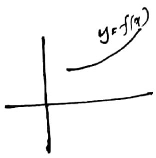
Volume

Volume of a Solid of Revolution about X-axis (Disk method)

Let f be a continuous non negative function on $[a, b]$ and let R be the region under the graph of f on the interval $[a, b]$. The volume of the solid of revolution generated by revolving R about the x-axis is

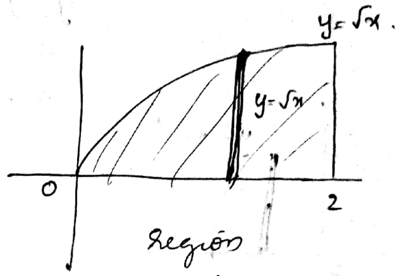
$$V = \int_a^b \pi \left[f(x)\right]^2 dx .$$

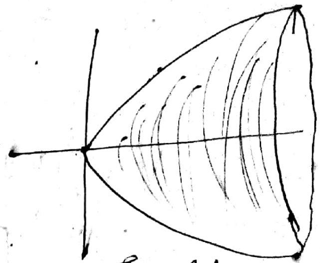


i.e. $V = \pi \int_a^b y^2 \, dx \quad , f \geq 0$

- Find the volume of the solid generated by ~~obtaining~~ revolving the region under the graph of $y = \sqrt{x}$ on $[0, 2]$ about the x-axis.

Solution :-





$$V = \int_a^b \pi \, y^2 dx .$$

$$= \pi \int_0^2 (\sqrt{x})^2 dx$$

$$= \pi \int_0^2 x \, dx .$$

$$= \pi \left[\frac{x^2}{2}\right]_0^2$$

$$= 2\pi$$

- By revolving the region under the graph of $y = \sqrt{r^2 - x^2}$ on $[-r, r]$. Show that the volume of a sphere of radius r is $V = \frac{4}{3}\pi r^3$.

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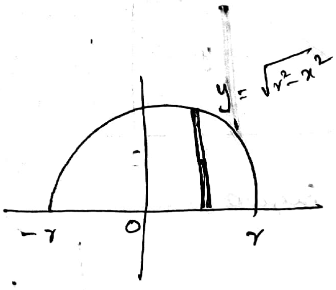
$y = \sqrt{r^2 - x^2}$

$y^2 = r^2 - x^2$

$x^2 + y^2 = r^2$



$$V = \int_a^b \pi y^2 \, dx$$

$$= \pi \int_{-r}^{r} \left[\sqrt{r^2 - x^2}\right]^2 dx$$

$$= \pi \int_{-r}^{r} (r^2 - x^2) \, dx$$

$$= \pi \, 2 \int_0^r (r^2 - x^2) \, dx$$

[The graph is ~~by~~ symmetry with respect to y - axis.

$$= 2\pi \left[r^2 x - \frac{x^3}{3} \right]_0^r$$

$$= 2\pi \left[r^3 - \frac{r^3}{3} \right]$$

$$= 4\pi/_3 \, r^3$$

$$= \frac{4}{3} \pi r^3$$

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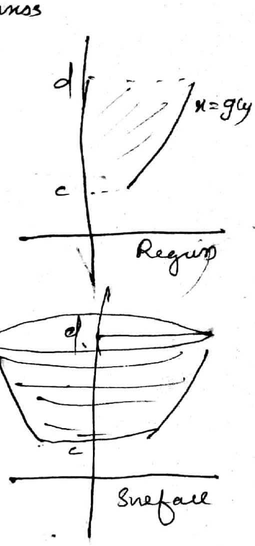
Volume of a Solid of Revolution (Disk Method) about y-axis

$$V = \int_c^d \pi [g(y)]^2 \, dy.$$

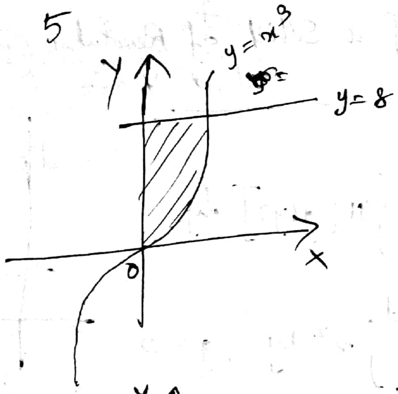
$$V = \pi \int_c^d x^2 \, dy \;, g \geq 0$$

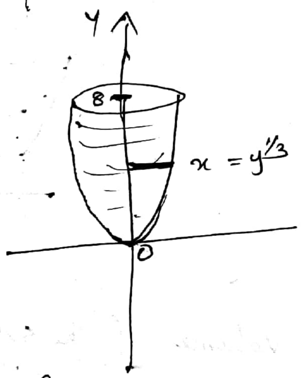


- Find the volume of the solid obtained by revolving the region bounded by the graphs of $y = x^3$, $y = 8$ and $x = 0$ about y-axis.

