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Philipp N. Baecker

Real Options and Intellectual Property

Capital Budgeting Under Imperfect Patent Protection

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Capital Budgeting Under Imperfect Patent Protection

With 47 Figures and 16 Tables

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Preface

As an obvious consequence of increasingly international competition and the rising importance of intangible as opposed to tangible assets, reliable methods for assessing returns and risk associated with intellectual property (IP) have moved to the center stage of value-based management in major corporations. While a variety of frameworks for dealing with IP portfolios are conceivable, this text approaches the issue from a capital budgeting perspective, devoting particular attention to methods commonly summarized under the label "real options." Also in line with a burgeoning interest in economic analyses of the law, new light is shed on the implications of imperfect patent protection, which has led to increased litigation activity in the pharmaceutical and software industries in particular.

Covering very basic as well as quite sophisticated models of investment under uncertainty in the presence of patent risk, this text provides a self-contained introduction to the valuation of intellectual property as real options and also offers some original contributions to current research in the field. Although written with a primarily academic audience in mind, it may therefore serve as a valuable resource for practitioners looking for an up-to-date and accessible overview.

This text draws heavily upon my Ph.D. work. It would not have been possible without the help of a number of inspired and thoughtful people keeping me company along the way. Specifically, I would like to thank my parents; my advisors Ulrich Hommel and Onno Lint for their support throughout this and many other projects; Karoline Jung-Senssfelder for being there when she is needed; Gudrun Fehler and the assistants at the Endowed Chair of Corporate Finance and Capital Markets for assisting me in my teaching activities and for their encouragement; the German National Academic Foundation for generous financial support as well as Jürgen Bunge and Rainer Strub-Röttgerding for supporting me morally and intellectually; the people at the Sol C. Snider Entrepreneurial Research Center at the University of Pennsylvania, in particular Ian C. MacMillan, for giving me the opportunity to experience an exceptionally research-friendly environment; his colleague Sidney G. Winter for having sparked thought about the strategic management of innovation and the patent system; Ernesto Mordecki (Centro de Matemática, University of the Republic Uruguay) for helpful comments on sect. 7.1.2; Yakov Amihud (Stern School of Business, New York University) and Holger Kraft (Department of Mathematics, Technical University of Kaiserslautern) for advice on conceptual and technical issues concerning part III. All remaining errors are, of course, my own.

Reflecting the dynamic nature of the material discussed and with the explicit aim of motivating more detailed investigations into specialized topics, each chapter also hints at a multitude of open research questions to be addressed in future studies. In this respect, this text is still work in progress. Comments, additions, and corrections are therefore highly appreciated. You can reach me at `philipp.baecker@ebs.edu`.

Oestrich-Winkel, *Philipp Baecker*
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1

Introduction

With this text, the author proposes an integrated approach to patent risk and capital budgeting in pharmaceutical research and development (R&D), developing an option-based view (OBV) of imperfect patent protection, which draws upon contingent-claims analysis,[1] stochastic game theory, as well as novel numerical methods. Bridging a widening gap between recent advances in the theory of financial analysis and current challenges faced by pharmaceutical companies, it aims at re-initiating a discussion about the contribution of quantitative frameworks to value-based R&D management.

1.1 Goal and Motivation

Over the last years, due to intensive competition in the knowledge economy, legal aspects surrounding IP rights—including litigation and settlement—have continuously gained in importance. Correspondingly, professional IP management has become an indispensable element of successful value-based management (VBM) in research-intensive firms [132, p. 1].

Moreover, since VBM is essentially about maximizing risk-adjusted returns, risk management clearly lies at the heart of any serious attempt at developing a reliable quantitative framework. The pharmaceutical industry, in many ways the stereotypical research-intensive industry, is chosen as an illustrative point of reference to clarify the benefits of combining technology-related, market-related, and legal risk, when

[1] While the application of such methods to capital budgeting, relying on a close analogy between real investment opportunities and derivative securities, is widely known as the "real option approach," the author prefers the slightly less descriptive and more general label "investment analysis under uncertainty."

assessing the attractiveness of investment opportunities and allocating scarce resources across a portfolio of interrelated R&D projects.

