discussing the dot product we mentioned that in physics it has another very important use in defining work. Theorem 1.2.5 enables us to describe this application in more detail. Indeed, if $\mathbf{F}$ is a constant force acting on some object, and if $\mathbf{r}$ is the object's displacement caused by $\mathbf{F}$, then the corresponding work W is given by $|\mathbf{r}|$ times the magnitude $|\mathbf{F}|\cos\theta$ of the orthogonal projection of $\mathbf{F}$ onto the line of motion, that is, $W = \mathbf{F}\cdot\mathbf{r}$. Note that $\mathbf{F}$ does not have to point in the same direction as $\mathbf{r}$. For example, if $\mathbf{F}$ denotes the force of gravity moving something down an incline as shown in Figure 1.19, then $\mathbf{F}$ can be decomposed into the sum of two forces: $\mathbf{F}_1$ along $\mathbf{r}$ and $\mathbf{F}_2$ orthogonal to $\mathbf{r}$. The force $\mathbf{F}_2$ does not cause any motion; it just presses the object to the slope. The force $\mathbf{F}_1$, on the other hand, is the sole cause of the motion and the work W is proportional to its magnitude $|\mathbf{F}|\cos\theta$. ♦

At this point we should mention certain unit vectors that are often used to make formulas simpler. These are the *coordinate unit vectors* or *standard vectors*

$$\mathbf{i} = (1,0),\ \mathbf{j} = (0,1) \text{ in } \mathbb{R}^2, \tag{1.44}$$

$$\mathbf{i} = (1,0,0),\ \mathbf{j} = (0,1,0),\ \mathbf{k} = (0,0,1) \text{ in } \mathbb{R}^3, \tag{1.45}$$

$$\mathbf{e}_1 = (1,0,0,\ldots,0),\ \mathbf{e}_2 = (0,1,0,\ldots,0),\ldots,\ \mathbf{e}_n = (0,0,\ldots,0,1) \text{ in } \mathbb{R}^n. \tag{1.46}$$

Every vector in these spaces can be decomposed into components along these unit vectors:

$$(x,y) = x\mathbf{i} + y\mathbf{j} \tag{1.47}$$

$$(x,y,z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \tag{1.48}$$

$$(x_1,x_2,\ldots,x_n) = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_n\mathbf{e}_n. \tag{1.49}$$

Writing $\mathbf{r} = (x, y, z)$, we can easily see that in $\mathbb{R}^3$

$$\mathbf{r} \cdot \mathbf{i} = x, \ \mathbf{r} \cdot \mathbf{j} = y, \ \mathbf{r} \cdot \mathbf{k} = z \tag{1.50}$$

hold, but just the first two of these equations hold in $\mathbb{R}^2$; and for $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ in $\mathbb{R}^n$ we have

$$\mathbf{x} \cdot \mathbf{e}_i = x_i \text{ for } i = 1, 2, \ldots, \ n. \tag{1.51}$$

Note that in $\mathbb{R}^n$, for any n, the standard vectors are orthogonal to each other, that is,

$$\mathbf{e}_i \cdot \mathbf{e}_j = 0 \text{ for all } i \neq j. \tag{1.52}$$

Combining this orthogonality with the fact that they are unit vectors, that is, that

$$\mathbf{e}_i \cdot \mathbf{e}_i = 1 \text{ for } i = 1, 2, \ldots, \ n, \tag{1.53}$$

they are said to be *orthonormal*.

Exercises

Exercise 1.2.1. Let $\mathbf{p} = (5, 5)$ and $\mathbf{q} = (1, -7)$.
a. Determine $\mathbf{p} + \mathbf{q}$ and $\mathbf{p} - \mathbf{q}$.
b. Represent $\mathbf{p}, \mathbf{q}, \mathbf{p} + \mathbf{q}$, and $\mathbf{p} - \mathbf{q}$ by arrows in a parallelogram.
c. Compute $|\mathbf{p}|, |\mathbf{q}|, |\mathbf{p} + \mathbf{q}|$, and $|\mathbf{p} - \mathbf{q}|$.
d. Is $|\mathbf{p} + \mathbf{q}|^2 = |\mathbf{p}|^2 + |\mathbf{q}|^2$?

Exercise 1.2.2. Let $\mathbf{p} = (2, -2, 1)$ and $\mathbf{q} = (2, 3, 2)$. Show that $|\mathbf{p} + \mathbf{q}|^2 = |\mathbf{p}|^2 + |\mathbf{q}|^2$ and $|\mathbf{p} - \mathbf{q}|^2 = |\mathbf{p}|^2 + |\mathbf{q}|^2$. Interpret geometrically.

Exercise 1.2.3. Let P, Q, and R be the vertices of a triangle in $\mathbb{R}^2$ or $\mathbb{R}^3$. Use vectors to show that the line segment joining the midpoints of any two sides of the triangle is parallel to and one half the length of the third side. (Note: two vectors are parallel if and only if one is a scalar multiple of the other.)

Exercise 1.2.4. Find the angle between the vectors $\mathbf{p} = (-2, 4)$ and $\mathbf{q} = (3, -5)$.

Exercise 1.2.5. Find the angle between the vectors $\mathbf{p} = (1, -2, 4)$ and $\mathbf{q} = (3, 5, 2)$.

Exercise 1.2.6. Find six different nonobtuse angles between various nonparallel diagonals of the unit cube (defined in Exercise 1.1.4) and between its edges and diagonals.

Exercise 1.2.7. The line segments joining the centers of the faces of the unit cube form a regular octahedron. Find the angles between its various edges, and try to draw it.

Exercise 1.2.8. Consider a triangle in the xy plane with vertices $A = (1,3)$, $B = (2,4)$, and $C = (4,-1)$. Find
a. the orthogonal projection of $\mathbf{p} = \overrightarrow{AB}$ onto the line of $\mathbf{q} = \overrightarrow{BC}$,
b. the distance of A from that line, and
c. the area of the triangle.

Exercise 1.2.9. Decompose the vector $\mathbf{p} = (2,-3,1)$ into components parallel and perpendicular to the vector $\mathbf{q} = (12,3,4)$.

Exercise 1.2.10. Prove the parallelogram law for the length:

$$|\mathbf{p}+\mathbf{q}|^2 + |\mathbf{p}-\mathbf{q}|^2 = 2|\mathbf{p}|^2 + 2|\mathbf{q}|^2 \tag{1.54}$$

for all vectors in $\mathbb{R}^n$. Interpret geometrically!

Exercise 1.2.11. Using dot products, prove the Theorem of Thales: If we take a point P on a circle and form a triangle by joining it to the opposite ends of an arbitrary diameter, then the angle at P is a right angle. (See Figure 1.20.)

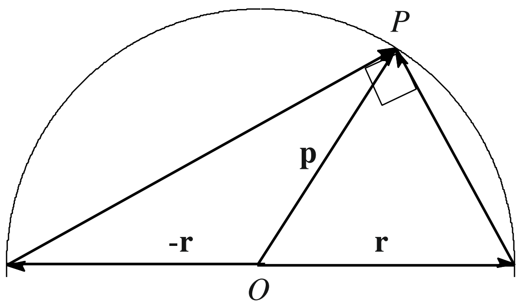

Fig. 1.20. The Theorem of Thales

***Exercise 1.2.12.** Fill in the details of the following, alternative proof of Cauchy's inequality (Theorem 1.2.6) in $\mathbb{R}^n$ for any $n \geq 2$:

By Part 1 of Theorem 1.2.1, $(\mathbf{p} - \lambda\mathbf{q}) \cdot (\mathbf{p} - \lambda\mathbf{q}) \geq 0$ for every scalar λ. Expand the left-hand side to obtain a quadratic function of λ. The graph of this function is a parabola above the λ-axis. Find the λ-value of the lowest point in terms of $\mathbf{p}$ and $\mathbf{q}$, substitute it into the inequality, and simplify.

***Exercise 1.2.13.**
a. Using the result of Theorem 1.2.6 prove the triangle inequality (Part 3 of Theorem 1.2.1).
b. Prove that equality occurs in the triangle inequality if and only if the vectors are parallel and point in the same direction.

***Exercise 1.2.14.**
a. Prove the inequality $||\mathbf{p}| - |\mathbf{q}|| \leq |\mathbf{p} - \mathbf{q}|$ for all vectors in $\mathbb{R}^n$.
b. When do we have equality in Part (a)? Explain!

Exercise 1.2.15. Let $\mathbf{p}$ be any nonzero vector in $\mathbb{R}^2$ and $\mathbf{u}_p$ the unit vector in its direction. Show
a. that the vector $\mathbf{p}$ can be written as $\mathbf{p} = |\mathbf{p}|(\cos\phi, \sin\phi)$, where ϕ is the angle from the positive x-axis to $\mathbf{p}$, and
b. that $\mathbf{u}_p = (\cos\phi, \sin\phi)$.

Exercise 1.2.16. Let $\mathbf{p}$ be any nonzero vector in $\mathbb{R}^3$ and $\mathbf{u}_p$ the unit vector in its direction. Show that
a. the components $\mathbf{u}_p \cdot \mathbf{i}$, $\mathbf{u}_p \cdot \mathbf{j}$, $\mathbf{u}_p \cdot \mathbf{k}$ of $\mathbf{u}_p$ equal the cosines of the angles $\alpha_1, \alpha_2, \alpha_3$ between $\mathbf{p}$ and the positive coordinate axes (these are called the *direction cosines* of $\mathbf{p}$),
b. $\cos^2\alpha_1 + \cos^2\alpha_2 + \cos^2\alpha_3 = 1$ (What familiar formula in $\mathbb{R}^2$ does this correspond to?),
c. $\mathbf{p} = |\mathbf{p}|(\cos\alpha_1, \cos\alpha_2, \cos\alpha_3)$.

Exercise 1.2.17. Find the direction cosines (see Exercise 1.2.16) of $\mathbf{p} = (3, -4, 12)$, and the angles $\alpha_1, \alpha_2, \alpha_3$.

Exercise 1.2.18. Prove the formula $\cos(\alpha - \beta) = \cos\alpha\, \cos\beta + \sin\alpha\, \sin\beta$ by considering the scalar product of two unit vectors $\mathbf{e}_a = (\cos\alpha, \sin\alpha)$ and $\mathbf{e}_b = (\cos\beta, \sin\beta)$.

***Exercise 1.2.19.** Show that in $\mathbb{R}^2$ an inner product may be defined by $\mathbf{p} \cdot \mathbf{q} = 2p_1q_1 + p_2q_2$, that is, show that this product also satisfies the first three properties of Theorem 1.2.2. What is the geometrical meaning of this product?

***Exercise 1.2.20.** Consider an *oblique coordinate system* in the plane with axes labeled ξ and η as shown in Figure 1.21. Given a vector $\mathbf{p}$, let p_1 and p_2 denote the *orthogonal* scalar components of $\mathbf{p}$, that is, the signed lengths of the orthogonal projections of $\mathbf{p}$ onto the axes, and let p^1 and p^2 denote the *parallel* scalar components of $\mathbf{p}$. (Note that in p^1 and p^2 the 1 and 2 are superscripts, not exponents!)
a. Show that if $\mathbf{p} \cdot \mathbf{q} = |\mathbf{p}||\mathbf{q}|\cos\theta$ as usual, then $\mathbf{p} \cdot \mathbf{q} = p_1q^1 + p_2q^2 = p^1q_1 + p^2q_2$, and
b. express $\mathbf{p} \cdot \mathbf{q}$ in the form $\sum_{ij} g_{ij}p^iq^j$, that is, find appropriate constants g_{ij}.

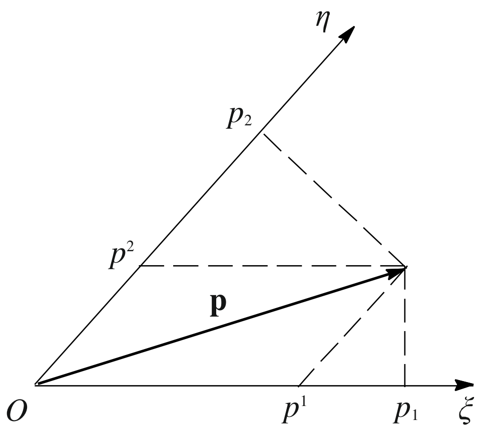

Fig. 1.21. Oblique coordinate system with parallel components p^i and orthogonal components p_i of a vector $\mathbf{p}$

(In differential geometry and in the theory of relativity such coordinates are very important. The quantities p^1 and p^2 are called the contravariant components and p_1 and p_2 the covariant components of $\mathbf{p}$, because of their behavior under coordinate transformations. In Cartesian coordinate systems they coincide.)

MATLAB Exercises

In MATLAB the functions $\mathbf{norm}(\mathbf{u})$ and $\mathbf{dot}(\mathbf{u}, \mathbf{v})$ return the length and dot product of vectors, respectively. The command $\mathbf{rand}(1, n)$ generates an n-vector with random components uniformly distributed between 0 and 1.

Exercise 1.2.21.
a. For $n = 2, 3, 4, 10, 20$, find the cosine of and the angle $\theta(n)$ between $\mathbf{u} = [1, 1, \ldots, 1]$ and $\mathbf{v} = [1, 0, 0, \ldots, 0]$. (To enter $\mathbf{u}$ and $\mathbf{v}$ use, for each n, $\mathbf{u} = \mathbf{ones}(1, n)$, $\mathbf{v} = \mathbf{zeros}(1, n)$, $\mathbf{v}(1) = 1$. The MATLAB function $\mathbf{acos}(x)$ gives the inverse cosine in radians.)
b. Make a conjecture for the value of $\lim_{n\to\infty} \theta(n)$ and prove it.

Exercise 1.2.22.
a. For $n = 10, 50, 100, 500$, find the angle $\theta(n)$ between $\mathbf{u} = \mathbf{rand}(1, n) - 1/2$ and $\mathbf{v} = \mathbf{rand}(1, n) - 1/2$. Use the up-arrow key to repeat this computation several times.
b. What do you observe? Can you give a heuristic explanation?

Exercise 1.2.23. Use MATLAB to decompose the vector $\mathbf{u} = [1, 1, 1, 1, 1]$ into the sum of a vector parallel to $\mathbf{v} = [1, 2, 3, 4, 5]$ and one orthogonal to it.

1.3 Lines and Planes

We now have the necessary machinery for developing equations for lines and planes in $\mathbb{R}^n$ and for computing corresponding intersections, distances, and angles.

We start with lines. Let $\mathbf{p}_0 = (x_0, y_0, z_0)$ and $\mathbf{v} = (v_1, v_2, v_3) \neq \mathbf{0}$ be arbitrary vectors of $\mathbb{R}^3$. We consider $\mathbf{p}_0$ as the position vector starting at O of a point P_0 in three-dimensional space, but place the representative arrow of $\mathbf{v}$ conveniently at P_0. Let L denote the line drawn through the point P_0 along $\mathbf{v}$. (See Figure 1.22.) Then obviously the position vector $\mathbf{p} = (x, y, z)$ of any point P on L can be written as $\mathbf{p}_0 + t\mathbf{v}$ for some appropriate number t. Conversely, for every number t, $\mathbf{p}_0 + t\mathbf{v}$ is the position vector of a point P on L. Thus

$$\mathbf{p} = \mathbf{p}_0 + t\mathbf{v} \tag{1.55}$$

is the desired equation of the line L, which is the set of points P whose position vectors satisfy 1.55, that is,

$$L = \{P : \mathbf{p} = \mathbf{p}_0 + t\mathbf{v} \text{ with } t \in \mathbb{R}\}. \tag{1.56}$$

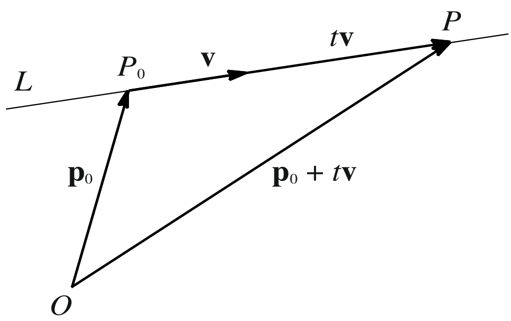

Fig. 1.22. Parametric vector representation of a line

The variable t is called a *parameter* and Equation 1.55 a *parametric vector equation* of the line L. It describes the line L *with a scale superimposed on it.* This scale has $t = 0$ at P_0 and $t = 1$ at the point with position vector $\mathbf{p}_0 + \mathbf{v}$. Clearly, the same line has many *parametric* representations, since there are many ways of putting a scale on it. For example, replacing t by $2t$ in Equation 1.55, we get a different parametric equation of the same line, in which only the scale has been changed to one with intervals of doubled lengths. The point P_0 can also be changed to any other point of the line; this change would just move the zero point of the scale, and would result in a different equation, but one that still describes the same line. (See, for instance, Example 1.3.1.)

Also, we may think of t as denoting time and then Equation 1.55 describes not only the line L but also the motion of a point along L, which is at the position P at time t and at P_0 at time 0. In this interpretation the vector $\mathbf{v}$ stands for the velocity of the moving point, but in general it is called a *direction vector* of L.

Since equality of vectors means equality of components, Equation 1.55 is equivalent to the three *parametric scalar equations*

$$\begin{aligned} x &= x_0 + tv_1, \\ y &= y_0 + tv_2, \\ z &= z_0 + tv_3. \end{aligned} \tag{1.57}$$

The situation is entirely analogous in $\mathbb{R}^n$. In fact, Equation 1.55 remains unchanged, but with the vectors reinterpreted as lying in $\mathbb{R}^n$.

If none of the components of $\mathbf{v}$ is zero, then we can solve each of Equations 1.57 for t and we obtain nonparametric equations (also called, somewhat inaptly, symmetric equations) for L:

$$\frac{x - x_0}{v_1} = \frac{y - y_0}{v_2} = \frac{z - z_0}{v_3}. \tag{1.58}$$

Notice that a line in three dimensions is given by two nonparametric equations (choose any two of the three equations implicit in 1.58), rather than by just one equation as in two dimensions. As will be seen shortly, the explanation for this difference is that each of the two equations describes a plane and the line is then represented as the intersection of these planes.

Example 1.3.1. (A Line Through Two Points). Let us find equations for the line L that passes through the points $A(2, -3, 5)$ and $B(6, 1, -8)$.

We may take either one of the given points as P_0, and the vector $\overrightarrow{AB}$ as $\mathbf{v}$. We put $\mathbf{p}_0 = (2, -3, 5)$ and $\mathbf{v} = (6 - 2, 1 + 3, -8 - 5) = (4, 4, -13)$, and so we obtain the parametric equation

$$\mathbf{p} = (2, -3, 5) + t(4, 4, -13), \tag{1.59}$$

or

$$(x, y, z) = (2 + 4t, -3 + 4t, 5 - 13t) \tag{1.60}$$

for the line L. The corresponding scalar equations are

$$x = 2 + 4t, \quad y = -3 + 4t, \quad z = 5 - 13t, \tag{1.61}$$

and eliminating t leads to the nonparametric equations

$$\frac{x - 2}{4} = \frac{y + 3}{4} = \frac{z - 5}{-13}. \tag{1.62}$$

If we take B as P_0, that is, put $\mathbf{p}_0 = (6, 1, -8)$, then we get a different parameterization of L:

$$\mathbf{p} = (6, 1, -8) + t(4, 4, -13), \tag{1.63}$$

or in scalar form

$$x = 6 + 4t, \quad y = 1 + 4t, \quad z = -8 - 13t. \tag{1.64}$$

Hence the new nonparametric equations become

$$\frac{x-6}{4} = \frac{y-1}{4} = \frac{z+8}{-13}. \tag{1.65}$$

Equations 1.62 and 1.65 look very different, except for the denominators. We can check that they represent the same line by showing that the coordinates of A and B satisfy both. Indeed,

$$\frac{2-2}{4} = \frac{-3+3}{4} = \frac{5-5}{-13} = 0, \quad \frac{6-2}{4} = \frac{1+3}{4} = \frac{-8-5}{-13} = -1 \tag{1.66}$$

and

$$\frac{2-6}{4} = \frac{-3-1}{4} = \frac{5+8}{-13} = -1, \quad \frac{6-6}{4} = \frac{1-1}{4} = \frac{-8+8}{-13} = 0. \tag{1.67}$$

Since two distinct points determine a unique line, and Equations 1.62 and 1.65 both represent a line, they must represent the same line. ♦

If a component of $\mathbf{v}$ is zero, then the corresponding equation is already in nonparametric form, and the line is parallel to one or two of the coordinate planes as in the following example.

Example 1.3.2. (A Line Through a Point with a Given Direction). Let us find equations for the line L that passes through the point $A(2, -3, 5)$ parallel to the z-axis.

In this case only the z coordinate varies, and the parametric vector equation can immediately be written as

$$\mathbf{p} = (2, -3, 5) + t(0, 0, 1). \tag{1.68}$$

In components:

$$x = 2, \quad y = -3, \quad z = 5 + t, \tag{1.69}$$

and the nonparametric equations are just the first two of these equations. ♦

Example 1.3.3. (Intersection of Two Lines). Let us find the intersection of the two lines L_1 and L_2 given by

$$\mathbf{p} = (4,3,9) + s(2,-3,7), \text{ and } \mathbf{p} = (3,2,0) + t(-1,4,2), \tag{1.70}$$

if there is one. (Notice that we used two different parameters s and t, since using only one would have meant looking not just for the point of intersection but, in the time interpretation of parameters, also for two moving points to be there at the same time, or in the static interpretation, also for two scales to match. See Exercise 1.3.9.)

The obvious way to attack this problem is to equate the corresponding scalar components of the two expressions for $\mathbf{p}$ and solve the resulting equations for s and t. Thus we can write the vector equation

$$(4,3,9) + s(2,-3,7) = (3,2,0) + t(-1,4,2) \tag{1.71}$$

in terms of components as

$$4 + 2s = 3 - t, \quad 3 - 3s = 2 + 4t, \quad 9 + 7s = 2t. \tag{1.72}$$

We can easily solve these equations to obtain $s = -1$ and $t = 1$. (Notice that in general we cannot expect to have solutions for two unknowns in three equations, which corresponds to the geometric fact that in three dimensions most lines avoid each other. More about this subject in Chapter 2.) Substituting these parameter values into either of Equations 1.70, we obtain the position vector of the point of intersection as

$$\mathbf{p} = (2,6,2). \tag{1.73}$$

♦

Example 1.3.4. (Distance Between a Point and a Line). Find the distance of the point $A(9,13,-1)$ from the line L given by

$$\mathbf{p} = (-1,-2,4) + t(3,1,-5). \tag{1.74}$$

The point A and the line L determine a plane. In this plane, we drop a perpendicular from A to L, and the desired distance is the length of this line segment. We can find this length as follows. First, pick any point Q on L, say $Q = (-1,-2,4)$. Second, decompose $\mathbf{r} = \overrightarrow{QA}$ into two components $\mathbf{r}_1$ and $\mathbf{r}_2$, respectively parallel and orthogonal to $\mathbf{v} = (3,1,-5)$. (See Corollary 1.2.1.) Then $\mathbf{r} = (10,15,-5)$ and $\mathbf{r}_1 = \frac{\mathbf{r}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v} = \frac{70}{35}(3,1,-5) = (6,2,-10)$. Consequently, $\mathbf{r}_2 = \mathbf{r} - \mathbf{r}_1 = (4,13,5)$ and the required distance is obtained as $|\mathbf{r}_2| = \sqrt{210}$. ♦

Let us now consider planes.

To obtain parametric equations we may proceed very much as for lines but with *two* vectors in place of $\mathbf{v}$ and *two* parameters s and t instead of one. Thus, let $\mathbf{p}_0 \in \mathbb{R}^3$ be regarded as the position vector of a fixed point P_0 of the plane S we want to describe, P a variable point with position vector $\mathbf{p}$, and $\mathbf{u}$ and $\mathbf{v}$ two nonparallel, nonzero vectors of $\mathbb{R}^3$ with their representative arrows drawn in S, as shown in Figure 1.23.

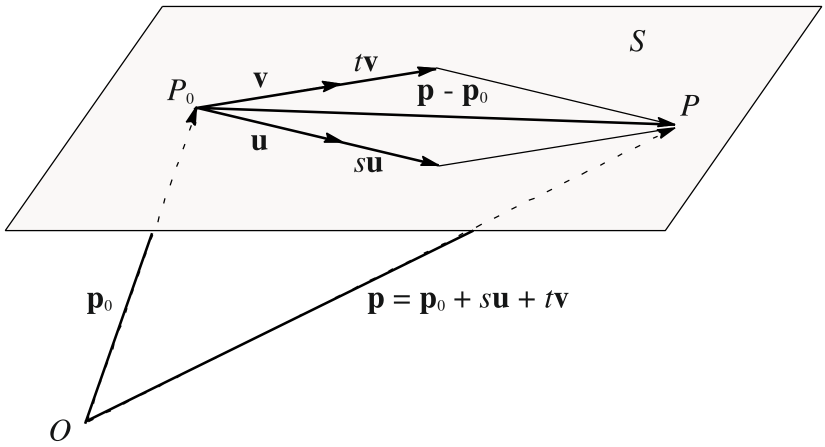

Fig. 1.23. Parametric vector representation of a plane

Then $\mathbf{p}$ can be expressed as

$$\mathbf{p} = \mathbf{p}_0 + s\mathbf{u} + t\mathbf{v}, \tag{1.75}$$

where s and t are appropriate real numbers. Conversely, any pair $s, t \in \mathbb{R}$ determines a point of S via Equation 1.75. So this is a parametric vector equation of S. In other words, S is the set of points P whose position vectors satisfy 1.75, that is,

$$S = \{P : \mathbf{p} = \mathbf{p}_0 + s\mathbf{u} + t\mathbf{v} \text{ with } t \in \mathbb{R}\}. \tag{1.76}$$

In components, the vector equation 1.75 becomes the set of three scalar equations:

$$x = x_0 + su_1 + tv_1,\ y = y_0 + su_2 + tv_2,\ z = z_0 + su_3 + tv_3. \tag{1.77}$$

Example 1.3.5. (A Plane Through Two Lines). Let us write a parametric vector equation for the plane S containing the two intersecting lines of Example 1.3.3.

We may take any point of either line to be P_0 and the same vectors for $\mathbf{u}$ and $\mathbf{v}$ as in Example 1.3.3. If we take $P_0 = (4, 3, 9)$, say, then S will be described by the equation

$$\mathbf{p} = (4, 3, 9) + s(2, -3, 7) + t(-1, 4, 2). \tag{1.78}$$

Let us write this equation out in components as in Equation 1.77 and eliminate s and t:

$$x = 4 + 2s - t,\ y = 3 - 3s + 4t,\ z = 9 + 7s + 2t. \tag{1.79}$$

Bring the constant terms to the left, multiply the first of these equations in turn by 4 and 2, and add the results to the second and third equations respectively, to get

$$4(x-4)+(y-3)=5s \text{ and } 2(x-4)+(z-9)=11s. \tag{1.80}$$

Now multiply the first of these equations by 11 and the second one by (-5), and add the results. Then we obtain the following nonparametric equation for S:

$$34(x-4)+11(y-3)-5(z-9)=0. \tag{1.81}$$

♦

In general, if we eliminate s and t from Equations 1.77 as in the example above, then we end up with an equation of the form

$$a(x-x_0)+b(y-y_0)+c(z-z_0)=0, \tag{1.82}$$

where $a,\ b,\ c$ are appropriate numbers arising from the elimination process.[6] It is reasonable to ask what their geometric meaning may be. Now $x-x_0$, $y-y_0$, $z-z_0$ are the components of the vector $\mathbf{p}-\mathbf{p}_0$ and if we consider $a,\ b,\ c$ to be the components of a vector $\mathbf{n}$, then Equation 1.82 may be written in vector form as

$$\mathbf{n}\cdot(\mathbf{p}-\mathbf{p}_0)=0. \tag{1.83}$$

This equation shows that the vector $\mathbf{n}$ is orthogonal to the variable vector $\mathbf{p}-\mathbf{p}_0$ lying in S (see Figure 1.24), and so it must be orthogonal to the

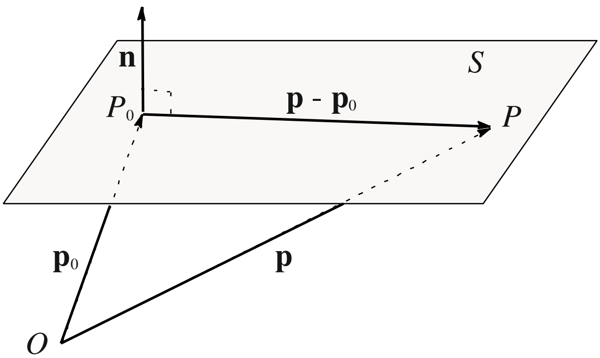

Fig. 1.24. Nonparametric representation of a plane

[6] In Chapter 6 (page 250) we discuss a shortcut method for such eliminations in three dimensions, which involves what is called the cross product of vectors.

plane S, that is, to every vector in S. Such a vector is called a *normal vector* of S, and one usually says it is *normal* to S rather than orthogonal to S.

Equation 1.83 may be rewritten as

$$\mathbf{n} \cdot \mathbf{p} = \mathbf{n} \cdot \mathbf{p}_0, \tag{1.84}$$

and if we set $d = \mathbf{n} \cdot \mathbf{p}_0$ and write Equation 1.84 in components, then we get

$$ax + by + cz = d \tag{1.85}$$

as the simplest type of equation for a plane.

It is also natural to ask what the geometric meaning of d is. From its definition we see that $d = |\mathbf{n}||\mathbf{p}_0| \cos\theta$ (see Figure 1.25), and this expression is $|\mathbf{n}|$ times the projection of $\mathbf{p}_0$ onto the line of $\mathbf{n}$. The vector $\mathbf{p}_0$ joins the origin to S. So the projection of $\mathbf{p}_0$ onto the line of $\mathbf{n}$, which is perpendicular to S, gives the distance of O from S if θ is acute, and the negative of this distance if θ is obtuse. Thus d equals $|\mathbf{n}|$ times the distance of O from S if $\mathbf{n}$ points from O towards S and the negative of this distance if $\mathbf{n}$ points from S towards O.

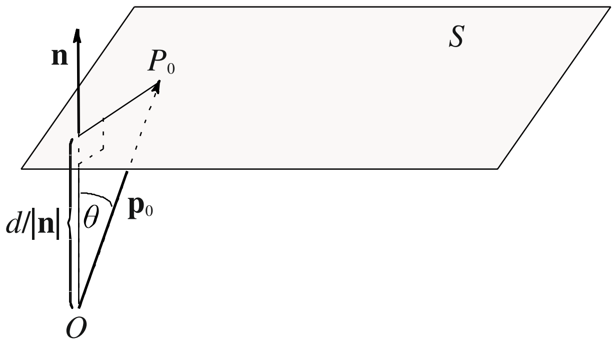

Fig. 1.25. The distance of a plane from the origin

Although planes can be visualized only in three dimensions, the vector equations just deduced remain valid in $\mathbb{R}^n$ for $n > 3$ as well, with the obvious reinterpretation of the vectors. There is a difference, however, in the meaning of the parametric and the nonparametric equations, which did not occur in $\mathbb{R}^3$, namely that the parametric Equation 1.75 describes two-dimensional sets in $\mathbb{R}^n$, which may justifiably still be called planes, but the nonparametric Equation 1.82 or 1.84 describes $(n-1)$-dimensional sets that are usually called *hyperplanes*.

Example 1.3.6. (Distance of a Point from a Plane). Given a plane S with equation $4x - 4y + 7z = 1$ and a point $A(5, 2, -3)$, find the distance D of A from S.

To solve this problem, we first pick an arbitrary point P of S. We can pick a point by choosing arbitrary values for x and y, say $x = 2$ and $y = 0$, and then solving the equation $4x - 4y + 7z = 1$ for z, to get $z = -1$. Thus, $P(2, 0, -1)$ is a point in S. Next, project the vector $\overrightarrow{AP} = (-3, -2, 2)$ onto $\mathbf{n}$. (See Figure 1.25 with A in place of O and P in place of P_0.) From the equation of S we can read off $\mathbf{n} = (4, -4, 7)$. Thus

$$D = \frac{|\overrightarrow{AP} \cdot \mathbf{n}|}{|\mathbf{n}|} = \frac{|-12 + 8 + 14|}{\sqrt{16 + 16 + 49}} = \frac{10}{9}. \tag{1.86}$$

♦

Example 1.3.7. (Distance Between Two Lines). Find the perpendicular distance D between the lines L_1 and L_2 given by the equations

$$x = 1 + 4s, \quad y = s, \quad z = -2 + 3s \tag{1.87}$$

and

$$x = 2 + 2t, \quad y = -1 + t, \quad z = 0. \tag{1.88}$$

We can solve this problem by first finding a vector $\mathbf{n}$ that is orthogonal to the direction vectors $\mathbf{u} = (4, 1, 3)$ and $\mathbf{v} = (2, 1, 0)$ of the given lines and, second, by projecting the vector $\overrightarrow{PQ}$ joining an arbitrary point P of L_1 to an arbitrary point Q of L_2 onto $\mathbf{n}$. To find an appropriate $\mathbf{n}$ we may consider a plane S that contains the vectors $\mathbf{u}$ and $\mathbf{v}$ through any point P_0 and find a normal vector of S as in Example 1.3.5. Whether L_1 and L_2 intersect or not, it is most convenient to choose $P_0 = O$ and the parametric equation

$$\mathbf{p} = s\mathbf{u} + t\mathbf{v}, \tag{1.89}$$

that is,

$$x = 4s + 2t, \quad y = s + t, \quad z = 3s \tag{1.90}$$

for the plane S. Eliminating the parameters yields

$$3x - 6y - 2z = 0, \tag{1.91}$$

and so we obtain $\mathbf{n} = (3, -6, -2)$. We may choose $P = (1, 0, -2)$ and $Q = (2, -1, 0)$ and then $\overrightarrow{PQ} = (1, -1, 2)$. Hence

$$D = \frac{|\overrightarrow{PQ} \cdot \mathbf{n}|}{|\mathbf{n}|} = \frac{|3 + 6 - 4|}{\sqrt{9 + 36 + 4}} = \frac{5}{7}. \tag{1.92}$$

In general, the solution above works only in three dimensions, because, for $n > 3$, the vector $\mathbf{n}$ is not unique. The following alternative solution, however, works for any $n \geq 3$. Write the equations of the two lines as

$$\mathbf{p} = \mathbf{p}_0 + s\mathbf{u} \tag{1.93}$$

and

$$\mathbf{q} = \mathbf{q}_0 + t\mathbf{v}. \tag{1.94}$$

Then an arbitrary transversal between the two lines is given by the vector

$$\mathbf{p} - \mathbf{q} = \mathbf{p}_0 - \mathbf{q}_0 + s\mathbf{u} - t\mathbf{v}. \tag{1.95}$$

The normal transversal, that is, the line segment joining the two lines orthogonally, is then characterized by the two scalar equations

$$(\mathbf{p} - \mathbf{q}) \cdot \mathbf{u} = (\mathbf{p}_0 - \mathbf{q}_0 + s\mathbf{u} - t\mathbf{v}) \cdot \mathbf{u} = 0, \tag{1.96}$$

and

$$(\mathbf{p} - \mathbf{q}) \cdot \mathbf{v} = (\mathbf{p}_0 - \mathbf{q}_0 + s\mathbf{u} - t\mathbf{v}) \cdot \mathbf{v} = 0. \tag{1.97}$$

These are two equations for the two unknowns s and t, which are easy to solve, and the distance between the lines is then $|\mathbf{p} - \mathbf{q}|$ with the solutions for s and t substituted in it. Thus, for the given lines, $\mathbf{p}_0 = (1, 0, -2)$, $\mathbf{q}_0 = (2, -1, 0)$, $\mathbf{u} = (4, 1, 3)$, and $\mathbf{v} = (2, 1, 0)$, and Equations 1.96 and 1.97 become

$$((-1, 1, -2) + s(4, 1, 3) - t(2, 1, 0)) \cdot (4, 1, 3) = 0, \tag{1.98}$$
$$((-1, 1, -2) + s(4, 1, 3) - t(2, 1, 0)) \cdot (2, 1, 0) = 0, \tag{1.99}$$

or in simplified form,

$$26s - 9t - 9 = 0, \tag{1.100}$$
$$9s - 5t - 1 = 0. \tag{1.101}$$

The solution is $s = \frac{36}{49}$ and $t = \frac{55}{49}$. Hence

$$\begin{aligned} D = |\mathbf{p} - \mathbf{q}| &= \left|(-1, 1, -2) + \frac{36}{49}(4, 1, 3) - \frac{55}{49}(2, 1, 0)\right| \\ &= \frac{1}{49}\left|(-15, 30, 10)\right| = \frac{1}{49}\sqrt{(-15)^2 + 30^2 + 10^2} = \frac{5}{7}, \end{aligned} \tag{1.102}$$

as before. ♦

Exercises

In the first eight exercises find equations for the indicated lines both in parametric and nonparametric forms.

Exercise 1.3.1. Through $P_0(1, -2, 4)$ and along $\mathbf{v} = (2, 3, -5)$.

Exercise 1.3.2. Through $P_0(1, -2, 4)$ and parallel to the y-axis.

Exercise 1.3.3. Through $P_0(7,-2,5)$ and $P_1(5,6,-3)$.

Exercise 1.3.4. Through $P_0(1,-2,4)$ and $P_1(1,6,-3)$.

Exercise 1.3.5. Through $P_0(1,-2,4)$ and $P_1(1,-2,-3)$.

Exercise 1.3.6. Through $P_0(5,4,-8)$ and normal to the plane given by $3x-4y+3z=7$.

Exercise 1.3.7. Through $P_0(5,4,-8)$ and parallel both to the plane given by $3x-4y+3z=7$ and to the xy-plane. *Hint*: The direction vector of the line sought must be orthogonal to the normal vectors of the two planes.

Exercise 1.3.8. Through $P_0(1,-2,4)$ and parallel to the planes given $3x-4y+3z=7$ and $-x+3y+4z=8$.

Exercise 1.3.9.
a. Plot the lines $\mathbf{p}=(4,3)+t(2,-3)$ and $\mathbf{p}=(3,2)+t(-1,4)$ in $\mathbb{R}^2$, indicating the points with $t=0,\pm1$ on each.
b. Explain why there is no common t value for the point of intersection.
c. Change the parameterization of each line (that is, write new equations for them, employing a new parameter) so that the new common parameter, say s, is 0 for both lines at the point of intersection.

Exercise 1.3.10. Show that in $\mathbb{R}^n$, for any n, $\mathbf{p}=t\mathbf{a}+(1-t)\mathbf{b}$ is a parametric vector equation of the line through the two points with position vectors $\mathbf{a},\mathbf{b}$. (The numbers t and $1-t$ are called *barycentric coordinates of* P, since for $0\leq t\leq 1$ this formula gives the center of mass of two point masses of size t and $1-t$ at A and B respectively (cf. Exercise 1.1.7).) What is the geometric meaning of t and $1-t$? *Hint:* Write $\mathbf{p}=\mathbf{b}+t(\mathbf{a}-\mathbf{b})$.

***Exercise 1.3.11.** Show that in $\mathbb{R}^n$, for any n, $\mathbf{p}=r\mathbf{a}+s\mathbf{b}+t\mathbf{c}$, with $0\leq r,s,t$ and $r+s+t=1$, represents the points P of a triangle with noncollinear vertices given by the position vectors $\mathbf{a},\mathbf{b},\mathbf{c}$. (The numbers r,s,t are again called barycentric coordinates of P, for a reason similar to the one in the previous exercise.) *Hint*: Generalize the solution of Exercise 1.1.9.

In the next eight exercises find equations for the indicated planes both in parametric and nonparametric forms.

Exercise 1.3.12. Through $P_0(1,-2,4)$ and containing the line given by $\mathbf{p}=(3,-2,1)+t(2,1,-3)$.

Exercise 1.3.13. Through O and containing the line given by $\mathbf{p}=(3,-2,1)+t(2,1,-3)$.

Exercise 1.3.14. Through O and orthogonal to the line given by $\mathbf{p}=(3,-2,1)+t(2,1,-3)$.

Exercise 1.3.15. Through $P_0(5, 4, -8)$ and orthogonal to the line given by $\mathbf{p} = (3, -2, 1) + t(2, 1, -3)$.

Exercise 1.3.16. Through $P_0(1, -2, 4)$ and parallel to the plane given by $3x - 4y + 3z = 7$.

Exercise 1.3.17. Through $P_0(5, 4, -8)$ and parallel to the plane given by $7x + y + 2z = 8$.

Exercise 1.3.18. Through the points O, $P_1(1, 6, -3)$, and $P_2(7, -2, 5)$.

Exercise 1.3.19. Through the points $P_0(5, 4, -8)$, $P_1(1, 6, -3)$, and $P_2(7, -2, 5)$.

In the next six exercises find the points of intersection.

Exercise 1.3.20. Of the two lines $\mathbf{p} = (5, 1, 1) + s(-2, 1, 6)$ and $\mathbf{p} = (3, -2, 1) + t(2, 1, -3)$.

Exercise 1.3.21. Of the two lines $\mathbf{p} = (-5, 4, -1) + s(2, 1, -7)$ and $\mathbf{p} = (9, -9, -2) + t(2, -4, 5)$.

Exercise 1.3.22. Of the line $\mathbf{p} = (5, 1, 1) + s(-2, 1, 6)$ and the plane $7x + y + 2z = 8$.

Exercise 1.3.23. Of the line $\mathbf{p} = (3, -2, 6) + s(-3, 5, 7)$ and the plane $3x + 2y - 2z = 3$.

Exercise 1.3.24. Of the line $\mathbf{p} = (3, -2, 6) + s(-3, 5, 7)$ and the plane $\mathbf{p} = (4, -2, 1) + s(-2, 1, 3) + t(1, 3, 2)$.

Exercise 1.3.25. Of the line $\mathbf{p} = (3, 2, -4) + s(7, -5, 4)$ and the plane $\mathbf{p} = (0, -2, 1) + s(-3, 0, 3) + t(2, -3, 4)$.

In the next six exercises find the distances.

Exercise 1.3.26. Between the point $P_0(1, -2, 4)$ and the plane $3x + 2y - 2z = 3$.

Exercise 1.3.27. Between the point $P_0(3, 4, 0)$ and the plane $y - 2z = 5$.

Exercise 1.3.28. Between the lines $\mathbf{p} = (3, -2, 6) + s(-3, 5, 7)$ and $\mathbf{p} = (5, 1, 1) + t(-2, 1, 6)$.

Exercise 1.3.29. Between the lines $\mathbf{p} = (2, 1, 5) + s(-4, 0, 3)$ and $\mathbf{p} = (0, -2, 3) + t(5, 0, -2)$.

Exercise 1.3.30. Between the point $P_0(3, 4, 0)$ and the line $L : \mathbf{p} = (3, -2, 6) + s(-3, 5, 7)$. (*Hint*: Pick an arbitrary point Q on L and decompose the vector $\overrightarrow{QP_0}$ into components parallel and perpendicular to L.)

Exercise 1.3.31. Between the point $P_0(1, -2, 4)$ and the line $\mathbf{p} = (3, 2, -4) + s(7, -5, 4)$. (See the hint in the previous exercise.)

Exercise 1.3.32. What is the geometric meaning of Equation 1.83 in $\mathbb{R}^2$? Make a drawing and explain.

Exercise 1.3.33. Let the equation of a plane S be given in $\mathbb{R}^3$ in the form $\mathbf{n} \cdot \mathbf{p} = d$, with $|\mathbf{n}| = 1$. Let us define a function by $f(\mathbf{q}) = \mathbf{n} \cdot \mathbf{q} - d$, where $\mathbf{q}$ is the position vector of any point Q in $\mathbb{R}^3$, whether in S or not. Show that the value of $f(\mathbf{q})$ equals the signed distance of Q from S, which is positive if $\mathbf{n}$ points from S towards Q, and negative if $\mathbf{n}$ points from Q towards S.

Exercise 1.3.34. Redo Exercise 1.3.26 by using the result of Exercise 1.3.33.

Exercise 1.3.35. Redo Exercise 1.3.27 by using the result of Exercise 1.3.33.

Exercise 1.3.36. Let P denote a variable point on a line L given by $\mathbf{p} = \mathbf{p}_0 + t\mathbf{v}$, and Q denote any fixed point in space not on L. Show that $\overrightarrow{QP}$ is orthogonal to $\mathbf{v}$ if and only if the distance between P and Q is minimized as a function of t.

Exercise 1.3.37. Find an equation for the *normal transverse* L of the lines given in Exercise 1.3.29. (This term means the line connecting the given ones orthogonally.) *Hint*: First, find the direction vector $\mathbf{v}$ of L, then a plane S containing $\mathbf{v}$ and one of the given lines, and last, the point of intersection of S and the other line.

MATLAB Exercises

In the next six exercises find the distances using MATLAB:

Exercise 1.3.38. Between the point $P_0(1, -2, 4)$ and the plane $3x + 2y - 2z = 3$.

Exercise 1.3.39. Between the point $P_0(1, -2, 4, 5)$ and the hyperplane $3x + 2y - 2z + w = 3$ in $\mathbb{R}^4$.

Exercise 1.3.40. Between the point $P_0(3, 4, 0)$ and the line $\mathbf{p} = (3, -2, 6) + s(-3, 5, 7)$.

Exercise 1.3.41. Between the point $P_0(3, 4, 0, 3, 4, 0)$ and the line $\mathbf{p} = (3, -2, 6, 3, -2, 6) + s(-3, 5, 7, -3, 5, 7)$ in $\mathbb{R}^6$.

Exercise 1.3.42. Between the lines $\mathbf{p} = (3, -2, 6, 4) + s(-3, 5, 7, 1)$ and $\mathbf{p} = (5, 1, 1, 2) + t(-2, 1, 6, 2)$ in $\mathbb{R}^4$.

Exercise 1.3.43. Between the lines $\mathbf{p} = (2, 1, 5, 2, 1, 5) + s(-4, 1, 3, -4, 1, 3)$ and $\mathbf{p} = (0, -2, 3, 0, -2, 3) + t(5, 0, -2, 5, 0, -2)$ in $\mathbb{R}^6$.

2. *Systems of Linear Equations, Matrices*

2.1 Gaussian Elimination

Equations of the form $\sum a_i x_i = b$, for unknowns x_i with arbitrary given numbers a_i and b, are called *linear*, and every set of simultaneous linear equations is called a *linear system*. They are generalizations of the equations of lines and planes which we have studied in Section 1.3. In this section, we begin to discuss how to solve them, that is, how to find numerical values for the x_i that satisfy all the equations of a given system. We also examine whether a given system has any solutions and, if so, then how we can describe the set of all solutions.

Linear systems arise in many applications. Examples in which they occur, in addition to lines and planes, are least-squares fitting of lines, planes, or curves to observed data, methods for obtaining approximate solutions of various differential equations, Kirchhoff's equations relating currents and potentials in electrical circuits, and various economic models. In many applications, the number of equations and unknowns can be quite large, sometimes in the hundreds or thousands. Thus it is very important to understand the structure of such systems and to apply systematic and efficient methods for their solution. Even more important is that, as we shall see, studying such systems leads to several new concepts and theories that are at the heart of linear algebra.

We begin with a simple example.

Example 2.1.1. (A System of Three Equations in Three Unknowns with a Unique Solution). Let us solve the following system:

$$\begin{aligned} 2x + 3y - z &= 8 \\ 4x - 2y + z &= 5 \\ x + 5y - 2z &= 9. \end{aligned} \tag{2.1}$$

(Geometrically this problem amounts to finding the point of intersection of three planes.)

We want to proceed as follows: multiply both sides of the first equation by 2 and subtract the result from the second equation to eliminate the $4x$,

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and subtract 1/2 times the first equation from the third equation to eliminate the x. The system is then changed into the new, equivalent[1] system:

$$\begin{aligned} 2x + 3y - z &= 8 \\ -8y + 3z &= -11 \\ \frac{7}{2}y - \frac{3}{2}z &= 5. \end{aligned} \tag{2.2}$$

As our next step we want to get rid of the $7y/2$ term in the last equation. We can achieve this elimination by multiplying the middle equation by $-7/16$ and subtracting the result from the last equation. Then we get

$$\begin{aligned} 2x + 3y - z &= 8 \\ -8y + 3z &= -11 \\ \frac{-3}{16}z &= \frac{3}{16}. \end{aligned} \tag{2.3}$$

At this point we can easily find the solution by starting with the last equation and working our way back up. First, we find $z = -1$, and second, substituting this value into the middle equation we get $-8y-3 = -11$, which yields $y = 1$. Last, we enter the values of y and z into the top equation and obtain $2x + 3 + 1 = 8$, hence $x = 2$.

Substituting these values for x, y, z into Equations 2.1 indeed confirms that they are solutions. ♦

The method of solving a linear system used in the example above is called *Gaussian elimination*,[2] and it is the foremost method of solving such systems. However, before discussing it more generally, let us mention that the way the computations were presented was the way a computer would be programmed to do them. For people, slight variations are preferable. We would rather avoid fractions, and if we want to eliminate, say, x from an equation beginning with bx by using an equation beginning with ax, with a and b nonzero integers, then we could multiply the first equation by a and the other by b to get abx in both. Also, we would sometimes add and sometimes subtract, depending on the signs of the terms involved, where computers always subtract. Last, we might rearrange the equations in a different order, if we see that doing so would result in simpler arithmetic.[3] For example, right at the start of the example above, we could have put the last equation on top because it begins

[1] Equivalence of systems will be discussed in detail on page 46.

[2] Named after Carl Friedrich Gauss (1777–1855). It is ironic that in spite of his many great achievements he is best remembered for this simple but widely used method and for the so-called Gaussian distribution in probability and statistics, which was mistakenly attributed to him but had been discovered by Abraham de Moivre in the 1730s.

[3] Computer programs have to reorder the equations sometimes but for different reasons, namely to avoid division by zero and to minimize roundoff errors.

with x rather than $2x$, and used that equation the way we have used the one beginning with $2x$.

The essence of the method is to subtract multiples of the first equation from the others so that the leftmost term in the first equation eliminates all the corresponding terms below it. Then we continue by similarly using the leftmost term in the new second equation to eliminate the corresponding term (or terms if there are more equations) below that, and so on, down to the last equation. Next, we work our way up by solving the last equation first, then substituting its solution into the previous equation, solving that, and so on. The two phases of the method are called *forward elimination* and *back substitution.* As will be seen shortly, a few complications can and do frequently arise, which make the theory that follows even more interesting and necessary. First, however, we introduce a crucial notational simplification.

Notice that in the forward elimination computations of Example 2.1.1 the variables x, y, z were not really used; they were needed only in the back substitution steps used to determine the solution. All the forward elimination computations were done on the coefficients only. In computer programs there is not even a way (and no need either) to enter the variables. In writing, the coefficients are usually arranged in a rectangular array enclosed in parentheses or brackets, called a *matrix* (plural: *matrices*) and designated by a capital letter, as in

$$A = \begin{bmatrix} 2 & 3 & -1 \\ 4 & -2 & 1 \\ 1 & 5 & -2 \end{bmatrix}. \tag{2.4}$$

This matrix contains the coefficients on the left side of system 2.1 in the same arrangement, and is therefore referred to as the *coefficient matrix* or just the *matrix* of that system. We may include the numbers on the right sides of the equations as well:

$$B = \begin{bmatrix} 2 & 3 & -1 & 8 \\ 4 & -2 & 1 & 5 \\ 1 & 5 & -2 & 9 \end{bmatrix}. \tag{2.5}$$

This is called the *augmented matrix* of the system. It is often written with a vertical line before its last column as

$$B = \left[\begin{array}{ccc|c} 2 & 3 & -1 & 8 \\ 4 & -2 & 1 & 5 \\ 1 & 5 & -2 & 9 \end{array}\right]. \tag{2.6}$$

Example 2.1.2. (Solving the $\mathbf{3\times3}$ System of Example 2.1.1, Using Augmented Matrix Notation). We write the computations of Example 2.1.1 as

$$\left[\begin{array}{rrr|r} 2 & 3 & -1 & 8 \\ 4 & -2 & 1 & 5 \\ 1 & 5 & -2 & 9 \end{array}\right] \rightarrow \left[\begin{array}{rrr|r} 2 & 3 & -1 & 8 \\ 0 & -8 & 3 & -11 \\ 0 & 7/2 & -3/2 & 5 \end{array}\right]$$
$$\rightarrow \left[\begin{array}{rrr|r} 2 & 3 & -1 & 8 \\ 0 & -8 & 3 & -11 \\ 0 & 0 & -3/16 & 3/16 \end{array}\right]. \tag{2.7}$$

The arrows between the matrices above do not designate equality, they just indicate the flow of the computation. For two matrices to be equal, all the corresponding entries must be equal, and here they are clearly not equal.

Next, we change from the last augmented matrix to the corresponding system

$$\begin{aligned} 2x + 3y \; - \; z &= \; 8 \\ -8y + 3z &= -11 \\ \frac{-3}{16} z &= \frac{3}{16}, \end{aligned} \tag{2.8}$$

which we solve as in Example 2.1.1. ♦

The matrix A in Equation 2.4 is a 3×3 (read: "three by three") matrix, and in Equation 2.5, B is a 3×4 matrix. Similarly, if a matrix has m rows and n columns, we call it an $m \times n$ matrix. In describing matrices, we always say *rows first, then columns.*

The general form of an $m \times n$ matrix is

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \tag{2.9}$$

where the $a_{11}, a_{12}, \ldots, a_{mn}$ (read "a sub one-one, a sub one-two," etc.) are arbitrary real numbers. They are called the entries of the matrix A, with a_{ij} denoting the entry at the intersection of the ith row and jth column. Thus in the double subscript ij the order is important. Also, the matrix A is often denoted by $[a_{ij}]$ or (a_{ij}).

Two matrices are said to be equal if they have the same shape, that is, the same numbers of rows and columns, and their corresponding entries are equal.

A matrix consisting of a single row is called a *row vector*, and that of a single column, a *column vector*, and, if we want to emphasize the size n, a row n-vector or a column n-vector.

By definition, a system of m linear equations for n unknowns $x_1, x_2, \ldots, x_n$ has the general form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned} \tag{2.10}$$

with the coefficient matrix A given in Equation 2.9 having arbitrary entries and the b_i denoting arbitrary numbers as well.[4] We shall frequently find it useful to collect the x_i and the b_i values into two *column* vectors and write such systems as

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \tag{2.11}$$

or abbreviated as

$$A\mathbf{x} = \mathbf{b}. \tag{2.12}$$

The expression $A\mathbf{x}$ will be discussed in detail in Section 2.3 and generalized in Section 2.4. For now, we shall just use $A\mathbf{x} = \mathbf{b}$ as a compact reference to the system.

The augmented matrix of this general system is written as

$$[A|\mathbf{b}] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]. \tag{2.13}$$

The reason for using *column* vectors $\mathbf{x}$ and $\mathbf{b}$ will be explained at the end of Section 2.3, although for $\mathbf{b}$ at least, the choice is rather natural since then the right sides of Equations 2.10 and 2.11 match.

Henceforth all vectors will be column vectors unless explicitly designated otherwise, and also $\mathbb{R}^n$, *for every* n, *will be regarded as a space of column vectors.*

In general, if we want to solve a system given as $A\mathbf{x} = \mathbf{b}$, we reduce the corresponding augmented matrix $[A|\mathbf{b}]$ to a simpler form $[U|\mathbf{c}]$ (details will follow), which we change back to a system of equations, $U\mathbf{x} = \mathbf{c}$. We then solve the latter by back substitution, that is, from the bottom up.

[4] Writing any quantity with a general subscript, like the i here in b_i, is general mathematical shorthand for the list of all such quantities, for all possible values of the subscript i, as in this case for the list $b_1, b_2, \ldots, b_m$. Also, it is customary to say "the b_i" instead of "the b_i's" to avoid any possible confusion.

Let us review the steps of Example 2.1.2. We copied the first row, then we took 4/2 times the entries of the first row in order to change the 2 into a 4, and subtracted those multiples from the corresponding entries of the second row. (We express this operation more briefly by saying that we subtracted 4/2 times the first row from the second row.) Then we took 1/2 times the entries of the first row to change the 2 into a 1 and subtracted them from the third row. In all this computation the entry 2 of the first row played a pivotal role and is therefore called the *pivot* for these operations. In general, a pivot is an entry whose multiples are used to obtain zeros below it, and the first nonzero entry remaining in the last nonzero row after the reduction is also called a pivot. (The precise definition will be given on page 53.) Thus, in this calculation the pivots are the numbers $2, -8, -3/16$.

The operations we used are called elementary row operations.

Definition 2.1.1. ***(Elementary Row Operations).*** *We call the following three types of operations on the augmented matrix of a system elementary row operations:*
1. Multiplying a row by a nonzero number.
2. Exchanging two rows.
3. Changing a row by subtracting a nonzero multiple of another row from it.

Definition 2.1.2. ***(Equivalence of Systems and of Matrices).*** *Two systems of equations are called equivalent if their solution sets are the same. Furthermore, the augmented matrices of equivalent systems are called equivalent to each other as well.*

All elimination steps in this section, like the ones above, have been designed to produce equivalent, but simpler, systems.

Theorem 2.1.1. ***(Row Equivalence).*** *Each elementary row operation changes the augmented matrix of every system of linear equations into the augmented matrix of an equivalent system.*

Proof. Consider any two rows of the augmented matrix of the $m \times n$ system 2.10, say the ith and the jth row:

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = b_i \tag{2.14}$$

and

$$a_{j1}x_1 + a_{j2}x_2 + \cdots + a_{jn}x_n = b_j. \tag{2.15}$$

1. If we form a new augmented matrix by multiplying the ith row of the augmented matrix 2.13 by any $c \neq 0$, then the ith row of the corresponding new system is

$$ca_{i1}x_1 + ca_{i2}x_2 + \cdots + ca_{in}x_n = cb_i, \tag{2.16}$$

which is clearly equivalent to Equation 2.14. Furthermore, since all the other equations of the system 2.10 remain unchanged, every solution $\mathbf{x}$ of the old system is also a solution of the new system and vice versa.
2. If we form a new augmented matrix by exchanging the ith row of the augmented matrix 2.13 by its jth row, then the corresponding system of equations remains the same, except that equations 2.14 and 2.15 are switched. Clearly, changing the order of equations does not change the solutions.
3. If we change the jth row of the augmented matrix 2.13 by subtracting c times the ith row from it, for any $c \neq 0$, then the jth row of the corresponding new system becomes

$$(a_{j1} - ca_{i1})\, x_1 + (a_{j2} - ca_{i2})\, x_2 + \cdots + (a_{jn} - ca_{in})\, x_n = b_j - cb_i. \tag{2.17}$$

The other equations of the system, including the ith one, remain unchanged. Clearly, every vector $\mathbf{x}$ that solves the old system, also solves Equation 2.17, and so it solves the whole new system as well. Conversely, if a vector $\mathbf{x}$ solves the new system, then it solves Equation 2.14, and hence also Equation 2.16, as well as Equation 2.17. Adding the latter two together, we find that it solves Equation 2.15, that is, it solves the old system. ■

Hence any two matrices obtainable from each other by a finite number of successive elementary row operations are equivalent, and to indicate that they are related by such row operations, they are said to be *row equivalent.* Column operations would also be possible, but they are rarely used, and we shall not discuss them at all.

We have used only the third type of elementary row operation so far. The first kind is not necessary for Gaussian elimination but will be used later in further reductions. The second kind must be used if we encounter a zero where we need a pivot, as in the following example.

Example 2.1.3. (A 4×3 System with a Unique Solution and Requiring a Row Exchange). Let us solve the following system of $m = 4$ equations in $n = 3$ unknowns:

$$\begin{aligned} x_1 + 2x_2 \qquad\quad &= 2 \\ 3x_1 + 6x_2 - \quad x_3 &= 8 \\ x_1 + 2x_2 + \quad x_3 &= 0 \\ 2x_1 + 5x_2 - 2x_3 &= 9. \end{aligned} \tag{2.18}$$

We do the computations in matrix form. We indicate the row operations in the rows between the matrices by arrows, which may be read as “becomes” or “is replaced by.” For example, $\mathbf{r}_2 \leftarrow \mathbf{r}_2 - 3\mathbf{r}_1$ means that row 2 is replaced by the old row 2 minus 3 times row 1. (The rows may be considered to be vectors, and so we designate them by boldface letters.)

$$\left[\begin{array}{rrr|r} 1 & 2 & 0 & 2 \\ 3 & 6 & -1 & 8 \\ 1 & 2 & 1 & 0 \\ 2 & 5 & -2 & 9 \end{array}\right] \begin{array}{l} \mathbf{r}_1 \leftarrow \mathbf{r}_1 \\ \mathbf{r}_2 \leftarrow \mathbf{r}_2 - 3\mathbf{r}_1 \\ \mathbf{r}_3 \leftarrow \mathbf{r}_3 - \mathbf{r}_1 \\ \mathbf{r}_4 \leftarrow \mathbf{r}_4 - 2\mathbf{r}_1 \end{array} \left[\begin{array}{rrr|r} 1 & 2 & 0 & 2 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 1 & -2 \\ 0 & 1 & -2 & 5 \end{array}\right]$$

$$\begin{array}{l} \mathbf{r}_1 \leftarrow \mathbf{r}_1 \\ \mathbf{r}_2 \leftarrow \mathbf{r}_4 \\ \mathbf{r}_3 \leftarrow \mathbf{r}_3 \\ \mathbf{r}_4 \leftarrow \mathbf{r}_2 \end{array} \left[\begin{array}{rrr|r} 1 & 2 & 0 & 2 \\ 0 & 1 & -2 & 5 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & -1 & 2 \end{array}\right] \begin{array}{l} \mathbf{r}_1 \leftarrow \mathbf{r}_1 \\ \mathbf{r}_2 \leftarrow \mathbf{r}_2 \\ \mathbf{r}_3 \leftarrow \mathbf{r}_3 \\ \mathbf{r}_4 \leftarrow \mathbf{r}_4 + \mathbf{r}_3 \end{array} \left[\begin{array}{rrr|r} 1 & 2 & 0 & 2 \\ 0 & 1 & -2 & 5 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right]. \tag{2.19}$$

The back substitution phase should start with the third row of the last matrix, since the fourth row just expresses the trivial equation $0 = 0$. The third row gives $x_3 = -2$, the second row corresponds to $x_2 - 2x_3 = 5$ and so $x_2 = 1$, and the first row yields $x_1 + 2x_2 = 2$, from which $x_1 = 0$. ♦

As the example above shows, the number m of equations and the number n of unknowns need not be the same. In this case the four equations described four planes in three-dimensional space, having a single point of intersection given by the *unique solution* we have found. Of course, in general, four planes need not have a point of intersection in common or may have an entire line or plane as their intersection (in the latter case the four equations would each describe the same plane). Systems with solutions are called *consistent*. On the other hand, if there is no intersection, then the system has no solution, and it is said to be *inconsistent*. Inconsistency of the system can happen with just two or three planes as well, for instance if two of them are parallel, and also in two dimensions with parallel lines. So before attacking the general theory, we discuss examples of inconsistent systems and systems with infinitely many solutions. Systems with more equations than unknowns are called *overdetermined,* and are usually (though not always, see Example 2.1.3) inconsistent. Systems with fewer equations than unknowns are called *underdetermined*, and they usually (though not always) have infinitely many solutions. For example, two planes in $\mathbb{R}^3$ would usually intersect in a line, but exceptionally they could be parallel and have no intersection. On the other hand, a system with the same number of equations as unknowns is called *determined* and usually (though not always) has a unique solution. For instance, three planes in $\mathbb{R}^3$ would usually intersect in a point, but by exception they could be parallel and have no intersection or intersect in a line or a plane.

Example 2.1.4. (A 3 × 3 Inconsistent System). Consider the system given by the matrix

$$[A|\mathbf{b}] = \left[\begin{array}{rrr|r} 1 & 2 & 0 & 2 \\ 3 & 6 & -1 & 8 \\ 1 & 2 & 1 & 4 \end{array}\right]. \tag{2.20}$$

Subtracting $3\mathbf{r}_1$ from $\mathbf{r}_2$, and $\mathbf{r}_1$ from $\mathbf{r}_3$, we get

$$[A'|\mathbf{b}'] = \left[\begin{array}{ccc|c} 1 & 2 & 0 & 2 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 1 & 2 \end{array}\right]. \tag{2.21}$$

The last two rows of $[A'|\mathbf{b}']$ represent the contradictory equations $-x_3 = 2$ and $x_3 = 2$. These two equations describe parallel planes. Thus $[A|\mathbf{b}]$ had to represent an inconsistent system.

The row operations above have produced two equations of new planes, which have turned out to be parallel to each other. The planes corresponding to the rows of the original $[A|\mathbf{b}]$ are, however, not parallel. Instead, only the three lines of intersection of pairs of them are (see Exercise 2.1.16), like the three parallel edges of a prism; that is why there is no point of intersection common to all three planes.

We may carry the matrix reduction one step further and obtain, by adding the second row to the third one,

$$[A''|\mathbf{b}''] = \left[\begin{array}{ccc|c} 1 & 2 & 0 & 2 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 4 \end{array}\right]. \tag{2.22}$$

This matrix provides the single self-contradictory equation $0 = 4$ from its last row. There is no geometrical interpretation for such an equation, but algebraically it is the best way of establishing the inconsistency. Thus, this is the typical pattern we shall obtain in the general case whenever there is no solution. ♦

Next we modify the matrix of the last example so that all three planes intersect in a single line.

Example 2.1.5. (A 3×3 System with a Line for Its Solution Set). Let

$$[A|\mathbf{b}] = \left[\begin{array}{ccc|c} 1 & 2 & 0 & 2 \\ 3 & 6 & -1 & 8 \\ 1 & 2 & 1 & 0 \end{array}\right]. \tag{2.23}$$

We can reduce this matrix to

$$[A'|\mathbf{b}'] = \left[\begin{array}{ccc|c} 1 & 2 & 0 & 2 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]. \tag{2.24}$$

The corresponding system is

$$\begin{aligned} x_1 + 2x_2 \qquad &= 2 \\ -x_3 &= 2 \\ 0 &= 0, \end{aligned} \tag{2.25}$$

which represents just two planes, since the last equation has become the trivial identity $0 = 0$. Algebraically, the second row gives $x_3 = -2$, and the first row relates x_1 to x_2. We can choose an arbitrary value for either x_1 or x_2 and solve the first equation of the system 2.25 for the other. In some other examples, however, we have no choice, as between x_1 and x_2 here. However, *since the pivot cannot be zero, we can always solve the pivot's row for the variable corresponding to the pivot, and that is what we always do.* Thus, we set x_2 equal to a parameter t and solve the first equation for x_1, to obtain $x_1 = 2 - 2t$. We can write the solutions in vector form as (remember: the convention is to use column vectors)

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix} + t \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}. \tag{2.26}$$

This is a parametric vector equation of the line of intersection L of the three planes defined by the rows of $[A|\mathbf{b}]$. The coordinates of each of L's points make up one of the infinitely many solutions of the system. ♦

Example 2.1.6. (A 3×4 System with a Line for its Solution Set). Let us solve the system

$$\begin{aligned} 2x_1 + 3x_2 - 2x_3 + 4x_4 &= 4 \\ -6x_1 - 8x_2 + 6x_3 - 2x_4 &= 1 \\ 4x_1 + 4x_2 - 4x_3 - x_4 &= -7. \end{aligned} \tag{2.27}$$

These equations represent three hyperplanes in four dimensions.[5] We can proceed as before:

$$\left[\begin{array}{cccc|c} 2 & 3 & -2 & 4 & 4 \\ -6 & -8 & 6 & -2 & 1 \\ 4 & 4 & -4 & -1 & -7 \end{array}\right] \begin{array}{l} \mathbf{r}_1 \leftarrow \mathbf{r}_1 \\ \mathbf{r}_2 \leftarrow \mathbf{r}_2 + 3\mathbf{r}_1 \\ \mathbf{r}_3 \leftarrow \mathbf{r}_3 - 2\mathbf{r}_1 \end{array} \left[\begin{array}{cccc|c} 2 & 3 & -2 & 4 & 4 \\ 0 & 1 & 0 & 10 & 13 \\ 0 & -2 & 0 & -9 & -15 \end{array}\right]$$

$$\begin{array}{l} \mathbf{r}_1 \leftarrow \mathbf{r}_1 \\ \mathbf{r}_2 \leftarrow \mathbf{r}_2 \\ \mathbf{r}_3 \leftarrow \mathbf{r}_3 + 2\mathbf{r}_2 \end{array} \left[\begin{array}{cccc|c} 2 & 3 & -2 & 4 & 4 \\ 0 & 1 & 0 & 10 & 13. \\ 0 & 0 & 0 & 11 & 11 \end{array}\right]. \tag{2.28}$$

The variables that have pivots as coefficients, x_1, x_2, x_4 in this case, are called *basic variables*. They can be obtained in terms of the other, so-called *free variables* that correspond to the pivot-free columns. The free variables are usually replaced by parameters, but this is just a formality to show that they can be chosen freely.

Thus, we set $x_3 = t$, and find the solutions again as the points of a line, now given by

---

[5] A hyperplane in $\mathbb{R}^4$ is a copy of $\mathbb{R}^3$, just as a plane in $\mathbb{R}^3$ is a copy of $\mathbb{R}^2$.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -9/2 \\ 3 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}. \tag{2.29}$$

♦

Example 2.1.7. (A 3×4 System with a Plane for Its Solution Set). Consider the system

$$\begin{aligned} 2x_1 + 3x_2 - 2x_3 + 4x_4 &= 2 \\ -6x_1 - 9x_2 + 7x_3 - 8x_4 &= -3 \\ 4x_1 + 6x_2 - x_3 + 20x_4 &= 13. \end{aligned} \tag{2.30}$$

We solve this system as follows:

$$\left[\begin{array}{rrrr|r} 2 & 3 & -2 & 4 & 2 \\ -6 & -9 & 7 & -8 & -3 \\ 4 & 6 & -1 & 20 & 13 \end{array}\right] \begin{array}{l} \mathbf{r}_1 \leftarrow \mathbf{r}_1 \\ \mathbf{r}_2 \leftarrow \mathbf{r}_2 + 3\mathbf{r}_1 \\ \mathbf{r}_3 \leftarrow \mathbf{r}_3 - 2\mathbf{r}_1 \end{array} \left[\begin{array}{rrrr|r} 2 & 3 & -2 & 4 & 2 \\ 0 & 0 & 1 & 4 & 3 \\ 0 & 0 & 3 & 12 & 9 \end{array}\right]$$

$$\begin{array}{l} \mathbf{r}_1 \leftarrow \mathbf{r}_1 \\ \mathbf{r}_2 \leftarrow \mathbf{r}_2 \\ \mathbf{r}_3 \leftarrow \mathbf{r}_3 - 3\mathbf{r}_2 \end{array} \left[\begin{array}{rrrr|r} 2 & 3 & -2 & 4 & 2 \\ 0 & 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]. \tag{2.31}$$

Since the pivots are in columns 1 and 3, the basic variables are x_1 and x_3 and the free variables x_2 and x_4. Thus we use two parameters and set $x_2 = s$ and $x_4 = t$. Then the second row of the last matrix leads to $x_3 = 3-4t$ and the first row to $2x_1 + 3s - 2(3-4t) + 4t = 2$, that is, to $2x_1 = 8 - 3s - 12t$. Putting all these results together, we obtain the two-parameter set of solutions

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 3 \\ 0 \end{bmatrix} + s \begin{bmatrix} -3/2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -6 \\ 0 \\ -4 \\ 1 \end{bmatrix}, \tag{2.32}$$

which is also a parametric vector equation of a plane in $\mathbb{R}^4$. ♦

Exercises

In the first four exercises, find all solutions of the systems by Gaussian elimination.

Exercise 2.1.1.
$$\begin{aligned} 2x_1 + 2x_2 - 3x_3 &= 0 \\ x_1 + 5x_2 + 2x_3 &= 1 \\ -4x_1 + 6x_3 &= 2 \end{aligned}$$

Exercise 2.1.2.
$$\begin{aligned} 2x_1 + 2x_2 - 3x_3 &= 0 \\ x_1 + 5x_2 + 2x_3 &= 0 \\ -4x_1 + 6x_3 &= 0 \end{aligned}$$

Exercise 2.1.3.
$$\begin{aligned} 2x_1 + 2x_2 - 3x_3 &= 0 \\ x_1 + 5x_2 + 2x_3 &= 1 \end{aligned}$$

Exercise 2.1.4. $2x_1 + 2x_2 - 3x_3 = 0$

In the next nine exercises use Gaussian elimination to find all solutions of the systems given by their augmented matrices.

Exercise 2.1.5.
$$\left[\begin{array}{rrr|r} 1 & 0 & -1 & 0 \\ -2 & 3 & -1 & 0 \\ -6 & 6 & 0 & 0 \end{array}\right]$$

Exercise 2.1.6.
$$\left[\begin{array}{rrr|r} 1 & 0 & -1 & 1 \\ -2 & 3 & -1 & 0 \\ -6 & 6 & 0 & -2 \end{array}\right]$$

Exercise 2.1.7.
$$\left[\begin{array}{rrr|r} 1 & 0 & -1 & 1 \\ -2 & 3 & -1 & 0 \\ -6 & 6 & 0 & 0 \end{array}\right]$$

Exercise 2.1.8.
$$\left[\begin{array}{rrrr|l} 3 & -6 & -1 & 1 & 12 \\ -1 & 2 & 2 & 3 & 1 \\ 6 & -8 & -3 & -2 & 9 \end{array}\right]$$

Exercise 2.1.9.
$$\left[\begin{array}{lrrr|r} 1 & 4 & 9 & 2 & 0 \\ 2 & 2 & 6 & -3 & 0 \\ 2 & 7 & 16 & 3 & 0 \end{array}\right]$$

Exercise 2.1.10.
$$\left[\begin{array}{lrr|r} 2 & 4 & 1 & 7 \\ 0 & 1 & 3 & 7 \\ 3 & 3 & -1 & 9 \\ 1 & 2 & 3 & 11 \end{array}\right]$$

Exercise 2.1.11.
$$\left[\begin{array}{rrrrr|r} 3 & -6 & -1 & 1 & 5 & 0 \\ -1 & 2 & 2 & 3 & 3 & 0 \\ 4 & -8 & -3 & -2 & 1 & 0 \end{array}\right]$$

Exercise 2.1.12.
$$\left[\begin{array}{rrrr|r} 3 & -6 & -1 & 1 & 7 \\ -1 & 2 & 2 & 3 & 1 \\ 4 & -8 & -3 & -2 & 6 \end{array}\right]$$

Exercise 2.1.13.
$$\left[\begin{array}{rrrr|r} 3 & -6 & -1 & 1 & 5 \\ -1 & 2 & 2 & 3 & 3 \\ 6 & -8 & -3 & -2 & 1 \end{array}\right]$$

Exercise 2.1.14. What is wrong with the following way of "solving" Exercise 2.1.13?

$$\left[\begin{array}{rrrr|r} 3 & -6 & -1 & 1 & 5 \\ -1 & 2 & 2 & 3 & 3 \\ 6 & -8 & -3 & -2 & 1 \end{array}\right] \mathbf{r}_1 \leftrightarrow \mathbf{r}_2 \left[\begin{array}{rrrr|r} -1 & 2 & 2 & 3 & 3 \\ 3 & -6 & -1 & 1 & 5 \\ 6 & -8 & -3 & -2 & 1 \end{array}\right]$$

$$\begin{array}{l} \mathbf{r}_1 \leftarrow \mathbf{r}_1 \\ \mathbf{r}_2 \leftarrow 2\mathbf{r}_2 - \mathbf{r}_3 \\ \mathbf{r}_3 \leftarrow \mathbf{r}_3 - 2\mathbf{r}_2 \end{array} \left[\begin{array}{rrrr|r} -1 & 2 & 2 & 3 & 3 \\ 0 & -4 & 1 & 4 & 9 \\ 0 & 4 & -1 & -4 & -9 \end{array}\right]$$

$$\begin{array}{l} \mathbf{r}_1 \leftarrow \mathbf{r}_1 \\ \mathbf{r}_2 \leftarrow \mathbf{r}_2 \\ \mathbf{r}_3 \leftarrow \mathbf{r}_3 + \mathbf{r}_2 \end{array} \left[\begin{array}{rrrr|r} -1 & 2 & 2 & 3 & 3 \\ 0 & -4 & 1 & 4 & 9 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right],$$

$x_3 = s$, $x_4 = t$, $-4x_2 + s + 4t = 9$, $x_2 = -\frac{9}{4} + \frac{1}{4}s + t$, $-x_1 + 2x_2 + 2s + 3t = 3$, $x_1 = -3 + 2\left(-\frac{9}{4} + \frac{1}{4}s + t\right) + 2s + 3t = \frac{5}{2}s + 5t - \frac{15}{2}$, and so

$$\mathbf{x} = \begin{bmatrix} -15/2 \\ -9/4 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 5/2 \\ 1/4 \\ 1 \\ 0 \end{bmatrix} s + \begin{bmatrix} 5 \\ 1 \\ 0 \\ 1 \end{bmatrix} t.$$

Exercise 2.1.15. Show that each pair of the three planes defined by the rows of the matrix in Example 2.1.5 on page 49 has the same line of intersection.

Exercise 2.1.16. Show that the three planes defined by the rows of the matrix in Equation 2.20 on page 48 have parallel lines of intersection.

2.2 The Theory of Gaussian Elimination

We are now at a point where we can summarize the lessons from our examples. Given m equations for n unknowns, we consider their augmented matrix,

$$[A|\mathbf{b}] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right], \tag{2.33}$$

and reduce it using elementary row operations according to the following algorithm:

1. Search the first column from the top down for the first nonzero entry. If all the entries in the first column are zero, then search the second column from the top down, then the third column for the first nonzero entry. Repeat with succeeding columns if necessary, until a nonzero entry is found. The entry thus found is called the current *pivot*. Stop, if no pivot can be found.
2. Put the row containing the current pivot on top (unless it is already there).

3. Subtract appropriate multiples of the first row from each of the lower rows to obtain all zeros below the current pivot in its column (unless there are all zeros there or no lower rows are left).
4. Repeat the previous steps on the submatrix[6] consisting of all those elements of the last matrix that lie lower than and to the right of the last pivot. Stop if no such submatrix is left.

These steps constitute the *forward elimination* phase of Gaussian elimination (the second phase will be discussed following Definition 2.2.2), and they lead to a matrix of the form described below.

Definition 2.2.1. ***(Echelon Matrix).*** *A matrix is said to be in echelon form*[7] *or an echelon matrix if it has a staircase-like pattern characterized by the following properties:*
a*. The all-zero rows (if any) are at the bottom.*
b*. The leftmost nonzero entry of each nonzero row, called a leading entry, is in a column to the right of the leading entry of every row above it.*

These properties imply that in an echelon matrix U all the entries of a column below a leading entry are zero. If U arises from the reduction of a matrix A by the forward elimination algorithm above, then the pivots of A become the leading entries of U. Also, if we were to apply the algorithm to an echelon matrix, then it would not be changed and we would find that its leading entries are its pivots.

Note that while a given matrix is row equivalent to many different echelon matrices (just multiply any nonzero row of an echelon matrix by 2, for example), the algorithm above leads to a single well-defined echelon matrix in each case. Furthermore, it will be proved in Section 3.3 that the number and locations, although not the values, of the pivots are unique for all echelon matrices obtainable from the same A. Consequently, the results of Theorem 2.2.1 below, even though they depend on the pivots, are valid unambiguously.

Here is a possible $m \times (n+1)$ echelon matrix obtainable from the matrix $[A|\mathbf{b}]$ above:

$$[U|\mathbf{c}] = \left[\begin{array}{ccccccc|c} p_1 & * & * & * & \cdots & * & * & c_1 \\ 0 & p_2 & * & * & \cdots & * & * & c_2 \\ 0 & 0 & 0 & p_3 & & * & * & c_3 \\ \vdots & & & & & & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & p_r & * & c_r \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & c_{r+1} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & & & & & & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \end{array}\right]. \tag{2.34}$$

[6] A submatrix of a given matrix A is a matrix obtained by deleting any number of rows and/or columns of A.

[7] "Echelon" in French means "rung of a ladder," and in English it is used for some ladder-like military formations and rankings.

The first n columns constitute the echelon matrix U obtained from A, and the last column is the corresponding reduction of $\mathbf{b}$. The p_i denote the pivots of A, while the entries denoted by * and by c_i denote numbers that may or may not be zero. The number r is very important, since it determines the character of the solutions, and has a special name.

Definition 2.2.2. ***(Rank).*** *The number r of nonzero rows of an echelon matrix U obtained by the forward elimination phase of the Gaussian elimination algorithm from a matrix A is called the rank of A and will be denoted by* $\operatorname{rank}(A)$.[8,9]

We can now describe the back substitution phase of Gaussian elimination, in which we change the augmented matrix $[U|\mathbf{c}]$ back to a system of equations $U\mathbf{x} = \mathbf{c}$:

5. If $r < m$ and $c_{r+1} \neq 0$ hold, then the row containing c_{r+1} corresponds to the self-contradictory equation $0 = c_{r+1}$, and so the system has no solutions or, in other words, it is *inconsistent.* (This case occurs in Example 2.1.4, where $m = 3,\ r = 2$ and $c_{r+1} = c_3 = 4$.)
6. If $r = m$ or $c_{r+1} = 0$, then the system is *consistent* and, for every i such that the ith column contains no pivot, the variable x_i is a free variable and we set it equal to a parameter s_i. (In Example 2.1.6, for instance, $r = m = 3$ and x_3 is free. In Example 2.1.7 we have $m = 3,\ r = 2$ and $c_{r+1} = c_3 = 0$ and the free variables are x_2 and x_4.) We need to distinguish two subcases here:
   a. If $r = n$, then there are no free variables and the system has a *unique solution.* (In Example 2.1.2, for instance, $r = m = n = 3$.)
   b. If $r < n$, then the system has *infinitely many solutions.* (In Examples 2.1.6 and 2.1.7, for instance, $r = 3$ and $n = 4$.)
7. In any of the cases of Part 6, we solve for the basic variables x_i corresponding to the pivots p_i, starting in the rth row and working our way up row by row.

The Gaussian elimination algorithm proves the following theorem:

Theorem 2.2.1. ***(Summary of Gaussian Elimination).*** *Consider the $m \times n$ system with A an $m \times n$ matrix and $\mathbf{b}$ an n-vector:*

$$A\mathbf{x} = \mathbf{b}. \tag{2.35}$$

Suppose the matrix $[A|\mathbf{b}]$ is reduced by the algorithm above to the echelon matrix $[U|\mathbf{c}]$ with $\operatorname{rank}(U) = r$.

[8] Of course, r is also the rank of U, since the algorithm applied to U would leave U unchanged.

[9] Some books call this quantity the *row rank* of A until they define the column rank and show that the two are equal.

If $r = m$*, that is, if* U *has no zero rows, then the system 2.35 is consistent. If* $r < m$*, then the system is consistent if and only if* $c_{r+1} = 0$*.*

For a consistent system,

a. there is a unique solution if and only if there are no free variables, that is, if $r = n$*;*

b. if $r < n$*, then there is an* $(n - r)$*-parameter infinite set of solutions of the form*

$$\mathbf{x} = \mathbf{x}_0 + \sum_{i=1}^{n-r} s_i \mathbf{v}_i. \tag{2.36}$$

We may state the uniqueness condition $r = n$ in another way by saying that the pivots are the diagonal entries $u_{11}, u_{22}, \ldots, u_{nn}$ of U, that is, that U has the form

$$U = \begin{bmatrix} p_1 & * & * & \cdots & * \\ 0 & p_2 & * & \cdots & * \\ 0 & 0 & p_3 & & * \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & p_r \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}. \tag{2.37}$$

A matrix of this form is called an *upper triangular matrix* and the p_i its diagonal entries. (The pivots are never 0, but in general, an upper triangular matrix is allowed to have 0 diagonal entries as well.)

Note that for every $m \times n$ matrix we have $0 \leq r \leq \min(m, n)$, because r equals the number of pivots and there can be only one pivot in each row and in each column. We have $r = 0$ only for zero matrices. At the other extreme, if, for a matrix A, $r = \min(m, n)$ holds, then A is said to have *full rank.* If $r < \min(m, n)$ holds, then A is said to be *rank deficient.*

Exercises

Exercise 2.2.1. List all possible forms of 2×2 echelon matrices in a manner similar to Equation 2.37, with p_i for the pivots and $*$ for the entries that may or may not be zero.

Exercise 2.2.2. List all possible forms of 3×3 echelon matrices in a manner similar to Equation 2.37, with p_i for the pivots and $*$ for the entries that may or may not be zero. (*Hint*: There are eight distinct forms.)

In the next four exercises find conditions on a general vector $\mathbf{b}$ that would make the equation $A\mathbf{x} = \mathbf{b}$ consistent for the given matrix A. (*Hint*: Reduce the augmented matrix using undetermined components b_i of $\mathbf{b}$, until the A in it is changed to echelon form, and set $c_{r+1} = c_{r+2} = \cdots = 0$.)

Exercise 2.2.3. $A = \begin{bmatrix} 1 & 0 & -1 \\ -2 & 3 & -1 \\ -6 & 6 & 0 \end{bmatrix}.$

Exercise 2.2.4. $A = \begin{bmatrix} 1 & -2 \\ 2 & -4 \\ -6 & 12 \end{bmatrix}.$

Exercise 2.2.5. $A = \begin{bmatrix} 1 & 2 & -6 \\ -2 & -4 & 12 \end{bmatrix}.$

Exercise 2.2.6. $A = \begin{bmatrix} 1 & 0 & -1 \\ -2 & 3 & -1 \\ 3 & -3 & 0 \\ 2 & 0 & -2 \end{bmatrix}.$

Exercise 2.2.7. Prove that the system $A\mathbf{x} = \mathbf{b}$ is consistent if and only if A and $[A|\mathbf{b}]$ have the same rank.

2.3 Homogeneous and Inhomogeneous Systems, Gauss–Jordan Elimination

In the sequel, we need to consider the expression $A\mathbf{x}$ as a new kind of product.[10]

Definition 2.3.1. ***(Matrix-Vector Product).*** *For every $m \times n$ matrix A and every column n-vector $\mathbf{x}$, we define $A\mathbf{x}$ as the column m-vector given by*

$$A\mathbf{x} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix}. \tag{2.38}$$

[10] The product $A\mathbf{x}$ is always written just by juxtaposing the two letters; we never use any multiplication sign in it.

Notice that, on the right, the components of the column vector $\mathbf{x}$ show up across every row of $A\mathbf{x}$; they are "flipped." Actually, the rows on the right are the dot products of the row vectors of A with the vector $\mathbf{x}$. It is customary to write $\mathbf{a}^i$ (with a superscript i) for the ith row of A, and $\mathbf{a}^i\mathbf{x}$ (without a dot) for the ith dot product on the right. Thus Equation 2.38 can also be written as

$$A\mathbf{x} = \begin{bmatrix} \mathbf{a}^1 \\ \mathbf{a}^2 \\ \vdots \\ \mathbf{a}^m \end{bmatrix} \mathbf{x} = \begin{bmatrix} \mathbf{a}^1\mathbf{x} \\ \mathbf{a}^2\mathbf{x} \\ \vdots \\ \mathbf{a}^m\mathbf{x} \end{bmatrix}. \tag{2.39}$$

We also need the following simple properties of $A\mathbf{x}$.

Theorem 2.3.1. ***(Properties of the Matrix-Vector Product).*** *If A is an $m \times n$ matrix, $\mathbf{x}$ and $\mathbf{y}$ column n-vectors, and c a scalar, then*

$$A(\mathbf{x}+\mathbf{y}) = A\mathbf{x} + A\mathbf{y} \text{ and } A(c\mathbf{x}) = c(A\mathbf{x}). \tag{2.40}$$

Proof. Using Equation 2.39 and the properties of vectors and dot products, we have

$$A(\mathbf{x}+\mathbf{y}) = \begin{bmatrix} \mathbf{a}^1 \\ \mathbf{a}^2 \\ \vdots \\ \mathbf{a}^m \end{bmatrix} (\mathbf{x}+\mathbf{y}) = \begin{bmatrix} \mathbf{a}^1(\mathbf{x}+\mathbf{y}) \\ \mathbf{a}^2(\mathbf{x}+\mathbf{y}) \\ \vdots \\ \mathbf{a}^m(\mathbf{x}+\mathbf{y}) \end{bmatrix} = \begin{bmatrix} \mathbf{a}^1\mathbf{x}+\mathbf{a}^1\mathbf{y} \\ \mathbf{a}^2\mathbf{x}+\mathbf{a}^2\mathbf{y} \\ \vdots \\ \mathbf{a}^m\mathbf{x}+\mathbf{a}^m\mathbf{y} \end{bmatrix} \tag{2.41}$$

$$= \begin{bmatrix} \mathbf{a}^1\mathbf{x} \\ \mathbf{a}^2\mathbf{x} \\ \vdots \\ \mathbf{a}^m\mathbf{x} \end{bmatrix} + \begin{bmatrix} \mathbf{a}^1\mathbf{y} \\ \mathbf{a}^2\mathbf{y} \\ \vdots \\ \mathbf{a}^m\mathbf{y} \end{bmatrix} = A\mathbf{x} + A\mathbf{y}. \tag{2.42}$$

Similarly,

$$A(c\mathbf{x}) = \begin{bmatrix} \mathbf{a}^1 \\ \mathbf{a}^2 \\ \vdots \\ \mathbf{a}^m \end{bmatrix} (c\mathbf{x}) = \begin{bmatrix} \mathbf{a}^1(c\mathbf{x}) \\ \mathbf{a}^2(c\mathbf{x}) \\ \vdots \\ \mathbf{a}^m(c\mathbf{x}) \end{bmatrix} = \begin{bmatrix} c(\mathbf{a}^1\mathbf{x}) \\ c(\mathbf{a}^2\mathbf{x}) \\ \vdots \\ c(\mathbf{a}^m\mathbf{x}) \end{bmatrix} = c \begin{bmatrix} \mathbf{a}^1\mathbf{x} \\ \mathbf{a}^2\mathbf{x} \\ \vdots \\ \mathbf{a}^m\mathbf{x} \end{bmatrix} = c(A\mathbf{x}). \tag{2.43}$$

∎

If the solutions of $A\mathbf{x} = \mathbf{b}$ are given by Equation 2.36, the latter is called the *general solution* of the system, as opposed to a *particular solution,* which is obtained by substituting particular values for the parameters into Equation 2.36.

It is customary and very useful to distinguish two types of linear systems depending on the choice of $\mathbf{b}$.

Definition 2.3.2. ***(Homogeneous Versus Inhomogeneous Systems).*** *A system of linear equations* $A\mathbf{x} = \mathbf{b}$ *is called homogeneous if* $\mathbf{b} = \mathbf{0}$*, and inhomogeneous if* $\mathbf{b} \neq \mathbf{0}$*.*

We may restate part of Theorem 2.2.1 for homogeneous systems as follows.

Theorem 2.3.2. ***(Solutions of Homogeneous Systems).*** *For any* $m \times n$ *matrix* A*, the homogeneous system*

$$A\mathbf{x} = \mathbf{0} \tag{2.44}$$

is always consistent: it always has the trivial solution $\mathbf{x} = \mathbf{0}$*. If* $r = n$*, then it has only this solution; and if* $m < n$ *or, more generally, if* $r < n$ *holds, then it has nontrivial solutions as well.*

There is an important relationship between the solutions of corresponding homogeneous and inhomogeneous systems, the analog of which is indispensable for solving many differential equations.

Theorem 2.3.3. ***(General and Particular Solutions).*** *For any* $m \times n$ *matrix* A *and any column* m*-vector* $\mathbf{b}$, *if* $\mathbf{x} = \mathbf{x}_b$ *is any particular solution of the inhomogeneous equation*

$$A\mathbf{x} = \mathbf{b}, \tag{2.45}$$

with $\mathbf{b} \neq \mathbf{0}$, *then*

$$\mathbf{x} = \mathbf{x}_b + \mathbf{v} \tag{2.46}$$

is its general solution if and only if

$$\mathbf{v} = \sum_{i=1}^{n-r} s_i \mathbf{v}_i \tag{2.47}$$

is the general solution of the corresponding homogeneous equation

$$A\mathbf{v} = \mathbf{0}. \tag{2.48}$$

Proof. Assume first that 2.47 is the general solution of Equation 2.48. (Certainly, the Gaussian elimination algorithm would give it in this form.) Then applying A to both sides of Equation 2.46, we get

$$A\mathbf{x} = A(\mathbf{x}_b + \mathbf{v}) = A\mathbf{x}_b + A\mathbf{v} = \mathbf{b} + \mathbf{0} = \mathbf{b}. \tag{2.49}$$

Thus, every solution of the homogeneous Equation 2.48 leads to a solution of the form 2.46 of the inhomogeneous equation 2.45.

Conversely, assume that 2.46 is a solution of the inhomogeneous equation 2.45. Then

$$A\mathbf{v} = A(\mathbf{x} - \mathbf{x}_b) = A\mathbf{x} - A\mathbf{x}_b = \mathbf{b} - \mathbf{b} = \mathbf{0}. \tag{2.50}$$

This equation shows that the $\mathbf{v}$ given by Equation 2.47 is indeed a solution of Equation 2.48, or, in other words, that a solution of the form 2.46 of the inhomogeneous equation 2.45 leads to a solution of the form 2.47 of the homogeneous equation 2.48. ■

This theorem establishes a one-to-one pairing of the solutions of the two equations 2.45 and 2.48. Geometrically this means that the solutions of $A\mathbf{v} = \mathbf{0}$ are the position vectors of the points of the hyperplane through the origin given by Equation 2.47, and the solutions of $A\mathbf{x} = \mathbf{b}$ are those of a parallel hyperplane obtained from the first one by shifting it by the vector $\mathbf{x}_b$. (See Figure 2.1.) Note that we could have shifted by the coordinate vector of any other point of the second hyperplane, that is, by any other particular solution $\mathbf{x}'_b$ of Equation 2.45 (see Figure 2.2), and we would have obtained the same new hyperplane.

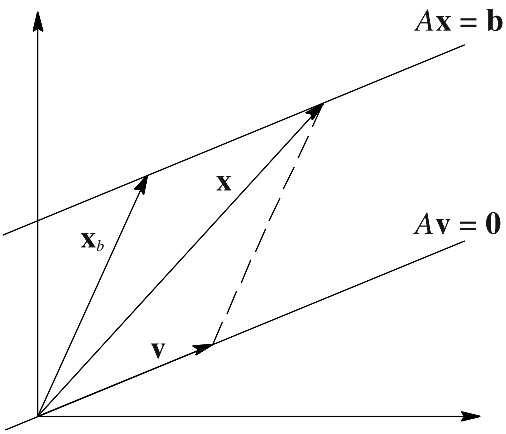

Fig. 2.1. The solution vector $\mathbf{x}$ of the inhomogeneous equation equals the sum of a particular solution and a solution of the corresponding homogeneous equation: $\mathbf{x} = \mathbf{x}_b + \mathbf{v}$

Sometimes the forward elimination procedure is carried further so as to obtain leading entries in the echelon matrix that equal 1, and to obtain 0 entries in the basic columns not just below but also above the pivots. This method is called *Gauss–Jordan elimination* and the final matrix a *reduced echelon matrix* or a *row–reduced echelon matrix*. We give one example of this method.

