System of Linear Algebraic Equations

An equation of the form $ax + by = c$, where a, b and c are real numbers is said to be a linear equation in the variable x and y.

The ~~gh~~ graph of a linear equation in two variables is a straight line.

$ax + by + cz = d$, $a, b, c, d \in \mathbb{R}$ is a linear equation in the variables x, y and z. and ~~d~~ is the equation of a plane in 3-space.

A system of m linear equations in n variables or unknowns has the general form

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \qquad\qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{array}\right\} \quad (1)$$

If $a_{ij} \rightarrow$ coefficients of the variables in the linear systems.

$b_1, b_2 \ldots b_m \longrightarrow$ Constants of the system.

If all constants are zero, then systems (1) is said to be homogeneous and otherwise non homogeneous.

eg: $x_1 - x_2 + x_3 = 0$
$6x_2 + x_3 = 0$
$4x_1 - x_2 = 0$
is a homogeneous system

$2x_1 + 5x_2 + 6x_3 = 1$
$4x_1 + 3x_2 - x_3 = 9$
is a non homogeneous system.

Solutions

A solution of a linear system (1) is a set of n numbers $x_1, x_2, \ldots, x_n$ that satisfies each equations in the system.

eg: $x_1 = 3,\ x_2 = -1$ is a solution of the system $3x_1 + 6x_2 = 3$
$x_1 - 4x_2 = 7$

A linear system of equation is said to be consistent if it has at least one solution and inconsistent if it has no solution.

- Verify that $x_1 = 14+7t$, $x_2 = 9+6t$, $x_3 = t$, $t \in \mathbb{R}$ is a solution of the system

$$2x_1 - 3x_2 + 4x_3 = 1$$
$$x_1 - x_2 - x_3 = 5$$

Solving Systems:-

We can transform a system of linear equations into an equivalent system using the following operations.

(I) Multiply an equation by a non zero ~~element~~ constant

(II) Interchange the positions of equations in the system

(III) Add a nonzero multiple of one equation to any other equation.

eg: Solve
$$2x_1 + 6x_2 + x_3 = 7$$
$$x_1 + 2x_2 - x_3 = -1$$
$$5x_1 + 7x_2 - 4x_3 = 9$$

$$2x_1 + 6x_2 + x_3 = 7$$
$$2x_1 + 4x_2 - 2x_3 = -2$$
$$2x_2 + 3x_3 = 9$$

$$x_1 + 2x_2 - x_3 = -1 \quad — (1)$$
$$2x_1 + 6x_2 + x_3 = 7 \quad — (2)$$
$$5x_1 + 7x_2 - 4x_3 = 9 \quad — (3)$$

$(1) \times -2 \Rightarrow -2x_1 - 4x_2 + 2x_3 = 2 \quad — (4)$

$(2) + (4) \Rightarrow 2x_2 + 3x_3 = 9$

$$x_1 + 2x_2 - x_3 = -1 \quad — (1)$$
$$2x_2 + 3x_3 = 9 \quad — (4)$$
$$5x_1 + 7x_2 - 4x_3 = 9 \quad — (3)$$

$(1) \times -5 + (3) \Rightarrow -3x_2 + x_3 = 14 \quad — (5)$

$$x_1 + 2x_2 - x_3 = -1 \quad — (1)$$
$$2x_2 + 3x_3 = 9 \quad — (4)$$
$$-3x_2 + x_3 = 14. \quad — (5)$$

$(1) - (4) \Rightarrow x_1 - 4x_3 = -10$
$$2x_2 + 3x_3 = 9$$
$$-3x_2 + x_3 = 14$$

$$2x_2 + 3x_3 = 9$$
$$-9x_2 + 3x_3 = 42$$

$x_2 = -33$

$x_2 = -33/11 = -3$

5) =>

$x_3 = 5$

$x_1 = 10$

$x_2 = -3$

The System

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$$

can be expressed as matrix

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right)$$

and is called Augmented matrix of the System.

eg: $x_1 + x_2 = 3$
$2x_1 + 5x_2 = 5$

the augmented matrix is

$$\left(\begin{array}{cc|c} 1 & 2 & 3 \\ 2 & 5 & 5 \end{array}\right)$$

Row Echelon form

- The first non-zero entry in a non zero row is 1.
- In consecutive non zero rows, the first entry 1 in the lower row appears to the right of the 1 in the higher row.
- Rows consisting of all zeros are at the bottom of the matrix.

Row Reduced Echelon form

the above three properties with an extra property, that is a column containing a first entry 1 has zeros else where.

Note

- In Gaussian elimination, the augmented matrix should be in row echelon form.
- In Gauss-Jordan Elimination the augmented matrix should be in row reduced echelon form.