On the one hand, both technology-related and market-related risk factors have already been the subject of numerous analyses. More recently, also the interaction between these components of risk has attracted the attention of researchers in the area of option pricing. Since technological risk is the main driver in R&D, when it comes to applying such techniques in a capital budgeting context, practitioners intuitively recognize the importance of an integrated perspective. On the other hand, only very few contributions explicitly account for patent risk or even set out to examine how imperfect patent protection changes the overall risk profile and, as a consequence, optimal capital allocation. Among the rare exceptions are mostly empirical analyses, by and large lacking a thorough microeconomic foundation.

Nevertheless, the potential value impact of litigation is enormous. Of all patents litigated in the pharmaceutical industry, around 50 percent are found to be invalid, including some of the most valuable ones. For instance, in 2002, following a lawsuit in which Chiron had sought over 1 billion dollars in damages from Genentech, Chiron's patent on monoclonal antibodies specific to breast cancer antigens was invalidated. In 2000, the United States Court of Appeals for the Federal Circuit (CAFC) invalidated an Eli Lilly patent on Prozac. Although this decision came less than two years before the patent was set to expire, it caused Eli Lilly's stock price to fall by 31 percent in a single day [193, p. 76].

As the title implies, particular emphasis will thus lie on the role of imperfect patent protection, and how it affects profitability and incentives in an economy, where value creation is increasingly attributed to intangible assets. A primary tool employed throughout the analysis will be the real option approach to investment valuation, drawing the most comprehensive picture of all relevant aspects of the challenging decision problems at hand. Therefore, the discussion will also touch upon some methodological issues, generating new insights into the intricacies of option-based decision making, its advantages and disadvantages. Nevertheless, the reader should keep in mind the ultimate objective of connecting the corners of a proposed *risk triangle,* encompassing market-related, technology-related, and patent-related elements. The author will demonstrate how this triangle may form the basis of truly value-based R&D management, which goes beyond an isolated treatment of risk factors.

Although not all types of intangible assets are and can in principle be protected by patents—even a casual analysis of value drivers in the pharmaceutical industry shows the paramount importance of IP,[2] more specifically the length, breadth, and *strength* of patent protection [219]. Contributing to the vast body of literature on IP management, the text also provides a self-contained introduction to relevant continuous-time and discrete-time option models, describing their sensitivities to some key characteristics commonly used to describe pharmaceutical R&D projects.

Fleshing out some important ideas in detail and hinting at several opportunities for future research, the analysis proceeds by developing various option-based formalizations of imperfect patent protection. This text thus not only contributes to and provides a brief overview of the existing literature on contingent-claims analysis in pharmaceutical R&D, but also extends the paradigm to settings of imperfect patent protection.

Furthermore, since closed-form solutions to more advanced option-based models are rarely available, a significant part of the exposition is devoted to procedures required to obtain numerical results. Following a review of the more familiar finite differencing schemes, the author develops and evaluates a variety of novel techniques involving Monte Carlo simulation in combination with genetic algorithms (GAs) and pattern search. In addition, this text advances applied research on option valuation, presenting one of the first implementations of a high-dimensional quasi-random number generator (QRNG) publicly available and demonstrating its performance in a variety of settings.

As a result of the multi-faceted nature of the R&D process, it becomes necessary to introduce a number of simplifications in order to capture the complexity of real-world decision problems using stylized formal models. Wherever appropriate, the author will shed light on the relationship between the methods described and quantitative techniques and heuristics commonly employed by practitioners in the field. Consequently, despite a strong focus on the state of the art in capital budgeting, the ultimate goal of this text remains to advance both the theory and the practice of resource allocation in industries that are driven by investments into promising but highly uncertain endeavors.