$y = x^3 \Rightarrow y^{1/3} = (x^3)^{1/3}$

i.e. $x = y^{1/3}$





$$V = \int_0^8 \pi \left(y^{1/3}\right)^2 dy$$

$$= \pi \int_0^8 y^{2/3}\, dy$$

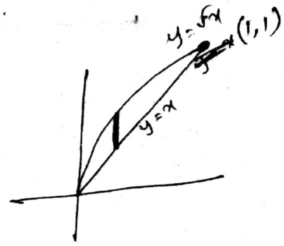
$$= \pi \left[\frac{y^{2/3+1}}{2/3+1}\right]_0^8 = \pi \left[\frac{y^{5/3}}{5/3}\right]_0^8$$

$$= \frac{3\pi}{5}\left[8^{5/3}\right] = \frac{96\pi}{5}$$

Volume by Washer Method (Region revolved about the x-axis)

$$V = \pi \int_a^b \left([f(x)]^2 - [g(x)]^2 \right) dx \; ; \; f \geq g \geq 0$$

- Find the volume of the solid obtained by revolving the region bounded by $y = \sqrt{x}$ and $y = x$ about the x-axis.



$f(x) = \sqrt{x}$; $g(x) = x$.

$$V = \pi \int_0^1 \left((\sqrt{x})^2 - x^2 \right) dx$$

$$= \pi \int_0^1 (x - x^2)\, dx.$$

$$= \pi \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1$$

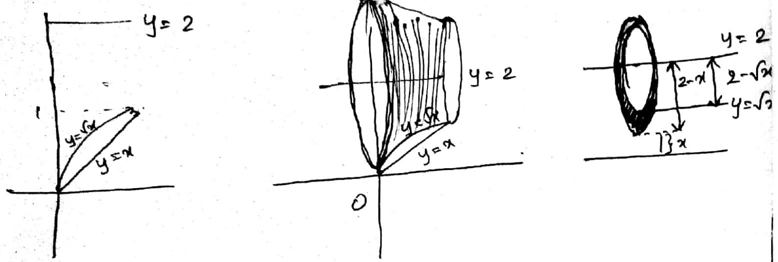
$$= \pi \left[\frac{1}{2} - \frac{1}{3}\right]$$

$$= \pi \times \frac{1}{6}$$

$$= \frac{\pi}{6}$$

$\sqrt{x} = x$
$x = x^2$
$x^2 - x = 0$
$x(x-1) = 0$
$x = 0, 1$
i.e. the graph intersect at 0 & 1

Find the volume of the solid generated by revolving the region bounded by $y=\sqrt{x}$ and $y = x$ about the line $y = 2$.



$$V = \int_0^1 \pi \left[x^2 - 5x + 4\sqrt{x}\right] dx$$

$$= \pi \left[\frac{x^3}{3} - \frac{5x^2}{2} + \frac{8}{3}x^{3/2}\right]_0^1$$

$$= \frac{\pi}{2}$$

Volume of a Solid with known Cross sections

Let S be a solid ~~generate~~ bounded by planes that are perpendicular to the x-axis at $x=a$ and $x=b$. If the cross sectional area of S at any point x in $[a,b]$ is $A(x)$, where A is continuous on $[a,b]$, then volume of S is

$$V = \int_a^b A(x)\, dx.$$

Arc length of a curve

A function f is smooth on an interval if its derivative f' is continuous on that interval. The graph of smooth function is called smooth curve.

Let f be smooth on $[a,b]$. Then the arc length of the graph of f from $(a, f(a))$ to $(b, f(b))$ is

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

Arc length differentials

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

or equivalently $(ds)^2 = (dy)^2 + (dx)^2$

Surface Area of a Surface of Revolution

Let f be a nonnegative smooth function on $[a,b]$. The surface area of the surface obtained by revolving the graph of f about x-axis is

$$S = 2\pi \int_a^b f(x)\sqrt{1+[f'(x)]^2}\,dx.$$

$$S = 2\pi \int_a^b y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

Work done by a Constant Force

The work W done by a constant force F in moving a body a distance d in the direction of the force is

$$W = F.d.$$

Work done by a Variable Force

Suppose that a force F, where F is continuous on $[a,b]$, acts on a body moving it along x-axis. Then the work done by the force in moving the body from $x=a$ to $x=b$ is

$$W = \int_a^b F(x)\,dx.$$

Definitions

Let S denote a system of n masses $m_1, m_2, \ldots, m_n$ located at $x_1, x_2, \ldots, x_n$, lying on a line respectively. and let $m = \sum_{k=1}^{n} m_i$ denote the total mass of the system.