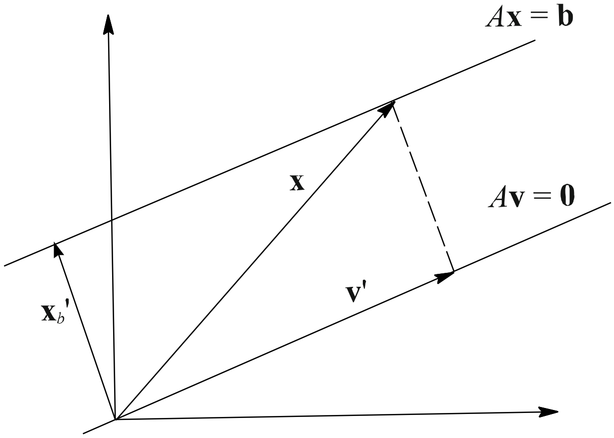

Fig. 2.2. The same solution vector $\mathbf{x}$ of the inhomogeneous equation also equals the sum of another particular solution and another solution of the corresponding homogeneous equation: $\mathbf{x} = \mathbf{x}_b' + \mathbf{v}'$

Example 2.3.1. (Solving Example 2.1.7 by Gauss–ordan Elimination). Let us continue the reduction of Example 2.1.7, starting with the echelon matrix obtained in the forward elimination phase:

$$\left[\begin{array}{cccc|c} 2 & 3 & -2 & 4 & 2 \\ 0 & 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \mathbf{r}_1 \leftarrow \mathbf{r}_1/2 \\ \mathbf{r}_2 \leftarrow \mathbf{r}_2 \\ \mathbf{r}_3 \leftarrow \mathbf{r}_3 \end{array} \left[\begin{array}{cccc|c} 1 & 3/2 & -1 & 2 & 1 \\ 0 & 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$
$$\begin{array}{l} \mathbf{r}_1 \leftarrow \mathbf{r}_1 + \mathbf{r}_2 \\ \mathbf{r}_2 \leftarrow \mathbf{r}_2 \\ \mathbf{r}_3 \leftarrow \mathbf{r}_3 \end{array} \left[\begin{array}{cccc|c} 1 & 3/2 & 0 & 6 & 4 \\ 0 & 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]. \tag{2.51}$$

From here on we proceed exactly as in the Gaussian elimination algorithm: we assign parameters s and t to the free variables x_2 and x_4, and solve for the basic variables x_1 and x_3. The latter step is now trivial, since all the work has already been done. The equations corresponding to the final matrix are

$$\begin{aligned} x_1 + \frac{3}{2}s + 6t &= 4 \\ x_3 + 4t &= 3. \end{aligned} \tag{2.52}$$

Thus we find the same general solution as before:

$$\begin{aligned} x_1 &= 4 - \frac{3}{2}s - 6t \\ x_2 &= \quad\;\; s \\ x_3 &= 3 \qquad\; - 4t \\ x_4 &= \qquad\qquad\; t \end{aligned} \tag{2.53}$$

or in vector form as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 3 \\ 0 \end{bmatrix} + s \begin{bmatrix} -3/2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -6 \\ 0 \\ -4 \\ 1 \end{bmatrix}. \tag{2.54}$$

Notice how the numbers in the first and third rows of this solution correspond to the entries of the last matrix in 2.51, which is in reduced echelon form. ♦

As can be seen from this example, the reduced echelon form combines the results of both the forward elimination and back substitution phases of Gaussian elimination, and the general solution can simply be read from it. In general, if $[R|\mathbf{c}]$ is the reduced echelon matrix corresponding to the system $A\mathbf{x} = \mathbf{b}$, then we assign parameters s_j to the free variables x_j; and if r_{ik} is a pivot of R, that is, a leading entry 1 in the ith row and kth column, then x_k is a basic variable, and the ith row of the reduced system $R\mathbf{x} = \mathbf{c}$ is

$$x_k + \sum_{j>k} r_{ij} s_j = c_i. \tag{2.55}$$

Thus the general solution is given by

$$\begin{array}{ll} x_j = s_j & \text{if } x_j \text{ is free, and} \\ x_k = c_i - \sum_{j>k} r_{ij} s_j & \text{if } x_k \text{ is basic and is in the } i\text{th row.} \end{array} \tag{2.56}$$

Gauss–Jordan elimination is rarely used for the solution of systems, because a variant of Gaussian elimination, which we shall study in Section 8.1, is usually more efficient. However, Gauss–Jordan elimination is the preferred method for inverting matrices, as we shall see in Section 2.3. Also, it is sometimes helpful that the reduced echelon form of a matrix is unique (see Theorem 3.4.2), and that the solution of every system is immediately visible in it.

We conclude this section with an application.

Example 2.3.2. (An Electrical Network). Consider the electrical network shown in Figure 2.3. Here the R_k are positive numbers denoting resistances (unit: ohm (Ω)), the i_k are currents (unit: ampere (A)), and V_1 and V_2 are the voltages (unit: volt (V)) of two batteries represented by the circles. These quantities are related by three laws of physics:

1. **Kirchhof's first law.** The sum of the currents entering a node equals the sum of the currents leaving it.
2. **Kirchhof's second law.** The sum of the voltage drops or potential differences around every loop equals zero.
3. **Ohm's law.** The voltage drop across a resistor R equals Ri, where i is the current flowing through the resistor.

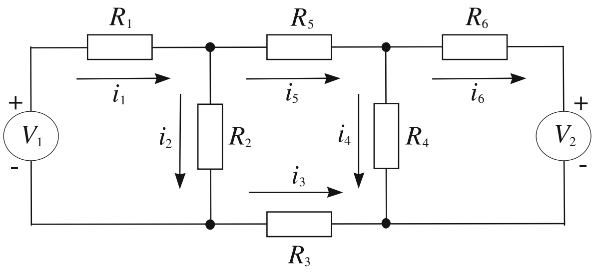

Fig. 2.3. An electrical network with resistors and two voltage sources

By convention, outside a battery the current that it generates flows from the positive terminal to the negative one.[11] However, in a multiloop circuit the directions of the currents are not so obvious. In the circuit above, for instance, the current i_6 is generated by both batteries, and although V_2 would make it flow from right to left, it is possible that the contribution of V_1 would make it flow as the arrow shows. In fact, the arrows for the direction of the currents can be chosen arbitrarily, and if the equations result in a negative value for an i_k, then the current flows in the direction opposite the arrow.

For the circuit of Figure 2.3, Kirchhof's laws give the following six equations for the six unknown currents:[12]

$$\begin{array}{rcl} i_1 - i_2 - i_5 &=& 0 \\ -i_4 + i_5 - i_6 &=& 0 \\ i_3 + i_4 + i_6 &=& 0 \\ R_1 i_1 + R_2 i_2 &=& V_1 \\ R_2 i_2 + R_3 i_3 - R_4 i_4 - R_5 i_5 &=& 0 \\ R_4 i_4 - R_6 i_6 &=& V_2 \end{array} \tag{2.57}$$

Assume that $R_1 = 4\,\Omega$, $R_2 = 24\,\Omega$, $R_3 = 1\,\Omega$, $R_4 = 3\,\Omega, R_5 = 2\,\Omega$, $R_6 = 8\,\Omega$, $V_1 = 80\,\mathrm{V}$, and $V_2 = 62\,\mathrm{V}$. Then the augmented matrix of the system becomes

$$\left[\begin{array}{cccccc|c} 1 & -1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 4 & 24 & 0 & 0 & 0 & 0 & 80 \\ 0 & 24 & 1 & -3 & -2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & -8 & 62 \end{array}\right], \tag{2.58}$$

[11] This convention was established before the discovery of electrons, which actually make up the flow by carrying a negative charge around the loop in the opposite direction.

[12] Actually, Kirchhof's laws give more equations than these six. In Example 3.5.7 we shall examine how to select a sufficient set.

and Gaussian elimination gives the solution $i_1 = 8\,\text{A}$, $i_2 = 2\,\text{A}$, $i_3 = -6\,\text{A}$, $i_4 = 10\,\text{A}$, $i_5 = 6\,\text{A}$, $i_6 = -4\,\text{A}$.

In order to give a transparent illustration of Kirchhof's laws, we show this solution in Figure 2.4, with the arrows pointing in the correct directions for the currents.

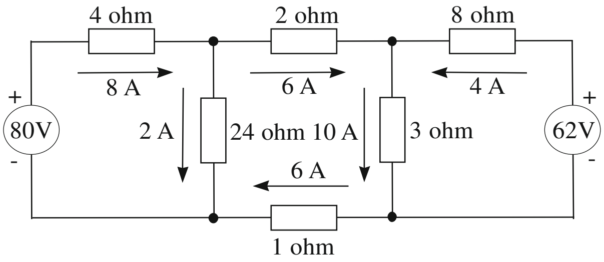

Fig. 2.4. The same circuit as in Figure 2.3, solved

The system above was obtained by what is called the branch method. Another possibility is to use the loop method, which we are now going to illustrate for the same circuit.

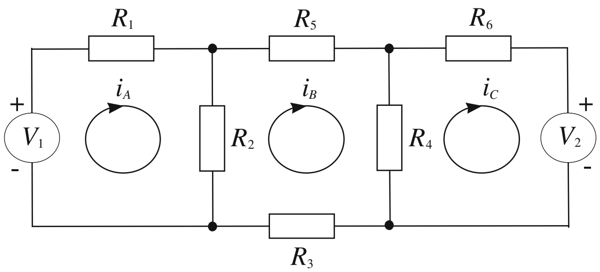

Fig. 2.5. The same circuit with loops shown

We consider only three unknown currents: i_A, i_B, i_C, one for each of the three small loops (see Figure 2.5), with arbitrarily assigned directions. Then, for the resistors shared by two loops, we must use the appropriate superposition of the currents of those loops. Thus the loop equations are

$$\begin{aligned} R_1 i_A + R_2(i_A - i_B) &= V_1 \\ R_2(i_B - i_A) + R_3 i_B + R_4(i_B - i_C) + R_5 i_B &= 0 \\ R_4(i_C - i_B) + R_6 i_C &= -V_2 \end{aligned} \tag{2.59}$$

or, equivalently,

$$\begin{aligned} (R_1 + R_2)\,i_A - R_2 i_B &= V_1 \\ -R_2 i_A + (R_2 + R_3 + R_4 + R_5)\,i_B - R_4 i_C &= 0 \\ -R_4 i_B + (R_4 + R_6)\,i_C &= -V_2 \end{aligned} \tag{2.60}$$

For the given numerical values, the augmented matrix of this system becomes

$$\left[\begin{array}{rrr|r} 28 & -24 & 0 & 80 \\ -24 & 30 & -3 & 0 \\ 0 & -3 & 11 & -62 \end{array}\right], \tag{2.61}$$

whose solution is $i_A = 8\,\mathrm{A}$, $i_B = 6\,\mathrm{A}$, and $i_C = -4\,\mathrm{A}$. From these loop currents we can easily recover the earlier branch currents as $i_1 = i_A = 8\,\mathrm{A}$, $i_2 = i_A - i_B = 2\,\mathrm{A}$, $i_3 = -i_B = -6\,\mathrm{A}$, $i_4 = i_B - i_C = 10\,\mathrm{A}$, $i_5 = i_B = 6\,\mathrm{A}$, $i_6 = i_C = -4\,\mathrm{A}$. ♦

Exercises

Exercise 2.3.1. List all possible forms of 2×2 reduced echelon matrices.

Exercise 2.3.2. List all possible forms of 3×3 reduced echelon matrices.

Exercise 2.3.3. Solve Exercise 2.1.5 by Gauss–Jordan elimination.

Exercise 2.3.4. Solve Exercise 2.1.8 by Gauss–Jordan elimination.

Exercise 2.3.5. Solve Exercise 2.1.11 by Gauss–Jordan elimination.

Exercise 2.3.6. Solve Exercise 2.1.12 by Gauss–Jordan elimination.

In each of the next two exercises find two particular solutions $\mathbf{x}_b$ and $\mathbf{x}'_b$ of the given system and the general solution $\mathbf{v}$ of the corresponding homogeneous system. Write the general solution of the given system as $\mathbf{x}_b + \mathbf{v}$ and also as $\mathbf{x}'_b + \mathbf{v}$, and show that the two forms are equivalent; that is, that the set of vectors of the form $\mathbf{x}_b + \mathbf{v}$ is identical with the set of vectors of the form $\mathbf{x}'_b + \mathbf{v}$.

Exercise 2.3.7.
$$\begin{aligned} 2x_1 + 3x_2 - 1x_3 &= 4 \\ 3x_1 + 5x_2 + 2x_3 &= 1 \end{aligned}$$

Exercise 2.3.8.
$$\begin{aligned} 2x_1 + 2x_2 - 3x_3 - 2x_4 &= 4 \\ 6x_1 + 6x_2 + 3x_3 + 6x_4 &= 0 \end{aligned}$$

MATLAB Exercises

In MATLAB, linear systems are entered in matrix form. We can enter a matrix by writing its entries between brackets, row by row from left to right, top to bottom, and separating row entries by spaces or commas, and rows by semicolons. For example the command $A = [2, 3; 1, -2]$ would produce the matrix

$$A = \begin{bmatrix} 2 & 3 \\ 1 & -2 \end{bmatrix}.$$

(The size of a matrix is automatic; no size declaration is needed or possible, unlike in most other computer languages.) The entry a_{ij} of the matrix A is denoted by $A(i,j)$ in MATLAB, the ith row by $A(i,:)$ and the jth column by $A(:,j)$.

The vector $\mathbf{b}$ must be entered as a column vector. This can be achieved either by separating its entries by semicolons or by writing a prime after the closing bracket, as in $\mathbf{b} = [1, 2]'$. This would result in the column vector

$$\mathbf{b} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

The augmented matrix can be formed by the command $[A\ \mathbf{b}]$. Sometimes we may wish to name it as, say, $A\_b = [A\ \mathbf{b}]$ or simply as $C = [A\ \mathbf{b}]$. The command $\mathbf{rref}(C)$ returns the reduced echelon form of C.

The command $\mathbf{x} = A\backslash\mathbf{b}$ always returns a solution of the system $A\mathbf{x} = \mathbf{b}$. This is the unique solution if there is only one; it is a certain particular solution with as many zeros as possible for components of $\mathbf{x}$ with the lowest subscripts, and is the least-squares "solution" (to be discussed in Section 5.1) if the system is inconsistent. This command is the most efficient method of finding a solution and is the one you should use whenever possible. On the other hand, to find the *general solution* of an underdetermined system this method does not work, and you should use $\mathbf{rref}([A\ \mathbf{b}])$ to obtain the reduced echelon matrix, and proceed as in Example 2.3.1 or Equations 2.56.

Exercise 2.3.9.
a. Write MATLAB commands to implement elementary row operations on a 3×6 matrix A.
b. Use these commands to reduce the matrix

$$A = \begin{bmatrix} 3 & -6 & -1 & 1 & 5 & 2 \\ -1 & 2 & 2 & 3 & 3 & 6 \\ 4 & -8 & -3 & -2 & 1 & 0 \end{bmatrix}$$

to reduced echelon form and compare your result to $\mathbf{rref}(A)$.
c. Write MATLAB commands to compute a matrix B with the same rows as the matrix A, but the first two rows switched.
d. Compare $\mathbf{rref}(B)$ with $\mathbf{rref}(A)$. Explain your result.

Exercise 2.3.10. Use MATLAB to find the general solution of $A\mathbf{x} = \mathbf{0}$ for

$$A = \begin{bmatrix} -1 & -2 & -1 & -1 & 1 \\ -1 & -2 & 0 & 3 & -1 \\ 1 & 2 & 1 & 1 & 1 \\ 0 & 0 & 2 & 8 & 2 \end{bmatrix}.$$

Exercise 2.3.11. Let A be the same matrix as in the previous exercise and let

$$\mathbf{b} = \begin{bmatrix} 9 \\ 1 \\ -5 \\ -4 \end{bmatrix}.$$

a. Find the general solution of $A\mathbf{x} = \mathbf{b}$ using **rref**$([A\ \mathbf{b}])$.
b. Verify that $\mathbf{x} = A\backslash\mathbf{b}$ is indeed a particular solution by computing $A\mathbf{x}$ from it.
c. Find the parameter values in the general solution obtained in Part (a), that give the particular solution of Part (b).
d. To verify the result of Theorem 2.3.3 for this case, show that the general solution of Part (a) equals $\mathbf{x} = A\backslash\mathbf{b}$ plus the general solution of the homogeneous equation found in the previous exercise.

Exercise 2.3.12. Let A and $\mathbf{b}$ be the same as in the last exercise. The command $\mathbf{x} = \mathbf{pinv}(A) * \mathbf{b}$ gives another particular solution of $A\mathbf{x} = \mathbf{b}$. (This solution will be explained in Section 5.1.) Verify Theorem 2.3.3 for this particular solution, as in Part (d) of the previous exercise.

Exercise 2.3.13. Let A be the same matrix as in Exercise 2.3.10 and let $\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$. Compute $\mathbf{x} = A\backslash\mathbf{b}$ and substitute this into $A\mathbf{x}$. Explain how your result is possible. (*Hint*: Look at **rref**$([A\ \mathbf{b}])$.)

2.4 The Algebra of Matrices

Just as for vectors, we can define algebraic operations for matrices, and these operations will vastly extend their utility.

In order to motivate the forthcoming definitions, it is helpful to interpret matrices as functions or mappings. Thus if A is an $m \times n$ matrix, the matrix-vector product $A\mathbf{u}$ may be regarded as describing a mapping $T_A : \mathbb{R}^n \to \mathbb{R}^m$ of every $\mathbf{u} \in \mathbb{R}^n$ to $A\mathbf{u} \in \mathbb{R}^m$, that is, as $T_A(\mathbf{u}) = A\mathbf{u}$. This is also reflected in the terminology: We frequently read $A\mathbf{u}$ as A being *applied* to $\mathbf{u}$ instead of A *times* $\mathbf{u}$. If $m = n$, we may consider T_A as a transformation of the vectors of $\mathbb{R}^n$ to corresponding vectors in the same space.

Example 2.4.1. (Rotation Matrix). The matrix

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \tag{2.62}$$

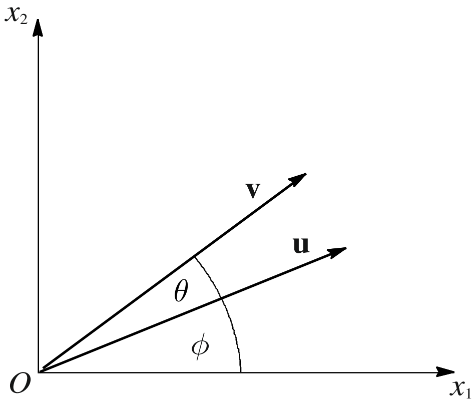

Fig. 2.6. Rotation of $\mathbf{u} \in \mathbb{R}^2$ by the angle θ

represents the rotation T_θ of $\mathbb{R}^2$ around O by the angle θ, as can be seen in the following way. (See Figure 2.6.) Let

$$\mathbf{u} = \begin{bmatrix} |\mathbf{u}|\cos\phi \\ |\mathbf{u}|\sin\phi \end{bmatrix} \tag{2.63}$$

be any nonzero vector in $\mathbb{R}^2$ (see Exercise 1.2.15 on page 26). Then, by Definition 2.3.1,

$$T_\theta(\mathbf{u}) = R_\theta\mathbf{u} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} |\mathbf{u}|\cos\phi \\ |\mathbf{u}|\sin\phi \end{bmatrix} = |\mathbf{u}| \begin{bmatrix} \cos\theta\cos\phi - \sin\theta\sin\phi \\ \sin\theta\cos\phi + \cos\theta\sin\phi \end{bmatrix}, \tag{2.64}$$

and so

$$T_\theta(\mathbf{u}) = R_\theta\mathbf{u} = |\mathbf{u}| \begin{bmatrix} \cos(\phi+\theta) \\ \sin(\phi+\theta) \end{bmatrix}. \tag{2.65}$$

This is indeed a vector of the same length as $\mathbf{u}$ and it encloses the angle $\phi+\theta$ with the x_1-axis. ♦

Such transformations will be discussed in detail in Chapter 4. Here we just present the concept briefly, in order to lay the groundwork for the definitions of matrix operations. These definitions are analogous to the familiar definitions for functions of real variables, where, given functions f and g, their sum $f+g$ is defined as the function such that $(f+g)(x) = f(x)+g(x)$ for every x, and, for any real number c, the product cf is defined as the function such that $(cf)(x) = cf(x)$ for every x.

Definition 2.4.1. ***(Sum and Scalar Multiple of Mappings).*** *Let T_A and T_B be two mappings of $\mathbb{R}^n$ to $\mathbb{R}^m$, for any positive integers m and n.*

We define their sum $T_A + T_B$ *as the mapping that maps every* $\mathbf{x} \in \mathbb{R}^n$ *to* $T_A(\mathbf{x}) + T_B(\mathbf{x}) \in \mathbb{R}^m$ *or, in other words, as the mapping given by*

$$(T_A + T_B)(\mathbf{x}) = T_A(\mathbf{x}) + T_B(\mathbf{x}) \textit{ for all } \mathbf{x} \in \mathbb{R}^n. \tag{2.66}$$

Furthermore, for any scalar c*, the mapping* cT_A *is defined as the mapping that maps every* $\mathbf{x}$ *to* $c(T_A(\mathbf{x})) \in \mathbb{R}^m$*, that is, the mapping for which*

$$(cT_A)(\mathbf{x}) = c(T_A(\mathbf{x})) \textit{ for all } \mathbf{x} \in \mathbb{R}^n. \tag{2.67}$$

Now, let T_A and T_B be two mappings that correspond to two matrices A and B respectively, that is, such that $T_A(\mathbf{x}) = A\mathbf{x}$ and $T_B(\mathbf{x}) = B\mathbf{x}$ for all appropriate $\mathbf{x}$. Then we can use Definition 2.4.1 to define $A + B$ and cA.

Definition 2.4.2. ***(Sum and Scalar Multiple of Matrices).*** *Let* A *and* B *be two* $m \times n$ *matrices, for any positive integers* m *and* n*. We define their sum* $A + B$ *as the matrix that corresponds to* $T_A + T_B$*, or, in other words, as the matrix for which we have*

$$(A + B)\mathbf{x} = A\mathbf{x} + B\mathbf{x} \textit{ for all } \mathbf{x} \in \mathbb{R}^n. \tag{2.68}$$

Similarly, for any scalar c*, the matrix* cA *is defined as the matrix that corresponds to* cT_A*, that is, as the matrix for which*

$$(cA)\mathbf{x} = c(A\mathbf{x}) \textit{ for all } \mathbf{x} \in \mathbb{R}^n. \tag{2.69}$$

The mappings $T_A + T_B$ and cT_A both clearly exist, but the existence of corresponding matrices $A + B$ and cA requires proof. Their existence will be proved by Theorem 2.4.1 below, where they will be computed explicitly.

Definition 2.4.2 can be paraphrased as requiring that the order of the operations be reversible: On the right-hand side of Equation 2.68 we first *apply* A and B separately to $\mathbf{x}$ and then *add*, and on the left we first *add* A to B and then *apply* the sum to $\mathbf{x}$. Similarly, on the right-hand side of Equation 2.69 we first apply A to $\mathbf{x}$ and then multiply by c, while on the left this is reversed: A is first multiplied by c and then cA is applied to $\mathbf{x}$. We may also regard Equation 2.68 as a new distributive rule and Equation 2.69 as a new associative rule. Note that Equation 2.69 enables us to drop the parentheses, that is, to write $cA\mathbf{x}$ for $c(A\mathbf{x})$.

Example 2.4.2. (A Matrix Sum). Let

$$A = \begin{bmatrix} 3 & 5 \\ 4 & 2 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 3 \\ 4 & 7 \end{bmatrix}. \tag{2.70}$$

Then, applying Definition 2.3.1, for any $\mathbf{x}$ we have

$$A\mathbf{x} = \begin{bmatrix} 3 & 5 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3x_1 + 5x_2 \\ 4x_1 + 2x_2 \end{bmatrix}, \tag{2.71}$$

and

$$B\mathbf{x} = \begin{bmatrix} 2 & 3 \\ 4 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_1 + 3x_2 \\ 4x_1 + 7x_2 \end{bmatrix}. \tag{2.72}$$

Hence

$$A\mathbf{x} + B\mathbf{x} = \begin{bmatrix} 3x_1 + 5x_2 \\ 4x_1 + 2x_2 \end{bmatrix} + \begin{bmatrix} 2x_1 + 3x_2 \\ 4x_1 + 7x_2 \end{bmatrix} = \begin{bmatrix} 5x_1 + 8x_2 \\ 8x_1 + 9x_2 \end{bmatrix} = \begin{bmatrix} 5 & 8 \\ 8 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \tag{2.73}$$

Thus, by Equation 2.68,

$$(A + B)\mathbf{x} = \begin{bmatrix} 5 & 8 \\ 8 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{2.74}$$

and so,

$$A + B = \begin{bmatrix} 5 & 8 \\ 8 & 9 \end{bmatrix}. \tag{2.75}$$

Here we see that $A + B$ is obtained from A and B by adding corresponding entries. That this addition rule is true in general, and not just for these particular matrices, will be part of Theorem 2.4.1. ♦

Example 2.4.3. (A Scalar Multiple of a Matrix). Let $c = 2$ and

$$A = \begin{bmatrix} 3 & 4 \\ 4 & 2 \end{bmatrix}. \tag{2.76}$$

Then, applying Definition 2.3.1, for every $\mathbf{x}$ we have

$$A\mathbf{x} = \begin{bmatrix} 3 & 4 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3x_1 + 4x_2 \\ 4x_1 + 2x_2 \end{bmatrix}, \tag{2.77}$$

and so,

$$c\,(A\mathbf{x}) = 2\begin{bmatrix} 3x_1 + 4x_2 \\ 4x_1 + 2x_2 \end{bmatrix} = \begin{bmatrix} 6x_1 + 8x_2 \\ 8x_1 + 4x_2 \end{bmatrix} = \begin{bmatrix} 6 & 8 \\ 8 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \tag{2.78}$$

Thus, by Equation 2.69,

$$(2A)\mathbf{x} = \begin{bmatrix} 6 & 8 \\ 8 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{2.79}$$

and so,

$$2A = \begin{bmatrix} 6 & 8 \\ 8 & 4 \end{bmatrix}. \tag{2.80}$$

Here we see that $2A$ is obtained by multiplying every entry of A by 2. The next theorem generalizes this multiplication rule to arbitrary c and A. ♦

Theorem 2.4.1. ***(The Sum and Scalar Multiple of Matrices in Terms of Entries).*** *For any two matrices $A = [a_{ik}]$ and $B = [b_{ik}]$ of the same shape, we have*

$$(A + B)_{ik} = a_{ik} + b_{ik} \text{ for all } i, k, \tag{2.81}$$

and for any scalar c, we have

$$(cA)_{ik} = ca_{ik} \text{ for all } i, k. \tag{2.82}$$

Proof. To write Equations 2.68 and 2.69 in terms of components, let us first recall from Definition 2.3.1 that, for all appropriate $\mathbf{x}$, the product $A\mathbf{x}$ is a column m-vector whose ith component is given, for each i, by

$$(A\mathbf{x})_i = a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n, \tag{2.83}$$

which can be abbreviated as

$$(A\mathbf{x})_i = \sum_{j=1}^{n} a_{ij}x_j. \tag{2.84}$$

Applying the same principle to B and $A + B$, we also have, for all i,

$$(B\mathbf{x})_i = \sum_j b_{ij}x_j \tag{2.85}$$

and

$$[(A + B)\mathbf{x}]_i = \sum_j (A + B)_{ij}x_j. \tag{2.86}$$

From Equation 2.68, the definition of vector addition, and the last three equations,

$$\begin{aligned}[(A + B)\mathbf{x}]_i &= (A\mathbf{x} + B\mathbf{x})_i = (A\mathbf{x})_i + (B\mathbf{x})_i \\ &= \sum_j a_{ij}x_j + \sum_j b_{ij}x_j = \sum_j (a_{ij} + b_{ij})x_j.\end{aligned} \tag{2.87}$$

Comparing the two evaluations of $[(A+B)\mathbf{x}]_i$ in Equations 2.86 and 2.87, we obtain

$$\sum_j (A + B)_{ij}x_j = \sum_j (a_{ij} + b_{ij})x_j. \tag{2.88}$$

Equation 2.88 must hold for every choice of $\mathbf{x}$. Choosing $x_k = 1$ for any fixed k, and $x_j = 0$ for all $j \neq k$, yields the first statement of the theorem:

$$(A + B)_{ik} = a_{ik} + b_{ik} \text{ for all } i, k. \tag{2.89}$$

Equation 2.82 can be obtained similarly, and its proof is left as Exercise 2.4.2. ∎

This theorem can be paraphrased as: Every entry of a sum of matrices equals the sum of the corresponding entries of the summands; and we multiply a matrix A by a scalar c, by multiplying every entry by c. Notice again, as for vectors, the reversal of operations: "every entry of a sum = sum of corresponding entries" and "every entry of $cA = c \times$ corresponding entry of A."

Let us emphasize that only matrices of the same shape can be added to each other, and that the sum has the same shape, in which case we call them *conformable for addition*. However, for matrices of differing shapes there is no reasonable way of defining a sum.

We can also define multiplication of matrices in certain cases and this will prove to be an enormously fruitful operation. For real-valued functions f and g, their composite $f \circ g$ was defined by $(f \circ g)(x) = f(g(x))$ for all x, and we first define the composite of two mappings similarly, to represent the performance of two mappings in succession.

Definition 2.4.3. ***(Composition of Mappings).*** *Let T_B be a mapping of $\mathbb{R}^n$ to $\mathbb{R}^p$ and T_A be a mapping of $\mathbb{R}^p$ to $\mathbb{R}^m$, for any positive integers m, p, and n. We define the composite $T_A \circ T_B$ as the mapping that maps every $\mathbf{x} \in \mathbb{R}^n$ to $T_A(T_B(\mathbf{x})) \in \mathbb{R}^m$ or, in other words, as the mapping given by*

$$(T_A \circ T_B)(\mathbf{x}) = T_A(T_B(\mathbf{x})) \text{ for all } \mathbf{x} \in \mathbb{R}^n. \tag{2.90}$$

Next, we define the product of two matrices as the matrix that corresponds to the composite mapping.

Definition 2.4.4. ***(Matrix Multiplication).*** *Let A be an $m \times p$ matrix and B a $p \times n$ matrix, for any positive integers m, p, and n. Let T_A and T_B be the corresponding mappings. That is, let T_B map every $\mathbf{x} \in \mathbb{R}^n$ to a vector $T_B(\mathbf{x}) = B\mathbf{x}$ of $\mathbb{R}^p$ and T_A map every $\mathbf{y} \in \mathbb{R}^p$ to $T_A(\mathbf{y}) = A\mathbf{y}$ of $\mathbb{R}^m$. We define the product AB as the $m \times n$ matrix that corresponds to the composite mapping $T_A \circ T_B$, that is, by the formula*

$$(AB)\mathbf{x} = (T_A \circ T_B)(\mathbf{x}) = A(B\mathbf{x}) \text{ for all } \mathbf{x} \in \mathbb{R}^n. \tag{2.91}$$

These mappings and Definition 2.4.4 are illustrated symbolically in Figure 2.7.

That such a matrix always exists will be proved by Theorem 2.4.2, where it will be computed explicitly.

Let us emphasize that only for matrices A and B such that the number of columns of A (the p in the definition) equals the number of rows of B can the product AB be formed, in which case we call them *conformable for multiplication*. Also, we never use any sign for this multiplication, we just write the factors next to each other.

Furthermore, Equation 2.91 can also be viewed as a new associative law or as a reversal of the order of the two multiplications (but not of the factors). Hence, we can drop the parentheses, that is, we can write $AB\mathbf{x}$ for $A(B\mathbf{x})$.

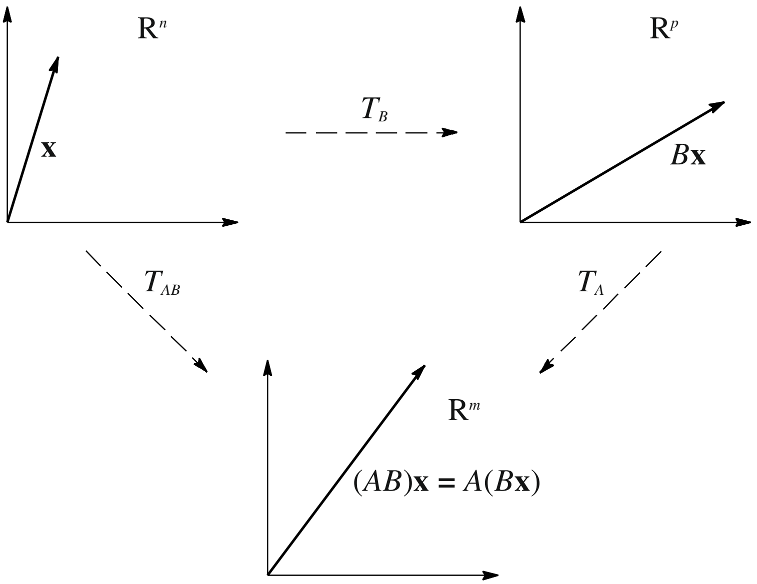

Fig. 2.7. The product of two matrices corresponding to two mappings in succession

Example 2.4.4. (A Matrix Multiplication). Let

$$A = \begin{bmatrix} 3 & 5 \\ 1 & 2 \\ 2 & 4 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}. \tag{2.92}$$

Then, applying Equation 2.85, for every $\mathbf{x}$ we have

$$B\mathbf{x} = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_1 + x_2 \\ 3x_2 \end{bmatrix}, \tag{2.93}$$

and similarly

$$A(B\mathbf{x}) = \begin{bmatrix} 3 & 5 \\ 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 2x_1 + x_2 \\ 3x_2 \end{bmatrix} = \begin{bmatrix} 6x_1 + 18x_2 \\ 2x_1 + 7x_2 \\ 4x_1 + 14x_2 \end{bmatrix} = \begin{bmatrix} 6 & 18 \\ 2 & 7 \\ 4 & 14 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \tag{2.94}$$

Thus

$$AB = \begin{bmatrix} 6 & 18 \\ 2 & 7 \\ 4 & 14 \end{bmatrix}. \tag{2.95}$$

♦

From the definition we can easily deduce the following rule that gives the entries of AB and shows that the vector $\mathbf{x}$ can be dispensed with in their computation.

Theorem 2.4.2. ***(Matrix Multiplication in Terms of Entries).*** *Let A be an $m \times p$ matrix and B a $p \times n$ matrix. Then the product AB is an $m \times n$ matrix whose entries are given by the formula*

$$(AB)_{ik} = \sum_{j=1}^{p} a_{ij} b_{jk} \text{ for } i = 1, \ldots, m \text{ and } k = 1, \ldots, n. \tag{2.96}$$

Proof. The components of $B\mathbf{x}$ can be written as

$$(B\mathbf{x})_j = \sum_{k=1}^{n} b_{jk} x_k \text{ for } j = 1, \ldots, p. \tag{2.97}$$

Also,

$$(A\mathbf{y})_i = \sum_{j=1}^{p} a_{ij} y_j \text{ for } i = 1, \ldots, m. \tag{2.98}$$

Substituting from Equation 2.97 into 2.98, we get

$$(A(B\mathbf{x}))_i = \sum_{j=1}^{p} a_{ij} \left(\sum_{k=1}^{n} b_{jk} x_k \right) = \sum_{k=1}^{n} \left(\sum_{j=1}^{p} a_{ij} b_{jk} \right) x_k. \tag{2.99}$$

On the other hand, we have

$$((AB)\mathbf{x})_i = \sum_{k=1}^{n} (AB)_{ik} x_k. \tag{2.100}$$

In view of Definition 2.4.4 the left-hand sides of Equations 2.99 and 2.100 must be equal, and since the vector $\mathbf{x}$ can be chosen arbitrarily, the coefficients of x_k on the right-hand sides of Equations 2.99 and 2.100 must be equal. This proves the theorem. ■

The special case of Theorem 2.4.2, in which $m = n = 1$, which is also a special case of the definition of $A\mathbf{x}$ (Definition 2.3.1), is worth stating separately:

Corollary 2.4.1. ***(Matrix Products with Row and Column Vectors).*** *If A is a $1 \times p$ matrix, that is, a row p-vector*

$$\mathbf{a} = (a_1, a_2, \ldots, a_p) \tag{2.101}$$

and B a $p \times 1$ matrix, that is, a column p-vector

$$\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_p \end{bmatrix}, \tag{2.102}$$

then their matrix product **ab** *is a scalar and is equal to their dot product as vectors, namely*

$$\mathbf{ab} = \sum_{j=1}^{p} a_j b_j. \tag{2.103}$$

Also, if $B = (\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n)$ *is a* $p \times n$ *matrix, where the* $\mathbf{b}_i$ *stand for the column* p*-vectors of* B, *then*

$$\mathbf{a}B = (\mathbf{ab}_1, \mathbf{ab}_2, \ldots, \mathbf{ab}_n), \tag{2.104}$$

which is a row n*-vector.*

It is very important to observe that matrix multiplication is *not commutative.* This will be seen by direct computations, but it also follows from the definition as two mappings in succession, since mappings are generally not commutative. The latter is true even in the case of transformations in the same space. Consider, for instance, the effect of a north-south stretch followed by a 90-degree rotation on a car facing north, and of the same operations performed in the reverse order. In the first case we end up with a longer car facing west, and in the second case with a wider car facing west.

In case of the two vectors in Corollary 2.4.1, the product **ba** is very different from **ab**. The latter is a scalar, as given by Equation 2.103. However, if the column vector comes first, then **a** and **b** do not even have to have the same number of entries. Changing **b** in Corollary 2.4.1 to a column m-vector and **a** to a row n-vector we get, by Theorem 2.4.2 with $p = 1$,

$$\mathbf{ba} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} (a_1, a_2, \ldots, a_n) = \begin{bmatrix} b_1a_1 & b_1a_2 & \cdots & b_1a_n \\ b_2a_1 & b_2a_2 & \cdots & b_2a_n \\ \vdots & \vdots & \vdots & \vdots \\ b_ma_1 & b_ma_n & \cdots & b_ma_n \end{bmatrix}. \tag{2.105}$$

If $m \neq n$, then **ab** does not exist. On the other hand, the **ba** above is called the *outer product* of the two vectors, in contrast to the much more important inner product given by Equation 2.103, presumably because the outer product is in the space of $m \times n$ matrices, which contains the spaces $\mathbb{R}^m$ and $\mathbb{R}^n$ of the factors, and those spaces, in turn, contain the space $\mathbb{R}^1$ of the inner product.

Even if the product AB is defined, often the product BA is not. For example, if A is 2×3, say, and B is 3×1, then AB is, by Definition 2.4.4, a 2×1 matrix, but BA is not defined since the inside numbers 1 and 2 in 3×1 and 2×3 do not match, as required by Definition 2.4.4.

The interpretation of the product in Corollary 2.4.1 as a dot product suggests that the formula of Theorem 2.4.2 can also be interpreted similarly.

Corollary 2.4.2. ***(Product of Two Matrices in Terms of Their Row and Column Vectors).*** *Let* A *be an* $m \times p$ *matrix and* B *a* $p \times n$ *matrix* *and let us denote the* ith *row of* A *by* $\mathbf{a}^i$ *and the* kth *column of* B *by* $\mathbf{b}_k$*, that is, let*[13]

$$\mathbf{a}^i = (a_{i1}, a_{i2}, \ldots, a_{ip}) \tag{2.106}$$

and

$$\mathbf{b}_k = \begin{bmatrix} b_{1k} \\ b_{2k} \\ \vdots \\ b_{pk} \end{bmatrix}. \tag{2.107}$$

Then we have

$$(AB)_{ik} = \mathbf{a}^i \mathbf{b}_k \text{ for } i = 1, \ldots, m \text{ and } k = 1, \ldots, n. \tag{2.108}$$

This result may be paraphrased as saying that the entry in the ith *row and* kth *column of* AB *equals the dot product of the* ith *row of* A *with the* kth *column of* B*. Consequently, we may write out the entire product matrix as*

$$AB = \begin{bmatrix} \mathbf{a}^1 \\ \mathbf{a}^2 \\ \vdots \\ \mathbf{a}^m \end{bmatrix} (\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n) = \begin{bmatrix} \mathbf{a}^1\mathbf{b}_1 & \mathbf{a}^1\mathbf{b}_2 & \cdots & \mathbf{a}^1\mathbf{b}_n \\ \mathbf{a}^2\mathbf{b}_1 & \mathbf{a}^2\mathbf{b}_2 & \cdots & \mathbf{a}^2\mathbf{b}_n \\ \vdots & \vdots & \vdots & \vdots \\ \mathbf{a}^m\mathbf{b}_1 & \mathbf{a}^m\mathbf{b}_2 & \cdots & \mathbf{a}^m\mathbf{b}_n \end{bmatrix}. \tag{2.109}$$

The last formula is analogous to the outer product in Equation 2.105, but the entries on the right are inner products of vectors rather than ordinary products of numbers. This corollary is very helpful in the evaluation of matrix products, as will be seen below.

Let us also comment on the use of superscripts and subscripts. The notation we follow for row and column vectors is standard in multilinear algebra (treated in more advanced courses) and will serve us well later, but we have stayed with the more elementary standard usage of just subscripts for matrix elements. Thus our notation is a mixture of two conventions. To be consistent, we should have used a^i_j instead of a_{ij} to denote an entry of A, since then a^i_j could have been properly interpreted as the jth component of the ith row $\mathbf{a}^i$, and also as the ith component of the jth column $\mathbf{a}_j$. However, since here we need no such sophistication, we have adopted the simpler convention.

Example 2.4.5. (A Matrix Product in Terms of Row and Column Vectors). Let

$$A = \begin{bmatrix} 2 & 4 \\ 3 & 7 \end{bmatrix} \text{ and } B = \begin{bmatrix} 3 & -1 \\ 5 & 6 \end{bmatrix}. \tag{2.110}$$

[13] The i here is a superscript to distinguish a row of a matrix from a column, which is denoted by a *sub*script, and must not be mistaken for an exponent.

Then

$$AB = \begin{bmatrix} (2\ \ 4)\begin{bmatrix}3\\5\end{bmatrix} & (2\ \ 4)\begin{bmatrix}-1\\6\end{bmatrix} \\ \\ (3\ \ 7)\begin{bmatrix}3\\5\end{bmatrix} & (3\ \ 7)\begin{bmatrix}-1\\6\end{bmatrix} \end{bmatrix} \tag{2.111}$$

and so

$$AB = \begin{bmatrix} 2\cdot 3 + 4\cdot 5 & 2\cdot(-1) + 4\cdot 6 \\ 3\cdot 3 + 7\cdot 5 & 3\cdot(-1) + 7\cdot 6 \end{bmatrix} = \begin{bmatrix} 26 & 22 \\ 44 & 39 \end{bmatrix}. \tag{2.112}$$

For further reference, note that we can factor out the column vectors $\begin{bmatrix}3\\5\end{bmatrix}$ and $\begin{bmatrix}-1\\6\end{bmatrix}$ in the columns of AB as given in Equation 2.111, and write AB as

$$AB = \left[\begin{bmatrix}2 & 4\\3 & 7\end{bmatrix}\begin{bmatrix}3\\5\end{bmatrix} \quad \begin{bmatrix}2 & 4\\3 & 7\end{bmatrix}\begin{bmatrix}-1\\6\end{bmatrix} \right] = \left[A\begin{bmatrix}3\\5\end{bmatrix} \quad A\begin{bmatrix}-1\\6\end{bmatrix} \right]. \tag{2.113}$$

Thus, in the product AB the matrix A can be distributed over the columns of B. Similarly, we can factor out the row vectors $(2\ \ 4)$ and $(3\ \ 7)$ from the rows of AB as given in Equation 2.111 and write AB also as

$$AB = \begin{bmatrix} (2\ \ 4)B \\ (3\ \ 7)B \end{bmatrix}, \tag{2.114}$$

that is, with the matrix B distributed over the rows of A. ♦

Example 2.4.6. (A Matrix Product in Terms of Entries). Let

$$A = \begin{bmatrix} 2 & -2 & 4 \\ 1 & 3 & 5 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & -1 \\ 4 & -2 \\ 6 & 3 \end{bmatrix}. \tag{2.115}$$

Then

$$AB = \begin{bmatrix} 2\cdot 2 - 2\cdot 4 + 4\cdot 6 & 2\cdot(-1) - 2\cdot(-2) + 4\cdot 3 \\ 1\cdot 2 + 3\cdot 4 + 5\cdot 6 & 1\cdot(-1) + 3\cdot(-2) + 5\cdot 3 \end{bmatrix} = \begin{bmatrix} 20 & 14 \\ 44 & 8 \end{bmatrix}. \tag{2.116}$$

♦

Example 2.4.7. (The Product of Two Rotation Matrices). The matrices

$$R_{30} = \begin{bmatrix} \cos 30^\circ & -\sin 30^\circ \\ \sin 30^\circ & \cos 30^\circ \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{bmatrix} \tag{2.117}$$

and

$$R_{60} = \begin{bmatrix} \cos 60^\circ & -\sin 60^\circ \\ \sin 60^\circ & \cos 60^\circ \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix} \tag{2.118}$$

represent rotations by 30° and 60° respectively, according to Example 2.4.1. Their product

$$R_{30}R_{60} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \cos 90^\circ & -\sin 90^\circ \\ \sin 90^\circ & \cos 90^\circ \end{bmatrix} = R_{90} \qquad (2.119)$$

represents the rotation by 90°, as it should. ♦

An interesting use of matrices and matrix operations is provided by the following example, typical of a large number of similar applications involving *incidence* or *connection matrices.*

Example 2.4.8. (A Connection Matrix for an Airline). Suppose that an airline has nonstop flights between cities A, B, C, D, E as described by the matrix

$$M = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}. \qquad (2.120)$$

Here the entry m_{ij} is 1 if there is a nonstop connection from city i to city j, and 0 if there is not, with the cities labeled $1, 2, \ldots, 5$ instead of $A, B, \ldots, E$. Then the entries of the matrix

$$M^2 = MM = \begin{bmatrix} 2 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 2 & 0 \\ 1 & 0 & 0 & 0 & 2 \end{bmatrix} \qquad (2.121)$$

show the one-stop connections. Why? Because, if we consider the entry

$$(M^2)_{ik} = \sum_{j=1}^{5} m_{ij}m_{jk} \qquad (2.122)$$

of M^2, then the jth term equals 1 in this sum if and only if $m_{ij} = 1$ and $m_{jk} = 1$, that is, if we have a nonstop flight from i to j and another from j to k. If there are two such j values, then the sum will be equal to 2, showing that there are two choices for one-stop flights from i to k. Thus, for instance, $(M^2)_{11} = 2$ shows that there are two one-stop routes from A to A: Indeed, from A one can fly to B or D and back. The entries of the matrix[14]

[14] In matrix expressions with several operations, the precedence rules are analogous to those for numbers: first powers, then products, and last addition and subtraction, unless otherwise indicated by parentheses.

$$M + M^2 = \begin{bmatrix} 2 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 & 1 \\ 1 & 0 & 1 & 1 & 2 \end{bmatrix} \tag{2.123}$$

show the number of ways of reaching one city from another with one-leg or two-leg flights. In particular, the zero entries show, for instance, that B and E are not so connected. Evaluating $(M^3)_{25} = (M^3)_{52}$, we would similarly find that even those two cities can be reached from each other with three-leg flights.

What are the vectors on which these matrices act, that is, what meaning can we give to an equation like $\mathbf{y} = M\mathbf{x}$? The answer is that if the components of $\mathbf{x}$ are restricted to just 0 and 1, then $\mathbf{x}$ may be regarded as representing a set of cities and $\mathbf{y}$ the set that can be reached nonstop from $\mathbf{x}$. Thus, for instance,

$$\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{2.124}$$

represents the set $\{A, B\}$, and then

$$\mathbf{y} = M\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \tag{2.125}$$

represents the set $\{A, B, D\}$ that can be reached nonstop from $\{A, B\}$. (Again, if a number greater than 1 were to show up in $\mathbf{y}$, that would indicate that the corresponding city can be reached in more than one way.) ♦

We present one more example, which is a simplified version of a large class of similar applications of matrices.

Example 2.4.9. (A Matrix Description of Population Changes). We want to describe how in a certain town two population groups, those younger than 50 and those 50 or older, change over time. We assume that over every decade, on the one hand, there is a net increase of 10% in the under fifty population, and on the other hand, 20% of the under fifty population becomes fifty or older, while 40% of the initial over fifty population dies. If x_1 and x_2 denote the numbers of people in the two groups at a given time, then their numbers a decade later will be given by the product

$$A\mathbf{x} = \begin{bmatrix} 1.1 & 0 \\ 0.2 & 0.6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \tag{2.126}$$

Similarly, two decades later the two population groups will be given by

$$A(A\mathbf{x}) = A^2\mathbf{x} = \begin{bmatrix} 1.21 & 0 \\ 0.34 & 0.36 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \tag{2.127}$$

and so on. (In Example 7.2.1 we will examine how the two populations change in the long run.) ♦

As we have seen, a matrix can be regarded as a row vector of its columns and also as a column vector of its rows. Making full use of this choice, we can rewrite the product of matrices two more ways, corresponding to the particular cases shown in Equations 2.113 and 2.114. We obtain these new formulas by factoring out the $\mathbf{b}_j$ coefficients in the columns of the matrix on the right of Equation 2.109 and the $\mathbf{a}^i$ coefficients in the rows:

$$\begin{aligned} AB &= \begin{bmatrix} \mathbf{a}^1\mathbf{b}_1 & \mathbf{a}^1\mathbf{b}_2 & \cdots & \mathbf{a}^1\mathbf{b}_n \\ \mathbf{a}^2\mathbf{b}_1 & \mathbf{a}^2\mathbf{b}_2 & \cdots & \mathbf{a}^2\mathbf{b}_n \\ \vdots & \vdots & \vdots & \vdots \\ \mathbf{a}^m\mathbf{b}_1 & \mathbf{a}^m\mathbf{b}_2 & \cdots & \mathbf{a}^m\mathbf{b}_n \end{bmatrix} \\ &= \left[\begin{bmatrix} \mathbf{a}^1 \\ \mathbf{a}^2 \\ \vdots \\ \mathbf{a}^m \end{bmatrix} \mathbf{b}_1, \begin{bmatrix} \mathbf{a}^1 \\ \mathbf{a}^2 \\ \vdots \\ \mathbf{a}^m \end{bmatrix} \mathbf{b}_2, \cdots, \begin{bmatrix} \mathbf{a}^1 \\ \mathbf{a}^2 \\ \vdots \\ \mathbf{a}^m \end{bmatrix} \mathbf{b}_n \right] \\ &= (A\mathbf{b}_1 \quad A\mathbf{b}_2 \quad \cdots \quad A\mathbf{b}_n) \end{aligned} \tag{2.128}$$

and

$$AB = \begin{bmatrix} \mathbf{a}^1\mathbf{b}_1 & \mathbf{a}^1\mathbf{b}_2 & \cdots & \mathbf{a}^1\mathbf{b}_n \\ \mathbf{a}^2\mathbf{b}_1 & \mathbf{a}^2\mathbf{b}_2 & \cdots & \mathbf{a}^2\mathbf{b}_n \\ \vdots & \vdots & \vdots & \vdots \\ \mathbf{a}^m\mathbf{b}_1 & \mathbf{a}^m\mathbf{b}_2 & \cdots & \mathbf{a}^m\mathbf{b}_n \end{bmatrix} = \begin{bmatrix} \mathbf{a}^1(\mathbf{b}_1 \quad \mathbf{b}_2 \quad \cdots \quad \mathbf{b}_n) \\ \mathbf{a}^2(\mathbf{b}_1 \quad \mathbf{b}_2 \quad \cdots \quad \mathbf{b}_n) \\ \vdots \\ \mathbf{a}^m(\mathbf{b}_1 \quad \mathbf{b}_2 \quad \cdots \quad \mathbf{b}_n) \end{bmatrix} = \begin{bmatrix} \mathbf{a}^1 B \\ \mathbf{a}^2 B \\ \vdots \\ \mathbf{a}^m B \end{bmatrix}. \tag{2.129}$$

We summarize these results as follows.

Corollary 2.4.3. ***(Product of Two Matrices in Terms of the Row or Column Vectors of One of Them).*** *Let A and B be as in Corollary 2.4.2. With the same notation for the rows and columns used there, we have*

$$AB = A(\mathbf{b}_1 \quad \mathbf{b}_2 \quad \cdots \quad \mathbf{b}_n) = (A\mathbf{b}_1 \quad A\mathbf{b}_2 \quad \cdots \quad A\mathbf{b}_n) \tag{2.130}$$

and

$$AB = \begin{bmatrix} \mathbf{a}^1 \\ \mathbf{a}^2 \\ \vdots \\ \mathbf{a}^m \end{bmatrix} B = \begin{bmatrix} \mathbf{a}^1 B \\ \mathbf{a}^2 B \\ \vdots \\ \mathbf{a}^m B \end{bmatrix}. \tag{2.131}$$

Although the matrix product is not commutative, it still has the other important properties expected of a product, namely associativity and distributivity.

Theorem 2.4.3. ***(Associativity and Distributivity of Matrix Multiplication).*** *Let A, B, and C be arbitrary matrices for which the expressions below all make sense. Then we have the associative law*

$$A(BC) = (AB)C \tag{2.132}$$

and the distributive law

$$A(B + C) = AB + AC. \tag{2.133}$$

Proof. Let A, B, and C be $m \times p$, $p \times q$, and $q \times n$ matrices respectively. Then we may evaluate the left side of Equation 2.132 using Equations 2.130 and Definition 2.4.4 as follows:

$$\begin{aligned} A(BC) &= A(B(\mathbf{c}_1\ \mathbf{c}_2\ \cdots\ \mathbf{c}_n)) = A(B\mathbf{c}_1\ B\mathbf{c}_2\ \cdots\ B\mathbf{c}_n) \\ &= (A(B\mathbf{c}_1)\ \cdots\ A(B\mathbf{c}_n)) = ((AB)\mathbf{c}_1\ \ \cdots\ \ (AB)\mathbf{c}_n) \\ &= (AB)(\mathbf{c}_1\ \mathbf{c}_2\ \cdots\ \mathbf{c}_n) = (AB)C. \end{aligned} \tag{2.134}$$

We leave the proof of the distributive law to the reader as Exercise 2.4.18. ■

Note that Equation 2.132 enables us to write ABC, without parentheses, for $A(BC)$ or $(AB)C$.

Once we have defined addition and multiplication of matrices, it is natural to ask what matrices take the place of the special numbers 0 and 1 in the algebra of numbers. Zero is easy: we take every matrix with all entries equal to zero to be a zero matrix. Denoting it by O, regardless of its shape, we have, for every A of the same shape,

$$A + O = A, \tag{2.135}$$

and whenever the product is defined,

$$AO = O \text{ and } OA = O. \tag{2.136}$$

Note that the zero matrices on either side of each of Equations 2.136 may be of different size, although they are usually denoted by the same letter O.

While a little less straightforward, it is still easy to see how to find analogs of 1. For every n, the $n \times n$ matrix

$$I = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & & \\ 0 & 0 & \cdots & 1 \end{bmatrix} \tag{2.137}$$

with 1's along its "main diagonal" and zeros everywhere else, has the properties

$$AI = A \text{ and } IA = A, \tag{2.138}$$

whenever the products are defined. This fact can be verified by direct computation in every one of the product's forms, with A in the general form (a_{ij}). We may do it as follows: We write $I = (\delta_{ij})$, where

$$\delta_{ij} = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases} \tag{2.139}$$

is called Kronecker's delta function and is the standard notation for the entries of the matrix I. With this notation, for A and I sized $m \times n$ and $n \times n$ respectively, Theorem 2.4.2 gives

$$(AI)_{ik} = \sum_{j=1}^{n} a_{ij}\delta_{jk} = a_{ik} \text{ for } i = 1, \ldots, m \text{ and } k = 1, \ldots, n, \tag{2.140}$$

since, by the definition of δ_{jk}, in the sum all terms are zero except the one with $j = k$ and that one gives a_{ik}. This result is, of course, equivalent to $AI = A$. We leave the proof of the other equation of 2.138 to the reader.

For every n the matrix I is called the *unit matrix* or the *identity matrix* of order n. We usually dispense with any indication of its order unless it is important and would be unclear from the context. In such cases we write it as I_n. Notice that the columns of I are the standard vectors $\mathbf{e}_i$ (regarded as column vectors, of course), that is,

$$I = (\mathbf{e}_1 \ \ \mathbf{e}_2 \ \ \ldots \ \ \mathbf{e}_n). \tag{2.141}$$

In Section 2.5 we shall see how the inverse of a matrix can be defined in some cases.

In closing this section, we just want to present briefly the promised explanation of the reason for using *column* vectors for $\mathbf{x}$ in the equation $A\mathbf{x} = \mathbf{b}$. In a nutshell, we used column vectors because otherwise the whole formalism of this section would have broken down. The product $A\mathbf{x}$ was used in Definition 2.4.4 of the general matrix product AB, which led to the formula of Theorem 2.4.2 for the components $(AB)_{ik}$. If we want to multiply AB by a third matrix C, we have no problem repeating the previous procedure, that is, form the products $(AB)_{ik}c_{kl}$ and sum over k. However, had we used a *row* vector $\mathbf{x}$ in the beginning, that would have led to the formula $(AB)_{ik} = \sum_{j=1}^{n} a_{ij}b_{kj}$ and then multiplying this result by c_{kl} or c_{lk}, and summing over k, we would have had to use the first subscript of b for summation in this second product, unlike in the first one. Thus the associative law could not be maintained and the nice formulas of Corollary 2.4.3 would also cease to hold. Basically, once we decided to use *rows* of A to multiply $\mathbf{x}$ in the product $A\mathbf{x}$, then we had to make $\mathbf{x}$ a column vector in order to end up with a reasonable formalism.

Exercises

Exercise 2.4.1. Let

$$A = \begin{bmatrix} 2 & 3 \\ 1 & -2 \end{bmatrix} \text{ and } B = \begin{bmatrix} 3 & -4 \\ 2 & 2 \end{bmatrix}.$$

Find the matrices a. $C = 2A + 3B$, and b. $D = 4A - 3B$.

Exercise 2.4.2. Prove Equation 2.82 of Theorem 2.4.1.

In the next six exercises find the products of the given matrices in both orders, that is, both AB and BA, if possible.

Exercise 2.4.3.

$$A = \begin{bmatrix} 1 & -2 & 3 \end{bmatrix} \text{ and } B = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}.$$

Exercise 2.4.4.

$$A = \begin{bmatrix} 2 & 3 & 5 \\ 1 & -2 & 3 \end{bmatrix} \text{ and } B = \begin{bmatrix} 3 & -4 \\ 2 & 2 \\ 1 & -3 \end{bmatrix}.$$

Exercise 2.4.5.

$$A = \begin{bmatrix} 2 & 3 & 5 \\ 1 & -2 & 3 \\ 3 & -4 & 2 \end{bmatrix} \text{ and } B = \begin{bmatrix} 3 & -4 \\ 2 & 2 \\ 1 & -3 \end{bmatrix}.$$

Exercise 2.4.6.

$$A = \begin{bmatrix} 1 & -2 & 3 & -4 \end{bmatrix} \text{ and } B = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}.$$

Exercise 2.4.7.

$$A = \begin{bmatrix} 1 & -2 & 3 & -4 \end{bmatrix} \text{ and } B = \begin{bmatrix} 3 & -4 \\ 2 & 2 \\ 1 & -3 \\ -2 & 5 \end{bmatrix}.$$

Exercise 2.4.8.

$$A = \begin{bmatrix} 2 & 3 & 5 \\ 1 & -2 & 3 \end{bmatrix} \text{ and } B = \begin{bmatrix} 3 & -4 \\ 2 & 2 \\ 1 & -3 \\ -2 & 5 \end{bmatrix}.$$

Exercise 2.4.9. Verify the associative law for the product of the matrices

$$A = [1 \ \ -2], \ B = \begin{bmatrix} 3 & -4 \\ 2 & 2 \end{bmatrix}, \text{ and } C = \begin{bmatrix} 1 & -3 \\ 3 & 0 \end{bmatrix}.$$

Exercise 2.4.10. With the notation of Example 2.4.1, prove that for every two rotation matrices R_α and R_β we have $R_\alpha R_\beta = R_{\alpha+\beta}$.

Exercise 2.4.11. Find two nonzero 2×2 matrices A and B such that $AB = O$.

Exercise 2.4.12. Show that the cancellation law does not hold for matrix products: Find nonzero 2×2 matrices A, B, C such that $AB = AC$ but $B \neq C$.

***Exercise 2.4.13.** Let A be an $m \times p$ matrix and B a $p \times n$ matrix. Show that the product AB can also be written in the following alternative forms:
a. $AB = \mathbf{a}_1\mathbf{b}^1 + \mathbf{a}_2\mathbf{b}^2 + \cdots + \mathbf{a}_p\mathbf{b}^p$,
b. $AB = (\sum_{i=1}^p \mathbf{a}_i b_{i1}, \sum_{i=1}^p \mathbf{a}_i b_{i2}, \ldots, \sum_{i=1}^p \mathbf{a}_i b_{in})$ or $(AB)_j = \sum_{i=1}^p \mathbf{a}_i b_{ij}$,
c. $AB = \begin{bmatrix} \sum_{j=1}^p a_{1j}\mathbf{b}^j \\ \sum_{j=1}^p a_{2j}\mathbf{b}^j \\ \vdots \\ \sum_{j=1}^p a_{mj}\mathbf{b}^j \end{bmatrix}$ or $(AB)^i = \sum_{j=1}^p a_{ij}\mathbf{b}^j$.

Exercise 2.4.14. Let A be any $n \times n$ matrix. Its powers, for all nonnegative integer exponents k, are defined by induction as $A^0 = I$ and $A^k = AA^{k-1}$. Show that the rules $A^kA^l = A^{k+l}$ and $(A^k)^l = A^{kl}$ hold, just as for real numbers.

Exercise 2.4.15. Find a nonzero 2×2 matrix A such that $A^2 = O$.

Exercise 2.4.16. Find a 3×3 matrix A such that $A^2 \neq O$ but $A^3 = O$.

Exercise 2.4.17. Find the number of three-leg flights connecting B and D in Example 2.4.8 by evaluating $(M^3)_{24} = (M^3)_{42}$.

Exercise 2.4.18. Prove Equation 2.133 of Theorem 2.4.3.

The next five exercises deal with *block multiplication.*

Exercise 2.4.19. Show that

$$\left[\begin{array}{cc|cc} 1 & -2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right]\left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \\ \hline 3 & 2 \\ 1 & -1 \end{array}\right]$$
$$= \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 1 & -1 \end{bmatrix}.$$

Exercise 2.4.20. Show that if two conformable matrices of any size are partitioned, as in the previous exercise, so that the products make sense, then

$$[A \;\; B]\begin{bmatrix} C \\ D \end{bmatrix} = [AC + BD].$$

Exercise 2.4.21. Show that if two conformable matrices of any size are partitioned into four submatrices each, so that the products and sums make sense, then

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}\begin{bmatrix} E & F \\ G & H \end{bmatrix} = \begin{bmatrix} AE + BG & AF + BH \\ CE + DG & CF + DH \end{bmatrix}.$$

Exercise 2.4.22. Compute the product by block multiplication, using the result of the previous exercise:

$$\left[\begin{array}{cc|cc} 1 & -2 & 1 & 0 \\ 3 & 4 & 0 & 1 \\ \hline -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right]\left[\begin{array}{cc|cc} 1 & -2 & 1 & 0 \\ 2 & 0 & -3 & 1 \\ \hline 0 & 0 & 2 & 3 \\ 0 & 0 & 7 & 4 \end{array}\right].$$

Exercise 2.4.23. Partition the first matrix of the previous exercise as

$$\left[\begin{array}{cc|cc} 1 & -2 & 1 & 0 \\ 3 & 4 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ \hline 0 & -1 & 0 & 0 \end{array}\right].$$

Find the appropriate corresponding partition of the second matrix, and evaluate the product by using these blocks.

MATLAB Exercises

In MATLAB, the product of matrices is denoted by $*$, and a power like A^k by $A\hat{\ }k$; both the same as for numbers. The unit matrix of order n is denoted by

eye(n), and the $m \times n$ zero matrix by **zeros**(m, n). The command **rand**(m, n) returns an $m \times n$ matrix with random entries uniformly distributed between 0 and 1. The command **round**(A) rounds each entry of A to the nearest integer.

Exercise 2.4.24. As in Example 2.4.1, let $\mathbf{v}$ denote the vector obtained from the vector $\mathbf{u}$ by a rotation through an angle θ.
a. Compute $\mathbf{v}$ for $\mathbf{u} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$ and each of $\theta = 15\,°, 30\,°, 45\,°, 60\,°, 75\,°$, and $90\,°$. (MATLAB will compute the trig functions if you use radians.)
b. Use MATLAB to verify that $R_{75^\circ} = R_{25^\circ} * R_{50^\circ}$.

Exercise 2.4.25. Let

$$M = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \end{bmatrix}$$

be the connection matrix of an airline network as in Example 2.4.8.
a. Which cities can be reached from A with exactly two stops?
b. Which cities can be reached from A with two stops or less?
c. What is the number of stops needed to reach all cities from all others?

Exercise 2.4.26. Let $\mathbf{a} = 10 * \mathbf{rand}(1, 4) - 5$ and $\mathbf{b} = 10 * \mathbf{rand}(1, 4) - 5$.
a. Compute $C = \mathbf{a} * \mathbf{b}'$ and **rank**(C) for ten instances of such $\mathbf{a}$ and $\mathbf{b}$. (Use the up-arrow key.)
b. Make a conjecture about **rank**(C) in general.
c. Prove your conjecture.

Exercise 2.4.27. Let $A = 10 * \mathbf{rand}(2, 4) - 5$ and $B = 10 * \mathbf{rand}(4, 2) - 5$.
a. Compute $C = A * B,\ D = B * A$, **rank**(C), and **rank**(D) for ten instances of such A and B.
b. Make a conjecture about **rank**(C) and **rank**(D) in general.

Exercise 2.4.28. In MATLAB you can enter blocks in a matrix in the same way as you enter scalars. Use this method to solve a. Exercise 2.4.19, and b. Exercise 2.4.22.

2.5 The Inverse and the Transpose of a Matrix

While for vectors it is impossible to define division, for matrices it is possible in some very important cases.

We may try to follow the same procedure as for numbers. The fraction b/a has been defined as the solution of the equation $ax = b$, or as b times $1/a$, where $1/a$ is the solution of $ax = 1$. For matrices we mimic the latter formula: To find the inverse of a matrix A, we look for the solution of the matrix equation

$$AX = I, \tag{2.142}$$

where I is the $n \times n$ unit matrix and X an unknown matrix. In terms of mappings, because I represents the identity mapping or no change, this equation means that if a mapping is given by the matrix A, we are looking for the matrix X of the (right) inverse mapping, that is, of the mapping that is undone if followed by the mapping A. (As it turns out, and as should be evident from the geometrical meaning, the order of the factors does not matter if A represents a mapping from $\mathbb{R}^n$ to itself.)

By the definition of the product, if I is $n \times n$, then A must be $n \times p$ and X of size $p \times n$ for some p. Then Equation 2.142 corresponds to n^2 scalar equations for np unknowns. Thus, if $p < n$ holds, then we have fewer unknowns than equations and generally no solutions apart from exceptional cases. On the other hand, if $p > n$ holds, then we have more unknowns than equations, and so generally infinitely many solutions. Since we are interested in finding unique solutions, we restrict our attention to those cases in which $p = n$ holds, or in other words to $n \times n$ or *square* matrices A. (Cases of $p \neq n$ are left to Exercises 2.5.8–2.5.11.) For a square matrix, n is called the order of A. For such A, Equation 2.142 may be written as

$$A(\mathbf{x}_1 \ \ \mathbf{x}_2 \ \dots \ \mathbf{x}_n) = (\mathbf{e}_1 \ \ \mathbf{e}_2 \ \dots \ \mathbf{e}_n) \tag{2.143}$$

and by Equation 2.130 we can decompose this equation into n separate systems

$$A\mathbf{x}_1 = \mathbf{e}_1, \quad A\mathbf{x}_2 = \mathbf{e}_2, \quad \dots, \quad A\mathbf{x}_n = \mathbf{e}_n \tag{2.144}$$

for the n unknown n-vectors $\mathbf{x}_1, \ \mathbf{x}_2, \dots, \ \mathbf{x}_n$.

Before proceeding further with the general theory, let us consider an example.

Example 2.5.1. (Finding the Inverse of a 2×2 Matrix by Solving Two Systems). Let

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \tag{2.145}$$

and so let us solve

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{2.146}$$

or equivalently the separate systems

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{21} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x_{12} \\ x_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{2.147}$$

Subtracting 3 times the first row from the second in both systems, we get

$$\begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{21} \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_{12} \\ x_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{2.148}$$

Adding the second row to the first and dividing the second row by –2, again in both systems, we obtain

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{21} \end{bmatrix} = \begin{bmatrix} -2 \\ 3/2 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_{12} \\ x_{22} \end{bmatrix} = \begin{bmatrix} 1 \\ -1/2 \end{bmatrix}. \tag{2.149}$$

Hence

$$x_{11} = -2, \; x_{21} = 3/2, \; x_{12} = 1, \; x_{22} = -1/2 \tag{2.150}$$

or in matrix form

$$X = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}. \tag{2.151}$$

It is easy to check that this X is a solution of $AX = I$, and in fact of $XA = I$, too. Furthermore, since the two systems given by Equations 2.147 have the same matrix A on their left sides, the row reduction steps were exactly the same for both, and can therefore be combined into the reduction of a single augmented matrix with the two columns of I on the right, that is, of $[A|I]$ as follows:

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -2 & -3 & 1 \end{array}\right] \to$$
$$\left[\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & -2 & -3 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & 1 & 3/2 & -1/2 \end{array}\right]. \tag{2.152}$$

♦

We can generalize the results of this example in part as a definition and in part as a theorem.

Definition 2.5.1. ***(The Inverse of a Matrix).*** *A matrix A is called invertible if it is a square matrix and there exists a unique square matrix X of the same size such that $AX = I$ and $XA = I$ hold. Such an X, if one exists, is called the inverse of A and is denoted by A^{-1}.*

Theorem 2.5.1. ***(Inverting a Matrix by Row Reduction).*** *A square matrix is invertible if and only if the augmented matrix $[A|I]$ can be reduced* *by elementary row operations to the form $[I|C]$, and in that case C is the inverse A^{-1} of A.*

Proof. The augmented matrix corresponding to the equation $AX = I$ is $[A|I]$. If the reduction of $[A|I]$ produces the form $[I|C]$, then the matrix equation corresponding to the latter augmented matrix is $IX = C$ or, equivalently, $X = C$. By Theorem 2.1.1, $IX = C$ has the same solution set as $AX = I$, and so C is the unique solution of $AX = I$.

By reversing the elementary row operations, we can undo the above reduction; that is, we can change $[I|C]$ back to $[A|I]$. But then the same steps would change $[C|I]$ into $[I|A]$, which corresponds to solving the matrix equation $CY = I$ for an unknown matrix Y uniquely as $IY = A$, or $Y = A$. Hence, $CA = I$. Thus, if C solves $AX = I$, then it also solves $XA = I$, and it is the only solution of both equations. Thus A is invertible, with C as its inverse A^{-1}.

On the other hand, if $[A|I]$ cannot be reduced to the form $[I|X]$, then the system $AX = I$ has no solution for the following reason: In this case the reduction of A must produce a zero row at the bottom of every corresponding echelon matrix U, because if U had no zero row, then it could be further reduced to I. The last row of every reduction of $[A|I]$ that reduces A to an echelon matrix U with a zero bottom row must be a sum of nonzero multiples of some rows (or maybe just a single row). Suppose this sum contains c times the ith row (with $c \neq 0$). Then the submatrix $[A|\mathbf{e}_i]$ (see footnote 6 on page 54) will be reduced to $[U|c\mathbf{e}_n]$: Since the zero entries of the $\mathbf{e}_i$ column cannot affect the single 1 of it, c times this 1 ends up at the bottom. The matrix $[U|c\mathbf{e}_n]$, however, represents an inconsistent system, because the last row of U is zero, but the last component of $c\mathbf{e}_n$ is not. ∎

Example 2.5.2. (Finding the Inverse of a 2 × 2 Matrix by Row Reduction). Let us find the inverse of the matrix

$$A = \begin{bmatrix} 2 & 3 \\ 1 & -2 \end{bmatrix} \tag{2.153}$$

if it exists.

We form the augmented matrix $[A|I]$ and reduce it as follows:

$$\left[\begin{array}{cc|cc} 2 & 3 & 1 & 0 \\ 1 & -2 & 0 & 1 \end{array}\right] \begin{array}{l} \mathbf{r}_1 \leftarrow \mathbf{r}_2 \\ \mathbf{r}_2 \leftarrow \mathbf{r}_1 \end{array} \left[\begin{array}{cc|cc} 1 & -2 & 0 & 1 \\ 2 & 3 & 1 & 0 \end{array}\right]$$
$$\begin{array}{l} \mathbf{r}_1 \leftarrow \mathbf{r}_1 \\ \mathbf{r}_2 \leftarrow \mathbf{r}_2 - 2\mathbf{r}_1 \end{array} \left[\begin{array}{cc|cc} 1 & -2 & 0 & 1 \\ 0 & 7 & 1 & -2 \end{array}\right] \begin{array}{l} \mathbf{r}_1 \leftarrow \mathbf{r}_1 \\ \mathbf{r}_2 \leftarrow \mathbf{r}_2/7 \end{array} \left[\begin{array}{cc|cc} 1 & -2 & 0 & 1 \\ 0 & 1 & 1/7 & -2/7 \end{array}\right]$$
$$\begin{array}{l} \mathbf{r}_1 \leftarrow \mathbf{r}_1 + 2\mathbf{r}_2 \\ \mathbf{r}_2 \leftarrow \mathbf{r}_2 \end{array} \left[\begin{array}{cc|cc} 1 & 0 & 2/7 & 3/7 \\ 0 & 1 & 1/7 & -2/7 \end{array}\right]. \tag{2.154}$$

Thus we can read off the inverse of A as

$$A^{-1} = \frac{1}{7}\begin{bmatrix} 2 & 3 \\ 1 & -2 \end{bmatrix}. \tag{2.155}$$

It is easy to check that we do indeed have $AA^{-1} = A^{-1}A = I$. ♦

Example 2.5.3. (Showing Noninvertibility of a 2×2 Matrix by Row Reduction). Here is an example of a noninvertible square matrix. Let us try to compute the inverse of

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}. \tag{2.156}$$

We form the augmented matrix $[A|I]$ and reduce it as follows:

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 2 & 4 & 0 & 1 \end{array}\right] \begin{array}{l} \mathbf{r}_1 \leftarrow \mathbf{r}_1 \\ \mathbf{r}_2 \leftarrow \mathbf{r}_2 - 2\mathbf{r}_1 \end{array} \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 0 & -2 & 1 \end{array}\right]. \tag{2.157}$$

The corresponding system is

$$\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}, \tag{2.158}$$

and so the second row of $[A|I]$ corresponds to the self-contradictory equations

$$0x_{11} + 0x_{21} = -2 \tag{2.159}$$

$$0x_{12} + 0x_{22} = 1. \tag{2.160}$$

Thus A has no inverse. ♦

Just as for numbers $b/a = a^{-1}b$ is the solution of $ax = b$, for matrix equations we have a similar consequence of Definition 2.5.1.

Theorem 2.5.2. ***(Using the Inverse to Solve Matrix Equations).*** *If A is an invertible $n \times n$ matrix and B an arbitrary $n \times p$ matrix, then the equation*

$$AX = B \tag{2.161}$$

has the unique solution

$$X = A^{-1}B. \tag{2.162}$$

Proof. That $X = A^{-1}B$ is a solution can be seen easily by substituting it into Equation 2.161:

$$A(A^{-1}B) = (AA^{-1})B = IB = B, \tag{2.163}$$

and that it is the only solution can be seen in this way. Assume that Y is another solution, so that

$$AY = B \tag{2.164}$$

holds. Multiplying both sides of this equation by A^{-1} we get

$$A^{-1}(AY) = A^{-1}B \tag{2.165}$$

and this equation reduces to

$$(A^{-1}A)Y = Y = A^{-1}B, \tag{2.166}$$

which shows that $Y = X$. ■

If $p = 1$ holds, Equation 2.161 becomes our old friend

$$A\mathbf{x} = \mathbf{b}, \tag{2.167}$$

where $\mathbf{x}$ and $\mathbf{b}$ are n-vectors. Thus Theorem 2.5.2 provides a new way of solving this equation. Unfortunately, this technique has little practical significance, since computing the inverse of A is generally more difficult than solving Equation 2.167 by Gaussian elimination. In some theoretical considerations, however, it is useful to know that the solution of Equation 2.167 can be written as

$$\mathbf{x} = A^{-1}\mathbf{b}, \tag{2.168}$$

and if we have several equations like 2.167 with the same left sides, then they can be combined into an equation of the form 2.161 with $p > 1$ and profitably solved by computing the inverse of A and using Theorem 2.5.2.

Example 2.5.4. (Solving an Equation for an Unknown 2 × 3 Matrix). Let us solve

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} X = \begin{bmatrix} 2 & 3 & -5 \\ 4 & -1 & 3 \end{bmatrix}. \tag{2.169}$$

From Example 2.5.1 we know that

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^{-1} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}. \tag{2.170}$$

Hence, by Theorem 2.5.2, we obtain

$$X = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix} \begin{bmatrix} 2 & 3 & -5 \\ 4 & -1 & 3 \end{bmatrix} = \begin{bmatrix} 0 & -7 & 13 \\ 1 & 5 & -9 \end{bmatrix}. \tag{2.171}$$

♦

As we have just seen, if A is invertible, then Equation 2.168 provides the solution of Equation 2.167 for every n-vector $\mathbf{b}$. It is then natural to ask whether the converse is true, that is, whether the existence of a solution of Equation 2.167 for *every* $\mathbf{b}$ implies the invertibility of A. (We know that a *single* $\mathbf{b}$ is not enough: Equation 2.167 may be solvable for some right-hand sides and not for others; see, e.g., Examples 2.1.4 and 2.1.5.) The answer is yes.

Theorem 2.5.3. ***(Existence of Solutions Criterion for the Invertibility of a Matrix).*** *An $n \times n$ matrix A is invertible if and only if $A\mathbf{x} = \mathbf{b}$ has a solution for every n-vector $\mathbf{b}$.*

Proof. The "only if" part of this statement has already been proved; we just included it for the sake of completeness. To prove the "if" part, let us assume that $A\mathbf{x} = \mathbf{b}$ has a solution for every n-vector $\mathbf{b}$. Then it has a solution for each standard vector $\mathbf{e}_i$ in the role of $\mathbf{b}$; that is, each of the equations

$$A\mathbf{x}_1 = \mathbf{e}_1,\ A\mathbf{x}_2 = \mathbf{e}_2, \ldots,\ A\mathbf{x}_n = \mathbf{e}_n \tag{2.172}$$

has a solution by assumption. These equations can, however, be combined into the single equation

$$A\left(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\right) = \left(\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\right), \tag{2.173}$$

which can be written as

$$AX = I \tag{2.174}$$

whose augmented matrix is $[A|I]$. From the proof of Theorem 2.5.1 we know that the solution of this equation, if one exists, must be $X = A^{-1}$, and since we have stipulated the existence of a solution, the invertibility of A follows. ■

The condition of solvability of $A\mathbf{x} = \mathbf{b}$ for *every* possible right side can be replaced by the requirement of uniqueness of the solution for a *single* $\mathbf{b}$.

Theorem 2.5.4. ***(Unique-Solution Criterion for the Invertibility of a Matrix).*** *An $n \times n$ matrix A is invertible if and only if $A\mathbf{x} = \mathbf{b}$ has a unique solution for some n-vector $\mathbf{b}$.*

Proof. If A is invertible, then, by Theorem 2.5.2, $\mathbf{x} = A^{-1}\mathbf{b}$ gives the unique solution of $A\mathbf{x} = \mathbf{b}$ for every $\mathbf{b}$. Conversely, if, for some $\mathbf{b}$, $A\mathbf{x} = \mathbf{b}$ has a unique solution, then Theorem 2.2.1 (page 55) shows that the rank of A equals n and consequently that $AX = I$ also has a unique solution. Of course, this solution must be A^{-1}. ■

The vector $\mathbf{b}$ in Theorem 2.5.4 may be taken to be the zero vector. This case is sufficiently important for special mention.

Corollary 2.5.1. ***(Trivial-Solution Criterion for the Invertibility of a Matrix).*** *A square matrix A is invertible if and only if $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.*

Definition 2.5.2. ***(Singular and Nonsingular Matrices).*** *An $n \times n$ matrix A for which the associated system $A\mathbf{x} = \mathbf{b}$ has a unique solution for every n-vector $\mathbf{b}$ is called nonsingular; otherwise, it is called singular.*

Let us collect some equivalent characterizations of nonsingular square matrices that follow from our considerations up to now.

Theorem 2.5.5. ***(Various Criteria for a Matrix to be Nonsingular).*** *An $n \times n$ matrix A is nonsingular if and only if it has any (and thus all) of the following properties:*
1. A is invertible.
2. The rank of A is n.
3. A is row equivalent to I.
4. $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b}$.
5. $A\mathbf{x} = \mathbf{b}$ has a unique solution for some $\mathbf{b}$.
6. The homogeneous equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

For numbers, the product and the inverse are connected by the formula $(ab)^{-1} = a^{-1}b^{-1} = b^{-1}a^{-1}$. For matrices, we have an analogous result, but with the significant difference that the product is noncommutative and the order of the factors on the right must be reversed.

Theorem 2.5.6. ***(Inverse of the Product of Two Matrices).*** *If A and B are invertible matrices of the same size, then so too is AB, and*

$$(AB)^{-1} = B^{-1}A^{-1}. \tag{2.175}$$

Proof. The proof is very simple: Repeated application of the associative law and the definition of I give

$$\begin{aligned}(AB)(B^{-1}A^{-1}) &= ((AB)B^{-1})A^{-1} = (A(BB^{-1}))A^{-1} = (AI)\,A^{-1} \\ &= AA^{-1} = I\end{aligned} \tag{2.176}$$

and similarly in the reverse order

$$(B^{-1}A^{-1})(AB) = I. \tag{2.177}$$

■

Another theorem for numbers, namely that $\left(a^{-1}\right)^{-1} = a$, also has an analog for matrices.

Theorem 2.5.7. ***(Inverse of the Inverse of a Matrix).*** *If A is an invertible matrix, then so too is A^{-1} and*

$$(A^{-1})^{-1} = A. \tag{2.178}$$

The proof is left as Exercise 2.5.19.

There exists another simple operation for matrices, one that has no analog for numbers. Although we will not need it until later, we present it here since it rounds out our discussion of the algebra of matrices.

Definition 2.5.3. ***(Transpose of a Matrix).*** *For every $m \times n$ matrix A, we define its transpose A^T as the $n \times m$ matrix obtained from A by making the jth column of A into the jth row of A^T for each j; that is, by defining the jth row of A^T as $(a_{1j}, a_{2j}, \ldots, a_{mj})$. Equivalently,*

$$(A^T)_{ji} = a_{ij} \tag{2.179}$$

for all $1 \leq i \leq m$ and $1 \leq j \leq n$.

From this definition it easily follows that the ith row of A becomes the ith column of A^T as well. Also, the transpose of a column n-vector is a row n-vector and vice versa. This fact is often used for avoiding the inconvenient appearance of tall column vectors by writing them as transposed rows:

Example 2.5.5. (Transpose of a Row Vector)

$$(x_1, x_2, \ldots, x_n)^T = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}. \tag{2.180}$$

♦

Example 2.5.6. (Transpose of a $\mathbf{2 \times 3}$ Matrix). Let

$$A = \begin{bmatrix} 2 & 3 & -5 \\ 4 & -1 & 3 \end{bmatrix}. \tag{2.181}$$

Then

$$A^T = \begin{bmatrix} 2 & 4 \\ 3 & -1 \\ -5 & 3 \end{bmatrix}. \tag{2.182}$$

♦

The transpose has some useful properties.

Theorem 2.5.8. ***(Transpose of the Product of Two Matrices and of the Inverse of a Matrix).*** *If A and B are matrices such that their product is defined, then*

$$(AB)^T = B^T A^T, \tag{2.183}$$

and if A is invertible, then so too is A^T and

$$(A^T)^{-1} = (A^{-1})^T. \tag{2.184}$$

Proof. AB is defined when A is $m \times p$ and B is $p \times n$, for arbitrary m, p, and n. Then B^T is $n \times p$ and A^T is $p \times m$, and so $B^T A^T$ is also defined and is $n \times m$, the same size as $(AB)^T$. To prove Equation 2.183, we need only show that corresponding elements of those two products are equal. Indeed, for every $i = 1, \ldots, n$ and $j = 1, \ldots, m$,

$$((AB)^T)_{ij} = (AB)_{ji} = \sum_{k=1}^{p} a_{jk} b_{ki} = \sum_{k=1}^{p} (B^T)_{ik} (A^T)_{kj} = (B^T A^T)_{ij}. \tag{2.185}$$

Hence $(AB)^T = B^T A^T$.

Next, we prove the second statement of the theorem. If A is invertible, then there is a matrix A^{-1} such that $AA^{-1} = A^{-1}A = I$. Applying Equation 2.183, with $B = A^{-1}$, we obtain

$$(A^{-1})^T A^T = (AA^{-1})^T = I^T = I, \tag{2.186}$$

and also

$$A^T (A^{-1})^T = (A^{-1}A)^T = I^T = I. \tag{2.187}$$

Hence A^T is invertible and $(A^T)^{-1} = (A^{-1})^T$. ■

Exercises

In the first six exercises find the inverse matrix if possible.

Exercise 2.5.1. $A = \begin{bmatrix} 2 & 3 \\ 4 & -1 \end{bmatrix}.$

Exercise 2.5.2. $A = \begin{bmatrix} 5 & 2 \\ 3 & 4 \end{bmatrix}.$

Exercise 2.5.3. $A = \begin{bmatrix} 2 & 3 & 5 \\ 4 & -1 & 1 \\ 3 & 2 & -2 \end{bmatrix}.$

Exercise 2.5.4. $A = \begin{bmatrix} 0 & -6 & 2 \\ 3 & -1 & 0 \\ 4 & 3 & -2 \end{bmatrix}.$

Exercise 2.5.5. $A = \begin{bmatrix} 2 & 3 & 5 \\ 4 & -1 & 3 \\ 3 & 2 & 5 \end{bmatrix}.$

Exercise 2.5.6. $A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & -1 & 1 \end{bmatrix}$.

Exercise 2.5.7. Find two invertible 2×2 matrices A and B such that $A \neq -B$ and $A + B$ is not invertible.

Exercise 2.5.8. a. Given the 2×3 matrix

$$A = \begin{bmatrix} 2 & 0 & 4 \\ 4 & -1 & 1 \end{bmatrix},$$

find all 3×2 matrices X by Gauss–Jordan elimination such that $AX = I$ holds. (Such a matrix is called a *right inverse of* A.)
b. Can you find a 3×2 matrix Y such that $YA = I$ holds?

Exercise 2.5.9. a. Given the 3×2 matrix

$$A = \begin{bmatrix} 2 & -1 \\ 4 & -1 \\ 2 & 2 \end{bmatrix},$$

find all 2×3 matrices X by Gauss–Jordan elimination such that $XA = I$ holds. (Such a matrix is called a *left inverse of* A.)
b. Can you find a 2×3 matrix Y such that $AY = I$ holds?

***Exercise 2.5.10.** a.. Try to formulate a general rule, based on the results of the last two exercises, for the existence of a right inverse and for the existence of a left inverse of a 2×3 and of a 3×2 matrix.
b. Same as above for an $m \times n$ matrix.
c. When would the right inverse and the left inverse be unique?

***Exercise 2.5.11.** Show that if a square matrix has a right inverse X and a left inverse Y, then $Y = X$ must hold. (*Hint*: Modify the second part of the proof of Theorem 2.5.2.)

Exercise 2.5.12. The matrix

$$E = \begin{bmatrix} 1 & 0 & 0 \\ c & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is obtained from the unit matrix I by the elementary row operation of adding c times its first row to its second row. Show that for every 3×3 matrix A the same elementary row operation performed on A results in the product matrix EA. Also, find E^{-1} and describe the elementary row operation it corresponds to. (A matrix that produces the same effect by multiplication as an elementary row operation, like this E and the matrices P in the next two exercises, is called an *elementary matrix*.)

Exercise 2.5.13. Find a matrix P such that, for every 3×3 matrix A, PA equals the matrix obtained from A by multiplying its first row by a nonzero scalar c. (*Hint*: Try $A = I$ first.) Find P^{-1}.

Exercise 2.5.14. Find a matrix P such that, for every 3×3 matrix A, PA equals the matrix obtained from A by exchanging its first and third rows. (*Hint*: Try $A = I$ first.) Find P^{-1}.

Exercise 2.5.15. If A is any invertible matrix and c any nonzero scalar, what is the inverse of cA? Prove your answer.

Exercise 2.5.16. For every invertible matrix A and every positive integer n we define $A^{-n} = (A^{-1})^n$. Show that in this case we also have $A^{-n} = (A^n)^{-1}$ and $A^{-m}A^{-n} = A^{-m-n}$ if m is a positive integer as well.

Exercise 2.5.17. A square matrix with a single 1 in each row and in each column and zeros everywhere else is called a *permutation matrix*.
a. List all six 3×3 permutation matrices P and their inverses.
b. Show that, for every such P and for every $3 \times n$ matrix A, PA equals the matrix obtained from A by the permutation of its rows that is the same as the permutation of the rows of I that results in P.
c. What is BP if B is $n \times 3$?

Exercise 2.5.18. State six conditions corresponding to those of Theorem 2.5.5 for a matrix to be singular.

Exercise 2.5.19. Prove Theorem 2.5.7. (*Hint*: Imitate the proof of Theorem 2.5.2 for the equation $A^{-1}X = I$.)

Exercise 2.5.20. Prove that if A, B, C are invertible matrices of the same size, then so is ABC, and $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$.

MATLAB Exercises

In MATLAB, the transpose of A is denoted by A'. The reduction of $[A|I]$ can be achieved by the command **rref**$([A$ **eye**$(n)])$ or **rref**$([A$ **eye**$($**size**$(A))])$. A^{-1} can also be obtained by writing **inv**(A). These commands or the command **rank**(A) can be used to determine whether A is singular or not.

Exercise 2.5.21. Let $A =$ **round**$(10 *$ **rand**$(4))$, $B =$ **triu**(A), and $C =$ **tril**(A).
a. Find the inverses of B and C in **format rat** by using **rref** if they exist, and verify that they are inverses indeed.
b. Repeat Part (a) five times. (Use the up-arrow key.)
c. Do you see any pattern? Make a conjecture and prove it.

Exercise 2.5.22. Let $A = \mathbf{round}(10 * \mathbf{rand}(3,5))$.
a. Find a solution for $AX = I$ by using **rref**, or show that no solution exists.
b. If you have found a solution, verify that it satisfies $AX = I$.
c. If there is a solution, compute $A\backslash\mathbf{eye}(3)$ and check whether it is a solution.
d. If there is a solution of $AX = I$, try to find a solution for $YA = I$ by using **rref**.
(*Hint*: Rewrite this equation as $A^T Y^T = I$ first.) Draw a conclusion.
e. Repeat all of the above three times.

3. *Vector Spaces and Subspaces*

3.1 General Vector Spaces

At the end of Section 1.1 we mentioned that various sets of functions have the same kind of structure as the Euclidean vector spaces we had studied, and are also called vector spaces. (By structure we mean the algebraic operations on these sets.) The precise definition will follow below. In subsequent sections, we are going to develop several concepts, such as subspaces, linear independence, and so on, that are common to all these cases. Thus it is advantageous to consider such spaces in general, before taking them up individually. Nevertheless, our focus will remain $\mathbb{R}^n$, but we also need to study its subspaces, which are vector spaces in their own right.

In this section, we define general vector spaces, study some of the implications of the definition, and list several examples and counterexamples, but we leave the most important examples, the subspaces of $\mathbb{R}^n$, to Section 3.2.

Definition 3.1.1. ***(Vector Space).*** *A set V is called a (real) vector space and its elements are called vectors if V is not empty and to each $\mathbf{p}, \mathbf{q} \in V$ and each real number c a unique sum $\mathbf{p}+\mathbf{q} \in V$ and a unique product $c\mathbf{p} \in V$ are associated, satisfying the eight rules below*[1] *for all $\mathbf{p}, \mathbf{q}, \mathbf{r}$, and real a and b:*
1. $\mathbf{p}+\mathbf{q} = \mathbf{q}+\mathbf{p}$ *(commutativity of addition),*
2. $(\mathbf{p}+\mathbf{q})+\mathbf{r} = \mathbf{p}+(\mathbf{q}+\mathbf{r})$ *(associativity of addition),*
3. There is a vector $\mathbf{0} \in V$ such that $\mathbf{p}+\mathbf{0} = \mathbf{p}$ for every $\mathbf{p}$ (existence of zero vector),
4. For every vector $\mathbf{p}$ there is an associated vector $-\mathbf{p} \in V$ such that $\mathbf{p}+(-\mathbf{p}) = \mathbf{0}$ (existence of additive inverse),
5. $1\mathbf{p} = \mathbf{p}$ *(rule of multiplication by 1),*
6. $a(b\mathbf{p}) = (ab)\mathbf{p}$ *(associativity of multiplication by scalars),*
7. $(a+b)\mathbf{p} = a\mathbf{p}+b\mathbf{p}$ *(first distributive law),*
8. $a(\mathbf{p}+\mathbf{q}) = a\mathbf{p}+a\mathbf{q}$ *(second distributive law).*

Example 3.1.1. ***($\mathbb{R}^n$ as a Vector Space).*** For every n, the Euclidean space $\mathbb{R}^n$ is a vector space by this new definition as well, since it was taken as the paradigm of all vector spaces: That $\mathbb{R}^n$ satisfies the eight axioms, was proved in Theorem 1.1.1. for $n = 2$. ♦

[1] Such rules in definitions are usually called axioms.

