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Rationality and Equilibrium

A Symposium in Honor of Marcel K. Richter

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Rationality and Equilibrium

Charalambos D. Aliprantis
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With 8 Figures
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Foreword to ‘Rationality and Equilibrium’ – A Symposium in honor of Marcel K. Richter

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This collection of papers is dedicated to Marcel K. Richter, in appreciation of the fundamental impact that his research, mentoring, and personality has had on economics and on economists.

Marcel K. Richter’s research has taken economic theory to places it needed to go, and along the way has left tight, crisp, important, and beautifully elegant results. Each paper is a destination, a result that is worth the trip, a stop that instructs the student on the effectiveness of mathematics and the liberating power of crystalline logic. No paper of his is carelessly written.

A good representative of Ket Richter’s work is his 1966 *Econometrica* paper “Revealed Preference Theory.” This paper has had a profound influence, not only on the problem of preference characterization, but also on the use of powerful logical tools in economic theory. Using set theory and mathematical logic, it provided a simple, clear, and general method to address the topic of consumer rationality, which strongly contrasted with the complex alternative literature on revealed preference and integrability theory. This was followed by “Rational Choice” and by the joint work with Leonid Hurwicz, “Revealed Preference Without Demand Continuity Assumptions,” both published in *Preferences, Utility and Demand*, edited by J. Chipman, L. Hurwicz, M.K. Richter, and H. Sonnenschein (1971).

Many other topics in economic theory benefited from Ket Richter’s lucidity. He developed fundamental relationships between preference, utility and demand, in, among others, “Continuous and Semicontinuous Utility” (IER 1980), “Duality and Rationality” (JET 1979), “An Integrability Condition with Applications to Utility Theory and Thermodynamics” (with Leonid Hurwicz, JME 1979), and “Ville Axioms and Consumer Theory” (with Leonid Hurwicz, *Econometrica* 1979). Together with G. Fuhrken, he wrote “Polynomial Utility” (ET 1991) and “Additive Utility” (ET 1991). With Taesung Kim he wrote “Nontransitive Nontotal Consumer Theory” (JET 1986). With Rosa L. Matzkin, he provided conditions for rationalization of finite demand observations, in “Testing Strictly Concave Rationality” (JET, 1991), and with Kam Chau Wong, he provided conditions for the existence of a concave utility function on finite sets, in “Concave Utility on Finite Sets”

(JET 2004). His "Cardinal Utility, Portfolio Selection and Taxation" (RES 1960) developed the theory of portfolio demand under the assumption that the utility of the investor depended only on characteristics of the probability distribution of the portfolio's uncertain earnings.

The theory of the core and Walrasian allocations benefited from Ket Richter's major contributions, such as "Existence of Nonatomic Core Walras Allocations" (JET 1986) and "The Core-Walras Equivalence" (JET 1984), both co-authored with Thomas Armstrong, and from "Coalitions, Core and Competition" (JET 1971). In "Invariance Axioms and Economic Indexes" (*Econometrica* 1966), he contributed to the axiomatic foundations of index number theory. With Leonid Hurwicz, Ket generalized constrained maximization and implicit function theory in "Optimization and Lagrange Multipliers" and in "Implicit Functions and Diffeomorphisms without C^1" (both published in *Advances in Mathematical Economics* 2003).

More recently, with Kam Chau Wong, Ket Richter has moved forward the theory of bounded rationality, by studying issues involving the computability and definability of utility, demand, and equilibrium. Some of the papers in this series are "Computable Preference and Utility" (JME 1999), "Noncomputability of Competitive Equilibrium" (ET 1999), and "Definable Utility in O-Minimal Structures" (JME 2000).

Revealed preference is, however, the topic with which Marcel K. Richter is most associated. Besides the papers mentioned above on this topic, other classics are his paper with Leonard Shapiro, "Revelations of a Gambler" (JME 1978); his well known paper with Daniel McFadden, "Stochastic Rationality and Revealed Stochastic Preference" (in Preference, Uncertainty and Rationality, edited by J. Chipman, D. McFadden, and M.K. Richter, 1990), which laid the foundation for the existence of a random utility rationalization of probabilistic choice; and his joint paper with Andreu MasColell, Rolf Mantel, and Daniel McFadden, "A Characterization of Community Excess Demand Functions" (JET 1974), in which revealed preference theory was used to demonstrate that a variant of the Sonnenschein-Debreu characterization held without the added restriction to a strictly positive closed price cone.

Marcel K. Richter's mentoring has been as unique as his research. The input and dedication he has demonstrated in his research has paralleled the input and dedication he has given to his students. For Ket Richter, no student thesis is ready to be defended until all ideas are clearly presented, all details are worked out, and all the lines of the thesis have undergone the close scrutiny of his red pen. In fact, Ket Richter has been a consummate mentor. All University of Minnesota students in economics have benefited from Ket's friendly help and open door, and his willingness to take up any topic. Ket's personality is as impressive as, and very much in line with, his research. He truly cares about each person and makes his or her happiness his own concern.

The papers assembled in this issue are by colleagues, students, and admirers of Marcel K. Richter. They deal with topics deeply connected to his work and interests, such as preferences, demand, equilibrium, core allocations, and testable restrictions. On behalf of everybody who has contributed to this symposium, we wish Ket all the best and thank him for his many contributions.

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Revealed stochastic preference: a synthesis*

Daniel L. McFadden

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Summary. The problem of revealed stochastic preference is whether probability distributions of observed choices in a population for various choice situations are consistent with a hypothesis of maximization of preference preorders by members of the population. This is a population analog of the classical revealed preference problem in economic consumer theory. This paper synthesizes the solutions to this problem that have been obtained by Marcel K. Richter and the author, and by J. C. Falmagne, in the case of finite sets of alternatives, and utilizes unpublished research of Richter and the author to give results for the non-finite choice sets encountered in economic consumer theory.

Keywords and Phrases: Choice, Stochastic preference, Revealed preference, Random utility maximization.

JEL Classification Numbers: D1, C6.

1 Introduction

The problem of revealed stochastic preference asks the question: Are the distributions of choices observed for a population of individuals in a variety of choice situations consistent with rational choice theory, which postulates that individuals maximize preferences? In economic consumer theory, each choice situation is defined by a budget set; in psychometrics, by the alternatives offered in an experiment; and in political voting behavior, by the issues presented in an election.

* The preparation of this paper was supported by the E. Morris Cox endowment at the University of California, Berkeley. I am indebted to Robert Anderson, Salvador Barbara, Werner Hildenbrand, Rosa L. Matzkin, and Aviv Nevo for useful suggestions and comments. I am especially indebted to Marcel K. Richter, who was the source of many of the ideas and arguments contained in this paper.

Distributions of responses arise because of taste heterogeneity in the population, or because of stochastic elements in individual preferences. The last possibility connects rational choice theory to psychometric models of choice based on random scale maximization. This paper synthesizes the relatively complete solutions to the revealed preference problem that have been obtained for finite choice sets, and extends these results to the non–finite choice sets commonly encountered in economic consumer behavior. This paper is based primarily on unpublished research that Marcel K. Richter and I did in 1971, and on subsequent published results for the finite case by Falmagne (1978) and by McFadden and Richter (1990). Ket Richter has had an impact on economic theory far beyond the papers published over his name. It is a fitting tribute to his career to draw upon his unpublished ideas and words to suggest the scope and significance of his influence.

The origin of the revealed stochastic preference problem is the classical economic theory of revealed preference, where the Samuelson–Houthaker *Strong Axiom of Revealed Preference* (SARP) and Richter's *Congruence Axiom* provide tight necessary and sufficient conditions for consistency of one individual's choices with preference maximization (see Samuelson, 1938; Houthaker, 1950; Richter, 1966,1971). Marschak (1960) connected this theory to the psychometric literature (Thurstone, 1927; Luce, 1959), posing the question of when observed choice probabilities could be rationalized as consistent with *random utility maximization* (RUM). Papers addressing the revealed stochastic preference problem include Block and Marschak (1960), McFadden and Richter (1971,1990), McFadden (1973,1975), Falmagne (1978), Fishbern (1978), Cohen (1980), Barbara and Pattanaik (1986), McLennan (1990), Fishburn and Falmagne (1989), Barbara (1990), Cohen and Falmagne (1990), Fishburn (1992), and Bandyopadhyay, Dasgupta, and Pattanaik (1999).[1]

The ingredients of a revealed preference problem are the universe of objects of choice, a family of feasible budget sets giving the alternatives from which a decision–maker must choose, a class of permissible decision rules consistent with a specified theory of choice behavior, and observations on the probabilities of choices made. Both the SARP and the Congruence Axiom consider classes of decision rules that maximize a preference preorder. They differ in that the SARP requires permissible decision–rules to produce unique maxima on feasible budget sets, and assumes a unique offer is observed, while the Congruence Axiom allows decision rules that yield multiple maxima, and assumes that decision–makers offer the sets of acceptable alternatives in the case of ties. One can generate a variety of revealed preference problems by varying the ingredients, particularly the family of feasible sets, the class of permissible decision rules, and the structure of observations. Some of the possibilities are discussed in the conclusion.

This paper is organized as follows. Section 2 sets notation and gives a formal statement of the revealed stochastic preference problem. Section 3 reviews

[1] There is a large literature in mathematical psychology dealing with concepts of stochastic transitivity, and their relationship to the RUM hypothesis; see Fishburn (1999). There is also a very extensive literature on the Luce Choice Axiom (Luce, 1959), which provided the foundation for the econometric theory of discrete choice behavior; see McFadden (1974), Halldin (1974), Manski (1977), McFadden (1981), McCausland (2002). McFadden (2001) surveys this subject and provides many references.

the revelation problem when the universe of alternatives is finite, and relates the necessary and sufficient conditions for this problem obtained by McFadden and Richter (1971, 1990) and by Falmagne (1978). Section 4 gives the McFadden and Richter (1971) results on the extension of set functions, together with new results on countable additivity. Section 5 draws upon this mathematical theory to obtain necessary and sufficient conditions for the revealed stochastic preference problem with a non−finite universe of alternatives that includes the classical economic consumer problem. Section 6 concludes with discussion of further extensions and problems.

2 The revealed stochastic preference problem

2.1 Notation

The following notation for the space of alternatives, the choice situations, observed behavior, and the hypothesis of rational behavior will be used throughout the paper:

- $(\mathbf{X}, \mathcal{X})$ a metric space $\mathbf{X}$ of possible objects of choice, and the Borel σ-algebra $\mathcal{X}$ of subsets of $\mathbf{X}$.
- $\mathbf{Q}$ a non−empty index set, a metric space interpreted as indexing the feasible choice situations.
- $\mathbf{B}(q)$ a non-empty set in $\mathcal{X}$ for $q \in \mathbf{Q}$, interpreted as the set of available alternatives, or "budget set", in choice situation q.
- $\mathbf{d} : \mathbf{Q} \to \mathcal{X}$ a decision rule that maps $\mathbf{Q}$ into subsets of $\mathbf{X}$, with $\emptyset \neq \mathbf{d}(q) \subseteq \mathbf{B}(q)$, interpreted as a behavior rule that designates the decision-maker's acceptable alternatives in $\mathbf{B}(q)$. The decision rule is *decisive* if $\mathbf{d}(q)$ is a singleton; a non-decisive choice is interpreted as the offer of a set of "tied" alternatives.
- $(q, \mathbf{C})$ a pair, termed a *trial*; composed of a feasible choice situation $q \in \mathbf{Q}$ and a set $\mathbf{C} \in \mathcal{X}$. The outcome of a trial is a *success* (*failure*) if $\mathbf{C}$ contains (excludes) the choice $\mathbf{d}(q)$ made by an individual in situation q. A trial can be a *partial success* if the decision rule is non-decisive and $\mathbf{d}(q)$ intersects $\mathbf{C}$ but is not contained in $\mathbf{C}$.
- $(\mathbf{D}.\mathcal{D}, \zeta)$ a probability space consisting of a set $\mathbf{D}$ of decision rules, a Boolean σ-algebra $\mathcal{D}$ of measurable subsets of $\mathbf{D}$, and a probability ζ on $\mathcal{D}$. This is interpreted as the universe of decision rules that could appear in a population of decision-makers.
- Π_q a choice probability on $\mathcal{X}$ for $q \in \mathbf{Q}$, with $\Pi_q(\mathbf{C})$ for $\mathbf{C} \in \mathcal{X}$ interpreted as the proportion of individuals in the population with choice functions satisfying $\mathbf{d}(q) \subseteq \mathbf{C}$. The algebra $\mathcal{D}$ contains the sets $\mathbf{D}(q, \mathbf{C}) = \{\mathbf{d} \in \mathbf{D} | \mathbf{d}(q) \subseteq \mathbf{C}\}$ for $q \in \mathbf{Q}$ and $\mathbf{C} \in \mathcal{X}$, so that the probability $\Pi_q(\mathbf{C}) \equiv \zeta(\mathbf{D}(q, \mathbf{C}))$ that the trial $(q, \mathbf{C})$ is a success is defined. The choice probability satisfies $\Pi_q(\mathbf{B}(q)) = 1$, and if the decision rules of the population are almost surely decisive, it satisfies $\Pi_q(\mathbf{C}) + \Pi_q(\mathbf{C}^c) = 1$. More generally, let Π_q^- and Π_q^+ denote set-valued bounds for $q \in \mathbf{Q}$, satisfying $\Pi_q^-(\mathbf{C}) \leq \zeta(\mathbf{D}(q, \mathbf{C})) \leq \Pi_q^+(\mathbf{C})$ for $\mathbf{C} \in \mathcal{X}$.

$\mathbf{t} =< (q_1, \mathbf{C}_1), ..., (q_m, \mathbf{C}_m) >$ a *trial sequence*, an ordered sequence with repetitions permitted, and elements $(q_i, \mathbf{C}_i) \in \mathbf{Q} \times \mathcal{X}$ for $i = 1, ..., m$, where m is a positive integer.

$\mathbf{H}$ a set of choice functions in $\mathbf{D}$, interpreted as the choice functions consistent with a specified hypothesis of rational choice behavior. The algebra $\mathcal{D}$ contains $\mathbf{H}$, so that the sets $\mathbf{H}(q, \mathbf{C}) = \mathbf{D}(q, \mathbf{C}) \cap \mathbf{H}$ are contained in the Boolean σ-algebra $\mathcal{H} = \{\mathbf{A} \cap \mathbf{H} | \mathbf{A} \in \mathcal{D}\}$ for $q \in \mathbf{Q}$ and $\mathbf{C} \in \mathcal{X}$, and $\zeta(\mathbf{H}(q, \mathbf{C}))$ is defined as the probability that the trial $(q, \mathbf{C})$ is a success for decision rules that satisfy the rational choice hypothesis.

$\alpha_{\mathbf{H}}(t)$ the H-intersection number of a trial sequence $\mathbf{t} = < (q_1, \mathbf{C}_1), ..., (q_m, \mathbf{C}_m) >$, defined to be the maximum number of successes for the sequence attainable by a choice function in $\mathbf{H}$:

$$\alpha_{\mathbf{H}}(t) = \max_{d \in \mathbf{H}} \sum_{i=1}^{m} \mathbf{1}(\mathbf{d}(q_i) \subseteq \mathbf{C}_i).$$

$u : \mathbf{X} \to \mathbb{R}$ a utility or scale function on $\mathbf{X}$, a representation of a preference preorder. A utility function u is *weakly decisive* if $\mathbf{d}(q) = \mathbf{d}(q; u) \equiv \{x \in \mathbf{B}(q) | u(x) \geq u(x') \text{ for all } x' \in \mathbf{B}(q)\}$ is non-empty for $q \in \mathbf{Q}$, and is *decisive* if $\mathbf{d}(q; u)$ is a singleton for $q \in \mathbf{Q}$.

$(\mathbf{U}, \mathcal{U}, \nu)$ a non-empty set of utility functions u specified by a hypothesis of rational choice behavior, a metric space, with $\mathcal{U}$ the Borel σ-algebra of subsets of $\mathbf{U}$, and ν a probability on $\mathcal{U}$, termed a *random utility maximization* (RUM) model. A space of decision rules $(\mathbf{H}, H, \zeta)$ and a space of utility functions $(\mathbf{U}, \mathcal{U}, \nu)$ are *consistent* (or, the set $\mathbf{H}$ of decision rules is $\mathbf{U}$-*rational*) if each $u \in \mathbf{U}$ is weakly decisive, and each $d \in \mathbf{H}$ is of the form $\mathbf{d}(q; u)$ for some $u \in \mathbf{U}$ and all $q \in \mathbf{Q}$, the inverse image of $\mathbf{H}(q, \mathbf{C})$ is in $\mathcal{U}$ for $q \in \mathbf{Q}, \mathbf{C} \in \mathcal{X}$ (i.e., $\mathbf{U}(q, \mathbf{C}) \equiv \{u \in \mathbf{U} | \mathbf{d}(q; u) \subseteq \mathbf{C} \in \mathcal{U}$), and $\zeta(\mathbf{H}(q, \mathbf{C})) = \nu(\mathbf{U}(q, \mathbf{C}))$.

$\alpha_{\mathbf{U}}(t)$ the U-intersection number of a trial sequence $\mathbf{t} =< (q_1, \mathbf{C}_1), ..., (q_m, \mathbf{C}_m) >$, defined to be the maximum number of successes for the sequence attainable by a utility function in $\mathbf{U}$; i.e., $\alpha_{\mathbf{U}}(t) = \max_{u \in \mathbf{U}} \sum_{i=1}^{m} \mathbf{1}(\mathbf{d}(q_i; u) \subseteq \mathbf{C}_i)$. If the space $\mathbf{H}$ of decision rules and the space $\mathbf{U}$ of utility functions are consistent, then the decision-rule and utility intersection numbers are the same.

2.2 Discussion

The central results in this paper concern random utility maximization, and utilize spaces $(\mathbf{U}, \mathcal{U}, \nu)$ of weakly decisive utility functions. These results will have equivalent restatements in terms of the consistent space of decision rules $(\mathbf{H}, \mathcal{H}, \zeta)$. We will also give some results directly for a space of hypothesized decision rules $(\mathbf{H}, \mathcal{H}, \zeta)$; these can be applied to theories of choice other than random utility maximization. The universe $(\mathbf{D}, \mathcal{D}, \zeta)$ of decision rules will play no direct role in our analysis; but is useful in interpreting revealed preference problems as null hypotheses $\mathbf{H}$ on this universe. In this interpretation, the revealed preference problem can be viewed as an extreme case of the econometric problem of estimating the probability measure ζ or bounding $\zeta(\mathbf{H})$.

In the classical theory of economic consumer demand, each alternative is a commodity vector represented by a point in a closed consumption set $\mathbf{X}$ contained in the non-negative orthant of $\mathbb{R}^n$; $\mathbf{X}$ is often assumed to be convex. The space of choice possibilities $\mathbf{Q}$ is a set of n-vectors of positive commodity prices $q = (q_1, ..., q_n)$, where income is normalized to one. Then, the possible choice sets are the budget sets $\mathbf{B}(q) = \{x = (x_1, ..., x_n) \in \mathbf{X} | q_1x_1 + ... + q_nx_n \leq 1\}$; with $\mathbf{Q}$ restricted so that $\mathbf{B}(q)$ is always non-empty. The admissible behavior rules $\mathbf{H} \subseteq \mathbf{D}$ under the theory of utility-maximizing choice behavior are those consistent with a specified family of weakly decisive utility functions $\mathbf{U}$. For this setup, it will often be natural to impose some combination of the following assumptions: [A1]. $\mathbf{X}$ is compact and convex; [A2] The feasible choice sets $\mathbf{B}(q)$ are closed and convex for $q \in \mathbf{Q}$; [A3] $\mathbf{Q}$ is a metric space, and the mapping $\mathbf{B}(q)$ from $\mathbf{Q}$ into non-empty subsets of $\mathbf{X}$ is a continuous, compact-valued, convex-valued correspondence;[2] [A4a] Utility functions u $\in \mathbf{U}$ are uniformly bounded, continuous and quasi-concave, or uniformly Lipschitz, and strictly quasi-concave, [A4b] Utility functions $u \in \mathbf{U}$ are defined on an open neighborhood of $\mathbf{X}$, and are uniformly bounded and concave.

A complete theory of choice behavior requires either (1) that the structure of the choice problem is such that decision rules are always decisive, if necessary through the introduction of explicit tie-breaking mechanisms, or (2) that decision-makers are observed to offer sets of "tied" acceptable alternatives and they passively accept assignments from their offers. An incomplete theory that does not specify tie-breaking mechanisms may nevertheless be empirically complete if in practice decision-rules are almost surely decisive. Shape restrictions may ensure that economic consumer choice is decisive; i.e., if budget sets are compact and convex, and admissible utility functions are continuous and strictly quasi-concave, then decision rules always yield singletons. However, more generally utility maximization does not rule out ties. We will assume that offer sets of admissible alternatives are observed, and define $\Pi_q(\mathbf{C})$ to be the probability that the observed offer set in choice situation q is contained in $\mathbf{C}$. The sum of the probabilities $\Pi_q(\mathbf{C}) + \Pi_q(\mathbf{C}^c)$ is less than one if the probability of a partial success (where $\mathbf{d}(q)$ intersects but is not contained in $\mathbf{C}$) is positive. In this case, we can consider observed lower bounds Π_q^- on the probabilities of success and upper bounds Π_q^+ on the probabilities of success or partial success (i.e., $\Pi_q^+(\mathbf{C}) = 1 - \Pi_q^-(\mathbf{C}^c)$). Alternately, if we observe $\Pi_q(\mathbf{C}) + \Pi_q(\mathbf{C}^c) = 1$ for all $\mathbf{C} \in \mathcal{X}$, then admissible decision rules are almost surely decisive at each $q \in \mathbf{Q}$, and Π_q is an almost surely complete description of the distribution of demand.

[2] A correspondence is continuous if it is upper and lower hemicontinuous in the terminology of Hildenbrand (1974, I.B.III). When the space of closed non-empty subsets of $(\mathbf{X}, \mathcal{X})$ is metrized by the Hausdorff distance, then an equivalent characterization is that $\mathbf{B}$(q) is a continuous function from $\mathbf{Q}$ into this metric space.

2.3 Revelation problems

We define the revelation problems we will consider.

2.3.1 The revealed distribution problem

If Π_q is a probability on $\mathcal{X}$ for $q \in \mathbf{Q}$, find a probability ζ on $\mathcal{H}$ (or, by extension, a probability ζ on $\mathcal{D}$ satisfying $\zeta(\mathbf{H}) = 1$) such that $\Pi_q(\mathbf{C}) = \zeta(\mathbf{H}(q, \mathbf{C}))$ for $\mathbf{C} \in \mathcal{X}, q \in \mathbf{Q}$. [Alternately, find a probability ν on $\mathcal{U}$ such that $\Pi_q(\mathbf{C}) = \nu(\mathbf{U}(q, \mathbf{C}))$ for $\mathbf{C} \in \mathcal{X}, q \in \mathbf{Q}$].

2.3.2 Revealed dominating distribution problem

If Π_q^- and Π_q^+ are non-negative bounded set functions on $\mathcal{X}$ for $q \in \mathbf{Q}$, find a probability ζ on $\mathcal{H}$ such that $\Pi_q^-(\mathbf{C}) \leq \zeta(\mathbf{H}(q, \mathbf{C})) \leq \Pi_q^+(\mathbf{C})$ for $\mathbf{C} \in \mathcal{X}, q \in \mathbf{Q}$. [Alternately, find a probability ν on $\mathcal{U}$ with $\Pi_q^-(\mathbf{C}) \leq \nu(\mathbf{U}(q, \mathbf{C})) \leq \Pi_q^+(\mathbf{C})$ for $\mathbf{C} \in \mathcal{X}, q \in \mathbf{Q}$].

2.3.3 The axiom of revealed stochastic preference [ARSP]

For a class $\mathbf{H}$ of hypothesized decision rules, or alternately, for a class $\mathbf{U}$ of hypothesized utility functions, and for each finite sequence of trials $\mathbf{t} = \;<(q_1, \mathbf{C}_1), ..., (q_m, \mathbf{C}_m)>$ with $\mathbf{C}_i \in \mathcal{X}$ and $q_i \in \mathbf{Q}$,

$$\sum_{i=1}^{m} \Pi_{q_i}(\mathbf{C}_i) \leq \alpha_{\mathbf{H}}(\mathbf{t}) = \max_{d \in \mathbf{H}} \sum_{i=1}^{m} \mathbf{1}(\mathbf{d}(q_i) \subseteq \mathbf{C}_i)\ , \tag{1}$$

or alternately,

$$\sum_{i=1}^{m} \Pi_{q_i}(\mathbf{C}_i) \leq \alpha_{\mathbf{U}}(\mathbf{t}) = \max_{u \in \mathbf{U}} \sum_{i=1}^{m} \mathbf{1}(\mathbf{d}(q_i; u) \subseteq \mathbf{C}_i)\ . \tag{2}$$

The expressions $\alpha_{\mathbf{H}}(\mathbf{t})$ and $\alpha_{\mathbf{U}}(\mathbf{t})$ are, respectively, the H-intersection number and $\mathbf{U}$-intersection number for the trial sequence $\mathbf{t}$. When $\mathbf{H}$ is $\mathbf{U}$-rational, these numbers coincide. More generally, the axiom may be applied to hypothesized decision rules $\mathbf{H}$ that are not necessarily obtained from utility maximization. ARSP says that the sum of choice probabilities over a finite sequence of trials is no larger than the maximum number of successes that an admissible decision rule [alternately, an admissible utility function] can produce. A central result for the revealed distribution problem, due to McFadden and Richter (1971), is that under some regularity conditions, ARSP is necessary and sufficient for consistency of observed choice probabilities with a specified theory of choice behavior.

3 Finite families of choice situations

3.1 Discussion

In psychometric and voting applications, as well as discrete choice applications in economics and marketing, it is natural to consider choice situations in which the space of possible alternatives is finite. The classical economic choice problem can also be interpreted as finite when the index set $\mathbf{Q}$ of budgets is finite, so that $\mathbf{X}$ can be partitioned into a finite family of subsets $\{\mathbf{X}_1, ..., \mathbf{X}_m\}$ with the property for each $i = 1, ..., m$ and $q \in \mathbf{Q}$, either $\mathbf{X}_i \subseteq \mathbf{B}(q)$ or $\mathbf{X}_i \cap \mathbf{B}(q) = \emptyset$, and $\mathcal{X}$ is the field generated by the partition. In this case, observations provide no information on choice behavior within partition sets, so that the partition sets can themselves be defined as the objects of choice.

Throughout this section, we will define the index set $\mathbf{Q}$ as the family of feasible "budget sets" in $\mathbf{X}$, and name singleton sets by their elements, so that $\Pi_{\mathbf{B}}(x)$ denotes a choice probability for $\mathbf{C} = \{x\}$ when $\mathbf{B}$ is a feasible choice set in the family $\mathbf{Q}$, and by construction $\Pi_{\mathbf{B}}(x) = 0$ for $x \notin \mathbf{B}$. Note that if $\mathbf{X}$ contains m elements, then there are $m!$ possible total orders of these elements. We will represent these orders by the finite family $\mathbf{U}$ of utility functions from $\mathbf{X}$ onto the integers $\{1, ..., m\}$; note that this definition excludes ties, so that utility-maximizing choice functions will be decisive.

The revealed stochastic preference problem was originally examined for the case of $\mathbf{X}$ finite by Marschak (1960), Block and Marschak (1960), and Luce and Suppes (1965), and it is for this case that the most complete characterizations of a solution have been given, by McFadden and Richter (1971, 1990), Falmagne (1978), and Barbara (1990). A closely related result with a different application was obtained by Freedman and Purves (1969). We will need several definitions.

3.2 Definitions

3.2.1. A set $\mathbf{Q}$ of choice situations forms a *net* if for every feasible set of alternatives, every larger set contained in $\mathbf{X}$ is also feasible; i.e., if $\mathbf{B} \in \mathbf{Q}$ and $\mathbf{A} \subseteq \mathbf{X} \backslash \mathbf{B}$, then $\mathbf{B} \cup \mathbf{A} \in \mathbf{Q}$. A set of choice situations is *exhaustive* if it forms a net and it contains each singleton in $\mathbf{X}$.

3.2.2. Suppose choice situations $\mathbf{Q}$ form a net. Let $\#(\mathbf{A})$ denote the number of elements in a subset $\mathbf{A}$ of $\mathbf{X}$. For $x \in \mathbf{X} \backslash \mathbf{A} \in \mathbf{Q}$, *the Block-Marschak polynomial* $\mathcal{K}_{x,\mathbf{A}}$ is the function

$$\mathcal{K}_{x,\mathbf{A}} = \sum_{i=0}^{\#(\mathbf{A})} (-1)^{\#(A)-i} \sum_{\mathbf{C} \subseteq \mathbf{A} \& \#(\mathbf{C})=i} \Pi_{\mathbf{X} \backslash \mathbf{C}}(x) \ . \tag{3}$$

The Block-Marschak polynomials can also be defined recursively, with

$$\mathcal{K}_{x,\emptyset} = \Pi_{\mathbf{X}}(x),$$

$$\mathcal{K}_{x,\mathbf{A}} = \Pi_{\mathbf{X}\backslash\mathbf{A}}(x) - \sum_{\mathbf{C}\subset\mathbf{A}} \mathcal{K}_{x,\mathbf{C}} \ ,$$

for all $\mathbf{A} \subseteq \mathbf{X}\backslash\mathbf{B}$ and $\mathbf{B} \in \mathbf{Q}$; see Falmagne (1978, Theorem 2). An implication of this construction is $\Pi_{\mathbf{B}}(x) = \sum_{\mathbf{C}\subseteq\mathbf{X}\backslash\mathbf{B}} \mathcal{K}_{x,\mathbf{C}}$ for $\mathbf{B} \in \mathbf{Q}$. When the choice probabilities are the result of utility maximization, Barbara and Pattanaik (1986) provide a useful interpretation of $\mathcal{K}_{x,\mathbf{A}}$ as the probability of the event that x is ranked behind the elements of $\mathbf{A}$ and ahead of all the remaining elements in $\mathbf{X}\backslash\mathbf{A}$.

For a trial $(\mathbf{B}, x)$ with $x \in \mathbf{B} \in \mathbf{Q}$, and for $u \in \mathbf{U}$, define $a_{\mathbf{B},x,u} = \mathbf{1}(x = \mathrm{argmax}_{\mathrm{x}'\in\mathbf{B}}\mathrm{u}(\mathrm{x}'))$. Form a column vector π composed of subvectors for each $\mathbf{B} \in \mathbf{Q}$, with each subvector composed of the choice probabilities $\Pi_{\mathbf{B}}(x)$ for $x \in \mathbf{B}$. Form the matrix $\mathbf{A}$ with element $a_{\mathbf{B},x,u}$ in the row corresponding to the trial $(\mathbf{B}, x)$ and column u for $u \in \mathbf{U}$. An element of $\mathbf{A}$ is one if the associated trial is a success for the specified utility function, and is zero otherwise. Then, integer-weighted row sums of $\mathbf{A}$ will be the number of successes attainable for a specified trial sequence (with repetitions given by the integer weights) for the various utility functions, and the maximum of these rows sums will be the $\mathbf{U}$-intersection number for the trial sequence.

3.3 Theorem

If $\mathbf{X}$ is finite, $\mathbf{U}$ is the class of utility functions that totally order $\mathbf{X}$, $\mathbf{Q}$ is a family of choice situations, with $\mathbf{B} \in \mathbf{Q}$ a non-empty subset of $\mathbf{X}$, and $\Pi_{\mathbf{B}}(x)$ is a choice probability for $x \in \mathbf{B} \in \mathbf{Q}$ satisfying $\Pi_{\mathbf{B}}(\mathbf{B}) = 1$, then the following conditions are equivalent:

(a) There exists a probability ν on $\mathcal{U}$ that rationalizes the choice probability; i.e.,

$$\Pi_{\mathbf{B}}(x) = \sum_{u\in\mathbf{U}} a_{\mathbf{B},x,u}\nu_u \ \text{ for } \ x \in \mathbf{B} \in \mathbf{Q} \ . \tag{4}$$

(b) The system of linear inequalities $\pi \le \mathbf{A}\nu, \nu \ge 0, \mathbf{1}'\nu \le 1$ has a solution.
(c) The linear program $\min_{\nu,s} \mathbf{1}'s$ subject to $\nu \ge 0$, $s \ge 0$, $\mathbf{A}\nu + s \ge \pi, \mathbf{1}'\nu \le 1$ has an optimal solution with $s = 0$.
(d) The linear program $\max_{r,t}(r'\pi - t)$ subject to $0 \le r \le 1, t \ge 0$, and $r'\mathbf{A} \le t\mathbf{1}'$ has no positive solution.
(e) The choice probabilities $\Pi_{\mathbf{B}}(x), x \in \mathbf{B} \in \mathbf{Q}$, satisfy ARSP [cf 2.4].
If the set $\mathbf{Q}$ of feasible choice situations forms a net, then (a)–(e) are equivalent to
(f) The Block-Marschak polynomials $\mathcal{K}_{x,\mathbf{X}\backslash\mathbf{B}}$ for $x \in \mathbf{B} \in \mathbf{Q}$, are non-negative.

Proof. If a probability ν satisfies (a), then it satisfies (b) with $\pi = \mathbf{A}\nu$. Conversely, if π satisfies (b), then $\pi = \mathbf{A}\nu$ since π satisfies $\Pi_{\mathbf{B}}(\mathbf{B}) = 1$, so that (a) is satisfied. But ν solves (b) if and only if ν and $s = 0$ solve (c). The linear program (d) is dual to the linear program (c), so that (c) has an optimal solution with $s = 0$ if and only if (d) has no positive solution; see Karlin (1959, V.4.1). An optimal solution to (d) satisfies $t = \max_u r'A_u$, where A_u is a column of $\mathbf{A}$. Thus, (d) has a positive optimal solution if and only if for some r satisfying $0 \leq r \leq 1$, one has $r'\pi > t = \max_u r'A_u$. But if this is true, then one can achieve the strict inequality with a vector r whose components are all rational numbers. Clear a common denominator so that r is a vector of non-negative integers. Then, $\max_u r'A_u$ is the intersection number of the sequence of trials with the components of r giving the number of repetitions for each trial, so that (d) has a positive solution if and only if ARSP in (e) is violated. This establishes that (a)−(e) are equivalent.

Consider condition (f), and suppose $\mathbf{Q}$ forms a net so that the Block-Marschak polynomials $\mathcal{K}_{x,\mathbf{A}}$ are defined for $x \in \mathbf{X}\backslash\mathbf{A} \in \mathbf{Q}$. Let $\mathbf{r} =< r_1, ..., r_k >$ denote an ordered sequence of the elements of a set $\mathbf{A} = \{r_1, ..., r_k\} \subseteq \mathbf{X}$, where $k = 0, ..., \#(\mathbf{X})$, and $\mathbf{R_A}$ denote the family of all ordered sequences $\mathbf{r}$ of the elements of $\mathbf{A}$. Let $\mathbf{B}\backslash\mathbf{r}$ denote the set of elements of $\mathbf{B}$ that are not contained in the sequence $\mathbf{r}$. For $\mathbf{r} \subseteq \mathbf{B}$, define

$$\mathbf{S_{r,B}} = \{u \in \mathbf{U} | u(r_1) > ... > u(r_k) > u(x) \text{ for } x \in \mathbf{B}\backslash\mathbf{r}\}.$$

Then, $\mathbf{S_{r,B}}$ contains the utility functions for which the elements in $\mathbf{r}$ are ranked in descending order and are better than any remaining elements in $\mathbf{B}$. If (a) holds, it is immediate from the construction of $\mathbf{S_{r,B}}$ that for $x \in \mathbf{B} \in \mathbf{Q}, \mathit{\Pi}_\mathbf{B}(x) = \nu(\mathbf{S_{<x>,B}})$. The sets $\mathbf{S_{r,B}}$ for $\mathbf{B} \in \mathbf{Q}$ have the property that $\mathbf{S_{<r,x>,B}}$ for $x \in \mathbf{B}\backslash\mathbf{r}$ is a partition of $\mathbf{S_{r,B}}$ (Falmagne, 1978, Lemma 1) and for $x \in \mathbf{B} \in \mathbf{Q}$ and $\mathbf{A} = \mathbf{X}\backslash\mathbf{B}, \mathbf{S_{<x>,B}} = \cup_{\mathbf{C}\leq\mathbf{A}} \cup_{\mathbf{r}\in\mathbf{R_C}} \mathbf{S_{<r,x>,X}}$, with the sets in this union disjoint (Falmagne, 1978, Lemma 2). Note that $\cup_{x\in\mathbf{X}}\mathbf{S_{<x>,X}} = \mathbf{U}$. The family of sets $T_0 = \{\mathbf{S_{<r,x>,X}} | x \in \mathbf{B} \in \mathbf{Q} \text{ and } \mathbf{r} \in \mathbf{R}_C \text{ for } \mathbf{C} \subseteq \mathbf{X}\backslash\mathbf{B}\}$ then form a Boolean semi-algebra (Neveu, 1965, 1.6.1). Consider the sets $\mathbf{M}_{x,\mathbf{A}} = \{u \in \mathbf{U} | u(x') > u(x) > u(x'') \text{ for } x' \in \mathbf{A} \text{ and } x \neq x'' \in \mathbf{X}\backslash\mathbf{A}\} = \cup_{\mathbf{r}\subseteq\mathbf{R_A}} \mathbf{S_{<r,x>,X}}$, and note that the sets in the last union are disjoint. Barbara and Pattanaik (1986, Theorem 2.1) utilize the recursive definition of $\mathcal{K}_{\mathbf{x},\mathbf{A}}$ to prove by induction for $x \in \mathbf{X}\backslash\mathbf{A} \in \mathbf{Q}$ that when (a) holds, $\mathcal{K}_{x,\mathbf{A}} = \nu(\mathbf{M}_{x,\mathbf{A}}) = \sum_{\mathbf{r}\subseteq\mathbf{R_A}} \nu(\mathbf{S_{<r,x>,X}}) \geq 0$.

Then, (a) implies (f).

Suppose that the Block-Marschak polynomials are non-negative for a class of feasible choice sets $\mathbf{Q}$ that forms a net, so that (f) holds. Following Falmagne (1978, Theorem 4), construct a set-valued function ν on T_0 in the following steps:

(1) For $x \in \mathbf{X}, \nu(\mathbf{S_{<x>,X}}) = \mathcal{K}_{x,\emptyset} \equiv \mathit{\Pi}_\mathbf{X}(x)$.
(2) For $x, y \in \mathbf{X}, x \neq y, \nu(\mathbf{S_{<y,x>,X}}) = \mathcal{K}_{x,\{y\}} \equiv \mathit{\Pi}_{\mathbf{X}\backslash\{y\}}(x) - \mathit{\Pi}_\mathbf{X}(x)$.
(3) Suppose ν has been defined for $\mathbf{S}_{r,X}$ with $\mathbf{r} \in \mathbf{R}_A$ for all $\mathbf{A}$ such that $\mathbf{X}\backslash\mathbf{A} \in \mathbf{Q}$ and $\#(\mathbf{A}) < k$. Suppose $\mathbf{A}$ meets this condition with $\#(\mathbf{A}) = k - 1$, and suppose $x \in \mathbf{X}\backslash\mathbf{A}$ satisfies $(\mathbf{X}\backslash\mathbf{A}) \cup x \in \mathbf{Q}$. Define $\mathit{\Delta} = \sum_{\mathbf{r}\in\mathbf{R_A}} \nu(\mathbf{S_{r,X}})$. Then, define $\nu(\mathbf{S_{<r,x>,X}})$ by the recursion $\nu(\mathbf{S_{<r,x>,X}}) = \mathcal{K}_{x,\mathbf{A}} \cdot \nu(\mathbf{S}_{r,\mathbf{X}})/\mathit{\Delta}$ if $\mathit{\Delta} > 0$, and otherwise $\nu(\mathbf{S_{<r,x>,X}}) = 0$.

It is immediate from this construction and the fact that $\mathbf{S_{<r,x>,X}}$ is a partition of $\mathbf{S}_{r,\mathbf{X}}$ for $x \in \mathbf{X}\backslash\mathbf{r}$ that ν is non-negative and additive on $\mathcal{I}_0$, with $\nu(\mathbf{U}) = 1$; see Falmagne (1978, Lemma 4). Then ν has a unique extension to a probability on the Boolean algebra $\mathcal{I}$ generated by $\mathcal{I}_0$ (Neveu, 1964, 1.6.1). Further,defining $\nu(\mathbf{A}) = sup\{\nu(\mathbf{B}) | \mathbf{B} \in T \& \mathbf{B} \subseteq \mathbf{A}\}$ for $\mathbf{A} \subseteq \mathbf{U}$ extends ν to a probability on the Boolean algebra of all subsets of $\mathbf{U}$; see Neveu (1965, I.6.2). The final step of the proof is to show that the constructed probability ν satisfies (a). Since $\Pi_{\mathbf{B}}(x) = \sum_{\mathbf{C}\subseteq\mathbf{X}\backslash\mathbf{B}} \mathcal{K}_{x,\mathbf{C}}$ for $\mathbf{B} \in \mathbf{Q}$, it is sufficient to show that $\mathcal{K}_{x,\mathbf{A}} = \nu(\mathbf{M}_{x,\mathbf{A}}) = \sum_{\mathbf{r}\subseteq\mathbf{A}} \nu(\mathbf{S}_{<\mathbf{r},\mathbf{x}>,\mathbf{X}})$ for $\mathbf{X}\backslash\mathbf{A} \in \mathbf{Q}$. But the construction $\nu(\mathbf{S}_{<\mathbf{r},\mathbf{x}>,\mathbf{X}}) = \mathcal{K}_{x,\mathbf{A}} \cdot \nu(\mathbf{S}_{r,\mathcal{X}})/\Delta$ implies $\sum_{\mathbf{r}\in\mathbf{R}_\mathbf{A}} \nu(\mathbf{S}_{<\mathbf{r},\mathbf{x}>,\mathbf{X}}) = \mathcal{K}_{x,\mathbf{A}}$. This completes the proof. □

3.4 Remarks

The equivalence of (a)–(e) was established by McFadden and Richter (1971,1990). The equivalence of (a) and (f) when the family of feasible choice sets is exhaustive was established by Falmagne (1978), with useful interpretation and refinements given by Barbara and Pattianiak (1986). Theorem 3.3 generalizes the Falmagne result slightly by noting that it is not necessary that the feasible choice sets be exhaustive, provided that they form a net so that the Block-Marschak polynomials are defined.

The linear programs (c) and (d) provide finite algorithms that can, in principle, determine if observed choice probabilities can be rationalized. These are, further, completely general, requiring no particular structure for the set of feasible choice situations. The construction for condition (f) is also a finite algorithm, with the advantage that each step in the recursive construction of the measure ν defines a probability on a Boolean semi-algebra of subsets of $\mathcal{U}$. In some applications, such as construction of bounds, this intermediate information may be directly useful. The primary limitation of the Block-Marschak polynomial condition is that it requires that the feasible choice sets form a net. This excludes some natural applications, such as those where only paired comparisons are observed, or those defined by economic budget sets for a finite number of price vectors.

Part of the literature on stochastic choice has concentrated on situations where decision-makers are faced with binary choice situations (see Luce, 1959; McLennon, 1991; Fishburn, 1999). Falmagne's condition on the Block−Marschak polynomials is not applicable to this case, and while ARSP is applicable, it does not fully exploit the geometry of the polytope containing the vectors of rationalizable-choice probabilities. Fishburn (1992) surveys the results on this problem, including the mathematical literature on the polytopes generated by the decisive preference preorders.

4 Extension of set functions

4.1 The dominance problem

The results of this paper for the non-finite case hinge on the following mathematical problem: If P is a non-negative bounded set function on a family $\mathcal{S}$ of subsets of a non-empty set $\mathbf{H}$, find a probability η on the Boolean algebra $\mathcal{I}$ generated by $\mathcal{S}$ such that $\eta(\mathbf{S}) \geq P(\mathbf{S})$ for $\mathbf{S} \in \mathcal{S}$. The following axiom is the key to the existence of a solution.

4.2 The dominance axiom

For each finite sequence $\mathbf{t} =< \mathbf{S}_1, ..., \mathbf{S}_m >$ in $\mathcal{S}$, with repetitions allowed,

$$\sum_{i=1}^{m} P(\mathcal{S}_i) \leq \alpha_{\mathbf{H}}(\mathbf{t}) = \max_{d\in\mathbf{H}} \sum_{i=1}^{m} \mathbf{1}(d \in \mathbf{S}_i) \ . \tag{5}$$

4.3 Finitely-additive extension theorem

P is a non-negative bounded set function satisfying the dominance axiom on a family $\mathcal{S}$ of subsets of a non-empty set $\mathbf{H}$ if and only if there exists a finitely additive probability η on a Boolean algebra $\mathcal{I}$ of subsets of $\mathbf{H}$ containing $\mathcal{S}$ such that $\eta(\mathbf{S}) \geq P(\mathbf{S})$ for $\mathbf{S} \in \mathcal{S}$. If, further, $\mathcal{S}$ is closed under complementation and contains $\mathbf{H}$, and P satisfies $P(\mathbf{S}) + P(\mathbf{S}^c) = 1$ for $\mathbf{S} \in \mathcal{S}$, then $\eta(\mathbf{S}) = P(\mathbf{S})$ for $\mathbf{S} \in \mathcal{S}$.

Proof. Necessity of the dominance axiom. Let $\mathbf{S}_1, ..., \mathbf{S}_n$ be a sequence of sets in $\mathcal{S}$, and $\mathbf{T}_1, ..., \mathbf{T}_m$ the partition of $\mathbf{H}$ that they induce. Then $\mathbf{T}_j \in \mathcal{I}$. Let k_j equal the number of sets $\mathbf{S}_i$ containing $\mathbf{T}_j$. Then

$$\sum_{i=1}^{n} P(\mathbf{S}_i) \leq \sum_{i=1}^{n} \eta(\mathbf{S}_i) = \sum_{j=1}^{m} k_j \cdot \eta(\mathbf{T}_j) \leq \max_{j\leq m} k_j \equiv \alpha_D(< \mathbf{S}_1, ..., \mathbf{S}_n >) \ .$$

Sufficiency of the dominance axiom. Suppose P satisfies the dominance axiom. Let $\mathbf{Y}$ denote the linear space spanned by the indicator functions $\mathbf{1_S}$ of the sets $\mathbf{S} \in \mathcal{I}$, and $\mathbf{Z}$ denote its linear subspace spanned by the indicator functions $\mathbf{1_S}$ of the sets $\mathbf{S} \in \mathcal{S}$. Define on $\mathbf{Y}$ the norm $\parallel f \parallel = sup_{d\in\mathbf{H}}|f(d)|$. Define a convex cone in $\mathbf{Z}$,

$$\mathbf{W} = \Big\{ f \in \mathbf{Z} | f {=} \sum_{i=1}^{m} k_i \mathbf{1}_{\mathbf{S}_i} \text{ for } m{>}0, \text{ non-negative scalars } k_i, \text{ and } \mathbf{S}_i \in \mathcal{S} \Big\},$$

and on $\mathbf{W}$ define the functional

$$p(f) = \sup \Big\{ \sum_{i=1}^{m} k_i P(\mathbf{S}_i) | f = \sum_{i=1}^{m} k_i \, \mathbf{1}_{\mathcal{S}_i} \text{ for m} > 0, \\ \text{non-negative scalars } k_i, \text{ and } \mathbf{S}_i \in \mathcal{S} \Big\} \ .$$

On the space $\mathbb{R} \times \mathbf{Z}$ with the norm $|r| + \parallel f \parallel$ for $(r, f) \in \mathbb{R} \times \mathbf{Z}$, define the sets $\mathbf{A}_1 = \{(r, f) \in \mathbb{R} \times \mathbf{Z} | r \geq \parallel f \parallel\}$ and $\mathbf{A}_2 = \{(r, f) \in \mathbb{R} \times \mathbf{W} | r < p(f)\}$. Then $\mathbf{A}_1$ and $\mathbf{A}_2$ are convex cones, and $\mathbf{A}_1$ has a non-empty interior in the norm topology of $\mathbf{Z}$. Suppose $\mathbf{A}_1$ and $\mathbf{A}_2$ have a common point (r^0, f^0). Then, $\parallel f^0 \parallel \leq r^0 < p(f^0) - \varepsilon$ for some positive ε. From the definition of $p(f)$, there exists a representation $f^0 = \sum_{i=1}^{m} k_i \, \mathbf{1}_{\mathcal{S}_i}$ such that $p(f^0) \leq \sum_{i=1}^{m} k_i P(\mathbf{S}_i) + \varepsilon/2$. Then $\sup_{d\in\mathbf{H}} \sum_{i=1}^{m} k_i \, \mathbf{1}_{\mathbf{S}_i}(d) < \sum_{i=1}^{m} k_i P(\mathbf{S}_i) - \varepsilon/2$. Since this inequality is continuous in the k_{i}, these numbers can be chosen to be rational, and a common denominator cleared so that the inequality $\sup_{d\in\mathbf{H}} \sum_{i=1}^{m} k_i \, \mathbf{1}_{\mathbf{S}_i}(d) < \sum_{i=1}^{m} k_i P(\mathbf{S}_i)$

holds for some k_i integral. Considering a sequence of sets $\mathbf{S}_i$ with repetitions k_i for $i = 1, ..., m$ gives a violation of the dominance axiom. Hence, $\mathbf{A}_1$ and $\mathbf{A}_2$ are disjoint. A separating hyperplane theorem (Dunford and Schwartz (1964, V.2.8) implies the existence of a non-zero continuous linear functional (λ, ζ) on $\mathbb{R} \times \mathbf{Z}$ such that $\lambda r - \zeta(f) \geq 0$ for $(r, f) \in \mathbf{A}_1$ and $\lambda r - \zeta(f) \leq 0$ for $(r, f) \in \mathbf{A}_2$. If $\lambda \leq 0$, the first inequality holds at (1,0) $\in \mathbf{A}_1$ only if $\lambda = 0$. However, $\lambda = 0$ requires $\zeta(f) \leq 0$ for all $f \in \mathbf{Z}$, implying $\zeta(f) \equiv 0$, a contradiction of (λ, ζ) non-zero. Hence, $\lambda > 0$, and we can normalize it to one. Then, the first inequality implies $\| f \| \geq \zeta(f)$ on $\mathbf{Z}$, while the second inequality implies $\zeta(\mathbf{1_S}) \geq P(\mathbf{S})$ for $\mathbf{S} \in \mathcal{S}$. The Hahn-Banach theorem implies ζ can be extended to a linear functional on $\mathbf{Y}$ satisfying $\zeta(f) \leq \| f \|$. Then $\eta(\mathbf{S}) = \zeta(\mathbf{1_S})$ is a finitely additive probability satisfying the dominance condition $\eta(\mathbf{S}) \geq P(\mathbf{S})$ for $\mathbf{S} \in \mathcal{S}$.

If $\mathbf{S}$ is closed under complementation and $P(\mathbf{S}) + P(\mathbf{S}^c) = 1$ for $\mathbf{S} \in \mathcal{S}$, then the inequality $1 = P(\mathbf{S}) + P(\mathbf{S}^c) \leq \eta(\mathbf{S}) + \eta(\mathbf{S}^c) = 1$ implies $\eta(\mathbf{S}) = P(\mathbf{S})$ for $\mathbf{S} \in \mathcal{S}$. □

4.4 Compact families

A family $\mathcal{K}$ of subsets of a set $\mathbf{H}$ is *compact* if every sequence of members with the finite intersection property has a non-empty intersection. The family formed from $\mathcal{K}$ by the operations of finite union and countable intersection is again compact (Neveu, 1965, 1.6.1).

4.5 Tightness

Suppose a non-negative bounded set function P is defined on a family $\mathcal{S}$ of subsets of a set $\mathbf{H}$. Suppose that $\mathcal{S}$ is closed under complementation and contains $\mathbf{H}$, and that $P(\mathbf{S}) + P(\mathbf{S}^c) = 1$ for $\mathbf{S} \in \mathcal{S}$. The function P is *tight* if there is a compact family of subsets $\mathcal{K}$ of $\mathbf{H}$ such that for each $\epsilon > 0$ and $\mathbf{S} \in \mathcal{S}$ there exist $\mathbf{S}' \in \mathcal{S}$ and $\mathbf{K} \in \mathcal{K}$ such that $\mathbf{S}' \subseteq \mathbf{K} \subseteq \mathbf{S}$ and $P(\mathbf{S}) - P(\mathbf{S}') < \epsilon$.

The definition does not require that $P(\mathbf{K})$ be defined for $\mathbf{K} \in \mathcal{K}$, but simplifies (to the requirement that $P(\mathbf{S}) - P(\mathbf{K}) < \epsilon$ for some $\mathbf{K} \subseteq \mathbf{S}$ and $\mathbf{K} \in \mathcal{K}$) when $\mathcal{K} \subseteq \mathcal{S}$. If $\mathbf{S}$ is (almost surely) finite, it is itself a compact class.[3] More generally, suppose $\mathbf{H}$ can be partitioned into "atoms" $\{\mathbf{H}_1, ..., \mathbf{H}_N\}$ plus a non-atomic set $\mathbf{H}_0, \mathcal{S}_0$ is a family of subsets of $\mathbf{H}_0$, and $\mathcal{S}$ is a family whose members can be written as finite unions of sets in $\mathcal{S}_0$ and the atoms $\mathbf{H}_1, ..., \mathbf{H}_N$, or complements of such sets. If $\mathcal{K}_0$ is a compact class of subsets of $\mathbf{H}_0$ that is closed under countable intersection, then $\mathcal{K}$ formed by finite unions of $\mathbf{H}_1, ..., \mathbf{H}_N$, and sets $\mathbf{K} \in \mathcal{K}_0$ is again a compact class. Thus, P can satisfy our definition of tightness even if it has a finite number of atoms. When $\mathcal{S}$ is a Boolean algebra and $\mathcal{K}$ is contained in $\mathcal{S}$, our definition of tightness coincides with that of Neveu (1965, I.6.3). The following result relates tightness and countable additivity.

[3] The class $\mathcal{S}$ is almost surely finite if it is countable, and $P(\mathbf{S}) = 0$ for all except a finite number of sets $\mathbf{S}$ in $\mathcal{S}$.

4.6. Lemma. If P is a non-negative, finitely additive set function defined on the Boolean algebra $\mathcal{I}_0$ generated by a family $\mathcal{S}$ of subsets of a non-empty set $\mathbf{H}$, and if P is tight on $\mathcal{S}$, then P is countably additive on $\mathcal{I}_0$, and has a unique countably additive extension to the Boolean σ-algebra $\mathcal{I}$ generated by $\mathcal{I}_0$. Conversely, if P is countably additive on a Boolean σ-algebra $\mathcal{I}$, then each of the following conditions is sufficient for it to be tight:

(a) $\mathbf{H}$ is a Polish space (i.e., a complete separable metric space) and $\mathcal{I}$ is its Borel $\sigma-$field.
(b) $\mathbf{H}$ is a compact Hausdorff space with a countable base, and $\mathcal{I}$ is its Borel σ-field.
(c) $\mathbf{H}$ is a countable space, and $\mathcal{I}$ is the field of all subsets of $\mathbf{H}$.

Proof. Suppose P is finitely additive on the Boolean algebra $\mathcal{I}_0$ generated by a family of sets $\mathcal{S}$, and P is tight on $\mathcal{S}$. We show that P is tight on $\mathcal{I}_0$, and consequently σ-additive. First let $\mathbf{T}_n = \cap_{i=1}^n \mathbf{S}_i$ be a finite intersection of sets $\mathbf{S}_i \in \mathcal{S}$. The tightness assumption on P implies there exists a compact class $\mathcal{K}$ of subsets of $\mathbf{H}$, which we take without loss of generality to be closed under finite union and countable intersection, such that given $\epsilon > 0$ there exist $\mathbf{S}'_i \subseteq \mathbf{C}_i \subseteq \mathbf{S}_i$ with $\mathbf{C}_i \in \mathcal{K}$ and $P(\mathbf{S}_i) - P(\mathbf{S}'_i) < \epsilon \cdot 2^{-i}$. The set $\cap_{i=1}^n \mathbf{C}_i$ is in $\mathcal{K}$. The set inclusion $(\cap_{i=1}^n \mathbf{S}_i)\backslash(\cap_{i=1}^n \mathbf{S}'_i) \subseteq \cup_{i=1}^n (\mathbf{S_i}\backslash\mathbf{S'_i})$ and the additivity and sub-additivity of P imply

$$P\left(\bigcap_{i=1}^n \mathbf{S}_i\right) - P\left(\bigcap_{i=1}^n \mathbf{S}'_i\right) = P\left(\left(\bigcap_{i=1}^n \mathbf{S}_i\right) \setminus \left(\bigcap_{i=1}^n \mathbf{S}'_i\right)\right)$$
$$\leq \sum_{i=1}^n P(\mathbf{S}_i\backslash\mathbf{S'_i}) = \sum_{i=1}^n [P(\mathbf{S}_i) - P(\mathbf{S'_i})] \leq \epsilon\ .$$

Then P satisfies the definition for tightness on the family $\mathcal{S}_1$ of all sets formed from $\mathcal{S}$ by the operation of countable intersection. Next, consider the family $\mathcal{S}_2$ of all sets formed from $\mathcal{S}_1$ by the operation of finite union $\mathbf{V} = \bigcup_{j=1}^N \mathbf{T}_j$ of pairwise disjoint sets $\mathbf{T}_j \in \mathcal{S}_1$. From the previous construction, there exist $\mathbf{C}_j \in \mathcal{K}$ and $\mathbf{T}'_j \in \mathcal{S}_1$ satisfying $\mathbf{T}'_j \subseteq \mathbf{C}_j \subseteq \mathbf{T}_j$ and $P(\mathbf{T}_j) - P(\mathbf{T}'_j) \leq \epsilon/N$, implying $\cup_{j=1}^N \mathbf{K}_j \in \mathcal{K}, \mathbf{V}' = \cup_{j=1}^N \mathbf{T}'_j \in \mathcal{S}_2$, and $P(\mathbf{V}) - P(\mathbf{V}') \leq \epsilon$. But $\mathcal{S}_2 = \mathcal{I}_0$ (Neveu, 1965 I.2.2), so that we have established that P is tight on $\mathcal{I}_0$.

Suppose sets $\mathbf{V}_n, \mathbf{V}'_n \in \mathcal{I}_0$ and $\mathbf{C}_n \in \mathcal{K}$ satisfy $\mathbf{V}'_n \subseteq \mathbf{C}_n \subseteq \mathbf{V}_n$, $P(\mathbf{V}_n) - P(\mathbf{V}'_n) \leq \epsilon$, and $\mathbf{V}_n \searrow \emptyset$. Then, $\mathbf{C}_n \searrow \emptyset$, and compactness implies there exists N such that $\mathbf{V}'_N \subseteq \mathbf{C}_N = \emptyset$. Then $P(\mathbf{V}'_N) = 0$, implying $P(\mathbf{V}_N) \leq \epsilon$, and P is continuous at $\emptyset$, and hence countably additive. The Hahn extension theorem (Dunford, 1964, III.5.8) establishes that P has a unique countably additive extension to the Boolean σ-algebra $\mathcal{I}$ generated by $\mathcal{I}_0$.

Consider the sufficient conditions for tightness. Condition (a) is given by Neveu (1965), II.7.3. Condition (b) reduces to condition (a) by the Urysohm metrization theorem. Condition (c) reduces to condition (a) by assigning $\mathbf{H}$ the metric $\rho(x, y) = \mathbf{1}(x \neq y)$. □

4.7 Countably additive extension theorem

Suppose $\mathcal{S}$ is a family of subsets of a non-empty set $\mathbf{H}$ that contains $\mathbf{H}$ and is closed under complementation, and P is a non-negative bounded set function on $\mathcal{S}$ that satisfies $P(\mathbf{S}) + P(\mathbf{S}^c) = 1$ for $\mathbf{S} \in \mathcal{S}$ and is tight. Then, P satisfies the dominance axiom if and only if there exists a countably additive probability η on the Boolean σ-algebra $\mathcal{I}$ of subsets of $\mathbf{H}$ generated by $\mathcal{S}$ such that $\eta(\mathbf{S}) = P(\mathbf{S})$ for $\mathbf{S} \in \mathcal{S}$.

Proof. The proof of Theorem 4.3 establishes the existence of η finitely additive on $\mathcal{I}$ and satisfying $\eta(\mathbf{S}) = P(\mathbf{S})$ for $\mathbf{S} \in \mathcal{S}$ if and only if the dominance axiom holds. This also establishes the necessity of the dominance axiom when η is countably additive. For the sufficiency of the dominance axiom, apply the first result in Lemma 4.6 to the finitely additive measure η. □

5 Solutions for general revealed stochastic preference problems

5.1 Discussion

Consider the revealed distribution problem of 2.3.1, where Π_q is a probability on $\mathcal{X}$ for $q \in \mathbf{Q}$, and one seeks a probability ζ on $\mathcal{H}$, or alternately a probability ν on $\mathcal{U}$, that rationalizes the observed choice probabilities. Theorem 5.2 establishes that the Axiom of Revealed Stochastic Preference (ARSP) in 2.4 is necessary and sufficient for the existence of a finitely additive probability solving the revealed distribution problem. Its corollaries extend this result to solve the revealed dominating distribution problem. Theorem 5.3 gives regularity conditions under which ARSP is necessary and sufficient for the existence of a countably additive representation solving the revealed distribution problem. Its corollaries show that these regularity conditions are met for a formulation of the classical economic consumer revealed preference problem.

5.2 Theorem

Suppose Π_q is a finitely additive probability on $\mathcal{X}, q \in \mathbf{Q}$, satisfying $\Pi_q(\mathbf{C}) + \Pi_q(\mathbf{C}^c) = 1$ for each $\mathbf{C} \in \mathcal{X}$. Then ARSP is necessary and sufficient for the existence of a finitely additive probability η on $\mathcal{H}$ solving the revealed distribution problem.

Proof. Recall that $(\mathbf{H}, \mathcal{H})$ is the measurable space of hypothesized decision rules, with $\mathbf{H}(q, \mathbf{C}) \equiv \{\mathbf{d} \in \mathbf{H} | \mathbf{d}(q) \subseteq \mathbf{C}\}$ for $q \in \mathbf{Q}$ and $\mathbf{C} \in \mathcal{X}$. Define the class of sets $\mathcal{S} = \{\mathbf{H}(q, \mathbf{C}) | q \in \mathbf{Q} \text{ and } \mathbf{C} \in \mathcal{X}\}$. By assumption, $\mathcal{H}$ contains $\mathcal{S}$.

Necessity. Suppose η is a finitely additive probability satisfying $\Pi_q(\mathbf{C}) = \eta(\mathbf{H}(q, \mathbf{C}))$ for $q \in \mathbf{Q}$ and $\mathbf{C} \in \mathcal{X}$. For a finite sequence of trials $\mathbf{t} = < (q_1, \mathbf{C}_1), ..., (q_m, \mathbf{C}_m) >$ with $\mathbf{C}_i \in \mathcal{X}, q_i \in \mathbf{Q}$, define $\mathbf{S}_i = \mathbf{H}(q_i, \mathbf{C}_i), i = 1, ..., m$, and $P(\mathbf{S}_i) = \eta(\mathbf{H}(q_i, \mathbf{C}_i))$. Theorem 4.3 then implies that η satisfies the dominance axiom; i.e., $\sum_{i=1}^{m} \Pi_{q_i}(\mathbf{C}_i) \leq \alpha_H(\mathbf{t})$. This condition coincides with ARSP.

Sufficiency. Suppose ARSP. If $\Pi_{q_1}(\mathbf{C}_1) \geq \Pi_{q_2}(\mathbf{C}_2)$ and $\mathbf{H}(q_1, \mathbf{C}_1) = \mathbf{H}(q_2, \mathbf{C}_2)$, then

$$\Pi_{q_1}(\mathbf{C}_1) + \Pi_{q_2}(\mathbf{C}_2{}^c) \leq \alpha_{\mathbf{H}}(< \mathbf{H}(q_1, \mathbf{C}_1), \mathbf{H}(q_2, \mathbf{C}_2^c) >) = 1,$$

implying $\Pi_{q_1}(\mathbf{C}_1) \leq \Pi_{q_2}(\mathbf{C}_2)$. Hence, one can define uniquely a set function P on $\mathcal{S}$ satisfying $P(\mathbf{H}(q, \mathbf{C})) = \Pi_q(\mathbf{C}), \mathbf{C} \in \mathcal{X}, q \in \mathbf{Q}$. By construction, P satisfies the dominance axiom. Theorem 4.3 then establishes that the dominance problem has a solution, and hence that there exists a finitely additive probability η on $\mathbf{H}$ such that $\Pi_q(\mathbf{C}) \leq \eta(\mathbf{H}(q, \mathbf{C}))$ for $\mathbf{C} \in \mathcal{X}, q \in \mathbf{Q}$. Since Π_q is a probability satisfying $\Pi_q(\mathbf{C}) + \Pi_q(\mathbf{C}^c) = 1$, this solution satisfies $\Pi_q(\mathbf{C}) = \eta(\mathbf{H}(q, \mathbf{C}))$ for $\mathbf{C} \in \mathcal{X}, q \in \mathbf{Q}$, and hence solves the revealed distribution problem. □

5.2.1 Corollary to Theorem 5.2. If Π_q is a non-negative bounded set function on $\mathcal{X}, q \in \mathbf{Q}$, then a necessary and sufficient condition for the existence of a finitely additive probability η on $\mathcal{H}$ satisfying $\Pi_q(\mathbf{C}) \leq \eta(\mathbf{H}(q, \mathbf{C}))$ for $\mathbf{C} \in \mathcal{X}, q \in \mathbf{Q}$, is that Π_q satisfy ARSP.

5.2.2 Corollary to Theorem 5.2. If Π_q^- and Π_q^+ are non-negative bounded set functions on $\mathcal{X}, q \in \mathbf{Q}$, then a necessary and sufficient condition for the existence of a finitely additive probability η on $\mathcal{H}$ solving the revealed dominating distribution problem is that the function Π_q on $\mathcal{X}, q \in \mathbf{Q}$ defined by $\Pi_q(\mathbf{C}) = \max\{\Pi_q^-(\mathbf{C}), 1 - \Pi_q^+(\mathbf{C}^c)\}$ satisfy ARSP.

Proof. Necessity of ARSP. If there exists a probability η on $\mathcal{H}$ such that $\Pi_q^-(\mathbf{C}) \leq \eta(\mathbf{H}(q, \mathbf{C})) \leq \Pi_q^+(\mathbf{C})$ for all $\mathbf{C} \in \mathcal{X}$, then $\eta(\mathbf{H}(q, \mathbf{C}^c)) \leq \Pi_q^+(\mathbf{C}^c)$, implying $1 - \Pi_q^+(\mathbf{C}^c) \leq \eta(\mathbf{H}(q, \mathbf{C}))$, and hence $\eta(\mathbf{H}(q, \mathbf{C}^c)) \geq \Pi_q(\mathbf{C})$. Corollary 5.2.2 then implies that Π_q satisfies ARSP.

Sufficiency of ARSP. If Π_q satisfies ARSP, then by Corollary 5.2.1, there exists η on $\mathcal{H}$ such that $\Pi_q(\mathbf{C}) \leq \eta(\mathbf{H}(q, \mathbf{C}))$. Then $\Pi_q^-(\mathbf{C}) \leq \eta(\mathbf{H}(q, \mathbf{C}))$ and $1 - \Pi_q^+(\mathbf{C}^c) \leq \eta(\mathbf{H}(q, \mathbf{C}^c))$ imply the result. □

5.3 Theorem

Suppose the universe of alternatives $\mathbf{X}$ is a complete separable metric space, and let $\mathcal{X}$ be its Borel $\sigma-$field. Suppose the feasible choice sets $\mathbf{B}(q)$ are non-empty compact subsets of $\mathbf{X}$. Suppose the set $\mathbf{H}$ of decision rules consistent with a hypothesis of rationality is given a topology whose basis are the sets $\mathbf{H}(q, \mathbf{C})$ for $q \in \mathbf{Q}$ and open $\mathbf{C} \in \mathcal{X}$. Suppose that $\mathbf{H}$ is a compact space in this topology, and let $\mathcal{H}$ be its Borel σ-field. Suppose Π_q is a countably additive probability on $\mathcal{X}, q \in \mathbf{Q}$, satisfying $\Pi_q(\mathbf{C}) + \Pi_q(\mathbf{C}^c) = 1$ for each $\mathbf{C} \in \mathcal{X}$, and $\Pi_q(\mathbf{B}(q)) = 1$. Then ARSP is necessary and sufficient for the existence of a countably additive probability η on $\mathcal{H}$ solving the revealed distribution problem.

Proof. The necessity of ARSP is immediate from Theorem 5.2. To prove sufficiency, suppose ARSP holds, and that η is a finitely additive probability, given by Theorem 5.2, that satisfies $\eta(\mathbf{H}(q, \mathbf{C})) = \Pi_q(\mathbf{C})$ for $q \in \mathbf{Q}, \mathbf{C} \in \mathcal{X}$. For $\mathbf{C} \in \mathcal{X}$ open, the set $\mathbf{H}(q, \mathbf{C})^c$ is closed by construction, and satisfies

$\eta(\mathbf{H}(q,\mathbf{C})^c) = 1 - \Pi_q(\mathbf{C}) = \Pi_q(\mathbf{C}^c)$. Then, the family $\mathcal{S}_0 = \{\mathbf{H}(q,\mathbf{C})^c | q \in \mathbf{Q}$, open $\mathbf{C} \in \mathcal{X}\}$ is a family of closed subsets of a compact space, and is therefore a compact class. On the family $\mathcal{S} = \{\mathbf{H}(q,\mathbf{C})^c | q \in \mathbf{Q}, \mathbf{C} \in \mathcal{X}\}$, η satisfies

$$\begin{aligned}\eta(\mathbf{H}(q,\mathbf{C})^c) &= \Pi_q(\mathbf{C}^c) = \sup\{\Pi_q(\mathbf{C}'^c)) | \mathbf{C}' \text{ open, } \mathbf{C} \subseteq \mathbf{C}'\} \\ &= \sup\{\eta(\mathbf{H}(q,\mathbf{C}')^c) | \mathbf{C}' \text{ open, } \mathbf{C} \subseteq \mathbf{C}'\}\ ,\end{aligned}$$

since by Lemma 4.6 Π_q is countably additive, hence tight, on the compact feasible choice sets $\mathbf{B}(q)$. Therefore, η is tight on $\mathcal{S}$, and Lemma 4.6 implies that it is countably additive on $\mathcal{H}$. □

5.3.1 Corollary. Suppose $\mathbf{X}$ is a convex compact metric space with metric ρ, and the feasible choice situations $\mathbf{B}(q)$ are convex closed non-empty subsets of $\mathbf{X}$. Suppose Π_q is a (countably additive) probability on $\mathcal{X}$, $q \in \mathbf{Q}$, satisfying $\Pi_q(\mathbf{C}) + \Pi_q(\mathbf{C}^c) = 1$ for each $\mathbf{C} \in \mathcal{X}$, and $\Pi_q(\mathbf{B}(q)) = 1$. Suppose decision-makers are hypothesized to maximize utilities from a family $\mathbf{U}$ of uniformly bounded functions on $\mathbf{X}$ that are equicontinuous; i.e., for each $\varepsilon > 0$ there exists $\delta > 0$ such that $x, x' \in \mathbf{X}$ and $\rho(x,x') < \delta$ implies $\sup_{u\in\mathbf{U}} |u(x) - u(x')| < \varepsilon$. Then ARSP is necessary and sufficient for the existence of a (countably additive) probability ν on $\mathcal{U}$ solving the revealed distribution problem.

Proof. The Arzela-Ascoli theorem (Dunford and Schwartz, 1964, IV.6.8) establishes that $\mathbf{U}$ is a compact subset of the space $\mathbf{C}(\mathbf{X})$ of continuous functions on $\mathbf{X}$, with $\| u \| = \sup_{x\in\mathbf{X}} |u(x)|$ for $u \in \mathbf{C}(\mathbf{X})$. For open $\mathbf{C} \in \mathcal{X}$, the set $\mathbf{U}(q,\mathbf{C}) = \{u \in \mathbf{U} | \sup_{x\in\mathbf{C}\cap B(q)} u(x) > \sup_{x'\in\mathbf{B}(q)\backslash\mathbf{C}} u(x')\}$ is open. To show this, suppose $u \in \mathbf{U}(q,\mathbf{C})$. Then there exists $x'' \in \mathbf{B}(q)$ with $u(x'') > \sup_{x'\in\mathbf{B}(q)\backslash\mathbf{C}} u(x') + \varepsilon$ for some $\varepsilon > 0$. Consider u' satisfying $\| u - u' \| < \varepsilon/3$. Then, $u'(x'') > \sup_{x'\in\mathbf{B}(q)\backslash\mathbf{C}} u'(x') + \varepsilon/3$, implying $u' \in \mathbf{U}(q,\mathbf{C})$. Hence, $\mathbf{U}(q,\mathbf{C})$ with $\mathbf{C}$ open is an open set in $\mathbf{U}$. Theorem 5.3 then gives the result. □

5.3.2 Corollary. Suppose $\mathbf{X}$ is a convex compact subset of a locally convex normed linear space $\mathbf{L}$, and the feasible choice situations $\mathbf{B}(q)$ are convex closed non-empty subsets of $\mathbf{X}$. Suppose Π_q is a countably additive probability on $\mathcal{X}$ for $q \in \mathbf{Q}$, satisfying $\Pi_q(\mathbf{C}) + \Pi_q(\mathbf{C}^c) = 1$ for each $\mathbf{C} \in \mathcal{X}$, and $\Pi_q(\mathbf{B}(q)) = 1$. Suppose decision-makers are hypothesized to maximize utilities from a family $\mathbf{U}$ of uniformly bounded and concave functions on an open set containing $\mathbf{X}$. Then ARSP is necessary and sufficient for the existence of a countably additive probability ν on $\mathcal{U}$ solving the revealed distribution problem.

Proof. Assume that the uniform bound on $u \in \mathbf{U}$ is $\| u \| \leq 1$. Each point in $\mathbf{X}$ has an open neighborhood that is contained in the open set on which utility functions are defined. Since $\mathbf{X}$ is compact, it has a maximum diameter μ. Also, one can extract from the open neighborhoods a finite sub-cover; let λ be the diameter of the smallest neighborhood in this sub-cover. Suppose $x, x' \in \mathbf{X}$. By construction, the domain of the functions in $\mathbf{U}$ contains $x - (x' - x)\lambda/\mu$. Then, for $0 < \theta < 1$, concavity implies

$$u((1-\theta)x + \theta x') \geq (1-\theta)u(x) + \theta u(x'),$$

$$\text{or } u(x+\theta(x'-x)) - u(x) \geq \theta[u(x') - u(x)] \geq -2\theta$$

and

$$\begin{aligned} u(x) &= u((\lambda/(\theta\mu+\lambda))(x+\theta(x'-x)) \\ &\quad +(\theta\mu/(\theta\mu+\lambda))(x-(x'-x)\lambda/\mu)) \\ &\geq (\lambda/(\theta\mu+\lambda))u(x+\theta(x'-x)) \\ &\quad +(\theta\mu/(\theta\mu+\lambda))u(x-(x'-x)\lambda/\mu), \text{ or} \\ u(x+\theta(x'-x)) - u(x) &\leq (\theta\mu/(\theta\mu+\lambda))[u(x+\theta(x'-x)) \\ &\quad -u(x-(x'-x)\lambda/\mu)] \leq 2\theta\mu/\lambda\ . \end{aligned}$$

Given $\varepsilon > 0$, choose $\theta = \varepsilon \cdot \min(1/2, \lambda/2\mu)$. Then, $\mathbf{U}$ satisfies the condition that $x, x' \in \mathbf{X}$ with $\rho(x, x') < \theta$ implies $|u(x) - u(x')| < \varepsilon$ for all $u \in \mathbf{U}$, and Corollary 5.3.1 gives the result. □

5.4 Remarks

Theorem 5.3 is difficult to apply without sufficient conditions for the compactness of the set $\mathbf{H}$ of hypothesized decision rules. Corollary 5.3.2, which was suggested by Rosa Matzkin, provides conditions which correspond to the classical revealed preference problem. The requirement that the utility functions $u \in \mathbf{U}$ be defined on an open set containing $\mathbf{X}$ can be replaced by a condition on the subgradient $\boldsymbol{\Gamma}(x,u) = \{p \in L^* | u(y) - u(x) \leq p(y-x) \text{ for } y \in \mathcal{X}\}$ that there exist a bound $\mathcal{K} > 0$ such that $\emptyset \neq \boldsymbol{\Gamma}(x,u) \cap \{p \in L^* | \parallel p \parallel < \mathcal{K}\}$ for $x \in \mathcal{X}, u \in \mathcal{U}$ (see Matzkin, 1992; Brown and Matzkin, 1996).

If in Corollary 5.3.1, $\mathbf{Q}$ is compact and $\mathbf{B}(q)$ is a continuous correspondence, then it is sufficient to test ARSP for trial sequences drawn from a countable subset of the set of possible trials, and if ARSP fails, this will be detected in a finite number of trials (see McFadden, 1979). Thus, under these regularity conditions, a test of the validity of ARSP is computable. Going further, one can consider a net formed by nests of trial sequences $\mathbf{t} =< (q_1, C_1), ..., (q_m, C_m) >$; i.e., sequences $\mathbf{t}_1 \subseteq \mathbf{t}_2 \subseteq ...$, and utilize the linear program in Theorem 3.1(c) to recover the convex closed sets $\mathbf{G}_t$ of rationalizing probabilities on the finite algebras of subsets of $\mathbf{H}$ induced by the trial sequences $\mathbf{t}_k$, provided ARSP holds. For each set $\mathbf{H}_1$ in the Boolean algebra generated by the $\mathbf{H}(q, \mathbf{C})^c$ for $\mathbf{C}$ open and $q \in \mathbf{Q}$, the net formed by the probabilities $\eta_{\mathbf{t}}(\mathbf{H}_1)$ for $\eta_{\mathbf{t}} \in \mathbf{G}_t$ and a net of trial sequences $\mathbf{t}$ containing the trials that enter the finite intersection and union operations that produce $\mathbf{H}_1$ will contain a sub-net that converges to $\eta(\mathbf{H}_1)$ for a probability η that solves the revealed distribution problem in Theorem 5.2. Thus, there is a sequence of finite linear programming problems that provide a computable test of ARSP, and computable bounds for the rationalizing probabilities.

Two published papers have considered somewhat different versions of the issue of countably additive rationalizations. McFadden (1975) examines the question of when a joint probability over endowments and a compact set of preferences can be found that rationalize observed moments, such as per capita mean market demands.

By restricting and redefining the observed moments, the general moment problem can be specialized to the revealed distribution problem. Cohen (1980) extends the finite analysis considered in Section 3 to the case where $\mathbf{X}$ is infinite, but all choice sets $\mathbf{B}(q), q \in \mathbf{Q}$, are finite. The Block-Marschak polynomials are defined for each finite restriction of $\mathbf{X}$, and a net of choice sets contained in this restriction. Now consider a net of nested restrictions of $\mathbf{X}$, and generalized sequences of the probability measures constructed by Falmagne's method, as described in Theorem 3.3. Conditions are then given under which a generalized subsequence has a countably additive limit. Cohen's proof is difficult, but the essential idea is that when choices can be rationalized for all nested sequences of finite $\mathbf{X}$, and compactness conditions hold in the limit so that there can be no countable union of disjoint sets with positive measure, then the Kolmogorov consistency theorem and the Caratheodory extension theorem apply to achieve countable additivity. Theorem 5.3 and its corollaries provide more easily checked conditions for countable additivity, and handle the economic choice application where choice sets are not finite.

6 Extensions

New revealed preference problems can be generated by varying the family of feasible choice sets, the class of permissible decision rules, and the structure of observations. For example, one could consider classes of permissible choice rules that are either more restrictive than classical preference maximization (e.g., optimization of smooth preferences, or preferences that are homothetic, have linear Engle curves, or are in parametric families) or less restrictive (e.g., incomplete optimization of preferences, preferences that are not preorders, or preferences that are context or perception-dependent).[4] One could also consider observational situations encountered in practice (e.g., composition of market and experimental choice data, conditional distributions or conditional moments of choices given observable consumer characteristics). The classical revealed preference problem is traditionally formulated under the assumption that an individual's choices are observed in a sequence of static budget situations *without* carry-over of durables, experience, or learning from one situation to the next. The revealed distribution problem assumes that individuals are not tracked and that information is collected only on a population's distributions of choices. However, our analysis of this problem has maintained the assumption that the budget situations are static, without dynamics introduced by intertemporal maximization and state dependence. A much broader class of revealed preference problems could be formulated that allow these dynamic elements, and account explicitly for the panel data structure implicit in observation of repeated choice situations. For example, the observed choices of an individual in repeated choice situations may be interpreted as a realization of a stochastic process indexed by the choice situations, and the distributionof the stochastic process in a population, or its moments, may constitute the observations that can be analyzed.

[4] Homotheticity restrictions permit stochastic preference versions of the computational tests of revealed preference theory developed by Varian (1982,1983). One could go further and formulate parametric or nonparametric econometric tests of ARSP for a variety of hypothesized decision models.

The Axiom of Revealed Stochastic Preference can be applied to many classes of permissible choice rules; the only modifications come in the properties and interpretation of the choice probabilities and the determination of intersection numbers for trial sequences. For an expanded menu of revealed preference problems, if observed choice data are consistent with the specified class of permissible choice rules, additional interesting questions arise: Do the observations identify a unique distribution, or identify bounds on the possible distributions (McFadden, 1975)? Can the analysis be made conditional on observed population characteristics, with solutions that reflect the systematic variation in choice distributions with these characteristics? Many of these extended revealed stochastic preference problems have not been studied, and deserve the attention of economic theorists.

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Communication in dynastic repeated games: 'Whitewashes' and 'coverups'*

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Summary. We ask whether communication can directly substitute for memory in *dynastic repeated games* in which short lived individuals care about the utility of their offspring who replace them in an infinitely repeated game. Each individual is unable to observe what happens before his entry in the game. Past information is therefore conveyed from one cohort to the next by means of communication.

When communication is costless and messages are sent simultaneously, communication mechanisms or *protocols* exist that sustain the same set of equilibrium payoffs as in the standard repeated game. When communication is costless but sequential, the incentives to "whitewash" the unobservable past history of play become pervasive. These incentives to whitewash can only be countered if some player serves as a "neutral historian" who verifies the truthfulness of others' reports while remaining indifferent in the process. By contrast, when communication is sequential and (lexicographically) costly, all protocols admit only equilibria that sustain stage Nash equilibrium payoffs.

We also analyze a centralized communication protocol in which history leaves a "footprint" that can only hidden by the current cohort by a unanimous "coverup." We show that in this case the set of payoffs that are sustainable in equilibrium coincides with the weakly renegotiation proof payoffs of the standard repeated game.

Keywords and Phrases: Dynastic repeated games, Communication, Whitewashing, Coverups.

JEL Classification Numbers: C72, C73, D82.

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"History is a pack of lies about events that never happened told by people who weren't there." – George Santayana

1 Introduction

1.1 Motivation

In any longstanding strategic relationship, history matters. The ability of the "players" to construct effective deterrents against "bad" behavior typically relies on accurate monitoring and recall of the history of play.

One chief interpretation of a long-term relationship is that of a stage game being repeated between "dynastic players" rather than between infinitely lived individuals. An infinitely repeated game is interpreted as an ongoing society populated by short lived individuals who care about the utility of their successors who replace them. Each successor then faces the same "types" of opponents as his predecessor.

Examples of repeated strategic interaction that would be modelled as dynastic repeated games abound. For example, in longstanding disputes between groups with competing claims (e.g., Catholics versus Protestants in Northern Ireland, Israelis versus Palestinians), the conflicts typically outlive any particular individual. Though the names of individuals involved change with time, the issues (payoffs) often remain the same. Other examples include electoral competition between political parties (e.g., Democrats versus Republicans) and strategic competition between firms. Firms, like political parties, are long lived organizations populated by short-lived managers, each of whom are periodically replaced. Putting agency issues aside, incentives may be structured so that each current manager acts in the long run interest of the firm, despite his relatively short tenure.

Since it seems unappealing to assume that any living individual observes something that takes place before he is "born," a natural problem arises with dynastic games. It is well known that if the players do not have the means to condition their current actions on the history of play, equilibrium behavior changes dramatically. In the extreme case in which players have no knowledge of the past, strategic behavior can only depend on payoff relevant information (i.e., players must use so-called Markov strategies). When this happens and when the environment is stationary, then only repetitions of the stage game Nash equilibria are possible, even in an infinitely repeated game.[1]

In a dynastic game, each new entrant cannot condition his behavior on history unless his "knowledge" of that history comes, directly or indirectly, from past participants. Often, that means that current players must rely on the historical accounts directly communicated by their predecessors.[2] This paper examines the properties of dynastic repeated games when participants do not observe history prior to their entry in to the game, and must therefore rely on accounts communicated by their predecessors.

[1] Hence, Santayana's other famous dictum: "Those who cannot remember the past are condemned to repeat it," is quite literally true.

[2] For a useful perspective on the ways in which history is transmitted and collective memories are formed, see Pennebaker, Paez, and Rime (1997).

For simplicity, we examine a model in which each member of a dynasty only lives one period. At the end of each period, individuals in the current cohort die, and are replaced by their successors in each dynasty. Each dynastic individual cares about his successor's utility as if it was his own discounted utility. Successors inherit the same preferences, but cannot observe prior behavior.

Since prior behavior is not directly observed, we assume that the only way current behavior can be linked to the past is through the reports of the previous cohort. We therefore augment the model to allow for messages to be sent at the end of each period from the current generation to the next. Communication is assumed to be publicly observed. Because the veracity of reports cannot be verified by neutral parties, the messages can also be manipulated. To see why incentives for manipulation may exist, suppose, for example, that two dynasties face off to play the Prisoners' Dilemma in Figure 1 below. Consider the Subgame Perfect equilibrium (SPE) which, for patient enough players, sustains perpetual mutual cooperation, (C, C) using "grim trigger" strategies. In this equilibrium, the dynastic players revert permanently to (D, D) if any defection is ever observed.

		Dynasty 2: C	Dynasty 2: D
Dynasty 1	C	2, 2	−2, 3
	D	3, −2	0, 0

Figure 1. Prisoners' dilemma

Now suppose that at some date t, the date t member of Dynasty 1 defects by choosing "D." Despite the fact that the individual in Dynasty 1 defected in the PD game, *both* individuals at date t may have an incentive to *whitewash* the defection by falsely reporting action (C, C) to the next generation. By lying, the current generation can insulate the next generation against the mutually destructive punishment phase. However, because lying precludes punishment, incentives for good behavior in the current stage are destroyed.

Unlike in standard communication (cheap talk) games,[3] individuals within a dynasty value future payoffs in the same way. The potential incentive to misreport exists not because of payoff differences, but rather from a desire to protect future generations from the consequences of past deviations. By *whitewashing* deviations, the current generation has the chance to give their successors a clean slate to start the game. In this sense, the environment we analyze is reminiscent of repeated game models with renegotiation.[4]

Our interest, therefore, is in the extent to which history is accurately conveyed from one generation to another. How does potential manipulation of information across generations distinguish dynastic repeated play from the "full memory"

[3] See Crawford and Sobel (1982) for a classic reference.

[4] See Farrell and Maskin (1989), Abreu, Pearce, and Stacchetti (1993), and Benoit and Krishna (1993).

model? To sensibly address this question, we adopt an implementation approach. We examine whether there exist useful *communication protocols*, i.e., mechanisms in which communication directly substitutes for memory.

We examine two models. The first is a model of potential *whitewashing*. Information transmission constitutes cheap talk. Each individual may misrepresent the information unobservable to the next generation. Misrepresentation is typically costless to the sender. We examine whether or to what extent members of the current generation may whitewash the past in equilibrium. The second model examines the potential for *coverups*: The aggregation mechanism utilizes, to some extent, observable information. Consequently, members of the current cohort may attempt to hide information which might otherwise be observable to the next generation. Hiding information is difficult, and may require widespread agreement among the senders.

Despite the incentive to whitewash or to coverup detrimental histories, protocols that support full and honest communication exist. In the whitewashing model, if the communication protocol is decentralized and if messages are sent simultaneously then standard Nash implementation logic can be used to show that Perfect Bayesian equilibria (PBE) exist in which no whitewashing takes place. The idea is familiar: since reports contain redundant information, each individual's report can be used to screen the veracity of others. Hence, if there are at least three players and all but one players' messages agree, then the next generation uses the agreed upon message as the "official version" of history.[5] If there are only two players, then the absence of an agreed upon message is treated as if a defection occurred. Hence, for any stage game the set of possible equilibrium strategies and payoffs is equivalent to that of the standard repeated game.

Ironically, the simultaneous moves protocol disciplines the players by instituting a coordination failure. For example, in our Prisoners' Dilemma example above, under the grim trigger strategy, both individuals in the present cohort would be better off by whitewashing a deviation, but neither can do it given the anticipated truthful message of the other. Clearly, if given the opportunity, one of the players would prefer to signal his intent to lie by moving first.[6] Indeed, suppose that the messages are sent sequentially, and that members of Dynasty 1 communicate first. If the date t player 1 whitewashes his own defection, then it is clearly a best response for the date t player 2 to confirm the lie. Sequential moves therefore allow the players to break the "coordination failure" that prevented whitewashing in the simultaneous case. But since whitewashing will occur, the mutual cooperation equilibrium using grim trigger strategies cannot be sustained in the first place.

We characterize the PBE payoff set in any game when messages are sequential rather than simultaneous. A necessary condition for play to differ from the repetition

[5] Of course, the original "cross-checking" argument goes back to Maskin (1999). Baliga, Corchon, and Sjostrom (1997) use a similar type of mechanism in another model of cheap talk communication when there are three or more players. Similar types of mechanisms have also been used in repeated games with private monitoring and communication. See Ben-Porath and Kahneman (1996), Compte (1998), and Kandori and Matsushima (1998).

[6] Lagunoff and Matsui (1997) analyze repeated coordination games in which the players move sequentially.

of stage Nash actions in any PBE is that some player serves as "a neutral historian." The neutral historian is an individual who screens and verifies the truthfulness of reports of others, while remaining indifferent in the process. This necessity of a neutral historian rules out some types of equilibria. Nevertheless, a wide array of payoffs approximating the original payoffs of the repeated game are shown to be sustainable. It turns out that rectangular, "self generating" subsets of the equilibrium payoff set are sustainable when communication is sequential.[7] The analysis of the sequential communication protocol is important as a robustness check. If individuals are unable to commit themselves to the timing structure of the protocol, then some individual may break the simultaneous communication structure by attempting to "speak first."

Hence, on the one hand, our results are reassuring. Rectangular self generation is broad enough to include many if not most payoffs of interest in the full memory (non-dynastic) repeated game. On the other hand, it turns out that protocols with "neutral historians" are fragile. We show that these constructs fail when communication is no longer costless. When individuals weigh the complexity costs of the reporting strategies they use, then for any sequential protocol, only the Nash equilibria of the stage game are sustainable in the dynastic game. This is the case even when the actual payoffs from the stage game are lexicographically more important than the costs associated with a more complex reporting strategy.

In many instances, the assumption that the past history of play cannot be verified at all by the current cohort may be too extreme. It is easy to think of situations in which the past should be, at least in part observable, unless a concerted effort to hide it is made. The remnants of the Jewish Holocaust and the Stalin Purges all too quickly come to mind. To begin to address this type of set-up, we analyze a model of dynastic repeated games with a different set of assumptions about the communication protocol between one cohort and the next.

We assume history leaves a marker or "footprint" for the new generation. Efforts to manipulate information now entail effort to hide or *coverup* these footprints. We examine a protocol in which the truth can only be hidden by the current cohort when all individuals agree to the coverup. Somewhat counterintuitively, the difficulty in achieving consensus to unanimously coverup the truth may actually *increase* the incentive to hide it. We are able to show that, when communication is sequential, the set of sustainable payoffs coincides with the set of weakly renegotiation proof payoffs of the standard repeated game.[8] The conclusion is that when the potential for *coverup* exists and when messages are sequenced, equilibria with strictly Pareto-ranked continuation payoffs cannot occur.

[7] Self generating payoff sets were first defined by Abreu, Pearce, and Stacchetti (1986) and Abreu, Pearce, and Stacchetti (1990) who used dynamic programming methods to characterize the equilibrium payoff set in a repeated game. Note also that we are somewhat abusing the meaning of the word "rectangular" here. What is required is that the lower contour of the self generating set is that of a rectangle. See the statement of Theorem 3 below.