Solve the system

$$2x_1 + 6x_2 + x_3 = 7$$
$$x_1 + 2x_2 - x_3 = -1$$
$$5x_1 + 7x_2 - 4x_3 = 9$$

using (1) Gaussian Elimination
(2) Gauss-Jordan Elimination

Augmented matrix is

$$\left(\begin{array}{ccc|c} 2 & 6 & 1 & 7 \\ 1 & 2 & -1 & -1 \\ 5 & 7 & -4 & 9 \end{array}\right)$$

(1) $\left(\begin{array}{ccc|c} 2 & 6 & 1 & 7 \\ 1 & 2 & -1 & -1 \\ 5 & 7 & -4 & 9 \end{array}\right)$

$R_1 \leftrightarrow R_2$

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & -1 \\ 2 & 6 & 1 & 7 \\ 5 & 7 & -4 & 9 \end{array}\right)$$

$R_2 \to R_2 - 2R_1,\ R_3 \to R_3 - 5R_1$

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & -1 \\ 0 & 2 & 3 & 9 \\ 0 & -3 & 1 & 14 \end{array}\right)$$

$R_2 \to R_2/2$,

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & -1 \\ 0 & 1 & 3/2 & 9/2 \\ 0 & -3 & 1 & 14 \end{array}\right)$$

$R_3 \to R_3 + 3R_2$

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & -1 \\ 0 & 1 & 3/2 & 9/2 \\ 0 & 0 & 11/2 & \frac{55}{2} \end{array}\right)$$

$R_3 \to \frac{2R_3}{11}$

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & -1 \\ 0 & 1 & 3/2 & 9/2 \\ 0 & 0 & 1 & 5 \end{array}\right)$$

The matrix is in row Echelon form

$$x_1 + 2x_2 - x_3 = -1$$
$$x_2 + 3/2\, x_3 = 9/2$$
$$x_3 = 5$$

$x_3 = 5$

$\Rightarrow x_2 = 9/2 - 3/2\, x_3$
$= \frac{1}{2}(9 - 3x_3)$
$= \frac{1}{2}(9 - 15)$
$= -6/2 = -3$

$x_1 = -1 - 2x_2 + x_3$
$= -1 + 6 + 5$
$= \underline{10}$

(2)
$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & -1 \\ 0 & 1 & 3/2 & 9/2 \\ 0 & 0 & 1 & 5 \end{array}\right)$$

$R_1 \rightarrow R_1 - 2R_2$

$$\left(\begin{array}{ccc|c} 1 & 0 & -4 & -10 \\ 0 & 1 & 3/2 & 9/2 \\ 0 & 0 & 1 & 5 \end{array}\right)$$

$9/2 - 15/2$

$R_2 \rightarrow R_2 - 3/2 R_3$

$$\left(\begin{array}{ccc|c} 1 & 0 & -4 & -10 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 5 \end{array}\right)$$

$R_1 \rightarrow R_1 + 4R_3$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 10 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 5 \end{array}\right)$$

$x_1 = 0, x_2 = -3, x_3 = 5$

- HW
Use Gauss-Jordan Elimination to Solve

$x_1 + 3x_2 - 2x_3 = -7$
$4x_1 + x_2 + 3x_3 = 5$
$2x_1 - 5x_2 + 7x_3 = 19$

Ans: $x_1 + 0x_2 + x_3 = 2$
$0x_1 + x_2 - x_3 = -3$

Theorem (Existence of Nontrivial Solution)
A homogeneous system posses nontrivial solution if the number m of equations is less than the number n of variables ($m<n$)

- Solve $2x_1 - 4x_2 + 3x_3 = 0$
$x_1 + x_2 - 2x_3 = 0.$

Ans: If $x_3 = t$,

$x_1 = 5/6\,t, \quad x_2 = 7/6\,t$

- Balance the chemical equation

$C_2H_6 + O_2 \longrightarrow CO_2 + H_2O$

we want to find positive integers x_1, x_2, x_3 and x_4 so that

$x_1 C_2H_6 + x_2 O_2 \longrightarrow x_3 CO_2 + x_4 H_2O$

Carbon (C)

$2x_1 = x_3 \qquad 2x_1 + 0x_2 - x_3 = 0$

Hydrogen

$6x_1 = 2x_4$ $\qquad$ $6x_1 + 0x_2 + 0x_3 - 2x_4 = 0$

Oxygen

$2x_2 = 2x_3 + x_4.$ $\qquad$ $0x_1 + 2x_2 - 2x_3 - x_4 = 0$

$$\left(\begin{array}{cccc|c} 2 & 0 & -1 & 0 & 0 \\ 6 & 0 & 0 & -2 & 0 \\ 0 & 2 & -2 & -1 & 0 \end{array}\right)$$