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Example 3.1.2. (The Set of m×n Matrices as a Vector Space). The set $\mathcal{M}_{m,n}$ of all $m \times n$ matrices, with the usual rules of addition and multiplication of matrices by scalars, has the structure of a vector space, which is basically the same as that of $\mathbb{R}^{mn}$, except for the insignificant detail of the components being arranged in a rectangular array, rather than in a column. ♦

In the examples below we exhibit various vector spaces of real-valued functions in which, for every f, g, the sum $f + g$ and the product cf are defined in the usual way by

$$(f + g)(x) = f(x) + g(x) \tag{3.1}$$

and

$$(cf)(x) = cf(x) \tag{3.2}$$

for all x for which the right-hand sides are defined. Thus the domain of $f + g$ is the intersection of the domains of f and g, and the domain of cf is the same as that of f.

Example 3.1.3. (Sets of Real Functions as Vector Spaces). Let D be any set of real numbers. The set $\mathcal{F}(D)$ of all real-valued functions on D with the above operations is easily shown to be a vector space. ♦

Definition 3.1.1 implicitly states that in every vector space the sum of two vectors and every scalar multiple of a vector must also be vectors *in the same space.* These are conditions that must also be checked to determine whether a given set is a vector space or not. While in the foregoing examples these conditions were obviously true, in many others they may not be, or the operations can lead out of the set we started with. For example, the sum of two numbers between 0 and 1 may well be more than 1. Thus the interval $(0, 1)$ with the usual operations is not a vector space. We have an explicit name for these implicitly included properties.

Definition 3.1.2. ***(Closure Under Addition and Under Multiplication by Scalars).*** *A set S is said to be closed under addition if for every pair of elements $\mathbf{p}, \mathbf{q} \in S$ the sum $\mathbf{p} + \mathbf{q}$ is defined and belongs to S. The set S is said to be closed under multiplication by scalars if for every scalar c and $\mathbf{p} \in S$ the product $c\mathbf{p}$ is defined and belongs to S.*

Remark 3.1.1. Equations 3.1 and 3.2 define operations on functions that satisfy the eight axioms of Definition 3.1.1 because the corresponding operations on the right-hand sides are the usual operations on $\mathbb{R}$, which satisfy the axioms for all values of x. Thus, to show that a given set S of *functions* is a vector space, all we need to show is that S is closed under these operations, and the axioms follow automatically. In particular, the closure of S under multiplication by all scalars implies $\mathbf{0} \in S$ and $-\mathbf{p} \in S$ if $\mathbf{p} \in S$, since $c = 0$ implies $c\mathbf{p} = \mathbf{0}$, and $c = -1$ implies $c\mathbf{p} = -\mathbf{p}$.

Example 3.1.4. (Sets of Polynomials as Vector Spaces). Let $\mathcal{P}_n = \{P | P(x) = p_0 + p_1 x + \cdots + p_n x^n; p_0, p_1, \ldots, p_n \in \mathbb{R}\}$ be the set of single-variable polynomials of degree n or less and the zero polynomial,[2] $0 = 0 + 0x + \cdots + 0x^n$ for all x, together with the rules of addition of functions and their multiplication by scalars given by Equations 3.1 and 3.2, which in this case mean

$$\begin{aligned}&(p_0 + p_1 x + \cdots + p_n x^n) + (q_0 + q_1 x + \cdots + q_n x^n)\\ &= (p_0 + q_0) + (p_1 + q_1)x + \cdots + (p_n + q_n)x^n \end{aligned} \tag{3.3}$$

and

$$c(p_0 + p_1 x + \cdots + p_n x^n) = cp_0 + cp_1 x + \cdots + cp_n x^n \tag{3.4}$$

for all x. ♦

With these rules the set $\mathcal{P}_n$ becomes a vector space: Clearly $\mathcal{P}_n$ is closed under addition and multiplication by scalars, and by the preceding remark, the polynomials of $\mathcal{P}_n$ satisfy the axioms of Definition 3.1.1 and can be regarded as vectors.

Note that $\mathcal{P}_m \subset \mathcal{P}_n$ if $m < n$, because then the set of polynomials of degree n *or less* includes the set of polynomials of degree m or less. This is in contrast to Euclidean spaces, where $\mathbb{R}^m \not\subseteq \mathbb{R}^n$ if $m < n$, since m-tuples are not also n-tuples. (However, if $m < n$, then $\mathbb{R}^m$ is in one-to-one correspondence with a *subset* of $\mathbb{R}^n$, which is also an m-dimensional vector space. For example, $\mathbb{R}^2$ is in one-to-one correspondence with the subset $\{(x_1, x_2, 0) : x_1, x_2 \in \mathbb{R}\}$ of $\mathbb{R}^3$.)

Example 3.1.5. (Sets of Continuous Functions as Vector Spaces). The set $C[a, b]$ of all continuous functions on the interval $[a, b]$ is also a vector space, for the following reasons. It is closed under addition and under multiplication by scalars, because the sum of two functions continuous on an interval is also continuous there, and every scalar multiple of such a function is continuous there as well. ♦

Example 3.1.6. (Sets of Discontinuous Functions Are Not Vector Spaces). Consider the set $D[a, b]$ of discontinuous functions on the interval $[a, b]$. This set is *not* a vector space because it is not closed under addition: The sum of two discontinuous functions need not be discontinuous. For example, if the only discontinuity of f is a jump of size 1 at some point $p \in [a, b]$, then the only discontinuity of $g = -f$ is a jump of size -1 at p, and $D[a, b]$ is not closed under addition since $f + g = f + (-f) = 0$ for all $x \in [a, b]$, and the zero function is *continuous.* Axiom 3 is also violated, because there is no zero vector in this set; the only candidate is outside $D[a, b]$. ♦

---

[2] By convention, the zero polynomial has no degree, because otherwise the additive rule for degrees in the multiplication of polynomials could not be maintained. Some authors, however, define the degree of the zero polynomial to be $-\infty$.

Example 3.1.7. (The Set of Ordered Pairs with an Unusual Addition Rule Is Not a Vector Space). The set of all ordered pairs of real numbers with addition and multiplication by scalars defined by $(p_1, p_2) + (q_1, q_2) = (p_1 + p_2, q_1 + q_2)$ and $c(p_1, p_2) = (cp_1, cp_2)$ is not a vector space, because Axiom 1 fails to hold. For example, let $(p_1, p_2) = (1, 2)$ and $(q_1, q_2) = (1, 3)$. Then, by the given addition rule, we have $(1, 2) + (1, 3) = (1 + 2, 1 + 3) = (3, 4)$, but $(1, 3) + (1, 2) = (1 + 3, 1 + 2) = (4, 3)$; thus this kind of addition is not commutative. ♦

Notice that, as in the preceding two examples, to disprove a general rule, all we need is one counterexample. On the other hand, to prove a general rule is more difficult: We must prove it for all cases. We usually do this by algebraically proving an arbitrary typical case using letter symbols, since proving a rule for *any* appropriate case proves it for *all* such cases. We illustrate this kind of proof in the next example, and many others will follow.

Example 3.1.8. (Solution Set of a Differential Equation as a Vector Space). The set $\mathcal{D}$ of all differentiable functions f on $\mathbb{R}$ for which $f'(x) + f(x) = 0$ for all x and addition and multiplication by scalars are defined by Equations 3.1 and 3.2 is a vector space. Indeed, $\mathcal{D}$ is closed under addition and multiplication by scalars, since if f and g are any functions in $\mathcal{D}$, then $f' + f = 0$ and $g' + g = 0$, whence $(f + g)' + (f + g) = 0$ and $(cf)' + (cf) = 0$, and so $f + g$ and cf are both in $\mathcal{D}$. The eight axioms follow automatically by Remark 3.1.1 above. ♦

Subtraction of vectors can be defined just as before.

Definition 3.1.3. ***(Subtraction of Vectors).*** *For all vectors* $\mathbf{p}$ *and* $\mathbf{q}$ *in a vector space* V*, we define*

$$\mathbf{p} - \mathbf{q} = \mathbf{p} + (-\mathbf{q}). \tag{3.5}$$

We have a list of further properties of vectors just as in $\mathbb{R}^n$.

Theorem 3.1.1. ***(Properties of* 0 *and Negatives).*** *For all vectors* $\mathbf{p}$, $\mathbf{q}$, $\mathbf{x}$ *in a vector space* V *and for all scalars* c *and* d, *we have*

1. $0\mathbf{p} = \mathbf{0}$,
2. $c\mathbf{0} = \mathbf{0}$,
3. $\mathbf{p} + \mathbf{x} = \mathbf{q}$ *if and only if* $\mathbf{x} = \mathbf{q} - \mathbf{p}$,
4. *If* $c\mathbf{p} = \mathbf{0}$ *then either* $c = 0$ *or* $\mathbf{p} = \mathbf{0}$ *or both,*
5. $-\mathbf{p} = (-1)\mathbf{p}$,
6. $(-c)\mathbf{p} = c(-\mathbf{p}) = -(c\mathbf{p})$,
7. $c(\mathbf{p} - \mathbf{q}) = c\mathbf{p} - c\mathbf{q}$,
8. $(c - d)\mathbf{p} = c\mathbf{p} - d\mathbf{p}$.

Proof. First we prove Property 1. By Axiom 5 we have $\mathbf{p} = 1\mathbf{p}$. By the definition of the number 0, Axiom 7, and Axiom 5 again, this equation can be changed to $\mathbf{p} = (1+0)\mathbf{p} = 1\mathbf{p} + 0\mathbf{p} = \mathbf{p} + 0\mathbf{p}$. Change the order of the terms on the right using Axiom 1 and add $-\mathbf{p}$ to both sides: Then we get $\mathbf{p}+(-\mathbf{p}) = (0\mathbf{p}+\mathbf{p})+(-\mathbf{p}) = 0\mathbf{p}+[\mathbf{p}+(-\mathbf{p})]$, where in the last step we used Axiom 2. Applying Axiom 4 on both sides, we obtain $\mathbf{0} = 0\mathbf{p} + \mathbf{0}$. Axiom 3 reduces this result to $\mathbf{0} = 0\mathbf{p}$, as was to be proved.

To prove Property 2, observe that $c\mathbf{0} = c(0\mathbf{p})$ by Property 1 and, by Axiom 6, the ordinary multiplication of numbers, and Property 1 again, $c(0\mathbf{p})$ becomes $(c0)\mathbf{p} = 0\mathbf{p} = \mathbf{0}$.

The proof of Property 3 runs as follows. Suppose first that $\mathbf{x} = \mathbf{q} - \mathbf{p}$. Then $\mathbf{p}+\mathbf{x} = \mathbf{p}+(\mathbf{q}-\mathbf{p}) = (\mathbf{q}-\mathbf{p})+\mathbf{p} = [\mathbf{q}+(-\mathbf{p})]+\mathbf{p} = \mathbf{q}+[(-\mathbf{p})+\mathbf{p}] = \mathbf{q}+[\mathbf{p}+(-\mathbf{p})] = \mathbf{q}+\mathbf{0} = \mathbf{q}$.

To prove the converse, assume that $\mathbf{p} + \mathbf{x} = \mathbf{q}$. Then subtracting $\mathbf{p}$ from both sides gives $(\mathbf{p} + \mathbf{x}) - \mathbf{p} = \mathbf{q} - \mathbf{p}$, and the left-hand side reduces to $(\mathbf{x} + \mathbf{p}) - \mathbf{p} = \mathbf{x} + [\mathbf{p} + (-\mathbf{p})] = \mathbf{x} + \mathbf{0} = \mathbf{x}$.

Property 4 may be proved by showing that $c \neq 0$ and $\mathbf{p} \neq \mathbf{0}$ cannot hold simultaneously if $c\mathbf{p} = \mathbf{0}$. Thus, suppose that $c\mathbf{p} = \mathbf{0}$ and $c \neq 0$ hold. Then $\mathbf{p} = \frac{1}{c}(c\mathbf{p}) = \frac{1}{c}\mathbf{0} = \mathbf{0}$, and so $\mathbf{p} \neq \mathbf{0}$ cannot be true in this case.

To prove Property 5, consider $\mathbf{p} + (-1)\mathbf{p} = 1\mathbf{p} + (-1)\mathbf{p} = [1 + (-1)]\mathbf{p} = 0\mathbf{p} = \mathbf{0}$. Subtracting $\mathbf{p}$ on both sides, we get $(-1)\mathbf{p} = -\mathbf{p}$ as in the proof of Property 3. ■

The proofs of the remaining statements are straightforward, and are left as exercises.

Exercises

In the first ten exercises determine whether the given set describes a vector space or not. For each function space the operations are defined by Equations 3.1 and 3.2. Explain your answers!

Exercise 3.1.1. The set of all polynomials of degree two and the zero polynomial.

Exercise 3.1.2. The set of all solutions (x, y) of the equation $2x + 3y = 0$, with addition and multiplication by scalars defined as in $\mathbb{R}^2$.

Exercise 3.1.3. The set of all solutions (x, y) of the equation $2x + 3y = 1$, with addition and multiplication by scalars defined as in $\mathbb{R}^2$.

Exercise 3.1.4. The set of all twice differentiable functions f for which $f''(x) + 2f(x) = 0$ holds.

Exercise 3.1.5. The set of all twice differentiable functions f for which $f''(x) + 2f(x) = 1$ holds.

Exercise 3.1.6. The set $\mathcal{P}$ of all polynomials in a single variable x.

Exercise 3.1.7. The set of all ordered pairs of real numbers with addition and multiplication by scalars defined by $(p_1, p_2) + (q_1, q_2) = (p_1 + q_2, p_2 + q_1)$ and $c(p_1, p_2) = (cp_1, cp_2)$.

Exercise 3.1.8. The set of all ordered pairs of real numbers with addition and multiplication by scalars defined by $(p_1, p_2) + (q_1, q_2) = (p_1 + q_2, 0)$ and $c(p_1, p_2) = (cp_1, cp_2)$.

Exercise 3.1.9. The set of all ordered pairs of real numbers with addition and multiplication by scalars defined by $(p_1, p_2) + (q_1, q_2) = (p_1 + q_1, 0)$ and $c(p_1, p_2) = (cp_1, cp_2)$.

Exercise 3.1.10. The set of all ordered pairs of real numbers with addition and multiplication by scalars defined by $(p_1, p_2) + (q_1, q_2) = (p_1 + q_1, p_2 + q_2)$ and $c(p_1, p_2) = (|c|p_1, |c|p_2)$.

***Exercise 3.1.11.** Prove the last three parts of Theorem 3.1.1.

***Exercise 3.1.12.** Prove that in every vector space if $\mathbf{p} + \mathbf{x} = \mathbf{p}$ holds for all $\mathbf{p}$, then $\mathbf{x} = \mathbf{0}$ must hold.

***Exercise 3.1.13.** Prove that in every vector space we have the following cancellation rule: If, for some $\mathbf{p}, \mathbf{q}, \mathbf{r}$ the equation $\mathbf{p} + \mathbf{q} = \mathbf{p} + \mathbf{r}$ holds, then $\mathbf{q} = \mathbf{r}$ must hold.

***Exercise 3.1.14.** Show that we could define vector spaces by just seven axioms instead of eight if we replace Axioms 3 and 4 by the single axiom:
3'. There is a vector $\mathbf{0}$ such that $0\mathbf{p} = \mathbf{0}$ holds for all vectors $\mathbf{p}$.
In other words, prove that if we define the zero vector by Axiom 3' instead of Axiom 3, then this axiom in conjunction with the other axioms implies both the additive property of the zero vector expressed in Axiom 3 and the existence of additive inverses expressed in Axiom 4, with $(-1)\mathbf{p}$ in the role of $-\mathbf{p}$.

MATLAB Exercises

Exercise 3.1.15. Let V denote the set of all ordered quintuples of 0's and 1's, with addition defined by the MATLAB command $\mathbf{p}|\mathbf{q}$ and multiplication by scalars by $c\&\mathbf{p}$.
a. Generate such vectors by the command $\mathbf{round}(\mathbf{rand}(1, 5))$ and use MATLAB to check whether each of the eight vector space axioms is satisfied for those vectors and selected scalars.
b. If you think this set is a vector space, prove it. If not, explain why.
c. Do you get a vector space if the scalars are also restricted to 0's and 1's and their addition and multiplication are also defined by $c|d$ and $c\&d$, respectively?

Exercise 3.1.16. Let V denote the set of all ordered quintuples of real numbers and define addition of vectors by the MATLAB command $\max(\mathbf{p}, \mathbf{q})$ and multiplication by scalars by componentwise multiplication as in $\mathbb{R}^5$.
a. Generate such vectors by the command $\mathbf{round}(10 * \mathbf{rand}(1,5) - 5)$ and use MATLAB to check whether each of the vector space axioms is satisfied for those vectors and selected scalars.
b. If you think this is a vector space, prove it. If not, explain why.

3.2 Subspaces

In the solution of linear systems and in the parametric description of planes and hyperplanes, we have encountered expressions like $s\mathbf{u} + t\mathbf{v}$ or more generally of the type

$$\sum_{i=1}^{n} s_i \mathbf{u}_i, \tag{3.6}$$

with n being any positive integer. Such expressions are called *linear combinations* of the vectors involved. In many applications, we need to consider the set of all linear combinations of the given vectors as the coefficients vary. Such sets describe lines, planes, etc., through the origin. We shall explore various questions concerning the vectors $\mathbf{u}_i$, which "generate" them, such as how many $\mathbf{u}_i$ we need and how some can be added, omitted, or changed. Because of this, it is useful to begin by characterizing these sets in a general way without involving the $\mathbf{u}_i$ vectors.

Definition 3.2.1. ***(Subspace).*** *A subset U of a vector space X is called a subspace of X if it is a vector space with addition of vectors and multiplication by scalars being the same as in X.*

Example 3.2.1. (A Vector Space Subset That Is Not a Subspace). Consider the interval $U = [0, 10)$ with addition and multiplication by any real c defined modulo 10.[3] This set with these operations is a vector space (we omit the proof), and although U is a subset of $\mathbb{R}^1$, it is not a subspace of it, because the operations are not the same. ♦

Remark 3.2.1. In the rest of the book we shall not consider operations like those in Example 3.2.1; we shall only use the usual arithmetic of $\mathbb{R}$.

[3] Modulo 10 operations on U mean that the results of ordinary addition and multiplication are reduced by subtracting or adding an appropriate multiple of 10, so that the final result ends up in U. For example: $5 + 7 = 2(\text{mod}10)$, $(-3) \cdot 4 = 8(\text{mod}10)$.

In the previous section we defined a vector space as a nonempty set U such that for every pair $\mathbf{p}, \mathbf{q}$ of vectors in U and every scalar c the sum $\mathbf{p}+\mathbf{q}$ and the product $c\mathbf{p}$ belong to U; that is, U is *closed* under these operations, and eight algebraic rules hold. Now, in the case of a subset U of a vector space, the algebraic rules holding in X remain valid for the vectors of U since the operations are the same in U as in X. *Thus, to test whether a subset U of a vector space X with the same operations as in X is a subspace, it is enough to test whether it is nonempty and closed under addition and under multiplication by scalars.* In particular, the nonemptiness and the closure of U under multiplication by scalars imply $\mathbf{0} \in U$ and $-\mathbf{p} \in U$ if $\mathbf{p} \in U$, since $c = 0$ and $\mathbf{p} \in U$ imply $c\mathbf{p} = \mathbf{0}$, and $c = -1$ and $\mathbf{p} \in U$ imply $c\mathbf{p} = -\mathbf{p}$.

Let us look at some examples.

Example 3.2.2. (A Subspace of $\mathbb{R}^3$***).*** Consider in the space $\mathbb{R}^3$ the set U of vectors[4] whose third coordinate is 0, that is, let $U = \{\mathbf{u} \in \mathbb{R}^3 | \mathbf{u} = (u_1, u_2, 0)^T\}$. It is easy to see that this set is nonempty, and is a subspace of $\mathbb{R}^3$, since if $\mathbf{u}, \mathbf{v} \in U$ and c is any scalar, then $\mathbf{u}+\mathbf{v} = (u_1, u_2, 0)^T + (v_1, v_2, 0)^T = (u_1+v_1, u_2+v_2, 0)^T$, and so the third component of $\mathbf{u}+\mathbf{v}$ being zero, $\mathbf{u}+\mathbf{v}$ also belongs to U and, similarly, $c\mathbf{u}$ does too. ♦

Example 3.2.3. (A Subset of $\mathbb{R}^3$ ***That Is Not a Subspace).*** Consider in the space $\mathbb{R}^3$ the set U of vectors whose third coordinate is 1, that is, let $U = \{\mathbf{u} \in \mathbb{R}^3 | \mathbf{u} = (u_1, u_2, 1)^T\}$. To see that this set is not a subspace of $\mathbb{R}^3$ all we have to do is to exhibit two vectors in U whose sum is not in U, or one vector u in U and a scalar c whose product $c\mathbf{u}$ is not in U. In this example any two vectors of U will add up to a vector with 2 for its third component, and so this sum vector will be outside of U. Alternatively, c times any vector $\mathbf{u}$ of U will result in a vector having c as its third component, and then $c\mathbf{u}$ is outside of U if $c \neq 1$. ♦

In general, to prove that a subset U of a vector space with the same operations *is* a subspace we have to prove the closure properties for all $\mathbf{u}, \mathbf{v} \in U$ and all scalars c (making sure also that U is not empty), while to prove that U is *not* a subspace all we need is a single counterexample to either of the closure requirements.

Note that in every vector space X, the set $\{\mathbf{0}\}$ consisting of the zero vector alone is a subspace of X and so too is the whole space X. These are called the *trivial subspaces* of X, while all the others are its *nontrivial* or *proper subspaces.*

Example 3.2.4. (The Subset of $\mathbb{R}^3$ ***Generated by Two Given Vectors).*** Let us find the smallest subspace U of $\mathbb{R}^3$ that contains the vectors $(1, 1, 1)^T$ and $(1, 2, 3)^T$.

[4] Remember the convention that all vectors are to be column vectors, but they may be written as row vectors transposed.

Since U must be closed under multiplication by scalars, it must contain all multiples of $(1,1,1)^T$ and $(1,2,3)^T$, and since it must also be closed under addition of its vectors, it must contain the sums of these multiples. In other words, U must contain all linear combinations $s(1,1,1)^T + t(1,2,3)^T$. This fact, however, is all we need: $U = \{\mathbf{u} \in \mathbb{R}^3 | \mathbf{u} = s(1,1,1)^T + t(1,2,3)^T; s,t \in \mathbb{R}\}$ is a subspace of $\mathbb{R}^3$, as we are going to show below. Furthermore, it is the smallest one that contains the vectors $(1,1,1)^T$ and $(1,2,3)^T$, since as we have just said, every subspace that contains these two vectors must contain all the linear combinations of which this U consists.

U is a subspace, because, first, it is clearly nonempty. Then, second, let $\mathbf{u} = s_1(1,1,1)^T + t_1(1,2,3)^T$, $\mathbf{v} = s_2(1,1,1)^T + t_2(1,2,3)^T$, and c be any scalar. Then both the sum $\mathbf{u}+\mathbf{v} = (s_1+s_2)(1,1,1)^T + (t_1+t_2)(1,2,3)^T$ and the product $c\mathbf{u} = cs_1(1,1,1)^T + ct_1(1,2,3)^T$ are linear combinations of $(1,1,1)^T$ and $(1,2,3)^T$, and consequently U is closed under addition and under multiplication by scalars.

We say that $(1,1,1)^T$ and $(1,2,3)^T$ *generate or span* U or that U is their span (see Definition 3.2.2 below). Geometrically U is the plane through the origin containing the given vectors. ♦

Example 3 2 5 (The Solution Set of a Homogeneous Equation as a Subspace of $\mathbb{R}^3$*).* Consider the set U in $\mathbb{R}^3$ of all solutions of the equation $x-2y+z=0$. If we solve this equation, we obtain the parametric form of U as the set of vectors $(x,y,z)^T = s(2,1,0)^T + t(-1,0,1)^T$. We can show this to be a subspace either by reasoning as in the previous example, or directly from the defining equation, without even solving it, as follows. Let $\mathbf{u} = (x_1,y_1,z_1)^T$ and $\mathbf{v} = (x_2,y_2,z_2)^T$ be two solutions of $x-2y+z=0$, that is, let $x_1-2y_1+z_1=0$ and $x_2-2y_2+z_2=0$ hold and let c be an arbitrary scalar. Then

$$(x_1+x_2)-2(y_1+y_2)+(z_1+z_2) = (x_1-2y_1+z_1)+(x_2-2y_2+z_2) = 0+0 = 0, \tag{3.7}$$

and so $\mathbf{u}+\mathbf{v}$ is also a solution and, similarly, $c\mathbf{u}$ is as well. Thus U is closed under both operations, clearly nonempty, and therefore a subspace. ♦

The constructions illustrated in the last two examples can be generalized, and constitute the two most important ways in which subspaces occur in applications. We state these as theorems.

Theorem 3.2.1. ***(The Set of all Finite Linear Combinations of Vectors of a Subset of a Vector Space Is a Subspace).*** *Let* X *be a vector space and* S *a nonempty subset of* X*. Then the set* $U = \{\mathbf{u} = \sum_{i=1}^n s_i\mathbf{a}_i | n$ *any positive integer;* $s_1, s_2, \ldots, s_n \in \mathbb{R};\ \mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n \in S\}$ *of all finite linear combinations of vectors of* S *is a subspace of* X*.*

Proof. Let $\mathbf{u} = \sum_{i=1}^n s_i\mathbf{a}_i$ and $\mathbf{v} = \sum_{j=1}^m t_j\mathbf{b}_j$ be arbitrary vectors in U and c any scalar. Then $\mathbf{u}+\mathbf{v} = \sum_{i=1}^n s_i\mathbf{a}_i + \sum_{j=1}^m t_j\mathbf{b}_j$ and $c\mathbf{u} = \sum_{i=1}^n cs_i\mathbf{a}_i$ are evidently also finite linear combinations of vectors of S and, as such, members of U. Thus U is closed under both operations, clearly nonempty, and is therefore a subspace of X. ■

Theorem 3.2.2. ***(The Solution Set of a Homogeneous Equation Is a Subspace of*** $\mathbb{R}^n$***).*** *Let A be any $m \times n$ matrix. The set U of all solutions of the homogeneous equation $A\mathbf{x} = \mathbf{0}$ is a subspace of $\mathbb{R}^n$.*

Proof. Let $\mathbf{u}$ and $\mathbf{v}$ be arbitrary vectors in U and c any scalar. Then $A\mathbf{u} = \mathbf{0}$ and $A\mathbf{v} = \mathbf{0}$ hold, and so $A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v} = \mathbf{0} + \mathbf{0} = \mathbf{0}$ and $A(c\mathbf{u}) = cA\mathbf{u} = c\mathbf{0} = \mathbf{0}$ as well. Thus U is closed under both operations, is nonempty since it always contains $\mathbf{0}$ even if nothing else, and is therefore a subspace of $\mathbb{R}^n$. ■

The subspaces occurring in these theorems have special names.

Definition 3.2.2. ***(Span of a Subset).*** *Given a vector space X and a nonempty subset S of X, the subspace of all linear combinations of all finite sets of vectors from S is called the span of S, or the subspace spanned or generated by S, and will be denoted by* Span (S)*. The span of the empty subset of X is defined to be the subspace $\{\mathbf{0}\}$ consisting of just the zero vector.*

Definition 3.2.3. ***(Null Space of a Matrix).*** *For every $m \times n$ matrix A, the set of all solutions of the homogeneous equation $A\mathbf{x} = \mathbf{0}$ is called the solution space of the equation or the null space*[5] *of A, and will be denoted by* Null(A).

Example 3.2.6. (The Subspaces of $\mathbb{R}^3$***).*** We can now easily describe all the subspaces of $\mathbb{R}^3$. If we consider all vectors as position vectors (see Example 1.1.1), then they are: the set $\{\mathbf{0}\}$, every set that consists of the position vectors of the points of a line or a plane through the origin, and $\mathbb{R}^3$ itself. For, clearly, every single nonzero vector spans such a line and any two nonparallel nonzero vectors span such a plane, while any three noncoplanar vectors span $\mathbb{R}^3$.

The last statement can be proved as follows. Suppose to the contrary that $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ are arbitrary noncoplanar vectors in $\mathbb{R}^3$ and there exists a vector $\mathbf{b} \in \mathbb{R}^3$ that is not in Span$\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$. Writing $A = (\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3)$ and $\sum_{i=1}^3 x_i \mathbf{a}_i = A\mathbf{x}$, we can say, equivalently, that $A\mathbf{x} = \mathbf{b}$ has no solution for some $\mathbf{b}$. Then, by Parts 4 and 6 of Theorem 2.5.5, $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution, which shows that $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ are coplanar, in contradiction to our assumption. Thus, any three noncoplanar vectors in $\mathbb{R}^3$ must span $\mathbb{R}^3$. ♦

Exercises

In the next eight exercises determine whether the given set is a subspace of the indicated vector space or not, and prove your statement.

[5] Note that the null space can also be viewed as the set of all vectors orthogonal to the rows of A. See Definition 3.5.3 and Corollary 3.5.2.

Exercise 3.2.1. $U = \{\mathbf{x}|x_1 = x_2 = x_3\}$ in $\mathbb{R}^3$.

Exercise 3.2.2. $U = \{\mathbf{x}|x_1 = x_2^2\}$ in $\mathbb{R}^3$.

Exercise 3.2.3. $U = \{\mathbf{x} : |x_1| = |x_2| = |x_3|\}$ in $\mathbb{R}^3$.

Exercise 3.2.4. $U = \{\mathbf{x}|x_1 + x_2 + x_3 = 0\}$ in $\mathbb{R}^4$.

Exercise 3.2.5. $U = \{\mathbf{x}|x_1 = x_2 \text{ or } x_3 = 0\}$ in $\mathbb{R}^3$.

Exercise 3.2.6. $U = \{\mathbf{x}|x_1 = x_2 \text{ and } x_3 = 0\}$ in $\mathbb{R}^3$.

Exercise 3.2.7. $U = \{\mathbf{x}|x_1 \geq 0\}$ in $\mathbb{R}^3$.

Exercise 3.2.8. $U = \{\mathbf{x} : |\mathbf{x}| = |x_1| + |x_2|\}$ in $\mathbb{R}^3$.

Exercise 3.2.9. Let U and V be subspaces of a vector space X. The set of all vectors belonging to both U and V is called the intersection of U and V and is denoted by $U \cap V$. Prove that $U \cap V$ is a subspace of X.

Exercise 3.2.10. Let U and V be subspaces of a vector space X. Show by an example that $U \cup V$ is not necessarily a subspace of X. When is it a subspace? Prove your answer.

Exercise 3.2.11. Let U be a subspace of a vector space X. Is its complement $\overline{U} = \{\mathbf{x} \in X|\mathbf{x} \notin U\}$ a subspace?

Exercise 3.2.12. Let $\mathbf{a}$ be an arbitrarily given vector in $\mathbb{R}^n$. Show that the set of all vectors orthogonal to $\mathbf{a}$ is a subspace of $\mathbb{R}^n$.

3.3 Span and Independence of Vectors

A frequently occurring problem is that of decomposing a given vector $\mathbf{b} \in \mathbb{R}^m$ into a linear combination of some other given vectors $\mathbf{a}_i \in \mathbb{R}^m$, if possible. This problem amounts to solving the linear vector equation $\mathbf{b} = \sum_{i=1}^n s_i \mathbf{a}_i$ for the unknown coefficients s_i. If we consider the given vectors $\mathbf{a}_i$ as columns of a matrix A as in Section 2.3, $\mathbf{b}$ as a column m-vector, and the s_i as entries of a column n-vector $\mathbf{s}$, then the above equation takes on the familiar form $A\mathbf{s} = \mathbf{b}$, since then

$$\sum_{i=1}^n s_i \mathbf{a}_i = (\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n) \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix} = A\mathbf{s} = \mathbf{b}. \tag{3.8}$$

Let us look at some examples.

Example 3.3.1. (Decomposing a Vector of $\mathbb{R}^4$ *into a Linear Combination of Three Given Vectors).* Write $\mathbf{b} = (7, 7, 9, 11)^T$ as a linear combination of $\mathbf{a}_1 = (2, 0, 3, 1)^T$, $\mathbf{a}_2 = (4, 1, 3, 2)^T$, and $\mathbf{a}_3 = (1, 3, -1, 3)^T$, if possible.

The system to be solved can be written as

$$\begin{bmatrix} 2 & 4 & 1 \\ 0 & 1 & 3 \\ 3 & 3 & -1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 7 \\ 9 \\ 11 \end{bmatrix}. \tag{3.9}$$

Gaussian elimination gives $s_1 = 6$, $s_2 = -2$, and $s_3 = 3$, and it is easy to check that $\mathbf{b} = 6\mathbf{a}_1 - 2\mathbf{a}_2 + 3\mathbf{a}_3$ is indeed true. ♦

Example 3.3.2. (Decomposing a Vector of $\mathbb{R}^3$ *into a Linear Combination of Three Given Vectors).* Write $\mathbf{b} = (2, 8, 0)^T$ as a linear combination of $\mathbf{a}_1 = (1, 3, 1)^T$, $\mathbf{a}_2 = (2, 6, 2)^T$, and $\mathbf{a}_3 = (0, -1, 1)^T$, if possible.

The system to be solved can be written as

$$\begin{bmatrix} 1 & 2 & 0 \\ 3 & 6 & -1 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \\ 0 \end{bmatrix}. \tag{3.10}$$

Gaussian elimination gives $s_1 = 2 - 2t$, $s_2 = t$, and $s_3 = -2$ and it is easy to check that $\mathbf{b} = (2 - 2t)\mathbf{a}_1 + t\mathbf{a}_2 - 2\mathbf{a}_3$ is true for every value of the parameter t. ♦

As the last example shows, sometimes the decomposition of a vector into linear combinations, if it exists, is not unique. The uniqueness depends solely on the $\mathbf{a}_i$ vectors: Given the $\mathbf{a}_i$, the decomposition is either unique for every vector $\mathbf{b}$ for which one exists, or it is not unique for any of them. This follows from the fact that every solution of Equation 3.8 is unique if in the row reduction there are only basic variables and nonunique if there are also free variables, and this does not depend on $\mathbf{b}$. (For a direct proof see Exercise 3.3.2.) The test for uniqueness of such decompositions is usually phrased in terms of the following definition in which the zero vector is taken for $\mathbf{b}$.

Definition 3.3.1. ***(Linear Independence).*** *Let n be any positive integer and $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ be arbitrary vectors in a vector space X. We call these vectors linearly independent of each other, or the set $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$ linearly independent, if the equation*

$$\sum_{i=1}^{n} s_i \mathbf{a}_i = \mathbf{0} \tag{3.11}$$

implies that the coefficients s_i are all zero. The linear combination with zero coefficients is called the trivial combination. If the above equation has nontrivial solutions as well, then the $\mathbf{a}_i$ vectors are said to be linearly dependent on each other and the set $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$ linearly dependent.

Let us reemphasize that independence of the vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ is equivalent to the uniqueness of the decomposition (Equation 3.8) of any $\mathbf{b}$ for which such a decomposition is possible. This is why the notion of independence is so important.

In the next two examples we reformulate the definition in case we have only two or three vectors to test for dependence and show what this means geometrically in $\mathbb{R}^2$ and $\mathbb{R}^3$.

Example 3.3.3. (Dependence of Two Vectors). Two vectors in a vector space are dependent if and only if one is a scalar multiple of the other.

Before proving the assertion, let us illustrate how it is possible that only one of two vectors is a scalar multiple of the other. In $\mathbb{R}^2$, let $\mathbf{a}_1 = \mathbf{0}$ and $\mathbf{a}_2 = (1, 2)^T$. Then $\mathbf{a}_1 = 0\mathbf{a}_2$, but there is no c for which $\mathbf{a}_2 = c\mathbf{a}_1$. Now $\mathbf{a}_1$ and $\mathbf{a}_2$ are, indeed, dependent, because $1\mathbf{a}_1 + 0\mathbf{a}_2$ is a nontrivial linear combination of the two vectors that equals $\mathbf{0}$, as required by Definition 3.3.1 for dependence.

Now, for the proof in general: First assume that $\mathbf{a}_1$ and $\mathbf{a}_2$ are arbitrary dependent vectors. Then, according to Definition 3.3.1 there exist two scalars s_1 and s_2, not both zero, such that

$$s_1\mathbf{a}_1 + s_2\mathbf{a}_2 = \mathbf{0}. \tag{3.12}$$

Say $s_1 \neq 0$. Then we can solve Equation 3.12 for $\mathbf{a}_1$, to obtain

$$\mathbf{a}_1 = -\frac{s_2}{s_1}\mathbf{a}_2, \tag{3.13}$$

which exhibits $\mathbf{a}_1$ as a multiple of $\mathbf{a}_2$. On the other hand, if $s_1 = 0$, then we must have $s_2 \neq 0$, and so, by Equation 3.12, $\mathbf{a}_2 = \mathbf{0}$. Thus, $\mathbf{a}_2$ is a scalar multiple of $\mathbf{a}_1$, namely,

$$\mathbf{a}_2 = 0\mathbf{a}_1. \tag{3.14}$$

Conversely, if one of the two vectors is a scalar multiple of the other, say

$$\mathbf{a}_2 = c\mathbf{a}_1, \tag{3.15}$$

then we can rewrite this equation as

$$c\mathbf{a}_1 + (-1)\mathbf{a}_2 = \mathbf{0},$$

which shows that Equation 3.12 is solved by $s_1 = c$ and $s_2 = -1$, and so at least one of these solutions is not zero. This fact is the condition in Definition 3.3.1 for the two vectors $\mathbf{a}_1$ and $\mathbf{a}_2$ to be dependent. The same conclusion follows by just exchanging the subscripts in the argument, if $\mathbf{a}_1$ is a multiple of $\mathbf{a}_2$.

What does this result mean geometrically in $\mathbb{R}^2$ and $\mathbb{R}^3$? It means that two vectors are dependent if and only if they are parallel, or if we consider bound vectors, collinear. (This statement includes the zero vector, which is considered to be both parallel and orthogonal to every vector.) ♦

Example 3.3.4. (Dependence of Three Vectors). In every vector space, three vectors are dependent if and only if one is a linear combination of the other two. We leave the proof of this statement as Exercise 3.3.1. Here we want to discuss only the geometric meaning in $\mathbb{R}^3$.

Let $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ be arbitrary nonzero linearly dependent bound vectors in $\mathbb{R}^3$. Then, by Definition 3.3.1, there exist scalars s_1, s_2, s_3, not all zero, such that $s_1\mathbf{a}_1 + s_2\mathbf{a}_2 + s_3\mathbf{a}_3 = \mathbf{0}$. Say, without loss of generality, that $s_3 \neq 0$. Then we can solve the above equation for $\mathbf{a}_3$ and so obtain $\mathbf{a}_3$ as a linear combination of $\mathbf{a}_1$ and $\mathbf{a}_2$. If they are considered to be bound vectors, then, by the geometrical interpretation of multiplication of vectors by scalars and by the parallelogram law for vector addition, we see that $\mathbf{a}_3$ is in the plane of $\mathbf{a}_1$ and $\mathbf{a}_2$. This result can also be stated symmetrically as saying that $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ as bound vectors are coplanar[6] if they are linearly dependent. The converse can also be seen easily, and both statements can be extended to include the zero vector. Thus the linear dependence of three bound vectors in $\mathbb{R}^3$ means that they are coplanar. ♦

Notice that if $X = \mathbb{R}^m$, then, letting the $\mathbf{a}_i$ vectors be the columns of a matrix A, we can rewrite Equation 3.11 as

$$A\mathbf{s} = \mathbf{0} \tag{3.16}$$

and so the vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ of $\mathbb{R}^m$ are independent if and only if this equation has only the trivial solution.

Example 3.3.5. (Testing the Independence of Three Vectors). Test the column vectors of the matrix

$$A = \begin{bmatrix} 2 & 3 & 5 \\ 1 & -2 & 3 \\ 3 & -4 & 2 \end{bmatrix} \tag{3.17}$$

for independence.

We need to solve the equation $A\mathbf{s} = \mathbf{0}$. Row reduction of $[A|\mathbf{0}]$ proceeds as follows:

$$\left[\begin{array}{ccc|c} 2 & 3 & 5 & 0 \\ 1 & -2 & 3 & 0 \\ 3 & -4 & 2 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -2 & 3 & 0 \\ 2 & 3 & 5 & 0 \\ 3 & -4 & 2 & 0 \end{array}\right] \to$$
$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 0 \\ 0 & 7 & -1 & 0 \\ 0 & 2 & -7 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -2 & 3 & 0 \\ 0 & 7 & -1 & 0 \\ 0 & 0 & -47/7 & 0 \end{array}\right]. \tag{3.18}$$

[6] Three or more bound vectors are said to be coplanar if they all lie in a plane, including the case when some of them are collinear.

Back substitution gives $(-47/7)s_3 = 0$ and so $s_3 = 0$, then $7s_2 + (-1) \cdot 0 = 0$, and consequently $s_2 = 0$, and finally $s_1 = 0$. Thus, $A\mathbf{s} = \mathbf{0}$ has only the trivial solution, and so the columns of A are independent. ♦

Example 3.3.6. (Showing the Independence of Three Vectors by Uniqueness of Solutions). By the argument just before Definition 3.3.1, the vectors $\mathbf{a}_1$, $\mathbf{a}_2, \mathbf{a}_3$ of Example 3.3.1 are independent if the equation $A\mathbf{s} = \mathbf{b}$ has a unique solution for some $\mathbf{b}$. Since the solution is unique for the $\mathbf{b}$ of Example 3.3.1, the solution must be unique for every other $\mathbf{b}$ for which it exists, including $\mathbf{b} = \mathbf{0}$. Thus Equation 3.11 has only the trivial solution. We could, of course, have verified this directly by substituting the given vectors into Equation 3.16 and solving it. ♦

Example 3.3.7. (Showing the Dependence of Three Vectors by Nonuniqueness of Solutions). The vectors $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ of Example 3.3.2 are linearly dependent, since if the solution of $A\mathbf{s} = \mathbf{b}$ is not unique for the $\mathbf{b}$ of Example 3.3.2, then it is not unique for $\mathbf{b} = \mathbf{0}$ either, and so Equation 3.11 has nontrivial solutions. (Find some!) ♦

Notice how the back substitution in Example 3.3.5 results in zeros for the unknown s_i values. The same procedure could prove all the following statements.

Theorem 3.3.1. ***(Independence of the Columns of Various Matrices)***

1. *The columns of every upper triangular matrix with nonzero diagonal entries are independent.*
2. *The basic columns of every echelon matrix (that is, the columns containing the pivots) are independent, and its nonzero rows are too.*
3. *If A is a square matrix, then its columns are independent if and only if A is invertible; and the same holds for the rows as well.*

Proof. If U is an upper triangular matrix with nonzero diagonal entries, then we can solve $U\mathbf{s} = \mathbf{0}$ by back substitution from the bottom up, and get the unique solution $\mathbf{s} = \mathbf{0}$, which means that the columns of U are independent. Alternatively, we can prove this fact by noting that an upper triangular U is also an echelon matrix without free columns, and so the obvious solution $\mathbf{s} = \mathbf{0}$ of the equation $U\mathbf{s} = \mathbf{0}$ must be unique. Similarly, the basic columns of every echelon matrix form an echelon matrix E without free columns, which again implies that the columns of E are independent.

For the rows of U, we have to solve from the top down. Letting $\mathbf{u}^i$, for $i = 1, \ldots, r$, denote the rows of U, we test their independence by solving

$$\sum_{i=1}^{r} s_i \mathbf{u}^i = \mathbf{0}. \tag{3.19}$$

If U looks like the U in Equation 2.34, then the first component of Equation 3.19 is $s_1p_1 = 0$, whence $s_1 = 0$. The second component will be $s_1 \cdot * + s_2p_2 = 0$, and so $s_2 = 0$. And so on.

Corollary 2.5.1 on page 92 shows that the columns of A are independent if and only if A is invertible. Applying this condition to A^T in place of A, we find that the rows of a square matrix A are independent if and only if A^T is invertible, which by the second half of Theorem 2.5.8 is equivalent to the invertibility of A. ■

In the next theorem we present a characterization of linear dependence for later use. We do this here because it also provides good practice with the basic concepts.

Theorem 3.3.2. ***(Dependence of Vectors of a List and Reducing a Spanning Set).*** *Any two or more nonzero vectors* $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ *in a vector space* X *are linearly dependent if and only if one of the vectors, say* $\mathbf{a}_k$*, for some* $k \geq 2$*, equals a linear combination of the previous vectors of the list.*[7] *Also, if some* $\mathbf{a}_k$ *is equal to such a linear combination, then* $\mathrm{Span}(\mathcal{A} - \{\mathbf{a}_k\}) = \mathrm{Span}(\mathcal{A})$*, where* $\mathcal{A} = \{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$ *and* $\mathcal{A} - \{\mathbf{a}_k\}$ *denotes the set of* $\mathbf{a}_i$ *vectors for all* $i \neq k$.

Proof. Suppose

$$\mathbf{a}_k = \sum_{i=1}^{k-1} s_i \mathbf{a}_i. \tag{3.20}$$

Then

$$\sum_{i=1}^{k-1} s_i \mathbf{a}_i + (-1)\mathbf{a}_k + \sum_{i=k+1}^{n} 0\mathbf{a}_i = \mathbf{0} \tag{3.21}$$

provides a nontrivial decomposition of $\mathbf{0}$, which proves the dependence of the vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$.

Conversely, if the $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ are linearly dependent, then

$$\sum_{i=1}^{n} s_i \mathbf{a}_i = \mathbf{0} \tag{3.22}$$

for some coefficients s_i, not all zero. Let s_k be the nonzero coefficient with largest index. Then we must have $k \geq 2$, since otherwise only s_1 would be nonzero and $s_1\mathbf{a}_1 = \mathbf{0}$ would lead to the contradictory fact $s_1 = 0$, because $\mathbf{a}_1$ was assumed to be nonzero. Multiplying both sides of Equation 3.22 by $1/s_k$, we get

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[7] We call a finite sequence or ordered n-tuple briefly a list, in contrast to a set, which is unordered.

$$\sum_{i=1}^{k-1} \frac{s_i}{s_k}\mathbf{a}_i + \mathbf{a}_k + \sum_{i=k+1}^{n} 0\mathbf{a}_i = \mathbf{0}, \tag{3.23}$$

where the last sum is empty if $k = n$. Thus,

$$\mathbf{a}_k = \sum_{i=1}^{k-1} \left(-\frac{s_i}{s_k}\mathbf{a}_i\right), \tag{3.24}$$

which expresses $\mathbf{a}_k$ as a linear combination of the preceding vectors in the list.

To prove the last statement of the theorem, let $\mathbf{a}_k$ be a linear combination as in Equation 3.20. If $\mathbf{b}$ is any vector in $\mathrm{Span}(\mathcal{A})$, then

$$\mathbf{b} = \sum_{i=1}^{n} t_\iota \mathbf{a}_i. \tag{3.25}$$

Eliminating the kth term here by using Equation 3.20 gives $\mathbf{b}$ as a linear combination of the vectors of $\mathcal{A} - \{\mathbf{a}_k\}$. Conversely, if $\mathbf{b}$ is any linear combination of the vectors of $\mathcal{A} - \{\mathbf{a}_k\}$, then it is in $\mathrm{Span}(\mathcal{A})$ as well. Thus $\mathrm{Span}\left(\mathcal{A} - \{\mathbf{a}_k\}\right) = \mathrm{Span}(\mathcal{A})$. ■

Exercises

Exercise 3.3.1. Prove that three vectors in every vector space are dependent if and only if one is a linear combination of the other two. (*Hint*: Imitate the proof in Example 3.3.3.)

***Exercise 3.3.2.** Show directly that, in every vector space X, if the decomposition of a vector $\mathbf{b}$ into a linear combination of given vectors $\mathbf{a}_i$ is not unique, then the decomposition of every other decomposable vector $\mathbf{c}$ is also not unique. (*Hint*: Write $\mathbf{c} = \mathbf{c} + \mathbf{b} - \mathbf{b}$ with different decompositions for the two $\mathbf{b}$ vectors.)

In the next four exercises write the vectors $\mathbf{b}$ as linear combinations of the vectors $\mathbf{a}_i$ if possible.

Exercise 3.3.3. $\mathbf{b} = (7, 32, 16, -3)^T$, $\mathbf{a}_1 = (4, 7, 2, 1)^T$, $\mathbf{a}_2 = (4, 0, -3, 2)^T$, $\mathbf{a}_3 = (1, 6, 3, -1)^T$.

Exercise 3.3.4. $\mathbf{b} = (7, 16, -3)^T$, $\mathbf{a}_1 = (4, 2, 1)^T$, $\mathbf{a}_2 = (4, -3, 2)^T$, $\mathbf{a}_3 = (1, 3, -1)^T$.

Exercise 3.3.5. $\mathbf{b} = (7, 16, -3)^T$, $\mathbf{a}_1 = (4, 2, 1)^T$, $\mathbf{a}_2 = (4, -3, 2)^T$, $\mathbf{a}_3 = (0, 5, -1)^T$.

Exercise 3.3.6. $\mathbf{b} = (4, 7, 0)^T$, $\mathbf{a}_1 = (4, 2, 1)^T$, $\mathbf{a}_2 = (4, -3, 2)^T$, $\mathbf{a}_3 = (0, 5, -1)^T$.

In the next five exercises determine whether the given vectors are independent or not, and prove your statement.

Exercise 3.3.7. The four vectors of Exercise 3.3.4.

Exercise 3.3.8. The three vectors $\mathbf{a}_1 = (4, 2, 1)^T$, $\mathbf{a}_2 = (4, -3, 2)^T$, $\mathbf{a}_3 = (1, 3, -1)^T$ from Exercise 3.3.4.

Exercise 3.3.9. The three vectors $\mathbf{a}_1 = (4, 2, 1)^T$, $\mathbf{a}_2 = (4, -3, 2)^T$, $\mathbf{a}_3 = (0, 5, -1)^T$ from Exercise 3.3.5.

Exercise 3.3.10. $\mathbf{a}_1 = (2, 1)^T$, $\mathbf{a}_2 = (-3, 2)^T$, $\mathbf{a}_3 = (1, -1)^T$.

Exercise 3.3.11. $\mathbf{a}_1 = (1, 0, 0, 1)^T$, $\mathbf{a}_2 = (0, 0, 1, 1)^T$, $\mathbf{a}_3 = (1, 1, 0, 0)^T$, $\mathbf{a}_4 = (1, 0, 1, 1)^T$.

Exercise 3.3.12. Let $\mathbf{a}_1$ and $\mathbf{a}_2$ be independent vectors in a vector space X, and $\mathbf{v}$ another vector of X, not in $\text{Span}\{\mathbf{a}_1, \mathbf{a}_2\}$. Prove, using Definition 3.3.1, that the vectors $\mathbf{v}, \mathbf{a}_1, \mathbf{a}_2$ are independent of each other. (*Hint*: Prove this indirectly by assuming that the three vectors are dependent and showing that that assumption leads to a contradiction.)

Exercise 3.3.13. State three different ways of characterizing the linear independence of n vectors.

Exercise 3.3.14. Prove that in every vector space a finite set of vectors that contains the zero vector is a dependent set.

Exercise 3.3.15. Prove that every set of more than three vectors in $\mathbb{R}^3$ is a dependent set. (*Hint*: Consider the $3 \times n$ matrix A with the given vectors as columns and apply Theorem 2.2.1 on page 55 to the equation $A\mathbf{s} = \mathbf{0}$.)

Exercise 3.3.16. Prove that every set of three independent vectors in $\mathbb{R}^3$ spans $\mathbb{R}^3$. (*Hint*: Consider the matrix A with the given vectors as columns and apply Theorem 2.5.5 on page 93.)

Exercise 3.3.17. Prove that any three vectors in $\mathbb{R}^3$ that span $\mathbb{R}^3$ are independent. (*Hint*: Consider the matrix A with the given vectors as columns and apply Theorem 2.5.5 on page 93.)

Exercise 3.3.18. Prove that a square matrix is singular if and only if its columns are dependent. (*Hint*: Apply Theorem 2.5.5 on page 93.)

***Exercise 3.3.19.** Prove that in $\mathbb{R}^n$ a set of m vectors $\mathbf{a}_i$ is independent if and only if $0 < m \leq n$ and the matrix A with the given vectors as columns has rank m.

***Exercise 3.3.20.** Prove that in $\mathbb{R}^n$ a set of m vectors $\mathbf{a}_i$ spans $\mathbb{R}^n$ if and only if $0 < n \leq m$ and the matrix A with the given vectors as columns has rank n.

MATLAB Exercises

Exercise 3.3.21. Solve Exercise 3.3.5 using MATLAB. (*Hint*: Reduce the associated system $\mathbf{b} = \sum_{i=1}^{n} s_i \mathbf{a}_i$ using **rref.**)

Exercise 3.3.22. Solve Exercise 3.3.6 using MATLAB.

Exercise 3.3.23. Use MATLAB to determine whether the four vectors of Exercise 3.3.3 span $\mathbb{R}^3$ or not:
a. by using the fact that the given vectors span $\mathbb{R}^3$ if and only if for their matrix A the equations $A\mathbf{x} = \mathbf{e}_i$ have a solution for each i.
b. and alternatively by using the MATLAB command **rank**(A). Explain.

Exercise 3.3.24. Solve Exercise 3.3.11 using MATLAB. Explain.

Exercise 3.3.25. Use MATLAB to find a spanning set for the solution space of $A\mathbf{x} = \mathbf{0}$, where

$$A = \begin{bmatrix} 1 & -2 & 1 & 0 \\ 3 & 4 & 0 & 1 \\ -1 & 0 & 2 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}.$$

(*Hint*: Find the general solution of $A\mathbf{x} = \mathbf{0}$ as in Example 2.3.1 on page 61.)

3.4 Bases

In any vector space or subspace, we are frequently interested in finding a minimal set of vectors whose linear combinations make up the space, as described below.

Definition 3.4.1. ***(Basis).*** *A finite subset $\mathcal{B}$ of a vector space X is called a basis for X if*
1. $\mathcal{B}$ spans X, and
2. $\mathcal{B}$ is a set of independent vectors.[8, 9]

[8] By a slight but useful abuse of language it is customary to say that the *set* $\mathcal{B}$ is independent and spans X rather than just that its vectors are and do so.

[9] There are many interesting, so-called infinite-dimensional vector spaces that do not possess finite spanning sets, that is, have no basis in the sense above. The notion of a basis can, however, be generalized in various ways to cover such spaces as well but, except for a few elementary examples, those are beyond the scope of this book.

Example 3.4.1. (Standard Basis). The standard vectors $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ form a basis, called the *standard basis* for $\mathbb{R}^n$, since (1) every vector in $\mathbb{R}^n$ can be written as a linear combination of these vectors in the usual way as $\mathbf{a} = \sum_{i=1}^{n} a_i \mathbf{e}_i$, and (2) the $\mathbf{e}_i$ are independent since their matrix is the unit matrix and $I\mathbf{s} = \mathbf{0}$ implies $\mathbf{s} = \mathbf{0}$. ♦

Example 3.4.2. (A Basis for $\mathbb{R}^3$). The columns of the matrix A of Example 3.3.5 on page 112 form a basis for $\mathbb{R}^3$, since (1) the same row operations as in Example 3.3.5 solve the equation $A\mathbf{s} = \mathbf{b}$ for every $\mathbf{b}$, and (2) the columns are independent as proved in Example 3.3.5. ♦

As in the last example and some earlier ones, we are often interested in the subspace generated by the columns of a matrix, and we give it a name.

Definition 3.4.2. ***(Column Space).*** *The subspace of $\mathbb{R}^m$ spanned by the columns of an $m \times n$ matrix A is called the column space of A and is denoted by* $\mathrm{Col}(A)$.

Example 3.4.3. (A Basis for a Column Space). Consider the columns of the matrix

$$A = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 6 & 2 \\ 0 & -1 & 1 \end{bmatrix}. \tag{3.26}$$

This matrix can be reduced to the echelon matrix

$$U = \begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}. \tag{3.27}$$

By Theorem 3.3.1 of page 113, the first two columns of U are independent and may therefore serve as a basis for $\mathrm{Col}(U)$. Indeed,

$$s_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + s_2 \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \tag{3.28}$$

is solved by $s_2 = -1$ and $s_1 = 4$. Hence

$$4 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \tag{3.29}$$

and this equation exhibits the last column of U as a linear combination of the first two. Consequently, in every linear combination of all three columns of U the last one can be eliminated by substituting its expression from Equation 3.29.

We were, however, interested in the column space of A rather than that of U, and now we can easily find a basis for that too, once we have found a basis for Col(U). We just have to undo in Equation 3.29 the row operations that have led from A to U. Thus, interchange the last two rows of 3.29, multiply the new last row by -1, and add two times the first row to the second row. This procedure results in

$$4\begin{bmatrix}1\\2\\0\end{bmatrix} - \begin{bmatrix}3\\6\\-1\end{bmatrix} = \begin{bmatrix}1\\2\\1\end{bmatrix}. \tag{3.30}$$

Hence the last column of A is the same linear combination of the first two columns of A as was the case for the columns of U. The same argument with the right-hand side of Equation 3.28 replaced by the zero vector shows that the first two columns of A are also independent. Thus, those columns form a basis for Col(A).

Let us remark that had we carried the reduction of A further, the coefficients $s_1 = 4$ and $s_2 = -1$ would have appeared in the reduced echelon matrix

$$R = \begin{bmatrix}1 & 0 & 4\\0 & 1 & -1\\0 & 0 & 0\end{bmatrix}, \tag{3.31}$$

obtainable from Equation 3.27 by subtracting three times the second row of U from its first row. In this matrix it is obvious that the last column equals four times the first column minus the second column. ♦

The procedure of this example can be generalized for every matrix and leads to the following theorem.

Theorem 3.4.1. ***(Finding a Basis for*** Col(A)***).*** *We can find a basis for the column space of an $m \times n$ matrix A by reducing A to an echelon matrix U and taking as basis vectors those columns of A that correspond to the basic columns of U.*

Proof. If U is obtained from A by row reduction, then the equation $U\mathbf{s} = \mathbf{0}$ has exactly the same set of solutions as $A\mathbf{s} = \mathbf{0}$. Therefore the columns of A are related by the same linear combinations as are the columns of U. For instance, if we set $s_i = 0$ for all free variables in $U\mathbf{s} = \mathbf{0}$, then, by the second statement of Theorem 3.3.1, all the basic variables must also be zero. This shows that the basic columns of U are independent, and then so are the corresponding columns of A as well. We call these the *basic columns of* A.

Similarly, every linear combination showing the dependence of the nonbasic columns of U on the basic ones has its counterpart, *with the same coefficients*, for the corresponding columns of A. Furthermore, if $A\mathbf{s} = \mathbf{b}$ is any element of the column space of A and $U\mathbf{s} = \mathbf{c}$ is the corresponding reduced form, then we can eliminate the nonbasic columns of U and write $\mathbf{c}$

as a linear combination of the basic columns of U; then $\mathbf{b}$ will be a linear combination, with the same coefficients, of the basic columns of A. Thus, the basic columns of A are independent and span $\mathrm{Col}(A)$. ■

Example 3.4.4. (Illustration of the Proof Above). Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 6 & 6 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 2 \\ 5 \\ 9 \end{bmatrix}. \tag{3.32}$$

Then the corresponding equation $A\mathbf{s} = \mathbf{b}$ can be written as

$$A\mathbf{s} = s_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + s_2 \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} + s_3 \begin{bmatrix} 3 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 9 \end{bmatrix}, \tag{3.33}$$

and can be reduced by subtracting $2\mathbf{r}_1$ from $\mathbf{r}_2$ and $3\mathbf{r}_1$ from $\mathbf{r}_3$ to

$$s_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + s_2 \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + s_3 \begin{bmatrix} 3 \\ -1 \\ -3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \tag{3.34}$$

and further to

$$U\mathbf{s} = s_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + s_2 \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + s_3 \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}. \tag{3.35}$$

Thus, an echelon matrix corresponding to A is

$$U = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \tag{3.36}$$

with basic columns $\mathbf{u}_1$ and $\mathbf{u}_3$, and the vector $\mathbf{b}$ is changed to $\mathbf{c} = (2, 1, 0)^T$. Hence the variable s_2 is free and we set $s_2 = t$. Then, from Equation 3.35, $s_3 = -1$ and $s_1 + 2t - 3 = 2$, and so $s_1 = 5 - 2t$. Combining these components, we can write the general solution of $U\mathbf{s} = \mathbf{c}$ as

$$\mathbf{s} = \begin{bmatrix} 5 \\ 0 \\ -1 \end{bmatrix} + t \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} = \mathbf{s}_0 + t\mathbf{v}_1, \tag{3.37}$$

where $\mathbf{s}_0 = (5, 0, -1)^T$ and $\mathbf{v}_1 = (-2, 1, 0)^T$. Indeed,

$$U\mathbf{s}_0 = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 5 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \mathbf{c} \tag{3.38}$$

and

$$U\mathbf{v}_1 = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \mathbf{0}, \tag{3.39}$$

or equivalently,

$$U\mathbf{s} = (5 - 2t) \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + (-1) \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \mathbf{c}. \tag{3.40}$$

The vector $\mathbf{s}$ is also the solution of $A\mathbf{s} = \mathbf{b}$ and so, corresponding to the equations above, we have

$$A\mathbf{s}_0 = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 6 & 6 \end{bmatrix} \begin{bmatrix} 5 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 9 \end{bmatrix} = \mathbf{b} \tag{3.41}$$

and

$$A\mathbf{v}_1 = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 6 & 6 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \mathbf{0}, \tag{3.42}$$

or equivalently,

$$A\mathbf{s} = (5 - 2t) \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + t \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} + (-1) \begin{bmatrix} 3 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 9 \end{bmatrix} = \mathbf{b}. \tag{3.43}$$

Notice also that $\mathbf{s}_0 = (5, 0, -1)^T$ is a particular solution of the inhomogeneous equation $A\mathbf{s} = \mathbf{b}$ and $t\mathbf{v}_1 = t\,(-2, 1, 0)^T$ is the general solution of the corresponding homogeneous equation $A\mathbf{s} = \mathbf{0}$, as required by Theorem 2.3.3.

Furthermore, $\mathbf{u}_1$ and $\mathbf{u}_3$ being the basic columns of U, the corresponding columns $\mathbf{a}_1$ and $\mathbf{a}_3$ are the basic columns of A and the set

$$\mathcal{B} = \{\mathbf{a}_1, \mathbf{a}_3\} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \\ 6 \end{bmatrix} \right\} \tag{3.44}$$

is a basis for $\operatorname{Col}(A)$. ♦

The argument of the proof of Theorem 3.4.1 can be reversed in a sense, to characterize echelon matrices.

Theorem 3.4.2. ***(Uniqueness of the Reduced Echelon Matrix).*** *The row-reduced echelon matrix R of every matrix A is unique, and all other* *echelon forms of A have nonzero numbers as their pivots in the same locations as R has its pivots.*

Proof. If A is a zero matrix, then $R = A$. Otherwise, proceeding from left to right, the first nonzero column of A is its first basic column. The second basic column is the next column that is independent of the first basic column, and we can find the other basic columns one after the other as those columns of A that are independent of the earlier basic columns. Since every basic column of a reduced echelon matrix R must be a standard vector, R must be the matrix of the same shape as A that has the standard vector $\mathbf{e}_k$ for its kth basic column. Also, by the proof of Theorem 3.4.1, each $\mathbf{e}_k$ must be in the same position as the kth basic column of A, for $k = 1, 2, \ldots, r$. Furthermore, the nonbasic columns of R must be the same linear combinations of the $\mathbf{e}_k$ vectors as the corresponding nonbasic columns of A are of the basic columns of A. Thus, this construction gives a unique reduced echelon matrix R for any given A.

Again by the proof of Theorem 3.4.1, every other echelon form U of A must have its basic columns in the same positions as R, and since multiplying any nonzero row of an echelon matrix by an arbitrary nonzero number leaves it in echelon form, the pivots of U must be in the same columns as those of R and can be arbitrary nonzero numbers. ∎

The column space of a matrix has an important application in the theory of linear systems.

Theorem 3.4.3. ***(A Condition for the Consistency of a Linear System).*** *The equation $A\mathbf{x} = \mathbf{b}$ is consistent if and only if the vector $\mathbf{b}$ lies in the column space of A.*

Proof. Since the expression $A\mathbf{x}$ equals the linear combination $\sum_{i=1}^{n} x_i \mathbf{a}_i$ of the columns of the matrix A, it is in the column space of A for every $\mathbf{x}$; and so it can equal a vector $\mathbf{b}$ if and only if $\mathbf{b}$ is in that column space too. ∎

We can deal with the rows of a matrix much as with the columns.

Definition 3.4.3. ***(Row Space).*** *The subspace of $\mathbb{R}^n$ spanned by the transposed*[10] *rows of an $m \times n$ matrix A is called the row space of A and is usually denoted by* $\text{Row}(A)$.

Example 3.4.5. (A Basis for a Row Space). Let us find a basis for the row space of the matrix A of Example 3.4.3:

$$A = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 6 & 2 \\ 0 & -1 & 1 \end{bmatrix}. \tag{3.45}$$

[10] This transposition of the rows is just a technicality, which makes some formulas simpler by adhering to the convention of using only column vectors.

The solution is very easy. Again, we consider the echelon matrix U corresponding to A:

$$U = \begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}. \tag{3.46}$$

By Theorem 3.3.1 of page 113 the first two rows of U are independent and, when transposed, may serve as a basis for $\mathrm{Row}(U)$. The rows of A, however, are linear combinations of the rows of U, since they can be recovered from the latter by elementary row operations, and so the first two transposed rows of U may serve as a basis for $\mathrm{Row}(A)$ as well. In fact, $\mathrm{Row}(A)$ is the same as $\mathrm{Row}(U)$. Thus the set

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\} \tag{3.47}$$

forms a basis for $\mathrm{Row}(A)$. It is easy to check that these two vectors are independent and the transposed rows of A are linear combinations of them. ♦

The procedure of the preceding example can again be generalized for any matrix and leads to the following theorem.

Theorem 3.4.4. ***(Finding a Basis for*** $\mathrm{Row}(A)$***).*** *We can find a basis for the row space of an $m \times n$ matrix A by reducing A to an echelon matrix U and taking as basis vectors the transposed nonzero rows of U.*[11]

The row and column spaces of a matrix are related to those of its transpose.

Theorem 3.4.5. ***(Row and Column Space of A^T).*** *For every matrix A,*

$$\mathrm{Row}(A) = \mathrm{Col}(A^T) \tag{3.48}$$

and

$$\mathrm{Col}(A) = \mathrm{Row}(A^T). \tag{3.49}$$

Proof. For any $m \times n$ matrix A, let $\mathbf{s} = (s_1, \ldots, s_m)^T$ be any vector in $\mathbb{R}^m$, and write $\mathbf{a}^i$ for the ith *row* of A. Then every vector $\mathbf{x} \in \mathrm{Row}(A)$ can be written as

$$\mathbf{x} = \left(\sum_{i=1}^{m} s_i \mathbf{a}^i \right)^T = \sum_{i=1}^{m} s_i \left(\mathbf{a}^i\right)^T = \sum_{i=1}^{m} s_i \left(A^T\right)_i = A^T \mathbf{s}, \tag{3.50}$$

[11] Note that in this case the transposed rows of U themselves form a basis of the row space rather than the corresponding transposed rows of A, in contrast to the case for columns. (See Exercise 3.4.5.) The reason for the asymmetry lies in our use of *row* operations for the reduction to echelon form.

where $\left(A^T\right)_i$ stands for the ith column of A^T. Since the expression on the right represents an arbitrary vector in the column space of A^T, we find that

$$\text{Row}(A) = \text{Col}(A^T), \tag{3.51}$$

and interchanging the roles of A and A^T, that

$$\text{Row}(A^T) = \text{Col}(A). \tag{3.52}$$

■

Applying Theorem 3.4.4 to A^T in place of A and using Equation 3.52, we obtain a new way of computing a basis for Col(A).

Corollary 3.4.1. ***(Using*** A^T ***to Find a Basis for*** Col(A)***).*** *We can find a basis for the column space of an* $m \times n$ *matrix* A *by reducing* A^T *to an echelon matrix* U *and taking as basis vectors the transposed nonzero rows of* U*.*

We have encountered still another subspace associated with a matrix: its null space (see Definition 3.2.3). How do we find a basis for it? The procedure is straightforward. We solve $A\mathbf{x} = \mathbf{0}$ in the usual manner by Gaussian elimination. The solution is always obtained as a linear combination of some vectors of $\mathbb{R}^n$, and these vectors are easily shown to be independent. They form a basis for Null(A), which is thus a subspace of $\mathbb{R}^n$. Let us look at an example.

Example 3.4.6. (A Basis for a Null Space). Let

$$A = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 4 & 2 & 6 & 6 \\ 8 & 2 & 8 & 6 \end{bmatrix}. \tag{3.53}$$

This matrix can be reduced to the echelon form

$$U = \begin{bmatrix} 2 & 1 & 3 & 3 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{3.54}$$

The solution of $A\mathbf{x} = \mathbf{0}$ can be obtained by back substitution from the matrix U as $x_3 = s_1,\ x_4 = s_2,\ x_2 = -2s_1 - 3s_2$, and $x_1 = -s_1/2$. In vector form we may write

$$\mathbf{x} = s_1 \begin{bmatrix} -1/2 \\ -2 \\ 1 \\ 0 \end{bmatrix} + s_2 \begin{bmatrix} 0 \\ -3 \\ 0 \\ 1 \end{bmatrix}. \tag{3.55}$$

Thus Null(A) is spanned by the two vectors on the right of the last equation. Furthermore, those are also independent, because if we set $\mathbf{x} = \mathbf{0}$, then the last two rows of Equation 3.55 correspond to the equations $1s_1 + 0s_2 = 0$ and $0s_1 + 1s_2 = 0$, from which $s_1 = s_2 = 0$ follows. Thus the set

$$\mathcal{B} = \left\{ \begin{bmatrix} -1/2 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\} \tag{3.56}$$

is a basis for $\text{Null}(A)$. Writing

$$X = \begin{bmatrix} -1/2 & 0 \\ -2 & -3 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \tag{3.57}$$

we can rephrase this result as saying that we have found a matrix X with independent columns such that $\text{Col}(X) = \text{Null}(A)$. ♦

The preceding example suggests the general way for finding a basis for the nullspace of a matrix.

Theorem 3.4.6. ***(A Basis for* Null (*A*)*).*** *For every $m \times n$ matrix A, with rank $r < n$ we can find a basis for* $\text{Null}(A)$ *by finding the general solution of $A\mathbf{x} = \mathbf{0}$ by Gaussian (or Gauss–Jordan) elimination in the form $\mathbf{x} = \sum_{i=1}^{n-r} s_i \mathbf{v}_i$. Then the $n - r$ vectors $\mathbf{v}_i$ form a basis for* $\text{Null}(A)$. *If $r = n$, then* $\text{Null}(A) = \{\mathbf{0}\}$ *and we say that its basis is the empty set.*

Proof. In Gaussian or Gauss–Jordan elimination, if $r < n$, we assign parameters s_i to each of the $n - r$ free variables and obtain the r basic variables as linear combinations of them. Hence we can write every solution vector in the form $\mathbf{x} = \sum_{i=1}^{n-r} s_i \mathbf{v}_i$, which shows that the vectors $\mathbf{v}_i$, for $i = 1, 2, \ldots, n - r$, span $\text{Null}(A)$. Now, each $\mathbf{v}_i$ vector has the component 1 in the position corresponding to the number of the column to which the parameter s_i belongs. (In the example above, the parameter s_1 belonged to column 3 and the parameter s_2 to column 4. Thus, $\mathbf{v}_1$ has a 1 in the third position and $\mathbf{v}_2$ in the fourth position.) Also, each $\mathbf{v}_i$ has 0 entries everywhere else in the rows of these 1's. Thus the equation $\sum_{i=1}^{n-r} s_i \mathbf{v}_i = \mathbf{0}$ has a row saying $s_i = 0$ for each i; that is, its only solution for the s_i is the trivial one and so the $\mathbf{v}_i$ vectors are independent. ■

The next theorem provides an alternative characterization of independence of a set of vectors as a minimal spanning set.

Theorem 3.4.7. ***(A Condition for Independence in Terms of Spanning Sets).*** *In any real vector space X, let the nonempty finite subset $\mathcal{A} = \{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$ of nonzero vectors span the subspace V. Then $\mathcal{A}$ is independent if and only if every spanning set of V has at least n elements.*

Proof. We prove both parts indirectly.
1. Suppose $\mathcal{A}$ is dependent, that is, $n \geq 2$ and $\sum_{i=1}^{n} s_i \mathbf{a}_i = \mathbf{0}$ with some $s_i \neq 0$. Say, $s_n \neq 0$. Then $\mathbf{a}_n = -\sum_{i=1}^{n-1} \frac{s_i}{s_n} \mathbf{a}_i$, and so in any linear combination of $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ the vector $\mathbf{a}_n$ can be eliminated. Thus the set $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_{n-1}\}$ spans V but has fewer than n elements.
2. Suppose $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_m\}$ with $m < n$ also spans V. Then we can write

$$\mathbf{a}_i = \sum_{j=1}^{m} a_{ij} \mathbf{b}_j \text{ for all } i = 1, 2, \ldots, n, \tag{3.58}$$

with appropriate coefficients a_{ij}. Write A for the $n \times m$ matrix (a_{ij}).

Test the $\mathbf{a}_i$ vectors for independence: We have

$$\sum_{i=1}^{n} x_i \mathbf{a}_i = \sum_{i=1}^{n} x_i \sum_{j=1}^{m} a_{ij} \mathbf{b}_j = \sum_{j=1}^{m} \left(\sum_{i=1}^{n} x_i a_{ij} \right) \mathbf{b}_j = \mathbf{0} \tag{3.59}$$

if

$$\sum_{i=1}^{n} x_i a_{ij} = 0 \text{ for all } j = 1, 2, \ldots, m. \tag{3.60}$$

The coefficient matrix A^T of this system is $m \times n$, with $m < n$. Hence its rank satisfies $r \leq m$, and so the number of free variables in Equation 3.60 satisfies $n - r \geq n - m > 0$. Thus Equation 3.59 has nontrivial solutions, which shows that $\mathcal{A}$ cannot be independent if such a $\mathcal{B}$ exists. ■

Call a finite nonempty spanning set $\mathcal{A}$ of a subspace V, if one exists, minimal if its cardinality[12] is less than or equal to the cardinality of every other spanning set of V. Then we have the following obvious corollaries.

Corollary 3.4.2. ***(Bases are Minimal Spanning Sets).*** *In any vector space X, every basis of a finitely generated nonzero subspace V is a minimal spanning set of V and vice versa. The cardinality of every basis of such a subspace V equals the minimum of the cardinalities of the spanning sets of V.*

Corollary 3.4.3. ***(Bases are Maximal Independent Sets).*** *In any vector space X, every basis of a finitely generated nonzero subspace V is a maximal independent subset of V and vice versa. The cardinality of every basis of such a subspace V equals the maximum of the cardinalities of the independent subsets of V.*

Since every subspace U of $\mathbb{R}^m$ is the column space of some $m \times n$ matrix, Theorem 3.4.1 can be interpreted as giving a procedure for the *reduction* of

[12] The cardinality of a finite set means the number of its elements.

a spanning set of a subspace U of $\mathbb{R}^m$ to a basis of U. The same construction can also be used to solve the related problem of *extending* an independent set $\{b_1, b_2, \ldots, b_k\}$ in a subspace U to a basis. First, however, we want to prove a theorem that shows that such an extension is always possible. This Exchange Theorem will also be used to conclude that all bases in a vector space have the same number of vectors.

Theorem 3.4.8. **(The Exchange Theorem). In any vector space X, let $A = (\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n)$ be a list of nonzero vectors that span X, and $B = (\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_k)$ a list of independent vectors of X. Then $k \leq n$ holds, and k of the spanning vectors a_i can be exchanged for the vectors of B. That is, X is spanned by the k vectors of B together with some $n - k$ vectors of A.*

Proof. Consider the vectors $\mathbf{b}_1, \mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$. They span X since the $\mathbf{a}_i$ themselves do, and adjoining $\mathbf{b}_1$ to the list does not change that. Furthermore, these vectors are linearly dependent since $\mathbf{b}_1$ can certainly be expressed as a linear combination of the spanning vectors $\mathbf{a}_i$, say as $\mathbf{b}_1 = \sum_{i=1}^n s_i \mathbf{a}_i$. Now, by Exercise 3.3.14, the independence of $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_k$ implies that $\mathbf{b}_1 \neq \mathbf{0}$, and so $\mathbf{b}_1 - \sum_{i=1}^n s_i \mathbf{a}_i = \mathbf{0}$ has a nontrivial linear combination on the left, which shows that the vectors $\mathbf{b}_1, \mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ are linearly dependent. Thus, we may apply Theorem 3.3.2 from page 114 to this list and omit one of the $\mathbf{a}_i$ vectors so that the remaining n vectors will still span X. Call the remaining $\mathbf{a}_i$ vectors $\mathbf{a}'_1, \mathbf{a}'_2, \ldots, \mathbf{a}'_{n-1}$.

Next, consider the list $\mathbf{b}_1, \mathbf{b}_2, \mathbf{a}'_1, \mathbf{a}'_2, \ldots, \mathbf{a}'_{n-1}$ of vectors, which are linearly dependent just as $\mathbf{b}_1, \mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ above, and apply Theorem 3.3.2 from page 114 to this list. Accordingly, one of the vectors $\mathbf{a}'_i$ can be omitted so that the remaining n vectors will still span X. (The omitted vector cannot be $\mathbf{b}_2$ because $\mathbf{b}_1$ and $\mathbf{b}_2$ were assumed to be independent and the omitted vector must depend on the previous vectors in the list.)

We can proceed similarly with the rest of the vectors of B, exchanging an $\mathbf{a}_i$ for a $\mathbf{b}_j$ in each step, until we exhaust A or B. If A were exhausted first, that is, if we had $k > n$, then at some point all vectors of A would be exchanged for the first n vectors of B, and the vectors $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n$ would span X. But then $\mathbf{b}_{n+1}$ (as any other vector) would be a linear combination of $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n$, and this fact contradicts the assumed independence of the $\mathbf{b}_j$ vectors. Thus A cannot be exhausted first, that is, we must have $k \leq n$, and in that case all the vectors of B can be brought into the spanning set. ∎

While this theorem ensures that an independent set can be extended to a basis, it does not give a practical method for doing so. But, in $\mathbb{R}^m$, the construction described in Theorem 3.4.1 does so, as follows. Suppose U is given as the span of $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$, and $\{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_k\}$ is an independent set in U with $k < n$. Then we construct, according to Theorem 3.4.1, a basis for the column space of the matrix $C = (\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_k, \mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n)$. Since $U = \mathrm{Col}(C)$, and the row reduction proceeds from left to right, the independent vectors $\mathbf{b}_j$ will be in the basis of U so found. Let us see an example.

Example 3.4.7. (Extending an Independent Set to a Basis). Let

$$U = \mathrm{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}, \tag{3.61}$$

where we call the columns on the right $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$, and let

$$\mathbf{b}_1 = \begin{bmatrix} 3 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \ \mathbf{b}_2 = \begin{bmatrix} 0 \\ 3 \\ 1 \\ 1 \end{bmatrix}. \tag{3.62}$$

It is easy to see that $\mathbf{b}_1 = 2\mathbf{a}_1 + \mathbf{a}_2 - \mathbf{a}_3$ and $\mathbf{b}_2 = -\mathbf{a}_1 + \mathbf{a}_2 + 2\mathbf{a}_3$, and so both $\mathbf{b}_1$ and $\mathbf{b}_2$ are in U and are clearly independent of each other. We want to extend the set $\{\mathbf{b}_1, \mathbf{b}_2\}$ to a basis for U.

We form the matrix

$$C = \begin{bmatrix} 3 & 0 & 1 & 1 & 0 \\ 0 & 3 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{bmatrix} \tag{3.63}$$

and reduce it to echelon form as follows:

$$\begin{bmatrix} 3 & 0 & 1 & 1 & 0 \\ 0 & 3 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 3 & 0 & 1 & 1 & 0 \\ 0 & 3 & 0 & 1 & 1 \end{bmatrix} \to$$
$$\begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & -3 & 1 & -2 & 0 \\ 0 & 3 & 0 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 0 & -3 & 1 & -2 & 0 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 3 & 0 & 1 & 1 \end{bmatrix} \to$$
$$\begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 0 & -3 & 1 & -2 & 0 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 0 & -3 & 1 & -2 & 0 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{3.64}$$

In the final form, the pivots are in the first three columns, and so the corresponding columns of C, that is, $\mathbf{b}_1, \mathbf{b}_2, \mathbf{a}_1$, form a basis for $U = \mathrm{Col}(C)$. ♦

Exercises

In the first four exercises find bases for Row(A), Col(A), Null(A).

Exercise 3.4.1. $A = \begin{bmatrix} 1 & 3 & 1 \\ 3 & 2 & 4 \\ 2 & -1 & 3 \end{bmatrix}.$

Exercise 3.4.2. $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 3 \\ 0 & 2 & 4 \end{bmatrix}.$

Exercise 3.4.3. $A = \begin{bmatrix} 1 & 3 & 1 \\ 3 & 2 & 4 \\ 2 & -1 & 3 \\ 0 & 1 & 1 \end{bmatrix}.$

Exercise 3.4.4. $A = \begin{bmatrix} 1 & 1 & 1 & 2 & -1 \\ 2 & 2 & 2 & 4 & -2 \\ 3 & 3 & 0 & 4 & 4 \end{bmatrix}.$

Exercise 3.4.5. Find a 3×3 matrix A whose first two rows transposed do not form a basis for Row(A), but for which only the first two transposed rows of any corresponding echelon matrix U do.

Exercise 3.4.6. Show that for the matrix of Exercise 3.4.1 the first two columns of any corresponding echelon matrix U do not form a basis for Col(A), but only the first two columns of A itself do.

Exercise 3.4.7. Determine whether each of the following vectors is in the column space of the matrix of Exercise 3.4.1, and if it is, then write it as a linear combination of the first two columns: $\mathbf{a} = (1, 4, 3)^T$, $\mathbf{b} = (-10, 1, 7)^T$, $\mathbf{c} = (9, -5, -10)^T$, $\mathbf{d} = (5, 9, 4)^T$.

***Exercise 3.4.8.** Show that for two matrices A and B we have $AB = O$ if and only if Col(B) is a subspace of Null(A).

Exercise 3.4.9. Show that for any two matrices A and B such that AB exists and A is invertible we have Null(B) = Null(AB).

Exercise 3.4.10. Prove that in $\mathbb{R}^n$ any set of n independent vectors forms a basis. (*Hint*: Either consider the matrix A with the given vectors as columns and apply Theorem 2.5.5 on page 93 or use the Exchange Theorem.)

***Exercise 3.4.11.** Prove that in $\mathbb{R}^n$ no set of fewer than n vectors spans $\mathbb{R}^n$. (*Hint*: Use the Exchange Theorem.)

***Exercise 3.4.12.** Prove that in $\mathbb{R}^n$ any set of n vectors that span $\mathbb{R}^n$ forms a basis. (*Hint*: Either consider the matrix A with the given vectors as columns and apply Theorem 2.5.5 on page 93 or use the result of Exercise 3.4.11.)

Exercise 3.4.13. Prove that in $\mathbb{R}^n$ every set of more than n vectors is a dependent set. (*Hint*: Use either Gaussian elimination or the Exchange Theorem.)

Exercise 3.4.14. Let

$$U = \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

Check that the vectors

$$\mathbf{b}_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \ \mathbf{b}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

are in U and extend the set $\{\mathbf{b}_1, \mathbf{b}_2\}$ to a basis for U.

MATLAB Exercises

Exercise 3.4.15. Let $A = \textbf{magic}(4)$

a. Use **rref** on A and A^T to find a basis for the row space and the column space, respectively. Extract and transpose the appropriate submatrix in each case to obtain two matrices B and C whose columns form the bases. (Such a matrix is called a *basis matrix*.)

b. The command $D = \textbf{orth}(A)$ returns a basis matrix for $\text{Col}(A)$ (usually different from the one obtained by **rref**). Show that the columns of D and of the matrix C computed in Part (a) span the same space.

c. Let $U = \textbf{orth}(A')$. Show that the columns of U and of the matrix B computed in Part (a) span the same space.

d. The command $N = \textbf{null}(A)$ returns a basis matrix for $\text{Null}(A)$. Compute $B^T N$ and explain your result.

e. Compute $M = \textbf{null}(A^T * A)$ and explain your result.

Exercise 3.4.16. Repeat Exercise 3.4.15 for the matrix

$$A = \begin{bmatrix} 1 & 3 & 5 & 7 \\ 2 & 4 & 6 & 8 \\ 0 & 2 & 4 & 6 \\ 3 & 5 & 7 & 9 \end{bmatrix}.$$

Exercise 3.4.17. Let $A = \textbf{round}(10 * \textbf{rand}(3, 4) - 5)$.

a. Compute the rank of each: A, $[A, A], [A, A, A], [A; A]$, $[A; A; A]$, and $[A, A; A, A]$.

b. Repeat the above for six instances of A.
c. Do you see any patterns? Make a conjecture and prove it.

3.5 Dimension, Orthogonal Complements

We wish to define the dimension of a vector space as the number of vectors in a basis. To ensure the consistency of such a definition, we start with a theorem that says that all bases of a vector space must have the same number of vectors.

Theorem 3.5.1. ***(The Dimension Theorem).*** *If a vector space X has two bases $\mathcal{A} = \{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m\}$ and $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ with m and n positive integers, then $m = n$ must hold.*

Proof. By the definition of a basis, the set $\mathcal{A}$ is an independent set and the set $\mathcal{B}$ spans X. Hence, by Theorem 3.4.8 (the Exchange Theorem), we must have $m \le n$. Reversing the roles of $\mathcal{A}$ and $\mathcal{B}$, we find that $n \le m$ too must hold. Thus $m = n$ follows.

Alternatively, Corollary 3.4.2 also proves this theorem, because it says that all bases have the minimum cardinality of spanning sets.

We give yet another, direct proof.[13]

Since $\mathcal{A}$ and $\mathcal{B}$ span X, we can write

$$\mathbf{a}_i = \sum_{j=1}^{n} a_{ij}\mathbf{b}_j \text{ for all } i = 1, 2, \ldots, m \tag{3.65}$$

and

$$\mathbf{b}_j = \sum_{k=1}^{m} b_{jk}\mathbf{a}_k \text{ for all } j = 1, 2, \ldots, n, \tag{3.66}$$

with appropriate coefficients a_{ij} and b_{jk}. Substituting from Equation 3.66 into Equation 3.65, we get

$$\mathbf{a}_i = \sum_{j=1}^{n} a_{ij} \sum_{k=1}^{m} b_{jk}\mathbf{a}_k = \sum_{j=1}^{n}\sum_{k=1}^{m} a_{ij}b_{jk}\mathbf{a}_k \text{ for all } i = 1, 2, \ldots, m. \tag{3.67}$$

Since $\mathcal{A}$, being a basis, is an independent set, all the coefficients on the right must be 0 except the coefficient of $\mathbf{a}_i$, which must be 1. Hence,

[13] Avoiding the Exchange Lemma, by James Ford, Amer. Math. Monthly, Vol. 102 (Apr. 1995).

$$\sum_{j=1}^{n} a_{ij}b_{ji} = 1 \text{ for all } i = 1, 2, \ldots, m. \tag{3.68}$$

Thus, summing over i gives

$$\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}b_{ji} = m. \tag{3.69}$$

Similarly,

$$\sum_{j=1}^{n}\sum_{i=1}^{m} b_{ji}a_{ij} = n. \tag{3.70}$$

Now, the two double sums are rearrangements of each other and so $m = n$. ■

This theorem enables us to make the following definition.

Definition 3.5.1. ***(Dimension).*** *If a vector space X has a basis of n vectors, where n is a positive integer, then n is called the dimension of X and we write $n = \dim(X)$. The dimension of the vector space $\{\mathbf{0}\}$ is defined to be zero, and we say it has the empty set for a basis. If X has no finite basis and is not $\{\mathbf{0}\}$, then it is said to be infinite dimensional.*

Example 3.5.1. (Dimension of $\mathbb{R}^n$***).*** Not unexpectedly, for any positive integer n, the dimension of $\mathbb{R}^n$ is n. ♦

Example 3.5.2. (Dimension of a Subspace of $\mathbb{R}^3$***).*** The subspace of $\mathbb{R}^3$ given by $U = \{\mathbf{u} \in \mathbb{R}^3 | \mathbf{u} = s(1,1,1)^T + t(1,2,3)^T\}$ has dimension two since the two vectors $(1,1,1)^T$ and $(1,2,3)^T$ form a basis for U. ♦

Example 3.5.3. (Dimension of $\mathcal{P}_n$***).*** The space $\mathcal{P}_n$ of all polynomials in a variable x of degree at most n, for any positive integer n, and the zero polynomial has the set $\{1, x, x^2, \ldots, x^n\}$ for a basis, and so is $n+1$ dimensional. To prove the independence of these monomials, let

$$P_n(x) = p_0 + p_1x + \cdots + p_nx^n = 0, \tag{3.71}$$

with $p_0, p_1, \ldots, p_n \in \mathbb{R}$. Then $P_n(0) = 0$ implies $p_0 = 0$. Differentiating, we get

$$P_n'(x) = p_1 + 2p_2x + \cdots + np_nx^{n-1} = 0, \tag{3.72}$$

and so $P_n'(0) = 0$ implies $p_1 = 0$. Similarly, setting $x = 0$ in the higher derivatives proves that Equation 3.71 implies that all coefficients p_i must be zero. ♦

Example 3.5.4. (Dimension of $\mathcal{P}$). The space $\mathcal{P}$ of all polynomials in a variable x has the infinite set $\{1, x, x^2, \ldots\}$ for a basis, in the sense that any polynomial is a finite linear combination of these vectors, and their independence (which too is defined using only finite linear combinations) follows as in the preceding example. Thus $\mathcal{P}$ is an infinite-dimensional vector space. ♦

Theorem 3.5.2. ***(The Dimensions of the Subspaces Associated with a Matrix).*** *Let A be an $m \times n$ matrix with rank r. Then the dimension of its column space*[14] *and the dimension of its row space both equal r, and the dimension of its null space equals $n - r$.*

Proof. Recall that the rank of a matrix was defined on page 55 as the number of nonzero rows in the corresponding echelon form U, which is the same as the number of pivots in the latter. Since those rows transposed form a basis for the row space, we have $\dim(\text{Row}(A)) = r$. Also, since the column space of A has for a basis the columns of A corresponding to those of U with pivots, $\dim(\text{Col}(A)) = r$ holds as well. Last, Theorem 3.4.6 provides a basis with $n - r$ vectors for $\text{Null}(A)$. ■

Notice how remarkable this theorem is. Looking at any matrix of any size, with rows and columns having very little to do with each other, who would have guessed that the row and column spaces have the same dimension?

The dimension of the null space of a matrix is sometimes called its *nullity*, and part of this theorem can be stated as follows.

Corollary 3.5.1. ***(Rank + Nullity = Number of Columns).*** *Let A be any $m \times n$ matrix. Then* $\text{rank}\,(A) + \text{nullity}\,(A) = n$.

There is more to this than meets the eye. The row space and the null space of A are both subspaces of $\mathbb{R}^n$, and since their dimensions add up to n, we may, in a sense, expect *them* to add up to $\mathbb{R}^n$ as well. This is indeed the case.

Theorem 3.5.3. ***(Decomposing a Vector of $\mathbb{R}^n$ into the Sum of a Vector from $\text{Null}(A)$ and a Vector from $\text{Row}(A)$).*** *Let A be any $m \times n$ matrix and $\mathbf{x}$ any vector in $\mathbb{R}^n$. Then $\mathbf{x}$ can be decomposed uniquely into a sum of a vector $\mathbf{x}_0$ from Null(A) and a vector $\mathbf{x}_R$ from Row(A), that is, $\mathbf{x} = \mathbf{x}_0 + \mathbf{x}_R$, with $\mathbf{x}_0$ and $\mathbf{x}_R$ uniquely determined by $\mathbf{x}$. Furthermore, every vector of $\text{Null}(A)$ is orthogonal to every vector of Row(A), and so, in particular, $\mathbf{x}_0$ and $\mathbf{x}_R$ are orthogonal to each other.*

Proof. Let us start by proving the orthogonality of the vectors of $\text{Row}(A)$ to those of $\text{Null}(A)$. Assume that $\mathbf{u}$ is in $\text{Null}(A)$. Then we have $A\mathbf{u} = \mathbf{0}$; and if we write $\mathbf{a}^i$ for the ith row of A, this equation implies $\mathbf{a}^i\mathbf{u} = 0$ for each i. From

[14] The dimension of the column space is sometimes called the *column rank of A*, and the first statement of the theorem is phrased as "column rank = row rank."

this, for any linear combination of the rows, we get $\sum_{i=1}^{m} s_i \mathbf{a}^i \mathbf{u} = 0$. This can be written as $\mathbf{s}^T A\mathbf{u} = 0$, and furthermore, by the rule for the transpose of a product (which requires the reversal of the factors), also as $\mathbf{u}^T A^T \mathbf{s} = 0$. Equation 3.50 shows that $\mathbf{v} = A^T \mathbf{s}$ is an arbitrary vector of Row(A), and so we have

$$\mathbf{u}^T \mathbf{v} = 0. \tag{3.73}$$

The matrix product of the row vector $\mathbf{u}^T$ and the column vector $\mathbf{v}$ corresponds to the dot product of the two column vectors $\mathbf{u}$ and $\mathbf{v}$, and so the above equation expresses the orthogonality of any $\mathbf{u} \in$ Null(A) to any $\mathbf{v} \in$ Row(A).

Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_r\}$ be a basis for Row(A) and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_{n-r}\}$ a basis for Null(A), and $B = (\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_r)$ and $C = (\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_{n-r})$ the corresponding matrices with the given vectors as columns. Then, by Equation 3.73 applied to the basis vectors,

$$(\mathbf{c}_i)^T \mathbf{b}_j = 0 \tag{3.74}$$

holds for each i and j, which may be written in matrix form as

$$C^T B = O. \tag{3.75}$$

We want to show that the n vectors of $\mathcal{B} \cup \mathcal{C}$ form a basis for $\mathbb{R}^n$. To this end, let us test the columns of the joint matrix $(B, C) = (\mathbf{b}_1, \mathbf{b}_2, \ldots \mathbf{b}_r, \mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_{n-r})$ for independence. Assume

$$B\mathbf{s} + C\mathbf{t} = \mathbf{0} \tag{3.76}$$

for $\mathbf{s} \in \mathbb{R}^r$ and $\mathbf{t} \in \mathbb{R}^{n-r}$. Left-multiply this equation by $(C\mathbf{t})^T$ to obtain

$$(C\mathbf{t})^T B\mathbf{s} + (C\mathbf{t})^T C\mathbf{t} = 0, \tag{3.77}$$

which can also be written as

$$\mathbf{t}^T C^T B\mathbf{s} + (C\mathbf{t})^T (C\mathbf{t}) = 0. \tag{3.78}$$

By Equation 3.75 the first term is zero, and the second term equals $|C\mathbf{t}|^2$. Since $\mathbf{0}$ is the only vector whose length is zero, Equation 3.78 implies

$$C\mathbf{t} = \mathbf{0}. \tag{3.79}$$

We have, however, assumed that the columns of C form a basis for Null(A), and so Equation 3.79 has only the trivial solution $\mathbf{t} = \mathbf{0}$. Substituting $\mathbf{t} = \mathbf{0}$ into Equation 3.76 and using the independence of the columns of B, we obtain $\mathbf{s} = \mathbf{0}$ as well. Thus, Equation 3.76 has only the trivial solution $\mathbf{s} = \mathbf{0}$ and $\mathbf{t} = \mathbf{0}$, and so the columns of (B, C) are independent. Then they also form a basis for $\mathbb{R}^n$, since any set of n independent vectors in $\mathbb{R}^n$ does so. (See Exercise 3.4.10.)

Now, let us write any $\mathbf{x} \in \mathbb{R}^n$ in terms of the basis $\mathcal{B} \cup \mathcal{C}$ as

$$\mathbf{x} = B\mathbf{s} + C\mathbf{t}. \tag{3.80}$$

Then

$$\mathbf{x}_R = B\mathbf{s} \text{ and } \mathbf{x}_0 = C\mathbf{t} \tag{3.81}$$

provide the claimed decomposition. Its uniqueness follows from the uniqueness of decompositions in any basis, ensured by the independence of the basis vectors. (See the discussion preceding Definition 3.3.1.) ■

Example 3.5.5. (A Decomposition). Let us again consider the matrix of Example 3.4.3:

$$A = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 6 & 2 \\ 0 & -1 & 1 \end{bmatrix}. \tag{3.82}$$

and find the decomposition described above of the vector $\mathbf{x} = (1, 2, 3)^T$.

The echelon form

$$E = \begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \tag{3.83}$$

provides us with the basis vectors $\mathbf{b}_1 = (1, 3, 1)^T$ and $\mathbf{b}_2 = (0, 1, -1)^T$ for the row space and with the single basis vector $\mathbf{c}_1 = (-4, 1, 1)^T$ for the null space. (Why?) It is easy to check that $\mathbf{c}_1$ is orthogonal to $\mathbf{b}_1$ and $\mathbf{b}_2$, as it should be. To decompose any $\mathbf{x} \in \mathbb{R}^n$ into row space and null space components, we should solve

$$s_1\mathbf{b}_1 + s_2\mathbf{b}_2 + t_1\mathbf{c}_1 = \mathbf{x} \tag{3.84}$$

for the unknown coefficients s_1, s_2, t_1. In our case this amounts to solving the system

$$\begin{bmatrix} 1 & 0 & -4 \\ 3 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ t_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \tag{3.85}$$

which we can do in the usual way by Gaussian elimination:

$$\left[\begin{array}{ccc|c} 1 & 0 & -4 & 1 \\ 3 & 1 & 1 & 2 \\ 1 & -1 & 1 & 3 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -4 & 1 \\ 0 & 1 & 13 & -1 \\ 0 & -1 & 5 & 2 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -4 & 1 \\ 0 & 1 & 13 & -1 \\ 0 & 0 & 18 & 1 \end{array}\right], \tag{3.86}$$

and then back substitution gives $t_1 = 1/18$, $s_2 = -31/18$, and $s_1 = 22/18$. Thus $\mathbf{x} = (1, 2, 3)^T$ is decomposed into the row space and null space components

$$\mathbf{x}_R = \frac{22}{18}\begin{bmatrix}1\\3\\1\end{bmatrix} - \frac{31}{18}\begin{bmatrix}0\\1\\-1\end{bmatrix} = \frac{1}{18}\begin{bmatrix}22\\35\\53\end{bmatrix} \text{ and } \mathbf{x}_0 = \frac{1}{18}\begin{bmatrix}-4\\1\\1\end{bmatrix}. \tag{3.87}$$

Note that we could have obtained $\mathbf{x}_0$ by utilizing the orthogonality of $\mathbf{c}_1$ to $\mathbf{b}_1$ and $\mathbf{b}_2$, and taking the dot product of both sides of Equation 3.84 with $\mathbf{c}_1$ to obtain

$$t_1\mathbf{c}_1^T\mathbf{c}_1 = \mathbf{c}_1^T\mathbf{x}. \tag{3.88}$$

This equation evaluates to $18t_1 = 1$, and gives the same value $t_1 = 1/18$ as in the previous computation but much more quickly. Then, once we have found $\mathbf{x}_0$, we could compute $\mathbf{x}_R$ as $\mathbf{x} - \mathbf{x}_0$. However, this shortcut works only if one of the subspaces Null(A) or Row(A) is one dimensional, as in this example. ♦

Theorem 3.5.3 established two relations between Null(A) and Row(A) that can be generalized: first, that the sums of their vectors make up all of $\mathbb{R}^n$ and, second, that those vectors are orthogonal to each other. Since these concepts occur in other contexts as well, we make corresponding definitions for arbitrary subspaces.

Definition 3.5.2. ***(Sum of Subspaces).*** *Let U and V be subspaces of a vector space X. Then the set $\{\mathbf{u} + \mathbf{v} | \mathbf{u} \in U, \mathbf{v} \in V\}$ is called the sum of U and V, and is denoted by $U + V$.*

It is easy to see that $U + V$ is a subspace of X (Exercise 3.5.8).

Definition 3.5.3. ***(Orthogonal Subspaces and Orthogonal Complement).*** *Let U and V be subspaces of a vector space X with an inner product (see page 17). They are said to be orthogonal to each other if every $\mathbf{u} \in U$ is orthogonal to every $\mathbf{v} \in V$. Furthermore, the set of all vectors $\mathbf{x}$ in X that are orthogonal to all vectors $\mathbf{u}$ in U is called the orthogonal complement of U, and is denoted by $U^\perp$ (read "U-perp"). In other words, $U^\perp = \{\mathbf{x} \in X | \mathbf{x} \perp \mathbf{u}$ for all $\mathbf{u} \in U\}$.*

Again, it is straightforward to verify that $U^\perp$ is a subspace of X (Exercise 3.5.13). Furthermore, with this notation we can write part of Theorem 3.5.3 as follows.

Corollary 3.5.2. ***(Relations Between* Row(A) *and* Null(A)*).*** *Let A be any $m \times n$ matrix. Then we have*

$$\text{Row}(A) + \text{Null}(A) = \mathbb{R}^n \tag{3.89}$$

$$\text{Row}(A) = \text{Null}(A)^\perp \tag{3.90}$$

and

$$\text{Null}(A) = \text{Row}(A)^{\perp}. \tag{3.91}$$

Let U be a subspace of $\mathbb{R}^n$. Then, by choosing a matrix A whose row space is U, the preceding formulas yield the following result.

Corollary 3.5.3. ***(The Sum of Orthogonal Complements).*** *Let U be a subspace of $\mathbb{R}^n$. Then we have*

$$U + U^{\perp} = \mathbb{R}^n. \tag{3.92}$$

It may seem strange that the column space of a matrix has been slighted in the discussion so far. This was due to the fact that we usually write linear systems as $A\mathbf{x} = \mathbf{b}$ rather than as $\mathbf{y}^T A = \mathbf{b}^T$. Nevertheless, the column space is very important, and occasionally we need yet another subspace associated with A. This new subspace will also remove the temporary asymmetry of the theory.

Definition 3.5.4. ***(Left Null Space).*** *Let A be any $m \times n$ matrix. The set of all vectors $\mathbf{y} \in \mathbb{R}^m$ such that $\mathbf{y}^T A = \mathbf{0}^T$ is called the left null space of A, and we denote it by* Left-null(A).[15]

For any matrix A, the four spaces Row(A), Col(A), Null(A), and Left-null(A) are sometimes referred to as the four *fundamental subspaces of A.*

As expected, we have relations between the left null space and the column space, analogous to those between the row space and the null space. Taking the transpose of both sides of the defining relation above, we get

$$A^T\mathbf{y} = \mathbf{0} \tag{3.93}$$

and so

$$\text{Left-null}(A) = \text{Null}(A^T). \tag{3.94}$$

Also, as we have seen in Equation 3.52 on page 124, we have

$$\text{Col}(A) = \text{Row}(A^T). \tag{3.95}$$

Consequently, the relations between the left null space and the column space can be obtained from the previous results simply by applying them to A^T. The main features are as follows.

Theorem 3.5.4. ***(Relations Between*** **Col(A)** ***and*** **Left-null(A)*).*** *Let A be any $m \times n$ matrix. Then*

$$\dim(\text{Left-null}(A)) + \dim(\text{Col}(A)) = m, \tag{3.96}$$

$$\text{Col}(A) + \text{Left-null}(A) = \mathbb{R}^m, \tag{3.97}$$

[15] This notation is our own; there is no standard one.

$$\text{Left-null}(A)^{\perp} = \text{Col}(A), \tag{3.98}$$

$$\text{Col}(A)^{\perp} = \text{Left-null}(A), \tag{3.99}$$

and any vector $\mathbf{y} \in \mathbb{R}^m$ *can be uniquely decomposed into the sum of a vector* $\mathbf{y}_C$ *in Col*(A) *and a vector* $\mathbf{y}_L$ *in Left-null*(A).

The left null space arises naturally in the following type of problem.

Corresponding to nonparametric and parametric representations of lines and planes, the two fundamental ways of representing a subspace V of $\mathbb{R}^n$ are:

1. as the solution space of a homogeneous system $A\mathbf{x} = \mathbf{0}$, that is, as $\text{Null}(A)$ for some $m \times n$ matrix A, and
2. as the set of all linear combinations $\mathbf{x} = \sum_{i=1}^{p} s_i \mathbf{b}_i = B\mathbf{s}$ of some vectors $\mathbf{b}_i \in \mathbb{R}^n$, that is, as $\text{Col}(B)$ for some $n \times p$ matrix B.

The question is how to pass from one representation to the other. In the direction $1 \to 2$ we have solved this problem in the previous section (see Example 3.4.6 and Theorem 3.4.6). In the other direction, given the matrix B, we need to find a matrix A such that $\text{Null}(A) = \text{Col}(B)$. Taking the orthogonal complement of each side and making use of Equations 3.90 and 3.99, we get $\text{Row}(A) = \text{Left-null}(B)$. Thus we can find a suitable A by finding a basis for $\text{Left-null}(B)$ and using the transposes of these basis vectors as the rows of A.

Example 3.5.6. (Finding a Matrix A for a Given Matrix B so that $\mathbf{Null}(\boldsymbol{A}) = \mathbf{Col}(\boldsymbol{B})$***).*** Let V be the column space of the matrix

$$B = \begin{bmatrix} 0 & 2 & -1 \\ 1 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{3.100}$$

and write V as the null space of a matrix A.

As stated above, to solve this problem we need to find a basis for $\text{Left-null}(B)$. This space is the same as $\text{Null}(B^T)$, and so we need to find all solutions of $B^T\mathbf{x} = \mathbf{0}$. We can do this in the usual way by reducing B^T as follows:

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 1 & 0 \\ -1 & 0 & -2 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -4 & 1 & 2 \end{bmatrix}. \tag{3.101}$$

The free variables are x_4 and x_5, and we set them equal to parameters: $x_4 = s_1$ and $x_5 = s_2$. Solving for the other components, we get $4x_3 = s_1 + 2s_2$, $x_2 = 0$, and $x_1 = -2x_3 + x_5 = -2\frac{s_1+2s_2}{4} + s_2 = -s_1/2$. Hence we can write the solution vectors as

$$\mathbf{x} = \frac{1}{4}\begin{bmatrix} -2 & 0 \\ 0 & 0 \\ 1 & 2 \\ 4 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} s_1 \\ s_2 \end{bmatrix}. \tag{3.102}$$

The columns of the matrix on the right form a basis for Left-null(B), and so its transpose is a solution to the problem, that is,

$$A = \begin{bmatrix} -2 & 0 & 1 & 4 & 0 \\ 0 & 0 & 2 & 0 & 4 \end{bmatrix}. \tag{3.103}$$

We leave it to the reader to check that indeed $\text{Null}(A) = \text{Col}(B)$. ♦

Still another numerical relation between subspaces of a matrix calls for deeper examination, namely that the row space and the column space both have the same dimension. If we consider any vector $\mathbf{x}_R$ in Row(A), then $A\mathbf{x}_R$ is in Col(A) because any $\mathbf{x}_R$ is in $\mathbb{R}^n$. Also, every vector in Col(A) can be obtained as $A\mathbf{x}_R$, since, using the decomposition $\mathbf{x} = \mathbf{x}_0 + \mathbf{x}_R$ for any $\mathbf{x} \in \mathbb{R}^n$ from Theorem 3.5.3, we find that $A\mathbf{x}_R = A\mathbf{x}$, and $\text{Col}(A) = \{A\mathbf{x} | \mathbf{x} \in \mathbb{R}^n\}$. Furthermore, $\mathbf{x}_R$ is uniquely determined by $A\mathbf{x}$, since if $A\mathbf{x}_R = A\mathbf{x}'_R$, then $A(\mathbf{x}_R - \mathbf{x}'_R) = \mathbf{0}$ and so $\mathbf{x}_R - \mathbf{x}'_R$ is in the null space as well as in the row space, and therefore must be the zero vector. This mapping of Row(A) to Col(A), given by $\mathbf{x}_R \to A\mathbf{x}_R$, can be represented, relative to any choice of bases in Row(A) and Col(A), by an $r \times r$ nonsingular matrix. We shall compute this matrix in Section 4.2. Also, the inverse of this mapping is discussed in Exercise 5.1.14 for the special case of A with independent rows, without reference to any bases.

We conclude this section with an application to electric networks.

Example 3.5.7. (The Interdependence of the Equations Expressing Kirchhof's Laws). In Example 2.3.2 we stated Kirchhof's laws for electric networks, but did not discuss the dependence of the equations obtained for the various nodes and loops of a circuit. This issue is seldom mentioned in physics books, but Sears and Zemansky state the following[16] rules without proof:

(1) If there are n branch points (nodes) in the network, apply the point rule (Kirchhof's first law) at $n-1$ of these points. Any points may be chosen. Application of the point rule at the nth point does not lead to an independent relation.

(2) Imagine the network to be separated into a number of simple loops, like the pieces of a jigsaw puzzle. Apply the loop rule (Kirchhof's second law) to each of these loops.

So, why are these rules valid?

[16] p. 523, Sears and Zemansky, *University Physics 2nd ed. 1955*, Addison-Wesley, Reading, Mass.

We are going to examine this question on the circuit of Example 2.3.2.

However, to analyze just Kirchhof's laws without Ohm's law, we do not need the resistors and the batteries, but only the nodes and edges as shown in Figure 3.1. Such a diagram is called a directed graph, where the arrows indicate the assigned direction of the currents.

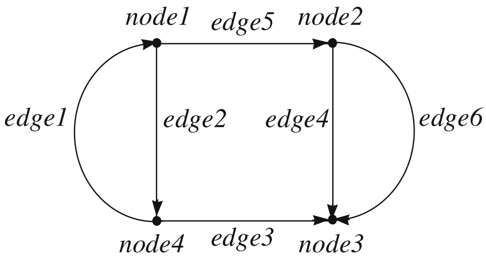

Fig. 3.1. A directed graph

We can characterize a graph by an *edge-node incidence matrix*. Such a matrix has a row for each edge and a column for each node. For an undirected graph, the entry a_{ij} is 1 if node j is on edge i, and 0 otherwise. For a directed graph, the entry a_{ij} is -1 if node j is the foot of the arrow of edge i, is 1 if node j is the tip of the arrow of edge i, and is 0 if node j is not on edge i. Thus the edge-node incidence matrix for the graph of Figure 3.1 is

$$A = \begin{bmatrix} 1 & 0 & 0 & -1 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & -1 & 1 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \end{bmatrix}. \tag{3.104}$$

Now Kirchhof's first law, for all four nodes together, can be expressed in terms of the matrix A as saying that the vector $\mathbf{y}$, whose components are the currents, must be in Left-null$\,(A)$, that is, that

$$\mathbf{y}^T A = \mathbf{0}^T. \tag{3.105}$$

(We write $\mathbf{y}$ for the current vector instead of $\mathbf{i}$, because the latter is reserved for the first standard vector in $\mathbb{R}^2$ or $\mathbb{R}^3$. Thus, we also use y_k to denote the current in edge k.) Indeed, the first component of this vector equation is $\mathbf{y}^T\mathbf{a}_1 = y_1 - y_2 - y_5 = 0$, which expresses Kirchhof's first law for node 1; the second component is $\mathbf{y}^T\mathbf{a}_2 = -y_4 + y_5 - y_6 = 0$, which is Kirchhof's first law for node 2; etc.

It is easy to see that the columns of A add up to $\mathbf{0}$, because each row sums to 0. This relation can also be expressed as $A\,(1,1,1,1)^T = \mathbf{0}$, which shows that $(1,1,1,1)^T$ is in Null (A). In fact, $\left\{(1,1,1,1)^T\right\}$ is a basis for Null (A), since $A\mathbf{x} = \mathbf{0}$ corresponds to the system

$$\begin{array}{rrrrl} x_1 & & & -x_4 & = 0 \\ -x_1 & & & +x_4 & = 0 \\ & & x_3 & -x_4 & = 0 \\ & -x_2 & +x_3 & & = 0 \\ -x_1 & x_2 & & & = 0 \\ & -x_2 & +x_3 & & = 0 \end{array} \tag{3.106}$$

whose general solution is $\mathbf{x} = s(1,1,1,1)^T$. Thus, Null (A) is one dimensional and so, by Corollary 3.5.1, rank $(A) = 3$ and, by Theorem 3.5.2, $\dim(\mathrm{Col}(A)) = 3$ as well. It is easy to see that actually *any* three columns form a basis for Col(A). Similarly, for any circuit with n nodes, rank $(A) = n-1$ and any $n-1$ columns $\mathbf{a}_j$ form a basis for Col(A). The equation $\mathbf{y}^T A = \mathbf{0}^T$, which expresses Kirchhof's first law for each node, can be written in components as $\mathbf{y}^T \mathbf{a}_j = 0$ for $j = 1, 2, \ldots, n$, and any one of these equations is a linear combination of the other $n-1$ of them.[17] This result proves Rule (1) of Sears and Zemansky quoted above and, for the network of Example 2.3.2, we have obtained the first three of the Equations 2.57.

Kirchhof's second law is also contained in the matrix A: Denoting the vector of the potentials at the nodes by $\mathbf{x} = (x_1, x_2, x_3, x_4)^T$, the vector

$$A\mathbf{x} = \begin{bmatrix} x_1 - x_4 \\ x_4 - x_1 \\ x_3 - x_4 \\ x_3 - x_2 \\ x_2 - x_1 \\ x_3 - x_2 \end{bmatrix} \tag{3.107}$$

represents the potential differences along the edges in the direction of the arrows. Now, for any $\mathbf{x}$, $A\mathbf{x}$ is a member of Col(A) which, by Equation 3.98, is the orthogonal complement of Left-null (A). Thus, for any $\mathbf{y} \in$ Left-null (A), $A\mathbf{x}$ must satisfy $\mathbf{y}^T A\mathbf{x} = \mathbf{0}$. Furthermore, it is easy to see that the directed loops of the graph generate some members $\mathbf{y}$ of Left-null (A) if we set $y_i = 1$ if edge i is in the given loop with its arrow in the loop's direction, $y_i = -1$ if edge i is in the given loop with its arrow opposite the loop's direction, and $y_i = 0$ if edge i is not in the given loop. For instance, the small loop on the left with a clockwise direction corresponds to $\mathbf{y} = (1,1,0,0,0,0)^T$ and, indeed,

$$(1,1,0,0,0,0)\,A\mathbf{x} = (x_1 - x_4) + (x_4 - x_1) = 0. \tag{3.108}$$

[17] We say that $n-1$ of these equations are independent of each other because, in general, we call a set of linear equations independent if their coefficient vectors are independent.

Similarly, the loop in the middle with a clockwise direction corresponds to $\mathbf{y} = (0, -1, -1, 1, 1, 0)^T$ and

$$(0, -1, -1, 1, 1, 0)\, A\mathbf{x} = -\,(x_4 - x_1) - (x_3 - x_4) + (x_3 - x_2) + (x_2 - x_1) = 0. \tag{3.109}$$

Finally, the small loop on the right with a clockwise direction corresponds to $\mathbf{y} = (0, 0, 0, -1, 0, 1,)^T$ and

$$(0, 0, 0, -1, 0, 1,)\, A\mathbf{x} = -\,(x_3 - x_2) + (x_3 - x_2) = 0. \tag{3.110}$$

Thus the three equations above represent Kirchhof's second law for the three simple loops. Since Left-null (A) is three dimensional, the three vectors above form a basis for it and so the three loop equations $\mathbf{y}^T A\mathbf{x} = \mathbf{0}$ completely determine the column space of A. Notice that these equations determine only the potential differences and not the potentials themselves. Hence it is customary to "ground" one of the nodes, that is, to set one of the x_i values equal to 0.

The last three equations illustrate Rule (2) of Sears and Zemansky quoted above. For general circuits with n nodes and m edges, $\dim(\text{Left-null}(A)) = m - r = m - n + 1$, and so we need to write Kirchhof's second law $\mathbf{y}^T A\mathbf{x} = \mathbf{0}$ for the $\mathbf{y}$ vectors of $m - n + 1$ loops. In the case of planar graphs we have $m - n + 1$ simple loops as described in Rule (2) of Sears and Zemansky quoted above,[18] but for nonplanar graphs (e.g., for the edges of a cube) this is not always the case.

To complete the analysis of any network with batteries and resistors, we need to relate the currents and the potential differences by making use of the voltage sources and Ohm's law. In the case of the network of Example 2.3.2 on which the graph of Figure 3.1 was based, we can write

$$A\mathbf{x} = \begin{bmatrix} x_1 - x_4 \\ x_4 - x_1 \\ x_3 - x_4 \\ x_3 - x_2 \\ x_2 - x_1 \\ x_3 - x_2 \end{bmatrix} = \begin{bmatrix} V_1 - R_1 i_1 \\ R_2 i_2 \\ R_3 i_3 \\ R_4 i_4 \\ R_5 i_5 \\ V_2 + R_6 i_6 \end{bmatrix}. \tag{3.111}$$

Substituting from here into Equations 3.108, 3.109, and 3.110, we get the last three of Equations 2.57. In general, the incidence matrix gives $n - 1$ independent node equations for the m currents and $m - n + 1$ loop equations for the potential differences, which can be converted to equations for the currents by inserting the applied voltages and using Ohm's law. Thus we get altogether $(n - 1) + (m - n + 1) = m$ independent equations for the m currents. ♦

[18] This result follows from Euler's polyhedral formula. See, e.g., http://en.wikipedia.org/wiki/Euler_characteristic

Exercises

Exercise 3.5.1. Let

$$A = \begin{bmatrix} 1 & 1 & 1 & 2 & 0 \\ 3 & 3 & 0 & 6 & 0 \end{bmatrix}.$$

a. Find the dimensions of the four subspaces associated with A.
b. Decompose $(-2, 0, 1, 4, 1)^T$ into the sum of an $\mathbf{x}_0 \in \text{Null}(A)$ and an $\mathbf{x}_R \in \text{Row}(A)$.

Exercise 3.5.2. Let

$$A = \begin{bmatrix} 1 & 1 & 1 & 2 & -1 \end{bmatrix}.$$

a. Find the dimensions of the four subspaces associated with A.
b. Decompose $(1, 1, 1, 1, 1)^T$ into the sum of an $\mathbf{x}_0 \in \text{Null}(A)$ and an $\mathbf{x}_R \in \text{Row}(A)$.

Exercise 3.5.3. Let

$$A = \begin{bmatrix} 3 & 3 & 0 & 4 & 4 \end{bmatrix}.$$

a. Find the dimensions of the four subspaces associated with A.
b. Decompose $(1, 1, 1, 1, 1)^T$ into the sum of an $\mathbf{x}_0 \in \text{Null}(A)$ and an $\mathbf{x}_R \in \text{Row}(A)$.

Exercise 3.5.4. Let

$$A = \begin{bmatrix} 1 & 0 & 0 & 2 & -1 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 2 \end{bmatrix}.$$

a. Find the dimensions of the four subspaces associated with A.
b. Decompose $(6, 2, 1, 4, 8)^T$ into the sum of an $\mathbf{x}_0 \in \text{Null}(A)$ and an $\mathbf{x}_R \in \text{Row}(A)$.

Exercise 3.5.5. Let

$$A = \begin{bmatrix} 0 & 2 & 0 & 0 & 4 \\ 0 & 0 & 0 & 2 & 2 \end{bmatrix}.$$

a. Find the dimensions of the four subspaces associated with A.
b. Decompose $(1, 2, 3, 4, 5)^T$ into the sum of an $\mathbf{x}_0 \in \text{Null}(A)$ and an $\mathbf{x}_R \in \text{Row}(A)$.

Exercise 3.5.6. Let

$$A = \begin{bmatrix} 1 & 1 & 1 & 2 & -1 \\ 0 & 2 & 0 & 0 & 4 \end{bmatrix}.$$

a. Find the dimensions of the four subspaces associated with A.
b. Decompose $(1, 3, 4, 2, 8)^T$ into the sum of an $\mathbf{x}_0 \in \text{Null}(A)$ and an $\mathbf{x}_R \in \text{Row}(A)$.

Exercise 3.5.7. Show, without referring to Theorem 3.5.3, that $\text{Row}(A) \cap \text{Null}(A) = \{\mathbf{0}\}$ for any matrix A.

Exercise 3.5.8. Let U and V be subspaces of a vector space X. Prove that $U + V$ is a subspace of X.

Exercise 3.5.9. Let U and V be subspaces of a vector space X. Prove that $U \cap V$ is a subspace of $U + V$.

Exercise 3.5.10. Show that if A and B have the same number of rows, then $\text{Col}(A) + \text{Col}(B) = \text{Col}[A \;\; B]$. (This fact can be used to find a basis for the sum of subspaces.)

Exercise 3.5.11. Show that if A and B have the same number of columns, then $\text{Null}(A) \cap \text{Null(B)} = \text{Null} \begin{bmatrix} A \\ B \end{bmatrix}$. (This fact can be used to find a basis for the intersection of subspaces.)

Exercise 3.5.12. Let

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}.$$

a. Find a basis for each of $\text{Col}(A)$, $\text{Col}(B)$, and $\text{Col}(A + B)$.
b. Find a basis for each of $\text{Col}(A) + \text{Col}(B)$ and $\text{Col}(A) \cap \text{Col}(B)$.
c. Is $\text{Col }(A + B) = \text{Col}(A) + \text{Col}(B)$?
d. Verify the formula $\dim(U + V) = \dim U + \dim V - \dim(U \cap V)$ of Exercise 3.5.23 for $U = \text{Col}(A)$ and $V = \text{Col}(B)$.

Exercise 3.5.13. Let U be a subspace of an inner product space X. (See page 17.) Prove that $U^\perp$ is a subspace of X.

Exercise 3.5.14. Show that if A and B are matrices such that AB is defined and A is a nonsingular square matrix, then B and AB have the same rank. (*Hint*: Use the result of Exercise 3.4.9.)

Exercise 3.5.15. Use the result of the previous exercise to show that row-equivalent matrices have the same rank. (*Hint*: Use also the results of Exercises 2.5.12, 2.5.13, and 2.5.14 of page 96, suitably generalized.)

Exercise 3.5.16. Show that if A and B are matrices such that AB is defined, then $\text{Col}(AB)$ is a subspace of $\text{Col}(A)$, and $\text{Row}(AB)$ is a subspace of $\text{Row}(B)$. What do these facts imply about the ranks and the nullities of the matrices?

Exercise 3.5.17. Show that if A and B are matrices such that AB is defined and the columns of B are linearly dependent, then the columns of AB are linearly dependent as well. Is the converse true?

Exercise 3.5.18. Let A be the matrix of Exercise 3.5.4 and let U be the subspace of $\mathbb{R}^3$ spanned by the first two columns of A, and V the subspace spanned by the last two columns. Find a basis for each of $U,\ V,\ U^\perp,\ V^\perp,\ U \cap V,\ U+V.$

Exercise 3.5.19. Let

$$A = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 2 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

and let U be the subspace of $\mathbb{R}^4$ spanned by the first three columns of A, and V the subspace spanned by the last three columns. Find a basis for each of $U,\ V,\ U^\perp,\ V^\perp,\ U \cap V,\ U+V.$

***Exercise 3.5.20.** Let U and V be finite-dimensional subspaces of a vector space X. Prove that $U+V$ is the subspace generated by $U \cup V$. (*Hint*: Consider a basis for each subspace.)

***Exercise 3.5.21.** For any subspaces U and V of a vector space X we call $U+V$ a *direct sum* if $U \cap V = \{\mathbf{0}\}$, and denote it in that case by $U \oplus V$. Show that for any finite-dimensional subspaces with $U \cap V = \{\mathbf{0}\}$ we have $\dim(U \oplus V) = \dim U + \dim V$. (*Hint*: Consider a basis for each subspace.)

***Exercise 3.5.22.** Show that for any finite-dimensional subspaces U and V of a vector space X the sum $U+V$ is direct if and only if every $\mathbf{x} \in U+V$ can be decomposed uniquely into a sum of a vector $\mathbf{u}$ from U and a vector $\mathbf{v}$ from V, that is, $\mathbf{x} = \mathbf{u} + \mathbf{v}$, with $\mathbf{u}$ and $\mathbf{v}$ uniquely determined by $\mathbf{x}$.

***Exercise 3.5.23.** Let U and V be subspaces of $\mathbb{R}^n$. Show that
$\dim(U+V) = \dim U + \dim V - \dim(U \cap V)$.

***Exercise 3.5.24.** Generalize the definition of sum to more than two subspaces as summands.

***Exercise 3.5.25.** Find a formula analogous to the one in Exercise 3.5.23 for the sum of three subspaces of $\mathbb{R}^n$.

Exercise 3.5.26. Let U and V be subspaces of $\mathbb{R}^n$. Show that if they are orthogonal to each other, then $U \cap V = \{\mathbf{0}\}$. Is the converse true? (Explain!)

***Exercise 3.5.27.** Let U and V be subspaces of $\mathbb{R}^n$. Show that $(U \cap V)^\perp = U^\perp + V^\perp$. (*Hint*: Extend a basis for $U \cap V$ to bases for $U,\ V$, and $\mathbb{R}^n$.)

***Exercise 3.5.28.** Let U and V be subspaces of $\mathbb{R}^n$. Show that $(U \cap V)^\perp = U^\perp \cap V^\perp$.

***Exercise 3.5.29.** Let U and V be subspaces of $\mathbb{R}^n$. Show that $U \subset V$ implies $V^\perp \subset U^\perp$.

***Exercise 3.5.30.** Let U be a subspace of $\mathbb{R}^n$. Show that $(U^\perp)^\perp = U$.

Exercise 3.5.31. Find a matrix A such that $\mathrm{Null}(A) = \mathrm{Col}(B)$, where

$$B = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \\ 2 & 1 & 0 \end{bmatrix}.$$

Exercise 3.5.32. Find a matrix A such that $\mathrm{Null}(A) = \mathrm{Row}(B)$, where

$$B = \begin{bmatrix} 1 & 1 & 1 & 2 & -1 \\ 0 & 2 & 0 & 0 & 4 \\ 0 & 0 & 0 & 2 & 2 \end{bmatrix}.$$

***Exercise 3.5.33.** Prove the following alternative algorithm for obtaining a basis for the left null space of a matrix:

Let A be any real $m \times n$ matrix of rank r. Consider the block matrix $[A \quad I]$, where I is the unit matrix of order m. Reduce this block matrix by elementary row operations to a form $\begin{bmatrix} U & L \\ O & M \end{bmatrix}$, in which U is an $r \times n$ echelon matrix and O the $(m-r) \times n$ zero matrix. Then the transposed rows of the $(m-r) \times m$ matrix M form a basis for the left null space of A. (*Hint*: For any $\mathbf{b} \in \mathrm{Col}(A)$ reduce $A\mathbf{x} = I\mathbf{b}$ by elementary row operations until A is in an echelon form $\begin{bmatrix} U \\ O \end{bmatrix}$ with U having no zero rows. On the right-hand side denote the result of this reduction of the matrix I by $\begin{bmatrix} L \\ M \end{bmatrix}$ and consider the condition for consistency.)

***Exercise 3.5.34.** Use the algorithm of the previous exercise to solve Exercise 3.5.31.

***Exercise 3.5.35.** Modify the algorithm of Exercise 3.5.33 to obtain a new algorithm for finding a basis for $\mathrm{Null}(A)$.

***Exercise 3.5.36.** Show that the result of Exercise 3.5.33 can be used to solve $A\mathbf{x} = \mathbf{b}$ by the following new algorithm:

Let A be any real $m \times n$ matrix and $\mathbf{b} \in \mathrm{Col}(A)$. Reduce the block matrix $\begin{bmatrix} A^T & I \\ -\mathbf{b}^T & \mathbf{0} \end{bmatrix}$ to the form $\begin{bmatrix} U & L \\ \mathbf{0} & \mathbf{x}^T \end{bmatrix}$ by elementary row operations without ever exchanging the last row or multiplying it by a scalar, where U is an $n \times m$ echelon matrix and $\mathbf{x}^T$ a row n-vector. Then the latter is the transpose of a particular solution $\mathbf{x}$. The general solution is given by the sum of this $\mathbf{x}$ and any linear combination of the transposed rows of L that correspond to the zero rows of U.

***Exercise 3.5.37.** Use the algorithm of the previous exercise to solve $A\mathbf{x} = \mathbf{b}$ with

$$A = \begin{bmatrix} 1 & -2 & 3 & 2 \\ 0 & 3 & -3 & 0 \\ -1 & -1 & 0 & -2 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} -5 \\ 6 \\ -1 \end{bmatrix}.$$

***Exercise 3.5.38.** Prove the following generalization of the algorithm of Exercise 3.5.36 to invert a matrix:

Let A be any real, nonsingular $n \times n$ matrix. Consider the block matrix $\begin{bmatrix} A & I \\ -I & O \end{bmatrix}$ where I is the unit matrix of order n and O is the $n \times n$ zero matrix. Row-reduce this block matrix, without exchanging or multiplying any of the last n rows in the process, to a form $\begin{bmatrix} U & L \\ O & M \end{bmatrix}$ in which U is upper triangular. Then $M = A^{-1}$.

***Exercise 3.5.39.** Use the algorithm of the previous exercise to invert

$$A = \begin{bmatrix} 1 & -2 \\ 4 & 3 \end{bmatrix}.$$

***Exercise 3.5.40.** Prove that in any n-dimensional vector space X any set of n independent vectors forms a basis. (*Hint*: Use the Exchange Theorem.)

***Exercise 3.5.41.** Prove that in any n-dimensional vector space X no set of fewer than n vectors spans X. (*Hint*: Use the Exchange Theorem.)

***Exercise 3.5.42.** Prove that in any n-dimensional vector space X any set of n vectors that spans X forms a basis. (*Hint*: Use the results of Exercise 3.5.41 and Theorem 3.5.1.)

***Exercise 3.5.43.** Prove that in any n-dimensional vector space X any set of more than n vectors is a dependent set. (*Hint*: Use the Exchange Theorem.)

MATLAB Exercises

In MATLAB, the method of Exercises 3.5.33 and 3.5.35 can be used to obtain bases for Left-null(A) and Null(A) by computing **rref**([A,**eye**(**size**$(A, 1)$)]) and the same for A' in place of A. The command $N =$ **null**(A) also returns a (usually different) basis matrix for Null(A).

Exercise 3.5.44. Let $A =$ **magic**(4)
a. Use **rref** as explained above to find a basis for Left-null(A) and for Null(A). Extract and transpose the appropriate submatrix in each case to obtain two matrices B and C whose columns form the bases.

b. Use $N = \textbf{null}(A)$ and $L = \textbf{null}(A')$ to obtain two different bases, and show that they span the same subspaces as B and C, respectively.
c. Use **rref** as in Exercise 3.4.15 to compute bases for Row(A) and Col(A).
d. Decompose $\mathbf{x} = (1, 2, 3, 4)^T$ into a sum of an $\mathbf{x}_0 \in$ Null(A) and an $\mathbf{x}_R \in$ Row(A).
e. Decompose $\mathbf{x}$ into a sum of an $\mathbf{x}_L \in$ Left-null(A) and an $\mathbf{x}_C \in$ Col(A).

Exercise 3.5.45. Repeat the previous exercise for the matrix $A = \textbf{magic}(8)$ and $\mathbf{x} = (1, 2, 3, 4, 1, 2, 3, 4)^T$.

Exercise 3.5.46. Let $A = \textbf{round}(10 * \textbf{rand}(3, 4) - 5)$.
a. Compute the nullity of A, A^T, A^TA, and AA^T.
b. Repeat the above for six instances of A.
c. Do you see any patterns? Make a conjecture and prove it.

3.6 Change of Basis

The notion of a basis is a generalization of that of a coordinate system. Just as we sometimes need to change coordinate systems, so too do we sometimes need to change bases to make certain equations simpler. For example, changing of bases will be used in the eigenvalue problems of Chapter 7, which arise in the evolution of many physical systems.

Let X be an n-dimensional vector space and $\mathcal{A} = \{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$ a basis for X. Then any vector $\mathbf{x}$ in X can be written uniquely as $\mathbf{x} = \sum_{i=1}^n x_{Ai}\mathbf{a}_i$. We have here an ordering of $\mathcal{A}$ implicit in the subscripts, and the ordered n-tuple $A = (\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n)$ is sometimes called an *ordered basis* for X. The transposed ordered n-tuple $\mathbf{x}_A = (x_{A1}, x_{A2}, \ldots, x_{An})^T$ is called the *coordinate vector of* $\mathbf{x}$ *relative to* A, and is a vector of $\mathbb{R}^n$. Its components x_{Ai} are called the *coordinates of* $\mathbf{x}$ *relative to* A.

We begin our discussion of changing bases in the special case of $X = \mathbb{R}^n$, and consider general vector spaces afterward. Working with $\mathbb{R}^n$ enables us to consider the ordered basis $A = (\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n)$ of $\mathbb{R}^n$ to be a matrix, called a *basis matrix.* It is, of course, of size $n \times n$. Similarly, let $B = (\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n)$ be another basis matrix of $\mathbb{R}^n$. Then any vector $\mathbf{x}$ of $\mathbb{R}^n$ can be decomposed uniquely as $\mathbf{x} = \sum_{i=1}^n x_{Ai}\mathbf{a}_i$ and also as $\mathbf{x} = \sum_{i=1}^n x_{Bi}\mathbf{b}_i$. Using the basis matrices, we may write the equations above as

$$\mathbf{x} = A\mathbf{x}_A \text{ and } \mathbf{x} = B\mathbf{x}_B. \tag{3.112}$$

For given A, B, and any $\mathbf{x}$, the Equations 3.112 can be solved for $\mathbf{x}_A$ and $\mathbf{x}_B$, and therefore, by Theorem 2.5.3 on page 92, the matrices A and B must be invertible. Thus we can solve $A\mathbf{x}_A = B\mathbf{x}_B$ as

$$\mathbf{x}_A = A^{-1}B\mathbf{x}_B. \tag{3.113}$$

The matrix

$$S = A^{-1}B \tag{3.114}$$

is usually referred to as the *transition matrix or change of basis matrix* from the basis A to the basis B, because Equation 3.114 can be solved to give this change as

$$B = AS. \tag{3.115}$$

The matrix S, being the product of two invertible matrices, must also be invertible, and we have $S^{-1} = B^{-1}A$.

Using the definition of S we can write Equation 3.113 as

$$\mathbf{x}_A = S\mathbf{x}_B \tag{3.116}$$

and, multiplying both sides by S^{-1}, we can also write

$$\mathbf{x}_B = S^{-1}\mathbf{x}_A. \tag{3.117}$$

Note the formal difference between Equations 3.115 and 3.117: whereas the basis matrix A transforms with S on the right, the corresponding coordinate vector x_A transforms with S^{-1} on the left.

Let us summarize the preceding discussion in a theorem.

Theorem 3.6.1. ***(Change of Basis of*** $\mathbb{R}^n$***).*** *If A and B are $n \times n$ matrices whose columns form two bases of $\mathbb{R}^n$, then A and B are invertible and there exists an invertible $n \times n$ matrix S such that $B = AS$ and so $S = A^{-1}B$. Furthermore, the coordinate vectors $\mathbf{x}_A$ and $\mathbf{x}_B$ of any $\mathbf{x} \in \mathbb{R}^n$ are related by Equations 3.116 and 3.117.*

Corollary 3.6.1. ***(Change from the Standard Basis of*** $\mathbb{R}^n$***).*** *In the important particular case of a transition from the standard basis I to a basis B, replacing A in Equations 3.112 and 3.115 by I, we get $\mathbf{x}_A = \mathbf{x}_I = \mathbf{x}$, $B = IS = S$ and Equations 3.116 and 3.117 become $\mathbf{x} = B\mathbf{x}_B$ and $\mathbf{x}_B = B^{-1}\mathbf{x}$. Let us emphasize that where the standard vectors transform with B, the coordinate vectors transform with B^{-1}, that is,*

$$\mathbf{b}_i = B\mathbf{e}_i \text{ and } \mathbf{x}_B = B^{-1}\mathbf{x}. \tag{3.118}$$

Example 3.6.1. (A Change of Basis of $\mathbb{R}^2$***).*** In $\mathbb{R}^2$ let us change from the standard basis $\{\mathbf{i}, \mathbf{j}\}$ to the basis $\{\mathbf{b}_1, \mathbf{b}_2\}$, where $\mathbf{b}_1 = \frac{\sqrt{2}}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\mathbf{b}_2 = \frac{\sqrt{2}}{2}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

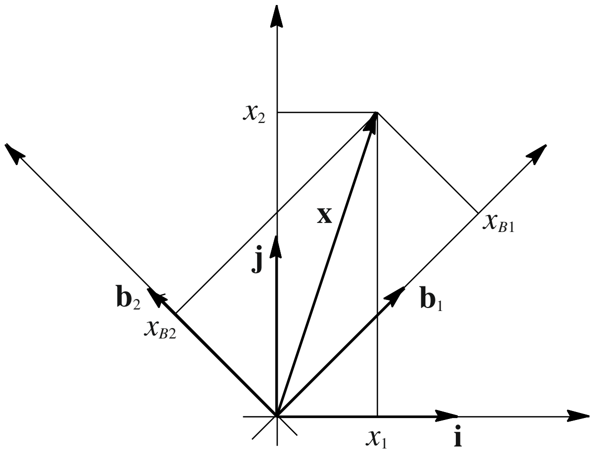

Fig. 3.2. Rotating the standard basis in $\mathbb{R}^2$ by 45°

This new basis is obtained from the old one by a 45° rotation. (See Figure 3.2.) Then the transition matrix is

$$B = \frac{\sqrt{2}}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \tag{3.119}$$

and

$$B^{-1} = \frac{\sqrt{2}}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}. \tag{3.120}$$

Hence for any vector $\mathbf{x} = (x_1, x_2)^T$ we have

$$\mathbf{x}_B = B^{-1}\mathbf{x} = \frac{\sqrt{2}}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{\sqrt{2}}{2}\begin{bmatrix} x_1 + x_2 \\ x_2 - x_1 \end{bmatrix}. \tag{3.121}$$

♦

Example 3.6.2. (A Change of Basis of $\mathbb{R}^3$***)***
a. Find the transition matrix S for the change of basis in $\mathbb{R}^3$ from $(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3)$ to $(\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3)$, where $\mathbf{a}_1 = (1, 2, 0)^T$, $\mathbf{a}_2 = (0, 1, 3)^T$, $\mathbf{a}_3 = (0, 0, 1)^T$, $\mathbf{b}_1 = (1, 0, 1)^T$, $\mathbf{b}_2 = (1, 1, 0)^T$, $\mathbf{b}_3 = (0, 1, 1)^T$.
b. Use S to write $\mathbf{x} = -\mathbf{b}_1 + 2\mathbf{b}_2 + 3\mathbf{b}_3$ as $x_{A1}\mathbf{a}_1 + x_{A2}\mathbf{a}_2 + x_{A3}\mathbf{a}_3$.
Solution: The basis matrix A is given by

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix} \tag{3.122}$$

and the basis matrix B by

$$B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}. \tag{3.123}$$

Thus

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 6 & -3 & 1 \end{bmatrix}, \tag{3.124}$$

and so the change of basis matrix is

$$S = A^{-1}B = \begin{bmatrix} 1 & 1 & 0 \\ -2 & -1 & 1 \\ 7 & 3 & -2 \end{bmatrix}. \tag{3.125}$$

Consequently,

$$\mathbf{x}_A = S\mathbf{x}_B = \begin{bmatrix} 1 & 1 & 0 \\ -2 & -1 & 1 \\ 7 & 3 & -2 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ -7 \end{bmatrix} \tag{3.126}$$

and

$$\mathbf{x} = \mathbf{a}_1 + 3\mathbf{a}_2 - 7\mathbf{a}_3. \tag{3.127}$$

♦

All the formulas in the previous discussion except those involving matrix products and inverses can also be given a meaning in an arbitrary finite-dimensional vector space X, not just in $\mathbb{R}^n$, by considering A and B not as matrices but as ordered bases of X, that is, as the n-tuples of vectors $A = (\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n)$ and $B = (\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n)$.

In $\mathbb{R}^n$, the formula $B = AS$, by Exercise 2.4.13 Part b, is equivalent to

$$\mathbf{b}_j = \sum_{i=1}^{n} \mathbf{a}_i s_{ij} \text{ for } j = 1, 2, \ldots, n. \tag{3.128}$$

This equation, however, unlike $B = AS$, is valid in any finite-dimensional X, not just in $\mathbb{R}^n$, and expresses each vector $\mathbf{b}_j$ as a linear combination of the $\mathbf{a}_i$ vectors, and shows that s_{ij} is the ith component of the coordinate vector $\mathbf{b}_{jA}$ of $\mathbf{b}_j$ relative to the basis A, that is, $s_{ij} = (\mathbf{b}_{jA})_i$. Thus, S is again an $n \times n$ matrix, whose jth column is given by

$$\mathbf{s}_j = \mathbf{b}_{jA}. \tag{3.129}$$

The expressions $\mathbf{x} = A\mathbf{x}_A$ and $\mathbf{x} = B\mathbf{x}_B$ of any vector in $\mathbb{R}^n$ relative to the bases A and B, in X become

$$\mathbf{x} = \sum_{i=1}^{n} x_{Ai}\mathbf{a}_i \tag{3.130}$$

and

$$\mathbf{x} = \sum_{j=1}^{n} x_{Bj}\mathbf{b}_j. \tag{3.131}$$

Substituting the expression of $\mathbf{b}_j$ from Equation 3.128 into this equation, we get

$$\mathbf{x} = \sum_{i=1}^{n}\sum_{j=1}^{n} s_{ij}x_{Bj}\mathbf{a}_i. \tag{3.132}$$

Since the $\mathbf{a}_i$ vectors form a basis, the coefficients in the two expressions 3.130 and 3.132 of $\mathbf{x}$ relative to the basis A must be equal:

$$x_{Ai} = \sum_{j=1}^{n} s_{ij}x_{Bj} \tag{3.133}$$

or equivalently

$$\mathbf{x}_A = S\mathbf{x}_B. \tag{3.134}$$

We can also show that the matrix S must again be invertible and S^{-1} gives the transformation in the reverse direction. To this end, write the $\mathbf{a}_k$ vectors as linear combinations of the $\mathbf{b}_j$ vectors:

$$\mathbf{a}_k = \sum_{j=1}^{n} \mathbf{b}_j t_{jk} \text{ for } k = 1, \ldots, n. \tag{3.135}$$

If we substitute the expression 3.128 of the $\mathbf{b}_j$ vectors into this equation and equate corresponding coefficients of the $\mathbf{a}_i$ vectors on the two sides, then we get[19]

$$\sum_{j=1}^{n} s_{ij}t_{jk} = \delta_{ik} \text{ for } i, k = 1, \ldots, n. \tag{3.136}$$

In matrix notation this equation becomes $ST = I$, and so S must be invertible and $T = S^{-1}$. Multiplying both sides of Equation 3.134 by S^{-1}, we get

[19] Recall that $\delta_{ik} = 1$ if $i = k$ and is 0 otherwise. (See page 82.)

$$\mathbf{x}_B = S^{-1}\mathbf{x}_A. \tag{3.137}$$

Also, using $T = S^{-1}$, we can abbreviate Equation 3.135 as $A = BS^{-1}$.

Thus we have proved the following generalization of Theorem 3.6.1 from $\mathbb{R}^n$ to any finite-dimensional vector space X.

Theorem 3.6.2. ***(Change of Basis in any Finite-Dimensional Vector Space).*** *If $A = (\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n)$ and $B = (\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n)$ are two ordered bases of a vector space X, then the matrix S whose columns are given by $\mathbf{s}_j = \mathbf{b}_{jA}$ is invertible and relates the coordinate vectors $\mathbf{x}_A$ and $\mathbf{x}_B$ of any $\mathbf{x} \in X$ by $\mathbf{x}_A = S\mathbf{x}_B$ and $\mathbf{x}_B = S^{-1}\mathbf{x}_A$.*

Example 3.6.3. (A Change of Basis in a Subspace of $\mathbb{R}^3$***).*** Consider the two-dimensional subspace V of $\mathbb{R}^3$ spanned by the columns of

$$A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 2 & -1 \end{bmatrix}. \tag{3.138}$$

Another basis for V is given by the columns of

$$B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 4 & 3 \end{bmatrix}. \tag{3.139}$$

Find the transition matrix S from the basis A to the basis B.

In this case, since we are working in $\mathbb{R}^3$, we have no problem with considering A and B to be matrices. Furthermore, by Equation 3.128, we must have $AS = B$ with S being 2×2. We can solve $AS = B$ for the unknown matrix S by Gauss–Jordan elimination (see Exercise 3.6.8), pretty much as we obtained the inverse of a matrix:

$$\left[\begin{array}{cc|cc} 1 & 0 & 1 & 1 \\ 2 & 1 & 0 & 1 \\ 2 & -1 & 4 & 3 \end{array}\right] \rightarrow \left[\begin{array}{cc|cc} 1 & 0 & 1 & 1 \\ 0 & 1 & -2 & -1 \\ 0 & -1 & 2 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cc|cc} 1 & 0 & 1 & 1 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]. \tag{3.140}$$

Thus

$$S = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}. \tag{3.141}$$

♦

Example 3.6.4. (A Change of Basis in $\mathcal{P}_3$***).*** In quantum mechanics, the simplest solutions of the differential equation of the harmonic oscillator involve what are called the Hermite polynomials. To find more general solutions, it is important to rewrite a given polynomial as a linear combination of Hermite polynomials. Although this problem is handled generally by introducing an appropriate inner product in $\mathcal{P}$ and exploiting the orthogonality of the Hermite polynomials, for the first few Hermite polynomials we can also use the present method.

The first four Hermite polynomials are given by $H_0(x) = 1$, $H_1(x) = 2x$, $H_2(x) = 4x^2 - 2$, and $H_3(x) = 8x^3 - 12x$. We want to find a formula that expresses an arbitrary third degree polynomial as a linear combination of these polynomials. In the space $\mathcal{P}_3 = \{\mathbf{p} = P : P(x) = p_0 + p_1x + p_2x^2 + p_3x^3; p_0, p_1, p_2, p_3 \in \mathbb{R}\}$, we choose the basis A to consist of the monomials, that is, $\mathbf{a}_i = M_i$, where $M_i(x) = x^i$ for $i = 0, \ldots, 3$, and the basis B to consist of the first four Hermite polynomials, that is, $\mathbf{b}_i = H_i$ for $i = 0, \ldots, 3$. (We could show directly that these $\mathbf{b}_i$ are independent and so form a basis for $\mathcal{P}_3$, but this fact also follows, by Theorem 2.5.3 on page 92, from the result shown below that we can find unique coordinates of any $\mathbf{p}$ relative to B.) Then, according to Theorem 3.6.2, the columns of the matrix S are given by the coordinates of the $\mathbf{b}_i$ vectors relative to A. These columns can be read off the definitions of the Hermite polynomials, to give (in ascending order of degrees)

$$S = \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 2 & 0 & -12 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 8 \end{bmatrix}. \tag{3.142}$$

The coordinate vector of any $\mathbf{p}$ relative to A is $\mathbf{p}_A = (p_0, p_1, p_2, p_3)^T$, and its coordinate vector relative to B is given by $\mathbf{p}_B = S^{-1}\mathbf{p}_A$. Thus, multiplication of $\mathbf{p}_A$ by

$$S^{-1} = \frac{1}{8}\begin{bmatrix} 8 & 0 & 4 & 0 \\ 0 & 4 & 0 & 6 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3.143}$$

will give the coordinates of any $\mathbf{p}$ relative to B. For instance, for the polynomial $P(x) = 1 + 2x + 16x^3$ we have

$$\frac{1}{8}\begin{bmatrix} 8 & 0 & 4 & 0 \\ 0 & 4 & 0 & 6 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 0 \\ 16 \end{bmatrix} = \begin{bmatrix} 1 \\ 13 \\ 0 \\ 2 \end{bmatrix}. \tag{3.144}$$

Thus

$$P(x) = H_0(x) + 13H_1(x) + 2H_3(x), \tag{3.145}$$

that is,

$$1 + 2x + 16x^3 = 1 + 13(2x) + 2(8x^3 - 12x), \tag{3.146}$$

which is obviously true. ♦

We have just seen how vectors can be described in terms of their components relative to a basis, and how a change of basis affects those components. We can similarly define the components of a matrix relative to a basis and examine how they change with a change of basis. We discuss square matrices only, since it is mainly for them that such changes become necessary in applications. (For nonsquare matrices see Exercise 3.6.13.)

Let M be an $n \times n$ matrix and consider the associated mapping

$$\mathbf{y} = M\mathbf{x} \tag{3.147}$$

of $\mathbb{R}^n$ to itself. Let us write $\mathbf{x}$ and $\mathbf{y}$ in terms of the ordered basis $A = (\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n)$ as $\mathbf{x} = A\mathbf{x}_A$ and $\mathbf{y} = A\mathbf{y}_A$. Substituting from these equations into $\mathbf{y} = M\mathbf{x}$ we get $A\mathbf{y}_A = MA\mathbf{x}_A$ and, multiplying both sides by A^{-1},

$$\mathbf{y}_A = A^{-1}MA\mathbf{x}_A. \tag{3.148}$$

We call the matrix

$$M_A = A^{-1}MA \tag{3.149}$$

the matrix *representing M with respect to the basis A*. This means that M_A represents the *same mapping* of $\mathbf{x}$ to $\mathbf{y}$ in terms of the basis A as M does in terms of the standard basis. Similarly, if we introduce a second ordered basis $B = (\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n)$ into $\mathbb{R}^n$, then

$$M_B = B^{-1}MB \tag{3.150}$$

represents M with respect to the basis B. With this notation we have

$$\mathbf{y}_A = M_A\mathbf{x}_A \tag{3.151}$$

and

$$\mathbf{y}_B = M_B\mathbf{x}_B. \tag{3.152}$$

Solving Equations 3.149 and 3.150 for M, we obtain

$$M = AM_AA^{-1} = BM_BB^{-1}. \tag{3.153}$$

Hence

$$M_B = B^{-1}AM_AA^{-1}B = \left(A^{-1}B\right)^{-1} M_AA^{-1}B, \tag{3.154}$$

which can be expressed in terms of the transition matrix $S = A^{-1}B$ (see Equation 3.114) as

$$M_B = S^{-1}M_AS. \tag{3.155}$$

Again, we restate our findings as a theorem.

Theorem 3.6.3. ***(The Matrix of a Linear Mapping Relative to a Basis and Its Change Corresponding to a Change of Basis).*** *If A and B are $n \times n$ matrices whose columns form two bases of $\mathbb{R}^n$ and M is an $n \times n$ matrix representing the mapping $\mathbf{y} = M\mathbf{x}$ of $\mathbb{R}^n$ to itself, then $M_A = A^{-1}MA$ is the matrix representing M with respect to the basis A and so too is $M_B = B^{-1}MB$ with respect to the basis B. Furthermore, M_A and M_B are related by Equation 3.155, where $S = A^{-1}B$.*

Example 3.6.5. (The Matrix of a Reflection in $\mathbb{R}^2$ Relative to a Rotated Basis). In $\mathbb{R}^2$ let us change from the standard basis $\{\mathbf{i}, \mathbf{j}\}$ to the basis $\{\mathbf{b}_1, \mathbf{b}_2\}$, where $\mathbf{b}_1 = \frac{\sqrt{2}}{2}\begin{bmatrix}1\\1\end{bmatrix}$ and $\mathbf{b}_2 = \frac{\sqrt{2}}{2}\begin{bmatrix}-1\\1\end{bmatrix}$, as in Example 3.6.1.

Let us see how the matrix

$$M = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix} \tag{3.156}$$

changes with the change from the standard basis to the rotated one described above. This matrix transforms any vector $\mathbf{x} = (x_1, x_2)^T$ to $M\mathbf{x} = (x_2, x_1)^T$, and so it represents the reflection across the line $x_2 = x_1$. Then

$$M_B = B^{-1}MB = \frac{1}{2}\begin{bmatrix}1 & 1\\-1 & 1\end{bmatrix}\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}1 & -1\\1 & 1\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}. \tag{3.157}$$

Applied to any vector $\mathbf{x}_B$ this matrix gives

$$\mathbf{y}_B = M_B\mathbf{x}_B = \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}\begin{bmatrix}x_{B1}\\x_{B2}\end{bmatrix} = \begin{bmatrix}x_{B1}\\-x_{B2}\end{bmatrix}, \tag{3.158}$$

which shows that M_B represents the reflection across the x_{B1}-axis, as it should, since that axis is the line $x_2 = x_1$. ♦

Example 3.6.6. (The Matrix of a Rotation in $\mathbb{R}^2$ Relative to a Changed Basis). In $\mathbb{R}^2$ let us change from the standard basis $\{\mathbf{i}, \mathbf{j}\}$ to the basis $\{\mathbf{a}_1, \mathbf{a}_2\}$, where $\mathbf{a}_1 = \mathbf{i} = (1, 0)^T$ and $\mathbf{a}_2 = (1, 1)^T$. Then the transition matrix is

$$A = \begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}, \tag{3.159}$$

and

$$A^{-1} = \begin{bmatrix}1 & -1\\0 & 1\end{bmatrix}. \tag{3.160}$$

Hence for any vector $\mathbf{x} = (x_1, x_2)^T$ we have

$$\mathbf{x}_A = A^{-1}\mathbf{x} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 - x_2 \\ x_2 \end{bmatrix} \tag{3.161}$$

and

$$\mathbf{x} = (x_1 - x_2)\mathbf{a}_1 + x_2\mathbf{a}_2. \tag{3.162}$$

We can check that, indeed,

$$(x_1 - x_2)\begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 - x_2 \\ 0 \end{bmatrix} + \begin{bmatrix} x_2 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{x}. \tag{3.163}$$

Next, let us see how the rotation matrix

$$M = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

of Example 2.4.1 on page 67 is represented in the new basis. We find that

$$\begin{aligned} M_A = A^{-1}MA &= \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \cos\theta - \sin\theta & -2\sin\theta \\ \sin\theta & \cos\theta + \sin\theta \end{bmatrix}. \end{aligned} \tag{3.164}$$

♦

The type of transformation shown for M_A to M_B in Equation 3.155 with any invertible matrix S is of sufficient importance to merit a special name.

Definition 3.6.1. ***(Similar Matrices).*** *Let X denote the set of $n \times n$ matrices. For matrices A and B of X, we say that B is similar to A if there exists an invertible matrix S in X such that $B = S^{-1}AS$. The corresponding transformation of X to itself with a given S is called a similarity transformation.*

Observe that the formula $M_B = B^{-1}MB$ of Equation 3.150 describes a similarity transformation, too. It transforms the representation M of a mapping in the standard basis to its representation M_B relative to the basis B. In this case the transition matrix S happens to be B, as we have seen in Corollary 3.6.1.

Several important properties of similarity are stated in Exercises 3.6.17 and 3.6.18.

Exercises

Exercise 3.6.1. Let $\mathbf{a}_1 = (1, 2)^T$ and $\mathbf{a}_2 = (-2, 1)^T$ be the vectors of the ordered basis A for $\mathbb{R}^2$.

a. Find the transition matrix S for the change from the standard basis to A and
b. use S to find the components of $\mathbf{x} = (3, 5)^T$ in the basis A.

Exercise 3.6.2. Let $\mathbf{a}_1 = (1, 1, 2)^T$, $\mathbf{a}_2 = (0, -2, 1)^T$, and $\mathbf{a}_3 = (1, 1, 0)^T$ be the vectors of the ordered basis A for $\mathbb{R}^3$.
a. Find the transition matrix S for the change from the standard basis to A.
b. What is the transition matrix for the change from the basis A to the standard basis?
c. Use S to find the components of $\mathbf{x} = (3, 4, 5)^T$ in the basis A.

Exercise 3.6.3. a. Find the transition matrix S corresponding to the change of basis in $\mathbb{R}^2$ from $(\mathbf{a}_1, \mathbf{a}_2)$ to $(\mathbf{b}_1, \mathbf{b}_2)$, where $\mathbf{a}_1 = (1, 2)^T$, $\mathbf{a}_2 = (-2, 1)^T$, $\mathbf{b}_1 = (3, 2)^T$, $\mathbf{b}_2 = (1, 1)^T$.
b. Use S to write $\mathbf{x} = 3\mathbf{b}_1 - 2\mathbf{b}_2$ as $x_{A1}\mathbf{a}_1 + x_{A2}\mathbf{a}_2$.
c. Use S to write $\mathbf{x} = 2\mathbf{a}_1 + 4\mathbf{a}_2$ as $x_{B1}\mathbf{b}_1 + x_{B2}\mathbf{b}_2$.

Exercise 3.6.4. a. Find the transition matrix S corresponding to the change of basis in $\mathbb{R}^3$ from $(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3)$ to $(\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3)$ where $\mathbf{a}_1 = (1, 2, 0)^T$, $\mathbf{a}_2 = (-2, 1, 0)^T$, $\mathbf{a}_3 = (0, 0, 1)^T$, $\mathbf{b}_1 = (3, 2, 0)^T$, $\mathbf{b}_2 = (1, 1, 0)^T$, $\mathbf{b}_3 = (0, 1, 1)^T$.
b. Use S to write $\mathbf{x} = 2\mathbf{a}_1 + 4\mathbf{a}_2 + 3\mathbf{a}_3$ as $x_{B1}\mathbf{b}_1 + x_{B2}\mathbf{b}_2 + x_{B3}\mathbf{b}_3$.

Exercise 3.6.5. In $\mathbb{R}^3$ let $x_{A1} = x_2 - x_3$, $x_{A2} = x_3 - x_1$, $x_{A3} = x_1 + x_2$ give the transformation of the coordinates of a vector $\mathbf{x}$ upon a change from the standard basis $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ to a new basis $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$. Find the new basis vectors $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$.

Exercise 3.6.6. In $\mathbb{R}^2$ the matrix

$$M = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

represents the projection onto the x_1-axis. Find its representation M_A relative to the basis $(\mathbf{a}_1, \mathbf{a}_2)$ obtained from the standard basis by a rotation through an angle θ.

Exercise 3.6.7. Let $\mathbf{a}_1 = \mathbf{e}_3$, $\mathbf{a}_2 = \mathbf{e}_1$, and $\mathbf{a}_3 = \mathbf{e}_2$ be the vectors of the ordered basis A for $\mathbb{R}^3$.
a. Find the transition matrix S for the change from the standard basis to A,
b. use S to find the components of $\mathbf{x} = (3, 4, 5)^T$ in the basis A, and
c. find the representation M_A of the matrix

$$M = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

relative to the basis A.

Exercise 3.6.8. Explain why the reduction of the augmented matrix $[A|B]$ in Example 3.6.3 solves $AS = B$.

Exercise 3.6.9. Let V be the subspace of $\mathbb{R}^3$ spanned by the columns of

$$A = \begin{bmatrix} 1 & 0 \\ 2 & 4 \\ 3 & -1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 3 \\ 0 & -2 \\ 7 & 11 \end{bmatrix}.$$

a. Find the transition matrix from the basis A to the basis B and
b. the transition matrix from B to A.

Exercise 3.6.10. Find a basis C for the space V of the previous exercise so that $\mathbf{c}_1 = \mathbf{a}_1$ and $\mathbf{c}_2 \perp \mathbf{a}_1$.

Exercise 3.6.11. a. Find a formula that gives an arbitrary third degree polynomial as a linear combination of the first four Legendre polynomials: $L_0(x) = 1$, $L_1(x) = x$, $L_2(x) = \frac{1}{2}(3x^2 - 1)$, and $L_3(x) = \frac{1}{2}(5x^3 - 3x)$.
b. Express the polynomial $P(x) = 1 - 2x + 3x^2 - 4x^3$ as a linear combination of these Legendre polynomials.

Exercise 3.6.12. Let L and M be $n \times n$ matrices and A a basis matrix of $\mathbb{R}^n$. Show that
a. $(L + M)_A = L_A + M_A$ and
b. $(LM)_A = L_A M_A$.

Exercise 3.6.13. Suppose M is an $m \times n$ matrix, representing a mapping from $\mathbb{R}^n$ to $\mathbb{R}^m$. If we change from the standard basis I_n in $\mathbb{R}^n$ to an arbitrary ordered basis A, and from I_m in $\mathbb{R}^m$ to some B, then find the representation $M_{A,B}$ of M relative to the new bases.

Exercise 3.6.14. Show that all matrices of the form

$$M(t) = \begin{bmatrix} 1 & t \\ 0 & 2 \end{bmatrix}$$

are similar to one another. (*Hint*: For any values t and t' try to find an invertible matrix S, depending on t and t', such that $SM(t) = M(t')S$ holds.)

Exercise 3.6.15. Show that $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is not similar to $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Exercise 3.6.16. Is $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ similar to $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$?

Exercise 3.6.17. Show that if A and B are similar matrices, then
a. A^T and B^T are similar,
b. A^k and B^k are similar for any positive integer k, and
c. if additionally A is invertible, then so is B, and A^{-1} and B^{-1} are similar as well.

***Exercise 3.6.18.** Show that similarity of matrices is an equivalence relation. (A relation $\sim$ is called an equivalence relation on a set X if it is:

1. Reflexive, that is, $A \sim A$ holds for all A in X,
2. Symmetric, that is, $A \sim B$ implies $B \sim A$ for all $A,\ B$ in X, and
3. Transitive, that is, $A \sim B$ and $B \sim C$ imply $A \sim C$ for all $A,\ B,\ C$ in X.)

Exercise 3.6.19. Show that if A and B are similar matrices, then $\operatorname{rank}(A) = \operatorname{rank}(B)$. (*Hint*: Show first that if A and B are similar, then their null spaces have the same dimension.)

Exercise 3.6.20. The sum of the diagonal elements of a square matrix is called its *trace*: For an $n \times n$ matrix $\operatorname{Tr}(A) = \sum_{i=1}^{n} a_{ii}$.
a. Show that if A and B are $n \times n$ matrices, then $\operatorname{Tr}(AB) = \operatorname{Tr}(BA)$.
b. Apply the result of Part a to S and BS^{-1} to show that if A and B are similar so that $A = SBS^{-1}$, then $\operatorname{Tr}(A) = \operatorname{Tr}(B)$.
c. Is $\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$ similar to $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$?

MATLAB Exercises

Exercise 3.6.21. Let

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 3 & 4 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$$

be a basis matrix for $\mathbb{R}^4$. Find the coordinate vector $\mathbf{x}_A$ relative to this basis for each of the $\mathbf{x}$ vectors $\mathbf{e}_1,\ \mathbf{e}_4,\ (1,1,1,1)^T$, and $(1,2,3,4)^T$.

Exercise 3.6.22. a. Find the transition matrix S from the basis A in the previous exercise to the standard basis.
b. Use this S to compute $\mathbf{x}$ if $\mathbf{x}_A$ is $\mathbf{e}_1,\ \mathbf{e}_4,\ (1,1,1,1)^T$, or $(1,2,3,4)^T$.

Exercise 3.6.23. Let

$$B = \begin{bmatrix} 1 & 0 & 1 & 4 \\ 3 & 4 & 0 & 3 \\ 2 & 4 & 4 & 6 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

be a basis matrix for $\mathbb{R}^4$.
a. Find the transition matrix S from the basis A in Exercise 3.6.21 to this basis.
b. Find the transition matrix S' from the basis B to the basis A.
c. Use S or S' to compute $\mathbf{x}_B$ if $\mathbf{x}_A$ is $\mathbf{e}_1,\ \mathbf{e}_4,\ (1,1,1,1)^T$, or $(1,2,3,4)^T$.

d. Find the representative matrix A_B of the matrix A relative to the basis B.
e. Find the representative matrix B_A of the matrix B relative to the basis A.

Exercise 3.6.24. Let

$$A = \begin{bmatrix} 1 & 0 & 1 & 4 & -1 & 0 \\ 3 & 4 & 0 & 3 & 2 & 1 \\ 1 & 2 & 2 & 3 & 0 & 3 \\ 0 & 0 & 2 & 2 & 0 & 0 \\ 2 & 4 & 4 & 6 & 0 & 6 \\ 0 & -1 & 0 & 1 & 1 & 0 \end{bmatrix}.$$

Use the MATLAB command **rref** to find a basis matrix B for Col(A). The command $C = \textbf{orth}(A)$ creates another basis matrix C for Col(A). Find the transition matrix S from the basis B to the basis C. (*Hint*: By Theorem 3.6.2 you need to solve $BS = C$.)

4. *Linear Transformations*

4.1 Representation of Linear Transformations by Matrices

Beginning with our first discussion of matrix operations, we have seen matrices being used to represent mappings or transformations.[1] For instance, the rotation matrix of Example 4.2.1 on page 67 was introduced for that purpose, and the notion of similarity of matrices in the previous section resulted from the fact that similar matrices represent the same transformation in different bases. In this section we want to explore the connections between matrices and mappings more systematically.

First, however, we define mappings on arbitrary vector spaces.

Definition 4.1.1. ***(Mappings from a Vector Space to a Vector Space).*** *Given two vector spaces U and V and a subset W of U, a mapping $T : W \to V$ is an association $\mathbf{y} = T(\mathbf{x})$ of all elements $\mathbf{x}$ of W to elements $\mathbf{y}$ of V. The set W is called the domain of T.*

The next question that arises quite naturally is: What kinds of mappings can be represented by matrices? The answer is fairly simple.

If A is an $m \times n$ matrix, then the equation $\mathbf{y} = A\mathbf{x}$ describes a mapping of $\mathbb{R}^n$ to $\mathbb{R}^m$. This mapping has two fundamental properties: First, if $\mathbf{x}_1$ and $\mathbf{x}_2 \in \mathbb{R}^n$ are mapped to $\mathbf{y}_1$ and $\mathbf{y}_2 \in \mathbb{R}^m$ respectively, then $\mathbf{x}_1 + \mathbf{x}_2$ is mapped to $\mathbf{y}_1 + \mathbf{y}_2$ and, second, $c\mathbf{x}_1$ is mapped to $c\mathbf{y}_1$, for every scalar c. Thus every mapping representable by a matrix A via the equation $\mathbf{y} = A\mathbf{x}$ must have these two properties. Luckily these properties are also sufficient; that is, any mapping of $\mathbb{R}^n$ to $\mathbb{R}^m$ with these two properties is representable by a matrix in this manner, as we shall see shortly. Such mappings have a special name.

Definition 4.1.2. ***(Linear Transformation).*** *A mapping or transformation T of a vector space U to a vector space V is called linear if it preserves vector addition and multiplication by scalars; that is, if for all $\mathbf{x}_1, \mathbf{x}_2 \in U$ and all scalars c*

$$T(\mathbf{x}_1 + \mathbf{x}_2) = T(\mathbf{x}_1) + T(\mathbf{x}_2) \tag{4.1}$$

[1] The two terms are used interchangeably.

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and

$$T(c\mathbf{x}_1) = cT(\mathbf{x}_1). \tag{4.2}$$

A mapping T satisfying the first condition is said to be additive; one satisfying the second condition is said to be homogeneous.[2, 3]

We can combine these two requirements into a single form.

Lemma 4.1.1. ***(Combining the Conditions for Linearity).*** *A mapping T of a vector space U to a vector space V is linear if and only if*

$$T(a\mathbf{x}_1 + b\mathbf{x}_2) = aT(\mathbf{x}_1) + bT(\mathbf{x}_2) \tag{4.3}$$

for all $\mathbf{x}_1, \mathbf{x}_2 \in U$ and all scalars a, b.

Lemma 4.1.1 can be generalized to linear combinations of several terms, not just of two.

Corollary 4.1.1. ***(Combined n-Term Condition for Linearity).*** *Let n be any integer ≥ 2. A mapping T of a vector space U to a vector space V is linear if and only if it preserves all linear combinations of n terms; that is, if and only if for all vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n \in U$ and all scalars $c_1, c_2, \ldots, c_n$,*

$$T\left(\sum_{i=1}^{n} c_i \mathbf{x}_i\right) = \sum_{i=1}^{n} c_i T(\mathbf{x}_i). \tag{4.4}$$

We leave the proofs of Lemma 4.1.1 and Corollary 4.1.1 to the reader as Exercise 4.1.1.

Before continuing with the properties of linear transformations, we list examples that arise more naturally without matrices, and some on arbitrary vector spaces, not just on Euclidean ones.

Example 4.1.1. (Zero Mapping). For any two vector spaces U and V, the mapping $T : U \to V$ defined by $T\mathbf{x} = \mathbf{0}$ for all $\mathbf{x} \in U$ is called the zero mapping and will be denoted by $T = O$. ♦

Example 4.1.2. (Identity Mapping). For any vector space U, the mapping $T : U \to U$ defined by $T\mathbf{x} = \mathbf{x}$ for all $\mathbf{x} \in U$ is called the identity mapping and will be denoted by $T = I$ or I_U. ♦

Example 4.1.3. (Scalar Mapping). For any vector space U and any fixed scalar c, the mapping $T : U \to U$ defined by $T\mathbf{x} = c\mathbf{x}$ for all $\mathbf{x} \in U$ is called a scalar mapping. The linearity of T follows from the axioms of a vector space. Clearly, O and I are scalar mappings. ♦

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[2] By convention, $cT(\mathbf{x}_1)$ stands for $c\,(T(\mathbf{x}_1))$.

[3] Note that the above use of the word *linear* is more restrictive than is customary in calculus, where a linear function is defined as one of the form $Ax + b$ rather than just Ax.

Example 4.1.4. (A Projection to a Subspace). The mapping $T:\mathbb{R}^n \to \mathbb{R}^n$ given by $T(x_1, x_2, \ldots, x_n) = (0, x_2, \ldots, x_n)$ is a linear mapping. ♦

Example 4.1.5. (Projection of $\mathbb{R}^n$ to the x_1-Axis). $T:\mathbb{R}^n \to \mathbb{R}^n$ given by $T(x_1, x_2, \ldots, x_n) = (x_1, 0, 0, \ldots, 0)$ is a linear mapping. ♦

Example 4.1.6. (A Projection of $\mathbb{R}^n$ to $\mathbb{R}^1$). The mapping $T:\mathbb{R}^n \to \mathbb{R}^1$ given by $T(x_1, x_2, \ldots, x_n) = x_1$ is a linear mapping. ♦

Example 4.1.7. (Differentiation Mapping). The differentiation mapping on the space of polynomials, $D:\mathcal{P} \to \mathcal{P}$ given by $DP(x) = P'(x)$ (derivative of P), is clearly linear. ♦

Example 4.1.8. (Integration Mapping). The integration mapping on the space of polynomials, $T:\mathcal{P} \to \mathcal{P}$ given by $TP(x) = \int_0^x P(t)\,dt$, is also linear. ♦

Corollary 4.1.1 can be combined with the definition of a basis to obtain the following observation.

Theorem 4.1.1. ***(A Linear Mapping Is Determined by Its Action on a Basis).*** *A linear mapping T of a finite-dimensional vector space U to a vector space V is completely determined by its action on the vectors of any basis of U. In other words, if $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ form a basis for U and $T(\mathbf{a}_1), T(\mathbf{a}_2), \ldots, T(\mathbf{a}_n)$ are arbitrarily prescribed vectors of V, then $T(\mathbf{x})$ is uniquely determined for every $\mathbf{x} \in U$.*

Proof. By the definition of a basis, if $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ form a basis for U, then we can write any $\mathbf{x} \in U$ uniquely as

$$\mathbf{x} = \sum_{i=1}^{n} x_{Ai}\mathbf{a}_i, \tag{4.5}$$

where the x_{Ai} are the coordinates of $\mathbf{x}$ relative to the ordered basis $A = (\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n)$. Then, from this equation and Corollary 4.1.1, the linearity of T determines $T(\mathbf{x})$ uniquely as

$$T(\mathbf{x}) = \sum_{i=1}^{n} x_{Ai}T(\mathbf{a}_i). \tag{4.6}$$

We leave it as Exercise 4.1.2 to show that, conversely, defining $T(\mathbf{x})$ by Equation 4.6 implies the linearity of T. ■

Let us specialize now to mappings from $\mathbb{R}^n$ to $\mathbb{R}^m$ and to the standard basis for $\mathbb{R}^n$. Denote the transform $T(\mathbf{e}_i) \in \mathbb{R}^m$ of the standard vector $\mathbf{e}_i \in \mathbb{R}^n$ by $\mathbf{t}_i$, for $i = 1, 2, \ldots, n$, and the $m \times n$ matrix with these vectors as columns by $[T]$. Then the above equations yield

$$\mathbf{x} = \sum_{i=1}^{n} x_i\mathbf{e}_i \tag{4.7}$$

and

$$T(\mathbf{x}) = \sum_{i=1}^{n} x_i T(\mathbf{e}_i) = \sum_{i=1}^{n} x_i \mathbf{t}_i = (\mathbf{t}_1, \mathbf{t}_2, \ldots, \mathbf{t}_n) \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = [T]\mathbf{x}. \tag{4.8}$$

Let us summarize our result as a theorem.

Theorem 4.1.2. ***(Representing a Linear Mapping in Euclidean Spaces by a Matrix).*** *If T is a linear mapping from $\mathbb{R}^n$ to $\mathbb{R}^m$ and $[T]$ is the $m \times n$ matrix whose columns are the vectors $\mathbf{t}_i = T(\mathbf{e}_i)$ of $\mathbb{R}^m$, for each standard vector $\mathbf{e}_i$ of $\mathbb{R}^n$, then the mapping T corresponds to multiplication by the matrix $[T]$, so that*

$$T(\mathbf{x}) = [T]\mathbf{x} \tag{4.9}$$

for every $\mathbf{x} \in \mathbb{R}^n$. The matrix $[T]$ can also be obtained by factoring $T(\mathbf{x})$ as in Equation 4.8.

Let us now look at some examples of linear transformations and their matrix representations.

Example 4.1.9. (Finding the Matrix Corresponding to a Certain Transformation from $\mathbb{R}^3$ to $\mathbb{R}^2$). Let T denote the transformation from $\mathbb{R}^3$ to $\mathbb{R}^2$ given by

$$T(\mathbf{x}) = \begin{bmatrix} x_1 - x_2 \\ x_1 + x_3 \end{bmatrix} \tag{4.10}$$

and find the matrix $[T]$ that represents this transformation.

By Theorem 4.1.2 all we need to do is to find the transforms of the standard vectors $\mathbf{e}_1 = (1, 0, 0)^T$, $\mathbf{e}_2 = (0, 1, 0)^T$, and $\mathbf{e}_3 = (0, 0, 1)^T$. Substituting these vectors for $\mathbf{x}$, one after the other, into Equation 4.10, we obtain

$$\mathbf{t}_1 = T(\mathbf{e}_1) = \begin{bmatrix} 1 - 0 \\ 1 + 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \tag{4.11}$$

$$\mathbf{t}_2 = T(\mathbf{e}_2) = \begin{bmatrix} 0 - 1 \\ 0 + 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \tag{4.12}$$

and

$$\mathbf{t}_3 = T(\mathbf{e}_3) = \begin{bmatrix} 0 + 0 \\ 0 + 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{4.13}$$

Thus

$$[T] = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \tag{4.14}$$

is the matrix that represents the given transformation. It is easy to see that $[T]\mathbf{x}$ is indeed the same as $T(\mathbf{x})$ for every $\mathbf{x}$, that is,

$$[T]\mathbf{x} = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 - x_2 \\ x_1 + x_3 \end{bmatrix} = T(\mathbf{x}). \tag{4.15}$$

As remarked at the end of Theorem 4.1.2, we may obtain $[T]$ alternatively by factoring $T(\mathbf{x})$ as follows:

$$T(\mathbf{x}) = \begin{bmatrix} x_1 - x_2 \\ x_1 + x_3 \end{bmatrix} = \begin{bmatrix} 1x_1 - 1x_2 + 0x_3 \\ 1x_1 + 0x_2 + 1x_3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \tag{4.16}$$

and we can read off here the same matrix for $[T]$ that we have found before. ♦

Example 4.1.10. (Finding the Matrix Corresponding to a Reflection in $\mathbb{R}^2$***).*** Let us find the matrix that represents reflection across the line $y = x$ in $\mathbb{R}^2$.

Again we need only find the action of the transformation on the standard vectors. This is obviously a transformation from $\mathbb{R}^2$ to itself, and from the description we find that

$$\mathbf{t}_1 = T(\mathbf{e}_1) = \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{4.17}$$

and

$$\mathbf{t}_2 = T(\mathbf{e}_2) = \mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{4.18}$$

Thus

$$[T] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \tag{4.19}$$

♦

Example 4.1.11. (Finding the Matrix Corresponding to a Reflection in $\mathbb{R}^2$ ***Followed by a Stretch).*** Let us find the matrix $[T]$ that represents reflection across the y-axis in $\mathbb{R}^2$ followed by a twofold stretch in the x direction.

The matrix $[R]$ for the reflection can be obtained, in a way similar to that used in the previous example, from

$$\mathbf{r}_1 = R(\mathbf{e}_1) = -\mathbf{e}_1 = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \tag{4.20}$$

and

$$\mathbf{r}_2 = R(\mathbf{e}_2) = \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{4.21}$$

as

$$[R] = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{4.22}$$

Similarly, the matrix of the stretch S is

$$[S] = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \tag{4.23}$$

and the matrix $[T]$ of the composite transformation is obtained, by the definition of the matrix product, from

$$S(R(\mathbf{x})) = S([R]\mathbf{x}) = [S]([R]\mathbf{x}) = [S][R]\mathbf{x} \tag{4.24}$$

as

$$[T] = [S][R] = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}. \tag{4.25}$$

Note that the action of $[S]$ follows that of $[R]$ even though $[S]$ is written first in going from left to right. This happens because $[S][R]$ is defined so that $([S][R])\mathbf{x} = [S]([R]\mathbf{x})$ holds for every vector $\mathbf{x}$. ♦

Linear transformations can be represented by matrices in terms of any bases, not just the standard bases, and such representations are possible in every finite-dimensional vector space, not just in $\mathbb{R}^n$ and $\mathbb{R}^m$.

Theorem 4.1.3. ***(Representing a Linear Mapping in Arbitrary Finite-Dimensional Vector Spaces by a Matrix).*** *Let T be a linear transformation from a finite-dimensional vector space U to a finite-dimensional vector space V, and let $A = (\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n)$ be an ordered basis*[4] *for U, and $B = (\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_m)$ an ordered basis for V. Write the vectors of U and V in terms of these bases as*

$$\mathbf{x} = \sum_{j=1}^{n} x_{Aj}\mathbf{a}_j \text{ and } \mathbf{y} = \sum_{i=1}^{m} y_{Bi}\mathbf{b}_i. \tag{4.26}$$

Then there exists a unique $m \times n$ matrix $T_{A,B}$ that represents T relative to these ordered bases so that $\mathbf{y} = T(\mathbf{x})$ becomes

$$\mathbf{y}_B = T_{A,B}\mathbf{x}_A. \tag{4.27}$$

[4] For terminology and notation see Section 3.6.

Here $\mathbf{x}_A \in \mathbb{R}^n$ *and* $\mathbf{y}_B \in \mathbb{R}^m$ *are given by* $\mathbf{x}_A = (x_{A1}, x_{A2}, \ldots, x_{An})^T$ *and* $\mathbf{y}_B = (y_{B1}, y_{B2}, \ldots, y_{Bm})^T$.

Proof. We have

$$T(\mathbf{x}) = T\left(\sum_{j=1}^{n} x_{Aj}\mathbf{a}_j\right) = \sum_{j=1}^{n} x_{Aj} T(\mathbf{a}_j) \tag{4.28}$$

and $T(\mathbf{a}_j)$, being an element of V, can be written with appropriate coefficients as

$$T(\mathbf{a}_j) = \sum_{i=1}^{m} [T_{A,B}]_{ij}\mathbf{b}_i. \tag{4.29}$$

Then

$$\begin{aligned}\mathbf{y} = T(\mathbf{x}) &= \sum_{j=1}^{n} x_{Aj} \sum_{i=1}^{m} [T_{A,B}]_{ij}\mathbf{b}_i \\ &= \sum_{i=1}^{m}\left(\sum_{j=1}^{n} [T_{A,B}]_{ij} x_{Aj}\right)\mathbf{b}_i = \sum_{i=1}^{m} [T_{A,B}\mathbf{x}_A]_i\, \mathbf{b}_i \text{ for all } \mathbf{x} \in U.\end{aligned} \tag{4.30}$$

Since, on the other hand,

$$\mathbf{y} = \sum_{i=1}^{m} y_{Bi}\mathbf{b}_i, \tag{4.31}$$

Equation 4.27 must hold. ∎

Corollary 4.1.2. ***(Representing a Linear Mapping in Euclidean Spaces by a Matrix Relative to Arbitrary Bases).*** *If* $U = \mathbb{R}^n$ *and* $V = \mathbb{R}^m$, *then the ordered bases in Theorem 4.1.3 can be considered as matrices* $A = (\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n)$ *and* $B = (\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_m)$, *and we have*

$$T_{A,B} = B^{-1}[T]A. \tag{4.32}$$

Proof. In this case

$$B\mathbf{y}_B = \mathbf{y} = T(\mathbf{x}) = T(A\mathbf{x}_A) = [T]A\mathbf{x}_A, \tag{4.33}$$

and multiplying through by B^{-1}, we get the statement. ∎

Example 4.1.12. (Finding the Matrix Corresponding to a Certain Transformation from $\mathbb{R}^3$ ***to*** $\mathbb{R}^2$ ***Relative to Given Bases).*** Let T denote the transformation from $\mathbb{R}^3$ to $\mathbb{R}^2$ given by

$$T(\mathbf{x}) = \begin{bmatrix} x_1 - x_2 \\ x_1 + x_3 \end{bmatrix} \tag{4.34}$$

as in Example 4.1.9, and consider the ordered bases given by the columns of

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \tag{4.35}$$

and

$$B = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}. \tag{4.36}$$

Find the matrix $T_{A,B}$ that represents this transformation.

Taking $[T]$ from Example 4.1.9, computing the inverse of B, and substituting into Equation 4.32, we get

$$T_{A,B} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -1 & -2 & -1 \\ 1 & 0 & 3 \end{bmatrix}. \tag{4.37}$$

♦

The formalism we discussed earlier of representing a transformation T from $\mathbb{R}^n$ to $\mathbb{R}^m$ by a matrix $[T]$ is a particular case of the present formalism, using the standard ordered bases with matrix $A = I_n$ and $B = I_m$ respectively. Thus in the present notation $[T] = T_{I_n,I_m}$, and $[T]$ is the matrix representing T relative to the standard ordered bases.

Sometimes the matrix $T_{A,B}$ can be obtained only directly from its definition by Equation 4.29, as in the following example.

Example 4.1.13. (Finding the Matrix Corresponding to Differentiation in $\mathcal{P}_n$). Let $U = \{\mathbf{p} : \mathbf{p} = p_0 + p_1x + \cdots + p_nx^n\}$ be the space $\mathcal{P}_n$ of polynomials of degree n or less together with the zero polynomial and $V = \{\mathbf{q} : \mathbf{q} = q_0 + q_1x + \cdots + q_{n-1}x^{n-1}\}$ the space $\mathcal{P}_{n-1}$ of polynomials of degree $n-1$ or less together with the zero polynomial. Define the *differentiation map* D from U to V by

$$D(\mathbf{p}) = D(p_0 + p_1x + \cdots + p_nx^n) = p_1 + 2p_2x + \cdots + np_nx^{n-1}. \tag{4.38}$$

It is easy to see that D is linear.

Consider the ordered bases $A = (1, x, \ldots, x^n)$ and $B = (1, x, \ldots, x^{n-1})$. Then, with the previous notation we have $\mathbf{a}_j = x^{j-1}$ and $\mathbf{b}_i = x^{i-1}$. Furthermore,

$$D(\mathbf{a}_j) = (j-1)x^{j-2} \tag{4.39}$$

for $j = 1, 2, \ldots, n+1$. Thus

$$\begin{aligned} D(\mathbf{a}_1) &= 0\mathbf{b}_1 + 0\mathbf{b}_2 + \cdots + 0\mathbf{b}_n, \\ D(\mathbf{a}_2) &= 1\mathbf{b}_1 + 0\mathbf{b}_2 + \cdots + 0\mathbf{b}_n, \\ D(\mathbf{a}_3) &= 0\mathbf{b}_1 + 2\mathbf{b}_2 + \cdots + 0\mathbf{b}_n, \\ &\vdots \\ D(\mathbf{a}_{n+1}) &= 0\mathbf{b}_1 + 0\mathbf{b}_2 + \cdots + n\mathbf{b}_n. \end{aligned} \tag{4.40}$$

According to Equation 4.29 the coefficients of the $\mathbf{b}_i$ here form the transpose of the matrix that represents D relative to these bases, which we now denote by $D_{A,B}$. Thus

$$D_{A,B} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & n \end{bmatrix}. \tag{4.41}$$

The coordinate vector $\mathbf{p}_A = (p_0, p_1, \ldots, p_n)^T$ of $\mathbf{p} = p_0 + p_1 x + \cdots + p_n x^n$ is transformed according to Equation 4.27 into

$$\mathbf{q}_B = D_{A,B}\mathbf{p}_A = (p_1, 2p_2, \ldots, np_n)^T. \tag{4.42}$$

Note that a standard basis is defined only for $\mathbb{R}^n$, and for $\mathcal{P}_n$ and $\mathcal{P}_{n-1}$ the bases A and B come closest to the notion of a standard basis. Also note that we could have defined D as a mapping from $\mathcal{P}_n$ to itself, and in that case $D_{A,B}$ would have had to be amended by an extra all-zero row. ♦

Exercises

Exercise 4.1.1.
a. Prove Lemma 4.1.1 and Corollary 4.1.1.
b. Prove that $T(\mathbf{0}) = \mathbf{0}$ for every linear transformation T.

Exercise 4.1.2. Show that if $T(\mathbf{x})$ is defined by Equation 4.6, then T is a linear transformation, that is, it satisfies Equations 4.1 and 4.2.

***Exercise 4.1.3.** Is it true that a mapping from $\mathbb{R}^n$ to $\mathbb{R}^m$ is linear if and only if it preserves straight lines, that is, if and only if, given any $\mathbf{x}_0$ and $\mathbf{a} \in \mathbb{R}^n$, the vectors $\mathbf{x} = \mathbf{x}_0 + t\mathbf{a}$, for every scalar t, are mapped into vectors $\mathbf{y} = \mathbf{y}_0 + t\mathbf{b}$ in $\mathbb{R}^m$? If this is true, prove it, and if not, then prove an appropriate modification.

Exercise 4.1.4. Determine whether each of the following transformations is linear or not and explain:
a. $T : \mathbb{R}^n \to \mathbb{R}$ such that $T(\mathbf{x}) = \mathbf{a} \cdot \mathbf{x}$, with $\mathbf{a}$ a fixed vector of $\mathbb{R}^n$.
b. $T : \mathbb{R}^n \to \mathbb{R}^n$ such that $T(\mathbf{x}) = (A - \lambda I)\mathbf{x}$, with A a fixed $n \times n$ matrix, I the $n \times n$ unit matrix, and λ any number. (This kind of operation will play an important role in Chapter 7.)
c. $T : \mathbb{R}^n \to \mathbb{R}^m$ such that $T(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$, with A a fixed $m \times n$ matrix and $\mathbf{b}$ a fixed nonzero vector of $\mathbb{R}^m$.
d. $T : \mathbb{R}^n \to \mathbb{R}$ such that $T(\mathbf{x}) = |\mathbf{x}|$.
e. $T : \mathbb{R}^n \to \mathbb{R}$ such that $T(\mathbf{x}) = \mathbf{x}^T\mathbf{a}$, with $\mathbf{a}$ a fixed vector of $\mathbb{R}^n$.
f. $T : \mathbb{R}^n \to \mathbb{R}^m$ such that $T(\mathbf{x}) = (\mathbf{a}^T\mathbf{x})\mathbf{b}$, with $\mathbf{a}$ a fixed vector of $\mathbb{R}^n$ and $\mathbf{b}$ a fixed vector of $\mathbb{R}^m$.

Exercise 4.1.5. Find the matrix $[T]$ that represents the transformation from $\mathbb{R}^2$ to $\mathbb{R}^3$ given by

$$T(\mathbf{x}) = \begin{bmatrix} x_1 - x_2 \\ 2x_1 + 3x_2 \\ 3x_1 + 2x_2 \end{bmatrix}.$$

Exercise 4.1.6. Find the matrix $[T]$ that represents the transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$ given by

$$T(\mathbf{x}) = \begin{bmatrix} x_1 - x_2 \\ x_2 - x_3 \\ x_3 - x_1 \end{bmatrix}.$$

Exercise 4.1.7. Find the matrix $[T]$ that represents the transformation of Exercise 4.1.4f above. (This matrix is called the *tensor product of* $\mathbf{b}$ *and* $\mathbf{a}$ and is usually denoted by $\mathbf{b} \otimes \mathbf{a}$.) It can also be expressed as an outer product (see Equation 2.105). How?

Exercise 4.1.8. Find the matrix $[T]$ and the corresponding linear transformation T from $\mathbb{R}^2$ to $\mathbb{R}^3$ that map the vector $(1, 1)^T$ to $(1, 1, 1)^T$ and $(1, -1)^T$ to $(1, -1, -1)^T$.

Exercise 4.1.9. Find the matrix $[T]$ and the corresponding linear transformation T from $\mathbb{R}^3$ to $\mathbb{R}^2$ that map the vector $(1, 1, 1)^T$ to $(1, 1)^T$, the vector $(1, -1, -1)^T$ to $(1, -1)^T$ and $(1, 1, 0)^T$ to $(1, 0)^T$.