[8] In the sense of Farrell and Maskin (1989).

1.2 Outline

The material in the paper is divided into 4 further sections. In Section 2 we describe the model in detail. This includes briefly setting up the standard repeated game notation and a complete description of the dynastic repeated game with communication. Section 3 is devoted to the analysis of our "whitewashing" model. We first analyze simultaneous and then sequential messages. We then move on to the case of lexicographic costs of more complex reporting strategies and show that only the stage game Nash equilibrium payoffs survive in this case. Section 4 is concerned with our model of "coverups." After describing the model, we go on to show that only weakly renegotiation-proof equilibria are viable in this case. Section 5 concludes the paper with a brief discussion putting our results in the context of existing literature.

For ease of exposition, all proofs are confined to an appendix. In the numbering of equations, Lemmas, Theorems etc. a prefix of "A" means that the corresponding item is located in the Appendix.

2 The model

2.1 A standard repeated game

We first describe a standard, n-player repeated game. We will then augment this structure to describe the dynastic repeated game with communication from one cohort to the next. The standard repeated game structure is of course familiar. We set it up below simply to establish the basic notation.

The stage game is described by the array $G = (S, u; I)$ where $I = \{1, \dots, n\}$ is the set of players, indexed by i. The n-fold cartesian product $S = \times_{i \in I} S_i$ is the set of pure action *profiles* $s = (s_1, \dots, s_n) \in S$, assumed to be finite. Stage game payoffs are defined by $u = (u_i)_{i \in I}$ where $u_i : S \to \mathbb{R}$ for each $i \in I$. Let $\sigma \in \Delta(S)$ denote a mixed action profile.[9] The corresponding payoff to player i, denoted by U_i, is defined in the usual way: $U_i(\sigma) = \sum_s \sigma(s) u_i(s)$. Dropping the i subscript and writing $U(\sigma)$ gives the entire profile of payoffs. Finally, we let $\mathcal{N}$ denote the set of Nash equilibria of the stage game.

In the repeated game, denote the behavior profile at time t by $\sigma(t) = (\sigma_1(t), \dots, \sigma_n(t))$. For $t \geq 1$, a period t *behavior history* (of length t) is an array $h^t \equiv (\sigma(0), \sigma(1), \dots, \sigma(t-1))$ of action profiles observed by time t. The null history is $h^0 = \emptyset$. Let $U_i(\sigma(t))$ denote the expected payoff at date t. The set of period t behavior histories is denoted by $H^t = \Delta(S)^t$. Let $H = \cup_{t=0}^{\infty} H^t$ denoting the collection of all (finite) behavior histories.[10]

The players' (for simplicity) common discount factor is denoted by $\delta \in (0, 1)$, so that for a given infinite history $h^\infty = (\sigma(0), \sigma(1), \dots)$, player i's payoff in the

[9] At the expense of some extra notation and further manipulations we could consider *correlated* action profiles in the stage game without altering the nature of our results below.

[10] Therefore we are assuming that actual mixed strategies are observed. This simplifies our framework, and particularly notation, considerably. However, we later argue that none of the results in the paper depend on the assumption that mixed strategies are observed.

repeated game is given by

$$V_i(h^\infty) \;=\; (1-\delta)\sum_{t=0}^{\infty}\delta^t\,U_i(\sigma(t)) \tag{1}$$

A behavior strategy in the repeated game is a map $f_i : H \to \Delta(S_i)$. Let $f = (f_1, \ldots, f_n)$ denote the profile of strategies in the repeated game. Given any finite history h^t, the mixed action at date t is given by $f(h^t) = (f_1(h^t), \ldots, f_n(h^t))$.

Given (1) the continuation payoff to i given strategy profile f after any history h^t follows the recursive equation

$$V_i(f|\,h^t) \;=\; (1-\delta)\,U_i(f(h^t)) \;+\; \delta\,V_i(f|\,h^t, f(h^t)) \tag{2}$$

where $(h^t, f(h^t))$ denotes the period $t+1$ history given by the concatenation of history h^t and period $t+1$ behavior profile $f(h^t)$. Dropping the player subscript in equations (1) and (2) will denote the corresponding payoff profiles in the repeated game.

A *subgame perfect equilibrium* (SPE), f^*, for the repeated game is defined in the usual way: for each i, and each finite history h^t, and each strategy f_i for i, we require that $V_i(f^*|\,h^t) \geq V_i(f_i, f^*_{-i}|\,h^t)$.[11]

We denote respectively by $\mathcal{F}(\delta)$ the set of SPE strategy profiles and by $\mathcal{E}(\delta)$ the set of SPE payoff profiles of the repeated game when the common discount factor is δ.

The standard model of repeated play we have just sketched out may be found in a myriad of sources. See, for example, Fudenberg and Maskin (1986) and the references contained therein. Hereafter, we refer to the standard repeated game model above as the *full memory repeated game*.

2.2 The dynastic repeated game

Now assume that each $i \in I$ indexes an entire progeny of individuals. We refer to each of these as a *dynasty*. Individuals in each dynasty are assumed to live one period. At the end of each period t (the beginning of period $t+1$), a new individual from each dynasty — the date $(t+1)$-lived individual — is born and replaces the date t lived individual in the same dynasty. Hence, $U_i(\sigma(t))$ now refers to the payoff received by the t-th individual in dynasty i. Each date t individual is altruistic in the sense that his payoff includes, as an additively separable argument, the utility of the $t+1$-th individual from the same dynasty. The weight given to his own payoff is $1-\delta$, while the weight given to his offspring (the $(t+1)$-th individual) is δ. Therefore, in the dynastic repeated game, the long-run payoffs retain the same recursive structure given in equation (2). This observation is of course sufficient to show that, if all individuals in each dynasty can observe the past history of play, then the dynastic repeated game is in fact identical to the full memory repeated game described in Subsection 2.1. In fact, this full information dynastic repeated game

[11] As is standard, here, and throughout the rest of the paper, a subscript of "$-i$" indicates an array with the i-th element taken out.

is one extremely appealing interpretation/justification of the standard full memory repeated game model.

2.3 Communication

If the t-th individuals in each dynasty cannot observe the history of play that took place before they are born, then their behavior cannot vary across distinct histories h^t. It follows immediately that, in the absence of communication from one cohort to the next, if all individuals in all dynasties are ignorant in this way, in each period of the dynastic repeated game only those payoffs that are Nash equilibria of the stage game can be attained.

The question then becomes: can communication substitute for memory? Assume that at each time t a cohort *can* observe the action profile that takes place at t. Assume also that they have the chance to communicate with the next cohort. Can they credibly convey sufficient information to the $t+1$-th cohort to attain payoffs beyond the Nash equilibria of the stage game?

Of course, communication can take place in a variety of different ways. For instance, as we anticipated above, whether messages are sent simultaneously or sequentially will have an impact on the outcome of the game. We begin by defining a model of the communication between one cohort and the next in which the individuals in each dynasty speak simultaneously to the individuals in the next cohort. This will be modified in Subsection 3.2 to allow for sequential communication.

Let A_i denote a set of payoff-irrelevant communication actions for dynasty i, with $A = \times_i A_i$ being the set of profiles of such actions. These need not be related to the stage game itself, but are choices that collectively determine a message sent from one generation to the next. The messages that can possibly be sent to the next cohort is given by the set M. We assume that both A and M are invariant across time.[12] At each date t, let $a(t) = (a_1(t), \dots, a_n(t))$ denote the profile of communication actions, and let $m(t) \in M$ denote the message (or profile of messages) sent in period t.

Unless otherwise noted, we will assume that any message(s) transmitted to the next cohort is commonly observed by all members of the next cohort.[13] A *communication protocol* in the dynastic repeated game is a list $C \equiv (A, M, \Phi)$ where $\Phi : A \to M$. For $a = (a_1, \dots, a_n) \in A$, $\Phi(a)$ is the message sent to the next generation, after the payoff-relevant behavior occurs in that period. Some "natural" examples of features that communication protocols may satisfy are:

[12] It is worth emphasizing here that this assumption implies that "calendar time" cannot be "verified" by any cohort. For instance, the time t and the time $2t$ individuals could generate exactly the same message to be transmitted to the next cohort. In this case, calendar time would clearly "look the same" to the individuals in cohort $t+1$ and in cohort $2t+1$. A rather strong feature of this assumption, which may seem unappealing to some, is that any date t individuals could in fact generate a message for the next cohort which matches the null history $\emptyset$. In effect they would be telling their offspring that "the game has just started, you are the first to play."
The qualitative nature of our results below would remain unaffected if somehow these "re-starting" messages were forbidden. In some cases (for instance Theorem 5 below), period $t = 0$ would have to be excluded from our characterization of equilibria which would only apply to periods $t \geq 1$.

[13] See our discussion of this assumption in the concluding section of the paper.

C^1 **Babbling**: $\Phi(a) = m$ for all $a \in A$. Clearly, the message m is uninformative.

C^2 **Dictatorial**: There exists a dynasty $i \in I$ such that $\Phi(a) = a_i$.

C^3 **Unanimity**: $A_i = M = H, \ \forall i,$ and

$$\Phi(a) = \begin{cases} h & if \ a_i = h, \ \forall i \\ h^* & otherwise. \end{cases}$$

Here, a single version of history is sent if everyone agrees. Otherwise, a default history is reported.

C^4 **Decentralized Communication**: $M = \times_{i \in I} A_i, \ H \subseteq A_i, \ \forall i,$ and $\Phi(m) = m$. In a decentralized communication protocol, everyone separately reports history to next generation.

While the first three examples fulfill an expository function, the last is a useful benchmark. It describes the least restrictive communication, and so it provides the most attractive environment for accurate transmission. We examine the case of decentralized communication protocols in detail below.

Every communication protocol identifies a dynastic repeated game with communication. A strategy for an individual in a dynasty is a pair consisting of an "action" strategy and a "communication" strategy. The former processes the messages received from the prior generation and determines current behavior in the stage game. The latter determines the individual's communication action, which, via the communication protocol, determines the message conveyed to the next cohort of individuals.

We begin by defining action strategies. For simplicity, we examine action strategies that can be written as a single repeated game strategy describing the plans of all individuals within a dynasty. When $H \subseteq M$, there is little loss of generality with this assumption since an individual from, say, the t-th cohort, need only use that part of the repeated game strategy which follows histories h^t of length t. Let $g_i : M \to \Delta(S_i)$ denote an action strategy for dynasty i. Let $g = (g_1, \dots g_n)$.

A communication strategy is a map μ_i from the prior generation's messages and current (observed) actions to current messages. Formally, $\mu_i(m, \sigma)$ denotes a communication action $a_i \in A_i$ by an individual from dynasty i given that the prior generation's message profile is m and that the current action profile is σ. The profile $(\mu_1(m, \sigma), \dots, \mu_n(m, \sigma))$ then maps to a message m' via Φ. This message m' is sent to the next generation. Let $\mu = (\mu_1, \dots, \mu_n)$. To summarize, date t individuals choose action profile $\sigma(t) = g(m(t-1))$ and take communication actions $a(t) = \mu(\ m(t-1), \sigma(t))$.

As with the full memory repeated game, something to start off play is needed. (In the full memory repeated game this is the empty history $h^0 = \emptyset$.) In the dynastic repeated game with communication we need to define which message the first (born at $t = 0$) cohort observes. Let this initial message be denoted by $m(-1) = h^0 = \emptyset$.

The pair (g, μ) describes all behavior in the dynastic repeated game with communication. An individual's dynamic payoff after receiving message m is expressed as $V_i(g, \mu | \ m)$. An individual's dynamic payoff after receiving message m and after action profile σ is expressed as $V_i(g, \mu | \ m, \sigma)$. In either case, an individual's payoff still follows the recursive form in (2). We can now define a Perfect Bayesian equilibrium (PBE) pair (g^*, μ^*) in the usual way: for each i, any

μ_i and g_i, and for any m and any σ, $V_i(g^*, \mu^*| \ m) \geq V_i(g^*_{-i}, g_i, \mu^*| \ m)$ and $V_i(g^*, \mu^*| \ m, \sigma) \geq V_i(g^*, \mu^*_{-i}, \mu_i| \ m, \sigma)$.[14]

Given a communication protocol C and a common discount factor δ, we denote by $\mathcal{F}^C(\delta)$ the set of PBE, and by $\mathcal{E}^C(\delta)$ the set of PBE payoff profiles. Let C^1 denote the babbling protocol described above with $m = \emptyset$, and recall that C^4 denotes the decentralized protocol. Then it is easy to see that

$$\mathcal{E}^{C^1}(\delta) \subseteq \mathcal{E}^C(\delta) \subseteq \mathcal{E}^{C^4}(\delta) \tag{3}$$

for all δ and all C. Moreover, $\mathcal{E}^{C^1}(\delta)$ coincides with Nash equilibrium payoffs of the stage game. Clearly, if there is no communication, or when communication is uninformative, there is no hope of attaining anything beyond payoffs of the stage game. Conversely, if there are no restrictions on communication, then the largest possible payoff set can be sustained.

3 Whitewashing

In this section we focus on the case in which the current cohort has no access at all to any direct information about the past history of play. We examine first the case of simultaneous messages. Then we move on to the case in which the members of the current cohort speak sequentially to the next cohort. Finally, we turn to the case of sequential communication in which a more complex reporting strategy is lexicographically more costly than a simpler one.

3.1 Decentralized communication

We now proceed to examine equilibrium behavior under the *decentralized communication protocol* defined in Section 2.3 above. Recall that in the decentralized communication protocol, all individuals in each cohort effectively report separately and simultaneously a history of play to the next generation, and all reports are commonly observed by all individuals in the next cohort. The messages are unrestricted in the sense that any finite history (of any length) can be conveyed by any individual (in other words $H \subseteq A_i$ for every i). Intuitively, a decentralized protocol corresponds to a world in which there is no attempt to collectively limit information, nor is there any direct trace left by the actual history of play. This is of course in contrast with the model of coverups that we will analyze in Section 4 below.

Our first characterization of the equilibrium set of dynastic games with communication tells us that in the case of decentralized simultaneous communication from one cohort to the next, the equilibrium set is the same as in the full memory standard repeated game. In other words, in this case *communication* can indeed *substitute*

[14] While there are no proper subgames beginning each date, all individuals have common knowledge regarding messages they receive. Hence, all Perfect Bayesian equilibria are Perfect *Public* equilibria of the dynastic game. Consequently, the definitions above make (partial) use of the "One-Shot Deviation Principle."

for memory in the dynastic repeated game. Those familiar with standard implementation logic will not find the following Theorem and its immediate Corollary very surprising.

It should be made clear at the outset that we are setting up Theorem 1 below as a benchmark case. Our task in the rest of this Section and in Section 4 will then be to show that this equivalence between the full memory standard repeated game and the dynastic repeated game is in fact not robust in more senses than one. Indeed Theorem 1 fails in a rather dramatic way when communication carries arbitrarily small costs and when the actual past history of play does leave a detectable trace.[15]

Theorem 1. *Assume that the number of players n is at least 3. Fix any common discount factor δ and any SPE $f^* \in \mathcal{F}(\delta)$ of the full memory game.*

Then, for any decentralized communication protocol C, there exists a PBE $(g^*, \mu^*) \in \mathcal{F}^C(\delta)$ of the dynastic game with communication protocol C that is equivalent to f^* in the following sense.

For each m such that $m = (h, \ldots, h)$ for some $h \in H$ and for every $\sigma \in \Delta(s)$

$$g_i^*(m) = f_i^*(h) \qquad \text{and} \qquad \mu_i^*(m, \sigma) = (h, \sigma) \qquad \forall\, i \tag{4}$$

In other words, provided that the messages sent by the previous cohort are all the same (equal to h), then the PBE (g^*, μ^*) prescribes the same actual behavior as the SPE f^* after h. Moreover, again provided that the messages sent by the previous cohort are all the same (equal to h), then in the PBE (g^*, μ^*), any profile of current behavior (equilibrium or not) is truthfully reported to the next cohort by all i.

Finally, notice that (4) since the first cohort all receive the same (empty) message, obviously implies that the outcome path generated by (g^*, μ^*) is the same as the outcome path generated by f^*.

The proof of Theorem 1 is in the Appendix. As we mentioned above, it runs along familiar lines. The argument requires building into the equilibrium of the dynastic repeated game the correct incentives for truthful reporting by all individuals in each cohort.[16] In turn, this of course requires a mechanism to detect and punish lies. Unilateral deviations from truthful reporting are easily identified when there

[15] Note also that we state Theorem 1 for the case of 3 or more players, ignoring the two-player case. This result can be generalized to the case of two players (a slightly weaker statement holds) for discount factors arbitrarily close to 1. Proceeding in this way saves a non-negligible amount of space. The proof of the result for the two-player case involves mimicking the "mutual minmax" argument used to prove Folk Theorems in two-player standard repeated games, and thus a substantial modification of the "cross-checking" argument that we outline below.
Since, as we just said, Theorem 1 largely plays the role of a straw man in what follows, we take the view that stating it only for the case of three or more players carries little or no cost for the sharpness of the overall message we are trying to convey.

[16] Notice that all the equilibria that we construct in the paper are "truthful" in the sense that, without loss of generality, provided that $H \subseteq A_i$ for every i (so that the space of message actions is rich enough), we can take it to be the case that, in equilibrium, the members of each cohort report truthfully to the next generation. Truthfully, of course, can only mean that a given one-to-one map of action messages into histories is used throughout. This is a weak version of the so-called "revelation principle" that obviously holds in our model.

are three dynasties or more. Since only single-player deviations from equilibrium ever need to be considered this is enough to induce truth-telling as required. Finally, since the argument constructs incentives for truthful reporting, taking as given the corresponding (action) incentives in the full memory game, Theorem 1 does not depend on our assumption that mixed strategies are observable.[17] Indeed, we require only that *whatever* is observable in the full memory game can be observed (period-by-period) and reported in the dynastic game.

It is trivial that any equilibrium of the dynastic game with communication is also an equilibrium of the full memory game. Therefore, an immediate corollary of Theorem 1 is that the sets of equilibrium payoff profiles are the same in the two cases.

Corollary 1. *Assume that $n \geq 3$ and let any decentralized communication protocol C and any common discount factor δ be given. Then $\mathcal{E}^C(\delta) = \mathcal{E}(\delta)$.*

Before we turn to sequential communication, some remarks about Theorem 1 and Corollary 1 are in order. First, the "cross-checking" aspect of the mechanism we construct for three or more players in the proof is by no means new. Baliga, Corchon, and Sjostrom (1997) use this type of mechanism in a (static) model of communication. It is also reminiscent of communication mechanisms in repeated games with private monitoring. For example, Ben-Porath and Kahneman (1996) prove a Folk Theorem when public communication is admissible in a repeated game with private monitoring.[18] Specifically, they show that the Folk Theorem applies in any private monitoring game in which individuals' behavior is (perfectly) observed by at least two others. Like ours, their proof also exploits a procedure whereby the deviator is identified as the one whose report fails to correspond to identical reports of at least two others.

Secondly, if we allow players in the dynastic game to (independently) *randomize* their communication actions, then Theorem 1 and Corollary 1 would have to be modified to take into account the fact that messages now become a potential coordination device. In particular, it is not hard to see that using random messages the players could achieve a randomization across different Nash equilibria of the stage game in a given period. This, of course, would alter (enlarge) the set of achievable long-run payoffs when δ is bounded away from 1.[19] However the effect of randomized messages would become negligible as the discount factor approaches 1 since playing the different Nash equilibria in sequence through time would have the same effect on long-run payoffs. For simplicity, and because our main question here is whether the messages can substitute for memory, we focus on pure message strategies throughout the paper.

[17] For exactly the same reasons this is also true in all our other results below. See also footnote 10 above.

[18] See also the papers by Compte (1998), and Kandori and Matsushima (1998).

[19] We are grateful to Dino Gerardi for pointing out this fact.

3.2 Coordination failure and sequential communication

Clearly, equilibria in the full memory game represent the outer bound of what is possible in the dynastic repeated game. As we know from Theorem 1 and Corollary 1 decentralized communication with simultaneous choice of message actions achieves this bound. In a sense, this equivalence relies on a dependence on the simultaneity of messages, which is unsettling for at least two reasons. First of all, simultaneous messages may be simply not feasible.

Secondly, there may be incentives for the players to depart from any simultaneous communication protocol. Consider for instance the repeated PD game in Figure 1. A standard way of supporting mutual cooperation utilizes a joint punishment (e.g., permanent reversion to the unique equilibrium of the stage game) as a way to deter deviations from the path of perpetual cooperation. However, once a defection in behavior takes place, everyone, including the "injured party," would prefer to whitewash the history of defection. It is clear that the simultaneous structure of the communication protocol prevents any agent from signaling his intent to falsify the truth. If communication were sequential then the intent to "whitewash" could be relayed by one member of the current cohort to the other. Consequently, incentives to sequence messages may arise, especially if the timing of communication is not observed by subsequent cohorts.

In this Section we explore the consequences of assuming that message actions are taken *sequentially* by individuals in a given cohort.

To keep matters simple, we examine the simplest class of protocols with sequential choice of action messages. We consider the class of communication protocols that are decentralized in the sense that we specified above, but modified so that the individuals in each cohort choose their message actions one after the other.

Definition 1. *We say that a communication protocol is a* sequential decentralized *communication protocol if it can be obtained as the following simple modification of a (simultaneous) communication protocol that is decentralized in the sense of protocol C^4 described above.*

There exists a permutation mapping $\theta : \{1, \dots , n\} \to \{1, \dots , n\}$ that describes the order in which the message actions are chosen.[20] *In other words, first individual $\theta(1)$ chooses a message action $a_{\theta(1)} \in A_{\theta(1)}$. Immediately after, all other individuals in the cohort observe $a_{\theta(1)}$. Then individual $\theta(2)$ chooses a message action $a_{\theta(2)} \in A_{\theta(2)}$ which is then observed by all other individuals in the same cohort. The choice of action messages then continues in this fashion until individual $\theta(n)$ makes his choice.*

The rest of the details of a sequential decentralized communication protocol are as in Example C^4 above.

With a sequential decentralized communication protocol, Theorem 1 no longer holds. Some equilibria of the full memory game are destroyed by the incentives to whitewash the past. For example, consider the stage game in Figure 2.

[20] The actual permutation mapping θ is irrelevant for all our results below. Unless we specify otherwise in what follows we assume that θ is in fact the natural order so that $\theta(i) = i$.

		Dynasty 2		
		L	M	R
	T	2, 2	0, 3	0, 0
Dynasty 1	M	3, 0	1, 1	0, 0
	B	0, 0	0, 0	0, 0

Figure 2. A 3×3 stage game

In the full memory game, every payoff profile in the strictly individually rational set $\{(v_1, v_2) :\ v_i > 0, \forall i = 1, 2\}$ is sustainable as an SPE if δ is close enough to one.[21]

Now suppose that two dynasties play this game with sequential decentralized communication. We assert that any equilibrium in which a deviation is countered with permanent reversion to the worst Nash equilibrium (B, R) cannot be sustained. Consider the perpetual repetition of (T, L) each period. Suppose further that at some date t, the date t member of Dynasty 1 chooses to "cheat" by deviating to M. Now both members of generation t communicate sequentially to date $t + 1$ individuals. Dynasty 1 communicates first. If the date t player 1 whitewashes by lying about his defection then player 2 will confirm the lie *unless* player 2 can be made at least as well off by telling the truth about (M, L) taken in the current period. This means that in the continuation, player 2 must receive a payoff of at least 2 for telling the truth. However, if he receives a payoff of more than 2, then player 2 will always report "(M, L)" even when "(T, L)" was the true action taken.

Notice that since communication is cheap talk, the structure of the reporting "subgame" remains the same after every history.[22] Hence, the set of continuation equilibria after the reporting stage must remain the same. Yet, the reporting "subgame" will typically have multiple "subgame perfect" equilibria, one for each possible history in the game. In this particular game, this means that player 2 must be *indifferent* between truthful reporting and whitewashing. Therefore, player 2's continuation payoff in the putative continuation is 2 regardless of whether he reports "(T, L)" or "(M, L)."

Moreover, since in the putative equilibrium, "$(0, 0)$" is the hypothesized continuation after a deviation, player 1 must have an incentive to truthfully report "(M, L)," after his own deviation. But since player 2 will truthfully report "(M, L)," should player 1 attempt to lie, the continuation payoff profile after the sequence of reports "$(\ (T, L), (M, L)\ )$" *must* be $(0, 2)$. That is, if player 1 attempts to lie, and player 2 reports the truth, then player 2 must be indifferent between his reports, and player 1 must no better off than if he told the truth. In the latter case, player 1 cannot be strictly worse off since his lowest feasible payoff is 0, and so he too is indifferent between his reports.

[21] Notice that, as is often the case, the (lower) boundary of the individually rational set is not sustainable, except of course for the origin $(0, 0)$ which is a Nash equilibrium of the stage game.

[22] We put "subgame" in quotes since the reporting stage is not literally a proper subgame of the dynastic game. Nevertheless one can refer to the subgame perfect equilibria of the extensive form reporting game, whose terminal payoffs are equilibrium continuations.

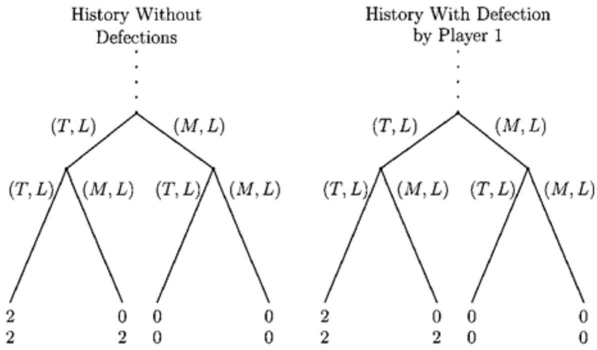

Figure 3

The problem, however, is that $(0, 2)$ is not a feasible SPE continuation payoff in the game. Hence, the mutual cooperation equilibrium using these particular punishments cannot be sustained. The relevant portion of the game tree in the message phase is represented in a schematic way in Figure 3.[23]

3.3 Sequential communication: the necessity of a "neutral historian"

The above example demonstrates that the potential for sequential moves to break the "coordination failure" that prevented whitewashing in the simultaneous case, is enough for whitewashing to occur in some cases. The example also demonstrates a general property of perfect equilibria when communication is sequential. To characterize this property, some further notation is required.

We will denote by a^{i-1} the $(i-1)$-tuple of message actions chosen by individuals $1, \ldots, i-1$ (by convention, set $a^0 = \emptyset$). In this way we can then write the strategy of individual i in the communication round as determining $a_i = \mu_i(m, \sigma, a^{i-1})$. In other words, i chooses his message action as a function of the message $m = (m_1, \ldots, m_n)$ sent by the previous cohort, the current action profile σ, and the message actions a^{i-1} chosen by individuals 1 through to $i-1$ in the current cohort. In this way μ determines an entire path a of message actions that can be written as

$$a_1 = \mu_1(m, \sigma, a^0), \quad a_2 = \mu_2(m, \sigma, a^1), \ldots,$$
$$a_i = \mu_i(m, \sigma, a^{i-1}), \ldots, a_n = \mu_n(m, \sigma, a^{n-1})$$

In any sequential decentralized protocol, the choice of profile (g, μ) determines a reporting "subgame" in the communication round. This "subgame" can be viewed as an extensive form game of perfect information. The "terminal nodes" of this extensive form game are the equilibrium continuations that begin with the next generation's play of the game. Since communication is cheap talk, the terminal nodes of this game do not vary with past reports and play. We write $v = V(g, \mu | \, a)$

[23] Clearly, each player has 9 choices at each node in the complete version of the tree drawn in Figure 3 — one for each possible outcome of play at $t-1$. We have only represented the two relevant ones purely for the sake of visual clarity.

to denote the payoff vector associated with the terminal node reached by path a given that the continuation of play is determined by (g, μ). When the path has only been determined up to, say, player j's report, i.e., when the path is $a^j = (a_1, \ldots, a_{j-1})$, then players' strategies $\mu_{j+1}, \ldots, \mu_n$ are used to determine the terminal node. Since these strategies depend, in turn, on prior history (m, σ), we must write $V(g, \mu| \, m, \sigma, a^j)$ to denote the terminal node reached by path a^j given history (m, σ) and strategies (g, μ). In the result below, the *reporting subgame* refers to the induced extensive form game in the message phase whose terminal payoffs are equilibrium continuation payoffs.

Theorem 2. *(Necessity of a "Neutral Historian") Fix a sequential, decentralized protocol C, and fix a PBE, $(g, \mu) \in \mathcal{F}^C$. Let (m', σ') and (m'', σ'') denote two prior histories of message and action profiles, respectively, each of which generate distinct message paths. Then, there is a path a, a player j, and a pair of distinct action messages, a'_j and a''_j, for that player such that $\mu_j(m', \sigma', a^{j-1}) = a'_j$ and $\mu_j(m'', \sigma'', a^{j-1}) = a''_j$ and*

$$V_j(g, \mu| \, m', \sigma', a^{j-1}, a'_j) \;=\; V_j(g, \mu| \, m'', \sigma'', a^{j-1}, a''_j)$$

In words, there exists some path a and some player (a "neutral historian") who distinguishes between (m', σ') and (m'', σ'') by using two action messages that generate payoff equivalent continuations.

The result gives necessary conditions for the existence of multiple equilibria of the message game. This is of crucial importance since the message game must have multiple equilibria in order to successfully punish deviations. That is, since the next generation has no independent verification of the actual history, both the continuation from prescribed play and the continuations after deviations must all be equilibrium continuations of the reporting game. The older generation must be able to coordinate either on the original equilibrium path continuation, or on the punishment continuation. Generally, there will be at least $n + 1$ equilibria of the message game. One for the equilibrium prescription, and one each for each individual's deviation from equilibrium.

However, because the reporting game cannot vary with prior history, the multiplicity of equilibria in the reporting game implies that subgame is nongeneric. Somewhere, there must be "ties," in someone's payoff, and these ties must have the structure described in Theorem 2. This structure implies that in every PBE (g^*, μ^*), the reporting subgame must utilize a "neutral historian."

Specifically, in any pair of equilibrium paths, there exists someone whose potential veto of one of the paths is governed by the later choice of a "neutral historian," player j. The neutral historian is one who uses his indifference between two terminal payoffs (i.e., his "neutrality") to influence the earlier messages of others. In essence, the neutral historian screens and verifies the truthfulness of others' reports, while remaining indifferent in the process. An explicit construction that identifies the behavior of this neutral historian is given in the proof of Theorem 3 in the next Section.

3.4 Self generation

Using a, by now standard, notion of self generating sets (Abreu 1988, Abreu, Pearce, and Stacchetti 1986, Abreu, Pearce, and Stacchetti 1990), one can construct equilibrium payoff sets for sequential protocols. Formally, a set $\mathcal{V}$ is *self-generating* if for each $v \in \mathcal{V}$ there exists a $\sigma \in \Delta(S)$ and a map $w : \Delta(S) \to \mathcal{V}$ such that (i) $v = (1-\delta)U(\sigma) + \delta w(\sigma)$, and (ii) $v_i \geq (1-\delta)U(\sigma_i', \sigma_{-i}) + \delta w_i(\sigma_i', \sigma_{-i})$ for each i and each σ_i'. It is not difficult to show that all bounded self generating sets are contained in the SPE profiles, $\mathcal{E}(\delta)$, of the standard repeated game. Indeed, $\mathcal{E}(\delta)$ itself, as well as any stage Nash payoff vector are self generating sets. The following Theorem justifies our interest in a special type of self generating sets: those that have the lower boundary of a rectangle in $\mathbb{R}^n$. It asserts that any such self generating subset is sustainable in any sequential, decentralized protocol provided that the protocol allows each individual to report histories and continuation payoffs to the next generation of players.

Theorem 3. *Fix any $0 < \delta < 1$, and in the full memory game let $\mathcal{V}$ be a compact, self-generating set with the following property. For any i, and any $\hat{v} \in \mathcal{V}$, let $\underline{v}_i = \min v_i$ such that $(v_i, v_{-i}) \in \mathcal{V}$ for some v_{-i}. Then for every $\hat{v} \in \mathcal{V}$ we have that*

$$(\underline{v}_i^*, \hat{v}_{-i}) \in \mathcal{V} \tag{5}$$

Then, for any sequential, decentralized communication protocol, C, in which $H \subseteq A_i$ for each i,

$$\mathcal{V} \subseteq \mathcal{E}^C(\delta)$$

The proof in the Appendix gives an explicit account of how the "neutral historian" is used to sustain payoffs in the self generating set. Roughly speaking, given any point in $\mathcal{V}$, a protocol is constructed in which player 2 (who speaks second), serves as the neutral historian. After player 1's (who speaks first) report, player 2 is asked to "confirm" it or not. (The messages of all other players are ignored.) If player 2 confirms 1's report, then play unfolds as dictated by a particular SPE of the full memory game. If on the other hand player 1 deviates from reporting the truth, this is treated as if he had behaviorally deviated from equilibrium play, and he is punished by being awarded the lowest possible payoff in $\mathcal{V}$. The equilibrium is also constructed so that player 2 is always *indifferent* between confirming 1's report or not. Thus, exploiting the properties of $\mathcal{V}$, both 1 and 2 are given the correct incentives to report the truth and to serve as the "neutral historian" respectively.

The question of whether any particular subset of the equilibrium payoff set $\mathcal{E}(\delta)$ that has the lower boundary of a rectangle is self generating is open. However, particular examples of such self generating sets are not difficult to construct. Clearly, individual Nash equilibrium payoff profiles are degenerate self-generating rectangles. To see that more interesting self generating sets that have the lower boundary of a rectangle are common, consider once again the Prisoner's Dilemma game in Figure 1 in the Introduction.

We claim that for $\delta \geq 4/7$ the rectangle $\{v : (1,1) \leq v \leq (2,2)\}$ is self generating. Notice that this includes mutual cooperation. While we don't verify the property for all profiles in the set, we can easily do so for the extremal ones. First consider $(2, 2)$. Consider the following is simple penal code. Any deviation from $(2, 2)$ by, say, a player 1, is followed by a one period punishment in which they play: $\sigma_1(C) = 4/5,\ \sigma_2(C) = 1/3$. Following this, the players revert to mutual cooperation. Further deviations by player 1 restart the punishment; deviations by player 2 are countered by the same punishments (switching the players' roles).

We now verify that this is a SPE penal code when $\delta = 4/7$. In fact, by extending the length of the punishment, this type of penal code also works for discount factors greater than 4/7 as well.[24]

While the one shot punishment gives player 1 a stage payoff of $-1/3$, his dynamic payoff in the punishment phase is 1 since $1 = (1 - \delta)(-1/3) + \delta 2$ when $\delta = 4/7$. The payoff to player 2 in player 1's punishment phase is $(1 - \delta)2 + \delta 2 = 2$. Note that player 1 obtains a payoff of unity from perpetual deviation. Hence, a (weak) best response is to submit to punishment in order to obtain 2 in the continuation. This penal code simultaneously verifies that the profiles $(2, 2)$, $(1, 2)$, and $(2, 1)$ are all SPE profiles (since punishment continuations are obviously SPE outcomes as well). To see that $(1, 1)$ is also a SPE profile, we construct a penal code which supports $(7/4, 7/4)$ as follows: play $\sigma_i(C) = .914$ (approximately) each period. Any deviation is met with one period reversion to the Nash equilibrium, after which time play resumes as before. This can be verified to be a SPE penal code. Since the value of the punishment profile is $(1, 1)$, it constitutes a SPE continuation. Other payoffs in the rectangle may be shown to be sustained more easily since the punishment will generally be more severe, and the one shot gain to deviation will generally be lower.[25]

3.5 Costly communication

Our discussion so far of sequential decentralized communication has yielded two insights. First of all the potential to whitewash *does* have an impact on the structure of equilibria of the repeated game. Theorem 1 no longer holds in this case. Some equilibria of the full memory game are not viable under sequential decentralized communication because they would leave one or more individuals with an incentive to whitewash the past after certain histories of play.

On the other hand, the logic of Theorem 3 demonstrates that whitewashing can be prevented, even if it makes all current and future generations better off, and even when individuals can signal their intentions to coordinate on the whitewashing of previous deviations. Since our examples suggest that most payoffs of interest (e.g., mutual cooperation in Prisoner's Dilemma) can be sustained by rectangular self generation if players are sufficiently patient, these payoffs are also sustainable by equilibria of sequential protocols. Our next step is to show that such equilibria are fragile.

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[24] Notice that we only establish $\delta = 4/7$ as a lower bound for self-generation when Nash reversion is the punishment.

[25] The same logic that sustains $(7/4, 7/4)$ can be applied to verify the self generating property for other interior points in the rectangle.

We modify the dynastic game with sequential decentralized communication in the following way. We assume that an "infinitesimal" cost is associated with communication strategies that are more "complex". In other words, we assume that there are costs attached to more complex communication strategies, but that they matter in the comparison between the payoffs that two strategies yield *only* if these two strategies yield actual payoffs (from actual play that is) that are the *same*. Using a standard term, we call these *lexicographic* costs of more complex communication strategies.

Once the model is modified to allow for lexicographic costs of more complex communication strategies, it behaves in a dramatically different way. Theorem 4 asserts that once complexity costs are taken into account the set of PBE of the dynastic game shrinks to $\mathcal{N}$ – the set of Nash equilibria of the stage game.

We begin with a definition of what it means for a strategy to be more complex than another at the communication stage. We want to deem a communication strategy to be more complex than another if it prescribes communication actions that depend "more finely" on the history of play. To describe formally what we mean by more finely some extra notation is required.

Recall that with sequential decentralized communication the message action of individual i is denoted by $a_i = \mu_i(m, \sigma, a^{i-1})$, where $m = (m_1, \ldots, m_n)$ is the message sent by the previous cohort, σ is the current action profile, and a^{i-1} is the profile of message actions chosen by individuals 1 through to $i - 1$ in the current cohort. Let $\mathcal{M}_i$ be the set of all possible tuples (m, σ, a^{i-1}).[26]

Given a communication strategy μ_i we can of course identify the way in which μ_i *partitions* $\mathcal{M}_i$. We let this partition be denoted by $\mathcal{P}_i(\mu_i)$. The "cell" of $\mathcal{P}_i(\mu_i)$ that contains any given $(m, \sigma, a^{i-1}) \in \mathcal{M}_i$ is denoted by $\lambda_i(m, \sigma, a^{i-1})$ and is defined as follows.

$$\begin{aligned}\lambda_i(m, \sigma, a^{i-1}) = \big\{(m', \sigma', a^{i-1\prime}) \in \mathcal{M}_i \;\mid\; & \mu_i(m', \sigma', a^{i-1\prime}) \\ & = \mu_i(m, \sigma, a^{i-1})\,\big\}\end{aligned} \tag{6}$$

Lastly, as is standard, if the partition $\mathcal{P}_i(\mu_i)$ is *coarser* than the partition $\mathcal{P}_i(\mu_i')$ we write $\mathcal{P}_i(\mu_i) \succ \mathcal{P}_i(\mu_i')$.[27]

We can now proceed with a formal definition of the assertion that a communication strategy is more complex than another.[28]

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[26] So, to be precise we have that $\mathcal{M}_i = M \times \Delta(S) \times A^{i-1}$, where $A^{i-1} = A_1 \times \ldots \times A_{i-1}$ if $i \geq 2$, and $A^{i-1} = \emptyset$ if $i = 1$.

[27] Of course in this case we may also say that $\mathcal{P}_i(\mu_i')$ is *finer* than $\mathcal{P}_i(\mu_i)$.

[28] The notion of complexity embodied in Definition 2 below is related to the definition of complexity based on the number of states in an automaton needed to implement a strategy (Rubinstein 1986, Abreu and Rubinstein 1988, Aumann and Sorin 1989, Piccione 1992, Rubinstein and Piccione 1993, Chatterjee and Sabourian 2000, among others). However, it should be noted that the two are not the same. The reasons are two-fold. First of all, we do not restrict attention to strategies that are at all implementable by a finite automaton. In this sense the "domain" of Definition 2 is broader than the ones based on counting states in a finite automaton. Secondly, our definition below is "weaker" than the automaton based ones in the following sense. Given two communication strategies that *are* implementable by a finite automaton, it is possible that one be less complex than the other in the sense that it requires fewer states, but that the two are *not comparable* in the sense of Definition 2 below. On the other hand, it is

Definition 2. *We say that communication strategy* μ_i' *is more complex than communication strategy* μ_i *if and only if* $\mathcal{P}_i(\mu_i) \succ \mathcal{P}_i(\mu_i')$.[29]

As we anticipated above, we assume that whenever the payoffs stemming from the (repeated) stage-game are equal, communication strategies that are less complex in the sense of Definition 2 are preferred. The easiest way to include our assumption of lexicographic costs of more complex reporting strategies is to modify the definition of an PBE for the dynastic repeated game with sequential decentralized communication.

Definition 3. *Consider an PBE* (g^*, μ^*) *for the dynastic repeated game with sequential decentralized communication. We say that* (g^*, μ^*) *is robust to lexicographic complexity costs of communication if and only if for every dynasty* i *and every* $(m, \sigma, a^{i-1}) \in \mathcal{M}_i$*, there does* not *exist a communication strategy* μ_i' *such that*

$$V_i(g^*, \mu_i', \mu_{-i}^* | \, m, \sigma, a^{i-1}) \;=\; V_i(g^*, \mu^* | \, m, \sigma, a^{i-1}) \tag{7}$$

and μ_i^* *is more complex than* μ_i' *in the sense of Definition 2.*

Given a sequential decentralized communication protocol C *and a common discount factor* δ*, the set of PBE that are robust to lexicographic complexity costs of communication will be denoted by* $\tilde{\mathcal{F}}^C(\delta)$*, while the corresponding set of payoffs will be denoted by* $\tilde{\mathcal{E}}^C(\delta)$ *throughout the rest of the paper.*

Notice once again that Definition 3 embodies the idea that complexity costs of communication *only matter* if the payoffs from the (repeated) stage game are the same. A PBE is robust to lexicographic complexity costs of communication if, given the strategies of the others, no individual can choose a communication strategy that leaves his basic payoff unaffected but which has a lower degree of complexity than the equilibrium one.

The idea is that players will not distinguish histories that have equal payoffs in the equilibrium continuation. Indeed, why would they expend energy to otherwise? But if a player does not make fine distinctions between otherwise identical histories, then he must play the same way after each such history of play.

Notice that the way we have incorporated the role of complexity costs into the equilibrium notion for our model is in some strong sense the *weakest* possible one. If we modelled the complexity costs of communication to be even small but positive, their impact on the equilibrium set could not be smaller than in the lexicographic case we are considering here. In this sense, Theorem 4 below refers to the *limit* case in which complexity costs of communication have been shrunk to zero.

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easy to check that in this case, if one strategy is less complex than the other in the sense used here, then it necessarily is less complex than the other in the sense of requiring fewer states. While counting states provides a complete order of strategies that are implementable by a finite automaton, Definition 2 only defines a partial order on this set. See also footnote 29 below.

[29] Notice that the complexity of a communication strategy only defines a partial order on the set of communication strategies in the sense that clearly there exist pairs μ_i and μ_i' such that both $\mathcal{P}_i(\mu_i) \not\succ \mathcal{P}_i(\mu_i')$ and $\mathcal{P}_i(\mu_i') \not\succ \mathcal{P}_i(\mu_i)$ hold.

Theorem 4. *Let any sequential decentralized communication protocol C and any common discount factor δ be given. Then any PBE (g^*, μ^*) that is robust to lexicographic complexity costs of communication has the following features. The action profile taken in any subgame $\sigma = g^*(m)$ is a Nash equilibrium of the stage game (in other words $g^*(m) \in \mathcal{N}$ for every $m \in M$). Moreover the action profile σ is the same in every period along the equilibrium path — except possibly in the first period.*[30]

The proof of Theorem 4 is in the Appendix. A brief outline of the argument is as follows.[31] We know from Theorem 2 that in order to generate distinct message paths in any PBE with sequential communication in the dynastic game it must be the case that at least one player is indifferent between two distinct action messages in terms of the continuation payoffs they generate. This immediately implies that at least one player can unilaterally deviate from this putative equilibrium and use a communication strategy that is less complex in the sense of Definition 2 and obtain the same continuation payoff. Hence, in any PBE of the dynastic game with sequential communication that is robust to lexicographic complexity costs according to Definition 3, the action messages of all players must be *the same* in every period.

Since the action messages chosen in equilibrium must be the same in every communication subgame, regardless of history, it now follows that the continuation payoffs to every individual cannot depend either on the current action profile or on the message received from the previous cohort. But then it follows immediately that the action profile σ chosen in every period cannot be anything other than a Nash equilibrium of the stage game.

The potential to whitewash is quite devastating when communication is sequential and decentralized and complexity costs of communication have even a lexicographic impact on payoffs. All deviations will be whitewashed by the current cohort. Continuation payoffs are therefore independent of current behavior, and only behavior that is equilibrium in a static sense will survive in any equilibrium of the dynastic game.

4 Coverups

So far we have analyzed the dynastic repeated game with communication protocols that ensured that *all* the information available to the current cohort is the result of message actions taken by the previous cohort. Of course, this is an extreme assumption. A more "realistic" view is that the information available to the current cohort is a mixture of the true history of play and of the communication behavior of the previous cohorts. The purpose of this section is to characterize equilibrium behav-

[30] Recall that each cohort cannot verify "calendar time" (see footnote 12 above). If, instead, we allowed calendar time to be verifiable it is relatively easy to see that different Nash equilibria of the stage game could be played in different periods in a PBE.

[31] We are grateful to an associate editor of this journal for pointing out that the proof of Theorem 4 could be shortened considerably by appealing to Theorem 2 in the way we do below.

ior in the dynastic repeated game under one such possible "mixed" communication protocol.

As we mentioned in Section 1 we examine a communication protocol in which the past history of play leaves a "footprint." This footprint will be enough to reveal the true behavior of the previous cohort, unless the individuals in the current cohort *unanimously* agree to report a different history to the next generation. As we anticipated in Section 2 we call this a model of "coverups".

To describe in detail the communication protocol with unanimous conspiracy to coverup, it is convenient to refer back to Example C^3 of Section 2.3 above. Essentially, we need to fill out the details of C^3 above: we need to specify the history h^* that is reported to the next generation in case of disagreement.

As in Example C^3, let $A_i = M = H$. Now define $\Phi : M \times \Delta(S) \times A \to M$ as

$$\Phi(m, \sigma, a) = \begin{cases} h & if \ \exists\, h \text{ such that } h = a_i \,\forall\, i \\ (m, \sigma) & \text{otherwise} \end{cases} \tag{8}$$

Using (8) we can now proceed to define our communication protocol with unanimous conspiracy to coverup.

Definition 4. *The dynastic repeated game with unanimous conspiracy to coverup is defined as follows.*

Individuals in each cohort choose their message actions sequentially as described in Section 3.2 above.

Consider a cohort that has received message m from the previous cohort. Assume that the individuals in the current cohort have chosen action profile σ. Let a be the profile of action messages chosen by individuals in the current cohort. Then the message received by the next cohort is given by $m = \Phi(m, \sigma, a)$, where Φ is as in (8).

In other words, the message sent to the next cohort is equal to (m, σ) where m is the message of the previous cohort and σ is the *true* current action profile, *unless* all individuals in the current cohort choose *identical* action messages $a_1 = \ldots\ a_n = h$. In the latter case the message passed on to the next generation is h.

It turns out that the set of equilibrium payoffs of the dynastic repeated game with unanimous conspiracy to coverup is the same as the set of payoffs generated by equilibria of the full memory game that satisfy a restriction that has been analyzed before — namely those equilibria of the full memory game that are Weakly Renegotiation Proof (henceforth WRP) in the sense of Farrell and Maskin (1989). This is of independent interest since it tells us that Theorem 5 can be viewed as providing non-cooperative foundations for restricting attention to the set of WRP equilibria in a repeated game.

Before we proceed any further, for completeness we give a definition of those SPE that are WRP in the full memory game.

Definition 5 (Farell and Maskin, 1989). *Consider the full memory game of Section 2. Let f^* denote an SPE of this game.*

We say that f^* is Weakly Renegotiation Proof *if and only if it has the property that no continuation equilibrium is strictly Pareto-dominated by another continuation equilibrium.*

In other words, an SPE f^* of the full memory game is WRP if and only if there exist no pair of finite histories h and h' such that

$$V_i(f^*|h) \;>\; V_i(f^*|h') \quad \forall\, i = 1, \ldots, n \tag{9}$$

Throughout the rest of the paper, given a common discount factor δ the set of SPE strategy profiles of the full memory game that are WRP is denoted by $\mathcal{F}^R(\delta)$, while the corresponding set of payoff vectors is denoted by $\mathcal{E}^R(\delta)$.

We are now ready to state our last result.[32]

Theorem 5. *Let a communication protocol C with unanimous conspiracy to coverup and a common discount factor δ be given. Then $\mathcal{E}^C(\delta) = \mathcal{E}^R(\delta)$. In other words, the set of PBE payoffs in the dynastic repeated game with unanimous conspiracy to coverup is the same as the set of SPE payoffs of the full memory game that are WRP.*

The proof of Theorem 5 is in the Appendix. Intuitively, the argument that makes it hold runs along the following lines.

Start with $\mathcal{E}^C(\delta) \subseteq \mathcal{E}^R(\delta)$. Consider a reporting subgame of the dynastic repeated game, in which the players choose their actions sequentially, from 1 to n. Suppose that the equilibrium prescribes that all individuals report the action profile σ, but that the continuation payoffs associated with these reports are strictly Pareto-dominated by the continuation payoffs associated with another profile of message actions that are different from the “true” σ. Then using backwards induction (on the set of individuals, within the reporting “subgame”) it is possible to show that the true reporting behavior could not be an equilibrium in the first place. Since every individual can unilaterally trigger the true σ to be communicated to the next cohort, it is also possible to show that it cannot be the case that equilibrium behavior prescribes unanimous reporting of a “false” σ that is associated with continuation payoffs that are strictly dominated by the continuation payoffs associated with the true profile σ. In this way, it is possible to show that the equilibrium behavior in the reporting subgame cannot be associated with a profile of continuation payoffs that are dominated by the continuation payoffs associated with another profile of message actions. Hence no continuation equilibrium can be strictly Pareto-dominated by another continuation equilibrium.

To see that $\mathcal{E}^C(\delta) \supseteq \mathcal{E}^R(\delta)$ consider a reporting subgame of the dynastic repeated game and the following communication strategy profile which serves to implement any WRP equilibrium of the full memory game.[33]

Player 1 starts by reporting the true history of play, and provided that there have been no deviations, all subsequent players also report the true history of play. If any deviations from truthful reporting have occurred (so that the true history is revealed through its “footprint”) then every subsequent player reports the true history of

[32] We are grateful to an Associate Editor of this journal for suggesting that a previous version of Theorem 5 could be strengthened to yield the result that we now report below. See footnote 33 below.

[33] This part of the statement and proof of Theorem 5 was suggested to us by an Associate Editor of this journal.

play. Now consider the possibility that, through a sequence of deviations, all players report some false history instead. Then, since the corresponding equilibrium of the full memory game is WRP there must be at least one player who (at least weakly) prefers the continuation payoffs associated with the true history of play. Since a single player disagreeing with the attempt to cover up the true history is sufficient to reveal it, this is enough to show that true reporting will in fact take place in equilibrium. Hence $\mathcal{E}^C(\delta) \supseteq \mathcal{E}^R(\delta)$.

5 Concluding remarks

This paper examines play in dynastic repeated games.[34] Since each new generation cannot (perfectly) observe prior play, they must rely on messages of the prior generation. When these messages constitute cheap talk, then communication protocols must guard against whitewashing. When some prior information is available, then protocols must deter coverups.

Our results show that standard mechanism designs in the protocol can easily sustain all outcomes that were available in the full memory repeated game. Even when reports in the communication phase are sequenced, protocols which necessarily utilize some "neutral historian" exist to sustain most if not all outcomes of the full memory game.

However, our results also suggest that these equilibria are fragile. If individuals' reports in any communication phase are sequenced, and if complexity matters even lexicographically, then only stage Nash equilibria can appear along the equilibrium path. In this world, the messages conveyed from one generation to the next are devoid of any real content.

Despite some similarities, the present model examines a very different type of communication than in typical sender-receiver, cheap talk models such as Crawford and Sobel (1982) and, more recently Krishna and Morgan (2001). The latter is representative of a more recent variety which, as in our model, features multiple senders of information. Yet, in all these models, difficulties in reporting incentives are due to different payoff functions between the sender and receiver. By contrast, in our model there are no payoff differences, at least between sender and receiver of the same dynasty. The incentive problems arise because of the requirement that the equilibria coordinate behavior on intertemporal sanctions. Sometimes these sanctions punish many or all individuals for the sins of one. This coordination on sanctions drives the necessary wedge between the senders and receivers of the hidden information.

Though no one in a cohort observes past history, the assumption of public messages means that all individuals of a given generation inherit the same "memory" from their predecessors. For this reason, the present model also bears some resemblance to repeated games with public monitoring. (Green and Porter (1984), Abreu, Pearce, and Stacchetti (1986), Abreu, Pearce, and Stacchetti (1990), and Fudenberg, Levine, and Maskin (1994), among others).

[34] Related issues arise in two recent papers (Kobayashi 2003, Lagunoff and Matsui 2002) that model organizations populated by dynastic overlapping generations of players.

The public observation assumption is motivated by our desire to bias things as much as possible against whitewashing. Public observation allows the adoption of standard techniques from Nash implementation (see, for example, Jackson (1999) or Maskin and Sjöström (2002) and the references contained therein). For this reason, the sensitivity of such mechanisms to sequencing and complexity is somewhat unexpected.

Nevertheless, one could imagine dropping the assumption of public observability. In that event, we do not yet know what happens. With private, intra-dynastic communication and with only two dynasties, it is possible that folk theorem-like results similar to those reported in Kobayashi (2003) could be proved. However, with more than two dynasties, this is less clear.

With private communication, the model would bear closer resemblance to repeated games with private monitoring since each member of each cohort would possess a private version of the history of play that is not common knowledge across players.[35]

One the one hand allowing private communication in the present set-up diminishes the scope for the use of "cross-checking" mechanisms to punish deviators. On the other hand, with private messages there seems to be a large degree of flexibility in defining out-of-equilibrium beliefs, which in turn suggests that a larger set of equilibrium outcomes may be sustainable.[36] It may well turn out to be the case that the role of out-of-equilibrium beliefs is, in fact, a key difference between dynastic games with private messages and repeated games with private monitoring.

The field of repeated games with private monitoring is an extremely important and currently active area of research in which general results have, by and large, eluded the best efforts of a formidable line-up of investigators. Though desirable, extending the model we have analyzed here to the general private-message case is evidently beyond the scope of the present paper.

Appendix

Proof of Theorem 1. Let f^* denote any SPE in the full memory game.

We now define (g^*, μ^*) as follows. For each profile $m = (m_1, \dots, m_n)$, and each i,

$$g_i^*(m) = \begin{cases} f_i^*(h) & if \exists h \text{ such that } m_j{=}h, \forall j{\in}J \subseteq I, \text{ with} |J| \geq n{-}1 \\ f_i^*(h^0) & \text{otherwise} \end{cases} \tag{A.1}$$

[35] See, for instance, Ben-Porath and Kahneman (1996), Sekiguchi (1997), Compte (1998), Kandori and Matsushima (1998), Mailath and Morris (1998), Bhaskar and van Damme (2002), Compte (2002a), Compte (2002b), Ely and Valimaki (2002), Kandori (2002), Mailath and Morris (2002), Matsushima (2002), and Piccione (2002). With few exceptions, this literature tends to examine outcomes of games that are "close" to those with public monitoring.

[36] We are grateful to Dino Gerardi for pointing out a two-player example of out-of-equilibrium beliefs in which each player believes that the other player has received the same message with probability one. These out-of-equilibrium beliefs, when available, are helpful in sustaining many of the same equilibrium outcomes of the dynastic game with public communication.

and for each m, each σ, and each i,

$$\mu_i^*(m,\sigma) = \begin{cases} (h,\sigma) & if \exists h \text{ such that } m_j{=}h, \forall j{\in}J \subseteq I, \text{ with} |J| \geq n{-}1 \\ (h^0,\sigma) & \text{otherwise} \end{cases} \tag{A.2}$$

It is now straightforward to verify that (g^*, μ^*) is a PBE for the dynastic repeated game. The details are therefore omitted. Since the profile f^* was taken to be an arbitrary SPE of the full memory game, this is clearly enough to prove our claim. □

Proof of Corollary 1. It is immediate to check that $\mathcal{E}^C(\delta) \subseteq \mathcal{E}(\delta)$. The details of this claim are omitted. Since Theorem 1 obviously implies that $\mathcal{E}(\delta) \subseteq \mathcal{E}^C(\delta)$ the claim is proved. □

Proof of Theorem 2. Let (g, μ) denote any PBE. As hypothesized in the Theorem, let (m', σ') and (m'', σ''), denote the prior histories of message and action profiles, respectively.

Now suppose that the Theorem is false. Then, for any path, a, and for every player j, we must have

EITHER

$$\mu_j(m', \sigma', a^{j-1}) = \mu_j(m'', \sigma'', a^{j-1}) \tag{A.3}$$

OR

$$\begin{gathered} \mu_j(m', \sigma', a^{j-1}) \neq \mu_j(m'', \sigma'', a^{j-1}), \text{ and} \\ V_j(g, \mu|\; m', \sigma', a^{j-1}, \mu_j(m', \sigma', a^{j-1})\;) \neq V_j(g, \mu|\; m'', \sigma'', a^{j-1}, \\ \mu_j(m'', \sigma'', a^{j-1})\;) \end{gathered} \tag{A.4}$$

We use the following backward induction argument. Suppose, first, that either (A.3) or (A.4) holds for all paths and for player n. We now argue that (A.4) cannot hold for player n. To see this, observe that if (A.4) did indeed hold then, without loss of generality, we have

$$\begin{gathered} \hat{v}_n \equiv V_n(g, \mu|\; m', \sigma', a^{n-1}, \mu_n(m', \sigma', a^{n-1})\;) \;> \\ V_n(g, \mu|\; m'', \sigma'', a^{n-1}, \mu_n(m'', \sigma'', a^{n-1})\;) \equiv v_n \end{gathered} \tag{A.5}$$

But since player n is the last mover to report, then at any node a^{n-1}, his preferences for $\hat{v}_n$ over v_n cannot depend on prior histories (m', σ') and (m'', σ''). That is, we can drop the notational dependence of V_n on (m', σ') and (m'', σ'') and rewrite (A.5) as

$$\hat{v}_n \equiv V_n(g, \mu|a^{n-1}, \mu_n(m', \sigma', a^{n-1})) {>} V_n(g, \mu|a^{n-1}, \mu_n(m'', \sigma'', a^{n-1})) \equiv v_n \tag{A.6}$$

From (A.6), it is easy to see that $\mu_n(m'', \sigma'', a^{n-1})$ is not a best response, violating the equilibrium property of μ. Therefore, for player n, (A.3) must hold, i.e.,

$$\mu_n(m', \sigma', a^{n-1}) = \mu_n(m'', \sigma'', a^{n-1}) \tag{A.7}$$

Notice that (A.7) implies that player n must play the same way on *every* path a, regardless of which prior history, (m', σ') or (m'', σ''), occurred.

Now suppose that (A.3) holds for players $i+1, \ldots, n$. As was true for player n, these players must play exactly the same way on every path. Said another way, along any path all subsequent players fail to distinguish between (m', σ') and (m'', σ'') in their reporting behavior. But if strategies $\mu_{i+1}, \ldots, \mu_n$ fail to distinguish between (m', σ') and (m'', σ'') in these subgames, then this must also be true for player i. For if, instead, (A.4) held, i.e., if

$$V_i(g, \mu | m', \sigma', a^{i-1}, \mu_i(m', \sigma', a^{i-1})) \neq V_i(g, \mu | m'', \sigma'', a^{i-1}, \mu_i(m'', \sigma'', a^{i-1}))$$

then either $\mu_i(m', \sigma', a^{i-1})$ or $\mu_i(m'', \sigma'', a^{i-1})$ can no longer be a best response. Hence, player i must satisfy (A.3).

But now consider the incentives of player $i = 1$. Observe that, as hypothesized, (m', σ') and (m'', σ'') each generate distinct paths. Let a' denote the path following (m', σ') and let a'' denote the path following (m'', σ''). Since equation (A.3) holds for all other players, $2, \ldots, n$, it must be true that player 1 plays differently after each of (m', σ') and (m'', σ'') in order to distinguish a' from a''. Specifically,

$$\mu_1(m', \sigma', a^0) \neq \mu_1(m'', \sigma'', a^0) \tag{A.8}$$

That is, player 1 must have distinct choices after (m', σ') and (m'', σ'') since no other player distinguishes between the two histories. But (A.8) contradicts (A.3) for player 1. Since we have already established that player 1, as well as all other players cannot satisfy (A.4), it must be the case that player 1 violates both (A.3) and (A.4), and so we have obtained our contradiction. This concludes the proof. □

The following Lemma will be used for the proof of Theorem 3.

Lemma A. 1. *Let $\mathcal{V}$ be a self-generating closed set of long-run payoffs for the full memory game that satisfies (5).*

For any vector $v \in \mathcal{V}$, we let $\mathcal{Z}(v)$ be the set of strategy profiles that sustain the vector of long-run payoffs v as an SPE of the full memory game, with continuation payoffs that lie entirely in $\mathcal{V}$. For any vector $v \in \mathcal{V}$, we let f_v denote a generic element of $\mathcal{Z}(v)$. Also, for any vector $v \in \mathcal{V}$ we let $P_i(v)$ be the projection of v on the lower boundary of $\mathcal{V}$ for player i. In other words, $P_i(v) = (v_1, \ldots, v_{i-1}, \underline{v}_i, v_{i+1} \ldots, v_n)$.

Now consider an arbitrary $v^* \in \mathcal{V}$. Then there exists an $f^* \in \mathcal{Z}(v^*)$ with the following properties.

For any history h^t, let $\sigma^*(h^t)$ be the mixed profile of actions prescribed by f^* at time $t+1$, conditional on history h^t taking place. Let also σ^i be any mixed action profile that agrees with $\sigma^*(h^t)$ on all components except *for player i. That is σ^i satisfies $\sigma^i_j = \sigma^*_j(h^t)$ for every $j \neq i$ and $\sigma^i_i \neq \sigma^*_i(h^t)$.*

Then, for any history h^t and for any i,

$$V(f^*|h^t, \sigma^i) \;=\; P_i[V(f^*|h^t, \sigma^*(h^t)] \tag{A.9}$$

Moreover, let $\hat{\sigma}$ be any mixed action profile that differs from $\sigma^*(h^t)$ on two or more *components. Let $\mathcal{D}$ be the set of players for which $\hat{\sigma}$ and $\sigma^*(h^t)$ differ. Then, for any history h^t*

$$V_i(f^*|h^t, \hat{\sigma}) \;=\begin{cases} \underline{v}_i & \text{if } i \in \mathcal{D} \\ V_i(f^*|h^t, \sigma^*(h^t)) & \text{otherwise} \end{cases} \tag{A.10}$$

In other words, without loss of generality, we can take f^* to have the property that any unilateral deviation by player i is punished by giving i a continuation payoff of $\underline{v}_i$ and leaving the continuation payoffs of all other players unchanged. Moreover, again without loss of generality, we can take f^* to have the property that any deviation by two or more players yields "bad" continuation payoffs for the deviating players only as in the right-hand side of (A.10).

Proof. Let any $\tilde{f} \in \mathcal{Z}(v^*)$ be given. We now construct f^* with the desired property as a modification of $\tilde{f}$. The construction is recursive.

On $h^0 = \emptyset$, f^* prescribes the same behavior as $\tilde{f}$. So long as no player deviates from the outcome path prescribed by $\tilde{f}$, the prescriptions of f^* are the same as those of $\tilde{f}$.

Suppose now that some history h^t (on the equilibrium path of $\tilde{f}$) has taken place and that a deviation by player i only has occurred at time t (we ignore deviations by more than one player for the time being). Let σ^i be the mixed action profile played at t which includes i's deviation. Let also $\tilde{\sigma}(h^t)$ be the equilibrium prescription of $\tilde{f}$ after history h^t, and let $v = V(\tilde{f}|h^t, \tilde{\sigma}(h^t))$ be the associated continuation payoff. Then, after i's deviation at t the prescriptions of f^* are the same as those of $f_{P_i(v)}|h^0$ (recall that, according to our notation, $f_{P_i(v)}$ denotes a strategy profile that sustains the vector of long-run payoffs $P_i(v)$, with continuation payoffs that lie entirely in $\mathcal{V}$). Notice that this implies that the continuation payoff vector implied by f^* after (h^t, σ^i) is $P_i(v)$.

So long as the prescriptions of $f_{P_i(v)}|h^0$ are observed after time t, the prescriptions of f^* remain as we have just described. Suppose now that a history $h^m = (h^t, \sigma^i, h^{m-t})$ (with $m > t$ and h^{m-t} on the equilibrium path of $f_{P_i(v)}$) has occurred and that at time m a deviation by player j takes place. Let σ^j be the mixed action profile played at m which includes j's deviation. Let also $\sigma^i(h^{m-t})$ be the equilibrium prescription of $f_{P_i(v)}$ after history h^{m-t}, and let $b^i = V(f_{P_i(v)}|h^m, \sigma^i(h^{m-t}))$ be the associated continuation payoff vector. Then, after j's deviation at m the prescriptions of f^* are the same as those of $f_{P_j(b^i)}|h^0$. Notice that this implies that the continuation payoff vector implied by f^* after (h^m, σ^j) is $P_j(b^i)$.

So long as the prescriptions of $f_{P_j(b^i)}$ are observed after time m, the prescriptions of f^* remain as we have just described. If a further deviation occurs then the players "switch" to a new "phase" in which the deviating player is pushed down to the lowest payoff available for him in $\mathcal{V}$ by playing the appropriate SPE from then on, in a way completely analogous to the one we have just described. Thus the description of f^* can be completed by recursing forward the construction we have given. The rest of the details are omitted.

Clearly, the profile of strategies f^* that we have constructed has the property described in (A.9) by construction. Evidently it is also the case that, by construction, all continuation payoff vectors of f^* lie in $\mathcal{V}$, as required.

We now show that f^* is an SPE strategy profile of the full memory repeated game. This is relatively straightforward to check since only one-shot single-player deviations need ever be considered. To verify that no such profitable deviations are possible, suppose that some history h^s has taken place, and let v_i be i's continuation payoff according to f^* after h^s. Thus, if i at time s adheres to the prescription of f^* he receives a payoff of v_i. Notice that v_i is also i's continuation payoff in the particular SPE that is being played in the "phase" that follows history h^s. If on the other hand he deviates in any way from what f^* prescribes he receives a payoff of $\underline{v}_i$. Since $\underline{v}_i$ is the lowest continuation payoff that i can get in any of the SPE that are used in the construction of f^* above, it is clear that this must be sufficient to deter i from deviating from the prescriptions of f^* after h^s has taken place.

Hence, we have shown that an SPE f^* satisfying (A.9) exists as required. It remains to show that f^* can be made to satisfy (A.10) as well. However, this is completely straightforward once we know that an SPE satisfying (A.9) exists since deviations by two or more players can always be ignored when checking if a given strategy profile is an SPE. The details are omitted for the sake of brevity. □

Proof of Theorem 3. Fix a set $\mathcal{V}$ satisfying the hypothesis of the Theorem. Now fix $v^* \in \mathcal{V}$. We must show that $v^* \in E^C(\delta)$ for any sequential protocol C with $H \subseteq A_i$. Without loss of generality, we consider the sequential protocol with the natural order: player 1 speaks first, player 2 speaks second, and so forth.

Let f^* be an SPE of the full memory game that sustains v^* as vector of long-run payoffs. Using Lemma A.1. we can assume without loss of generality that f^* has the properties described in (A.9) and (A.10), and that all its continuation payoffs lie in $\mathcal{V}$.

We now construct the pair (g^*, μ^*) that sustains the arbitrary payoff vector $v^* \in \mathcal{V}$ as a PBE. Loosely speaking our construction of (g^*, μ^*) runs along the following lines. Only the messages of players 1 and 2 are ever taken into account. Player 1 is asked to report the history of play, then player 2 is asked to "confirm" 1's report. If player 1 reports the truth, then player 2 confirms 1's report and play unfolds according to f^*. If, on the other hand 1 ever issues a false report, then player 2 does not confirm and reports 1's deviation from the truth. In this case the continuation of play unfolds as if player 1 had *behaviorally* deviated from f^*, using the punishments prescribed by f^*. Since for player 1's deviations, f^* punishes 1 in a way that leaves 2 *indifferent*, player 2 always has the correct incentives to report 1's deviation from truthful reporting. Given the punishment for behavioral deviations built into f^*, 1 now also has the correct incentives to always report the true history of play.

Recall that we set $m(-1) = h^0 = \emptyset$. To set the system in motion, let $g^*(h^0) = f^*|h^0$. The rest of the equilibrium is constructed recursively forward in the following way. In every period $t \geq 0$, player 1 reports the truth in the sense that $\mu_1^* = (h^{t-1}, \sigma)$ where h^{t-1} is the history reported by player 2 in period $t-1$ and σ is the true mixed profile that was played by the current cohort.

In every period $t \geq 0$, player 2's message depends on the veracity of the report of player 1 in the following way. If player 1's report is truthful (as defined above) then player 2 issues an identical report (h^{t-1}, σ) where h^{t-1} is the history reported by player 2 in period $t-1$ (h^0 if $t = 0$) and σ is the true mixed profile that was played by the current cohort. If on the other hand player 1's report is not truthful, player 2 issues a report (h^{t-1}, σ') where h^{t-1} is the history reported by player 2 at $t-1$, and σ' is a mixed action profile that is *different* from that reported by player 1, *and* that records a behavioral deviation by player 1 only in period t. In other words, $\sigma_1' \neq f_1^*|h^{t-1}$ and $\sigma_i' = f_i^*|h^{t-1}$ for every $i = 2, \ldots, n$. Notice that these two conditions can clearly always be satisfied simultaneously.

The reports of all players i with $i \geq 3$ (if $n \geq 3$) are ignored. Therefore we simply set them equal to a fixed message m_i regardless of the history of play.

The g^* component of the equilibrium is easy to describe. In period t, if the reports of players 1 and 2 are the *same*, then all players behave according to f^*, conditional on the reported (h^{t-1}, σ). If, on the other hand, the reports of players 1 and 2 differ, then all players behave according to f^* conditional on $\hat{h}^t = (\hat{h}^{t-1}, \hat{\sigma})$ defined as follows. We set $\hat{h}^{t-1}$ equal to the $t-1$ history reported by player 1. Moreover, we set $\hat{\sigma}_1$ – the first component of $\hat{\sigma}$ – equal to the report of player 2, and all other components $(\hat{\sigma}_2, \ldots, \hat{\sigma}_n)$ equal to the report of player 1.

Clearly the proposed equilibrium yields a vector of long-run payoffs v^* as required. Of course, it remains to show that f^* is indeed a PBE of the repeated game with decentralized communication protocol C. We need to verify that no player ever has an incentive to unilaterally deviate in any period, either at the communication stage or at the behavior stage.

All players $i \geq 3$ (if any) clearly have no incentive to deviate in any period. Their messages are ignored, and hence they cannot gain by deviating at the communication stage. At the behavior stage, since f^* is an SPE of the full memory game, and histories are reported truthfully, no individual deviation can be profitable.

Consider now player 1, at the reporting stage after some history h^{t-1} has been reported by player 2 of the previous cohort, and the mixed profile σ has taken place in the current period. If he reports the truth (h^{t-1}, σ) as required, he receives a continuation payoff of corresponding to f^*, conditional on (h^{t-1}, σ). If on the other hand he reports anything else, he receives a payoff of $\underline{v}_i$. Since all continuation payoffs of f^* lie in $\mathcal{V}$, this cannot be a profitable deviation by player 1. Of course, at the behavior stage player 1 has no incentive to deviate simply because f^* is an SPE of the full memory game and histories are reported truthfully.

Lastly, consider player 2 at the reporting stage after some history h^{t-1} has been reported by player 2 of the previous cohort, the mixed profile σ has taken place in the current period, and player 1 has reported some (possibly false) $(\tilde{h}^{t-1}, \tilde{\sigma})$. Clearly, using (A.9) and (A.10) and because of the way we have defined $\hat{\sigma}$ above, the continuation payoff of player 2 is the same regardless of his report. Hence he cannot profitably deviate at this stage. Again, at the behavior stage player 2 has no incentive to deviate simply because f^* is an SPE of the full memory game and histories are reported truthfully. This is clearly enough to conclude the proof. □

Proof of Theorem 4. Consider an PBE (g^*, μ^*) of the dynastic repeated game with sequential decentralized communication protocol C, and assume that (g^*, μ^*) is robust to lexicographic complexity costs of communication.

By Definition 3, of course (g^*, μ^*) must be a PBE of the dynastic repeated game with sequential decentralized communication protocol C. Hence Theorem 2 applies to tell us that if $a' \neq a''$ are two distinct action paths following any (m', σ') and (m'', σ'') respectively, then for some player i we must have that

$$V_i(g^*, \mu^* | \, m', \sigma', a^{i-1'}, a_i') \;=\; V_i(g^*, \mu^* | \, m'', \sigma'', a^{i-1''}, a_i'') \qquad \text{(A.11)}$$

Then, using (A.11) it is clear that we could find a strategy $\mu_i \neq \mu_i^*$ such that $\mu(m', \sigma', a^{i-1'}) = \mu(m'', \sigma'', a^{i-1''})$ and

$$\begin{aligned} & V_i(g^*, \mu^* | \, m', \sigma, a^{i-1'}, \mu(m', \sigma', a^{i-1'})) \\ & = V_i(g^*, \mu^* | \, m'', \sigma'', a^{i-1''}, \mu(m'', \sigma'', a^{i-1''})) \end{aligned} \qquad \text{(A.12)}$$

and $\mathcal{P}(\mu^*) \succ \mathcal{P}(\mu)$. Therefore, using Definitions 2 and 3 we conclude if (g^*, μ^*) is robust to lexicographic complexity costs of communication we must have that

$$\mu_i^*(m', \sigma', a^{i-1'}) = \mu_i^*(m'', \sigma'', a^{i-1''}) \; \forall m', m'', \sigma', \sigma'', a^{i-1'}, a^{i-1''} \qquad \text{(A.13)}$$

Using (A.13) we can now define *the* sequence of message actions that every individual in every cohort (except possibly the first one) will take. Recursively forward from individual 1 we set

$$a_1^* \;=\; \mu_i^*(m, \sigma, \emptyset) \quad \forall \, m, \, \sigma \qquad \text{(A.14)}$$

and (letting $a^{i-1*} = (a_1^*, \ldots, a_{i-1}^*)$, for $i = 2, \ldots, n$)

$$a_i^* \;=\; \mu_i^*(m, \sigma, a^{i-1*}) \quad \forall \, m, \, \sigma \qquad \text{(A.15)}$$

Lastly, we let $m^* = (a_1^*, \ldots, a_n^*)$. This is the message that every cohort will receive in any subgame of the PBE (g^*, μ^*), except of course for the first cohort that will, by assumption, receive a message $m = \emptyset$.

It now follows from (A.14) and (A.15) that the continuation payoff to individual i after any message m has been received from the previous cohort can be written as a function of his choice σ_i as

$$(1 - \delta) U_i(\sigma_i, g_{-i}^*(m)) \; + \delta V_i(g^*, \mu^* | m^*) \qquad \text{(A.16)}$$

Since all cohorts, except for the first one, receive message m^* from the previous cohort, the statement of the theorem now follows immediately from (A.16). The rest of the details are omitted. □

Lemma A. 2. *Let a communication protocol C with unanimous conspiracy to coverup and a common discount factor δ be given. Let also $\mathcal{E}^R(\delta)$ be as in Definition 5. Then $\mathcal{E}^C(\delta) \subseteq \mathcal{E}^R(\delta)$.*

Proof. Fix δ. Suppose, by contradiction, that $v^* \in \mathcal{E}^C(\delta)$ while $v^* \notin \mathcal{E}^R(\delta)$. Since $v^* \notin \mathcal{E}^R(\delta)$ then for all f^* that sustain v^* in the full memory repeated game, there exists some pair of histories, h', h'', such that

$$v' \equiv V(f^*|\, h') >> V(f^*|\, h'') \equiv v'' \tag{A.17}$$

Now let (g^*, μ^*) sustain v^* under protocol C with unanimous conspiracy to coverup. Clearly, since (A.17) must hold for *every* SPE of the full memory game that sustains v^*, we must have that for some pair (m', σ') and (m'', σ''), corresponding to h' and h'' respectively, the following holds

$$v' = V(g^*, \mu^*|\, m', \sigma') \;>>\; V(g^*, \mu^*|\, m'', \sigma'') = v'' \tag{A.18}$$

To derive the contradiction, suppose now that (m'', σ'') has in fact occurred. We proceed to show that v'' cannot be the equilibrium continuation of the communication phase. To verify this, we proceed by induction. Consider the incentives of player n, when all others have reported $a'_i = (m', \sigma'),\ \forall i \neq n$. That is, all others have (falsely) reported prior path (m', σ'). According to the protocol for unanimous conspiracy to coverup, if n also reports $a'_n = (m', \sigma')$ then v' is attained. However, if player n vetoes v' by reporting any other a_n, then the true history (m'', σ'') is revealed, and so continuation v'' occurs. But then (A.18) immediately implies that n's best response is to in fact report a'_n.

Proceeding by induction, using the same argument as for player n, it is now easy to show that every player i's ($i > 1$) best response to all preceding players $j = 1, \ldots, i-1$ having chosen a'_j is in fact to report a'_i. Finally, consider the choice of player 1. Clearly, if he reports a'_1 (given the best responses of all other players) he achieves a payoff of v'_1 while if he reports a''_1 he gets a payoff of v''_1. Hence, using (A.18) again, reporting a''_1 cannot be player 1's equilibrium behavior in the reporting subgame. Moreover, given the equilibrium strategies of the other players in the reporting subgame, it is clear that player 1 (by choosing a'_1) can achieve a continuation payoff of v'_1. Hence, v'' cannot be a continuation equilibrium payoff vector of the reporting subgame, as is in fact required. This contradiction is clearly enough to establish the result. □

Lemma A. 3. *Let a communication protocol C with unanimous conspiracy to coverup and a common discount factor δ be given. Let also $\mathcal{E}^R(\delta)$ be as in Definition 5. Then $\mathcal{E}^C(\delta) \supseteq \mathcal{E}^R(\delta)$.*

Proof. Fix δ and any $v^* \in \mathcal{E}^R(\delta)$. Let f^* be any WRP SPE strategy profile that sustains v^* in the full memory game.

Notice that since f^* is a WRP SPE we know that there is *no* pair of finite histories h' and h'' for which (A.17) holds. In other words, for any pair h' and h''

$$\exists i \text{ such that } V_i(f^*|h') > V_i(f^*|h'') \Rightarrow \exists j \text{ such that } V_j(f^*|h') \leq V_j(f^*|h'') \quad \text{(A.19)}$$

We now specify a pair (g^*, μ^*) that sustains v^* as a PBE payoff in the dynastic game with unanimous conspiracy to coverup. The action strategy profile g^* is the same as f^* in the full memory game. The communication strategy profile is as follows. Player 1 always reports the true history of play. That is μ_1^* prescribes a_1 $= (m, \sigma)$ for every possible (m, σ).

For each $i = 2, \ldots, n$ we let μ_i^* be defined as

$$\mu_i^*(m,\sigma) = \begin{cases} h & \textit{if } \exists h \text{ such that } a_j = h,\ \forall j < i \\ & \qquad \text{and} V_i(g^*, \mu^*|\, m, \sigma) < V_i(g^*, \mu^*|\, h) \\ (m, \sigma) & \text{otherwise} \end{cases} \quad \text{(A.20)}$$

In other words, i reports the true history of play unless all players before him have agreed on a particular report which induces a continuation payoff that he prefers to the continuation payoff following the true history of play.

It is easy to verify that the profile μ^* we have just defined yields an equilibrium path in which all players report the true history of play. Hence (g^*, μ^*) yields payoffs v^* in the dynastic game. Of course, we still need to verify that it constitutes a PBE of the dynastic game.

We need to check that no player has an incentive to deviate from μ_i^* in the communication stage. Consider first players $i = 2, \ldots, n$. If we are in the case in which (A.20) prescribes the report (m, σ), then either two players before i have made conflicting reports, or all players before i have reported a history h that induces a continuation payoff which gives leaves i no better off than the continuation following the true history of play. By sticking to the prescription of μ_i^* player i obtains a continuation payoff of $V_i(g^*, \mu^*|\, m, \sigma)$. Any deviation will yield either a payoff of $V_i(g^*, \mu^*|\, m, \sigma)$ or of $V_i(g^*, \mu^*|h)$ (depending on whether the deviation is to h or to some other history, and on whether all subsequent players agree to h or not). Hence no profitable deviation is possible in this case. If we are in the case described by the top line of (A.20), then any deviation leads to a continuation payoff of $V_i(g^*, \mu^*|\ m, \sigma)$. On the other hand abiding by the prescription of μ_i^* leads to a payoff of $V_i(g^*, \mu^*|h)$ if all subsequent players agree to h, ad to a payoff of $V_i(g^*, \mu^*|\ m, \sigma)$ otherwise. Hence no profitable deviation is possible in this case.

It then remains to check that player 1 cannot profitably deviate from μ_1^*. Of course, this is only possible if he can report a false history that all subsequent players agree upon. However, because (A.19) holds this is impossible. Some subsequent player must prefer (at least weakly) the continuation payoff that follows the true history of play. Hence, according to the bottom line of (A.20) he will report the true history of play. This is clearly sufficient to prove the claim. □

Proof of Theorem 5. The claim is a direct consequence of Lemma A.2 and Lemma A.3. □

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The structure of the Nash equilibrium sets of standard 2-player games

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Summary. In this paper I study a class of two-player games, in which both players' action sets are $[0,1]$ and their payoff functions are continuous in joint actions and quasi-concave in own actions. I show that a no-improper-crossing condition is both necessary and sufficient for a finite subset A of $[0,1] \times [0,1]$ to be the set of Nash equilibria of such a game.

Keywords and Phrases: Nash equilibrium, revealed preferences

JEL Classification Numbers: C65, C72

1 Introduction

In this paper I consider the structure of the Nash equilibrium sets of a class of two-player games. We call a two-player game *standard* if each player chooses independently from the unit interval $[0,1]$ and each player has a payoff function over $[0,1] \times [0,1]$ that is quasi-concave in own action and continuous in joint actions. Although it is well-known that any standard game has at least one Nash equilibrium, the characterization of the Nash equilibrium set of a standard game has never been attempted previously. This is the issue I will address here. More precisely, I will study the problem:

(P) Given an arbitrary finite subset A of $[0,1] \times [0,1]$, what condition must A satisfy so that A is the set of Nash equilibria of some standard game?

My inquiry follows the tradition of the revealed preference theory. While the classical revealed preference theory focuses on implications of individual utility maximization only, several authors have recently established some revealed preference results in the game theory context (Sprumont 2000 and Ray and Zhou 2001).

I will have a more detailed discussion on the relationship between our main result here and the other existing results in the literature at the end of the paper.

2 The main result

Let Γ_c denote the class of all standard two-player games. There are two basic results concerning the Nash equilibria of standard games: First, every standard game has at least one Nash equilibrium; Second, the set of Nash equilibria of a standard game is generically finite.

Suppose the actual choices of the two players in a standard game are observed and they are represented by a finite subset $A \subset [0,1] \times [0,1]$. Without knowing players' payoff functions, we cannot tell if they are playing the Nash equilibria of the game. However, we can ask a weaker question: Are their actions at least consistent with the Nash equilibrium theory? To answer this question, we try to identify a condition on A that is both necessary and sufficient for A to be the Nash equilibrium set of some standard game. Since A represents the observed players' actions, we cannot *a priori* impose any condition on A other than its finiteness. However, for simplicity, we first assume that A contains no duplicated strategies for either player, i.e.,

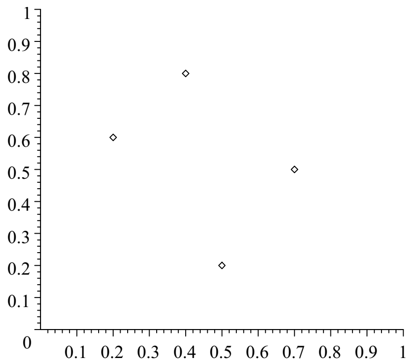
$$A = \{(x_1, y_1), (x_2, y_2), ..., (x_n, y_n)\},$$

in which $x_i \neq x_j$ and $y_i \neq y_j$ for all $i \neq j$.[1]

We say that two pairs of strategy profiles $\{(p,q),(p',q')\}$ and $\{(r,s),(r',s')\}$ cross each other if

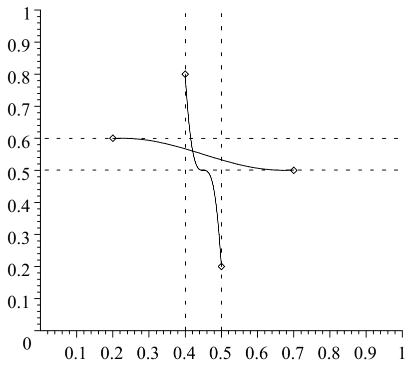
1. $min\{r,r'\} \leq min\{p,p'\} \leq max\{p,p'\} \leq max\{r,r'\}$, and
2. $min\{q,q'\} \leq min\{s,s'\} \leq max\{s,s'\} \leq max\{q,q'\}$.

Consider, for example, four strategy profiles $(.2,.6),(.4,.8),(.5,.2)$, and $(.7,.5)$. If we divide them into two pairs, $\{(.2,.6),(.7,.5)\}$ and $\{(.4,.8),(.5,.2)\}$, then these two pairs cross each other.



[1] This assumption will be removed later.

The key observation is that these four points alone cannot be the set of Nash equilibria for any game in Γ_c. The argument is simple. The set of Nash equilibria of any game is the intersection of the "reaction curves" of players 1 and 2. Suppose these four points were the set of Nash equilibria for some game in Γ_c. Then the reaction curve of player 1 would pass $(.2, .6)$ and $(.7, .5)$, and the reaction curve of player 2 would pass $(.4, .8)$ and $(.5, .2)$. As a result, these two curves would intersect at least one more time inside the square $[.4, .5] \times [.5, .6]$, yielding another Nash equilibrium in addition to the original four strategy profiles. This is a contradiction.



Hence, for a set A to be the set of Nash equilibria of a game in Γ_c, it is crucial that it does not contain strategy profiles in similar positions. To formalize this, we need to introduce two orderings σ and τ on $[0,1] \times [0,1]$: σ orders everything according to player 1's strategies, i.e., $(x, y)\ \sigma(x', y')$ if $x < x'$, and τ orders everything according to player 2's strategies, i.e., $(x, y)\ \tau(x', y')$ if $y < y'$. Now we can state the main condition.

Definition 1. *The no-improper-crossing Condition. A set A satisfies the no-improper-crossing condition if none of the pairs $\{(x_{\sigma(i)}, y_{\sigma(i)}),$ $(x_{\sigma(i+1)}, y_{\sigma(i+1)})\}$ and $\{(x_{\tau(j)}, y_{\tau(j)}),\ (x_{\tau(j+1)}, y_{\tau(j+1)})\}$ cross each other, in which the first pair consist of two strategy profiles that are consecutive according to τ, and the second pair consecutive according to σ.*

It turns out that this condition is not only necessary, but also sufficient for A to be the set of Nash equilibria of some game in $G \in \Gamma_c$.

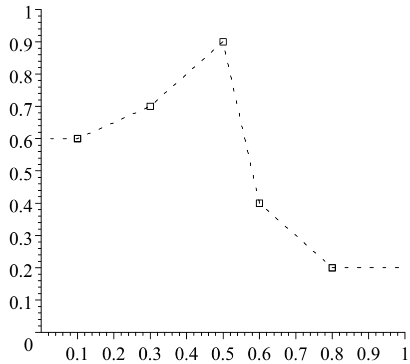
Theorem 1. *A finite subset A of $[0,1] \times [0,1]$ is the set of Nash equilibria of a game $G \in \Gamma_c$ if and if it satisfies the no-improper-crossing condition.*

Proof. First, suppose A is the Nash equilibrium set of some standard game and A does not satisfy the no-improper-crossing condition. According to the Berge theorem: the best response correspondence for any standard payoff function is upper-hemi-continuous and convex-valued. Hence, we can see why the no-improper-crossing condition is necessary. Since player 2's payoff function is standard, the graph of his best response correspondence R_2 must be path connected. Hence, we can find a continuous curve r_2, which is a selection of R_2, that passes from

$(x_{\sigma(i)}, y_{\sigma(i)})$ to $(x_{\sigma(i+1)}, y_{\sigma(i+1)})$. Similarly, we can also find another continuous curve r_1, a selection of R_1, for player 1 that passes from $(x_{\tau(j)}, y_{\tau(j)})$ to $(x_{\tau(j+1)}, y_{\tau(j+1)})$. These two curves have to intersect at some point (x, y) in the middle. This must be a Nash equilibrium of the game, yet it does not belong to A. A contradiction.

On the other hand, we assume that A satisfies the no-improper-crossing condition. We now demonstrate that A is the set of Nash equilibria of some game $G \in \Gamma_c$. First, let us construct two "reaction" curves, one for each player, so that the intersection of these two curves is exactly A. For player 2, we simply construct the reaction curve r_2 by connecting $(x_{\sigma(i)}, y_{\sigma(i)})$ and $(x_{\sigma(i+1)}, y_{\sigma(i+1)})$ for all i. More precisely,

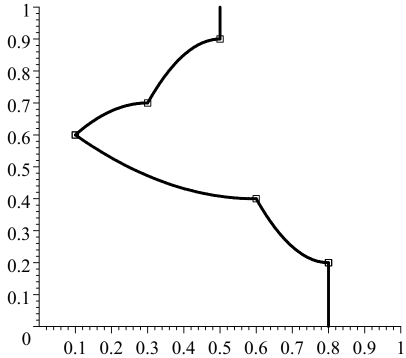
$$r_2(x) = \begin{cases} y_{\sigma(1)}, & 0 \le x \le x_{\sigma(1)} \\ \dfrac{\left(x_{\sigma(i+1)} - x\right) y_{\sigma(i)} + \left(x - x_{\sigma(i)}\right) y_{\sigma(i+1)}}{x_{\sigma(i+1)} - x_{\sigma(i)}}, & x_{\sigma(i)} \le x \le x_{\sigma(i+1)} \\ y_{\sigma(n)}, & x_{\sigma(n)} \le x \le 1 \end{cases}.$$



The construction of r_2 with $A = \{(.1, .6), (.3, .7), (.5, .9), (.6, .4), (.8, .2)\}$

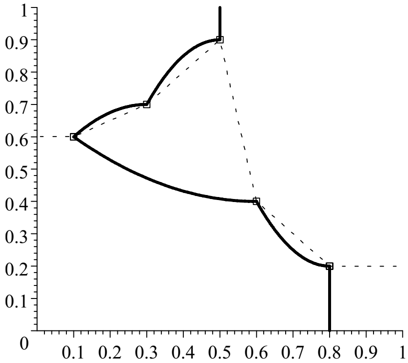
Similarly, we can construct the reaction curve r_1. However, after constructing r_1, we need to curve slightly the line segment between $(x_{\tau(j)}, y_{\tau(j)})$ and $(x_{\tau(j+1)}, y_{\tau(j+1)})$. This modification is made so that r_1 and r_2 will not touch each other in the middle in case they connect two points that are consecutive for both σ and τ.