Coming from a more theoretical angle and beyond the narrower research question of how to best implement value-based R&D manage-

[2] A closer look reveals that only a small fraction of value-generating intangibles actually enjoys patent protection. It is a widely-known fact that the intellectual capital of a firm is very difficult to identify, let alone to quantify or contract upon.

ment in the presence of patent risk, the OBV outlined in this text could be regarded as a cornerstone of an equally ambitious and innovative reconceptualization of uncertain property rights. Again, capital budgeting in pharmaceutical R&D will serve as an illustrative example, demonstrating the wide applicability of the concepts presented.

Figure 1.1 illustrates graphically how uncertainty (1) during the interrelated phases of R&D and commercialization (2) drives patent and project value (3). However, when trying to capture opportunities by choosing an optimal investment policy in response to market-related, technology-related, and patent-related risk factors (4), it is important to anticipate competition, which has a profound impact on uncertainty as well as investment policy (5), eventually causing substantial variations in profitability. Throughout the following chapters, selected aspects of this complex decision problem will be the subject of intensive analysis.

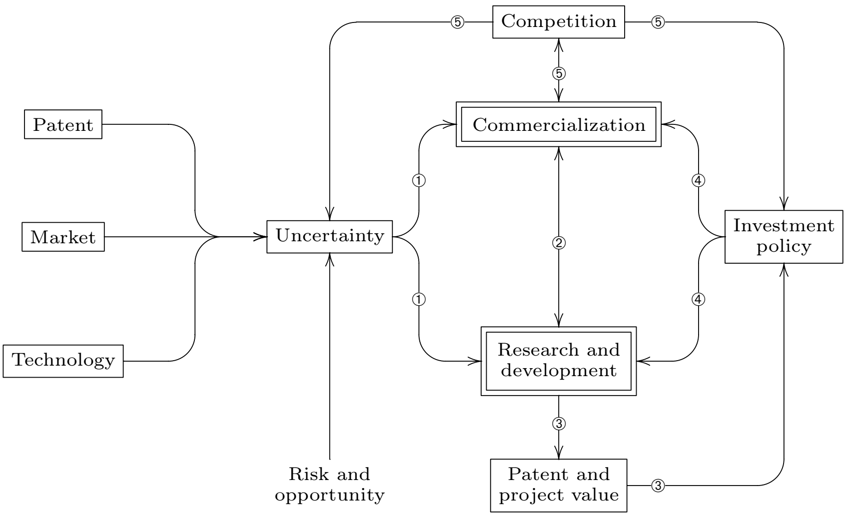

Fig. 1.1. A framework for analyzing R&D under combined market-related, technology-related, and patent-related uncertainty.

In summary, four interrelated research questions are to be addressed, which were identified as gaps in the extant literature on option-based capital budgeting in R&D.

How does patent risk change patent value and investment policy?

Neglecting imperfect patent protection means ignoring an important driver of patent value. Hence, determining optimal investment policies is impossible unless one accounts for patent risk. Nevertheless, common methods for R&D valuation are characterized by a lack of systematic approaches to patent risk assessment. A variety of ways to overcome the obvious limitations of these methods within the framework of option-based capital budgeting are presented and described in detail.

What is the source of patent risk?

Simply including patent risk as some exogenous parameter in the valuation process fails to explain its determinants and how these are tied to other key value drivers. So far, no convincing formalization based on the theory of investment under uncertainty has been proposed. A novel game-theoretic approach for the formal treatment of imperfect patent protection is developed and analyzed both analytically and numerically.

How do market, technology, and patent risk interact?

Although, in the parlance of option theory, both technological and patent risk can be regarded as *technical* risk, they affect the value of R&D in distinct ways and should thus be treated differently. With few noteworthy exceptions, researchers have failed to do so and, for this reason, have also paid little attention to how the various types of risk interact. The author introduces a variety of formal models which may serve as blueprints for comprehensive risk management tools to be devised in the future.

Are option-based models of imperfect patent protection ready for practice?

There is a widening gap between valuation tools popular in practice and the highly sophisticated techniques found in the academic literature. This gap complicates a fruitful dialog between practitioners and academics, thereby inhibiting significant progress in the field of quantitative methods for IP management. Several advanced models are described in detail, employing a level of formality that renders the discussion accessible to a wider audience.