Exercise 4.1.10. Let $A = (\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n)$ be a basis matrix of $\mathbb{R}^n$ and $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n$ arbitrary vectors of $\mathbb{R}^m$. Let T be the linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ for which $T(\mathbf{a}_i) = \mathbf{b}_i$. Find the corresponding matrix $[T]$. (*Hint*: Use $(T(\mathbf{a}_1), T(\mathbf{a}_2), \ldots, T(\mathbf{a}_n)) = [T]A$.)

Exercise 4.1.11. Find the matrix $[T]$ that represents a twofold stretch of $\mathbb{R}^2$ in the $y = x$ direction. (*Hint*: Rotate by 45°, stretch, and rotate by −45 °.)

***Exercise 4.1.12.** Find the matrix $[T]$ that represents the reflection of $\mathbb{R}^2$ across the $ax + by = 0$ line.

***Exercise 4.1.13.** Use the result of the previous exercise to show that the composition of two reflections of $\mathbb{R}^2$ across any two lines through the origin equals a rotation. Describe this rotation geometrically.

Exercise 4.1.14. Show that every linear transformation transforms parallel lines into parallel lines.

***Exercise 4.1.15.** Find all linear transformations T from $\mathbb{R}^2$ to $\mathbb{R}^2$ that map perpendicular lines into perpendicular lines.

Exercise 4.1.16. Verify Equation 4.27 for the result of Example 4.1.12.

Exercise 4.1.17. Find the matrix $T_{A,B}$ that represents the transformation T of Exercise 4.1.6 relative to the ordered bases given by

$$A = B = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

Exercise 4.1.18. Find on the space $\mathcal{P}_n$ the representative matrix of the operation D relative to the ordered basis

$$A = B = (1, 1 + x, 1 + x + x^2, \ldots, 1 + x + \cdots + x^n).$$

(Here D is the transformation of Example 4.1.13.)

Exercise 4.1.19. Let $U = \{\mathbf{p} : \mathbf{p} = p_0 + p_1 x + \cdots + p_n x^n\}$ be the space $\mathcal{P}_n$ of polynomials of degree n or less together with the zero polynomial and $V = \{q : q = q_0 + q_1 x + \cdots + q_{n+1} x^{n+1}\}$ the space $\mathcal{P}_{n+1}$ of polynomials of degree $n+1$ or less together with the zero polynomial. Define the *integration map* T from U to V by

$$T(p) = T(p_0 + p_1 x + \cdots + p_n x^n) = p_0 x + \frac{p_1 x^2}{2} + \cdots + \frac{p_n x^{n+1}}{n+1}.$$

Find the matrix $T_{A,B}$ that represents this transformation relative to the ordered bases $A = (1, x, \ldots, x^n)$ and $B = (1, x, \ldots, x^{n+1})$.

Exercise 4.1.20. For the same spaces with the same bases as in the previous exercise, find the representative matrix of the transformation X corresponding to multiplication by x, that is, of X such that $X(p_0 + p_1 x + \cdots + p_n x^n) = p_0 x + p_1 x^2 + \cdots + p_n x^{n+1}$.

MATLAB Exercises

Exercise 4.1.21. Let

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 3 & 4 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$$

be a basis matrix for $\mathbb{R}^4$. Find the matrix $[T]$ of the linear transformation that transforms the columns of this matrix into the corresponding columns of

$$B = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 2 & 2 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}.$$

Exercise 4.1.22. Let $A = \mathbf{magic}(4)$.
a. Use **rref** on A and A^T to find a basis B for A's row space and a basis C for its column space.
b. Find the matrix $[T]$ of the linear transformation from $\text{Row}(A)$ to $\text{Col}(A)$ that maps the basis B to the basis C, relative to the standard basis.
c. Find the matrix $T_{B,C}$ of the same linear transformation relative to the bases B and C.

Exercise 4.1.23. In MATLAB, polynomials are stored as row vectors of their coefficients in order of descending powers. For example, the polynomial $f(x) = 3x^2 - 2x + 1$ can be entered as $\mathbf{f} = [3\ {-2}\ 1]$. The product of polynomials is computed by the function **conv**. Thus, if $g(x) = 2x^2 + x$ and $\mathbf{g} = [2\ 1\ 0]$, then $\mathbf{conv}(\mathbf{f}, \mathbf{g})$ produces $[6\ {-1}\ 2\ 1\ 0]$, corresponding to $f(x)g(x) = 6x^4 - x^3 + 2x^2 + x$.
a. Show by hand that if f is the polynomial above and $\mathbf{p} \in \mathcal{P}_4$, then the mapping T from $\mathcal{P}_4$ to $\mathcal{P}_6$ given by $\mathbf{p} \to \mathbf{conv}(\mathbf{f}, \mathbf{p})$ is linear.
b. Use MATLAB and judiciously selected values of $\mathbf{p}$ to find the matrix $[T]$ of this transformation, that is, a matrix $[T]$ such that $[T]\mathbf{p} = \mathbf{conv}(\mathbf{f}, \mathbf{p})$ for every $\mathbf{p} \in \mathcal{P}_4$.

4.2 Properties of Linear Transformations

Since linear transformations are particular types of functions, all the various concepts relevant to functions in general, apply to linear transformations. Also, they have some properties that are significant for linear functions only. We first define these concepts one by one, then discuss and illustrate them, and last, describe their relationships to corresponding concepts for the representative matrices $T_{A,B}$.

Definition 4.2.1. ***(Properties of Transformations).*** *Let T be a transformation from a vector space U to a vector space V.*
1. The range of T, denoted by $\mathrm{Range}(T)$, *is the subset* $\{\mathbf{y} : \mathbf{y} = T(\mathbf{x}),\ \mathbf{x} \in U\}$ *of V.*
2. T is said to be a mapping onto V if $\mathrm{Range}(T) = V$.
3. T is said to be one-to-one if $T(\mathbf{x}) = T(\mathbf{y})$ implies $\mathbf{x} = \mathbf{y}$ for all $\mathbf{x}, \mathbf{y}$.
4. T is said to be an isomorphism if it is linear and both one-to-one and onto. Furthermore, two vector spaces U and V are said to be isomorphic to each other if there exists an isomorphism from U to V.[5]
5. The kernel of T, denoted by $\mathrm{Ker}(T)$, *is the subset* $\{\mathbf{x} : T(\mathbf{x}) = \mathbf{0}, \mathbf{x} \in \mathrm{Domain}\,(T)\}$ *of U.*
6. T is said to be invertible if there exists a transformation $S : V \to U$ such that $S\,(T(\mathbf{x})) = \mathbf{x}$ for all $\mathbf{x} \in U$ and $T\,(S(\mathbf{y})) = \mathbf{y}$ for all $\mathbf{y} \in V$. In this case S is called an inverse of T.

Before turning to examples, we present some theorems.

Theorem 4.2.1. ***(The Domain of a Linear Transformation is a Subspace).*** *Let T be a linear transformation from a nonempty subset W of a vector space U to a vector space V. Then W is a subspace of U.*

The proof is left as Exercise 4.2.1. Thus, in light of this theorem, we usually take U as $\mathrm{Domain}\,(T)$.

Theorem 4.2.2. ***(The Range of a Linear Transformation is a Subspace).*** *Let T be a linear transformation from a vector space U to a vector space V. Then* $\mathrm{Range}(T)$ *is a subspace of V.*

The proof is left as Exercise 4.2.6.

Theorem 4.2.3. ***(If T Is One-to-One, Then $T(\mathbf{x}) = T(\mathbf{y})$ Is Equivalent to $\mathbf{x} = \mathbf{y}$).*** *Let T be a transformation from a vector space U to a vector space V. If T is one-to-one, then $T(\mathbf{x}) = T(\mathbf{y})$ is equivalent to $\mathbf{x} = \mathbf{y}$.*

Proof. By the definition of any mapping, $\mathbf{x} = \mathbf{y}$ implies $T(\mathbf{x}) = T(\mathbf{y})$, and for a one-to-one mapping, by Part 3 of Definition 4.2.1, $T(\mathbf{x}) = T(\mathbf{y})$ implies $\mathbf{x} = \mathbf{y}$. ■

Theorem 4.2.4. ***(The Kernel of a Linear Transformation Is a Subspace).*** *Let T be a linear transformation from a vector space U to a vector space V. Then* $\mathrm{Ker}\,(T)$ *is a subspace of U.*

The proof is left as Exercise 4.2.7.

[5] In general, an isomorphism between algebraic structures means a mapping that is one-to-one, onto and preserves all algebraic operations. For vector spaces the linearity of T expresses the preservation of the algebraic operations.

Example 4.2.1. (A Linear Transformation $\boldsymbol{T}$: $\mathbb{R}^3 \to \mathbb{R}^2$*).* The transformation T of Example 4.1.9 of page 166 from $\mathbb{R}^3$ to $\mathbb{R}^2$, given by

$$T(\mathbf{x}) = \begin{bmatrix} x_1 - x_2 \\ x_1 + x_3 \end{bmatrix}, \tag{4.43}$$

has range $\mathbb{R}^2$ and is a mapping *onto* $\mathbb{R}^2$ because, for every $\mathbf{y} = (y_1, y_2)^T$, the general solution of $T(\mathbf{x}) = \mathbf{y}$ is

$$\mathbf{x} = \begin{bmatrix} y_2 - s \\ y_2 - y_1 - s \\ s \end{bmatrix}, \tag{4.44}$$

where s is a parameter. So the mapping T transforms every $\mathbf{x}$ of this form into the arbitrarily given $\mathbf{y}$. Furthermore, since all the $\mathbf{x}$ vectors of Equation 4.44, for fixed $\mathbf{y}$ and different values of the parameter s, are mapped to the same $\mathbf{y}$, T is not one-to-one. Equation 4.44 also shows, with $\mathbf{y} = \mathbf{0}$, that $\operatorname{Ker}(T) = \{\mathbf{x} : \mathbf{x} = (-s, -s, s)^T,\ s \in \mathbb{R}\}$. ♦

Theorem 4.2.5. ***(A Linear Transformation Is Invertible If and Only If It Is an Isomorphism).*** *A linear transformation T from a vector space U to a vector space V is invertible if and only if it is an isomorphism. Furthermore, an invertible linear transformation has a unique inverse, denoted by T^{-1}, which is also linear and an isomorphism.*

Proof. Assume that $T : U \to V$ is invertible. For any $\mathbf{x} \in U$, let $\mathbf{y} = T(\mathbf{x})$ be the corresponding element of V. Then, by the invertibility of T, there exists a mapping $S : V \to U$ such that $S(\mathbf{y}) = S(T(\mathbf{x})) = \mathbf{x}$, and $T(S(\mathbf{y})) = T(\mathbf{x}) = \mathbf{y}$. Also, for any $\mathbf{y} \in V$ there exists an $\mathbf{x} \in U$ such that $S(\mathbf{y}) = \mathbf{x}$ and so $T(S(\mathbf{y})) = T(\mathbf{x}) = \mathbf{y}$. Thus, the invertibility of T implies that it is onto. Furthermore, if T is invertible, then applying S to both sides of $T(\mathbf{x}_1) = T(\mathbf{x}_2)$, for any $\mathbf{x}_1, \mathbf{x}_2 \in U$, results in $S(T(\mathbf{x}_1)) = S(T(\mathbf{x}_2))$, or equivalently, in $\mathbf{x}_1 = \mathbf{x}_2$. Thus, the invertibility of T implies that it is also one-to-one. Consequently, an invertible linear T is an isomorphism.

Conversely, if $T : U \to V$ is an isomorphism, then, given any $\mathbf{y} \in V$, there is exactly one $\mathbf{x} \in U$ such that $T(\mathbf{x}) = \mathbf{y}$. We define $S : V \to U$ by $S(\mathbf{y}) = \mathbf{x}$. Hence $S(T(\mathbf{x})) = S(\mathbf{y}) = \mathbf{x}$ and $T(S(\mathbf{y})) = T(\mathbf{x}) = \mathbf{y}$. Thus, if T is an isomorphism, then it is invertible. Clearly, this S is unique and also invertible.

To prove that if T is an isomorphism, then $S = T^{-1}$ is linear, we apply Lemma 4.1.1. Letting $\mathbf{y}_1, \mathbf{y}_2 \in V$ and with a, b any scalars, we have, by the definition of S and by the linearity of T,

$$\begin{aligned} T(S(a\mathbf{y}_1 + b\mathbf{y}_2)) &= a\mathbf{y}_1 + b\mathbf{y}_2 = aT(S(\mathbf{y}_1)) + bT(S(\mathbf{y}_2)) \\ &= T(aS(\mathbf{y}_1) + bS(\mathbf{y}_2)). \end{aligned} \tag{4.45}$$

Hence, since T is one-to-one,

$$S(a\mathbf{y}_1 + b\mathbf{y}_2) = aS(\mathbf{y}_1) + bS(\mathbf{y}_2). \tag{4.46}$$

■

Definition 4.2.2. ***(Rank).*** *For any linear transformation T, the dimension of its range is called the rank of T, denoted* rank(T).

Definition 4.2.3. ***(Nullity).*** *For any linear transformation T, the dimension of its kernel is called the nullity of T, denoted* nullity(T).

Theorem 4.2.6. ***(Rank + Nullity = Dimension of the Domain).*** *Let T be a linear transformation from a finite-dimensional vector space U to a vector space V. Then*

$$\operatorname{rank}(T) + \operatorname{nullity}(T) = \dim(U). \tag{4.47}$$

Proof. Let $\dim(U) = n$, and $\operatorname{nullity}(T) = k$, with $0 \le k \le n$.

If $0 < k < n$, then let $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k\}$ be a basis for $\operatorname{Ker}(T)$ and extend it to a basis $\mathcal{A} = \{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k, \mathbf{a}_{k+1}, \ldots, \mathbf{a}_n\}$ for U. Then $\{T\mathbf{a}_1, T\mathbf{a}_2, \ldots, T\mathbf{a}_k, T\mathbf{a}_{k+1}, \ldots, T\mathbf{a}_n\} = \{\mathbf{0}, \mathbf{0}, \ldots, \mathbf{0}, T\mathbf{a}_{k+1}, \ldots, T\mathbf{a}_n\}$ span Range(T). We are going to show that the vectors $T\mathbf{a}_{k+1}, \ldots, T\mathbf{a}_n$ are also independent: Assume that, for some scalars c_i,

$$\sum_{i=k+1}^{n} c_i T\mathbf{a}_i = T\left(\sum_{i=k+1}^{n} c_i \mathbf{a}_i\right) = \mathbf{0}. \tag{4.48}$$

Hence

$$\sum_{i=k+1}^{n} c_i \mathbf{a}_i \in \operatorname{Ker}(T). \tag{4.49}$$

Since $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k\}$ is a basis for $\operatorname{Ker}(T)$, we can express the sum above as a linear combination of these basis vectors, that is, as

$$\sum_{i=k+1}^{n} c_i \mathbf{a}_i = \sum_{i=1}^{k} b_i \mathbf{a}_i. \tag{4.50}$$

Equivalently,

$$\sum_{i=1}^{k} b_i \mathbf{a}_i - \sum_{i=k+1}^{n} c_i \mathbf{a}_i = \mathbf{0}. \tag{4.51}$$

By the linear independence of the $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ vectors, all the coefficients b_i and c_i must equal 0. Thus, the vectors $T\mathbf{a}_{k+1}, \ldots, T\mathbf{a}_n$ are also independent and form a basis for Range(T). Hence $\operatorname{rank}(T) = n - k = \dim(U) - \operatorname{nullity}(T)$.

If $k = 0$, then $\operatorname{Ker}(T)$ has the empty set for a basis, and letting $\mathcal{A} = \{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$ be a basis for U, we can show similarly as above that it is also a basis for Range(T). So, in this case, nullity$(T) = 0$ and rank$(T) = n = \dim(U)$.

If $k = n$, then $\operatorname{Ker}(T) = U$, and $\operatorname{Range}(T) = \{\mathbf{0}\}$, and so nullity$(T) = n = \dim(U)$, while rank$(T) = 0$. ■

Theorem 4.2.7. ***(Finite-Dimensional Vector Spaces Are Isomorphic If and Only If They Have the Same Dimension).*** *Two finite-dimensional vector spaces U and V are isomorphic to each other if and only if* $\dim(U) = \dim(V)$.

Proof. Let $\dim(U) = n$ and $\dim(V) = m$, and let $T : U \to V$ be an isomorphism. Then, T being one-to-one, $\operatorname{Ker}(T) = \{\mathbf{0}\}$ and nullity$(T) = 0$. Thus, by Theorem 4.2.6, rank$(T) = n$, and since T is onto, $\operatorname{Range}(T) = V$ and $m = \dim(V) = n$.

Conversely, if $\dim(U) = \dim(V) = n$, then choose a basis $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$ for U and a basis $\{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ for V. The linear mapping $T : U \to V$ determined by $T(\mathbf{a}_i) = \mathbf{b}_i$ for $i = 1, 2, \ldots, n$, is clearly an isomorphism. ■

Example 4.2.2. (The Mapping $T : \mathbb{R}^n \to X$ Given by $T(\mathbf{x}_A) = \mathbf{x}$). Consider an arbitrary n-dimensional vector space X and an ordered basis $A = (\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n)$ for X. The mapping T of $\mathbb{R}^n$ to X, given by associating with each coordinate vector $\mathbf{x}_A$ the corresponding $\mathbf{x} \in X$, is an isomorphism. Indeed, the defining equation $T(\mathbf{x}_A) = \sum_{i=1}^{n} x_{Ai}\mathbf{a}_i$ shows that T is linear, and A, being a basis, spans X, which shows that T is onto; and the $\mathbf{a}_i$ are linearly independent, which shows that T is one-to-one. The corresponding mapping T^{-1} in the reverse direction, that is, from X to $\mathbb{R}^n$, given by $T^{-1}(\mathbf{x}) = \mathbf{x}_A$, is the inverse of T, and is also an isomorphism. (See Exercise 4.2.10.) ♦

That X and $\mathbb{R}^n$ above are isomorphic shows that for each value of n there is essentially only one n-dimensional vector space: $\mathbb{R}^n$. For instance, a five-dimensional subspace of $\mathcal{P}_{10}$ and every five-dimensional subspace of every other vector space are all isomorphic to $\mathbb{R}^5$.

Example 4.2.3. (A Linear Transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$). Consider the transformation T from $\mathbb{R}^3$ to $\mathbb{R}^3$ given by

$$T(\mathbf{x}) = \begin{bmatrix} x_1 - x_2 \\ x_1 + x_2 \\ 2x_1 - 3x_2 \end{bmatrix}. \tag{4.52}$$

We can also write this transformation as

$$T(\mathbf{x}) = x_1 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}. \tag{4.53}$$

Thus the range of T is the two-dimensional subspace of such vectors in $\mathbb{R}^3$, and so T is not onto. Also, our T is represented by the matrix

$$[T] = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 2 & -3 & 0 \end{bmatrix} \tag{4.54}$$

and we can see that $\mathrm{Range}(T) = \mathrm{Col}([T])$.

Furthermore, T is not one-to-one, because $\mathrm{Ker}(T) = \{\mathbf{x} : \mathbf{x} = (0,0,s)^T, s \in \mathbb{R}\}$, and all the infinitely many vectors in $\mathrm{Ker}(T)$ are mapped to the single vector $\mathbf{0}$. ♦

We can see from this example, and in general from Theorem 4.1.2 of page 166, that for a transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ the range of T is exactly the same as the column space of the representative matrix $[T]$ relative to the standard bases, and the kernel of T the same as the null space of $[T]$; and so the rank of T equals the rank of $[T]$. The same sort of relationships hold for general vector spaces and bases as well.

Theorem 4.2.8. ***(*** $\mathrm{Col}(T_{A,B})$ ***and*** $\mathrm{Range}(T)$ ***Are Isomorphic).*** *Let T be a linear transformation from an n-dimensional vector space U to an m-dimensional vector space V, $A = (\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n)$ an ordered basis for U, $B = (\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_m)$ an ordered basis for V, and $T_{A,B}$ the $m \times n$ matrix that represents T relative to these bases. Then* $\mathrm{rank}(T_{A,B}) = \mathrm{rank}(T)$, *and Equation 4.30 on page 169 establishes an isomorphism from* $\mathrm{Col}(T_{A,B})$ *to* $\mathrm{Range}(T)$.

Proof. The matrix $T_{A,B}$ is $m \times n$, and denoting $\mathrm{rank}(T_{A,B})$ by r, we have $r \le \min(m,n)$. Let $C = (\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_r)$ be an $m \times r$ matrix whose columns form a basis for $\mathrm{Col}(T_{A,B})$. Then, for every $\mathbf{x} \in U$, we have a vector $\mathbf{t}_C = (t_{C1}, t_{C2}, \ldots, t_{Cr})^T \in \mathbb{R}^r$ such that

$$T_{A,B}\mathbf{x}_A = \sum_{k=1}^{r} t_{Ck}\mathbf{c}_k = C\mathbf{t}_C. \tag{4.55}$$

Substituting this expression into Equation 4.30, we get

$$T(\mathbf{x}) = \sum_{i=1}^{m} [T_{A,B}\mathbf{x}_A]_i \, \mathbf{b}_i = \left(\sum_{i=1}^{m} c_{i1}\mathbf{b}_i, \ldots, \sum_{i=1}^{m} c_{ir}\mathbf{b}_i \right) \begin{bmatrix} t_{C1} \\ \vdots \\ t_{Cr} \end{bmatrix}. \tag{4.56}$$

Thus $T(\mathbf{x})$ lies in the span of the r vectors $\sum_{i=1}^{m} c_{i1}\mathbf{b}_i, \ldots, \sum_{i=1}^{m} c_{ir}\mathbf{b}_i$. These vectors not only span the range of T but are also independent, because from Equation 4.56 we can write

$$T(\mathbf{x}) = \sum_{i=1}^{m}\sum_{k=1}^{r} c_{ik}t_{Ck}\mathbf{b}_i = \sum_{i=1}^{m} (C\mathbf{t}_C)_i\mathbf{b}_i, \tag{4.57}$$

and if $T(\mathbf{x}) = \mathbf{0}$, then, by the independence of the $\mathbf{b}_i$, we must have $C\mathbf{t}_C = \mathbf{0}$, and so, by the independence of the columns of C, also $\mathbf{t}_C = \mathbf{0}$. Thus the r vectors $\sum_{i=1}^m c_{i1}\mathbf{b}_i, \ldots, \sum_{i=1}^m c_{ir}\mathbf{b}_i$ form a basis for the range of T. So the rank of T equals the rank r of $T_{A,B}$. The first half of Equation 4.56 maps every vector $T_{A,B}\mathbf{x}_A$ of the column space of $T_{A,B}$ to a vector $T(\mathbf{x})$ in the range of T. Furthermore, by the independence of the $\mathbf{b}_i$, this mapping of the column space of $T_{A,B}$ to the range of T is one-to-one, and then, by the result of Exercise 4.2.9, it is also an isomorphism. ■

We can define the sum and scalar multiple of transformations for general vector spaces much as we did for Euclidean spaces.

Definition 4.2.4. ***(Sum and Scalar Multiple of Transformations).*** *Let S and T be transformations from a vector space U to a vector space V and let c be any scalar. Then $Q = S + T$ and $R = cT$ are defined as the transformations from U to V that satisfy*

$$Q(\mathbf{x}) = S(\mathbf{x}) + T(\mathbf{x}) \tag{4.58}$$

and

$$R(\mathbf{x}) = cT(\mathbf{x}) \tag{4.59}$$

for every $\mathbf{x} \in U$.

Theorem 4.2.9. ***(Sums and Scalar Multiples of Linear Transformations Are Linear).*** *If S and T are linear, then the transformations Q and R defined above are linear.*

Proof. For $\mathbf{x}, \mathbf{y} \in U$ and a, b arbitrary scalars,

$$\begin{aligned} Q(a\mathbf{x} + b\mathbf{y}) &= S(a\mathbf{x} + b\mathbf{y}) + T(a\mathbf{x} + b\mathbf{y}) \\ &= aS(\mathbf{x}) + bS(\mathbf{y}) + aT(\mathbf{x}) + bT(\mathbf{y}) \\ &= a\left(S(\mathbf{x}) + T(\mathbf{x})\right) + b\left(S(\mathbf{y}) + T(\mathbf{y})\right) = aQ(\mathbf{x}) + bQ(\mathbf{y}). \end{aligned} \tag{4.60}$$

Thus, by Lemma 4.1.1, Q is linear.

Similarly,

$$\begin{aligned} R(a\mathbf{x} + b\mathbf{y}) &= cT(a\mathbf{x} + b\mathbf{y}) = c\left(aT(\mathbf{x}) + bT(\mathbf{y})\right) \\ &= acT(\mathbf{x}) + bcT(\mathbf{y}) = aR(\mathbf{x}) + bR(\mathbf{y}), \end{aligned} \tag{4.61}$$

which shows that R is linear. ■

Example 4.2.4. (Sum and Scalar Multiple of Certain Transformations). Let S and T be the linear transformations from $\mathbb{R}^3$ to $\mathbb{R}^2$ given by

$$S(\mathbf{x}) = \begin{bmatrix} x_2 + x_3 \\ x_3 \end{bmatrix} \tag{4.62}$$

and

$$T(\mathbf{x}) = \begin{bmatrix} x_1 - x_2 \\ x_1 + x_3 \end{bmatrix}. \tag{4.63}$$

Then $Q = S + T$ and $R = cT$ are the linear transformations given by

$$Q(\mathbf{x}) = \begin{bmatrix} x_1 + x_3 \\ x_1 + 2x_3 \end{bmatrix} \tag{4.64}$$

and

$$R(\mathbf{x}) = \begin{bmatrix} c\,(x_1 - x_2) \\ c\,(x_1 + x_3) \end{bmatrix}. \tag{4.65}$$

♦

With Definition 4.2.4 Theorems 4.2.8 and 4.2.9 lead to the following theorem.

Theorem 4.2.10. ***(The Linear Transformations from*** U ***to*** V ***Form a Vector Space, Which Is Isomorphic to*** $\mathcal{M}_{m,n}$***).*** *Let* $L(U,V)$ *denote the set of all linear transformations from an* n*-dimensional vector space* U *to an* m*-dimensional vector space* V*, and let* $A = (\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n)$ *be an ordered basis for* U*,* $B = (\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_m)$ *an ordered basis for* V*, and* $T_{A,B}$ *the* $m \times n$ *matrix that represents any* $T \in L(U,V)$ *relative to these bases. Then*
1. $L(U,V)$*, together with addition and scalar multiple of transformations as in Definition 4.2.4, is a vector space.*
2. The mapping M *from* $L(U,V)$ *to the vector space* $\mathcal{M}_{m,n}$ *of all* $m \times n$ *matrices*[6] *given by* $M(T) = T_{A,B}$ *is linear and an isomorphism. Hence* $L(U,V)$ *is mn-dimensional.*

Proof. 1. $L(U,V)$ is clearly nonempty: the zero mapping O is in it. Theorem 4.2.9 shows that $L(U,V)$ is closed under addition and multiplication by scalars. The vector space axioms for $L(U,V)$ follow from the corresponding ones in V for every $\mathbf{x}$ in $\mathbf{y} = T(\mathbf{x})$. In particular, the zero element is the zero mapping O, and the element $-T$ is the mapping $(-1)\,T$.

2. Let $S, T \in L(U,V)$ and a, b any scalars. Then, by Theorem 4.1.3, for all $\mathbf{x} \in U$, $\mathbf{y} = T(\mathbf{x})$ becomes in terms of coordinates $\mathbf{y}_B = T_{A,B}\mathbf{x}_A = M(T)\,\mathbf{x}_A$. Similarly $\mathbf{y} = (aS + bT)(\mathbf{x})$ becomes

$$\mathbf{y}_B = (aS + bT)_{A,B}\,\mathbf{x}_A = M(aS + bT)\,\mathbf{x}_A. \tag{4.66}$$

[6] See Example 3.1.2.

On the other hand, $\mathbf{y} = (aS + bT)(\mathbf{x}) = aS(\mathbf{x}) + bT(\mathbf{x})$ also becomes

$$\begin{aligned}\mathbf{y}_B &= aS_{A,B}\mathbf{x}_A + bT_{A,B}\mathbf{x}_A \\ &= (aS_{A,B} + bT_{A,B})\,\mathbf{x}_A = (aM(S) + bM(T))\,\mathbf{x}_A.\end{aligned} \tag{4.67}$$

Since these equations hold for all $\mathbf{x} \in U$, and so for all $\mathbf{x}_A$ as well, we must have

$$M(aS + bT) = aM(S) + bM(T), \tag{4.68}$$

showing that M is linear.

Let C be any $m \times n$ matrix and define a linear mapping T by $T(\mathbf{a}_k) = \sum_{i=1}^{m} c_{ik}\mathbf{b}_i$ for $k = 1, 2, \ldots, n$. (See Theorem 4.1.1.) Then, for any $\mathbf{x} = \sum_{k=1}^{n} x_{Ak}\mathbf{a}_k$ of U,

$$T(\mathbf{x}) = \sum_{k=1}^{n} x_{Ak}T(\mathbf{a}_k) = \sum_{k=1}^{n} x_{Ak}\sum_{i=1}^{m} c_{ik}\mathbf{b}_i = \sum_{i=1}^{m}\left(\sum_{k=1}^{n} c_{ik}x_{Ak}\right)\mathbf{b}_i \tag{4.69}$$

defines the components: $y_{Bi} = \sum_{k=1}^{n} c_{ik}x_{Ak}$ of a coordinate vector $\mathbf{y}_B = C\mathbf{x}_A$ relative to the given ordered basis B of V. Thus C is the matrix $T_{A,B}$ that corresponds to this linear transformation T, and so M is onto $\mathcal{M}_{m,n}$.

M is also one-to one: Let $S, T \in L(U, V)$. Assume that $M(S) = M(T)$, that is, $S_{A,B} = T_{A,B}$. Then, Equation 4.69, with $C = S_{A,B} = T_{A,B}$, determines T uniquely. ■

Just as for functions of a single variable, we define the composition of transformations as follows.

Definition 4.2.5. ***(Composition of Transformations).*** *Let R be a transformation from a vector space U to a vector space V and S a transformation from V to a vector space W. Then the composite $T = S \circ R$ is defined as the transformation from U to W that satisfies*

$$T(\mathbf{x}) = S(R(\mathbf{x})) \tag{4.70}$$

for every $\mathbf{x} \in U$.

Theorem 4.2.11. ***(The Composite of Linear Transformations is Linear).*** *Let R be a linear transformation from a vector space U to a vector space V and S a linear transformation from V to a vector space W. Then $T = S \circ R$ is linear.*

Proof. Letting $\mathbf{x}_1, \mathbf{x}_2 \in U$ and with a, b any scalars, we have, first by the linearity of R, and then by the linearity of S,

$$\begin{aligned}T(a\mathbf{x}_1 + b\mathbf{x}_2) &= S(R(a\mathbf{x}_1 + b\mathbf{x}_2)) = S(aR(\mathbf{x}_1) + bR(\mathbf{x}_2)) \\ &= aS(R(\mathbf{x}_1)) + bS(R(\mathbf{x}_2)) = aT(\mathbf{x}_1) + bT(\mathbf{x}_2).\end{aligned} \tag{4.71}$$

■

As we have seen in Example 4.1.11 on page 167, the composition of transformations induces the multiplication of the corresponding standard representative matrices, that is, $T = S \circ R$ implies

$$[T] = [S][R]. \tag{4.72}$$

To obtain such a relation was, of course, the motivation behind the definition of matrix multiplication. We have analogous relations for the representative matrices relative to arbitrary bases, but we do not go into this subject any further.

From the first half of Equation 4.56 we can also see, by the independence of the $\mathbf{b}_i$, that $T_{A,B}\mathbf{x}_A = \mathbf{0}$ is equivalent to $T(\mathbf{x}) = \mathbf{0}$. Then, for every $\mathbf{x}_A$ in the null space of $T_{A,B}$, the linear mapping of $\mathbf{x}_A$ to $\mathbf{x}$ given by

$$\mathbf{x} = \sum_{j=1}^{n} x_{Aj}\mathbf{a}_j \tag{4.73}$$

is a mapping onto the kernel of T. Thus, by the result of Exercise 4.2.8, we have the following theorem.

Theorem 4.2.12. ***(The Kernel of a Linear Transformation Is Isomorphic to the Null Space of Its Representative Matrix).*** *Let T be a linear transformation from a vector space U to a vector space V, $A = (\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n)$ an ordered basis for U, $B = (\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_m)$ an ordered basis for V, and $T_{A,B}$ the matrix that represents T relative to these bases. Then the mapping given by Equation 4.73 restricted to the null space of $T_{A,B}$ is an isomorphism to the kernel of T. Thus* $\dim(\text{Null}(T_{A,B})) = \dim(\text{Ker}(T))$.

The earlier Theorem 4.2.6 reflects the fact, stated in the next theorem, that the action of the transformation T splits its domain into two parts, analogously to the decomposition for matrices in Theorem 3.5.3 on page 133.

Theorem 4.2.13. ***(A Linear Transformation Splits Its Domain into the Direct Sum of Its Kernel and a Subspace Isomorphic to Its Range).*** *Let T be a linear transformation of rank r from a vector space U to a vector space V and let us denote its range and kernel by R and K respectively. If $\overline{K}$ is a subspace of U complementary to K, that is, one that satisfies*

$$U = K + \overline{K} \tag{4.74}$$

and

$$K \cap \overline{K} = \{\mathbf{0}\}, \tag{4.75}$$

then T maps K to $\{\mathbf{0}\}$, while it maps the r-dimensional $\overline{K}$ isomorphically onto the subspace R of V.

Proof. Let $\dim(U) = n$, and let $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_{n-r}\}$ be a basis for K. Then we can extend this set to a basis $\mathcal{A} = \{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$ for U. Let $\overline{K} = \text{Span}\{\mathbf{a}_{n-r+1}, \ldots, \mathbf{a}_n\}$. Then every $\mathbf{x}$ in U can be uniquely decomposed as

$$\mathbf{x} = \sum_{i=1}^{n-r} x_{Ai}\mathbf{a}_i + \sum_{i=n-r+1}^{n} x_{Ai}\mathbf{a}_i. \tag{4.76}$$

Writing $\mathbf{x}_K$ for the first sum and $\mathbf{x}_{\overline{K}}$ for the second, we have $\mathbf{x}_K \in K$ and $\mathbf{x}_{\overline{K}} \in \overline{K}$, and

$$T(\mathbf{x}) = T(\mathbf{x}_K) + T(\mathbf{x}_{\overline{K}}) = \mathbf{0} + T(\mathbf{x}_{\overline{K}}) = T(\mathbf{x}_{\overline{K}}). \tag{4.77}$$

Since by the definition of range, every $\mathbf{y}$ of R can be written as $T(\mathbf{x})$ for some $\mathbf{x}$, Equation 4.77 shows that such a $\mathbf{y}$ can also be written as $T(\mathbf{x}_{\overline{K}})$. Thus T maps $\overline{K}$ onto R. Since both $\overline{K}$ and R have dimension r, the result of Exercise 4.2.8 shows that T is an isomorphism of $\overline{K}$ onto R. ■

The subspace $\overline{K}$ above is generally not unique, because the extension of the basis $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_{n-r}\}$ is not unique, and different extensions result in different subspaces $\overline{K}$. (See Exercise 4.2.12.) However, if U has an inner product, then $\overline{K}$ may be taken as the unique orthogonal complement of K.

At the end of Section 3.5 we mentioned, in effect, that for every $\mathbf{x}_R$ in the row space of a matrix M the mapping given by $\mathbf{x}_R \to M\mathbf{x}_R$ is an isomorphism from $\text{Row}(M)$ to $\text{Col}(M)$. As promised there, let us now show how to represent this isomorphism by an $r \times r$ matrix.

Theorem 4.2.14. ***(The*** $r \times r$ ***Matrix That Represents the Action of a Matrix*** M ***on Row***(M)***).*** *Let* M *be an* $m \times n$ *matrix of rank* r, *and* $A = (\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_r)$ *a basis matrix for Row*(M), *and* $B = (\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_r)$ *one for Col*(M). *Then*

$$M_{A,B} = (B^T B)^{-1} B^T M A \tag{4.78}$$

is the $r \times r$ *matrix that represents, relative to the bases* A *and* B, *the mapping of Row*(M) *to Col*(M) *given by* $\mathbf{x}_R \to M\mathbf{x}_R$.

Proof. Let M, A, and B be as stated in the theorem. Then every $\mathbf{x}$ in $\text{Row}(M)$ can be written as

$$\mathbf{x} = \sum_{i=1}^{r} x_{Ai}\mathbf{a}_i = A\mathbf{x}_A \tag{4.79}$$

and every $\mathbf{y}$ in $\text{Col}(M)$ similarly as

$$\mathbf{y} = \sum_{i=1}^{r} y_{Bi}\mathbf{b}_i = B\mathbf{y}_B. \tag{4.80}$$

Let $T_M : \mathrm{Row}(M) \to \mathrm{Col}(M)$ be the isomorphism given by $\mathbf{y} = M\mathbf{x}$ or, equivalently, by

$$B\mathbf{y}_B = MA\mathbf{x}_A. \tag{4.81}$$

Since we know that T_M is an isomorphism, there must exist a unique solution $\mathbf{y}_B$ of this equation, and comparing that solution with Equation 4.27 we can obtain $M_{A,B}$. This solution is best found by Gaussian elimination, but we can also write a formula for it. However, in trying to solve Equation 4.81 for $\mathbf{y}_B$ explicitly, we encounter a problem, namely that B is an $m \times r$ matrix and generally not square. Thus it has no inverse unless $r = m$. We can, however, apply the following trick: We left-multiply Equation 4.81 by B^T, and now B^TB is $r \times r$ and will be shown to have an inverse by Lemma 5.1.3 on page 202. Thus we obtain

$$\mathbf{y}_B = (B^TB)^{-1}B^TMA\mathbf{x}_A. \tag{4.82}$$

Comparing this result with Equation 4.27, we find

$$M_{A,B} = (B^TB)^{-1}B^TMA \tag{4.83}$$

for the matrix that represents M relative to the bases A for $\mathrm{Row}(M)$ and B for $\mathrm{Col}(M)$. ■

Example 4.2.5. (A Mapping of* Row(*M*) *to* Col(*M*)*). Let the given matrix be

$$M = \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 1 & 2 & 1 & 3 \end{bmatrix}. \tag{4.84}$$

The reduced echelon form of this matrix is

$$R = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{4.85}$$

Thus the vectors $\mathbf{a}_1 = (1,0,1,1)^T$ and $\mathbf{a}_2 = (0,1,0,1)^T$ form a basis for the row space of M, and $\mathbf{b}_1 = (1,0,1)^T$ and $\mathbf{b}_2 = (1,1,2)^T$ a basis for its column space. Equation 4.81 now becomes

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} y_{B1} \\ y_{B2} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 1 & 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_{A1} \\ x_{A2} \end{bmatrix} \tag{4.86}$$

or

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} y_{B1} \\ y_{B2} \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ 1 & 2 \\ 5 & 5 \end{bmatrix} \begin{bmatrix} x_{A1} \\ x_{A2} \end{bmatrix}. \tag{4.87}$$

This equation can be reduced to (instead of multiplying by $\left(B^TB\right)^{-1}B^T$)

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y_{B1} \\ y_{B2} \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_{A1} \\ x_{A2} \end{bmatrix}, \tag{4.88}$$

and so

$$M_{A,B} = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}. \tag{4.89}$$

Alternatively, we can obtain this matrix much more laboriously by substituting into Equation 4.83:

$$B^TB = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 3 & 6 \end{bmatrix}. \tag{4.90}$$

Thus

$$\left(B^TB\right)^{-1} = \begin{bmatrix} 2 & -1 \\ -1 & 2/3 \end{bmatrix} \tag{4.91}$$

and

$$\begin{aligned} M_{A,B} &= (B^TB)^{-1}B^TMA \\ &= \begin{bmatrix} 2 & -1 \\ -1 & 2/3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 1 & 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}. \end{aligned} \tag{4.92}$$

To check this result, consider, for example, $\mathbf{x} = \mathbf{a}_1$. Then $\mathbf{x}_A = (1, 0)^T$ and, on the one hand, M maps $\mathbf{x}$ to

$$M\mathbf{x} = \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 1 & 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 5 \end{bmatrix} \tag{4.93}$$

and, on the other hand, $M_{A,B}$ maps $\mathbf{x}_A$ to

$$\mathbf{y}_B = M_{AB}\mathbf{x}_A = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}. \tag{4.94}$$

This coordinate vector corresponds to the $\mathbf{y}$ in the range of M given by

$$\mathbf{y} = B\mathbf{y}_B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 5 \end{bmatrix}, \tag{4.95}$$

the same as $M\mathbf{x}$ before. ♦

Exercises

Exercise 4.2.1. Show that if a transformation T from a nonempty subset W of a vector space U to a vector space V satisfies Equations 4.1 and 4.2 for all $\mathbf{x}_1$ and $\mathbf{x}_2 \in W$ and all scalars c, then its domain W must be a subspace of U.

Exercise 4.2.2. For each of the transformations of Exercise 4.1.4, determine the range, the kernel, and whether it is one-to-one or onto. (These concepts apply to nonlinear transformations as well.)

***Exercise 4.2.3.** Prove that any linear transformation T is one-to-one if and only if $\mathrm{Ker}(T) = \{\mathbf{0}\}$.

Exercise 4.2.4. For each of the transformations of Exercises 4.1.5 through 4.1.9, determine the range, the kernel, and whether it is one-to-one or onto.

Exercise 4.2.5. Let $N = M^T$, where M is the matrix of Equation 4.84. Find a basis matrix A for $\mathrm{Row}(N)$ and a basis matrix B for $\mathrm{Col}\ (N)$, and find the representative matrix $N_{A,B}$ for the mapping N from $\mathrm{Row}(N)$ to $\mathrm{Col}(N)$ given by $\mathbf{y} = N\mathbf{x}$.

Exercise 4.2.6. Prove that if T is a linear transformation from a vector space U to a vector space V, then $\mathrm{Range}(T)$ is a subspace of V.

Exercise 4.2.7. Prove that if T is a linear transformation from a vector space U to a vector space V, then $\mathrm{Ker}(T)$ is a subspace of U.

***Exercise 4.2.8.** Prove that if T is a linear transformation from a vector space U onto a vector space V, and $\dim(U) = \dim(V)$, then T is an isomorphism. (*Hint*: First show that if $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ is a basis for V and $T(\mathbf{a}_i) = \mathbf{b}_i$, for $i = 1, 2, \ldots, n$, then the $\mathbf{a}_i$ form a basis for U.)

***Exercise 4.2.9.** Prove that if T is a one-to-one linear transformation from a vector space U to a vector space V, and $\dim(U) = \dim(V)$, then T is an isomorphism. (*Hint*: First show that if $\mathcal{A} = \{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$ is a basis for U, then the vectors $T(\mathbf{a}_i) = \mathbf{b}_i$, for $i = 1, 2, \ldots, n$, form a basis for V.)

Exercise 4.2.10. Prove that if $T_{A,B}$ represents an isomorphism, then it is nonsingular. What does $(T_{A,B})^{-1}$ represent?

Exercise 4.2.11. Find on the space $\mathcal{P}_n$ the representative matrix of the operation $X \circ D - D \circ X$ relative to the ordered basis $A = B = (1, x, \ldots, x^n)$. (Here X is the transformation of multiplication by x as in Exercise 4.1.20, and D is the differentiation map.) Determine the range and the kernel of this mapping and whether it is one-to-one or onto.

Exercise 4.2.12. Let K be the $\mathbf{x}$-axis in $\mathbb{R}^3$, that is, $K = \{\mathbf{x} : \mathbf{x} = x\mathbf{e}_1, x \in \mathbb{R}\}$. Show that the complementary subspace $\overline{K}$ of this K as defined in Theorem 4.2.13 is not unique, by exhibiting two different complementary subspaces that both satisfy Equations 4.74 and 4.75.

MATLAB Exercises

Exercise 4.2.13. Let

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 4 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

be a basis matrix for a subspace U of $\mathbb{R}^4$. Let

$$[T] = \begin{bmatrix} 0 & 3 & 1 & 0 \\ 4 & 2 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 1 & 3 & 0 & 4 \\ 2 & 2 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

be the matrix of a linear transformation T from U into $\mathbb{R}^6$. Find the range and kernel of T. (Note that T is considered only on U and not on $\mathbb{R}^4$.)

Exercise 4.2.14. a. Let T denote the mapping of Exercise 4.1.23 from $\mathcal{P}_4$ to $\mathcal{P}_6$ given by $\mathbf{p} \to \text{conv}(\mathbf{f}, \mathbf{p})$, with $\mathbf{f} = [3, -2, 1]$. Find the range and kernel of T.
b. What are the range and kernel of the analogous mapping with $\mathbf{f} = [3, -2, 0]$?

4.3 Applications of Linear Transformations in Computer Graphics

One of the most important tasks in computer graphics is the programming of motion. For example, we may want to picture a robot as it moves across the screen, or as its arm rotates, or as its legs move, etc. Whether we want to represent two-dimensional motion, or motion in three dimensions, the computation is usually done by applying the matrices that represent the desired transformations, namely rotations, reflections, translations, projections, or stretching, to the position vectors of points of the picture. (Which points to transform is a technical matter that we do not consider.) A computer can do these computations so quickly that it can create the illusion of continuous motion by displaying 50 or 60 slightly altered versions of a picture per second.

There is, however, a problem with one of the needed transformations: translation, which is not linear. This can be seen in either $\mathbb{R}^2$ or $\mathbb{R}^3$ by writing the translation of every vector $\mathbf{p}$ by the fixed vector $\mathbf{t}$ as $\mathbf{p}' = \mathbf{p} + \mathbf{t}$. Then for every scalar c we have $(c\mathbf{p})' = c\mathbf{p} + \mathbf{t}$ which, if this were a linear transformation, should equal $c\mathbf{p}' = c(\mathbf{p} + \mathbf{t})$ for every c and not just for certain values.

It would be very convenient if translations could also be made into linear transformations. This can indeed be done. To this end, in the two-dimensional case, we regard $\mathbb{R}^2$ as the plane $x_3 = 1$ in $\mathbb{R}^3$; that is, we associate with every vector $(x_1, x_2)^T \in \mathbb{R}^2$ the vector $(x_1, x_2, 1)^T \in \mathbb{R}^3$. The components $x_1, x_2, 1$ are called *homogeneous coordinates* of the point given by (x_1, x_2), which in general mean every triple of the form (x_1x_3, x_2x_3, x_3) with $x_3 \neq 0$. These coordinates were originally introduced in the middle of the nineteenth century to unify the treatment of parallel and nonparallel lines in projective geometry, but have recently found a new application to the problem at hand. A similar construction can be given in three dimensions as well.

Let us then consider translation in $\mathbb{R}^2$. We can make it into a linear transformation in $\mathbb{R}^3$ by using the method described in the following theorem.

Theorem 4.3.1. ***(Representing Translation by a Matrix in Homogeneous Coordinates).*** *Let $\mathbf{t} = (t_1, t_2, 0)^T$ be a fixed vector in $\mathbb{R}^3$ and let $\mathbf{p}' = \mathbf{p} + \mathbf{t}$ represent translation by $\mathbf{t}$ in the plane $x_3 = 1$, that is, for vectors of the form $\mathbf{p} = (p_1, p_2, 1)^T$. Then there is a unique linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ that coincides in the plane $x_3 = 1$ with the given translation or, in other words, one for which $T(\mathbf{p}) = \mathbf{p}' = \mathbf{p} + \mathbf{t}$, when $\mathbf{p}$ is of the form above. This T has the representative matrix*

$$T(t_1, t_2) = \begin{bmatrix} 1 & 0 & t_1 \\ 0 & 1 & t_2 \\ 0 & 0 & 1 \end{bmatrix}. \tag{4.96}$$

Proof. Indeed, the above matrix provides a linear transformation that has the desired property of coinciding in the plane $x_3 = 1$ with translation by $\mathbf{t}$:

$$T(t_1, t_2)\mathbf{p} = \begin{bmatrix} 1 & 0 & t_1 \\ 0 & 1 & t_2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ 1 \end{bmatrix} = \begin{bmatrix} p_1 + t_1 \\ p_2 + t_2 \\ 1 \end{bmatrix} = \mathbf{p} + \mathbf{t}. \tag{4.97}$$

Next, we need to prove that this linear transformation is unique.

Thus, let $L : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation such that $L(\mathbf{p}) = \mathbf{p}' = \mathbf{p} + \mathbf{t}$ for all $\mathbf{p} = (p_1, p_2, 1)^T$, with $\mathbf{t}$ being a fixed vector of the form $\mathbf{t} = (t_1, t_2, 0)^T$. Let $\mathbf{x} = (x_1, x_2, x_3)^T \in \mathbb{R}^3$ be any vector with $x_3 \neq 0$. Then $\mathbf{x}$ can be expressed as

$$\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3 = x_3\left[\frac{x_1}{x_3}\mathbf{e}_1 + \frac{x_2}{x_3}\mathbf{e}_2 + \mathbf{e}_3\right]. \tag{4.98}$$

Then, using the linearity of L and the fact that the vector in brackets on the right-hand side of Equation 4.98 is in the plane $x_3 = 1$, we get

$$\begin{aligned}
L(\mathbf{x}) &= x_3 L\left[\frac{x_1}{x_3}\mathbf{e}_1 + \frac{x_2}{x_3}\mathbf{e}_2 + \mathbf{e}_3\right] \\
&= x_3\left[\left(\frac{x_1}{x_3} + t_1\right)\mathbf{e}_1 + \left(\frac{x_2}{x_3} + t_2\right)\mathbf{e}_2 + \mathbf{e}_3\right] \\
&= (x_1 + t_1x_3)\mathbf{e}_1 + (x_2 + t_2x_3)\mathbf{e}_2 + x_3\mathbf{e}_3 \\
&= \begin{bmatrix} 1 & 0 & t_1 \\ 0 & 1 & t_2 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = T(t_1, t_2)\mathbf{x}.
\end{aligned} \tag{4.99}$$

We have thus shown that $L(\mathbf{x}) = T(t_1, t_2)\mathbf{x}$ for every $\mathbf{x} \in \mathbb{R}^3$ with $x_3 \neq 0$. On the other hand, if $x_3 = 0$, then we can rewrite $\mathbf{x}$ by adding and subtracting a term with $x_3 = 1$:

$$\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 = (x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \mathbf{e}_3) - \mathbf{e}_3. \tag{4.100}$$

Here, we can use the linearity of L and the fact that for the two terms on the right, L coincides with translation by $\mathbf{t}$, to write

$$\begin{aligned}
L(\mathbf{x}) &= L(x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \mathbf{e}_3) - L(\mathbf{e}_3) = (x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \mathbf{e}_3 + \mathbf{t}) - (\mathbf{e}_3 + \mathbf{t}) \\
&= x_1\mathbf{e}_1 + x_2\mathbf{e}_2 = \mathbf{x} = \begin{bmatrix} 1 & 0 & t_1 \\ 0 & 1 & t_2 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix} = T(t_1, t_2)\mathbf{x}.
\end{aligned} \tag{4.101}$$

■

The matrices of the other basic geometric transformations can also be represented by corresponding matrices in homogeneous coordinates as follows.

The matrix of rotation by an angle θ around the origin of $\mathbb{R}^2$ becomes the matrix

$$R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{4.102}$$

representing rotation in $\mathbb{R}^3$ around the x_3-axis. Similarly, the matrix

$$S(a, b) = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{4.103}$$

represents scaling by the factor a in the x_1-direction and by the factor b in the x_2-direction.

Example 4.3.1. (Rotating the Arm of a Robot). Let us consider a greatly simplified robot consisting of a rectangular body and an arm that is just a line segment, as shown in Figure 4.1. We want to find the matrix $\overline{R}(\theta)$ that represents lifting the arm by an angle θ from its normally horizontal position. Since the arm rotates about the point $(2,4)$, and the matrix $R(\theta)$ above represents rotation about the origin, we first apply $T(-2,-4)$ to move the point $(2,4)$ to the origin, then rotate, and then use $T(2,4)$ to move the "shoulder" back to $(2,4)$.

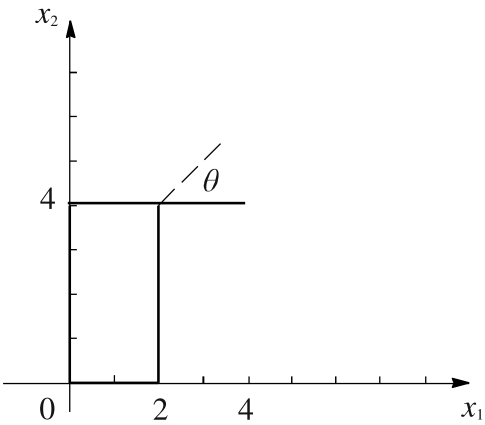

Fig. 4.1. A one-armed robot

Thus the required matrix is given by

$$\begin{aligned}\overline{R}(\theta) &= T(2,4)R(\theta)T(-2,-4)\\ &= \begin{bmatrix}1 & 0 & 2\\ 0 & 1 & 4\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & -2\\ 0 & 1 & -4\\ 0 & 0 & 1\end{bmatrix}\\ &= \begin{bmatrix}\cos\theta & -\sin\theta & -2\cos\theta+4\sin\theta+2\\ \sin\theta & \cos\theta & -4\cos\theta-2\sin\theta+4\\ 0 & 0 & 1\end{bmatrix}.\end{aligned} \tag{4.104}$$

This matrix is to be applied to the homogeneous position vectors of the points of the arm. Notice that it leaves the center of rotation fixed, as it should, that is,

$$\begin{bmatrix}\cos\theta & -\sin\theta & -2\cos\theta+4\sin\theta+2\\ \sin\theta & \cos\theta & -4\cos\theta-2\sin\theta+4\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}2\\ 4\\ 1\end{bmatrix} = \begin{bmatrix}2\\ 4\\ 1\end{bmatrix}. \tag{4.105}$$

It changes the homogeneous coordinates of the endpoint $(4, 4)$ into

$$\begin{bmatrix} \cos\theta & -\sin\theta & -2\cos\theta + 4\sin\theta + 2 \\ \sin\theta & \cos\theta & -4\cos\theta - 2\sin\theta + 4 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 2\cos\theta + 2 \\ 2\sin\theta + 4 \\ 1 \end{bmatrix}. \quad (4.106)$$

This result could have been obtained much more easily by elementary means, but they would have required custom tailoring for different problems. In contrast, the generality of the procedure shown above makes it more suitable for computer calculations. ♦

In $\mathbb{R}^3$ we could develop the handling of translations in a similar manner, but we leave that for the exercises and deduce the matrix of a particular rotation instead.

Example 4.3.2. (A Rotation in $\mathbb{R}^3$). In $\mathbb{R}^3$ let us find the matrix of the rotation by an angle θ about the vector $\mathbf{p} = (1, 1, 1)^T$. (The pictures of rotating objects on computer screens are obtained by applying this kind of a matrix, with small changes in θ, to the position vectors of points of the object about 50 times a second.)

The plan we want to follow is this: First, we apply a matrix R_1 that rotates the whole space by $\pi/4$ around the z-axis. This step rotates the vector $\mathbf{p}$ into the yz-plane, so that $R_1\mathbf{p}$ is at an angle α from the z-axis, with $\sin\alpha = \sqrt{2/3}$ and $\cos\alpha = \sqrt{1/3}$. (Why?) Next, we apply a matrix R_2 that rotates the whole space by α around the x-axis. Then $R_2R_1\mathbf{p}$ will lie in the z-axis, and we can now multiply by a matrix R_3 that expresses rotation by the angle θ around the z-axis, and which is easy to write down. Finally, we undo the first two rotations to get the vector $\mathbf{p}$ back to its original position.

For the matrix R_3 we have

$$R_3 = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad (4.107)$$

and setting $\theta = \pi/4$ we get the matrix

$$R_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix}. \quad (4.108)$$

Similarly,

$$R_2 = \frac{1}{\sqrt{3}} \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & -\sqrt{2} \\ 0 & \sqrt{2} & 1 \end{bmatrix}. \quad (4.109)$$

Then

$$R_2R_1 = \frac{1}{\sqrt{6}}\begin{bmatrix} \sqrt{3} & -\sqrt{3} & 0 \\ 1 & 1 & -2 \\ \sqrt{2} & \sqrt{2} & \sqrt{2} \end{bmatrix}. \tag{4.110}$$

Because this is an orthogonal matrix,[7] its inverse equals its transpose, and so

$$R_1^{-1}R_2^{-1} = \frac{1}{\sqrt{6}}\begin{bmatrix} \sqrt{3} & 1 & \sqrt{2} \\ -\sqrt{3} & 1 & \sqrt{2} \\ 0 & -2 & \sqrt{2} \end{bmatrix}. \tag{4.111}$$

Thus the matrix we sought is given by

$$R_\theta = R_1^{-1}R_2^{-1}R_3R_2R_1 = \tag{4.112}$$

$$\frac{1}{6}\begin{bmatrix} 4\cos\theta + 2 & -2\cos\theta - 2(\sin\theta)\sqrt{3} + 2 & -2\cos\theta + 2(\sin\theta)\sqrt{3} + 2 \\ 2(\sin\theta)\sqrt{3} - 2\cos\theta + 2 & 4\cos\theta + 2 & -2\cos\theta - 2(\sin\theta)\sqrt{3} + 2 \\ -2\cos\theta - 2(\sin\theta)\sqrt{3} + 2 & -2\cos\theta + 2(\sin\theta)\sqrt{3} + 2 & 4\cos\theta + 2 \end{bmatrix}.$$

Note that for $\theta = 2\pi/3$ this becomes the permutation matrix

$$R_{2\pi/3} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \tag{4.113}$$

which takes the standard basis vectors $\mathbf{i}$ into $\mathbf{j}$, $\mathbf{j}$ into $\mathbf{k}$, and $\mathbf{k}$ into $\mathbf{i}$, as it should.

Let us also note that we can interpret R_1 and R_2 as describing so-called passive transformations, that is, changes of basis (or changes of coordinate system), in which the space does not move, as opposed to the active rotations of the whole space that we just described. Indeed, with the notation of Theorem 3.6.3 on page 156, we may consider the ordered basis A to be the standard basis, that is, $A = I$, and the new ordered basis B as the one given by $B = R_1^{-1}R_2^{-1}$. Then Corollary 3.6.1 on page 149 gives $S = B$, and for the vector $\mathbf{x} = \mathbf{p} = [1, 1, 1]^T$, together with this definition of B, it gives $\mathbf{x}_B = \sqrt{3}[0, 0, 1]^T$. This vector shows that in the B basis the axis of the rotation by θ is the z'-axis. Thus we may take $M_B = R_3$. Then $M_A = R_\theta$ and Equation 3.153 on page 156 yields Equation 4.112 above. ◆

The last subject we want to discuss in this section is that of projecting three-dimensional images onto *viewplanes*. We consider only *orthographic* (or orthogonal) projections, that is, projections by rays orthogonal to the viewplane. Perspective projections are also widely used, but we do not discuss

[7] Such matrices will be discussed in Section 5.2. For now, you may just accept the given inverse, which could also, of course, be computed by the usual elimination method.

them. (They are nonlinear, but they can be linearized by using homogeneous coordinates as we did for translations.) On the other hand, orthogonal projections in arbitrary dimensions and relative to general bases will be discussed in the next chapter.

The simplest orthographic projections are those onto the coordinate planes. These produce top, bottom, or side views. The components of such projections can be obtained by just omitting one of the coordinates. For instance, the top view of the point (x, y, z) would be the point (x, y) in the xy-plane. This projection is given by the matrix

$$P(\mathbf{i}, \mathbf{j}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}. \tag{4.114}$$

The somewhat more difficult question is how to find the matrix that projects onto an arbitrary plane. Let the desired viewplane V be spanned by the orthogonal unit vectors $\mathbf{u}, \mathbf{v}$, and let $\mathbf{n}$ be a normal unit vector of V. We want to decompose an arbitrary vector $\mathbf{p} = (p_1, p_2, p_3)^T$ as

$$\mathbf{p} = r\mathbf{u} + s\mathbf{v} + t\mathbf{n}, \tag{4.115}$$

where r, s, t are undetermined coefficients. Taking scalar products of both sides of Equation 4.115 in turn by $\mathbf{u}, \mathbf{v}$, and $\mathbf{n}$, we get

$$r = \mathbf{p} \cdot \mathbf{u}, \; s = \mathbf{p} \cdot \mathbf{v}, \text{ and } t = \mathbf{p} \cdot \mathbf{n}. \tag{4.116}$$

The projection of $\mathbf{p}$ onto V is the vector obtained from Equation 4.115 by omitting the $\mathbf{n}$ component, and is thus

$$\mathbf{p}_V = (\mathbf{p} \cdot \mathbf{u})\mathbf{u} + (\mathbf{p} \cdot \mathbf{v})\mathbf{v}. \tag{4.117}$$

The vector $\begin{bmatrix} \mathbf{p} \cdot \mathbf{u} \\ \mathbf{p} \cdot \mathbf{v} \end{bmatrix}$ of the coefficients here is the coordinate vector (see page 148) of the projection $\mathbf{p}_V$ relative to the vectors $\mathbf{u}$ and $\mathbf{v}$ that span the viewplane V. We have

$$\begin{bmatrix} r \\ s \end{bmatrix} = \begin{bmatrix} \mathbf{p} \cdot \mathbf{u} \\ \mathbf{p} \cdot \mathbf{v} \end{bmatrix} = \begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} \tag{4.118}$$

and so the projection into the rs coordinate system of $\mathbf{u}, \mathbf{v}$ is given by the matrix

$$P_V(\mathbf{u}, \mathbf{v}) = \begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix}. \tag{4.119}$$

In components, Equation 4.117 is

$$p_{V1} = (p_1u_1 + p_2u_2 + p_3u_3)u_1 + (p_1v_1 + p_2v_2 + p_3v_3)v_1, \tag{4.120}$$

$$p_{V2} = (p_1u_1 + p_2u_2 + p_3u_3)u_2 + (p_1v_1 + p_2v_2 + p_3v_3)v_2, \quad (4.121)$$

$$p_{V3} = (p_1u_1 + p_2u_2 + p_3u_3)u_3 + (p_1v_1 + p_2v_2 + p_3v_3)v_3, \quad (4.122)$$

or in matrix form

$$\mathbf{p}_V = \begin{bmatrix} u_1^2 + v_1^2 & u_1u_2 + v_1v_2 & u_1u_3 + v_1v_3 \\ u_1u_2 + v_1v_2 & u_2^2 + v_2^2 & u_2u_3 + v_2v_3 \\ u_1u_3 + v_1v_3 & u_2u_3 + v_2v_3 & u_3^2 + v_3^2 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}. \quad (4.123)$$

From this equation we can read off the *projection matrix onto V relative to the standard vectors of* $\mathbb{R}^3$ as

$$P(\mathbf{u}, \mathbf{v}) = \begin{bmatrix} u_1^2 + v_1^2 & u_1u_2 + v_1v_2 & u_1u_3 + v_1v_3 \\ u_1u_2 + v_1v_2 & u_2^2 + v_2^2 & u_2u_3 + v_2v_3 \\ u_1u_3 + v_1v_3 & u_2u_3 + v_2v_3 & u_3^2 + v_3^2 \end{bmatrix}. \quad (4.124)$$

Example 4.3.3. (A Projection onto a Viewplane). Let us consider the house shown in front and side views in Figure 4.2 and find its view on the $x + y + z = 0$ plane.

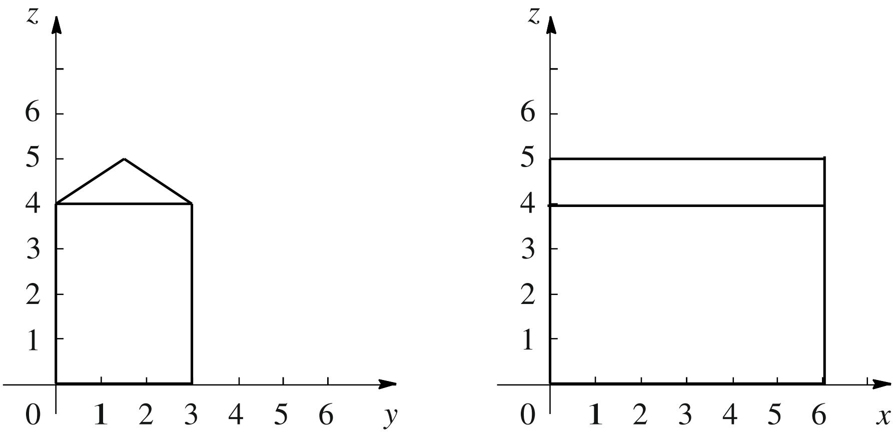

Fig. 4.2. Front and side views of a house

First, we must choose a coordinate system in the viewplane, that is, choose the vectors **u** and **v**. A good choice is $\mathbf{u} = \frac{1}{\sqrt{2}}(-1, 1, 0)^T$ and $\mathbf{v} = \frac{1}{\sqrt{6}}(-1, -1, 2)^T$. (Why?) Thus

$$P_V(\mathbf{u}, \mathbf{v}) = \begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{-1}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \end{bmatrix} \quad (4.125)$$

and the rs coordinates of each point are given, according to Equation 4.118, by applying this matrix to the column vector of their xyz coordinates.

Thus, for instance, the top right corner in the first view corresponds to the two corners $(0,3,4)^T$ and $(6,3,4)^T$ in the xyz coordinates, and so to the points with position vectors $P_V(\mathbf{u},\mathbf{v})(0,3,4)^T = \left(\frac{3}{\sqrt{2}}, \frac{5}{\sqrt{6}}\right)^T$ and $P_V(\mathbf{u},\mathbf{v})(6,3,4)^T = \left(\frac{-3}{\sqrt{2}}, \frac{-1}{\sqrt{6}}\right)^T$ in the rs system. Similarly, the bottom right corner in the first view corresponds to the two corners $(0,3,0)^T$ and $(6,3,0)^T$ in the xyz coordinates, and so to the points with position vectors $P_V(\mathbf{u},\mathbf{v})(0,3,0)^T = \left(\frac{3}{\sqrt{2}}, \frac{-3}{\sqrt{6}}\right)^T$ and $P_V(\mathbf{u},\mathbf{v})(6,3,0)^T = \left(\frac{-3}{\sqrt{2}}, \frac{-9}{\sqrt{6}}\right)^T$ in the rs system. Proceeding in a like manner for all vertices and joining those that are connected by edges, we get Figure 4.3. ♦

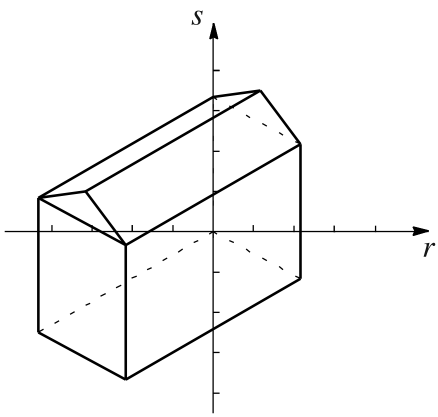

Fig. 4.3. Oblique view of the house

Exercises

Exercise 4.3.1. Find the matrix in homogeneous coordinates that represents a 30 degree rotation about the point $(1,-2)$.

Exercise 4.3.2. Find the matrix in homogeneous coordinates that maps the rectangle with vertices $(1,-2)$, $(1,2)$, $(4,2)$, $(4,-2)$ onto the unit square.

Exercise 4.3.3. Find the inverse of the matrix in the previous exercise. Describe the mapping it represents.

Exercise 4.3.4. Find the matrix that rotates an arbitrary vector $\mathbf{p}$ of $\mathbb{R}^3$ into the z-axis

a. by first rotating it about the z-axis into the yz-*plane* and then about the x-axis into the z-axis, and

b. by first rotating it about the x-axis into the xz-*plane* and then about the y-axis into the z-axis.

Exercise 4.3.5. Using the result of the previous exercise, find the matrix that represents rotation by an angle θ about an arbitrary vector $\mathbf{p}$ of $\mathbb{R}^3$.

Exercise 4.3.6. Find a 4×4 matrix, analogous to that of Equation 4.96, that represents translation in the $x_4 = 1$ plane by an arbitrary vector $\mathbf{t} = (t_1, t_2, t_3, 0)^T$.

Exercise 4.3.7. Find a 4×4 matrix that represents in homogeneous coordinates the rotation by an angle θ about the $x = y = 1$, $z = 0$ line of $\mathbb{R}^3$.

Exercise 4.3.8. Find a 4×4 matrix that represents in homogeneous coordinates the rotation by an angle θ about the $\mathbf{p} = t(1,1,1)^T + (1,0,0)^T$ line of $\mathbb{R}^3$.

Exercise 4.3.9. Find the view of the house of Example 4.3.3 on the $x+2y = 0$ plane by choosing an appropriate basis in the latter, computing the rs coordinates of the vertices relative to this basis and plotting them.

Exercise 4.3.10. Rederive Equation 4.124 by changing from the standard basis to the basis $(\mathbf{u}, \mathbf{v}, \mathbf{n})$, dropping the $\mathbf{n}$-component, and returning to the standard basis.

5. *Orthogonal Projections and Bases*

5.1 Orthogonal Projections and Least-Squares Approximations

In this section we discuss the very practical problem of fitting a line, a plane, or a curve to a set of given points when this can only be done approximately. For example, we may expect some observed data to be the coordinates of points on a straight line, but they turn out to be only approximately so. Then our problem is to find a line that fits them best in some sense. The criterion generally used is the least-squares principle, which we shall describe shortly. First, however, we need to discuss the following problem.

Given a point P in $\mathbb{R}^m$ and a subspace[1] V, we wish to find the point Q in V that is closest to P. The solution has the following geometric property (see Figure 5.1).

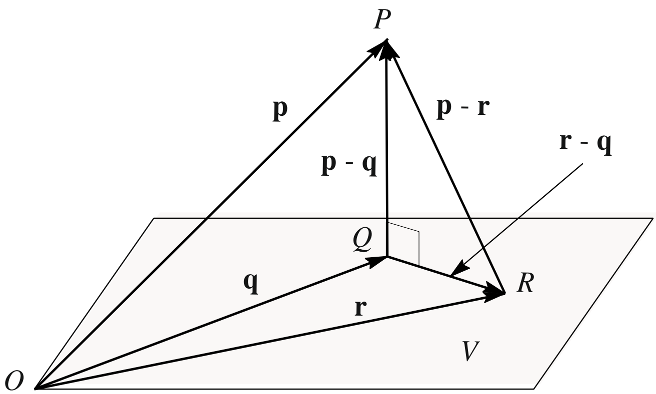

Fig. 5.1. Finding the point Q in a given plane V that is nearest to a given point P

Lemma 5.1.1. ***(Minimizing the Distance Between a Point and a Subspace in*** $\mathbb{R}^m$***).*** *The point of* V *closest to* P *is* Q *if and only if* $\mathbf{p}-\mathbf{q} \in V^{\perp}$.

[1] We view $\mathbb{R}^m$ and this subspace both as sets of points and as vector spaces.

The original version of this chapter was revised. An erratum can be found at https://doi.org/10.1007/978-0-8176-8325-2_9

G. Schay, *A Concise Introduction to Linear Algebra*,
DOI 10.1007/978-0-8176-8325-2_5,

Proof. The statement is trivially true if P is in V: then $Q = P$ and $\mathbf{p} - \mathbf{q} = \mathbf{0}$. If $P \notin V$, then let Q be a point in V such that $\mathbf{p} - \mathbf{q} \in V^{\perp}$ (such a point does exist; why?), and let R be any point of V other than Q. Then we have $\mathbf{r} - \mathbf{q} \in V$ and so $\mathbf{p} - \mathbf{q} \perp \mathbf{r} - \mathbf{q}$. Thus the PQR triangle is a right triangle and, by the Theorem of Pythagoras (which is valid in $\mathbb{R}^m$, too), the side PQ is shorter than the hypotenuse PR. In other words, the distance $|\mathbf{p} - \mathbf{q}|$ is less than the distance $|\mathbf{p} - \mathbf{r}|$ for every $\mathbf{r} \neq \mathbf{q}$ in V.

Conversely, if R is any point in V such that $\mathbf{p} - \mathbf{r}$ is not in $V^{\perp}$, then, by the above argument, the point Q for which $\mathbf{p} - \mathbf{q} \in V^{\perp}$ is nearer to P than is the point R, and so such an R is not the point in V closest to P. ■

In connection with this lemma we use the following terminology.

Definition 5.1.1. ***(Projections onto Subspaces in*** $\mathbb{R}^m$***).*** *If a vector* $\mathbf{p}$ *in* $\mathbb{R}^m$ *is decomposed into the sum of a vector* $\mathbf{q}$ *in a subspace* V *of* $\mathbb{R}^m$ *and a vector* $\mathbf{p} - \mathbf{q} \in V^{\perp}$*, then we call* $\mathbf{q}$ *and* $\mathbf{p} - \mathbf{q}$ *the (orthogonal) projections of* $\mathbf{p}$ *onto* V *and* $V^{\perp}$ *respectively.*[2]

The next question is: How do we find the point Q? This question is fairly easy to answer if we consider (without any loss of generality) V to be the column space of an $m \times n$ matrix A with independent columns, and so with $m \geq n$ as well. Then we could use the theory of Section 3.4, but it is more efficient to proceed as follows.

If $\mathbf{q}$ and $\mathbf{r}$ are vectors in the column space of the matrix A, then we may write them as $\mathbf{q} = A\mathbf{x}$ and $\mathbf{r} = A\mathbf{y}$ for some n-vectors $\mathbf{x}$ and $\mathbf{y}$. The condition $\mathbf{p} - \mathbf{q} \in \mathrm{Col}(A)^{\perp}$ implies that $\mathbf{q} - \mathbf{p}$ must be orthogonal to every such $\mathbf{r}$, and this orthogonality can be written as

$$\mathbf{r}^T(\mathbf{q} - \mathbf{p}) = (A\mathbf{y})^T(A\mathbf{x} - \mathbf{p}) = 0 \tag{5.1}$$

for every $\mathbf{y}$. Equivalently, writing here $(A\mathbf{y})^T = \mathbf{y}^T A^T$ and distributing A^T, we find that

$$\mathbf{y}^T(A^T A\mathbf{x} - A^T\mathbf{p}) = 0 \tag{5.2}$$

must hold for every $\mathbf{y}$, and that can happen only if the vector in the parentheses is the zero vector. Then

$$A^T A\mathbf{x} = A^T\mathbf{p}. \tag{5.3}$$

In least-squares theory the corresponding scalar equations are called the *normal equations* or the *normal system*. Equation 5.3 is easy to remember: Just multiply the usual equation $A\mathbf{x} = \mathbf{p}$ by A^T from the left on both sides. The interpretation is, however, entirely different if $\mathbf{p}$ is not in $\mathrm{Col}(A)$. Then

[2] This decomposition is unique by the last statement in Theorem 3.5.4 on page 137, since V may always be considered to be the column space of a matrix.

$A\mathbf{x} = \mathbf{p}$ has no solution, while Equation 5.3 always has one, as will be proved shortly. Also, $A\mathbf{x}$ and $\mathbf{p}$ are in $\mathbb{R}^m$, but $A^T A\mathbf{x}$ and $A^T\mathbf{p}$ are in $\mathbb{R}^n$.

The projection of $\mathbf{p}$ onto the column space of A can be obtained by computing $\mathbf{q} = A\mathbf{x}$, once we have determined $\mathbf{x}$ from the solution of Equation 5.3. We shall write an explicit formula for this projection, but it is more efficient to obtain it by Gaussian elimination, as in the following example.

Example 5.1.1. (Projecting onto* $\mathbf{Col}(A)$ *and* $\mathbf{Col}(A)^{\perp}$*). Let V be the columns space of the matrix

$$A = \begin{bmatrix} 1 & 0 \\ 3 & -1 \\ 1 & 1 \end{bmatrix} \tag{5.4}$$

and $\mathbf{p} = (1, 2, 3)^T$. Then $\text{Col}(A)$ is a plane in $\mathbb{R}^3$ just as the V in Figure 5.1 is. Furthermore,

$$A^T = \begin{bmatrix} 1 & 3 & 1 \\ 0 & -1 & 1 \end{bmatrix}, \tag{5.5}$$

$$A^T A = \begin{bmatrix} 11 & -2 \\ -2 & 2 \end{bmatrix}, \tag{5.6}$$

and

$$A^T\mathbf{p} = \begin{bmatrix} 10 \\ 1 \end{bmatrix}. \tag{5.7}$$

Hence the normal equations are given by

$$\begin{bmatrix} 11 & -2 \\ -2 & 2 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 10 \\ 1 \end{bmatrix}. \tag{5.8}$$

We solve this equation by Gaussian elimination as follows:

$$\left[\begin{array}{cc|c} 11 & -2 & 10 \\ -2 & 2 & 1 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 8 & 15 \\ -2 & 2 & 1 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 8 & 15 \\ 0 & 18 & 31 \end{array}\right]. \tag{5.9}$$

Thus,

$$\mathbf{x} = \frac{1}{18} \begin{bmatrix} 22 \\ 31 \end{bmatrix} \tag{5.10}$$

and

$$\mathbf{q} = A\mathbf{x} = \frac{1}{18} \begin{bmatrix} 1 & 0 \\ 3 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 22 \\ 31 \end{bmatrix} = \frac{1}{18} \begin{bmatrix} 22 \\ 35 \\ 53 \end{bmatrix}. \tag{5.11}$$

This vector is the orthogonal projection of $\mathbf{p}$ onto $\operatorname{Col}(A)$. The vector $\mathbf{p}-\mathbf{q}$ in $\operatorname{Col}(A)^{\perp}$, shown in Figure 5.1, is given by

$$\mathbf{p}-\mathbf{q} = \begin{bmatrix}1\\2\\3\end{bmatrix} - \frac{1}{18}\begin{bmatrix}22\\35\\53\end{bmatrix} = \frac{1}{18}\begin{bmatrix}-4\\1\\1\end{bmatrix}. \tag{5.12}$$

(Notice that $\mathbf{q}$ and $\mathbf{p}-\mathbf{q}$ are the same as $\mathbf{x}_R$ and $\mathbf{x}_0$ in Equation 3.87 on page 136. Why?) ♦

We wish to show now that the normal equations always have a unique solution, as would be expected from the geometry. This result will follow from the following lemmas.

Lemma 5.1.2. ***(Rank of*** A^TA***).*** *If* A *is any* $m \times n$ *matrix, then* $\operatorname{rank}(A^TA) = \operatorname{rank}(A)$.

Proof. We may prove this statement by showing that A and A^TA have the same null space, since then, by Corollary 3.5.1 on page 133, they must have the same rank as well.

Let $\mathbf{x}$ be in the null space of A. Then $A\mathbf{x} = \mathbf{0}$ holds, and multiplying this equation by A^T from the left we obtain $A^TA\mathbf{x} = \mathbf{0}$, which shows that such an $\mathbf{x}$ is also in the null space of A^TA. Conversely, if $\mathbf{x}$ is in the null space of A^TA, then $A^TA\mathbf{x} = \mathbf{0}$ holds and, multiplying this equation by $\mathbf{x}^T$ from the left, we get

$$\mathbf{x}^TA^TA\mathbf{x} = (A\mathbf{x})^T(A\mathbf{x}) = |A\mathbf{x}|^2 = 0. \tag{5.13}$$

Since $\mathbf{0}$ is the only vector of length zero, Equation 5.13 implies $A\mathbf{x} = \mathbf{0}$, and so $\mathbf{x}$ is in the null space of A. ■

Lemma 5.1.3. ***(***A^TA ***Is Invertible If*** $\operatorname{rank}(A) = n$***).*** *If* A *is an* $m \times n$ *matrix with independent columns, then* A^TA *is invertible.*

Proof. If A is an $m\times n$ matrix with independent columns, then the n columns form a basis for $\operatorname{Col}(A)$, and so $\operatorname{rank}(A) = n$ and $n \le m$ must hold. On the other hand, A^TA is an $n \times n$ matrix and, by the previous lemma, its rank is the same as that of A, that is, n. Hence, by Parts 1 and 2 of Theorem 2.5.5 on page 93, A^TA is invertible. ■

By the last lemma, A^TA is invertible and so we may solve Equation 5.3 by multiplying both sides of it by $(A^TA)^{-1}$ from the left. Thus, we may summarize our discussion of projections in the following theorem.

Theorem 5.1.1. ***(A Formula for Projections).*** *Let* $\mathbf{p}$ *be a vector in* $\mathbb{R}^m$*,* V *a subspace of* $\mathbb{R}^m$*, and* A *an* $m \times n$ *matrix with independent columns such that* $V = \operatorname{Col}(A)$*. Then the orthogonal projection* $\mathbf{q}$ *of* $\mathbf{p}$ *onto* V *can be obtained by solving the normal system*

$$A^T A\mathbf{x} = A^T \mathbf{p} \tag{5.14}$$

for $\mathbf{x}$ *and setting* $\mathbf{q} = A\mathbf{x}$*. The solution can be written explicitly as*

$$\mathbf{x} = (A^T A)^{-1} A^T \mathbf{p} \tag{5.15}$$

and

$$\mathbf{q} = A(A^T A)^{-1} A^T \mathbf{p}. \tag{5.16}$$

In view of the application of Theorem 5.1.1 to least-squares problems (see the discussion following Theorem 5.1.3 below), it is customary to use the following terminology.

Definition 5.1.2. ***(Least-Squares Solution).*** *With the notation of Theorem 5.1.1, the solution* $\mathbf{x}$ *of the normal system, Equation 5.14, is called the least-squares solution of the possibly inconsistent system* $A\mathbf{x} = \mathbf{p}$.

Of course, if $A\mathbf{x} = \mathbf{p}$ is inconsistent, then it has no solution, and its least-squares "solution" is not really a solution to it, but only to the normal system. On the other hand, if $A\mathbf{x} = \mathbf{p}$ is consistent, then the two solutions coincide (see Exercise 5.1.5).

The matrix $P = A(A^T A)^{-1} A^T$ in Equation 5.16 is called the *projection matrix* representing the projection of $\mathbb{R}^m$ onto $\mathrm{Col}(A)$. In general, projection matrices are defined as follows.

Definition 5.1.3. ***(Projection Matrix).*** *A matrix* P *is called a projection matrix if it is square and has the following two properties:*
1. It is idempotent: $P^2 = P$ *and*
2. It is symmetric: $P^T = P$.

The proof that $P = A(A^T A)^{-1} A^T$ of Equation 5.16 has indeed the two properties above, is left as Exercise 5.1.11. Before turning to examples, we state two important properties of projection matrices.

Theorem 5.1.2. ***(Two Properties of Projection Matrices).*** *If* P *is a projection matrix, then*
1. $\mathbf{x} \in \mathrm{Col}(P)$ *is equivalent to* $P\mathbf{x} = \mathbf{x}$.
2. $\mathrm{Null}(P) = \mathrm{Col}(P)^{\perp}$.

Proof. Assume that P is $m \times m$. Then
1. $\mathbf{x} \in \mathrm{Col}(P)$ if and only if there is a $\mathbf{y} \in \mathbb{R}^m$ such that $\mathbf{x} = P\mathbf{y}$. (See Theorem 3.5.3.) Applying P on both sides and using the idempotency of P, we get

$$P\mathbf{x} = P^2\mathbf{y} = P\mathbf{y} = \mathbf{x}. \tag{5.17}$$

Conversely, if $P\mathbf{x} = \mathbf{x}$, then, since $P\mathbf{x} \in \mathrm{Col}(P)$, we get $\mathbf{x} \in \mathrm{Col}(P)$.

2. Let $\mathbf{x} \in \mathrm{Col}(P)$ and $\mathbf{y} \in \mathrm{Null}(P)$. Then

$$\mathbf{x}^T\mathbf{y} = (P\mathbf{x})^T\mathbf{y} = \mathbf{x}^T P^T \mathbf{y} = \mathbf{x}^T P\mathbf{y} = \mathbf{x}^T\mathbf{0} = 0. \tag{5.18}$$

Thus $\mathbf{x}$ and $\mathbf{y}$ are orthogonal to each other. Furthermore, if $\dim(\mathrm{Col}(P)) = r$, then $\dim(\mathrm{Null}(P)) = m - r$. Hence $\mathrm{Col}(P)$ and $\mathrm{Null}(P)$ are orthogonal complements of each other. ■

Example 5.1.2. (Computing a Projection Matrix). Let us compute the projection matrix that represents the projection onto the column space of the matrix A of Example 5.1.1 and use it to recompute the projection of the vector $\mathbf{p}$.

From Equations 5.6 and 5.7,

$$(A^TA)^{-1} = \begin{bmatrix} 11 & -2 \\ -2 & 2 \end{bmatrix}^{-1} = \frac{1}{18}\begin{bmatrix} 2 & 2 \\ 2 & 11 \end{bmatrix} \tag{5.19}$$

and

$$(A^TA)^{-1}A^T = \frac{1}{18}\begin{bmatrix} 2 & 2 \\ 2 & 11 \end{bmatrix}\begin{bmatrix} 1 & 3 & 1 \\ 0 & -1 & 1 \end{bmatrix} = \frac{1}{18}\begin{bmatrix} 2 & 4 & 4 \\ 2 & -5 & 13 \end{bmatrix}. \tag{5.20}$$

Thus the projection matrix is

$$P = A(A^TA)^{-1}A^T = \frac{1}{18}\begin{bmatrix} 2 & 4 & 4 \\ 4 & 17 & -1 \\ 4 & -1 & 17 \end{bmatrix}, \tag{5.21}$$

from which we can now obtain the projection of the vector $\mathbf{p}$ as

$$\mathbf{q} = P\mathbf{p} = \frac{1}{18}\begin{bmatrix} 2 & 4 & 4 \\ 4 & 17 & -1 \\ 4 & -1 & 17 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \frac{1}{18}\begin{bmatrix} 22 \\ 35 \\ 53 \end{bmatrix}, \tag{5.22}$$

just as in Example 5.1.1. ♦

We now present another theorem about projection matrices.

Theorem 5.1.3. ***(A Projection Matrix*** P ***Projects onto*** $\mathrm{Col}(P)$***).*** *A projection matrix P represents the projection onto its own column space.*

Proof. Let P be an $m \times m$ projection matrix and $\mathbf{p}$ be any vector in $\mathbb{R}^m$. We are going to show that $\mathbf{p} - P\mathbf{p}$ is orthogonal to the column space of P, and then, $P\mathbf{p}$ being in $\mathrm{Col}(P)$, the vectors $P\mathbf{p}$ and $\mathbf{p} - P\mathbf{p}$ provide the decomposition of $\mathbf{p}$ into its projections onto $\mathrm{Col}(P)$ and $\mathrm{Null}(P) = \mathrm{Col}(P)^\perp$ respectively. By Definition 5.1.1, this decomposition shows that P represents the projection onto $\mathrm{Col}(P)$.

Since every vector of Col(P) can be written as $P\mathbf{x}$, we test the orthogonality of $\mathbf{p} - P\mathbf{p}$ to Col(P) by computing the dot product of these two vectors, using the assumed properties $P^T = P$ and $P^2 = P$:

$$(P\mathbf{x})^T(\mathbf{p} - P\mathbf{p}) = \mathbf{x}^T P^T(\mathbf{p} - P\mathbf{p}) = \mathbf{x}^T P\mathbf{p} - \mathbf{x}^T P^2\mathbf{p} = 0. \tag{5.23}$$

■

We are now ready to discuss least-squares problems.

Suppose we are given m points (x_i, y_i), $i = 1, 2, \ldots, m$, in the xy-plane and we want to find the equation of a straight line, in the form $y = ax + b$, such that the sum of the squared vertical distances from the points to the line is minimized. (Hence the name least-squares.) In other words, we want to minimize the function

$$f(a, b) = \sum_{i=1}^{m} d_i^2 = \sum_{i=1}^{m} (ax_i + b - y_i)^2. \tag{5.24}$$

(See Figure 5.2.) The solution line is called the least-squares line for the given points or the line of best fit in the least-squares sense.

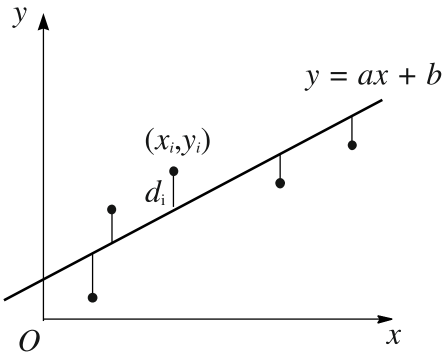

Fig. 5.2. A line and vertical distances to it from given points

We could solve this problem by differentiating $f(a, b)$ with respect to both a and b, setting the partial derivatives equal to zero, and solving the resulting equations, but we prefer to reformulate this as a projection problem in $\mathbb{R}^m$ as follows: Define

$$\mathbf{s} = \begin{bmatrix} a \\ b \end{bmatrix}, \quad A = \begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ \vdots & \vdots \\ x_m & 1 \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}. \tag{5.25}$$

Then we have

$$A\mathbf{s}-\mathbf{y}=\begin{bmatrix} ax_1+b-y_1 \\ ax_2+b-y_2 \\ \vdots \\ ax_m+b-y_m \end{bmatrix} \tag{5.26}$$

and so

$$f(a,b)=|A\mathbf{s}-\mathbf{y}|^2. \tag{5.27}$$

Thus the problem of minimizing $f(a,b)$, which was originally posed in the xy-plane, can be reinterpreted as that of finding the vector $\mathbf{s}$ that minimizes the distance[3] in $\mathbb{R}^m$ from the point given by $\mathbf{y}$ to the column space of A. That is, we find the least-squares solution to the generally inconsistent equation $A\mathbf{s}=\mathbf{y}$, which is consistent only if all the points (x_i, y_i) fall on a straight line. By Lemma 5.1.1, this minimum is achieved by the vector $\mathbf{s}$ for which $A\mathbf{s}$ is the orthogonal projection of $\mathbf{y}$ onto the column space of A, and the solution is given, according to Theorem 5.1.1, by the solution of the normal system

$$A^TA\mathbf{s}=A^T\mathbf{y}. \tag{5.28}$$

Using the definitions of A and $\mathbf{y}$ from Equation 5.25, for the least-squares line we have

$$A^TA=\begin{bmatrix} x_1 & x_2 & \cdots & x_m \\ 1 & 1 & \cdots & 1 \end{bmatrix}\begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ \vdots & \vdots \\ x_m & 1 \end{bmatrix}=\begin{bmatrix} \sum x_i^2 & \sum x_i \\ \sum x_i & m \end{bmatrix} \tag{5.29}$$

and

$$A^T\mathbf{y}=\begin{bmatrix} x_1 & x_2 & \cdots & x_m \\ 1 & 1 & \cdots & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}=\begin{bmatrix} \sum x_iy_i \\ \sum y_i \end{bmatrix}. \tag{5.30}$$

Thus Equation 5.28 becomes

$$\begin{bmatrix} \sum x_i^2 & \sum x_i \\ \sum x_i & m \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix}=\begin{bmatrix} \sum x_iy_i \\ \sum y_i \end{bmatrix}. \tag{5.31}$$

Let us look at an example.

[3] In Equation 5.27 we have the *square* of the distance and not the distance itself, but this makes no difference since the two are minimized simultaneously. (Why?)

Example 5.1.3. (Finding a Least-Squares Line). Find the least-squares line for the points $(-1,0),\ (1,1),\ (2,1),\ (3,2)$, and $(5,3)$.

First, we compute the expressions in the last two formulas:

$$\sum x_i^2 = (-1)^2 + 1^2 + 2^2 + 3^2 + 5^2 = 40, \tag{5.32}$$

$$\sum x_i = -1 + 1 + 2 + 3 + 5 = 10, \tag{5.33}$$

$$\sum y_i = 0 + 1 + 1 + 2 + 3 = 7, \tag{5.34}$$

and

$$\sum x_i y_i = (-1)\cdot 0 + 1\cdot 1 + 2\cdot 1 + 3\cdot 2 + 5\cdot 3 = 24. \tag{5.35}$$

Hence the normal system is given by

$$\begin{bmatrix} 40 & 10 \\ 10 & 5 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 24 \\ 7 \end{bmatrix} \tag{5.36}$$

and its solution is $a = \frac{1}{2}$ and $b = \frac{2}{5}$. So the least-squares line is given by

$$y = \frac{1}{2}x + \frac{2}{5}. \tag{5.37}$$

♦

The problem of fitting a least-squares plane to m data points (x_i, y_i, z_i) in $\mathbb{R}^3$ is similar to the one above, except that we now have three unknowns $a,\ b,\ c$ instead of the previous two. We are looking for the best fitting plane with equation $z = ax + by + c$ such that the function

$$f(a,b,c) = \sum_{i=1}^{m} d_i^2 = \sum_{i=1}^{m} (ax_i + by_i + c - z_i)^2 \tag{5.38}$$

is minimized. This problem can again be reformulated as a projection problem in $\mathbb{R}^m$: If we define

$$\mathbf{s} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}, \quad A = \begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ \vdots & \vdots & \vdots \\ x_m & y_m & 1 \end{bmatrix}, \quad \mathbf{z} = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_m \end{bmatrix}, \tag{5.39}$$

then we can write

$$f(a,b,c) = |A\mathbf{s} - \mathbf{z}|^2, \tag{5.40}$$

which can be minimized by solving the 3×3 normal system

$$A^T A\mathbf{s} = A^T \mathbf{z}. \tag{5.41}$$

We leave it as Exercise 5.1.18 to write Equation 5.41 in a form analogous to Equation 5.31.

The method of least squares is applicable to curves and curved surfaces as well, as long as we know what the form of the equation should be and if we need to find only unknown coefficients that occur linearly. For example, if we believe that some data points (x_i, y_i) fall approximately on a parabola whose equation is of the form $y = ax^2 + bx + c$, then we can set up the problem for the unknown coefficients $a,\ b,\ c$ much as in the previous cases and find the best fitting parabola in the same way. If the coefficients do not occur linearly, we may still be able to transform the problem to one in which they do. For instance, if we want to fit a curve with equation $y = ae^{bx}$, then we can take logarithms on each side to obtain an equation of the form $\ln y = \ln a + bx$, and this is linear in the unknown coefficients $\ln a$ and b. Thus, we can proceed exactly as before if we just replace a by $\ln a$ and y by $\ln y$.

Exercises

Exercise 5.1.1. In $\mathbb{R}^3$ find a matrix A with independent columns whose column space is the x-axis.

Exercise 5.1.2. In $\mathbb{R}^3$ find a matrix A with independent columns whose column space is the xy-plane.

Exercise 5.1.3. In $\mathbb{R}^3$ find a matrix A with independent columns whose column space is the $x = y$ plane.