The intersection of r_1 and r_2 must contain A since both go through all points in A. Suppose that the intersection of r_1 and r_2 also contains a point $x \notin A$. In this case, suppose that x is on the r_1-segment connecting $(x_{\tau(j)}, y_{\tau(j)})$ and $(x_{\tau(j+1)}, y_{\tau(j+1)})$ and the r_2-segment connecting $(x_{\sigma(i)}, y_{\sigma(i)})$ and $(x_{\sigma(i+1)}, y_{\sigma(i+1)})$. By construction, all these four points are different. Since $x_{\sigma(i)}$ and $x_{\sigma(i+1)}$ are consecutive for σ, $x_{\tau(j)}$ and $x_{\tau(j+1)}$ must be at the two different ends outside interval $\overline{x_{\sigma(i)} x_{\sigma(i+1)}}$. Similarly, $y_{\sigma(i)}$ and $y_{\sigma(i+1)}$ must be at



The construction of r_1 with $A =$
$\{(.1, .6), (.3, .7), (.5, .9), (.6, .4), (.8, .2)\}$

the two different ends outside interval $\overline{y_{\tau(j)} y_{\tau(j+1)}}$. But now $(x_{\sigma(i)}, y_{\sigma(i)})$ and $(x_{\sigma(i+1)}, y_{\sigma(i+1)})$, and $(x_{\tau(j)}, y_{\tau(j)})$ and $(x_{\tau(j+1)}, y_{\tau(j+1)})$ cross improperly. A contradiction. Hence, the intersection of r_1 and r_2 is exactly A.



The intersection of r_1 and r_2 is exactly A

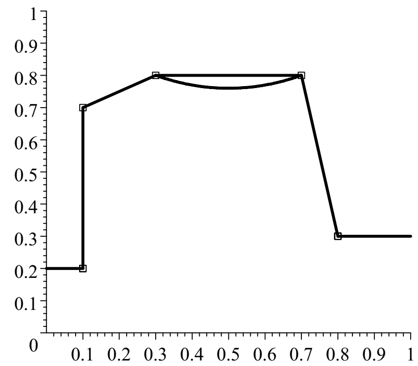
Finally, let us construct utility functions u_1 and u_2 that have r_1 and r_2 as reaction functions respectively. Since both r_1 and r_2 are graphs of single-valued continuous functions, we can let $u_1(x_1, x_2) = -(x_1 - r_1(x_2))^2$ and $u_2(x_1, x_2) = -(x_2 - r_2(x_1))^2$. □

The no-improper-crossing condition may seem cumbersome, but it does encompass several intuitive cases. For example, when a set A is monotonic, either increasing or decreasing, it satisfies the no-improper-crossing condition. (A set $A = \{(x_1, y_1), (x_2, y_2), ..., (x_n, y_n)\}$ is monotonically increasing if $(x_i - y_i)(x_j - y_j) > 0$ for all $i \neq j$, and it is monotonically decreasing if $(x_i - y_i)(x_j - y_j) < 0$ for all $i \neq j$.)

Before closing, let us remove the extra assumption that A contains no duplicated strategies for either player. When A may contain several profiles that assign the same strategy for player 1 or several profiles that assign the same strategy for player 2, we need to define more carefully the orders σ and τ: σ orders A lexicographically according to player 1's strategies first and player 2's strategies second, i.e., (x, y) $\sigma(x', y')$ if $x < x'$, or $y < y'$ when $x = x'$, and τ orders A lexicographically according to player 2's strategies first and player 1's strategies second, i.e., (x, y) $\tau(x', y')$ if $y < y'$, or $x < x'$ when $y = y'$. With this modification, we can restate the no improper cross condition and the main result.

Definition 2. *The no-improper-crossing Condition. A set A satisfies the no-improper-crossing condition if none of the pairs $\{(x, y), (x', y')\}$ and $\{(u, v), (u', v')\}$ cross each other, in which the first pair consist of two strategy profiles that are consecutive according to σ, and the second pair consist of two strategy profiles that are consecutive according to τ.*

Let us go through Theorem 1 again. The proof of the necessity of the no-improper-crossing condition is virtually the same. To prove its sufficiency, we need to construct the best response correspondences more carefully. We first connect $(0, y_{\sigma(1)})$ and $(x_{\sigma(1)}, y_{\sigma(1)})$, $(x_{\sigma(i)}, y_{\sigma(i)})$ and $(x_{\sigma(i+1)}, y_{\sigma(i+1)})$ for all i, $(x_{\sigma(n)}, y_{\sigma(n)})$ and $(1, y_{\sigma(n)})$ by straight lines. If $y_{\sigma(i)} = y_{\sigma(i+1)}$ for any pair $(x_{\sigma(i)}, y_{\sigma(i)})$ and $(x_{\sigma(i+1)}, y_{\sigma(i+1)})$, we will curve the segment slightly so that it touches the original segment only at $(x_{\sigma(i)}, y_{\sigma(i)})$ and $(x_{\sigma(i+1)}, y_{\sigma(i+1)})$. (This is crucial for not creating additional "equilibria" because the original straight segment must be in the graph of R_1.) At the end, we obtain a curve that can be viewed as the graph of a correspondence R_2 that is upper-hemi-continuous and always convex-valued. We take R_2 as the candidate for player 2's best response correspondence.



Similarly we can construct R_1, the candidate best response correspondence for player 1. The resulting R_1 is also upper-hemi-continuous and convex-valued. Given that A satisfies the no-improper-crossing condition, the intersection of the graphs of R_1 and R_2 is exactly A. To complete our proof, we need to identify two payoff functions u_1 and u_2, for players 1 and 2 respectively, that are continuous and quasi-concave in own actions such that R_1 and R_2 are the best response correspondences of u_1 and u_2. This is indeed true due to an inverse of the Berge theorem, which we discuss in a separate section.

3 An inverse of the Berge theorem

In this section I prove an inverse of the Berge theorem, which is needed in the proof of our main result. I thank Nicholas Yannelis for bringing it to my attention that this result has been proved in a paper by Hidetoshi Komiya published in Economic Theory (1997). Here I include a simple alternative proof. The result holds in a rather general setting.

Suppose that X and Y are two convex compact subsets of some finite Euclidean spaces R^d and R^r.

Theorem 2. *(Berge) Suppose that $u(x, y)$ is quasi-concave in x and continuous in (x, y). Let B be the best response correspondence of u, i.e., for all $y \in Y$,*

$$B(y) = \{x \in X | u(x, y) \geqslant u(x', y), \forall x' \in X\} .$$

Then $B(y)$ is a nonempty convex compact set for every y, and the correspondence B is upper-hemi-continuous in y.

Now I prove an inverse of the Berge theorem.

Theorem 3. *Let B be a correspondence from Y to X. Suppose that $B(y)$ is a nonempty convex set for every y, and the correspondence B is upper-hemi-continuous in y. Then there exists a function u on $X \times Y$ that is quasi-concave in x and continuous in (x, y) such that B is the best response correspondence of u, i.e., for every y,*

$$B(y) = \{x \in X | u(x, y) \geqslant u(x', y), \forall x' \in X\} .$$

Proof. The construction is trivial when $B(y)$ is always single-valued. In this case, we can take $u(x, y) = -dist(x, B(y))$. When $B(y)$ is multi-valued, however, this function is not necessarily continuous and we have to modify our construction. First, define a function v by:

$$v(x, y) = -dist\left((x, y), graph(B)\right) .$$

Since B is upper-hemi-continuous, $graph(B)$ is closed. Hence, $(x, y) \in graph(B)$ if and only if $v(x, y) = 0$. Moreover, v is continuous. However, since $graph(B)$ is not a convex set, $v(x, y)$ is not necessarily quasi-concave in x. We now find a function u, which is quasi-concave in x, through a partial convexification of v. For any fixed y,

$$u(x, y) = sup\left\{\alpha | x \in con\{x' | v(x', y) \geqslant \alpha\}\right\} .$$

Note that if $u(x, y) = \alpha$, then there are $d + 1$ numbers $\lambda_j \geqslant 0$ with $\sum \lambda_j = 1$ and $d + 1$ points $x_j \in X$ with $v(x_j, y) \geqslant \alpha$ such that:

$$x = \sum_{j=1}^{j=d+1} \lambda_j x_j .$$

This follows from the continuity of v and an application of the Caratheodory theorem. Hence, for any real number α,

$$\{x|u(x,y) \geqslant \alpha\} = con\{x'|v(x',y) \geqslant \alpha\}.$$

This means $u(x,y)$ is quasi-concave in x. Next, we show that $u(x,y)$ is continuous in (x,y).

Take $(x,y) \in X \times Y$. Suppose that (x^k, y^k) is an arbitrary sequence that converges to (x,y). If we can show that $u(x,y) = \overline{lim}\, u(x^k, y^k)$ for every such sequence, then we must have $u(x,y) = lim\, u(x^k, y^k)$, which means u is continuous.

Let $\alpha = \overline{lim}\, u(x^k, y^k)$. By taking a subsequence if necessary, we may assume that $u(x^k, y^k) = \alpha^k \to \alpha$. For each k, there are $d+1$ numbers $\lambda_j^k \geqslant 0$ with $\sum \lambda_j^k = 1$ and $d+1$ points $x_j^k \in X$ with $v(x_j^k, y^k) \geqslant \alpha^k$ such that:

$$x^k = \sum_{j=1}^{j=d+1} \lambda_j^k x_j^k.$$

Again, we may assume, w.l.o.g., that all λ_j^k and x_j^k converge to some λ_j and x_j. Then, since v is continuous, we have:

$$x = \sum_{j=1}^{j=d+1} \lambda_j x_j,$$

with $v(x_j, y) \geqslant \alpha$. This implies $u(x,y) \geqslant \alpha = \overline{lim}\, u(x^k, y^k)$.

Now suppose $u(x,y) > \alpha = \overline{lim}\, u(x^k, y^k)$. There would exist an $\varepsilon > 0$, and $d+1$ numbers $\lambda_j \geqslant 0$ with $\sum \lambda_j = 1$ and $d+1$ points $x_j \in X$ with $v(x_j, y) > \alpha + \varepsilon$ such that:

$$x = \sum_{j=1}^{j=d+1} \lambda_j x_j.$$

Since X is a convex set, the fact that x is a convex combination of x_j implies that the convex combination of the neighborhoods of x_j's is also a neighborhood of x. Hence, we can find sequences of x_j^k such that $x_j^k \to x_j$ for each j, and

$$x^k = \sum_{j=1}^{j=d+1} \lambda_j x_j^k.$$

Since $(x_j^k, y^k) \to (x_j, y)$, all $v(x_j^k, y^k) \to v(x_j, y) > \alpha + \varepsilon$ for all j. Hence, there exists some large K such that for all j and all $k > K$,

$$v(x_j^k, y^k) > \alpha + \frac{\varepsilon}{2}.$$

Hence, $u(x^k, y^k) > \alpha + \frac{\varepsilon}{2}$ for all $k > K$, which contradicts the assumption that $\alpha = \overline{lim}\, u(x^k, y^k)$. This proves $u(x,y) = \overline{lim}\, u(x^k, y^k)$.

Finally, it is clear that the maximum value of u is zero for any $y \in Y$. Since we have

$$\{x|u(x,y) \geqslant 0\} = con\{x'|v(x',y) \geqslant 0\} = conB(y) = B(y),$$

B is then the best response correspondence for u. □

4 Conclusion

My inquiry in this paper follows the tradition of the revealed preference theory. Samuelson (1947) first developed the theory of revealed preferences in the context of consumer demand theory. Later on other authors, such as for Houthakker (1950), Richter (1966), and Sen (1971), extended Samuelson's work in more general settings. However, the focus of the classical theory is on implications of individual rationality hypothesis. In the general equilibrium framework Sonnenschein (1973) studied the implication of individual utility maximization on the aggregate excess demand function. His work was further improved by Mantel (1974) and Debreu (1974). In 1990s, Brown and Matzkin (1996) initiated the study of testable implications of general equilibrium theory on the equilibrium manifold.

Recently, Sprumont (2000) and Ray and Zhou (2001) established some revealed preference results in the game theory context, which explore explicitly the implications of collective rationality hypothesis embodied in various game theory equilibrium concepts. Their basic model is as follows. Consider a standard game $\{1,2;[0,1] \times [0,1]\}$. The players' preferences are unknown to an outside observer. However, the observer can let the players choose actions from different sets X and Y, where both X and Y are subsets of $[0,1]$, and observe their actions $(x(X \times Y), y(X \times Y))$.[2] The question Sprumont, Ray and Zhou ask is: what condition must these observed actions satisfy so that one can find preferences $\succeq_1$ and $\succeq_2$ on $[0,1] \times [0,1]$ that can rationalize the observed actions $(x(X \times Y), y(X \times Y))$ as Nash equilibria, i.e., $(x(X \times Y), y(X \times Y)) = NE\{1,2; X \times Y; \succeq_1, \succeq_2\}$ for all X's and Y's?

In Sprumont-Ray-Zhou's setting, players' preferences over $[0,1] \times [0,1]$ remain the same while game forms $X \times Y$ vary. This "invariant preference" assumption, however, has been criticized by some scholars. One can argue that players' preferences might depend on the particular game form they face. Think of the above game as the game of dividing-a-dollar. Player 1 makes a demand of x and player 2 makes a demand of y. If $x + y \leq 1$, the outcome is x to player 1 and y to player 2. If $x + y > 1$, both players receive zero dollar. In this case player 2 may feel very differently towards player 1's action $x = .9$ when player 1's choice set is $X = [.9, 1]$ than when 1's choice set is $X = [0,1]$. In the first situation, she might prefer $(x,y) = (0.9, 0.1)$ to $(x,y) = (0.9, 0.2)$; but in the second situation, she might prefer $(0.9, 0.2)$ to $(0.9, 0.1)$. This critique motivates me to study the problem (P) in which the outcomes of one particular game only are rationalized. In this paper I show that the no-improper-crossing condition is both necessary and

[2] This is parallel to the assumption in the (individual) revealed preference theory that the observer can vary the choice sets of the individual and observe his choice in each of these restricted choice problems.

sufficient for a finite subset A of $[0,1] \times [0,1]$ to be the set of Nash equilibria of a standard two-player game. If the observed actions by two players do not satisfy the no-improper-crossing condition, then they cannot be rationalized as the set of Nash equilibria of any standard game.

We should also study the same problem for wider classes of games. Yet it is extremely difficult, if not impossible, to solve this problem in full generality. Of course, given the inverse of the Berge maximum theorem, the study of the set of Nash equilibria of games is equivalent to the study of the intersection of graphs of correspondences that are upper-hemi-continuous and convex-valued. It is still unclear what mathematical tools are most suitable for tackling this problem. While the main result reported here has a limited scope, I hope it will stimulate further research that eventually leads to more general results along this line.

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Nash equilibrium in games with incomplete preferences★

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Summary. This paper investigates Nash equilibrium under the possibility that preferences may be incomplete. I characterize the Nash-equilibrium-set of such a game as the union of the Nash-equilibrium-sets of certain derived games with complete preferences. These games with complete preferences can be derived from the original game by a simple linear procedure, provided that preferences admit a concave vector-representation. These theorems extend some results on finite games by Shapley and Aumann. The applicability of the theoretical results is illustrated with examples from oligopolistic theory, where firms are modelled to aim at maximizing *both* profits and sales (and thus have multiple objectives). Mixed strategy and trembling hand perfect equilibria are also discussed.

Keywords and Phrases: Incomplete preferences, Nash equilibrium, multi-objective programming, Cournot Equilibrium.

JEL Classification Numbers: D11, C72, D43.

1 Introduction

The theory of incomplete preferences is an important subfield of decision theory, which is designed to include in its realm statements such as "I don't know if I prefer alternative a or b" *in addition* to the statements "I prefer a to b," and "I am indifferent between a and b". The fundamentals of this theory have been laid out in the seminal contributions of Aumann (1962) and Bewley (1986), and it has been

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pursued further in the recent literature.[1] The importance of this theory becomes even more apparent when one considers the behavior of economic agents made up of collections of individuals (such as coalitions). However, a very large fraction of the work on incomplete preferences concerns only individual choice problems; only few authors have studied strategic interaction between agents with incomplete preferences. The present paper focuses on precisely this issue, and proposes a way to study strategic interactions in which agents sometimes remain indecisive.

An immediate observation is that, since incomplete preferences can leave many options unranked, equilibrium sets in games that allow for incomplete preferences can be considerably larger than in games with complete preferences. While this may at first seem undesirable, we should note that if one picks some plausible, but incorrect, complete preferences to represent the ranking of an indecisive agent, a good range of equilibria may be overlooked. The relevance of this issue becomes even more apparent when considering games that do not have any equilibria. Maybe some of the players' preferences are, in reality, not as complete as the model would have us believe. Incorporating this indecision to the model might, in fact, enlarge the original (empty) equilibrium set, thereby solving the nonexistence problem.[2]

The potential lack of information on the part of the modeler is another reason for modeling the preferences of some players as incomplete. An outside observer might only be able to establish some baseline about an agent's preferences, such as that the agent would prefer a lottery that stochastically dominates another, or that her preferences are single peaked with a certain bliss point. However, it may be difficult to go beyond such a baseline assumption, and ascertain the precise trade-offs an agent would be willing to make. It is a consequence of one of our main results (Theorem 1) that games with incomplete preferences can be used as a tool for *robust* modelling of such strategic situations. It may be useful to assume that agents effectively possess incomplete preferences, in a way to encompass all plausible formulations of their actual preference profiles. With this conservative modelling strategy one would be able to identify those action profiles that arise as an equilibrium for the actual complete preferences of the agents.

After we develop the basic Nash equilibrium theory with incomplete preferences (Sects. 3 and 4), we illustrate the workings of this theory by means of examples from oligopoly theory. Motivated by the long-standing debate on the modelling of the objectives of the firm, we investigate a scenario in which firms have incomplete preferences: we assume that they might not be able to rank all combinations of profits, revenues, sales and possibly other variables. We show that this model allows us to substantially mitigate a well known nonexistence problem of oligopoly theory, the Edgeworth paradox of capacity-constrained Bertrand competition. Using the theory developed in this paper, we are also able to give an upper bound on the set

[1] Among the recent papers that develop a utility theory for incomplete preferences are Dubra, Maccheroni, and Ok (2004), Mandler (2001), Ok (2002), and Sagi (2003). The choice theoretic foundations of incomplete preferences are, on the other hand, examined in Danan (2003), Eliaz and Ok (2004), and Mandler (2004).

[2] Bade (2003) and Roemer (1999, 2001) all tackle the nonexistence problem of the models of multidimensional political competition between two parties by assuming that parties' preferences are incomplete.

of all reasonable Cournot equilibria when firms care not only about their profits but also about revenues and sales. Finally, we apply the assumption of incomplete preferences to the celebrated Kreps-Scheinkman model of oligopolistic competition, and show that this modified model has a pure strategy equilibrium, whereas the equilibrium of the original model involves complicated off-the-equilibrium path mixing.

In passing, we note that most work on games with incomplete preferences focuses only on the problem of the existence of equilibrium (cf. Ding, 2000; Shafer and Sonnenschein, 1975; Yu and Yuan, 1998; and the references cited therein). By contrast, our objective here is to obtain operational characterizations of Nash equilibrium sets of such games. In this sense, our paper is closer in spirit to that of Shapley (1959), who characterizes the set of all mixed strategy Nash equilibria in vector-valued two-player zero-sum games. This characterization has been extended by Aumann (1962) to a larger class of matrix games. In particular, we show here that the set of Nash equilibria of *any* game with incomplete preferences can be characterized in terms of certain derived games with complete preferences. Provided that all players' preferences can be represented by concave functions, we can sharpen this result further; in this case it suffices for the characterization of the equilibrium set to look at games with complete preferences that are derived from the original game by a simple *linear* procedure. We conclude with a discussion of trembling hand perfect equilibria in games with incomplete preferences.

2 Preliminaries

Throughout this paper $G = \{(A_i, \succsim_i)_{i\in I}\}$ will denote an arbitrary (normal-form) game. Where I is a (finite) set of players, player i's nonempty action space is denoted by A_i and $\succsim_i$ is player i's preference relation on the outcome space $A := \underset{i\in I}{\times} A_i$. Each preference relation $\succsim_i$ is assumed to be transitive and reflexive but, in contrast to the standard theory, need not be complete. Some player i is *indifferent* between a and b, denoted by $a \sim_i b$, if and only if $a \succsim_i b$ and $b \succsim_i a$. Player i *strictly prefers* an outcome a to b, denoted by $a \succ_i b$, if and only if $a \succsim_i b$ but not $b \succsim_i a$.

We say that a preference relation $\succsim'$ on A is a *completion* of another preference relation $\succsim$ on A, if $\succsim'$ is complete, and if $a \succsim b$ implies $a \succsim' b$ and $a \succ b$ implies $a \succ' b$. We say that a game $G' = \left\{(A_i, \succsim_i')_{i\in I}\right\}$ is a *completion* of a game $G = \{(A_i, \succsim_i)_{i\in I}\}$ if $\succsim_i'$ is a completion of $\succsim_i$ for each i. A transitive and reflexive relation $\succsim'$ is called a *transitive closure* of a reflexive relation $\succsim$ if $\succsim'$ is the smallest transitive and reflexive relation such that $a \succsim b$ implies $a \succsim' b$, we write $\succsim' = tc(\succsim)$. It is easy to show that $tc(\succsim) = \bigcup_{i=0}^{\infty} \succsim^i$ where $a \succsim^0 b$ if, and only if, $a \succsim b$ and $a \succsim^i b$ (for $i > 0$) if, and only if, there exist $a_1, a_2, ..., a_i$ such that $a \succsim a^1 \succsim ... a^i \succsim b$.

There is a natural way of extending the standard notion of Nash equilibrium to the present framework. An action profile $a = (a_1, ..., a_{|I|})$ is a Nash equilibrium if and only if no agent has an incentive to deviate from her own action given every one else's action. More formally, the profile a is a *Nash equilibrium* if for no player

i there exists an action $a_i' \in A_i$ such that $(a_i', a_{-i}) \succ_i (a_i, a_{-i})$. If each player's preference relation is complete, this definition reduces to the common definition of the Nash equilibrium. In what follows, we denote the set of all Nash equilibria of a game G by $N(G)$.

3 A general characterization result

In this section we shall characterize the set of all Nash equilibria of a game G with incomplete preferences as the union of all Nash-equilibrium sets of all completions of G. This characterization reduces the problem of identifying Nash equilibria of a game with incomplete preferences to the familiar problem of obtaining Nash equilibria of a collection of games with complete preferences. The proof of the theorem is based on the following Lemma 1 which amounts to a generalization of the classical theorem by Szpilrajn (1930).

Lemma 1. Let $\succsim$ be a preference relation on some nonempty set A, let B be a nonempty subset of A, and let a^* be a maximal point of $\succsim$ in B. Then there exists a completion $\succsim'$ of $\succsim$ such that a^* is a maximal point of $\succsim'$ in B.

Proof. Define the following two partial orders on the quotient set $A/_\sim$: Define $\succsim^q$ by $[a] \succsim^q [b]$ iff $a \succsim b$. Define $\succsim^*$ by $[a] \succsim^* [b]$ iff $[a] = [a^*]$ and there exists a b' in B such that $[b] = [b']$. Define $\succsim^u$ as the union of $\succsim^q$ and $\succsim^*$ and $\succsim^t$ as the transitive closure of $\succsim^u$, $\succsim^t := tc(\succsim^u)$. We need to show that $[a] \succ^q [b]$ implies $[a] \succ^t [b]$. Suppose not, that is, suppose that for some $[a], [b]$ in $A/_\sim$ and some $n \in \mathbb{N}$ we have $[a] \succ^q [b]$ and there exist some distinct $[a^1], [a^2], ..., [a^n] \in A/_\sim$ with $[b] := [a^0] \succ^u [a^1] \succ^u [a^2] \succ^u ... \succ^u [a^n] \succ^u [a] := [a^{n+1}]$, where all inequalities $[a^{i-1}] \succ [a^i]$ are strict, since we have $[a^{i-1}] \neq [a^i]$ for all i. Since $\succsim^q$ is transitive, there must be (exactly) one $1 \leq i \leq n+1$ such that $[a^{i-1}] \succ^* [a^i]$. So $[a^{i-1}] = [a^*]$ and $[a^i] = [b']$ for some b' in B. Let us rearrange the above chain as $[b'] \succ^q [a^{i+1}] \succ^q .... \succ^q [a] \succ^q [b] \succ^q [a^1] \succ^q ..... \succ^q [a^*]$. The transitivity of $\succsim^q$ implies in turn that $[b'] \succ^q [a^*]$, a contradiction. It follows from Szpilrajn's theorem that there exists a completion of $\succsim^t$, call it $\succsim''$. By construction $\succsim''$ is also a completion of $\succsim^q$ and $\succsim^*$which implies that also with respect to $\succsim''$, $[a^*]$ is a maximum in the set of all $[b]$ for $b \in B$. Finally define $\succsim'$ on A by $a \succsim' b$ if and only if $[a] \succsim'' [b]$. It is easily checked that $\succsim'$ fulfills our requirements. □

Observe that Lemma 1 applies to sets A with infinitely many elements. If A were finite we would not need Szpilrajn's theorem for the proof. The following fact is now easy to obtain.

Theorem 1. *Let $G = \{(A_i, \succsim_i)_{i \in I}\}$ be any game. Then*

$$N(G) = \bigcup \{N(G') : G' \textit{is a completion of } G\} .$$

Proof. The "$\supseteq$" part is obvious. To see the "$\subseteq$" part of the claim, let a^* be a Nash equilibrium of G, and define $B_i := \{(a_i, a_{-i}^*) : a_i \in A_i\}$ for all players i. So for any player i, a^* is a maximal point of $\succsim_i$ in B_i. By Lemma 1, there exists a completion $\succsim_i'$ of $\succsim_i$ for each player i such that a^* is maximal point of $\succsim_i'$ in B_i. Consequently a^* is a Nash equilibrium of the completion $G' = \{(A_i, \succsim_i')_{i \in I}\}$. □

Theorem 1 establishes a strong relation between games with complete preferences and games with incomplete preferences. In particular, it allows us to commute back and forth between games with complete and incomplete preferences when calculating equilibrium sets. The problem of equilibria computation in a game with incomplete preferences is thus reduced to a known problem: the computation of equilibria in games with complete preferences.

But at times it can also be useful to model a situation as a game with incomplete preferences, even though we suspect that the preferences of all agents are complete. This case arises when one does not know the preferences of players precisely. We can then specify the preferences of the players as incomplete preorders, consisting only of the preference statements we feel safe to posit. Theorem 1 then says that the equilibria of any completion of a game must lie within the equilibrium set of the game with incomplete preferences. In other words, non-equilibria are robust under improvements in our knowledge about the preferences of the players. So Theorem 1 on the one hand allows us to simplify the solution of games with incomplete preferences, on the other hand it justifies the use of models with incomplete preferences as tools of robust modelling, when the preferences of the players are not known in detail to the modeler.[3]

4 The case of vector-valued utility functions

The preceding characterization theorem relies on the concept of a completion. In general, however, the set of all completions of a game is not easy to determine. To develop a more operational theory we shall now restrict our attention to games in which all players have representable preference relations. Following the recent literature on the representation of incomplete preferences, we shall consider preference relations $\succsim$ that are representable in the sense that there exists a function $u : A \to \mathbb{R}^n$ such that $a \succsim b$ iff $u(a) \geq u(b)$.[4,5] Such vector-valued utility functions are convenient since the problem of maximizing a utility is formally equivalent to the well studied problem of Pareto-optimization. Any n-person Pareto-optimization problem can simply be mapped to a problem of maximizing a preference relation that is representable in the $\mathbb{R}^n$ by identifying the utility-vector representing the incomplete preference relation with the vector of all the n persons' utilities. The utility possibility frontier then corresponds to the set of all maximal points of the incomplete preference relation.

[3] This approach presupposes that only the modeller does not know the preferences of the players. If we assume that the players are equally ignorant about the preferences of the other players the robustness result of Theorem 1 breaks down: Given certain priors about the other players, some player might chose an action in equilibrium, that she would never choose when preferences where common knowledge.

[4] See Ok (2002) for an axiomatic treatment of such a vector-valued utility representation.

[5] Notation: For any $n \in \mathbb{N}$ and $a, b \in \mathbb{R}^n$ $a \geq b$ signifies $a_i \geq b_i$ for all i; $a > b$ signifies $a \geq b$ but not $b \geq a$. Finally $a \gg b$ iff $a_i > b_i$ for all i.

In what follows, by $G = \left\{(A_i, u^i)_{i\in I}\right\}$ we mean the game $G = \{(A_i, \succsim_i)_{i\in I}\}$ where $u^i : A \to \mathbb{R}^{m_i}$ represents $\succsim_i$ in the sense as defined above. For any set of vectors $\beta = \{\beta^1, ..., \beta^{|I|}\}$ with $\beta^i \in \mathbb{R}^{m_i}$ we define the game

$$G_\beta := \left\{(A_i, \beta^i u^i)_{i\in I}\right\}$$

where $\beta^i u^i : A \to \mathbb{R}$ is defined as the dot product of β^i and u^i,that is $\beta^i u^i := \sum_{j=1}^{m_i} \beta^i_j u^i_j$. To simplify our notation we let

$$\Delta := \left\{\{\beta^1, ..., \beta^{|I|}\} : \beta^i \in \Delta^{m_i} \text{ for all } i\right\} \text{ and } \Delta_+ := \Delta \cap \mathbb{R}^{\Sigma m_i}_{++}$$

where Δ^{m_i} denotes the $m_i - 1$ dimensional simplex.

Since in this section we are switching back and forth between the game $G = \left\{(A_i, u^i)_{i\in I}\right\}$ and the derived games G_β, it makes sense to indicate the utility function in the definition of the best response correspondence. So we denote player i's *best response correspondence with respect to his utility function* u^i as BR_{u^i}, that is, $BR_{u^i} : A_{-i} \rightrightarrows A_i$ is defined by

$$BR_{u^i}(a_{-i}) := \arg\max_{b_i \in A_i} u^i(b_i, a_{-i}).$$

For any $\beta^i \gg 0$, the function $\beta^i u^i$ represents a completion of the preferences represented by u^i. So, for any $\beta \in \Delta_+$, the game G_β is a *(linear)* completion of the game G. Therefore, by applying Theorem 1 we know that $N(G_\beta)$ is a subset of $N(G)$ for all $\beta \in \Delta_+$. In some cases we can also use such collections of vectors to describe an upper bound on $N(G)$, or even the full set $N(G)$. If we restrict the utility functions of all players to be concave, and all action spaces to be convex, we can derive such an upper bound. The arguments that are commonly being used to defend concavity (such as decreasing marginal utility) also apply to multidimensional utilities. Alternatively, when considering some coalition whose utility is simply the vector of the utilities of its members, the concavity of the coalition's utility can be a consequence of the concavity of the utility functions of its constituents. Following its proof, we will show that Theorem 2 does not extend to quasiconcave utility functions.

Theorem 2. *Let $G = \left\{(A_i, u^i)_{i\in I}\right\}$ be a game such that each A_i is a nonempty convex subset of some finite dimensional Euclidean space and each u^i is concave in a_i. Then*

$$\bigcup \{N(G_\beta) : \beta \in \Delta_+\} \subseteq N(G) \subseteq \bigcup \{N(G_\beta) : \beta \in \Delta\}.$$

Proof. The first inclusion is clear from the discussion above. To see the second inclusion, pick any $a \in N(G)$, and fix any $i \in I$. Then we have $a_i \in BR_{u^i}(a_{-i}) = \arg\max_{b_i \in A_i} u^i(b_i, a_{-i})$. Define

$$Y(a_{-i}) := \left\{x \in \mathbb{R}^{m_i} : u^i(b_i, a_{-i}) \geq x \text{ for some } b_i\right\}.$$

and

$$X(u^i(a)) := \{x \in \mathbb{R}^{m_i} : u^i(a) < x\}$$

Observe that both of these sets are convex. To see that $Y(a_{-i})$ is convex pick any $x^i, x^{i\prime} \in Y(a_{-i})$ and any $\lambda \in (0,1)$. Then there exist some $b_i, b_i' \in A_i$ with $x^i \leq u_i(b_i, a_{-i})$ and $x^{i\prime} \leq u_i(b_i', a_{-i})$. While the convexity of all A_i implies $\lambda b_i + (1-\lambda)b_i' \in A_i$, the concavity of all u^i implies $u^i(\lambda b_i + (1-\lambda)b_i', a_{-i}) \geq \lambda x^i + (1-\lambda)x^{i\prime}$. Observe furthermore that by the maximality of $u^i(a)$ in $Y(a_{-i})$, we have $X(u^i(a)) \cap Y(a_{-i}) = \emptyset$. So by Minkowski's separating hyperplane theorem, there exists some vector $p^i \in \mathbb{R}^{m_i}$ and some constant c such that $p^i x \leq c$ for all $x \in Y(a_{-i})$ and $p^i x \geq c$ for all $x \in X(u^i(a))$. Since $x < u^i(a)$ implies $x \in Y(a_{-i})$ and $x > u^i(a)$ implies $x \in X(u^i(a))$, we have $p^i \geq 0$ and $p^i u^i(a) = c$. So there exists some $\beta^i \in \Delta^{m_i}$ such that $\beta^i u^i(a) \geq \beta^i x$ for all $x \in Y(a_{-i})$. Since $u^i(A_i, a_{-i})$ a subset of $Y(a_{-i})$, we also have that $\beta^i u^i(a) \geq \beta^i u^i(b_i, a_{-i})$ which implies that $a_i \in BR_{\beta^i u^i}(a_{-i})$. Since $i \in I$ is arbitrary, this yields a β in Δ such that $a \in N(G_\beta)$. □

The proof of Theorem 2 does not extend to quasiconcave functions, as the sum of two quasiconcave functions is not necessarily itself quasiconcave. Take the following trivial one-player game with a convex action space $A = \{(x,y) \in [0,1]^2 : x + y = 1\}$ and a quasiconcave utility defined by $u(x,y) = (x^2, y^2)$. While $(\frac{1}{2}, \frac{1}{2})$ is an equilibrium in this game, there does not exist any $\beta \in \Delta$ such that $(\frac{1}{2}, \frac{1}{2}) \in \arg\max_{x,y \in A} \beta u$.

Theorem 2 gives an upper and a lower bound on the set of all Nash equilibria. In applications we would not expect that there would be many elements in $\bigcup \{N(G_\beta) : \beta \in \Delta\}$ that are not contained in $\bigcup \{N(G_\beta) : \beta \in \Delta_+\}$, that is, the bulk of $N(G)$ is likely to be contained in $\bigcup \{N(G_\beta) : \beta \in \Delta_+\}$. In particular, if we assume componentwise strict concavity - as would be for example reasonable when we investigate players that are made up of individuals that each have strictly concave utility functions - the characterization at hand provides one with a full description of the set of all Nash equilibria of a game. This claim is proved next.

Lemma 2. *Let $G = \{(A_i, u^i)_{i \in I}\}$ be a game such that, for some i, A_i is a convex subset of some finite dimensional Euclidean space and u^i is concave in a_i. For some component $j \in \{1, ..., m_i\}$ let $u^i_j : A \to R$ be strictly concave in a_i. Then for any $\beta^i \in \Delta$ such that $\beta^i_j > 0$ and any $a_{-i} \in A_{-i}$ we have $BR_{\beta^i u^i}(a_{-i}) \subseteq BR_{u^i}(a_{-i},)$.*

Proof. Fix any a_{-i} and any $\beta^i \in \Delta$ such that $\beta^i_j > 0$. Assume that there exist an a_i' and some $\beta^i \geq 0$ with $\beta^i_j > 0$ such that $a_i' \in BR_{\beta^i u^i}(a_{-i})$ but $a_i' \notin BR_{u^i}(a_{-i})$. This implies that there exists an $a_i'' \in A_i$ such that $u_i(a_i'', a_{-i}) > u_i(a_i', a_{-i})$. It follows that $\beta^i u_i(a_i'', a_{-i}) \geq \beta^i u_i(a_i', a_{-i})$, and since $a_i' \in BR_{\beta^i u^i}(a_{-i}, A_i)$, we have $\beta^i u_i(a_i'', a_{-i}) = \beta^i u_i(a_i', a_{-i})$. The strict concavity of $u^i_j(., a_{-i})$ and positivity of β^i_j together with the concavity of $u^i(., a_{-i})$ imply that $\beta^i u^i(., a_{-i})$ is a strictly concave function. Since a strictly concave function is maximized at a unique point, we conclude that $a_i'' = a_i'$ contradicting our assumption that $u^i(a_i'', a_{-i}) > u^i(a_i', a_{-i})$. □

Theorem 3. *Let $G = \{(A_i, u^i)_{i \in I}\}$ be a game such that each A_i is a nonempty convex subset of a finite dimensional Euclidean space and every component of each u^i is strictly concave in a_i. Then $N(G) = \bigcup \{N(G_\beta) : \beta \in \Delta\}$.*

Proof. Since every component of each player's utility functions is strictly concave in her own action, we know that

$$\varDelta=\{\beta\in\varDelta\text{: for all } i, \beta_k^i>0 \text{ for at least one } k \text{ for which } u_k^i \text{ is strictly concave}\}.$$

So combining Lemma 2 and Theorem 2, we have

$$\bigcup\left\{BR_{\beta^i u^i}(a_{-i}) : \beta^i \in \varDelta\right\} \subseteq BR_{u^i}(a_{-i}) \subseteq \bigcup\left\{BR_{\beta^i u^i}(a_{-i}) : \beta^i \in \varDelta\right\}.$$

It follows that $N(G) = \bigcup\{N(G_\beta) : \beta \in \varDelta\}$. □

5 Applications to oligopoly theory

We now illustrate the theory developed so far by studying how to incorporate multiple objectives in some standard models of oligopolistic competition. The following alternatives to profits have been suggested as objectives for the oligopolistic firm. Firms might concentrate on maximizing revenues or sales, possibly as imperfect proxies for long run profits. Due to the difficulty in evaluating managerial efforts, the executives of a firm might be judged according to the relative performance of the firm, and this might compel managers to focus on the market share in terms of profits, sales and revenues which suggests another set of possible objectives of the firm.[6]

At the very least, it seems worthwhile to explore the implications of the hypothesis that objectives other than profits play a role in the firms decision making, when profits are above a certain threshold (e.g. nonnegative). We model the preferences of firms such that they depend only on profits and sales[7]. More precisely, we investigate the following preference structure on the part of the firms: When making profits, a firm prefers a situation a to a situation b if in a it has at least as much profit and sales as in b. If, however, situation a is better according to either one of the criteria, while b is better according to the other criterion, then the firm is undecided between these two options. If the firm is making losses in situation a or b it prefers the one with the lower losses (or equivalently higher profits) no matter how these two situations compare according to the sales of the firm. For ease of presentation, we focus in what follows on duopolies. We assume that both firms produce a homogenous good at constant marginal cost $c > 0$, and that at a price p the market demand is $1 + c - p$ as long as this expression is positive, otherwise market demand is 0. By convention, i and j denote two *different* firms in what follows.

[6] See, for instance, Baumol (1959), Fershtman and Judd (1987), Galbraith (1967), Holmstrom (1982), Marris (1964), Simon (1964), and Sklivas (1987) for arguments in favor of modelling firms as pursuing objectives that deviate from profit maximization.

[7] Revenues and market shares can w.l.o.g. be dropped from consideration as they are monotone transformations of profits and sales (at least as long as profits are positive).

5.1 Cournot competition

Consider the Cournot model in which firms choose their production levels. Here $A_i = \mathbb{R}_+$, and the utility function of firm i is defined on $\mathbb{R}^2_+$ by

$$u^i(q) := (\pi^i(q), v^i(q)),$$

where π^i denotes the common profit function, and $v^i(q) := q_i$ if $q_i \leq 1 - q_j$ and $v^i(q) := 1 - q_j$ otherwise. Here $v^i(q)$ represents the sales of firm i as long as profits are nonnegative. If firm i incurs losses at the output profile q, then $v^i(q)$ takes a constant value; the particular value of this constant, $1 - q_j$, is chosen to obtain a utility function that is concave and continuous in the firm's own action q_i. We denote the resulting game $\{(A_i, u^i)_{i=1,2}\}$ by G^C.[8]

We now compute $N(G^C)$. Since each firm's objective function $u^i(q)$ is concave in the firm's own quantity, and since the first component $\pi^i(q)$ is even strictly concave Theorem 2 and Lemma 2 readily yield

$$\bigcup \left\{N(G^C_\beta) : \beta \in \Delta, \beta^i_1 > 0\right\} \subseteq N(G^C) \subseteq \bigcup \left\{N(G^C_\beta) : \beta \in \Delta\right\}. \quad (*)$$

In the linearly completed game G^C_β, firm i's objective function is $\beta^i u^i$, where

$$\beta^i u^i(q) = \begin{cases} \beta^i_1 \pi^i(q) + \beta^i_2 q_i, & \text{if } q_i \leq 1 - q_j \\ \beta^i_1 \pi^i(q) - \beta^i_2 (1 - q_j), & \text{if } q_i > 1 - q_j \end{cases}.$$

So, in the completed game G^C_β, firm i's best response to q_j is: $\frac{1-q_j}{2} + \frac{\beta^i_2}{2\beta^i_1}$ if this expression is in the interval $[0, 1 - q_j]$, if this expression is smaller than any value in this interval then the best response is not to sell anything, otherwise firm i's best response is $1 - q_j$. This implies that

$$\begin{aligned} \bigcup \left\{N(G^C_\beta) : \beta{\in}\Delta, \beta^i_1{>}0\right\} &= \left\{q{\in}\mathbb{R}^2_+ : \frac{1}{2}(1{-}q_j) \leq q_i {\leq} 1{-}q_j \text{ for } i = 1,2\right\} \\ &= \bigcup \left\{N(G^C_\beta) : \beta \in \Delta\right\}. \end{aligned}$$

Combining this with (*), we conclude that

$$N(G^C) = \left\{q \in \mathbb{R}^2_+ : \frac{1}{2}(1 - q_j) \leq q_i \leq 1 - q_j \text{ for } i = 1,2\right\}.$$

Figure 1 illustrates this analysis: the lines AB and CD represent the reaction curves when firms maximize profits. The line AD represents the reaction curves of the two firms when they maximize sales subject to nonnegativity of profits. All points in the triangle ADE represent Nash equilibria of G^C. For any point q in this triangle, there exists a β such that $q \in N(G^C_\beta)$. The line AGH represents reaction

[8] A range of alternative formulations of this game $\{(A_i, v^i)_{i \in I}\}$, with v^i being a monotone transformation of u^i for $i = 1, 2$, yield the same set of Nash equilibria. The present formulation in convenient, for it allows Theorem 2 and Lemma 2 to be applied. Finally, observe that profits play an important role in the preferences of the firm, as they enter u^i both directly and also indirectly (via sales).

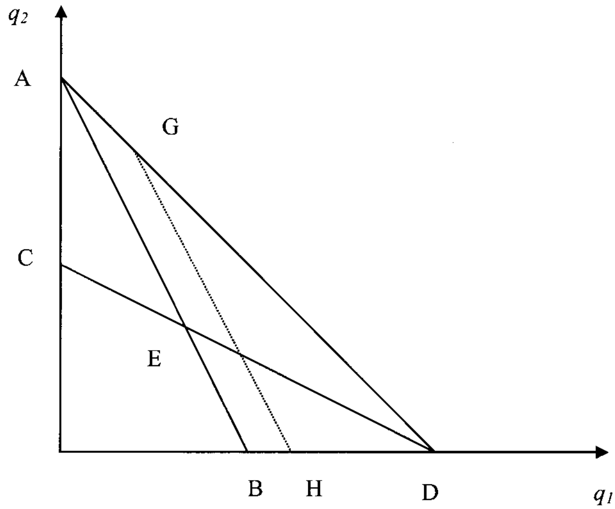

Figure 1

curve of firm 1 for its completed utility $\beta^1 u^1$. For small q_2, firm 1 best-responds with $\frac{1-q_2}{2} + \frac{\beta_2^i}{2\beta_1^i}$; observe that this line (the portion GH of the line AGH) has the same slope as the firm's reaction curve according to the traditional model (the line AB). For large q_2, the response with a quantity $q_1 = \frac{1-q_2}{2} + \frac{\beta_2^i}{2\beta_1^i}$ would yield negative profits and firm 1 is best off by setting its quantity to $1 - q_2$; this explains the portion AG of its reaction curve. Observe that if more weight is placed on profits in the completion, the line AGH would shift closer to the line AB.[9]

If we assume that the objective function of a firm should be some positive combination of their revenues, sales, and profits (as long as the firm does not make any losses, and if it does, it only tries try to reduce them), we know by Theorem 1 that the equilibrium outcomes of the Cournot model must lie within this triangle. This application of Theorem 1 helps us to exclude a wide range of action profiles that cannot be equilibria for any reasonable complete formulation of the game.

5.2 Bertrand competition

To show that Nash equilibrium sets in games with incomplete preferences need not be large (as opposed to what the previous example might suggest), we now study the above multi-objective model within Bertrand competition. We again assume that firms have incomplete preferences as long as profits are nonnegative, preferring higher sales and profits. The unique Nash equilibrium in this case accords perfectly with the standard case: the firms set their prices equal to marginal cost. This is not

[9] Notice that in this case $N(G^C)$ coincides with the convex hull of the Nash equilibria in the "extreme" games $G^C_{\pi^1,\pi^2}, G^C_{\pi^1,v^2}, G^C_{v^1,\pi^2}$ and $G^C_{v^1,v^2}$ (where $G^C_{f^1,f^2}$ denotes the game in which firm i's utility is the function f^i). This is a peculiar consequence of the fact that best response correspondences in the "extreme" games are linear. The observation does not generalize.

surprising since no sales or revenue motive could give a firm an incentive to deviate from the unique equilibrium of the price competition game played amongst profit maximizing duopolists. The uniqueness of this equilibrium is also not unexpected since in all relevant cases profits and sales move in the same direction as the own price of a firm changes. This can be interpreted as a confirmation of the robustness of the Bertrand equilibrium: By Theorem 1, any completion of the present Bertrand model either has no equilibrium or its only equilibrium is the classical one. To deviate from the classical equilibrium firms must have preferences that cannot be represented as increasing combinations of their profits, revenues and/or sales.

There is, however, more to this story. Edgeworth (1925) showed in his famous critique of Bertrand-competition that games played amongst capacity constrained price setters can fail to have any Nash equilibria. In the literature on oligopolistic competition, this observation is called the *Edgeworth paradox.*[10] Interestingly, this paradox does not arise in the present multi-objective Bertrand-model. To demonstrate, let both firms face some identical capacity constraint $K \leq \frac{1}{2}$. As usual, we assume that the firm with the lower price serves the customers that are willing to pay most for the good, the firm with the higher price serves the rest if there is any. More precisely, given prices p_i and p_j, the demand is shared in the following way. If $p_i < p_j$, then firm i faces the full market demand, that is, $D^i(p_i, p_j) = 1+c-p_i$, where $D^i(p_i, p_j)$ denotes the demand for firm i given the prices p_i and p_j. If $p_i > p_j$ and $K \geq 1+c-p_j$, then no residual demand remains for firm i: $D^i(p_i, p_j) = 0$. If, on the other hand, $1+c-p_j > K$ (and still $p_i > p_j$) firm i faces a residual demand of $D^i(p_i, p_j) = 1 + c - p_i - K$. If both firms set an equal price $p_i = p_j = p$ they share the market-demand equally: $D^i(p_i, p_j) = \frac{1+c-p}{2}$. The duopolists play the game $G^B = \left\{(A_i, u^i)_{i=1,2}\right\}$, where $A_i = \mathbb{R}_+$ is the price space, and

$$u^i := \left(\min\left\{D^i(p_i, p_j), K\right\}(p_i - c), v^i(p_i, p_j)\right)$$

with $v^i(p_i, p_j) := \min\left\{D^i(p_i, p_j), K\right\}$ as long as $p_i \geq c$ and $v^i(p_i, p_j) := -1$ otherwise. Observe that these preferences over price profiles are not convex, so we cannot apply Theorem 2.

The main result of this section is that there is a unique equilibrium in this Bertrand-game:

$$N(G^B) = \{(1 - 2K + c, 1 - 2K + c)\}\,. \qquad (**)$$

Let us first show that in any Nash equilibrium we have $p_i = p_j$. Suppose $p_i < p_j$ in equilibrium. If $1 - p_i + c \leq K$, then $D^j(p_i p_j) = 0$, so by dropping its price to p_i firm j could get positive profits and sales. If, on the other hand, $1 - p_i + c > K$, then firm i sells K units of the good on the market, whereas it could also sell K at any higher price p_i' for which $p_i' < p_j$ and $1 - p_i' + c > K$. Charging such a price p_i' firm i could increase its profits while keeping its quantity sold constant. So, in any equilibrium, both firms must charge the same price, say p.

[10] Maskin (1986) shows that for a very large range of such capacity constrained Betrand games mixed strategy equilibria exist. However, such mixed strategy equilibria are generally very difficult to calculate. Moreover, the supports of the firms' equilibrium strategies tend to be extremly large (see, for example, Osborne and Pitchik, 1986).

If $p < 1 - 2K + c$, then either firm could increase its profits, while keeping its quantity sold constant, by increasing its price by a small amount. If $p > 1-2K+c$, then there exists some $p_i' < p$ at which firm i sells a higher quantity yielding higher profits, given that firm j continues to play p. Thus, *if an equilibrium exists*, it must equal $(1 - 2K + c, 1 - 2K + c)$.

What is more, any deviation of a firm either lowers this firm's profits or its quantity sold. The essential difference between the classical model and the model advanced here is this last step. Certain deviations from $(1-2K+c, 1-2K+c)$ may raise profits, but only at the expense of sales. Under the classical profit maximization hypothesis these deviations are of course beneficial, and lead to the non-existence of equilibrium. Since sales decrease as a consequence, these deviations are, however, not beneficial in our multi-objective model: (**) holds. One can similarly show that the equilibrium of the model is (c, c) when $K > \frac{1}{2}$ Thus if we denote the present Bertrand model with capacity constraint K by $G^B(K)$, then we have $N(G^B(K)) = (p_K^*, p_K^*)$ where $p_K^* := \max\{1 - 2K + c, c\}, K \geq 0$.[11]

It is important to note that this result is robust in the sense that it remains valid even when a certain level of trade-off between profits and sales is allowed. To see this, instead of assuming that firms cannot rank any two price profiles, one with higher sales and the other with higher (positive) profits, let us change the firms utilities to

$$u_\kappa^i := (\pi^i, \kappa\pi^i + v^i) \qquad i = 1, 2,$$

where $\kappa > 0$. According to this alternative model, a change that considerably increases profits, while decreasing sales only to a small extent, is preferred by the firm if κ is big enough.

For instance, take the profit and sales profile (π^*, v^*) in Figure 2. In our initial formulation of the preferences, a firm prefers only those profiles (π, v),for which we have $(\pi, v) \geq (\pi^*, v^*)$ holds. According to our new formulation, the objective profile of firm i is $u_\kappa^i := (\pi^i, \kappa\pi^i + v^i)$ that is firm i strictly prefers any (π, v) in Figure 2 that lies left of the line π^* and above the line AB (the line through (π^*, v^*) with slope $-\kappa$). Observe that the cone of options that cannot be ranked according to $u_\kappa^i(p)$ (the striped areas in Figure 2) decreases with κ, in the limit as $\kappa \to \infty$ the firms preferences converge to the standard scenario. Let us simplify the analysis by assuming $c = 0$ (all the results reported below hold for any $c > 0$ as well), and let us find an upper bound on κ such that a firm with the utility function $u_\kappa^i(p)$ has no incentive to deviate from the equilibrium of $G^B(K)$. We restrict our attention to all linear completions of $u_\kappa^i(p)$, that is, all functions $\gamma\pi^i(p) + v^i(p)$ with $\gamma \in (\kappa, \infty)$. We distinguish three cases in which K belongs to $[0, \frac{1}{3}]$, $(\frac{1}{3}, \frac{1}{2}]$ or $(\frac{1}{2}, \infty)$.

If $K \leq \frac{1}{3}$, any deviation from the equilibrium price $p_K^* := \max\{1 - 2K + c, c\}$ lowers profits and sales at the same time. If $\frac{1}{3} < K \leq \frac{1}{2}$, then there does not exist a preferred deviation from p_K^*, if for all $p > 1 - 2K$, we have $\gamma(1-2K)K + 1 - 2K \geq \gamma(1 - K - p)p + 1 - K - p$. The latter inequality holds

[11] The argument establishing the unique equilibrium for multi-objective competition easily extends to the case of different capacities $K_1, K_2 > 0$, any strictly decreasing continuous demand $D : \mathbb{R}_+ \to \mathbb{R}_+$, and different sharing rules. The unique equilibrium is (p^*, p^*) with $p^* = \max\{c, D^{-1}(K_1 + K_2)\}$.

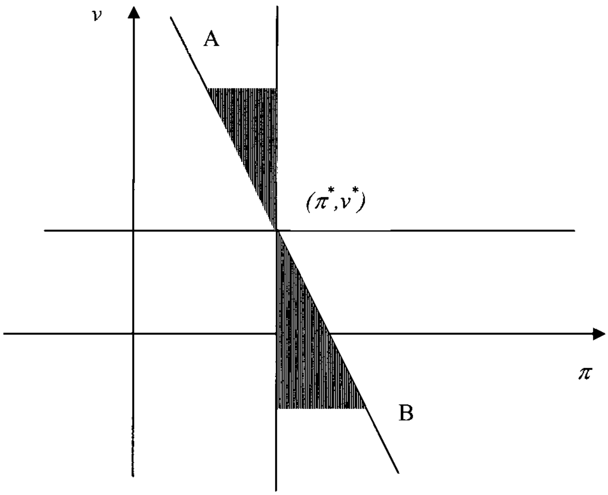

Figure 2

only if $\frac{1}{3K-1} \geq \gamma$. So if $2 \geq \gamma$, then there does not exist any profitable deviation from (p_K^*, p_K^*) for any $K \leq \frac{1}{2}$. Let us now show that if we impose $2 \geq \gamma$ there also does not exist any preferred deviation from $(p_K^*, p_K^*) = (0, 0)$ for $K > \frac{1}{2}$. Given that the other firm charges $p_j = 0$ the set of feasible profit-sales for firm i is smaller when $K > \frac{1}{2}$ than it is when $K = \frac{1}{2}$. We just showed that, given $\gamma \leq 2$, the profit sales profile $(0, \frac{1}{2})$ is maximal in the larger set when $\gamma \leq 2$. But $(0, \frac{1}{2})$ is also contained (and therefore maximal) in the smaller set, so $p_i = 0$ is a best response to $p_j = 0$, when $K > \frac{1}{2}$. We conclude that if $\gamma \in (0, 2]$, (p_K^*, p_K^*) remains an equilibrium of $G^B(K)$ for *any* $K \geq 0$. In other words (p_K^*, p_K^*) is the unique equilibrium of the present multi-objective Bertrand game with capacity constraint $K \geq 0$ as long as both firms assign a "weight" lower than $\frac{2}{3}$ to profits.[12]

5.3 The Kreps-Scheinkman model

There exists a marked tension between the models of Cournot and Bertrand competition. While the mechanism assumed in the Bertrand setup (i.e. price competition), has more empirical support, the prediction of the Cournot model (i.e. oligopolistic rents exist), seems more realistic. In a famous article, Kreps and Scheinkman (1983) reconcile these conflicting intuitions in a sequential setup, where they provide a model in which firms reap oligopolistic rents even though they compete in prices. In this section we introduce the incomplete preferences discussed in the previous two subsections into the Kreps-Scheinkman model. We show that the Kreps-Scheinkman result is robust to this modification. There is, however, a considerable

[12] If we allow the capacites of both firms to differ, then we have

$$N(G^B(K_1, K_2)) = \{(\max\{1 - K_1 - K_2 + c, c\}, \max\{1 - K_1 - K_2 + c, c\})\}$$

is a Nash equilibrium provided that the weight on profits is less than one half (that is $\kappa < 1$).

advantage of the present approach: While some complicated off-the-equilibrium path mixing is necessary to solve the original game of Kreps and Scheinkman (1983) the incomplete preferences version of the model has straightforward pure strategy equilibria.

In the first stage of the game the two competing firms simultaneously build their capacities, incurring a cost c for each unit. Subsequently, each firm can produce a homogenous good up to its capacity level at zero marginal cost. Having observed the capacities both firms compete in the market by setting prices. The market demand is $1 + c - p$ as long as this expression is positive and 0 otherwise. The firms share the demand as explained above in the discussion of Bertrand competition. The only difference between this model and that of Kreps and Scheinkman (1983) is that the firms are not only motivated by profits but also by sales and revenues; they have the same type of incomplete preferences as discussed above. We denote the resulting game as G^{KS}, and a subgame that obtains after firm i chooses capacity K_i, by $G^{KS}(K_1, K_2)$. A strategy of firm $i = 1, 2$ in this game consists of a capacity $K_i \geq 0$ and a function $f_i : \mathbb{R}^2_+ \to \mathbb{R}_+$ that specifies a price for every $G^{KS}(K_1, K_2)$. As is standard, we call a strategy profile a *subgame perfect* Nash equilibrium, if it induces a Nash equilibrium in every subgame.

Proposition 1. *The set of all subgame perfect Nash equilibria of G^{KS} consists of all (K_1, f_1, K_2, f_2) such that*

$$\begin{aligned}\frac{1}{2}(1 - K_j) \leq K_i \leq 1 - K_j \quad \textit{and} \quad & f_i(K_i, K_j) \\ &= \max\{1 - K_i - K_j + c, c\},\ i, j = 1, 2,\ j \neq i.\end{aligned}$$

Before proving this proposition we need to remark that in games with incomplete preferences there can be action profiles that survive backward induction even though they are not subgame perfect Nash equilibria. For instance, consider the game presented in Figure 3.

In this game player 1 has two objectives (and hence incomplete preferences) while player 2 has complete preferences. Observe that the strategy profile raR survives backward induction, but that raR is not a Nash equilibrium, for player 1 is better off playing lL. Since the focus of the present paper is on static games, we will not pursue this matter further here. Suffice it to say that Proposition 1 cannot be proved by backward induction; we need to check directly if the strategy profiles in Proposition 1 induce Nash equilibria in every subgame.[13]

Proof of Proposition 1. Let us first show that indeed all action profiles named in Proposition 1 are subgame perfect Nash equilibria. Pick any (K_1, f_1, K_2, f_2) that satisfies the conditions given in Proposition 1. We need to show that by deviating from its choice of K_1, f_1 firm 1 can neither raise sales nor profits without lowering the other for the given K_2, f_2. Suppose an alternative strategy K', f' that raises sales or profits while keeping the other at least constant existed. Since

[13] The game in discussion actually belongs to a certain class of games with incomplete preferences in which backward induction can be used to solve for subgame perfect Nash equilibria, to define this class and give the proof would however go beyond the scope of this paper. Questions like this one will be dealt with in a companion paper on dynamic games.

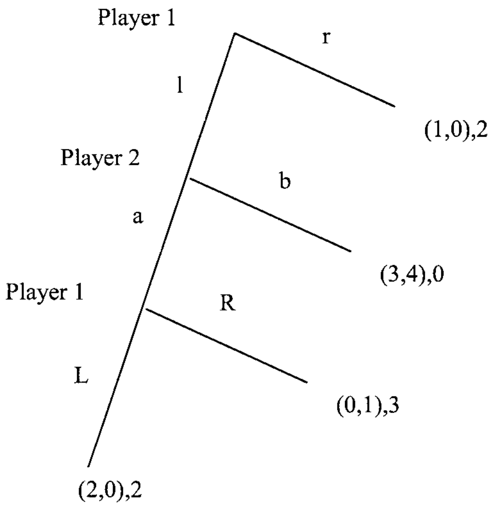

Figure 3

$v^1(K_1, f_1, K_2, f_2) = K_1$[14] we need to have $K' \geq K_1$. Keeping firm 2's strategy fixed this again implies that $f_2(K', K_2) = \max(c, 1 - K' - K_2 + c) \leq 1 - K_1 - K_2 + c = f_2(K_1, K_2)$ or in words firm 2 will not raise its price given firm 1's altered strategy. We proceed by distinguishing three cases: The price firm 1 would charge in subgame $G^{KS}(K', K_2) : f'(K', K_2)$ might be higher equal or lower than $f^1(K_1, K_2)$, the price firm 1 charges according to the given strategy profile (K_1, f_1, K_2, f_2).

Case 1. $f'(K', K_2) > f^1(K_1, K_2)$. Since $f^1(K_1, K_2) = f^2(K_1, K_2) \geq f_2(K', K_2)$ firm 1 charges a higher price than firm 2 and only faces the residual demand $1 - K_2 - f'(K^{'}, K_2) < K_1$. So under this deviation sales decrease and we can rule out such a deviation. *Case 2*: If $f'(K', K_2) = f^1(K_1, K_2)$ the firm spends more on building capacity while selling the same quantity, so we can rule out this case. *Case 3*: Observe that if $f'(K', K_2) < f^1(K_1, K_2)$ for all deviations f', K' there exists a deviation f'', K'' that is at least as good for firm 1 and has $f''(K'', K_2) = 1 - K'' - K_2 + c$ (if this equation does not already hold for f', K' price can either be raised or capacity reduced to increase profits while keeping the quantity sold constant). But this means we need to find some $K'' \leq 1 - K_2$ such that $(K'', (1 - K'' - K_2)K'') > (K_1, (1 - K_1 - K_2)K_1)$. In our discussion of Cournot equilibria we saw that no such deviation exists. So we conclude that indeed all strategy profiles named in Proposition 1 are Nash equilibria. We can draw the argument that any of these strategy profiles induces a Nash equilibrium in any proper subgame $G^{KS}(K_1, K_2)$ from our discussion of Bertrand competition.[15] So we conclude that indeed all the strategy profiles defined in Proposition 1 subgame perfect equilibria.

[14] Observe the slight change in notation: the functions v^i and π^i now map vectors of two real variables (K_1, K_2) and two functions f_1, f_2 to the reals.

[15] This was actually only shown for $K_1 = K_2 \in [0, 1]$; the generalization needed here is easy.

We only need to check that we did not overlook any subgame perfect Nash equilibria. First, again by our calculation of Bertrand equilibria we do know that in any proper subgame $G^{KS}(K_1, K_2)$ there is a *unique* Nash equilibrium$\{(p_1, p_2) :\ p_i = \max\{1 - K_i - K_j, c\}\}$. It follows that for any alternative candidate of a subgame perfect Nash equilibrium (K_1, f_1, K_2, f_2) we must have that $f_i(K_i, K_j) = \max\{1 - K_i - K_j + c, c\}$. Taking this into account the firms' utility vectors reduce to the same payoff vectors we had specified in the Cournot game. And we saw above that $\left\{(K_1, K_2) : \frac{1}{2}\left(1 - K_j\right) \le K_i \le 1 - K_j\right\}$ is the set of Cournot equilibria. So indeed Proposition 1 describes the set of all Nash equilibria. □

Proposition 1 shows that the Kreps-Scheinkman compromise between quantity and price competition translates to the case where firms have multiple objectives, profits and sales. As in the case of firms motivated only by profits, the *outcomes* of the Cournot model of Section 5.1. and the two stage game of capacity and price setting are equivalent given the present variation of the firms objectives. One advantage of the incomplete preference formulation of the Kreps-Scheinkman model is that it has a pure strategy equilibrium, while the solution of the original model depends on some rather complex mixing off-the-equilibrium path.

5.4 Owners and mangers in the Kreps-Scheinkman model

Following the literature on the objectives of the firm more closely we can actually single out one of all these equilibrium outcomes, the classical Cournot equilibrium. We investigate the option that the quantity and the price of a firm need not be set by the same agent. They might be set by different agents with different preferences. Actually, a large part of the literature on the goals of firms focuses on the possible differences between the goals of owners and managers. It is generally held that owners wish to maximize profits, while there remains much larger debate around the goals of managers. It is furthermore typically assumed that owners set up the game for managers which then make the day to day decisions. Following this tradition, it seems warranted that we introduce two different agents in the above formulation of the Kreps-Scheinkman model. We assume that capacities are set by profit maximizing owners while managers with incomplete preferences over profits and sales choose prices.[16] It is easy to see that in this modified game the set of subgame perfect Nash equilibria reduces to

$$\Bigg\{(K_1, f_1, K_2, f_2) : K_i = \frac{1}{3} \text{ and } f_i(K_i, K_j) = \max\{1 - K_i - K_j + c, c\},\ i, j = 1, 2,\ j \neq i\Bigg\}.$$

[16] In this model we ignore the contractual stage of the game. This should be more carefully adressed elsewhere where strategic delegation matters would be discussed.

The Nash equilibrium outcome is then found as:

$$(K_1, f_1(K_1, K_2), K_2, f_2(K_1, K_2)) = \left(\frac{1}{3}, \frac{1}{3} + c, \frac{1}{3}, \frac{1}{3} + c\right),$$

which is also the Nash equilibrium outcome in the original Kreps-Scheinkman model and of the classical Cournot model. So, we obtain the same prediction as Kreps and Scheinkman (1983), albeit in a slightly different setup. By introducing two different decisionmakers for the two stages of the game and by assuming that the second decision maker, the manager, has incomplete preferences (in the sense of having two objectives, profits and sales), we find that the *unique* pure strategy subgame perfect equilibrium of the resulting game induces the classical Cournot equilibrium outcome.[17]

6 Remarks on the existence of equilibrium

A few remarks on the existence of equilibria for games with incomplete preferences are in order. In this section we compare and discuss the two prevalent approaches in the literature for establishing conditions under which games with incomplete preferences have equilibria. We start by the observation that given Theorem 1, we can simply import existence results for games with complete preferences to the theory of games with incomplete preferences. If we can show that some completion of a game has an equilibrium, then the game itself has an equilibrium. With representable preferences, linear completions provide a natural starting point, since they are easy to calculate. And indeed this technique is applied in the literature on multicriteria decision-making to establish conditions under which equilibria exist.[18]

To obtain completions that are covered by some suitable fixed point theorem, these papers assume in general that there exists some $\beta \in \Delta_+$ such that, for each player i, the function $\beta^i u^i$ is quasiconcave in her own action. This condition guarantees that, in the completion G_β, the best response correspondence of each player is convex-valued, which again makes standard fixed point arguments applicable. In general, however, this condition is not easy to verify, for the set of quasiconcave functions is not closed under addition, and hence it does not suffice to check that every component of the utility of each player is quasiconcave in her own action. A more restrictive but substantially more operational requirement is, on the other hand, that all component utilities u^i are concave in the players' own action, for then $\beta^i u^i$ must be concave (and therefore quasiconcave) in the players' own action for all $\beta^i \gg 0$.

We should also note that there is an alternative approach to establish existence of equilibria. Shafer and Sonnenschein (1975), for instance, focus directly on games

[17] Our prior discussion on the extent of incompleteness of a managers preferences that is necessary for $(\max\{1 - 2K + c, c\}, \max\{1 - 2K + c, c\})$ to be an equilibrium, applies also here. Again, it is not necessary that managers cannot rank *any* two price profiles such that one yields the firm higher sales the other higher profits.

[18] Ding (2000), Wang (1991), and Yu and Yuan (1998) among others apply this technique in their existence proofs.

with incomplete preferences in which the preferences of each player are convex in her own action.[19] Below we illustrate the difference between the two approaches to the existence of equilibrium by means of two examples. The first is an example of a game to which the Shafer-Sonnenschein existence theorem applies, while no linear completion of this game has a Nash equilibrium. On the other hand, the Shafer-Sonnenschein existence theorem does not cover the game considered in the second example, but we can establish the existence of an equilibrium using the linear completion method.

Example 1. Consider a two player game $G = \{(A_i, u_i)_{i=1,2}\}$ where $A_i := \{x \in [0,1]^2 : x_1 + x_2 \leq 1\}$ for $i = 1, 2$, and

$$u^1(x,y) := \begin{cases} \begin{pmatrix} x_1^2 \\ x_2^2 \end{pmatrix}, & \text{if } y_1 = \frac{1}{2} \\ \begin{pmatrix} -|\frac{1}{2} - x_1| \\ 0 \end{pmatrix}, & \text{otherwise,} \end{cases}$$

$$u^2(x,y) := \begin{cases} \begin{pmatrix} -|\frac{1}{2} - y_1| \\ 0 \end{pmatrix}, & \text{if } x_1 = \frac{1}{2} \\ \begin{pmatrix} y_1^2 \\ y_2^2 \end{pmatrix}, & \text{otherwise.} \end{cases}$$

Clearly both u^1 and u^2 are continuous and quasiconcave in the respective own action, and A_1 and A_2 are both convex and compact, so by the Shafer-Sonnenschein existence theorem there exists an equilibrium of G. (Indeed, $((.3,.7),(\frac{1}{2}\frac{1}{2}))$ constitutes a Nash equilibrium.) Observe, however, that for no $\beta \in \Delta_+$ is $\beta^1 u^1$ quasiconcave in x and we even have that $N(G_\beta)$ is empty for all $\beta \in \Delta_+$.[20]

Example 2. Take a two player game $G = \{(A_i, u_i)_{i=1,2}\}$ with $A_1 := [0,1]^2$, $A_2 := [0,1]$,

$$u^1(x,y) := \begin{pmatrix} x_1^2 + x_2^2 + y \\ 2x_1x_2 \end{pmatrix} \quad \text{and} \quad u^2(x,y) := |x_1 + x_2 - y|.$$

Observe that u^1 is not quasiconcave in x. Holding player 2's action fixed observe that player 1's utility from choosing either $(1,0)$ or $(0,1)$ is $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, whereas the utility of playing the linear combination $(\frac{1}{2}, \frac{1}{2})$ is $\begin{pmatrix} \frac{1}{2} + y \\ \frac{1}{2} \end{pmatrix}$. On the other hand, for $\beta = (\frac{1}{2}, \frac{1}{2})$, the function $\beta u^1(\cdot, y)$ is quasiconcave for all y. Therefore G_β, and hence G, has an equilibrium.

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[19] Actually their requirement is even more general than that, but for the purpose of our discussion we only cover the case of convex preferences.

[20] To see this, observe that for any $\beta \in \Delta_+$ the game G_β reduces to some kind of matching-pennies game. For any $\beta \in \Delta_+$, the x_1, y_1- best responses of both players map to the set $\{0, \frac{1}{2}, 1\}$. But player 1 prefers an extreme action (0 or 1) if, and only if, player two plays the middle ($\frac{1}{2}$). Player 2, on the other hand, prefers extreme actions if, and only if, also player 1 plays an extreme action.

7 Mixed extensions

7.1 Mixed strategy equilibria

In this section we investigate the mixed strategy extension of normal form games with incomplete preferences. We will see that, with some additional arguments, Theorem 2 in fact yields a full characterization of the set of all mixed strategy equilibria.

As in the standard context of games with complete preferences, we define the *mixed strategy extension* of a game $G = \{(A_i, u^i)_{i\in I}\}$ (with each A_i being a non-empty subset of a metric space and each u_i being bounded and Borel measurable) as $G^{mix} := \{(\triangle A_i, U^i)_{i\in I}\}$, where ΔA_i is the set of all Borel probability measures on A_i and U^i is a real function on $\triangle A := \underset{i\in I}{\times} \triangle A_i$ defined by $U^i(\sigma) = \int_A u^i d\sigma$ for some $u^i : A \to \mathbb{R}^{m_i}$. A strategy profile σ is a *mixed strategy Nash equilibrium* of $G = \{(A_i, u^i)_{i\in I}\}$, if it is a Nash equilibrium of $G^{mix} = \{(\triangle A_i, U^i)_{i\in I}\}$.[21] We denote the set of mixed strategy Nash equilibria of a game G by $N^{mix}(G)$.

Theorem 4 (Shapley, 1959; Aumann, 1962). *Let $G = \{(A_i, u^i)_{i\in I}\}$ be a finite game. Then*

$$\bigcup \{N^{mix}(G_\beta) : \beta \in \Delta_+\} = N^{mix}(G).$$

Proof. By Theorem 2 we readily have $\bigcup \{N^{mix}(G_\beta) : \beta \in \Delta_+\} \subseteq N^{mix}(G)$. Moreover, since all A_i are finite, for all maximal points x in $u^i(A_i, a_j)$, there exists a $\beta^i \gg 0$ such that $x \in \max_{y\in u^i(A_i,a_j)} \beta^i y$. Consequently $\bigcup \{N^{mix}(G_\beta) : \beta \in \Delta_+\} = N^{mix}(G)$. □

Theorem 4 is closely related to the earlier results of Shapley (1959) and Aumann (1962). Aumann shows that σ is a mixed strategy Nash equilibrium of a finite game $G = \{(A_i, \succsim_i)_{i\in I}\}$ if, and only if, σ is a Nash equilibrium of a completion of G, such that each completed preference relation admits a von Neumann-Morgenstern representation.[22] This result stands in between Theorems 1 and 4. Given that Aumann assumes certain properties about each player's preferences, he does not need to look at *all* completions (as in Theorem 1) but only at those that obey the von Neumann-Morgenstern axioms to determine the set of all Nash equilibria. Theorem 4 imposes more assumptions on the preferences of the players, in particular we assume finite dimensional representability, and thus a smaller set of completions suffices to capture all Nash equilibria: we find that an action profile is a Nash equilibrium in a game with incomplete preferences if, and only, if it is a Nash equilibrium in a *linear* completion of that game. Shapley's (1959) result only covers

[21] Aumann's (1962) definition of a mixed strategy equilibrium in a game with incomplete preferences reduces to the present one, when the preferences of each player can be represented by a finite dimensional utility function.

[22] Aumann 's (1962) result is actually more general than that, the result, for instance, does not assume finite dimensional representability of the incomplete preference relations. The precise statement of his result would go beyond the scope of this paper.

two-person zero-sum matrix games. He furthermore assumes a peculiar kind of preferences that directly admit finite dimensional representation, so that also in his case linear completions of the game suffice to describe the set of all Nash equilibria.

We conclude by noting that a version of Theorem 4 still holds when we relax the finiteness assumption:

Theorem 5. *Let* $G = \left\{(A_i, u^i)_{i \in I}\right\}$ *be a game such that each* A_i *is a convex subset of some finite dimensional Euclidean space. Then*

$$\bigcup \left\{N^{mix}(G_\beta) : \beta \in \Delta_+\right\} \subseteq N^{mix}(G) \subseteq \bigcup \left\{N^{mix}(G_\beta) : \beta \in \Delta\right\}.$$

The proof of Theorem 5 is analogous to the proof of Theorem 2, and is therefore omitted here.

7.2 Trembling hand perfect equilibria

Since Nash equilibrium sets of games with incomplete preferences can be large, it is of interest to consider refinements of equilibria such as "trembling hand perfection". As in the standard theory, we say that a strategy profile σ in a finite strategic game $G = \{(A_i, u^i)_{i \in I}\}$ is a *trembling hand perfect equilibrium*, if there exists a sequence $(\sigma^k)_{k=0}^{\infty}$ of completely mixed strategy profiles that converges to σ and $\sigma_i \in BR_{u^i}(\sigma^k_{-i})$ for all $k \in \mathbb{N}$ and all $i \in I$.

While in the context of games with complete preferences it is trivial to show that all trembling hand perfect equilibria are Nash equilibria, this implication does not hold true for games with incomplete preferences. Consider, for instance, the following two-player-game:

	L	R
A	$\binom{1}{1}, 1$	$\binom{\frac{1}{2}}{0}, 0$
B	$\binom{0}{1}, 1$	$\binom{1}{1}, 0$

The row player (1) has incomplete preferences over the outcomes; her preferences are represented by the two-component-vectors. The preferences of the column player (player 2), on the other hand, are complete. It is easily checked that the only Nash equilibrium in this game is (A, L). However (B, L) is trembling hand perfect according to the above definition. To see this, take any sequence of completely mixed strategies $(\sigma^k)_{k=0}^{\infty}$ that converges to σ, where $\sigma_1(B) = 1 = \sigma_2(L)$. Given that player 2 plays the completely mixed strategy σ_2^k player 1 compares $U^1(A, \sigma_2^k) = (1 - 0.5\sigma_2^k(R), 1 - \sigma_2^k(R))$ to $U^1(B, \sigma_2^k) = (\sigma_2^k(R), 1)$. For no $\sigma_2^k(R) > 0$ can these utilities be ranked. Therefore we have that $B \in BR_{u^1}(\sigma_2^k)$ (and of course $\{L\} = BR_{u^2}(\sigma_1^k)$). We conclude that (B, L) is a trembling hand perfect equilibrium, while it is not even a Nash equilibrium.

Since we are interested in the concept of trembling hand perfection only as a refinement of Nash equilibrium, in what follows we restrict our attention to trembling hand perfect Nash equilibria. To state the following result we define a game

$G' := \{(A_i, v^i)_{i\in I}\}$ a *representable completion* of a game $G = \{(A_i, u^i)_{i\in I}\}$ if for every player i the utility $v^i : A \to \mathbb{R}$ represents a completion of the preferences represented by u^i.

Lemma 3. *Let $G = \{(A_i, u^i)_{i\in I}\}$ be a finite game. If σ is a trembling hand perfect equilibrium of some representable completion of G, then σ is a trembling hand perfect Nash equilibrium in G.*

Proof. Let the strategy profile σ be a trembling hand perfect equilibrium of some representable completion $G' := \{(A_i, v^i)_{i\in I}\}$ of G. By Theorem 1 σ is a Nash equilibrium of the game G. On the other hand, since σ is a trembling hand perfect equilibrium of G', there exists a sequence $(\sigma^k)_{k=0}^{\infty}$ of completely mixed strategy profiles such that $\sigma_i \in BR_{u^{i\prime}}(\sigma^k_{-i})$ for all i, k and $\sigma^k \to \sigma$. But since v^i a completion of u^i, we have $\sigma_i \in BR_{u^i}(\sigma^k_{-i})$ for all i and k. So σ is a trembling hand perfect Nash equilibrium of G. □

Corollary 1 is a direct consequence of Lemma 3.

Corollary 1. *Let $G = \{(A_i, u^i)_{i\in I}\}$ be a finite game. Then G has a trembling hand perfect Nash equilibrium.*

Unfortunately, however, the converse of Lemma 3 is not true: Trembling hand perfect Nash equilibria in games with incomplete preferences need not be trembling hand perfect equilibria in any completion of that game. Consider, for instance, the strategy profile $\sigma_1(B) = \frac{1}{2}, \sigma_2(L) = 1$ in the following game:

	L	R
A	$\binom{0}{2}, 1$	$\binom{0}{3}, 0$
B	$\binom{1}{1}, 1$	$\binom{0}{2}, 0$

First, notice that σ is a Nash equilibrium in this game. Secondly, it is also trembling hand perfect. For, since $U^1(A, \sigma_2^k) = \big(0, 2 + \sigma_2^k(R)\big)$ and $U^1(B, \sigma_2^k) = (1 - \sigma_2^k(R), 1 + \sigma_2^k(R))$ are not comparable we have $A, B \in BR_{u^1}(\sigma_2^k)$ for any completely mixed strategy profile σ_2^k, and hence σ is a trembling hand perfect Nash equilibrium. We now argue that in any completion of G for which σ is a Nash equilibrium playing B is a weakly dominated strategy for player 1. For any such completion we must have $(A, L) \sim_1' (B, L)$ since player 1 is playing A and B with positive probability in the Nash equilibrium σ. On the other hand, $u^1(A, R) > u^1(B, R)$ so for any completion of the preferences of player 1 must have $(A, R) \succ_1' (B, R)$. We conclude that B indeed is a weakly dominated strategy in any completion of G for which σ is a Nash equilibrium. Therefore, there is no completion of G such that σ is trembling hand perfect in that completion.

Observe that this example arises only because definition of mixed strategy equilibria for games with incomplete preferences does not require that players be indifferent between all actions that they play with a positive probability in equilibrium. All these actions need to be best responses to all other players strategies, but in the case of incomplete preferences, $a_i, a_i' \in BR(a_{-i})$ does not imply $a_i \sim_i a_i'$, for the two actions might be unranked.

As in the context of complete preferences, it can be shown for games with incomplete preferences that a strategy profile in a finite two-player game is a trembling hand perfect Nash equilibrium if, and only if, it is a Nash equilibrium in which neither player plays a weakly dominated strategy.[23] This is proved in the same way as it is proved under the assumption of completeness of preferences.

8 Conclusion

The goal of this paper was to develop an operational theory of games with incomplete preferences, and to demonstrate the applicability of this theory by means of some economic examples. We started out by showing a fundamental similarity between the theory of games with incomplete preferences and the existing theory of games with complete preferences. In Theorem 1 we showed that for every Nash equilibrium of a game with incomplete preferences, there exists a completion of that game such that this action profile is a Nash equilibrium of the completed game. This result permits us to move back and forth between games with complete and incomplete preferences. For the calculation of equilibrium sets, we can choose whichever form of the game is more convenient, with complete or incomplete preferences. Equivalently, we can without loss of generality model the preferences of some players as incomplete, if we do not know them precisely.

Restricting our attention to games with preferences that can be represented by vector valued utility functions, we can obtain more operational results. In particular, we showed here that under certain restrictions on the players action spaces and preferences, linear completions of a game suffice to characterize the set of all equilibria. Finite mixed strategy games do, for example, fit these restrictions, thereby allowing one to characterize the set of all equilibria in mixed strategies of a finite game in terms of its linear completions. Unfortunately, it turns out that with respect to some concepts it does not suffice to look at the completions of a given game. For instance we cannot characterize all trembling hand perfect equilibria by using the completion method.

We discussed our results at the hand of three well-known models of oligopolistic competition in which firms have incomplete preferences over profits and sales. We established a maximal set of equilibria in the case of quantity competition between firms and we showed that the classical nonexistence problem of capacity constrained Bertrand competition can be solved by assuming that firms cannot rank all sales and output profiles. Finally, we showed that our pure strategy equilibrium solution of capacity constrained Bertrand competition yields the same result as the mixed strategy solution in the context of the celebrated sequential model of quantity and price competition by Kreps and Scheinkman (1983).

[23] The definition of weak domination is a direct application of the standard definition to games with incomplete preferences. We say that a strategy σ_i weakly dominates another strategy σ_i' if $u_i(\sigma_i, \sigma_{-i}) \geq u_i(\sigma_i', \sigma_{-i})$ for all σ_{-i} and $u_i(\sigma_i, \sigma'_{-i}) > u_i(\sigma_i', \sigma'_{-i})$ for at least one σ'_{-i}. A strategy is called weakly dominated if there exists another strategy that weakly dominates it.

With the exception of our discussion of the incomplete preferences version of the model by Kreps and Scheinkman (1983), the present study only covered normal-form games. In our treatment of the latter model, we pointed out in section 5.3. that a set of peculiar problems arise in extensive form games with incomplete preferences. In particular, in this context we can no longer use backward induction to solve for subgame perfect equilibria. Further investigation of extensive form games is needed to make the theory of games with incomplete preferences applicable to a wider range of economic problems. It would also be interesting to study games with incomplete preferences and incomplete information.

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Remarks concerning concave utility functions on finite sets ⋆

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Summary. A direct construction of concave utility functions representing convex preferences on finite sets is presented. An alternative construction in which at first directions of supergradients ("prices") are found, and then utility levels and lengths of those supergradients are computed, is exhibited as well. The concept of a least concave utility function is problematic in this context.

Keywords and Phrases: Concave utility, Finite sets, Supergradients, Afriat-Varian algorithm, Least concavity.

JEL Classification Numbers: D11, C60.

1 Introduction

Richter and Wong have recently [10] constructed a concave utility function representing a "convex" preference ordering $\succeq$ defined on a finite subset K of R^l, using duality (a Theorem of the Alternative). In the present note we exhibit a simple, direct construction of such a utility. A two-step alternative construction, in which at first directions of gradient vectors (supergradients, "prices") are determined, and at the second step utility levels and lengths of those supergradients are computed, is exhibited as well. Furthermore, we show that in many cases there is no least concave utility function representing the given preference ordering.

Thus, let $\succeq$ denote a complete, transitive and reflexive binary relation on K. Richter and Wong [10] show that a testable "convexity" condition (condition G in Section 2) is sufficient (its necessity is obvious) for the existence of a concave (i.e., a restriction to K of a concave function defined on all of R^l) utility function

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representing $\succeq$ on K. Richter and Wong assign to each point in K a real number (which eventually turns out to be the utility level at this point) and a real vector (which plays the role of the gradient, or rather of a supergradient, of the utility function at the point). A system of linear inequalities involving these numbers and vectors is introduced, and existence of solutions to this system (condition E in Sect. 5) is shown to be equivalent to condition G, via a duality argument, involving the Theorem of the Alternative.

A consistent system of linear inequalities may be solved effectively [8]. However, standard solution methods yield values for all unknowns at once. Both utility levels and supergradients (for all points) are obtained at one and the same time. There is a certain redundancy here – the values of the utility function are actually all that we are after, the supergradients were not asked for. Moreover, the geometry of the preference relation is not clearly visible during the solution process.

We suggest a procedure for computing utility levels successively from the bottom up, one indifference level at a time. The inductive argument is similar to ones given in [4] and in [9]. While this procedure may not be computationally superior to the one put forward in [10], it possesses a geometric-intuitive appeal. Knowledge of supergradients in addition to the values has the advantage of facilitating an easy extension of the utility function from K to all of R^l. However, one may either calculate supergradients once utility levels are determined, or, alternatively, one may consider the upper boundary of the convex hull in R^{l+1} of the graph of the utility functions defined on K as being the graph of a concave function defined on the convex hull of K. (Such a function is easily extendible into all of R^l.) The methods are compared in more detail in Remark 1, Section 5.

An alternative procedure is suggested, where we start by determining, for each point x in K, a vector perpendicular to a hyperplane supporting the convex hull of the upper set $\{x \in K : y \succeq x\}$ at x. Interpreting these vectors as prices, we realize that the original ordering $\succeq$ is an extension of the resulting (indirectly) revealed preference ordering. Employing a variant of the Varian algorithm [12], (a modification of the Afriat algorithm [1]), we assign lengths to these vectors and scalar values (levels) to the points of K, so as to obtain a concave utility function (given as a minimum of affine functions, see (4), (22)), whose value at a point of K is given by the computed level, and the appropriately normalized price vectors are supergradients. Once again, this method has a geometric-intuitive appeal and proceeds successively. The two methods may be thought of as each emphasizing "one half" of the data called for in [10] (utility levels and supergradients).

Least concave utility functions play an important role both in the mathematical theory and in economic applications ([3–5]). In Section 4 we show that in the finite case least concave utilities do not necessarily exist. Actually, in many cases there exist no minimal elements with respect to the relation "more concave than" (introduced in [3]). Even when minimal elements do exist, they are not unique (unlike the infinite case where least concave utility functions are unique up to increasing affine transformations). Put differently, the minimal elements are not cardinal.

One might inquire as to what happens to the constructed utility functions when the set K gets larger, eventually becoming dense in a convex subset Ω of R^l.

Unfortunately, one cannot say much in an explicit form – the utility functions (or their supergradients) should satisfy certain uniform boundedness conditions similar to those put forward in Section 2 of [4] or to the A-*restricted multipliers* condition introduced in [7]. We will elaborate on these issues in Section 5.

In the next section we describe more precisely the procedures suggested for constructing utility functions. The following section deals with the proofs. Least concave utility functions are discussed in Section 4.

It should be clear that the contribution of the present note is rather technical. Here, as in many other instances, the conceptual breakthrough is due to Ket Richter (with his collaborators).

2 Descriptions

Let $\succeq$ denote a complete, transitive and reflexive binary relation on K such that

$$\begin{array}{c}\text{CONDITION G}:\\ \text{For all } n \leq l+1, \text{for all distinct } x, y^1, \cdots, y^n \in K, \text{ and for}\\ \text{all real numbers } \lambda_1, \cdots, \lambda_n > 0:\\ \text{if } \sum_{i=1}^n \lambda_i = 1 \text{ and } x = \sum_{i=1}^n \lambda_i y^i,\\ \text{then either (a) or (b) holds}:\\ \text{a) } x \sim y^i \text{ for all } i = 1, \cdots, n,\\ \text{b) } x \succ y^i \text{ for some } i = 1, \cdots, n.\end{array}$$

A function $u : K \to R$ is *concave* if for all $y, x^1, \cdots, x^n \in K$: if $y = \sum_{i=1}^n \lambda_i x^i$ for some $\lambda_i \geq 0$ with $\sum_{i=1}^n \lambda_i = 1$, then $u(y) \geq \sum_{i=1}^n \lambda_i u(x^i)$. Note that u is concave on a finite K if and only if there exists a concave extension $v : R^l \to R$ with $v|_K = u$, and that we may restrict attention to the case $n \leq l+1$ (compare [10]). Theorem 1 of [10] asserts that if the ordering $\succeq$ satisfies condition G then there exists a concave utility function on K representing $\succeq$ (and conversely).

Let us denote the (finitely many) indifference classes by $L_1, \cdots, L_m$ (so that $x \succeq y$ if and only of $x \in L_i,\ y \in L_j$ with $i \geq j$). A (non-concave) utility function w representing $\succeq$ is given by $w(x) = j$ for $x \in L_j$. For $1 \leq k \leq m$, set $w_k(x) = \min(w(x), k)$, and let $\succeq_k$ denote the preference relation determined by w_k. Theorem 1 of [10] would follow, once we proved that for every $1 \leq k \leq m$ there exists a concave function u_k on K representing the preference $\succeq_k$. For constructing u_k, we consider, for every $1 \leq k \leq m$, the set C_k of those points $x \in K$ such that x is a strict convex combination of $n \leq l+1$ elements $y^1, \cdots, y^n$ of K for which the possibility (b) of condition G holds with respect to $\succeq_k$, i.e., $x \succ_k y^i$ for some $i = 1, \cdots, n$ (see Fig. 1). In the next section we prove the following proposition.

Proposition 1. There exist real numbers $U_1, \cdots, U_m$, such that: (i) for every $1 \leq k \leq m$ the function u_k defined on K by $u_k(x) = U_j$ for $x \in L_j$, $1 \leq j \leq k-1$, $u_k(x) = U_k$ if $x \in L_j, k \leq j \leq m$, is a concave utility function representing the preference $\succeq_k$ on K, and (ii): if $j \leq k,\ x \in C_j,\ x = \sum_{i=1}^n \lambda_i y^i$ with

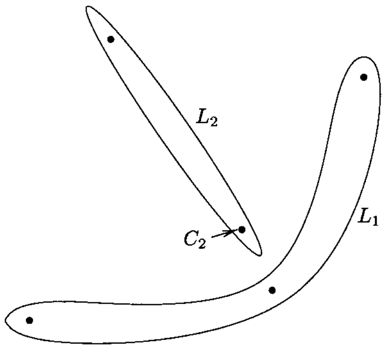

Figure 1

$\lambda_1, \cdots, \lambda_n > 0$ and $\sum_{i=1}^{n} \lambda_i = 1$, then

$$u_k(x) > \sum_{i=1}^{n} \lambda_i u_k(y^i). \tag{1}$$

In particular, the function u_m is a concave utility function representing $\succeq$ $(=\succeq_m)$ on K.

The (easy) inductive proof is constructive.

In the construction described above we compute directly the utility levels without bothering about (super)gradients. We now suggest an alternative construction, in which directions of supergradients are determined first. Set $M_j = conv(\cup_{k=j}^{m} L_k)$, where "$conv(S)$" denotes the convex hull of a set S. With every point $x \in L_j$ we may associate a hyperplane supporting M_j at x. As the next proposition asserts, we may find a hyperplane separating x *strongly* from the set $\{y : y \succ x\}$, hence also from its convex hull M_{j+1}.

Proposition 2. For every $x \in K$ there exists a vector $p \in R^l$ such that (i) $\langle p, y - x \rangle \geq 0$ for every $y \succeq x$, and (ii) $\langle p, y - x \rangle > 0$ for every $y \succ x$.

Thus we may associate with every $x \in K$ a vector satisfying (i) and (ii) (see Fig. 2, where K and the ordering $\succeq$ are the same as in Fig. 1). We fix such a vector and denote it by $p(x)$. We may normalize the vector $p(x)$ to be a unit vector.

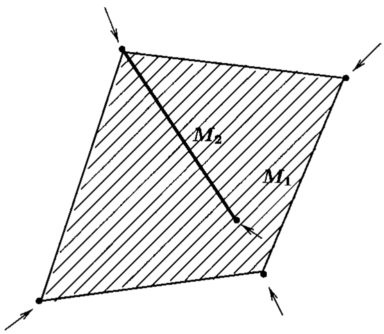

Figure 2

The vectors $p(x)$ might be loosely interpreted as price vectors, the collection of pairs $(p(x), x)$ (for $x \in K$) may be viewed as an expenditure configuration ([1, 7]), and the following holds:

Proposition 3. There exist positive numbers λ_i, U_i for $1 \leq i \leq m$ such that

$$U_i < U_j \text{ if } i < j, \tag{2}$$

and for every $x, y \in K,\ x \in L_i,\ y \in L_j$ we have

$$U_j \leq U_i + \lambda_i \langle p(x), y - x \rangle. \tag{3}$$

It is well known that Proposition 3 implies that the function $u(x)$ defined as

$$u(x) = \min_{1 \leq i \leq m} \min_{z \in L_i} u_i + \lambda_i \langle p(z), x - z \rangle \tag{4}$$

is a concave utility function representing $\succeq$ on K (such that $u(x) = u_i$ if $x \in L_i$). The details will be explained in Section 3.