Owing to the necessarily limited scope of the exposition, one also needs to point out some aspects that, regardless of their obvious relevance to capital budgeting in pharmaceutical R&D in general and the issue of imperfect patent protection in particular, have been excluded from the formal part of the analysis or might not receive their due share of attention in the discussion. Specifically, these aspects include the peculiarities of technology platforms, the wide field of quantitative portfolio management, as well as decision problems to be dealt with by generics manufacturers.[3] Also, ethical as well as legal issues surrounding the patentability of certain pharmaceutical innovations are not discussed in detail. The interested reader is referred to the numerous sources provided throughout the text.

1.2 Method and Outline

As explained previously, analyses are in line with, but also improve upon modern theory on value-maximizing investment strategies under uncertainty, which hinges on a close analogy between real investment opportunities and derivative securities.

Roughly speaking, the material following this introduction is divided into three parts. Part I ("Patenting Under Uncertainty") introduces the reader to the economic significance of patents, investment under uncertainty, and the specific challenges faced by research-driven, pharmaceutical firms. Advancing financial and game-theoretic research into the nature of R&D, parts II ("Exogenous Patent Risk") and III ("Endogenous Patent Risk") offer two alternative approaches to capturing patent risk in formal models. Both lead to a more thorough understanding of dynamic investment policies and are, despite fundamental differences, closely related. The text concludes with a brief summary of important findings and suggestions for future research.

Section 3.2 in particular as well as parts II and III are slightly more formal than the remaining discussion. Nevertheless, an attempt was made to construct models from the most basic of building blocks and keep mathematics to a necessary minimum. The appendix contains several proofs not included in the text as well as additional technical details on numerical methods.

[3] On several occasions, however, it will not be difficult to draw far-reaching conclusions based on the general mechanisms underlying the models presented. The author will highlight such connections whenever possible without obscuring the line of argument.

Part I

Patenting Under Uncertainty

2

Patent Protection, the Firm, and the Economy

This chapter serves as an introduction to the issue of patenting from a manager's as well as an economist's perspective. Since it is beyond the scope of the analysis to review relevant strategic management research or the economics of the patent system in its entirety,[1] the discussion will be limited to selected arguments, sufficient to shed light on the two sides of IP, which is both a strategic resource and an incentive mechanism.

2.1 Patents as a Strategic Resource

Observation 2.1. From a manager's perspective, patents constitute a *strategic resource,* which can be employed to gain competitive advantage.

Contrary to the industry-centered approach of popular strategic analysis, which is exemplified by the widely-known works of Porter [272, 273], contemporary management research has increasingly shifted its focus from monopoly rents[2] originating on the industry level to the factors that determine how a firm gains competitive advantage by acquiring, developing, and retaining valuable resources [320]. More precisely, only resources that are valuable, rare, inimitatable, and non-substitutable (VRIN) can be considered a source of economic rents or, to put it differently, competitive advantage [23].

[1] For a more extensive overview of the economics of patents see for example Langinier and Moschini [179].

[2] Under which conditions firms may benefit from such rents follows from a careful analysis of structure, conduct, and performance in a particular industry. Strategic management within this framework boils down to the mirror image of antitrust policy [225].

Due to its partly inward-looking perspective [212], the resource-based view (RBV) of the firm, originally developed by Lippman and Rumelt [205], Barney [22], and Wernerfelt [335], is more or less complementary to external analyses relying on Porter's "five forces." Possibly for exactly this reason, it has evolved into one of the most influential ideas in the field of strategic management.[3]

Nevertheless, the RBV has been criticized as tautological on several occasions,[4] mainly because it does not fully explain why some firms are more successful in building their portfolio of resources than others. This shortcoming motivates a rising interest in the capability-based view of the firm. According to the capability-based view, competitive advantage is obtained and defended by effectively leveraging existing capabilities to transform resources and developing new ones as the need arises [319].