Exercise 5.1.4. In $\mathbb{R}^3$ find the projection of the vector $(1, -1, 2)^T$ onto the column space of the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 1 & -1 \\ 2 & 1 \end{bmatrix}.$$

Exercise 5.1.5. Show that, for a matrix A with independent columns, if $\mathbf{p}$ is in $\mathrm{Col}(A)$, then the equations $A\mathbf{x} = \mathbf{p}$ and $A^TA\mathbf{x} = A^T\mathbf{p}$ both have the same unique solution $\mathbf{x}$.

Exercise 5.1.6. In $\mathbb{R}^3$ find a projection matrix P that represents the projection onto the x-axis.

Exercise 5.1.7. In $\mathbb{R}^3$ find a projection matrix P that represents the projection onto the xy-plane.

Exercise 5.1.8. In $\mathbb{R}^3$ find a projection matrix P that represents the projection onto the line through the origin given by $x = at,\ y = bt,\ z = ct$. Compare with Corollary 1.2.1 on page 21.

Exercise 5.1.9. In $\mathbb{R}^3$ find a projection matrix P that represents the projection onto the $x = y$ plane and find the projections of the vectors $(1, -1, 2)^T$, $(1, 2, 3)^T$, $(2, -1, -2)^T$ onto the $x = y$ plane.

Exercise 5.1.10. In $\mathbb{R}^3$ what is the orthogonal complement of the $x = y$ plane? Find a projection matrix Q that represents the projection onto it.

Exercise 5.1.11. Prove that, for every matrix A with independent columns, the matrix $A(A^TA)^{-1}A^T$ is idempotent and symmetric.

Exercise 5.1.12. Prove that, for every projection matrix P that represents the projection onto the column space of a matrix A, the matrix $I - P$ is also a projection matrix and it represents the projection onto Left-null$(A) =$ Col$(A)^\perp$.

Exercise 5.1.13. What is wrong with the following "proof" of Theorem 5.1.3?
As we have seen, $A(A^TA)^{-1}A^T$ represents the projection onto Col(A). Substituting P for A into this expression and making use of Properties 1 and 2 of projection matrices in Definition 5.1.3, we get the projection matrix representing the projection onto Col(P) as $P(P^TP)^{-1}P^T = P(PP)^{-1}P = PP^{-1}P = P$.

***Exercise 5.1.14.** Let A be a matrix with independent rows.
a. Show that AA^T is invertible.
b. Show that if $\mathbf{b}$ is any vector in Col(A), then the equation $A\mathbf{x} = \mathbf{b}$ has a solution in Row(A) given by $\mathbf{x}_R = A^T(AA^T)^{-1}\mathbf{b}$. (The matrix $A^T(AA^T)^{-1}$ is called the *pseudoinverse* of A, and is usually denoted by A^+. It is a right inverse of A, and coincides with the two-sided inverse A^{-1} if A is square.)
c. The mapping of Col(A) to Row(A) given by $\mathbf{x}_R = A^T(AA^T)^{-1}\mathbf{b}$ is an isomorphism.

Exercise 5.1.15. Let A be a matrix with independent rows. Find a formula for the matrix of the projection onto Null(A).

Exercise 5.1.16. Find the least-squares line for the points $(1, -2)$, $(-3, 1)$, $(2, 0)$, $(3, -2)$, and $(-5, 3)$ in $\mathbb{R}^2$.

Exercise 5.1.17. Using Equations 5.28, 5.29, and 5.30, prove that every least-squares line in $\mathbb{R}^2$ passes through the centroid of the given points (x_i, y_i).

Exercise 5.1.18. Find a formula analogous to Equation 5.31 for the normal system 5.41 of the least-squares plane to m data points (x_i, y_i, z_i) in $\mathbb{R}^3$.

Exercise 5.1.19. Using the formula obtained in the previous exercise, prove that every least-squares plane passes through the centroid of the given points (x_i, y_i, z_i).

Exercise 5.1.20. Find formulas analogous to Equations 5.29 and 5.30 for the coefficients in the normal system for the least-squares parabola $y = ax^2 + bx + c$ to m data points (x_i, y_i) in $\mathbb{R}^2$.

MATLAB Exercises

Exercise 5.1.21. Let

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 3 & 4 & 0 \\ 2 & 3 & 2 \\ 0 & -1 & 0 \end{bmatrix}.$$

a. Use MATLAB to find the matrix P of the projection onto $\mathrm{Col}(A)$.
b. Verify that P is a projection matrix.
c. The command $B = \mathbf{orth}(A)$ computes a basis matrix for $\mathrm{Col}(A)$. Verify that the matrix P projects onto $\mathrm{Col}(A)$ by showing that $P\mathbf{e}_i$ is a linear combination of the columns of B for each standard vector $\mathbf{e}_i$ of $\mathbb{R}^4$.

Exercise 5.1.22. If the linear system $A\mathbf{s} = \mathbf{y}$ is overdetermined, that is, there are more equations than unknowns, then the MATLAB command $\mathbf{s} = A\backslash\mathbf{y}$ returns the solution of the corresponding normal system $A^T A\mathbf{s} = A^T\mathbf{y}$. Use this command and Equations 5.24 through 5.28 to solve the problem of Example 5.1.3 with MATLAB. Plot the result by also computing $z = \mathbf{s}(1)^*\mathbf{x} + \mathbf{s}(2)$ or $z = \mathbf{polyval}(\mathbf{s}, \mathbf{x})$ and entering the command $\mathbf{plot}(\mathbf{x}, \mathbf{y}, 'o', \mathbf{x}, \mathbf{z})$.

Exercise 5.1.23. Use the MATLAB routine **polyfit** instead of $A\backslash\mathbf{y}$ to find the vector $\mathbf{s}$ of the previous exercise.

Exercise 5.1.24. Use the method of Exercise 5.1.22 and Equations 5.38 through 5.41 to find the plane that best fits the points $(1, 2, 3)$, $(2, 2, 4)$, $(-1, 0, 3)$, $(5, -2, 2)$, and $(7, 0, -1)$. (You may also want to use MATLAB to plot these points and the plane in a three-dimensional coordinate system.)

Exercise 5.1.25. Use **polyfit** to do Exercise 5.1.16.

Exercise 5.1.26. Use **polyfit** to find the parabola that fits best to the points $(1, 2)$, $(2, 4)$, $(-1, 0)$, $(-2, 5)$, and $(4, 14)$, and use MATLAB to plot it, together with the given points.

5.2 Orthogonal Bases

In Section 3.2 we saw how to decompose vectors into linear combinations of given vectors and how to test the latter for independence. Both of these procedures required the solution of linear systems. If, however, the given vectors are orthogonal to each other, then their independence becomes automatic, and the decomposition of other vectors can be achieved much more simply by taking dot products, as in the following example.

Example 5.2.1. (Decomposing a Vector in $\mathbb{R}^2$ into Orthogonal Components). In $\mathbb{R}^2$ let us consider the basis $\{\mathbf{a}_1, \mathbf{a}_2\}$ with $\mathbf{a}_1 = (1,1)^T$and $\mathbf{a}_2 = (-1,1)^T$. We want to write an arbitrary vector $\mathbf{x} = (x_1, x_2)^T$ as

$$\mathbf{x} = x_{A1}\mathbf{a}_1 + x_{A2}\mathbf{a}_2. \tag{5.42}$$

Taking the dot product of both sides of the last equation by $\mathbf{a}_1$ and noting that $\mathbf{a}_1 \cdot \mathbf{a}_2 = 0$, we get

$$\mathbf{a}_1 \cdot \mathbf{x} = x_{A1}\mathbf{a}_1 \cdot \mathbf{a}_1, \tag{5.43}$$

which reduces to

$$x_1 + x_2 = 2x_{A1} \tag{5.44}$$

and yields

$$x_{A1} = \frac{x_1 + x_2}{2}. \tag{5.45}$$

Similarly, taking the dot product of both sides of Equation 5.41 by $\mathbf{a}_2$, we get

$$x_{A2} = \frac{x_2 - x_1}{2}. \tag{5.46}$$

♦

The procedure illustrated in the example above can be stated in general terms as follows.

Theorem 5.2.1. ***(Orthogonal Decompositions).*** *Let $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ be mutually orthogonal nonzero vectors in a vector space X equipped with an inner product. Then*

1. *Every vector $\mathbf{x} \in \operatorname{Span}\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$ can be decomposed uniquely as*

$$\mathbf{x} = \sum_{i=1}^{n} x_{Ai}\mathbf{a}_i \tag{5.47}$$

with the coefficients given by

$$x_{Ai} = \frac{\mathbf{a}_i \cdot \mathbf{x}}{\mathbf{a}_i \cdot \mathbf{a}_i} \text{ for } i = 1, 2, \ldots, n. \tag{5.48}$$

2. *The vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ are independent.*

Proof. Since $\mathbf{x} \in \operatorname{Span}\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$, there exists a decomposition of $\mathbf{x}$ in the form of Equation 5.47 with some coefficients x_{Ai}. Taking the dot product of both sides of Equation 5.47 with $\mathbf{a}_i$ and utilizing the assumed orthogonality $\mathbf{a}_i \cdot \mathbf{a}_j = 0$ for all $i \neq j$, we get Equation 5.48. Also, if we take $\mathbf{x} = \mathbf{0}$, then Equation 5.48 shows that each x_{Ai} equals zero, and so the vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ are independent. ■

Comparing Theorem 5.2.1 with Corollary 1.2.1 on page 21 we see that each component $x_{A\iota}\mathbf{a}_i$ above is the orthogonal projection of $\mathbf{x}$ onto $\mathbf{a}_i$.

Frequently the vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ are also "*normalized,*" so that they are replaced with $\mathbf{q}_i = \mathbf{a}_i / |\mathbf{a}_i|$ for $i = 1, 2, \ldots, n$, that is, they are replaced with mutually orthogonal unit vectors. In this case, the $\mathbf{q}_i$ are said to constitute an *orthonormal* set, and Equation 5.47 takes on an especially simple form.

Corollary 5.2.1. ***(Orthonormal Decompositions).*** *If* $\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_n$ *are mutually orthonormal nonzero vectors in an inner product space* X, *then*

$$\mathbf{x} = \sum_{i=1}^{n} (\mathbf{q}_i \cdot \mathbf{x})\, \mathbf{q}_i \tag{5.49}$$

for any $\mathbf{x} \in \operatorname{Span}\{\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_n\}$.

The formula giving the projection matrix that represents the projection onto the column space of a matrix A also becomes much simpler if the columns of A are orthonormal. We have already seen a particular case of this simplification on page 194 in Chapter 4. In general we had the formula $P = A(A^TA)^{-1}A^T$ (see page 203 in this chapter) for such a projection matrix, and if the columns of A are orthonormal, then $A^TA = I$ holds (see Theorem 5.2.2 below), and the formula reduces to $P = AA^T$. If we write $A = (\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n)$ in this equation, then we can write the projection of every $\mathbf{x}$ onto the column space of A as

$$P\mathbf{x} = AA^T\mathbf{x} = (\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n) \begin{bmatrix} \mathbf{a}_1^T\mathbf{x} \\ \vdots \\ \mathbf{a}_n^T\mathbf{x} \end{bmatrix} = (\mathbf{a}_1 \cdot \mathbf{x})\mathbf{a}_1 + \cdots + (\mathbf{a}_n \cdot \mathbf{x})\mathbf{a}_n. \tag{5.50}$$

Thus, in this special case of orthonormal columns, the projection onto the column space of the matrix A is the sum of the projections onto the individual columns.

Notice that the right-hand sides of Equations 5.49 and 5.50 are the same, apart from notation (the $\mathbf{a}_i$ in the latter are also orthonormal vectors, the same as the $\mathbf{q}_i$). The difference between the two equations is that, in Equation 5.49, $\mathbf{x} \in \operatorname{Span}\{\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_n\}$, but in Equation 5.50, $\mathbf{x}$ may be outside $\operatorname{Span}\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$. If $\mathbf{x}$ is in $\operatorname{Span}\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$, then $P\mathbf{x} = \mathbf{x}$. (See Theorem 5.1.2.)

There is still another very important simplification that results from orthonormality. To wit: the computation of the inverse of a square matrix with orthonormal columns, called an *orthogonal matrix,*[4] becomes trivial.

Theorem 5.2.2. ***(The Inverse of an Orthogonal Matrix Is Its Transpose).*** *If Q is an orthogonal matrix, then it is invertible and*

$$Q^{-1} = Q^T. \tag{5.51}$$

Furthermore, in this case the rows of Q are orthonormal as well as its columns.

Proof. If Q is an $n \times n$ orthogonal matrix, then by Theorem 5.2.1 its n columns are independent, and so its rank is n. Thus it is invertible. Also, the assumed orthogonality can be written as $Q^TQ = I$, and multiplying both sides by Q^{-1} from the right gives $Q^T = Q^{-1}$. Now, multiplying by Q from the left results in $QQ^T = I$, which shows the orthonormality of the rows. ■

Orthogonal matrices occur in many applications because they represent distance-preserving transformations called isometries.

Theorem 5.2.3. ***(A Transformation with an Orthogonal Matrix Preserves Length).*** *For every $n \times n$ orthogonal matrix Q and every n-vector* $\mathbf{x}$,

$$|Q\mathbf{x}| = |\mathbf{x}|. \tag{5.52}$$

Proof. $|Q\mathbf{x}|^2 = (Q\mathbf{x})^T Q\mathbf{x} = \mathbf{x}^TQ^TQ\mathbf{x} = \mathbf{x}^TI\mathbf{x} = \mathbf{x}^T\mathbf{x} = |\mathbf{x}|^2$. Hence $|Q\mathbf{x}| = |\mathbf{x}|$. ■

Matrices representing rotations and reflections are orthogonal.

Since orthogonality of basis vectors is such a useful property, in some applications where we have a natural basis that is not orthogonal, we often make a changeover to an orthogonal or orthonormal basis, as in the following example.

Example 5.2.2. (Finding an Orthonormal Basis). Find an orthonormal basis for the subspace U of $\mathbb{R}^4$ spanned by the vectors $\mathbf{a}_1 = (2, 0, -1, 1)^T$, $\mathbf{a}_2 = (1, 1, 0, 1)^T$, and $\mathbf{a}_3 = (1, -3, -1, 3)^T$.

To avoid dealing with unpleasant fractions, first we just find an orthogonal basis $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$, and normalize its vectors afterward. We may take any one of the given vectors as $\mathbf{b}_1$; say, we take $\mathbf{b}_1 = \mathbf{a}_1$. Next, we compute the orthogonal projection $\mathbf{p}_2$ of $\mathbf{a}_2$ onto $\mathbf{b}_1$, and take $\mathbf{b}_2$ as some scalar multiple of $\mathbf{a}_2 - \mathbf{p}_2$, since then $\mathbf{b}_2$ will be in the plane of $\mathbf{a}_1$ and $\mathbf{a}_2$, and will be orthogonal to $\mathbf{b}_1$. Thus

[4] Orthonormal matrix would be a better name, but we have no choice.

$$\mathbf{a}_2 - \mathbf{p}_2 = \mathbf{a}_2 - \frac{\mathbf{a}_2 \cdot \mathbf{b}_1}{\mathbf{b}_1 \cdot \mathbf{b}_1}\mathbf{b}_1 = (1,1,0,1)^T - \frac{3}{6}(2,0,-1,1)^T. \tag{5.53}$$

To avoid fractions, let $\mathbf{b}_2$ be 2 times this vector:

$$\mathbf{b}_2 = (2,2,0,2)^T - (2,0,-1,1)^T = (0,2,1,1)^T. \tag{5.54}$$

Next, take the projection $\mathbf{p}_3$ of $\mathbf{a}_3$ onto the plane of $\mathbf{b}_1$ and $\mathbf{b}_2$ and subtract it from $\mathbf{a}_3$. We get

$$\begin{aligned}\mathbf{a}_3 - \mathbf{p}_3 &= \mathbf{a}_3 - \frac{\mathbf{a}_3 \cdot \mathbf{b}_1}{\mathbf{b}_1 \cdot \mathbf{b}_1}\mathbf{b}_1 - \frac{\mathbf{a}_3 \cdot \mathbf{b}_2}{\mathbf{b}_2 \cdot \mathbf{b}_2}\mathbf{b}_2 \\ &= (1,-3,-1,3)^T - \frac{6}{6}(2,0,-1,1)^T - \frac{-4}{6}(0,2,1,1)^T \end{aligned} \tag{5.55}$$

and taking $\mathbf{b}_3$ as 3 times this vector, we obtain

$$\mathbf{b}_3 = (3,-9,-3,9)^T - (6,0,-3,3)^T + (0,4,2,2)^T = (-3,-5,2,8)^T. \tag{5.56}$$

Normalizing the $\mathbf{b}_i$ vectors we have found, we get an orthonormal basis for U consisting of the vectors

$$\mathbf{q}_1 = \frac{1}{\sqrt{6}}(2,0,-1,1)^T,\ \mathbf{q}_2 = \frac{1}{\sqrt{6}}(0,2,1,1)^T,\ \mathbf{q}_3 = \frac{1}{\sqrt{102}}(-3,-5,2,8)^T. \tag{5.57}$$

♦

The procedure illustrated in the above example is called the *Gram–Schmidt orthogonalization procedure*. It is used mostly in function spaces such as the space of polynomials mentioned in Example 3.5.4 on page 133, in which an inner product can be defined by a suitable integral. We shall not pursue this subject here. In general, the algorithm can be described as follows.

Theorem 5.2.4. ***(Gram–Schmidt Orthogonalization Procedure).*** *Let U be an inner product space with basis $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$. Define new vectors $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n$ successively as $\mathbf{b}_1 = \mathbf{a}_1$ and*

$$\mathbf{b}_k = \mathbf{a}_k - \sum_{i=1}^{k-1} \frac{\mathbf{a}_k \cdot \mathbf{b}_i}{\mathbf{b}_i \cdot \mathbf{b}_i}\mathbf{b}_i \text{ for } k = 2,3,\ldots,n. \tag{5.58}$$

Then these $\mathbf{b}_i$ vectors form an orthogonal basis for U, and if we also normalize them, then the unit vectors $\mathbf{q}_i = \mathbf{b}_i/|\mathbf{b}_i|$ form an orthonormal basis for U.

Proof. We prove this theorem by mathematical induction; that is, we first prove it for $n = 2$, and second, we prove that if it is true for any $n \geq 2$, then it must also be true for $n + 1$. These two parts together prove the theorem for every $n \geq 2$, because from $n = 2$, the second part shows it for $n = 3$, then from that for $n = 4$, etc.

1. First, let $n = 2$. Then $\mathbf{b}_1 = \mathbf{a}_1 \neq \mathbf{0}$, since $\{\mathbf{a}_1, \mathbf{a}_2\}$ is a basis, hence an independent set, and every finite set containing $\mathbf{0}$ is a dependent set (see Exercise 3.3.14). For $n = 2$, Equation 5.58 reduces to

$$\mathbf{b}_2 = \mathbf{a}_2 - \frac{\mathbf{a}_2 \cdot \mathbf{b}_1}{\mathbf{b}_1 \cdot \mathbf{b}_1}\mathbf{b}_1. \tag{5.59}$$

Now, $\mathbf{b}_2 \neq \mathbf{0}$, since if we had $\mathbf{b}_2 = \mathbf{0}$, then Equation 5.59 would express $\mathbf{a}_2$ as a multiple of $\mathbf{b}_1 = \mathbf{a}_1 \neq \mathbf{0}$, contradicting the assumption that $\{\mathbf{a}_1, \mathbf{a}_2\}$ is a basis. Taking the dot product of both sides of Equation 5.59 by $\mathbf{b}_1$, we get

$$\mathbf{b}_2 \cdot \mathbf{b}_1 = \mathbf{a}_2 \cdot \mathbf{b}_1 - \frac{\mathbf{a}_2 \cdot \mathbf{b}_1}{\mathbf{b}_1 \cdot \mathbf{b}_1}\mathbf{b}_1 \cdot \mathbf{b}_1 = \mathbf{a}_2 \cdot \mathbf{b}_1 - \mathbf{a}_2 \cdot \mathbf{b}_1 = 0. \tag{5.60}$$

Thus $\mathbf{b}_1 \perp \mathbf{b}_2$, neither is $\mathbf{0}$, and so, by Theorem 5.2.1, $\mathbf{b}_1$ and $\mathbf{b}_2$ form a basis for U.

2. Assume that Theorem 5.2.4 holds for any given $n \geq 2$. We show that this assumption implies that it also holds for $n + 1$. Let U be an inner product space with basis $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_{n+1}\}$. Define $\mathbf{b}_1 = \mathbf{a}_1$ and $\mathbf{b}_k$ as in Equation 5.58 for all $k = 2, 3, \ldots, n+1$. Then, by the induction assumption above, $\{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ form an orthogonal basis for the subspace of U spanned by $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$. So $\mathbf{b}_i \neq \mathbf{0}$ for $i = 1, 2, \ldots, n$, (since the $\mathbf{b}_i$ vectors form a basis) and $\mathbf{b}_i \cdot \mathbf{b}_j = 0$ for $i, j = 1, 2, \ldots, n$ and $i \neq j$. For $k = n + 1$, Equation 5.58 defines $\mathbf{b}_{n+1}$ as

$$\mathbf{b}_{n+1} = \mathbf{a}_{n+1} - \sum_{i=1}^{n} \frac{\mathbf{a}_{n+1} \cdot \mathbf{b}_i}{\mathbf{b}_i \cdot \mathbf{b}_i}\mathbf{b}_i. \tag{5.61}$$

Now, $\mathbf{b}_{n+1} \neq \mathbf{0}$, since if it were, then Equation 5.61 would express $\mathbf{a}_{n+1}$ as a linear combination of the $\mathbf{b}_i$ vectors, hence also of the $\mathbf{a}_i$ vectors, for $i = 1, 2, \ldots, n$, contradicting the assumption that $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_{n+1}\}$ is a basis for U. Taking the dot product of both sides of Equation 5.61 by $\mathbf{b}_j$, for every $j = 1, 2, \ldots, n$, and utilizing $\mathbf{b}_i \cdot \mathbf{b}_j = 0$ for all $i, j = 1, 2, \ldots, n$ and $i \neq j$, we get

$$\mathbf{b}_{n+1} \cdot \mathbf{b}_j = \mathbf{a}_{n+1} \cdot \mathbf{b}_j - \sum_{i=1}^{n} \frac{\mathbf{a}_{n+1} \cdot \mathbf{b}_i}{\mathbf{b}_i \cdot \mathbf{b}_i}\mathbf{b}_i \cdot \mathbf{b}_j = \mathbf{a}_{n+1} \cdot \mathbf{b}_j - \frac{\mathbf{a}_{n+1} \cdot \mathbf{b}_j}{\mathbf{b}_j \cdot \mathbf{b}_j}\mathbf{b}_j \cdot \mathbf{b}_j = 0. \tag{5.62}$$

Thus $\mathbf{b}_{n+1}$ too is orthogonal to all $\mathbf{b}_j$ for $j = 1, 2, \ldots, n$, and so the vectors $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_{n+1}$ are mutually orthogonal and, as shown above, nonzero. Hence, by Theorem 5.2.1, they form a basis for $U = \operatorname{Span}\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_{n+1}\}$. ■

If in the Gram–Schmidt procedure U is an n-dimensional subspace of $\mathbb{R}^m$ with $m \geq n$, then we can consider the given independent $\mathbf{a}_i$ vectors as the columns of a basis matrix $A = (\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n)$ for U, and the procedure leads to a factoring of A as follows.

Theorem 5.2.5. ***(QR Factorization).*** *If A is an $m \times n$ matrix with $m \geq n$, and with independent columns, then it can be factored as $A = QR$, where Q is an $m \times n$ matrix with orthonormal columns that span* $\operatorname{Col}(A)$, *and R is an $n \times n$ invertible, upper triangular matrix.*

Proof. Equation 5.58 can be reformulated in terms of the unit vectors $\mathbf{q}_i = \mathbf{b}_i / |\mathbf{b}_i|$ as

$$\mathbf{a}_k = \mathbf{b}_k + \sum_{i=1}^{k-1} (\mathbf{a}_k \cdot \mathbf{q}_i)\, \mathbf{q}_i \text{ for } k = 2, 3, \dots, n. \tag{5.63}$$

We can write $\mathbf{b}_k$ too in a form like that of the other terms: $\mathbf{b}_k = (\mathbf{a}_k \cdot \mathbf{q}_k)\, \mathbf{q}_k$, since $\mathbf{b}_k = |\mathbf{b}_k|\, \mathbf{q}_k$ and taking the dot product of Equation 5.63 with $\mathbf{q}_k$ and using the orthogonality of $\mathbf{q}_k$ to $\mathbf{q}_i$ for $i = 1, 2, \dots k-1$, result in

$$\mathbf{a}_k \cdot \mathbf{q}_k = \mathbf{b}_k \cdot \mathbf{q}_k = |\mathbf{b}_k|\, \mathbf{q}_k \cdot \mathbf{q}_k = |\mathbf{b}_k| \text{ for } k = 2, 3, \dots, n. \tag{5.64}$$

Thus, bringing the $\mathbf{b}_k = (\mathbf{a}_k \cdot \mathbf{q}_k)\, \mathbf{q}_k$ term in Equation 5.63 into the sum and writing the dot products in matrix form, we obtain

$$\mathbf{a}_k = \sum_{i=1}^{k} (\mathbf{a}_k \cdot \mathbf{q}_i)\, \mathbf{q}_i = \sum_{i=1}^{k} \left(\mathbf{q}_i^T \mathbf{a}_k\right) \mathbf{q}_i \text{ for } k = 2, 3, \dots, n. \tag{5.65}$$

(Note that this formula is the same as Equation 5.49 with $\mathbf{x} = \mathbf{a}_k$; we just had to establish that $\mathbf{a}_k \in \operatorname{Span}\{\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_k\}$.) Similarly, we also have

$$\mathbf{a}_1 \cdot \mathbf{q}_1 = \mathbf{a}_1 \cdot \frac{\mathbf{a}_1}{|\mathbf{a}_1|} = \frac{|\mathbf{a}_1|^2}{|\mathbf{a}_1|} = |\mathbf{a}_1| \tag{5.66}$$

and

$$\mathbf{a}_1 = |\mathbf{a}_1|\, \mathbf{q}_1 = (\mathbf{a}_1 \cdot \mathbf{q}_1)\, \mathbf{q}_1 = \left(\mathbf{q}_1^T \mathbf{a}_1\right) \mathbf{q}_1. \tag{5.67}$$

We can combine the left-hand sides of Equations 5.66 and 5.64 into the columns of the matrix A, and the right-hand sides into a matrix product (see Exercise 2.4.13, Part b):

$$A = (\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n) = (\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_n) \begin{bmatrix} \mathbf{q}_1^T \mathbf{a}_1 & \mathbf{q}_1^T \mathbf{a}_2 & \mathbf{q}_1^T \mathbf{a}_3 & \cdots & \mathbf{q}_1^T \mathbf{a}_n \\ 0 & \mathbf{q}_2^T \mathbf{a}_2 & \mathbf{q}_2^T \mathbf{a}_3 & \cdots & \mathbf{q}_2^T \mathbf{a}_n \\ 0 & 0 & \mathbf{q}_3^T \mathbf{a}_3 & \cdots & \mathbf{q}_3^T \mathbf{a}_n \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \mathbf{q}_n^T \mathbf{a}_n \end{bmatrix}. \tag{5.68}$$

The matrices on the right are called Q and R, respectively. In the proof of the Gram–Schmidt algorithm, we saw that $\mathbf{b}_k \neq \mathbf{0}$ for $k = 1, 2, \ldots, n$. So, by Equation 5.64, $\mathbf{q}_k^T \mathbf{a}_k = \mathbf{a}_k \cdot \mathbf{q}_k = |\mathbf{b}_k| \neq 0$ for $k = 1, 2, \ldots, n$. These products are the diagonal entries of R; and since they are nonzero, R is invertible. ■

Example 5.2.3. (QR Factorization of the Matrix of Example 5.2.2). Let

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & -3 \\ -1 & 0 & -1 \\ 1 & 1 & 3 \end{bmatrix} \tag{5.69}$$

and, as computed in Example 5.2.2,

$$\mathbf{q}_1 = \frac{1}{\sqrt{6}}(2, 0, -1, 1)^T, \ \mathbf{q}_2 = \frac{1}{\sqrt{6}}(0, 2, 1, 1)^T, \ \mathbf{q}_3 = \frac{1}{\sqrt{102}}(-3, -5, 2, 8)^T. \tag{5.70}$$

Hence $\mathbf{q}_1^T \mathbf{a}_1 = \sqrt{6}$, $\mathbf{q}_1^T \mathbf{a}_2 = \frac{1}{2}\sqrt{6}$, $\mathbf{q}_1^T \mathbf{a}_3 = \sqrt{6}$, $\mathbf{q}_2^T \mathbf{a}_2 = \frac{1}{2}\sqrt{6}$, $\mathbf{q}_2^T \mathbf{a}_3 = -\frac{2}{3}\sqrt{6}$, $\mathbf{q}_3^T \mathbf{a}_3 = \frac{1}{3}\sqrt{102}$, and so the required factorization is

$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & -3 \\ -1 & 0 & -1 \\ 1 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 2/\sqrt{6} & 0 & -3/\sqrt{102} \\ 0 & 2/\sqrt{6} & -5/\sqrt{102} \\ -1/\sqrt{6} & 1/\sqrt{6} & 2/\sqrt{102} \\ 1/\sqrt{6} & 1/\sqrt{6} & 8/\sqrt{102} \end{bmatrix} \begin{bmatrix} \sqrt{6} & \sqrt{6}/2 & \sqrt{6} \\ 0 & \sqrt{6}/2 & -2\sqrt{6}/3 \\ 0 & 0 & \sqrt{102}/3 \end{bmatrix}. \tag{5.71}$$

The QR factorization is used to simplify the computations in least-squares problems: If we substitute $A = QR$ into the equations of Theorem 5.1.1, then the orthonormality of the columns of Q implies that $Q^T Q = I_n$, and we get

$$A^T A = R^T Q^T Q R = R^T R. \tag{5.72}$$

Thus the normal system $A^T A\mathbf{x} = A^T \mathbf{p}$ becomes

$$R^T R\mathbf{x} = R^T Q^T \mathbf{p} \tag{5.73}$$

or, since R^T is invertible,

$$R\mathbf{x} = Q^T \mathbf{p}. \tag{5.74}$$

This system can be solved very easily, because R is upper triangular. ♦

Exercises

Exercise 5.2.1. In $\mathbb{R}^3$ find the projection of the vector $\mathbf{x} = (2, 3, 4)^T$ into the plane of the vectors $(2, 1, 2)^T$ and $(1, 0, -1)^T$.

Exercise 5.2.2. Use Equation 5.50 to rederive Equation 4.124 of page 195 for the matrix giving the projection into the plane spanned by the orthonormal unit vectors $\mathbf{u}$, $\mathbf{v}$ in $\mathbb{R}^3$.

Exercise 5.2.3. In $\mathbb{R}^4$ find
a. the projection of the vector $\mathbf{x} = (1, 2, 3, -1)^T$ into the subspace U of Example 5.2.2,
b. the projection of $\mathbf{x}$ into $U^\perp$,
c. a basis for $U^\perp$, and
d. a vector $\mathbf{q}_4$ such that $\{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_4\}$ is an orthonormal basis for $\mathbb{R}^4$.

Exercise 5.2.4. Are orthogonal matrices not just distance preserving but angle preserving as well? Prove your answer.

Exercise 5.2.5. Show that if Equation 5.52 holds for a given $n \times n$ matrix Q and every n-vector $\mathbf{x}$, then Q is orthogonal.

Exercise 5.2.6. Prove that if P and Q are orthogonal matrices of the same size, then so too is PQ.

Exercise 5.2.7. Find an orthonormal basis for $\mathbb{R}^3$ that includes the vectors $\mathbf{q}_1 = \frac{1}{3}(-1, 2, 2)^T$ and $\mathbf{q}_2 = \frac{1}{3}(2, -1, 2)^T$.

Exercise 5.2.8.
1. Find an orthonormal basis for the subspace U of $\mathbb{R}^4$ spanned by the vectors $\mathbf{a}_1 = (0, 0, -1, 1)^T$, $\mathbf{a}_2 = (1, 0, 0, 1)^T$, and $\mathbf{a}_3 = (1, 0, -1, 0)^T$.
2. Extend the orthonormal basis above to an orthonormal basis for $\mathbb{R}^4$.
3. Find the QR factorization of the matrix $A = (\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3)$.

Exercise 5.2.9. Show that if $\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_n$ denote the columns of an orthogonal matrix, then $\sum_{i=1}^{n} \mathbf{q}_i \mathbf{q}_i^T = I$.

MATLAB Exercises

In MATLAB, the command $C = \textbf{orth}(A)$ returns an orthonormal basis matrix for $\text{Col}(A)$ and so, by Equation 5.50, $P = CC^T$ is the matrix of the projection onto $\text{Col}(A)$. Similarly, the command $N = \textbf{null}(A)$ returns an orthonormal basis matrix for $\text{Null}(A)$ and so $Q = NN^T$ is the matrix of the projection onto $\text{Null}(A)$. Use these commands and matrices to solve the following exercises.

Exercise 5.2.10. In $\mathbb{R}^3$ let a plane S be given by the equation $2x_1 + 3x_2 - x_3 = 0$.
a. Find the matrix of the projection onto S and the matrix of the projection onto the normal vector of S.
b. Check that the sum of the projection matrices found in Part a is I.
c. Use the projection matrices found in Part a to decompose the vector $\mathbf{x} = (2, 3, 4)^T$ into a component in S and one orthogonal to S.
d. Find the distance of the point $A = (2, 3, 4)$ from S.

Exercise 5.2.11. In $\mathbb{R}^3$ let a line L be given by the parametric equation $\mathbf{p} = (1, 3, 5)^T t$.
a. Find the matrix of the projection onto L and the matrix of the projection onto the orthogonal complement of L.
b. Check that the sum of the projection matrices found in Part a is I.
c. Use the projection matrices found in Part a to decompose the vector $\mathbf{x} = (2, 3, 4)^T$ into a component in L and one orthogonal to L.
d. Find the distance of the point $A = (2, 3, 4)$ from L.

Exercise 5.2.12. In $\mathbb{R}^3$ let a plane S be given by the parametric equation $\mathbf{p} = (1, 2, 0)^T s + (1, 3, 5)^T t$.
a. Find the matrix of the projection onto S and the matrix of the projection onto the normal vector of S.
b. Check that the sum of the projection matrices found in Part a is I.
c. Use the projection matrices found in Part a to decompose the vector $\mathbf{x} = (2, 3, 4)^T$ into a component in S and one orthogonal to S.
d. Find the distance of the point $A = (2, 3, 4)$ from S.

Exercise 5.2.13. In $\mathbb{R}^3$ let a line L be given by the equations $2x_1 + 3x_2 - x_3 = 0$ and $-x_1 + 2x_2 + 5x_3 = 0$.
a. Find the matrix of the projection onto L and the matrix of the projection onto the orthogonal complement of L.
b. Check that the sum of the projection matrices found in Part a is I.
c. Use the projection matrices found in Part a to decompose the vector $\mathbf{x} = (2, 3, 4)^T$ into a component in L and one orthogonal to L.
d. Find the distance of the point $A = (2, 3, 4)$ from L.

6. *Determinants*

6.1 Determinants: Definition and Basic Properties

Determinants are certain complicated functions of square matrices (or, equivalently, of their column vectors or of their entries). Their usefulness follows mainly from two of their properties: first, they can be used to compute areas and volumes and second, a zero determinant characterizes singular matrices. Computing areas and volumes brings determinants into the formulas for changing variables in multiple integrals, and their vanishing for singular matrices is at the heart of Chapter 7 for evaluating what are called eigenvalues of matrices, which occur in many geometrical and physical applications.

Rather than giving an explicit formula right away, we define determinants by three very simple properties, from which we derive others. Only then do we turn to their evaluation. The defining properties will be obtained by examining how areas of parallelograms and volumes of parallelepipeds depend on their edge vectors.

First, the area of a parallelogram is the absolute value of a linear function of each of two edge vectors if the other edge vector is fixed. More precisely, if we denote the area of the parallelogram spanned by the vectors $\mathbf{a}$ and $\mathbf{b}$ in $\mathbb{R}^2$ by $A(\mathbf{a}, \mathbf{b})$, then we have $A(\mathbf{a}, \lambda\mathbf{b}) = |\lambda| A(\mathbf{a}, \mathbf{b})$ for every real λ, and $A(\mathbf{a}, \mathbf{b}) + A(\mathbf{a}, \mathbf{c}) = A(\mathbf{a}, \mathbf{b} + \mathbf{c})$ for all vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ such that $\mathbf{b}$ and $\mathbf{c}$ point to the same side of $\mathbf{a}$. These equations follow at a glance from Figures 6.1 and 6.2. (The latter is a genuinely two-dimensional figure, try not to see it as a wedge in three dimensions.) In Figure 6.2 all three parallelograms have

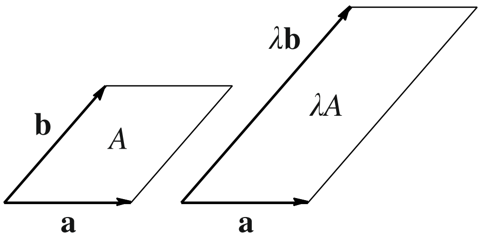

Fig. 6.1. Dependence of the area of a parallelogram on a scalar multiple of an edge vector

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the same base length $|\mathbf{a}|$, and the heights of the shaded ones add up to the height of the unshaded one.

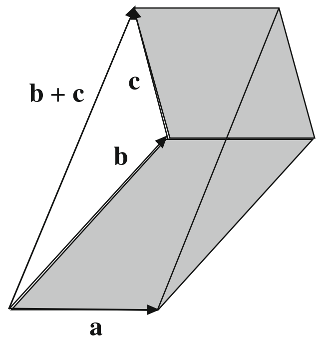

Fig. 6.2. Dependence of the area of a parallelogram on a sum of counterclockwise-oriented edge vectors $\mathbf{b}, \mathbf{c}$

If, however, $\mathbf{b}$ and $\mathbf{c}$ point to opposite sides of $\mathbf{a}$, then we have $A(\mathbf{a}, \mathbf{b} + \mathbf{c}) = A(\mathbf{a}, \mathbf{b}) - A(\mathbf{a}, \mathbf{c})$, with subtraction instead of addition, as can be seen from Figure 6.3. Nevertheless, we can turn the right side of this equation into a sum for a *signed* area function $D(\mathbf{a}, \mathbf{b}) = \pm A(\mathbf{a}, \mathbf{b})$, if we choose the plus sign when $\mathbf{a}$ followed by $\mathbf{b}$ indicates a counterclockwise traversal of the parallelogram spanned by $\mathbf{a}$ and $\mathbf{b}$, and choose the minus sign otherwise. Furthermore, the function D is homogeneous; that is, $D(\mathbf{a}, \lambda\mathbf{b}) = \lambda D(\mathbf{a}, \mathbf{b})$ for every real λ, without the absolute value that was present in the formula for $A(\mathbf{a}, \lambda\mathbf{b})$. We do not go into this subject any further here, but we shall return to the relationship between areas and determinants at the end of Section 6.2. Relations similar to those above for areas hold for volumes of parallelepipeds in $\mathbb{R}^3$ as well.

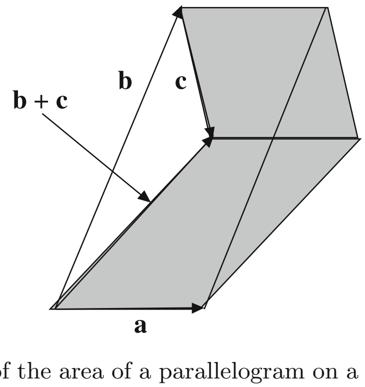

Fig. 6.3. Dependence of the area of a parallelogram on a sum of edge vectors $\mathbf{b}, \mathbf{c}$, with $\mathbf{b}$ following $\mathbf{a}$ counterclockwise and $\mathbf{c}$ following $\mathbf{a}$ clockwise

Definition 6.1.1. ***(Determinants).*** *For any positive integer* n,[1] *the determinant of order* n *is a function that assigns to every* $n \times n$ *matrix*

$$A = (\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n) \tag{6.1}$$

a number, denoted by

$$\det(A) = \det(\mathbf{a}_1\ \mathbf{a}_2\ \cdots\ \mathbf{a}_n)\ \text{or}\ |A| = |\mathbf{a}_1\ \mathbf{a}_2\ \cdots\ \mathbf{a}_n|, \tag{6.2}$$

such that

1. *It is a multilinear function of the columns; that is,*

$$\begin{aligned}&|\mathbf{a}_1\ \cdots\ \mathbf{a}_{i-1}\ s\mathbf{a}_i + t\mathbf{a}'_i\ \mathbf{a}_{i+1}\ \cdots\ \mathbf{a}_n|\\ &= s|\mathbf{a}_1\ \cdots\ \mathbf{a}_{i-1}\ \mathbf{a}_i\ \mathbf{a}_{i+1}\ \cdots\ \mathbf{a}_n| + t|\mathbf{a}_1\ \cdots\ \mathbf{a}_{i-1}\ \mathbf{a}'_i\ \mathbf{a}_{i+1}\ \cdots\ \mathbf{a}_n|\end{aligned} \tag{6.3}$$

   holds for all real s, t *and every positive integer* $i \le n$.
2. *It is zero if any two columns are equal, that is,*

$$|\cdots \mathbf{a}_i \cdots \mathbf{a}_j \cdots| = 0 \quad \text{if} \quad \mathbf{a}_i = \mathbf{a}_j \tag{6.4}$$

   holds for any i, j, *with* $i \neq j$, *and*
3. *The "volume" of the unit hypercube is 1; that is,*

$$|\mathbf{e}_1\ \mathbf{e}_2\ \cdots\ \mathbf{e}_n| = 1, \tag{6.5}$$

   or, equivalently, $\det(I) = 1$, *where* I *is the* $n \times n$ *identity matrix.*

We are going to show shortly that these properties do indeed define a unique function for every n, by giving formulas for it in Theorems 6.1.2, 6.1.3, and 6.1.4. But before computing any determinant, we need one more property, which follows from those above.

Theorem 6.1.1. ***(Exchanging Columns Changes the Sign of a Determinant).*** *If the matrix* A' *is obtained from* A *by interchanging any two columns, and their determinants exist, then* $|A'| = -|A|$.

Proof. Suppose we have

$$|A| = |\cdots \mathbf{a}_i \cdots \mathbf{a}_j \cdots| \tag{6.6}$$

and

$$|A'| = |\cdots \mathbf{a}_j \cdots \mathbf{a}_i \cdots|. \tag{6.7}$$

[1] For $n = 1$, see Exercise 6.1.5.

Consider the determinant in which the ith and jth columns are both equal to $\mathbf{a}_i + \mathbf{a}_j$. By Axiom 2 of Definition 6.1.1, its value is 0 and, by Axiom 1, it can be reduced as follows:

$$\begin{aligned} 0 &= |\cdots \mathbf{a}_i + \mathbf{a}_j \cdots \mathbf{a}_i + \mathbf{a}_j \cdots| \\ &= |\cdots \mathbf{a}_i \cdots \mathbf{a}_i + \mathbf{a}_j \cdots| \; + \; |\cdots \mathbf{a}_j \cdots \mathbf{a}_i + \mathbf{a}_j \cdots| \\ &= |\cdots \mathbf{a}_i \cdots \mathbf{a}_i \cdots| \; + \; |\cdots \mathbf{a}_i \cdots \mathbf{a}_j \cdots| \\ &\quad + \; |\cdots \mathbf{a}_j \cdots \mathbf{a}_i \cdots| \; + \; |\cdots \mathbf{a}_j \cdots \mathbf{a}_j \cdots|. \end{aligned} \tag{6.8}$$

On the right side the first and last terms are zero by Axiom 2, and so the sum of the middle terms is also zero, that is,

$$|A| + |A'| = 0. \tag{6.9}$$

This equation is equivalent to the statement of the theorem. ■

We now have enough information to evaluate determinants in the case $n = 2$.

Theorem 6.1.2. ***(**2×2 **Determinant).*** *The determinant of order two is given by the formula*

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}. \tag{6.10}$$

Proof. If $|A|$ exists, then by the linearity assumption we have

$$\begin{aligned} |A| &= |\mathbf{a}_1 \ \mathbf{a}_2| = |a_{11}\mathbf{e}_1 + a_{21}\mathbf{e}_2, \ a_{12}\mathbf{e}_1 + a_{22}\mathbf{e}_2| \\ &= a_{11}|\mathbf{e}_1, \ a_{12}\mathbf{e}_1 + a_{22}\mathbf{e}_2| + a_{21}|\mathbf{e}_2, \ a_{12}\mathbf{e}_1 + a_{22}\mathbf{e}_2| \\ &= a_{11}(a_{12}|\mathbf{e}_1 \ \mathbf{e}_1| + a_{22}|\mathbf{e}_1 \ \mathbf{e}_2|) + a_{21}(a_{12}|\mathbf{e}_2 \ \mathbf{e}_1| + a_{22}|\mathbf{e}_2 \ \mathbf{e}_2|), \end{aligned} \tag{6.11}$$

where, according to Axioms 2 and 3 and Theorem 6.1.1, the determinants of the standard vectors equal $0, 1, -1$, and 0 respectively. Thus Equation 6.11 reduces to the formula $|A| = a_{11}a_{22} - a_{21}a_{12}$.

It is straightforward to check that this expression indeed satisfies the axioms and thereby prove the existence of the determinant of order two:

By Equation 6.10 and simple algebra,

1.

$$\begin{aligned} |\mathbf{a}_1 \ s\mathbf{a}_2 + t\mathbf{a}_2'| &= \begin{vmatrix} a_{11} & sa_{12} + ta_{12}' \\ a_{21} & sa_{22} + ta_{22}' \end{vmatrix} \\ &= a_{11}\left(sa_{22} + ta_{22}'\right) - a_{21}\left(sa_{12} + ta_{12}'\right) \\ &= s\left(a_{11}a_{22} - a_{21}a_{12}\right) + t\left(a_{11}a_{22}' - a_{21}a_{12}'\right) \\ &= s|\mathbf{a}_1 \ \mathbf{a}_2| + t|\mathbf{a}_1 \ \mathbf{a}_2'|. \end{aligned} \tag{6.12}$$

A linear combination in the first column can be handled similarly.

2.

$$|\mathbf{a}_1\ \mathbf{a}_1| = \begin{vmatrix} a_{11} & a_{11} \\ a_{21} & a_{21} \end{vmatrix} = a_{11}a_{21} - a_{21}a_{11} = 0. \tag{6.13}$$

3.

$$|\mathbf{e}_1\ \mathbf{e}_2| = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \cdot 1 - 0 \cdot 0 = 1. \tag{6.14}$$

■

Example 6.1.1. (Evaluating a 2 × 2 Determinant)

$$\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = 2 \cdot 5 - 4 \cdot 3 = -2. \tag{6.15}$$

♦

Let us now find a formula for determinants of order three.

Theorem 6.1.3. (3×3 ***Determinant).*** *For any* 3×3 *matrix* A, *we have*

$$\begin{aligned} |A| = &\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \\ &= a_{11}a_{22}a_{33} + a_{21}a_{32}a_{13} + a_{31}a_{12}a_{23} \\ &\quad - a_{11}a_{32}a_{23} - a_{21}a_{12}a_{33} - a_{31}a_{22}a_{13}. \end{aligned} \tag{6.16}$$

Proof. The proof of this theorem is much like that of the 2×2 case. If $|A|$ exists, then by the linearity property of determinants we have

$$|A| = |\mathbf{a}_1\ \mathbf{a}_2\ \mathbf{a}_3| = \left| \sum_{i=1}^{3} a_{i1}\mathbf{e}_i,\ \sum_{j=1}^{3} a_{j2}\mathbf{e}_j,\ \sum_{k=1}^{3} a_{k3}\mathbf{e}_k \right| = \sum_{i,j,k=1}^{3} a_{i1}a_{j2}a_{k3}|\mathbf{e}_i\ \mathbf{e}_j\ \mathbf{e}_k|. \tag{6.17}$$

Now each determinant on the right is zero whenever two of the standard vectors are equal, is $+1$ when $|\mathbf{e}_i\ \mathbf{e}_j\ \mathbf{e}_k|$ is $|\mathbf{e}_1\ \mathbf{e}_2\ \mathbf{e}_3|$ or can be obtained from the latter by two column exchanges, and is -1 when $|\mathbf{e}_i\ \mathbf{e}_j\ \mathbf{e}_k|$ can be obtained from $|\mathbf{e}_1\ \mathbf{e}_2\ \mathbf{e}_3|$ by just one exchange, since every column exchange changes only the sign of a determinant. (For example, in the second term of Equation 6.16 $i = 2,\ j = 3,\ k = 1$ and, if we exchange $\mathbf{e}_1$ and $\mathbf{e}_2$ in $|\mathbf{e}_1\ \mathbf{e}_2\ \mathbf{e}_3| = 1$, we get $|\mathbf{e}_2\ \mathbf{e}_1\ \mathbf{e}_3| = -1$ and, if next we exchange $\mathbf{e}_1$ and $\mathbf{e}_3$, we obtain $|\mathbf{e}_2\ \mathbf{e}_3\ \mathbf{e}_1| = 1$ for the desired values of $i,\ j,\ k$.) ■

We could again verify the existence of $|A|$ by checking that the formula of the theorem satisfies the axioms, but we relegate this task to the general $n \times n$ case below.

Example 6.1.2. (Evaluating a* $\mathbf{3 \times 3}$ *Determinant). Let us evaluate such a determinant:

$$\begin{vmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{vmatrix} = 1 \cdot 5 \cdot 9 + 2 \cdot 6 \cdot 7 + 3 \cdot 4 \cdot 8 - 1 \cdot 6 \cdot 8$$

$$-2 \cdot 4 \cdot 9 - 3 \cdot 5 \cdot 7 = 45 + 84 + 96 - 48 - 72 - 105 = 0. \tag{6.18}$$

That the value turned out to be zero is a result of the three column vectors being linearly dependent, as will be seen shortly. Soon we shall also discuss better methods for evaluating determinants. ♦

In the general case we proceed much as in those above. We use linearity to write any nth order determinant as a linear combination of determinants of standard vectors analogously to Equation 6.17, and reduce the nonzero ones among the latter to ± 1 by exchanges of adjacent $\mathbf{e}_i$ vectors.

For any determinant of the n standard vectors of $\mathbb{R}^n$, every exchange of *adjacent* $\mathbf{e}_i$ vectors, called a *transposition*, changes the sign. Consequently, if in any such determinant the number of transpositions we use to bring the standard vectors into their natural order is odd, then the determinant is -1, and if the number of transpositions used is even, then it is $+1$. The natural order can be obtained from a given arrangement through various different sequences of transpositions; nevertheless, as will be seen below, it does not matter which of these sequences we choose.

Let us consider the set of all possible arrangements of the integers $1, 2, \ldots, n$ in a row. Any such arrangement is called a *permutation* of the natural arrangement, or *natural permutation*, $(1, 2, \ldots, n)$.

Lemma 6.1.1. ***(Number of Permutations).*** *The number of permutations of n elements is $n!$.*

Proof. For $n = 1$ the only permutation is (1) and so its number is $1! = 1$. For $n = 2$ we have the two permutations $(1, 2)$ and $(2, 1)$. These can be thought of as arising from the previous (1) by placing the digit 2 on either side of the 1 and so their number is $2 \cdot 1! = 2 \cdot 1 = 2!$. For $n = 3$ we can obtain all permutations by placing the digit 3 in all the possible places in the previous permutations of two digits, that is, to the left of the first digit, between the two digits, or after them. Thus their number is $3 \cdot 2! = 3 \cdot 2 \cdot 1 = 3!$. This process can be continued to arbitrary n, using mathematical induction. ■

We define an *inversion* in a permutation as an ordered pair of digits such that the larger number precedes the smaller one in the permutation. Thus, for example, the permutation $(3, 2, 4, 5, 1)$ has the five inversions $(3, 2)$, $(3, 1)$, $(2, 1)$, $(4, 1)$, $(5, 1)$.

A permutation is called *even* if it has an even number of inversions and *odd* otherwise. In particular, the natural permutation is even because it has zero inversions.

Any transposition changes the number of inversions by 1, since it affects only the relative order of the two transposed digits, and so it changes every even permutation into an odd one and vice versa. Hence the number of even permutations must equal the number of odd permutations, since if we consider all even permutations and transpose their first two digits, then we get odd permutations, which shows that the number of even permutations is less than or equal to the number of odd permutations. Similarly, transposing the first two digits of all odd permutations, we find that the number of odd permutations is less than or equal to the number of even permutations. Thus the numbers of even and of odd permutations both equal $n!/2$.

Obviously, every permutation can be changed into the natural permutation by a sequence of transpositions, and each transposition changes the number of inversions by one. Therefore, every odd permutation requires an odd number of transpositions for it to be changed into the natural permutation (which is even), and every even permutation requires an even number of transpositions for it to be changed into the natural permutation, regardless of the particular transpositions used.

Let P denote any permutation of the integers $1, 2, \ldots, n$; that is, let $P = (p_1, p_2, \ldots, p_n)$, where the p_i are the numbers $1, 2, \ldots, n$ permuted. Define a function ϵ on the set of all such permutations by

$$\epsilon(P) = \begin{cases} 1 \text{ if } P \text{ is even} \\ -1 \text{ if } P \text{ is odd} \end{cases}. \tag{6.19}$$

With this notation we are led to the following theorem.

Theorem 6.1.4. *($n \times n$ **Determinant**). The determinant of any $n \times n$ matrix A is given by*

$$\det(A) = \sum_P \epsilon(P) a_{p_1 1} a_{p_2 2} \cdots a_{p_n n}, \tag{6.20}$$

where the sum runs through all permutations of $1, 2, \ldots, n$.

Proof. The discussion above has already proved that if $\det(A)$ exists, it must have this form. To prove its existence, we just need to show, similarly to the 2×2 case, that the sum in the theorem satisfies the three defining axioms.

1. By Equation 6.20 and simple algebra,

$$\begin{aligned} &|\mathbf{a}_1 \ \cdots \ \mathbf{a}_{i-1} \ s\mathbf{a}_i + t\mathbf{a}'_i \ \mathbf{a}_{i+1} \ \cdots \ \mathbf{a}_n| \\ &\quad = \sum_P \epsilon(P) a_{p_1 1} a_{p_2 2} \cdots \left(s a_{p_i i} + t a'_{p_i i}\right) \cdots a_{p_n n} \\ &\quad = s \sum_P \epsilon(P) a_{p_1 1} a_{p_2 2} \cdots a_{p_i i} \cdots a_{p_n n} + t \sum_P \epsilon(P) a_{p_1 1} a_{p_2 2} \cdots a'_{p_i i} \cdots a_{p_n n} \\ &\quad = s|\mathbf{a}_1 \ \cdots \ \mathbf{a}_{i-1} \ \mathbf{a}_i \ \mathbf{a}_{i+1} \cdots \ \mathbf{a}_n| + t|\mathbf{a}_1 \ \cdots \ \mathbf{a}_{i-1} \ \mathbf{a}'_i \ \mathbf{a}_{i+1} \ \cdots \ \mathbf{a}_n|. \end{aligned} \tag{6.21}$$

2. Let

$$\det(A) = |\cdots \mathbf{a}_i \cdots \mathbf{a}_j \cdots| = \sum_P \epsilon(P) a_{p_1 1} a_{p_2 2} \cdots a_{p_n n}, \tag{6.22}$$

$$\det(A') = |\cdots \mathbf{a}_j \cdots \mathbf{a}_i \cdots| = \sum_{P'} \epsilon(P') a_{p'_1 1} a_{p'_2 2} \cdots a_{p'_n n}, \tag{6.23}$$

and $\mathbf{a}_i = \mathbf{a}_j$. Then, of course,

$$\det(A) = \det(A'). \tag{6.24}$$

On the other hand, ignoring $\mathbf{a}_i = \mathbf{a}_j$, let us change A into A' by transpositions of columns. If there are k numbers between i and j, then $k+1$ transpositions would move $\mathbf{a}_i$ to the right of $\mathbf{a}_j$ and after that k transpositions would move $\mathbf{a}_j$ to the old place of $\mathbf{a}_i$, resulting in a total of $2k+1$ transpositions. Thus, if P is even, then P' is odd and vice versa, implying that $\epsilon(P') = -\epsilon(P)$ and, since the other factors remain unchanged,

$$\det(A) = -\det(A'). \tag{6.25}$$

Equations 6.24 and 6.25 together imply:

$$\det(A) = 0. \tag{6.26}$$

3. If $A = I$, then the sum in Equation 6.20 reduces, all other terms being 0, to the single term

$$\epsilon(P_0) a_{11} a_{22} \cdots a_{nn} = 1, \tag{6.27}$$

where P_0 is the natural permutation, which is even, and each $a_{ii} = 1$. ∎

This formula is hopelessly inefficient for computing determinants for large n, since the number of terms is $n!$, which grows very fast. Already for $n = 5$ or 6 we have $5! = 120$ and $6! = 720$. We shall, however, use this formula to prove some other properties that will lead to better evaluation methods. Also, it is theoretically very important to know that the defining axioms are satisfied by a unique expression.

The next three theorems will show how determinants can be evaluated by elementary operations on the rows or columns.

Theorem 6.1.5. ***(Combining Columns in a Determinant).*** *1. If the matrix A' is obtained from A by adding any scalar c times one column to another, then $|A'| = |A|$.*
2. If a matrix A has a zero column, then $|A| = 0$.

Proof. 1. Let

$$|A| = |\cdots \mathbf{a}_i \cdots \mathbf{a}_j \cdots| \tag{6.28}$$

and assume that A' is obtained by adding c times the jth column of A to the ith one. Then by Axioms 1 and 2 we have

$$\begin{aligned}|A'| &= |\cdots \mathbf{a}_i + c\mathbf{a}_j \cdots \mathbf{a}_j \cdots| \\ &= |\cdots \mathbf{a}_i \cdots \mathbf{a}_j \cdots| \ + \ c|\cdots \mathbf{a}_j \cdots \mathbf{a}_j \cdots| = |A| + 0 = |A|. \qquad (6.29)\end{aligned}$$

2. Add any other column to the zero column. Then, by Part 1, the determinant does not change and, because the resulting determinant has two equal columns, it equals zero. ■

Theorem 6.1.6. ***(Determinant of Transposed Matrix).*** *For every square matrix A, we have*

$$\det(A^T) = \det(A). \qquad (6.30)$$

Proof. From Theorem 6.1.4 we have the formula

$$\det(A) = \sum_P \epsilon(P) a_{p_1 1} a_{p_2 2} \cdots a_{p_n n}, \qquad (6.31)$$

where the sum runs through all permutations of $1, 2, \ldots, n$. Now every term in this sum contains one matrix element from each row and one from each column and they are arranged in the order of the columns. Their product in each term remains unchanged if we rearrange them in the order of the rows, that is, if we rearrange the product $a_{p_1 1} a_{p_2 2} \cdots a_{p_n n}$ to $a_{1q_1} a_{2q_2} \cdots a_{nq_n}$. Now $Q = (q_1, q_2, \ldots, q_n)$ is the inverse of the permutation P; that is, it is the permutation that brings $P = (p_1, p_2, \ldots, p_n)$ back to the natural order. Thus Q can be obtained from the natural permutation with exactly as many transpositions as P can, and so it is even when P is even and odd when P is odd. Therefore,

$$\det(A) = \sum_P \epsilon(P) a_{p_1 1} a_{p_2 2} \cdots a_{p_n n} = \sum_Q \epsilon(Q) a_{1q_1} a_{2q_2} \cdots a_{nq_n} = \det(A^T). \qquad (6.32)$$

■

Corollary 6.1.1. ***(Rows in a Determinant).*** *Every property of columns of determinants is also valid for the rows.*

This result will enable us to compute determinants by the more familiar row reductions rather than by column operations. The next theorem tells us what the last step of the reduction is in most cases.

Theorem 6.1.7. ***(Determinant of an Upper Triangular Matrix).*** *If A is an upper triangular matrix, then $|A|$ equals the product of its diagonal elements.*

Proof. The product of the diagonal elements is, of course, one of the terms in the expansion of Theorem 6.1.4. Thus we want to show that in this case all the other terms are necessarily zero.

Consider any one of the terms in the sum on the right of Equation 6.20, say $T = \epsilon(P)a_{p_1 1}a_{p_2 2}\cdots a_{p_n n}$. Call the 0 entries in A below the main diagonal the bad entries. How can we avoid having a bad entry as a factor in T? Since the product in T contains exactly one matrix element from each row and one from each column, from the first column it must contain a_{11}, since all the other entries in the first column are bad. Next, T cannot contain a_{12}, since it already has a factor from the first row. From the second column it must contain a_{22}, since it cannot contain a_{12} nor any of the bad entries below a_{22}. Similarly, from the third column it must contain a_{33}, since it cannot contain a_{13} or a_{23} because it already contains elements of the first and second rows and the entries below a_{33} are all bad. This argument could be continued for the remaining factors, and it shows that the product of the diagonal elements is the only possibly nonzero term in the expansion of Theorem 6.1.4.

If one or more of the diagonal elements are zero, then in the expansion even the term consisting of the product of the diagonal elements is zero, and so the product of the diagonal elements and the determinant are both zero. ■

We can use the preceding properties to compute determinants by row reductions as in the following examples.

Example 6.1.3. (Evaluating a Determinant by Row Reduction). Let us again evaluate the determinant

$$D = \begin{vmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{vmatrix}. \tag{6.33}$$

If we subtract the second row from the third row, and the first row from the second row, then, by Part 1 of Theorem 6.1.5 applied to rows, the value of D remains unchanged and we get

$$D = \begin{vmatrix} 1 & 4 & 7 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix}. \tag{6.34}$$

Hence, by Axiom 2 applied to rows, we obtain $D = 0$. ◆

Example 6.1.4. (Evaluating Another Determinant by Row Reduction). Let us evaluate the determinant

$$D = \begin{vmatrix} 1 & 1 & 2 \\ 2 & 5 & 4 \\ 3 & 6 & 9 \end{vmatrix}. \tag{6.35}$$

By Axiom 1 applied to rows we can factor out a 3 from the third row to get

$$D = 3 \cdot \begin{vmatrix} 1 & 1 & 2 \\ 2 & 5 & 4 \\ 1 & 2 & 3 \end{vmatrix}. \tag{6.36}$$

By Part 1 of Theorem 6.1.5 applied to rows we can subtract twice the first row from the second row and the first row from the third row to get

$$D = 3 \cdot \begin{vmatrix} 1 & 1 & 2 \\ 0 & 3 & 0 \\ 0 & 1 & 1 \end{vmatrix}. \tag{6.37}$$

Exchanging the last two rows we obtain, by Theorem 6.1.1 applied to rows, the equation

$$D = -3 \cdot \begin{vmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 3 & 0 \end{vmatrix}. \tag{6.38}$$

Finally, if we subtract 3 times the second row from the last row and apply Theorem 6.1.7, then we find

$$D = -3 \cdot \begin{vmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & -3 \end{vmatrix} = -3 \cdot 1 \cdot 1 \cdot (-3) = 9. \tag{6.39}$$

♦

From the properties of determinants above, a very important relation between a matrix and its determinant can be deduced.

Theorem 6.1.8. ***(A Is Singular If and Only If*** $|A| = 0$***).*** *A square matrix A is singular if and only if its determinant equals zero.*

Proof. First, if the square matrix A is singular, then it can be reduced by elementary row operations to an upper triangular matrix U with a zero last row. If we perform the same row operations on the determinant $|A|$, then in each step the determinant either remains unchanged or we can factor out some number. We have $|U| = 0$ and so $|A| = c|U| = 0$.

If, on the other hand, the square matrix A is nonsingular, then it can be reduced by elementary row operations to the identity matrix I. We have $|I| = 1$ and, if we perform the same row operations on the determinant $|A|$, then in each step the determinant either remains unchanged or we can take out a nonzero factor. Hence $|A| = c|I| = c$ for some nonzero scalar c. ■

There exists another useful relation between matrices and their determinants.

Theorem 6.1.9. ***(The Determinant of a Product Equals the Product of the Determinants).*** *If A and B are square matrices of the same size, then $|AB| = |A||B|$.*

Proof. If we write $C = AB$ and $\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n$ for the columns of C, then we can express C in terms of the columns of B as

$$C = \begin{pmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 & \cdots & A\mathbf{b}_n \end{pmatrix} \tag{6.40}$$

and each $\mathbf{c}_k = A\mathbf{b}_k$ here can also be written as a linear combination of the columns of A, that is, as

$$\mathbf{c}_k = \sum_i b_{ik}\mathbf{a}_i. \tag{6.41}$$

Then we can write the determinant of the $\mathbf{c}_i$ vectors as

$$\begin{aligned} |\mathbf{c}_1 \; \mathbf{c}_2 \; \cdots \; \mathbf{c}_n| &= \left| \sum_i b_{i1}\mathbf{a}_i \quad \sum_j b_{j2}\mathbf{a}_j \quad \cdots \quad \sum_z b_{zn}\mathbf{a}_z \right| \\ &= \sum_{i,j,\ldots,z} b_{i1}b_{j2}\cdots b_{zn}|\mathbf{a}_i \; \mathbf{a}_j \; \cdots \; \mathbf{a}_z|. \end{aligned} \tag{6.42}$$

The determinants on the right can be evaluated in much the same way as those of the standard vectors in Theorem 6.1.3; that is, whenever two of the subscripts are the same, the determinant of the $\mathbf{a}_i$ vectors vanishes, and otherwise it reduces to $\pm|A|$, with the sign given by $\epsilon(P)$, where P stands for the permutation $(i, j, \ldots, z)$ of $(1, 2, \ldots, n)$. This reduction results in

$$|C| = \sum_P \epsilon(P) b_{i1}b_{j2}\cdots b_{zn}|\mathbf{a}_1 \; \mathbf{a}_2 \; \cdots \; \mathbf{a}_n| = |A||B|. \tag{6.43}$$

■

An immediate consequence of this theorem is the following result, whose proof is left as Exercise 6.1.9.

Corollary 6.1.2. ***(The Determinant of A^{-1} Equals the Reciprocal of the Determinant of A).*** *If A is an invertible matrix, then*

$$|A^{-1}| = \frac{1}{|A|}.$$

Exercises

In the first four exercises evaluate the determinants by row reduction.

Exercise 6.1.1. $D = \begin{vmatrix} 2 & -3 & 2 \\ 1 & 4 & 0 \\ 0 & 1 & -5 \end{vmatrix}.$

Exercise 6.1.2. $D = \begin{vmatrix} 0 & 1 & 2 \\ 4 & 0 & 3 \\ 3 & 2 & 1 \end{vmatrix}$.

Exercise 6.1.3. $D = \begin{vmatrix} -1 & 1 & 2 & 3 \\ 2 & 0 & -5 & 0 \\ 0 & 0 & 0 & -1 \\ 3 & -3 & 1 & 4 \end{vmatrix}$.

Exercise 6.1.4. $D = \begin{vmatrix} -1 & -2 & 2 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 1 & -1 & 1 & -1 \end{vmatrix}$.

Exercise 6.1.5. Prove that in the trivial case of $n = 1$, Axiom 2 of Definition 6.1.1 does not apply, and the other properties give $\det(A) = \det(a_{11}) = a_{11}$.

Exercise 6.1.6. Use Definition 6.1.1 directly to prove that for every $n \times n$ matrix A and every scalar c we have $\det(cA) = c^n \det(A)$.

Exercise 6.1.7. Show that the result of Theorem 6.1.7 holds for lower triangular matrices as well, that is, that

$$\begin{vmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33}$$

and similar relations hold for every n.

Exercise 6.1.8. Is the analog of Theorem 6.1.7 true for matrices lower triangular with respect to the secondary diagonal, that is, does

$$\begin{vmatrix} 0 & 0 & a_{13} \\ 0 & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = -a_{13}a_{22}a_{31}$$

hold, and similar relations for $n \neq 3$? Prove your result!

Exercise 6.1.9. Prove Corollary 6.1.2.

Exercise 6.1.10. Show that if A and B are similar matrices, then $\det(A) = \det(B)$.

Exercise 6.1.11. Use Theorems 6.1.8 and 6.1.9 to show that for all $n \times n$ matrices A and B the product AB is invertible if and only if both A and B are.

Exercise 6.1.12. A matrix A is called *skew-symmetric* if $A^T = -A$. Show that for every 3×3 skew symmetric matrix $\det(A) = 0$. Is this true for other values of n?

Exercise 6.1.13. Prove that for every orthogonal matrix Q we have $\det(Q) = \pm 1$.

Exercise 6.1.14. Show that for the Vandermonde determinant of order three,

$$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (b-a)(c-b)(c-a). \tag{6.44}$$

***Exercise 6.1.15.** Generalize the result of the previous exercise to $n > 3$ and prove your formula.

Exercise 6.1.16. Show, using Theorem 6.1.8 and Equation 6.44, that the monomials $1, x, x^2$ are linearly independent in the vector space $\mathcal{P}$ of all polynomials over $\mathbb{R}$. (*Hint*: Substitute three numbers a, b, c for x into the definition of linear independence applied to these functions.)

MATLAB Exercises

In MATLAB the determinant of a matrix A is given by $\mathbf{det}(A)$. Note that it is preferable to use $\mathbf{rank}(A)$ rather than $\mathbf{det}(A)$ to determine whether A is singular or not, because MATLAB's computation of the latter is more affected by roundoff errors.

Exercise 6.1.17. Let $A = \mathbf{round}(10 * \mathbf{rand}(5))$ and $\mathbf{x} = \mathbf{round}(10 * \mathbf{rand}(5,1))$.
a. Create a matrix B by entering $B = A$; $B(:,4) = \mathbf{x}$. What does this do?
b. Create matrices as above, in which the third column of A is replaced by multiples of random vectors and use these to illustrate the linear dependence of the det function on the third column.
c. Is $\det(A) + \det(B)$ equal to $\det(A + B)$ in general, when A and B are the same size? Experiment with random matrices. Explain.
d. Let A be as above and compute $B = A$; $B(:,4) = B(:,4) + 3 * B(:,1)$. Compare $\det(A)$ and $\det(B)$. Explain.

Exercise 6.1.18. Enter the following program and explain what it does:
$\mathbf{x} = 1:6$
$\mathbf{y} = \mathbf{randperm}(6)$
$A = \mathbf{vander}(\mathbf{x})'$
$B = \mathbf{vander}(\mathbf{y})'$
$P = B \backslash A$
$\mathbf{det}(P)$

Exercise 6.1.19. The following program achieves the same result as the one above but in a much more efficient way. Enter it and explain what it does:

$n = 6$,
$\mathbf{x} = \mathbf{randperm}(n)$,
$I = \mathbf{eye}(n)$,
$P = I$,
for $i = 1 : n$
$P(i, :) = I(\mathbf{x}(i), :)$
end
$\mathbf{det}(P)$

6.2 Further Properties of Determinants

Frequently determinants are evaluated by reduction formulas, called *expansions along a row or a column*, in which an nth order determinant for every $n \geq 2$ is[2] expressed in terms of certain determinants of order $n-1$. The latter have special names.

Definition 6.2.1. ***(Minors and Cofactors).*** *Given any $n \times n$ matrix A, we define its ijth minor M_{ij} as the determinant of the submatrix S_{ij} obtained from A by deleting the ith row and jth column. The quantity $A_{ij} = (-1)^{i+j} M_{ij}$ is called the cofactor of a_{ij}.*

Before stating the general reduction theorem, let us examine the 3×3 case. Then

$$\begin{aligned} |A| &= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \\ &= a_{11}a_{22}a_{33} + a_{21}a_{32}a_{13} + a_{31}a_{12}a_{23} \\ &\quad - a_{11}a_{32}a_{23} - a_{21}a_{12}a_{33} - a_{31}a_{22}a_{13}. \end{aligned} \tag{6.45}$$

If we factor out the elements of the first column and apply the definition of the 2×2 determinant, then we get

$$|A| = a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{21}(a_{12}a_{33} - a_{32}a_{13}) + a_{31}(a_{12}a_{23} - a_{22}a_{13}) \tag{6.46}$$

and

$$|A\,| = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{21}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31}\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}. \tag{6.47}$$

The determinants in this formula are the minors $M_{11},\ M_{21},\ M_{31}$ respectively, and the same determinants with the signs included (in this case it makes a

[2] We shall assume throughout this section without further mention that $n > 1$ holds.

difference only in the second term) are the corresponding cofactors. Equation 6.47 is the expansion of $|A|$ along its first column. By factoring out the elements of any other row or column we would obtain analogous expansions with respect to those.

Theorem 6.2.1. ***(Evaluating Determinants by Cofactor Expansion).*** *The determinant of every $n \times n$ matrix A can, for any fixed i, be evaluated as*

$$|A| = a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in} \tag{6.48}$$

and also, for any fixed j, as

$$|A| = a_{1j}A_{1j} + a_{2j}A_{2j} + \cdots + a_{nj}A_{nj}. \tag{6.49}$$

Proof. Let us consider the expansion along the jth column; that is, let us start with the proof of the second formula. Write $\sum_i a_{ij}\mathbf{e}_i$ for $\mathbf{a}_j$ in $|A|$,

$$|A| = \left|\mathbf{a}_1 \cdots \mathbf{a}_{j-1}\ \sum_i a_{ij}\mathbf{e}_i\ \mathbf{a}_{j+1} \cdots \mathbf{a}_n\right| \tag{6.50}$$

and by linearity rewrite this equation as

$$|A| = \sum_i a_{ij}|\mathbf{a}_1 \cdots \mathbf{a}_{j-1}\ \mathbf{e}_i\ \mathbf{a}_{j+1} \cdots \mathbf{a}_n|. \tag{6.51}$$

We are going to show that the determinants in the sum on the right are exactly the cofactors A_{ij}. We write the determinant of the ith term with the components of the column vectors as

$$|\mathbf{a}_1 \cdots \mathbf{a}_{j-1}\ \mathbf{e}_i\ \mathbf{a}_{j+1} \cdots \mathbf{a}_n| = \begin{vmatrix} a_{11} & a_{12} & \cdots & 0 & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & 0 & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & 1 & \cdots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{31} & a_{32} & \cdots & 0 & \cdots & a_{nn} \end{vmatrix}. \tag{6.52}$$

By subtracting appropriate multiples of the jth column from the others, we can change this determinant to

$$|\mathbf{a}_1 \cdots \mathbf{a}_{j-1}\ \mathbf{e}_i\ \mathbf{a}_{j+1} \cdots \mathbf{a}_n| = \begin{vmatrix} a_{11} & a_{12} & \cdots & 0 & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & 0 & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{31} & a_{32} & \cdots & 0 & \cdots & a_{nn} \end{vmatrix} \tag{6.53}$$

where the 0's and the 1 replace the original ith row and jth column. We can move the ith row to the top with $i-1$ transpositions and the jth column to the left with $j-1$ transpositions. Since each transposition multiplies the determinant by -1, these moves result in

$$|\mathbf{a}_1 \cdots \mathbf{a}_{j-1}\ \mathbf{e}_i\ \mathbf{a}_{j+1} \cdots \mathbf{a}_n| = (-1)^{i+j} \begin{vmatrix} 1 & \mathbf{0}^T \\ \mathbf{0} & S_{ij} \end{vmatrix}, \tag{6.54}$$

where S_{ij} is the submatrix obtained from A by deleting the ith row and jth column. The determinant on the right equals the minor $M_{ij} = \det(S_{ij})$ because reducing it to upper triangular form we can use the same elementary row operations as we would on the corresponding rows of $|S_{ij}|$, and we would get the same multipliers factored out and the same products of the diagonal elements. Thus

$$|\mathbf{a}_1 \cdots \mathbf{a}_{j-1}\ \mathbf{e}_i\ \mathbf{a}_{j+1} \cdots \mathbf{a}_n| = (-1)^{i+j} |S_{ij}| = A_{ij}. \tag{6.55}$$

Substituting this expression into Equation 6.51 we obtain the Equation 6.49 of the theorem. Equation 6.48, that is, the expansion along any row, follows from Equation 6.49 and Theorem 6.1.6. ■

Before giving an example, let us remark that the signs $(-1)^{i+j}$ in the definition of the cofactors alternate in a checkerboard-like pattern, as shown in Table 6.1, and are usually taken from there rather than from the formula $(-1)^{i+j}$.

+	−	+	−	···
−	+	−	+	···
+	−	+	−	···
−	+	−	+	−
···		···		···

Table 6.1. The signs of the minors

Example 6.2.1. (Evaluating a Determinant by Cofactor Expansion). Evaluate the determinant

$$D = \begin{vmatrix} 1 & 1 & 0 & 2 \\ 2 & 3 & 5 & 4 \\ 1 & 0 & 0 & 1 \\ 0 & 6 & 2 & 3 \end{vmatrix} \tag{6.56}$$

using Theorem 6.2.1.

Since the third row and the third column have the most zeros, it is simplest to expand along one of those. Let us choose the third row. Then we get

$$D = 1 \cdot \begin{vmatrix} 1 & 0 & 2 \\ 3 & 5 & 4 \\ 6 & 2 & 3 \end{vmatrix} - 1 \cdot \begin{vmatrix} 1 & 1 & 0 \\ 2 & 3 & 5 \\ 0 & 6 & 2 \end{vmatrix}. \tag{6.57}$$

The 3×3 determinants here can be expanded similarly, along their first rows, say. Thus we obtain

$$D = 1 \cdot \begin{vmatrix} 5 & 4 \\ 2 & 3 \end{vmatrix} + 2 \cdot \begin{vmatrix} 3 & 5 \\ 6 & 2 \end{vmatrix} - 1 \cdot \begin{vmatrix} 3 & 5 \\ 6 & 2 \end{vmatrix} + 1 \cdot \begin{vmatrix} 2 & 5 \\ 0 & 2 \end{vmatrix}. \tag{6.58}$$

Finally, this expression can be evaluated as

$$D = (5 \cdot 3 - 2 \cdot 4) + 2 \cdot (3 \cdot 2 - 6 \cdot 5) - (3 \cdot 2 - 6 \cdot 5) + 2 \cdot 2 = -13. \tag{6.59}$$

♦

We are now in a position to be able to state the solution of every $n \times n$ system of linear equations in closed form using determinants.

Theorem 6.2.2. ***(Cramer's Rule).***[3] *Let A be a nonsingular $n \times n$ matrix. The solution of $A\mathbf{x} = \mathbf{b}$ for any $\mathbf{b} \in \mathbb{R}^n$ is given by*

$$x_i = \frac{|\mathbf{a}_1 \cdots \mathbf{a}_{i-1}\ \mathbf{b}\ \mathbf{a}_{i+1} \cdots \mathbf{a}_n|}{|A|} \text{ for } i = 1, 2, \ldots,\ n. \tag{6.60}$$

Proof. Write the system to be solved in the form

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b}. \tag{6.61}$$

Using each side of this equation to replace the ith column of the determinant of the $\mathbf{a}_j$ vectors, we get

$$\begin{aligned} &|\mathbf{a}_1 \cdots \mathbf{a}_{i-1}\ x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n\ \mathbf{a}_{i+1} \cdots \mathbf{a}_n| \\ &\quad = |\mathbf{a}_1 \cdots \mathbf{a}_{i-1}\ \mathbf{b}\ \mathbf{a}_{i+1} \cdots \mathbf{a}_n|. \end{aligned} \tag{6.62}$$

On the left, subtract from the ith column x_1 times the first column, x_2 times the second column, and so on. This operation results in

$$|\mathbf{a}_1 \cdots \mathbf{a}_{i-1}\ x_i\mathbf{a}_i\ \mathbf{a}_{i+1} \cdots \mathbf{a}_n| = |\mathbf{a}_1 \cdots \mathbf{a}_{i-1}\ \mathbf{b}\ \mathbf{a}_{i+1} \cdots \mathbf{a}_n|, \tag{6.63}$$

which can be changed to

$$x_i|\mathbf{a}_1 \cdots \mathbf{a}_{i-1}\ \mathbf{a}_i\ \mathbf{a}_{i+1} \cdots \mathbf{a}_n| = |\mathbf{a}_1 \cdots \mathbf{a}_{i-1}\ \mathbf{b}\ \mathbf{a}_{i+1} \cdots \mathbf{a}_n|. \tag{6.64}$$

Dividing by $|A|$, which is not zero by the assumed invertibility of A, we obtain the formula of the theorem. Since we know that $A\mathbf{x} = \mathbf{b}$ has a unique solution for every invertible A, it must be the one with these values of the x_i. (A direct check is left as Exercise 6.2.5.) ■

[3] Named after Gabriel Cramer (1704–1752).

Note that in Equation 6.60 for x_i, the matrix in the numerator on the right is obtained from A by replacing its ith column by $\mathbf{b}$.

In addition to Cramer's rule for the solution of linear systems, there also exists an explicit formula involving determinants for the inverse of a matrix. However, before presenting it we state an interesting intermediate result that we shall need in the proof.

Lemma 6.2.1. ***(Mismatched Rows or Columns in a Cofactor Expansion Yield Zero).*** *For every $n \times n$ matrix A, if we add the elements of one row (column) multiplied by the cofactors of another row (column), then we get zero, that is,*

$$a_{k1}A_{j1} + a_{k2}A_{j2} + \cdots + a_kA_{jn} = 0 \text{ for } k \neq j \tag{6.65}$$

and

$$a_{1k}A_{1j} + a_{2d_{2j}} + \cdots + a_{nk}A_{nj} = 0 \text{ for } k \neq j. \tag{6.66}$$

Proof. The proof of this lemma is similar to that of Theorem 6.2.1. Consider the determinant of the matrix A with the jth column replaced by $\mathbf{a}_k$ for some $k \neq j$. Then, because two columns are equal, we have

$$|\mathbf{a}_1 \cdots \mathbf{a}_{j-1} \ \mathbf{a}_k \ \mathbf{a}_{j+1} \cdots \mathbf{a}_n| = 0 \tag{6.67}$$

and, expanding the $\mathbf{a}_k$ vector that is in the jth place, we can write this equation as

$$\left|\mathbf{a}_1 \cdots \mathbf{a}_{j-1} \ \sum_i a_{ik}\mathbf{e}_i \ \mathbf{a}_{j+1} \cdots \mathbf{a}_n\right| = 0, \tag{6.68}$$

which is equivalent to

$$\sum_i a_{ik}|\mathbf{a}_1 \cdots \mathbf{a}_{j-1} \ \mathbf{e}_i \ \mathbf{a}_{j+1} \cdots \mathbf{a}_n| = 0. \tag{6.69}$$

Equation 6.55 tells us that the determinant in the ith term here is the cofactor A_{ij}. Thus we have

$$\sum_i a_{ik}A_{ij} = 0, \tag{6.70}$$

which is the same as Equation 6.66. Equation 6.65 then follows by Theorem 6.1.6. ∎

In Equation 6.65 we have a dot product of the ith row of A with the jth row of a matrix whose elements are the cofactors. We want to make use of this fact in the next theorem, but we generally prefer to write such dot products as products of a row by a column of a matrix, and so we define the second matrix as follows.

Definition 6.2.2. ***(Adjoint Matrix).*** *The transposed matrix of the cofactors of A is called the adjoint matrix of A and is written as*

$$\operatorname{adj}(A) = (A_{ij})^T. \tag{6.71}$$

Theorem 6.2.3. ***(A Formula for A^{-1}).*** *The inverse of an invertible matrix A is given by*

$$A^{-1} = \frac{\operatorname{adj}(A)}{|A|}. \tag{6.72}$$

Proof. If we take the matrix product of A and $\operatorname{adj}(A)$ in this order, then by Equation 6.65 the off-diagonal elements all vanish, and by Theorem 6.2.1 the diagonal elements are all equal to $|A|$. Thus $|A|$ can be factored out, and we get

$$A \operatorname{adj}(A) = |A|I. \tag{6.73}$$

This result implies the statement of the theorem. ■

Example 6.2.2. (Evaluating the Inverse of a Matrix, Using the Adjoint). Let us use Theorem 6.2.3 to find the inverse of

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 5 & 4 \\ 3 & 6 & 9 \end{bmatrix}. \tag{6.74}$$

From Example 6.1.4 we know that $|A| = 9$, and the minors are $A_{11} = 5 \cdot 9 - 6 \cdot 4 = 21$, $A_{12} = -(2 \cdot 9 - 3.4) = -6$, $A_{13} = 2 \cdot 6 - 3 \cdot 5 = -3$, $A_{21} = -(1 \cdot 9 - 6.2) = 3$, $A_{22} = 1 \cdot 9 - 3 \cdot 2 = 3$, etc. Thus,

$$A^{-1} = \frac{1}{9} \begin{bmatrix} 21 & 3 & -6 \\ -6 & 3 & 0 \\ -3 & -3 & 3 \end{bmatrix}. \tag{6.75}$$

♦

There remains only one property of determinants to discuss: the one that we used to motivate their definition at the beginning of the previous section, namely their relationship to areas and volumes. The only problem that we need to clear up is how the nonnegativity of the latter can be reconciled with the linearity of determinants. We are going to show in a somewhat indirect way that the area of the parallelogram spanned by the vectors $\mathbf{a}_1$ and $\mathbf{a}_2$ in $\mathbb{R}^2$ is given by $|\det(\mathbf{a}_1, \mathbf{a}_2)|$.

The area properties that we use will be based in part on the introductory discussion in Section 6.1, but instead of the linearity property we employ the first property of Theorem 6.1.5, that is, the equation

$$|\mathbf{a}_1, \mathbf{a}_2| = |\mathbf{a}_1, \mathbf{a}_2 + \lambda \mathbf{a}_1|. \tag{6.76}$$

That a corresponding equation is valid for areas spanned by any vectors $\mathbf{a}_1$ and $\mathbf{a}_2$ in $\mathbb{R}^2$ and every real λ can be seen from Figure 6.4, where the area of the shaded parallelogram equals the area of the one spanned by $\mathbf{a}_1$ and $\mathbf{a}_2$. Thus we define the notion of an area function as follows.

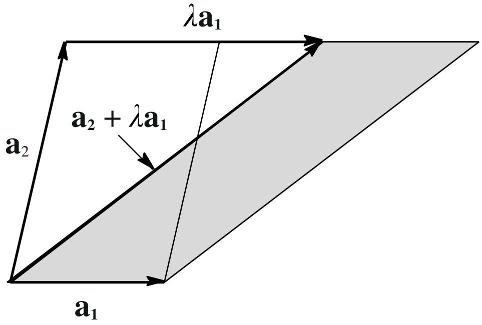

Fig. 6.4. The area of a parallelogram does not change if $\mathbf{a}_2$ is replaced by $\mathbf{a}_2 + \lambda\mathbf{a}_1$

Definition 6.2.3. ***(Area Function).*** *An area function on* $\mathbb{R}^2$ *is a function* F *that assigns to every pair of vectors* $\mathbf{a}_1$ *and* $\mathbf{a}_2$ *in* $\mathbb{R}^2$ *a number* $F(\mathbf{a}_1, \mathbf{a}_2)$*, such that*

$$F(\mathbf{a}_1, \mathbf{a}_2) = F(\mathbf{a}_1, \mathbf{a}_2 + \lambda\mathbf{a}_1) = F(\mathbf{a}_1 + \lambda\mathbf{a}_2, \mathbf{a}_2), \tag{6.77}$$

$$F(\mathbf{a}_1, \lambda\mathbf{a}_2) = F(\lambda\mathbf{a}_1, \mathbf{a}_2) = |\lambda| F(\mathbf{a}_1, \mathbf{a}_2), \tag{6.78}$$

$$F(\mathbf{a}_1, \mathbf{a}_2) = F(\mathbf{a}_2, \mathbf{a}_1) \tag{6.79}$$

and

$$F(\mathbf{e}_1, \mathbf{e}_2) = 1 \tag{6.80}$$

for every real λ*.*

It is clear that $|\det|$ is an area function, and the next theorem shows that it is the only one.

Theorem 6.2.4. ***($|\det|$ Is the Only Area Function).*** *The only function on* $\mathbb{R}^2$ *with the properties above is the absolute value of the determinant.*

Proof. Assume that F is an area function on $\mathbb{R}^2$. Then first we see that $F(\mathbf{a}_1, \mathbf{0}) = 0$, since by Equation 6.78 we have

$$F(\mathbf{a}_1, \mathbf{0}) = F(\mathbf{a}_1, 0\mathbf{0}) = 0F(\mathbf{a}_1, \mathbf{0}) = 0. \tag{6.81}$$

In this case $\det(\mathbf{a}_1, \mathbf{0}) = 0$ holds, too, and so, for $\mathbf{a}_2 = \mathbf{0}$, F is the same as $|\det|$.

If, on the other hand, $\mathbf{a}_2 \neq \mathbf{0}$, then, writing $\mathbf{a}_1 = a_{11}\mathbf{e}_1 + a_{21}\mathbf{e}_2$ and $\mathbf{a}_2 = a_{12}\mathbf{e}_1 + a_{22}\mathbf{e}_2$, and assuming $a_{22} \neq 0$, we can reduce $F(\mathbf{a}_1, \mathbf{a}_2)$ as follows:

$$\begin{aligned} F(\mathbf{a}_1, \mathbf{a}_2) &= F\left(\mathbf{a}_1 - \frac{a_{21}}{a_{22}}\mathbf{a}_2, \mathbf{a}_2\right) = F\left(\left(a_{11} - \frac{a_{21}}{a_{22}}a_{12}\right)\mathbf{e}_1, \mathbf{a}_2\right) \\ &= F\left(\frac{|A|}{a_{22}}\mathbf{e}_1, \mathbf{a}_2\right) = \left|\frac{|A|}{a_{22}}\right| F\left(\mathbf{e}_1, \mathbf{a}_2\right) = \left|\frac{|A|}{a_{22}}\right| F(\mathbf{e}_1, \mathbf{a}_2 - a_{12}\mathbf{e}_1) \\ &= \left|\frac{|A|}{a_{22}}\right| F(\mathbf{e}_1, a_{22}\mathbf{e}_2) = |\det(A)| F(\mathbf{e}_1, \mathbf{e}_2) = |\det(A)|. \end{aligned} \tag{6.82}$$

Thus F is the same as $|\det|$ in this case as well.

In the remaining case of $\mathbf{a}_2 \neq \mathbf{0}$ and $a_{22} = 0$, we must have $a_{12} \neq 0$, and we may proceed similarly as above, but use a_{12} as we used a_{22} before. ■

Theorem 6.2.4 does not explain the geometric significance of the sign of the determinant. What we have is that $\det(\mathbf{a}_1, \mathbf{a}_2)$ is positive if $\mathbf{a}_1$ followed by $\mathbf{a}_2$ indicates a counterclockwise traversal of the parallelogram spanned by $\mathbf{a}_1$ and $\mathbf{a}_2$, and is negative otherwise. We do not prove this fact here.

In three dimensions we could define a volume function for parallelepipeds by properties similar to those in Definition 6.2.3 and prove the theorem, analogous to the last one, that the only volume function is the absolute value of the third order determinant. Since all this work would be very much like the discussion above, only more involved, we do not present it. Furthermore, the sign of $\det(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3)$ is positive if $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ form a right-handed triple in this order, and negative otherwise. Again, this result will not be discussed here any further.

Exercises

In the first four exercises evaluate the determinants of the corresponding exercises of the previous section by expansion along any row or column.

Exercise 6.2.1. $D = \begin{vmatrix} 2 & -3 & 2 \\ 1 & 4 & 0 \\ 0 & 1 & -5 \end{vmatrix}.$

Exercise 6.2.2. $D = \begin{vmatrix} 0 & 1 & 2 \\ 4 & 0 & 3 \\ 3 & 2 & 1 \end{vmatrix}.$

Exercise 6.2.3. $D = \begin{vmatrix} -1 & 1 & 2 & 3 \\ 2 & 0 & -5 & 0 \\ 0 & 0 & 0 & -1 \\ 3 & -3 & 1 & 4 \end{vmatrix}.$

Exercise 6.2.4. $D = \begin{vmatrix} -1 & -2 & 2 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 1 & -1 & 1 & -1 \end{vmatrix}$.

Exercise 6.2.5. Check Cramer's rule by direct substitution from Equation 6.60 into Equation 6.61.

In the next three exercises solve $A\mathbf{x} = \mathbf{b}$ by Cramer's rule with the given data, if possible.

Exercise 6.2.6.

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}.$$

Exercise 6.2.7.

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 0 & 4 \\ 0 & 3 & 1 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

Exercise 6.2.8.

$$A = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 2 & 0 & 4 & 1 \\ 1 & 3 & 4 & 0 \\ 0 & 3 & 1 & 1 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 4 \\ 3 \end{bmatrix}.$$

Exercise 6.2.9. Use Theorem 6.2.3 to find the inverse of the matrix A in Exercise 6.2.6.

Exercise 6.2.10. Use Theorem 6.2.3 to find the inverse of the matrix A in Exercise 6.2.7.

Exercise 6.2.11. Show that for every invertible $n \times n$ matrix A we have

$$\det(\operatorname{adj}(A)) = (\det(A))^{n-1}. \tag{6.83}$$

Exercise 6.2.12. Show that for every invertible $n \times n$ matrix A the matrix $\operatorname{adj}(A)$ is also invertible and satisfies

$$(\operatorname{adj}(A))^{-1} = \operatorname{adj}(A^{-1}). \tag{6.84}$$

Exercise 6.2.13. Show that in the plane the area of the triangle with vertices (a_1, a_2), (b_1, b_2), (c_1, c_2) is given by the absolute value of

$$\frac{1}{2}\begin{vmatrix} a_1 & a_2 & 1 \\ b_1 & b_2 & 1 \\ c_1 & c_2 & 1 \end{vmatrix}. \tag{6.85}$$

Exercise 6.2.14. Show that in the plane an equation of the line through the points (x_1, y_1), (x_2, y_2) is given by

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0. \tag{6.86}$$

Exercise 6.2.15. Show that in the plane an equation of the form $y = ax^2 + bx + c$ for the parabola through the distinct points (x_1, y_1), (x_2, y_2), (x_3, y_3) is equivalent to

$$\begin{vmatrix} y & x^2 & x & 1 \\ y_1 & x_1^2 & x_1 & 1 \\ y_2 & x_2^2 & x_2 & 1 \\ y_3 & x_3^2 & x_3 & 1 \end{vmatrix} = 0. \tag{6.87}$$

Exercise 6.2.16. Show that in the plane an equation of the form $x^2 + y^2 + ax + by + c = 0$ for a circle through the noncollinear points (x_1, y_1), (x_2, y_2), (x_3, y_3) is equivalent to

$$\begin{vmatrix} x^2 + y^2 & x & y & 1 \\ x_1^2 + y_1^2 & x_1 & y_1 & 1 \\ x_2^2 + y_2^2 & x_2 & y_2 & 1 \\ x_3^2 + y_3^2 & x_3 & y_3 & 1 \end{vmatrix} = 0. \tag{6.88}$$

Exercise 6.2.17. Use the result of the previous exercise to find an equation of the circle through the points $(0, 0), (2, -1), (4, 0)$ and find its center and radius as well.