3 Proofs

Proof of Proposition 1 The claim is obviously true for $k = 1$. For $k = 2$ set $u_2(x) = 0$ for $x \in L_1$ and $u_2(x) = 1$ otherwise, so that $U_1 = 0,\ U_2 = 1$. Suppose that $k < m$ and that there exists a function u_k satisfying the induction hypothesis. Set $u_{k+1}(x) = u_k(x) = U_j$ for $x \in L_j,\ 1 \leq j \leq k$. There are finitely many points $x \in C_j,\ j \leq k$ and every such point x may be written in finitely many ways in the form $x = \sum_{i=1}^n \lambda_i y^i$ with $n \leq l+1,\ \lambda_1, \cdots, \lambda_n > 0,\ \sum_{i=1}^n \lambda_i = 1$. Set $I_l = \{i : y^i \in L_j,\ j \leq k\}$ and $I_u = \{i : y^i \in L_j,\ j \geq k+1\}$. By the induction hypothesis,

$$\begin{aligned} u_k(x) > \sum_{i=1}^n \lambda_i u_k(y^i) &= \sum_{i \in I_l} \lambda_i u_k(y^i) + \sum_{i \in I_u} \lambda_i u_k(y^i) \\ &= \sum_{i \in I_l} \lambda_i u_k(y^i) + U_k \sum_{i \in I_u} \lambda_i \end{aligned} \tag{5}$$

We may find a real number $U_{k+1} > U_k$ such that all the inequalities

$$u_k(x) > \sum_{i \in I_l} \lambda_i u_k(y^i) + U_{k+1} \sum_{i \in I_u} \lambda_i \tag{6}$$

will hold. Set $u_{k+1}(x) = U_{k+1}$ for $x \in L_p,\ p \geq k+1$. It follows from (5) and (6) that (1) holds for $x \in C_j,\ j \leq k$. The function u_{k+1} is clearly a utility representation for the preference $\succeq_{k+1}$ on K. If $x \in C_{k+1}$ then there exists at least one index $i \leq n$ such that $y^i \in L_j$ with $j \leq k$ and (1) follows (with k replaced by $k+1$).

To prove concavity of u_{k+1}, assume that $y, x^1, \cdots, x^n \in K$ and y is a convex combination of $x^1, \cdots, x^n$. Set

$$\Lambda = \{\boldsymbol{\lambda} = (\lambda_1, \cdots, \lambda_n) : \lambda_i \geq 0,\ \sum_{i=1}^n \lambda_i = 1, y = \sum_{i=1}^n \lambda_i x^i\} \tag{7}$$

Then Λ is a compact convex polyhedron, hence a convex combination of its (finitely many) extreme points Λ_e. Let $\boldsymbol{\lambda} = (\lambda_1, \cdots, \lambda_n)$ be an element of Λ_e. Then at most $l+1$ of the numbers $\lambda_1, \cdots, \lambda_n$ are non-zero. Assume, on the contrary, that the set $I = \{i : \lambda_i > 0\}$ contains more than $l+1$ elements. Then the vectors $\{x^i : i \in I\}$ are affinely dependent, so that there exist real numbers $\alpha_i : i \in I$, not all zero, such that $\sum_{i\in I} \alpha_i = 0, \sum_{i\in I} \alpha_i x^i = 0$. Setting $\alpha_i = 0$ if $i \notin I$ and choosing $t > 0$ small enough, we see that $\boldsymbol{\lambda}_+ = (\lambda_1 + t\alpha_1, \cdots, \lambda_n + t\alpha_n)$ and $\boldsymbol{\lambda}_- = (\lambda_1 - t\alpha_1, \cdots, \lambda_n - t\alpha_n)$ are two distinct elements of Λ with $\boldsymbol{\lambda} = \frac{1}{2}(\boldsymbol{\lambda}_+ + \boldsymbol{\lambda}_-)$, contradicting $\boldsymbol{\lambda} \in \Lambda_e$. Hence every $\boldsymbol{\lambda} \in \Lambda$ may be written in the form $\boldsymbol{\lambda} = \sum_{p=1}^q \beta_p \boldsymbol{\lambda}^{\mathbf{p}}$ where $\beta_p > 0, 1 \le p \le q, \sum_{p=1}^q \beta_p = 1$, and at most $l+1$ of the numbers $\lambda_1^p, \cdots, \lambda_n^p$ are non-zero, where $\boldsymbol{\lambda}^{\mathbf{p}} = (\lambda_1^p, \cdots, \lambda_n^p), 1 \le p \le q$. It follows that for every function v defined on K,

$$\sum_{i=1}^n \lambda_i v(x^i) = \sum_{p=1}^q \beta_p \sum_{i=1}^n \lambda_i^p v(x^i) \tag{8}$$

so that if we prove that $v(y) \ge \sum_{i=1}^n \lambda_i^p v(x^i)$ for every $1 \le p \le q$, then $v(y) \ge \sum_{i=1}^n \lambda_i v(x^i)$. Thus it suffices to prove that $u_{k+1}(y) \ge \sum_{i=1}^n \lambda_i u_{k+1}(x^i)$ for $n \le l+1$. If $y \in \cup_{j=k+1}^m L_j$ then $u_{k+1}(y) \ge u_{k+1}(x)$ for all $x \in K$. If $y \in C_j$ for $j \le k$ then $u_{k+1}(y) \ge \sum_{i=1}^n \lambda_i u_{k+1}(x^i)$ follows from the inequality (1), proved already for u_{k+1}. If $y \in L_j \setminus C_j$ for $j \le k$ then $u_{k+1}(y) = u_{k+1}(x^i)$ for all $1 \le i \le n$. □

Proof of Proposition 2 Let $x \in K$ be non-maximal with respect to $\succeq$ (otherwise there is nothing to prove). Set $U = \{y - x : y \succ x\}$ and denote by $\mathcal{C}$ the convex cone generated by the finite collection $\{x - z : z \sim x\}$. We claim that $conv(U) \cap \mathcal{C} = \emptyset$. Otherwise, there exist points $y_1, \cdots, y_n, z_1, \cdots, z_q$ and non-negative numbers $\alpha_1, \cdots, \alpha_n, \beta_1, \cdots, \beta_q$ such that $y_i - x \in U$ for $1 \le i \le n$, $z_j \sim x$ for $1 \le j \le q$, $\sum_{i=1}^n \alpha_i = 1$, and $\sum_{i=1}^n \alpha_i(y_i - x) = \sum_{j=1}^q \beta_j(x - z_j)$. Hence

$$\sum_{i=1}^n \alpha_i y_i + \sum_{j=1}^q \beta_j z_j = \left(1 + \sum_{j=1}^q \beta_j\right) x. \tag{9}$$

At least one of the coefficients α_i is positive, and we may assume without loss of generality that all the coefficients α_i, β_j appearing in (9) are positive. Hence x is a strict convex combination of $y_1, \cdots, y_n, z_1, \cdots, z_q$ (here q might vanish) with $y_i, z_j \succeq x$ and $y_1 \succ x$, contradicting condition G, if $n + q \le l + 1$. It is proved in [10] that condition G implies that x cannot be such a convex combination even if $n + q > l + 1$ (see Remark 1 in [10]).

It follows that the compact set $conv(U)$ may be separated strongly from the convex cone $\mathcal{C}$, so that there exists a vector $p \in R^{l+1}$ such that $\langle p, w\rangle > 0$ for all $w \in U$, $\langle p, w\rangle \le 0$ for all $x \in \mathcal{C}$. Hence $\langle p, y\rangle > \langle p, x\rangle$ if $y \succ x$ and $\langle p, y\rangle \ge \langle p, x\rangle$ if $y \sim x$. □

Proof of Proposition 3 (i) Note that there exists (constructively, see [8]) a sufficiently large positive number t such that for every $x \in K$ the vector $\mathbf{x} \in \mathbf{K} \subset R^{l+1}$ given by $\mathbf{x}_i = x_i + t$ for $1 \le i \le l$ and $\mathbf{x}_{l+1} = t - \sum_{i=1}^{l} x_i$ has positive components and is contained in the hyperplane $H = \{\mathbf{y} \in R^{l+1} : \sum_{i=1}^{l+1} \mathbf{y}_i = (l+1)t\}$ (thus $\mathbf{K} \subset H$). For every vector $p \in R^l$ normalized by $\sum_{i=1}^{l} p_i = 1$, define $\mathbf{p} \in R^{l+1}$ by $\mathbf{p}_i = p_i + s$ for $1 \le i \le l$ and $\mathbf{p}_{l+1} = s$. Then all components of $\mathbf{p}$ are positive if s is sufficiently large, and for every p and all $x, y \in K$ we have

$$\langle \mathbf{p}, \mathbf{x} - \mathbf{y} \rangle = \langle p, x - y \rangle. \tag{10}$$

(ii) The construction of $p(x)$ for $x \in K$ implies that if $y \succeq x$ then $\langle p(x), y \rangle \ge \langle p(x), x \rangle$. It follows from (10) that

$$y \succeq x \Longrightarrow \langle \mathbf{p}(\mathbf{x}), \mathbf{x} - \mathbf{y} \rangle \le 0. \tag{11}$$

Hence the inequality $\langle \mathbf{p}(\mathbf{x}), \mathbf{x} - \mathbf{y} \rangle > 0$ implies that $x \succ y$. In other words, if $\mathbf{x}$ is *directly revealed strictly preferred* to $\mathbf{y}$ then $x \succ y$. The inequality $\langle \mathbf{p}(\mathbf{x}), \mathbf{x} - \mathbf{y} \rangle \ge 0$ is equivalent by (10) to $\langle p(x), x - y \rangle \ge 0$. By Proposition 2 this is inconsistent with $y \succ x$, hence $x \succeq y$. Put differently, if $\mathbf{x}$ is *directly revealed preferred* to $\mathbf{y}$ then $x \succeq y$.

(iii) Set $\mathbf{x} \succeq \mathbf{y}$ if $x \succeq y$. Then (ii) implies that the complete order $\succeq$ defined on $\mathbf{K}$ is an extension (a refinement, in the terminology of [1]) of the revealed preference ordering determined by the expenditure configuration $(\mathbf{p}(\mathbf{x}), \mathbf{x}),\ \mathbf{x} \in \mathbf{K}$. We may now use a variant of the Varian algorithm 3 [12] (a modification of the Afriat algorithm [1]) and construct inductively, for $1 \le i \le m$, positive numbers λ_i, U_i such that (2) and (3) hold. In the construction we also use auxiliary positive sequences ϵ_i and μ_i with ϵ_i strictly decreasing. We start by setting $U_1 = \lambda_1 = \epsilon_1 = \mu_1 = 1$. Assuming that U_j, λ_j and ϵ_j have been defined already for all $1 \le j \le i$, the quantities $U_j + \lambda_j \langle \mathbf{p}(\mathbf{x}), \mathbf{y} - \mathbf{x} \rangle$ are well defined for all $1 \le j \le i$, $x \in L_j$, $y \in K$. To simplify notations, set

$$a_{i,j} = \min_{k>i} \min_{x \in L_j} \min_{y \in L_k} \lambda_j \langle \mathbf{p}(\mathbf{x}), \mathbf{y} - \mathbf{x} \rangle, \tag{12}$$

$$V_{i+1} = \min_{j \le i} U_j + a_{i,j}, \tag{13}$$

and define

$$\epsilon_{i+1} = \min\left(\frac{\epsilon_i}{2}, \frac{a_{i,i}}{2}\right), \tag{14}$$

$$U_{i+1} = V_{i+1} - \epsilon_{i+1}. \tag{15}$$

Let $I(x, y)$ denote the indicator function (characteristic function) of the set $\{x, y : \langle \mathbf{p}(\mathbf{x}), \mathbf{y} - \mathbf{x} \rangle < 0\}$, and set (using (15))

$$\mu_{i+1} = \min_{j \le i} \min_{x \in L_{i+1}} \min_{y \in L_j} \frac{U_{i+1} - U_i}{-\langle \mathbf{p}(\mathbf{x}), \mathbf{y} - \mathbf{x} \rangle} I(x, y), \tag{16}$$

$$\lambda_{i+1} = \min(\mu_{i+1}, \lambda_i). \tag{17}$$

(iv) Statement (ii) of Proposition 2 implies that the quantities $a_{i,j}$ as defined in (12) are positive. Hence $\epsilon_i > 0$ for all i. Note that $a_{i-1,j} \le a_{i,j}$ by definition (12), so that $U_j + a_{i-1,j} \leq U_j + a_{i,j}$ (for $j < i$). Moreover, $U_i + a_{i,i} = V_i - \epsilon_i + a_{i,i}$. Hence (13) implies that

$$V_{i+1} \geq \min(V_i, V_i - \epsilon_i + a_{i,i}), \tag{18}$$

so that

$$U_{i+1} \geq \min(V_i - \epsilon_{i+1}, V_i - \epsilon_i + a_{i,i} - \epsilon_{i+1}). \tag{19}$$

But each term in the right hand side of (19) is strictly bigger (by (14) and (15)) than $U_i = V_i - \epsilon_i$, implying that $U_{i+1} > U_i$, so that the inductively defined sequence $\{U_i\}$ is a strictly increasing monotone sequence of positive numbers.

(v) It remains to prove (3). By (10) it suffices to verify the inequality

$$U_j \leq U_i + \lambda_i \langle \mathbf{p}(\mathbf{x}), \mathbf{y} - \mathbf{x} \rangle. \tag{20}$$

for every $x, y \in K,\ x \in L_i,\ y \in L_j$.

We distinguish three cases:

(1) If $x \sim y$ then $i = j$ and the inequality follows from statement (i) in Proposition 2.
(2) If $y \succ x$ then $x \in L_{\tilde{j}}$ and $y \in L_{\tilde{i}+1}$ where $\tilde{j} = i,\ \tilde{i} = j - 1$, and $\tilde{j} \leq \tilde{i}$. Then by (13) and (15) (with $\tilde{i}, \tilde{j}$ replacing i, j) we know that $U_{\tilde{i}+1} < V_{\tilde{i}+1} \leq U_{\tilde{j}} + \lambda_{\tilde{j}} \langle \mathbf{p}(\mathbf{x}), \mathbf{y} - \mathbf{x} \rangle$.
(3) If $x \succ y$ then $x \in L_{\tilde{i}+1}$ and $y \in L_j$ with $\tilde{i} = i - 1$ and $j \leq \tilde{i}$. If $\langle \mathbf{p}(\mathbf{x}), \mathbf{y} - \mathbf{x} \rangle$ is non-negative then (20) holds trivially. If $\langle \mathbf{p}(\mathbf{x}), \mathbf{y} - \mathbf{x} \rangle < 0$ then $I(x, y) = 1$ and (16) and (17) imply that

$$\lambda_{\tilde{i}+1} \leq \frac{U_{\tilde{i}+1} - U_j}{-\langle \mathbf{p}(\mathbf{x}), \mathbf{y} - \mathbf{x} \rangle} = \frac{U_j - U_{\tilde{i}+1}}{\langle \mathbf{p}(\mathbf{x}), \mathbf{y} - \mathbf{x} \rangle} \tag{21}$$

so that $\langle \mathbf{p}(\mathbf{x}), \mathbf{y} - \mathbf{x} \rangle \lambda_{\tilde{i}+1} \geq U_j - U_{\tilde{i}+1}$.

□

Consider now the function $u(x)$ defined over all of R^l by (4). As is well known (see e.g. [1,7,10]), Proposition 3 implies that u is a concave utility function representing $\succeq$ on K. In fact, u is given as a minimum of affine function, hence is concave. Replacing x by z and y by x in (3), we see from (4) that $u(x) = U_j$ whenever $x \in L_j$. That u represents $\succeq$ follows from (2).

4 Least concave utility functions

Let X be an arbitrary set, and let $\succeq$ be a preference relation on X. Recall that if u and v are two utility functions representing $\succeq$ on X, so that $u(x) = f(v(x))$ where $f = u \circ v^{-1}$ is a real increasing function defined on $v(X)$ (the range of v), we say that u is *more concave than* v if f is concave in $v(X)$ [3]. Let $\mathcal{U}$ denote the set of concave utility functions representing $\succeq$ on X. If X is a *convex* subset of R^l and $\succeq$ is concavifiable (i.e. $\mathcal{U} \neq \emptyset$), then there exist least elements in $\mathcal{U}$ with respect to the partial ordering "more concave than" ([3,4]). (Here and in the sequel the terms "least" and "minimal" with respect to a partial ordering have the same meaning as in [2].) Least concave utility representations are cardinal and play a role in many applications, see e.g. [3,4], and [5].

What happens if X contains only a finite number of elements? (Note that if $X \subset R^l$ and $\succeq$ satisfies condition G then the set $\mathcal{U}$ is not empty.) Recall that a utility function v representing $\succeq$ on X is least concave if for *every* concave utility function representing $\succeq$ on X there exists a real concave function f defined on the set $v(X) \subset R^1$ such that f is strictly increasing on $v(X)$ and $u(x) = f(v(x))$ for all $x \in X$. (If X contains only a finite number of elements then the set $v(X)$ is finite and and the functions $u, \ v, \ \text{and} \ f$ may be regarded as vectors.)

Example 1. Let X be a finite subset of the unit interval. Then there exist least concave utility functions representing the order $\succeq=\geq$ on X and they are all given by $u(x) = \alpha x + \beta$ with $\alpha > 0$.

Example 2. Set $K = \{(0,0), (.5,.5), (0,1)\} \subset R^2$, and define a preference ordering on K by $(0,1) \succ (.5,.5) \succ (0,0)$. Then the function $u_\beta(x) = \beta x_1 + x_2$ is a concave utility function representing $\succeq$ on K if and only if $-1 < \beta < 1$, and u_{β_1} is more concave than u_{β_2} if and only if $\beta_1 \geq \beta_2$. Every concave utility function for $\succeq$ on K is obtainable from a certain u_β by an increasing affine transformation. Hence there exist no minimal (and certainly no least) elements in $\mathcal{U}$. The same applies more generally if there exists an extreme point of K which is neither maximal nor minimal with respect to $\succeq$ and is indifferent to no other point of K. As the next example shows, least concave utility functions may fail to exist in other cases as well.

Example 3. $K = \{(0,0), (1,0), (.5,0), (.5,.5), (0,1)\} \subset R^2$ and define a preference relation on K by $(0,1) \succ (.5,.5) \succ (.5,0) \succ (1,0) \sim (0,0)$. The function $u_\alpha : K \to R^1$ defined by $u_\alpha(0,0) = u_\alpha(1,0) = 0, \ u_\alpha(.5,0) = 1, \ u_\alpha(.5,.5) = \alpha, \ u_\alpha(0,1) = 2$ is concave on K and represents $\succeq$ if $1 < \alpha < 2$. Here u_{α_1} is more concave than u_{α_2} if and only if $\alpha_1 \geq \alpha_2$. Once again, there exist no least elements in $\mathcal{U}$.

As a way to remedy the difficulty exhibited in Examples 2 and 3, one may try modifying the relation "more concave than" by considering the convex hull X of K and extending $\succeq$. Recall that it is shown in [10] (and explained in Section 1) that the function u is concave on K if and only if there exists a concave extension of u to X. Let u, v be two concave functions on X representing $\succeq$ on K such that the preferences induced by u and v on X coincide. Then the real function $f = u \circ v^{-1}$ is well-defined defined on $v(X)$, and we say that u *mct* v if and only if f is concave.

It is easy to see that every function u_β of the type considered in Example 2 is a least element with respect to the relation mct. However, the functions u_{β_1} and u_{β_2} are mct incomparable if $\beta_1 \neq \beta_2$. (Similarly, every function u_α of the type considered in Example 3 is a least element with respect to the relation mct. However, the functions u_{α_1} and u_{α_2} are mct incomparable if $\beta_1 \neq \beta_2$.)

Least concave utility functions do not necessarily exist if the range is a non-standard extension of R [6]. It appears that not only is least concavity inherently not an elementary (first order) concept, but that it belongs essentially to analysis and as such is not a very appropriate concept for sets containing only finitely many elements.

5 Remarks

Remark 1. Richter and Wong [10] show that condition G is equivalent to the following condition:

$$\text{CONDITION E :}$$
$$\text{There exist numbers } u^i \ (i = 1, \cdots, N), \text{ and vectors } \mathbf{u}^i \in R^l \ (i = 1, \cdots, N),$$
$$\text{that solve the linear system}$$
$$u^i - u^j > 0 \quad \text{for all } i, j = 1, \cdots, N \text{ with } x^i \succ x^j,$$
$$u^i - u^j - \langle \mathbf{u}^i, x^i - x^j \rangle \geq 0 \text{ for all } i, j = 1, \cdots, N,$$
$$u^i - u^j = 0 \text{ for all } i, j = 1, \cdots, N \text{ with } x^i \sim x^j.$$

Here $K = \{x^1, \cdots, x^N\}$.

The equivalence of conditions G and E is demonstrated by means of a duality argument (the Theorem of the Alternative). It is observed that the numbers u^i are the values of a candidate concave utility function, while the vectors $\mathbf{u}^i$ play the role of super-gradients. There exist "effective" (i.e., polynomial time) algorithms for solving a linear system [8]. Thus, the method of [10] leaves nothing do be desired from the computational-complexity point of view. An additional advantage is that an explicit formula for the values of (an extension of) a convex utility function u is given by

$$u(x) = \min_{1 \leq i \leq N} u_i + \langle \mathbf{u}^i, x - x^i \rangle, \tag{22}$$

compare [11,7], and equation (3) in [10].

In the first method proposed in this note (see Proposition 1) the numbers u^i are computed successively using a geometric argument in R^l, and the vectors $\mathbf{u}^i$ do not appear. Loosely, this is more "efficient", as the numbers u^i are the values of the utility function on K. The utility levels are computed one at a time and one could stop upon reaching the particular value at a point of interest. The inequality (1) needs to be checked at most N^{l+1} times at each level, so that while the method is not superior to (not less complex than) a one used for solving a linear system, it is not inferior either. On the other hand, the computationally effective algorithms for solving a linear system yield the values of all unknowns at the end (when the process terminates), and the geometric structures of K and of the preference ordering are not clearly visible.

With extra work one may construct an extension $u(x)$ after the numbers u^i were computed. In fact, as outlined in Richter and Wong [10], footnote 5, it is possible to determine real vectors playing the role of the supergradients $\mathbf{u}^i$ and satisfying the system of linear inequalities using (for example) a separating hyperplane argument. Alternatively, one may regard the upper boundary of the convex hull in R^{l+1} of the graph of the utility functions defined on K (i.e., the convex hull of the set (x^i, u^i) for $1 \leq i \leq N$) as the graph of a concave function defined on the convex hull of

K. (Such a function is easily extendible to all of R^l.) Once again, this involves polynomial algorithms [8].

Remark 2. In the second method proposed in this note (see Propositions 2 and 3) both utility levels *and* supergradients are computed successively, the supergradient $\mathbf{u}^i$ being equal to $\lambda_j p(x^i)$ if $x^i \in L_j$. The directions of the vectors are determined first (Proposition 2) via a geometric argument involving separation of convex sets. Then a variant of the Varian algorithm [12] is employed (Proof of Proposition 3) in order to compute successively utility levels and appropriate lengths for the vectors. Once again, this algorithm is about as hard as solving a linear system (polynomial time) such as the one appearing in condition E. Afriat [1] computes first (successively) lengths for the price vectors; utility levels are determined in a later step. This could be done because Afriat has to "rationalize" only the revealed preference ordering, while we have to account for the ordering $\succeq$ (which may contain properly the revealed preference ordering). The Varian version is suitable for our purposes. It might be interesting to find a way of assigning lengths to the vectors $p(x)$ without calculating utility levels at the same time.

The set K and the ordering $\succeq$ were lifted to R^{l+1} in order to achieve positivity of all "price" vectors and gain non-satiation. It was shown that the lifted preference ordering $\succeq$ is a refinement of the resulting revealed preference ordering. While these steps are not absolutely necessary for the argument, they help to clarify it and to put it in context.

Remark 3. It was shown in Section 4 that least concave utility representations do not exist in many instances if K is finite. Observe that if K is a finite subset of R^l, the method used in [3] (considering the infimum of a certain family of concave utility function) does not work, nor does any of the three methods offered in [4]. The strict inequalities $u^i - u^j > 0$ for all $i, j = 1, \cdots, N$ with $x^i \succ x^j$, do not necessarily survive the infimum. The success of the constructions put forward in [4] depends on the possibility of carrying out various steps in the "best" possible way (e.g., computing least values of certain quantities compatible with desired inequalities). This cannot be done in general in the finite case. Thus, we have to find values so that the *strict* inequality (1) holds, and in the k-th induction step we cannot use the infimum of the allowable values for U_{k+1} compatible with (6). Similarly, the numbers ϵ_i in the proof of Proposition 3 cannot chosen in an optimal fashion.

Remark 4. It may be interesting to inquire what happens if the set K gets larger end eventually becomes dense in an open subset of R^l. Thus, consider an increasing sequence K_m of finite subsets of R^l such that $\cup_{m=1}^{\infty} K_m$ is dense in a convex set Ω and the ordering on each K_m is induced by a convex ordering $\succeq$ defined on Ω. Assume for simplicity that Ω is compact. It is well-known [7] that $\succeq$ is concavifiable on Ω if and only if there exists a sequence u_m of concave utility representations of the order induced by $\succeq$ on K_m such that 1) $u_m(x) \to 1$ if $x \in \Omega$ is maximal with respect to $\succeq$, 2) $u_m(x) \to 0$ if $x \in \Omega$ is minimal with respect to $\succeq$, and for every non-maximal $x \in \Omega$ we have $\sup_m u_m(x) < 1$. This condition is unfortunately very implicit and cannot be put easily in an explicit form. (One may ask for the condition to be verified for utility functions constructed by either one of our procedures.) In analogy to Theorem 3 of [7] it is easy to derive the following (still very implicit) modification of condition E:

CONDITION $\tilde{\mathrm{E}}$:

The following statements are equivalent:

(i) There exists a concave utility function representing $\succeq$ in Ω;
(ii) For every compact set L of non-maximal elements of Ω there exists a positive constant A such that for every m there exist numbers u^i $(i = 1, \cdots, N(m))$, and vectors $\mathbf{u}^i \in R^l$ $(i = 1, \cdots, N(m))$, (here $N(m)$ denotes the cardinality of $K_m \cap L$) satisfying condition E with respect to $K_m \cap L$, such that $A^{-1} < \|\mathbf{u}^i\| < A$ for all $i = 1, \cdots, N(m)$;
(iii) For every compact set L of non-maximal elements of Ω there exists a positive constant A such that for every m there exist vectors $\mathbf{u}^i \in R^l$ $(i = 1, \cdots, N(m))$ satisfying $A^{-1} < \|\mathbf{u}^i\| < A$ so that for any sequence $x_1, \cdots, x_{k+1}$ of elements of $K_m \cap L$ with $x_{k+1} = x_1$ we have $\sum_{j=1}^{k} \langle \mathbf{u}^j, x_{j+1} - x_j \rangle \geq 0$;
(iv) For every m there exist numbers u_m^i $(i = 1, \cdots, N(m))$, and vectors $\mathbf{u_m}^i \in R^l$ $(i = 1, \cdots, N(m))$ satisfying condition E, such that 1) lim $u_m^i = 1$ if lim x_m^i is maximal with respect to $\succeq$ in Ω, 2) lim $u_m^i = 0$ if lim x_m^i is minimal with respect to $\succeq$ in Ω, and 3) if $x \in \Omega$ is non-maximal then

$$\overline{\lim}_{\lim\, x_m^i = x} u_m^i < 1$$

.

Condition G is qualitative, and it is not clear whether one may formulate a quantitative, uniform (in m) version equivalent to concavifiability of $\succeq$ in Ω.

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Walrasian versus quasi-competitive equilibrium and the core of a production economy

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Summary. This paper presents very general conditions guaranteeing that a quasi-competitive equilibrium is a Walrasian equilibrium. We also develop a generalization (and a simplified proof) of Nikaido's and McKenzie's extensions of the classic Debreu-Scarf theorem on core convergence, and apply the first result to obtain an equivalence between the set of Edgeworth equilibria and the set of Walrasian equilibria in a production economy.

Keywords and Phrases: Core convergence, Indecomposability, Irreducibility, Quasi-competitive and Walrasian equilibrium.

JEL Classification Numbers: C71, D50, D51.

1 Introduction

There are numerous results in the general equilibrium and welfare economics literature where investigators have been looking for conditions sufficient to establish either the existence of a Walrasian (competitive) equilibrium, or the equivalence between a normative criterion and a Walrasian equilibrium; but where what is actually established is the existence or equivalence of a quasi-competitive equilibrium. In particular, the most general versions of the 'Second Fundamental Theorem of Welfare Economics' assert that, with appropriate convexity conditions, any Pareto efficient allocation can be supported as a quasi-competitive equilbrium.[1] Similarly, core convergence results typically establish convergence of the core in, say, replica economies, to a subset of the set of quasi-competitive equilibria. Moreover,

[1] A notable exception to this is the recent paper by Hurwicz and Richter (2001). However, their approach is quite different from mine, and a discussion of the relationship between their approach and mine will have to await a later work.

while authors very often state simple conditions sufficient to ensure that the quasi-competitive equilibrium obtained will actually be a Walrasian equilbrium, these simple conditions are typically patently unrealistic. For example, it is very often assumed in an exchange economy setting that each consumer has a strictly positive endowment of each commodity in the commodity space.

My primary objective in this article has been to develop a more realistic and acceptable condition which will imply that a known quasi-competitive equilibrium is actually a Walrasian equilibrium. Several such conditions, the most general of which I am calling 'indecomposability,' are presented in Section 4 of this paper; and it is there established that if an economy satisfies indecomposability, together with a very general production condition, then any quasi-competitive equilibrium for the economy will be a Walrasian equilibrium. This result can, of course, be combined with many already-published results to obtain a new and stronger conclusion. However, I had particularly wanted to apply it to establish the equality between what Aliprantis et al. (1987a,b) call the set of 'Edgeworth allocations' and the set of Walrasian allocations, in a finite economy setting. In looking for a convenient such result to use in such an application, I realized that an argument very like a combination of the proofs of the well-known results of Nikaido (1968) and McKenzie (1988), which show that the set of Edgeworth allocations is a subset of the set of quasi-competitive allocations, could establish a generalization of their results. This theorem, which is presented in Section 3, generalizes their results by allowing for production (which McKenzie's result does not), dropping all transitivity requirement on preferences (Nikaido assumes that preferences are continuous weak orders), and assuming only that preferences are weakly convex, lower semi-continuous, and locally non-saturating. The proof of this result is also considerably shorter and more transparent than Nikaido's proof, and the theorem can be, and is combined with the indecomposability material to establish sufficient conditions for the equality of the set of Edgeworth allocations and the set of Walrasian allocations.

Notation and concepts regarding the core are presented in the next section.

2 Notational framework

We will be dealing with a private ownership economy, $\mathcal{E} = (\langle X_i, P_i\rangle, \langle Y_k\rangle, \langle \boldsymbol{r}_i\rangle, [s_{ik}])$, where X_i $(i = 1, \ldots, m)$ and Y_k $(k = 1, \ldots, \ell)$ are non-empty subsets of $\mathbb{R}^n$; P_i is an irreflexive binary relation on X_i and $\boldsymbol{r}_i \in \mathbb{R}^n$, for each $i \in M$; while $s_{ik} \in \mathbb{R}_+$ for $i = 1, \ldots, m, k = 1, \ldots, \ell$ and for each $k \in L$:

$$\sum\nolimits_{i \in M} s_{ik} = 1,$$

where we define $M = \{1, \ldots, m\}$ and $L = \{1, \ldots, \ell\}$; and when we say the $\mathcal{E}$ is an economy, we will mean that $\mathcal{E}$ is a tuple of this form.

For an economy, $\mathcal{E}$ we use the generic notation '$(\langle \boldsymbol{x}_i\rangle_{i\in M}, \langle \boldsymbol{y}_k\rangle_{k\in L})$, $(\langle \boldsymbol{x}_i^*\rangle_{i\in M}, \langle \boldsymbol{y}_k^*\rangle_{k\in L})$,' etc., to denote allocations fo $\mathcal{E}$; and '$\langle \boldsymbol{x}_i\rangle_{i\in M}, \langle \boldsymbol{x}_i^*\rangle_{i\in M}$,' etc., to denote consumption allocations in $\boldsymbol{X}(\mathcal{E}) \stackrel{\text{def}}{=} \prod_{i\in M} X_i$.

2.1 Definitions. If $\mathcal{E}$ is an economy, we will say that $\big(\langle \boldsymbol{x}_i\rangle_{i\in M}, \langle \boldsymbol{y}_k\rangle_{k\in L}\big)$ is a *feasible* (or *attainable*) *allocation for $\mathcal{E}$* iff:

$$\boldsymbol{x}_i \in X_i \text{ for } i = 1, \dots, m \quad \text{and} \quad \boldsymbol{y}_k \in Y_k \text{ for } k = 1, \dots, \ell;$$

and:

$$\sum\nolimits_{i\in M} \boldsymbol{x}_i = \sum\nolimits_{i\in M} \boldsymbol{r}_i + \sum\nolimits_{k\in L} \boldsymbol{y}_k. \tag{1}$$

We will denote the set of all such allocations by '$A(\mathcal{E})$,' that is:

$$A(\mathcal{E}) = \Big\{\big(\langle \boldsymbol{x}_i\rangle_{i\in M}, \langle \boldsymbol{y}_k\rangle_{k\in L}\big) \in \mathbb{R}^{mn+\ell n} \mid \sum\nolimits_{i\in M} \boldsymbol{x}_i = \sum\nolimits_{i\in M} \boldsymbol{r}_i + \sum\nolimits_{k\in L} \boldsymbol{y}_k\Big\}. \tag{2}$$

We will say that $\langle \boldsymbol{x}_i\rangle_{i\in M}$ is an attainable consumption allocation for $\mathcal{E}$ iff there exists $\langle \boldsymbol{y}_k\rangle_{k\in L}$ such that $\big(\langle \boldsymbol{x}_i\rangle_{i\in M}, \langle \boldsymbol{y}_k\rangle_{k\in L}\big) \in A(\mathcal{E})$, and we will denote the set of all attainable consumption allocations for $\mathcal{E}$ by '$\boldsymbol{X}^*(\mathcal{E})$,' or simply by '$\boldsymbol{X}^*$.'

Further bits of notation are the following. We define the sets Π_k and Π by:

$$\Pi_k = \{\boldsymbol{p} \in \mathbb{R}^n \mid (\exists \boldsymbol{y}^* \in Y_k)(\forall \boldsymbol{y} \in Y_k)\colon \boldsymbol{p}\cdot\boldsymbol{y}^* \geq \boldsymbol{p}\cdot\boldsymbol{y}\} \text{ for } k = 1, \dots, \ell;$$

and:

$$\Pi = \bigcap\nolimits_{k\in L} \Pi_k,$$

respectively, and the functions $\pi_k\colon \Pi_k \to \mathbb{R}$ by:

$$\pi_k(\boldsymbol{p}) = \max_{\boldsymbol{y}\in Y_k} \boldsymbol{p}\cdot\boldsymbol{y} \quad \text{for } k \in L.$$

We will also need a couple of fairly standard definitions, as well as a 'well-known' proposition, as follows.

2.2 Definitions. If P_i is an irreflexive binary relation on the non-empty set $X_i \subseteq \mathbb{R}^n$, we shall say that P_i is:

1. *weakly convex* iff X_i is a convex set, and, for each $\boldsymbol{x}_i^* \in X_i$, the set:

$$P_i\boldsymbol{x}_i^* = \{\boldsymbol{x}_i \in X_i \mid \boldsymbol{x}_i P_i \boldsymbol{x}_i^*\},$$

   is convex.
2. *lower semi-continuous* iff, for each $\boldsymbol{x}_i^* \in X_i$ and each $\boldsymbol{x}_i' \in P_i\boldsymbol{x}_i^*$, there exists a neighborhood of $\boldsymbol{x}_i'$, $N(\boldsymbol{x}_i')$, such that $\boldsymbol{x}_i P_i \boldsymbol{x}_i^*$, for all $\boldsymbol{x}_i \in N(\boldsymbol{x}_i')$.

2.3 Proposition. *If P_i is a lower semi-continuous binary relation on a convex set, $X_i \subseteq \mathbb{R}^n$, and $\boldsymbol{x}_i^* \in X_i$, $\boldsymbol{p}^* \in \mathbb{R}^n$, and $w_i^* \in \mathbb{R}$ satisfy:*

$$w_i^* > \min \boldsymbol{p}^*\cdot\boldsymbol{X}_i \overset{def}{=} \min_{\boldsymbol{x}_i\in X_i} \boldsymbol{p}^*\cdot\boldsymbol{x}_i \text{ and } (\forall \boldsymbol{x}_i \in X_i)\colon \boldsymbol{x}_i P_i \boldsymbol{x}_i^* \Rightarrow \boldsymbol{p}^*\cdot\boldsymbol{x}_i \geq w_i^*,$$

then:

$$(\forall \boldsymbol{x}_i \in X_i)\colon \boldsymbol{x}_i P_i \boldsymbol{x}_i^* \Rightarrow \boldsymbol{p}^*\cdot\boldsymbol{x}_i > w_i^*.$$

We will be considering possible actions of coalitions of consumers, where a coalition of consumers can be identified with a subset, S, of M; the collection of all such coalitions, that is, the collection of all non-empty subsets of M, will be denoted by '$\boldsymbol{\mathcal{S}}$.'

In order to define the production possibilities available to a coalition, $S \subseteq M$, we begin by defining the sets Z_{ik}, for $(i,k) \in M \times L$, by:

$$Z_{ik} = s_{ik}Y_k \stackrel{\text{def}}{=} \{\boldsymbol{z} \in \mathbb{R}^n \mid (\exists \boldsymbol{y}_k \in Y_k) \colon \boldsymbol{z} = s_{ik}\boldsymbol{y}_k\}. \tag{3}$$

We then define the set $\varPi_{ik}$ and the function $\widehat{\pi}_{ik} \colon \varPi_{ik} \to \mathbb{R}$ by:

$$\varPi_{ik} = \{\boldsymbol{p} \in \mathbb{R}^n \mid (\exists \boldsymbol{z}^* \in Z_{ik})(\forall \boldsymbol{z} \in Z_{ik}) \colon \boldsymbol{p} \cdot \boldsymbol{z}^* \geq \boldsymbol{p} \cdot \boldsymbol{z}\}$$
$$\text{and } \widehat{\pi}_{ik}(\boldsymbol{p}) = \max_{\boldsymbol{z} \in Z_{ik}} \boldsymbol{p} \cdot \boldsymbol{z}.$$

Finally, we define the i^{th} consumer's production set, Z_i, as:

$$Z_i = \sum\nolimits_{k \in L} Z_{ik} = \sum\nolimits_{k \in L} s_{ik}Y_k. \tag{4}$$

With these definitions, it is easy to prove the following two propositions.

2.4 Proposition. *Let $\mathcal{E}$ be an economy, and $\boldsymbol{p}^* \in \mathbb{R}^n$. Then:*

1. *if $\boldsymbol{y}_k^*$ maximizes $\boldsymbol{p}^* \cdot \boldsymbol{y}$ on Y_k, then $\boldsymbol{z}_{ik}^* \stackrel{def}{=} s_{ik}\boldsymbol{y}_k^*$ maximizes $\boldsymbol{p}^* \cdot \boldsymbol{z}$ on Z_{ik}; and:*
2. *for all $(i,k) \in M \times L$:, $\varPi_k \subseteq \varPi_{ik}$ and for any $\boldsymbol{p} \in \varPi_k$:*

$$\widehat{\pi}_{ik}(\boldsymbol{p}) = s_{ik}\pi_k(\boldsymbol{p}), \tag{5}$$

3. *and, for all $i \in M$, and for any $\boldsymbol{z}_i \in Z_i$ and any $\boldsymbol{p} \in \varPi$:*

$$\boldsymbol{p} \cdot \boldsymbol{z}_i \leq \sum\nolimits_{k \in L} s_{ik}\pi_k(\boldsymbol{p}). \tag{6}$$

2.5 Proposition. *If $\mathcal{E}$ is an economy, then, given the definitions of this section:*

$$Y \equiv \sum_{k \in L} Y_k \subseteq \sum_{i \in M} Z_i. \tag{7}$$

Moreover, if Y_k is convex and contains **0**, *for each $k \in L$, then for each $S \in \boldsymbol{\mathcal{S}}$ we have:*

$$\sum_{i \in S} Z_i \subseteq Y. \tag{8}$$

In the remainder of this section we will state a number of fairly standard and familiar definitions. *We will assume throughout this discussion that the following condition holds:*

$$X_i \cap [\boldsymbol{r}_i + Z_i] \neq \emptyset \quad \text{for } i = 1, \ldots, m; \tag{9}$$

in other words, for each $i \in M$, we suppose that there exist $\bar{\boldsymbol{x}}_i \in X_i$ and $\bar{\boldsymbol{z}}_i \in Z_i$ such that:

$$\bar{\boldsymbol{x}}_i = \boldsymbol{r}_i + \bar{\boldsymbol{z}}_i. \tag{10}$$

The assumption expressed as equation (9) is fairly restrictive; in a modern industrialized society, individuals specialize in the expectation of being able to purchase (or trade for) necessities which they themselves do not produce. On the other hand, similar conditions are used throughout the literature, and it greatly simplifies our analysis, by making the following definition apply to any $S \in \boldsymbol{\mathcal{S}}$.[2]

2.6 Definition. We will say that $\langle(\boldsymbol{x}_i, \boldsymbol{z}_i)\rangle_{i \in S}$ is *attainable for the coalition* $S \in \boldsymbol{\mathcal{S}}$ iff:

$$\boldsymbol{x}_i \in X_i \text{ and } \boldsymbol{z}_i \in Z_i \text{ for all } i \in S, \tag{11}$$

and:

$$\sum\nolimits_{i \in S} \boldsymbol{x}_i = \sum\nolimits_{i \in S} \boldsymbol{r}_i + \sum\nolimits_{i \in S} \boldsymbol{z}_i. \tag{12}$$

2.7 Definition. Let $\langle \boldsymbol{x}_i^* \rangle_{i \in M}$ be a consumption allocation for $\mathcal{E}$, and let $S \in \boldsymbol{\mathcal{S}}$ be a coalition. We shall say that $\langle \boldsymbol{x}_i^* \rangle_{i \in M}$ *can be improved upon by the coalition* $\boldsymbol{S}$ iff there exists an allocation, $\langle(\boldsymbol{x}_i, \boldsymbol{z}_i)\rangle_{i \in S}$, which is attainable for S, and satisfies:

$$(\forall i \in S)\colon \boldsymbol{x}_i P_i \boldsymbol{x}_i^*. \tag{13}$$

2.8 Definition. The *core of an economy* $\mathcal{E}$ is defined as the set of all attainable consumption allocations for $\mathcal{E}$ which cannot by improved upon by any coalition, $S \in \boldsymbol{\mathcal{S}}$. We shall denote the set of all core allocations for $\mathcal{E}$ by '$\boldsymbol{C}(\mathcal{E})$.'

2.9 Definition. An $(m + \ell + 1)n$-tuple, $(\langle \boldsymbol{x}_i^* \rangle, \langle \boldsymbol{y}_k^* \rangle, \boldsymbol{p}^*)$ is a *Walrasian* (or *competitive*) *equilibrium for the economy* $\boldsymbol{\mathcal{E}}$ iff:

1. $\boldsymbol{p}^* \neq \boldsymbol{0}$,
2. $\big(\langle \boldsymbol{x}_i^* \rangle, \langle \boldsymbol{y}_k^* \rangle\big) \in A(\mathcal{E})$,
3. for each k ($k = 1, \ldots, \ell$), we have: $\boldsymbol{p}^* \cdot \boldsymbol{y}_k^* = \pi_k(\boldsymbol{p}^*)$, and
4. for each i ($i = 1, \ldots, m$), we have:
   a. $\boldsymbol{p}^* \cdot \boldsymbol{x}_i^* \leq w_i(\boldsymbol{p}^*) \stackrel{\text{def}}{=} \boldsymbol{p}^* \cdot \boldsymbol{r}_i + \sum_{k \in L} s_{ik}\pi_{ik}(\boldsymbol{p}^*)$, and:
   b. $(\forall \boldsymbol{x}_i \in X_i)\colon \boldsymbol{x}_i P_i \boldsymbol{x}_i^* \Rightarrow \boldsymbol{p}^* \cdot \boldsymbol{x}_i > w_i(\boldsymbol{p}^*)$.

2.10 Definitions. We define the set of all **Walrasian allocations for $\boldsymbol{\mathcal{E}}$, $\boldsymbol{\mathcal{W}}(\mathcal{E})$**, by:

$$\boldsymbol{\mathcal{W}}(\mathcal{E}) = \big\{\big(\langle \boldsymbol{x}_i^* \rangle, \langle \boldsymbol{y}_k^* \rangle\big) \in A(\mathcal{E}) \mid \big(\exists \boldsymbol{p}^* \in \mathbb{R}^n\big)\colon \\ (\langle \boldsymbol{x}_i^* \rangle, \langle \boldsymbol{y}_k^* \rangle, \boldsymbol{p}^*) \text{ is a Walrasian equilibrium for } \mathcal{E}\big\}$$

We then define the set of *Walrasian consumption allocations for* $\boldsymbol{\mathcal{E}}$, $\boldsymbol{W}(\mathcal{E})$, by:

$$\boldsymbol{W}(\mathcal{E}) = \big\{\langle \boldsymbol{x}_i^* \rangle_{i \in M} \in \boldsymbol{X}^*(\mathcal{E}) \mid (\exists \langle \boldsymbol{y}_k^* \rangle_{k \in L})\colon \big(\langle \boldsymbol{x}_i^* \rangle, \langle \boldsymbol{y}_k^* \rangle\big) \in \boldsymbol{\mathcal{W}}(\mathcal{E})\big\}.$$

[2] It should be noted that if the set of *viable* coalitions, $\boldsymbol{\mathcal{S}}^*$, is a proper subset of $\boldsymbol{\mathcal{S}}$, then Theorem 3.1, below, remains essentially intact.

3 The core in replicated economies

Given an economy, $\mathcal{E} = (\langle X_i, P_i\rangle, \langle Y_k\rangle, \langle \boldsymbol{r}_i\rangle, [s_{ik}])$, we consider the sequence of related economies, $\mathcal{E}_q$, defined in the following way.

$$\begin{aligned} \mathcal{E}_1 &= \mathcal{E}, \\ \dots & \quad \dots \\ \mathcal{E}_q &= \langle (X_{hi}, P_{hi}, \boldsymbol{r}_{hi}, Z_{hi})\rangle_{(h,i)\in, Q\times M}, \text{ where :} \\ Q &= \{1\dots, q\}, X_{hi} = X_i, P_{hi} = P_i,\ \boldsymbol{r}_{hi} = \boldsymbol{r}_i \\ \text{and } Z_{hi} &= Z_i \ \text{ for } h = 1,\dots,q; i = 1,\dots,m. \end{aligned}$$

Thus, in $\mathcal{E}_q$, the agents (consumers) have a double index; agent (h, i) is the h^{th} agent of the i^{th} type; and each agent of the i^{th} type has the same economic characteristics as does the i^{th} agent in the original economy. We will refer to $\mathcal{E}_q$ as the *q-fold replication of* $\boldsymbol{\mathcal{E}}$. In dealing with $\mathcal{E}_q$, we will use the notation '$\langle(\boldsymbol{x}_{hi}, \boldsymbol{z}_{hi})\rangle_{(h,i)\in Q\times M}$' to denote allocations for $\mathcal{E}_q$

We will follow the basic approach introduced by Debreu and Scarf (1963) in considering the sets $\boldsymbol{C}_q$, defined as the set of all feasible allocations, $\langle \boldsymbol{x}_i\rangle_{i\in M} \in \boldsymbol{X}^*(\mathcal{E})$ such that the allocation $\langle \boldsymbol{x}_{hi}\rangle_{(h,i)\in Q\times M}$ given by:

$$\boldsymbol{x}_{hi} = \boldsymbol{x}_i \quad \text{for } h = 1,\dots,q; i = 1,\dots,m; \tag{14}$$

is in $\boldsymbol{C}(\mathcal{E}_q)$. The following result then generalizes a 'well-known' version of the 'First Fundamental Theorem of Welfare Economics.' Moreover, it establishes the fact that if $\boldsymbol{W}(\mathcal{E}) \neq \emptyset$, then $\boldsymbol{C}_q \neq \emptyset$, for $q = 1, 2, \dots$. I have stated it here without proof, since it can be proved by fairly standard arguments.

3.1 Theorem. *For any economy, $\mathcal{E}$, we have:*

1. $\boldsymbol{W}(\mathcal{E}) \subseteq \boldsymbol{C}_q$, and
2. $\boldsymbol{C}_{q+1} \subseteq \boldsymbol{C}_q$,
for $q = 1, 2, \dots$.

Debreu and Scarf (1963) showed that given any exchange economy, $\mathcal{E} = \langle(P_i, \boldsymbol{r}_i)\rangle_{i\in M}$, satisfying certain assumptions, we will have:

$$\bigcap_{q=1}^{\infty} \boldsymbol{C}_q \subseteq \boldsymbol{W}(\mathcal{E});$$

which, given Theorem 3.1, means that under the Debreu-Scarf conditions, we have:

$$\lim_{q\to\infty} \boldsymbol{C}_q \overset{\text{def}}{=} \bigcap_{q=1}^{\infty} \boldsymbol{C}_q = \boldsymbol{W}(\mathcal{E}). \tag{15}$$

We will prove a generalization of their result; one which applies to a private ownership economy with production, and which dispenses with most of their assumptions regarding consumer preferences. However, we will begin by introducing the idea of a 'quasi-competitive equilibrium,' and proving that an Edgeworth allocation is a quasi-competitive equilibrium allocation. We follow Aliprantis et. al. (1987a,b) in defining the set of *Edgeworth Allocations for* $\boldsymbol{\mathcal{E}}$, $\boldsymbol{X}^E(\mathcal{E})$, by:

$$\boldsymbol{X}^E(\mathcal{E}) = \bigcap\nolimits_{q=1}^{\infty} C_q. \tag{16}$$

3.2 Definition. We shall say that $(\langle \boldsymbol{x}_i^* \rangle, \langle \boldsymbol{y}_k^* \rangle, \boldsymbol{p}^*)$ is a *quasi-competitive equilibrium for the economy* $\mathcal{E}$, iff $(\langle \boldsymbol{x}_i^* \rangle, \langle \boldsymbol{y}_k^* \rangle, \boldsymbol{p}^*)$ satisfies conditions 1–3 of Definition 2.9, and:

$4'$. for each $i \in M$, we have:

a. $\boldsymbol{p}^* \cdot \boldsymbol{x}_i^* \leq w_i(\boldsymbol{p}^*) \equiv \boldsymbol{p}^* \cdot \boldsymbol{r}_i + \sum_{k \in L} s_{ik}\pi(\boldsymbol{p}^*)$, and:

b. either:

$$w_i(\boldsymbol{p}^*) = \min \boldsymbol{p}^* \cdot X_i,$$

or:

$$(\forall \boldsymbol{x}_i \in X_i)\colon \boldsymbol{x}_i P_i \boldsymbol{x}_i^* \Rightarrow \boldsymbol{p}^* \cdot \boldsymbol{x}_i > w_i(\boldsymbol{p}^*).$$

We will denote the set of all consumption allocations, $\langle \boldsymbol{x}_i^* \rangle \in \boldsymbol{X}^*(\mathcal{E})$, for which there exists a production allocation $\langle \boldsymbol{y}_k^* \rangle_{k \in L}$ and a price vector $\boldsymbol{p}^*$ such that $(\langle \boldsymbol{x}_i^* \rangle, \langle \boldsymbol{y}_k^* \rangle, \boldsymbol{p}^*)$ is a quasi-competitive equilibrium for $\mathcal{E}$ by '$\boldsymbol{W}^\dagger(\mathcal{E})$.'

In our initial result, we will establish conditions sufficient to ensure that $\boldsymbol{X}^E(\mathcal{E}) \subseteq \boldsymbol{W}^\dagger(\mathcal{E})$. In our proof, which owes a great deal to McKenzie (1988) and Nikaido (1968, Theorem 17.4, p. 291), we will need to make use of the following mathematical result; the proof of which is omitted, since it is fairly 'well-known.'

3.3 Proposition. *If* $C_i \subseteq \mathbb{R}^n$ *is convex and non-empty, for* $i = 1, \ldots, m$, *then the convex hull of* $C \stackrel{def}{=} \bigcup_{i=1}^m C_i$, $co(C)$, *is given by:*

$$co(C) = \Big\{\boldsymbol{x} \in \mathbb{R}^n \mid (\exists \boldsymbol{a} \in \Delta_m \,\&\, \boldsymbol{x}_i \in C_i, \text{ for } i{=}1,\ldots,m)\colon \boldsymbol{x} = \sum\nolimits_{i=1}^m a_i \boldsymbol{x}_i\Big\}; \tag{17}$$

where we denote the standard unit simplex in $\mathbb{R}^m$ *by '*Δ_m*,' that is:*

$$\Delta_m = \Big\{\boldsymbol{a} \in \mathbb{R}_+^m \mid \sum\nolimits_{i=1}^m a_i = 1\Big\}. \tag{18}$$

While the above result may seem obvious, it should be noted that the conclusion no longer holds if the C_i's are not all convex; that is, the convex hull of C is not generally given by the formula in equation (17) unless the sets C_i are all convex.

3.4 Theorem. *If* $\mathcal{E}$ *is an economy such that:*

1. Y_k *is convex, for* $k = 1, \ldots, \ell$;
   and, for each $i \in M$:
2. P_i *is locally non-saturating, lower semi-continuous, and weakly convex, and:*

3. $X_i \cap [\boldsymbol{r}_i + Z_i] \neq \emptyset$,
then:

$$\boldsymbol{X}^E(\mathcal{E}) \equiv \bigcap_{q=1}^{\infty} \boldsymbol{C}_q \subseteq \boldsymbol{W}^{\dagger}(\mathcal{E}).$$

Proof. Suppose $\langle \boldsymbol{x}_i^* \rangle_{i \in M} \in \boldsymbol{C}_q$ for all q, define $\mathbb{P}_i = P_i \boldsymbol{x}_i^* - \boldsymbol{r}_i - Z_i$, for each $i \in M$,[3] and:

$$\mathbb{P} = co\Big(\bigcup_{i=1}^{m} \mathbb{P}_i\Big); \tag{19}$$

that is, $\mathbb{P}$ is the convex hull of the union of the $\mathbb{P}_i$'s. The tricky part of the proof is to establish the fact that $\mathbf{0} \notin \mathbb{P}$.

Suppose, by way of establishing a contradiction, that $\mathbf{0} \in \mathbb{P}$. Then, since each $\mathbb{P}_i$ is a convex set (and non-empty, by the assumption that each P_i is locally non-saturating), it follows from Proposition 3.3 that there exist $\boldsymbol{a} \in \Delta_m$, $\boldsymbol{x}_i \in X_i$, and $\boldsymbol{z}_i \in Z_i$ for $i = 1, \ldots, m$, such that:

$$\sum_{i=1}^{m} a_i(\boldsymbol{x}_i - \boldsymbol{r}_i - \boldsymbol{z}_i) = \mathbf{0}, \tag{20}$$

and:

$$\boldsymbol{x}_i P_i \boldsymbol{x}_i^* \quad \text{for } i = 1, \ldots, m. \tag{21}$$

We will show that these two conditions allow us to construct a coalition in $\mathcal{E}_{q^*}$, for some (finite) integer, q^*, which can improve upon $\langle \boldsymbol{x}_i^* \rangle_{i \in M}$; contradicting the assumption that $\langle \boldsymbol{x}_i^* \rangle_{i \in M} \in \boldsymbol{C}_q$, for all q.

Accordingly, we begin by noting that (20) implies:

$$\sum_{i=1}^{m} a_i(\boldsymbol{x}_i - \boldsymbol{r}_i) = \sum_{i=1}^{m} a_i \boldsymbol{z}_i. \tag{22}$$

We then define $I = \{i \in M \mid a_i > 0\}$, and, for each $i \in I$ and each positive integer, q, we let b_i^q be the smallest integer greater than or equal to qa_i. Now, by assumption 3, for each $i \in I$ there exist $\widehat{\boldsymbol{x}}_i \in X_i$ and $\widehat{\boldsymbol{z}}_i \in Z_i$ such that:

$$\widehat{\boldsymbol{x}}_i = \boldsymbol{r}_i + \widehat{\boldsymbol{z}}_i. \tag{23}$$

We make use of the $\widehat{\boldsymbol{x}}_i$ to define, for each $i \in I$ and each positive integer, q:

$$\boldsymbol{x}_i^q = \Big(\frac{qa_i}{b_i^q}\Big)\boldsymbol{x}_i + \Big[1 - \Big(\frac{qa_i}{b_i^q}\Big)\Big]\widehat{\boldsymbol{x}}_i; \tag{24}$$

and note that, since each P_i is lower semi-continuous, and since:

$$\frac{qa_i}{b_i^q} \to 1 \text{ as } q \to \infty,$$

[3] That is:

$$\mathbb{P}_i = \big\{\boldsymbol{v} \in \mathbb{R}^n \mid (\exists \boldsymbol{x}_i \in X_i \;\&\; \boldsymbol{z}_i \in Z_i) \colon \boldsymbol{x}_i P_i \boldsymbol{x}_i^* \;\&\; \boldsymbol{v} = \boldsymbol{x}_i - \boldsymbol{r}_i - \boldsymbol{z}_i\big\}.$$

it follows from (21) that for each $i \in I$, there exists a positive integer, q_i such that for all $q \geq q_i$,

$$\boldsymbol{x}_i^q P_i \boldsymbol{x}_i^*. \tag{25}$$

But now let:

$$q^* = \max_{i \in I} q_i,$$

let $b^* = \max\{b_1^{q^*}, \ldots, b_m^{q^*}\}$ and consider the coalition, S, in $\mathcal{E}_{b^*}$ consisting of $b_i^{q^*}$ consumers of each type $i \in I$, and the allocation $\langle \bar{\boldsymbol{x}}_{hi} \rangle_{(h,i) \in S}$ defined by:

$$\bar{\boldsymbol{x}}_{hi} = \boldsymbol{x}_i^{q^*} \quad \text{for } h = 1, \ldots, b_i^{q^*}, \text{ and each } i \in I. \tag{26}$$

We have $\bar{\boldsymbol{x}}_{hi} P_{hi} \boldsymbol{x}_i^*$ for each h and each $i \in I$; while by using (24), (23) and (22) in turn, we have:

$$\begin{aligned}
\sum\nolimits_{i \in I} \sum\nolimits_{h=1}^{b_i^{q^*}} \bar{\boldsymbol{x}}_{hi} &= \sum\nolimits_{i \in I} b_i^{q^*} \boldsymbol{x}_i^{q^*} = \sum\nolimits_{i \in I} \left[(q^* a_i) \boldsymbol{x}_i + (b_i^{q^*} - q^* a_i) \widehat{\boldsymbol{x}}_i \right] \\
&= q^* \Big(\sum\nolimits_{i \in I} a_i (\boldsymbol{x}_i - \boldsymbol{r}_i) \Big) - q^* \sum\nolimits_{i \in I} a_i \widehat{\boldsymbol{z}}_i + \sum\nolimits_{i \in I} b_i^{q^*} (\boldsymbol{r}_i + \widehat{\boldsymbol{z}}_i) \\
&= q^* \Big(\sum\nolimits_{i \in I} a_i \boldsymbol{z}_i \Big) - q^* \sum\nolimits_{i \in I} a_i \widehat{\boldsymbol{z}}_i + \sum\nolimits_{i \in I} b_i^{q^*} (\boldsymbol{r}_i + \widehat{\boldsymbol{z}}_i) \\
&= \sum\nolimits_{i \in I} b_i^{q^*} \boldsymbol{r}_i + \sum\nolimits_{i \in I} b_i^{q^*} \left[\Big(\frac{q^* a_i}{b_i^{q^*}} \Big) \boldsymbol{z}_i + \widehat{\boldsymbol{z}}_i - \Big(\frac{q^* a_i}{b_i^{q^*}} \Big) \widehat{\boldsymbol{z}}_i \right].
\end{aligned}$$

Thus, since each Z_i is convex, it follows that the coalition S can improve upon $\langle \boldsymbol{x}_i^* \rangle_{i \in M}$; contradicting the assumption that $\langle \boldsymbol{x}_i^* \rangle_{i \in M} \in C_q$ for all positive integers, q. Therefore $\mathbf{0} \notin \mathbb{P}$.

Since we have now established the fact that $\mathbf{0} \notin \mathbb{P}$, it follows from the 'Separating Hyperplane Theorem' that there exists a non-zero $\boldsymbol{p}^* \in \mathbb{R}^n$ satisfying:

$$(\forall \boldsymbol{v} \in \mathbb{P}) \colon \boldsymbol{p}^* \cdot \boldsymbol{v} \geq 0. \tag{27}$$

From the definition of $\mathbb{P}$, it then follows immediately that for each $i \in M$, we have:

$$(\forall \boldsymbol{x}_i \in X_i \;\&\; \boldsymbol{z}_i \in Z_i) \colon \boldsymbol{x}_i P_i \boldsymbol{x}_i^* \Rightarrow \boldsymbol{p}^* \cdot \boldsymbol{x}_i \geq \boldsymbol{p}^* \cdot (\boldsymbol{r}_i + \boldsymbol{z}_i); \tag{28}$$

and since P_i is locally non-saturating it then follows that, for each i and each $\boldsymbol{z}_i \in Z_i$:

$$\boldsymbol{p}^* \cdot \boldsymbol{x}_i^* \geq \boldsymbol{p}^* \cdot \boldsymbol{r}_i + \boldsymbol{p}^* \cdot \boldsymbol{z}_i. \tag{29}$$

Now, since $\langle \boldsymbol{x}_i^* \rangle_{i \in M} \in \boldsymbol{C}_q$, for each q, it follows from the definitions that there exists $\langle \boldsymbol{y}_k^* \rangle_{k \in L}$ such that:

$$\sum_{i \in M} \boldsymbol{x}_i^* = \sum_{i \in M} \boldsymbol{r}_i + \sum_{k \in L} \boldsymbol{y}_k^*. \tag{30}$$

Defining:

$$\boldsymbol{z}_i^* = \sum_{k \in L} s_{ik} \boldsymbol{y}_k^*, \tag{31}$$

we then have from (29) that:

$$\boldsymbol{p}^* \cdot (\boldsymbol{x}_i^* - \boldsymbol{r}_i - \boldsymbol{z}_i^*) \geq 0 \quad \text{for } i = 1, \ldots, m. \tag{32}$$

However, since $\sum_{i \in M} \boldsymbol{z}_i^* = \sum_{k \in L} \boldsymbol{y}_k^*$, it follows from (30) that:

$$\sum_{i \in M} \boldsymbol{p}^* \cdot (\boldsymbol{x}_i^* - \boldsymbol{r}_i - \boldsymbol{z}_i^*) = \boldsymbol{p}^* \cdot \Big(\sum_{i \in M} \boldsymbol{x}_i^* - \sum_{i \in M} \boldsymbol{r}_i - \sum_{k \in L} \boldsymbol{y}_k^* \Big) = 0;$$

and then, from (32) and our definitions we see that:

$$\boldsymbol{p}^* \cdot \boldsymbol{x}_i^* = \boldsymbol{p}^* \cdot \boldsymbol{r}_i + \boldsymbol{p}^* \cdot \boldsymbol{z}_i^* = \boldsymbol{p}^* \cdot \boldsymbol{r}_i + \sum_{k \in L} s_{ik} \boldsymbol{p}^* \cdot \boldsymbol{y}_k^* \quad \text{for } i = 1, \ldots, m. \tag{33}$$

Now, let $j \in L$ be arbitary, let $\boldsymbol{y}_j \in Y_j$, and define, for each i:

$$\boldsymbol{z}_i = \sum_{k \neq j} s_{ik} \boldsymbol{y}_k^* + s_{ij} \boldsymbol{y}_j. \tag{34}$$

Then we see that $\boldsymbol{z}_i \in Z_i$ for each i, so that by (29), we have:

$$\boldsymbol{p}^* \cdot \boldsymbol{x}_i^* \geq \boldsymbol{p}^* \cdot \boldsymbol{r}_i + \boldsymbol{p}^* \cdot \boldsymbol{z}_i \quad \text{for } i = 1, \ldots, m. \tag{35}$$

Adding the inequalities in (35), and making use of our definitions of the $\boldsymbol{z}_i$, we have:

$$\begin{aligned} \sum_{i \in M} \boldsymbol{p}^* \cdot \boldsymbol{x}_i^* = \boldsymbol{p}^* \cdot \sum_{i \in M} \boldsymbol{x}_i^* &\geq \boldsymbol{p}^* \cdot \sum_{i \in M} \boldsymbol{r}_i + \boldsymbol{p}^* \cdot \sum_{i \in M} \boldsymbol{z}_i \\ &= \boldsymbol{p}^* \cdot \sum_{i \in M} \boldsymbol{r}_i + \boldsymbol{p}^* \cdot \Big[\sum_{i \in M} \Big(\sum_{k \neq j} s_{ik} \boldsymbol{y}_k^* + s_{ij} \boldsymbol{y}_j \Big) \Big] \\ &= \boldsymbol{p}^* \cdot \sum_{i \in M} \boldsymbol{r}_i + \sum_{k \neq j} \boldsymbol{p}^* \cdot \boldsymbol{y}_k^* + \boldsymbol{p}^* \cdot \boldsymbol{y}_j. \end{aligned} \tag{36}$$

From (30) and (36) we then see that $\boldsymbol{p}^* \cdot \boldsymbol{y}_j^* \geq \boldsymbol{p}^* \cdot \boldsymbol{y}_j$, and thus $\boldsymbol{y}_k^*$ maximizes $\boldsymbol{p}^* \cdot \boldsymbol{y}_k$ on Y_k, for $k = 1, \ldots, \ell$.

Finally, let $i \in M$ be arbitrary. Then from (33) and the conclusion of the previous paragraph, we have:

$$\boldsymbol{p}^* \cdot \boldsymbol{x}_i^* = w_i(\boldsymbol{p}^*) \stackrel{\text{def}}{=} \boldsymbol{p}^* \cdot \boldsymbol{r}_i + \sum\nolimits_{k=1}^{\ell} s_{ik} \pi(\boldsymbol{p}^*).$$

Furthermore, it follows from (28) that:

$$(\forall \boldsymbol{x}_i \in X_i) \colon \boldsymbol{x}_i P_i \boldsymbol{x}_i^* \Rightarrow \boldsymbol{p}^* \cdot \boldsymbol{x}_i \geq w_i(\boldsymbol{p}^*);$$

and thus from Proposition 2.3, we see that either:

$$w_i(\boldsymbol{p}^*) = \min \boldsymbol{p}^* \cdot X_i,$$

or:

$$(\forall \boldsymbol{x}_i \in X_i)\colon \boldsymbol{x}_i P_i \boldsymbol{x}_i^* \Rightarrow \boldsymbol{p}^* \cdot \boldsymbol{x}_i > w_i(\boldsymbol{p}^*).$$

Therefore, $(\langle \boldsymbol{x}_i^* \rangle, \langle \boldsymbol{y}_k^* \rangle, \boldsymbol{p}^*)$ is a quasi-competitive equilibrium for $\mathcal{E}$. □

In the next section we will strengthen the conclusion of Theorem 3.4 to conclude that $\boldsymbol{X}^E(\mathcal{E}) = \boldsymbol{W}(\mathcal{E})$.

4 Quasi-competitive and Walrasian equilibrium

In this section we will be examining the implications of the following condition.

4.1 Definition. We say that $\mathcal{E}$ is *indecomposable at the allocation* $\big(\langle \boldsymbol{x}_i^* \rangle_{i\in M}, \langle \boldsymbol{y}_k^* \rangle_{k\in L}\big)$ iff, given any partition of the consumers into two groups, $\{S_1, S_2\}$,[4] there exists $(\widehat{\boldsymbol{x}}_i, \widehat{\boldsymbol{z}}_i) \in X_i \times Z_i$, for each $i \in M$, $\mu_i \in \mathbb{R}_+$ for each $i \in S_2$, and $\overline{\boldsymbol{y}} \in \mathbb{A}Y$,[5] such that:

$$\sum_{i\in S_1} (\widehat{\boldsymbol{x}}_i - \boldsymbol{r}_i - \widehat{\boldsymbol{z}}_i) = \sum_{i\in S_2} \mu_i(\boldsymbol{r}_i + \widehat{\boldsymbol{z}}_i - \widehat{\boldsymbol{x}}_i) + \overline{\boldsymbol{y}}, \tag{37}$$

and:

$$(\forall i \in S_1)\colon \widehat{\boldsymbol{x}}_i P_i \boldsymbol{x}_i^*.$$

We will say that $\boldsymbol{\mathcal{E}}$ *is indecomposable* (or that it is **globally indecomposable**) iff it is indecomposable at each attainable allocation, $\big(\langle \boldsymbol{x}_i^* \rangle_{i\in M}, \langle \boldsymbol{y}_k^* \rangle_{k\in L}\big) \in A(\mathcal{E})$.

This condition generalizes the 'irreducibility' condition introduced in McKenzie (1961), and which was generalized somewhat in Moore (1975, 1999). Notice that it says that, given any attainable allocation in the economy, and any coalition, $S_1 \neq M$, the coalition could improve upon the given allocation for each of its members if they were allowed to choose amounts to be given up by a coalition consisiting of replicas (possibly fractional) of the consumers not in S_1, and add in a production vector from $\mathbb{A}Y$. It is easily shown that the following condition, which is the irreducibility condition used in Moore (1999), implies the indecomposability condition just introduced.

4.2 Definition. We shall say that the economy, $\mathcal{E}$ is *irreducible at the consumption allocation* $\langle \boldsymbol{x}_i^* \rangle \in \boldsymbol{X}^*(\boldsymbol{\mathcal{E}})$ iff, given any partition of the consumers, $\{S_1, S_2\}$, there exists $\langle(\boldsymbol{x}_i, \boldsymbol{z}_i)\rangle_{i\in M}$ such that:

$$\begin{aligned} &\boldsymbol{x}_i \in X_i \;\&\; \boldsymbol{z}_i \in Z_i \quad \text{for } i = 1, \ldots, m, \\ &\sum_{i\in S_1} (\boldsymbol{x}_i - \boldsymbol{r}_i - \boldsymbol{z}_i) = \sum_{i\in S_2} (\boldsymbol{r}_i + \boldsymbol{z}_i - \boldsymbol{x}_i), \end{aligned} \tag{38}$$

[4] By a partition of the consumers, $\{S_1, S_2\}$, we mean $S_j \subseteq M$ & $S_j \neq \emptyset$, for $j = 1, 2$, $S_1 \cap S_2 = \emptyset$, and $S_1 \cup S_2 = M$.

[5] Where '$\mathbb{A}Y$' denotes the asymptotic cone of Y. See Debreu (1959, p. 22).

and:

$$(\forall i \in S_1)\colon \boldsymbol{x}_i P_i \boldsymbol{x}_i^*. \tag{39}$$

In particular, notice that this latter condition (and thus the indecomposability condition) is automatically satisfied in any economy, $\mathcal{E}$ at attainable allocations which are strongly Pareto-dominated by another feasible allocation for $\mathcal{E}$. In effect, the economy $\mathcal{E}$ is irreducible at $\langle \boldsymbol{x}_i^* \rangle \in \boldsymbol{X}^*(\mathcal{E})$ iff, given any partition of the consumers into two groups, S_1 and S_2, there is a feasible trade between the two groups which would make each of the consumers in S_1 better off than they are at $\langle \boldsymbol{x}_i^* \rangle$. Of course, this same trade may make each of the consumers in S_2 worse off than they are at $\langle \boldsymbol{x}_i^* \rangle$!

4.3 Theorem. *If* $(\langle \boldsymbol{x}_i^* \rangle, \langle \boldsymbol{y}_k^* \rangle, \boldsymbol{p}^*)$ *is a quasi-competitive equilibrium for the economy,* $\mathcal{E}$*, and:*

1. $\mathcal{E}$ *is indecomposable at* $\big(\langle \boldsymbol{x}_i^* \rangle, \langle \boldsymbol{y}_k^* \rangle\big)$*, and*
2. $int(X) \cap [\boldsymbol{r} + Y] \neq \emptyset$*,*
   Then $(\langle \boldsymbol{x}_i^* \rangle, \langle \boldsymbol{y}_k^* \rangle, \boldsymbol{p}^*)$ *is a Walrasian equilibrium for* $\mathcal{E}$*.*

Proof. From assumption 2 we see that there exists $\widehat{\boldsymbol{x}} \in X^*$ and $\theta \in \mathbb{R}_{++}$ such that:

$$\boldsymbol{x}^\dagger \stackrel{\text{def}}{=} \widehat{\boldsymbol{x}} - \theta \boldsymbol{p}^* \in X^*, \tag{40}$$

and thus:

$$\boldsymbol{p}^* \cdot \boldsymbol{x}^\dagger = \boldsymbol{p}^* \cdot [\widehat{\boldsymbol{x}} - \theta \boldsymbol{p}^*] = \boldsymbol{p}^* \cdot \widehat{\boldsymbol{x}} - \theta \boldsymbol{p}^* \cdot \boldsymbol{p}^* < \boldsymbol{p}^* \cdot \widehat{\boldsymbol{x}} \leq \boldsymbol{p}^* \cdot \boldsymbol{x}^*,$$

where the last inequality follows easily from the definition of a quasi-competitive equilibrium. Therefore, since our definition of attainable allocations implies that at a quasi-competitive equilibrium, each consumer's consumption expenditure must be equal to wealth:

$$(\exists i \in M)\colon w_i(\boldsymbol{p}^*) = \boldsymbol{p}^* \cdot \boldsymbol{x}_i^* > \min \boldsymbol{p}^* \cdot X_i. \tag{41}$$

Now, define $S_i \subseteq M \ (i = 1, 2)$ by:

$$S_1 = \{i \in M \mid w_i(\boldsymbol{p}^*) > \min \boldsymbol{p}^* \cdot X_i\},$$

and:

$$S_2 = \{i \in M \mid w_i(\boldsymbol{p}^*) = \min \boldsymbol{p}^* \cdot X_i\},$$

respectively. By (41), $S_1 \neq \emptyset$. Suppose by way of obtaining a contradiction, that $S_2 \neq \emptyset$ as well. Then by the indecomposability condition, there exists $\langle (\boldsymbol{x}_i, \boldsymbol{z}_i) \rangle_{i \in M}$, $\mu_i \geq 0$ for each $i \in S_2$, and $\overline{\boldsymbol{y}} \in \mathbb{A}Y$ such that:

$$\sum_{i \in S_1} (\boldsymbol{x}_i - \boldsymbol{r}_i - \boldsymbol{z}_i) = \sum_{i \in S_2} \mu_i (\boldsymbol{r}_i + \boldsymbol{z}_i - \boldsymbol{x}_i) + \overline{\boldsymbol{y}}, \tag{42}$$

and:

$$(\forall i \in S_1)\colon \boldsymbol{x}_i P_i \boldsymbol{x}_i^*. \tag{43}$$

However, by definition of S_1 and the fact that $(\langle \boldsymbol{x}_i^* \rangle, \langle \boldsymbol{y}_k^* \rangle, \boldsymbol{p}^*)$ is a quasi-competitive equilibrium, we have, for each $i \in S_1$:

$$\boldsymbol{p}^* \cdot \boldsymbol{x}_i > w_i(\boldsymbol{p}^*) = \boldsymbol{p}^* \cdot \boldsymbol{r}_i + \sum\nolimits_{k \in L} s_{ik} \boldsymbol{p}^* \cdot \boldsymbol{y}_k^*. \tag{44}$$

Moreover, it follows from profit maximization and Proposition 2.4.4 that for each $i \in M$:

$$\sum\nolimits_{k \in L} s_{ik} \boldsymbol{p}^* \cdot \boldsymbol{y}_k^* \geq \boldsymbol{p}^* \cdot \boldsymbol{z}_i, \tag{45}$$

where $\boldsymbol{z}_i \in Z_i$ is from (42). Thus, from (44) and (45) we see that, for each $i \in S_1$:

$$\boldsymbol{p}^* \cdot (\boldsymbol{x}_i - \boldsymbol{r}_i - \boldsymbol{z}_i) > 0;$$

so that, from (42):

$$\boldsymbol{p}^* \cdot \Big(\sum_{i \in S_2} \mu_i (\boldsymbol{r}_i + \boldsymbol{z}_i - \boldsymbol{x}_i) \Big) + \boldsymbol{p}^* \cdot \overline{\boldsymbol{y}} > 0.$$

However, since $\boldsymbol{p}^* \in \Pi$, it must be the case that $\boldsymbol{p}^* \cdot \overline{\boldsymbol{y}} \leq 0$. Consequently, it then follows that for at least one $i \in S_2$, we must have:

$$\boldsymbol{p}^* \cdot \boldsymbol{x}_i < \boldsymbol{p}^* \cdot \boldsymbol{r}_i + \boldsymbol{p}^* \cdot \boldsymbol{z}_i \leq \boldsymbol{p}^* \cdot \boldsymbol{r}_i + \sum\nolimits_{k=1}^{\ell} s_{ik} \boldsymbol{p}^* \cdot \boldsymbol{y}_k^* = w_i(\boldsymbol{p}^*);$$

which contradictions our definition of S_2. Therefore, $S_2 = \emptyset$, and thus $(\langle \boldsymbol{x}_i^* \rangle, \langle \boldsymbol{y}_k^* \rangle, \boldsymbol{p}^*)$ is a Walrasian (competitive) equilibrium for $\mathcal{E}$. □

We can now make use of this last theorem, together with Theorems 3.1 and 3.4 to deduce some results regarding core convergence. The first of these two results follows more or less immediately, and will be stated without formal proof.

4.4 Theorem. *If $\mathcal{E}$ is an indecomposable economy such that:*

1. *Y_k is convex, for $k = 1, \ldots, \ell$;*
2. *$int(X) \cap [\boldsymbol{r} + Y] \neq \emptyset$,*
   and, for each $i \in M$:
3. *P_i is locally non-saturating, weakly convex and lower semi-continuous, and:*
4. *$X_i \cap [\boldsymbol{r}_i + Z_i] \neq \emptyset$,*
   then:

$$\boldsymbol{X}^E(\mathcal{E}) \equiv \big[\bigcap\nolimits_{q=1}^{\infty} \boldsymbol{C}_q \big] = \boldsymbol{W}(\mathcal{E}).$$

Our final result makes use of the following definition.

4.5 Definitions. We will say that the j^{th} commodity is a **numéraire good for $\boldsymbol{P_i}$** iff for all $\boldsymbol{x} \in X_i$ and all $\theta \in \mathbb{R}_{++}$,[6] we have:

$$\boldsymbol{x} + \theta \boldsymbol{e}_j \in X_i \text{ and } (\boldsymbol{x} + \theta \boldsymbol{e}_j) P_i \boldsymbol{x}, \tag{46}$$

where $\boldsymbol{e}_j$ is the j^{th} unit coordinate vector.[7] We shall say that the j^{th} commodity is a *numéraire good for the economy*, $\mathcal{E} = (\langle X_i, P_i \rangle, \langle Y_k \rangle, \langle \boldsymbol{r}_i \rangle, [s_{ik}])$ iff it is a numéraire good for each $i \in M$, and for each $i \in M$ there exists $\theta_i > 0$ such that:

$$X_i \cap [(\boldsymbol{r}_i - \theta_i \boldsymbol{e}_j) + Z_i] \neq \emptyset. \tag{47}$$

Since it is easily seen that the numéraire good assumption implies that $\mathcal{E}$ is indecomposable and that each preference relation is locally non-saturating, our last result is a corollary of Theorem 4.4.

4.6 Corollary. *If $\mathcal{E} = (\langle X_i, P_i \rangle, \langle Y_k \rangle, \langle \boldsymbol{r}_i \rangle, [s_{ik}])$ is an economy such that:*

1. *Y_k is convex, for $k = 1, \ldots, \ell$;*
2. *$int(X) \cap [\boldsymbol{r} + Y] \neq \emptyset$,*
3. *for some $j' \in \{1, \ldots, n\}$, the commodity j' is a numéraire good for $\mathcal{E}$, and, for each $i \in M$:*
4. *P_i is weakly convex and lower semi-continuous, then:*

$$\boldsymbol{X}^E(\mathcal{E}) \equiv \left[\bigcap_{q=1}^{\infty} \boldsymbol{C}_q\right] = \boldsymbol{W}(\mathcal{E}).$$

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[6] Where $\mathbb{R}_{++} = \{x \in \mathbb{R} \mid x > 0\}$.

[7] The vector having all coordinates equal to zero except for the j^{th} coordinate, which is equal to one.

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Characterization and incentive compatibility of Walrasian expectations equilibrium in infinite dimensional commodity spaces*

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Summary. We consider a differential information economy with infinitely many commodities and analyze the veto power of the grand coalition with respect the ability of blocking non-Walrasian expectations equilibrium allocations. We provide two different Walrasian expectations equilibrium equivalence results. First by perturbing the initial endowments in a precise direction we show that an allocation is a Walrasian expectations equilibrium if and only if it is not "privately dominated" by the grand coalition. The second characterization deals with the fuzzy veto in the sense of Aubin but within a differential information setting. This second equivalence result provides a different characterization for the Walrasian expectations equilibrium and shows that the grand coalition privately blocks in the sense of Aubin any non Walrasian expectations equilibrium allocation with endowment participation rate arbitrarily close to the total initial endowment participation for every individual. Finally, we show that any no free disposal Walrasian expectations equilibria is coalitional Bayesian incentive compatible. Since the deterministic Arrow-Debreu-McKenzie model is a special case of the differential information economy model, one derives new characterizations of the Walrasian equilibria in economies with infinitely many commodities.

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1 Introduction

A differential (asymmetric) information economy consists of a set of agents, each of whom is characterized by a random utility function, a random initial endowment, a private information set and a prior. Such an economy is a generalization of the classical Walrasian deterministic economy as formulated rigorously by Arrow-Debreu and McKenzie. A natural extension of the competitive (Walrasian) equilibrium concept, which is appropriate for an asymmetric information economy is the *Walrasian expectations equilibrium* or *Radner equilibrium* introduced by Radner (1968).

The Radner equilibrium, like the Walrasian equilibrium, is a non cooperative solution concept capturing the idea that if each agent maximizes her ex ante utility function subject to her budget constraint by taking into account her own private information, then, this individualistic behavior will lead to a feasible redistribution of the initial endowments for each state of nature, (i.e., the total demand will balance the total initial endowment for each state of nature). It is important to notice that since agents make decisions before the state of nature is realized, (i.e., agents maximize ex-ante expected utility), prices do not reveal any private information ex ante. However, the equilibrium price reflects the private information as it has been obtained by maximizing expected utility subject to the budget constraint and also considering the private information of each agent.

Thus, the Radner equilibrium takes into account the private information of each agent, i.e., a change in the private information changes the Radner equilibrium. This is in sharp contrast with the traditional rational expectation equilibrium (REE) which, as it is well known, by now is not "sensitive" (does not take into account) to the private information of an agent (see Allen and Yannelis, 2001, and the reference there for a discussion of those issues).

The aim of this paper is to characterize the Radner equilibrium by means of cooperative solutions and also to analyze the incentive compatibility of the Radner equilibrium within an infinite dimensional commodity space setting.

Dealing with cooperative solution concepts with differential information, the basic problem which arises is, how agents within a coalition share their private information. Yannelis (1991) introduced the *private core* concept which is based on individual measurability requirements (i.e., when a coalition blocks an allocation each member in the coalition uses only her own private information - thus, we refer to this blocking as "private blocking"). The Radner equilibrium allocations have the property that they are not privately blocked by any coalition of agents and, therefore, the private core contains as a strictly subset the set of Radner equilibrium allocations. Throughout this paper, we consider that the way in which a coalition shares the information is the one leading to the private core solution, that is, every member in a coalition takes into account only her own private information. It turns out, that allowing individuals to make redistributions of their initial endowments based on their own private information results in allocations that are always Bayesian incentive compatible and also take into account the informational advantage of an individual (see Koutsougeras and Yannelis, 1993).

If one enlarges the number of coalitions, the possibilities of blocking an allocation increases and, then, the set of allocations which are not privately blocked is reduced. Addressing a finite set of agents and complete information economies, Debreu and Scarf (1963) enlarge the set of coalitions by replicating the original economy. By identifying the core allocations of each replicated economy with allocations in the initial economy, Debreu-Scarf showed that the set of non blocked allocations in every replicated economy converges to the set of Walrasian equilibria. A second development, was proposed by Aubin (1979) who also addresses a finite set of agents and complete information economies, stated an essentially similar approach although formally different than the one by Debreu and Scarf. By considering that, when forming a coalition, the agents in the economy can participate with any proportion of their endowments, the number of coalitions that may block an allocation is infinitely enlarge. This veto mechanism is referred in the literature to the confusing term fuzzy veto. Aubin (1979) showed that the allocations belonging to the core solution derived from this veto mechanism, called in this paper *Aubin core*, coincides with the Walrasian equilibrium allocations. Debreu-Scarf core convergence result and Aubin's result can be extended to differential information economies (see Meo, 2002).

The approaches of Debreu-Scarf and Aubin, enlarge the possibilities of blocking in order to obtain that the allocations which are not blocked by any coalition are precisely the Walrasian (or competitive) equilibrium allocations. In a companion paper Hervés-Beloso et al. (2003) showed that in a differential information economy with a finite dimensional commodity space, the veto power of just one coalition (the grand coalition) characterizes the Radner equilibrium. This result, not only corresponds to an extension of the Debreu-Scarf (1963) deterministic result to a differential information economy, but also has a different flavor. In particular, Debreu-Scarf in order to characterize the Walrasian equilibrium, replicate the economy i.e., enlarge the number of coalitions that agents can form. Hervés-Beloso et al. (2003) provide a characterization of the Radner equilibrium by considering the veto power (blocking power) of the grand coalition, by enlarging the possible redistribution of the initial endowments.

The main purpose of this paper is threefold: First, we provide a characterization of the Radner equilibrium for a differential information economy with an infinite dimensional commodity space, generalizing the Hervés-Beloso et al. (2003) finite dimensional commodity space characterization of the Radner equilibrium. It should be noted, that such a characterization is in general false for the deterministic Walrasian equilibrium in infinite dimensional spaces, as it was shown in Tourky and Yannelis (2001) and Podczeck (2003). In particular, the key observation is that unless the infinite dimensional commodity space is separable, one is bound not to obtain a characterization of the Walrasian equilibrium by means of the core and a fortiori of the Radner equilibrium. To overcome this difficulty, our commodity space in this paper is chosen to be the bounded sequence space ℓ^∞, endowed with

Mackey topology, which is separable. Moreover, the random utility function of each individual is assumed to be Mackey continuous. This is the standard set up for which Bewley (1972, 1973) has proved the existence of Walrasian equilibrium and its equivalence to the core. Recall that in this set up prices are in ℓ_1, and therefore one has a well defined price valuation of commodities. Indeed, for such a set up we show that an allocation x is a Radner equilibrium allocation of and only if x is not privately blocked by the grand coalition in any of the economies obtained by perturbing the original initial endowments in the direction of x (Theorem 4.1). The proof of this first equivalence theorem relies on an extension of the core-Walras equivalence showed by Bewley (1973) to differential information economies (Theorem 3.2) and on an extension of Vind's (1972) result to economies with infinitely many commodities and differential information (Theorem 3.3).

Second, we provide another characterization of Radner equilibria (Theorem 4.2) which deals with the Aubin veto mechanism within a differential information framework. Following this veto mechanism each agent in a coalition uses her own private information and can participate with a determined weight in the coalition. If we consider (as in the original definition by Aubin) the possibility of null weights or contributions, the grand coalition contains implicitly any other coalition. In this case, consider the veto power of the grand coalition as equivalent to the veto power of all coalitions. This is the reason why, in this paper, we modify Aubin's definition by requiring any participation (representing the contribution of an agent in a coalition) to be strictly positive. Even with non-null participation, the intuition underlying Aubin's result suggests that the grand coalition is able to block any non equilibrium allocation with arbitrarily small participation of some of the agents. However, this equivalence result provides a second characterization for the Radner equilibria and shows that the grand coalition privately blocks any non Radner equilibrium allocation with participation as close to the total participation as one wants for every individual.

Since the deterministic Arrow-Debreu-McKenzie model is a special case of the differential information economy model, Theorems 4.1 and 4.2 yield to new characterizations of the Walrasian equilibria in economies with infinitely many commodities.

Thirdly, we analyze the Bayesian incentive compatibility of the Radner equilibria. As it was shown in Glycopantis et al. (2002), the finite dimensional Radner equilibrium need not be Bayesian incentive compatible because of the free disposal requirement that Radner (1968) imposes. By redefining the Radner equilibrium to exclude free disposal, we show that any no free disposal Radner equilibrium allocation is Coalitional Bayesian Incentive Compatible (CBIC). Note that the private core is always CBIC (see Koutsougeras and Yannelis (1993)) and in finite dimensional commodity spaces the CBIC of the no free disposal Radner equilibrium follows from Koutsougeras and Yannelis. However, in this paper not only we use a stronger definition of CBIC, and allow for infinitely many commodities but also prove directly that the no free disposal Radner equilibrium is CBIC.

The paper is organized as follows. Section 2 states the model of a differential information economies with infinitely many commodities, contains the main concepts and a discussion of the assumptions. Section 3 focuses on the interpretation of the economy stated as a continuum differential information economy with a finite number of types of agents. Moreover, in this section, a private core-Walras expectations equilibrium equivalence and an extension of Vind's (1972) result are given for a differential information economy with infinitely many commodities. Section 4 contains two different characterizations of Radner equilibrium by using the private blocking power of the grand coalition. Finally, Section 5 shows the Bayesian incentive compatibility property of the free disposal Radner equilibria.

2 The model

Consider a differential information economy $\mathcal{E}$ with n consumers. Let $(\Omega, \mathcal{F})$ be a measurable space, where Ω denotes the states of nature of the world and the algebra $\mathcal{F}$ denotes the set of all events. Hence, $(\Omega, \mathcal{F})$ describes the exogenous *uncertainty*. The set of states of nature, Ω, is finite and there are infinitely many commodities in each state. $N = \{1, \ldots, n\}$ is the *set of n traders* or agents and ℓ^∞ will denote the *commodity space* which is the set of all bounded sequences.

The economy extends over two time periods $\tau = 0, 1$. Consumption takes place at $\tau = 1$. At $\tau = 0$ there is uncertainty over the states of nature and agents make contracts (agreements) that may be contingent on the realized state of nature at period $\tau = 1$ (that is, ex ante contract arrangement).

In this paper, we consider that for every state of nature $\omega \in \Omega$ and for every agent $i \in N$, the consumption set is ℓ^∞_+ which is the positive cone of the set of all bounded sequences ℓ^∞. Note that infinitely many commodities arise whenever one allows an infinite variation in any of the characteristics describing commodities. This characteristics could be physical properties, locations or the time of delivery. In fact, an infinite variation in time could arise if an infinite time horizon is allowed by considering the case of infinitely many time periods in each state of nature. Hence, it is economically natural to restrict commodity bundles to ℓ^∞_+, in each state, since we can assume that only bounded bundles would ever appear in an economy. For instance, if we restrict our economy to earth, then the availability of primary resources puts an upper bound on the quantity of any single commodity that can be produced. If an infinite number of physical commodities appear in the economy, then the units of these commodities can be chosen in such a way that only bounded bundles are possible.

Thus, a *differential information exchange economy* $\mathcal{E}$ with a finite number of agents and infinitely many commodities in every state of nature is defined by

$$\mathcal{E} = \{((\Omega, \mathcal{F}), \ell^\infty_+, \mathcal{F}_i, U_i, e_i, q) : i = 1, \ldots, n\}, \quad \text{where:}$$

1. ℓ^∞_+ is the *consumption set* for every state of nature ω and for every agent $i = 1, \ldots, n$.
2. $\mathcal{F}_i$ is a partition of Ω, denoting the *private information* of agent i;
3. $U_i : \Omega \times \ell^\infty_+ \to \mathbb{R}$ is the *random utility function* of agent i;
4. $e_i : \Omega \to \ell^\infty_+$ is the *random initial endowment* of agent i, assumed to be constant on elements of $\mathcal{F}_i$.