$R_1 \to R_1/2$, $\cancel{R_2 \to}$, $R_2 \leftrightarrow R_3$

$$\left(\begin{array}{cccc|c} 1 & 0 & -1/2 & 0 & 0 \\ 0 & 2 & -2 & -1 & 0 \\ 6 & 0 & 0 & -2 & 0 \end{array}\right)$$

$R_3 \to R_3 - 6R_1$

$$\left(\begin{array}{cccc|c} 1 & 0 & -1/2 & 0 & 0 \\ 0 & 2 & -2 & -1 & 0 \\ 0 & 0 & 3 & -2 & 0 \end{array}\right)$$

$R_2 \to R_2/2$, $R_3 \to R_3/3$

$$\left(\begin{array}{cccc|c} 1 & 0 & -1/2 & 0 & 0 \\ 0 & 1 & -1 & -1/2 & 0 \\ 0 & 0 & 1 & -2/3 & 0 \end{array}\right)$$

$x_3 - 2/3\, x_4 = 0$

$x_2 - x_3 - 1/2\, x_4 = 0.$

$x_1 - 1/2\, x_3 = 0.$

$x_3 - 2/3\, x_4 = 0 \Rightarrow x_3 = 2/3\, x_4$

If $x_4 = t$

$x_3 = 2/3\, t.$

$x_1 = 1/2\, x_3$
$= 1/2(2/3\, t)$
$= 1/3\, t.$

$x_2 = 1/2\, x_4 + x_3$
$= 1/2\, t + 2/3\, t.$

$x_2 = \frac{7}{6} t.$

for $t = 6$, $x_4 = 6$, $x_1 = 2$, $x_3 = 4$, $x_2 = 7$
the balanced chemical equation is
$2C_2H_6 + 7O_2 \to 4CO_2 + 6H_2O.$

Properties of Homogeneous System

Let $AX = 0$ denote a homogeneous system of linear equations.

(I) If x_1 is a Solution of $AX=0$ then cx_1 is also " " for any constant c.

(II) If x_1 & x_2 are Solutions of $AX=0$ then so is X_1+X_2.

Proof:

(I) If x_1 is a Solution of $Ax=0$. then $AX_1=0$.

Now $A(cX_1) = c(AX_1) = c\cdot 0 = 0$.

$\Rightarrow cX_1$ is also a Solution of $AX=0$.

(II) Given X_1 & X_2 are Solutions of $AX=0$ i.e $AX_1=0$ and $AX_2=0$

then $A(X_1+X_2) = AX_1 + AX_2 = 0+0=0$

$\Rightarrow X_1+X_2$ is also a Solution of $AX=0$

- $$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

$u_1 = (a_{11}\ a_{12} \cdots a_{1n})$ $u_2 = (a_{21}\ a_{22} \cdots a_{2n}) \cdots$

$u_m = (a_{m1}\ a_{m2} \cdots a_{mn})$ are called Row vectors

$$V_1 = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} \quad V_2 = \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} \cdots V_n = \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}$$

are ...

Rank of a matrix

The rank of an $m \times n$ matrix A, denoted by rank(A) is the maximum no. of L.I Row vectors in A.

Find the rank of the 3×4 matrix

$$\begin{pmatrix} 1 & 1 & -1 & 3 \\ 2 & -2 & 6 & 8 \\ 3 & 5 & -7 & 8 \end{pmatrix}$$

rank = 2.

Row Space

Span $(u_1, u_2, \ldots, u_n)$ of coefficient matrix is the Subspace of R^4

Theorem: Rank of a matrix by Row Reduction

If a matrix A is Row equivalent to a Row-echelon form B, then

(1) the row space of A = The row space of B

(2) the nonzero rows of B form a basis for the row space of A

(3) rank (A) = the no of nonzero rows in B

- Find rank of

$$A = \begin{pmatrix} 1 & 1 & -1 & 3 \\ 2 & -2 & 6 & 8 \\ 3 & 5 & -7 & 8 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 1 & -1 & 3 \\ 0 & 1 & -2 & -1/2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

rank (A) = no. of non zero rows in row echelon form = 2.

i.e rank $(A) = 2$

Determine whether the set of vectors $u_1 = (2,1,1)$, $u_2 = (0,3,0)$, $u_3 = (3,1,2)$ is L.I or L.D

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 3 & 0 \\ 3 & 1 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is L.I.

Consistence of $AX = B$

$AX = B$ is consistent iff rank (A) = rank $(A|B)$

No of Parameters in Solution

Let $A_{m \times n}$ Suppose $AX = B$ is consistent. If rank $(A) = r$ then solution contains $n - r$ parameters

Determinants (Skip 1 — —)

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}.$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$