***Exercise 6.2.18.** Show that the volume of a tetrahedron with vertices (a_1, a_2, a_3), (b_1, b_2, b_3), (c_1, c_2, c_3), (d_1, d_2, d_3) is given by the absolute value of

$$\frac{1}{6}\begin{vmatrix} a_1 & a_2 & a_3 & 1 \\ b_1 & b_2 & b_3 & 1 \\ c_1 & c_2 & c_3 & 1 \\ d_1 & d_2 & d_3 & 1 \end{vmatrix}. \tag{6.89}$$

Exercise 6.2.19. Show that in $\mathbb{R}^3$ an equation of the plane through the points (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) is given by

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0.$$

Exercise 6.2.20. Show that if $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation, then it maps the unit square to a parallelogram of area $|\det([T])|$.

6.3 The Cross Product of Vectors in $\mathbb{R}^3$

Consider two arbitrary vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^3$. Project the corresponding parallelogram onto the coordinate planes and define a vector $\mathbf{u} \times \mathbf{v}$ with components equal to the areas of these projections with appropriate signs. (See Figure 6.5 for noncollinear $\mathbf{u}$ and $\mathbf{v}$. If they are collinear, the areas are zero and $\mathbf{u} \times \mathbf{v} = \mathbf{0}$.) The formal definition is given below. As we shall see shortly, this definition makes $\mathbf{u} \times \mathbf{v}$ perpendicular to both $\mathbf{u}$ and $\mathbf{v}$ and its length equal to the area of the parallelogram spanned by $\mathbf{u}$ and $\mathbf{v}$. We could define $\mathbf{u} \times \mathbf{v}$ geometrically, using this property, but it is easier to define it in terms of the components.

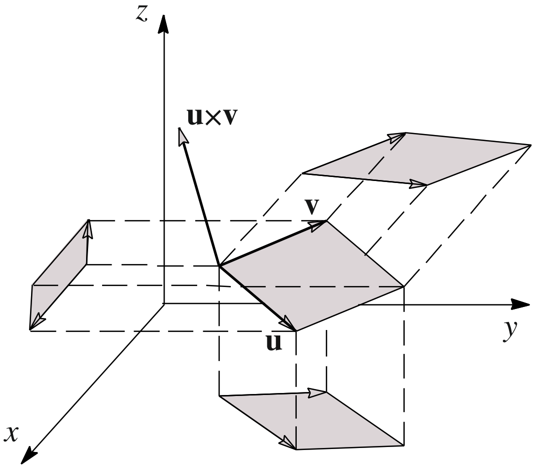

Fig. 6.5. Projecting a parallelogram onto the coordinate planes

Definition 6.3.1. ***(Cross Product).*** *We define the vector product or cross product for all vectors* $\mathbf{u}$ *and* $\mathbf{v}$ *of* $\mathbb{R}^3$ *as the vector of* $\mathbb{R}^3$ *given by*

$$\mathbf{u} \times \mathbf{v} = (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}. \tag{6.90}$$

This equation is usually abbreviated as

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}. \tag{6.91}$$

This construction works only in three dimensions, because in other dimensions the number of coordinate planes is different from the number of coordinate axes. In four dimensions, for example, we have six coordinate planes. (Why?) But, since our world is three dimensional, this is a very important special case. Also, in this context it is customary to write $\mathbf{i}, \mathbf{j}, \mathbf{k}$ for the standard vectors rather than $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$.

Example 6.3.1. The cross product of $\mathbf{u} = (1, 2, 3)^T$ and $\mathbf{v} = (4, 5, 6)^T$ is given by

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix}$$
$$= (2 \cdot 6 - 3 \cdot 5)\mathbf{i} - (1 \cdot 6 - 3 \cdot 4)\mathbf{j} + (1 \cdot 5 - 2 \cdot 4)\mathbf{k} = -3\mathbf{i} + 6\mathbf{j} - 3\mathbf{k}, \tag{6.92}$$

where the determinant is evaluated by using the formal cofactor expansion along the first row. ♦

Example 6.3.2. (Evaluating a Cross Product). The cross products of the standard vectors can easily be computed from the definition. For instance, since $\mathbf{i} = (1, 0, 0)^T$ and $\mathbf{j} = (0, 1, 0)^T$, we have

$$\mathbf{i} \times \mathbf{j} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = \mathbf{k}. \tag{6.93}$$

Similarly,

$$\mathbf{i} \times \mathbf{j} = -\mathbf{j} \times \mathbf{i} = \mathbf{k},\ \mathbf{j} \times \mathbf{k} = -\mathbf{k} \times \mathbf{j} = \mathbf{i},\ \mathbf{k} \times \mathbf{i} = -\mathbf{i} \times \mathbf{k} = \mathbf{j}, \tag{6.94}$$

and

$$\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = \mathbf{0}. \tag{6.95}$$

♦

In the following theorem we list the most useful properties of the cross product.

Theorem 6.3.1. ***(Properties of the Cross Product).*** *For all vectors* $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ *of* $\mathbb{R}^3$ *and every scalar* c *we have*

1. $(c\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (c\mathbf{v}) = c(\mathbf{u} \times \mathbf{v})$,
2. $\mathbf{v} \times \mathbf{u} = -\mathbf{u} \times \mathbf{v}$,
3. $\mathbf{u} \times \mathbf{u} = \mathbf{0}$,
4. $\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = 0$ and $\mathbf{v} \cdot (\mathbf{u} \times \mathbf{v}) = 0$,
5. $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{u} \times \mathbf{v}$, in this order, form a right-handed triple,
6. $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$,
7. $(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = \mathbf{u} \times \mathbf{w} + \mathbf{v} \times \mathbf{w}$,
8. $|\mathbf{u} \times \mathbf{v}|^2 = |\mathbf{u}|^2|\mathbf{v}|^2 - (\mathbf{u} \cdot \mathbf{v})^2$,

9. $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}| \sin\theta$, where $\theta \in [0, \pi]$ denotes the angle between $\mathbf{u}$ and $\mathbf{v}$, and the right-hand side gives the area of the parallelogram spanned by $\mathbf{u}$ and $\mathbf{v}$,
10. $\mathbf{u} \times \mathbf{v} = \mathbf{0}$ if and only if $\mathbf{u}$ and $\mathbf{v}$ are linearly dependent,
11. $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \mathbf{v} \cdot (\mathbf{w} \times \mathbf{u}) = \mathbf{w} \cdot (\mathbf{u} \times \mathbf{v}) = \det(\mathbf{u}, \mathbf{v}, \mathbf{w})$,
12. $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{v} \cdot \mathbf{w})\mathbf{u}$,
13. $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}$.

Proof. Statements 1, 2, and 3 follow by straightforward substitution from Definition 6.3.1. Notice that Statement 2 says that the cross product is not commutative but, as we say, *anticommutative.* Furthermore, Statement 3 coupled with Statement 1 for $\mathbf{v} = \mathbf{u}$ says that the cross product of collinear vectors is zero, as mentioned earlier.

The first part of Statement 4 can be proved as follows: From Equation 6.90 we have

$$\begin{aligned}\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) &= (u_2v_3 - u_3v_2)\mathbf{u} \cdot \mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{u} \cdot \mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{u} \cdot \mathbf{k} \\ &= (u_2v_3 - u_3v_2)u_1 - (u_1v_3 - u_3v_1)u_2 + (u_1v_2 - u_2v_1)u_3 \\ &= \begin{vmatrix} u_1 & u_2 & u_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = 0. \end{aligned} \tag{6.96}$$

The second part of Statement 4 can be established similarly. Geometrically these two equations mean that the cross product is orthogonal to its factors. This is a very important property of the cross product, which makes it useful in many applications.

Statement 5 is true for the standard vectors, as can be seen from the formulas of Example 6.3.2. For other vectors it could be proved by rotating and stretching or shrinking them into the standard vectors, because these operations do not change the "handedness" of a triple. We omit the details.

Statements 6 and 7 follow again by straightforward substitution from Definition 6.3.1.

Statement 8 can be proved as follows:

$$\begin{aligned}|\mathbf{u} \times \mathbf{v}|^2 &= \sum_{i<j} (u_iv_j - u_jv_i)^2 \\ &= \frac{1}{2}\sum_{i \neq j} (u_iv_j - u_jv_i)^2 = \frac{1}{2}\sum_{i,j=1}^{3} (u_iv_j - u_jv_i)^2 \\ &= \frac{1}{2}\sum_{i,j=1}^{3} \left[(u_iv_j)^2 - 2u_iv_ju_jv_i + (u_jv_i)^2\right] \\ &= \frac{1}{2}\sum u_i^2 \sum v_j^2 - \sum u_iv_i \sum u_jv_j + \frac{1}{2}\sum u_j^2 \sum v_i^2 \\ &= |\mathbf{u}|^2|\mathbf{v}|^2 - (\mathbf{u} \cdot \mathbf{v})^2. \end{aligned} \tag{6.97}$$

Statement 9 follows from Statement 8, because

$$|\mathbf{u}|^2|\mathbf{v}|^2 - (\mathbf{u}\cdot\mathbf{v})^2 = |\mathbf{u}|^2|\mathbf{v}|^2 - (|\mathbf{u}||\mathbf{v}|\cos\theta)^2 = (|\mathbf{u}||\mathbf{v}|\sin\theta)^2 \tag{6.98}$$

and, both $|\mathbf{u}\times\mathbf{v}|$ and $|\mathbf{u}||\mathbf{v}|\sin\theta$ being nonnegative, we can take square roots in Equations 6.97 and 6.98. The expression $|\mathbf{u}||\mathbf{v}|\sin\theta$ gives the area of the parallelogram spanned by $\mathbf{u}$ and $\mathbf{v}$ since, as Figure 6.6 shows, $|\mathbf{u}|$ is the parallelogram's base length and $|\mathbf{v}|\sin\theta$ its height.

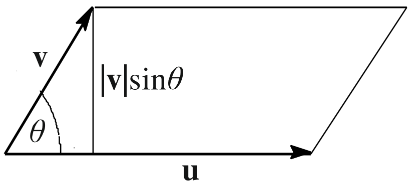

Fig. 6.6. Finding the area of a triangle

The "if" part of Statement 10 follows from the fact that two vectors are linearly dependent if and only if they are collinear (the zero vector is collinear with every vector by definition), and for collinear vectors we already know that the cross product is zero. The "only if" part follows from Statement 9, since if $\mathbf{u}$ and $\mathbf{v}$ are not collinear, then none of the factors in $|\mathbf{u}||\mathbf{v}|\sin\theta$ is zero, and so $|\mathbf{u}\times\mathbf{v}| \neq 0$.

The proof of Statement 11 is similar to Equation 6.96 above. We leave this for Exercise 6.3.4.

Statements 12 and 13 could be proved simply by writing out each expression in terms of the components u_i, v_i, w_i, but we gain a little more insight if we proceed as follows. First, if $\mathbf{u}$ and $\mathbf{v}$ are linearly dependent, then both sides of Statement 12 are $\mathbf{0}$. Otherwise, $\mathbf{u}\times\mathbf{v}$ being orthogonal to $\mathbf{u}$ and $\mathbf{v}$, all vectors orthogonal to $\mathbf{u}\times\mathbf{v}$ must lie in the plane of $\mathbf{u}$ and $\mathbf{v}$, that is, must be linear combinations of $\mathbf{u}$ and $\mathbf{v}$. Now, $(\mathbf{u}\times\mathbf{v})\times\mathbf{w}$ *is* orthogonal to $\mathbf{u}\times\mathbf{v}$, and so it must satisfy

$$(\mathbf{u}\times\mathbf{v})\times\mathbf{w} = a\mathbf{u} + b\mathbf{v} \tag{6.99}$$

for some scalars a and b. Taking the dot product of both sides with $\mathbf{w}$, and considering that the left side is orthogonal to it, we obtain

$$a(\mathbf{u}\cdot\mathbf{w}) + b(\mathbf{v}\cdot\mathbf{w}) = 0. \tag{6.100}$$

The solutions of this equation for the unknown a and b can be written in the form

$$a = c(\mathbf{v}\cdot\mathbf{w}) \text{ and } b = -c(\mathbf{u}\cdot\mathbf{w}) \tag{6.101}$$

with still another scalar c to be determined. Substituting these values into Equation 6.99, we get

$$(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = c(\mathbf{v} \cdot \mathbf{w})\mathbf{u} - c(\mathbf{u} \cdot \mathbf{w})\mathbf{v}. \tag{6.102}$$

Since both sides are homogeneous linear expressions of the u_i, v_i, w_i components and this is an identity, which must hold for every choice of the vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, the scalar c must be independent of this choice. Thus we can evaluate it conveniently by setting $\mathbf{u} = \mathbf{w} = \mathbf{i}$ and $\mathbf{v} = \mathbf{j}$ in Equation 6.102. This substitution leads to $c = -1$, which proves Statement 12. Statement 13 could be proved similarly.

Incidentally, Statements 12 and 13 show that the cross product is not associative. ■

Let us now look at some applications of the cross product.

Example 6.3.3. (Using the Cross Product for Finding an Equation of a Plane). Find a nonparametric equation of the plane S through the three points $A = (1, 0, 2)$, $B = (3, 1, 2)$, $C = (2, 1, 4)$.

We can find a normal vector of S by taking the cross product of any two independent vectors lying in it, say, $\overrightarrow{AB} = (2, 1, 0)^T$ and $\overrightarrow{AC} = (1, 1, 2)^T$. Thus we may take

$$\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 0 \\ 1 & 1 & 2 \end{vmatrix} = 2\mathbf{i} - 4\mathbf{j} + \mathbf{k}. \tag{6.103}$$

If we denote the position vector of a general point of S by $\mathbf{p} = (x, y, z)^T$ and the position vector of A by $\mathbf{a}$, then the desired equation can be written in general as

$$\mathbf{n} \cdot (\mathbf{p} - \mathbf{a}) = 0, \tag{6.104}$$

and for this particular plane as

$$2(x - 1) - 4(y - 0) + 1(z - 2) = 0. \tag{6.105}$$

♦

In general, the parametric vector equation of a plane S was written on page 32 (Equation 1.75) as

$$\mathbf{p} = \mathbf{p}_0 + s\mathbf{u} + t\mathbf{v}, \tag{6.106}$$

where $\mathbf{u}$ and $\mathbf{v}$ denote two noncollinear vectors lying in S, $\mathbf{p}$ and $\mathbf{p}_0$ the position vectors of a variable and a fixed point of S, and s and t two parameters. Taking the dot product of both sides with the vector $\mathbf{n} = \mathbf{u} \times \mathbf{v}$, which is orthogonal to $\mathbf{u}$ and $\mathbf{v}$, we obtain the general nonparametric equation of a plane as

$$\mathbf{n} \cdot \mathbf{p} = \mathbf{n} \cdot \mathbf{p}_0. \tag{6.107}$$

This is the simple way of eliminating s and t that we alluded to in footnote 6 in Chapter 1, on page 33.

Example 6.3.4. (Using the Cross Product for Finding a Vector Normal to a Triangle). Find a vector $\mathbf{n}$ normal to the triangle T with vertices $A = (0, -2, 2)$, $B = (0, 2, 3)$, $C = (2, 0, 2)$, and whose length equals the area of T.

We can find a normal vector of T by taking the cross product of any two of its edge vectors, say, $\overrightarrow{AB} = (0, 4, 1)^T$ and $\overrightarrow{AC} = (2, 2, 0)^T$. Thus we may take

$$\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 4 & 1 \\ 2 & 2 & 0 \end{vmatrix} = -2\mathbf{i} + 2\mathbf{j} - 8\mathbf{k}. \tag{6.108}$$

The area of T is given, according to Statement 9, by $\frac{1}{2}|\mathbf{n}|$. Thus the normal vector whose length equals the area of T is $\frac{1}{2}\mathbf{n}$, and

$$\text{Area}(T) = \sqrt{1^2 + 1^2 + 4^2} = \sqrt{18}. \tag{6.109}$$

♦

A vector normal to a plane figure S and having length equal to the area of S, like the vector $\frac{1}{2}\mathbf{n}$ above, is sometimes called an *area vector of* S.

The cross product has many applications in physics. We discuss just two of them briefly.

Example 6.3.5. (Coriolis Force). Any object rotating relative to the universe experiences some forces due to its inertia, that is, its tendency to move uniformly in a straight line. A well-known example is the centrifugal force pushing us outward in a turning car. If the object is moving relative to a rotating frame of reference, then there is an additional such force acting on it, called the *Coriolis force*, which is shown in physics to be given by a cross product:

$$\mathbf{F} = 2m\mathbf{v} \times \boldsymbol{\omega} \tag{6.110}$$

where m denotes the mass of the object, $\mathbf{v}$ its velocity vector relative to the rotating frame, and $\boldsymbol{\omega}$ the angular velocity vector of the frame's rotation relative to the universe. In the case of the Earth, $\boldsymbol{\omega}$ is a vector parallel to the Earth's axis pointing from the South Pole to the North Pole and having length $2\pi/(24\,\text{h})$. ♦

Note that the right-hand rule of Statement 5 of Theorem 6.3.1 reflects an essential property of this force, in contrast to the geometrical examples above, where it played no role.

The Coriolis force has powerful effects in meteorology. For example, the hot climate near the equator makes the air rise, which then cools off and descends at moderate latitudes. In Figure 6.7 the vector $\mathbf{v}$ represents the velocity of this descending air somewhere in the middle of the northern hemisphere, and the vector $\mathbf{F}$ is the Coriolis force, which, according to the right-hand rule, points to the east. This is the reason for the prevailing westerly winds there.

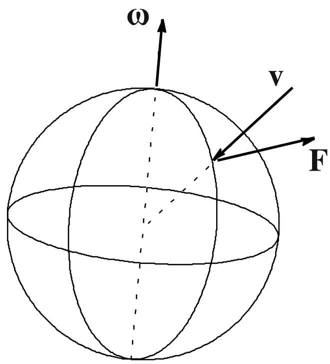

Fig. 6.7. The Coriolis force creates westerly winds in the northern hemisphere of the Earth

Similarly, a hurricane starts around a region where the barometric pressure is very low, and consequently the outside air starts moving towards it. This movement creates a Coriolis force, which is perpendicular to this radial flow and starts the circulation of the hurricane. There are other forces involved as well, and with the changed wind direction the direction of the Coriolis force changes too, but it is still the major cause of the hurricane. The right-hand rule shows that in the northern hemisphere the circulation must be counterclockwise (we leave the explanation of this direction for Exercise 6.3.10).

Example 6.3.6. (Lorentz Force). Another instance of a cross product occurs in the formula for the Lorentz force. This is the force exerted by a magnetic field $\mathbf{B}$ on a particle with an electric charge q and moving with velocity $\mathbf{v}$ relative to the field. It is given by

$$\mathbf{F} = q\mathbf{v} \times \mathbf{B}. \tag{6.111}$$

The effect of this force can be seen, for example, in cloud chamber photographs of the tracks of elementary particles. In a transverse magnetic field this force changes the particles' straight line paths to circles. This is also the force that drives electric motors by acting on the electrons comprising the current in the motor's coils. ♦

Exercises

Exercise 6.3.1. Find the cross product of the vectors $\mathbf{u} = (1, -1, 0)^T$ and $\mathbf{v} = (1, 2, 0)^T$, and verify by elementary geometry that $|\mathbf{u} \times \mathbf{v}|$ equals the area of the parallelogram spanned by $\mathbf{u}$ and $\mathbf{v}$.

Exercise 6.3.2. Use the cross product to find an equation of the plane S through the three points $A = (1, -1, 2),\ B = (0, -1, 3),\ C = (3, 0, 2)$.

Exercise 6.3.3. Verify Statements 11, 12, and 13 of Theorem 6.3.1 for the vectors $\mathbf{u} = (1, -1, 0)^T$, $\mathbf{v} = (1, 2, 0)^T$, and $\mathbf{w} = (1, 0, 3)^T$.

Exercise 6.3.4. Prove Statement 11 of Theorem 6.3.1.

Exercise 6.3.5. The expression $\mathbf{u}\cdot(\mathbf{v} \times \mathbf{w})$ of Statement 11 of Theorem 6.3.1 is called the *triple product* of these vectors. Show geometrically that its absolute value equals the volume of the parallelepiped spanned by the vectors $\mathbf{u},\ \mathbf{v}$, and $\mathbf{w}$.

***Exercise 6.3.6.** Let $\mathbf{n}_1,\ \mathbf{n}_2,\ \mathbf{n}_3,\ \mathbf{n}_4$ denote the outward pointing area vectors of a tetrahedron. Prove that their sum equals $\mathbf{0}$. (*Hint*: Let $\mathbf{a}_1, \ldots, \mathbf{a}_6$ denote the edge vectors, write each area vector as a cross product of these, and apply the appropriate properties from Theorem 6.3.1 to the sum.)

***Exercise 6.3.7.** Prove that for all vectors $\mathbf{a},\ \mathbf{b},\ \mathbf{c},\ \mathbf{d}$ of $\mathbb{R}^3$ we have

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c}). \tag{6.112}$$

***Exercise 6.3.8.** Prove that for all vectors $\mathbf{a},\ \mathbf{b},\ \mathbf{c},\ \mathbf{d}$ of $\mathbb{R}^3$ we have

$$(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) = [\det(\mathbf{a}, \mathbf{c}, \mathbf{d})]\mathbf{b} - [\det(\mathbf{b}, \mathbf{c}, \mathbf{d})]\mathbf{a} \tag{6.113}$$

and

$$(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) = [\det(\mathbf{a}, \mathbf{b}, \mathbf{d})]\mathbf{c} - [\det(\mathbf{a}, \mathbf{b}, \mathbf{c})]\mathbf{d}. \tag{6.114}$$

Exercise 6.3.9. Let $\mathbf{a}$ be any fixed nonzero vector of $\mathbb{R}^3$. Define the transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ by $T(\mathbf{x}) = \mathbf{a} \times \mathbf{x}$.

a. Show that T is linear.
b. Find the matrix $[T]$ that represents T relative to the standard basis.
c. Find the null space of the matrix $[T]$, and describe it geometrically.
d. What is the rank of $[T]$?

Exercise 6.3.10. Explain, using Equation 6.110, why in the northern hemisphere the circulation of hurricanes must be counterclockwise. What is it in the southern hemisphere and why?

7. *Eigenvalues and Eigenvectors*

7.1 Eigenvalues and Eigenvectors, Basic Properties

In this chapter we study another major branch of linear algebra, very different from what we have seen so far. The problems in this area arise in many applications in physics, economics, statistics, and other fields. The main reason for this phenomenon can be explained roughly as follows.

Frequently the states of a physical system can be described by an n-dimensional vector $\mathbf{x}$ and the latter's change in time by an $n \times n$ matrix A, so that the state of the system at some later time will be given by the vector $\mathbf{y} = A\mathbf{x}$. Often such changes are described by differential equations, that is, the vector $\mathbf{x}$ is an unknown differentiable vector function of time satisfying an equation of the form $\mathbf{x}' = A\mathbf{x}$. Such differential equations also lead to the same basic situation that we want to explain for the simpler case of $\mathbf{y} = A\mathbf{x}$.

Consider the two-dimensional case, for which we can express $\mathbf{y} = A\mathbf{x}$ as the pair of scalar equations

$$y_1 = a_{11}x_1 + a_{12}x_2 \tag{7.1}$$

and

$$y_2 = a_{21}x_1 + a_{22}x_2. \tag{7.2}$$

As these equations show, generally the matrix A mixes up the components of $\mathbf{x}$ and, over many such time steps (especially in higher-dimensional cases), this mixing can be quite involved. The question we ask, therefore, is whether it is possible to find a new basis $\{\mathbf{s}_1, \mathbf{s}_2\}$ for $\mathbb{R}^2$ in which such mixing does not occur, that is, in which the components are decoupled and develop separately as

$$y_{S1} = \lambda_1 x_{S1} \tag{7.3}$$

and

$$y_{S2} = \lambda_2 x_{S2}, \tag{7.4}$$

where λ_1 and λ_2 are appropriate scalars depending on the matrix A. The answer is frequently yes, and it leads to greatly simplified computations when many such steps follow each other. Let us illustrate this decoupling with an example.

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Example 7.1.1. (A Decoupling of Coordinates in the Action of a Matrix). Consider the matrix

$$A = \begin{bmatrix} 1/2 & 3/2 \\ 3/2 & 1/2 \end{bmatrix} \tag{7.5}$$

and its action on an arbitrary vector $\mathbf{x}$. Let $\mathbf{s}_1 = \frac{1}{\sqrt{2}}(1,1)^T$ and $\mathbf{s}_2 = \frac{1}{\sqrt{2}}(-1,1)^T$. Then

$$A\mathbf{s}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1/2 & 3/2 \\ 3/2 & 1/2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 2 \\ 2 \end{bmatrix} = 2\mathbf{s}_1 \tag{7.6}$$

and

$$A\mathbf{s}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1/2 & 3/2 \\ 3/2 & 1/2 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = -\mathbf{s}_2. \tag{7.7}$$

So if we write $\mathbf{x}$ and $\mathbf{y} = A\mathbf{x}$ in terms of the basis $\{\mathbf{s}_1, \mathbf{s}_2\}$ as

$$\mathbf{x} = x_{S1}\mathbf{s}_1 + x_{S2}\mathbf{s}_2 \text{ and } \mathbf{y} = y_{S1}\mathbf{s}_1 + y_{S2}\mathbf{s}_2, \tag{7.8}$$

then we get

$$\mathbf{y} = A\mathbf{x} = x_{S1}A\mathbf{s}_1 + x_{S2}A\mathbf{s}_2 = 2x_{S1}\mathbf{s}_1 - x_{S2}\mathbf{s}_2, \tag{7.9}$$

and from this

$$y_{S1} = 2x_{S1} \tag{7.10}$$

and

$$y_{S2} = -x_{S2}. \tag{7.11}$$

Thus the action of A on the S-components of $\mathbf{x}$ is simple multiplication (see Figure 7.1), while in the standard basis it would involve linear combinations. Similarly, the S-components of $A^2\mathbf{x}$ would be $2^2 x_{S1}$ and $(-1)^2 x_{S1}$, and so on for higher powers; much simpler expressions than those used in the standard basis. ♦

In general, how can we find a basis such as the one in the example above? We must find nontrivial $\{\mathbf{s}_1, \mathbf{s}_2\}$ and corresponding λ_1 and λ_2 such that

$$A\mathbf{s}_1 = \lambda_1\mathbf{s}_1 \text{ and } A\mathbf{s}_2 = \lambda_2\mathbf{s}_2. \tag{7.12}$$

Indeed, in that case, substituting

$$\mathbf{x} = x_{S1}\mathbf{s}_1 + x_{S2}\mathbf{s}_2 \text{ and } \mathbf{y} = y_{S1}\mathbf{s}_1 + y_{S2}\mathbf{s}_2 \tag{7.13}$$

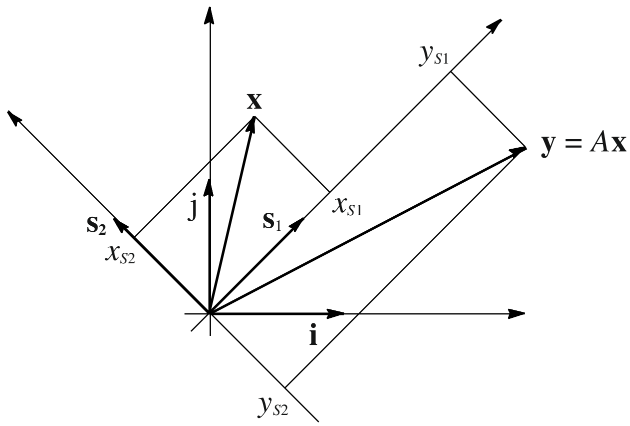

Fig. 7.1. The action of the matrix A in a new basis is multiplication by a scalar for each component

into the equation $\mathbf{y} = A\mathbf{x}$, we get

$$\begin{aligned} y_{S1}\mathbf{s}_1 + y_{S2}\mathbf{s}_2 &= A(x_{S1}\mathbf{s}_1 + x_{S2}\mathbf{s}_2) \\ &= x_{S1}A\mathbf{s}_1 + x_{S2}A\mathbf{s}_2 = x_{S1}\lambda_1\mathbf{s}_1 + x_{S2}\lambda_2\mathbf{s}_2, \end{aligned} \tag{7.14}$$

from which it follows that

$$y_{S1} = \lambda_1 x_{S1} \tag{7.15}$$

and

$$y_{S2} = \lambda_2 x_{S2}. \tag{7.16}$$

We shall see more detailed examples later. For now, we just want to base our fundamental definition on Equations 7.12, which lie at the heart of this whole theory.

Definition 7.1.1. ***(Eigenvalues and Eigenvectors).*** *For any $n \times n$ matrix A, a scalar λ is called an eigenvalue of A if there is a nonzero vector* $\mathbf{s}$ *such that*

$$A\mathbf{s} = \lambda\mathbf{s}. \tag{7.17}$$

Such a vector $\mathbf{s}$ *is called an eigenvector of A corresponding, or belonging, to λ. Furthermore, for every eigenvalue λ the zero vector is always a solution of Equation 7.17, and is called the trivial eigenvector of A belonging to λ.*

The word *eigen* is German for "own" and was adopted into English, although some authors use *proper value* and *proper vector* or *characteristic* *value* and *characteristic vector* instead. The Greek letter *lambda* is traditional in this context.

Geometrically, Equation 7.17 expresses the fact that an eigenvector of a matrix A is a nonzero vector whose direction is preserved by A, and the corresponding eigenvalue is the multiplier by which the matrix scales this eigenvector.

Definition 7.1.1 can be generalized to apply to linear transformations $T : V \to V$ from any nontrivial vector space to itself, which is important in infinite-dimensional cases. The matrix definition is a special case of this more general one.

So how do we solve Equation 7.17? The usual procedure is the following: We rewrite it as

$$(A-\lambda I)\,\mathbf{s} = \mathbf{0}, \tag{7.18}$$

since in this form we have an equation closely resembling the familiar $A\mathbf{s} = \mathbf{0}$, except that the matrix A is replaced by the matrix $A-\lambda I$. Recall that a homogeneous equation has nontrivial solutions if and only if its matrix is singular. By Theorem 6.1.8, for Equation 7.18, this condition is equivalent to

$$\det(A-\lambda I) = 0. \tag{7.19}$$

This is an equation for the unknown λ and is called the *characteristic equation* of the matrix A. The left side of the equation is called the *characteristic polynomial* of A. Equation 7.19 does not contain the unknown vector $\mathbf{s}$ and is therefore used to find the eigenvalues.[1] For an $n \times n$ matrix it is an algebraic equation of degree n. Once a value of λ is known, we substitute it into Equation 7.18, and solve the latter for $\mathbf{s}$ by Gaussian elimination. The solution set of Equation 7.18 for a given λ is $\mathrm{Null}\,(A-\lambda I)$, and is called the *eigenspace* of A belonging to λ.

Example 7.1.2. (A 2 × 2 Matrix with Two Eigenvalues and Two One-Dimensional Eigenspaces). Find the eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}. \tag{7.20}$$

The corresponding characteristic equation is

$$|A - \lambda I| = \det\left(\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = \begin{vmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} = 0. \tag{7.21}$$

Expanding the determinant, we get

$$(1-\lambda)^2 - 2^2 = 0, \tag{7.22}$$

[1] While of paramount theoretical importance, for large values of n this method is hopelessly inefficient. There exist fast, approximate methods for finding eigenvalues; we shall discuss some of them in Chapter 8.

$$(1-\lambda-2)(1-\lambda+2)=0, \tag{7.23}$$

or

$$(\lambda+1)(\lambda-3)=0. \tag{7.24}$$

Thus the eigenvalues are $\lambda_1 = -1$ and $\lambda_2 = 3$.

To find the eigenvectors we substitute these eigenvalues, one at a time, into Equation 7.18, and solve for the unknown vector $\mathbf{s}$. Let us start with $\lambda_1 = -1$. Then Equation 7.18 becomes

$$\begin{bmatrix} 1-(-1) & 2 \\ 2 & 1-(-1) \end{bmatrix} \mathbf{s} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \mathbf{s} = \mathbf{0}. \tag{7.25}$$

The solutions of this equation are of the form $\mathbf{s} = s(1,-1)^T$. Thus the eigenvectors belonging to the eigenvalue $\lambda_1 = -1$ form a one-dimensional subspace with basis vector $\mathbf{s}_1 = (1,-1)^T$.

Substituting $\lambda_2 = 3$ into Equation 7.18 we obtain

$$\begin{bmatrix} 1-3 & 2 \\ 2 & 1-3 \end{bmatrix} \mathbf{s} = \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} \mathbf{s} = \mathbf{0}. \tag{7.26}$$

The solutions of this equation are all the multiples of $\mathbf{s}_2 = (1,1)^T$. ♦

In the example above we had two one-dimensional eigenspaces, but in others they can be of higher dimensions as in the following example.

Example 7.1.3. (A 3×3 Matrix with Two Eigenvalues and a One-Dimensional and a Two-Dimensional Eigenspace). Find the eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{7.27}$$

The corresponding characteristic equation is

$$|A-\lambda I| = \begin{vmatrix} 3-\lambda & 0 & 1 \\ 0 & 3-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0. \tag{7.28}$$

Expanding the determinant, we get

$$(\lambda-1)(\lambda-3)^2 = 0. \tag{7.29}$$

The solutions are $\lambda_1 = 1$ and $\lambda_2 = 3$. Since $\lambda - 3$ occurs squared in Equation 7.29, we call $\lambda_2 = 3$ a *double eigenvalue* and we say that its (*algebraic*) *multiplicity* is 2.

Substituting $\lambda_1 = 1$ into Equation 7.18, we obtain

$$\begin{bmatrix} 3-1 & 0 & 1 \\ 0 & 3-1 & 0 \\ 0 & 0 & 1-1 \end{bmatrix} \mathbf{s} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mathbf{s} = \mathbf{0}. \tag{7.30}$$

The solutions of this equation are all the multiples of $\mathbf{s}_1 = (-1, 0, 2)^T$.

Substituting $\lambda_2 = 3$ into Equation 7.18, we obtain

$$\begin{bmatrix} 3-3 & 0 & 1 \\ 0 & 3-3 & 0 \\ 0 & 0 & 1-3 \end{bmatrix} \mathbf{s} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix} \mathbf{s} = \mathbf{0}. \tag{7.31}$$

The solutions of this equation are all the linear combinations of $\mathbf{s}_2 = (1, 0, 0)^T$ and $\mathbf{s}_3 = (0, 1, 0)^T$, which thus form a two-dimensional eigenspace. ♦

As we have seen, for an $n \times n$ matrix the characteristic equation is of degree n, and as such, according to a theorem known as the Fundamental Theorem of Algebra, it always has n roots, provided we count multiplicities and allow complex numbers. Hence, in principle at least, the characteristic equation can be reformulated as

$$(\lambda - \lambda_1)^{k_1}(\lambda - \lambda_2)^{k_2} \cdots (\lambda - \lambda_r)^{k_r} = 0, \tag{7.32}$$

where r is the number of distinct roots, k_i is the algebraic multiplicity of the root λ_i, and

$$k_1 + k_2 + \cdots + k_r = n. \tag{7.33}$$

The case of complex solutions is very important in many applications—for instance, rotation matrices have complex eigenvalues—and we shall discuss this case in Section 7.3. Also, the solution of higher degree equations is generally very difficult, and for $n > 4$ it cannot even be done in a finite number of algebraic steps except in some special cases. We want to avoid such difficulties and shall only consider examples in which the factorization of the characteristic polynomial is easy.

The foregoing examples seem to suggest that not only do we always have n eigenvalues, but that the sum of the dimensions of the eigenspaces is n. Unfortunately, this is not true in general, as the next example shows.

Example 7.1.4. (A Defective 2×2 Matrix with an Algebraically Double Eigenvalue and a One-Dimensional Eigenspace). Find the eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 4 & 1 \\ -1 & 2 \end{bmatrix}. \tag{7.34}$$

The corresponding characteristic equation is

$$|A - \lambda I| = \det\left(\begin{bmatrix} 4 & 1 \\ -1 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = \begin{vmatrix} 4-\lambda & 1 \\ -1 & 2-\lambda \end{vmatrix} = 0. \quad (7.35)$$

Expanding the determinant, we get

$$(4 - \lambda)(2 - \lambda) + 1 = 0, \quad (7.36)$$

$$\lambda^2 - 6\lambda + 9 = 0, \quad (7.37)$$

and so

$$(\lambda - 3)^2 = 0. \quad (7.38)$$

The only solution of this equation is $\lambda_1 = 3$, which is thus a double eigenvalue.

Substituting $\lambda_1 = 3$ into Equation 7.18, we obtain

$$\begin{bmatrix} 4-3 & 1 \\ -1 & 2-3 \end{bmatrix} \mathbf{s} = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \mathbf{s} = \mathbf{0}.$$

The solutions of this equation are all the multiples of $\mathbf{s}_1 = (1, -1)^T$, and so the sole eigenspace is one dimensional, although $n = 2$. We say that the matrix A is *defective*, and that the *geometric multiplicity* of λ_1 is 1 while its algebraic multiplicity is 2. ♦

In general there is no way to predict whether a matrix is defective or not. We have to compute the eigenvalues and eigenvectors and see. However, there are some important special cases in which we can be assured of a "full set" of eigenvectors. We now discuss two of these.

Theorem 7.1.1. ***(If A Has n Distinct Eigenvalues, Then Each One Has a One-Dimensional Eigenspace).*** *If the $n \times n$ matrix A has n eigenvalues of algebraic multiplicity 1 each, then there is a one-dimensional eigenspace for each eigenvalue. If $\mathbf{s}_i$ is any nonzero eigenvector in the ith eigenspace, for $i = 1, 2, \ldots, n$, then the vectors $\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_n$ are linearly independent and form a basis for $\mathbb{R}^n$.*

Proof. If λ_i is any eigenvalue of A, then by Definition 7.1.1 the equation

$$(A - \lambda_i I)\mathbf{s}_i = \mathbf{0} \quad (7.39)$$

has a nontrivial solution $\mathbf{s}_i$. Thus there is associated with each eigenvalue an eigenspace of dimension at least 1.

To see that the $\mathbf{s}_i$ vectors are independent, let $c_1, c_2, \ldots, c_n$ be scalars such that

$$c_1\mathbf{s}_1 + c_2\mathbf{s}_2 + \cdots + c_n\mathbf{s}_n = \mathbf{0}. \quad (7.40)$$

Multiply both sides of this equation by $A - \lambda_i I$. Then, by Equation 7.39, the ith term will be annihilated. If we then also multiply by $A - \lambda_j I$, then the jth term will also go away, because

$$\begin{aligned}(A - \lambda_j I)(A - \lambda_i I)\mathbf{s}_j &= A^2 - \lambda_j A - \lambda_i A - \lambda_j \lambda_i I \\ &= (A - \lambda_i I)(A - \lambda_j I)\mathbf{s}_j = (A - \lambda_i I)\mathbf{0} = \mathbf{0}. \quad (7.41)\end{aligned}$$

Continuing in this fashion, we can annihilate all terms of Equation 7.40 but one. Say, we keep the kth term. Then we are left with

$$\prod_{i=1, i\neq k}^{n} (A - \lambda_i I)c_k \mathbf{s}_k = \mathbf{0}. \quad (7.42)$$

Using the fact that $\mathbf{s}_k \neq \mathbf{0}$ is an eigenvector of A corresponding to the eigenvalue λ_k, we have

$$(A - \lambda_i I)c_k \mathbf{s}_k = c_k (A\mathbf{s}_k - \lambda_i \mathbf{s}_k) = c_k (\lambda_k \mathbf{s}_k - \lambda_i \mathbf{s}_k) = c_k(\lambda_k - \lambda_i)\mathbf{s}_k. \quad (7.43)$$

Successive application of this result to each factor in Equation 7.42 transforms it into

$$c_k \prod_{i=1, i\neq k}^{n} (\lambda_k - \lambda_i)\mathbf{s}_k = \mathbf{0}. \quad (7.44)$$

The assumption that each eigenvalue has multiplicity one implies that $\lambda_k - \lambda_i \neq 0$ for any $i \neq k$, and so $c_k = 0$ must hold for every k. This proves the independence of the $\mathbf{s}_k$ vectors.

The only statement left to prove is that each eigenspace is one dimensional. This follows from the fact that the n vectors $\mathbf{s}_k$, being independent, form a basis for $\mathbb{R}^n$. Consequently, if any eigenspace were of dimension greater than one, then it would contain a second basis vector, which would also be independent of the other eigenvectors, resulting in either $n + 1$ independent vectors in $\mathbb{R}^n$, or in overlapping eigenspaces. Both results are impossible, because, by Exercise 3.4.13, in $\mathbb{R}^n$, every set of more than n vectors is a dependent set, and, by Exercise 7.1.19, no nonzero eigenvector can belong to two different eigenvalues. ■

If the matrix A is symmetric, we can say even more.

Theorem 7.1.2. ***(The Eigenspaces of a Symmetric Matrix Are Orthogonal to Each Other).*** *Any two eigenvectors of a symmetric matrix that correspond to different eigenvalues are orthogonal to each other.*

Proof. Let $\mathbf{s}_1$ and $\mathbf{s}_2$ be two eigenvectors of the symmetric matrix A that correspond to the eigenvalues λ_1 and λ_2 respectively, with $\lambda_1 \neq \lambda_2$. Then using the fact that $\mathbf{s}_2^T A\mathbf{s}_1$ is a scalar and that $A^T = A$, we have

$$\mathbf{s}_2^T A\mathbf{s}_1 = (\mathbf{s}_2^T A\mathbf{s}_1)^T = \mathbf{s}_1^T A^T\mathbf{s}_2 = \mathbf{s}_1^T A\mathbf{s}_2 \tag{7.45}$$

and hence

$$\lambda_1\mathbf{s}_2^T\mathbf{s}_1 = \lambda_2\mathbf{s}_1^T\mathbf{s}_2. \tag{7.46}$$

Since $\mathbf{s}_2^T\mathbf{s}_1 = \mathbf{s}_1^T\mathbf{s}_2$ (each being just a different expression of the dot product), and since $\lambda_1 \neq \lambda_2$, Equation 7.46 implies that $\mathbf{s}_2^T\mathbf{s}_1 = 0$, that is, that the vectors $\mathbf{s}_1$ and $\mathbf{s}_2$ are orthogonal to each other. ■

In Section 7.3 we are going to prove that for every $n\times n$ symmetric matrix there actually exists an orthonormal set of eigenvectors spanning $\mathbb{R}^n$.

Exercises

In the first eight exercises find all eigenvalues and associated eigenvectors for the given matrices.

Exercise 7.1.1. $A = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}$.

Exercise 7.1.2. $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$.

Exercise 7.1.3. $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

Exercise 7.1.4. $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$.

Exercise 7.1.5. $A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$.

Exercise 7.1.6. $A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

Exercise 7.1.7. $A = \begin{bmatrix} 1 & 0 & -1 \\ -2 & 3 & -1 \\ -6 & 6 & 0 \end{bmatrix}$.

Exercise 7.1.8. $A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$.

Exercise 7.1.9. Can you find a relationship between the eigenvalues and eigenvectors of a square matrix A and those of the matrix cA, where c is any scalar?

Exercise 7.1.10. Can you find a relationship between the eigenvalues and eigenvectors of a square matrix A and those of the matrix $A + cI$, where c is any scalar?

Exercise 7.1.11. Prove that a square matrix is singular if and only if one of its eigenvalues is zero.

Exercise 7.1.12. Prove that if $\mathbf{s}$ is an eigenvector of a matrix A, then it is also an eigenvector of the matrix A^2. How are the associated eigenvalues related?

Exercise 7.1.13. Prove that if $\mathbf{s}$ is an eigenvector of a nonsingular matrix A, then it is also an eigenvector of the matrix A^{-1}. How are the associated eigenvalues related?

Exercise 7.1.14. Show that every square matrix A and its transpose A^T have the same eigenvalues.

Exercise 7.1.15. Let $\mathbf{u}$ be any unit vector in $\mathbb{R}^n$. Show that the matrix $A = \mathbf{u}\mathbf{u}^T$ represents the projection onto the line of $\mathbf{u}$ and find its eigenvalues and eigenspaces.

***Exercise 7.1.16.** Find the eigenvalues and eigenspaces of any projection matrix P.

Exercise 7.1.17. A row vector $\mathbf{s}^T$ is called a *left eigenvector* of a matrix A belonging to the eigenvalue λ if the equation $\mathbf{s}^T A = \lambda \mathbf{s}^T$ holds. Show that $\mathbf{s}^T$ is a left eigenvector of a matrix A belonging to the eigenvalue λ if and only if $\mathbf{s}$ is an eigenvector of A^T belonging to the eigenvalue λ.

Exercise 7.1.18. Show that if $\mathbf{u}$ and $\mathbf{v}$ are eigenvectors belonging to different eigenvalues of a matrix A and its transpose A^T respectively, then they are orthogonal to each other.

Exercise 7.1.19. Prove that no nonzero eigenvector can belong to two different eigenvalues.

MATLAB Exercises

The MATLAB command $\mathbf{c} = \mathbf{poly}(A)$ returns the coefficients in descending order of the characteristic polynomial of the matrix A, and the command $\mathbf{d} = \mathbf{roots}(\mathbf{c})$ returns the roots of this polynomial, that is, the eigenvalues of the matrix A. In the following exercises use these and the command $A\backslash\mathbf{b}$ to solve the appropriate linear systems to find the eigenvalues and eigenvectors of each matrix A. Also, compare these results to the solutions obtained by using the MATLAB routine $\mathbf{eig}(A)$.

Exercise 7.1.20. Let A be the matrix of Exercise 7.1.5.

Exercise 7.1.21. Let A be the matrix of Exercise 7.1.8.

Exercise 7.1.22. $A = \mathbf{hilb}(3)$.

Exercise 7.1.23. $A = \mathbf{hilb}(4)$.

Exercise 7.1.24. $A = \mathbf{ones}(3)$.

Exercise 7.1.25. $A = \mathbf{ones}(4)$.

Exercise 7.1.26. What conjectures can you make about the eigenvalues and eigenvectors of matrices consisting of ones, on the basis of the last two exercises? Can you prove these conjectures? (Notice that every such matrix is n times a projection matrix.)

Exercise 7.1.27. $A = \mathbf{hadamard}(4)$.

Exercise 7.1.28. $A = \mathbf{hadamard}(8)$.

Exercise 7.1.29. Orthogonal matrices with all entries equal to ± 1 are called Hadamard matrices. What conjectures can you make about their eigenvalues and eigenvectors on the basis of the last two exercises? Can you prove any of them?

7.2 Diagonalization of Matrices

If an $n \times n$ matrix A has n linearly independent eigenvectors $\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_n$, then we may use them as a basis for $\mathbb{R}^n$. Writing S for the matrix with these vectors as columns in the order indicated, that is, $S = (\mathbf{s}_1\ \mathbf{s}_2\ \ldots\ \mathbf{s}_n)$, we find

$$AS = (A\mathbf{s}_1\ A\mathbf{s}_2\ \ldots\ A\mathbf{s}_n) = (\lambda_1\mathbf{s}_1\ \lambda_2\mathbf{s}_2\ \ldots\ \lambda_n\mathbf{s}_n) = S\Lambda, \tag{7.47}$$

where

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \tag{7.48}$$

is the diagonal matrix whose diagonal entries are the eigenvalues corresponding to the columns of S in the same order. Since the columns of S form a basis for $\mathbb{R}^n$, the matrix S must be invertible and, multiplying Equation 7.47 by S^{-1}, we obtain

$$S^{-1}AS = \Lambda. \tag{7.49}$$

Now the left side of this equation is, by Theorem 3.6.3 on page 156, the matrix A_S that represents A in the basis S. In the terminology of Section 3.6, the matrices A and Λ are similar and Equation 7.49 specifies the similarity transformation connecting them.

We call the above process the *diagonalization* of A, and S a *diagonalizing matrix* for A. As is obvious from the foregoing discussion, the diagonalizing matrix S for a given diagonalizable A is not unique: the eigenvectors may be permuted, multiplied by arbitrary nonzero scalars, and linearly combined within higher dimensional eigenspaces if there are any. However, this is all the latitude we are permitted: the columns of every diagonalizing S must be independent because we need S^{-1} and they must be eigenvectors of A. Consequently, A is diagonalizable if and only if it has a full set of n linearly independent eigenvectors. Note that this requirement places no restriction on the multiplicities of the eigenvalues but, by Theorem 7.1.1, every matrix with distinct eigenvalues is diagonalizable.

Equation 7.49 has the useful consequence that A^k and Λ^k, for every positive integer k, are related by the same similarity transformation as were A and Λ:

$$\Lambda^k = S^{-1}ASS^{-1}AS \ \cdots \ S^{-1}AS = S^{-1}AIAI \ \cdots \ AS = S^{-1}A^kS. \tag{7.50}$$

The powers of Λ are very easy to compute:

$$\Lambda^k = \begin{bmatrix} \lambda_1^k & 0 & \cdots & 0 \\ 0 & \lambda_2^k & \cdots & 0 \\ & & \cdots & \\ 0 & 0 & \cdots & \lambda_n^k \end{bmatrix}, \tag{7.51}$$

and then A^k can be recovered as

$$A^k = S\Lambda^k S^{-1}. \tag{7.52}$$

Let us summarize the preceding discussion in a theorem.

Theorem 7.2.1. ***(Diagonalization).*** *If an $n \times n$ matrix A has n linearly independent eigenvectors, then, writing S for the matrix with these vectors as columns, we find that A is similar with transition matrix S to a diagonal matrix Λ, whose diagonal entries are the eigenvalues of A corresponding to the columns of S. The same similarity transforms A^k to Λ^k for every positive integer k.*

As mentioned in the introduction of Section 7.1, the simplification resulting from the change to a basis of eigenvectors is the main reason for their usefulness. In matrix form this simplification means that instead of multiplying by the general matrix A^k, which is usually very difficult to compute, in the new basis we multiply by the very simple matrix Λ^k. Let us now look at a concrete example.

Example 7.2.1. (Population Growth). Let us assume that a certain town was originally settled by 100 people, all under fifty years old. We want to investigate how the age distribution changes over a long time between two age groups: under fifty, and fifty or older. We must, of course, make some quantitative assumptions about the changes from one time period to the next. So let us say that over any decade, on the one hand, there is a net increase of 10% in the under fifty population, and on the other hand, 20% of the under fifty population becomes fifty or older, while 40% of the initial over fifty population dies. Denoting the number of people under fifty in the nth decade by $x_1(n)$ and the number of those fifty or over by $x_2(n)$, for $n = 0, 1, 2, \ldots$, we can write the following equations:

$$x_1(0) = 100, \tag{7.53}$$

$$x_2(0) = 0, \tag{7.54}$$

$$x_1(n+1) = 1.1x_1(n), \tag{7.55}$$

$$x_2(n+1) = 0.2x_1(n) + 0.6x_2(n). \tag{7.56}$$

In matrix form these equations become

$$\mathbf{x}(0) = \begin{bmatrix} 100 \\ 0 \end{bmatrix} \tag{7.57}$$

and

$$\mathbf{x}(n+1) = A\mathbf{x}(n), \tag{7.58}$$

where

$$A = \begin{bmatrix} 1.1 & 0 \\ 0.2 & 0.6 \end{bmatrix}. \tag{7.59}$$

Substituting $n = 0, 1, 2, \ldots$ into Equation 7.58, we find

$$\mathbf{x}(1) = A\mathbf{x}(0),\ \mathbf{x}(2) = A\mathbf{x}(1) = A^2\mathbf{x}(0), \ldots \tag{7.60}$$

and so

$$\mathbf{x}(n) = A^n\mathbf{x}(0) \tag{7.61}$$

for every positive integer n.

We want to diagonalize A to compute A^n here.

The characteristic equation for this A is

$$|A - \lambda I| = \begin{vmatrix} 1.1-\lambda & 0 \\ 0.2 & 0.6-\lambda \end{vmatrix} = (1.1-\lambda)(0.6-\lambda) = 0. \tag{7.62}$$

The solutions are $\lambda_1 = 1.1$ and $\lambda_2 = 0.6$. The corresponding eigenvectors can be found by substituting these values into $(A - \lambda I)\mathbf{s} = \mathbf{0}$:

$$(A - \lambda_1 I)\mathbf{s}_1 = \begin{bmatrix} 1.1 - 1.1 & 0 \\ 0.2 & 0.6 - 1.1 \end{bmatrix} \mathbf{s}_1 = \begin{bmatrix} 0 & 0 \\ 0.2 & -0.5 \end{bmatrix} \mathbf{s}_1 = \mathbf{0}. \tag{7.63}$$

A solution of this equation is $\mathbf{s}_1 = (5, 2)^T$. For the other eigenvector we have the equation

$$(A - \lambda_2 I)\mathbf{s}_2 = \begin{bmatrix} 1.1 - 0.6 & 0 \\ 0.2 & 0.6 - 0.6 \end{bmatrix} \mathbf{s}_2 = \begin{bmatrix} 0.5 & 0 \\ 0.2 & 0 \end{bmatrix} \mathbf{s}_2 = \mathbf{0}. \tag{7.64}$$

A solution of this equation is $\mathbf{s}_2 = (0, 1)^T$.

Thus

$$S = \begin{bmatrix} 5 & 0 \\ 2 & 1 \end{bmatrix}, \tag{7.65}$$

$$\Lambda = \begin{bmatrix} 1.1 & 0 \\ 0 & 0.6 \end{bmatrix}, \tag{7.66}$$

and

$$S^{-1} = \frac{1}{5} \begin{bmatrix} 1 & 0 \\ -2 & 5 \end{bmatrix}. \tag{7.67}$$

According to Corollary 3.6.1 on page 149, the coordinate vectors of each $\mathbf{x}(n)$ for $n = 0, 1, \ldots$, relative to the basis S, are given by

$$\mathbf{x}_S(0) = S^{-1}\mathbf{x}(0) = \frac{1}{5} \begin{bmatrix} 1 & 0 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} 100 \\ 0 \end{bmatrix} = \begin{bmatrix} 20 \\ -40 \end{bmatrix} \tag{7.68}$$

and

$$\begin{aligned} \mathbf{x}_S(n) &= S^{-1}\mathbf{x}(n) = S^{-1}A^n\mathbf{x}(0) = S^{-1}A^nS\mathbf{x}_S(0) \\ &= \Lambda^n \mathbf{x}_S(0) = \begin{bmatrix} 1.1^n & 0 \\ 0 & 0.6^n \end{bmatrix} \begin{bmatrix} 20 \\ -40 \end{bmatrix} = \begin{bmatrix} 20 \cdot 1.1^n \\ -40 \cdot 0.6^n \end{bmatrix}. \end{aligned} \tag{7.69}$$

Hence the solution in the standard basis is given by

$$\mathbf{x}(n) = S\mathbf{x}_S(n) = \begin{bmatrix} 5 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 20 \cdot 1.1^n \\ -40 \cdot 0.6^n \end{bmatrix} = \begin{bmatrix} 100 \cdot 1.1^n \\ 40 \cdot (1.1^n - 0.6^n) \end{bmatrix}. \tag{7.70}$$

For large values of n the term 0.6^n can be neglected, and we get

$$\mathbf{x}(n) \approx 1.1^n \cdot \begin{bmatrix} 100 \\ 40 \end{bmatrix}. \tag{7.71}$$

Thus, in the long run both the under fifty and the over fifty populations will increase 10% per decade, and there will be 40 people over fifty for every 100 under fifty. ♦

A system of equations like the one in the example above is called a system *of first order linear difference equations.* Equation 7.58 shows their general vector form. The word "difference" indicates that the equation involves an unknown function that occurs at different values of an integer-valued variable n, and "first order" refers to the fact that only n and $n+1$ occur and no additional $n+2$ and such. Such equations can always be solved by the method shown, provided the matrix A is diagonalizable.

Systems of *first order linear differential equations*, as defined below, with constant, diagonalizable matrices can also be solved in a similar manner.

Suppose we are to find a vector-valued function $\mathbf{u}$ of a scalar variable t such that

$$\frac{d\mathbf{u}}{dt} = A\mathbf{u}. \tag{7.72}$$

Here $\mathbf{u}(t)$ is in $\mathbb{R}^n$ for every real t, and A is a constant, diagonalizable $n \times n$ matrix. The derivative is defined componentwise, that is, $\mathbf{u}'(t) = (u_1'(t), \ldots, u_n'(t))$. Equation 7.72 is equivalent to the following system of first order linear differential equations:

$$\frac{du_i}{dt} = \sum_{i=1}^{n} a_{ij}u_j \text{ for } i = 1, 2, \ldots, n. \tag{7.73}$$

Let us consider a change of basis such that $\mathbf{u} = S\mathbf{v}$ for some constant, invertible $n \times n$ matrix S and $\mathbf{v}(t) \in \mathbb{R}^n$ for every real t. Then

$$\frac{dS\mathbf{v}}{dt} = AS\mathbf{v} \tag{7.74}$$

and

$$\frac{d\mathbf{v}}{dt} = S^{-1}AS\mathbf{v}. \tag{7.75}$$

If S diagonalizes A, so that $S^{-1}AS = \Lambda$ is a diagonal matrix, then Equation 7.75 becomes

$$\frac{d\mathbf{v}}{dt} = \Lambda\mathbf{v}, \tag{7.76}$$

which can be written in components as

$$\frac{dv_i}{dt} = \lambda_i v_i \text{ for } i = 1, 2, \ldots, n. \tag{7.77}$$

In contrast to Equations 7.73, *each equation here contains only one of the unknown functions*, and can therefore easily be solved as follows.

For each i and all t such that $v_i(t) \neq 0$ we rewrite Equations 7.77 as

$$\frac{1}{v_i}\frac{dv_i}{dt} = \lambda_i, \tag{7.78}$$

or, equivalently, as

$$\frac{d\ln|v_i|}{dt} = \lambda_i. \tag{7.79}$$

Hence,

$$\ln|v_i| = \lambda_i t + c_i, \tag{7.80}$$

where each c_i is an arbitrary constant. Thus the general solution of Equation 7.78 is

$$v_i = C_i e^{\lambda_i t}, \tag{7.81}$$

where $C_i = \pm e^{c_i}$. Allowing $C_i = 0$ too, we thus have the general solution of Equations 7.77 as well. From Equation 7.81 we obtain the general solution of the original Equation 7.72 by calculating $\mathbf{u} = S\mathbf{v}$.

Example 7.2.2. (An Electric Circuit with Resistor, Condenser, and Coil). In physics it is shown that the electric circuit of Figure 7.2 is governed by the equation

$$Ri + \frac{1}{C}q + L\frac{di}{dt} = E \tag{7.82}$$

where $R,\ L,\ C$ are positive numbers denoting the resistance, the inductance, and the capacitance of the indicated elements, $E(t)$ is the applied electromotive force or voltage, $q(t)$ is the charge on the capacitor at time t, and $i(t) = q'(t)$ is the current at time t.

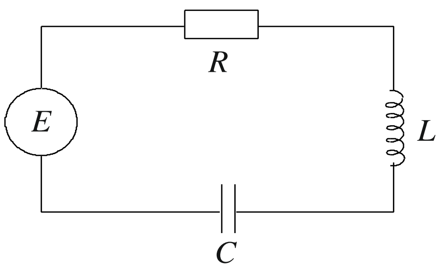

Fig. 7.2. An electric circuit

Equation 7.82 is an inhomogeneous differential equation, which means that it contains a term not involving the unknown functions i and q. The general solution of an inhomogeneous linear differential equation or system can be obtained, just as for ordinary equations (see Theorem 2.3.3 on page 59), by adding any particular solution of it to the general solution of the corresponding homogeneous equation or system. Thus, it is preferable to consider the latter first, by setting $E = 0$. This condition has a physical meaning as well: For instance, $E(t)$ can be a pulse, which is nonzero for some time, but becomes zero afterward, while the circuit is closed. During the latter period the homogeneous equation rules, but because of the initial pulse it may very well have nonzero solutions.

We have two unknown functions i and q and two equations: One is Equation 7.82 and the other the equation $i(t) = q'(t)$. We can write the homogeneous system in the form corresponding to the general case of Equations 7.73 as

$$\frac{di}{dt} = -\frac{R}{L}i - \frac{1}{LC}q \tag{7.83}$$

and

$$\frac{dq}{dt} = i. \tag{7.84}$$

Thus, in this case, the matrix A is of the form

$$A = \begin{bmatrix} -R/L & -1/LC \\ 1 & 0 \end{bmatrix}. \tag{7.85}$$

The characteristic equation is

$$\left(-\frac{R}{L} - \lambda\right)(-\lambda) + \frac{1}{LC} = 0 \tag{7.86}$$

or equivalently

$$\lambda^2 + \frac{R}{L}\lambda + \frac{1}{LC} = 0. \tag{7.87}$$

The discriminant of the last equation is

$$D = \frac{R^2}{L^2} - \frac{4}{LC}, \tag{7.88}$$

and if $D \neq 0$, then Equation 7.87 has the two solutions

$$\lambda_1 = \frac{1}{2}\left(-\frac{R}{L} - \sqrt{D}\right) \tag{7.89}$$

and

$$\lambda_2 = \frac{1}{2}\left(-\frac{R}{L} + \sqrt{D}\right). \tag{7.90}$$

The next step would be to compute the corresponding eigenvectors, but it is easier to avoid it by proceeding as follows.

First, observe that in the general formulation each u_k is a linear combination of the v_k, which are given by Equation 7.81. In the present case $u_1 = i$ and $u_2 = q$, and so the general solution must be of the form

$$i(t) = c_{11}e^{\lambda_1 t} + c_{12}e^{\lambda_2 t} \tag{7.91}$$

and

$$q(t) = c_{21}e^{\lambda_1 t} + c_{22}e^{\lambda_2 t} \tag{7.92}$$

with unknown coefficients c_{jk}. Substituting these expressions of $i(t)$ and $q(t)$ into Equations 7.83 and 7.84, we get two equations for the unknown c_{jk}, and we can prescribe two initial conditions, that is, values for $i(0)$ and $q(0)$, to have the necessary four equations for the four unknowns. For example, if there is no current at $t = 0$ but there is a charge Q on the capacitor, then the initial conditions are

$$i(0) = 0 \tag{7.93}$$

and

$$q(0) = Q, \tag{7.94}$$

and the coefficients turn out to be (Exercise 7.2.10)

$$c_{11} = -c_{12} = \frac{Q}{LC\sqrt{D}} \tag{7.95}$$

and

$$c_{21} = Q - c_{22} = \frac{Q\lambda_2}{\sqrt{D}}. \tag{7.96}$$

Notice that, if $D > 0$ holds, then both λ_1 and λ_2 are negative, and so both $i(t)$ and $q(t)$ decay to zero as $t \to \infty$. However, since $|\lambda_1| > |\lambda_2|$ holds, for large t the $e^{\lambda_1 t}$ terms will be negligible next to the $e^{\lambda_2 t}$ terms, and so the approach to zero will be like the latter.

The case $D = 0$ has to be treated differently, because then the matrix A is defective. We suggest a possible approach in Exercise 7.2.12d.

The remaining case of $D < 0$ will be treated in Section 7.4. This case is very important, since it occurs in many circuits and leads to oscillating solutions that are entirely different from the foregoing ones.

The discussion of the inhomogeneous case is left to other courses. ♦

Exercises

Exercise 7.2.1. Prove that a diagonalizable matrix A is invertible if and only if all of its eigenvalues are different from zero, and in that case Equations 7.51 and 7.52 are valid for all negative integers k as well.

Exercise 7.2.2. Define A^x for every real x and every diagonalizable A with positive eigenvalues.

Exercise 7.2.3. Show that if A is symmetric and diagonalizable with non-negative eigenvalues, then $\sqrt{A}$ exists and is also symmetric.

Exercise 7.2.4. Find A^{100} and $A^{1/2}$ for the matrix

$$A = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}.$$

Exercise 7.2.5. Find A^{100} for the matrix A of Exercise 7.1.5:

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}.$$

Exercise 7.2.6. Find A^4 for the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

of Exercise 7.1.8, using the eigenvalues and eigenvectors of A.

Exercise 7.2.7. Prove that similar matrices have the same characteristic polynomial, that is, if A and B are similar, then $\det(A - \lambda I) = \det(B - \lambda I)$ for every λ.

Exercise 7.2.8. Let A and B be similar matrices with $B = TAT^{-1}$. Prove that $\mathbf{s}$ is an eigenvector of A belonging to the eigenvalue λ if and only if $T\mathbf{s}$ is an eigenvector of B belonging to the same eigenvalue λ.

Exercise 7.2.9. Prove the converse of Theorem 7.3.2: If A is orthogonally similar to a diagonal matrix Λ, that is, $S^{-1}AS = \Lambda$ for some orthogonal matrix S, then A must be symmetric.

Exercise 7.2.10. Prove Equations 7.95 and 7.96.

Exercise 7.2.11. In biology the following type of simplified models for predator-prey populations are sometimes considered: Assume that in a certain area the number of animals of a certain predator species is $x_1(k)$ in year k, and the number of its prey is $x_2(k)$. Furthermore, the number of predators in the next year decreases in proportion to $x_1(k)$ and increases in proportion to the available food $x_2(k)$, while the number of prey animals decreases in proportion to the number of predators and increases in proportion to their own numbers. Thus, if we ignore other factors, we may for instance assume

$$x_1(k+1) = 0.8x_1(k) + 0.4x_2(k) \tag{7.97}$$

and

$$x_2(k+1) \ = -0.8x_1(k) + 2.0x_2(k). \tag{7.98}$$

Solve these equations for all k, assuming also that initially there were 1000 animals of each kind. What happens as $k \to \infty$?

***Exercise 7.2.12.** In this exercise we outline an alternative approach to the solution of differential equations like

$$\frac{d\mathbf{u}}{dt} = A\mathbf{u}. \tag{7.99}$$

Define, for every square matrix A and every real t,

$$e^{At} = I + At + \frac{(At)^2}{2!} + \frac{(At)^3}{3!} + \cdots, \tag{7.100}$$

assuming convergence and term-by-term differentiability.
a. Show that

$$\mathbf{u}(t) = e^{At}\mathbf{u}_0 \tag{7.101}$$

is the solution of Equation 7.99 satisfying the initial condition

$$\mathbf{u}(0) = \mathbf{u}_0. \tag{7.102}$$

b. Show that if A is diagonalizable so that $A = S\Lambda S^{-1}$, then

$$e^{At} = Se^{\Lambda t}S^{-1}, \tag{7.103}$$

where

$$e^{\Lambda t} = \begin{bmatrix} e^{\lambda_1 t} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2 t} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & e^{\lambda_n t} \end{bmatrix} \tag{7.104}$$

and Equation 7.101 becomes

$$\mathbf{u}(t) = Se^{\Lambda t}S^{-1}\mathbf{u}_0 = \sum_{k=1}^{n} c_k e^{\lambda_k t}, \tag{7.105}$$

where the c_k are appropriate constant vectors.

c. Use the above formalism to solve Equations 7.83 and 7.84 with $R = 5$, $L = 1$, $C = 1/4$, and $\mathbf{u}_0 = (0, 10)^T$. Plot the graphs of the solution for i and q.
d. Use Equations 7.100 and 7.101 to solve Equations 7.83 and 7.84 with $R = 2$, $L = 1$, $C = 1$, and $\mathbf{u}_0 = (0, 10)^T$ Plot the graphs of the solution for i and q.

MATLAB Exercises

Exercise 7.2.13. Consider the problem of Example 7.2.1 again.
a. Enter the matrix A from Equation 7.59 and use MATLAB to verify that Equation 7.61 leads to Equation 7.70.
b. Experiment with different death rates for the over fifty population, in place of the given 40%, to see what rates would lead to eventual extinction and what rate would lead to a steady population in the long run.
c. For what death rate would one of the eigenvalues equal 1? Compare this A to those examined in Part b and explain.

Exercise 7.2.14. In Exercise 7.2.11 the coefficient $r = -0.8$ represents the predation rate, that is, the number of prey caught per predator per month. Experiment with different values of r to find one for which there is a stable limiting population. What is the split between the two kinds of animals in the limit for this r as $k \to \infty$?

Exercise 7.2.15. If A is an $m \times r$ matrix and has rank r, then AA^T is an $m \times m$ symmetric matrix of rank r. (Why?)
a. Use the fact above to generate random symmetric matrices of ranks 1, 2, and 3 for $m = 4, 5, 6$.
b. Use **eig** to find the eigenvalues of each matrix generated in Part a and note the multiplicity of the eigenvalue 0.
c. Make a conjecture about the dependence of the multiplicity of the eigenvalue 0 on m and r, and prove it.

7.3 Principal Axes

We return now to theoretical considerations and discuss the diagonalization of symmetric matrices mentioned at the end of Section 7.1. We consider this topic because it is important in many applications and fairly easy to prove, while the general case lies beyond our scope.

First, however, we state a theorem whose proof is relegated to Section 7.4 because it requires complex numbers.

Theorem 7.3.1. ***(Real Eigenvalues).*** *The eigenvalues of a symmetric matrix are real.*

The next theorem contains the main result of this section and is variously called the *Principal Axis Theorem* and the *Spectral Theorem for Symmetric Matrices.* These names come from applications of the theorem to the determination of the principal axes of ellipsoids and the color spectra of light sources. For the same reason the set of eigenvalues of every matrix is called its *spectrum.*

Theorem 7.3.2. ***(Principal Axis Theorem).*** *For every symmetric matrix A there exists an orthogonal matrix S such that $S^{-1}AS = \Lambda$ is a diagonal matrix. The columns of S are eigenvectors of A, and the diagonal entries of Λ are the corresponding eigenvalues.*

Proof. Every $n \times n$ matrix A has at least one eigenvalue because its characteristic equation must have at least one solution according to the Fundamental Theorem of Algebra. Call such an eigenvalue λ_1. By Theorem 7.3.1, λ_1 is real and so there must exist a corresponding real unit eigenvector $\mathbf{s}_1$. The Gram–Schmidt algorithm guarantees that we can construct an $n \times n$ orthogonal matrix S_1 whose first column is $\mathbf{s}_1$. For such an S_1, $S_1^{-1} = S_1^T$, and so the first column of $S_1^{-1}AS_1$ is given by

$$S_1^{-1}A\mathbf{s}_1 = \lambda_1 S_1^{-1}\mathbf{s}_1 = \lambda_1 S_1^T\mathbf{s}_1 = \lambda_1\mathbf{e}_1. \tag{7.106}$$

Furthermore, $S_1^{-1}AS_1$ is also symmetric since

$$(S_1^{-1}AS_1)^T = (S_1^TAS_1)^T = S_1^TA^T\left(S_1^T\right)^T = S_1^{-1}AS_1. \tag{7.107}$$

Thus $S_1^{-1}AS_1$ has the form

$$S_1^{-1}AS_1 = \left[\begin{array}{c|ccc} \lambda_1 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & A_1 & \\ 0 & & & \end{array}\right], \tag{7.108}$$

where A_1 is an $(n-1) \times (n-1)$ symmetric matrix.

Now we can repeat the above argument with A_1 in place of A: Then A_1 has an eigenvalue λ_2 and a corresponding unit eigenvector $\mathbf{s}_2 \in \mathbb{R}^{n-1}$, and there exists an $(n-1) \times (n-1)$ orthogonal matrix S_2' with $\mathbf{s}_2$ as its first column. If we set

$$S_2 = \left[\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & S_2' & \\ 0 & & & \end{array}\right], \tag{7.109}$$

then this matrix is easily seen to be orthogonal as well, and we obtain

$$S_2^{-1}S_1^{-1}AS_1S_2 = \left[\begin{array}{cc|ccc} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ \hline 0 & 0 & & & \\ \vdots & \vdots & & A_2 & \\ 0 & 0 & & & \end{array}\right], \tag{7.110}$$

where the size of A_2 is $(n-2) \times (n-2)$.

Continuing in the same fashion, we can reduce A to a diagonal matrix Λ by applying n similarity transformations like these. Writing

$$S = S_1 S_2 \cdots S_n \tag{7.111}$$

we thus have

$$S^{-1}AS = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_n \end{bmatrix} = \Lambda. \tag{7.112}$$

The matrix S is orthogonal because it is the product of orthogonal matrices. Furthermore, from Equation 7.47 we can see that if there exists a nonsingular S that transforms A to a diagonal matrix Λ as above, then, conversely to Theorem 7.2.1, the columns of S must be eigenvectors of A corresponding to the diagonal entries of Λ as eigenvalues. ■

Before giving an example of the use of this theorem, we need some terminology.

Definition 7.3.1. ***(Form).*** *Let $n > 1$. A function from $\mathbb{R}^n$ to $\mathbb{R}$ is called a form. A form Q given by the formula $Q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$, where A is an arbitrary, symmetric $n \times n$ matrix, is called a quadratic form.*

Note that the use of a symmetric matrix A in the definition of a quadratic form involves no loss of generality. Assume that A is not necessarily symmetric. Then, since $\mathbf{x}^T A\mathbf{x}$ is a scalar, it equals its own transpose, and so, on the one hand,

$$Q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}, \tag{7.113}$$

and on the other hand,

$$Q(\mathbf{x}) = (\mathbf{x}^T A\mathbf{x})^T = \mathbf{x}^T A^T (\mathbf{x}^T)^T = \mathbf{x}^T A^T \mathbf{x}. \tag{7.114}$$

Adding Equations 7.113 and 7.114, we get

$$2Q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x} + \mathbf{x}^T A^T \mathbf{x}, \tag{7.115}$$

and so

$$Q(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T (A + A^T)\mathbf{x}. \tag{7.116}$$

Thus $Q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$ can be expressed in terms of the symmetric matrix $\frac{1}{2}(A + A^T)$ in place of the possibly nonsymmetric A, and therefore it is no restriction on Q to assume that A is symmetric to begin with.

If we make a change of basis with the orthogonal matrix S made up of eigenvectors of A, whose existence is guaranteed by Theorem 7.3.2, and write $\mathbf{x} = S\mathbf{y}$ (to make the notation simpler, we write $\mathbf{y}$ for the coordinate vector $\mathbf{x}_S$ of $\mathbf{x}$ relative to the basis S), then we get

$$\mathbf{x}^T A\mathbf{x} = \mathbf{y}^T S^T A S\mathbf{y} = \mathbf{y}^T \Lambda\mathbf{y}. \tag{7.117}$$

In component form this equation becomes

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_i x_j = \sum_{i=1}^{n} \lambda_i y_i^2, \tag{7.118}$$

and so we have transformed the general quadratic form to a sum of squares weighted with the eigenvalues. The orthonormal standard vectors $\mathbf{e}_i$ are eigenvectors of the matrix Λ, since $\Lambda\mathbf{e}_i = \lambda_i\mathbf{e}_i$. They correspond to the orthonormal eigenvectors $\mathbf{s}_i$ of the matrix A, because if $\mathbf{y} = \mathbf{e}_i$, then $\mathbf{x} = S\mathbf{y} = S\mathbf{e}_i = \mathbf{s}_i$.

We know that in $\mathbb{R}^2$ the equation $\sum_{i=1}^{2} \lambda_i y_i^2 = 1$ describes a conic section, for all values of the λ_i, and that the transformation by an orthogonal matrix is a rotation or a reflection. Therefore the equation

$$a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2 = \sum_{i=1}^{2}\sum_{j=1}^{2} a_{ij}x_i x_j = 1 \tag{7.119}$$

represents a conic section in a rotated position. The type of this conic section is determined by the eigenvalues. For instance, if $0 < \lambda_1 < \lambda_2$, then we have an ellipse. Setting $\lambda_1 = 1/a^2$ and $\lambda_2 = 1/b^2$, its equation is transformed into the standard form

$$\frac{y_1^2}{a^2} + \frac{y_2^2}{b^2} = 1. \tag{7.120}$$

The major and minor axes point in the directions of the vectors $\mathbf{y}_1 = (1,0)^T$ and $\mathbf{y}_2 = (0,1)^T$, and have half-lengths $a = 1/\sqrt{\lambda_1}$ and $b = 1/\sqrt{\lambda_2}$, respectively. (Just set successively $y_2 = 0$ and $y_1 = 0$ in Equation 7.120.) The vectors $\mathbf{y}_1$ and $\mathbf{y}_2$ correspond to the vectors $\mathbf{x}_1 = S\mathbf{y}_1 = (\mathbf{s}_1, \mathbf{s}_2)(1, 0)^T = \mathbf{s}_1$ and $\mathbf{x}_2 = S\mathbf{y}_2 = \mathbf{s}_2$ in the original basis. This result shows that the principal axes point in the directions of the eigenvectors, with the major axis corresponding to the smaller eigenvalue. Other conic sections and quadric surfaces in higher dimensions can be analyzed similarly. We consider some of them in the exercises.

Example 7.3.1. (An Ellipse). Discuss the conic section given by

$$8x_1^2 - 12x_1x_2 + 17x_2^2 = 20. \tag{7.121}$$

This equation can be written in the standard form

$$\mathbf{x}^T A\mathbf{x} = 1$$

with

$$A = \frac{1}{20}\begin{bmatrix} 8 & -6 \\ -6 & 17 \end{bmatrix}. \tag{7.122}$$

The eigenvalues and corresponding unit eigenvectors of this matrix are $\lambda_1 = 1/4$, $\lambda_2 = 1$, $\mathbf{s}_1 = \frac{1}{\sqrt{5}}(2, 1)^T$, $\mathbf{s}_2 = \frac{1}{\sqrt{5}}(-1, 2)^T$. Hence, according to the previous discussion, Equation 7.121 represents the ellipse of Figure 7.3, which is centered at the origin, whose major axis has half-length $1/\sqrt{\lambda_1} = 2$ and points in the direction of the eigenvector $\mathbf{s}_1$, and whose minor axis has half-length $1/\sqrt{\lambda_2} = 1$ and points in the direction of the eigenvector $\mathbf{s}_2$. ♦

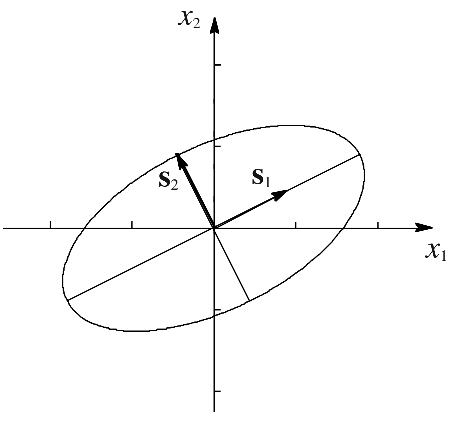

Fig. 7.3. An ellipse in nonstandard position

Exercises

Exercise 7.3.1. Find the direction and length of each principal axis of the ellipse given by the equation below, and sketch its graph.

$$13x_1^2 - 8x_1x_2 + 7x_2^2 = 30.$$

Exercise 7.3.2. Find the direction and length of each principal axis of the hyperbola given by the equation below, and sketch its graph.

$$7x_1^2 + 48x_1x_2 - 7x_2^2 = 25.$$

Exercise 7.3.3. Find the direction and length of each principal axis of the hyperbola given by the equation below, and sketch its graph.

$$2x_1^2 + 4x_1x_2 - x_2^2 = 12.$$

Exercise 7.3.4. Find the principal axes of the ellipsoid given by the equation below, change its equation to standard form so that the left side becomes a sum of squares weighted with the eigenvalues as in Equation 7.118, and describe its position and shape.

$$3x_1^2 + 3x_2^2 + 3x_3^2 - 2x_1x_2 - 2x_1x_3 - 2x_2x_3 = 4.$$

***Exercise 7.3.5.** a. Show that if A is a symmetric matrix, then $\nabla(\mathbf{x}^T A\mathbf{x}) = 2(A\mathbf{x})^T$. (We have the transpose on the right because ∇f, for every f, is usually considered to be a row vector.)
b. Use the method of Lagrange multipliers to show that the extreme values of $\mathbf{x}^T A\mathbf{x}$ subject to the constraint $\mathbf{x}^T\mathbf{x} = 1$ are eigenvalues of A. (This property of A can be developed into a practical method for computing eigenvalues.)

Exercise 7.3.6. Find the principal axes of the quadric surface given by the equation below, change its equation to standard form so that the left side becomes a sum of squares weighted with the eigenvalues as in Equation 7.118, and describe its position and shape.

$$2xy + 2xz + 2yz = 1.$$

***Exercise 7.3.7.** Prove the following theorem called Schur's lemma: For every real square matrix A with only real eigenvalues there exists an orthogonal matrix S such that $S^{-1}AS = T$ is upper triangular. (*Hint*: Modify the proof of Theorem 7.3.2 to account for the possible lack of symmetry.)

MATLAB Exercises

Exercise 7.3.8. The ellipse of Example 7.3.1, together with the corresponding one in standard position, can be plotted in MATLAB by using polar coordinates.

a. Enter the following program and explain the steps.

$\mathbf{t} = 0 : .1 : 2 * \mathbf{pi};$
$\mathbf{c} = \cos(\mathbf{t})\ ;\ \mathbf{s} = \sin(\mathbf{t});$
$\mathbf{ra} = 1./\mathbf{sqrt}(.4 * \mathbf{c}.\hat{}2 - .6 * \mathbf{c}. * \mathbf{s} + .85 * \mathbf{s}.\hat{}2);$
$\mathbf{rb} = 1./\mathbf{sqrt}(\mathbf{c}.\hat{}2/4 + \mathbf{s}.\hat{}2);$
$\mathbf{polar}(\mathbf{t}, \mathbf{ra})$
$\mathbf{hold}$
$\mathbf{polar}(\mathbf{t}, \mathbf{rb})$

b. An alternative program to plot the original ellipse directly from the matrix A is the following. Enter it and explain the steps.

$A = [8, -6; -6, 17]/20;\ \mathbf{t} = 0 :\ .2 : 2 * \mathbf{pi};$
$\mathbf{x} = [\cos(\mathbf{t}); \sin(\mathbf{t})];$
$\mathbf{q} = \mathbf{diag}(\mathbf{x}' * A * \mathbf{x})';$
$\mathbf{r} = 1./\mathbf{sqrt}(\mathbf{q});$
$\mathbf{polar}(\mathbf{t}, \mathbf{r})$

Exercise 7.3.9. Use MATLAB as in the previous exercise to solve Exercise 7.3.2.

Exercise 7.3.10. Use MATLAB to plot the conic section $\mathbf{x}^T A\mathbf{x} = 1$ with

$$A = \begin{bmatrix} 4 & 1 \\ -1 & 2 \end{bmatrix}.$$

As we have seen in Example 7.1.4 on page 258, this is a defective matrix. How can you square this fact with your result?

7.4 Complex Matrices

As we have seen in the preceding sections, complex eigenvalues may well occur even for matrices with real entries, and in some applications, e.g., in quantum physics, we must deal with complex-valued matrix components too. Consequently, we devote this section to such matters. We assume that the reader is familiar with complex numbers and exponential functions, but in an appendix at the end of the book we briefly review them.

Definition 7.4.1. ***(Complex Vector Space).*** *The complex vector space* $\mathbb{C}^n$ *is defined, for every positive integer* n*, as the set of ordered* n*-tuples* $\mathbf{z} = (z_1, z_2, \ldots, z_n)^T$ *of complex numbers, written as columns, and with addition of vectors and multiplication of vectors by scalars defined so that for all such* *vectors* $\mathbf{x} = (x_1, x_2, \dots, x_n)^T$, $\mathbf{y} = (y_1, y_2, \dots, y_n)^T$ *and all complex scalars* c *we have*

$$(x_1, x_2, \dots, x_n)^T + (y_1, y_2, \dots, y_n)^T = (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n)^T \quad (7.123)$$

and

$$c(z_1, z_2, \dots, z_n)^T = (cz_1, cz_2, \dots, cz_n)^T. \quad (7.124)$$

Example 7.4.1. (Operations on Certain Vectors). Let

$$\mathbf{x} = \begin{bmatrix} 2+3i \\ 1-4i \end{bmatrix} \text{ and } \mathbf{y} = \begin{bmatrix} 2-5i \\ 4+6i \end{bmatrix} \quad (7.125)$$

be vectors of $\mathbb{C}^2$, and $c = 5 - 6i$ a scalar. Then

$$\mathbf{x} + \mathbf{y} = \begin{bmatrix} 4-2i \\ 5+2i \end{bmatrix} \text{ and } c\mathbf{x} = \begin{bmatrix} 28+3i \\ -19-26i \end{bmatrix}. \quad (7.126)$$

♦

The length of such a vector cannot be defined in terms of the sum of the squares of the components, because such squares are generally complex, and we want a length to be real. This situation is, however, easy to remedy: We square the *absolute values* of the components.

Definition 7.4.2. ***(Length).*** *The length or norm of a vector* $\mathbf{z} = (z_1, z_2, \dots, z_n)^T \in \mathbb{C}^n$ *is defined as*

$$|\mathbf{z}| = \sqrt{|z_1|^2 + |z_2|^2 + \cdots + |z_n|^2}. \quad (7.127)$$

We can put this formula in a much simpler form, analogous to $|\mathbf{x}|^2 = \mathbf{x}^T\mathbf{x}$ in the real case, by expressing each absolute value on the right in terms involving complex conjugates, as follows:

$$|\mathbf{z}|^2 = \overline{z}_1 z_1 + \overline{z}_2 z_2 + \cdots + \overline{z}_n z_n. \quad (7.128)$$

Writing $\overline{\mathbf{z}}^T = (\overline{z}_1, \overline{z}_2, \dots, \overline{z}_n)$ and defining matrix multiplication exactly as in the real case, we may write this equation as

$$|\mathbf{z}|^2 = \overline{\mathbf{z}}^T\mathbf{z}. \quad (7.129)$$

We could have written $|\mathbf{z}|^2 = \mathbf{z}^T\overline{\mathbf{z}}$ as well, but this form is never used. In fact, there is a special name and notation for $\overline{\mathbf{z}}^T$.

Definition 7.4.3. ***(Hermitian Conjugate).*** *For every* $\mathbf{z} \in \mathbb{C}^n$ *the row vector* $\overline{\mathbf{z}}^T$ *is called the Hermitian conjugate*[2] *of* $\mathbf{z}$ *and is denoted by* $\mathbf{z}^H$. *Similarly, for every matrix* A *with complex entries we define its Hermitian conjugate as* $A^H = \overline{A}^T$ *(read: "A-Hermitian"), where* $\overline{A} = (\overline{a}_{ij})$.

[2] After Charles Hermite (1822–1901).

Equation 7.129 suggests the following generalization of the dot product.

Definition 7.4.4. ***(Inner Product).*** *For all* $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$ *the scalar* $\mathbf{x}^H\mathbf{y}$ *is called their inner product.*

Note that this definition is not commutative:

$$\mathbf{x}^H\mathbf{y} = \overline{x}_1 y_1 + \overline{x}_2 y_2 + \cdots + \overline{x}_n y_n, \tag{7.130}$$

while

$$\mathbf{y}^H\mathbf{x} = \overline{y}_1 x_1 + \overline{y}_2 x_2 + \cdots + \overline{y}_n x_n. \tag{7.131}$$

Thus,

$$\mathbf{y}^H\mathbf{x} = \overline{\mathbf{x}^H\mathbf{y}}, \tag{7.132}$$

and $\mathbf{x}^H\mathbf{y} = \mathbf{y}^H\mathbf{x}$ if and only if $\mathbf{x}^H\mathbf{y}$ is real.

Example 7.4.2. (Some Inner Products). For the vectors of Example 7.4.1 we have

$$\mathbf{x}^H = (2 - 3i, 1 + 4i) \text{ and } \mathbf{y}^H = (2 + 5i, 4 - 6i), \tag{7.133}$$

$$\begin{aligned} |\mathbf{x}|^2 &= \mathbf{x}^H\mathbf{x} = (2 - 3i, 1 + 4i)\begin{bmatrix} 2 + 3i \\ 1 - 4i \end{bmatrix} \\ &= (2 - 3i)(2 + 3i) + (1 + 4i)(1 - 4i) = \left(2^2 + 3^2\right) + \left(1^2 + 4^2\right) = 30, \end{aligned} \tag{7.134}$$

and

$$\begin{aligned} |\mathbf{y}|^2 &= \mathbf{y}^H\mathbf{y} = (2 + 5i, 4 - 6i)\begin{bmatrix} 2 - 5i \\ 4 + 6i \end{bmatrix} \\ &= (2 + 5i)(2 - 5i) + (4 - 6i)(4 + 6i) = \left(2^2 + 5^2\right) + \left(4^2 + 6^2\right) = 81. \end{aligned} \tag{7.135}$$

Similarly, the inner products can be computed as

$$\begin{aligned} \mathbf{x}^H\mathbf{y} &= (2 - 3i, 1 + 4i)\begin{bmatrix} 2 - 5i \\ 4 + 6i \end{bmatrix} = (2 - 3i)(2 - 5i) + (1 + 4i)(4 + 6i) \\ &= (-11 - 16i) \ + (-20 + 22i) = -31 + 6i, \end{aligned} \tag{7.136}$$

and

$$\begin{aligned} \mathbf{y}^H\mathbf{x} &= (2 + 5i, 4 - 6i)\begin{bmatrix} 2 + 3i \\ 1 - 4i \end{bmatrix} = (2 + 5i)(2 + 3i) + (4 - 6i)(1 - 4i) \\ &= (-11 + 16i) \ + (-20 - 22i) = -31 - 6i. \end{aligned} \tag{7.137}$$

♦

Two vectors of $\mathbb{C}^n$ are still called *orthogonal* if their inner product is zero, although the geometric meaning is lost.

For the Hermitian conjugate of a matrix product the expansion involves the reversal of the factors, just as for transposes.

Theorem 7.4.1. ***(The Hermitian Conjugate of Matrix Products).*** *For all matrices A and B for which AB is defined, we have*

$$(AB)^H = B^H A^H. \tag{7.138}$$

The Hermitian conjugate is used in place of the transpose to generalize the notions of symmetric and orthogonal matrices.

Definition 7.4.5. ***(Hermitian Matrix and Unitary Matrix).*** *A matrix A is called Hermitian if*

$$A^H = A, \tag{7.139}$$

and U is called unitary if

$$U^H U = I. \tag{7.140}$$

Example 7.4.3. (A Certain Hermitian Matrix). The matrix

$$A = \begin{bmatrix} 1 & 3+i \\ 3-i & 4 \end{bmatrix} \tag{7.141}$$

is Hermitian, since

$$A^H = \begin{bmatrix} \overline{1} & \overline{3-i} \\ \overline{3+i} & \overline{4} \end{bmatrix} = \begin{bmatrix} 1 & 3+i \\ 3-i & 4 \end{bmatrix} = A. \tag{7.142}$$

Note that the diagonal entries are real, as they must be in every Hermitian matrix. ♦

Example 7.4.4. (A Certain Unitary Matrix). The matrix

$$U = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} \tag{7.143}$$

is unitary, since

$$U^H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix} \tag{7.144}$$

and

$$U^H U = \frac{1}{2} \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix} \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I. \tag{7.145}$$

♦

The usefulness of Hermitian matrices rests on their following property:

Theorem 7.4.2. ***(Eigenvalues of a Hermitian Matrix).*** *A Hermitian matrix has only real eigenvalues.*

Proof. Suppose A is Hermitian and λ is an eigenvalue of A corresponding to a nonzero eigenvector $\mathbf{s}$. Then

$$A\mathbf{s} = \lambda\mathbf{s} \tag{7.146}$$

and

$$\mathbf{s}^H A\mathbf{s} = \lambda\mathbf{s}^H\mathbf{s}. \tag{7.147}$$

Here $\mathbf{s}^H\mathbf{s}$ is real because it equals $|\mathbf{s}|^2$. The left side is real as well, because for scalars, complex conjugation and Hermitian conjugation are the same, and consequently

$$\overline{\mathbf{s}^H A\mathbf{s}} = (\mathbf{s}^H A\mathbf{s})^H = \mathbf{s}^H A^H \left(\mathbf{s}^H\right)^H = \mathbf{s}^H A\mathbf{s}. \tag{7.148}$$

Thus λ must be real. ■

Example 7.4.5. (Eigenvalues of a Certain Hermitian Matrix). For the matrix of Example 7.4.3 we have the characteristic equation

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 3+i \\ 3-i & 4-\lambda \end{vmatrix} = (1-\lambda)(4-\lambda) - 10 = 0, \tag{7.149}$$

which gives the eigenvalues $\lambda_1 = 6$ and $\lambda_2 = -1$. ♦

Since for a real matrix, Hermitian conjugation is the same as transposition, we have now proved Theorem 7.3.1, which we restate here as a corollary.

Corollary 7.4.1. ***(Eigenvalues of a Real Symmetric Matrix).*** *The eigenvalues of a real symmetric matrix are real.*

For the eigenvalues of a unitary matrix we have an analogous theorem.

Theorem 7.4.3. ***(Eigenvalues of a Unitary Matrix).*** *The eigenvalues of a unitary matrix have absolute value 1.*

Proof. Let U be a unitary matrix, λ one of its eigenvalues, and $\mathbf{s}$ a nontrivial eigenvector belonging to λ. Then taking the Hermitian conjugate of both sides of

$$U\mathbf{s} = \lambda\mathbf{s}, \tag{7.150}$$

we get

$$\mathbf{s}^H U^H = \overline{\lambda}\mathbf{s}^H, \tag{7.151}$$

and multiplying corresponding sides:

$$\mathbf{s}^H U^H U \mathbf{s} = \overline{\lambda}\lambda \mathbf{s}^H \mathbf{s}. \tag{7.152}$$

Since U is unitary, we have $U^H U = I$, and so the left side above reduces to $\mathbf{s}^H \mathbf{s}$, which can then be canceled, leaving $\overline{\lambda}\lambda = 1$. This equation can be written as $|\lambda|^2 = 1$ and, since $|\lambda| \geq 0$, we must have $|\lambda| = 1$. ■

Just as for real symmetric matrices, we have the following analogous theorems for Hermitian matrices.

Theorem 7.4.4. ***(The Eigenspaces of a Hermitian Matrix Are Orthogonal to Each Other).*** *Any two eigenvectors of a Hermitian matrix that belong to different eigenvalues are orthogonal to each other.*

We leave the proof as Exercise 7.4.13.

Theorem 7.4.5. ***(The Spectral Theorem).*** *For every Hermitian matrix A there exists a unitary matrix U such that $U^H AU = \Lambda$ is a real diagonal matrix. The columns of U are eigenvectors of A, and the diagonal entries of Λ are the corresponding eigenvalues.*

The proof is similar to the one in the real case and is omitted. We just illustrate the procedure with some examples.