5. q is a probability function on Ω giving the (common) *prior* of every agent. It is assumed that q is positive on all elements of Ω.

   For any $x : \Omega \to \ell_+^\infty$, the *ex ante expected utility* of agent i is given by

$$V_i(x) = \sum_{\omega \in \Omega} U_i(\omega, x(\omega))q(\omega).$$

An *allocation* is a function $x = (x_1, \ldots, x_n)$ which associates to every agent i a random consumption bundle $x_i \in (\ell_+^\infty)^\Omega$.

We will refer to a function with domain Ω, constant on elements of $\mathcal{F}_i$, as $\mathcal{F}_i$*-measurable*, although, strictly speaking, measurability is with respect to the σ-algebra generated by the partition. We can think of such a function as delivering information to trader i, who can not discriminate between the states of nature belonging to any element of $\mathcal{F}_i$.

Let $\mathcal{X}_i$ denote the *set of all* $\mathcal{F}_i$*-measurable selections from the random consumption set of agent* i, that is:

$$\mathcal{X}_i = \left\{ x_i : \Omega \to \ell_+^\infty, \text{ such that } x_i \text{ is } \mathcal{F}_i\text{-measurable} \right\}.$$

Let $\mathcal{X} = \prod_{i=1}^n \mathcal{X}_i$. Any allocation x in $\mathcal{X}$ is called an *informationally feasible* allocation. An allocation x is said to be *physically feasible* if $\sum_{i=1}^n x_i \le \sum_{i=1}^n e_i$. An allocation x is *feasible* if it is both informationally and physically feasible.

A coalition $S \subset N$ *privately blocks* an allocation $x \in \mathcal{X}$ if there exists $(y_i)_{i \in S} \in \prod_{i \in S} \mathcal{X}_i$ such that $\sum_{i \in S} y_i \le \sum_{i \in S} e_i$ and $V_i(y_i) > V_i(x_i)$ for every $i \in S$.

The *private core* of the differential information exchange economy $\mathcal{E}$ is the set of all feasible allocations which are not privately blocked by any coalition (see Yannelis (1991)).

Next we shall define a Walrasian equilibrium notion in the sense of Radner (see Radner (1968, 1982)). For this, we need the following notations and definitions. Let ℓ_1 denote the space of absolutely summable sequences and let ℓ_1^+ denote the positive cone of ℓ_1. For any $a = (a_j)_{j=1}^\infty \in \ell_+^\infty,\ b = (b_j)_{j=1}^\infty \in \ell_1$, let $a \cdot b = \sum_{j=1}^\infty a_j b_j$. A *price system* is a non-zero function $p : \Omega \to \ell_1^+$. For a price system p, the *budget set* of agent i is given by

$$B_i(p) = \left\{ x_i \in \mathcal{X}_i, \text{ such that } \sum_{\omega \in \Omega} p(\omega) \cdot x_i(\omega) \le \sum_{\omega \in \Omega} p(\omega) \cdot e_i(\omega) \right\}.$$

Notice that traders must balance the budget ex-ante.

Definition 2.1 *A pair* (p, x)*, where* p *is a price system and* $x = (x_1, \ldots, x_n) \in \mathcal{X}$ *is an allocation, is a Walrasian expectations equilibrium (or a Radner equilibrium) if*

(i) *for all* i *the consumption function* x_i *maximizes* V_i *on* $B_i(p)$,
(ii) $\sum_{i=1}^n x_i \le \sum_{i=1}^n e_i$ *(free disposal), and*
(iii) $\sum_{\omega \in \Omega} p(\omega) \cdot \sum_{i=1}^n x_i(\omega) = \sum_{\omega \in \Omega} p(\omega) \cdot \sum_{i=1}^n e_i(\omega)$.

Throughout this paper we will refer explicitly to the following assumptions on preferences and endowments:

(A.1) *Continuity.* For every consumer i, her utility function $U_i(\omega, \cdot) : \ell^\infty_+ \to \mathbb{R}$ is Mackey continuous for every state ω.
(A.2) *Monotonicity.* For every consumer i, her utility function $U_i(\omega, \cdot) : \ell^\infty_+ \to \mathbb{R}$ is monotone for every state ω, that is, for every individual i, if $x, y \in \ell^\infty_+$ and $y \gg 0$, then $U_i(\omega, x + y) > U_i(x)$.
(A.3) *Convexity.* For every consumer i, her utility function $U_i(\omega, \cdot) : \ell^\infty_+ \to \mathbb{R}$ is concave for every state ω.
(A.4) *Interiority of initial endowments.* For every i and w, $e_i(w)$ belongs to the interior of ℓ^∞_+, i.e., there exists $a > 0$ such that $e_{ij}(w) > a$ for all $j \geq 1$ and for every $i = 1, \dots, n$.

The hypothesis (A.3) and (A.4) requiring monotonicity and convexity of preferences will be used in the proof of our main results where we will refer explicitly to them. Note also, that assumption (A.3) is a weak monotonicity condition; given a consumption bundle in some state of nature, if the amount of every coordinate increases then the utility increases. This assumption is required, for instance, in Section 3 in which a result by Vind (1972) is extended to differential information economies with infinitely many commodities. Vind's (1972) result was stated for economies with complete information and a finite number of commodities, under a stronger monotonicity assumption: If the amount of only one commodity increases, then the utility increases. In our setting, the stronger requirement (A.5) on initial endowments allows us to use a weaker monotonicity condition. This assumption (A.5) requiring that initial endowments are strictly positive was also used by Araujo (1985) addressing complete information economies with ℓ^∞ as commodity space.

The topological dual of ℓ^∞ depends, of course, on the topology considered on ℓ^∞. It is well known that the Mackey topology is the strongest of the locally convex topologies on ℓ^∞ having ℓ_1 as dual space. The stronger the topology is chosen, the larger the set of preference relations continuous with respect to it. Assumption (A.2) is stronger than the norm continuity of the utility functions, but, as Araujo (1985) remarked, if we relax this assumption, allowing in this way for a larger class of preferences, the equilibrium might fail to exist. On the other hand, Bewley (1972) proved, within a complete information scenario, an existence of equilibrium theorem for economies with ℓ^∞ as commodity space. Besides the usual assumptions for the existence of equilibrium, Bewley assumes preferences to be Mackey continuous. The Mackey topology is sufficiently strong to admit interesting preference relations. In words of Araujo (1985) "we can say that continuity with respect the Mackey topology is the best assumption of this kind."

The Mackey continuity of preferences can be interpreted in terms of impatience of consumers. To see this, given a consumption bundle $z \in \ell^\infty_+$ let $z^{(m)}$ denote the m-tail of z, i.e., the bundle defined as $z_j^{(m)} = 0,\ 1 \leq j \leq m$, and $z_j^{(m)} = z_j$ for $j > m$. It is known that if a consumer has a preference that is Mackey continuous then she exhibits impatience behavior in the sense that if x is preferred to y, then x is also preferred to $y + z^{(m)}$ for any sufficiently large m; that is, Mackey upper semicontinuity (usc) of preferences implies upper myopia. On the other hand, Mackey lower semicontinuity (lsc) of preferences implies lower myopia; that is, if x is preferred to y and $x - z^{(n)}$ belongs to the consumption set, then $x - z^{(n)}$ is also preferred to y for any sufficiently large n.

In particular, let y be a consumption bundle and, for a small $\varepsilon > 0$, let $x = y + \overline{\varepsilon}$ (where $\overline{\varepsilon}_n = \varepsilon$ for every n). Then, x is strictly preferred to y. Let $A_n = \{n+1, n+2, \dots\}$. Then, $\bigcap_{n=1}^{\infty} A_n = \emptyset$. Hence, χ_{A_n} converges to zero when n goes to ∞; where $\chi_{A_n}(\omega) \subset \ell_+^\infty$ is the function which is one on A_n and zero elsewhere. From the Mackey continuity of preferences it follows that x is also preferred to $y + \chi_{A_n}$ for any sufficiently large n. In other words, a little bit more in the near future would be preferred to a large constant amount more in every period after some date in the distant future.

For this kind of myopic preferences and addressing continuum economies with infinitely many commodities and complete information, Hervés-Beloso et al. (2000) showed that in order to get the core it is enough to consider, for any ε, the veto power of coalitions with measure less than ε. On the other hand, Bewley (1973) showed that Aumann's (1964) theorem on the equality of the core and the set of equilibria in atomless markets can be made to apply to complete information economies whose commodity space is ℓ^∞, under monotonicity, convexity and Mackey continuity of preferences. Hence, loosely speaking, existence and core equivalence of equilibria as well as blocking efficacy of small coalitions, for complete information economies with ℓ^∞ as commodity space, tend to be hold only in situations where the consumers "discount" the future in the sense that gains in the distant future are negligible.

In the next section we will deal with the continuum economies introduced by Aumann (1964, 1966). Our aim is to use this continuum approach in order to obtain the main results for the economy described in our model.

3 A continuum approach

In this section, we interpret differential information economies with n agents and infinitely many commodities as continuum or atomless economies with differential information and infinitely many commodities in which only a finite number of different characteristics can be distinguished (see García-Cutrín and Hervés-Beloso, 1993, for the deterministic case).

Given the economy $\mathcal{E} = \{((\Omega, \mathcal{F}), \ell_+^\infty, \mathcal{F}_i, U_i, e_i, q) : i = 1, \dots, n\}$ with a finite number of agents, we define a continuum economy $\mathcal{E}_c$, where the ith agent is the representative of infinitely many identical agents, as follows. The set of agents is represented by the real interval $[0,1]$, with the Lebesgue measure μ. We write $I = [0,1] = \bigcup_{i=1}^n I_i$, where $I_i = \left[\frac{i-1}{n}, \frac{i}{n}\right)$, if $i \neq n$, and $I_n = \left[\frac{n-1}{n}, 1\right]$. Each consumer $t \in I_i$ is characterized by her *private information* which is described by $\mathcal{F}_t = \mathcal{F}_i$, her *consumption set* ℓ_+^∞, for every $\omega \in \Omega$; her *random initial endowment* $e(t, \cdot) = e_i \in (\ell_+^\infty)^\Omega$ and her *expected utility function* $V_t = V_i$. We will refer to I_i as the set of agents of type i in the atomless economy $\mathcal{E}_c$. Then, the *continuum economy* $\mathcal{E}_c$ with a finite number of types is given by $\mathcal{E}_c = \left\{(\Omega, \mathcal{F}), \ell_+^\infty, I = \bigcup_{i=1}^n I_i,\ \mathcal{F}_i,\ e_i, V_i, q) : \ i = 1, \dots, n\right\}$.

An *allocation* in the continuum economy $\mathcal{E}_c$ is a Bochner integrable function $f : I \to (\ell_+^\infty)^\Omega$ or, alternatively, $f : I \times \Omega \to \ell_+^\infty$, where $f(t, \omega) \in \ell_+^\infty$ is the consumption bundle for agent t associated to the state of nature ω (see Diestel and Uhl, 1977, for the definition of Bochner integral as an extension of the Lebesgue integral to Banach spaces).

An allocation f is *feasible* in the economy $\mathcal{E}_c$ if: *(i)* for almost all $t \in I$ the function $f(t, \cdot)$ is $\mathcal{F}_t$-measurable, and *(ii)* $\int_I f(t, \omega) d\mu(t) \leq \int_I e(t, \omega) d\mu(t)$ for all $\omega \in \Omega$.

A *coalition* S is a measurable subset $S \subset I$, with $\mu(S) > 0$. An allocation f is *privately blocked* by a coalition S in the economy $\mathcal{E}_c$ if there exists $g : S \times \Omega \to \ell_+^\infty$ such that $g(t, \cdot)$ is $\mathcal{F}_t$-measurable for every $t \in S$, $\int_S f(t, \cdot) d\mu(t) \leq \int_S e(t, \cdot) d\mu(t)$ and $V_t(g(t, \cdot)) > V_t(f(t, \cdot))$ for every $t \in S$. The set of all feasible allocations that are not privately blocked by any coalition of agents is the *private core* of the economy $\mathcal{E}_c$.

A *Walrasian expectations equilibrium* in the sense of Radner (or a *Radner equilibrium*) in the associated continuum economy $\mathcal{E}_c$ is a pair (f, p) where f is a feasible allocation and $p \neq 0$ is a price system such that, for every consumer $t \in I$, the consumption bundle $f(t, \cdot)$ maximizes the expected utility function V_t on the budget set $B_t(p) = \{y \in (\ell_+^\infty)^\Omega$ such that $\sum_{\omega \in \Omega} p(\omega) \cdot y(\omega) \leq \sum_{\omega \in \Omega} p(\omega) \cdot e(t, \omega)\}$.

An allocation f in $\mathcal{E}_c$ can be interpreted as an allocation $x = (x_1, \ldots, x_n)$ in $\mathcal{E}$, where $x_i = \frac{1}{\mu(I_i)} \int_{I_i} f(t, \cdot) d\mu(t)$. Reciprocally, an allocation x in $\mathcal{E}$ can be interpreted as an allocation f in $\mathcal{E}_c$, where f is the step function given by $f(t, \cdot) = x_i$, if $t \in I_i$.

Next result shows that the continuum and the discrete approach can be considered equivalent with respect to Radner equilibria.

Theorem 3.1 *Under assumptions (A.2) and (A.4) the following statements hold:*

If (x, p) is a Radner equilibrium for the economy $\mathcal{E}$, then (f, p) is a Radner equilibrium for the continuum economy $\mathcal{E}_c$, where $f(t, \cdot) = x_i$ if $t \in I_i$.

Reciprocally, if (f, p) is a Radner equilibrium for the atomless economy $\mathcal{E}_c$, then (x, p) is a Radner equilibrium for $\mathcal{E}$, where $x_i = \frac{1}{\mu(I_i)} \int_{I_i} f(t, \cdot) d\mu(t)$.

Proof. Let $((x_1, \ldots, x_n), p) \in (\ell_+^\infty)^{\Omega \times n} \times \ell_1$ be a Radner equilibrium for $\mathcal{E}$. Then, $\int_I f(t, \omega) d\mu(t) = \sum_{i=1}^n \mu(I_i) x_i(\omega) \leq \sum_{i=1}^n \mu(I_i) e_i(\omega) = \int_I e(t, \omega) d\mu(t)$ for every state $\omega \in \Omega$; and the consumption function $f(t, \cdot)$ maximizes V_t on $B_t(p) = B_i(p)$ for all $t \in I_i$. Therefore, (f, p) is a Radner equilibrium for the continuum economy $\mathcal{E}_c$.

Conversely, let (f, p) be a Radner equilibrium for $\mathcal{E}_c$. Then, $x = (x_1, \ldots, x_n)$, with $x_i = \frac{1}{\mu(I_i)} \int_{I_i} f(t, \cdot) d\mu(t)$, is a feasible allocation in the economy $\mathcal{E}$. Since $\sum_{\omega \in \Omega} p(\omega) \cdot x_i(\omega) = \sum_{\omega \in \Omega} \frac{1}{\mu(I_i)} \int_{I_i} p(\omega) \cdot f(t, \omega) d\mu(t) \leq \sum_{\omega \in \Omega} p(\omega) \cdot e_i(\omega)$, we can deduce that $x_i \in B_i(p)$ for every agent i. Let $z \in (\ell_+^\infty)^\Omega$ be a random consumption bundle such that $V_i(z) > V_i(x_i)$. Then, since V_i is a concave and continuous function, there exists $S \subset I_i$, with $\mu(S) > 0$, such that $V_i(z) > V_i(f(t, \cdot))$ for every $t \in S$; and thus $\sum_{\omega \in \Omega} p(\omega) \cdot z(\omega) > \sum_{\omega \in \Omega} p(\omega) \cdot e_i(\omega)$. Otherwise, observe that if $V_i(z) \leq V_i(f(t, \cdot))$ for almost all $t \in I_i$ then $V_i(z) \leq V_i(x_i)$ (see Lemma in García-Cutrín and Hervés-Beloso, 1993, p. 582). □

3.1 Equal treatment private core equivalence

Considering complete information economies, different papers point out the core-Walras equivalence in continuum economies. Aumann (1964) showed the equivalence between the core and the Walrasian equilibria for atomless economies with a finite dimensional commodity space. Bewley (1973) proved a core-Walras equivalence for economies in which the commodity space is the space of essentially bounded, real-valued, measurable functions on a measure space. Rustichini and Yannelis (1991, 1992) generalized Aumann's result for economies in which the commodity space is an ordered separable Banach space.

Addressing economies with differential information, Einy et al. (2001) showed the equivalence between the Walrasian expectations equilibria (in the sense of Radner) and the private core for continuum economies with a finite number of commodities.

Next we state a result which shows that the set of Walrasian expectations equilibrium allocations with the equal treatment property coincides with the private core for the differential information continuum economy $\mathcal{E}_c$ (associated to the economy $\mathcal{E}$ with a finite number of agents) with infinitely many commodities.

Theorem 3.2 *Consider the differential information economy $\mathcal{E}$ under assumptions (A.1)–(A.4). Let f be a feasible allocation in the associated continuum economy $\mathcal{E}_c$ with $f(t,\cdot) = f_i$ for every $t \in I_i$. Then, f is a Walrasian expectations allocation if and only if f belongs to the private core of $\mathcal{E}_c$.*

Proof. Let (f, p) be a Walrasian expectations equilibrium in $\mathcal{E}_c$. Assume that f does not belong to the private core of $\mathcal{E}_c$. Then, there exists a coalition $S \subset I$ which privately blocks f via y. By concavity of the expected utility functions, we can consider that y is an equal treatment allocation, i.e., $y(t,\cdot) = y_i \in \mathcal{X}_i$ for every $t \in S_i = S \bigcap I_i$, $\sum_{i=1}^n \mu(S_i) y_i \leq \sum_{i=1}^n \mu(S_i) e_i$ and $V_i(y_i) > V_i(f_i)$ for every i with $\mu(S_i) > 0$. This implies that $\sum_{\omega\in\Omega} p(\omega)\cdot y_i(\omega) > \sum_{\omega\in\Omega} p(\omega)\cdot e_i(\omega)$, for every i with $\mu(S_i) > 0$; which is a contradiction to the physical feasibility of y for the coalition S.

Now, let f be an equal treatment allocation belonging to the private core of the continuum economy $\mathcal{E}_c$. Let $\mathcal{H}$ be the set of finite dimensional subspaces of ℓ^∞ containing $f_i(\omega)$ and $e_i(\omega)$, for every $\omega \in \Omega$ and every $i = 1, \ldots, n$. For any $H \in \mathcal{H}$ let $\mathcal{E}_c^H$ denote the continuum economy obtained from $\mathcal{E}_c$ by restriction of the consumption sets to $\ell_+^\infty \bigcap H$. That is,

$$\mathcal{E}_c^H = \left\{ (\Omega, \mathcal{F}), \ell_+^\infty \bigcap H, I = \bigcup_{i=1}^n I_i,\ \mathcal{F}_i,\ e_i, V_i^H, q,\ i = 1, \ldots, n \right\},$$

where V_i^H is the expected utility function of the agents of type i restricted to the finite dimensional positive cone $(\ell_+^\infty \bigcap H)^\Omega$. Obviously, since f belongs to the private core of $\mathcal{E}_c$, the allocation f belongs also to the private core of $\mathcal{E}_c^H$ for any $H \in \mathcal{H}$. On the other hand, the economy $\mathcal{E}_c^H$ satisfies the assumptions which guarantee that the set of Walrasian expectations equilibrium allocations coincides with the private core (see Einy et al., 2001). For each $H \in \mathcal{H}$, let p^H, with $\|p^H\| = 1$, be the price system such that (p^H, f) is a competitive equilibrium for the economy $\mathcal{E}_c^H$. By the Hahn-Banach theorem (see Aliprantis and Burkinshaw, 1985), p^H can be extended to the whole ℓ^∞. In this way, we obtain a bounded subset $\{p^H, H \in \mathcal{H}\}$ of ℓ_1. By the Alaoglu theorem (see Aliprantis and Burkinshaw, 1985) the set of prices $\{p^H, H \in \mathcal{H}\}$ is relatively compact in the weak* topology denoted by $\sigma(ba, \ell^\infty)$. Then there exists a $\sigma(ba, \ell^\infty)$ convergent subnet of $\{p^H, H \in \mathcal{H}\}$. Let p be the point to which it converges. Let us show that (p, f) is a Walrasian expectations equilibrium for the economy $\mathcal{E}_c$. Since the positive cone of ba is $\sigma(ba, \ell^\infty)$ closed, $p \geq 0$; and since $\|p^H\| = 1$ for every $H \in \mathcal{H}$, we deduce that $p \neq 0$. Assume that (p, f) is not a Walrasian expectations equilibrium for $\mathcal{E}_c$. Then, for some i, there exists $g \in \mathcal{X}_i$ such that $\sum_{\omega \in \Omega} p(\omega) \cdot g(\omega) \leq \sum_{\omega \in \Omega} p(\omega) \cdot e_i(\omega)$ and $V_i(g) > V_i(f)$. Actually, by assumption (A.4) and continuity of preferences, we can take g such that $\sum_{\omega \in \Omega} p(\omega) \cdot g(\omega) < \sum_{\omega \in \Omega} p(\omega) \cdot e_i(\omega)$. Then, there exists a subspace $\hat{H}$, such that $g \in \hat{H}$ and g belongs to the budget set of agent i for the price p^H for every H containing $\hat{H}$, which is a contradiction to the fact that (p^H, f) is a Walrasian expectations equilibrium for the economy $\mathcal{E}_c^H$. Finally, as in the proof of Theorem 2 in Bewley (1972), monotonicity and Mackey continuity of preferences allow us to conclude that $p \in \ell_1$. □

3.2 Infinite dimensional extension of Vind's theorem

In the case of considering $\mathbb{R}^\ell$ as commodity space and a complete information framework, Schmeidler (1972) and Grodal (1972) enforced the Aumann's (1964) core equivalence theorem. Vind (1972) completed the previous results by Schmeidler and by Grodal and showed that, for atomless economies, it is enough to consider the veto power of coalitions of any measure, in order to obtain the core; in particular, the blocking power of arbitrarily big coalitions is enough to get the core. Next we state an extension of this result to differential information continuum economies with infinitely many commodities and a finite number of types of agents. For this, we need some notation. Given $x = (x_h)_{h=1}^\infty \in \ell_+^\infty$ and $n \in \mathbb{N}$, we denote by x^n the element of ℓ^∞ defined by $x_h^n = x_h$ if $1 \leq h \leq n$ and $x_h^n = 0$ if $h > n$. Given a set $J \subset I = [0, 1]$, we denote by J_i the set of agents of type i belonging to J, that is, $J_i = J \bigcap I_i$.

Theorem 3.3 *Consider the differential information economy $\mathcal{E}$ under assumptions (A.1)–(A.4). Let f be a step function defined by $f(t, \cdot) = f_i$ if $t \in I_i$. Suppose that f is a feasible allocation in the associated atomless economy $\mathcal{E}_c$ and does not belong to the private core of $\mathcal{E}_c$. Then, for any ε, with $0 < \varepsilon < 1$, there exists a coalition S, with $\mu(S) = \varepsilon$, privately blocking the allocation f.*

Proof. Let f an equal treatment allocation which does not belong to the private core of the economy $\mathcal{E}_c$. Then, there exist a coalition of agents $A \subset I$ and an allocation $\widetilde{g} : A \to (\ell_+^\infty)^\Omega$ such that $\widetilde{g}(t, \cdot) \in \mathcal{X}_i$ for every $t \in A_i$ (i.e, $\widetilde{g}$ is informationally feasible for the coalition A), $\int_A \widetilde{g}(t, \cdot) d\mu(t) \leq \int_A e(t, \cdot) d\mu(t)$ (i.e, $\widetilde{g}$ is physically feasible for the coalition A) and $V_t(\widetilde{g}(t, \cdot)) > V_t(f(t, \cdot))$ for every $t \in A$.

For each state $\omega \in \Omega$, let $g_i(\omega) = \frac{1}{\mu(A_i)} \int_{A_i} \widetilde{g}(t,\omega) d\mu(t)$. Consider the allocation g given by $g(t,\cdot) = g_i$ if $t \in A_i = A \bigcap I_i$. Note that $g_i \in \mathcal{X}_i$ and $\int_A g(t,\cdot) d\mu(t) \leq \int_A e(t,\cdot) d\mu(t)$. Furthermore, by the convexity property of preferences, $V_i(g(t,\cdot)) > U_i(f_i)$ for every $t \in A_i$. On the other hand, observe that, by the Mackey continuity of preferences and assumption (A.4), we can take g such that $\int_A (e(t,w) - g(t,w))\, d\mu(t) \leq z(\omega) = z \gg 0$, for every $\omega \in \Omega$. Therefore, the coalition A privately blocks the allocation f via the allocation g which is constant on types and $\sum_{i=1}^n \mu(A_i) g_i \leq \sum_{i=1}^n \mu(A_i) e_i - z$, where z is a non null constant sequence.

Since g_i^n converges to g_i for the Mackey topology , Mackey continuity of preferences implies that there exists n_0 such that for every $n \geq n_0$ $V_i(g_i^n) > V_i(f_i)$ for every i with $\mu(A_i) > 0$. Hence, coalition A privately blocks f via g^n for every $n \geq n_0$. In particular, we have the following inequality between n-dimensional Lebesgue integrals, $\int_A g^n(t,\cdot) d\mu(t) \leq \int_A e^n(t,\cdot) d\mu(t)$, where $g^n(t,\omega) = g_i^n(\omega)$ for every $t \in A_i$.

Let the atomless measure $\eta(H) = \left(\mu(H), \int_H e^n(t,\cdot) d\mu(t), \int_H g^n(t,\cdot)) d\mu(t)\right)$, restricted to A. Applying Lyapunov theorem to η, we obtain that for any α, with $0 < \alpha < 1$, there exists a coalition $\overline{A} \subset A$, with $\mu(\overline{A}) = \alpha \mu(A)$, that privately blocks f via g^n. This proves the result for $\varepsilon \leq \mu(A)$.

Then, if $\mu(A) = 1$ the proof is complete. Otherwise, we have that $\mu(I \setminus A) > 0$. In this case, given $\varepsilon > 0$, consider the allocation $g_\varepsilon : A \times \Omega \to \ell_+^\infty$ defined by

$$g_\varepsilon(t,\omega) = \varepsilon g(t,\omega) + (1-\varepsilon) f(t,\omega).$$

By convexity of preferences (assumption (A.3)), $V_t(g_\varepsilon(t,\cdot)) > V_t(f(t,\cdot))$ for every $t \in A$. Moreover, by continuity of preferences, there exists n_1 such that $V_t(g_\varepsilon^n(t,\cdot)) > V_t(f(t,\cdot))$ for every $t \in A$ and for every $n \geq n_1$. Consider also the consumption bundle given by

$$h_i(\omega) = f_i(\omega) + \frac{\varepsilon \mu(A)}{\mu(I \setminus A)}\, z, \quad \text{for each } \omega \in \Omega.$$

By monotonicity of preferences (assumption (A.2)), we have that $V_i(h_i) > V_i(f_i)$. Again by Mackey continuity of preferences, there exists n_2 such that for every $n \geq n_2$ one has that $V_i(h_i^n) > V_i(f_i)$ for every $i = 1, \ldots, n$.

Consider now $n > \max\{n_1, n_2, n_3\}$ and the vector measure ν restricted to $I \setminus A$ and defined by

$$\nu(C) = \left(\mu(C), \int_C e^n(t,\cdot) d\mu(t), \int_C f^n(t,\cdot) d\mu(t)\right) \in \mathbb{R}^{2nk+1} \text{ for each } C \subset I \setminus A,$$

where k is the number of states of nature, that is, the cardinal of Ω.

Applying Lyapunov's convexity theorem to the atomless measure ν, we obtain that, given $\varepsilon > 0$, there exists $B \subset I \setminus A$ such that

(i) $\mu(B) = (1-\varepsilon)\mu(I \setminus A)$ and
(ii) $\int_B (e^n(t,\cdot) - f^n(t,\cdot))\, d\mu(t) = (1-\varepsilon) \int_{I \setminus A} (e^n(t,\cdot) - f^n(t,\cdot))\, d\mu(t)$.

Consider the coalition $S = A \bigcup B$. Note that $\mu(S) = \mu(A) + (1-\varepsilon)\mu(I \setminus A)$. It remains to show that the coalition S blocks the allocation f. For this, let $y : S \times \Omega \to \ell_+^\infty$ be the allocation given by:

$$y(t,\cdot) = \begin{cases} g_\varepsilon^n(t,\cdot) = \varepsilon g_i^n + (1-\varepsilon) f_i^n \text{ if } t \in A_i = A \bigcap I_i \\ \\ \quad y_i = f_i^n + \frac{\varepsilon\mu(A)}{\mu(B)}\, z^n \qquad \text{if } t \in B_i = B \bigcap I_i. \end{cases}$$

Observe that $h_i^n = f_i^n + \frac{\varepsilon\mu(A)}{\mu(I\setminus A)}\, z^n \le y_i = f_i^n + \frac{\varepsilon\mu(A)}{\mu(B)}\, z^n$ for every i. Thus, by construction, the members in the coalition S prefer the allocation y to the allocation f, that is, $V_i(y(t,\cdot)) > V_i(f_i)$ for every $t \in S_i,\ i = 1,\ldots,n$. Since, g_i and f_i belong to $\mathcal{X}_i$ and z is a constant sequence, we have that $y(t,\cdot) \in \mathcal{X}_t$ for every $t \in S$. In order to conclude that S privately blocks f via y it remains to show that y is physically feasible for the coalition S. Actually, we have the following inequalities:

$$\int_S (e(t,\cdot) - y(t,\cdot))\, d\mu(t) \ge \int_S (e^n(t,\cdot) - y(t,\cdot))\, d\mu(t)$$

$$\ge \int_A (e^n(t,\cdot) - g_\varepsilon^n(t,\cdot))\, d\mu(t) + \int_B (e^n(t,\cdot) - f^n(t,\cdot))\, d\mu(t) - \varepsilon\mu(A)\, z^n(\cdot)$$

$$\ge (1-\varepsilon)\int_A (e^n(t,\cdot) - f^n(t,\cdot))\, d\mu(t) + (1-\varepsilon)\int_{I\setminus A} (e^n(t,\cdot) - f^n(t,\cdot))\, d\mu(t)$$

$$= (1-\varepsilon)\int_I (e^n(t,\cdot) - f^n(t,\cdot))\, d\mu(t) \ge 0.$$

Therefore, the coalition S, with $\mu(S) = \mu(A) + (1-\varepsilon)\mu(I \setminus A)$, blocks the allocation f via the allocation y. Since ε is arbitrary, we have construct an arbitrarily large coalition privately blocking f. □

4 Equivalence results

In this section, we provide two different characterizations of the Walrasian expectations equilibria (Radner equilibrium). Both characterizations are obtained in terms of the private blocking power of the grand coalition. In order to obtain the Radner equilibrium equivalence theorems, the veto power of the coalition formed by all the agents is strengthened. In the first characterization, the blocking power of the grand coalition is made stronger by considering perturbations of the original initial endowments. The second characterization is obtained by considering that agents in a coalition can participate with a fraction of their resources, instead. Since the deterministic Arrow-Debreu-McKenzie model is a special case of the differential information economy model, one derives insights which yield to new characterizations of the Walrasian equilibria in economies with infinitely many conmodities.

4.1 Non-dominated allocations and equilibria

In this subsection, we obtain a first characterization of Walrasian expectations equilibrium in differential information economies with a finite number of traders and infinitely many commodities. This characterization is obtained by exploiting the veto power of only one coalition, i.e., the coalition formed by all the agents in the economy. Precisely, the main result stated in this subsection, Theorem 4.1, shows that an allocation is a Walrasian expectations allocation if and only if it is non dominated by the grand coalition in any economy which results from altering the initial endowments, as slightly as one wants, in a precise direction. Welfare theorems become particular cases of our main result.

Consider the differential information economy $\mathcal{E}=\{((\Omega,\mathcal{F}),\ell^{\infty}_{+},\mathcal{F}_i,U_i,e_i,q) : i = 1,\ldots,n\}$ defined in Section 2, with n consumers and infinitely many commodities.

In order to obtain our first equivalence result, we introduce some additional notation. Given an allocation $x = (x_1,\ldots,x_n)$ in the economy $\mathcal{E}$ and a vector $a = (a_1,\ldots,a_n)$, with $0 \leq a_i \leq 1$, let $\mathcal{E}(a,x)$ be a differential information economy which coincides with $\mathcal{E}$ except for the random initial endowment of each agent i that is given by the following convex combination of e_i and x_i.

$$e_i(a_i,x_i) = a_i e_i + (1 - a_i)x_i,$$

i.e., given the state $\omega \in \Omega,\ e_i(a_i,x_i)(\omega) = a_i e_i(\omega) + (1 - a_i)x_i(\omega) \in \ell^{\infty}_{+}$.

That is, $\mathcal{E}(a,x) \equiv \{((\Omega,\mathcal{F}),\ell^{\infty}_{+},\mathcal{F}_i,U_i,e_i(a_i,x_i) = a_i e_i + (1 - a_i)x_i,q) : i = 1,\ldots,n\}$.

Definition 4.1 *An allocation $z \in \mathcal{X}$ is privately dominated (or privately blocked by the grand coalition) in the economy $\mathcal{E}(a,x)$ if there exists a feasible allocation y in $\mathcal{E}(a,x)$ such that $V_i(y_i) > V_i(z_i)$ for every $i = 1,\ldots,n$.*

The meaning of the definition is clear. An allocation z is dominated in an economy if the total resources can be distributed in such a way that every agent is strictly better off with respect to z. That is, z is dominated if it is blocked by the grand coalition.

Observe that to be physically feasible and to be dominated are independent conditions for an allocation $z \in \mathcal{X}$. According to the definition above, a (privately) Pareto optimal allocation is a feasible and non-dominated allocation. That is, if z is feasible in an economy and it is not dominated then z is a Pareto optimum.

The next theorem states that a feasible allocation x in the economy $\mathcal{E}$ is a Radner equilibrium allocation if and only if it is not blocked by the grand coalition in any economy $\mathcal{E}(a,x)$ obtained by perturbing the initial endowments in the direction of x. In this way, we provide a characterization of Walrasian expectations equilibria by means of the veto power of the coalition formed by all the agents in a set of economies, which are defined from the initial economy by altering the original endowments following a precise direction.

Theorem 4.1 *Let x be a feasible allocation in the differential information economy $\mathcal{E}$ satisfying assumptions (A.1)–(A.4). Then x is a Walrasian expectations equilibrium allocation in $\mathcal{E}$ if and only if x is a non privately dominated allocation for every economy $\mathcal{E}(a,x)$.*

Proof. Let (p, x) be a Walrasian expectations equilibrium for the economy $\mathcal{E}$. Suppose that there exists $a = (a_1, \ldots, a_n)$, such that x is privately dominated in the economy $\mathcal{E}(a, x)$. Then, there exists $y = (y_1, \ldots, y_n) \in \mathcal{X}$ such that

(i) $\sum_{i=1}^{n} y_i \leq \sum_{i=1}^{n} e_i(a_i, x_i)$ and
(ii) $V_i(y_i) > V_i(x_i)$ for every agent $i \in \{1, \ldots, n\}$.

Since x is a Walrasian expectations equilibrium allocation in the economy $\mathcal{E}$, we have that $p \cdot x_i = \sum_{w \in \Omega} p(\omega) \cdot x_i(\omega) \leq \sum_{w \in \Omega} p(\omega) \cdot e_i(\omega) = p \cdot e_i$, for every agent i; and from *(ii)* we deduce that $p \cdot y_i = \sum_{w \in \Omega} p(\omega) \cdot y_i(\omega) > \sum_{w \in \Omega} p(\omega) \cdot e_i(\omega)$, for every agent $i = 1, \ldots, n$. Multiplying these inequalities by $(1 - a_i)$ and a_i, respectively, we obtain that $p \cdot (1 - a_i) y_i > p \cdot (1 - a_i) x_i$ and $p \cdot a_i y_i > p \cdot a_i e_i$. Thus, $p \cdot y_i > p \cdot a_i e_i + p \cdot (1 - a_i) x_i$, for every agent i. Therefore, $\sum_{i=1}^{n} p \cdot y_i > \sum_{i=1}^{n} p \cdot e_i(a_i, x_i)$, which is a contradiction to *(i)*, that is, a contradiction with the physical feasibility of y in the economy $\mathcal{E}(a, x)$.

Now, let $x \in \mathcal{X}$ be a non privately dominated allocation for every economy $\mathcal{E}(a, x)$. Let f be a step function on the real interval $I = [0, 1]$, defined by $f(t, \cdot) = x_i$ if $t \in I_i = \left[\frac{i-1}{n}, \frac{i}{n}\right)$, if $i \neq n$, and $f(t, \cdot) = x_n$ if $t \in I_n = \left[\frac{n-1}{n}, 1\right]$.

Assume that x is not an equilibrium allocation for the economy $\mathcal{E}$. Then, by Theorem 3.1, the step allocation f given by x is not an equilibrium allocation for the associated continuum economy $\mathcal{E}_c$ with n different types of agents. Applying Theorem 3.2, we have that f does not belong to the private core of the associated continuum economy. Furthermore, by Theorem 3.3, there exists a coalition $S \subset I = [0, 1]$, with $\mu(S) > 1 - \frac{1}{n}$, privately blocking the allocation f via an allocation $g : S \to (\ell_{+}^{\infty})^{\Omega}$, such that for each state of nature $\omega \in \Omega$, $g(t, \omega) = g_i(\omega)$ for every $t \in S_i = S \bigcap I_i$. That is, $\int_S g(t, \cdot) d\mu(t) = \sum_{i=1}^{n} \mu(S_i) g_i \leq \int_S e(t, \cdot) d\mu(t) = \sum_{i=1}^{n} \mu(S_i) e_i$ and $V_i(g_i) > V_i(x_i)$ for all $i = 1, \ldots, n$. Let $a_i = n\mu(S_i)$. Notice that, since $\mu(S) > 1 - \frac{1}{n}$, we obtain that $a_i > 0$ for every i.

In the economy $\mathcal{E}$ with a finite number of agents, let us consider the allocation $(g_1, \ldots, g_n)$. Let $z_i = a_i g_i + (1 - a_i) x_i$. By construction, $\sum_{i=1}^{n} z_i \leq \sum_{i=1}^{n} a_i e_i + (1 - a_i) x_i$ and $z_i \in \mathcal{X}_i$ for every i. By convexity of preferences, $V_i(z_i) > V_i(x_i)$, for every agent $i \in \{1, \ldots, n\}$.

Therefore, the grand coalition privately blocks x via z in the economy $\mathcal{E}(a, x)$, which is a contradiction. □

It should be noted that we characterize the equilibrium allocations as those non-dominated allocations in the economies given by infinitesimal perturbations in a precise direction of the original random initial endowments. In fact, the parameters a_i in the statement of Theorem 4.1 can be chosen arbitrarily close to one for every agent i. Indeed, note that given δ, with $0 < \delta < 1$, it is enough to consider the privately blocking coalition S such that $\mu(S) > 1 - \frac{\delta}{n}$ in order to guarantee $a_i = n\mu(S_i) > 1 - \delta$ for every i.

Notice also that the first welfare theorem is an immediate consequence of Theorem 4.1. In fact, if x is a Radner equilibrium allocation in the economy $\mathcal{E}$, then x is a Pareto optimal allocation not only in the economy $\mathcal{E}$ but also in any economy $\mathcal{E}(a, x)$ where x is feasible.

Moreover, observe that if x is a privately Pareto optimal allocation in $\mathcal{E}$, then x is also a privately Pareto optimal allocation in the economy in which the initial endowment allocation is x, that is, in the economy $\mathcal{E}(0, x)$. Thus, by taking $x_i = e_i$, for all i, all the economies $\mathcal{E}(a, x)$ are equal to $\mathcal{E}(0, x)$ and x is not privately blocked by the grand coalition. Then, if $x \gg 0$, we can apply Theorem 4.1 to the economy $\mathcal{E}(0, x)$ and we obtain, exactly, the second welfare theorem.

Therefore, Theorem 4.1 not only provides a characterization of equilibria in terms of the blocking power of the grand coalition but also allows us to obtain both welfare theorems as particular cases.

4.2 Fuzzy core and equilibria

Aubin (1979), addressing complete information economies with a finite number of agents and commodities, introduced the pondered veto concept and showed that the core obtained by this veto mechanism coincides with the Walrasian equilibria (see also Florenzano, 1990, for more general economies). The veto system proposed by Aubin extends the notion of ordinary veto in the sense that it is allowed a participation of the agents with a fraction of their endowments when forming a coalition. This veto mechanism is referred in the literature to fuzzy veto. On the other hand, the term fuzzy is usually used when elements belong to a set with certain probability. Then, this term may lead the reader to situate within another different scenario. In fact, regarding the veto mechanism introduced by Aubin, the agents actually (and not probably) participate in a coalition with a fraction of their endowments . Thus, as it is known, this veto mechanism is equivalent to the classical (Debreu-Scarf) veto system applied to the sequence of replicated economies. Therefore, we will refer this veto system as Aubin veto or veto in the sense of Aubin.

Following Aubin (1979), we define the privately Aubin blocking for differential information economies as follows.

Definition 4.2 *An allocation x is privately blocked in the sense of Aubin by the coalition S via the allocation y if there exist $\alpha_s \in (0, 1]$, for each $s \in S$, such that $\sum_{s\in S}\alpha_s y_s \leq \sum_{s\in S}\alpha_s e_s$, and $V_s(y_s) > V_s(x_s)$, for every $s \in S$.*

The Aubin private core of the economy $\mathcal{E}$ is the set of all feasible allocations which cannot be privately blocked in the sense of Aubin.

This definition of Aubin private veto and the consequent Aubin private core solution extend the notion of veto mechanism due to Aubin (1979) to a differential information setting. However, as it was noticed in the introduction, it is important to remark that we require the coefficients α_i to be strictly positive for every agent forming the coalition. Otherwise, the grand coalition contains implicitly the set of all possible coalitions.

Definition 4.3 *A feasible allocation x is Aubin dominated (or* dominated in the sense of Aubin*) in the differential information economy $\mathcal{E}$ if x is privately blocked in the sense of Aubin, by the grand coalition.*

The next result shows that the set of Radner equilibrium allocations for the economy $\mathcal{E} = \{((\Omega, \mathcal{F}), \ell_+^\infty, \mathcal{F}_i, U_i, e_i, q) : i = 1, \dots, n\}$, coincides with the set of allocations which are not Aubin dominated. Therefore, in order to obtain the Walrasian equilibria in the sense of Radner it suffices to consider the privately Aubin blocking power of just one coalition, namely, the grand coalition. Moreover, as we will show, from the proof we can deduce that the participation of every agent i can be taken as close to one as one wants.

Theorem 4.2 *Let $\mathcal{E}$ be an economy under assumptions (A.1)–(A.4). Then x is a Walrasian expectations equilibrium allocation in $\mathcal{E}$ if and only if x is not a dominated allocation in the sense of Aubin in the economy $\mathcal{E}$.*

Proof. Let x be dominated allocation in the sense of Aubin in $\mathcal{E}$. Then, the corresponding step function f given by x does not belong to the private core of the associated continuum economy $\mathcal{E}_c$. Hence, by Theorem 3.2, the step function f is not a Radner equilibrium in the continuum economy $\mathcal{E}_c$. Therefore, applying Theorem 3.1, x is not a Radner equilibrium allocation in $\mathcal{E}$.

Reciprocally, let x be a non Radner allocation in the economy $\mathcal{E}$. Then, by Theorem 3.1., the step function f defined by x is not a Radner allocation in the continuum economy $\mathcal{E}_c$. Hence, f does not belong to the private core of $\mathcal{E}_c$, that is, there exists a coalition S, with $\mu(S) > 0$ blocking f. By Theorem 3.3, the coalition S can be chosen such that $\mu(S_i) > 0$, for every $i = 1, \dots, n$. Then, there exists an allocation $g : S \times \Omega \to \ell_+^\infty$, with $g(t, \cdot) \in \mathcal{X}_i$ for every $t \in S_i$, such that

(i) $\int_S g(t, \cdot) d\mu(t) \leq \int_S e(t, \cdot) d\mu(t) = \sum_{i=1}^n \mu(S_i) e_i$ and
(ii) $V_i(g(t, \cdot)) > V_i(x_i)$ for every $t \in S_i$ and for every $i = 1, \dots, n$.

Consider the allocation $y : S \times \Omega \to \ell_+^\infty$ given by

$$y(t, \cdot) = y_i = \frac{1}{\mu(S_i)} \int_{S_i} g(t, \cdot) d\mu(t) \quad \text{for every } t \in S_i.$$

Observe that $y_i \in \mathcal{X}_i$ because $g(t, \cdot) \in \mathcal{X}_i$ for every $t \in S_i$. Then, taking $\alpha_i = n\mu(S_i) \in (0, 1]$ for every $i = 1, \dots, n$, we have that

(i) $\sum_{i=1}^n \alpha_i y_i \leq \sum_{i=1}^n \alpha_i e_i$ and
(ii) $V_i(y_i) > V_i(x_i)$ for every $i = 1, \dots, n$.

Condition *(i)* comes from the construction of the allocation y whereas condition *(ii)* is a consequence of convexity of preferences. Therefore, we conclude that x is privately dominated in the sense of Aubin. □

Remark. If we interpret that the participation of an agent i in the grand coalition is close to the total or complete participation when the corresponding coefficient α_i is close to one ($\alpha_i > 1 - \delta$, for any small δ), we will show that in Theorem 4.2 the participation of each agent can actually be required to be close to the total participation:

Given a positive real number $\delta < 1$, by Theorem 3.3, we can take the coalition S blocking the allocation f such that $\mu(S) > 1 - \frac{\delta}{n}$. Therefore, the coefficient $\alpha_i = n\mu(S_i) = n\mu(S \bigcap I_i) > 1 - \delta$ for every $i = 1, \dots, n$.

Note that as an immediate consequence of the equivalence result above and the characterization stated in Theorem 4.1, we obtain the following corollary.

Corollary 4.1 *Let $\mathcal{E}$ be an economy under assumptions (A.1)–(A.4) and let x be a feasible allocation in $\mathcal{E}$. The following statements are equivalent:*

1. *The allocation x is a Radner equilibrium allocation.*
2. *The allocation x is not privately blocked in the sense of Aubin.*
3. *The allocation x is not privately blocked in the sense of Aubin by the grand coalition.*
4. *The allocation x is not privately blocked in the sense of Aubin by the grand coalition with a participation of each agent as close as the total participation as one wants.*
5. *The allocation x is a non-dominated allocation in every economy $\mathcal{E}(a, x)$.*
6. *The allocation x is not dominated in any economy $\mathcal{E}(a, x)$ with coefficients a_i as close to the unit as one wants.*

5 Radner equilibrium and Bayesian incentive compatibility

Consider the differential information economy $\mathcal{E}$ described in Section 2:

$$\mathcal{E} = \{((\Omega, \mathcal{F}), \ell_+^\infty, \mathcal{F}_i, U_i, e_i, q) : i = 1, \dots, n\}$$

Definition 5.1 *A no-free disposal Radner equilibrium for the economy $\mathcal{E}$ is a pair (p, x), where $p \in \ell_1, p \neq 0$ is a price system and $x = (x_1, \dots, x_n) \in \mathcal{X}$ is an allocation, such that*

(i) for all i the consumption function x_i maximizes V_i on $B_i(p)$,
(ii) $\sum_{i=1}^n x_i = \sum_{i=1}^n e_i$ (no-free disposal).

Denote by $E_i(\omega)$ the event of agent i which contains the realized state of nature $\omega \in \Omega$. Obviously, $E_i(\omega)$ is an element of $\mathcal{F}_i$.

Definition 5.2 *An allocation $x \in \mathcal{X} = \prod_{i=1}^n \mathcal{X}_i$ is said to be Coalitional Bayesian Incentive Compatible (CBIC) if the following is not true:*

There exists a coalition $S \subset \{1, \dots, n\}$ and states ω, ω', $\omega \neq \omega'$ with $\omega' \in E_i(\omega)$ for all $i \notin S$, such that

$$U_i(\omega, e_i(\omega) + x_i(\omega') - e_i(\omega')) > U_i(\omega, x_i(\omega)) \text{ for every } i \in S.$$

The above definition of CBIC is related to the one in Koutsougeras and Yannelis (1993) and Krasa and Yannelis (1994), but we don't need to assume that the event $E_i(\omega)$ is an element of the[1] $\wedge_{C \in S}\mathcal{F}_i$, i.e., the event $E_i(\omega)$ is known to every member of the coalition S. Thus, our concept is slightly stronger than the one in the above papers. In essence, this notion of CBIC states that it is not possible for a coalition of agents S to benefit by announcing to the members of the complementary coalition $I \setminus S$, a false state that all members in $I \setminus S$ cannot distinguish from the true trade of

[1] The symbol $\wedge$ denotes the "meet" and $\wedge_{i \in S}\mathcal{F}_i$. is the finest partition contained in each $\mathcal{F}_i$

nature. Since the Radner equilibrium allows for multilateral contracts we insist on a coalitional notion of incentive compatibility since a contract which is individual Bayesian incentive compatible may not be CBIC; of course the reverse is always true.

We remark (see also Glycopantis et al., 2002) that the Radner equilibrium with free disposal is not CBIC, as the next example shows:

Example 5.1 Let $\Omega = \{a, b, c\},\ N = \{1, 2\},\ U_i(\omega, x) = x^{1/2}$ for every $x \in \mathbb{R}_+$, for each state of nature $\omega \in \Omega$ and for every agent $i = 1, 2$; $q(a) = q(b) = q(c) = 1/3$; $\mathcal{F}_1 = \{\{a, b\}, \{c\}\}$, $\mathcal{F}_2 = \{\{a, c\}, \{b\}\}$ and $e_1 = (15, 15, 0),\ e_2 = (15, 0, 15)$.

One can compute the Radner equilibrium for the economy above and find that $x_1 = (12, 12, 3)$ and $x_2 = (12, 3, 12)$ is an equilibrium allocation (with free disposal). Notice that this allocation is not CBIC because if a is the realized state of nature, $c \in E_2(a)$ (i.e., agent 2 can not distinguish a from c), then agent 1 reports c and if agent 2 believes her, agent 1 is better off, that is,

$$U_1(a, e_1(a) + x_1(c) - e_1(c)) = U_1(a, 12 + 3 + 0) > U_1(a, x_1(a)) = U_1(a, 12).$$

In other words, state a has occurred and agent 1 reports that it is state c. Thus, agent 1 keeps the initial endowment in state a (notice that she can even consume 15 units instead of 12 because nobody can verify that she wasted 3 units) and adds the 3 units she received in state c from agent 2 who believes that c has occurred and gives agent 1, 3 units.

The theorem below shows that if in the Radner equilibrium allocation we do not allow for free disposal, then it is always CBIC. Notice that without free disposal the Radner equilibrium in the example above is no trade and thus it is CBIC.

Theorem 5.1 *Let $\mathcal{E}$ be a differential information economy satisfying the assumptions (A.1) and (A.2). Then, any no free disposal Radner equilibrium allocation is Coalitional Bayesian Incentive Compatible.*

Proof. Let $x \in \mathcal{X}$ be a Radner equilibrium allocation and by way of contradiction, suppose that x is not CBIC. Then, there exist $S,\ \omega, \omega',\ \omega \neq \omega'$, with $\omega \in E_i(\omega')$ for every $i \notin S$, such that

$$U_i(\omega, e_i(\omega) + x_i(\omega') - e_i(\omega')) > U_i(\omega, x_i(\omega)) \text{ for every } i \in S. \quad (1)$$

Since for all i net trades are $\mathcal{F}_i$-measurable and $\omega \in E_i(\omega')$, for every $i \notin S$, it follows that $x_i(\omega) - e_i(\omega) = z_i(\omega) = x_i(\omega') - e_i(\omega') = z_i(\omega')$ for all $i \notin S$.

Hence,

$$U_i(\omega, e_i(\omega) + z_i(\omega)) = U_i(\omega, x_i(\omega)) \text{ for all } i \notin S. \quad (2)$$

It follows from (1) and the continuity of U_i that there exists a positive $\varepsilon \in \ell_+^\infty$ such that

$$U_i(\omega, e_i(\omega) + z_i(\omega) - \varepsilon) > U_i(\omega, x_i(\omega)) \text{ for every } i \in S. \quad (3)$$

Define for each agent i the function $y_i : \Omega \to \ell^\infty_+$ as

$$y_i(\omega) = \begin{cases} e_i(\omega) + z_i(\omega') - \varepsilon & \text{for } i \in S \\ e_i(\omega) + z_i(\omega') + \frac{|S|}{|N|-|S|}\,\varepsilon & \text{for } i \notin S, \end{cases}$$

where $\mid S \mid$ denotes the cardinality of the set S.

It can be easily checked that $y = (y_1, \dots, y_n)$ is feasible and also $\mathcal{F}_i$-measurable for every i.

It follows from (3) and from the definition of y that

$$U_i(\omega, y_i(\omega)) > U_i(\omega, x_i(\omega)) \text{ for every } i \in S. \tag{4}$$

From (2) and taking into account monotonicity we obtain that

$$\begin{aligned} U_i(\omega, y_i(\omega)) &= U_i(\omega, e_i(\omega) + z_i(\omega') + \frac{\mid S \mid}{\mid N \mid - \mid S \mid}\,\varepsilon) \\ &> U_i(\omega, x_i(\omega)) \text{ for all } i \notin S. \end{aligned} \tag{5}$$

Thus, (4) and (5) imply that $U_i(\omega, y_i(\omega)) > U_i(\omega, x_i(\omega))$ for every agent $i, (i = 1, \dots, n)$ and consequently for every i

$$V_i(y_i) = \sum_{\omega \in \Omega} U_i(\omega, y_i(\omega)) > \sum_{\omega \in \Omega} U_i(\omega, x_i(\omega)) = V_i(x_i).$$

Since $\sum_{i=1}^n y_i = \sum_{i=1}^n e_i = \sum_{i=1}^n x_i$, one has that $p \cdot \sum_{i=1}^n y_i = p \cdot \sum_{i=1}^n e_i$ for any $p \neq 0$ and consequently $p \cdot y_j \leq p \cdot e_j$ for some agent j. Thus, y_j is $\mathcal{F}_i$-measurable, it belongs to the budget set for j and it yields higher expected utility to agent j than $V_j(x_j)$, a contradiction to the fact that x is a Radner equilibrium allocation. □

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Comparative statics and laws of scarcity for games*

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Summary. A "law of scarcity" is that scarceness is rewarded. We demonstrate laws of scarcity for cores and approximate cores of games. Furthermore, we demonstrate conditions under which all payoffs in the core of any game in a parameterized collection have an equal treatment property and show that equal treatment core payoff vectors satisfy a condition of cyclic monotonicity. Our results are developed for parameterized collections of games and exact bounds on the maximum possible deviation of approximate core payoff vectors from satisfying a law of scarcity are stated in terms of the parameters describing the games. We note that the parameters can, in principle, be estimated. Results are compared to the developments in the literature on matching markets, pregames, and general equilibrium. This paper expands on results published in Kovalenkov and Wooders, *Economic Theory* (26, 383–396, 2005).

Keywords and Phrases: Monotonicity, cooperative games, clubs, games with side payments (TU games), cyclic monotonicity, law of scarcity, law of demand, approximate cores, effective small groups, parameterized collections of games

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1 Laws of scarcity, parameterized collections of games and equal treatment cores

The importance of the scarcity of a commodity in determining its value in exchange was already recognized by Adam Smith in the paradox that diamonds, although used only for adornment, were expensive, while water, essential to human life, was cheap. This apparent paradox has been much explained in the context of general equilibrium models of economies with private goods. The current paper[1] treats the problem from the perspective of cooperative game theory and demonstrates that if gains to population size are nearly exhausted, then numbers of players who are similar to each other and core payoffs respond in opposite directions. The players could be units of commodities, or people who are endowed with bundles of commodities, or people who just like to get together in groups for the pleasure of each other's company. We stress that our framework encompasses games derived from diverse economies, including economies with clubs, with endogenous choice of skills and club/group/jurisdiction formation, with pure or local public goods (or both), and also economies with production and exchange.[2] The only qualification is that the games generated are with side payments.

Stating our results more precisely, within the context of parameterized collections of games, we obtain analogues of the celebrated Laws of Demand and of Supply of general equilibrium theory. Roughly, the Law of Demand states that prices and quantities demanded change in the opposite directions while, with inputs signed negatively, the Law of Supply states that quantities demanded as inputs and produced as outputs change in the same direction as price changes.[3] In the framework of a cooperative game, supply and demand are not distinct concepts. Thus, following [43] we refer to our results for games as Laws of Scarcity. If player types are thought of as commodity types while payoffs to players are thought of as prices for commodities, our Laws of Scarcity are closely related to comparative statics results for general equilibrium models with quasi-linear utilities. As we discuss in a section relating our paper to the literature, our results extend the literature in several directions.

Games in a parameterized collection are described by certain parameters: (a) the number of approximate types of players and the goodness of the approximation and (b) the size of nearly effective groups of players and their distance from exact effectiveness.[4] An equal treatment payoff vector is defined to be a payoff vector that assigns the same payoff to all players of the same approximate type. Our laws of scarcity demonstrate that equal treatment ε-cores satisfy the property that numbers of players who are similar to each other and equal treatment ε-core payoffs respond

[1] A shorter version of the current paper, with fewer results, is [19].

[2] See, for example, Cole and Prescott (1997); Conley and Wooders (1997) or, for endogenous skill formation, Conley and Wooders (1996, 2001). See also Boehm (1974) and Shubik and Wooders (1983) for coalition production economies and Shubik and Wooders (1982) for a general model with coalition production and local public goods. We note that some of these papers permit nontransferable utility.

[3] The Law of Demand therefore rules out "Giffen goods" or treats compensated demands; see Mas-Colell, Whinston and Green [22], Sections 2.F and 4.C. This volume also provides a very clear exposition and further references.

[4] Parameterized collections of games were introduced in [15, 16], and [17].

in nearly opposite directions; specifically, we establish an *exact* upper bound on the extent to which equal treatment ε-core payoffs may respond in the same direction and this bound will, under some conditions, be small. We actually demonstrate a stronger result – equal treatment ε-core vectors and vectors of numbers of players of each approximate type satisfy cyclic monotonicity.[5] In addition to cyclic monotonicity, we demonstrate a closely related comparative statics result: When the relative size of a group of players who are all similar to each other increases, then equal treatment ε-core payoffs to members of that group will not significantly increase and may decrease.

The conditions required on a game to obtain our results are that (i) each player has many close substitutes (a thickness condition) and (ii) almost all gains to collective activities can be realized by groups of players bounded in size (a form of small group effectiveness – SGE). The first condition is frequently employed in economic theory. The second condition may appear to be restrictive, but in fact, if there are sufficiently many players of each type, then per capita boundedness (PCB) – finiteness of the supremum of average payoff over all games considered – and SGE are equivalent.[6] Our results yield explicit bounds, in terms of the parameters describing the games, on the maximal deviation of equal treatment ε-core payoffs from satisfying exact monotonicity. Moreover, our framework allows some latitude in the exact specification of approximate types. These two considerations suggest that in principle our results can be well applied to estimate the effects on equal treatment ε-core payoffs of changes in the composition of the total player set. Note that all the bounds we obtain are exact, and depend on the parameters describing the games and on the ε of the ε-core.

Our results also contribute to a literature relating games, markets and clubs. An advantage of the framework of cooperative games over detailed models of economies is that models of games can accommodate the entire spectrum from games derived from economies with only private goods to games derived from economies with pure public goods. Thus, it is of interest to determine conditions on games ensuring that they are 'market-like' – that they satisfy analogues of well known properties of competitive economies. Important papers in this direction include Shubik [37], which introduced the study of large games as models of large private-goods economies, Shapley and Shubik [36], which demonstrated an equivalence between markets and totally balanced games, and Wooders [43,44] demonstrating that games with many players are market games. Further motivation for the framework of cooperative games comes from Buchanan [2], who stressed the need for a general theory, including as extreme cases both purely private and purely public goods economies and the need for "a theory of clubs, a theory of cooperative membership."

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[5] Cyclic monotonicity relates to monotonicity in the same way as the Strong Axiom of Revealed Preference relates to the Weak Axiom of Revealed Preference (see, for example, Richter [27,28]).

[6] This is shown for "pregames" in Wooders [44], Theorem 4. Per capita boundedness and small group effectiveness were introduced as conditions limiting returns to coalition size in the study of large games in Wooders [41,42] respectively, where nonemptiness of approximate cores, the equal treatment property of cores, and other properties of large games were investigated.

For our results characterizing ε-cores of games to be interesting, it is important that under some reasonably broad set of conditions, ε-cores of large games are nonempty. Since Shapley and Shubik [35], showing nonemptiness of approximate cores of exchange economies with many players and quasi-linear utilities and Wooders [40,41], showing nonemptiness of approximate cores of game with many players with and without side payments, there has been a number of further results. For parameterized collections of games, such results are demonstrated in [15–17] and [45]. Importantly, in Kovalenkov and Wooders [16] equal treatment ε-cores of games with side payments are also shown to be nonempty. The interest of our monotonicity results is further enhanced by results showing that approximate cores have the equal treatment property; in this regard, note that [43] shows that approximate cores of large games treat most similar players nearly equally.[7] In this paper we present an equal treatment result for the "base case" of games with strictly effective small groups. In Kovalenkov and Wooders [15] further equal treatment results are demonstrated for parameterized collections of games.

In the next section we define parameterized collections of games. In Section 3, the results are presented. Section 4 consists of an example, applying our results to a matching model with hospitals and interns. Section 5 further relates the current paper to the literature and concludes the paper. In Appendix A we prove that the bounds cannot be tightened. In Appendix B, for the convenience of the reader we describe the pregame framework of the prior literature and make some connections to the framework of parameterized collections of games.

2 Cooperative games

Let (N, v) be a pair consisting of a finite set N, called the *player set*, and a function v, called the *characteristic function*, from subsets of N to the non-negative real numbers with $v(\emptyset) = 0$. The pair (N, v) is a *game* (*with side payments* or a *TU game*). Non-empty subsets of N are called *coalitions* or *groups*. A game (N, v) is *superadditive* if $v(S) \geq \sum_k v(S^k)$ for all groups $S \subset N$ and for all partitions $\{S^k\}$ of S.

In games and economies where the realization of maximum total payoff may require that a group of players sub-divide into smaller coalitions, superadditivity does not necessarily hold. If, however, a game is *essentially superadditive*, that is, a possibility open to a group S is to divide into subgroups and achieve the total payoff realizable by the subgroups, it is natural to apply solution concepts such as the core and approximate cores to the superadditive cover. (See Lemma 0 below.) Thus, we define the *superadditive cover* (N, v^s) of the game (N, v) where:

$$v^s(S) \stackrel{\text{def}}{=} \max \sum_k v(S^k)$$

and the maximum is taken over all partitions $\{S^k\}$ of S. Our results apply to both superadditive games and to superadditive cover games.

[7] These results extend prior results for sequences of games with a fixed distribution of player types in [40,41] and [39].

Given a nonnegative real number $\delta \geq 0$, two players i and j are *δ-substitutes* if for all groups $S \subset N$ with $i, j \notin S$, it holds that

$$|v(S \cup \{i\}) - v(S \cup \{j\})| \leq \delta.$$

When $\delta = 0$, players i and j are *exact substitutes*.

2.1 Parameterized collections of games

δ – substitute partitions. In our approach we approximate games with many players, all of whom may be distinct, by games with player types. This extends the prior model for pregames since the assumption of a compact metric space of player types is not required.

Let (N, v) be a game and let $\delta \geq 0$ be a non-negative real number. Informally, a δ-substitute partition is a partition of the player set N into subsets with the property that any two players in the same subset are "within δ" of being substitutes for each other. That is, if all players in a coalition are replaced by δ-substitutes, the payoff to that coalition changes by no more than δ per capita. Formally, a partition $\{N[t]\}$ of N into subsets is a *δ-substitute partition* if all players in each subset are δ-substitutes for each other.[8] The set $N[t]$ is interpreted as an *approximate type*. Note that in general a δ-substitute partition of N is not uniquely determined. Moreover, two games, say (N, v) and (N, v'), may have the same partitions into δ-substitutes but have no other relationship to each other (in contrast to games derived from a pregame). Examples are provided at the end of this subsection.

(δ, T) – type games. The notion of a (δ,T)-type game is an extension of the notion of a game with a finite number of types to a game with approximate types.

Let δ be a non-negative real number and let T be a positive integer. A game (N, v) is a *(δ, T)-type game* if there exists a T-member δ-substitute partition $\{N[t] : t = 1, .., T\}$ of N.

profiles . Profiles of player sets are defined relative to partitions of player sets into approximate types.

Let $\delta \geq 0$ be a non-negative real number, let (N, v) be a game and let $\{N[t] : t = 1, .., T\}$ be a partition of N into δ-substitutes. A *profile* relative to $\{N[t]\}$ is a vector of non-negative integers $f \in Z_+^T$. Given $S \subset N$ the *profile of* S is a profile, say $s \in Z_+^T$, where $s_t = |S \cap N[t]|$. A profile describes a group of players in terms of the numbers of players of each approximate type in the group. Let $\|f\|$ denote the number of players in a group described by f, that is, $\|f\| = \sum f_t$.

[8] The definition of δ-substitutes in our prior papers, including [19], is slightly less restrictive but more complicated.

β – effective B-bounded groups. The following notion formulates the idea of small group effectiveness, SGE, precisely defined in Appendix B, in the context of parameterized collections of games. Informally, groups of players containing no more than B members are β-effective if, by restricting coalitions to having fewer than B members, the per capita loss is no more than β.

Let β be a given non-negative real number, and let B be a given integer. A game (N, v) has *β-effective B-bounded groups* if for every group $S \subset N$ there is a partition $\{S^k\}$ of S into subgroups with $|S^k| \leq B$ for each k and

$$v(S) - \sum_k v(S^k) \leq \beta \, |S| \, .$$

When $\beta = 0$, 0-effective B-bounded groups are called *strictly effective B-bounded groups*.

parametrized collections of games $\Gamma((\delta, T), (\beta, B))$. Let T and B be positive integers, let δ and β be non-negative real numbers. Define

$$\Gamma((\delta, T), (\beta, B))$$

to be the collection of all (δ, T)-type games that have β-effective B-bounded groups.

Example 1. The following games illustrate the ideas of a δ-substitute partition and β-effective B-bounded groups. Let N be a finite set of players. Suppose that players can be ranked in the $[0, 1]$ interval so that if $i, j \in N$ and $i > j$ then i has a higher rank. We consider three different games, all with the same player set and the same ranking.

Let (N, v) be a game where the total payoff to any two players is the sum of their ranks. Suppose also that the payoff $v(S)$ to any other group S is zero. Then for any $\beta \geq 0$ and any $B \geq 2$, the game has β-effective B-bounded groups. Given $\delta \geq 0$, if the distance between the ranks of players i and j is less than δ, then i and j are δ-substitutes, both for the game (N, v) and for the superadditive cover game (N, v^s).

To see that there may be other games with the same partitions of the total player set into δ-substitutes, consider another game (N, v') but where the payoff $v'(\{i\})$ to player i is equal to his rank and the payoff to any other coalition is the given by the superadditive cover of v'. Here for any $\beta \geq 0$ and any $B \geq 1$, B-bounded groups are effective and if the distance between the ranks of i and j less than δ, then i and j are δ-substitutes.

Alternatively, let the payoff to any group consisting of two players be the square of the sum of the ranks of the members of the group (and again take the superadditive cover to create a superadditive game). Then if the distance between the ranks of players i and j is less than δ then i and j are $\delta^2 + 4\delta$ substitutes.

Example 2. This example serves to illustrate how the framework of parameterized collections of games allows new insights that may be hidden within the pregame framework. (Recall that pregames are formally defined in Appendix B.)

Let (N, v) be a game with buyers and sellers, where all sellers sell an identical product, each seller owns one unit of this product and has a reservation price for the unit he owns. Suppose that each buyer only wants to purchase at most one unit of the product and has a reservation price for the unit of product. Suppose that the reservation prices of the buyers are higher than the reservation prices of the sellers, so that always there exist some gains from trade, but the maximal gain from trade is bounded by some constant a. Then for any $\delta > 0$ the game (N, v) belongs to the collection $\mathit{\Gamma}((\delta, T_\delta), (0, 2))$ where T_δ is the smallest integer greater than a/δ.

Now consider instead a production game (N, v') where only two person coalitions are effective and where the worth of any two person coalition is the sum of the fixed productivities assigned to these two players and is less than or equal to a. In spite of the fact that the two games are quite different, the game (N, v') also belongs to the collection $\mathit{\Gamma}((\delta, T_\delta), (0, 2))$.

To put both these sorts of games within one pregame framework would require a topology on the space of player types and would require that the pregame is really just the union of two distinct pregames, one with buyers and sellers and another with production. Also, although it may be intuitive, the pregame framework does not make precise the similarities between the games that drive results, stated in terms of the parameters, applying to both games.

2.2 *Equal treatment ε-core*

the core and ε-cores. Let (N, v) be a game and let ε be a non-negative real number. A payoff vector x is in the *ε-core* of (N, v) if and only if it is *feasible*, that is, $\sum_{a \in N} x_a \leq v(N)$ and $\sum_{a \in S} x_a \geq v(S) - \varepsilon|S|$ for all $S \subset N$. When $\varepsilon = 0$, the ε-core is the *core*.

Lemma 0. Let (N, v) be a not-necessarily superadditive game and let (N, v^s) be its superadditive cover. Let $\varepsilon \geq 0$ be given. Then if x is a payoff vector in the ε-core of (N, v), then x is in the ε-core of (N, v^s).

Lemma 0 demonstrates that to study properties of the core of essentially superadditive games, we can assume, without loss of generality, that the game is superadditive. We leave the easy proof of Lemma 0 to the reader.

the equal treatment ε-core. Given non-negative real numbers ε and δ, we will define the *equal treatment ε-core* of a game (N, v) relative to a δ-substitute partition $\{N[t]\}$ of the player set as the set of payoff vectors x in the ε-core with the property that for each t and all i and j in $N[t]$, it holds that $x_i = x_j$.

Our notion of the equal treatment core is motivated by standard economic theory. All units of a commodity may differ; no two workers have exactly the same fingerprints or DNA for example. But yet, nonidentical commodities, if sufficiently similar, are treated as one commodity. The equal-treatment core may be viewed as a stand-in for the competitive equilibrium where similar items are grouped together as the same commodity.

For our comparative statics and monotonicity results, we restrict to payoffs in equal treatment ε-cores. As is well known, even with strictly effective small groups ε-cores do not necessarily treat identical players identically. For example, suppose that (N,v) is an inessential game where $v(S)=|S|$ for all groups $S\subset N$. Then, for any player $i\in N$, the payoff $x\in\mathbb{R}^N$ where $x_i=1+\varepsilon(|N|-1)$ and $x_j=1-\varepsilon$ for all $j\neq i$ is in the ε-core.[9] A number of results, however, have shown that under the assumption of per capita boundedness and thickness, bounding the percentages of players of each type strictly away from zero, approximate cores treat most similar players nearly identically.[10] The central result is that with strictly effective groups and sufficiently many players of each type, the core treats identical players identically.[11] We provide a version of this result below for parameterized collections of games with strictly effective small groups.

Proposition 0. Let $(N,v)\in\Gamma((\delta,T),(0,B))$. Let $z\in\mathbb{R}^N_+$ be in the core of (N,v). Suppose that there are more than B δ-substitutes for each player in the game. Then if i,$j\in N$ and i and j are δ-substitutes, it holds that

$$|z_i-z_j|\leq 2\delta.$$

Proof. The proof of this proposition is essentially the same as the proof of Theorem 3 of Wooders [41]), for NTU games. If $\delta=0$ then, for the special case of TU games, the proof is exactly the same as in the prior paper.

For any $S\subset N$ let $z(S)$ denote $\sum_{a\in S}z_a$. From the assumption that groups bounded in size by B are strictly effective, it holds that for some partition $\{S^k\}$ of N into groups with $|S^k|\leq B$ for each k,

$$v(N)-\sum v(S^k)=0.$$

Therefore, since z is in the core,

$$\sum v(S^k)-\sum z(S^k)=0$$

and

$$z(S^k)\geq v(S^k)\ \text{ for each }\ k.$$

It follows that

$$z(S^k)=v(S^k)\ \text{ for each }\ k.$$

For any $i\in N$ let $S^k(i)$ denote the member of $\{S^k\}$ containing player i. Now suppose, for some players i_0,$j_0\in N$, that i_0 and j_0 are δ-substitutes and

$$z_{i_0}-z_{j_0}>2\delta.$$

[9] Such examples go back to earliest versions of [40].

[10] See [40,39,43] and [45]. Another related result appears in Kovalenkov and Wooders [15], where conditions are demonstrated under which all payoffs in approximate cores treat similar players equally. These conditions hold for NTU games but not for TU games with unlimited side payments.

[11] Proofs of the more general results in the cited papers follow from "approximating" ε-cores by exact cores of games with admissible sizes of coalitions truncated.

Let us first show that then there exist two players i_1, $j_1 \in N$, such that i_1 and j_1 are δ-substitutes, $i_1 \notin S^k(j_1)$ and

$$z_{i_1} - z_{j_1} > \delta.$$

If $i_0 \notin S^k(j_0)$ then i_0, j_0 are such two players i_1, j_1. Otherwise, since $|S^k(j_0)| \leq B$ and since there are more than B δ-substitutes for each player it holds that there is some player l who is a δ-substitute for i_0 and j_0, and $l \notin S^k(j_0)$. Then it follows from the triangle inequality that either $z_{i_0} - z_l > \delta$ or $z_l - z_{j_0} > \delta$. Thus either i_0, l or l, j_0 are such two players i_1, j_1.

Now let us consider $S^* = S^k(i_1) \cup \{j_1\} \setminus \{i_1\}$. Since i_1 and j_1 are δ-substitutes, it holds that

$$v(S^*) \geq v(S^k(i_1)) - \delta.$$

But $z(S^*) < z(S^k(i_1)) - \delta \leq v(S^*)$ and we have a contradiction to the assumption that z is in the core. □

With the definition of the equal treatment ε-core in hand, we can next address monotonicity properties and comparative statics for this concept. In the following we will simply assume the nonemptiness of equal treatment ε-cores. With SGE along with PCB, for $\varepsilon > 0$ this assumption is satisfied for all sufficiently large games in parameterized collections (see Kovalenkov and Wooders result [16, 18]).

3 Laws of scarcity

A technical lemma is required. For $x, y \in \mathbf{R}^T$, let $x \cdot y$ denote the scalar product of x and y, i.e. $x \cdot y := \sum_{t=1}^{T} x_t y_t$.

Lemma 1. Let (N, v) be in $\Gamma((\delta, T), (\beta, B))$ and let (S^1, v), (S^2, v) be subgames of (N, v). Let $\{N[t]\}$ denote a partition of N into types and, for $k = 1, 2$, let f^k denote the profile of S^k relative to $\{N[t]\}$. Assume that $f_t^k \geq B$ for each k and each t. For each k, let $x^k \in R^T$ represent a payoff vector in the equal treatment ε-core of (S^k, v). Then

$$(x^1 - x^2) \cdot f^1 \leq (\varepsilon + \delta + \beta) \left\| f^1 \right\|.$$

Proof. Since (N, v) has β-effective B-bounded groups, there exists a partition $\{G^{1\ell}\}$ of S^1, such that $|G^{1\ell}| \leq B$ for any ℓ and

$$\sum_{\ell} v(G^{1\ell}) \geq v(S^1) - \beta \left\| f^1 \right\|.$$

Let us denote the profiles of $G^{1\ell}$ by g^ℓ. Observe that $\sum_\ell g^\ell = f^1$.

Since $f_t^2 \geq B$ for each t, it holds that $g^\ell \leq f^2$ for each ℓ. Therefore for each ℓ there exists a subset $G^{2\ell} \subset S^2$ with profile g^ℓ. Observe that since both $G^{1\ell}$ and $G^{2\ell}$ have profile g^ℓ, it holds that

$$\left| v(G^{1\ell}) - v(G^{2\ell}) \right| \leq \delta \left\| g^\ell \right\|.$$

Since x^2 represents a payoff vector in the equal treatment ε-core of (S^2, v) and $G^{2\ell} \subset S^2$ has profile g^ℓ, the total payoff $x^2 \cdot g^\ell$ cannot be improved on by the coalition $G^{2\ell}$ by more than $\varepsilon \|g^\ell\|$. Thus, for each set $G^{2\ell} \subset S^2$ with profile g^ℓ, it holds that

$$x^2 \cdot g^\ell \geq v(G^{2\ell}) - \varepsilon \left\|g^\ell\right\| \geq v(G^{1\ell}) - (\varepsilon + \delta) \left\|g^\ell\right\|.$$

Adding these inequalities we have

$$x^2 \cdot f^1 \geq \sum_\ell v(G^{1\ell}) - (\varepsilon + \delta) \left\|f^1\right\|.$$

It then follows that

$$x^2 \cdot f^1 \geq v(S^1) - (\varepsilon + \delta + \beta) \left\|f^1\right\|.$$

Since x^1 represents a payoff vector in the equal treatment ε-core of (S^1, v), $x^1 \cdot f^1$ is feasible for (S^1, v), that is, $x^1 \cdot f^1 \leq v(S^1)$. Combining these inequalities we have

$$(x^1 - x^2) \cdot f^1 \leq (\varepsilon + \delta + \beta) \left\|f^1\right\|.$$

□

Now we can state and prove our main results.

3.1 Approximate cyclic monotonicity

We derive an exact bound on the amount by which an approximate core payoff vector for a given game can deviate from satisfying exact cyclic monotonicity. The bound depends on:

δ, the extent to which players within each of T types may differ from being exact substitutes for each other;
β, the maximal loss of per capita payoff from restricting effective coalitions to contain no more than B players; and
ε, a measure of the extent to which the ε-core differs from the core.

Our result is stated both for absolute numbers and for proportions of players of each type. If exact cyclic monotonicity were satisfied, then the right hand sides of the equations (1) and (2) below could both be set equal to zero.

Proposition 1. Let (N, v) be in $\Gamma((\delta, T), (\beta, B))$ and let $(S^1, v), .., (S^K, v)$ be subgames of (N, v). Let $\{N[t]\}$ denote a partition of N into types and for each k let f^k denote the profile of S^k relative to $\{N[t]\}$. Assume that $f_t^k \geq B$ for each k and each t. For each k, let $x^k \in R^T$ represent a payoff vector in the equal treatment ε-core of (S^k, v). Then

$$\begin{aligned}(x^1 - x^2) \cdot f^1 + (x^2 - x^3) \cdot f^2 + .. + (x^K - x^1)& \\ \cdot f^K \leq (\varepsilon + \delta + \beta) \left\|f^1 + f^2 + .. + f^K\right\|&\end{aligned} \quad (1)$$

and

$$(x^1 - x^2) \cdot \frac{f^1}{\|f^1\|} + (x^2 - x^3) \cdot \frac{f^2}{\|f^2\|} + .. + (x^K - x^1) \cdot \frac{f^K}{\|f^K\|} \leq K(\varepsilon + \delta + \beta). \tag{2}$$

That is, the equal treatment ε-core correspondence approximately satisfies cyclic monotonicity both in terms of numbers of players of each type and percentages of players of each type.

Proof. From Lemma 1 we have

$$(x^k - x^{k+1}) \cdot f^k \leq (\varepsilon + \delta + \beta) \left\|f^k\right\|$$

for $k = 1, .., K-1$ and $(x^K - x^1) \cdot f^K \leq (\varepsilon + \delta + \beta)\|f^K\|$. Summing these inequalities we get (1).

Alternatively we have

$$(x^k - x^{k+1}) \cdot \frac{f^k}{\|f^k\|} \leq (\varepsilon + \delta + \beta)$$

for $k = 1, .., K-1$ and

$$(x^K - x^1) \cdot \frac{f^K}{\|f^K\|} \leq (\varepsilon + \delta + \beta).$$

Summing these inequalities we obtain (2). □

Remark. When $K = 2$, Proposition 1 implies that

$$(x^1 - x^2) \cdot (f^1 - f^2) \leq (\varepsilon + \delta + \beta) \left\|f^1 + f^2\right\|.$$

This form of monotonicity is typically called simply *monotonicity* or *weak monotonicity*.

Note that weak monotonicity does not imply cyclic monotonicity.

Corollary. When $K = 2$, Proposition 1 implies that

$$(x^1 - x^2) \cdot (f^1 - f^2) \leq (\varepsilon + \delta + \beta) \left\|f^1 + f^2\right\|$$

and

$$(x^1 - x^2) \cdot (\tfrac{f^1}{\|f^1\|} - \tfrac{f^2}{\|f^2\|}) \leq 2(\varepsilon + \delta + \beta).$$

That is, the equal treatment ε-core correspondence is approximately monotonic.

Note that the bound of Proposition 1 and its Corollary holds for any partition of the player set into δ-substitutes.

3.2 Comparative statics

For $j = 1, .., T$ let us define $e^j \in \mathbf{R}^T$ such that $e^j_l = 1$ for $l = j$ and 0 otherwise. Our comparative statics results relate to changes in the abundances of players of a particular type.

Proposition 2. Let (N, v) be in $\Gamma((\delta, T), (\beta, B))$ and let $(S^1, v), (S^2, v)$ be subgames of (N, v). Let $\{N[t]\}$ denote a partition of N into types and for each k let f^k denote the profile of S^k relative to $\{N[t]\}$. Assume that $f^k_t \geq B$ for each k and each t. For each k, let $x^k \in R^T$ represent a payoff vector in the equal treatment ε-core of (S^k, v). Then the following holds:

(A) If $f^2 = f^1 + me^j$ for some positive integer m (i.e., the second game has more players of approximate type j but the same numbers of players of other types) then

$$(x^2_j - x^1_j) \leq (\varepsilon + \delta + \beta)\frac{\left\|f^1 + f^2\right\|}{\|f^2 - f^1\|} = (\varepsilon + \delta + \beta)\frac{2\left\|f^2\right\| - m}{m}.$$

(B) If $\frac{f^2}{\|f^2\|} = (1 - \mu)\frac{f^1}{\|f^1\|} + \mu e^j$ for some $\mu \in (0, 1)$ (i.e., the second game has proportionally more players of approximate type j but the same proportions between the numbers of players of other types) then

$$(x^2_j - x^1_j) \leq (\varepsilon + \delta + \beta)\frac{2 - \mu}{\mu}.$$

That is, approximately the equal treatment ε-core correspondence provides lower payoffs for players of a type that is more abundant.

Proof. (A): Applying Corollary we get

$$(x^2 - x^1) \cdot me^j \leq (\varepsilon + \delta + \beta)\left\|f^1 + f^2\right\|.$$

Since $\|f^2\| = \|f^1\| + m$, this inequality implies our first result.

(B): From Lemma 1 we have

$$(1 - \mu)(x^1 - x^2) \cdot \frac{f^1}{\|f^1\|} \leq (1 - \mu)(\varepsilon + \delta + \beta)$$

and similarly

$$(x^2 - x^1) \cdot \frac{f^2}{\|f^2\|} \leq (\varepsilon + \delta + \beta).$$

Summing these inequalities we obtain

$$(x^2 - x^1) \cdot (\frac{f^2}{\|f^2\|} - (1 - \mu)\frac{f^1}{\|f^1\|}) \leq (2 - \mu)(\varepsilon + \delta + \beta).$$

Thus we get that

$$(x^2 - x^1) \cdot \mu e^j \leq (2 - \mu)(\varepsilon + \delta + \beta).$$

This inequality implies our second result. □

Obviously, again the bounds provided by Proposition 2 are independent of the specific partition of the player set into δ-substitutes.

3.3 Further remarks

1. (B) of Proposition 2 is a strict generalization of (A). ((A) follows from (B) for $\mu = \frac{m}{\|f^2\|}$.) We choose to present (A) in addition to (B) since (A) may be more intuitive. Notice also that although (A) is an immediate consequence of Proposition 1, (B) formally does not follow from Proposition 1.

2. Note that the bounds on the closeness of all our results are computable for a given game and depend only on the parameters describing the game. In the Appendix we demonstrate that all the bounds obtained are *exact*, that is, they cannot be made smaller.

3. For $\varepsilon = \beta = \delta = 0$ the bounds on the closeness of all our approximation results equals zero. Thus for games with finite number of player types and strictly effective small groups (e.g. for matching games with types) we demonstrate that the equal treatment core satisfies cyclic monotonicity and when a type becomes more abundant, players of that type receive (weakly) lower payoffs.

4. The results stated all require that there be at least B players of each type in each game under consideration. With other notions of approximate cores, specifically, the ε-remainder core and the ε_1-remainder ε_2-core, which allow a small percentage of players to be ignored, it may only be required that there are many substitutes for most players in the game; we leave the details to the interested reader. See [17] for definitions and further references.

5. We also leave it to the interested reader to show that results similar to those herein could be obtained for the strong ε-core. This approximate core notion requires that no group of agents can improve on a given payoff by ε in total, that is, given a game (N, v) and $\varepsilon \geq 0$, a payoff vector x is in the *strong ε-core* of (N, v) if and only if $\sum_{a \in N} x_a \leq v(N)$ and $\sum_{a \in S} x_a \geq v(S) - \varepsilon$ for all $S \subset N$. For strong ε-cores, the goodness of the approximation improves.

6. In the context of a pregame, as noted earlier, when there are sufficiently many players of each type in the games, then small group effectiveness, SGE, and per capita boundedness, PCB, are equivalent but, in the context of parameterized collections of games, this equivalence no longer holds. SGE, introduced in Wooders ([42–44]),[12] is a relaxation of "minimum efficient scale," MES ([41]). MES dictates that *all* gains rather than *almost all* gains to improvement can be realized by groups of players bounded in size.[13] As indicated already by the techniques of Wooders ([41]), when there are sufficiently many players of each type present in the games, sequences of games derived from a pregame satisfying PCB can be approximated by games satisfying MES. (In fact, [39] suggestively calls PCB *near minimum efficient scale.*) This is very useful in proving various results since games

[12] A condition closely related to SGE appears in Wooders and Zame [48]. There, to obtain one of their results, the authors assume that almost all gains to improvement can be realized by groups of players bounded in absolute size. An equivalence between this condition and SGE is demonstrated in [43].

[13] MES has also been called *strict* small group effectiveness. For pregames with side payments it is equivalent to the "exhaustion of gains to scale" in [33] and to the "0-exhaustion of blocking opportunities" [9,10].

satisfying MES are especially tractable. It is noteworthy that the results of the current paper do not depend on PCB – a parameterized collection of games does not necessarily have bounded average payoffs. Consider, for example, the collection of games where all players are identical, two-player coalitions are effective, and the per-capita payoff to a two-person coalition in any game in the collection equals the number of players in the game. Clearly, without any loss, coalitions can be restricted to have no more than two players and, even though per capita payoffs are unbounded, our results apply to all the games in the collection. Thus, the crucial property is SGE.

7. Cyclic monotonicity has appeared in several papers in economics; see, for example, Kusomoto [20], Epstein [11] and Jorgenson and Lau [13]. All these papers address duality theory in models of private goods economies. Kusomoto's also provides some more general treatment.

4 Matching hospitals and interns: An example

Given the great importance of matching models (see, for example, Roth and Sotomayor [32] for an excellent study and numerous references to related papers), we present an application of our results to a model of matching interns and hospitals. Our example is highly stylized. For a more complete discussion of the matching interns and hospitals problem, we refer the reader to Roth [31].

The problem consists of the assignment of a set of interns $\mathcal{I} = \{1, .., i, .., I\}$ to hospitals. The set of hospitals is $\mathcal{H} = \{1, .., h, .., H\}$. The total player set N is given by $N = \mathcal{I} \bigcup \mathcal{H}$. Each hospital h has a preference ordering over the interns and a maximum number of interns $\overline{I}(h)$ that it wishes to employ. Interns also have preferences over hospitals. We'll assume $\overline{I}(h) \leq 9$ for all $h \in H$. This gives us a bound of 10 on the size of strictly effective groups ($\beta = 0$). For simplicity, we'll assume that both hospitals and interns can be ordered by the real numbers so that players with higher numbers in the ordering are more desirable. The rank held by a player will be referred to as the player's *quality*. More than one player may share the same rank in the ordering. In fact, we assume that the total payoff to a group consisting of a hospital and no more than nine interns is given by the sum of the rankings attached to the hospital and to the interns. Let us also assume that the rank assigned to any intern is between 0 and 1 and the rank assigned to any hospital is between 1 and 2. Thus, if the hospital is ranked 1.3 for example and is assigned 5 interns of quality .2 each, then the total payoff to that group is 2.3.

Since all interns have qualities in the interval $[0, 1)$ and similarly, all hospitals have qualities in the interval $[1, 2]$, given any positive real number $\delta = \frac{1}{n}$ for some positive integer n we can partition the interval $[0, 2]$ into $2n$ intervals, $[0, \frac{1}{n}), .., [\frac{j-1}{n}, \frac{j}{n}), .., [\frac{2n-1}{n}, 2]$, each of measure $\frac{1}{n}$. Assume that if there is a player with rank in the jth interval, then there are at least 10 players with ranks in the same interval.

Given $\varepsilon \geq 0$, let x^1 represent a payoff vector in the ε-core that treats all interns with ranks in the same interval equally and all hospitals with ranks in the same interval equally (that is, x^1 is equal treatment relative to the given partition of the total player set into types). Let us now increase the abundance of some type of intern that appears in N with rank in the jth interval for some j. We could imagine, for example, that some university training medical students increases the number of type j interns by admitting more students from another country. Let x^2 represent an equal treatment payoff vector in the ε-core after the increase in type j interns. It then holds, from result (A) of Proposition 2 that

$$(x_j^2 - x_j^1) \leq (\varepsilon + \frac{1}{n})\frac{\|f^1 + f^2\|}{\|f^2 - f^1\|}.$$

Of course this is not the most general application of our results – we could increase the proportions of players of one type by reducing the numbers of players of other types. Then part (B) of our Proposition could be applied.

It is remarkable that our results apply so easily. For this simple sort of example, it is probably the case that a sharper result can be obtained. This is beyond the scope of our current paper, however. Research in progress considers whether sharper results are obtainable with assortative matching of the kind illustrated by this example – that is, where players can be ordered so that players with higher ranks in the orderings are superior in terms of their marginal contributions to coalitions.

Finally, the parameter values that we have used in this example were chosen for convenience and simplicity. In principle, these could be estimated and various questions addressed. For example, are payoffs to interns approximately competitive? Do non-market characteristics such as ethnic background or gender make significant differences to payoffs?

5 Relationships to the literature and conclusions

5.1 Matching markets

Our results may be viewed as a contribution to the literature on comparative statics properties of solutions of games. As noted by Crawford [7], the first suggestion of the sort of results obtained in this paper may be in Shapley [34], who showed that in a linear optimal-assignment problem the marginal product of a player on one side of a market weakly decreases when another player is added to that side of the market and weakly increases when a player is added to the other side of the market. Kelso and Crawford [14], building on the model of Crawford and Knoer [8], show that, for a many-to-one matching market with firms and workers, adding one or more firms to the market makes the firm-optimal stable outcome weakly better for all workers and adding one or more workers makes the firm-optimal stable outcome weakly better for all firms. Crawford [7] extends these results to both sides of the market and to many-to-many matchings.[14] In contrast to this literature, our results are not restricted to matching markets and treat all outcomes in equal treatment ε-cores. Moreover, we demonstrate cyclic monotonicity. Instead of the assumptions of "substitutability" of Kelso and Crawford [14], however, we require our thickness

[14] And also to pair-wise stable outcomes but this is apparently not so directly related to our paper.

condition and SGE. Unlike [14] and [7], our current results are limited to games with side payments – we plan to consider this limitation in future research.

Note that our results imply a certain *continuity* of comparative statics results with respect to changes in the descriptors of the total player set. In particular, the results are independent of the exact partition of players into approximate types. Specifically, given a number T of approximate types and a measure of the required closeness of the approximation, subject to the condition that players of each type are approximate substitutes for each other, our results apply independently of exactly where the boundary lines between types are drawn. Suppose, for example, that we wished to partition candidates for positions as hospital interns into three categories – say "good," "better" and "best." It may be that there is more than one way to partition the set of players into these categories while retaining the property that all players in each member of the partition are approximate substitutes for each other; the exact partition does not affect the results. Relating this feature of our work to general equilibrium theory, a finite set of commodities is typically considered to be an approximation to the real-world situation that all units of each commodity may differ. Descriptions of commodities are incomplete and a "commodity" is a group of objects that satisfy the description. For example, models of labor markets may have two types of workers, "skilled" and "unskilled" but no two workers (or two loaves of bread, or two oranges) may be exactly identical. In the differentiated commodities literature, results addressing this problem show that prices are continuous functions of attributes of commodities (cf., [21]). Since our framework does not require a topology on the space of player types, continuity takes a different but valid form and is more directly apparent.