- The moment of S about the origin is

$$M = \sum_{k=1}^{n} m_k x_k.$$

- The center of mass of S is located at

$$\bar{x} = \frac{M}{m} = \frac{1}{m}\sum_{k=1}^{n} m_k x_k.$$

Q. Find the center of mass of a system of four objects located at the points $-3, -1, 2$ and 4 on the x-axis (x in meters) with masses $3, 2, 4$ and 6 kilograms, respectively.

$$\bar{x} = \frac{1}{n}\sum_{k=1}^{n} m_k x_k$$

$$= \frac{3(-3)+2(-1)+4(2)+6(4)}{3+2+4+6} = \frac{21}{15} = \frac{7}{5}$$

Center of mass of a system of n particles in a plane :-

Let S denote a system of n particles with masses $m_1, m_2, \dots, m_n$ located at the points $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$ respectively. and let $m = \sum_{k=1}^{n} m_k$ denote the total mass of the system.

- The moment of S about the x-axis is
$$M_x = \sum_{k=1}^{n} m_k y_k .$$

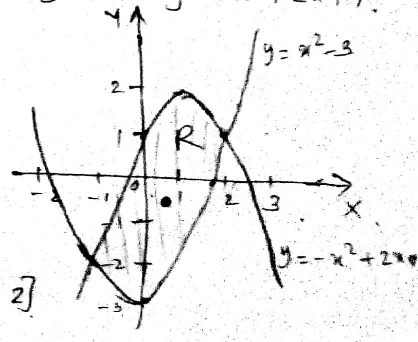
- The moment of S about the Y-axis is
$$M_y = \sum_{k=1}^{n} m_k x_k .$$

- The center of mass of S is located at the point $(\bar{x}, \bar{y})$ where
$$\bar{x} = \frac{M_y}{M} = \frac{1}{m} \sum_{k=1}^{n} m_k x_k \quad \text{and}$$
$$\bar{y} = \frac{1}{m} \sum_{k=1}^{n} m_k y_k .$$

Definition :(The centroid of a Region between two curves :)

Let R be a region bounded by the graphs of two continuous functions f and g on $[a,b]$, where $f(x) \geq g(x)$ $\forall$ x in $[a,b]$. Then the centroid of R is given by
$$\bar{x} = \frac{1}{A} \int_a^b x[f(x) - g(x)]\, dx .$$
and
$$\bar{y} = \frac{1}{A} \int_a^b \frac{1}{2}\left[(f(x))^2 - (g(x))^2\right] dx$$
where
$$A = \int_a^b [f(x) - g(x)]\, dx .$$

- Find the centroid of the region bounded by the graphs of $y = x^2 - 3$ and $y = -x^2 + 2x + 1$.



The intersecting points are $(-1,-2)$ and $(2,1)$.
Let $f(x) = -x^2 + 2x + 1$
and $g(x) = x^2 - 3$.
then $f(x) \geq g(x)$ on $[-1, 2]$

13.

$$A = \int_{-1}^{2} [f(x) - g(x)]\,dx$$

$$= \int_{-1}^{2} (-2x^2 + 2x + 4)\,dx$$

$$= \left[-\frac{2}{3}x^3 + x^2 + 4x\right]_{-1}^{2}$$

$$= \underline{\underline{9}}$$

$$\bar{x} = \frac{1}{A}\int_{-1}^{2} x[f(x) - g(x)]\,dx$$

$$= \frac{1}{9}\int_{-1}^{2} (-2x^3 + 2x^2 + 4x)\,dx$$

$$= \frac{1}{9}\left(-\frac{1}{2}x^4 + \frac{2}{3}x^3 + 2x^2\right]_{-1}^{2}$$

$$= \frac{1}{2}$$

$$\bar{y} = \frac{1}{A}\int_{-1}^{2} \frac{1}{2}[(-x^2 + 2x + 1)^2 - (x^2 - 3)^2]\,dx$$

$$= \frac{1}{9}\int_{-1}^{2} (-2x^3 + 4x^2 + 2x - 4)\,dx$$

$$= \frac{1}{9}\left[-\frac{1}{2}x^4 + \frac{4}{3}x^3 + x^2 - 4x\right]_{-1}^{2}$$

$$= \underline{\underline{\frac{1}{2}}}$$