Example 7.4.6. (Diagonalization of a Hermitian Matrix). In Example 7.4.5 we have found the eigenvalues $\lambda_1 = 6$ and $\lambda_2 = -1$ for the Hermitian matrix

$$A = \begin{bmatrix} 1 & 3+i \\ 3-i & 4 \end{bmatrix}. \tag{7.153}$$

The corresponding eigenvectors can be obtained by solving $(A - \lambda I)\mathbf{s} = \mathbf{0}$ with the values above for λ. For $\lambda_1 = 6$ this equation becomes

$$\begin{bmatrix} -5 & 3+i \\ 3-i & -2 \end{bmatrix} \begin{bmatrix} s_{11} \\ s_{21} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{7.154}$$

The two rows are dependent, as they should be, because the first one equals $-(3+i)/2$ times the second one. One solution is obviously $\mathbf{s} = (3+i, 5)^T$. Hence $\mathbf{s}^H = (3-i, 5)$, and so $\mathbf{s}^H\mathbf{s} = (3-i)(3+i)+5^2 = 35$. Thus a normalized solution is given by

$$\mathbf{s}_1 = \frac{1}{\sqrt{35}} \begin{bmatrix} 3+i \\ 5 \end{bmatrix}. \tag{7.155}$$

For $\lambda_2 = -1$ the equation $(A - \lambda I)\mathbf{s} = \mathbf{0}$ becomes

$$\begin{bmatrix} 2 & 3+i \\ 3-i & 5 \end{bmatrix} \begin{bmatrix} s_{12} \\ s_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{7.156}$$

and a normalized solution of this equation is

$$\mathbf{s}_2 = \frac{1}{\sqrt{35}} \begin{bmatrix} -5 \\ 3-i \end{bmatrix}. \tag{7.157}$$

We combine the two eigenvectors above into the matrix

$$S = \frac{1}{\sqrt{35}} \begin{bmatrix} 3+i & -5 \\ 5 & 3-i \end{bmatrix}. \tag{7.158}$$

This S is unitary, since

$$S^H = \frac{1}{\sqrt{35}} \begin{bmatrix} 3-i & +5 \\ -5 & 3+i \end{bmatrix} \tag{7.159}$$

and $S^H S = I$, as can be checked easily. Here we have denoted the unitary matrix U of Theorem 7.4.5 by S, in keeping with our earlier notation of $\mathbf{s}$ for eigenvectors. We leave it to the reader to check that $S^H AS = \Lambda$ holds; that is, that

$$S^H AS = \frac{1}{35} \begin{bmatrix} 3-i & +5 \\ -5 & 3+i \end{bmatrix} \begin{bmatrix} 1 & 3+i \\ 3-i & 4 \end{bmatrix} \begin{bmatrix} 3+i & -5 \\ 5 & 3-i \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ 0 & -1 \end{bmatrix}, \tag{7.160}$$

as required by Theorem 7.4.5. ♦

Example 7.4.7. (An Electric Circuit with Complex Eigenvalues). This is a continuation of the electric circuit problem of Example 7.2.2 of page 268 for the case of negative D, which we had to omit previously. This case does occur in many real-life electric circuits and needs to be solved just as much as the earlier cases did.

The whole formalism as presented in Section 7.2 remains valid; we just have to carry it somewhat further to obtain the solutions in real rather than complex form. We can do so whenever the initial conditions are real, although the matrix is not Hermitian and the eigenvalues are complex.

Before proceeding, let us mention an unfortunate notational collision between the traditional uses of the letter i for $\sqrt{-1}$ by mathematicians and for electric currents (from *i*ntensity) by physicists and engineers. The latter avoid this difficulty generally by using j for $\sqrt{-1}$. We shall stay with $i = \sqrt{-1}$, and in this section use only $i(t)$ for currents, not i as in Section 7.2.

Thus, let $D < 0$. Then the eigenvalues from Equations 7.89 and 7.90 may be written as

$$\lambda_1 = \frac{1}{2}\left(-\frac{R}{L} - i\sqrt{|D|}\right) \tag{7.161}$$

and

$$\lambda_2 = \frac{1}{2}\left(-\frac{R}{L} + i\sqrt{|D|}\right). \tag{7.162}$$

Since these are complex conjugates of each other, we can drop the subscripts and write $\lambda = \lambda_2$ and $\overline{\lambda} = \lambda_1$. Also, we write

$$\lambda = -a + i\omega, \tag{7.163}$$

where

$$a = \frac{R}{2L} \text{ and } \omega = \frac{\sqrt{|D|}}{2} \tag{7.164}$$

are nonnegative real numbers. With this notation the general solutions 7.91 and 7.92 become

$$i(t) = e^{-at}(c_{11}e^{-i\omega t} + c_{12}e^{i\omega t}) \tag{7.165}$$

and

$$q(t) = e^{-at}(c_{21}e^{-i\omega t} + c_{22}e^{i\omega t}). \tag{7.166}$$

In view of Euler's formula (see Equation A.28) these equations represent damped oscillations; that is, oscillations with angular frequency ω and exponentially decaying amplitudes.

With the initial conditions 7.93 and 7.94 that represent an initial charge Q and no initial current, we obtain from Equations 7.95 and 7.96

$$c_{11} = -c_{12} = \frac{Q}{LCi\sqrt{|D|}} \tag{7.167}$$

and

$$c_{21} = Q - c_{22} = \frac{Q\lambda_2}{i\sqrt{|D|}}. \tag{7.168}$$

Substituting these values into the general solution above, we get the corresponding particular solution as

$$i(t) = \frac{Q}{LC\sqrt{|D|}}e^{-at}\frac{e^{-i\omega t} - e^{i\omega t}}{i} = \frac{-2Q}{LC\sqrt{|D|}}e^{-at}\sin\omega t \tag{7.169}$$

and

$$q(t) = e^{-at}\left(Q\cos\omega t + \frac{QR}{L\sqrt{|D|}}\sin\omega t\right). \tag{7.170}$$

Notice that the imaginary parts of these functions have vanished, as they ought to. ♦

Exercises

In the first four exercises find and simplify, for the given vectors, (a) their Hermitian conjugates, (b) their lengths, (c) their inner products in both orders.

Exercise 7.4.1.

$$\mathbf{x} = \begin{bmatrix} 2 \\ 2i \end{bmatrix} \text{ and } \mathbf{y} = \begin{bmatrix} 5i \\ 4+i \end{bmatrix}.$$

Exercise 7.4.2.

$$\mathbf{x} = \begin{bmatrix} 2+4i \\ 1-2i \end{bmatrix} \text{ and } \mathbf{y} = \begin{bmatrix} 1-5i \\ 4+2i \end{bmatrix}.$$

Exercise 7.4.3.

$$\mathbf{x} = \begin{bmatrix} 2e^{i\pi/4} \\ 2i \end{bmatrix} \text{ and } \mathbf{y} = \begin{bmatrix} e^{i\pi/4} \\ e^{-i\pi/4} \end{bmatrix}.$$

Exercise 7.4.4.

$$\mathbf{x} = \begin{bmatrix} 2 \\ 2i \\ 1+i \end{bmatrix} \text{ and } \mathbf{y} = \begin{bmatrix} 5i \\ 4+i \\ 4-i \end{bmatrix}.$$

Exercise 7.4.5. Let $\mathbf{u}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ i \end{bmatrix}$.

a. Find another vector $\mathbf{u}_2 \in \mathbb{C}^2$ so that $\mathbf{u}_1$, $\mathbf{u}_2$ form an orthonormal basis for $\mathbb{C}^2$.

b. Find the coordinates of an arbitrary vector $\mathbf{x}$ with respect to this basis, that is, the coefficients x_{U1}, x_{U2} in the decomposition $\mathbf{x} = x_{U1}\mathbf{u}_1 + x_{U2}\mathbf{u}_2$.

c. Find the coordinates of the vector $\mathbf{x} = \begin{bmatrix} 2+4i \\ 1-2i \end{bmatrix}$ with respect to the basis above.

Exercise 7.4.6. Find the eigenvalues and eigenvectors of the rotation matrix

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

Exercise 7.4.7. Show that for every matrix A we have $A^{HH} = A$.

Exercise 7.4.8. Show that for every matrix A the product $A^H A$ is Hermitian.

Exercise 7.4.9. Show that a square matrix A is Hermitian if and only if $\mathbf{x}^H A\mathbf{x}$ is real for every vector $\mathbf{x}$ (of the right size, of course).

Exercise 7.4.10. Show that if U is unitary, then $|U\mathbf{x}| = |\mathbf{x}|$ for every $\mathbf{x}$ (of the right size).

Exercise 7.4.11. Show that if the matrix A is Hermitian, then $U = e^{itA}$ is unitary for every real t. (This is important in physics, since U provides the solutions to Schrödinger's differential equation $\frac{d\mathbf{u}}{dt} = iA\mathbf{u}$. Cf. Exercise 7.2.12.)

Exercise 7.4.12. Verify that Equations 7.169 and 7.170 give the particular solutions of Example 7.4.7 for the initial conditions $q(0) = Q$ and $i(0) = 0$.

Exercise 7.4.13. Prove Theorem 7.4.4.

Exercise 7.4.14. Prove that the determinant of every Hermitian matrix is real.

Exercise 7.4.15. Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}.$$

Exercise 7.4.16. Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix}.$$

MATLAB Exercises

In MATLAB, we can create orthogonal matrices as follows: For every real matrix A, the command $[Q, R] = \mathbf{qr}(A)$ returns an orthogonal matrix Q and an upper triangular matrix R such that $A = QR$. (Here we use this command only to obtain Q and discard the matrices A and R.) In the next exercise we want to show that every 3×3 orthogonal matrix represents a rotation or the product of a rotation and a reflection, and find the axis and angle of the rotation. (Cf. also Example 4.3.2 on page 192.)

Exercise 7.4.17. a. Enter the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$$

and find the corresponding orthogonal matrix Q. Let $[X, D] = \mathbf{eig}(Q)$. This command returns the eigenvectors of Q in X and the eigenvalues in D. Notice that one of the eigenvalues is 1 and check that the other two have absolute value 1. Let $t = \textbf{angle}(D(2,2))$ and show that the matrix Q represents a rotation by the angle t around the first eigenvector, as follows. Let $\textbf{sl} = X(:,1)$. Then the command $[S,T] = \textbf{qr}(\textbf{sl})$ creates an orthogonal matrix S whose first column is **s1**. Thus the columns of S are mutually orthogonal unit vectors, and so S represents a rotation or -1 times a rotation of the standard basis to the columns of S. The command $R = S' * Q * S$ transforms the matrix Q to the basis S. Show that the matrix R represents a rotation by angle t around the first vector of the new basis. (Compare R with the matrix in Exercise 7.4.6.)

b. Prove that in general, if Q is an orthogonal matrix with eigenvalues $\pm 1,\ e^{i\theta},\ e^{-i\theta}$, then Q can be written as $Q = \pm S * R * S'$, where S is an orthogonal matrix whose first column is the first eigenvector of Q, and

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}.$$

8. *Numerical Methods*

8.1 *LU* Factorization

In this section we consider a variant of Gaussian elimination in which the coefficient matrix A is written as a product of a lower triangular matrix L and an upper triangular or echelon matrix U. The main advantage of this method over the straightforward algorithm is that it is considerably more economical when we need to solve several systems of the form $A\mathbf{x} = \mathbf{b}$ with the same A but different right-hand sides $\mathbf{b}$. An additional, though less practical, advantage is that we gain some insight into the structure of Gaussian elimination in terms of matrix products.

Definition 8.1.1. ***(Lower Triangular Matrix).*** *A square matrix is called lower triangular if all the entries above its main diagonal are zero.*

The idea behind the new procedure is very simple: As we have seen in Chapter 2, forward elimination changes $A\mathbf{x} = \mathbf{b}$ into an equivalent system

$$U\mathbf{x} = \mathbf{c}, \tag{8.1}$$

where U is an echelon matrix. If we can write A as a product LU, then $A\mathbf{x} = \mathbf{b}$ becomes $LU\mathbf{x} = \mathbf{b}$, and multiplying $U\mathbf{x} = \mathbf{c}$ by L on both sides, we get $LU\mathbf{x} = L\mathbf{c}$. Hence we must have

$$L\mathbf{c} = \mathbf{b}. \tag{8.2}$$

Since L turns out to be lower triangular, it is very easy to solve this equation for $\mathbf{c}$ by "forward substitution," once L is known. Thus if we know L and U, then the solution of the system $A\mathbf{x} = \mathbf{b}$ is reduced to finding the solutions of the two extremely simple systems $L\mathbf{c} = \mathbf{b}$ and $U\mathbf{x} = \mathbf{c}$, which express the forward elimination and the back substitution phases of Gaussian elimination, respectively.

To see how to find L and how to apply this new approach, let us consider some examples.

Example 8.1.1. (LU Factorization of a Certain* $\mathbf{2 \times 2}$ *System). Let us find L and U for

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$$A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \tag{8.3}$$

and use those to solve $A\mathbf{x} = \mathbf{b}$ with $\mathbf{b} = (5, 6)^T$.

Here the first step of Gaussian elimination is that of subtracting twice the first row of A from the second row. This move is equivalent to multiplying A by the elementary matrix[1]

$$E = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}. \tag{8.4}$$

Indeed, if we write $A = (\mathbf{a}^1, \mathbf{a}^2)^T$, then

$$U = EA = \begin{bmatrix} 1\mathbf{a}^1 + 0\mathbf{a}^2 \\ -2\mathbf{a}^1 + 1\mathbf{a}^2 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 0 & -2 \end{bmatrix}, \tag{8.5}$$

and so the first row of this matrix is the same as the first row of A, and the second row is (-2) times the first row of A plus the second row of A; just what we needed.

We can now proceed in two ways to obtain the vector $\mathbf{c}$ of the reduced system $U\mathbf{x} = \mathbf{c}$. First, we can simply compute it the old way as

$$\mathbf{c} = E\mathbf{b} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 5 \\ -4 \end{bmatrix}. \tag{8.6}$$

Second, we can compute $\mathbf{c}$ in a new way by finding the matrix L for which $A = LU$ holds, and solving Equation 8.2. In this simple example the two methods are equally easy, but for larger systems with various right sides the second one is preferable. So let us see how the new method works in this case.

From the equation $U = EA$ we obtain $A = E^{-1}U$, since E is invertible. Thus

$$L = E^{-1} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}. \tag{8.7}$$

Notice that for this L we have

$$LU = L(EA) = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{a}^1 \\ -2\mathbf{a}^1 + \mathbf{a}^2 \end{bmatrix} = \begin{bmatrix} \mathbf{a}^1 \\ 2\mathbf{a}^1 + \left(-2\mathbf{a}^1 + \mathbf{a}^2\right) \end{bmatrix} = A; \tag{8.8}$$

that is, L has the desired effect of adding back the $2\mathbf{a}_1$ subtracted by E in the second row of A. Thus an LU decomposition of A is

$$\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & -2 \end{bmatrix}. \tag{8.9}$$

[1] An elementary matrix is a matrix that corresponds to an elementary row operation. See Exercise 2.5.12.

Hence the equation $L\mathbf{c} = \mathbf{b}$ is now

$$\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}. \tag{8.10}$$

We solve this system from the top down by forward substitution as $c_1 = 5$ and $2 \cdot 5 + c_2 = 6,\ c_2 = -4$. Thus we get the same $\mathbf{c}$, of course, as before.

There is nothing new in the rest of the computation: We solve $U\mathbf{x} = \mathbf{c}$ by back substitution; that is, from

$$\begin{bmatrix} 1 & 3 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ -4 \end{bmatrix} \tag{8.11}$$

we compute $-2x_2 = -4,\ x_2 = 2$ and $x_1 + 3 \cdot 2 = 5,\ x_1 = -1$. ♦

Example 8.1.2. (LU Factorization of a Certain* 3 × 3 *System). Let

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 6 & -1 \\ 1 & 2 & 1 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 2 \\ 8 \\ 0 \end{bmatrix} \tag{8.12}$$

as in Example 2.1.5 on page 49.

Multiplying A by

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{8.13}$$

we annihilate the $a_{21} = 3$ entry and obtain[2]

$$E_{21}A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & -1 \\ 1 & 2 & 1 \end{bmatrix}. \tag{8.14}$$

Next we multiply by

$$E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \tag{8.15}$$

to make the $a_{31} = 1$ entry 0, and produce

$$E_{31}E_{21}A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}. \tag{8.16}$$

[2] We denote this matrix by E_{21} to indicate the location of its sole nonzero off-diagonal entry.

Finally, multiplication by

$$E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \tag{8.17}$$

gives the echelon matrix

$$U = E_{32}E_{31}E_{21}A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}. \tag{8.18}$$

Now each of the matrices E_{21}, E_{31}, and E_{32} is invertible, with the inverse obtained simply by changing the sign of the nonzero off-diagonal entry. Thus their product is also invertible, and from the first part of Equation 8.18 we obtain $A = LU$ with

$$\begin{aligned} L = E_{21}^{-1}E_{31}^{-1}E_{32}^{-1} &= \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix}. \end{aligned} \tag{8.19}$$

Notice that very luckily this product too is lower diagonal. Also, the entries l_{ij} below the diagonal are exactly the multipliers of the rows occurring in forward elimination; that is, it is $l_{ij}\mathbf{a}^j$ that we would subtract from $\mathbf{a}^i$ in forward elimination. *This is always the case, and so we never need to compute L separately, we can just assemble it from the coefficients that occur in forward elimination.* (This is generally not the case for the product of the E matrices as in Equation 8.18, but fortunately we do not need that product anyway. See also Exercise 8.1.1.)

Now let us use the matrix L we have found to obtain $\mathbf{c}$: The equation $L\mathbf{c} = \mathbf{b}$ becomes

$$\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \\ 0 \end{bmatrix}. \tag{8.20}$$

Hence $c_1 = 2$, $3 \cdot 2 + c_2 = 8$, $c_2 = 2$, $1 \cdot 2 - 1 \cdot 2 + c_3 = 0$, and $c_3 = 0$.

Thus the equation $U\mathbf{x} = \mathbf{c}$ becomes

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}. \tag{8.21}$$

This is the same as Equation 2.25 of page 49 and has the solution

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix} + t \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \tag{8.22}$$

given there. ♦

We can now summarize the main points of the foregoing discussion in the following theorem.

Theorem 8.1.1. ***(LU Factorization).*** *If in the forward phase of the Gaussian elimination algorithm for an $m \times n$ matrix A no row exchanges are used, then A can be written as a product LU, where L is an $m \times m$ lower triangular matrix with 1's along its main diagonal, and U is the $m \times n$ echelon matrix obtained by the algorithm.*

Furthermore, each entry l_{ij} of L below the main diagonal is the coefficient of the row vector $\mathbf{a}^j$ in the product $l_{ij}\mathbf{a}^j$ that is subtracted from the row vector $\mathbf{a}^i$ in forward elimination.

Also, once L and U are known, the system $A\mathbf{x} = \mathbf{b}$, for any $\mathbf{b}$, can be replaced by the two simpler systems $L\mathbf{c} = \mathbf{b}$ and $U\mathbf{x} = \mathbf{c}$.

Proof. We prove only the case of A being $3 \times n$ and each E_{ij} and the matrix L being 3×3. (E_{ij} and L are always square, even when A is not.) For other dimensions the argument would be similar.

The elementary matrix representing the first step of the elimination algorithm is

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -l_{21} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{8.23}$$

because

$$E_{21}A = \begin{bmatrix} 1 & 0 & 0 \\ -l_{21} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{a}^1 \\ \mathbf{a}^2 \\ \mathbf{a}^3 \end{bmatrix} = \begin{bmatrix} \mathbf{a}^1 \\ \mathbf{a}^2 - l_{21}\mathbf{a}^1 \\ \mathbf{a}^3 \end{bmatrix}. \tag{8.24}$$

Thus, on the one hand, the l_{21} in the matrix E_{21} is the coefficient of $\mathbf{a}^1$ in the product that is subtracted from $\mathbf{a}^2$ in forward elimination and, similarly, the l_{ij} in the matrix E_{ij} is the coefficient of $\mathbf{a}^j$ in the product that is subtracted from $\mathbf{a}^i$.

On the other hand, to construct L and to show that each l_{ij} is also the appropriate entry of L, we can proceed as follows: From Equation 8.23 we get

$$E_{21}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{8.25}$$

and similarly we have

$$E_{31}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ l_{31} & 0 & 1 \end{bmatrix} \tag{8.26}$$

and

$$E_{32}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & l_{32} & 1 \end{bmatrix}. \tag{8.27}$$

Since U is defined as the echelon matrix obtained by the forward elimination algorithm embodied in the E_{ij} matrices, we must have $U = E_{32}E_{31}E_{21}A$. Hence we obtain $A = E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}U$ and, since L is defined as a lower triangular matrix for which $A = LU$, we find that $L = E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}$ is such a matrix. Thus we compute LU as follows: First we write

$$E_{32}^{-1}U = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & l_{32} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{u}^1 \\ \mathbf{u}^2 \\ \mathbf{u}^3 \end{bmatrix} = \begin{bmatrix} \mathbf{u}^1 \\ \mathbf{u}^2 \\ l_{32}\mathbf{u}^2 + \mathbf{u}^3 \end{bmatrix}. \tag{8.28}$$

Next

$$E_{31}^{-1}E_{32}^{-1}U = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ l_{31} & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{u}^1 \\ \mathbf{u}^2 \\ l_{32}\mathbf{u}^2 + \mathbf{u}^3 \end{bmatrix} = \begin{bmatrix} \mathbf{u}^1 \\ \mathbf{u}^2 \\ l_{31}\mathbf{u}^1 + l_{32}\mathbf{u}^2 + \mathbf{u}^3 \end{bmatrix} \tag{8.29}$$

and finally

$$\begin{aligned} LU = E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}U &= \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{u}^1 \\ \mathbf{u}^2 \\ l_{31}\mathbf{u}^1 + l_{32}\mathbf{u}^2 + \mathbf{u}^3 \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{u}^1 \\ l_{21}\mathbf{u}^1 + \mathbf{u}^2 \\ l_{31}\mathbf{u}^1 + l_{32}\mathbf{u}^2 + \mathbf{u}^3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{u}^1 \\ \mathbf{u}^2 \\ \mathbf{u}^3 \end{bmatrix}. \end{aligned} \tag{8.30}$$

Thus indeed

$$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}. \tag{8.31}$$

The preceding calculation shows why the l_{ij} coefficients from the forward elimination process appear intact in L: In the course of the multiplications above, we first added $l_{32}\mathbf{u}^2$ to $\mathbf{u}^3$, then $l_{31}\mathbf{u}^1$ to the sum, without disturbing anything else. Then we added $l_{21}\mathbf{u}^1$ to $\mathbf{u}^2$, again without disturbing anything else.

This step finishes the proof of the first two statements of Theorem 8.1.1. The last statement has been proved in the second paragraph of this section on page 291. ■

So far in this section we have used only one kind of elementary row operation on matrices: subtracting a multiple of one row from another. We are now going to discuss briefly how the other two kinds are sometimes incorporated into the LU factorization.

The elementary row operation of multiplying a row by some nonzero number (without subtraction from another row) is only necessary in Gauss–Jordan elimination to obtain l's as pivots. Corresponding to this observation, we write the matrix U of the LU factorization as $U = DU'$, where D is a diagonal matrix with the pivots of U as its diagonal elements, and U' is an echelon matrix with l's as pivots. Since the effect of multiplication of U' by D is multiplication of each row of U' by the corresponding diagonal element of D, the rows of U' are obtained from those of U by factoring out the pivots. The entries of U' are the coefficients that appear in Gauss–Jordan elimination when the entries of A above the pivots are annihilated, as the l_{ij} coefficients show up in the proof above.

It is customary to omit the prime from U' and to speak of the LDU factorization of A.

Example 8.1.3. (LDU Factorization of the A of Example 8.1.1). From Example 8.1.1 we have

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & -2 \end{bmatrix}. \tag{8.32}$$

Now we can factor out the -2 from the U, to get

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} = LDU. \tag{8.33}$$

♦

We summarize the LDU factorization as follows.

Corollary 8.1.1. ***(LDU Factorization).*** *If in the forward phase of the Gaussian elimination algorithm for an $m \times n$ matrix A no row exchanges are used, then A can be written as a product LDU, where L is an $m \times m$ lower triangular matrix with 1's along its main diagonal, D is an $m \times m$ diagonal matrix, and U an $m \times n$ echelon matrix with 1's as pivots. The D and U matrices here can be obtained by appropriately factoring the U of the LU factorization of A.*

The third elementary row operation, the exchange of rows, is necessary if we encounter a zero when looking for a pivot. In this case we can imagine all the necessary row exchanges to be done first. If P is the permutation matrix that represents these row exchanges (see Exercise 2.5.17), then we can apply the LU or LDU factorization to PA, instead of to A, since for PA no more row exchanges are needed.

To conclude this section, we present a brief quantitative discussion of the efficiency of LU factorization versus that of straightforward Gaussian elimination for an $n \times n$ matrix A.

When n is large, even computers may need considerable time to perform the necessary calculations, and so it is of great practical importance to know the length of time needed for any algorithm. Present-day computers take about the same time for every multiplication, division, and multiplication-addition combination. We call these *long* operations as opposed to the *short* operations of addition, subtraction, and comparison. For all practical purposes the length of time needed for our algorithms is proportional to the number of long operations, and so we want to count these.

In the forward phase of Gaussian elimination, assuming no row exchanges are needed, to get a 0 in place of a_{21}, we compute $l_{21} = a_{21}/a_{11}$ and subtract $l_{21}a_{1j}$ from each element a_{2j} of the second row, for $j = 2, 3, \ldots, n$. (The 0 we do not need to compute.) This procedure uses n long operations on the left side of $A\mathbf{x} = \mathbf{b}$.

Next, we do the same for each of the other rows below the first row. Thus to get all the $n-1$ zeros in the first column requires $n(n-1) = n^2 - n$ long operations in the worst case, that is, if all the entries in the first column are nonzero.

Now we do the same for the $(n-1) \times (n-1)$ submatrix below and to the right of a_{11}. For this computation we need $(n-1)^2 - (n-1)$ long operations.

Continuing in this manner, we find that the total number of long operations needed for forward elimination on the left side of $A\mathbf{x} = \mathbf{b}$ is

$$\sum_{k=1}^{n}(k^2 - k) = \frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2} = \frac{n^3 - n}{3} \approx \frac{n^3}{3}. \tag{8.34}$$

Since the same calculations produce L and U as well, this is also the number of long operations needed for the LU factorization of A.

To reduce the right-hand side of $A\mathbf{x} = \mathbf{b}$ along with A, we do $n-1$ multiplications $l_{k1}\mathbf{b}_1$ and subtractions from $\mathbf{b}_k$ when we produce the zeros in the first column of A. We then do $n-2$ such operations, when we produce the zeros in the second column, and so on. Thus altogether the right-hand side requires

$$\sum_{k=1}^{n-1} k = \frac{n(n-1)}{2} \approx \frac{n^2}{2} \tag{8.35}$$

long operations. If n is large, this number is negligible next to $n^3/3$, and so we usually consider $n^3/3$ as the approximate number of long operations needed for the whole of Gaussian elimination.

The number of long operations is the same, $n^2/2$, whether we reduce the right-hand side of $A\mathbf{x} = \mathbf{b}$ along with A or we solve $L\mathbf{c} = \mathbf{b}$ only afterward. Clearly, the number of long operations needed to solve $U\mathbf{x} = \mathbf{c}$ is also $n^2/2$, and so, once L and U are known, we can obtain $\mathbf{x}$ for a new $\mathbf{b}$ in just n^2 long operations instead of $n^3/3$.

Exercises

Exercise 8.1.1. Compute the product of the E matrices in Equation 8.18 and compare it to L.

Exercise 8.1.2. Compute the LDU factorization of $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$.

Exercise 8.1.3. Show that for a symmetric matrix A the matrix U in the LDU factorization satisfies $U = L^T$.

Exercise 8.1.4. Compute the time needed for a computer to solve an $n \times n$ system by Gaussian elimination for $n = 1000$ if it can do 10^5 long operations per second.

Exercise 8.1.5. Compute the number of long operations needed for the LU factorization of an $m \times n$ matrix A.

Exercise 8.1.6. Compute the number of long operations needed to solve $L\mathbf{c} = \mathbf{b}$ and $U\mathbf{x} = \mathbf{c}$, once L and U are known and A is $m \times n$.

Exercise 8.1.7. Show that the number of long operations in solving an $n \times n$ system $A\mathbf{x} = \mathbf{b}$ by Gauss–Jordan elimination is approximately $n^3/3$ when n is large, provided A is first changed to echelon form and then to reduced echelon form from the bottom up.

Exercise 8.1.8. Show that the number of long operations in inverting an $n \times n$ matrix A by the Gauss–Jordan elimination algorithm of Section 2.3 is approximately n^3 when n is large.

Exercise 8.1.9. Show that the number of long operations in computing the determinant of an $n \times n$ matrix A by reducing it to upper triangular form and multiplying the diagonal entries is approximately $n^3/3$ when n is large.

MATLAB Exercises

In MATLAB the LU factorization is provided by the command $[L, U] = \mathbf{lu}(A)$. However, the matrix L in this command usually is only a product of a permutation matrix and a lower triangular matrix because of row exchanges. The latter are introduced to minimize roundoff errors, as will be explained in the next section. Thus, if a genuine lower triangular matrix is required, then it is better to use the command $[L, U, P] = \mathbf{lu}(A)$. This command produces a lower triangular matrix L, an upper triangular matrix U, and a permutation matrix P such that $LU = PA$.

Exercise 8.1.10. For five instances of $A = \mathbf{round}(10 * \mathbf{rand}(4, 5))$ find an LU factorization of A, using $[L, U, P] = \mathbf{lu}(A)$, and change it, using the **diag** command, to the corresponding LDU factorization. Check that $LDU = PA$ holds.

8.2 Scaled Partial Pivoting

As we have seen, in Gaussian elimination we need a row exchange whenever a candidate for a pivot is zero. In machines, because of roundoff errors, we need an exchange also when such an entry is *near* zero, not just when it is exactly zero. The objective of this section is to present the standard procedure for dealing with this problem, but first we give an example of the kind of trouble we may encounter when an entry is near zero.

Example 8.2.1. (A Rounding Error in Solving a 2 × 2 System Can Result in a Wrong Solution). Let us imagine that we have a machine that rounds every number to two significant decimal digits, that is, to a number of the form $\pm 0.a_1 a_2 \times 10^n$, where a_1 and a_2 are single digits with $a_1 \neq 0$, and n is an arbitrary integer. (Although actual machines compute with much greater accuracy and round to a fixed number of binary rather than decimal digits, this setting illustrates the phenomenon quite well and avoids technical complications.) Let us see how our machine would solve the system $A\mathbf{x} = \mathbf{b}$, with

$$[A|\mathbf{b}] = \left[\begin{array}{cc|c} 0.001 & 1 & 1 \\ 1 & 1 & 2 \end{array}\right]. \tag{8.36}$$

The first step of Gaussian elimination would produce

$$\left[\begin{array}{cc|c} 0.001 & 1 & 1 \\ 0 & -999 & -998 \end{array}\right] \tag{8.37}$$

and our machine would round both 999 and 998 to 1000, resulting in

$$\left[\begin{array}{cc|c} 0.001 & 1 & 1 \\ 0 & -1000 & -1000 \end{array}\right]. \tag{8.38}$$

The machine would solve this system by back substitution and obtain $x_2 = 1$ and $x_1 = 0$. *But this solution is wrong.* The correct solution, from the matrix in 8.37, is

$$x_2 = \frac{998}{999} = 0.9989\ldots \text{ and } x_1 = \frac{1}{0.999} = 1.001\ldots. \tag{8.39}$$

Thus, while the machine's answer for x_2 is close enough, for x_1 it is way off.

So what has happened? It is this: In the first step of the back substitution the machine rounded $x_2 = 0.9989\ldots$ to 1. This step, in itself, is certainly all right, but in the next step we had to divide x_2 by 0.001 in solving for x_1. Here the small roundoff error, hidden in taking x_2 as 1, became magnified a thousandfold. Thus, somehow, we must avoid dividing a rounded number by a very small quantity, or multiplying it by a large quantity. In the present example we can achieve this goal by switching the two rows: If we reduce the matrix

$$[A|\mathbf{b}]' = \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0.001 & 1 & 1 \end{array}\right], \tag{8.40}$$

we get

$$\left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 0.999 & 0.998 \end{array}\right]. \tag{8.41}$$

This matrix is rounded by the machine to

$$\left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & 1 \end{array}\right], \tag{8.42}$$

which leads to the correct approximate solution $x_2 = 1$ and $x_1 = 1$. This time the pivot in the first row was large and did not magnify the roundoff error in x_2 when we solved for x_1.

Now one may think that all we need is a large pivot in the first row, and that can be achieved more simply by multiplying the first row of the matrix in Equation 8.36 by 1000. That would result in

$$\left[\begin{array}{cc|c} 1 & 1000 & 1000 \\ 1 & 1 & 2 \end{array}\right], \tag{8.43}$$

which would then be reduced to

$$\left[\begin{array}{cc|c} 1 & 1000 & 1000 \\ 0 & -999 & -998 \end{array}\right], \tag{8.44}$$

The machine would round this result to

$$\left[\begin{array}{cc|c} 1 & 1000 & 1000 \\ 0 & -1000 & -1000 \end{array}\right], \tag{8.45}$$

from which we get $x_2 = 1$ and $x_1 + 1000x_2 = 1000$. Thus, in solving this for x_1, the small roundoff error in $x_2 = 1$ is again magnified by a factor of 1000 and results in the same wrong answer of $x_1 = 0$ as before. This shows that in general it is the small value of a_{11}/a_{12} that magnifies the roundoff error, not just the small value of a_{11} alone. Since in the second row the corresponding ratio a_{21}/a_{22} is big, we can avoid the problem, as we have seen, by putting that row on top. ♦

Considerations like those in the foregoing example have led to the following strategy to minimize the magnification of roundoff errors in Gaussian elimination.

Proposition 8.2.1. ***(Scaled Partial Pivoting).*** *For every $m \times n$ matrix A,*

1. *Compute a scale factor for each row as the largest absolute value of the entries of that row. In other words, compute*

$$s_i = \max_{1 \leq j \leq n} |a_{ij}| \tag{8.46}$$

   for each i.
2. *Compute the ratio of the absolute value of the first entry in each row to its scale factor, that is, compute*

$$r_i = |a_{i1}|/s_i \tag{8.47}$$

   for each i. (Ignore rows with $s_i = 0$.)
3. *Find a row for which r_i is maximal and put it on top. Use the first entry of this row as the pivot to produce zeros below it as usual. (If all r_i are 0, then go to the next column, etc.) Note that in actual machine programs we do not really move the rows, just keep track of which one is to be used as the pivot row.*
4. *Repeat the above steps on the submatrix obtained by deleting the first row and the first column, until we run out of rows or columns.*

The above procedure is called *scaled partial pivoting.* The word "scaled" refers to the scaling used in Step 2 above, "pivoting" refers to the whole pivot-selection procedure by reordering the rows, and the adjective "partial" indicates that we do not consider a reordering of the columns as well. (The latter has been tried, but did not result in significant improvements.)

Example 8.2.2. (Scaled Partial Pivoting for a 4×4 System). To see how this procedure works, consider the system given by

$$[A|\mathbf{b}] = \left[\begin{array}{rrrr|r} 15 & 13 & -22 & 1 & 2 \\ -7 & -11 & 53 & 32 & 12 \\ 12 & 7 & 4 & 8 & 44 \\ 0 & 12 & -7 & 1 & 11 \end{array}\right]. \tag{8.48}$$

Then the scale factors are $s_1 = 22,\ s_2 = 53,\ s_3 = 12$, and $s_4 = 12$. The corresponding ratios in the first column are $r_1 = 15/22$, $r_2 = 7/53$, $r_3 = 12/12 = 1$, and $r_4 = 0$. Since r_3 is the biggest of these, we put the third row on top, and then proceed with the reduction of the first column as usual:

$$[A|\mathbf{b}]' = \left[\begin{array}{rrrr|r} 12 & 7 & 4 & 8 & 44 \\ 15 & 13 & -22 & 1 & 2 \\ -7 & -11 & 53 & 32 & 12 \\ 0 & 12 & -7 & 1 & 11 \end{array}\right]. \tag{8.49}$$

Subtracting appropriate multiples of the first row from the others, we first reduce this matrix to

$$\left[\begin{array}{rrrr|r} 12 & 7 & 4 & 8 & 44 \\ 0 & 17/4 & -27 & -9 & -53 \\ 0 & -83/12 & 166/3 & 110/3 & 113/3 \\ 0 & 12 & -7 & 1 & 11 \end{array}\right]. \tag{8.50}$$

Next, we should rescale the last three rows. But in practice this is usually not done because people have observed that it is not worth the effort. In this example, as the reader could easily check, rescaling would lead to the same result, namely that the fourth row should be the next pivot row. Thus we put the fourth row in second place, and proceed as follows:

$$\to \left[\begin{array}{rrrr|r} 12 & 7 & 4 & 8 & 44 \\ 0 & 12 & -7 & 1 & 11 \\ 0 & 0 & -1177/48 & -449/48 & -2731/48 \\ 0 & 0 & 7387/144 & 5363/144 & 6337/144 \end{array}\right]. \tag{8.51}$$

Now $7387/144 > 1177/48$, and so we swap the last two rows to get

$$\to \left[\begin{array}{rrrr|r} 12 & 7 & 4 & 8 & 44 \\ 0 & 12 & -7 & 1 & 11 \\ 0 & 0 & 7387/144 & 5363/144 & 6337/144 \\ 0 & 0 & -1177/48 & -449/48 & -2731/48 \end{array}\right] \tag{8.52}$$

and

$$\to \left[\begin{array}{rrrr|r} 12 & 7 & 4 & 8 & 44 \\ 0 & 12 & -7 & 1 & 11 \\ 0 & 0 & 7387/144 & 5363/144 & 6337/144 \\ 0 & 0 & 0 & 62406/7387 & -264901/7387 \end{array}\right]. \tag{8.53}$$

From here we proceed with regular back substitution to obtain $x_4 = -264901/62406$, $x_3 = 245855/62406$, $x_2 = 12372/3467$, and $x_1 = 193565/62406$. ♦

It should be obvious that no algorithm can completely prevent the magnification of roundoff errors in the solution of linear systems. However, J. M. Wilkinson proved in the 1960s that Gaussian elimination with partial pivoting is as good as we can get; that is, in this procedure the magnification depends only on the matrix A, and can be characterized by what is called the *condition number of* A. If A is a symmetric nonsingular matrix, then the condition number is given by the simple formula $c = |\lambda_n/\lambda_1|$, where λ_n is the eigenvalue of largest absolute value and λ_1 of the smallest. For other types of matrices the condition number is more difficult to compute; we do not go into this. In general, c is the factor by which a relative error $|\Delta\mathbf{b}|/|\mathbf{b}|$ in the right-hand side of $A\mathbf{x} = \mathbf{b}$ is magnified to produce the corresponding relative error $|\Delta\mathbf{x}|/|\mathbf{x}|$ in the solution $\mathbf{x}$.

Exercises

Exercise 8.2.1. Show how the machine of Example 8.2.1 would solve the system $A\mathbf{x} = \mathbf{b}$ with

$$[A|\mathbf{b}] = \left[\begin{array}{cc|c} 0.002 & 1 & 4 \\ 6 & -1 & 2 \end{array}\right], \tag{8.54}$$

compare the result to the correct solution, and explain the discrepancy.

Exercise 8.2.2. Show how the machine of Example 8.2.1 would solve the system $A\mathbf{x} = \mathbf{b}$ with

$$[A|\mathbf{b}] = \left[\begin{array}{cc|c} 2 & 1000 & 4000 \\ 6 & -1 & 2 \end{array}\right], \tag{8.55}$$

compare the result to the correct solution, and explain the discrepancy.

Exercise 8.2.3. Solve the system of Exercise 8.2.1 by the method of partial pivoting. Show all intermediate results, including the scale factors s_i and the ratios r_i. Compare the result to the correct solution, and explain why it is a good approximation.

Exercise 8.2.4. Solve the system

$$\begin{bmatrix} 2 & 4 & -2 \\ 1 & 3 & 4 \\ 5 & 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ -1 \\ 2 \end{bmatrix} \tag{8.56}$$

by the method of partial pivoting. Show all intermediate results, including the scale factors s_i and the ratios r_i.

Exercise 8.2.5. Solve the system

$$\begin{bmatrix} 1 & 2 & 5 \\ 4 & -7 & 11 \\ 5 & 8 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \tag{8.57}$$

by the method of partial pivoting. Show all intermediate results, including the scale factors s_i and the ratios r_i.

Exercise 8.2.6. a. Show that the condition number c of a symmetric matrix satisfies $c \geq 1$.
b. Find all symmetric 2×2 matrices with $c = 1$.
c. Find the condition number of the matrix A of Equation 8.36.
d. Find the condition number of the matrix A of Equation 8.40.
e. What conclusions can you draw from the answers to Parts a–d?

MATLAB Exercises

Exercise 8.2.7. Enter

$$A = \begin{bmatrix} 1 & 999999 & 999999 \\ 1 & 1 & 2 \end{bmatrix} \tag{8.58}$$

and

$$B = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 999999 & 999999 \end{bmatrix}.$$

Run the commands **rrefmovie**(A) and **rrefmovie**(B). Which one gives the better solution to the system represented by these as augmented matrices? Why?

Exercise 8.2.8. Enter the augmented matrix of Equation 8.49 as A in MATLAB and run the command **rrefmovie**(A).
a. Compare the observed sequence of operations to achieve Gauss–Jordan reduction to the one suggested in Exercise 8.1.7 on page 299. Which one is more efficient? How many long operations are needed in this method? (You can actually count *all* operations by using the MATLAB command **flops**. See **help flops**.)
b. Are the rows used in the same order as in Example 8.2.2? What is the difference? Which method is preferable, in general?

Exercise 8.2.9. Enter the MATLAB commands $A = \mathbf{hilb}(12)$, $\mathbf{c} = \mathbf{ones}(12, 1)$ and $\mathbf{b} = A * \mathbf{c}$. The matrix A is the Hilbert matrix of order 12, defined by $a_{ij} = (i + j - 1)^{-1}$ for $i, j = 1, \ldots, 12$. It is extremely ill-conditioned; that is, it has a very high condition number. You can check this by entering **cond**(A). The equation $A\mathbf{x} = \mathbf{b}$ should obviously have the solution $\mathbf{x} = \mathbf{c}$. See what the MATLAB commands $x = A\backslash\mathbf{b}$ and $\mathbf{x} = \mathbf{rref}[A \;\; \mathbf{b}]$ produce.

8.3 The Computation of Eigenvalues and Eigenvectors

In Chapter 7 the eigenvalues of a matrix A were always computed from the characteristic equation of A. Though indispensable for the theory, this is a very inefficient procedure for almost all but the very smallest matrices. There are several reasons for this. First, the expansion of an $n \times n$ determinant has $n!$ terms, which is already enormous for moderately large values of n. Second, the characteristic equation is an algebraic equation of degree n, and nth degree equations can be solved only by approximate methods anyway if n is 5 or more, so it might be better to use approximate methods designed directly for computing eigenvalues. Third, the solutions of high degree equations are usually very dependent on roundoff errors in the coefficients.

There exist several numerical procedures for the computation of eigenvalues and eigenvectors. We shall consider only the power method and some of its variants. In this method we reverse the procedure of using diagonalization to compute powers of a matrix, and use the powers to obtain the eigenvalues. For $n \leq 100$ or so, this technique is quite feasible with modern computers, which can compute such powers directly with great speed. To begin our discussion, let us take another look at Example 7.1.1.

Example 8.3.1. (The Powers of a Certain Matrix in the Diagonalizing Basis). The matrix

$$A = \begin{bmatrix} 1/2 & 3/2 \\ 3/2 & 1/2 \end{bmatrix} \tag{8.59}$$

has unit eigenvectors $\mathbf{s}_1 = \frac{1}{\sqrt{2}}(1,1)^T$ and $\mathbf{s}_2 = \frac{1}{\sqrt{2}}(-1,1)^T$ with corresponding eigenvalues $\lambda_1 = 2$ and $\lambda_2 = -1$.

So, if we write $\mathbf{x}$ in terms of the basis $\{s_1,s_2\}$ as

$$\mathbf{x} = x_{S1}\mathbf{s}_1 + x_{S2}\mathbf{s}_2, \tag{8.60}$$

then we get

$$A^n\mathbf{x} = x_{S1}A^n\mathbf{s}_1 + x_{S2}A^n\mathbf{s}_2 = 2^n x_{S1}\mathbf{s}_1 + (-1)^n x_{S2}\mathbf{s}_2. \tag{8.61}$$

Thus, for large values of n, the first term dominates (we say $\lambda_1 = 2$ is a dominant eigenvalue), and $A^n\mathbf{x}$ will point approximately in the direction of $\mathbf{s}_1$ and will have length $2^n x_{S1}$. This can also be seen from Figure 8.1, by observing that the direction of the vectors $\mathbf{x}$, $A\mathbf{x}$, $A^2\mathbf{x}$, $\ldots$ approaches that of $\mathbf{s}_1$, and their length nearly doubles with each step.

If the eigenvalue λ_1 were not known, we could use Equation 8.61 in various ways to give us an approximation for it. For instance, if we consider the ratio of the first components of the vectors $A^{n+1}\mathbf{x}$ and $A^n\mathbf{x}$, then we find that

$$\begin{aligned} \lim_{n\to\infty} \frac{(A^{n+1}\mathbf{x})_1}{(A^n\mathbf{x})_1} &= \lim_{n\to\infty} \frac{2^{n+1}x_{S1}/\sqrt{2} + (-1)^{n+2}x_{S2}/\sqrt{2}}{2^n x_{S1}/\sqrt{2} + (-1)^{n+1}x_{S2}/\sqrt{2}} \\ &= \lim_{n\to\infty} \frac{2x_{S1} + (-1/2)^n x_{S2}}{x_{S1} - (-1/2)^n x_{S2}} = 2 \text{ if } x_{S1} \neq 0. \end{aligned} \tag{8.62}$$

Thus the above ratio provides an approximation to $\lambda_1 = 2$ when n is large. ♦

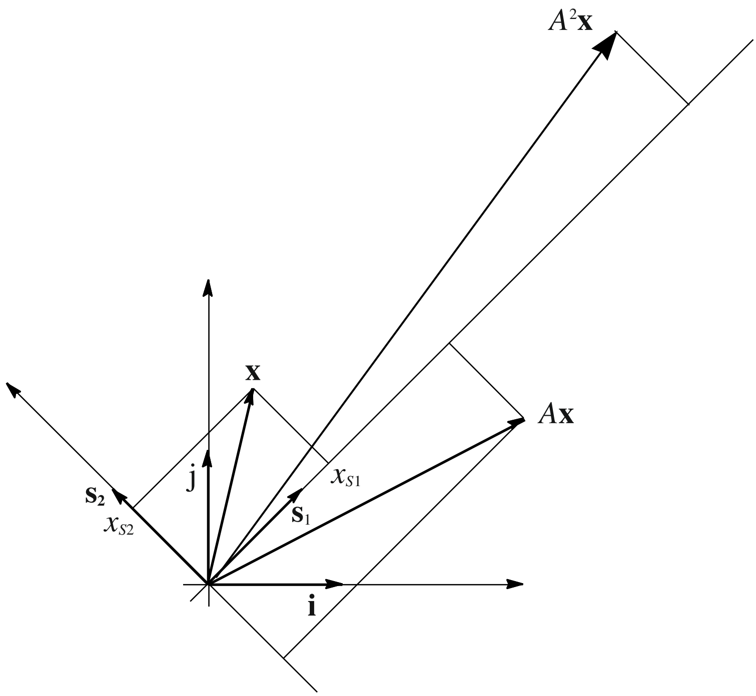

Fig. 8.1. The action of the matrix A and its powers in the diagonalizing basis

In the preceding example we could have used the second components or the lengths of the same vectors. However, because the eigenvectors are not known, the preferred procedure is to scale the vectors so that, for each value of n, we divide through by a selected component relative to the standard basis. It is this method that we summarize in the next theorem.

Theorem 8.3.1. ***(Direct Power Method for Obtaining the Dominant Eigenvalue and Its Eigenvector by Iteration).*** *Let A be a diagonalizable matrix with a dominant eigenvalue λ_1, that is, an eigenvalue such that $|\lambda_1| > |\lambda_j|$ for $j \neq 1$. Assume also that λ_1 has multiplicity 1, and that $\mathbf{s}_1$ is a corresponding eigenvector with a nonzero* kth *component, for some fixed k. Choose an arbitrary vector $\mathbf{x}_0$ with a nonzero* kth *component, and set successively $\mathbf{x}'_i = \mathbf{x}_i/(\mathbf{x}_i)_k$ and $\mathbf{x}_{i+1} = A\mathbf{x}'_i$ for $i = 0, 1, 2, \ldots$, assuming also that $(\mathbf{x}_i)_k \neq 0$ for every i. This makes $(\mathbf{x}'_i)_k = 1$ in each step, and $(\mathbf{x}_i)_k$ will approach the dominant eigenvalue λ_1, while the vectors $\mathbf{x}'_i$ and $\mathbf{x}_i$ will approach multiples of $\mathbf{s}_1$, unless the decomposition of the initial vector $\mathbf{x}_0$ relative to a basis of eigenvectors had no component in the direction of $\mathbf{s}_1$.*

Proof. Let A be $n \times n$ and $\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_n$ a complete set of eigenvectors. Then every $\mathbf{x}_0$ can be written as

$$\mathbf{x}_0 = x_{0S_1}\mathbf{s}_1 + x_{0S_2}\mathbf{s}_2 + \cdots + x_{0S_n}\mathbf{s}_n, \tag{8.63}$$

and

$$\mathbf{x}'_0 = \frac{1}{x_{0k}}(x_{0S_1}\mathbf{s}_1 + x_{0S_2}\mathbf{s}_2 + \cdots + x_{0S_n}\mathbf{s}_n), \tag{8.64}$$

where x_{0k} stands for the kth component of $\mathbf{x}_0$ relative to the standard basis.

Then

$$\mathbf{x}_1 = A\mathbf{x}'_0 = \frac{1}{x_{0k}}\left(\lambda_1 x_{0S_1}\mathbf{s}_1 + \lambda_2 x_{0S_2}\mathbf{s}_2 + \cdots + \lambda_n x_{0S_n}\mathbf{s}_n\right), \tag{8.65}$$

and similarly

$$\begin{aligned}\mathbf{x}_{i+1} = A\mathbf{x}'_i &= \frac{1}{x_{0k}x_{1k}\cdots x_{ik}}\left(\lambda_1^i x_{0S_1}\mathbf{s}_1 + \lambda_2^i x_{0S_2}\mathbf{s}_2 + \cdots + \lambda^i x_{0S_n}\mathbf{s}_n\right)\\ &= \frac{\lambda_1^i}{x_0\mu_{1k}\cdots x_{ik}}\left(x_{0S_1}\mathbf{s}_1 + \frac{\lambda_2^i}{\lambda_1^i}x_{0S_2}\mathbf{s}_2 + \cdots + \frac{\lambda_n^i}{\lambda_1^i}x_{0S_n}\mathbf{s}_n\right)\\ &\approx \frac{\lambda_1^i x_{0S_1}}{x_{0k}x_{1k}\cdots x_{ik}}\mathbf{s}_1\end{aligned} \tag{8.66}$$

for large i, because of the dominance of λ_1. (We can see from this result that the speed of convergence is determined by the magnitude of $|\lambda_2/\lambda_1|$, if λ_2 is the second largest eigenvalue in absolute value.)

Denoting the first component of $\mathbf{s}_1$ by s_{11}, we get

$$x_{i+1,1} \approx \frac{\lambda_1^i x_{0S_1} s_{11}}{x_{0k}x_{1k}\cdots x_{ik}} \tag{8.67}$$

and

$$\mathbf{x}'_{i+1} = \frac{\mathbf{x}_{i+1}}{x_{i+1,1}} \approx \frac{\mathbf{s}_1}{s_{11}}. \tag{8.68}$$

Hence

$$\mathbf{x}_{i+2} = A\mathbf{x}'_{i+1} \approx \frac{\lambda_1\mathbf{s}_1}{s_{11}} \tag{8.69}$$

and

$$x_{i+2,1} \approx \lambda_1. \tag{8.70}$$

■

As we have mentioned, the method seems to fail if by bad luck the decomposition of the initial vector $\mathbf{x}_0$ relative to a basis of eigenvectors has no component in the direction of $\mathbf{s}_1$. Although this failure is certainly true in theory, in practice after a few steps, roundoff errors will usually introduce a sufficiently large component in the required direction, which will eventually swamp the other components.

We illustrate the method by an example, in which we used MATLAB for the computations.

Example 8.3.2. (Obtaining the Dominant Eigenvalue and Its Eigenvector for a Certain* $\mathbf{2 \times 2}$ *Matrix by the Direct Power Method). Let $k = 1$ and

$$A = \begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix} \text{ and } \mathbf{x}_0 = \mathbf{x}'_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \tag{8.71}$$

Then

$$\mathbf{x}_1 = A\mathbf{x}'_0 = \begin{bmatrix} 6.0000 \\ 2.0000 \end{bmatrix} \text{ and } \mathbf{x}'_1 = \begin{bmatrix} 1 \\ 0.3333 \end{bmatrix},$$

$$\mathbf{x}_2 = A\mathbf{x}'_1 = \begin{bmatrix} 4.6667 \\ 2.6667 \end{bmatrix} \text{ and } \mathbf{x}'_2 = \begin{bmatrix} 1 \\ 0.5714 \end{bmatrix},$$

$$\mathbf{x}_3 = A\mathbf{x}'_2 = \begin{bmatrix} 5.1429 \\ 2.4286 \end{bmatrix} \text{ and } \mathbf{x}'_3 = \begin{bmatrix} 1 \\ 0.4722 \end{bmatrix},$$

$$\mathbf{x}_4 = A\mathbf{x}'_3 = \begin{bmatrix} 4.9444 \\ 2.5278 \end{bmatrix} \text{ and } \mathbf{x}'_4 = \begin{bmatrix} 1 \\ 0.5112 \end{bmatrix},$$

$$\mathbf{x}_5 = A\mathbf{x}'_4 = \begin{bmatrix} 5.0224 \\ 2.4888 \end{bmatrix} \text{ and } \mathbf{x}'_5 = \begin{bmatrix} 1 \\ 0.4955 \end{bmatrix}. \tag{8.72}$$

Thus we see that $(\mathbf{x}_i)_1 \to 5$ approximately, as $i \to \infty$ and so $\lambda_1 \approx 5$. Similarly, $\mathbf{s}_1 \approx (1, 0.5)^T$. ♦

The direct power method has the obvious drawback that it computes only dominant eigenvalues and corresponding eigenvectors. This problem can be alleviated by observing that, for each eigenvalue λ of A, the matrix $B = A - cI$ has an eigenvalue $\lambda - c$ with the same eigenvectors as those of A belonging to λ. (Clearly, if $A\mathbf{s} = \lambda\mathbf{s}$, then $(A - cI)\mathbf{s} = (\lambda - c)\mathbf{s}$ and vice versa.) Thus we can undo the dominance of any eigenvalue λ_1 by changing over to the matrix $B = A - \lambda_1 I$. It is, however, possible that the matrix B will not have a dominant eigenvalue, just as A itself did not have to have one. (This situation occurs if there are several eigenvalues with the same maximal absolute value.) Also, we may not be able to make every eigenvalue dominant by this method (see Exercise 8.3.2), and so we cannot compute such an eigenvalue this way.

We can, however, modify the power method to yield the eigenvalue *nearest to* 0, if it is unique. This *inverse power method* consists of defining the recursion by

$$A\mathbf{x}_{i+1} = \mathbf{x}'_i = \frac{\mathbf{x}_i}{(\mathbf{x}_i)_k} \tag{8.73}$$

with A on the left rather than on the right. Indeed, for nonsingular A, Equation 8.73 is equivalent to

$$\mathbf{x}_{i+1} = A^{-1}\mathbf{x}'_i, \tag{8.74}$$

and we know that the eigenvalues of A^{-1} are the reciprocals of the eigenvalues of A with the same eigenvectors. (See Exercise 7.1.13.) Thus, if μ_1 is a dominant eigenvalue of A^{-1}, then we can compute it with the direct power method of Theorem 8.3.1. Then A will have a unique eigenvalue nearest to 0, given by $\lambda_1 = 1/\mu_1$. In practice we use Equation 8.73, rather than 8.74, since A^{-1} is difficult to compute and it is more efficient to solve Equation 8.73 for each i, using LU factorization.

Coupled with shifting by an appropriate c, the inverse power method enables us to compute every eigenvalue of a diagonalizable matrix; because if c is nearest to a given eigenvalue λ of A, but $c \neq \lambda$, then $B = A - cI$ will be nonsingular (see Exercise 8.3.3) and $\lambda - c$ will be the unique eigenvalue of B nearest to 0. The only problem is how to find good values for c. There are some prescriptions for this choice, but, except for one, rather special suggestion in Example 8.3.4 below, they are left to more advanced texts. So too is the most popular iterative method, called the QR algorithm. (See, e.g., Gilbert Strang, *Linear Algebra and Its Applications*, Brooks Cole; 4th edition, 2005).

Example 8.3.3. (Obtaining the Nondominant Eigenvalue and Its Eigenvector for the* 2×2 *Matrix of Example 8.3.2 by the Inverse Power Method). As in Example 8.3.2, let $k = 1$ and

$$A = \begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix} \text{ and } \mathbf{x}_0 = \mathbf{x}'_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \tag{8.75}$$

and use Equation 8.73 to find the smaller eigenvalue λ_2 of A and a corresponding eigenvector $\mathbf{s}_2$. Then

$$\begin{aligned}
A\mathbf{x}_1 = \mathbf{x}'_0,\ \mathbf{x}_1 &= \begin{bmatrix} 0.3000 \\ -0.1000 \end{bmatrix} \text{ and } \mathbf{x}'_1 = \begin{bmatrix} 1 \\ -0.3333 \end{bmatrix}, \\
A\mathbf{x}_2 = \mathbf{x}'_1,\ \mathbf{x}_2 &= \begin{bmatrix} 0.0333 \\ 0.4333 \end{bmatrix} \text{ and } \mathbf{x}'_2 = \begin{bmatrix} 1.0000 \\ 13.0000 \end{bmatrix}, \\
A\mathbf{x}_3 = \mathbf{x}'_2,\ \mathbf{x}_3 &= \begin{bmatrix} 2.7000 \\ -4.9000 \end{bmatrix} \text{ and } \mathbf{x}'_3 = \begin{bmatrix} 1 \\ -1.8148 \end{bmatrix}, \\
A\mathbf{x}_4 = \mathbf{x}'_3,\ \mathbf{x}_4 &= \begin{bmatrix} -0.2630 \\ 1.0259 \end{bmatrix} \text{ and } \mathbf{x}'_4 = \begin{bmatrix} 1 \\ -3.9014 \end{bmatrix}, \\
A\mathbf{x}_5 = \mathbf{x}'_4,\ \mathbf{x}_5 &= \begin{bmatrix} -0.6803 \\ 1.8606 \end{bmatrix} \text{ and } \mathbf{x}'_5 = \begin{bmatrix} 1 \\ -2.7350 \end{bmatrix}, \\
A\mathbf{x}_6 = \mathbf{x}'_5,\ \mathbf{x}_6 &= \begin{bmatrix} -0.4470 \\ 1.3940 \end{bmatrix} \text{ and } \mathbf{x}'_6 = \begin{bmatrix} 1 \\ -3.1186 \end{bmatrix}, \\
A\mathbf{x}_7 = \mathbf{x}'_6,\ \mathbf{x}_7 &= \begin{bmatrix} -0.5237 \\ 1.5474 \end{bmatrix} \text{ and } \mathbf{x}'_7 = \begin{bmatrix} 1 \\ -2.9547 \end{bmatrix}.
\end{aligned} \tag{8.76}$$

Thus we see that $(\mathbf{x}_i)_1 \to -0.5$ approximately, as $i \to \infty$ and so $\mu_1 \approx -.5$ is the dominant eigenvalue of A^{-1} and $\lambda_1 \approx -2$ is the corresponding eigenvalue of A. Clearly, $\mathbf{s}_2 \approx (1, -3)^T$. ♦

Example 8.3.4. (Obtaining the Eigenvalues and Eigenvectors for a certain 3 × 3 Matrix by the Power Method). Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \\ 3 & 2 & 1 \end{bmatrix}. \tag{8.77}$$

First we use the direct method to obtain the largest eigenvalue, as in Example 8.3.2 above. Letting

$$\mathbf{x}_0 = \mathbf{x}'_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \tag{8.78}$$

we find

$$\mathbf{x}_1 = A\mathbf{x}'_0 = \begin{bmatrix} 6 \\ 5 \\ 6 \end{bmatrix} \text{ and } \mathbf{x}'_1 = \begin{bmatrix} 1 \\ 0.8333 \\ 1 \end{bmatrix},$$

$$\mathbf{x}_2 = A\mathbf{x}'_1 = \begin{bmatrix} 5.6667 \\ 4.8333 \\ 5.6667 \end{bmatrix} \text{ and } \mathbf{x}'_2 = \begin{bmatrix} 1 \\ 0.8529 \\ 1 \end{bmatrix},$$

$$\mathbf{x}_3 = A\mathbf{x}'_2 = \begin{bmatrix} 5.7059 \\ 4.8529 \\ 5.7059 \end{bmatrix} \text{ and } \mathbf{x}'_3 = \begin{bmatrix} 1 \\ 0.8505 \\ 1 \end{bmatrix},$$

$$\mathbf{x}_4 = A\mathbf{x}'_3 = \begin{bmatrix} 5.7010 \\ 4.8505 \\ 5.7010 \end{bmatrix} \text{ and } \mathbf{x}'_4 = \begin{bmatrix} 1 \\ 0.8508 \\ 1 \end{bmatrix},$$

$$\mathbf{x}_5 = A\mathbf{x}'_4 = \begin{bmatrix} 5.7016 \\ 4.8508 \\ 5.7016 \end{bmatrix} \text{ and } \mathbf{x}'_5 = \begin{bmatrix} 1 \\ 0.8508 \\ 1 \end{bmatrix}. \tag{8.79}$$

Thus $\lambda_1 \approx 5.7016$ and $\mathbf{s}_1 \approx (1, 0.8508, 1)^T$.

Next, we compute the eigenvalue of smallest absolute value by the inverse power method. Using the same $\mathbf{x}_0$ as above, we get

$$A\mathbf{x}_1 = \mathbf{x}'_0,\ \mathbf{x}_1 = \begin{bmatrix} 0.2500 \\ 0 \\ 0.2500 \end{bmatrix} \text{ and } \mathbf{x}'_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix},$$

$$A\mathbf{x}_2 = \mathbf{x}_1', \ \mathbf{x}_2 = \begin{bmatrix} -0.2500 \\ 1 \\ -0.2500 \end{bmatrix} \text{ and } \mathbf{x}_2' = \begin{bmatrix} -1 \\ 4 \\ -1 \end{bmatrix},$$

$$A\mathbf{x}_3 = \mathbf{x}_2', \ \mathbf{x}_3 = \begin{bmatrix} 2.2500 \\ -5 \\ 2.2500 \end{bmatrix} \text{ and } \mathbf{x}_3' = \begin{bmatrix} 1 \\ -2.2222 \\ 1 \end{bmatrix},$$

$$A\mathbf{x}_4 = \mathbf{x}_3^f, \ \mathbf{x}_4 = \begin{bmatrix} -1.3611 \\ 3.2222 \\ -1.3611 \end{bmatrix} \text{ and } \mathbf{x}_4' = \begin{bmatrix} 1 \\ -2.3673 \\ 1 \end{bmatrix},$$

$$A\mathbf{x}_5 = \mathbf{x}_4', \ \mathbf{x}_5 = \begin{bmatrix} -1.4337 \\ 3.3673 \\ -1.4337 \end{bmatrix} \text{ and } \mathbf{x}_5' = \begin{bmatrix} 1 \\ -2.3488 \\ 1 \end{bmatrix},$$

$$A\mathbf{x}_6 = \mathbf{x}_5', \ \mathbf{x}_6 = \begin{bmatrix} -1.4244 \\ 3.3488 \\ -1.4244 \end{bmatrix} \text{ and } \mathbf{x}_6' = \begin{bmatrix} 1 \\ -2.3510 \\ 1 \end{bmatrix},$$

$$A\mathbf{x}_7 = \mathbf{x}_6', \ \mathbf{x}_7 = \begin{bmatrix} -1.4255 \\ 3.3510 \\ -1.4255 \end{bmatrix} \text{ and } \mathbf{x}_7' = \begin{bmatrix} 1 \\ -2.3508 \\ 1 \end{bmatrix}. \tag{8.80}$$

Thus $\lambda_3 \approx -1/1.4255 \approx -0.7015$ and $\mathbf{s}_3 \approx (1, -2.3508, 1)^T$.

Experimentation with different initial vectors would show that the above eigenvalues are simple. (See Exercise 8.3.1.) Thus we still have to find a third eigenvalue whose absolute value must fall between 0.7016 and 5.7016. Since A is symmetric, we know that its eigenvalues are real, and so we can look for one near 3 or -3. Using $c = 3$ for shifting would yield λ_1 again, and so we use $c = -3$. Another argument that suggests a negative λ_2 is the following: We know that the eigenvectors of a symmetric matrix belonging to different eigenvectors are orthogonal to each other, and the vector $\mathbf{x}_0 = (1, 0.5, -1.5)^T$ is easily seen to be approximately orthogonal to the $\mathbf{s}_1$ and $\mathbf{s}_3$ found above. (We could, of course, find a vector exactly orthogonal to $\mathbf{s}_1$ and $\mathbf{s}_3$, which would be an appropriate $\mathbf{s}_2$, but we want to illustrate the shifted power method.) Now $A\mathbf{x}_0 \approx -\mathbf{x}_0$, and so λ_2 should be negative. Thus we apply the inverse power method to the matrix

$$B = A + 3I = \begin{bmatrix} 4 & 2 & 3 \\ 2 & 4 & 2 \\ 3 & 2 & 4 \end{bmatrix}. \tag{8.81}$$

Setting

$$\mathbf{x}_0 = \mathbf{x}_0' = \begin{bmatrix} 1.0 \\ 0.5 \\ -1.5 \end{bmatrix}, \tag{8.82}$$

we get

$$B\mathbf{x}_1 = \mathbf{x}_0',\ \mathbf{x}_1 = \begin{bmatrix} 1.1500 \\ 0.2250 \\ -1.3500 \end{bmatrix} \text{ and } \mathbf{x}_1' = \begin{bmatrix} 1 \\ 0.1957 \\ -1.1739 \end{bmatrix},$$

$$B\mathbf{x}_2 = \mathbf{x}_1,\ \mathbf{x}_2 = \begin{bmatrix} 1.0500 \\ 0.0859 \\ -1.1239 \end{bmatrix} \text{ and } \mathbf{x}_2' = \begin{bmatrix} 1 \\ 0.0818 \\ -1.0704 \end{bmatrix},$$

$$B\mathbf{x}_3 = \mathbf{x}_2',\ \mathbf{x}_3 = \begin{bmatrix} 1.0200 \\ 0.0357 \\ -1.0504 \end{bmatrix} \text{ and } \mathbf{x}_3' = \begin{bmatrix} 1 \\ 0.0350 \\ -1.0298 \end{bmatrix},$$

$$B\mathbf{x}_4 = \mathbf{x}_3',\ \mathbf{x}_4 = \begin{bmatrix} 1.0084 \\ 0.0152 \\ -1.0214 \end{bmatrix} \text{ and } \mathbf{x}_4' = \begin{bmatrix} 1 \\ 0.0151 \\ -1.0129 \end{bmatrix},$$

$$B\mathbf{x}_5 = \mathbf{x}_4',\ \mathbf{x}_5 = \begin{bmatrix} 1.0036 \\ 0.0066 \\ -1.0092 \end{bmatrix} \text{ and } \mathbf{x}_5' = \begin{bmatrix} 1 \\ 0.0065 \\ -1.0056 \end{bmatrix},$$

$$B\mathbf{x}_6 = \mathbf{x}_5',\ \mathbf{x}_6 = \begin{bmatrix} 1.0016 \\ 0.0028 \\ -1.0400 \end{bmatrix} \text{ and } \mathbf{x}_6' = \begin{bmatrix} 1 \\ 0.0028 \\ -1.0240 \end{bmatrix}. \tag{8.83}$$

Hence we find $\lambda_2(B) \approx 1$, and so $\lambda_2(A) \approx -2$ and $\mathbf{s}_2 \approx (1, 0, -1)^T$. ♦

Exercises

Exercise 8.3.1.
a. Apply the method of Theorem 8.3.1 to the matrix

$$A = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

of Example 7.1.3 of page 257. Use $k = 1$, first with the initial vector $\mathbf{x}_0 = (1, 1, 1)^T$ and second, with the initial vector $\mathbf{x}_0 = (1, -1, 1)^T$.
b. How would you modify Theorem 8.3.1 in the case in which there is a dominant eigenvalue of multiplicity 2 or more?

Exercise 8.3.2. Show that if the eigenvalues of A are 1, 2, and 3, then there is no c, whether real or complex, that would make $2-c$ a dominant eigenvalue of $B = A - cI$.

Exercise 8.3.3. Show that if c is not an eigenvalue of a square matrix A, then $B = A - cI$ is nonsingular.

MATLAB Exercises

Exercise 8.3.4. Let A = **hilb**(3) and compute Aˆ8./Aˆ7. Explain what you get.

Exercise 8.3.5. Use the methods of this section to compute the eigenvalues and eigenvectors of
a. A = **hilb**(3),
b. A = **ones**(3),
c. A = **ones**(4),
d. A = **hadamard**(4).

Erratum to: A Concise Introduction to Linear Algebra

Géza Schay

Erratum to:
G Schay, *A Concise Introduction to Linear Algebra*, https://doi.org/10.1007/978-0-8176-8325-2

Due to mostly post-production errors, a number of author corrections were not incorporated into the first edition of this book. With the author's approval, the following amendments have been made to the text, as well as fixes for other small typos. The publisher and the author apologize for these errors.

The following is a list of the substantive corrections that have been made in the 2018 reprinting

- On line 2 of the proof for Theorem 1.1.1., a capital P has been replaced with a lowercase p
- In Equation 1.31, a bolded subscript q has been unbolded and set italic
- The first line of Exercise 1.2.13 has been changed to read "Using the result of Theorem 1.2.6"
- The QED symbol for Example 1.3.3. has been moved to after Equation 1.73
- The paragraph before Definition 2.1.1. has been changed to reflect the fact that a definition for a *pivot* is explained on page 53
- In Fig. 2.6, the θ has been unbolded
- The paragraph after Definition 2.4.1. has been changed to refer to Definition 2.4.1., not Definition 4.2.4.
- The text immediately before the rules in Definition 3.1.1. has been changed to read "*satisfying the eight rules below[1] for all* **p**, **q**, **r**, *and real a and b*"

The updated online versions of these chapters can be found at
https://doi.org/10.1007/978-0-8176-8325-2_1
https://doi.org/10.1007/978-0-8176-8325-2_2
https://doi.org/10.1007/978-0-8176-8325-2_3
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- The text that ends Example 3.1.1. has been changed to read "was proved in Theorem 1.1.1. for $n = 2$."
- In the proof of Property 3 for Theorem 3.1.1., a 1 has been deleted from the second equation.
- For Equation 3.50, the upper limit of the summations have been specified as m's
- In Definition 3.5.4, the set for all vectors has been changed to $\mathbb{R}^m$
- The sentence before Equation 3.153 now refers to Equations 3.149 and 3.150
- The end of Theorem 3.6.3 now refers to Equation 3.155
- The sentence after Equation 3.164 now refers to Equation 3.155
- In the formula used after Definition 3.6.1 has been changed to refer to Equation 3.150
- Property 4 in Definition 4.2.1 now begins "T *is said to be an isomorphism if it is linear and both one-to-one and onto*."
- The paragraph after Example 4.2.2. now starts "That X and $\mathbb{R}^n$ above are isomorphic"
- The paragraph after Equation 5.37 now refers to $\mathbb{R}^3$
- After Equation 5.41, the author meant to leave the proof of Equation 5.41 for Exercise 5.1.18., not Equation 5.40
- Exercise 5.1.18 now refers to "the normal system of 5.41 of the least-squares plane", rather than 5.40
- Exercise 5.1.22. now asks students to use Equations 5.24 through 5.28
- Exercise 5.1.24 now asks students to use Equations 5.38 through 5.41
- The Proof of Theorem 5.2.1. now reads "Since $\mathbf{x} \in \text{Span}\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$, there exists a decomposition of $\mathbf{x}$ in the form of Equation 5.47 with some coefficients x_{Ai}. Taking the dot product of both sides of Equation 5.47 with $\mathbf{a}_i$ and utilizing the assumed orthogonality $\mathbf{a}_i \cdot \mathbf{a}_j = 0$ for all $i \neq j$, we get Equation 5.48. Also, if we take $\mathbf{x} = 0$, then Equation 5.48 shows that each x_{Ai} equals zero, and so the vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ are independent."
- The sentence before Corollary 5.2.1. now refers to Equation 5.47
- Part of the paragraph at the bottom of page 212 now reads "the right-hand sides of Equations 5.49 and 5.50" and "in Equation 5.49, $\mathbf{x} \in \text{Span}\{\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_n\}$, but in Equation 5.50, $\mathbf{x}$ may be outside"
- In the first step for the Proof of Theorem 5.2.4., the references to equations are now to Equation 5.58, Equation 5.59, and Equation 5.59, respectively
- The second step for the Proof of Theorem 5.2.4., the references to equations are now to Equation 5.58, Equation 5.58, Equation 5.61, and Equation 5.61, respectively

- The Proof for Theorem 5.2.5., the references to equations are now to Equation 5.57, Equation 5.63, Equation 5.63, Equation 5.49, and Equation 5.63, respectively
- Exercise 5.2.2. now asks students to use Equation 5.50
- Exercise 5.2.5. now asks students to prove Equation 5.52 in a particular case
- The beginning of the MATLAB Exercises for chapter 5 now refers to Equation 5.50
- In the second property for Definition 6.1.1., Equation 6.4 holds for "*any i, j with* $i \neq j$"
- The two sentences before Equation 7.19 now read "Recall that a homogenous equation has nontrivial solutions if and only if its matrix is singular. By Theorem 6.1.8. for Equation 7.17, this condition is equivalent to"
- In the first part of Equation 7.164, α has been replaced with *a*
- The ψ's in Equation A.32 have been replaced with ϕ's

A. *Appendices*

A.1 Implication and Equivalence

In this section we shall discuss in a very informal manner two basic relations of logic and the ways in which they are used in mathematics. These relations apply to statements that are either true or false, such as "$2+2=4$," "the sun is shining," "$x+1=4$." Here the first statement is always true, the second one is occasionally true, and "$x+1=4$" is true if x is 3 and is false for all other values of x. However, in mathematics, as in common speech, the statements we make are generally considered to be true, in contrast to some discussions in formal logic where they may be either true or false. So the equation above is usually considered to mean that x is 3.

Statements can be connected by various logical operations to form new statements. The connectives "and," "or," "not," "if... then" indicate the simplest of these. The first three are fairly straightforward and well known, but the last one is frequently misused and misunderstood. Also, since it is used in the theorems and proofs throughout the book, we now examine it in detail.

Let p and q denote arbitrary statements. We call the compound statement "if p then q" an *implication*. Note that when we make such a statement, we mean the implication to be true, but that does not say anything about the truth of p and q themselves. For instance, "if it rains, I will take my umbrella" does not say anything about rain or shine on any particular day, nor whether I will take my umbrella tomorrow.

It is sometimes helpful to illustrate implications by an Euler diagram[1] as shown in Figure A.1.

This diagram is meant to be interpreted as saying that for the points of the set P the statement p is true, and for those of Q the statement q is true. Clearly, for this configuration, if a point is in P, then it is also in Q. The diagram also shows that for the points outside the set P the statement "not p" is true, and for the points outside the set Q the statement "not q" is true.

[1] Euler first used these diagrams around 1770, more than a century before Venn's more familiar diagrams were published.

The original version of this chapter was revised. An erratum can be found at
https://doi.org/10.1007/978-0-8176-8325-2_9

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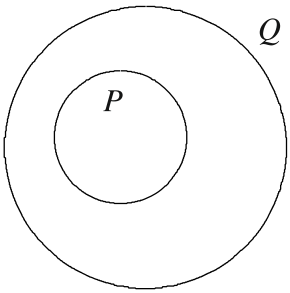

Fig. A.1. Euler diagram for implication

There are various equivalent ways of expressing an implication. Here is a list:

1. If p then q,
2. q if p,
3. $p \Rightarrow q$,
4. p implies q,
5. p only if q (look at the diagram),
6. q is a necessary condition for p,
7. p is a sufficient condition for q,
8. q follows from p.

An important fact is that implication is *transitive*; that is, if p implies q, and q implies r, then p implies r.

While this list merely showed different possible language constructions for the same relation, we can also make a logical change to produce a new relation that will be true exactly when the above implication is true. We obtain this new relation by interchanging p and q with (not q) and (not p), respectively. Thus $p \Rightarrow q$ becomes (not q) $\Rightarrow$ (not p). The latter statement is called the *contrapositive* of the former and is logically equivalent to it, as can perhaps best be seen by looking at the Euler diagram. Of course, the contrapositive can also be expressed in all the various equivalent ways that we had for the original implication.

Example A.1.1. In calculus we have encountered the simple theorem:

If $\sum_{n=1}^{\infty} a_n$ *converges, then* $\lim_{n\to\infty} a_n$ *exists and equals* 0.

The contrapositive of this statement is the equivalent statement:

If it is not true that $\lim_{n\to\infty} a_n$ *exists and equals* 0*, then* $\sum_{n=1}^{\infty} a_n$ *diverges.* ♦

Note that, because of the logical equivalence of contrapositives, it is sufficient to prove only one of these statements to establish the truth of the other one as well. The first statement is easy to prove, while the second one would be hard to prove directly, but is the one we need in most applications.

Implication is not symmetric. That is, $p \Rightarrow q$ is not equivalent to $q \Rightarrow p$; they say different things. The latter is called the *converse* of the former and vice versa. Depending on what p and q stand for, it can happen that one of the two implications is true and the other is not, that both are true, or that both are false. For instance, the statement "If an animal is a dog, then it is a canine" is true, but its converse "If an animal is a canine, then it is a dog" is false, because it could be a fox. Note that the converse too can be expressed in all the equivalent ways that were possible for the original implication.

Example A.1.2. The statement of Example A.1.1 above also illustrates the case in which the original implication is true but its converse is not. The original statement is:

If $\sum_{n=1}^{\infty} a_n$ *converges, then* $\lim_{n\to\infty} a_n$ *exists and equals* 0.

The converse is:

If $\lim_{n\to\infty} a_n$ *exists and equals* 0*, then* $\sum_{n=1}^{\infty} a_n$ *converges.*

As we know, this statement is false, since the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ provides a counterexample. ♦

Example A.1.3. The Theorem of Pythagoras illustrates the case of both a statement and its converse being true. Letting a, b, c stand for the lengths of the sides and the hypotenuse, respectively, of a triangle, "*If the triangle is a right triangle, then* $a^2 + b^2 = c^2$" and its converse "*If* $a^2 + b^2 = c^2$*, then the triangle is a right triangle*" are both true. ♦

The case of both a statement and its converse being true occurs so often that it has a new name and new language associated with it: The new relation between p and q is called *equivalence*, and it can be expressed by combining any of the forms in the list of expressions for implication for the two statements $p \Rightarrow q$ and $q \Rightarrow p$ and for their contrapositives. The most common ways are[2]

1. p is equivalent to q,
2. $p \Leftrightarrow q$,
3. p if and only if q,
4. p iff q,
5. $(p \Rightarrow q)$ and $(q \Rightarrow p)$,
6. p is a necessary and sufficient condition for q.
7. $(p \Rightarrow q)$ and $((\text{not } p) \Rightarrow (\text{not } q))$.

[2] The numbers in this list have only a vague connection to the numbers in the preceding list.

Observe that "p if and only if q" is an abbreviation for "(p if q) and (p only if q)," which is equivalent to "(p if q) and (q if p)." This statement can be written symbolically as "$(q \Rightarrow p)$ and $(p \Rightarrow q)$," which means the same as Statement 5 above, since the "and" operation is commutative. Thus, when we want to prove an equivalence, we must prove two implications. We usually do this in either the form 5 or in the form 7, which is obtained from 5 by replacing the second part by its contrapositive.

A.2 Complex Numbers

In this appendix we give a brief review of complex numbers and exponential functions.

In 1545 an Italian mathematician called Gerolamo Cardano published a general formula for the solution of cubic equations, building on earlier work of special cases by others. Cardano's formula has the interesting property that, even when the equation has three real roots, it gives those roots only if the square roots of negative numbers are used in the computations. This discovery started the exploration of such numbers, which were named *imaginary numbers*, as opposed to the familiar *real* numbers, and of sums of the two kinds, named *complex numbers.* These unfortunate names have stuck, although we now regard imaginary numbers as no more imaginary than reals or even than natural numbers. The theory of complex numbers was fully developed only in the beginning of the nineteenth century and that of complex matrices at the end of it.

Definition A.2.1. *The set $\mathbb{C}$ of complex numbers is a two-dimensional vector space*[3] *with multiplication also defined as follows: Two basis vectors of $\mathbb{C}$ are denoted by 1 and i, and each element of $\mathbb{C}$ is usually written as $z = a + bi$, where a and b are real numbers and a is an abbreviation for $a \cdot 1$. The vector 1 is identified with the real number 1, and its real multiples with the real numbers. The complex numbers of the form bi, for any real b, are called imaginary numbers. The product of two complex numbers is defined to be commutative, associative, and distributive, and for the basis vectors as $1^2 = 1$, $1 \cdot i = i$ and $i^2 = -1$.*

This definition has several simple consequences.

First, the original problem of the square roots of negative numbers is now solved: We define $\sqrt{c}$, for any complex number c, as any complex number z whose square is c, that is, $z = \sqrt{c}$ if $c = z^2$. Then $\sqrt{-1} = \pm i$, since $(\pm i)^2 = (\pm 1)^2 \cdot i^2 = 1 \cdot (-1) = -1$. The square root of any other negative

[3] To be more precise, it is a two-dimensional vector space *over the reals*, meaning that it is two dimensional with real numbers as the scalars in the definition of a vector space.

number can now be computed as follows: If a is any positive number, then $\sqrt{-a} = \pm i\sqrt{a}$, since then $(\pm i\sqrt{a})^2 = (\pm i)^2 (\sqrt{a})^2 = (-1)a = -a$. We shall see shortly that every complex number except zero has exactly two square roots, and exactly n nth roots. (Every root of 0 is 0.) Note that, when dealing with complex numbers, the symbol $\sqrt{c}$ is used ambiguously for either of the two roots, unlike in the case of real positive c, where it stands for the positive root. A similar convention holds for nth roots. The correct meaning should always be clear from the context.

The set $\mathbb{C}$ is closed under multiplication:

$$(a+bi)(c+di) = (ac-bd) + (ad+bc)\,i. \tag{A.1}$$

Division of complex numbers is also possible within $\mathbb{C}$, with the not unexpected exception of division by zero. For any complex numbers $a,\ b$, with $b \neq 0$, we say $a/b = z$ if $a = bz$. To find z let us write $a = a_1 + a_2 i,\ b = b_1 + b_2 i$, and $z = x + yi$, where $a_1,\ a_2,\ b_1,\ b_2, x, y$ are real. Then $a = bz$ becomes

$$a_1 + a_2 i = (b_1 + b_2 i)(x + yi) = b_1 x - b_2 y + (b_1 y + b_2 x)i. \tag{A.2}$$

This complex equation is equivalent to the two real equations

$$b_1 x - b_2 y = a_1 \tag{A.3}$$

and

$$b_2 x + b_1 y = a_2. \tag{A.4}$$

Their solution is

$$x = \frac{a_1 b_1 + a_2 b_2}{b_1^2 + b_2^2} \text{ and } y = \frac{a_2 b_1 - a_1 b_2}{b_1^2 + b_2^2}. \tag{A.5}$$

These fractions always exist, since $b \neq 0$ implies $b_1^2 + b_2^2 \neq 0$.

Now that we know that $a/b = z$ is well defined, we can obtain it more simply by multiplying both numerator and denominator of

$$\frac{a}{b} = \frac{a_1 + a_2 i}{b_1 + b_2 i} \tag{A.6}$$

by $\overline{b} = b_1 - b_2 i$ as in the following example.

Example A.2.1.

$$\frac{1+2i}{3+4i} = \frac{(1+2i)(3-4i)}{(3+4i)(3-4i)} = \frac{1\cdot 3 + 2\cdot 4 + (2\cdot 3 - 1\cdot 4)i}{3^2+4^2} = \frac{11}{25} + \frac{2}{25}i. \tag{A.7}$$

♦

The multiplier we used in the denominator occurs in many other situations as well, and so it has a name: For any complex number $z = x + yi$ we call the complex number $\overline{z} = x - yi$ the *complex conjugate of* z.

The main properties of the complex conjugate are:

$$z\overline{z} = x^2 + y^2, \tag{A.8}$$

$$z + \overline{z} = 2x, \tag{A.9}$$

$$z - \overline{z} = 2yi, \tag{A.10}$$

$$\overline{\overline{z}} = z, \tag{A.11}$$

$$\overline{z_1 + z_2} = \overline{z}_1 + \overline{z}_2, \tag{A.12}$$

and

$$\overline{z_1 z_2} = \overline{z}_1 \overline{z}_2. \tag{A.13}$$

Since $\mathbb{C}$ is a two-dimensional vector space, it can be represented by the points or vectors of a plane. Length, addition, subtraction, and multiplication by reals have the same geometrical meaning as in $\mathbb{R}^2$. We have, however, new constructions for products, quotients, powers, and roots of complex numbers, as will now be described.

For the complex number $z = x + yi$ we write $x = \Re z$ and $y = \Im z$ for the real and imaginary parts of z, respectively. The *absolute value* or *modulus* of z is defined as

$$|z| = \sqrt{x^2 + y^2}, \tag{A.14}$$

and any one of the angles from the positive real axis to the vector z is called the *argument* of z and is denoted by $\arg z$ or arc z. Thus, in polar coordinates, any $z \neq 0$ is represented by

$$z = r(\cos\phi + i \sin\phi), \tag{A.15}$$

where

$$r = |z| \text{ and } \phi = \arg z. \tag{A.16}$$

The polar form of z leads easily to the geometric meaning of the product. Let

$$z_1 = r_1(\cos\phi_1 + i \sin\phi_1), \tag{A.17}$$

and

$$z_2 = r_2(\cos\phi_2 + i \sin\phi_2). \tag{A.18}$$

Then

$$z_1 z_2 = r_1 r_2[(\cos\phi_1 \cos\phi_2 - \sin\phi_1 \sin\phi_2) + i(\sin\phi_1 \cos\phi_2 + \cos\phi_1 \sin\phi_2)] \tag{A.19}$$

and so

$$z_1 z_2 = r_1 r_2[\cos(\phi_1 + \phi_2) + i \sin(\phi_1 + \phi_2)]. \tag{A.20}$$

Thus, in the multiplication of complex numbers, the absolute values are multiplied and the arguments are added.

The following properties of the absolute value can easily be deduced:

$$|z|^2 = z\overline{z}, \tag{A.21}$$

$$|z_1 z_2| = r_1 r_2 = |z_1||z_2|, \tag{A.22}$$

and

$$||z_1| - |z_2|| \le |z_1 + z_2| \le |z_1| + |z_2|. \tag{A.23}$$

For a sequence of complex numbers z_n we say that

$$z_n \to z \text{ as } n \to \infty \text{ if } |z_n - z| \to 0 \text{ as } n \to \infty, \tag{A.24}$$

that is, if the real-valued sequence $|z_n - z|$ converges to zero.

As for reals, a series $\sum_{n=0}^{\infty} z_n$ is said to be convergent if the sequence of its partial sums $s_k = \sum_{n=0}^{k} z_n$ converges, and absolutely convergent if $\sum_{n=0}^{\infty} |z_n|$ is convergent. It is easy to prove that absolute convergence of a series implies its convergence, just as for reals. (See Exercises A.2.4 and A.2.5.)

The series $\sum_{n=0}^{\infty} \frac{z^n}{n!}$ is absolutely convergent for every value of z, since we know that the real series $\sum_{n=0}^{\infty} \frac{|z|^n}{n}$ is convergent for every value of $|z|$. The sum of the former series is e^z when z is real, and so we use it to define e^z for complex z:

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \tag{A.25}$$

for every $z \in \mathbb{C}$. This definition preserves the multiplication property $e^{z_1} e^{z_2} = e^{z_1+z_2}$ for complex exponents too:[4]

$$\begin{aligned} e^{z_1} e^{z_2} &= \left(1 + z_1 + \frac{z_1^2}{2!} + \frac{z_1^3}{3!} + \cdots\right)\left(1 + z_2 + \frac{z_2^2}{2!} + \frac{z_2^3}{3!} + \cdots\right) \\ &= 1 + (z_1 + z_2) + \frac{1}{2!}\left(z_1^2 + 2z_1 z_2 + z_2^2\right) \\ &\quad + \frac{1}{3!}\left(z_1^3 + 3z_1^2 z_2 + 3z_1 z_2^2 + z_2^3\right) + \cdots \\ &= 1 + (z_1 + z_2) + \frac{1}{2!}(z_1 + z_2)^2 + \frac{1}{3!}(z_1 + z_2)^3 + \cdots = e^{z_1+z_2}. \end{aligned} \tag{A.26}$$

[4] In this derivation we make use of the fact (without proof) that absolutely convergent series can be multiplied term by term and the terms may be rearranged arbitrarily.

If we use the definition above of the exponential function with $z = i\phi$, where ϕ is real, then we get Euler's formula:

$$e^{i\phi} = 1 + i\phi + \frac{1}{2!}(i\phi)^2 + \frac{1}{3!}(i\phi)^3 + \cdots = 1 + i\phi - \frac{\phi^2}{2!} - i\frac{\phi^3}{3!} + \frac{\phi^4}{4!} + \cdots$$
$$= \left(1 - \frac{\phi^2}{2!} + \frac{\phi^4}{4!} + \cdots\right) + i\left(\phi - \frac{\phi^3}{3!} + \frac{\phi^5}{5!} + \cdots\right); \tag{A.27}$$

that is,

$$e^{i\phi} = \cos\phi + i\sin\phi. \tag{A.28}$$

From this equation we get

$$\cos\phi = \frac{e^{i\phi} + e^{-i\phi}}{2} \text{ and } \sin\phi = \frac{e^{i\phi} - e^{-i\phi}}{2i} \tag{A.29}$$

and the polar form of any z as

$$z = re^{i\phi}. \tag{A.30}$$

The polar form of $\overline{z}$ can be written similarly as

$$\overline{z} = re^{-i\phi}. \tag{A.31}$$

From Equations A.26 and A.30 we obtain the following important rule for the multiplication of complex numbers: If $z_1 = r_1 e^{i\phi_1}$ and $z_2 = r_2 e^{i\phi_2}$, then

$$z_1 z_2 = r_1 r_2 e^{i(\phi_1 + \phi_2)}. \tag{A.32}$$

Using Euler's formula (Equation A.28) for the exponential here, we can reproduce Equation A.20. Thus the above derivation of Equation A.32 provides a new proof of the trigonometric formulas for the sine and cosine of the sum of two angles, based on the Taylor series. Alternatively, Equation A.32 could be obtained from Equation A.20 and Euler's formula.

Repeated application of Equation A.32 to the same $z = re^{i\phi}$ leads to the power rule

$$z^n = r^n e^{in\phi} \tag{A.33}$$

for any positive integer n. In Exercise A.2.6 this will be extended to roots as well.

Exercises

Exercise A.2.1. Prove Equation A.22.

Exercise A.2.2. Prove inequality A.23.

Exercise A.2.3. Let $P(z) = \sum_{k=0}^{n} a_k z^k$ be a polynomial with real coefficients a_n. Show that if z_0 is a zero of P, that is, if $P(z_0) = 0$, then so too is $\overline{z}_0$. In other words, complex roots of algebraic equations with real coefficients always come in complex conjugate pairs.

Exercise A.2.4. Prove that a complex series $\sum_{n=0}^{\infty} z_n$, with $z_n = x_n + iy_n$, converges if and only if the series of the real and imaginary parts $\sum_{n=0}^{\infty} x_n$ and $\sum_{n=0}^{\infty} y_n$ both converge.

Exercise A.2.5. Prove that absolute convergence of a complex series $\sum_{n=0}^{\infty} z_n$ implies its convergence. (*Hint*: Use the result of the previous exercise, the facts that $|z_n| \geq |x_n|$ and $|z_n| \geq |y_n|$, the comparison test, and the corresponding theorem for the real series $\sum_{n=0}^{\infty} x_n$ and $\sum_{n=0}^{\infty} y_n$.)

Exercise A.2.6. Invert Equation A.33 to obtain a formula for nth roots: If $z = w^n$, then we call w an nth root of z and write $w = z^{1/n}$. Letting $z = re^{i(\phi+2k\pi)}$ here, for any integer k, and $w = Re^{i\Phi}$, show that any nth root of z must satisfy

$$z^{1/n} = r^{1/n} e^{i(\phi+2k\pi)/n}, \tag{A.34}$$

and different values of k result in exactly n distinct nth roots for any $z \neq 0$.

Exercise A.2.7. Use Formula A.34 to find and plot
a. all square roots of i,
b. all square roots of $1 + i$,
c. all cube roots of 1,
d. all cube roots of -1,
e. all fourth roots of i.

Further Reading

The following book is similar in spirit to this one.

Malcolm Adams and Theodore Shifrin, *Linear Algebra: A Geometric Approach*, 2nd ed. W. H. Freeman, 2010.

The following three books are introductions to linear algebra but contain many more applications.

David C. Lay: *Linear Algebra and Its Applications*, 4th ed. Addison-Wesley, 1993.

Steven J. Leon: *Linear Algebra with Applications*, 8th ed. Prentice-Hall, 2009.

Gilbert Strang: *Linear Algebra and its Applications*, 3rd ed. Harcourt Brace, 1988.

The next four books are more advanced.

Lorenzo Sadun: *Applied Linear Algebra: The Decoupling Principle*, 2nd ed. Prentice-Hall, 2008. This book deals with Fourier series, differential equations, and infinite vector spaces.

Stephen H. Friedberg, Arnold J. Insel, and Lawrence E. Spence: *Linear Algebra*, 3rd ed. Prentice-Hall, 2001.

Sterling K. Berberian and T.J.I. Bromwich: *Introduction to Hilbert Space*, 2nd ed. Chelsea, 1999.

David G. Luenberger: *Introduction to Linear and Nonlinear Programming*, 2nd ed. Addison-Wesley 2003.

The next three references discuss computer algebra tools for linear algebra.

Elias Y. Deeba and Ananda D. Gunawardena: *Interactive Linear Algebra With Maple V*. Springer-Verlag, 1998.

John R. Wicks: *Linear Algebra: An Interactive Laboratory Approach With Mathematica*, Addison-Wesley, 1996.

http://www.umassd.edu/specialPrograms/atlast/ (MATLAB).

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