5.2 Pregames

Besides the matching literature, our results are related to prior results obtained within the context of a pregame, cf. [44, 43].[15] A pregame specifies a set of compact metric space of player types and a *single* worth function, assigning a worth to each finite list of attributes (repetitions allowed). (Recall that precise definitions appear in Appendix B.) Since there is only one worth function, all games derived from a pregame are related and, given the attributes of the members of a coalition, the payoff to that coalition is independent of the total player set in which the coalition is embedded; widespread externalities are not allowed. In contrast, our results apply to given games and, as in the earlier results for matching models, there is no requisite topological structure on the space of players types. While our results for a given game hold for all games in a collection described by the same parameters, there are no necessary relationships between games. For example, consider the collection of games where two-player coalitions are effective and there are only two types of players. This collection includes two-sided assignment games, such as marriage games and buyer-seller games, and also games where *any* two-player coalition is effective. There appears to be no way in which one pregame can accommodate all

[15] The formulation of a pregame was introduced in [40].

the games in the collection. These considerations indicate that the framework of parameterized collections of games is significantly broader than that of a pregame.[16]

In the context of pregames under conditions roughly equivalent to those of Wooders [40] – that *all* gains to coalition formation can be exhausted by coalitions bounded in size – a proof of the comparative statics result and weak monotonicity of core payoffs was provided in Scotchmer and Wooders [33]. Wooders [42,43] extended the *monotonicity* analysis of Scotchmer and Wooders to hold for arbitrary changes in abundances of players of each type in games satisfying SGE and made the connection to the Law of Demand of economic theory (cf., Hildenbrand 1994). Engl and Scotchmer [9,10] extended the *comparative statics* analysis of Scotchmer and Wooders to hold for proportions of players of each type and further addressed the relationships between monotonicity and the Law of Demand. All of these results, unlike the matching literature, require a fixed set of player types (or a fixed finite set of attributes of players and a single worth function defined over these attributes). The major difference between the results of these papers and those of the current paper are that our assumptions and results (a) treat more general collections of games, (b) apply to individual games, and (c) apply uniformly to all games described by the same parameters.

A major advantage of our approach over the prior approach using pregames is that, except for the special case of pregames satisfying strict small group effectiveness (or, in other words, 'exhaustion of gains to scale by coalitions bounded in size') with a finite number of exact types, *the conditions used in the prior literature cannot be verified* for any finite game.[17] That is, since the conditions are stated on the worth function of the entire pregame, which includes specification of the worths of arbitrarily large groups, or on the closeness of the worth function to the limiting per capita utility function, it is not possible to determine whether the conditions are satisfied. In contrast, given any game, values of parameters describing that game can be computed.[18]

Another major advantage of our approach is that we provide *exact bounds*, in terms of the parameters describing a game, on the amounts by which equal treatment ε-core payoff vectors can differ from satisfying cyclic monotonicity. We are unaware of any comparable results in the literature. The prior literature does not indicate the *sensitivity* of the results to specifications of bounds on group sizes and of types of players. Such an analysis is important for empirical testing since, in fact, few commodities are completely standardized. (This may be especially true in estimating hedonic prices as in Rosen 1978,1986 and more recent literature.) Nor

[16] A short survey discussing parameterized collections of games and their relationships to pregames appears in [46].

[17] *Strict* small group efectiveness dictates that *all* gains to coalition formation can be realized by partitioning the total player set, no matter how large, into coalitions bounded in size. This condition was introduced in Wooders [40] (condition *) and, for NTU games, in Wooders [41], where, as noted earlier, it was called "minimum efficient scale."

[18] Since there may be many but a *finite* number of coalitions, in fact determining the required sizes of δ and T, β and B may be time-consuming but it is possible. In contrast, to verify that a pregame satisfies SGE or PCB requires consideration of an *infinite* number of payoff sets or, even more demanding, a limiting set of equal treatment payoffs.

does the prior literature provide *empirically testable conclusions* on approximate monotonicity or comparative statics.

5.3 Attribute games and production games

One interpretation of a game with side payments, common in the literature, is to regard commodities or inputs or attributes of players more generally as themselves players in a game. We call this an *attribute game* and the equal treatment core is called the *attribute core*.[19] The motivation for the attribute core is rather obvious; we each may belong to only a few coalitions of people, but we put our money into mutual funds, the stock market or houses, our time into a firm or into family and leisure, and so on. The results of this paper immediately apply to attribute games.

To the best of our knowledge, attribute games have their origins in Owen (1975), which treats linear production games. A set of players N who each own a bundle of resources is taken as given. Coalitions of players can use their resources to produce outputs according to a common linear technology Λ^O. Prices for outputs are taken as given. This information generates a game (N, v) that has, due to the linearity of the technology, a nonempty core. Let $b_i \in \mathbb{R}^m$ denote the endowment of resources of player i.

Owen defines an equilibrium concept, which we shall call an *Owen equilibrium* – a set of prices for resources/ commodities/attributes $p^* = (p_1^*, ..., p_t^*, ..., p_m^*) \in \mathbb{R}_+^m$ with the property that there are no profits possible in production. As Owen writes (with a slight change in notation for consistency)

– “The components $p_1^*, ..., p_t^*, ..., p_M^*$ can be thought of as equilibrium prices for the resources. Each of the players is then paid for his resources according to the equilibrium price vector p^*; the resulting payments always give a vector in the core.”

The following simple example illustrates the Owen equilibrium and the fact that the equilibrium outcomes may be a strict subset of the core.

Example (Owen 1975). Owen's example is essentially a ‘glove game’ where one player owns two gloves. Let $N = \{1, 2\}$ and let the endowment vectors for players 1 and 2 be $b^1 = (1, 0)$ and $b^2 = (0, 2)$ respectively. Let the technology be given by

$$\Lambda^O(x, y) = \min\{x, y\}.$$

Suppose output sells for \$1 per unit. Then

$$v(N) = 1$$

$$v(\{1\}) = v(\{2\}) = 0.$$

The core of the game is the set of points

$$\{(u_1, u_2) \in \mathbb{R}_+^2 : u_1 + u_2 = 1\}.$$

[19] Of course this simply gives a name to a familiar concept. The equal-treatment core of a game goes back to some of the first papers introducing the core, cf. Shubik [37].

The Owen equilibrium prices solve the following minimization problem:

$$\begin{gathered}\text{minimize } \; p_1 + 2p_2 \\ \text{subject to} \\ p_1 + p_2 \geq 1,\end{gathered}$$

which has the unique solution:

$$p_1^* = 1, \; p_2^* = 0$$

and gives equilibrium utilities of $\{u_1 = 1, u_2 = 0\}$.[20]

For this example, the Owen equilibrium prices correspond to the equal treatment core of a game where the players are units of resources (also called the *attribute core*).[21]

For sequences of games with a fixed distribution of player types, Owen demonstrates that for all replications of the player set (and their endowments), all payoffs in the core have the equal-treatment property and if $u = (u_1, ..., u_n)$ represents a payoff in the core of the r-fold replicated game for all r, then u is obtained from an (Owen) equilibrium price vector p^*.

The *attribute core*, defined to be the core of a game where units of commodities are players, extends the notion of an Owen equilibrium to situations with nonlinear technologies. Thus, a vector p^* is in the attribute core if:

$$\begin{gathered}p^* \cdot \textstyle\sum_{i \in N} b_i = v(N) \\ \text{and} \\ p^* \cdot \textstyle\sum_{i \in S} b_i \geq v(S) \;\; \text{for all} \;\; S \subset N.\end{gathered}$$

Note that the attribute core is simply the core of a game where the players are commodities. This differs from the usual core notion of economies where coalitions consist of players who own bundles of commodities. As Owen's example already demonstrates, these two concepts do not necessarily have the same set of utility outcomes.

Continuing Owen's example, consider a glove game where each player is a RH glove or a LH glove and the payoff to a coalition consisting of n_1 RH glove players and n_2 LH glove players is $\Psi(n_1, n_2) := \min\{n_1, n_2\}$. Suppose that in total, there are f_1 RH gloves and f_2 left hand gloves. Our laws of scarcity apply equally well to this interpretation of a game. Note that this game is a member of the collection $\Gamma((0,2),(0,2))$.

If ownership of *bundles* of commodities is assigned to individual units (teams or divisions within a firm in the literature on subsidy-free pricing or endowments of individual consumers of commodities in the exchange economy interpretation),

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[20] The set of utility payoffs that arise from the values of endowments at Owen equilibrium prices has come to be called the *Owen set*, cf. Gellekom et al. (2000).

[21] We note that the Owen equilibrium is distinct from the so called 'hedonic core,', whose elements are price vectors for attributes but where improvement is carried out by coalitions of players, owning bundles of attributes. For this example, the hedonic core would consist of the set of vectors $\{(p_1, p_2) \in \mathbb{R}^2_+ : p_1 + 2p_2 = 1\}$.

then another cooperative game is generated. In this game, essentially some players in the original game are "syndicated," glued together to become one player.

From the data given above, we can construct games where players may be endowed with bundles of gloves. By endowing players in this game with various numbers of RH gloves and LH gloves, we create another game with possibly several types of players. For specificity, suppose:

1. m_1 players of type 1 are endowed with two right hand gloves each;
2. m_2 players of type 2 are endowed with a RH glove and;
3. m_3 players of type 3 are endowed with a LH glove.

For consistency, it must hold that $2m_1 + m_2 = f_1$ and $m_3 = f_2$. (Of course this is only one of many possible games that could be constructed.) Now it is not so immediate that our main results can be applied. However, from the data given, with the three possible endowments of gloves given by 1-3 above, we can determine a number of types T and a bound B so that the game constructed, say (M, w), is a member of the collection $\Gamma((0, T), (0, B))$. It is immediate that $T = 3$. It is fairly obvious and we leave for the reader to verify that $B = 3$ suffices; the largest coalitions that need form in realizing all gains to collective activities consist of one player of type 1 and two players of type 3. Thus, we have that $(M, w) \in \Gamma((0, 3), (0, 3))$ and our comparative statics and monotonicity results apply to the games in $\Gamma((0, 3), (0, 3))$.

5.4 Relationship to the literature on general equilibrium

The class of economies treated in the current paper could be considered as a generalization of the standard competitive model by Arrow-Debreu-McKenzie. Moreover we treat the equal treatment ε-core as a "stand-in" for the competitive equilibrium in the general context of the cooperative game theory. Hence, if player types are thought of as commodity types while payoffs to players are thought of as prices for commodities, as in the above subsection, our Laws of Scarcity are closely related to comparative statics results for general equilibrium models.

Indeed, Nachbar [23] has established conditions for a general equilibrium model under which the inner product of endowment changes and normalized competitive equilibrium price changes is negative.[22] The conditions are that (a) the general equilibrium version of Law of Demand holds and (b) goods are normal. A limitation of Nachbar's result is that the normalization have to be a very specific and unusual. However in case of quasi-linear utilities, which corresponds to the case of games with side payments treated in the current paper, both conditions (a) and (b) are satisfied and normalization became a natural one with a price of numeraire commodity set to one. Thus for quasi-linear utilities Nachbar's result implies a negative monotonicity relation between endowment changes and equilibrium price changes. The results of the current paper show the robustness of this monotonicity conclusion. More precisely our paper considers economies modelled as games with side payments and identifies conditions that ensure approximate negative monotonicity of payoffs in the equal treatment ε-core with respect to endowment changes.

[22] This result was generalized independently by Nachbar [24] and Quah [26] to allow discrete changes.

5.5 An intuition behind the results

Numerous examples of games derived from pregames may lead one to expect our comparative statics result. Consider a glove game, for example where the payoff function can be written as $u(x, y) = \min\{x, y\}$. Suppose initially that the number of RH gloves, say x, is equal to the number of LH gloves, y, and both x and y are greater than one. Then the equal-treatment core can be described by the set $\{(p_x, p_y) \in \mathbb{R}^2_+ : p_x + p_y = 1\}$; each RH glove is assigned p_x and each LH glove is assigned p_y and a pair of gloves is assigned 1. Now increase the number of players with RH gloves. The equal treatment core is now described by $\{(0, 1)\}$; each RH glove is assigned 0 and each LH glove is assigned 1.

In games with a finite set of player types, defining the core via linear programming also leads to a law of scarcity, quite immediately. Let (N, v) be a game with a finite number T of player types and with m_t players of type t, $t = 1, ..., T$. We take v as a mapping from subprofiles s of m ($s \in \mathbb{Z}^T_+$, $s \leq m$). Then, following Wooders [40], consider the following LP problem[23]:

$$\begin{aligned} &\text{minimize}_{p\geq 0} \;\; p \cdot m \\ &\text{subject to} \;\; p \cdot s \geq v(s) \;\; \text{for all} \;\; s \leq m \end{aligned}$$

If the game has a nonempty core, then the solution p^* satisfies $v(m) = p^* \cdot m$. Now consider the same problem but with an increased number of players of type $\widehat{t}$ in the objective function for some $\widehat{t} \in \{1, ..., T\}$. Assume that the same inequalities are the only constraints; this imposes a form of strict small group effectiveness on the game – only groups with profiles $s \leq m$ are effective. It is clear that the payoff to players of type $\widehat{t}$ will not increase with the increase in the number of players of that type in the objective function since the constraint set has not changed – the payoff to type $\widehat{t}$ can only decrease. This suggests some of the initial intuition underlying comparative statics results for games.

6 Appendix A: Exact bounds

We construct some sequences of games to demonstrates that all the bounds we obtained in our results are *exact*, that is, the bound cannot be decreased.

I). Let us concentrate first on the central case $\delta = \beta = 0$. Consider a game (N, v) where any player can get only 1 unit or less in any coalition and there are no gains to forming coalitions. This game has strictly effective 1-bounded groups and all agents are identical. Formally, however, we may partition the set of players into many types. Thus $(N, v) \in \Gamma((0, \tau), (0, 1))$ for any integer τ, $1 \leq \tau \leq |N|$. Notice also that for any $\varepsilon \geq 0$ the ε-core of the game is nonempty and very simple: it includes all payoff vectors that are feasible and provide at least $1 - \varepsilon$ for each of the players. All the games that we are going to construct will be subgames of a game (N, v).

[23] The core has been described as an outcome of a linear programming problem since the seminal works of Gilles and Shapley. Wooders [40] introduces the linear programming formulation with player types.

a). For the bound in Lemma 1 we can present even a single game with two payoffs vectors that realize this bound. Namely, let $\tau = 1$ (all players are of one type) and let us consider any two subgames S^1, S^2 with the same number of players and the equal treatment payoffs $x^1 = 1$ and $x^2 = 1-\varepsilon$. Then $(x^1-x^2)\cdot f^1 = \varepsilon \|f^1\|$.

b). For the bound in Proposition 1, for $K \leq |N|$ and some nonnegative integer $l \leq |N| - K$, let us consider $\tau = K$ and the subgroups $S^1, .., S^K$ with the profiles $f^1, .., f^K$ where $f^k_t = l + 1$ for $t = k$ and 1 otherwise. Let also consider payoff vectors x^k where $x^k_t = 1$ for $t = k$ and $1 - \varepsilon$ otherwise. Then $(x^i - x^j) \cdot f^i = \varepsilon l$ for any $i \neq j$. Hence

$$(x^1 - x^2) \cdot f^1 + (x^2 - x^3) \cdot f^2 + .. + (x^K - x^1) \cdot f^K$$
$$= \varepsilon l K = \varepsilon \left\| f^1 + f^2 + .. + f^K \right\| \frac{l}{l+K}$$

and

$$(x^1 - x^2) \cdot \frac{f^1}{\|f^1\|} + (x^2 - x^3) \cdot \frac{f^2}{\|f^2\|} + .. + (x^K - x^1)$$
$$\cdot \frac{f^K}{\|f^K\|} = K\varepsilon \frac{l}{l+K}.$$

It is straightforward to verify that for any fixed K both our bounds in Proposition 2 can not be improved for sequences of games (N, v), with $|N|$ going to infinity, for subgames constructed as above with l going to infinity.

c). For the bound in Proposition 2 it is enough to concentrate on (A) since it is a special case of the result (B). For $|N| \geq 2$ let us consider $\tau = 2$ and $l \leq |N| - 2$. Then consider the subgroups S^1, S^2 with the profiles $f^1 = (1, 1)$ and $f^2 = (l+1, 1)$ and payoff vectors $x^1 = (1 - \varepsilon, 1)$ and $x^2 = (1, 1)$. Then

$$(x^2_1 - x^1_1) = \varepsilon = \varepsilon \frac{\left\| f^1 + f^2 \right\|}{\left\| f^2 - f^1 \right\|} \frac{l}{l+4}.$$

It follows that both our bounds in Proposition 2 can not be improved for sequences of games (N, v), with $|N|$ going to infinity, for subgames constructed as above with l going to infinity.

II). It is easy to modify our example to allow for non-zero δ and β in a such a way that we will have the same profiles as in Part I, but will use the payoffs of $1 + \delta + \beta$ and $1 - \varepsilon$ instead of 1 and $1 - \varepsilon$. This will lead us to the appearance of $\varepsilon + \delta + \beta$ on the places of ε in all bound in Part I. We leave it as a simple exercise for the interested reader.

7 Appendix B: Pregames

In this appendix, for the convenience of the reader in comparing the concepts and in evaluating the contribution of this paper, we review the concept of a pregame.

Let Ω be a compact metric space, interpreted as a set of player "types" or attributes. A *profile* on Ω, interpreted as a description of a group of players in terms of numbers of players of each type in the group, is a function f from Ω to the set Z_+ of nonnegative integers for which the *support* $\sigma(f)$ of f, given by

$$\sigma(f) = \{\omega \in \ \Omega : f(\omega) \neq 0\},$$

is finite. A profile is simply a function f from Ω to the nonnegative integers with the property that $f(\omega) \neq 0$ for only a finite number of elements ω in Ω. For each $\omega \epsilon \Omega$, we interpret $f(\omega)$ as the number of players of type ω or, in other words, with attributes ω, in the group of players described by f. The set of profiles on Ω is denoted by $P(\Omega)$**.** We write $f \leq g$ if $f(\omega) \leq g(\omega)$ for each ω in Ω.

By the *norm* of a profile, we mean

$$\|f\| = \sum_{\omega \epsilon \sigma(f)} f(\omega),$$

which is simply the number of players in a group represented by f. This is a finite sum since f has finite support.

A *pregame* is a pair (Ω, Ψ) where Ω is a compact metric space, called the *space of attributes* and $\Psi : P(\Omega) \to R_+$, called the *characteristic function (of the pregame)*, is a function with the following properties:

(a) $\Psi(0) = 0$;

(b) given any $\epsilon > 0$ there is a $\delta > 0$ such that

for each pair of player types ω_1 and , ω_2 with $dist(\omega_1, \omega_2) < \delta$

it holds that , $|\Psi(f + \omega_1) - \Psi(f + \omega_2)| < \epsilon$ (continuity);

(c) $\Psi(f) + \Psi(g) \leq \Psi(f + g)$ for all profiles f and g , and;

The first condition means that zero players can realize nothing. The second is that players with similar attributes are nearly substitutes. The third expresses the idea that an option open to a group is to split into several smaller groups.

We frequently refer to the elements of Ω as "types". Players of the same type are substitutes.

7.1 Games induced by pregames

To derive a game from a pregame (Ω, Ψ), we specify a finite set N and a function $\alpha : N \to \Omega$, called an *attribute function*. With any subset S of N we can then associate a profile, $prof(\alpha|S)$, given by

$$prof(\alpha|S)(\omega) = |\alpha^{-1}(\omega) \cap S|.$$

The profile $prof(\alpha|S)(\omega)$ simply lists the numbers of players of each type in the subset S. We have now determined a game (N, v_α) where

$$\nu_\alpha(S) = \Psi(prof(\alpha|S))$$

for each $S \subset N$. Let $n = prof(\alpha|N)$. An *equal treatment payoff* is a function $x : \Omega \to R_+$. An equal treatment payoff assigns the same value to all players of the same type. The payoff x is *feasible for* n if

$$\sum_{\omega \in \sigma(n)} n(\omega)\, x(\omega) \leq \Psi(n).$$

For ease in notation, given a profile f and an equal-treatment payoff x define $x(f)$ by

$$x(f) = \sum_{\omega \in \sigma(f)} f(\omega)\, x(\omega).$$

Let f be a profile. When $\sum f^k = f$ for some collection of profiles $f^1, ..., f^K$, not necessary distinct, we say that the collection is *a partition of* f and each member of the collection is called a *subprofile of* f. Obviously, a partition of a profile is related to a partition of a set of players. If (N, v_α) is a game derived from (Ω, Ψ), and $\{S_1, ..., S_K\}$ is a partition of N, then

$$\{f^k : prof(\alpha|S_k) = f^k, k = 1, ..., K\}$$

is a partition of $prof(\alpha|N)$.

7.2 Small group effectiveness

A pregame (Ω, Ψ) satisfies *small group effectiveness*, (SGE), if for each positive real number $\beta > 0$ there is an integer $\eta_1(\beta)$ such that for each profile f, for some partition $\{f^k\}$ of f:

(a) $\|f^k\| \leq \eta_1(\beta)$ for each profile f^k in the partition,

(b) $\Psi(f) - \sum_k \Psi(f^k) < \beta \|f\|$.

Small group effectiveness means that given a measure of per capita approximation (a $\beta > 0$) there is an absolute bound on group sizes with the property that almost all gains to collective activities can be realized by groups of players smaller in size than that bound, that is, bounded group sizes nearly exhaust all gains to scale of collective activities.

Let (Ω, Ψ) satisfy small group effectiveness and let β and $\eta_1(\beta)$ satisfy the condition of the definition of SGE. Then it is immediate that any game generated by the pregame has β-effective $\eta_1(\beta)$-bounded groups. Since Ω is a compact metric space it holds that given $\delta > 0$ we can partition Ω into a finite number T of subsets so that all players with attributes in each subset are δ-substitutes. Thus, all games derived from (Ω, Ψ) are in the collection $\Gamma((\delta, T), (\beta, B))$.

When games are required to have many substitutes for each player, small group effectiveness is equivalent to per capita boundedness. A pregame (Ω, Ψ) satisfies *per capita boundedness* if there is a constant A such that

$$\frac{\Psi(f)}{\|f\|} \leq A \text{ for all profiles } f \in P(\Omega).$$

The following result holds more generally but is proven for the case where Ω is a finite set.

Wooders 1994b, *Econometrica,* Theorem 4. *With "thickness," SGE=PCB.*

(1) Let (T, Ψ) be a pregame satisfying SGE. Then the pregame satisfies PCB.

(2) Let (T, Ψ) be a pregame satisfying PCB. Then given any positive real number ρ, construct a new pregame (T, Ψ_ρ) where the domain of Ψ_ρ is restricted to profiles f where, for each $t = 1, \cdots, T$, either $\frac{f_t}{\|f\|} > \rho$ or $f_t = 0$ (thickness). Then (T, Ψ_ρ) satisfies SGE on its domain.

The equivalence, with thickness, of small group effectiveness with per capita boundedness indicates that SGE is an apparently mild yet powerful condition. But, as we see above, if a pregame satisfies SGE then, given $\beta > 0$, for appropriate choice of δ and T it holds that all games generated by the pregame belong to a parameterized collection of games $\Gamma((\delta, T), (\beta, B))$. Thus, our conditions on parameterized collections of games are less restrictive than those on pregames (as in [47,44], or [42]) and, with thickness, less restrictive than the earlier condition of PCB.

The concept of small group effectiveness requires that almost all feasible gains to collective activities can be achieved by groups bounded in absolute size. A related concept requires that almost all improvement be feasible for groups bounded in absolute size. A pregame (Ω, Ψ) satisfies *small group effectiveness for improvement* if for each positive real number $\epsilon > 0$ there is an integer $\eta_2(\epsilon)$ with the following property:

For any profile f and any payoff function $x : \sigma(f) \to R_+$
if $x(f) + \epsilon \|f\| < \Psi(f)$ then there is a subprofile g of f such that
$\|g\| \leq \eta_2(\epsilon)$ and $x(g) + \frac{\epsilon}{2} \|g\| < \Psi(g)$.

The pregame framework may also hide what makes the results work – the facts that there are many close substitutes for most players and that groups bounded is size can nearly exhaust gains to collective activities. In addition, since the pregame framework specifies payoffs for all groups, no matter how large, in general it is difficult, if not impossible to estimate the pregame function Ψ. In contrast, within the framework of parameterized collections, there are only four parameters to be estimated – δ, T, β, and B. The notion of β-effective B-bounded groups makes explicit how close coalitions bounded in size by B are to being able to realize all gains to collective activities for a given game.

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Existence of equilibria for economies with externalities and a measure space of consumers*

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Summary. This paper considers an exchange economy with a measure space of agents and consumption externalities, which take into account two possible external effects on consumers' preferences: dependence upon prices and dependence upon other agents' consumption. We first consider a model with a general externality mapping and we then treat the particular case of reference coalition externalities, in which the preferences of each agent a are influenced by prices and by the global or the mean consumption of the agents in finitely many (exogenously given) reference coalitions associated with agent a. Our paper provides existence results of equilibria in both models when consumers have transitive preferences. It extends in exchange economies the standard results by Aumann [2], Schmeidler [16], Hildenbrand [12], and previous results by Greenberg et al. [11] for price dependent preferences, Schmeidler [17] for fixed reference coalitions and Noguchi [15] for a more particular concept of reference coalitions. We also mention related results obtained independently by Balder [4].

Keywords and Phrases: Externalities, Reference coalitions, Measure space of agents, Equilibrium.

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1 Introduction

This paper considers an exchange economy with a measure space of agents and consumption externalities, which take into account two possible external effects on consumers' preferences: dependence upon prices and dependence upon other agents' consumptions.

The price dependence externality is a long recognized problem, which recently found new applications in the study of financial markets, where a two-period temporary equilibrium model has a reduced form as a Walrasian model with price dependent preferences. For the existence of equilibria in economies with a measure space of agents and price externalities we refer to Greenberg et al. [11], who use ordered preferences and use a game-theoretical approach, which exploits Debreu's original idea of introducing a price-setting player.

The dependence upon other agents' consumptions has also been considered in recent years, with attempts to have the same level of generality for a measure space of agents as for the case of finitely many agents. In this paper, we will consider agents with transitive strict preferences which are not necessarily complete as in Schmeidler [16]. The question arises if one can drop transitivity and completeness simultaneously but our approach does not cover this case, for which we refer to Khan and Vohra [14] and Yannelis [18]. Our treatment of the existence problem differs from [14] and [18] in considering a weaker convexity assumption on preferences that allows us to encompass the results of Aumann [2], Greenberg et al. [11] and Schmeidler [17]; indeed the way they model externalities does not allow for a convexifying effect on aggregation, even if the measure space is nonatomic.

We first present the model with a general externality mapping. The preference relation of each agent a, which may depend upon the externality e in a given *externality space* E, is denoted by $\prec_{a,e}$ and the influence of the externality on agents' preferences is represented by a given (exogenous) *externality mapping* Φ, which associates to each agent a, each price p and each (integrable) consumption allocation f, the externality $e = \Phi(a, p, f) \in E$. Thus, given the price p and the allocation f the choices of agent a will be made with the preference relation $\prec_{a,\Phi(a,p,f)}$. Our model considers "finitely many externality effects"; that is, formally, the *externality space* E is assumed to be a subset of a finite dimensional Euclidean space. This makes an explicit restriction on the couples (p, f) of prices and (integrable) consumption allocation that can influence agents' preferences via the externality mapping Φ.

The previous model allows us to consider the particular case of reference coalitions externalities, in which the preferences of each agent a are influenced by prices and by the global or the mean consumption of the agents in finitely many (exogenously given) reference coalitions associated with agent a. Let $(A, \mathcal{A}, \nu)$ be the measure space of consumers, and for each agent $a \in A$ and each price p, let $C_k(a, p) \in \mathcal{A}$ $(k = 1, \ldots, K)$ be finitely many (exogenously given) *reference coalitions*. Each coalition $C_k(a, p) \in \mathcal{A}$ can be considered as the reference class of agent a for a particular group of commodities, say clothes, music, housing or travel. The externality dependence operates via *reference consumption vectors* (for the particular group of commodities) which can be obtained either as the global or as the mean consumption of agents in the reference coalition of agent a. With a single reference coalition (i.e., $K = 1$), the externality mappings Φ_1 and Φ_2 corresponding to the global and the mean consumption are defined, respectively, by:

$$\Phi_1(a,p,f) = \int_{C(a,p)} f(\alpha)d\nu(\alpha);$$

$$\Phi_2(a,p,f) := \begin{cases} \dfrac{1}{\nu[C(a,p)]} \displaystyle\int_{C(a,p)} f(\alpha)d\nu(\alpha) & \text{if } \nu[C(a,p)] > 0, \\ 0 & \text{if } \nu[C(a,p)] = 0. \end{cases}$$

Both models consider finitely many external effects, with externality space $E{=}\mathbf{R}^H_+$, the closed positive orthant of the commodity space $\mathbf{R}^H$, denoting by H the number of commodities in the economy. The first case can be illustrated by network effects, i.e., the number of persons connected to a network (internet or mobile phone) and the global consumption in the reference coalition may be important for some agent to decide to connect herself. In the second model, only the mean consumption is used to define the "reference trend".

The aim of this paper is to provide an existence result of equilibria in the model with a general externality mapping and then to deduce from it an existence result in the reference coalitions model both for global and mean dependence. In an exchange economy, we extend the classical results by Aumann [2], Schmeidler [16], Hildenbrand [12], and previous results by Greenberg et al. [11], for price dependent preferences. In the reference coalition model, our result encompasses the result by Schmeidler [17] in the case of constant reference coalitions (i.e., when the coalition does not depend on agent a and the price system). We also generalize the existence result by Noguchi [15] who considers, for each agent a, a particular reference coalition, which consists of all the agents who belong to a certain income range associated with agent a (see Sect. 3.3). Finally, we mention the existence results obtained independently by Balder [4], which also generalizes those of [2, 11] and [17], without being directly comparable with ours since the externality dependence is defined in a different way and agents have ordered preferences.

The paper is organized as follows. In Section 2, we present the model with a general externality mapping and the associated concept of equilibrium [Sect. 2.1] and we state the main existence results. Our first existence result is stated under a strong convexity assumption on preferences [Sect. 2.2]. We then weaken this convexity assumption [Sect. 2.3] so that we can encompass the Aumann-Schmeidler-Hildenbrand existence result in exchange economies. In Section 3, we present the reference coalitions model [Sect. 3.1], and we deduce from our main result the existence of equilibria in this model [Sect. 3.2]. Finally, we present the particular case of a reference coalitions model considered by Noguchi [15] and deduce his existence result. In Section 4 we give the proof of our main existence result [Theorem 2] . We first prove an existence result [Theorem 4] under the additional assumption that the consumption set correspondences are integrably bounded [Sect. 4.1]. We then deduce the main result in the general case from it [Sect. 4.2]. Finally, in the Appendix we present the main properties of the individual quasi-demand correspondence that are used in the proof of the existence theorem, along with some properties of the Noguchi's reference coalition and a counterexample due to Balder [4].

2 The model and the existence result

2.1 The model and the equilibrium notion

We consider an exchange economy with a finite set H of commodities. The commodity space[1] is represented by the vector space $\mathbf{R}^H$.

The set of consumers is defined by a measure space $(A, \mathcal{A}, \nu)$, where $\mathcal{A}$ is a σ-algebra of subsets in A and ν is a measure on $\mathcal{A}$. An element $C \in \mathcal{A}$ is a possible group of consumers, also called a coalition. Each consumer a is endowed with a consumption set $X(a) \subset \mathbf{R}^H$, an initial endowment $\omega(a) \in \mathbf{R}^H$ and a strict preference relation $\prec_{a,e}$ on $X(a)$, which allows for dependence on externalities $e \in E$ (called the *externality space*), in a way which will be specified hereafter. The set $X(a)$ represents the possible consumptions of consumer a. A consumption allocation of the economy specifies the possible consumption of each consumer, and is formally a selection of the correspondence $a \to X(a)$, which is assumed to be integrable. The set of consumption allocations is denoted by $\mathcal{L}_X$. We assume also that the initial endowment mapping $\omega : A \to \mathbf{R}^H$ is integrable and thus the total initial endowment of the economy is $\int_A \omega(a) d\nu(a)$.

Specific to this economy is the fact that price externalities and consumption externalities can influence the preference relation of each agent a. Thus, given the price $p \in \mathbf{R}^H$ and the allocation $f \in \mathcal{L}_X$, the choices of agent a will be made with the strict preference relation $\prec_{a,\Phi(a,p,f)}$, where $\Phi : A \times \mathbf{R}^H \times \mathcal{L}_X \to E$ is a given mapping, called the *externality mapping*.

In the presence of externalities, the exchange economy is completely summarized by the couple $(\mathcal{E}, \Phi)$, where the externality space E and the externality mapping Φ are defined as above and $\mathcal{E}$ specifies the characteristics of the consumers

$$\mathcal{E} = \{\mathbf{R}^H, E, (A, \mathcal{A}, \nu), (X(a), (\prec_{a,e})_{e\in E}, \omega(a))_{a\in A}\}.$$

We now give the definition of an equilibrium in this economy.

Definition 1 *An equilibrium of the economy $(\mathcal{E}, \Phi)$ is an element $(f^*, p^*) \in \mathcal{L}_X \times \mathbf{R}^H$ such that $p^* \neq 0$ and*
(a) [*Preference Maximization*] *for a.e. $a \in A$, $f^*(a)$ is a maximal element for $\prec_{a,e^*{}_a}$ in the budget set $B(a, p^*) := \{x \in X(a) \mid p^* \cdot x \leq p^* \cdot \omega(a)\}$, where*

[1] For a finite set H we denote by $\mathbf{R}^H$ the set of all mappings from H to $\mathbf{R}$. An element x of $\mathbf{R}^H$ will be denoted by $(x_h)_{h\in H}$ or simply by (x_h) when no confusion is possible. For two elements $x = (x_h)$, $x' = (x'_h)$ in $\mathbf{R}^H$, we denote by $x \cdot x' = \sum_{h\in H} x_h x'_h$ the scalar product, by $\|x\| = \sqrt{x \cdot x}$ the Euclidean norm and by $B(x_0, r) = \{x \in \mathbf{R}^H \mid \|x - x_0\| \leq r\}$ the closed ball. For $X \subset \mathbf{R}^H$, we denote by int$X$, $\overline{X}$ and co$X$, respectively, the interior, the closure and the convex hull of X. The notations: $x \leq x'$, $x < x'$, $x << x'$ mean, respectively, that for all $h \in H$, $x_h \leq x'_h$, [$x \leq x'$ and $x \neq x'$], and $x_h < x'_h$; we let $\mathbf{R}^H_+ := \{x \in \mathbf{R}^H \mid 0 \leq x\}$ and $\mathbf{R}^H_{++} := \{x \in \mathbf{R}^H \mid 0 << x\}$. We also let $\mathbf{1} := (1, \ldots, 1) \in \mathbf{R}^H$ and the canonical basis $\{\mathbf{e^i} \mid i \in H\}$ of $\mathbf{R}^H$ be defined by $e^i_h = 1$, if $h = i$ and $e^i_h = 0$, if $h \neq i$.

$e^*{}_a := \Phi(a, p^*, f^*)$, *that is,* $f^*(a) \in B(a, p^*)$ *and there is no* $x \in B(a, p^*)$ *such that* $f^*(a) \prec_{a,e^*_a} x$;
(b) [*Market Clearing*] $\int_A f^*(a) d\nu(a) = \int_A \omega(a) d\nu(a)$.

2.2 A first existence result for general externality mappings

We present the list of assumptions that the economy $(\mathcal{E}, \Phi)$ will be required to satisfy. Let $(A, \mathcal{A}, \nu)$ be a measure space, we recall that a measurable set $\bar{A} \in \mathcal{A}$ is an atom if $\nu(\bar{A}) > 0$ and for every $C \in \mathcal{A}$ such that $C \subset \bar{A}$, one has $[\nu(C) = 0$ or $\nu(\bar{A} \setminus C) = 0]$ and we denote by A_{na} the nonatomic part of A; that is, the complementary in A of the union of all the atoms of A. We denote by $L^1(A, \mathbf{R}^H)$ the space of equivalence classes of integrable mappings from A to $\mathbf{R}^H$ and we let $\|f\|_1 := \int_A \|f(a)\| d\nu(a)$, which defines a norm on $L^1(A, \mathbf{R}^H)$. The space $L^1(A, \mathbf{R}^H)$ will be endowed with two different topologies, the norm topology defined by the norm $\|f\|_1$ and the weak topology $\sigma(L^1, L^\infty)$; we recall that a sequence $\{f^n\}$ converges weakly to f if and only if $\sup_n \|f^n\|_1 < \infty$ and $\int_C f^n(a) d\nu(a) \to \int_C f(a) d\nu(a)$, for every $C \in \mathcal{A}$ (see Dunford and Schwartz [8], p. 291).

Assumption A *The measure space* $(A, \mathcal{A}, \nu)$ *is positive, finite, complete and* $L^1(A, \mathbf{R}^H)$ *is separable for the norm topology;*

Assumption C *For a.e.* $a \in A$, *every* $(e, x) \in E \times X(a)$:
(i) E *is a closed subset of* $\mathbf{R}^N$ *and* $X(a)$ *is a closed, convex subset of* $\mathbf{R}^H_+$;
(ii)*[Irreflexivity and transitivity]* $\prec_{a,e}$ *is irreflexive*[2] *and transitive*[3];
(iii)*[Convexity of preferences on atoms] if* $a \in A \setminus A_{na}$ *the set* $\{x' \in X(a) \mid$ *not*$[x' \prec_{a,e} x]\}$ *is convex*;
(iv)*[Continuity] the sets:*

$$\{x' \in X(a) \mid x \prec_{a,e} x'\} \text{ and } \{(x', e') \in X(a) \times E \mid x' \prec_{a,e'} x\}$$

are open, respectively, in $X(a)$ *and in* $X(a) \times E$ *(for their relative topologies);*
(v)*[Measurability] the consumption set correspondence* $a' \to X(a')$ *and the preference correspondence* $(a', e') \to \prec_{a',e'}$ *are measurable*[4];
(vi) $\omega \in \mathcal{L}_X$, *i.e.,* $\omega : A \to \mathbf{R}^H$ *is integrable and* $\omega(a') \in X(a')$ *for a.e.* $a' \in A$;

Assumption M (i)*[Monotonicity] For a.e.* $a \in A$, $X(a) := \mathbf{R}^H_+$ *and for every* $e \in E$ *and every* x, x' *in* $X(a)$, $x < x'$ *implies* $x \prec_{a,e} x'$;
(ii)*[Strong survival]* $\int_A \omega(\alpha) d\nu(\alpha) >> 0$.

The above assumptions are standard and need no special comments. In a model without externalities (say $E = \{0\}$), they coincide with Aumann-Schmeidler's assumptions, as discussed in the next section.

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[2] That is, for every $x \in X(a)$, not$[x \prec_{a,e} x]$.

[3] That is, for every $x, x', x'' \in X(a)$, $x \prec_{a,e} x'$ and $x' \prec_{a,e} x''$ imply $x \prec_{a,e} x''$.

[4] We recall that a correspondence F, from a measurable space $(A, \mathcal{A})$ to $\mathbf{R}^n$, is said to be $\mathcal{A}-$measurable, or simply measurable, if its graph is a measurable set, i.e., $G_F := \{(a, x) \in A \times \mathbf{R}^H \mid x \in F(a)\}$ belongs to $\mathcal{A} \otimes \mathcal{B}(\mathbf{R}^n)$, where $\mathcal{B}(\mathbf{R}^n)$ denotes the $\sigma-$algebra of Borel subsets of $\mathbf{R}^n$ and $\mathcal{A} \otimes \mathcal{B}(\mathbf{R}^n)$ denotes the $\sigma-$algebra product. The preference correspondence $(a, e) \to \prec_{a,e}$ is said to be measurable if the correspondence $(a, e) \to \{(x, x') \in X(a) \times X(a) \mid x \prec_{a,e} x'\}$ is $\mathcal{A} \otimes \mathcal{B}(E)-$measurable.

The next assumptions concern the externality side. Hereafter, we suppose that for every $(a,p) \in A \times \mathbf{R}^H$, $\Phi(a,p,f) = \Phi(a,p,g)$ if $f = g$ almost everywhere on A. Without any risk of confusion, this allows us to consider Φ as a mapping $\Phi : A \times \mathbf{R}^H \times L_X \to E$, where

$$L_X := \{f \in L^1(A, \mathbf{R}^H) \mid f(a) \in X(a) \text{ a.e. } a \in A\}.$$

Assumption E *[Caratheodory]***(i)** *E is a closed subset of $\mathbf{R}^N$;*
(ii) *for all $(p,f) \in \mathbf{R}^H_+ \times L_X$, the mapping $a \to \Phi(a,p,f)$ is measurable;*
(iii) *for a.e. $a \in A$, for every sequence $\{p^n\} \subset \mathbf{R}^H_+$ converging to p and every integrably bounded[5] sequence $\{f^n\} \subset L_X$ converging weakly to f, the sequence $\{\Phi(a,p^n,f^n)\}$ converges to $\Phi(a,p,f)$;*

Assumption EB *[Boundedness] For all bounded sequence $\{(p^n, f^n)\} \subset \mathbf{R}^H_+ \times L_X$ and for a.e. $a \in A$, the sequence $\{\Phi(a,p^n,f^n)\}$ is bounded in E.*

Assumption EC$_0$ *[Convexity of preferences on the nonatomic part] For a.e. $a \in A_{na}$ and every $(e,x) \in E \times X(a)$, the set $\{x' \in X(a) \mid$ not$[x' \prec_{a,e} x]\}$ is convex.*

The above Caratheodory assumption is a standard regularity assumption. The boundedness assumption **EB** will be satisfied, in particular, in the reference coalitions model presented hereafter. We also point out that **EB** is satisfied when the correspondence $a \to X(a)$ is integrably bounded (see Assumption **IB** hereafter) and **C** and **E** hold.

The last assumption additionally assumes the convexity of preferences on the nonatomic part of A (whereas in **C** it was only assumed on the atomic part). This assumption will be discussed and weakened in Section 2.3.

We can now state our first existence result.

Theorem 1 *The exchange economy with externalities $(\mathcal{E},\Phi)$ admits an equilibrium (f^*, p^*) with $p^* >> 0$, if it satisfies Assumptions* **A**, **C**, **M**, **E**, **EB** *and* **EC$_0$**.

Theorem 1 is a direct consequence of a more general result [Theorem 2] that will be stated in the following section, which is devoted to the weakening of the convexity assumption **EC$_0$**.

2.3 Weakening the convexity assumption EC$_0$

Since Aumann [2], most of the existence results in models without externalities do not assume convexity of preferences on the nonatomic part A_{na} of the measure space of consumers (i.e., Assumption **EC$_0$**). To be able to cover Aumann's existence result, we will now weaken the convexity assumption **EC$_0$**. This will allow us to encompass the known existence results in the three following important cases.

E$_1$: No externalities [Aumann [2], Schmeidler [16], Hildenbrand [13]] $E_1 = \{0\}$ *and the mapping $\Phi_1 : A \times \mathbf{R}^H_+ \times L_X \to E_1$ is defined by $\Phi_1(a,p,f) = 0$.*

[5] That is, there is some integrable function $\rho : A \to \mathbf{R}_+$, such that $\sup_n \|f^n(a)\| \le \rho(a)$ for a.e. $a \in A$.

$\mathbf{E}_2$: Price dependent preferences [Greenberg et al. [11]] $E_2 = \mathbf{R}_+^H$ *and the mapping* $\Phi_2 : A \times \mathbf{R}_+^H \times L_X \to E_2$ *is defined by* $\Phi_2(a, p, f) = p$.

$\mathbf{E}_3$: Constant reference coalitions [Schmeidler [17]] $E_3 = (\mathbf{R}_+^H)^K$ *and the mapping* $\Phi_3 : A \times \mathbf{R}_+^H \times L_X \to E_3$ *is defined by*

$$\Phi_3(a, p, f) := \left(\int_{C_1} f(a) d\nu(a), \dots , \int_{C_K} f(a) d\nu(a) \right),$$

where the sets C_k $(k = 1, \dots , K)$ *are nonempty measurable subsets of* A_{na}, *which are pairwise disjoint, i.e.,* $C_j \cap C_k = \emptyset$ *for every* $j \neq k$.

In the three above cases, the externality mappings Φ_i $(i = 1, 2, 3)$ are "convex" on A_{na} in the sense of the following definition (see Proposition 1 below):

Definition 2 *We say that the externality mapping* $\Phi : A \times \mathbf{R}_+^H \times L_X \to E$ *is "convex" on the measurable set* $C \subset A$ *if for every* $p \in \mathbf{R}_+^H$, *for every* $\{f_i\}_{i \in I} \subset L_X$ *(I finite) and every* $f \in L_X$ *such that*

for a.e. $\alpha \in C$, $f(\alpha) \in co\{f_i(\alpha) \mid i \in I\}$,

there exists $f^* \in L_X$ *such that:*

for a.e. $\alpha \in C$, $f^*(\alpha) \in \{f_i(\alpha) \mid i \in I\}$,
for a.e. $\alpha \in A \setminus C$, $f^*(\alpha) = f(\alpha)$,
for a.e. $a \in A$, $\Phi(a, p, f) = \Phi(a, p, f^*)$ *and* $\int_A f(\alpha) d\nu(\alpha) = \int_A f^*(\alpha) d\nu(\alpha)$.

We now can state our main existence result, which extends Theorem 1 and allows us to cover the three above cases $\mathbf{E}_1$, $\mathbf{E}_2$, $\mathbf{E}_3$. For this, we need to introduce a new Convexity Assumption **EC** , which is clearly satisfied in the two important cases: (i) convexity of the preferences on A_{na} (i.e., Assumption $\mathbf{EC}_0$ of Theorem 1), and (ii) "convexity" of Φ on A_{na}.

Theorem 2 *The exchange economy with externalities* $(\mathcal{E}, \Phi)$ *admits an equilibrium* (f^*, p^*) *with* $p^* >> 0$, *if it satisfies Assumptions* **A**, **C**, **M**, **E**, **EB**, *together with the following one:*

Assumption EC *There exists a measurable set* $C \subset A_{na}$ *such that:*
(i) *for a.e.* $a \in A_{na} \setminus C$, *the preferences are convex, that is, for every* $(e, x) \in E \times X(a)$, *the set* $\{x' \in X(a) \mid not[x' \prec_{a,e} x]\}$ *is convex, and*
(ii) *the externality mapping* Φ *is "convex" on* C.

The proof of Theorem 2 is given in Section 4 and relies on an intermediary result (Theorem 4) in which the monotonicity assumption **M** is replaced by the assumption that the consumption correspondence $a \to X(a)$ is integrably bounded (which is clearly stronger than **EB** holds). In this case, it is worth pointing out that without Assumption **EC**, the corresponding existence result (Theorem 4) may not hold as shown in the Appendix with a counterexample due to Balder [4].

We end this section by showing that the three above externality mappings Φ_i $(i = 1, 2, 3)$ satisfy Assumption **EC**, and also a stronger Assumption $\mathbf{EC}_1$ (in which no convexity assumption on preferences is made).

Proposition 1 *(a) In the three above cases* $\mathbf{E}_1$, $\mathbf{E}_2$, $\mathbf{E}_3$, *the externality mappings* $\Phi = \Phi_i$ $(i = 1, 2, 3)$ *satisfy the following assumption:*

$\mathbf{EC}_1$ *There exists a measurable set* $C \subset A_{na}$ *such that:* **(i)** *the externality mapping* Φ *only depends on* $f_{|C}$, *in the sense that,* $\Phi(a, p, f) = \Phi(a, p, g)$, *if* $f_{|C} = g_{|C}$, *and* **(ii)** *the externality mapping* Φ *is "convex" on* C.
(b) If Assumption $\mathbf{EC}_1$ *holds, then* Φ *is "convex" on* A_{na}, *hence Assumption* $\mathbf{EC}$ *holds.*

Proof.
(a) Assumption $\mathbf{EC}_1$ is satisfied for $C = A_{na}$ for the cases $\mathbf{E}_1$ and $\mathbf{E}_2$ and for $C = \cup_{k=1}^{K} C_k$ for $\mathbf{E}_3$. This is a consequence of Lyapunov's theorem, applied to A_{na} in the first two cases and applied successively to each C_k $(k = 1, \ldots, K)$ in the latter case.

(b) We show that the externality mapping Φ is "convex" on A_{na}. Indeed, for every $p \in \mathbf{R}_+^H$, let $\{f_i\}_{i \in I} \subset L_X$ (I finite) and $f \in L_X$ such that, for a.e. $\alpha \in A_{na}$, $f(\alpha) \in \text{co}\{f_i(\alpha) \mid i \in I\}$. Since Φ is "convex" on $C \subset A_{na}$ (by $\mathbf{EC}_1$), there exists an integrable mapping $f' : A \to \mathbf{R}^H$ such that, for a.e. $\alpha \in C$, $f'(\alpha) \in \{f_i(\alpha) \mid i \in I\}$, for a.e. $a \in A$, $\Phi(a, p, f) = \Phi(a, p, f')$ and $\int_C f(\alpha) d\nu(\alpha) = \int_C f'(\alpha) d\nu(\alpha)$. From above, for a.e. $\alpha \in A_{na} \setminus C$, $f(\alpha) \in \text{co}\{f_i(\alpha) \mid i \in I\}$, hence, from Lyapunov's theorem, there exists an integrable mapping $f'' : A_{na} \setminus C \to \mathbf{R}^H$ such that $f''(\alpha) \in \{f_i(\alpha) \mid i \in I\}$ and $\int_{A_{na} \setminus C} f(\alpha) d\nu(\alpha) = \int_{A_{na} \setminus C} f''(\alpha) d\nu(\alpha)$. We consider now the mapping $f^* : A \to \mathbf{R}^H$ defined by $f^*(\alpha) = f'(\alpha)$ for every $\alpha \in C$, $f^*(\alpha) = f''(\alpha)$ for every $\alpha \in A_{na} \setminus C$ and $f^*(\alpha) = f(\alpha)$ for every $\alpha \in A \setminus A_{na}$ and we note that, for a.e. $\alpha \in A_{na}$, $f^*(\alpha) \in \{f_i(\alpha) \mid i \in I\}$ and for a.e. $\alpha \in A \setminus A_{na}$, $f^*(\alpha) = f(\alpha)$. Moreover, from above, for a.e. $a \in A$, $\Phi(a, p, f) = \Phi(a, p, f') = \Phi(a, p, f^*)$ (since $f'_{|C} = f^*_{|C}$) and $\int_A f(\alpha) d\nu(\alpha) = \int_A f^*(\alpha) d\nu(\alpha)$. □

3 The reference coalitions model

3.1 The model and the existence result

The general model of an exchange economy with externalities $(\mathcal{E}, \Phi)$ allows us to consider the reference coalitions model that we now present as an extension of Schmeidler's model.

We suppose that, given a price $p \in \mathbf{R}_+^H$, each agent a has finitely many reference coalitions of agents, $C_k(a, p) \in \mathcal{A}$ $(k = 1 \ldots K)$, whose consumption choices influence the preferences of agent a in a way defined precisely hereafter. Hence, the reference coalitions may depend upon the agent and also on the price that prevails; this differs from Schmeidler's model, in which the reference coalitions are constant. We will assume that each agent a is influenced either by the global consumption or by the mean consumption of agents in the coalition $C_k(a, p)$.

The "global dependence" case is characterized by the externality space $E := (\mathbf{R}^H_+)^K$ and the externality mapping $\Phi_1^{\mathcal{C}} : A \times \mathbf{R}^H_+ \times L_X \to E$ defined by

$$\Phi_1^{\mathcal{C}}(a,p,f) = \left(\int_{C_1(a,p)} f(\alpha)d\nu(\alpha), \ldots , \int_{C_K(a,p)} f(\alpha)d\nu(\alpha) \right).$$

The "mean dependence" case, is characterized by the externality space $E := (\mathbf{R}^H_+)^K$ and the externality mapping $\Phi_2^{\mathcal{C}} : A \times \mathbf{R}^H_+ \times L_X \to E$, defined by

$$\Phi_2^{\mathcal{C}}(a,p,f) = (\Phi_{21}^{\mathcal{C}}(a,p,f), \ldots , \Phi_{2K}^{\mathcal{C}}(a,p,f)),$$
$$\Phi_{2k}^{\mathcal{C}}(a,p,f) := \begin{cases} \dfrac{1}{\nu[C_k(a,p)]} \displaystyle\int_{C_k(a,p)} f(\alpha)d\nu(\alpha) & \text{if } \nu[C_k(a,p)] > 0 \\ 0 & \text{if } \nu[C_k(a,p)] = 0. \end{cases}$$

The reference coalitions model can thus be summarized by the exchange economies with externalities $(\mathcal{E}, \Phi_1^{\mathcal{C}})$ and $(\mathcal{E}, \Phi_2^{\mathcal{C}})$, where

$$\mathcal{E} = \{\mathbf{R}^H, (\mathbf{R}^H_+)^K, (A, \mathcal{A}, \nu), (X(a), (\prec_{a,e})_{e \in (\mathbf{R}^H_+)^K}, \omega(a))_{a \in A}\},$$
$$\mathcal{C} := (C_1(a,p), \ldots , C_K(a,p))_{(a,p) \in A \times \mathbf{R}^H_+},$$

and the externality mappings $\Phi_1^{\mathcal{C}}$ and $\Phi_2^{\mathcal{C}}$ are defined as above (and correspond, respectively, to the global and the mean dependence).

Before stating the existence result, we recall the following notations; for C_1, C_2 in $\mathcal{A}$, we let $C_1 \Delta C_2 := (C_1 \setminus C_2) \cup (C_2 \setminus C_1)$ and we let the characteristic function $\chi_{C_1} : A \to \mathbf{R}$ be defined by $\chi_{C_1}(a) = 1$ if $a \in C_1$ and $\chi_{C_1}(a) = 0$ if $a \notin C_1$.

Theorem 3 *The exchange economy with reference coalitions externalities* $(\mathcal{E}, \mathcal{C})$ *admits an equilibrium* (p_1^*, f_1^*) *with* $p_1^* >> 0$ *for global dependence and an equilibrium* (p_2^*, f_2^*) *with* $p_2^* >> 0$ *for mean dependence (i.e.,* $(\mathcal{E}, \Phi_i^{\mathcal{C}})$ *admits an equilibrium* (p_i^*, f_i^*) $(i = 1, 2)$*), if it satisfies Assumptions* **A**, **C**, **M**, $\mathbf{EC}_0$, *together with:*

Assumption R [Reference Coalition Side]
For every $k = 1, \ldots , K$ *for a.e.* $a \in A$ *and every* $p \in \mathbf{R}^H_+$*:*
(i) $\nu[C_k(a,p)] > 0$*;* **(ii)** *for every* $\lambda > 0$, $C_k(a, \lambda p) = C_k(a,p)$*;*
(iii) *for every sequence* $p^n \to p$ *in* $\mathbf{R}^H_+$, $\nu[C_k(a,p^n) \Delta C_k(a,p)] \to 0$*;*
(iv) *the set* $\{(a', a'') \in A \times A \mid a'' \in C_k(a', p)\} \in \mathcal{A} \otimes \mathcal{A}$.

The proof of Theorem 3 is given in Section 3.2.

3.2 Proof of Theorem 3

It is a consequence of Theorem 1 and we only have to prove that the externality mappings $\Phi_i^{\mathcal{C}}$ $(i = 1, 2)$ satisfy Assumptions **E** and **EB**. This will be done in the following three steps, noticing first that **[E(i)]** is satisfied.

Step 1 **[E(ii)]:** *For every* $(p, f) \in \mathbf{R}_+^H \times L_X$*, the mapping* $a \to \Phi_i^{\mathcal{C}}(a, p, f)$ $(i = 1, 2)$ *is measurable on* A*.*

Proof. Let $(p, f) \in \mathbf{R}_+^H \times L_X$. We first show that the mapping

$$a \to \Phi_1^{\mathcal{C}}(a, p, f) := \int_{C(a,p)} f(\alpha) d\nu(\alpha)$$

is measurable. We notice that the mappings $(a, \alpha) \to f(\alpha)$ and $(a, \alpha) \to \chi_{C(a,p)}(\alpha)$ are both measurable on $A \times A$ (endowed with the product $\sigma-$algebra $\mathcal{A} \otimes \mathcal{A}$), from the fact that $f \in L^1(A, \mathbf{R}_+^H)$ and Assumption **R(iv)** respectively. Hence, the mapping $(a, \alpha) \to \chi_{C(a,p)}(\alpha) f(\alpha)$ is measurable on $A \times A$.

Since $\chi_{C(a,p)}(\alpha) f(\alpha) \leq f(\alpha)$ for a.e. $(a, \alpha) \in A \times A$ and $f \in L^1(A, \mathbf{R}_+^H)$, applying the measurability part of Fubini's theorem, the mapping

$$a \to \int_A \chi_{C(a,p)}(\alpha) f(\alpha) d\nu(\alpha) = \int_{C(a,p)} f(\alpha) d\nu(\alpha)$$

is correctly defined and is measurable on A. Hence, the mapping $\Phi_1^{\mathcal{C}}$ satisfies Assumption **E(ii)**.

We now show that the mapping

$$\Phi_2^{\mathcal{C}}(a, p, f) := \begin{cases} \dfrac{1}{\nu[C(a,p)]} \Phi_1^{\mathcal{C}}(a, p, f) & \text{if } \nu[C_k(a,p)] > 0 \\ 0 & \text{if } \nu[C_k(a,p)] = 0 \end{cases}$$

is measurable on A. Using the above argument for $f = 1$, we deduce that the mapping $a \to \nu[C(a, p)]$ is measurable on A. Since $\nu[C(a, p)] > 0$ for a.e. $a \in A$ (by **R(i)**), in view of the measurability property of $\Phi_1^{\mathcal{C}}$, the mapping $\Phi_2^{\mathcal{C}}$ satisfies Assumption **E(ii)**. □

Step 2 **[E(iii)]:** *For a.e.* $a \in A$ *and for every sequence* $\{p^n\}$ *converging to* p *in* $\mathbf{R}_+^H$ *and every integrably bounded sequence* $\{f^n\}$ *converging weakly to* f *in* L_X*, the sequence* $\{\Phi_i^{\mathcal{C}}(a, p^n, f^n)\}$ *converges to* $\Phi_i^{\mathcal{C}}(a, p, f)$ $(i = 1, 2)$*.*

Proof. Let $\{(p^n, f^n)\}$ as above. We first prove that $\Phi_1^{\mathcal{C}}$ satisfies **[E(iii)]**, i.e., for a.e. $a \in A$

$$\Phi_1^{\mathcal{C}}(a, p^n, f^n) = \int_{C(a,p^n)} f^n(\alpha) d\nu(\alpha) \to \int_{C(a,p)} f(\alpha) d\nu(\alpha) = \Phi_1^{\mathcal{C}}(a, p, f).$$

For this, one notices that

$$\left\| \int_{C(a,p^n)} f^n(\alpha) d\nu(\alpha) - \int_{C(a,p)} f(\alpha) d\nu(\alpha) \right\| \leq$$
$$\left\| \int_{C(a,p^n)} f^n(\alpha) d\nu(\alpha) - \int_{C(a,p)} f^n(\alpha) d\nu(\alpha) \right\| + \left\| \int_{C(a,p)} [f^n(\alpha) - f(\alpha)] d\nu(\alpha) \right\|.$$

For the second term, since $\{f^n\}$ converges weakly to f, one has

$$\left\| \int_{C(a,p)} [f^n(\alpha) - f(\alpha)] d\nu(\alpha) \right\| \to 0.$$

For the first term we have

$$\left\| \int_A \chi_{C(a,p^n)}(\alpha) f^n(\alpha) d\nu(\alpha) - \int_A \chi_{C(a,p)}(\alpha) f^n(\alpha) d\nu(\alpha) \right\| \leq$$
$$\int_A |\chi_{C(a,p^n)}(\alpha) - \chi_{C(a,p)}(\alpha)| \| f^n(\alpha) \| d\nu(\alpha) \leq$$
$$\int_A |\chi_{C(a,p^n)}(\alpha) - \chi_{C(a,p)}(\alpha)| \rho(\alpha) d\nu(\alpha) = \int_{C(a,p^n) \Delta C(a,p)} \rho(\alpha) d\nu(\alpha),$$

recalling that the sequence $\{f^n\}$ is integrably bounded, hence, for some integrably function $\rho : A \to \mathbf{R}_+$, one has $\sup_n \|f^n(a)\| \leq \rho(a)$ for a.e. $a \in A$. Moreover, for a.e. $a \in A$, $\nu[C(a,p^n) \Delta C(a,p)] \to 0$ when $p^n \to p$ (by **R(iii)**), hence

$$\int_{C(a,p^n) \Delta C(a,p)} \rho(\alpha) d\nu(\alpha) \to 0,$$

since the mapping $C \to \int_C \rho(\alpha) d\nu(\alpha)$, from $\mathcal{A}$ to $\mathbf{R}_+$, is a positive measure, absolutely continuous with respect to ν. This implies that the first term converges to zero[6] and ends the proof that $\Phi_1^{\mathcal{C}}$ satisfies **[E(iii)]**.

We now prove that $\Phi_2^{\mathcal{C}}$ satisfies **[E(iii)]**. Since, for a.e. $a \in A$, $\Phi_1^{\mathcal{C}}(a, p^n, f^n) \to \Phi_1^{\mathcal{C}}(a, p, f)$ and $\nu[C(a,p)] > 0$ (by **R(i)**), it suffices to show that $\nu[C(a,p^n)] \to \nu[C(a,p)]$. Indeed, one has

$$|\nu[C(a,p^n)] - \nu[C(a,p)]| = \left| \int_A \chi_{C(a,p^n)}(\alpha) d\nu(\alpha) - \int_A \chi_{C(a,p)}(\alpha) d\nu(\alpha) \right|$$
$$\leq \int_A |\chi_{C(a,p^n)}(\alpha) - \chi_{C(a,p)}(\alpha)| d\nu(\alpha) = \nu[C(a,p^n) \Delta C(a,p)],$$

which converges to zero (by **R(iii)**) when $p^n \to p$. □

Step 3 **[EB]:** *If* $\{(p^n, f^n)\} \subset \mathbf{R}_+^H \times L_X$ *is a (norm-)bounded sequence, then for a.e.* $a \in A$ *the sequence* $\{\Phi_i^{\mathcal{C}}(a, p^n, f^n)\}$ $(i = 1, 2)$ *is bounded in* $(\mathbf{R}_+^H)^K$.

Proof. Let $\{(p^n, f^n)\}$ as above. For a.e. $a \in A$ and for every n, one has

$$0 \leq \Phi_1^{\mathcal{C}}(a, p^n, f^n) = \int_{C(a,p^n)} f^n(\alpha) d\nu(\alpha) \leq \int_A f^n(\alpha) d\nu(\alpha).$$

[6] Note: We don't need to use the fact that the sequence $\{f^n\}$ is integrably bounded. Indeed, if $\{f^n\}$ converges weakly to f and $\nu[C(a,p^n) \Delta C(a,p)] \to 0$, one has directly

$$\int_{C(a,p^n) \Delta C(a,p)} f^n(\alpha) d\nu(\alpha) \to 0.$$

For details, see Dunford and Schwartz [8] p. 294. Thanks to E. Balder for this remark.

Since $\{f^n\}$ is norm-bounded and $f^n \geq 0$, we deduce that for some $m \geq 0$

$$\sup_n \|\Phi_1^{\mathcal{C}}(a, p^n, f^n)\| \leq m.$$

We now prove that $\Phi_2^{\mathcal{C}}(a, p^n, f^n)$ is bounded. Indeed, from above, for a.e. $a \in A$ and every n, we get

$$\|\Phi_2^{\mathcal{C}}(a, p^n, f^n)\| = \left\| \frac{1}{\nu[C(a, p^n)]} \Phi_1^{\mathcal{C}}(a, p^n, f^n) \right\| \leq m \frac{1}{\nu[C(a, p^n)]},$$

since $\nu[C(a, p^n)] > 0$ for a.e. $a \in A$. Recalling now that the sequence $\{p^n\}$ is bounded and that in the previous step we have proved that, for a.e. $a \in A$, the mapping $p \to \frac{1}{\nu[C(a,p)]}$ is continuous on $\mathbf{R}_+^H$, we get that, for a.e. $a \in A$, there exists $m'_a > 0$ such that $\frac{1}{\nu[C(a,p^n)]} \leq m'_a$ for every n. It suffices to take for a.e. $a \in A$

$$m_a := \frac{1}{\min_{\{p \in \overline{\{p^n\}} \mid \nu[C(a,p)] > 0\}} \nu[C(a, p)]}.$$

Hence, for a.e. $a \in A$, $\sup_n \|\Phi_2^{\mathcal{C}}(a, p^n, f^n)\| \leq m'_a.m.$ □

3.3 Noguchi's reference coalitions model

We now present Noguchi's model (see [15]) and we deduce his existence result from Theorem 3. It can be described by a reference coalition model, with a unique reference coalition $C_N(a, p)$, defined, for each consumer a at price system p, by

$$C_N(a, p) := \{\alpha \in A \mid p \cdot \omega(\alpha) \in I(\omega(a), \delta(a), p)\},$$

where $\delta : A \to \mathbf{R}_+^H$ is a fixed function and $I(\omega(a), \delta(a), p)$ is a subset of $\mathbf{R}$. Quoting Noguchi [15], "intuitively speaking, $I(\omega(a), \delta(a), p)$ represents (for agent a) an income range in the income-scale, relative to income $p \cdot \omega(a)$ and with magnitude $p \cdot \delta(a)$" and among the examples given, we point out the following one defined by the interval $I(\omega(a), \delta(a), p) = (p \cdot \omega(a) + p \cdot \delta(a), \infty)$.

We now state the existence result.

Corollary 1 [Noguchi] *The economy $(\mathcal{E}, \Phi_2^{\mathcal{C}_N})$ admits an equilibrium, if it satisfies Assumptions* **A**, **C**, **M**, $\mathbf{EC}_0$ *together with:*

Assumption N *For every $(a, w, d, p, t) \in A \times (\mathbf{R}_+^H)^3 \times \mathbf{R}_+$:*
(i) *$I(w, d, p)$ is an open subset of $(0, \infty)$;*
(ii) *$\nu[C_N(a, p)] > 0$*[7]*;* **(iii)** *the function $\delta : A \to \mathbf{R}_+^H$ is measurable;*
(iv) *for every $\lambda > 0$, $I(w, d, \lambda p) = \lambda I(w, d, p)$;*
(v) *for every sequence $\{(p_n, t_n)\} \subset \mathbf{R}_+^H \times \mathbf{R}$, $(p_n, t_n) \to (p, t)$, if $t \in I(w, d, p)$, then $t_n \in I(w, d, p_n)$ for n large enough;*

[7] In fact, Noguchi [15] only assumed that $\nu[C(a, p)] > 0$ for every $(a, p) \in A \times \mathbf{R}_+^H$ such that $p \cdot \omega(a) > 0$. To be able to get Noguchi's existence result in the more general case, we need to weaken Assumption **R** of Theorem 3 and, also, Assumptions **E** and **EB** of Theorem 2 as in the working paper [7].

(vi) *for every sequence* $\{(p_n, t_n)\} \subset \mathbf{R}_+^H \times \mathbf{R}$, $(p_n, t_n) \to (p, t)$, $t_n \in I(w, d, p_n)$ *implies* $t \in \overline{I(w, d, p)}$;
(vii) *for every sequence* $\{(w_n, d_n)\} \subset \mathbf{R}_+^H \times \mathbf{R}_+^H$, $(w_n, d_n) \to (w, d)$, *if* $t \in I(w, d, p)$, *then* $t \in I(w_n, d_n, p)$ *for* n *large enough;*
(viii) *for every sequence* $(w_n, d_n) \to (w, d)$ *in* $\mathbf{R}_+^H \times \mathbf{R}_+^H$, $t \in I(w_n, d_n, p)$ *implies* $t \in \overline{I(w, d, p)}$;
(ix) *the set* $\overline{I(w, d, p)} \setminus I(w, d, p)$ *is countable and* $c \in \overline{I(\omega(a), \delta(a), p)} \setminus I(\omega(a), \delta(a), p)$ *implies* $\nu[\{a \in A \mid p \cdot \omega(a) = c\}] = 0$.

Proof. We define the reference coalitions $\mathcal{C} := (C(a, p))_{(a,p) \in A \times \mathbf{R}_+^H}$ by

$$C(a, p) := \{\alpha \in A \mid p \cdot \omega(\alpha) \in \overline{I(\omega(a), \delta(a), p)}\}.$$

From Assumption **N(ix)**, for every $(a, p) \in A \times \mathbf{R}_+^H$, we get

$$C_N(a, p) \subset C(a, p) \text{ and } \nu(C(a, p) \setminus C_N(a, p)) = 0,$$

hence, $\int_{C_N(a,p)} f(\alpha) d\nu(\alpha) = \int_{C(a,p)} f(\alpha) d\nu(\alpha)$ for every $f \in L_X$.

Consequently, every equilibrium of $(\mathcal{E}, \Phi_2^{\mathcal{C}})$ is an equilibrium for $(\mathcal{E}, \Phi_2^{\mathcal{C}_N})$. We now obtain the existence of equilibria of $(\mathcal{E}, \Phi_2^{\mathcal{C}})$ from Theorem 3 ($K = 1$) and it suffices to prove that the reference coalitions $\mathcal{C}$, defined above, satisfy Assumption **R** of Theorem 3. This is proved in Section 5.2 of the Appendix.

4 Proof of the existence theorem

4.1 Proof of Theorem 2 in the integrably bounded case

In this section, we provide an intermediary existence result, also of interest for itself, under the following additional assumption:

IB *[Integrably Bounded] The correspondence* $a \to X(a)$, *from* A *to* $\mathbf{R}_+^H$, *is integrably bounded, that is, for some integrable function* $\rho : A \to \mathbf{R}_+$, $\sup_{x \in X(a)} \|x\| \leq \rho(a)$ *for a.e.* $a \in A$.

Theorem 4 *Under Assumptions* **A**, **C**, **E**, **EC** *and* **IB**, *the economy* $(\mathcal{E}, \Phi)$ *admits a free-disposal quasi-equilibrium* $(f^*, p^*) \in \mathcal{L}_X \times \mathbf{R}^H$ *with* $p^* > 0$, *in the sense that:*

(a) [*Preference Maximization*] *for a.e.* $a \in A$, $f^*(a) \in B(a, p^*)$ *and for a.e.* $a \in A$ *such that* $p^* \cdot \omega(a) > \inf p^* \cdot X(a)$, $f^*(a)$ *is a maximal element for* $\prec_{a, e^*_a}$ *in the budget set* $B(a, p^*)$ *where* $e^*_a := \Phi(a, p^*, f^*)$;
(b) [*Market Clearing*] $\int_A f^*(a) d\nu(a) \leq \int_A \omega(a) d\nu(a)$.

To prepare the proof of Theorem 4, we define the "quasi-demand" correspondence D, from $A \times \mathbf{R}_+^H \times E$ to $\mathbf{R}_+^H$, by

$$D(a, p, e) := \begin{cases} \{x \in B(a, p) \mid \nexists x' \in B(a, p), x \prec_{a,e} x'\} & \\ & \text{if } \inf p \cdot X(a) < w(a, p) \\ B(a, p) & \text{if } \inf p \cdot X(a) = w(a, p). \end{cases}$$

We let $\Delta := \{p \in \mathbf{R}^H_+ \mid \sum_h p_h = 1\}$ and we define the correspondence Γ, from $\Delta \times L_X$ to $\Delta \times L_X$, by $\Gamma(p, f) = \Gamma_1(p, f) \times \Gamma_2(p, f)$, where

$$\Gamma_1(p, f) := \{p \in \Delta \mid (p - q) \cdot \int_A (f(a) - \omega(a))d\nu(a) \geq 0 \; \forall q \in \Delta\} \subset \Delta$$

$$\Gamma_2(p, f) := \{g \in L_X \mid g(a) \in \text{co}D(a, p, \Phi(a, p, f)) \text{ for a.e. } a \in A\} \subset L_X.$$

The next lemmas summarize the properties of the set L_X and of the correspondence Γ.

Lemma 1 *The set L_X, endowed with the weak topology of the (locally convex) space $L^1(A, \mathbf{R}^H)$, is nonempty, convex, compact and metrizable.*

Proof. First, the set L_X is nonempty, since it contains the mapping ω; indeed $\omega \in L^1(A, \mathbf{R}^H)$ and, for a.e. $a \in A$, $\omega(a) \in X(a)$ (by **C(vi)**). The set L_X is also convex, since for a.e. $a \in A$, $X(a)$ is a convex set (by **C(i)**).

We show now that L_X is compact for the weak topology of $L^1(A, \mathbf{R}^H)$. From the fact that the correspondence $a \to X(a)$ is integrably bounded (by **IB**), one has $\lim_{\nu(C)\to 0} \int_C f(a)d\nu(a) = 0$ uniformly for $f \in L_X$. Consequently, since $\nu(A) < \infty$, the set L_X, which is (norm-)bounded, is also weakly sequentially compact (see, for example, Dunford and Schwartz [8], p. 294). In view of Eberlein-Smulian's Theorem, this is equivalent to the fact that the weak closure of L_X is weakly compact. The proof will be complete if we show that L_X is weakly closed. But in the normed space $L^1(A, \mathbf{R}^H)$, the convex set L_X is weakly closed if and only if it is closed in the norm topology of $L^1(A, \mathbf{R}^H)$ (see, for example, Dunford and Schwartz [8], p. 422). To show that L_X is closed, we consider a sequence $\{f^n\} \subset L_X$ which converges to some $f \in L^1(A, \mathbf{R}^H)$ for the norm topology of $L^1(A, \mathbf{R}^H)$, then there exists a subsequence $\{f^{n_k}\}$, which converges almost everywhere to f. But, for a.e. $a \in A$, $f^{n_k}(a) \in X(a)$, since $f^{n_k} \in L_X$. Taking the limit when $k \to \infty$, for a.e. $a \in A$, $f(a) \in X(a)$, since $X(a)$ is a closed set (by **C(i)**). This ends the proof that L_X is weakly compact.

Finally, L_X is metrizable (for the weak topology) since, in a separable Banach space, the weak topology on a weakly compact set is metrizable (see, for example, Dunford and Schwartz [8], p. 434). □

Lemma 2 *The correspondences Γ_1 and Γ_2 defined on $\Delta \times L_X$ with values, respectively in Δ and L_X, have both a closed graph and non-empty, convex, compact values.*

Proof. For the correspondence Γ_1, the proof is a classical argument using Berge's Maximum Theorem (see Berge [5] p. 123) and proving that the function $(p, f) \to p \cdot (\int_A f(a)d\nu(a) - \int_A \omega(a)d\nu(a))$ is continuous on $\Delta \times L_X$. Indeed, this is clearly the case since the scalar product (of $\mathbf{R}^H$) $(p, x) \to p \cdot x$ is continuous and the real-valued functions $(p, f) \to p$ and $(p, f) \to \int_A f(a)d\nu(a)$ are continuous on $\Delta \times L_X$, when L_X is endowed with the weak topology of $L^1(A, \mathbf{R}^H)$ (recalling that L_X is metrizable).

Consider now the correspondence Γ_2. It has clearly convex values and we show hereafter that it has nonempty values. For every $(p, f) \in \Delta \times L_X$

$$\{g \in L_X \mid g(a) \in D(a, p, \Phi(a, p, f)), \text{ for a.e. } a \in A\} \subset \Gamma_2(p, f).$$

The existence of a measurable selection of the correspondence

$$a \to D(a) := D(a, p, \Phi(a, p, f)) \subset B(0, \rho(a))$$

is a consequence of Aumann's theorem and it suffices to show that (i) for a.e. $a \in A$, $D(a) \neq \emptyset$ and (ii) the correspondence $D(.)$ is measurable. The first assertion is a consequence of Proposition 2 of the Appendix. We now prove the second assertion. Indeed,

$$\begin{aligned} \mathrm{G}_D &= \{(a, z) \in A \times \mathbf{R}^H \mid z \in D(a)\} \\ &= \{(a, z) \in A \times \mathbf{R}^H \mid (a, \Phi(a, p, f), z) \in G\} = h^{-1}(G), \end{aligned}$$

where $G := \{(a, e, z) \in A \times E \times \mathbf{R}^H \mid z \in D(a, p, e)\}$ and $h : A \times \mathbf{R}^H \to A \times E \times \mathbf{R}^H$ is defined by $h(a, z) = (a, \Phi(a, p, f), z)$. But the mapping h is clearly measurable, since the mapping $a \to \Phi(a, p, f)$ is measurable (by **E(ii)**), and G$\in \mathcal{A} \otimes \mathcal{B}(E) \otimes \mathcal{B}(\mathbf{R}^H)$, since the correspondence $(a, e) \to D(a, p, e)$ is measurable [Proposition 2 of the Appendix]. Consequently, $\mathrm{G}_D = h^{-1}(G) \in \mathcal{A} \otimes \mathcal{B}(\mathbf{R}^H)$, which ends the proof of Assertion (ii).

Finally, every measurable selection of the correspondence $a \to D(a)$ is integrable, since from Assumption **IB**, for a.e. $a \in A$, $D(a) \subset B(0, \rho(a))$ for some integrable function ρ. This shows that $\Gamma_2(p, f)$ is nonempty.

We now show that the correspondence Γ_2 has a closed graph. Indeed (recalling that L_X is metrizable), let $\{(p^n, f^n, g^n)\}$ be a sequence converging to some element (p, f, g) in $\Delta \times L_X \times L_X$ such that $g^n \in \Gamma_2(p^n, f^n) \subset L_X$ for all n. Since the sequence $\{g^n\}$ is integrably bounded (by **IB**) and converges weakly to g in $L^1(A, \mathbf{R}^H)$, it is a standard result (see, for example, Yannelis [19]) that

$$\text{for a.e. } a \in A,\ g(a) \in \overline{\text{co}}\, Ls\{g^n(a)\}.$$

But, for a.e. $a \in A$, the correspondence $(p, f) \to coD(a, p, \Phi(a, p, f))$ has a closed graph and convex values, since the correspondence $(p, e) \to D(a, p, e)$ has a closed graph [Proposition 2 of Appendix] and the mapping $(p, f) \to \Phi(a, p, f)$ is continuous on $\Delta \times L_X$ (by **E(iii)**, **IB** and the metrizability of L_X). Hence, recalling that, for a.e. $a \in A$, $g^n(a) \in coD(a, p^n, \Phi(a, p^n, f^n))$ for all n, the closed graph property implies

$$Ls\{g^n(a)\} \subset coD(a, p, \Phi(a, p, f)).$$

Consequently, for a.e. $a \in A$

$$g(a) \in \overline{\text{co}}\, Ls\{g^n(a)\} \subset coD(a, p, \Phi(a, p, f)),$$

which shows that $g \in \Gamma_2(p, f)$ and ends the proof of the lemma. □

From the two above lemmas, recalling that the Cartesian product of two correspondences with closed graph and non-empty, convex, compact values is a correspondence with closed graph and non-empty, convex, compact values (see Berge

[5] p. 121), the space $L := \mathbf{R}^H \times L^1(A, \mathbf{R}^H)$, the set $K := \Delta \times L_X$ and the correspondence Γ satisfy all the assumptions of the following fixed-point theorem (see, for example, Fan [9] and Glicksberg [10]).

Theorem 5 (Fan-Glicksberg) *Let K be a non-empty, convex, compact subset of a Hausdorff locally convex space L and let Γ be a correspondence, from K to K, with a closed graph and non-empty, convex, compact values. Then there exists $\bar{x} \in K$ such that $\bar{x} \in \Gamma(\bar{x})$.*

Consequently, there exists an element $(\bar{p}, \bar{f}) \in \Delta \times L_X$ satisfying:

$$(\bar{p} - p) \cdot \int_A (\bar{f}(a) - \omega(a)) d\nu(a) \geq 0 \text{ for all } p \in \Delta, \tag{1}$$

$$\bar{f}(a) \in \text{co} D(a, \bar{p}, \Phi(a, \bar{p}, \bar{f})) \text{ for a.e. } a \in A. \tag{2}$$

The following lemma shows that we can remove the convex hull in the above assertion, by eventually modifying the function $\bar{f}$.

Lemma 3 *There exists $f^* \in L_X$ satisfying:*

$$(\bar{p} - p) \cdot \int_A (f^*(a) - \omega(a)) d\nu(a) \geq 0 \textit{ for all } p \in \Delta, \tag{3}$$

$$f^*(a) \in D(a, \bar{p}, \Phi(a, \bar{p}, f^*)) \textit{ for a.e. } a \in A. \tag{4}$$

Proof. From Assertion (2) and the fact that the correspondence $a \to D(a) := D(a, \bar{p}, \Phi(a, \bar{p}, \bar{f}))$, from A to $\mathbf{R}^H$, is measurable [Proposition 2 of the Appendix], there exist finitely many measurable selections f_i $(i \in I)$ of the correspondence $a \to D(a)$ such that, for a.e. $a \in A$, $\bar{f}(a) \in \text{co}\{f_i(a) \mid i \in I\}$. Indeed, consider the correspondence F, from A to $(\mathbf{R}^H \times \mathbf{R})^{\#H+1}$, defined by

$$F(a) := \{(f_i, \lambda_i)_{i=1,..,\#H+1} \mid (f_i, \lambda_i) \in D(a) \times \mathbf{R}_+, \text{ for all } i \\ \sum_i \lambda_i = 1 \text{ and } \bar{f}(a) = \sum_i \lambda_i f_i\}.$$

Then, clearly F is measurable and nonempty valued, from Caratheodory's theorem and the fact that $\bar{f}(a) \in \text{co}D(a)$. Consequently, from Aumann's theorem, there exists a measurable selection of the correspondence F, which defines the measurable selections f_i of the correspondence $a \to D(a)$.

From Assumption **EC**, there exists a measurable set $C \subset A_{na}$ such that:
(i) for a.e. $a \in A_{na} \setminus C$, the preference relation $\prec_{a, \Phi(a, \bar{p}, \bar{f})}$ is convex and
(ii) there exists $f^* \in L_X$ such that,

$$\text{for a.e. } a \in A, \Phi(a, \bar{p}, \bar{f}) = \Phi(a, \bar{p}, f^*)$$

$$\text{for a.e. } a \in C, f^*(a) \in \{f_i(a) \mid i \in I\} \subset D(a, \bar{p}, \Phi(a, \bar{p}, \bar{f})) \\ = D(a, \bar{p}, \Phi(a, \bar{p}, f^*)), \tag{5}$$

$$\text{for a.e. } a \in A \setminus C, f^*(a) = \bar{f}(a) \text{ and } \int_A \bar{f}(\alpha) d\nu(\alpha) = \int_A f^*(\alpha) d\nu(\alpha). \tag{6}$$

Since the preference relation $\prec_{a,\Phi(a,\bar{p},\bar{f})}$ is convex for a.e. $a \in A \setminus C$ (first, for a.e. $a \in A \setminus A_{na}$ by **C(iii)** and, second, for a.e. $a \in A_{na} \setminus C$ by **EC(i)**), the set $D(a, \bar{p}, \Phi(a, \bar{p}, \bar{f}))$ is convex. Then, from above

$$\text{for a.e. } a \in A \setminus C,\ f^*(a) = \bar{f}(a) \in coD(a, \bar{p}, \Phi(a, \bar{p}, \bar{f})) \\ = D(a, \bar{p}, \Phi(a, \bar{p}, f^*)). \tag{7}$$

The Assertions (3) and (4) of the lemma follow from Assertions (1),(6) and (5),(7) respectively. □

We come back to the proof of Theorem 4 and we show that, for $p^* = \bar{p}$, (p^*, f^*) is a free disposal quasi-equilibrium of $(\mathcal{E}, \Phi)$. Indeed, from Assertion (4), for a.e. $a \in A$, $f^*(a) \in D(a, p^*, \Phi(a, p^*, f^*))$, hence the equilibrium preference maximization condition is satisfied. This implies, in particular, that for a.e. $a \in A$, $f^*(a) \in B(a, p^*)$, hence $p^* \cdot f^*(a) \leq p^* \cdot \omega(a)$. Integrating over A, one gets $p^* \cdot \int_A (f^*(a) - \omega(a)) d\nu(a) \leq 0$. Using Assertion (3), one deduces that

$$p \cdot \int_A (f^*(a) - \omega(a)) d\nu(a) \leq 0 \text{ for all } p \in \Delta,$$

which implies the equilibrium market clearing condition

$$\int_A f^*(a) d\nu(a) \leq \int_A \omega(a) d\nu(a).$$

4.2 Proof of Theorem 2 in the general case

4.2.1 Truncation of the economy

For each integer $k > 1$ and for every $a \in A$, we let

$$X^k(a) := \{x \in X(a) \mid x \leq k[\mathbf{1} \cdot \omega(a)]\mathbf{1}\}$$

and, for every $e \in E$, we consider the restriction of the preference relation $\prec_{a,e}$ on the set $X^k(a)$, which will be denoted identically $\prec_{a,e}$ in the following. We define the truncated economy $\mathcal{E}^k$, by

$$\mathcal{E}^k = \{\mathbf{R}^H, E, (A, \mathcal{A}, \nu), (X^k(a), (\prec_{a,e})_{e \in E}, \omega(a))_{a \in A}\},$$

where the characteristics of $\mathcal{E}^k$ are the same as in the economy $\mathcal{E}$, but the consumption sets $X^k(a)$ and the preferences $(\prec_{a,e})_{e \in E}$ of the agents, which are defined as above.

The externality mapping $\Phi^k : A \times \mathbf{R}^H \times L_X^k \to E$ is defined as the restriction of Φ to $A \times \mathbf{R}^H \times L_X^k$, where

$$L_X^k := \{f \in L^1(A, \mathbf{R}^H) \mid f(a) \in X^k(a), \text{ for a.e. } a \in A\}.$$

It is easy to see that if $(\mathcal{E}, \Phi)$ satisfies the assumption of Theorem 2, then for every k, $(\mathcal{E}^k, \Phi^k)$ satisfies all the assumptions of Theorem 4. Consequently, from Theorem 4, for every k there exists a free-disposal quasi-equilibrium (p^k, f^k) of $(\mathcal{E}^k, \Phi^k)$ with $p^k > 0$.

4.2.2 For k large enough, $p^k >> 0$

Lemma 4 *There exists* $\delta > 0$ *such that* $p^k \geq \delta \mathbf{1}$ *for* k *large enough.*

Proof. Without any loss of generality, we can assume that the sequence $\{p^k\}$ converges to some element p^* in the compact set Δ. To prove the lemma it suffices to show that $p^* >> 0$.

We first show that, for a.e. $a \in A$

$$\exists (f(a), e(a)) \in \mathbf{R}^H_+ \times E,\ (p^*, f(a), e(a)) \in \text{ Ls}\{(p^k, f^k(a), e^k(a))\}. \quad (8)$$

Indeed, since (p^k, f^k) is a free-disposal quasi-equilibrium of $(\mathcal{E}^k, \Phi^k)$ for every k, one has

$$\text{for a.e. } a \in A,\ 0 \leq f^k(a) \text{ and } \int_A f^k(a) d\nu(a) \leq \int_A \omega(a) d\nu(a),$$

hence the sequence $\{\int_A f^k(a) d\nu(a)\}$ is bounded in $\mathbf{R}^H$ and we shall deduce that $\sup_k \|f^k\|_1 < \infty$. Defining in $\mathbf{R}^H$, $\|x\|_1 = \sum_h |x_h|$ and, recalling that, for some $m > 0$, $\|x\| \leq m\|x\|_1$ for every x, we get

$$\|f^k\|_1 := \int_A \|f^k(a)\| d\nu(a) \leq m \int_A \sum_{h \in H} f^k_h(a) d\nu(a)$$
$$= m \left\| \int_A f^k(a) d\nu(a) \right\|_1,$$

since $f^k(a) \geq 0$, for a.e. $a \in A$. Consequently, $\sup_k \|f^k\|_1 < \infty$, since the sequence $\{\int_A f^k(a) d\nu(a)\}$ is bounded.

Since the sequence $\{(p^k, f^k)\}$ is (norm-)bounded in $\mathbf{R}^H \times L^1(A, \mathbf{R}^H)$, from Assumption **EB**, there exists a set $N_1 \in \mathcal{A}$ with $\nu(N_1) = 0$ such that, for all $a \in A \setminus N_1$ the sequence $\{e^k(a)\}$ is bounded in E, where $e^k(a) := \Phi(a, p^k, f^k)$.

In view of the standard version of Fatou's lemma, one has

$$\int_A \liminf \|f^k(a)\| d\nu(a) \leq \liminf \int_A \|f^k(a)\| d\nu(a) = \liminf \|f^k\|_1 < \infty,$$

since from above $\sup_n \|f^k\|_1 < \infty$. Consequently, there exists $N_2 \in \mathcal{A}$ with $\nu(N_2) = 0$ such that, for all $a \in A \setminus N_2$, $\liminf \|f^k(a)\| < \infty$, which implies that, for all $a \in A \setminus N_2$, there exists a subsequence $\{k_n(a)\}$ such that the sequence $\{f^{k_n(a)}(a)\}$ is bounded in $\mathbf{R}^H_+$.

Let $a \in A \setminus (N_1 \cup N_2)$, noticing that the sequence $\{(f^{k_n(a)}(a), e^{k_n(a)}(a))\}$ is bounded in $\mathbf{R}^H_+ \times E$, without any loss of generality we can assume that the sequence $\{(f^{k_n(a)}(a), e^{k_n(a)}(a))\}$ converges to some element $(f(a), e(a)) \in \mathbf{R}^H_+ \times E$. Hence, Assertion (8) holds for all $a \in A \setminus [N_1 \cup N_2]$. □

We now come back to the proof of Lemma 4. We choose a particular agent $a_0 \in A$ for whom the following properties hold: (i) the preferences of agent a_0 are continuous, (ii) the preferences of agent a_0 are monotonic; (iii) $p^* \cdot \omega(a_0) > 0$ and there exists a subsequence $\{k_n\}$, depending on a_0, such that

(iv) $(p^{k_n}, f^{k_n}(a_0), e^{k_n}(a_0)) \to (p^*, f(a_0), e(a_0))$, for some $(f(a_0), e(a_0)) \in \mathbf{R}^H_+ \times E$, (v) for every n, $f^{k_n}(a_0) \in D^{k_n}(a_0, p^{k_n}, e^{k_n}(a_0))$ with $e^{k_n}(a_0) = \Phi(a_0, p^{k_n}, f^{k_n})$. Such an agent a_0 clearly exists, since each of the above Assertions $(i) - (v)$ hold for a.e. $a \in A$; they correspond, respectively, to Assumption **C(iv)**, **M(i)**, **M(ii)**, Assertion (8) and the equilibrium preference maximization condition for (p^{k_n}, f^{k_n}) for every n.

We will now show that $p^* >> 0$. Suppose it is not true, then there exists h such that $p^*_h = 0$. From the above properties of agent a_0, for all n, $p^{k_n} \cdot f^{k_n}(a_0) \leq p^{k_n} \cdot \omega(a_0)$, $p^{k_n} \to p^*$ and $f^{k_n}(a_0) \to f(a_0)$, and at the limit one gets $p^* \cdot f(a_0) \leq p^* \cdot \omega(a_0)$. Since agent a_0 has monotonic preferences, there exists $z = f(a_0) + t\mathbf{e^h}$, for some $t > 0$ such that $f(a_0) \prec_{a_0, e(a_0)} z$ and clearly $p^* \cdot z = p^* \cdot f(a_0) \leq p^* \cdot \omega(a_0)$. We now show that

$$\exists\, z' \in \mathbf{R}^H_+,\ p^* \cdot z' < p^* \cdot \omega(a_0),\ f(a_0) \prec_{a_0, e(a_0)} z'. \tag{9}$$

Indeed, if $p^* \cdot z < p^* \cdot \omega(a_0)$, we take $z' = z$. If $p^* \cdot z = p^* \cdot \omega(a_0) > 0$, we can choose $i \in H$ such that $p^*_i > 0$ and $z_i > 0$. Since agent a_0 has continuous preferences, there exists $\varepsilon > 0$ such that $z' = z - \varepsilon \mathbf{e^i} \in \mathbf{R}^H_+$ and $f(a_0) \prec_{a_0, e(a_0)} z'$. We have also $p^* \cdot z' = p^* \cdot z - \varepsilon p^*_i < p^* \cdot \omega(a_0)$. This ends the proof of Assertion (9).

We end the proof by contradicting the fact that $f^{k_n}(a_0)$ belongs to $D^k(a_0, p^{k_n}, e^{k_n}(a_0))$. Indeed, from $p^* \cdot \omega(a_0) > 0$ (by (iii)) and Assertion (9), recalling that the sequence $\{(p^{k_n}, f^{k_n}(a_0), e^{k_n}(a_0))\}$ converges to $(p^*, f(a_0), e(a_0))$ (by (iv)) and using the continuity of preferences of agent a_0, for n large enough, we get $p^{k_n} \cdot \omega(a_0) > 0$, $z' \in \mathbf{R}^H_+$, $p^{k_n} \cdot z' \leq p^{k_n} \cdot \omega(a_0)$ and $x^{k_n}(a_0) \prec_{a_0, e^{k_n}(a_0)} z'$. Moreover, we can also assume that $z' \in X^{k_n}(a_0)$. All together, these conditions contradict the fact that $f^{k_n}(a_0) \in D^k(a_0, p^{k_n}, e^{k_n}(a_0))$ and this ends the proof of the lemma. □

4.2.3 For k large enough, (p^k, f^k) is an equilibrium for $(\mathcal{E}, \Phi)$

It is a consequence of the following lemma.

Lemma 5 *For every k large enough and for a.e. $a \in A$, one has:*
(i) $B(a, p^k) \subset X^k(a)$;
(ii) $f^k(a)$ is a maximal element in $B(a, p^k)$ for $\prec_{a, e^k(a)}$,
where $e^k(a) = \Phi(a, p^k, f^k)$;
(iii) $p^k \cdot f^k(a) = p^k \cdot \omega(a)$;
(iv) $\int_A f^k(a) d\nu(a) = \int_A \omega(a) d\nu(a)$.

Proof. From Lemma 4, there exists K such that, for every $k \geq K$

$$p^k_h > \delta \text{ for each } h \in H \text{ and } \frac{1}{\delta} \leq k.$$

In the following we fix $k \geq K$.
(i) For a.e. $a \in A$, let $x \in B(a, p^k)$, i.e., $x \in \mathbf{R}^H_+$ and $p^k \cdot x \leq p^k \cdot \omega(a)$. From above, recalling that $p^k \in \Delta$, one gets

$$\delta x_h \leq p^k_h x_h \leq p^k \cdot x \leq p^k \cdot \omega(a) \leq \sum_h \omega_h(a) = \mathbf{1} \cdot \omega(a),$$

which implies that

$$0 \le x \le \frac{1}{\delta}[\mathbf{1} \cdot \omega(a)]\mathbf{1} \le k[\mathbf{1} \cdot \omega(a)]\mathbf{1}$$

or equivalently $x \in X^k(a)$.
(ii) For a.e. $a \in A$ such that $p^k \cdot \omega(a) > 0$, $f^k(a)$ is a maximal element in $B(a, p^k)$ for $\prec_{a,e^k(a)}$, since $B(a, p^k) \subset X^k(a)$ (by Part (i)) and the fact that (p^k, f^k) is a free-disposal quasi-equilibrium for $(\mathcal{E}^k, \Phi^k)$. For a.e. $a \in A$ such that $p^k \cdot \omega(a) = 0$, recalling that $p^k >> 0$ (by Lemma 4), we get $B(a, p^k) = \{0\}$ and the result follows from the Irreflexivity Assumption **C(ii)**.
(iii) The result is obvious for a.e. $a \in A$ such that $p^k \cdot \omega(a) = 0$. Assume now that $p^k \cdot \omega(a) > 0$. From the Monotonicity Assumption **M(i)**, there exists a sequence $\{f^n(a)\} \subset \mathbf{R}^H_+$ such that $f^n(a) \to f^k(a)$ and $f^k(a) \prec_{a,e^k(a)} f^n(a)$. From Part (ii), $f^k(a)$ is a maximal element of $\prec_{a,e^k(a)}$ in $B(a, p^k)$, consequently $p^k \cdot f^n(a) > p^k \cdot \omega(a)$. Passing to the limit one gets $p^k \cdot f^k(a) \ge p^k \cdot \omega(a)$, which together with $f^k(a) \in B(a, p^k)$ implies that $p^k \cdot f^k(a) = p^k \cdot \omega(a)$.
(iv) Integrating over A the equalities of Part (iii), one gets

$$p^k \cdot \left(\int_A f^k(a) d\nu(a) - \int_A \omega(a) d\nu(a) \right) = 0.$$

Since (p^k, f^k) is a free-disposal quasi-equilibrium for $(\mathcal{E}^k, \Phi^k)$, one has

$$\int_A f^k(a) d\nu(a) \le \int_A \omega(a) d\nu(a)$$

and, recalling that $p^k >> 0$ (by Lemma 4), we get

$$\int_A f^k(a) d\nu(a) = \int_A \omega(a) d\nu(a).$$

5 Appendix

5.1 Properties of the quasi-demand correspondence

Let $(A, \mathcal{A}, \nu)$ be a measure space of consumers, and assume that each consumer a is endowed with a consumption set $X(a) \subset \mathbf{R}^H$, a preference relation $\prec_{a,e}$ on $X(a)$ (for each externality $e \in E$) and a wealth mapping $w : A \times \mathbf{R}^H \to \mathbf{R}$. In the following, we let

$$P := \{p \in \mathbf{R}^H \mid \inf p \cdot X(a) \le w(a, p) \text{ for a.e. } a \in A\},$$

$$B(a, p) := \{x \in X(a) \mid p \cdot x \le w(a, p)\},$$

$$D(a, p, e) := \begin{cases} \{x \in B(a, p) \mid \nexists x' \in B(a, p), x \prec_{a,e} x'\} & \\ & \text{if } \inf p \cdot X(a) < w(a, p) \\ B(a, p) & \text{if } \inf p \cdot X(a) = w(a, p). \end{cases}$$

The properties of the quasi-demand correspondence D are summarized in the following proposition, which extends standard results (see, for example, Hildenbrand [12]) in the no-externality case (say $E = \{0\}$).

Proposition 2 *Let* $\{(A, \mathcal{A}, \nu), E, (X(a), (\prec_{a,e})_{e\in E},)_{a\in A}, w\}$ *satisfy Assumptions* **A**, **C** *and* **IB** *and assume that the wealth distribution* $w : A \times \mathbf{R}^H \to \mathbf{R}$ *is a Caratheodory function*[8]*. Then:*
(i) *for every* $p \in P$ *the correspondence* $(a, e) \to D(a, p, e)$*, from* $A \times E$ *to* $\mathbf{R}^H$*, is measurable;*
(ii) *for a.e.* $a \in A$ *the correspondence* $(p, e) \to D(a, p, e)$*, from* $P \times E$ *to* $\mathbf{R}^H$*, has a closed graph and nonempty, compact values.*

Proof. In the following, for a.e. $a \in A$ and for every $p \in P$, we let

$$P_a := \{p \in P \mid \inf p \cdot X(a) < w(a, p)\},$$
$$A_p := \{a \in A \mid \inf p \cdot X(a) < w(a, p)\}.$$

Proof of (i). Let $p \in P$, we prove that

$$G := \{(a, e, d) \in A \times E \times \mathbf{R}^H \mid d \in D(a, p, e)\} \in \mathcal{A} \otimes B(E) \otimes \mathcal{B}(\mathbf{R}^H)$$

and we first notice that $\mathrm{G} = \mathrm{G}_1 \cup \mathrm{G}_2$, where

$$\mathrm{G}_1 := \{(a, e, d) \in (A \setminus A_p) \times E \times \mathbf{R}^H \mid d \in X(a),\ p \cdot d \leq w(a, p)\},$$

$$\mathrm{G}_2 := \{(a, e, d) \in A_p \times E \times \mathbf{R}^H \mid d \in D(a, p, e)\}.$$

We notice that $\mathrm{G}_1 \in \mathcal{A} \otimes B(E) \otimes \mathcal{B}(\mathbf{R}^H)$, since the mapping $(a, d) \to p \cdot d - w(a, p)$ and the correspondence $a \to X(a)$ are measurable and $A_p \in \mathcal{A}$.

To show that $\mathrm{G}_2 \in \mathcal{A} \otimes \mathcal{B}(E) \otimes \mathcal{B}(\mathbf{R}^H)$, we apply the argument used by Hildenbrand [12]. Since the correspondence $B(., p)$, from A_p to $\mathbf{R}^H$, has nonempty values and is measurable, there exists a sequence of measurable mappings $\{f_n\}$, from A_p to $\mathbf{R}^H$, such that for a.e. $a \in A_p$, $\{f_n(a)\}$ is dense in $B(a, p)$ (see, for example, [6]). We now define the correspondences ξ_n, from $A_p \times E$ to $\mathbf{R}^H$, by

$$\xi_n(a, e) = \{x \in B(a, p) \mid \text{not}[x \prec_{a,e} f_n(a)]\}$$

and we claim that: $D(a, p, e) = \cap_{n=1}^{\infty} \xi_n(a, e)$, for a.e. $a \in A_p$.

Clearly, for every n, $D(a, p, e) \subset \xi_n(a, e)$. Conversely, let $x \in \cap_{n=1}^{\infty} \xi_n(a, e)$ and suppose that $x \notin D(a, p, e)$. Then, the set $U = \{x' \in B(a, p) \mid x \prec_{a,e} x'\}$ is nonempty and is open relative to $B(a, p)$ (by **C(iv)**). Since the sequence $\{f_n(a)\}$ is dense in $B(a, p)$, we deduce that for some n_0, $x \prec_{a,e} f_{n_0}(a)$, but this contradicts the fact that $x \in \xi_{n_0}(a, e)$. Thus, we have

$$\begin{aligned}\mathrm{G}_2 &= \{(a, e, d) \in A_p \times E \times \mathbf{R}^H \mid d \in D(a, p, e)\}\\ &= \cap_{n=1}^{\infty} \{(a, e, d) \in A_p \times E \times \mathbf{R}^H \mid d \in \xi_n(a, e)\}.\end{aligned}$$

Hence, the set G_2 is measurable, since $\prec_{a,e}$ is measurable (by **C(v)**), the mappings f_n and the correspondence $a \to B(a, p)$ are measurable and recalling that $A_p \in \mathcal{A}$.

[8] That is, for every $p \in \mathbf{R}^H$, the function $a \to w(a, p)$ is measurable and, for a.e. $a \in A$, the function $p \to w(a, p)$ is continuous. We note that the wealth distribution considered in our model $w(a, p) := p \cdot \omega(a)$ satisfies this property, when ω is assume to be measurable.

Proof of (ii). We first show that $D(a,p,e) \neq \emptyset$ for a.e. $a \in A$ and every $(p,e) \in P \times E$. For a.e. $a \in A \setminus A_p$, $D(a,p,e) = B(a,p) \neq \emptyset$ since $\inf p \cdot X(a) \leq w(a,p)$. We now consider $a \in A_p$ and we simply denote $B := B(a,p)$, which is clearly a nonempty, compact set (by **IB**). We suppose, by contraposition, that $D(a,p,e) = \emptyset$, that is, for every $x \in B$, there exists $x' \in B$, $x \prec_{a,e} x'$. Then $B = \cup_{x' \in B} V_{x'}$, where $V_{x'} = \{x \in B \mid x \prec_{a,e} x'\}$ is open in B (by **C(iv)**). Since B is compact, there exists a finite subset $\{x'_i \mid i \in N\} \subset B$ such that $B = \cup_{i \in N} V_{x'_i}$. We now claim that there exists $i \in N$ such that not $[x'_i \prec_{a,e} x'_j]$ for every $j \in N$. Indeed, if such a maximal element does not exist, for every $i \in N$, there exists $\sigma(i) \in N$ such that $x'_i \prec_{a,e} x'_{\sigma(i)}$. The mappings $\sigma : N \to N$ clearly admits a cycle, that is, for some i and some integer k one has $i = \sigma^k(i)$ (the composition of σ with itself k times). The transitivity (by **C(ii)**) of $\prec_{a,e}$ implies that $x'_i \prec_{a,e} x'_{\sigma^k(i)} = x'_i$ which contradicts the irreflexivity (by **C(ii)**) of $\prec_{a,e}$. We end the proof by considering such a maximal element $x'_i \in B$, which belongs to some set $V_{x'_j}$ $(j \in N)$, that is, $x'_i \prec_{a,e} x'_j$ for some $j \in J$. But this is in contradiction with the maximality of x'_i. This ends the proof that $D(a,p,e)$ is nonempty.

We now show that, for a.e. $a \in A$, the correspondence $(p,e) \to D(a,p,e)$, from $P \times E$ to $\mathbf{R}^H$, has a closed graph. Let $(p^n, e^n, x^n) \to (p,e,x)$ in $P \times E \times \mathbf{R}^H$ such that, for all n, $x^n \in D(a,p^n,e^n)$. From $p^n \cdot x^n \leq w(a,p^n)$, passing to the limit and recalling that the mapping $w(a,.)$ is continuous, one gets $p \cdot x \leq w(a,p)$. Recalling that $X(a)$ is closed, we get that $x \in B(a,p)$. Thus, if $\inf p \cdot X(a) = w(a,p)$, we have $x \in D(a,p,e) = B(a,p)$. We assume now that $\inf p \cdot X(a) < w(a,p)$. Since $p^n \to p$, for n large enough, $w(a,p^n) > \inf p^n \cdot X(a)$. Suppose now that $x \notin D(a,p,e)$. This implies that there exists $x' \in B(a,p)$ such that $x \prec_{a,e} x'$. From the fact that $w(a,p) > \inf p \cdot X(a)$ and the Continuity Assumption **C(iv)**, we can find $x'' \in X(a)$ such that $x \prec_{a,e} x''$ and $p \cdot x'' < w(a,p)$. Since $p^n \to p$, for n large enough, $p^n \cdot x'' < w(a,p^n)$. Since $e^n \to e$, from the Continuity Assumption **C(iv)**, for n large enough, $x^n \prec_{a,e^n} x''$. Consequently, we can choose n (large enough) such that $w(a,p^n) > \inf p^n \cdot X(a)$, $x'' \in B(a,p^n)$ and $x^n \prec_{a,e^n} x''$, but this contradicts the fact that $x^n \in D(a,p^n,e^n)$. □

5.2 Properties of Noguchi's reference coalitions

In this section, we end the proof of Corollary 1 (of Sect. 3.3) and it only remains to show that the reference coalitions, defined by

$$C(a,p) = \{\alpha \in A \mid p \cdot \omega(\alpha) \in \overline{I(\omega(a), \delta(a), p)}\}$$

satisfy Assumption **R** of Theorem 3.

Proof. **R(i)** is a consequence of **N(ii)** since $C_N(a,p) \subset C(a,p)$ and **R(ii)** is a direct consequence of **N(iv)**. □

Proof of **R(iii)**. Let $(a,p) \in A \times \mathbf{R}^H_+$, we define

$$W(a,p) := \{\omega' \in \mathbf{R}^H_+ \mid p \cdot \omega' \in I(\omega(a), \delta(a), p)\}.$$

Clearly, one has

$$\overline{W(a,p)} \setminus W(a,p) \subset \{\omega' \in \mathbf{R}_+^H \mid p \cdot \omega' \in \overline{I(\omega(a),\delta(a),p)} \setminus I(\omega(a),\delta(a),p)\}$$
$$\omega^{-1}(\overline{W(a,p)} \setminus W(a,p)) \subset \cup_{c \in \overline{I(\omega(a),\delta(a),p)} \setminus I(\omega(a),\delta(a),p)} \{\alpha \in A \mid p \cdot \omega(\alpha) = c\}$$

and using Assumption **N(ix)**, one gets

$$\nu[\omega^{-1}(\overline{W(a,p)} \setminus W(a,p))] = 0.$$

Since the measure $\tau := \nu o \omega^{-1}$ is a finite Borel measure on $\mathbf{R}_+^H$, from Noguchi [15] (see Lemma 2), for every sequence $\{p_n\} \subset \mathbf{R}_+^H$ converging to p, one has

$$\tau(W(a,p_n) \Delta W(a,p)) := \nu[\omega^{-1}(W(a,p_n) \Delta W(a,p))] \to 0.$$

Noticing that $C_N(a,p^n) = \omega^{-1}(W(a,p_n))$ and $C_N(a,p) = \omega^{-1}(W(a,p))$, one gets

$$\begin{aligned}\nu[C_N(a,p^n) \Delta C_N(a,p)] &= \nu[\omega^{-1}(W(a,p_n)) \Delta \omega^{-1}(W(a,p))] \\ &= \nu[\omega^{-1}(W(a,p_n) \Delta W(a,p))] \to 0.\end{aligned}$$

Recalling now that $\nu[C(a,p)/C_N(a,p)] = 0$ for every $(a,p) \in A \times \mathbf{R}_+^H$, from above, we get $\nu[C(a,p^n) \Delta C(a,p)] \to 0$ when $p_n \to p$ in $\mathbf{R}_+^H$. □

Proof of **R(iv)**. It is a consequence of the following lemma, defining, for a fixed $p \in \mathbf{R}_+^H$, the mappings $f : A \to (\mathbf{R}_+^H)^2$, $g : A \to \mathbf{R}_+^H$ and the correspondence F, from $(\mathbf{R}_+^H)^2$ to $\mathbf{R}$, by $f(a) = (\omega(a),\delta(a))$, $g(\alpha) = p \cdot \omega(\alpha)$ and $F(\omega,\delta) = I(\omega,\delta,p)$ and noticing that

$$C(a,p) = \{\alpha \in A \mid g(\alpha) \in \overline{F(f(a))}\}$$

and that, Condition **N** implies that f, g and F satisfy the assumption of the lemma. (We only notice that, **N(vii)** implies that for every $t \in \mathbf{R}_+^H$, the set $F^{-1}(t) := \{(\omega,\delta) \in (\mathbf{R}_+^H)^2 \mid t \in I(\omega,\delta,p)\}$ is open, hence measurable.)

Lemma 6 *Let $f : A \to \mathbf{R}^m$, $g : A \to \mathbf{R}^n$ be two measurable mappings and let F be a correspondence, from $\mathbf{R}^m$ to $\mathbf{R}^n$, such that, for every $(x,t) \in \times\mathbf{R}^m \times \mathbf{R}^n$, $F(x)$ is open and $F^{-1}(t)$ is measurable. Then the set*

$$G := \{(a,\alpha) \in A \times A \mid g(\alpha) \in \overline{F(f(a))}\}$$

is measurable.

Proof. Note that $(a,\alpha) \in G$ if and only if

$$\forall k \in \mathbb{N},\ B\left(g(\alpha), \frac{1}{k}\right) \cap F(f(a)) \neq \emptyset$$

and, using the fact that $F(f(a))$ is an open set, if and only if

$$\forall k \in \mathbb{N},\ \exists t_k \in \mathbb{Q}^n,\ \|t_k - g(\alpha)\| < \frac{1}{k} \text{ and } t_k \in F(f(a)).$$

Consequently

$$G = \cap_k \cup_{t \in \mathbb{Q}^n} \left[A \times \left\{ \alpha \in A \mid \|t - g(\alpha)\| < \frac{1}{k} \right\} \cap \{ a \in A \mid t \in F(f(a)) \} \times A \right],$$

which is measurable since the set $\{\alpha \in A \mid \|t - g(\alpha)\| < \frac{1}{k}\}$ is measurable (since the mapping g is measurable) and the set $\{a \in A \mid t \in F(f(a))\}$ is measurable (since the set $F^{-1}(t)$ is measurable and the mapping f is measurable). □

5.3 Balder's counterexample of nonexistence of equilibria

Theorem 4 may not hold if we remove the Convexity Assumption **EC**.

We consider the following example, due to Balder [4], of an economy $\mathcal{E}$ with a single commodity, $A = [0, 1]$ endowed with the Lebesgue measure ν and the Lebesgue $\sigma-$algebra (i.e. the completion of the Borel $\sigma-$algebra). For each agent $a \in [0, 1]$, the consumption set and the initial endowment are given by $X(a) = [0, 2]$ and $\omega(a) = 2$; the preference relation $\prec_{a,e}$ is defined by the utility function $u_{a,e}(x) := |x + e|$. The simplex of prices for this economy is $\Delta = \{1\}$, the externality space is R and the externality mapping $\Phi : A \times \Delta \times L_X \to \mathbf{R}$ is defined by

$$\Phi(a, 1, f) := a - 1 - \int_0^a f(\alpha) d\nu(\alpha).$$

It is easy to check that this economy satisfies all the assumptions of Theorem 4, but the convexity assumption **EC** (i.e., it satisfies Assumptions **A**, **C**, **E** and **IB**).

From Balder [4], this economy does not admit a free disposal quasi-equilibrium. For the sake of completeness the argument goes as follows. Assume that $\mathcal{E}$ admits such an equilibrium, denoted $(f^*, 1)$. For a.e. $a \in A$, we have $B(a, 1) = [0, 2]$, and from the equilibrium consumer condition we deduce that $f^*(a) = 0$ if $\int_0^a f^*(\alpha) d\nu(\alpha) > a$ and $f^*(a) = 2$ if $\int_0^a f^*(\alpha) d\nu(\alpha) < a$. We now consider the absolutely continuous function $\Psi : A \to \mathbf{R}_+$ defined by $\Psi(a) := [\int_0^a (f^*(\alpha) - 1) d\nu(\alpha)]^2$. From above, we deduce that

$$\Psi'(a) = 2(f^*(a) - 1) \int_0^a (f^*(\alpha) - 1) d\nu(\alpha) \leq 0 \text{ for a.e. } a \in A.$$

Hence, for every $a \in A$ $\Psi(a) \leq \Psi(0) = 0$, which together with $0 \leq \Psi(a)$, implies that $\int_0^a (f^*(\alpha) - 1) d\nu(\alpha) = 0$. Consequently, $f^* = 1$, which contradicts the above assertion that $f^*(a) \in \{0, 2\}$ for a.e. $a \in A$.

Finally, we show that the Convexity Assumption **EC** does not hold. Indeed, assume that **EC** holds. We first notice that for every $a \in A$, for $e = -1$ (which is for example obtained by Φ with $f = 1$) and for $x = \frac{1}{2}$, the set

$$\{x' \in [0, 2] \mid not[x' \prec_{a,e} x]\} = \left[0, \frac{1}{2}\right] \cup \left[\frac{3}{2}, 2\right],$$

is not convex. Consequently, by Assumption **EC**, the externality mapping Φ must be convex on A (in the sense of Definition 2). So, let $\{f_i\}_{i=1,2}$ defined by $f_1 = 0$

and $f_2 = 2$, then the function $f = 1$ satisfies $f(\alpha) \in co\{f_1(\alpha), f_2(\alpha)\}$ for a.e. $\alpha \in A$. Since Φ is convex on A, there exists $f^* \in L_X$ such that, for a.e. $\alpha \in A$, $f^*(\alpha) \in \{f_1(\alpha), f_2(\alpha)\}$ and $\Phi(a, 1, f^*) = \Phi(a, 1, f)$. From this last relation one gets that $\int_0^a f^*(\alpha)d\nu(\alpha) = \int_0^a f(\alpha)d\nu(\alpha)$ for a.e. $a \in A$, which implies that $f^*(\alpha) - f(\alpha) = 0$ for a.e. $\alpha \in A$. So, $f^* = f = 1$, which contradicts the above assertion that $f^*(\alpha) \in \{0, 2\}$ for a.e. $\alpha \in A$.

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Identification of consumers' preferences when their choices are unobservable*

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Summary. We provide conditions under which the heterogenous, deterministic preferences of consumers in a pure exchange economy can be identified from the equilibrium manifold of the economy. We extend those conditions to consider exchange economies, with two commodities, where consumers' preferences are random. For the latter, we provide conditions under which consumers' heterogenous random preferences can be identified from the joint distribution of equilibrium prices and endowments. The results can be applied to infer consumers' preferences when their demands are unobservable.

Keywords and Phrases: Preferences, Random utility, Pure exchange economies, Identification, Equilibrium correpsondence

JEL Classification Numbers: D12, D51

1 Introduction

A large body of work in economics has dealt with aggregation of agents' behavior. The use of a representative consumer has been common in macroeconomics, due to its tractability, but, at the same time, it has been recognized that only very strong

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assumptions on the preferences of the consumers or on the distribution of incomes are consistent with such a model (See Gorman (1953), Samuelson (1956), Eisenberg (1961), Chipman (1974), Chipman and Moore (1979), and Polemarchakis (1983), for theoretical results. For relevant empirical approaches, see Lewbel (1989), Stoker (1993), and their references.) When such strong conditions are not satisfied, one may consider studying conditions that will guarantee only a particular aggregate behavior of interest. For example, the research initiated by Hildenbrand (1983), and followed by Chiappori (1985), Grandmont (1987, 1992), Marhuenda (1995), and Quah (2000), among others, provides conditions on the shape of the distribution of income or the shape of agents' characteristics under which aggregate demand is monotone in prices. Another alternative, is, of course, to study the full disaggregated model, which specifies an individual demand function for each consumer. This allows for general types of consumer demands and distributions of incomes, but requires much more knowledge about the consumers. In particular, most predictions derived from such a model would require being able to first identify the demand functions of each of the individuals in the economy.

The identification of underlying behavior from observable behavior has attracted the attention of economic theorists and econometricians for a long time. The theory of revealed preference, which studies whether an individual's choices are generated by the maximization of preferences within a certain type, integrability theory, which provides conditions under which one can identify individual preferences from individual demand functions, and the econometric problem of identifying structural equations from reduced form equations, are all examples of questions of this type that have attracted the attention of many economists. Thanks to their work, we now have methods that allow us to identify preferences of consumers and production technologies of firms purely from their individual market behavior, and we have methods to identify aggregate demand functions and aggregate supply functions from only equilibrium observations. This identification is essential if, for example, one wants to evaluate the change in a consumer's welfare due to some new income tax or a new price policy, or if one wants to predict changes in the production plans of a firm due to some new legislation, or if one wants to predict a new market equilibrium in some new environment. The fact that these underlying functions and relationships can be identified making use of restrictions derived from economic models, such as optimization behavior by the individuals and the firms, or market equilibrium conditions, provides strong evidence about the usefulness of economic models.

For some time, the power of optimization and equilibrium restrictions, in the sense described above, had been contrasted with the weakness of aggregation restrictions. The path-breaking work of Sonnenschein (1973,1974), together with Debreu (1974), Mantel (1974, 1976), McFadden, Mas-Colell, Mantel and Richter (1974), and later results by Mas-Colell (1977a), Geanakoplos and Polemarchakis (1980), Andreu (1983) and, more recently, Chiappori and Ekeland (1999) have been widely interpreted to mean that aggregate observable behavior contains no strong implications, derived from the individual behavior that generated it. More specifically, their result is that, if the number of consumers is sufficiently large, any function satisfying some weak properties can be the aggregate demand function of an economy, or, in other words, the restrictions on individual demand that are generated by the optimization of individual preferences essentially vanish when these demand functions are aggregated. As Mas-Colell (1977a) showed, these results imply that any set of prices can be the equilibrium prices for some economy. This interpretation was challenged by Brown and Matzkin (1996), who showed that the restrictions of consumer demand that are generated by preference optimization are effectively translated into the equilibrium manifold of the economy. Brown and Matzkin (1996) showed that if individual endowments are observed, the aggregate behavior of the consumers satisfies restrictions that are derived from individual preference maximization.[1] In an unpublished paper, Brown and Matzkin (1990) showed that given the equilibrium manifold of a pure exchange economy, one can identify the demand functions of all the consumers in the economy. Later work by Balasko (1999) and by Chiappori, Ekeland, Kubler, and Polemarchakis (2002) provided constructive ways of identifying the individual demand functions from the equilibrium manifold. Unlike Brown and Matzkin (1990), Balasko (1999) used the condition that one can observe equilibrium prices when the endowments of all but one individual are zero, and Chiappori, Ekeland, Kubler, and Polemarchakis (2002) restricted the individual demand functions to be differentiable in income.

These identification results show that, without observing the choices that individuals make, one can still identify their preferences, as long as their endowments are observed and the aggregation of their behavior is also observed. In fact, a stronger statement is true. To identify individual preferences one only needs to observe the incomes of the individuals and the aggregate behavior, e.g. the aggregate endowment and equilibrium price. Since in most economic situations it is much easier to observe the incomes of the consumers than to observe their endowments, these results are important for empirical work. One could combine these identification results with prior results about the existence of representative consumers to derive a model with a small number of community groups. In such a model, the behavior of each community group could be required to be consistent with the existence of a representative consumer for the group, but no restrictions would be imposed across representative consumers. The identification results in Brown and Matzkin (1990) could then be used to identify the preferences of each of the representative consumers using only observations on the aggregate endowment, the equilibrium prices, and the aggregate income of each community.

When one is interested in using observational data to apply these results, however, one typically encounters the problem that it is rarely the case that the primitives of an economy stay fixed across observations. Some unobservable random shock may affect consumer preferences, generating a *distribution* of equilibrium prices, instead of a deterministic set of prices. The relevant question of interest in this context is then whether one can identify the random demand or random preferences of the individuals from the distribution of equilibrium prices, when the distributions of the individual demands are not observable. For the case where a distribution of demand is observable, McFadden (1975, 2002), McElroy (1981), Brown and Walker

[1] Earlier works that presented restrictions are McFadden, Mas-Colell, Mantel and Richter (1974), McFadden (1975), Diewert (1977), Mas-Colell and Neuefeind (1977), and Hildenbrand (1983), among others.

(1989), Lewbel (1996), and Brown and Calsamiglia (2003) consider restrictions on the distribution of demand generated from a distribution of preferences, and, starting from Barten (1968), there is a substantial literature on the identification of the distribution of preferences from an observable distribution of demand. The latter literature includes Heckman (1974), Dubin and McFadden (1984), McElroy (1987), and recent work by Brown and Matzkin (1998) and Beckert (2000)[2]. For the case where the distribution of demands are unobservable, Carvajal (2002) considers restrictions on the distribution of equilibrium prices.

This paper has two objectives. The first objective is to present and develop the identification results for pure exchange deterministic economies of Brown and Matzkin (1990). The second objective is to develop identification results for stochastic economies, where the preferences of consumers are random. We present the results for deterministic economies in the next section. In Section 3, we present those for stochastic economies.

2 Deterministic economies

In this section, we present identification results for pure exchange economies with nonrandom preferences. Since in many situations, consumers' incomes are easier to observe than consumer's endowments, we first express our identification results in terms of incomes. This requires defining the aggregate demand and the equilibrium correspondence over income tuples and aggregate endowments. Later on, we show that similar results can be obtained when the aggregate demand and the equilibrium correspondence are defined over tuples of individual endowments.

We consider an economy with J consumers and K commodities. To each commodity k, there corresponds a price p_k. We let $\Delta = \{p = (p_1, ..., p_K) \in R_+^K \,|\, \sum_{k=1}^K p_k = 1\}$ denote the set of normalized prices, $\Upsilon \subset R_+$ denote a set of incomes, and $\Upsilon^J = \Pi_{j=1}^J \Upsilon$ denote a set of $J-tuples$ of incomes. We will assume that to each consumer j, there corresponds a *demand function* $D_j : \Delta \times \Upsilon \to R_+^K$, which, for the time being, is defined just as a function that assigns to each price vector $p \in \Delta$ and income $I_j \in \Upsilon$, an element of the *budget hyperplane* $B(p, I) = \{x \in R_+^K \mid p \cdot x = I\}$. We let $\text{Đ}= (D_1, ..., D_J)$ denote the $J-$tuple of demand functions, and denote the *aggregate demand* function generated by Đ by a function $\underline{D} : \Delta \times \Upsilon^J \to R_+^K$, defined for each $(p, I_1, ..., I_J) \in \Delta \times \Upsilon^J$ by

$$\underline{D}\,(p, I_1, ..., I_J; \text{Đ}) = \sum_{j=1}^{J} D_j\,(p, I_j)$$

The vector $\underline{\omega} \in R_+^K$ will denote the *aggregate endowment.* An *equilibrium price* for an economy with demand functions Đ and aggregate endowment $\underline{\omega}$ is defined to be

[2] A large literature exists also for the case where observed individual behavior is generated by the maximization of a random preference over a finite, discrete choice set. The study of the recoverability of preferences, in this case, was introduced by McFadden (1974). (See Matzkin 1992, 1993 for later work on recoverability results under weak conditions.) McFadden and Richter (1991) characterized the restrictions that random optimization generates in this case. (See McFadden 2002 and the references mentioned in that paper for other work along this line.)

any $p \in \Delta$ such that for some $J-$tuple of endowment vectors $(\omega_1, ..., \omega_J) \in R_+^{JK}$

$$(1) \quad \sum_{j=1}^{J} \omega_j = \underline{\omega} \qquad \& \qquad \underline{D}\left(p, p \cdot \omega_1, ..., p \cdot \omega_J; \text{Đ}\right) = \underline{\omega}$$

The *equilibrium correspondence generated by* Đ, which assigns to each vector of aggregate endowments and J-tuple of incomes the set of equilibrium prices, will be denoted by $\Gamma : R_+^K \times \Upsilon^J \twoheadrightarrow \Delta$, and defined for all $(\underline{\omega}, I_1, ..., I_J) \in R_+^K \times \Upsilon^J$ by

$$\begin{aligned} &\Gamma\left(\underline{\omega}, I_1, ..., I_J; \text{Đ}\right) \\ &= \left\{ p \in \Delta | \text{ for } (\omega_1, ..., \omega_J) \in R_+^{JK} \text{ with } p \cdot \omega_j = I_j (j = 1, ..., J), \text{ (1) is satisfied} \right\} \end{aligned}$$

(We allow for the possibility that $\Gamma(\underline{\omega}, I_1, ..., I_J; \text{Đ})$ is empty-valued, for some $(\underline{\omega}, I_1, ..., I_J; \text{Đ})$.)

The analysis of any type of identification result for an underlying function requires a specification of the set to which this function belongs. The specification may require properties such as continuity and differentiability, or some other type of restrictions. For example, any result about the identification of a utility function from a demand function will need to specify the utility function to belong to a set such that no two utility functions in that set are strictly increasing transformations of each other. For our identification result of individual demand functions, we will specify a set of $J - tuples$ of demand functions Đ$= (D_1, ..., D_J)$ to be such that, for each consumer j, if D_j and D'_j are in two $J - tuples$ within this set and for some $(\widetilde{p}, \widetilde{I}_j)$ in their domain, $D_j(\widetilde{p}, \widetilde{I}_j) \neq D'_j(\widetilde{p}, \widetilde{I}_j)$, then the income expansion path generated from D_j, when the price is $\widetilde{p}$, is not a translation of the income expansion path of D'_j, when the price is $\widetilde{p}$. Clearly, we need to impose such a restriction to be able to identify each individual demand from observable variables that only depend on the sum of these individual demands. To see this, suppose that the set of allowable $J - tuples$ of demand functions includes Đ$= (D_1, ..., D_J)$ and Đ$' = (D'_1, ..., D'_J)$ where Đ$'$ is exactly the same as Đ, except that, at some value of (p, I), and for some vector a, $D'_1(p, I) = D_1(p, I) + a$ and $D'_2(p, I) = D'_2(p, I) - a$. Then, the aggregate demand generated from Đ will be identical to that generated from Đ$'$, even though Đ$\neq$ Đ$'$. We next formally specify this set, and then, in Theorem 1, we show that this condition is sufficient to identify the individual demand functions from either the aggregate demand or from the equilibrium correspondence, as defined above.

Definition. *Φ_I will denote the set of all $J-$tuples of demand functions, $(D_1, ..., D_J)$, such that for all $(D_1, ..., D_J)$, $(D'_1, ..., D'_J)$ in Φ_I, for all j and all $p \in \Delta$, either there exits $I_j \in \Upsilon$ such that $D_j(p, I_j) = D'_j(p, I_j)$ or there exist values $I_j, I'_j \in \Upsilon$ such that*

$$\Phi_I(i) : \qquad D_j\left(p, I_j\right) - D_j\left(p, I'_j\right) \neq D'_j\left(p, I_j\right) - D'_j\left(p, I'_j\right)$$

To see the restriction that this definition implies on the elements of Φ_I, consider, for example, the subset of individual demand functions of the Gorman type

$$\begin{aligned} \Theta_j &= \{D_j\left(\cdot, \cdot; a, b\right) | D_j(p, I_j; a, b) \\ &= a(p) + b(p) I_j, \text{ for some functions } a(\cdot) \in A, \ b(\cdot) \in B\} \end{aligned}$$

where A and B are set of functions defined on the set of prices. The definition of Φ_I implies that the subset of admissible demand functions D_j within Θ_j is

$$\begin{aligned}\overline{\Theta}_j &= \{D_j(\cdot,\cdot;a^*,b)\,|D_j(p,I_j;a^*,b) \\ &= a^*(p)+b(p)I_j,\ \text{for some function } b(\cdot)\in B\}\end{aligned}$$

where a^* is an element of A. In contrast to the standard results on aggregation of demand, the restriction is not imposed across the demands of the different consumers; instead, it is imposed across the set of demands permissible for any particular consumer.

In Theorem 1, we show that different demand tuples in Φ_I generate different aggregate demand functions and different equilibrium correspondences. In other words, this theorem shows that given an aggregate demand or an equilibrium correspondence, there is a unique $J-$tuple of demand functions in Φ_I that could have generated it.

Theorem 1. *If* $Đ, Đ' \in \Phi_I$ *and* $Đ \neq Đ'$,

(1.i) there exists $(p, I_1, ..., I_J) \in \Delta \times I^J$ *such that*

$$\underline{D}(p, I_1, ..., I_J; Đ) \neq \underline{D}(p, I_1, ..., I_J; Đ'),\ \text{and}$$

(1.ii) there exists $(\underline{\omega}, I_1, ..., I_J) \in R^K_+ \times I^J$ *such that*

$$\Gamma(\underline{\omega}, I_1, ..., I_J; Đ) \neq \Gamma(\underline{\omega}, I_1, ..., I_J; Đ')$$

Proof. Let $Đ = (D_1, ..., D_J)$ and $Đ' = (D'_1, ..., D'_J)$. Since $Đ \neq Đ'$, there exists $j \in \{1, ..., J\}$, $\widetilde{p} \in \Delta$, and $I_j \in \Upsilon$ such that $D_j(\widetilde{p}, I_j) \neq D'_j(\widetilde{p}, I_j)$. Suppose, without loss of generality, that $j = 1$. Then,

$$(1.1)\quad D_1(\widetilde{p}, I_1) \neq D'_1(\widetilde{p}, I_1)$$

Since $(D_1, ..., D_J), (D'_1, ..., D'_J) \in \Phi_I$, either for some $\widetilde{I}_2 \in \Upsilon$

$$(1.2)\quad D_2\left(\widetilde{p}, \widetilde{I}_2\right) = D'_2\left(\widetilde{p}, \widetilde{I}_2\right)$$

or there exist I_2, I'_2 such that

$$(1.3)\quad D_2(\widetilde{p}, I_2) - D_2(\widetilde{p}, I'_2) \neq D'_2(\widetilde{p}, I_2) - D'_2(\widetilde{p}, I'_2)$$

If (1.2) holds, then

$$D_1(\widetilde{p}, I_1) + D_2\left(\widetilde{p}, \widetilde{I}_2\right) \neq D'_1(\widetilde{p}, I_1) + D'_2\left(\widetilde{p}, \widetilde{I}_2\right)$$

If (1.3) holds, then either

$$(1.4)\quad D_2(\widetilde{p}, I_2) - D'_2(\widetilde{p}, I_2) \neq D_1(\widetilde{p}, I_1) - D'_1(\widetilde{p}, I_1)$$

or

$$(1.5)\quad D_2(\widetilde{p}, I'_2) - D'_2(\widetilde{p}, I'_2) \neq D_1(\widetilde{p}, I_1) - D'_1(\widetilde{p}, I_1)$$

Suppose w.l.o.g. that (1.4) holds, then

$$(1.6)\quad D_1(\widetilde{p}, I_1) + D_2(\widetilde{p}, I_2) \neq D'_1(\widetilde{p}, I_1) + D'_2(\widetilde{p}, I_2)$$

Hence, by (1.3) and (1.6), we have established the existence of $I_2 \in \Upsilon$ such that

$$D_1(\widetilde{p}, I_1) + D_2(\widetilde{p}, I_2) \neq D'_1(\widetilde{p}, I_1) + D'_2(\widetilde{p}, I_2)$$

Using the same argument, we can establish that there exist I_3 such that

$$D_1(\widetilde{p}, I_1) + D_2(\widetilde{p}, I_2) + D_3(\widetilde{p}, I_3) \neq D'_1(\widetilde{p}, I_1) + D'_2(\widetilde{p}, I_2) + D'_3(\widetilde{p}, I_3)$$

Continuing in this fashion, we can determine the existence of $I_2, I_{3,}..., I_J$ and $I'_2, I'_{3,}..., I'_J$ such that

$$(1.7)\quad D_1(\widetilde{p}, I_1) + \sum_{j=2}^{J} D_j(\widetilde{p}, I_j) \neq D'_1(\widetilde{p}, I_1) + \sum_{j=2}^{J} D'_j(\widetilde{p}, I_j)$$

Hence,

$$\underline{D}(\widetilde{p}, I_1, ..., I_J; \mathrm{Đ}) \neq \underline{D}(\widetilde{p}, I_1, ..., I_J; \mathrm{Đ}').$$

This proves (1.i).

To prove (1.ii), let $\omega_j = D_j(p, I_j) \quad (j = 1, ..., J)$ and $\underline{\omega} = \sum_{j=1}^{J} D_j(p, I_j)$. Then, $\underline{\omega} = \sum_{j=1}^{J} \omega_j$. By (1.7),

$$\sum_{j=1}^{J} D_j(p, p \cdot \omega_j) = \underline{\omega} \neq \sum_{j=1}^{J} D'_j(p, p \cdot \omega_j)$$

This implies that $p \in \Gamma(\underline{\omega}, I_1, ..., I_J; \mathrm{Đ})$ and $p \notin \Gamma(\underline{\omega}, I_1, ..., I_J; \mathrm{Đ}')$. Hence

$$\Gamma(\underline{\omega}, I_1, ..., I_J; \mathrm{Đ}) \neq \Gamma(\underline{\omega}, I_1, ..., I_J; \mathrm{Đ}')$$

This completes the proof.

The above theorem does not require any particular assumptions on the preferences of the consumers. In fact, the individual demand functions are not even required to be generated by the maximization of some preferences. All that is required is that to each budget set, each individual demand assigns only one element, and that element is on the budget hyperplane. In particular, and in contrast to the result in Chiappori, Ekeland, Kubler, and Polemarchakis (2002), demand functions do not need to be differentiable.

To compare our result with that in Balasko (1999), we note that, in the above theorem, the set Υ of incomes over which the demand functions are defined is not required to include the value 0. In contrast, Balasko's argument is based on the fact that, when only one consumer has positive income, the equilibrium manifold is the inverse demand function of that consumer. Hence, being able to observe the equilibrium manifold when consumers are given 0 income is critical for his argument. Balasko's argument does not require restricting the set of possible demand functions like we did above when we defined the set Φ_I. But, if we allow 0 to belong to Υ, then the restriction $(\Phi_I(i))$ is trivially satisfied, and Φ_I becomes the set of all demand functions. To see this, note that when $I'_j = 0$, condition $(\Phi_I(i))$ is satisfied by any demand functions that are different. Hence, Theorem 1 implies that when $0 \in \Upsilon$, the individual demands can be identified from either the aggregate demand or from the equilibrium correspondence, without imposing any other restrictions.

It is easy to obtain a result analogous to that of Theorem 1, when the individual demands, the aggregate demand, and the equilibrium correspondence are defined over the set of individual endowments, instead of over incomes. Let $W \subset R^K_+$ denote a set of individual endowments and denote $\Pi_{j=1}^J W$ by W^J. Abusing notation, we will now define for each consumer j, the *demand function* $D_j : \Delta \times W \to R^K_+$, as a function which assigns to each price vector $p \in \Delta$ and endowment vector $\omega_j \in W$, an element of the *budget hyperplane* $B(p, \omega_j) = \{x \in R^K_+ \mid p{\cdot}x = p{\cdot}\omega_j\}$. We let $Ð = (D_1, ..., D_J)$ denote the $J-$tuple of demand functions, and define the *aggregate demand* function generated by $Ð$ by

$$\underline{D}\,(p, \omega_1, ..., \omega_J; Ð) = \sum_{j=1}^{J} D_j\,(p, \omega_j)$$

We say that p is an *equilibrium price* if

$$\underline{D}\,(p, \omega_1, ..., \omega_J; Ð) = \sum_{j=1}^{J} \omega_j$$

and we define the (possibly empty-valued) *equilibrium correspondence* $\Gamma : W^J \twoheadrightarrow \Delta$ for all $(\omega_1, ..., \omega_J) \in W^J$ by

$$\Gamma\,(\omega_1, ..., \omega_J; Ð) = \left\{ p \in \Delta | \underline{D}\,(p, \omega_1, ..., \omega_J; Ð) = \sum_{j=1}^{J} \omega_j \right\}$$

Definition. *Φ_ω will denote the set of all $J-$tuples of demand functions, $(D_1, ..., D_J)$, such that for all $(D_1, ..., D_J)$, $(D'_1, ..., D'_J)$ in Φ_ω, for all j and $\widetilde{p} \in \Delta$, either there exists $\omega_j \in W$ such that $D_j(\widetilde{p}, \omega_j) = D'_j(\widetilde{p}, \omega_j)$ or there exist vectors $\omega_j, \omega'_j \in W$, such that*

$$\Phi_\omega(i): \quad D_j\,(\widetilde{p}, \omega_j) - D_j\,(\widetilde{p}, \omega'_j) \neq D'_j\,(\widetilde{p}, \omega_j) - D'_j\left(\widetilde{p}, \omega'_j\right)$$

(Note that if $0 \in W$, then Φ_ω is the set all demand functions, since then for all D_j, D'_j, $D_j(\widetilde{p}, 0) = D'_j(\widetilde{p}, 0)$.) .

Then, using the arguments in the proof of Theorem 1 and the fact that if we let $\omega'_j = D_j(p, \omega_j)$, then $D_j(p, \omega_j) = D_j(p, \omega'_j)$ and $D'_j(p, \omega'_j) = D'_j(p, \omega_j)$, for any p, ω_j, D_j, and D'_j, we have

Theorem 2. *If $Ð, Ð' \in \Phi_\omega$ and $Ð \neq Ð'$,* *(2.i) there exists* $(p, \omega_1, ..., \omega_J) \in \Delta \times W^J$ *such that*

$$\underline{D}(p, \omega_1, ..., \omega_J; \text{Đ}) \neq \underline{D}(p, \omega_1, ..., \omega_J; \text{Đ}'), \text{ and}$$

(2.ii) there exists $(\omega_1, ..., \omega_J) \in R^K_+ \times W^J$ *such that*

$$\Gamma(\omega_1, ..., \omega_J; \text{Đ}) \neq \Gamma(\omega_1, ..., \omega_J; \text{Đ}')$$

Theorems 1 and 2 show that consumers demands can be identified from aggregate behavior. From the individual demands we can identify the preferences of each of the individuals, imposing additional restrictions. We establish this in the following theorem. Assume that $W = R^K_+$. Let $\Phi_{\succsim}$ denote the set of all $J-tuples$ $\underline{\succsim} = (\succsim_1, ..., \succsim_J)$ of preference relations on $R^K_+ \times R^K_+$ that generate a $J-$tuple in Φ_ω and are such that (i) for each j, $\succsim_j$ can be represented by a monotone, continuous, concave, and strictly quasiconcave utility function, and (ii) the set of all bundles in the range of the demand function generated by $\succsim_j$ is R^K_+. For each $\underline{\succsim} \in \Phi_{\succsim}$, let $\underline{D}(p, \omega_1, ..., \omega_J; \underline{\succsim})$ and $\Gamma(\omega_1, ..., \omega_J; \underline{\succsim})$ denote, respectively, the aggregate demand and the value of the equilibrium correspondence generated by $\underline{\succsim}$. Then

Theorem 3. *If* $\underline{\succsim}, \underline{\succsim}' \in \Phi_{\succsim}$ *and* $\underline{\succsim} \neq \underline{\succsim}'$,

(3.i) there exists $(p, \omega_1, ..., \omega_J) \in \Delta \times W^J$ *such that*

$$\underline{D}(p, \omega_1, ..., \omega_J; \underline{\succsim}) \neq \underline{D}(p, \omega_1, ..., \omega_J; \underline{\succsim}'), \text{ and}$$

(3.ii) there exists $(\omega_1, ..., \omega_J) \in R^K_+ \times W^J$ *such that*

$$\Gamma(\omega_1, ..., \omega_J; \underline{\succsim}) \neq \Gamma(\omega_1, ..., \omega_J; \underline{\succsim}\prime)$$

Proof. Let $\underline{\succsim}, \underline{\succsim}' \in \Phi_{\succsim}$ be such that $\underline{\succsim} \neq \underline{\succsim}'$, and for each j, let D_j and D'_j denote the demand functions generated, respectively, by $\succsim_j$ and $\succsim'_j$. Since $\underline{\succsim} \neq \underline{\succsim}'$, for at least one $j \in \{1, ..., J\}$, $\succsim_j \neq \succsim'_j$. Since concavifiable preferences are lipschitizian (see Corollary in Mas-Colell (1977b, pp. 1412)) and since $\succsim_j$ and $\succsim'_j$ can be represented by monotone, continuous, concave, and strictly quasiconcave utility functions, it follows from Theorem 2 in Mas-Colell (1977b, pp. 1413) that $D_j \neq D'_j$. Hence, $(D_1, ..., D_J) \neq (D'_1, ..., D'_J)$, and, by assumption, $(D_1, ..., D_J), (D'_1, ..., D'_J) \in \Phi_\omega$. The statements in (3.i) and (3.ii) then follow by Theorem 2.

A similar result can be obtained, using Theorem 1, when the aggregate demand and the equilibrium correspondence are defined over a vector of aggregate endowments and a $J-$tuple of incomes.

To illustrate the results in this section, we next consider the equilibrium function of an economy with two commodities, 1 and 2, and two individuals, A and B, which possess Cobb-Douglas utility functions. We show that, when it is known a-priori that the utility functions are Cobb-Douglas, but the values of the parameters of the Cobb-Douglas utilities are unknown, one can identify the values of those parameters when the equilibrium price is observed at only two points of its domain. More specifically, suppose that it is known that the utility function of individual A is $U^A(x_1, x_2) = x_1^{\alpha}\, x_2^{1-\alpha}$, for some unknown value of α, and the utility function of individual B is $U^B(x_1, x_2) = x_1^{\beta}\, x_2^{1-\beta}$, for some unknown value of β. Suppose that the equilibrium price is observed at two points, (I^A, I^B, ω) and $(\overline{I}^A, \overline{I}^B, \overline{\omega})$, such that

$$\det \begin{bmatrix} I^A & I^B \\ \overline{I}^A & \overline{I}^B \end{bmatrix} \neq 0$$

where I^A and $\overline{I}^A$ denote values for the income of individual A, I^B and $\overline{I}^B$ denote values for the income of individual B, and ω and $\overline{\omega}$ denote values for the vector of aggregate endowment. Normalize the price of the second commodity to 1. Let p_1 denote the equilibirum price at (I^A, I^B, ω) and let $\overline{p}_1$ denote the equilibrium price at $(\overline{I}^A, \overline{I}^B, \overline{\omega})$. Using the properties of Cobb-Douglass utility functions, it is easy to verify that p_1, (I^A, I^B, ω) and $\overline{p}_1$, $(\overline{I}^A, \overline{I}^B, \overline{\omega})$ satisfy

$$p_1 \cdot \omega_1 = \alpha \cdot I^A + \beta \cdot I^B$$

and

$$\overline{p}_1 \cdot \overline{\omega}_1 = \alpha \cdot \overline{I}^A + \beta \cdot \overline{I}^B$$

It is then clear that there is a unique solution for α and β.

3 Random economies

In many cases, and in particular when one is dealing with real data, there are random elements that affect the preferences of the consumers and which therefore generate a *distribution* of prices, for a same vector of endowments. In this section, we show how the results in the previous section can be extended to guarantee identification of preferences from the equilibrium prices when these preferences are random. We consider economies with 2 commodities, where we normalize the price of the second commodity to equal 1. We will restrict the set of individual demand functions that we consider to be such that the demand for the first commodity is strictly decreasing in its price, p. Under these conditions, for every realization of the random elements, an equilibrium price, if it exists, is unique. Hence, we will be dealing with an *equilibrium function* instead of an equilibrium correspondence, as in Section 2. We concentrate on the case where the individual demand functions, the aggregate demand function, and the equilibrium correspondence are all defined on a set of $J-$tuples of individual endowments. To incorporate randomness into the model, we will assume that the preferences of the consumers in an economy depend on unobservable variables. We will show that, in this case, one can still identify the demand functions of each of the individual consumers, when their choices are not observed. This will require either specifying the distribution of the unobservable variables, or restricting the way in which the demand functions depend on these unobservable variables.

We consider first the case where the random shock is univariate and affects, in not necessarily the same way, the demand of all the consumers in the economy. Let $E \subset R$ denote the support of an unobservable random variable, ε. We define a *random demand function* $\widetilde{D} : R_+ \times W \times E \to R^K_+$ to be any function that assigns, to each price $p \in R_+$, endowment vector $\omega_j \in W$, and realization of ε, an element in the budget hyperplane $B(p, I) = \{x \in R^K_+ \mid p \cdot x = I\}$. We let $\widetilde{\mathbf{D}} = (\widetilde{D}_1, ..., \widetilde{D}_J)$ denote the $J-$tuple of random demand functions, and we denote the *aggregate random demand* function generated by $\mathbf{D}$ by the function $\underline{\widetilde{D}}$: $R_+ \times W^J \times E \to R^K_+$, defined for each $(p, \omega_1, ..., \omega_J, \varepsilon) \in R_+ \times W^J \times E$ by

$$\underline{\widetilde{D}}\left(p, \omega_1, ..., \omega_{J,}\varepsilon; \widetilde{\mathbf{D}}\right) = \sum_{j=1}^{J} \widetilde{D}_j\left(p, \omega_j, \varepsilon\right)$$

An *equilibrium price* for an economy with demand functions $\widetilde{\mathbf{D}}$, J-tuple of endowment vectors $(\omega_1, ..., \omega_J)$, and realization of the unobservable random variable ε, is defined to be any $p \in R_+$ such that

$$\underline{\widetilde{D}}\left(p, \omega_1, ..., \omega_{J,}\varepsilon; \widetilde{\mathbf{D}}\right) = \sum_{j=1}^{J} \omega_j$$

The (possibly empty-valued) *random equilibrium correspondence generated by* $\widetilde{\mathbf{D}}$, which to each $(p, \omega_1, ..., \omega_{J,}\varepsilon)$ assigns the set of equilibrium prices will be denoted by $\widetilde{\varGamma} : W^J \times E \twoheadrightarrow R_+$. This correspondence, together with any specified distribution $F_{\varepsilon,(\omega_1,...,\omega_J)}$ of the unobservable random shock and the $J-$tuple of endowment vectors, generates a distribution $F_{p,(\omega_1,...,\omega_J)}$ of the equilibrium price and the $J-$tuple of endowment vectors. We want to determine whether, from $F_{p,(\omega_1,...,\omega_J)}$, we can identify the $J-$tuple of individual random demand functions that generated $F_{p,(\omega_1,...,\omega_J)}$. Clearly, this is a more demanding result than the one in Section 2, where ε and therefore p had degenerate distributions, conditional on $(\omega_1, ..., \omega_J)$. In this new case, ε is unobservable, and the individual demand functions depend on it. We will assume, throughout, that ε is distributed independently of $(\omega_1, .., \omega_J)$ and that the support of the distribution of $(\omega_1, .., \omega_J)$ is W^J. Since even for the case where individual choices are observed and monotonicity properties are imposed, one can not jointly identify the distribution of ε and the demand function without any further restrictions (Matzkin (2003)), we will need to make some additional assumptions to achieve our results. For any J-tuple of random demand functions $\widetilde{\mathbf{D}}$, let $F_{p|(\omega_1,...,\omega_J)}(\cdot; \widetilde{\mathbf{D}}, F_\varepsilon)$ denote the conditional distribution of the equilibrium price p generated by $\widetilde{\mathbf{D}}$ and F_ε. We will impose restrictions that will either specify the distribution for ε, or will specify a restriction in the way that the demand functions depend on the unobservable shock. Our first identification result will assume that ε is distributed independently of $(\omega_1, ..., \omega_J)$, with a specified distribution F_ε that possesses a continuous density f_ε.

Definition. *$\Phi_{\omega,\varepsilon}$ will denote the set of $J-$tuples of continuous random demand functions, $\widetilde{\mathbf{D}} = (\widetilde{D}_1, ..., \widetilde{D}_J)$, such that*

$\Phi_{\omega,\varepsilon}(i)$: For each j, the first coordinate of $\widetilde{D}_j$ is continuous in $(p, \omega_j, \varepsilon)$, strictly decreasing in p, and strictly increasing in ε, and

$\Phi_{\omega,\varepsilon}(ii)$: For all $\widetilde{\mathbf{D}}, \widetilde{\mathbf{D}}' \in \Phi_{\omega,\varepsilon}$, for all j, for all (p, ε), either for some $\omega_j \in W$,

$$\widetilde{D}_j(p, \omega_j, \varepsilon) = \widetilde{D}_j'(p, \omega_j, \varepsilon)$$

or there exist $\omega_j, \omega_j' \in W$ such that

$$\widetilde{D}_j(p, \omega_j, \varepsilon) - \widetilde{D}_j(p, \omega_j', \varepsilon) \neq \widetilde{D}_j'(p, \omega_j, \varepsilon) - \widetilde{D}_j'(p, \omega_j', \varepsilon)$$

Condition ($\Phi_{\omega,\varepsilon}(i)$) is made to guarantee the uniqueness of the equilibrium price, for any given $J-$tuple of endowments and value of ε, and the monotonicity in ε of the equilibrium price, for any given $J - tuple$ of endowments. Condition ($\Phi_{\omega,\varepsilon}(ii)$) is a condition similar to ($\Phi_\omega(i)$). It is used to eliminate from the set of possible $J - tuples$ those that possess demand functions that generate income expansion paths that are translations of each other. Note that when $0 \in W$, $\Phi_{\omega,\varepsilon}$ consists of the set of all $J - tuples$ of random demand functions satisfying only ($\Phi_{\omega,\varepsilon}(i)$), since ($\Phi_{\omega,\varepsilon}(ii)$) will always be satisfied by letting $\omega_j' = 0$. The following theorem shows that, under these conditions, we can identify the random demand functions of each of the consumers in an economy from the distribution of the equilibrium prices, conditional on the vector of individual endowments.

Theorem 4. *Suppose that ε is distributed independently of $(\omega_1, ..., \omega_J)$ with a specified distribution F_ε, which possesses a continuous density, f_ε, and whose support is the bounded set E. Suppose that the distribution of $(\omega_1, ..., \omega_J)$ has support W^J. Then, if $\widetilde{\mathbf{D}}, \widetilde{\mathbf{D}}' \in \Phi_{\omega,\varepsilon}$ and $\widetilde{\mathbf{D}} \neq \widetilde{\mathbf{D}}'$*

$$F_{p,(\omega_1,...,\omega_J)}(\cdot; \widetilde{\mathbf{D}}, F_\varepsilon) \neq F_{p,(\omega_1,...,\omega_J)}(\cdot; \widetilde{\mathbf{D}}', F_\varepsilon)$$

Proof Suppose that $\widetilde{\mathbf{D}}, \widetilde{\mathbf{D}}' \in \Phi_{\omega,\varepsilon}$ and $\widetilde{\mathbf{D}} \neq \widetilde{\mathbf{D}}'$. Then, for some j and some $(\widetilde{p}, \omega_j, \widetilde{\varepsilon}) \in R_+ \times W \times E$, $\widetilde{D}_j(\widetilde{p}, \omega_j, \widetilde{\varepsilon}) \neq \widetilde{D}_j'(\widetilde{p}, \omega_j, \widetilde{\varepsilon})$. By the continuity of the demand functions in ε, we can assume that $f_\varepsilon(\widetilde{\varepsilon}) > 0$. Suppose, without loss of generality that $j = 1$. Then,

$$\widetilde{D}_1(\widetilde{p}, \omega_1, \widetilde{\varepsilon}) \neq \widetilde{D}_1'(\widetilde{p}, \omega_1, \widetilde{\varepsilon})$$

By the definition of $\Phi_{\omega,\varepsilon}$, either for some ω_2, $\widetilde{D}_2(\widetilde{p}, \omega_2, \widetilde{\varepsilon}) = \widetilde{D}_2'(\widetilde{p}, \omega_2, \widetilde{\varepsilon})$, or for some ω_2, ω_2', $\widetilde{D}_2(\widetilde{p}, \omega_2, \widetilde{\varepsilon}) - \widetilde{D}_2(\widetilde{p}, \omega_2', \widetilde{\varepsilon}) \neq \widetilde{D}_2'(\widetilde{p}, \omega_2, \widetilde{\varepsilon}) - \widetilde{D}_2'(\widetilde{p}, \omega_2', \widetilde{\varepsilon})$. In either case, we can establish the existence of a ω_2 such that

$$\widetilde{D}_1(\widetilde{p}, \omega_1, \widetilde{\varepsilon}) + \widetilde{D}_2(\widetilde{p}, \omega_2, \widetilde{\varepsilon}) \neq \widetilde{D}_1'(\widetilde{p}, \omega_1, \widetilde{\varepsilon}) + \widetilde{D}_2'(\widetilde{p}, \omega_2, \widetilde{\varepsilon})$$

Continuing in this fashion, we can find $\omega_1, \omega_2, ..., \omega_J$ such that

$$\sum_{j=1}^{J} \widetilde{D}_j(\widetilde{p}, \omega_j, \widetilde{\varepsilon}) \neq \sum_{j=1}^{J} \widetilde{D}_j'(\widetilde{p}, \omega_j, \widetilde{\varepsilon})$$

Suppose, without loss of generality, that

$$\sum_{j=1}^{J} \widetilde{D}_j^{(1)}(\widetilde{p}, \omega_j, \widetilde{\varepsilon}) < \sum_{j=1}^{J} \widetilde{D}_j'^{(1)}(\widetilde{p}, \omega_j, \widetilde{\varepsilon})$$

where $\widetilde{D}_j^{(1)}$ and $\widetilde{D}_j'^{(1)}$ denote the first coordinates of, respectively, $\widetilde{D}_j$ and $\widetilde{D}_j'$. For each j, let $\widetilde{\omega}_j = \widetilde{D}_j(\widetilde{p}, \omega_j, \widetilde{\varepsilon})$. Then, since $\widetilde{p} \cdot \widetilde{\omega}_j = \widetilde{p} \cdot \widetilde{D}_j(\widetilde{p}, \omega_j, \widetilde{\varepsilon}) = \widetilde{p} \cdot \omega_j$,

$$\widetilde{D}_j(\widetilde{p}, \widetilde{\omega}_j, \widetilde{\varepsilon}) = \widetilde{\omega}_j = \widetilde{D}_j(\widetilde{p}, \omega_j, \widetilde{\varepsilon}) \quad \text{and} \quad \widetilde{D}_j'(\widetilde{p}, \widetilde{\omega}_j, \widetilde{\varepsilon}) = \widetilde{D}_j'(\widetilde{p}, \omega_j, \widetilde{\varepsilon})$$

Hence, when the endowment vector is $(\widetilde{\omega}_1, ..., \widetilde{\omega}_J)$ and the value of the random shock is $\widetilde{\varepsilon}$, $\widetilde{p}$ is an equilibrium price when the $J-$tuple of demand functions is $\widetilde{\text{Đ}}$ and $\widetilde{p}$ is not an equilibrium price when the $J-$tuple of demand functions is $\widetilde{\text{Đ}}'$. Since for any $\widetilde{\text{Đ}}, \widetilde{\text{Đ}}' \in \Phi_{\omega,\varepsilon}$, the first coordinate of the random aggregate demand function generated from any $J - tuple$ of demand functions in $\Phi_{x,\varepsilon}$ is strictly decreasing in the price of the first commodity, the equilibrium price, if it exists, is unique, given any J-tuple of endowment vectors and any value of the unobservable random term. By the continuity of the $\widetilde{D}_j'^{(1)}$ functions in ε, the continuity of f_ε, and the fact that $f_\varepsilon(\widetilde{\varepsilon}) > 0$, it follows that there exists a neighborhood of $\widetilde{\varepsilon}$ in E such that for all values ε' in that neighborhood, $f_\varepsilon(\varepsilon') > 0$ and

$$\sum_{j=1}^{J} \widetilde{\omega}_j^{(1)} < \sum_{j=1}^{J} \widetilde{D}_j'^{(1)}(\widetilde{p}, \widetilde{\omega}_j, \varepsilon')$$

For any $\widetilde{\text{Đ}} \in \Phi_{\omega,\varepsilon}$ and any $p \in R_+$ and $\omega = (\omega_1, ..., \omega_J)$ define

$$e(p,\omega;\widetilde{\text{Đ}}) = \left\{ \begin{array}{ll} \sup\left\{\varepsilon \in E \,|\, \sum_{j=1}^{J} \widetilde{D}_j^{(1)}(p,\omega_j,\varepsilon) \le \sum_{j=1}^{J} \omega_j\right\} & \\ \quad \text{if } \left\{\varepsilon \in E \,|\, \sum_{j=1}^{J} \widetilde{D}_j^{(1)}(p,\omega_j,\varepsilon) \le \sum_{j=1}^{J} \omega_j\right\} \neq \varnothing & \\ \inf(E) \qquad \text{otherwise} & \end{array} \right\}$$

Then, $e(p, \omega; \widetilde{\text{Đ}})$ denotes the value of ε for which p is an equilibrium price when the vector of endowments is ω, if such a value exists; it equals $\inf(E)$ if for all values of ε in E, $\sum_{j=1}^{J} \omega_j < \sum_{j=1}^{J} \widetilde{D}_j^{(1)}(\widetilde{p}, \omega_j, \varepsilon)$; and it equals $\sup\{\varepsilon \in E \mid \sum_{j=1}^{J} \widetilde{D}_j^{(1)}(p, \omega_j, \varepsilon) < \sum_{j=1}^{J} \omega_j\}$ otherwise.

Since $f_\varepsilon(\widetilde{\varepsilon}) > 0$, the first coordinate of the aggregate demand generated by $\widetilde{\text{Đ}}'$ is strictly increasing in the value of the unobservable variable, and, from above, $\sum_{j=1}^{J} \widetilde{\omega}_j^{(1)} < \sum_{j=1}^{J} \widetilde{D}_j'^{(1)}(\widetilde{p}, \widetilde{\omega}_j, \varepsilon')$ for all ε' in a neighborhood of $\widetilde{\varepsilon}$, it follows that $e(\widetilde{p}, \widetilde{\omega}; \widetilde{\text{Đ}}') < \varepsilon' < \widetilde{\varepsilon}$ for all ε' in a neighborhood that possesses positive probability. By the definition of $e(p, \omega; \widetilde{\text{Đ}})$ and the fact that $\widetilde{p}$ is the equilibrium price when the endowment vector is $\widetilde{\omega}$, the value of ε is $\widetilde{\varepsilon}$, and the vector of demand functions is $\widetilde{\text{Đ}}$, it follows that $e(\widetilde{p}, \widetilde{\omega}; \widetilde{\text{Đ}}) = \widetilde{\varepsilon}$. Hence,

$$\widetilde{\varepsilon} = e(\widetilde{p}, \widetilde{\omega}; \widetilde{\text{Đ}}) > e(\widetilde{p}, \widetilde{\omega}; \widetilde{\text{Đ}}')$$

and

$$\Pr\left(\varepsilon \leq e(\widetilde{p}, \widetilde{\omega}; \widetilde{\mathbf{D}})\right) > \Pr\left(\varepsilon \leq e(\widetilde{p}, \widetilde{\omega}; \widetilde{\mathbf{D}}')\right)$$

Note that

$$\begin{aligned} F_{p|(\widetilde{\omega}_1,\ldots,\widetilde{\omega}_J)}(\widetilde{p}; \widetilde{\mathbf{D}}, F_\varepsilon) &= \Pr\left(p \leq \widetilde{p} | \omega = (\widetilde{\omega}_1, ..., \widetilde{\omega}_J); \widetilde{\mathbf{D}}, F_\varepsilon\right) \\ &= \Pr\left(\varepsilon \leq e(\widetilde{p}, \widetilde{\omega}; \widetilde{\mathbf{D}}) \mid \omega = (\widetilde{\omega}_1, ..., \widetilde{\omega}_J); \widetilde{\mathbf{D}}, F_\varepsilon\right) \\ &= \Pr\left(\varepsilon \leq \widetilde{\varepsilon}; \widetilde{\mathbf{D}}, F_\varepsilon\right) \\ &= F_\varepsilon\left(\widetilde{\varepsilon}\right) \end{aligned}$$

and

$$\begin{aligned} F_{p|(\widetilde{\omega}_1,\ldots,\widetilde{\omega}_J)}(\widetilde{p}; \widetilde{\mathbf{D}}', F_\varepsilon) &= \Pr\left(p \leq \widetilde{p} | (\widetilde{\omega}_1, ..., \widetilde{\omega}_J); \widetilde{\mathbf{D}}', F_\varepsilon\right) \\ &= \Pr\left(\varepsilon \leq e(\widetilde{p}, \widetilde{\omega}; \widetilde{\mathbf{D}}') \mid (\widetilde{\omega}_1, ..., \widetilde{\omega}_J); \widetilde{\mathbf{D}}', F_\varepsilon\right) \\ &= \Pr\left(\varepsilon \leq e(\widetilde{p}, \widetilde{\omega}; \widetilde{\mathbf{D}}'); \widetilde{\mathbf{D}}, F_\varepsilon\right) \\ &= F_\varepsilon\left(e(\widetilde{p}, \widetilde{\omega}; \widetilde{\mathbf{D}}')\right) \end{aligned}$$

where $F_{p|(\widetilde{\omega}_1,\ldots,\widetilde{\omega}_J)}(\widetilde{p}; \widetilde{\mathbf{D}}, F_\varepsilon)$ and $F_{p|(\widetilde{\omega}_1,\ldots,\widetilde{\omega}_J)}(\widetilde{p}; \widetilde{\mathbf{D}}', F_\varepsilon)$ are the conditional distributions of the equilibrium price, given $\omega = (\widetilde{\omega}_1, ..., \widetilde{\omega}_J)$, when the $J-$tuple of demand functions are, respectively, $\widetilde{\mathbf{D}}$ and $\widetilde{\mathbf{D}}'$. Hence, it follows that

$$F_{p|(\widetilde{\omega}_1,\ldots,\widetilde{\omega}_J)}(\widetilde{p}; \widetilde{\mathbf{D}}, F_\varepsilon) \neq F_{p|(\widetilde{\omega}_1,\ldots,\widetilde{\omega}_J)}(\widetilde{p}; \widetilde{\mathbf{D}}', F_\varepsilon)$$

This completes the proof.

Theorem 4 showed that when the distribution of ε is specified, we can identify the individual random demand functions from the distribution of the equilibrium price, conditional on the J-tuple of endowment vectors. The next theorem relaxes this assumption. Instead, a restriction on the demand functions is imposed. To describe this restriction, we will denote for each j, $\omega_j = (\omega_{1,j}, \omega_{2,j})$, and we will define the set $\widetilde{W}$ by

$$\widetilde{W} = \{(\omega_{1,j}, t) | \text{for some } \varepsilon \in E \text{ and } \omega_{2,j},\ (\omega_{1,j}, \omega_{2,j}) \in W \text{ and } t = \omega_{2,j} - \varepsilon\,\}.$$

Definition. *$\Phi'_{\omega,\varepsilon}$ will denote the set of $J-$tuples $\widetilde{\mathbf{D}} = (\widetilde{D}_1, ..., \widetilde{D}_J)$ of continuous random demand functions, $\widetilde{D}_j : R_+ \times \widetilde{W} \to R_+^K$ such that*

$\Phi'_{\omega,\varepsilon}(i)$: For each j, the first coordinate of $\widetilde{D}_j$ is strictly decreasing in p and strictly increasing in $\omega_{2,j} - \varepsilon$,

$\Phi'_{\omega,\varepsilon}(ii)$:For all $\widetilde{\mathbf{D}}, \widetilde{\mathbf{D}}' \in \Phi_{\omega,\varepsilon}$, all j, and all (p, ε), either for some $\omega_j \in W$,

$$D_j(p, \omega_{1,j}, \omega_{2,j} - \varepsilon) = \widetilde{D}'_j(p, \omega_{1,j}, \omega_{2,j} - \varepsilon),$$

or there exist $\omega_j, \omega_j' \in W$ *such that*

$$\widetilde{D}_j(p,\omega_{1,j},\omega_{2,j}-\varepsilon)-\widetilde{D}_j(p,\omega_{1,j}',\omega_{2,j}'-\varepsilon)$$
$$\neq \widetilde{D}_j'(p,\omega_{1,j},\omega_{2,j}-\varepsilon)-\widetilde{D}_j'(p,\omega_{1,j}',\omega_{2,j}'-\varepsilon)$$

$\Phi'_{\omega,\varepsilon}(iii)$: *For some* $\overline{p} \in R_+$, *there exist, for all* j, $(\overline{\omega}_{1,j},\overline{t}_j) \in \widetilde{W}$ *such that for all* $\widetilde{\mathbf{D}}, \widetilde{\mathbf{D}}' \in \Phi_{\omega,\varepsilon}$,

$$\widetilde{D}_j(\overline{p},\overline{\omega}_{1j},\overline{t}_j) = \widetilde{D}_j'(\overline{p},\overline{\omega}_{1j},\overline{t}_j)$$

Condition $(\Phi'_{\omega,\varepsilon}(i))$ is analogous to condition $(\Phi_{\omega,\varepsilon}(i))$. It is made to guarantee the uniqueness of the equilibrium price, for any given $J-$tuple of endowments and value of ε, and the monotonicity in ε of the equilibrium price, for any given $J-tuple$ of endowments. Note that the monotonicity of the equilibrium price in ε is decreasing. Condition $(\Phi'_{\omega,\varepsilon}(ii))$ involves two types of restrictions. The first is analogous to condition $(\Phi_{\omega,\varepsilon}(ii))$ in that it eliminates from the set $\Phi'_{\omega,\varepsilon}$ any $J-tuples$ with demand functions that generate income expansion paths that are translations of each other. The second restriction imposes a particular type of weak separability in the demand function. If, for example, the preferences of each consumer j are represented by a utility function of the form $U_j(x_1, x_2-\varepsilon)$, then, it is easy to verify that when the price of x_2 is normalized to 1, the demand function generated from this utility function will satisfy the special type of weak separability required in condition $(\Phi'_{\omega,\varepsilon}(ii))$. Condition $(\Phi'_{\omega,\varepsilon}(iii))$ fixes the values of the demand functions of each consumer at one point. If, for each j, we could observe the distribution of choices made by j, given p and ω_j, then condition $(\Phi'_{\omega,\varepsilon}(iii))$ together with the special type of weak separability condition imposed in $(\Phi_{\omega,\varepsilon}(ii))$ and the monotonicity with respect to $\omega_{2,j}-\varepsilon$ imposed in condition $(\Phi'_{\omega,\varepsilon}(i))$ would be enough to identify the distribution of ε and the demand function $\widetilde{D}_j$ (see Matzkin (2003)). Since, in our case, the distribution of consumer $j's$ choices is not observed, we need to require the additional conditions on the set $\Phi'_{\omega,\varepsilon}$. The following theorem establishes that, from the joint distribution of equilibrium prices and $J-tuples$ of endowment vectors, we can identify the distribution of ε and the random demand functions of each of the consumers in the economy.

Theorem 5. *Suppose that* ε *is distributed independently of* $(\omega_1,...,\omega_J)$ *with an unknown distribution function,* F_ε*, which possesses a continuous density,* f_ε*, and whose support is the bounded set* E*. Suppose that the the distribution of* $(\omega_1,...,\omega_J)$ *has support* W^J*. Then, if* $\widetilde{\mathbf{D}}, \widetilde{\mathbf{D}}' \in \Phi'_{\omega,\varepsilon}$ *and either* $\widetilde{\mathbf{D}} \neq \widetilde{\mathbf{D}}'$ *or* $F_\varepsilon \neq F_\varepsilon'$ *(or both)*

$$F_{p,(\omega_1,\ldots,\omega_J)}(\cdot;\widetilde{\mathbf{D}},F_\varepsilon) \neq F_{p,(\omega_1,\ldots,\omega_J)}(\cdot;\widetilde{\mathbf{D}}',F_\varepsilon')$$

Proof. Suppose first that $F_\varepsilon \neq F_\varepsilon'$. Then, for some $\widetilde{\varepsilon} \in E$, $F_\varepsilon(\widetilde{\varepsilon}) \neq F_\varepsilon'(\widetilde{\varepsilon})$. By $\Phi'_{\omega,\varepsilon}$(iii),

$$\sum_{j=1}^{J}\widetilde{D}_j(\overline{p},\overline{\omega}_{1,j},\overline{t}_j) = \sum_{j=1}^{J}\widetilde{D}_j'(\overline{p},\overline{\omega}_{1,j},\overline{t}_j)$$

Hence, $\overline{p}$ is an equilibrium price generated from both, $\widetilde{\mathbf{D}}, \widetilde{\mathbf{D}}'$, given $\widetilde{\varepsilon}$ and the endowment vector $(\omega_1, ..., \omega_J) = ((\overline{\omega}_{1,1}, \overline{\omega}_{2,1}), ..., (\overline{\omega}_{1,J}, \overline{\omega}_{2,J})) = ((\overline{\omega}_{1,1}, \overline{t}_1 + \widetilde{\varepsilon}), ..., (\overline{\omega}_{1,J}, \overline{t}_J + \widetilde{\varepsilon}))$. By $\Phi'_{\omega,\varepsilon}$(i) the equilibrium price is unique and decreasing in the value of ε. Hence,

$$
\begin{aligned}
&F_{p|\left(\left(\overline{\omega}_{1,1}, \overline{t}_1 + \widetilde{\varepsilon}\right), ..., \left(\overline{\omega}_{1,J}, \overline{t}_J + \widetilde{\varepsilon}\right)\right)} \left(\overline{p}; \widetilde{\mathbf{D}}, F_\varepsilon\right) \\
&= \Pr\left(p \leq \overline{p} | \omega = \left(\left(\overline{\omega}_{1,1}, \overline{t}_1 + \widetilde{\varepsilon}\right), ..., \left(\overline{\omega}_{1,J}, \overline{t}_J + \widetilde{\varepsilon}\right)\right); \widetilde{\mathbf{D}}, F_\varepsilon\right) \\
&= \Pr\left(\varepsilon \geq \widetilde{\varepsilon}; \widetilde{\mathbf{D}}, F_\varepsilon\right) \\
&= 1 - F_\varepsilon\left(\widetilde{\varepsilon}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
&F_{p|\left(\left(\overline{\omega}_{1,1}, \overline{t}_1 + \widetilde{\varepsilon}\right), ..., \left(\overline{\omega}_{1,J}, \overline{t}_J + \widetilde{\varepsilon}\right)\right)} \left(\overline{p}; \widetilde{\mathbf{D}}', F'_\varepsilon\right) \\
&= \Pr\left(p \leq \overline{p} | \omega = \left(\left(\overline{\omega}_{1,1}, \overline{t}_1 + \widetilde{\varepsilon}\right), ..., \left(\overline{\omega}_{1,J}, \overline{t}_J + \widetilde{\varepsilon}\right)\right); \widetilde{\mathbf{D}}', F'_\varepsilon\right) \\
&= \Pr\left(\varepsilon \geq \widetilde{\varepsilon}; \widetilde{\mathbf{D}}', F'_\varepsilon\right) \\
&= 1 - F'_\varepsilon\left(\widetilde{\varepsilon}\right)
\end{aligned}
$$

Since $F_\varepsilon(\widetilde{\varepsilon}) \neq F'_\varepsilon(\widetilde{\varepsilon})$,

$$
\begin{aligned}
&F_{p|\left(\left(\overline{\omega}_{1,1}, \overline{t}_1 + \widetilde{\varepsilon}\right), ..., \left(\overline{\omega}_{1,J}, \overline{t}_J + \widetilde{\varepsilon}\right)\right)} \left(\overline{p}; \widetilde{\mathbf{D}}, F_\varepsilon\right) \\
&\neq F_{p|\left(\left(\overline{\omega}_{1,1}, \overline{t}_1 + \widetilde{\varepsilon}\right), ..., \left(\overline{\omega}_{1,J}, \overline{t}_J + \widetilde{\varepsilon}\right)\right)} \left(\overline{p}; \widetilde{\mathbf{D}}', F'_\varepsilon\right)
\end{aligned}
$$

Suppose, next, that $F_\varepsilon = F'_\varepsilon$. Then, $\widetilde{\mathbf{D}} \neq \widetilde{\mathbf{D}}'$, where $\widetilde{\mathbf{D}}, \widetilde{\mathbf{D}}' \in \Phi_{\omega,\varepsilon}$. Hence, for some j and some $(\widetilde{p}, \omega_{1,j}, \omega_{2,j} - \widetilde{\varepsilon})$, $\widetilde{D}_j(\widetilde{p}, \omega_{1,j}, \omega_{2,j} - \widetilde{\varepsilon}) \neq \widetilde{D}'_j(\widetilde{p}, \omega_{1,j}, \omega_{2,j} - \widetilde{\varepsilon})$. Then, using $(\Phi'_{\omega,\varepsilon}(ii))$ and following arguments very similar to those used in the proof of Theorem 4, we can show that

$$
F_{p|(\widetilde{\omega}_1, ..., \widetilde{\omega}_J)}(\widetilde{p}; \widetilde{\mathbf{D}}, F_\varepsilon) \neq F_{p|(\widetilde{\omega}_1, ..., \widetilde{\omega}_J)}(\widetilde{p}; \widetilde{\mathbf{D}}', F_\varepsilon)
$$

Hence, different distributions of ε generate different conditional distributions of equilibrium prices given endowment vectors. This completes the proof.

Theorems 4 and 5 establish that in 2-commodity economies where the individual demands of the consumers are monotone in an unobservable random term, one can identify these individual demands, and, under some additional restrictions, also the distribution of the random term, solely from the conditional distribution of the equilibrium price, given the $J - tuples$ of individual endowments. These results assumed that a common unobservable variable was an argument in each of the individual demand functions. In many situations, however, it may be more reasonable to assume that to each individual consumer there corresponds a different unobservable random term. We next show that, restricting the demand functions further, we can still identify the individual demand functions also in this situation.

Definition. $\Phi_{\omega,\varepsilon_1,...,\varepsilon_J}$ *will denote the set of* $J-tuples$*,* $\widetilde{\mathbf{D}} = (\widetilde{D}_1,...,\widetilde{D}_J)$*, of continuous random demand functions* $\widetilde{D}_j : R_+ \times W \times E \to R_+^K$ *such that*

$\Phi_{\omega,\varepsilon_1,...,\varepsilon_J}(i)$ *: For all all* $p \in R_+$ *and all* j*, there exists* $\overline{\omega}_j \in W$ *and* α_j *such that for all* $\widetilde{\mathbf{D}}, \widetilde{\mathbf{D}}' \in \Phi_{\omega,\varepsilon_1,...,\varepsilon_J}$*, and all* $\varepsilon_j \in E$*,*

$$\widetilde{D}_j(p,\overline{\omega}_j,\varepsilon_j) = \widetilde{D}'_j(p,\overline{\omega}_j,\varepsilon_j) = \alpha_j$$

$\Phi_{\omega,\varepsilon_1,...,\varepsilon_J}(ii)$ *: For each* j*, the first coordinate of* $\widetilde{D}_j$ *is strictly decreasing in* p *and, except at vectors* $(p,\omega_j,\varepsilon_j)$ *such that* $p\cdot\omega_j = p\cdot\overline{\omega}_j$*, where* $\overline{\omega}_j$ *is as specified in* $\Phi_{\omega,\varepsilon_1,...,\varepsilon_J}(i)$*, the first coordinate of* $\widetilde{D}_j$ *is strictly increasing in* ε_j*.*

The effect of condition $(\Phi_{\omega,\varepsilon_1,...,\varepsilon_J}(i))$ is to eliminate the randomness of ε_j at some points. Note that when $0 \in W$, condition $(\Phi_{\omega,\varepsilon_1,...,\varepsilon_J}(i))$ is always satisfied by letting $\overline{\omega}_j = 0$. Condition $(\Phi_{\omega,\varepsilon_1,...,\varepsilon_J}(ii))$ plays a role similar to that played by condition $(\Phi_{\omega,\varepsilon}(i))$ in Theorem 4. For each j, let ε_{-j} denote the $J-1$ dimensional vector $(\varepsilon_1,..,\varepsilon_{j-1},\varepsilon_{j+1},...,\varepsilon_J)$. Assuming that the $\varepsilon'_j s$ are independent across j and, for each j, F_{ε_j} is a specified distribution, we can show that the demand functions of each of the individual consumers can be identified from the distribution of prices.

Theorem 6. *Suppose that for each* j*,* ε_j *is distributed independently of* $(\omega_1,...,\omega_J)$ *and of* ε_{-j} *with a specified distribution,* F_{ε_j}*, which possesses a continuous density,* f_{ε_j}*, and whose support is the bounded set* E*. Suppose that the distribution of* $(\omega_1,...,\omega_J)$ *has support* W^J*. Then, if* $\widetilde{\mathbf{D}}, \widetilde{\mathbf{D}}' \in \Phi_{\omega,\varepsilon_1,...,\varepsilon_J}$ *and* $\widetilde{\mathbf{D}} \neq \widetilde{\mathbf{D}}'$

$$F_{p,(\omega_1,...,\omega_J)}(\cdot;\widetilde{\mathbf{D}},F_\varepsilon) \neq F_{p,(\omega_1,...,\omega_J)}(\cdot;\widetilde{\mathbf{D}}',F_\varepsilon)$$

Proof. Suppose that $\widetilde{\mathbf{D}}, \widetilde{\mathbf{D}}' \in \Phi_{\omega,\varepsilon_1,...,\varepsilon_J}$ and $\widetilde{\mathbf{D}} \neq \widetilde{\mathbf{D}}'$. Then, for some j and some $(\widetilde{p},\omega_j,\widetilde{\varepsilon}_j) \in R_+ \times W \times E$, $\widetilde{D}_j(\widetilde{p},\omega_j,\widetilde{\varepsilon}_j) \neq \widetilde{D}'_j(\widetilde{p},\omega_j,\widetilde{\varepsilon}_j)$. By the continuity of $\widetilde{D}_j$ and $\widetilde{D}'_j$, we can assume that $f_{\varepsilon_j}(\widetilde{\varepsilon}_j) > 0$. Suppose, w.l.o.g. that $j=1$ and $\widetilde{D}_j^{(1)}(\widetilde{p},\omega_j,\widetilde{\varepsilon}_j) < \widetilde{D}_j'^{(1)}(\widetilde{p},\omega_j,\widetilde{\varepsilon}_j)$, where, as in the proofs of previous theorems, $\widetilde{D}_j^{(1)}$ and $\widetilde{D}_j'^{(1)}$ denote the first coordinate of $\widetilde{D}_j$ and $\widetilde{D}'_j$, respectively. Then,

$$\widetilde{D}_1^{(1)}(\widetilde{p},\omega_1,\widetilde{\varepsilon}_1) < \widetilde{D}_1'^{(1)}(\widetilde{p},\omega_1,\widetilde{\varepsilon}_1)$$

By the definition of $\Phi_{\omega,\varepsilon_1,...,\varepsilon_J}$, there exists $\overline{\omega}_2,...,\overline{\omega}_J$ and $\alpha_2,...,\alpha_J$ such that for all $k=2,...,J$ and all $\varepsilon_k \in E$, $\widetilde{D}_k(\widetilde{p},\overline{\omega}_k,\varepsilon_k) = \widetilde{D}'_k(\widetilde{p},\overline{\omega}_k,\varepsilon_k) = \alpha_k$. Hence, for all $\widetilde{\varepsilon}_2, ..., \widetilde{\varepsilon}_J$

$$\begin{aligned} & \widetilde{D}_1^{(1)}(\widetilde{p}, \omega_1, \widetilde{\varepsilon}_1) + \sum_{k=2}^{J} \widetilde{D}_k^{(1)}(\widetilde{p}, \overline{\omega}_k, \widetilde{\varepsilon}_k) \\ &= \widetilde{D}_1^{(1)}(\widetilde{p}, \overline{\omega}_1, \widetilde{\varepsilon}_1) + \sum_{k=1}^{J} \alpha_k^{(1)} \\ &< \widetilde{D}_1^{\prime(1)}(\widetilde{p}, \omega_1, \widetilde{\varepsilon}_1) + \sum_{k=1}^{J} \alpha_k^{(1)} \\ &= \widetilde{D}_1^{\prime(1)}(\widetilde{p}, \omega_1, \widetilde{\varepsilon}_1) + \sum_{k=2}^{J} \widetilde{D}_k^{\prime(1)}(\widetilde{p}, \overline{\omega}_k, \widetilde{\varepsilon}_k) \end{aligned}$$

Let $\widetilde{\omega}_1 = \widetilde{D}_1(\widetilde{p}, \omega_1, \widetilde{\varepsilon}_1)$ and for each $k = 2, ..., J$, let $\widetilde{\omega}_k = \alpha_k$. Then, since $\widetilde{p} \cdot \widetilde{\omega}_k = \widetilde{p} \cdot \widetilde{D}_k(\widetilde{p}, \overline{\omega}_k, \widetilde{\varepsilon}_k) = \widetilde{p} \cdot \overline{\omega}_k$,

$$\widetilde{D}_k(\widetilde{p}, \widetilde{\omega}_k, \widetilde{\varepsilon}_k) = \widetilde{\omega}_k = \widetilde{D}_k(\widetilde{p}, \overline{\omega}_k, \widetilde{\varepsilon}_k) \quad and \quad \widetilde{D}_1'(\widetilde{p}, \widetilde{\omega}_1, \widetilde{\varepsilon}_1) = \widetilde{D}_1'(\widetilde{p}, \omega_1, \widetilde{\varepsilon}_1)$$

Hence, when the endowment vector is $(\widetilde{\omega}_1, ..., \widetilde{\omega}_J)$ and the value of ε_1 is $\widetilde{\varepsilon}_1$, $\widetilde{p}$ is an equilibrium price for all values of $(\varepsilon_2, ..., \varepsilon_J)$, when the $J-$tuple of demand functions is $\widetilde{\mathbf{D}}$, and $\widetilde{p}$ is not an equilibrium price, for any value of $(\varepsilon_2, ..., \varepsilon_J)$, when the $J-$tuple of demand functions is $\widetilde{\mathbf{D}}'$. By $\Phi_{\omega,\varepsilon_1,...,\varepsilon_J}(ii)$, $\widetilde{p}$ is the unique such equilibrium price, when the $J-$tuple of demand functions is $\widetilde{\mathbf{D}}$. By the continuity of $\widetilde{D}_1^{\prime(1)}$ and f_{ε_1} in ε_1 and the fact that $f_{\varepsilon_1}(\widetilde{\varepsilon}_1) > 0$, it follows that there exists a neighborhood of $\widetilde{\varepsilon}_1$ such that for all values ε_1' in that neighborhood, $f_{\varepsilon_1}(\varepsilon_1') > 0$ and

$$\sum_{j=1}^{J} \widetilde{\omega}_j^{(1)} < \widetilde{D}_1^{\prime(1)}(\widetilde{p}, \widetilde{\omega}_1, \varepsilon_1') + \sum_{k=2}^{J} \widetilde{D}_k^{\prime(1)}(\widetilde{p}, \widetilde{\omega}_k, \widetilde{\varepsilon}_k)$$

For any $\widetilde{\mathbf{D}} \in \Phi_{\omega,\varepsilon_1,...,\varepsilon_J}$, define

$$e_1(\widetilde{p}, \omega_1; \widetilde{\mathbf{D}}) = \left\{ \begin{array}{cc} \sup \left\{ \varepsilon_1 \in E | \widetilde{D}_1^{(1)}(\widetilde{p}, \omega_1, \varepsilon_1) + \sum_{k=2}^{J} \widetilde{D}_k^{(1)}(\widetilde{p}, \widetilde{\omega}_k, \varepsilon_k) \leq \omega_1 + \sum_{k=2}^{J} \widetilde{\omega}_k \right\} & \\ \text{if } \left\{ \varepsilon_1 \in E | \widetilde{D}_1^{(1)}(\widetilde{p}, \omega_1, \varepsilon_1) + \sum_{k=2}^{J} \widetilde{D}_k^{(1)}(\widetilde{p}, \widetilde{\omega}_k, \varepsilon_k) \leq \omega_1 + \sum_{k=2}^{J} \widetilde{\omega}_k \right\} \neq \varnothing & \\ \inf(E) & \text{otherwise} \end{array} \right\}$$

Since $f_{\varepsilon_1}(\widetilde{\varepsilon}_1) > 0$, $\widetilde{D}_1^{\prime(1)}(\widetilde{p}, \widetilde{\omega}_1, \varepsilon_1)$ is strictly increasing in the value of the unobservable variable, and, from above, $\sum_{k=1}^{J} \widetilde{\omega}_k^{(1)} < \widetilde{D}_1^{\prime(1)}(\widetilde{p}, \widetilde{\omega}_1, \varepsilon_1') + \sum_{k=2}^{J} \widetilde{D}_k^{\prime(1)}(\widetilde{p}, \widetilde{\omega}_k, \widetilde{\varepsilon}_k)$ for all ε_1' in a neighborhood of $\widetilde{\varepsilon}_1$, it follows that $e_1(\widetilde{p}, \widetilde{\omega}_1; \widetilde{\mathbf{D}}') < \varepsilon_1' < \widetilde{\varepsilon}_1$ for all ε_1' in a neighborhood that possesses positive probability. By the definition of $e_1(\widetilde{p}, \omega_1; \widetilde{\mathbf{D}})$ and the fact that $\widetilde{p}$ is the equilibrium price when the endowment vector is $\widetilde{\omega}$, the value of ε_1 is $\widetilde{\varepsilon}_1$, and the vector of demand functions is $\widetilde{\text{Đ}}$, it follows that $e_1(\widetilde{p}, \widetilde{\omega}_1; \widetilde{\text{Đ}}) = \widetilde{\varepsilon}_1$. Hence,

$$\widetilde{\varepsilon}_1 = e_1(\widetilde{p}, \widetilde{\omega}_1; \widetilde{\text{Đ}}) > e_1(\widetilde{p}, \widetilde{\omega}_1; \widetilde{\text{Đ}}')$$

and

$$\Pr\left(\varepsilon_1 \leq e_1(\widetilde{p}, \widetilde{\omega}_1; \widetilde{\text{Đ}})\right) > \Pr\left(\varepsilon_1 \leq e_1(\widetilde{p}, \widetilde{\omega}_1; \widetilde{\text{Đ}}')\right)$$

Since

$$\begin{aligned} F_{p|(\widetilde{\omega}_1,...,\widetilde{\omega}_J)}(\widetilde{p}; \widetilde{\text{Đ}}, F_\varepsilon) &= \Pr\left(p \leq \widetilde{p} | \omega = (\widetilde{\omega}_1, ..., \widetilde{\omega}_J); \widetilde{\text{Đ}}, F_\varepsilon\right) \\ &= \Pr\left(\varepsilon_1 \leq e_1(\widetilde{p}, \widetilde{\omega}_1; \widetilde{\text{Đ}}) \mid \omega = (\widetilde{\omega}_1, ..., \widetilde{\omega}_J); \widetilde{\text{Đ}}, F_\varepsilon\right) \\ &= \Pr\left(\varepsilon_1 \leq \widetilde{\varepsilon}_1; \widetilde{\text{Đ}}, F_\varepsilon\right) \\ &= F_{\varepsilon_1}\left(\widetilde{\varepsilon}_1\right), \end{aligned}$$

$$\begin{aligned} F_{p|(\widetilde{\omega}_1,...,\widetilde{\omega}_J)}(\widetilde{p}; \widetilde{\text{Đ}}', F_\varepsilon) &= \Pr\left(p \leq \widetilde{p} | \omega = (\widetilde{\omega}_1, ..., \widetilde{\omega}_J); \widetilde{\text{Đ}}', F_\varepsilon\right) \\ &= \Pr\left(\varepsilon_1 \leq e_1(\widetilde{p}, \widetilde{\omega}_1; \widetilde{\text{Đ}}') \mid \omega = (\widetilde{\omega}_1, ..., \widetilde{\omega}_J); \widetilde{\text{Đ}}', F_\varepsilon\right) \\ &= \Pr\left(\varepsilon_1 \leq e_1(\widetilde{p}, \widetilde{\omega}_1; \widetilde{\text{Đ}}'); \widetilde{\text{Đ}}, F_\varepsilon\right) \\ &= F_{\varepsilon_1}\left(e_1(\widetilde{p}, \widetilde{\omega}_1; \widetilde{\text{Đ}}')\right) \end{aligned}$$

and

$$F_{\varepsilon_1}\left(\widetilde{\varepsilon}_1\right) > F_{\varepsilon_1}\left(e_1(\widetilde{p}, \widetilde{\omega}; \widetilde{\text{Đ}}')\right),$$

it follows that

$$F_{p|(\widetilde{\omega}_1,...,\widetilde{\omega}_J)}(\widetilde{p}; \widetilde{\text{Đ}}, F_\varepsilon) \neq F_{p|(\widetilde{\omega}_1,...,\widetilde{\omega}_J)}(\widetilde{p}; \widetilde{\text{Đ}}', F_\varepsilon)$$

This completes the proof.

If instead of specifying each distributions F_{ε_j} we would have required that, for each j, $\widetilde{D}_j$ satisfies properties such as those in $\Phi'_{\omega,\varepsilon}$, then we would have been able to establish the identification of also the distributions F_{ε_j}.

In Theorems 4–6, we established that from the distribution of equilibrium prices and $J-tuples$ of endowment vectors, we can identify the demand function of each of the J individual consumers. Using each of these demand functions, $\widetilde{D}_j$, together with the results in Mas-Colell (1977b), we can identify, for each value of ε, a unique preference relation generating the demand function $\widetilde{D}_j(\cdot, \cdot, \varepsilon)$, in the same way that in the proof of Theorem 3 we used the results in Mas-Colell (1977b) to establish the identification of the preferences of the J consumers from the demand functions of these consumers. An alternative way of identifying these preferences, which might be preferred in some circumstances, would be to first identify from the distribution of equilibrium prices the distribution of the demand function of each individual consumer, and then use results in Brown and Matzkin (1998) to identify the random utility functions that generate each of the demand distributions.

4 Conclusions

We have provided very weak conditions under which either from the aggregate demand function or from the equilibrium correspondence of a pure exchange economy one can identify the preferences of the consumers in the economy. We considered the case where the preferences of the consumers are deterministic, and cases where they are stochastic. In the latter case, we provided conditions under which from the conditional distribution of equilibrium prices, given endowments, one can identify both, the random demand functions and the distribution of an unobservable random terms which generate the randomness in the demand functions.

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Log-concave probability and its applications*

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Summary. In many applications, assumptions about the log-concavity of a probability distribution allow just enough special structure to yield a workable theory. This paper catalogs a series of theorems relating log-concavity and/or log-convexity of probability density functions, distribution functions, reliability functions, and their integrals. We list a large number of commonly-used probability distributions and report the log-concavity or log-convexity of their density functions and their integrals. We also discuss a variety of applications of log-concavity that have appeared in the literature.

Keywords and Phrases: Log-concavity. Reliability, Hazard functions, Probability distributions, Failure rates, Costly appraisals, Mean residual lifetime.

JEL Classification Numbers: C40, D40, D80.

1 Introduction

A function f that maps a concave set into the positive real numbers is said to be *log-concave* if the function $\ln f$ is concave and *log-convex* if $\ln f$ is a convex function. The log-concavity or log-convexity of probability densities and their integrals has interesting qualitative implications in many areas of economics, in political science, in biology, and in industrial engineering.

This paper records and proves a series of related theorems on the log-concavity or log-convexity of univariate probability density functions, cumulative distribution functions, and their integrals. We examine the invariance of these properties under integration, truncations, and other transformations. We relate the properties

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of density functions to those of reliability functions, failure rates, and the monotonicity of the "mean-residual-lifetime function." We define the "mean-advantage-over-inferiors function" for truncated distributions and relate monotonicity of this function to log-concavity or log-convexity of the probability density function and its integral. We examine a large number of commonly-used probability distributions and record the log-concavity or log-convexity of density functions and their integrals. Finally, we discuss a variety of applications of log-concavity that have appeared in the literature.

Most of the results found in this paper have appeared somewhere in the literature of statistics, economics, and industrial engineering. The purpose of this paper is to offer a unified exposition of related results on the log-concavity and log-convexity of univariate probability distributions and to sample some applications of this theory. An earlier draft of this paper has been available on the web since 1989. The current version streamlines the exposition and proofs and makes note of several related papers that have appeared since 1989.

2 From densities to distribution functions

2.1 Log-concavity begets log-concavity

The results in this paper include a bag of tricks that can be used to identify log-concave distribution functions when more straightforward methods fail. Many familiar probability distributions lack closed-form cumulative distribution functions, but have *density functions* that are represented by simple algebraic expressions. Often, straightforward application of calculus determines whether the density function is log-concave or log-convex. Conveniently, it turns out that log-concavity of the density function implies log-concavity of the cumulative distribution function. Moreover, log-concavity of the c.d.f. is a sufficient condition for log-concavity of the integral of the c.d.f. We do not have to look far to find a useful application of this result. The cumulative normal distribution does not have a closed-form representation and direct verification of its log-concavity is difficult. But the normal density function is easily seen to be log-concave, since its natural logarithm is a concave quadratic function.

The fact that log-concavity is passed from functions to their integrals was proved by Prèkopa [32]. Prèkopa finds this result as a corollary of a general theorem that requires a great deal of mathematical apparatus. Theorem 1, which applies to the case of differentiable functions of a single real variable this result has a simple calculus proof which we present in the Appendix.[1]

Theorem 1 *Let f be a probability density function whose support is the interval (a, b), and let F be the corresponding cumulative distribution function:*

- *If f is continuously differentiable and log-concave on (a, b), then F is also log-concave on (a, b).*

[1] The proof used here is due to Dierker [15]. There is a useful extension of Theorem 1 to higher dimensions. Prèkopa shows that if f is a log-concave probability density function defined on R^n, then the "marginal density functions" will also be log-concave. See also An [3]

– *If F is log-concave on (a, b), then the left hand integral G, defined by $G(x) = \int_a^x F(x)$, is also a log-concave function on on (a, b).*

The following corollary of Theorem 1 is often useful for diagnosing log-concavity.

Corollary 1 *If the density function f is monotone decreasing, then F is log-concave and so is its left hand integral G.*

Proof. Since F is a c.d.f., it must be that F is monotone increasing. Therefore if f is monotone decreasing, it must be that $f(x)/F(x)$ is monotone decreasing. But $(\frac{f(x)}{F(x)})' = (\ln F(x))''$. Therefore if f is monotone decreasing, F must be log-concave. Log-concavity of G follows from Theorem 1 □

2.2 Log-convexity (sometimes) Begets log-convexity

Log-convexity, unlike log-concavity, is not always inherited by the cumulative distribution function F from the density function f. Table 3 below lists examples of distribution functions that have strictly log-convex density functions and strictly *log-concave* distribution functions. But there is an easily diagnosed subset of log-convex density functions whose cdf's must also be log-convex. Let us define $f(a) = \lim_{x \to a} f(x)$. Then if $f(a) = 0$, the cdf F will inherit log-convexity from the density function.[2] Moreover, if F is log-convex, the left hand integral G, defined so that $G(x) = \int_a^x F(t)dt$, is also log-convex. A proof appears in the appendix.

Theorem 2 *Let f be a probability density function whose support is the interval (a, b), and let F be the corresponding cumulative distribution function:*

– *If f is continuously differentiable and log-convex on (a, b), and if $f(a) = 0$, then F is also log-convex on (a, b).*
– *If F is log-convex on (a, b), then the left hand integral G, defined by $G(x) = \int_a^x F(x)$, is also log-convex on (a, b).*

3 From densities to reliability functions

3.1 Reliability theory

Reliability theory is concerned with the time pattern of survival probability of a machine or an organism.[3] Let us consider a machine that will break down and be discarded at some time in the interval (a, b). The survival density function f is defined so that $f(x)$ is the probability that a machine breaks down at age x. The probability that the machine breaks down before reaching age x is given by $F(x)$, where F is the cumulative distribution function defined by $F(x) = \int_a^x f(t)dt$.

[2] Mark Yuying An [3] showed that F inherits log-convexity from f if $a = -\infty$. An's observation follows from our result, since for f to be a probability density function it must be that $f(-\infty) = 0$.

[3] A thorough and interesting treatment of reliability theory is found in Barlow and Proschan [7].

The *reliability function*, (also known as the *survival function*) $\bar{F}$, is defined so that $\bar{F}(x) = 1 - F(x)$ is the probability that the machine does *not* break down before reaching x. It follows from the definitions that $\bar{F}(x) = \int_x^b f(t)dt$. The conditional probability that a machine which has survived to time x will break down at time x is given by the *failure rate* (also known as the *hazard function*), which is defined by $r(x) = f(x)/\bar{F}(x)$. Let us also define a function H which is the right hand integral of the reliability function, so that $H(x) = \int_x^b \bar{F}(t)dt$.

3.2 Reliability functions inherit log-concavity

Theorem 3 mirrors Theorem 1 by establishing that log-concavity is inherited by right-hand integrals as well as by left-hand integrals. According to Theorem 3, if the density function is log-concave, the reliability function, as well as the cumulative distribution function, will be log-concave. Furthermore, log-concavity of the reliability function is inherited by its right-hand integral.

Theorem 3 *Let f be a probability density function whose support is the interval (a, b), and let $\bar{F}$ be the corresponding reliability function:*

- *If the density function f is continuously differentiable and log-concave on (a, b), then $\bar{F}$ is also log-concave on (a, b).*
- *If $\bar{F}$ is log-concave on (a, b), then the right hand integral H of the reliability function, defined by $H(x) = \int_x^b \bar{F}(t)dt$, is also log-concave on (a, b).*

Corollaries 2 and 3 are useful consequences of Theorem 3.

Corollary 2 *If the density function f is log-concave on (a, b), then the failure rate $r(x)$ is monotone increasing on (a, b).*

Proof. The failure rate is $r(x) = f(x)/\bar{F}(x) = -\bar{F}'(x)/\bar{F}(x)$. From Theorem 3, it follows that if f is log-concave, then $\bar{F}$ is also log-concave, and hence $\bar{F}'(x)/\bar{F}(x) = -r(x)$ is decreasing in x, so that $r(x)$ is increasing in x. □

Corollary 3 *If the density function f is monotone increasing, then the reliability function, $\bar{F}$, is log-concave and the failure rate is monotone increasing.*

Proof. Since $\bar{F}$ is a reliability function, it must be monotone decreasing. Therefore if f is monotone increasing, the failure rate $f/\bar{F}$ must be monotone increasing. But increasing failure rate is equivalent to a log-concave reliability function, which implies that the failure rate is monotone increasing and mean-residual-lifetime is monotone decreasing. □

Remark 1 The converse of Corollary 2 is not true. There exist probability distributions with monotone increasing failure rates but without log-concave density functions.

The "Mirror-image Pareto distribution," which is presented later in this paper, is an example of a distribution with monotone increasing failure rate, but with a density function that is log-convex rather than log-concave.

3.3 Reliability functions (sometimes) inherit log-convexity

Theorem 4 does for right hand integrals what Theorem 2 does for left hand integrals. The reliability function will inherit log-convexity from the density function if the density function approaches zero at the *upper* end of the interval (a, b).

Theorem 4 *Let f be a probability density function whose support is the interval (a, b), and let $\bar{F}$ be the corresponding reliability function:*

- *If f is continuously differentiable and log-convex on (a, b) and if $f(b) = 0$, then $\bar{F}$ is also log-convex on (a, b).*
- *If $\bar{F}$ is log-convex on (a, b), then the right hand integral H, defined by $H(x) = \int_x^b \bar{F}(t)dt$, is also log-convex on (a, b).*

4 Log-concavity begets Monotonicity

4.1 The mean-residual-lifetime function

In the industrial engineering literature, the *mean-residual-lifetime function* MRL is defined so that $MRL(x)$ is the expected length of time before a machine that is currently of age x will break down. Suppose that the density function of length of life is given by a function f with support (a, b) and the corresponding reliability function is $\bar{F}$. Then the probability that a machine which has survived to age x will survive to age $t > x$ is $f(t)/\bar{F}(x)$. The mean residual lifetime function is therefore given by:

$$MRL(x) = \int_x^b t\frac{f(t)}{\bar{F}(x)}dt - x.$$

If $MRL(x)$ is a monotone decreasing function, then a machine will “age” with the passage of time, in the sense that it’s expected remaining lifetime will diminish as it gets older. This property has been studied by Muth [27] and Swarz [37].

4.2 The mean-advantage-over-inferiors function

The *mean-residual-lifetime function* has a mirror image, which we will call the *mean-advantage-over-inferiors function*.[4] In the case of length of life, the mean advantage over inferiors is the difference between the age x of a machine that has not broken down and the average age at breakdown of the machines that it has outlasted. Suppose that the survival density function f for machines has support (a, b). For any x and t, the conditional probability that a machine broke down at age t, given that it did not survive to age x, is $f(t)/F(x)$. The average age at breakdown of machines that broke down before age x is therefore $\int_a^x t(f(t)/F(x))dt$. The mean

[4] We find it a bit surprising that our invidious civilization has not created a common English word for this idea, but we haven’t been able to find such a word.

advantage over inferiors of a machine that survives to exactly age x is defined to be:

$$\delta(x) = x - \int_a^x t\frac{f(t)}{F(x)}dt.$$

We are particularly interested in the question of when the function $\delta(x)$ is monotone increasing in x. As we will demonstrate, this property has important implications in the economics of information and product quality. The application explored here is a variant of George Akerlof's "lemons" model, in which credible appraisal is possible but costly. [1]

4.3 Log-concavity and monotonic differences

One reason to be interested in log-concavity of the left hand integral of the cumulative distribution function $G(x) = \int_a^x F(t)dt$ and of the right hand integral of the reliability function $H(x) = \int_x^b \bar{F}(t)dt$ is that these properties are equivalent to monotonicity of the mean-advantage-over-inferiors and mean-residual-lifetime functions, respectively.

Lemma 1 *The mean-advantage-over-inferiors function $\delta(x)$ is monotone increasing if and only if $G(x)$ is log-concave.*[5]

Proof. Integrating

$$\delta(x) = x - \int_a^x t\frac{f(t)}{F(x)}dt$$

by parts, we have

$$\delta(x) = x - \frac{xF(x) - \int_a^x F(t)dt}{F(x)} = \frac{\int_a^x F(t)dt}{F(x)} = \frac{G(x)}{G'(x)}.$$

Therefore $\delta(x)$ is monotone increasing if and only if $G'(x)/G(x)$ is monotone decreasing. The conclusion of Lemma 1 follows immediately from Remark 2. □

Combining the results of Lemma 1 and Theorem 1, we have the following.

Theorem 5 *The mean-advantage-over-inferiors function $\delta(x)$ is monotone increasing if either the density function f or the cumulative distribution function F is log-concave.*

Lemma 2 *The mean-residual-lifetime function $MRL(x)$ is monotone decreasing if and only if $H(x)$ is log-concave.*

[5] This result was previously reported and proved by Arthur Goldberger [19]. Goldberger attributes his proof to Gary Chamberlin.

Proof. Integrating

$$MRL(x) = \int_x^h \bar{f}(t)dt/\bar{F}(x) - x$$

by parts and noticing that $f(t) = -\bar{F}'(t)$, one finds that

$$MRL(x) = \frac{x\bar{F}(x) - \int_x^b \bar{F}(t)dt}{\bar{F}(x)} - x = \frac{-H(x)}{H'(x)}.$$

It follows that $MRL(x)$ is monotone increasing if and only if $H'(x)/H(x)$ is monotone decreasing. But $H'(x)/H(x)$ is monotone decreasing if and only if H is log-concave. □

Combining the results of Lemma 2 and Theorem 3, we have Theorem 6.

Theorem 6 *The mean residual lifetime function $MRL(x)$ will be monotone decreasing if the density function $f(x)$ is log-concave or if the reliability function $\bar{F}$ is log-concave.*

Since $\bar{F}$ is log-concave if and only if the failure rate is increasing, the following is an immediate consequence of Theorem 6.[6]

Corollary 4 *If the failure rate is monotone increasing, then the mean-residual lifetime function is monotone decreasing.*

4.4 Lemons with costly appraisals – an application

Consider a population of used cars of varying quality all of which must be sold by their current owners. The current owner of each used car knows its quality, but buyers know only the probability density function f of quality in the population. At a cost of $\$c$, any used-car owner can have it credibly and accurately appraised, so that buyers will know its actual value. There is a large number of potential buyers, and a used car of quality x is worth $\$x$ to any of these buyers.

In equilibrium for this market, there will be a pivotal quality, x^*, such that the owners of used cars of quality $x > x^*$ choose to have their objects appraised, in which case they can sell their used cars for their actual values x and receive a net return of $\$x - c$. Owners of used cars worse than x^* will not have them appraised and will be able to sell them for the average value of unappraised used cars, which in this case is the average value of used cars that are no better than x^*. The owner of a used car of quality x^* will be indifferent between appraising and not appraising. If the owner of a used car of quality x^* has it appraised, she will get a net revenue of $x^* - c$. If she does not have her object appraised, she will be able to sell it for $\int_a^{x^*} tf(t)/F(x^*)dt$. Since this owner is indifferent between appraising and not appraising, it must be that

$$x^* - c = \int_a^{x^*} t\frac{f(t)}{F(x^*)}dt,$$

[6] This result is proved, in the industrial engineering literature, by Muth [27].

or equivalently that $\delta(x^*) = c$. If the function $\delta(\cdot)$ is monotone increasing, there will be a unique solution for the pivotal quality x^*. Moreover, if δ is not monotone increasing, there will be multiple equilibria for at least some values of c. [7]

5 Transformations, truncations, and mirror images

5.1 Transformations

Some commonly-used distribution functions are defined by applying a simpler distribution to a transformed variable. For example, the lognormal distribution is defined on $(0, \infty)$ by the cumulative distribution function $F(x) = N(\ln(x))$ where N is the c.d.f. of the normal distribution. It happens that the normal distribution has a log-concave density function, and the transformation function $\ln(x)$ is a monotone increasing concave function. These two facts turn out to be sufficient to imply that the c.d.f. of the lognormal distribution is log-concave. On the other hand, the density function of the log-normal distribution is not log-concave.

Theorem 7 establishes the inheritance of log-concavity and log-convexity under concave and convex transformations of variables.

Theorem 7 *Let F be a positive-valued, twice-differentiable function with support (a, b) and let t be a monotonic, twice-differentiable function from (a', b') to $(a, b) = (t(a'), t(b'))$ Define the function $\hat{F}$ with support (a', b') so that for all $x \in (a', b')$, $\hat{F}(x) = F(t(x))$.*

- *If F is log-concave and t is a concave function, then $\hat{F}$ is log-concave.*
- *If F is log-convex and t is a convex function, then $\hat{F}$ is log-convex.*

Proof. Calculation shows that $(\ln F(x))''$ is of the same sign as $\frac{F''(x)}{F'(x)} - \frac{F'(x)}{F(x)}$, and $(\ln \hat{F}(x))''$ is of the same sign as $\frac{F''(x)}{F'(x)} + \frac{t''(x)}{t'(x)} - \frac{F'(x)}{F(x)}$.

If t is a concave function, then $\frac{t''(x)}{t'(x)} \leq 0$ and therefore if F is log-concave, it must be that $\frac{F''(x)}{F'(x)} + \frac{t''(x)}{t'(x)} - \frac{F'(x)}{F(x)} \leq 0$, which implies that $\hat{F}$ is log-concave.

If t is a convex function, then $\frac{t''(x)}{t'(x)} \geq 0$ and therefore if F is log-convex, it must be that $\frac{F''(x)}{F'(x)} + \frac{t''(x)}{t'(x)} - \frac{F'(x)}{F(x)} \geq 0$, which implies that $\hat{F}$ is log-convex. □

Linear transformations are both concave and convex. Therefore, as a corollary of Theorem 7, we can conclude that both log-concavity and log-convexity are preserved under linear transformations of variables, as described in Corollary 5. This result will be seen to have many useful applications.

Corollary 5 *Let F be a function with support (a, b). Let t be a linear transformation from the real line to itself and define a function $\hat{F}$ with support $(t(a), t(b))$ so that $\hat{F}(x) = F(t(x))$.*

[7] The function $\delta(x)$ must be increasing over some range, since $\delta(a) = 0$ and $\delta(b) > 0$. Therefore if δ is not a monotone increasing function, it will be increasing over some range and decreasing over other ranges and hence for at least some values of c there will be multiple solutions.

– *If F is log-concave, then $\hat{F}$ is log-concave.*
– *If F is log-convex, then $\hat{F}$ is log-convex.*

5.2 Mirror-image transformations

Consider a cumulative distribution function F and support (a, b). This distribution can be used to define another cumulative distribution function F^*, with support $(-b, -a)$, by setting $F^*(x) = \bar{F}(-x) = 1 - F(-x)$. The function F^*, defined in this way will be called the "mirror-image" of F, since the graphs of their density functions will be mirror-images, reflected around $x = 0$.

Theorem 8 *Let F and F^* be mirror-image cumulative distribution functions:.*

– *If the density function for either F or F^* is log-concave (log-convex), then so is the density function for the other.*
– *The c.d.f. for one of these functions is log-concave if and only if the reliability function of the other is log-concave.*
– *The mean-advantage-over-inferiors function for F^* is increasing (decreasing) if and only if the mean-residual-lifetime function for F is decreasing (increasing).*

Proof. Since $F^*(x) = 1 - F(-x) = \bar{F}(-x)$, it must be that $F^{*\prime}(x) = F'(x)$. Therefore where f^* and f are the density functions for F^* and F, respectively, $f^*(x) = f(-x)$ for all x. Since $f^*(x) = f(-x)$, these two densities are related by a linear transformation of the variable x. It follows from Corollary 5 that f^* is log-concave (log-convex) if and only if f is log-concave (log-convex).

Since $F^*(x) = \bar{F}(-x)$, it also follows from Corollary 5 that F^* is log-concave (log-convex) if and only $\bar{F}$ is log-concave (log-convex).

The mean-advantage-over-inferiors function for F is monotone increasing (decreasing) in x if and only if G is a log-concave (log-convex) function of x, where $G(x) = \int_a^x F(t)dt$. The mean-residual-lifetime function for F^* is monotone decreasing (increasing) in x if and only if H^* is log-concave (log-convex), where $H^*(x) = \int_{-x}^{-a} \bar{F}^*(t)dt$. But $\bar{F}^*(x) = F(-x)$, so that $H^*(x) = \int_{-x}^{-a} F(-t)dt = \int_a^x F(t)dt = G(x)$. Since $H^*(x) = G(x)$, for all x, it must be that H^* is log-convex (log-concave) if and only if G is log-convex (log-concave). □

If a probability distribution has a density function that is symmetric around zero, then this distribution will be its own mirror-image. In this case Theorem 8 has the following consequence.

Corollary 6 *If a probability distribution has a density function that is symmetric around zero, then*

– *The c.d.f. will be log-concave (log-convex) if and only if the reliability function is log-concave (log-convex).*
– *The mean-advantage over-inferiors function will be monotone increasing if and only if the mean-residual-lifetime is monotone decreasing.*

5.3 Truncations

Suppose that a probability distribution with support (a, b) is "truncated" to construct a new distribution function in which the probability mass is restricted to a subinterval, (a^*, b^*), of (a, b) while the relative probability density of any two points in this subinterval is unchanged. If F is the c.d.f. of the original distribution and F^* is the density function of the truncated distribution, then it must be that

$$F^*(x) = \frac{F(x) - F(a^*)}{F(b^*) - F(a^*)}.$$

But this means that the distribution function F^* is just a linear transformation of the F. It follows that the corresponding density functions are also linear transformations of each other, as are the left and right hand integrals of F and F^*. Applying Corollary 5, we can conclude the following.

Theorem 9 *If a probability distribution has a log-concave (log-convex) density function (cumulative distribution function), then any truncation of this probability distribution will also have a log-concave (log-convex) density function (cumulative distribution function).*

6 Log-concavity of some common distributions

This section contains a catalog of information about the log-concavity and log-convexity of density functions, distribution functions, reliability functions, and of the integrals of the distribution functions and reliability functions. Descriptions and discussions of these distributions can be found in reference works by Patel, Kapadia, and Owen [30], Johnson and Kotz [22], and Patil, Boswell, and Ratnaparkhi [31], and Evans, Hastings, and Peacock [16]. None of these references deal extensively with log-concavity. Patel et. al. report results on the monotonicity of failure rates and mean residual lifetime functions for some of the distributions that are most commonly studied by reliability theorists.

Whatever we learn about log-concavity of distributions applies immediately to truncations of these distributions, since log-concavity of a density function or of its integrals is inherited under truncation. [8]

In the tables below, we usually describe distributions in a "standardized form," where the linear transformation that sets the scale and the "zero" of random variable is chosen for simplicity of the expression. Recall from Theorem 7 that log-concavity is preserved under linear transformations, so that the results listed here apply to the entire family of distributions defined by linear transformations of the random variable x in any of these distributions.

[8] Reliability theorists normally concern themselves only with distributions that are bounded from below by zero. It may therefore seem surprising that we apply the definitions of reliability theory to distributions whose support may be unbounded from below. For our purposes, this is justified, since log-concavity is preserved under truncations of random variables. If we find that a distribution has, for example, a log-concave reliability function with a support that is unbounded from below, then we know that any truncation of this distribution from below is log-concave and has a support with a lower bound.

Table 1. Distributions with log-concave density functions (distribution functions marked $*$ lack a closed-form representation)

Name of distribution	Support	Density function f(x)	Cumulative dist function F(x)	$(\ln f(x))''$
Uniform	$[0, 1]$	1	x	0
Normal	$(-\infty, \infty)$	$\frac{1}{\sqrt{2\pi}} e^{-x^2/2}$	$*$	-1
Exponential	$(0, \infty)$	$\lambda e^{-\lambda x}$	$1 - e^{-\lambda x}$	0
Logistic	$(-\infty, \infty)$	$\frac{e^{-x}}{(1+e^{-x})^2}$	$\frac{1}{(1+e^{-x})^2}$	$-2f(x)$
Extreme Value	$(-\infty, \infty)$	$e^{-x} \exp\{-e^{-x}\}$	$\exp\{-e^{-x}\}$	$-e^{-x}$
Laplace (Double Exponential)	$(-\infty, \infty)$	$\frac{1}{2} e^{-\|x\|}$	$e^{\lambda x}$ if $x \leq 0$ $1 - \frac{1}{2}e^{-x}$ if $x \geq 0$	0 for $x \neq 0$
Power Function $(c \geq 1)$	$(0, 1]$	cx^{c-1}	x^c	$\frac{1-c}{x^2}$
Weibull $(c \geq 1)$	$[0, \infty)$	$cx^{c-1}e^{-x^c}$	$1 - e^{-x^c}$	$\frac{1-c}{x^2}(1 + cx^c)$
Gamma $(c \geq 1)$	$[0, \infty)$	$\frac{x^{c-1}e^{-x}}{\Gamma(c)}$	$*$	$\frac{1-c}{x^2}$
Chi-Squared $(c \geq 2)$	$[0, \infty)$	$\frac{x^{(c-2)/2}e^{-x/2}}{2^{c/2}\Gamma(c/2)}$	$*$	$\frac{2-c}{2x^2}$
Chi $(c \geq 1)$	$[0, \infty)$	$\frac{x^{c-1}e^{-x^2/2}}{2^{(c-2)/2}\Gamma(c)}$	$*$	$\frac{1-c}{x^2} - 1$
Beta $(\nu \geq 1, \omega \geq 1)$	$[0, 1]$	$\frac{x^{\nu-1}(1-x)^{\omega-1}}{B(\nu,\omega)}$	$*$	$\frac{1-\nu}{x^2} + \frac{1-\omega}{(1-x)^2}$
Maxwell		This is a Chi distribution with $c = 3$		
Rayleigh		This is a Weibull distribution with $c = 2$		

6.1 Distributions with log-concave density functions

For distributions that have log-concave density functions, it is easy to determine the log-concavity of the distribution function and reliability function and the monotonicity of failure rates, of mean-advantage-over-inferiors, and of mean-residual-lifetime functions. If the density function f is log-concave, then we know from Theorem 1 that the cumulative distribution function F and the left-hand integral of the cumulative distribution function G are also log-concave. From Theorem 3 and its corollary, we know that the reliability function $\bar{F}$ and its right-hand integral H are log-concave, and that the failure rate (hazard function) $r(x)$ is monotone increasing. From Theorem 5 we know that the mean-advantage-over-inferiors function $\delta(x)$ is monotone increasing, and from Theorem 6, we know that the mean-residual-lifetime function $MRL(x)$is monotone decreasing.

Table 1 lists several commonly-used continuous, univariate probability distributions that have log-concave density functions. For all of these distributions except the Laplace distribution, we can verify log-concavity of the density function f by checking that $(\ln f(x))'' \leq 0$ for all x in the support of f. Some distributions, such as the Weibull distribution, the power function distribution, the beta function, and the gamma function have log-concave density functions only if their parameters fall into certain ranges. The parameter ranges where these distributions are log-concave are indicated in Table 1.

Table 2. Distributions without log-concave density functions

Name of distribution	Support	Density function f(x)	c.d.f. F(x)	$(\ln f(x))''$
Power $(c < 1)$	$(0, 1]$	cx^{c-1}	x^c	$\frac{1-c}{x^2}$
Weibull $(c < 1)$	$(0, \infty)$	$cx^{c-1}e^{-x^c}$	$1 - e^{-x^c}$	$\frac{1-c}{x^2}(1 + cx^c)$
Gamma $(c < 1)$	$(0, \infty)$	$\frac{x^{c-1}e^{-x}}{\Gamma(c)}$	*	$\frac{1-c}{x^2}$
Beta ($\nu > 1$ or $\omega > 1$)	$[0, 1]$	$\frac{x^{\nu-1}(1-x)^{\omega-1}}{B(\nu,\omega)}$	*	$\frac{1-\nu}{x^2} + \frac{1-\omega}{(1-x)^2}$
Arc-sine	$[0, 1]$	$\frac{1}{\pi\sqrt{x(1-x)}}$	$\frac{2}{\pi}\sin^{-1}(x)$	$\frac{1-2x}{2x^2(1-x^2)}$
Pareto	$[1, \infty)$	$\beta x^{-\beta-1}$	$1 - x^{-\beta}$	$(\frac{\beta+1}{x})^2$
Lognormal	$(0, \infty)$	$\frac{1}{x\sqrt{2\pi}}e^{-(\ln x)^2/2}$	*	$\frac{\ln x}{x^2}$
Student's t	$(-\infty, \infty)$	$\frac{(1+\frac{x^2}{n})^{-n+1/2}}{\sqrt{n}B(.5,n/2)}$	*	$(1 - 2n)\frac{n-x^2}{(n+x^2)^2}$
Cauchy	$(-\infty, \infty)$	$\frac{1}{\pi(1+x^2)}$	$\frac{1}{2} + \frac{tan^{-1}(x)}{\pi}$	$2\frac{x^2-1}{(x^2+1)^2}$
F distribution	$(0, \infty)$	See discussion of F distribution below		
Mirror-image of Pareto dist.	$(-\infty, -1)$	$\beta x^{-\beta-1}$	$(-x)^\beta$	$(\frac{\beta+1}{x})^2$

6.2 Distributions whose density functions are not log-concave

Where the density function is not log-concave, determining the properties of the the cumulative distribution function F, the reliability function, $\bar{F}$, the mean-advantage-over inferiors function, and the mean-residual-lifetime function is a more complicated task.

One possible outcome is that f is log-convex. As is shown by the examples below, some distributions with log-convex density functions have log-concave c.d.f.'s, some have log-convex c.d.f.'s and some have c.d.f.'s which are neither log-concave nor log-convex.

For some probability distribution functions, f is neither log-concave nor log-convex but is log-concave over some interval of its support and log-convex over another interval.

Table 2 describes several distributions that do not have log-concave density functions.

Table 3 reports the log-convexity or log-concavity of density functions, distribution functions, and reliability functions, as well as the monotonicity of the mean-advantage-over-inferiors function $\delta(x)$ and the mean-residual-lifetime function $MRL(x)$.

6.3 Remarks on specific distributions

The Uniform Distribution For the uniform distribution, there are simple algebraic expressions for all of the functions studied in this paper. The mean-advantage-over-inferiors function is $\delta(x) = \int_0^x F(t)dt/F(x) = x/2$, the failure rate

Table 3. Properties of distributions without log-concave density

Name of distribution	Density function	c.d.f	$\delta(x)$	Reliability function	$MRL(x)$
Power function ($0 < c < 1$)	log-convex	log-concave	increasing	neither	nonmonotonic
Weibull ($0 < c < 1$)	log-convex	log-concave	increasing	log-convex	decreasing
Gamma ($0 < c < 1$)	log-convex	log-concave	increasing	log-convex	decreasing
Arc-Sine	log-convex	neither	nonmonotonic	neither	nonmonotonic
Pareto	log-convex	log-concave	increasing	log-convex	increasing
Lognormal	neither	log-concave	increasing	neither	nonmonotonic
Student's t	neither	neither	nonmonotonic	neither	nonmonotonic
Cauchy	neither	neither	undefined	neither	nonmonotonic
Mirror-image of Pareto dist.	log-convex	log-convex	decreasing	log-concave	decreasing
Beta ($\nu > 1$ or $\omega > 1$)	See discussion of Beta distribution below				
F distribution	See discussion of F distribution below				

(hazard function) is $r(x) = f(x)/\bar{F}(x) = \frac{1}{1-x}$, and the mean-residual-lifetime function is $MRL(x) = \int_x^1 \bar{F}(t)dt/\bar{F}(t) = (1-x)/2$.

The Normal Distribution The normal cumulative distribution function illustrates the usefulness of Theorems 1-4, since there do not exist closed-form expression for the c.d.f. or for the functions, $\delta(x)$, $r(x)$, and $MRL(x)$. Thus we are able to determine that the functions $\delta(x)$ and $r(x)$ are monotone increasing and that $MRL(x)$ is monotone decreasing, despite the fact that we can not write out these functions and calculate their derivatives.

The Extreme-Value Distribution The extreme value distribution arises as the limit as $n \to \infty$ of the greatest value among n independent random variables. This is sometimes known as the *Gumbel distribution*, or as a Type 1 Extreme Value distribution. In demography, this distribution is known as the *Gompertz distribution* and is frequently used to model the distribution of the length of human lives.

The Exponential Distribution Barlow and Proschan [7] point out that the exponential distribution is the only distribution for which the failure rate and the mean residual lifetime are constant. In most applications, the exponential distribution is written with the decay parameter λ. The failure rate is $f(x)/\bar{F}(x) = \lambda$. The mean residual lifetime function is $MRL(x) = \int_x^h \bar{F}(t)dt/\bar{F}(x) = \frac{1}{\lambda}$. If the lifetime of an object has an exponential distribution, then it does not "wear out" over time. That is to say, the probability of failure and the expected remaining lifetime remain constant so long as the object "survives".

The Laplace Distribution The Laplace density function is sometimes known as the *double exponential distribution*, since it is proportional to the exponential density for positive x and to the mirror-image of the exponential distribution for negative x. For the Laplace distribution, $\ln f(x) = -\lambda|x|$. The derivative of $\ln f(x)$ does not exist at $x = 0$, so that we can not verify log-concavity from the second derivative. However, concavity of the function $-\lambda|x|$ can be verified directly from the definition.

The Power Function Distribution The power function distribution has support $(a, b] = (0, 1]$, density function $f(x) = cX^{c-1}$, and c.d.f. $F(x) = x^c$. The mean-advantage-over-inferiors function is

$$\delta(x) = \frac{\int_a^x F(t)dt}{F(x)} = \frac{x}{1+c}.$$

Since $(\ln f(x))'' = \frac{c-1}{x^2}$, we see that f is strictly log-concave if $c > 1$, strictly log-convex if $0 < c < 1$, and log-linear (and hence both log-concave and log-convex) if $c = 1$.

If $0 < c < 1$, $f(a) = f(0) = \infty$ and $f(b) = f(1) = c$. Therefore neither Theorem 2 nor Theorem 4 applies, and we cannot use these theorems to conclude that either F or $\bar{F}$ inherits log-convexity from f. In fact, we can verify that F is log-concave by observing that $(\ln F(x))'' = \frac{-c}{x^2} < 0$. We also see by inspection that $\delta(x) = \frac{x}{1+c}$ is monotone increasing in x.

Since $\bar{F}(x) = 1 - x^c$, calculation shows that

$$(\ln \bar{F}(x))'' = \frac{cx^{c-2}(1-c-x^c)}{(1-x^c)^2}.$$

Therefore $(\ln \bar{F}(x))''$ is negative for x close to 1 and positive for x close to 0, and hence $\bar{F}$ is neither log-concave nor log-convex. The right hand integral of the reliability function is $H(x) = \frac{c+x^{c+1}}{1+c} - x$ This function is found to be neither log-concave nor log-convex. Therefore the mean-residual-lifetime function is neither monotone decreasing nor monotone increasing.

The Weibull Distribution The Weibull distribution has support $(a, b) = (0, \infty)$ and density function,

$$f(x) = cx^{c-1}e^{-x^c}.$$

Calculation shows that $(\ln f(x))' = \frac{c-1}{x} - cx^{c-1}e^{-x^c}$, and $(\ln f(x))'' = \frac{1-c}{x^2}(1+cx^c)$. The sign of $(\ln f(x))''$ is negative, zero, or positive, respectively, as $c > 1$, $c = 1$, or $c < 1$. Therefore the Weibull distribution is log-concave if $c > 1$, log-linear if $c = 1$, and log-convex if $c < 1$.

If $0 < c < 1$, $f(a) = f(0) = \infty$ and $f(b) = f(\infty) = 0$. Since $f(b) = 0$, we can conclude from Theorem 4 that $\bar{F}$ is log-convex. Therefore the failure rate $r(x)$ is monotone decreasing and, by Theorem 6, mean residual lifetime is an *increasing* function of age.

Since $f(a) \neq 0$, we cannot conclude from Theorem 2 that F inherits log-convexity from f for $0 < c < 1$. In fact, we can establish by other means that in this case F is log-concave, rather than log-convex. If $0 < c < 1$, $(\ln f(x))' < 0$ for all $x > 0$. Therefore $f(x)$ is seen to be a monotone decreasing function, and by Corollary 1, it must be that F is log-concave, the left hand integral G is log-concave. From Theorem 5, it follows that $\delta(x)$ is monotone increasing.

The Gamma Distribution The Gamma distribution has support $(a, b) = (0, \infty)$ and density function $f(x) = \frac{x^{c-1}e^{-x}}{\Gamma(c)}$. Calculation shows that $(\ln f(x))' = \frac{(c-1)}{x} - 1$, and $(\ln f(x))'' = \frac{1-c}{x^2}$. The sign of $(\ln f(x))''$ is negative, zero, or positive, respectively, as $c > 1$, $c = 1$, or $c < 1$. Therefore the Gamma distribution is log-concave if $c > 1$, log-linear if $c = 1$, and log-convex if $c < 1$.

For the Gamma distribution with $c < 1$, we have $f(a) = f(0) = \infty$ and $f(b) = f(\infty) = 0$. Since $f(b) = 0$, it follows from Theorem 4 that if $c < 1$ the reliability function $\bar{F}$ and its right hand integral H both inherit log-convexity from f. Since $\bar{F}$ and H are log-convex, the failure rate must be decreasing in x, and the mean-residual-lifetime function must be increasing in x.

Since $f(a) \neq 0$, Theorem 2 does not establish log-convexity of the cumulative distribution function F. In fact, when $0 < c < 1$, we see that $(\ln f(x))' < 0$ for all $x > 0$, so that f is monotone decreasing on (a, b). It follows from Corollary 1 that the cumulative distribution function F is log-concave and from Theorem 1 it follows that G, the left hand integral of F is also log-concave. Theorem 5, therefore implies that $\delta(x)$ is monotone increasing.

The Chi-squared Distribution The Chi-square distribution with c degrees of freedom is a gamma distribution with parameter $c/2$. The most common application of the Chi-squared distribution comes from the fact that the sum of the squares of c independent standard normal random variables has a chi-square distribution with c degrees of freedom. Since the gamma distribution has a log-concave density function for $c \geq 1$, it must be that the sum of the squares of two or more independent standard normal random variables has a log-concave density function.

The Chi Distribution Since $(\ln f(x))'' = -\frac{c-1}{x^2} - 1$, the chi distribution has a log-concave density function for $c \geq 1$.

The sample standard deviation from the sum of n independent standard normal variables has a chi distribution with $c = n/2$. Therefore the distribution of the sum of two or more independent standard normal variables is necessarily log-concave.

The chi distribution with $c = 2$ is sometimes known as the *Rayleigh distribution* amd the chi distribution with $c = 3$ is sometimes known as the *Maxwell distribution*.

The Beta Distribution The Beta distribution has support $(a, b) = (0, 1)$ and density function

$$f(x) = \frac{x^{\nu-1}(1-x)^{\omega-1}}{B(a, b)}.$$

Calculation shows that $(\ln f(x))'' = \frac{1-\nu}{x} + \frac{1-\omega}{x}$. Therefore if $\nu \geq 1$ and $\omega \geq 1$, then the density function is log-concave.

If $\nu < 1$ and $\omega < 1$, then the density function is log-convex. But in this case, Theorems 2 and 4 are of no assistance in determining log-convexity of F or $\bar{F}$, since $f(a) = f(b) = \infty$. More definite results apply for the special case of the Beta distribution where $\nu = \omega = .5$, which is known as the Arc-sine distribution and is discussed below.

If $\nu < 1$ and $\omega > 1$, the density function is neither log-convex nor log-concave on $(0, 1)$. In this case, however, the density function is monotone decreasing on $(0, 1)$, and therefore from Corollary 1 it follows that the distribution function F is log-concave and the mean-advantage-over-inferiors function δ is monotone decreasing.

If $\nu > 1$ and $\omega < 1$, the density function is again neither log-convex nor log-concave. In this case, the density function is monotone increasing on $(0, 1)$, and therefore by Corollary 3, the reliability function $\bar{F}$ is log-concave, the failure rate is monotone increasing, and mean-residual-lifetime is monotone decreasing.

The Arc-sine Distribution The Arc-sine distribution is the special case of the Beta distribution where $\nu = \omega = .5$. The cumulative distribution function has the closed-form expression, $F(x) = \frac{2}{\pi} \sin^{-1}(x)$. For this distribution,

$$(\ln f(x))'' = \frac{1 - 2x}{2x^2(1 - x^2)},$$

which is positive for $x < 1/2$ and negative for $x > 1/2$. The Arc-sine distribution is therefore neither log-concave nor log-convex, but is log-convex on the interval, $(0, 1/2)$ and log-concave on the interval $(1/2, 0)$. It follows that on the interval $(1/2, 1)$, the cumulative distribution is log-concave and $\delta(x)$ is monotone decreasing.

The Arc-sine distribution has the property that $\bar{F}(x) = F(1-x)$. Since $1-x < 1/2$ when $x > 1/2$ and vice versa, it must be that on the interval $(0, 1/2)$ $\bar{F}$ is log-concave and $MRL(x)$ is monotone decreasing.

The Pareto Distribution For the Pareto distribution $(\ln(f(x))' = -\frac{\beta+1}{x}$ and $(\ln f(x))'' = \frac{\beta+1}{x^2} > 0$. Thus the density function is monotone decreasing and log-convex for all x. Although f is log-convex, the condition of theorem 2 does not apply (since $f(a) = \beta > 0$) and the c.d.f is not log-convex. In fact, since f is a decreasing function, it follows from Corollary 1 that the c.d.f, $F(x)$, is log-concave and therefore from Lemma 1 it must also be that δ is monotone increasing.

The reliability function for the Pareto distribution is $\bar{F}(x) = x^{-\beta}$. Therefore $(\ln \bar{F}(x))'' = \beta/x^2 > 0$. Therefore the reliability function is log-convex. The right hand integral, $H(x) = \int_x^\infty F(t)dt$, converges if and only if $\beta > 1$ and in this case, $H(x) = \frac{1}{\beta-1}x^{1-\beta}$. In this case, $(\ln H(x))'' = \frac{\beta-1}{x^2} > 0$. Therefore $H(x)$ is log-convex and the mean residual lifetime is a decreasing function of x.

The Lognormal Distribution The log-normal distribution has support $(0, \infty)$ and a cumulative distribution function $F(x) = N(\ln(x))$ where N is the c.d.f. of the normal distribution.

Since the normal distribution has a log-concave c.d.f., it follows from Theorem 7, that the lognormal distribution also has a concave c.d.f. From Theorem 5 it then follows that $\delta(x)$ is increasing.

Unlike the normal distribution, the lognormal distribution does *not* have a log-concave density function. The lognormal density function is

$$f(x) = \frac{1}{x\sqrt{2\pi}} e^{-(\ln x)^2/2}.$$

A bit of calculation shows that

$$(\ln f(x))'' = \frac{\ln x}{x^2}.$$

Since $\ln x$ is negative for $0 < x < 1$ and positive for $x > 1$, it must be that $f(x)$ is neither log-concave nor log-convex on its entire domain, but log-concave on the interval $(0, 1)$ and log-convex on the interval $(1, \infty)$.

The failure rate of a log normally distributed random variable is neither monotone increasing nor monotone decreasing. (Patel, et.al. [30]). Furthermore the mean residual lifetime for the lognormal distribution is not monotonic, but is increasing for small values and decreasing for large values of x. (see Muth [27]). We have not found an analytic proof of either of these last two propositions. As far as we can tell, they have only been demonstrated by numerical calculation and computer graphics.

Student's t Distribution Student's t distribution is defined on the entire real line with density function

$$f(x) = \frac{(1 + \frac{x^2}{n})^{-n+1/2}}{\sqrt{n}B(.5, n/2)}$$

where $B(a, b)$ is the incomplete beta function and n is referred to as the number of degrees of freedom. For the t distribution $(\ln f(x))'' = -(n + 1)\frac{n-x^2}{(n+x^2)^2}$. Therefore the density function of the t distribution is log-concave on the central interval $[-\sqrt{n}, \sqrt{n}]$ and log-convex on each of the outer intervals, $[-\infty, -\sqrt{n}]$ and $[\sqrt{n}, \infty]$. Although the t distribution itself is not log-concave, a truncated t distribution will be log-concave if the truncation is restricted to a subset of the interval $[-\sqrt{n}, \sqrt{n}]$.

We do not have a general, analytic proof of the concavity or non-concavity of the c.d.f. of the t distribution. But numerical calculations show that the c.d.f is neither log-concave nor log-convex for the cases of n =1, 2, 3, 4, and 24. Since the t distribution is symmetric, the reliability function is the mirror-image of the c.d.f. Therefore if the c.d.f. is neither log-concave nor log-convex, the reliability function must also be neither concave nor convex.

The Cauchy Distribution The Cauchy distribution is a Student's t distribution with 1 degree of freedom. It is equal to the distribution of the ratio of two independent standard normal random variables.

The Cauchy distribution has density function $f(x) = \frac{1}{\pi(1+x^2)}$ and c.d.f $F(x) = 1/2 + \frac{tan^{-1}(x)}{\pi}$. Then $(\ln f(x))'' = -2\frac{x^2-1}{(x^2+1)^2}$. This expression is negative if $|x| < 1$ and positive if $|x| > 1$. Like the rest of the family of t distributions, the density function of the Cauchy distribution is neither log-concave, nor log-convex.

The integral $\int_{-\infty}^{x} F(t)dt$ does not converge for the Cauchy distribution, and therefore the function G is not well-defined.

The F Distribution The F distribution arises in statistical applications as the distribution of the ratio of two independent chi-square distributions with m_1 and m_2 degrees of freedom. The parameters m_1 and m_2, known as "degrees of freedom". The density function of an F distribution with m_1 and m_2 degrees of freedom is

$$f(x) = cx^{(m_1/2)-1}(1 + (m_1/m_2)x)^{-(m_1+m_2)/2}$$

where c is a constant that depends only on m_1 and m_2. The F distribution has support (a, b)=$(0, \infty)$.

For the F distribution,

$$(\ln f(x))'' = -(m_1/2 - 1)/x^2 + (m_1/m_2)^2(m_1 + m_2)/2(1 + m_1/m_2x)^{-2}.$$

If $m_1 > 2$, then $(\ln f(x))''$ is positive or negative depending on whether x is greater than or less than

$$\frac{m_2\sqrt{\frac{m_1-2}{m_1+m_2}}}{1 - \sqrt{\frac{m_1-2}{m_1+m_2}}}.$$

Therefore the density function is neither log-concave nor log-convex when $m_1 > 2$.

If $m_1 \leq 2$, then the density function is log-convex. Since $f(b) = f(\infty) = 0$, it follows from Theorem 4 that if $m_1 \leq 2$, the reliability function $\bar{F}$ is log-convex and the mean-residual-lifetime function $MRL(x)$ is monotone increasing.

Mirror-image of the Pareto Distribution None of the examples listed so far has a monotone increasing mean-advantage-over-inferiors function, $\delta(x)$. Indeed, we have not come across a "named" distribution that has this property. But, according to Theorem 8, the mirror-image of a distribution that has monotone increasing mean-residual-lifetime must have monotone decreasing $\delta(x)$.

A simple probability distribution with increasing mean-residual-lifetime is the Pareto distribution. The mirror-image of the Pareto distribution has support $(-\infty, -1)$ and c.d.f. $F(x) = (-x)^{-\beta}$ where $\beta > 0$. For $\beta > 1$, $G(x) = \int_{-\infty}^{x} F(t)dt$ converges and $G(x) = (\beta - 1)^{-1}(-x)^{1-\beta}$. Then $\delta(x) = G(x)/G'(x) = G(x)/F(x) = \frac{x}{1-\beta}$ and $\delta'(x) = \frac{1}{1-\beta} < 0$.

7 Notes on related literature

As far as we know, the earliest application of the assumption of log-concavity in the economics literature is due to Flinn and Heckman [18]. Economic applications can also be found in the industrial engineering literature in the context of *reliability theory*; see for example, Barlow and Proschan [7] and Muth, [27]. A pair of remarkable papers by Caplin and Nalebuff [11], [12] introduced Prèkopa's theorems on log-concave probability to the economics literature and applied them to voting theory and the theory of imperfect competition. Two useful theoretical papers by Mark Yuying An [2] and [3] discuss properties of log-concave and log-convex probability distributions. His papers contain several results not found here.[9] The main contributions are: 1) He shows that the standard results on inheritance of log-concavity can be established without the assumption that density functions are differentiable. 2) He pays more systematic attention to results concerning log-convexity than had been done previously. 3) He discusses the log-concavity of multivariate distributions.

Applications to labor economics and search theory

Flinn and Heckman [18] consider a model of job search in which job offers arrive as a Poisson process and where the wage associated with a job offer is drawn from a random variable with distribution function F. They show that if the right hand integral of the reliability function, $H(x) = \int_x^\infty (1 - F(t))dt$ is log-concave, then with optimal search strategies, an increase in the rate of arrivals of job offers will increase the exit rate from unemployment.

Heckman and Honore [21] discuss a labor market in which workers have differing comparative advantage in each of two sectors of the economy. They show that if the distribution of differences of skills is log-concave, then incomes of workers who are able to choose occupations according to comparative advantage in a competitive market will be more equally distributed than they would be if workers were randomly assigned to sectors and paid their marginal products.

Applications to monopoly theory

Consider a product whose consumers buy either one unit or none at all, and suppose that $F(\cdot)$ is the distribution function of consumers' reservation prices for this product. Then the quantity demanded at price p is proportional to $\bar{F}(p) = 1 - F(p)$ and a monopolistic seller's expected revenue $R(p)$ at price p is proportional to $p\bar{F}(p)$. Comparative statics is greatly simplified if the revenue function $R(\cdot)$ is quasi-concave. It is easy to show that log-concavity of the reliability function $\bar{F}(\cdot)$ implies quasi-concavity of $R(\cdot)$.[10] This fact finds frequent application in the economics literature. It is applied to the distribution of reservation demands for houses in Bagnoli and Khanna [6] and in a study of firm takeovers by Jegadeesh and Chowdry [13]. Segal [36] uses this assumption in his study of an optimal pricing mechanism for a monopolist who faces an unknown demand curve.

The assumption that willingness to pay is log-concavely distributed also plays a central part in the theory of price-competition with differentiated products. Dierker [15] develops foundations for a theory of price competition with differentiated products by showing that log-concavity of the distribution of certain preference

[9] An generously acknowledges an early draft of this paper, which predated his studies. In turn, our current paper has benefited from An's work. In particular, An's discussion motivated us to treat the inheritance theorems for log-convex distributions in a more systematic way. Our treatment of log-convexity is a slight generalization of that of An.

[10] In fact, as Caplin and Nalebuff [11] point out, quasi-concavity of the revenue function is implied by the condition that $1/\bar{F}(p)$ is a convex function of p, a condition which is weaker than log-concavity.

parameters implies quasi-concavity of a firm's profits in its own price. Caplin and Nalebuff [11] are able to establish existence and uniqueness of equilibrium under assumptions that the density functions of the population distribution of certain preference parameters satisfy assumptions that are weaker than log-concavity. Further development of the relation between log-concavity and equlibrium in spatial markets can be found in Anderson, de Palma, and Thisse [4].

Fang and Norman [17] have discovered an important application of log-concave probability distributions to the theory of commodity bundling. They show that if a monopolist sells several goods and if each consumer's demand for any one of the bundled goods is uncorrelated with his demand for the others, then it will be more profitable for the seller to bundle these goods rather than sell them separately under the following conditions: a) the mean willingness to pay for each good exceeds marginal cost of that good b) the probability density of willingness to pay for each good is log-concave. It is well understood (see Armstrong [5]) that in the limit as bundles get large (and demands are independent), the distribution of average willingness to pay becomes highly concentrated about the mean willingness to pay and thus a bundling monopolist can capture almost all of consumers' surplus. Fang and Norman note that in order to ensure that bundling is profitable when only a small number of independently demanded commodities is available, one needs a stronger convergence result than the law of large numbers. The desired property is that the probability that the sample mean deviates from the population mean by a specified amount is *monotonically decreasing* in sample size. Not all probability distributions have this property, but using a theorem of Proschan [33], Fang and Norman show that if the density function is log-concave, then the sample means converge monotonically as required.

Mechanism design theory

With games of incomplete information, it is customary to convert the game into a game of imperfect, but complete, information by assuming that an opponent of unknown characteristics is drawn from a probability distribution over a set of possible "types" of player. For example, in the literature on contracts, it is assumed that the principal does not know a relevant characteristic of an agent. From the principal's point of view the agent's type is a random variable, with distribution function, F. It is standard to assume, as do Laffont and Tirole [23] or Corbett and de Groote [14] that F is log-concave. This assumption is required to make the optimal incentive contract invertible in the agent's type and thus to ensure a separating equilibrium. In the theory of regulation, the regulator does not know the firm's costs. Baron and Myerson [8] show that a sufficient condition for existence of a separating equilibrium is that the distribution function of types is log-concave. Rob [35] in a study of pollution claim settlements, Lewis and Sappington [24] in a study of regulatory theory, and Riordan and Sappington [34], in a study of government procurement, use essentially the same condition.

Log-concavity also arises in the analysis of auctions. Myerson and Satterthwaite [28], Matthews[26], and Maskin and Riley[25], impose conditions that are implied by log-concavity of the distribution function in order to characterize efficient auctions.

Applications to political science and law

Many results from the theory of spatially differentiated markets have counterparts in the theory of voting and elections. An important paper by Nalebuff and Caplin [12] introduces powerful mathematical results that generalize the inheritance theorems for log-concave distributions and apply these concepts to voting theory and to the theory of income distribution. Weber [40] uses the assumption that individuals have single-peaked preferences and that the distribution of ideal points among individuals is log-concave to show the existence and uniqueness of equilibrium in a theory of “hierarchical" voting, where incumbents act as Stackelberg leaders with respect to potential entrants. primary elections are followed by general elections. Haimanko, LeBreton, and Weber [20] use similar assumptions to analyze equilibrium in a model where central governments use interregional redistribution to prevent succession of subgroups with divergent interests.

Cameron, Segal and Songer [10] study the transmission of information in a hierarchical court system. Their model has a lower court and a high court. The lower court hears the case, learns information that will not be directly available to the high court, and makes a decision. The high court’s utility function differs from the lower court’s and the high court tries to infer what the lower court learned from the decision it made. The high court must decide whether to incur the costs of reviewing the lower court’s decision. There is a close parallel in logical structure to that found in the mechanism design literature.

Costly signalling

As noted in Theorem 1, the mean-advantage-over-inferiors function $\delta(x)$ is increasing if and only if the left hand integral of the c.d.f. function is log-concave. The assumption that the distribution of quality has this property plays a critical role in theories of costly signalling and has found a variety of applications. Bergstrom and Bagnoli [9] develop a marriage market model in which there is asymmetric information about the quality of persons as potential marriage partners and where quality is revealed with the passage of time. In this model there is a unique equilibrium distribution of marriages by age and quality of the partners if $\delta(x)$ is increasing.

In Verrecchia [38], [39], a manager who wishes to maximize the market value of a firm must decide whether to incur a proprietary cost to disclose his information about the firm’s prospects. Thus, the manager compares the market’s expected value of the firm given his disclosure (less the cost of the disclosure) to the market’s expected value of the firm given that the manager chooses to not disclose his private information. The resulting theory is essentially the same as that illustrated in section 4.4 of this paper.

Nöldeke and Samuelson [29] explore an evolutionary model in which males engage in costly signaling (as exemplified by the peacock's tail) to convince females that they are superior mates. The authors ask whether there can be a costly signaling equilibrium if females care about the net value of males after they have paid the cost of their signals. They assume that females choose from among n competing males. Where F is the cumulative distribution function of initial male quality, it turns out there exists an equilibrium with costly signaling if and only if the right hand integral of F^{n-1} is a log-concave function. A sufficient condition for this function to be log-concave is that the distribution function F is log-concave.

Appendix – Proofs of inheritance theorems

Proof of Theorems 1 and 2

We apply two Remarks based on elementary calculus to prove Lemma 3, from which Theorems 1 and 2 are almost immediate.

Remark 2 A continuously differentiable function $f : \Re \rightarrow \Re^{+}$ is log-concave (log-convex) if and only if $\frac{f'(x)}{f(x)}$ is a non-increasing (non-decreasing) function of x in (a, b).

Proof. The function $\ln f$ is concave (convex) if and only if

$$(\ln f(x))'' = \frac{d}{dx}\frac{f'(x)}{f(x)}$$

is non-positive (non-negative) for all x in (a, b). □

Remark 3 Where $F(x) = \int_a^x f(t)dt$, the function F is log-concave (log-convex) if and only if $f'(x)F(x) - f(x)^2$ is non-positive (non-negative) for all x in (a, b).

Proof. The function $\ln F$ is concave (convex) if and only if the expression

$$(\ln F(x))'' = \frac{d}{dx}\left(\frac{f(x)}{F(x)}\right) = \frac{f'(x)F(x) - f(x)^2}{F(x)^2}$$

is non-positive (non-negative) for all x in (a, b). □

Lemma 3 *Let f be a continuously-differentiable function, mapping the interval (a, b) into the positive real numbers, let $F(x) = \int_a^x f(t)dt$ for all x in (a, b), and define $f(a) = \lim_{x\to a} f(x)$. Then:*

- *If f is log-concave on (a, b), then F is also log concave on (a, b).*
- *If f is log-convex on (a, b) and if $f(a) = 0$, then F is also log convex on (a, b).*

Proof. If f is log-concave, then for all $x \in (a, b)$,

$$\frac{f'(x)}{f(x)}F(x) = \frac{f'(x)}{f(x)}\int_a^x f(t)dt \leq \int_a^x \frac{f'(t)}{f(t)}f(t)dt = \int_a^x f'(t)dt = f(x) - f(a),$$

where the inequality follows from Remark 2. Since $f(a) \geq 0$, it follows that

$$\frac{f'(x)}{f(x)} F(x) \leq f(x) - f(a) \leq f(x).$$

and therefore

$$f'(x)F(x) - f(x)^2 \leq 0.$$

From Remark 3 it follows that F is log-concave.

Reasoning similar to that of the previous paragraph leads to the conclusion that if f is log-convex and if $f(a) = 0$, then

$$\frac{f'(x)}{f(x)} F(x) \geq f(x) - f(a) \geq f(x).$$

It follows that $f'(x)F(x) - f(x)^2 \geq 0$, and then from Remark 3, it follows that F is log-convex. □

7.1 Proof of Theorems 3 and 4

We now apply Remarks 2 and 4 to prove Lemma 4, from which Theorems 3 and 4 are almost immediate.

Remark 4 Where $\bar{F}(x) = \int_x^b f(t)dt$, the function $\bar{F}$ is log-concave (log-convex) if and only if $f'(x)\bar{F}(x) + f(x)^2$ is non-negative (non-positive) for all x in (a, b).

Proof. The function $\ln \bar{F}$ is concave (convex) if and only if the expression

$$(\ln \bar{F}(x))'' = \frac{d}{dx}\left(\frac{-f(x)}{\bar{F}(x)}\right) = -\frac{f'(x)\bar{F}(x) + f(x)^2}{\bar{F}(x)^2}$$

is non-positive (non-negative) for all x in (a, b). □

Lemma 4 *Let f be a continuously-differentiable function, mapping the interval (a, b) into the positive real numbers, let $\bar{F}(x) = \int_x^b f(t)dt$ for all x in (a, b), and define $f(b) = \lim_{x \to b} f(x)$. Then:*

- *If f is log-concave on (a, b), then $\bar{F}$ is also log concave on (a, b).*
- *If f is log-convex on (a, b) and if $f(b) = 0$, then $\bar{F}$ is also log convex on (a, b).*

Proof. If f is log-concave, then for all $x \in (a, b)$,

$$\frac{f'(x)}{f(x)} \bar{F}(x) = \frac{f'(x)}{f(x)} \int_x^b f(t)dt \geq \int_x^b \frac{f'(t)}{f(t)} f(t)dt = \int_x^b f'(t)dt = f(b) - f(x),$$

where the inequality follows from Remark 2. Since $f(b) \geq 0$, it must be that

$$\frac{f'(x)}{f(x)} \bar{F}(x) \geq f(b) - f(x) \geq -f(x).$$

Therefore $f'(x)F(x) + f(x)^2 \leq 0$, and from Remark 3 it follows that $\bar{F}$ is log-concave.

Reasoning similar to that of the previous paragraph shows that if f is log-convex and if $f(b) = 0$, then

$$\frac{f'(x)}{f(x)}\bar{F}(x) \leq f(b) - f(x) = -f(x).$$

It follows that $f'(x)\bar{F}(x) + f(x)^2 \leq 0$, and from Remark 3, it then follows that $\bar{F}$ is log-convex. □

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Notes on stochastic choice

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Prologue (2005)

The notes on Stochastic choice that follow were presented at a meeting held in San Sebastian in June of 1983 and organised by Salvador Barberà. It was research in progress that, alas, was never pursued. But it seems, by its subject, a most indicated contribution to a volume to honour Ket Richter. Obviously, I have the hope, but not the certainty, that something is still of interest in them. Or simply that there will be something to catch the sharp analytical eye of Ket. With my best regards to Ket, a model for us all of how theory should be done, here they go. I have corrected some obvious inaccuracies and, occasionally, tightened some looseness of language. I have also added some references (in particular, Falmagne,1978, Fishburn,1998, Barberà and Pattanaik,1986, McFadden and Richter,1991, McFadden,2004, are very relevant to the subject matter of these notes) and taken into account the remarks of a referee(whom I thank). Otherwise the text is as in 1983.

I. A general formalism

A very general setting for the stochastic choice problem can be described thus (see also Manski, 1977). There is given as data:

1. A set of alternatives X. It is convenient to think of X as finite.
2. A set of "budgets" $\mathcal{B} \subset 2^X$. Put $Y = \prod_{B \in \mathcal{B}} B$. A point of Y is a selection of an alternative in every budget. Denote by $\mathcal{M}$ the probability measures on Y.
3. A set of admissible statistics $f_j : \mathcal{M} \to R, j \in J$.
4. A set of observed values $a_j, j \in J$ of the statistics.

As an example, the usual stochastic choice problem corresponds to the above where the admissible statistics in (3) are the marginal distributions. [Precisely: f

is admissible if and only if it is of the form $f(v) = \int \psi(y) dv$ where $\psi(y)$ is the projection on one variable]. Even more restricted, if the admissible statistics are the mean of every marginal we have as data a sort of aggregate demand. Another situation falling in the above setting would be one where for every alternative x we are given the probability that x is chosen for some budget, etc.

Denote by $\mathcal{P}$ the set of linear orders on X.

Every probability measure μ on $\mathcal{P}$ induces a probability measure v_μ on Y by the rule $v_\mu(A) = \mu\{\succ \in \mathcal{P}$: denoting by $x(B)$ the $\succ$-maximal element on $B \in \mathcal{B}$ we have $\{x(B)\}_{B \in \mathcal{B}} \in A\}$.

That is to say, v_μ is the measure generated on Y by the choice vectors induced by preferences.

We then have two problems:

Rationalizability problem. A stochastic choice situation (described by (1)–(4) above) can be rationalized if there is a probability measure μ on $\mathcal{P}$ such that $f_j(v_\mu) = a_j$ for every j.

Which conditions must the data of the problem satisfy in order for a rationalization to exist?

Recoverability (or identification) problem. Assuming that the data are rationalizable, when is the rationalization unique?

Remark. Strictly speaking there is still a third problem, previous to the rationalizability one and vacuously non-restrictive in the usual stochastic choice model. It could be called the compatibility problem, namely, under which conditions there is a probability measure v on Y such that $f_j(v) = a_j$ for every j.

II. A particular case

After so much generality I become very concrete. I concentrate for the rest of the Notes on the particular case where there is a distinguished alternative, denoted 0, every $B \in \mathcal{B}$ includes 0 and for each $B \in \mathcal{B}$ there is an admissible statistic which is the probability that 0 is not selected in B. In other words, the data of the problem is an array $p(B)$, $B \in \mathcal{B}$, to be interpreted as asserting that given B the probability that 0 be the preferred element is $1 - p(B)$. We always put $p(\{0\}) = 0$.

Define the equivalence relation $\approx$ on $\mathcal{P}$ by $\succ \approx \succ'$ iff "$x \succ 0 \Leftrightarrow x \succ' 0$". Obviously, if $\succ \approx \succ'$ then the data of the problem will never be able to distinguish between $\succ$ and $\succ'$. Therefore, the rationalizability and, above all, the recoverability problem should properly be posed with respect to $\mathcal{P}^* = \mathcal{P}/_{\approx}$. Note that for the elements of $\mathcal{P}^*$ the transitivity requirement has no strength. Avoiding the transitivity issue is the main advantage of analyzing the particular case of a distinguished alternative.

I briefly discuss three subcases that differ by the nature of the admissible $\mathcal{B}$. Take X finite, with $\#X = n + 1$.

(a) $\mathcal{B} = \{B \in 2^X : 0 \in B\}$

The rationalization problem for this subcase has been extensively treated and is completely solved. See Falmagne, 1978, Barberà and Pattanaik, 1986, Cohen and Falmagne, 1990, Barberà, 1991..

Every preference in P^* can be identified with a set $B \in \mathcal{B}$, i.e. B is the set of alternatives at least as good as 0. Then a probability on $\mathcal{P}^*$ can be identified with a list $0 \leq \pi(B) \leq 1$, $\sum_{B \in \mathcal{B}} \pi(B) = 1$. If $\pi(\ )$ rationalizes $p(\ )$ then we must have $1 - p(B) = \sum_{\substack{A \in B \\ A \cap B = \{0\}}} \pi(A)$ for every $B \in \mathcal{B}$. Therefore, $p(\ )$ can be rationalized if and only if the following recursion process (see Barberà and Pattanaik, 1986)yields a probability measure. Put first. $\pi(\{0\}) = 1 - p(X)$. Suppose now that $\pi(C)$ has been computed for any C up to size $m+1$. Put then $\pi(B) = 1 - p((X \backslash B) \cup \{0\}) - \sum_{C \not\subset B} \pi(C)$ for B of size m + 2. Obviously, this recursion process gives us a complete list $\pi(B)$, $B \in \mathcal{B}$. Also, $\sum_{B \in \mathcal{B}} \pi(B) = 1$ by construction. Therefore, π *is a probability measure, i.e. in admissible rationalization, if and only if* $\pi(B)$ *is non-negative for all* B. Those are the conditions obtained in the above references. Note that if $p(\ \ )$ is rationalizable then the rationalization is unique and can be recovered by the previous recursion.

Recoverability, i.e. uniqueness, is not surprising in view of the fact that one gets from π to p by a linear transformation and that there are as many equations (one for each B) as unknowns (one for each B).

(b) $B = \{B \in 2^X : 0 \in B, \#B = 2\}$

This is in a sense the polar opposite to subcase (a). Here we only have the outcome of the pairwise matching of 0 against every $x \neq 0$. We write $p(\{0, x\}) = p(x)$.

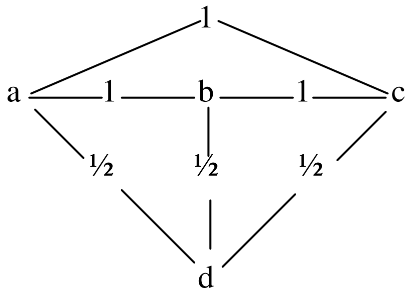
It is obvious that in this subcase, where there is much less information than in (a), any $p(\ \ )$ can be rationalized. Indeed, any $p(\ \ )$ can be looked at as a point in $[0, 1]^n$. Every extreme point of this convex set is of the form $p(\ ) \in \{0, 1\}$ and can be rationalized (by ordering "$x \succ 0 \Leftrightarrow p(x) = 1$)". Therefore, the entire $[0, 1]^n$ can be rationalized.

The counterpart to the above pleasant fact is that the preference distribution cannot be recovered. This is clear counting equations (n) and unknowns (2^n, one for every $B \in 2^X$, $0 \in B$).

(c) Intermediate subcases

To get clean results for families $\mathcal{B}$ intermediate between subcases (b) and (a) is hard. Consider, for example, the subcase $\mathcal{B} = \{B \in 2^X : 0 \in B, \#B = 3\}$. It is a good exercise to verify that for the instance represented in the figure below (with $x = 0, a, b, c, d$) there is no rationalizing preferences.

From now on I limit myself to subcase (b), i.e. our data is the probability $p(x)$ of any $x \in X$ winning against 0. The common fact in the next two sections is that restrictions are imposed on underlying permissible preferences. In Section III



I study a rationalization problem with a convexity hypothesis on preferences. In Section IV I sketch and discuss an analytic treatment of the recoverability problem.

III. Rationalizability with convex preferences

With 0 a distinguished alternative in X we are given, for every $x \in X$, $x \neq 0$ a number $0 \leq p(x) \leq 1$ which is interpreted as the probability of x winning over 0. We have seen (subcase (b) in II) that p can always be rationalized by a distribution μ on $\mathcal{P}$. In applications, however, it may be important that μ give positive weight only to preferences satisfying some restrictions.

Suppose, for example, that $X^* = X \backslash \{0\}$ is a subset of a linear space. Say $X^* \subset R^m$. Then we may be interested in rationalizing by members of the set of convex (or, more precisely, convex-compatible) preferences, i.e. $\mathcal{P}_C = \{\succ \in \mathcal{P}$: if "$A \subset X^*, x \succ 0$ for every $x \in A$, and $y \in$ (convex hull A) $\cap X^*$" then $y \succ 0\}$.

It is no longer true that any $p(\ )$ can be rationalized by a μ concentrated on $\mathcal{P}_C$. The problem of characterizing the set of admissible $p(\ )$ seems pretty hard indeed. But for the simplest case, i.e. $m = 1$ (the set X^* lies in the real line) the solution is fairly trivial.

Let $X^* \subset R$. Put $X^* = \{x_1, ..., x_n\}$ where $x_i > x_j$ for $i > j$. Denote $p(i) = p(x_i)$.

Proposition. *The function* $p : X^* \to [0, 1]$ *can be rationalized by a* μ *on* $\mathcal{P}_C$ *if and only if* $p_1 + \sum_{i=2}^{i=n} \max\{0, p_i - p_{i-1}\} \leq 1$.

Remark. Presumably the proposition can be extended to the case where $X^* \subset R$ is compact. The general statement would then be along the lines: "The function $p : X^* \to [0, 1]$ can be rationalized by a μ on $\mathcal{P}_C$ if and only if it is of bounded variation and has variation norm ≤ 1".

Proof of the Proposition.

(1) *Necessity.* Identifying sets with preferences let $\mathcal{B}_C$ be the set of convex preferences. For every $i = 1, ..., n$ denote $\mathcal{B}_i = \{B \in \mathcal{B}_C : x_{i-1} \notin B, x_i \in B\}$. These sets constitute, by the convexity hypothesis, a partition of $\mathcal{B}_C$. So, if π is a probability measure concentrated on $\mathcal{B}_C$ we have $\sum_{i=1}^{n} \pi(\mathcal{B}_i) \leq 1$. Suppose now that π generates p. Then $\pi(\mathcal{B}_1) = p_1$. Consider any $i > 1$. We have $p_i = \pi(\{B \in \mathcal{B}_C : x_i \in B\})$. But $\{B \in \mathcal{B}_C : x_i \in B\} = \mathcal{B}_i \cup \{B \in \mathcal{B}_C : x_i \in B$

and $x_j \in B$ for some j less than $i\}$. This is a disjoint union and, by convexity, the second set is a subset of $\{B \in \mathcal{B}_C : x_{i-1} \in B\}$ which probability is p_{i-1}. Therefore $p_i \leq \pi(\mathcal{B}_i) + p_{i-1}$, or $\max\{0, p_i - p_{i-1}\} \leq \pi(\mathcal{B}_i)$. Hence, $p_1 + \sum_{i=2}^{n} \max\{0, p_i - p_{i-1}\} \leq 1$ and necessity is established.

(2) *Sufficiency.* We shall actually show that: "There is always a π such that $\sum_{B \neq \phi} \pi(B) = p_1 + \sum_{i=2}^{n} \max\{0, p_i - p_{i-1}\}$". So, let the bracketed statement be an induction hypothesis on n. It is obviously true for $n = 1$. Let it be true for $n - 1$. In particular for the set $\{x_1, ..., x_{n-1}\}$, i.e. there is a probability measure π on $\mathcal{B}_C^{n-1} = \{B \in \mathcal{B}_C : x_n \notin B\}$ such that:

(a) $$\sum_{\substack{B \in \mathcal{B}_C^{n-1} \\ B \neq \phi}} \pi(B) = p_1 + \sum_{i=2}^{n-1} \max\{0, p_i - p_{i-1}\} \leq 1, \text{ and}$$

(b) for every $i \leq n - 1, p_i = \sum_{\substack{x_i \in B \\ B \in \mathcal{B}_C^{n-1}}} \pi(B)$.

Now we extend π to X^* as follows. Let $q_n = \min\{p_{n-1}, p_n\}$.

For any $B \in \mathcal{B}_C^{n-1}$ such that $x_{n-1} \in B$ consider the rule $B \to B \cup \{x_n\}$. Under *this rule* transfer a probability weight q_n from $\{B \in \mathcal{B}_C^{n-1} : x_{n-1} \in B\}$ to $\mathcal{B}_C$. If $q_n = p_n$ then we are done: the equality in (a) has not been altered and (b) also holds for $i = n$. If $q_n = p_{n-1} < p_n$ then we in addition transfer a probability weight $p_n - p_{n-1}$ from the set ϕ to the set $\{x_n\}$. This can be done because by the induction hypothesis $\pi(\phi) = 1 - (p_1 + \sum_{i=2}^{n-1} (p_i - p_{i-1})) \geq p_n - p_{n-1}$. Then again the equality in (a) remains and (b) has been extended to $i = n$. This concludes the induction step.

Remark. As it should be expected if the condition of the proposition holds then the admissible probability on preferences need not be unique. Suppose that $X^* = \{1, 2\}$ and $p_1 = \frac{1}{3}$, $p_2 = \frac{1}{3}$. Then two admissible π are "$\pi(\{1\}) = \pi(\{2\}) = \frac{1}{3}$, $\pi(\phi) = \frac{1}{3}$" and "$\pi'(\{1, 2\}) = \frac{1}{3}$, $\pi'(\phi) = \frac{2}{3}$". The π obtained by construction in the proof of the proposition would be π' in this example, namely, it is the one that maximizes the probability that 0 be the overall maximum, i.e. $\pi(\phi)$. The construction of the proof seems to indicate that this maximizing probability measure is unique.

IV. Analytic treatment of the recoverability problem

We keep studying the distinguished alternative case. We now take X to be an Euclidean space R^n. The distinguished alternative is the origin 0. The function $p : X \to [0, 1]$ gives the probability $p(x)$ that x wins against 0. For convenience, p is left undefined at 0.

For an analytic treatment it is important (or, at least, convenient) that the set of admissible preferences be somehow restricted to depend on a finite number of parameters. So, we assume that we have given a parameter set Q which, to make life simple, we identify with some Euclidean space R^m. For every parameter value $q \in Q$ preferences are expressed by a utility function $U(x, q)$, normalized to equal zero whenever $x = 0$. It is assumed that $U : R^n \times R^m \to R$ is a "nice" function (continuous, differentiable, analytic, ...).

Given a probability measure μ on Q a probability choice function $P : X \to [0, 1]$ is generated as follows: $p(x) = \mu\{q : U(x, q) > 0\} = \int_{\{q:U(x,q)>o\}} f(q)dq$ where the second equality applies only if M has a density f. From now on we shall assume that all μ we deal with have densities which, moreover, are sufficiently nice (say of class C^∞ and equal to zero outside of a compact set, or, at least, "rapidly decreasing").

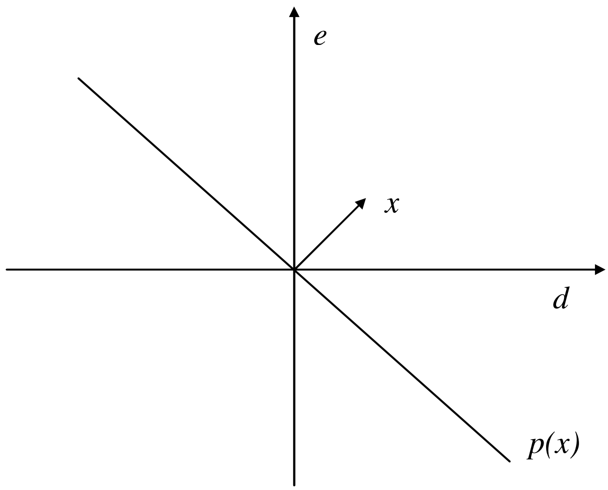
The recoverability problem is then: Assuming that p is generated as above, can f be uniquely recovered from p (in the class of "nice" densities)?

Remark. Strictly speaking the recoverability problem should be posed only for the family of indifference curves passing through the origin because this is all the information that p uses. But, in the parameterized world we are now working in, recovering the indifference curve will usually be tantamount, (i.e. except perhaps for a normalizing parameter) to recovering the entire utility function. So, I do not worry about the distinction.

For the remaining of this section I discuss an extended example with $n = 2$ and $U(x, q)$ a general quadratic: $U(x, q) = ax_1^2 + bx_2^2 + cx_1x_2 + dx_1 + ex_2$. So, without further a priori restrictions we have five parameters, i.e. $m = 5$. I consider a sequence of three subcases, which differ by the type of a priori restriction imposed.

Example 1. Take $a = b = c = 0$ as a priori restrictions.
(The same qualitative features of the example are obtained with other combinations of three zero restrictions, eg. $c = d = e = 0$, or $b = d = e = 0$). In this case $U(x, q)$ reduces to $U(x, q) \equiv U(x, d, e) = dx_1 + ex_2$.

This model *is not identified*. Take, for example, $p(x) = \frac{1}{2}$ for all x. Any symmetric density f on the $d - e$ plane will generate p because, for any $x, \{d, e : dx_1 + ex_2 > 0\}$ is just the half space above the hyperplane with normal x and the integral of a symmetric density on a half space is $\frac{1}{2}$. See the figure.



Example 2. As in example 1, $a = b = c = 0$. But suppose now that in addition there is another restriction in the form of a non-homogeneous linear equation. For instance, $d + e = 1$. The origin of this restriction could be, for example, a normalization convention.

Then the model is obviously identified because given any underlying density f we can use p to compute the distribution function of f on the line defined by $d + e = 1$ on the $d - e$ plane.

Observe also that f can be recovered by using only the information contained in the p function in any arbitrarily small neighbourhood of zero.

After discussing two more examples I shall present, in the next section, a recoverability proposition for arbitrary m and n which generalizes Example 2.

Example 3. $a = 1, b = 1, c = 0$.
In this case for given d, e the indifference curves of the utility function $U(x, e, d) = x_1^2 + x_2^2 + dx_1 + ex_2$ are concentric circles around the vector $(-\frac{d}{2}, -\frac{e}{2})$. i.e., x is preferred to 0 according to if $(-\frac{d}{2}, -\frac{e}{2})$ is closer to x than to 0.

So, in the obvious way we can identify the variable and the parameter space and think of densities f as being defined on the x space itself (think of the parameter as the peak of the preferences). Note that $p(x)$ is the integral of f on the half space of vectors to the side that includes x of the line perpendicular to x and cutting the segment $[0, x]$ in its middle point (this is the half space of vectors closer to x than to 0).

[*Remark.* The similarities of this with the well known majority voting model are intended].]

Now a mathematical disgression.

Let S^1 be the 1-dimensional sphere in two dimensional Euclidean space. Given f we can define a function $\psi : S^1 \times R \to R$ by letting $\psi(v, t)$ be the integral of the f function on the line (more generally, affine subspace) $\{y : v.y = t\}$ endowed with the usual Lebesgue measure. In Fourier analysis this function (as well as its obvious higher dimensional generalizations $\psi : S^{n-1} \times R \to R$) is known as the *Radon transform* of f and, not surprisingly, it is useful in things like X-ray reconstruction. The fact is that there is an inversion formula such that if f is "nice" then starting with $\psi(v, t)$ we recover f.

The inversion formula is particularly simple for the case at hand where f is defined on the plane and the Radon transform (also called in this case the X-ray transform) evaluates integrals on lines. For any x and $s > 0$ let $n_s(x)$ be the average value of $\psi(v, t)$ on lines which are at a distance s from x, i.e. $n_s(x) = \frac{1}{2\pi} \int_{S^1} \psi(v, v.x + s)dv$. Then it turns out that if f is continuous and has a compact support $f(x)$ can be recovered by the formula $f(x) = \frac{1}{\pi} \int_0^\infty \frac{dn_s(x)}{s}$ where the integral is in the sense of Stieltjes. More precisely, and integrating the above formula by parts:

$$f(x) = \lim_{\varepsilon \to 0} \frac{1}{\pi} \left(\frac{\eta_\varepsilon(x)}{\varepsilon} - \int_\varepsilon^\infty \frac{n_s(x)}{s^2} ds \right)$$

See Shepp and Kruskal (1978), Helgason, (1980), or Dym and McKean (1972) for these Fourier analysis techniques. Their relevance for recoverability problems in economics has been noted in another context by Ph. Dybvig and A. McLennan. I would also like to thank A. Grunbaum for the mathematical references.

Back to Example 3. The relevance of the mathematical disgression to our problem is that the Radon Transform of the density f can be computed from the choice probabilities $p(x)$. As it is clear from the geometric discussion:

$$\psi(v,t) = -\frac{\partial}{\partial t} p(2tv)$$

(Strictly speaking the above applies to $t \neq 0$. For $t = 0$ just let $\psi(v,0) = \lim_{t\to 0} \psi(v,t)$)

Summing up: the model of Example 3 has the recoverability property. Note however that, in contrast to Example 2, it is now essential to use all the information contained in $p(x)$. Restricting oneself to a small neighbourhood of 0 will not do.

Remark. Given an arbitrary $p(\ )$ we can compute $\psi(v,t)$ as above by means of the inversion formula to get a $f(x)$. That f be a well defined (and "nice") density function (i.e. $f(x) \geq 0$ and $\int f(x)dx = 1$) is, therefore, the necessary and sufficient condition for rationalizability within the restrictions of Example 3. What one gets, unfortunately, is not precisely a transparent condition.

Example 4. This is not a quadratic but a cubic example: $U(x,q) = x_2 - ax_1^3 - bx_1^2 - cx_1$. For given a, b, c the equation $x_2 = ax_1^3 - bx_1^2 - cx_1$ yields a non-linear indifference curve through the origin. Actually, I have no idea if this model is identified or not. Since we only have two variables but three parameters the guess is that it is not but . . .

In the next section I present the promised generalization of Example 2.

V. A generalization of Example 2.

Let's go back to the original set-up of Section IV with

$$U : R^n \times R^m \to R$$

Suppose first that U takes the additive form: $U(x,q) = g_1(x)q_1 + ... + g_m(x)q_m = g(x).q$. This covers all the polynomial cases and, pushed to the limit, could cover all the analytic utility functions. If q^3 lies in the segment $[q^1, q^2] \subset R^n$ then $U(., q^3)$ is intermediate between $U(\ .\ , q^1)$ and $U(., q^2)$ (or, rather, their preference relations are) in the sense used by Chichilnisky and Grandmont. In fact, one could wonder if for $m \geq 3$ the concept of intermediate preferences provides a characterization of the above additive form. We assume that the function $g : R^n \to R^m$ is C^1.

Suppose that in the space of parameter R^m there are some a priori given identifying restrictions in the form of a system of s linear equations:

$$B_{s\times m} q - c = 0$$

The density f is supported in the set of solutions to the above system. Hence, it is in the nature of the problem that solutions exists.

Proposition. *A sufficient condition for the model to be identified, i.e., for every nice f to be recoverable, is that:*

$$rank \begin{bmatrix} \underset{n\times m}{(\partial g(0))^T} & \underset{n\times 1}{0} \\ \underset{s\times m}{B} & \underset{s\times 1}{-c} \end{bmatrix} = m+1$$

Moreover, only the values of p( ) on a neighbourhood of 0 matter.

Sketch of proof. Denote

$$\begin{aligned} L &= \partial g(0)(R^n), \\ M &= \{q : Bq = \alpha c \text{ for some } \alpha\} \\ N &= \{q : Bq = c\}. \end{aligned}$$

The three are subspaces of R^m (N is affine). It is a simple exercise to verify that if the rank condition is satisfied then the dimension of L is not smaller than the dimension of M and, in fact, that the projection of L on M is onto.

We now argue that any affine half space in N i.e. any set of the form $A = \{q \in N : q.y < \beta\}$, $y \in R^m$, can be realized by taking a y belonging to L and putting $\beta = 0$. Indeed, we can first realize A in the form of $A = \{q \in N : q.z < \bar{q}.z\}$, where $\bar{q} \in A$ and z belongs to the translate of N to the origin. If the rank condition is satisfied then $c \neq 0$. So, $0 \notin N$ and therefore $\{q \in N : q.z = \bar{q}.z\}$ spans a hyperplane in M. By the observation in the previous paragraph this hyperplane is realized for some $y \in L$. This y does the job.

Appealing now (with some care) to the Implicit Function Theorem we conclude that any affine half space in N can be realized in the form $\{q \in N : q.g(x) < 0\}$ for an arbitrarily small x.

Because the density function f lies in N and $p(x) = \int_{\{q\in N:q.g(x)\leq 0\}} f(x)dx$ we can finally recover f from p by using the Fourier analysis techniques discussed in Example 3. This ends the sketch of proof.

Example 3 shows that the rank condition is sufficient but not necessary for identification. The ability to use any x not limited to a neighbourhood of the origin, may make up for insufficient variation of g at 0. Nevertheless, it can be presumed (?) that a more general condition will again revolve on a counting of effective parameters versus independent directions of variations of $g(x)$.

Remark. The entire analysis of this section uses only the information contained in $p(x)$, i.e. only on the pairwise comparison that include the origin. It stands to reason that if more information was available, eg. on all pairwise comparisons, then fewer identifying restrictions would suffice.

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