The notion of a "core competence" [274] represents a useful heuristic for selecting desirable capabilities in line with the strategic intent formulated by senior management [294]. A significant danger lies in core competencies becoming core rigidities and, eventually, core in-competencies. Sustainable competitive advantage is thus rooted in *dynamic* capabilities, that is capabilities of higher order, which facilitate the integration, generation, and reconfiguration of capabilities in a highly uncertain world [223, 322].

There is an obvious link between the capability-based view of the firm and the concept of *intangible assets,* which is inspired by the stock-taking approach characteristic of traditional accounting. Like tangible fixed assets, intangible assets are long-lived and used in the production of goods and services. However, they lack physical properties. Typically, intangible assets represent legal rights or competitive advantages developed or acquired by a firm. Simply put, there is a direct correspondence between the intangible assets and capabilities of a firm. As the term "asset" suggests, most of these capabilities are assumed to be static in nature. In contrast, the management of intangible assets necessarily involves *dynamic* capabilities.[5]

When managing intangible assets, one needs to account for considerable differences with respect to their useful lives and other key characteristics: patents, copyright, trademarks, and similar intangible assets can be specifically identified with reasonably descriptive names (iden-

[3] For more recent contributions see Amit and Schoemaker [5], Barney [23], Dierickx and Cool [88].

[4] For an introduction to the controvery regarding tautology in the RBV see Barney [24], Foss [108], Priem and Butler [278, 279].

[5] For a more detailed assessment of the relationship between industrial organization, the RBV, and property rights see Foss and Foss [107].

tifiability); intangible assets may be purchased or developed internally (manner of acquisition); some intangible assets have a definite life established by law, contract, or economic behavior (definite or indefinite life); since the right to a patent, copyright, or franchise can usually be identified separately, it can also be transferred by selling it to an interested counterparty (transferability).

For valuation purposes, intangible assets must be readily identifiable and separable from other assets used in the same firm. Legally speaking, an intangible asset can be defined based on practical considerations, such as whether it is supported by a contract (contractual-or-legal criterion) or whether its economic value can be measured objectively over a definite lifetime.[6] Intangible assets that cannot be separated from other assets of the firm are included in goodwill (separability criterion) [220].

IP is a subset of intangible assets. Consequently, valuation methods applicable to intangible assets also apply to IP—specifically patents, copyrights, trademarks, and identifiable know-how [3]. The ability to determine the market value of a piece of IP is a prerequisite for optimal patent strategies and, more specifically, value-based IP management.[7]

Of course, intellectual property rights (IPRs) are not the only means of appropriating returns from innovation [199]. Other possibilities include complementary assets [321], first-mover advantage [200], uncertain imitability [205], learning curve advantages, and secrecy. Therefore, the effectiveness of IPRs also varies by industry [318].

However, what makes patent protection in particular such a worthy subject of current research, is the growing use of patents as instruments to avoid litigation, increasingly in contrast to their original constitutional and statutory basis. This development is seen by some experts as evidence of the abandonment of the foundations of an effective patent system. Instead of the patents themselves, budgets for litigation and patent enforcement are the true determinants of monopoly power.

While, before the advent of the knowledge economy, IP protection used to be a rather arcane subject and the exclusive domain of legal experts, it has since appeared on the daily agenda of senior management in many industries, including pharmaceuticals. This change of mind

[6] IAS 38 ("Intangible Assets") now states that intangible assets may have an indefinite useful life when there is no foreseeable limit on the period over which the asset is expected to generate net cash inflows for the entity in question. However, the reader should bear in mind that the main focus of this analysis are intangible assets in the economic sense of the term, which does not necessarily coincide with legal definitions.

[7] For a comprehensive treatment of IP management in practice see Nermien [256].

reflects the empirical fact that IP typically accounts for the lion's share of value growth among market leaders [222, p. 5]

Heightened litigation activity and declining patent quality are at least partially due to the current regulatory environment. Step by step, patent offices in the US, Europe, Japan, and other industrialized countries, have developed from protectors of the public against excessive monopoly power to facilitators of patent propagation [193, p. 79]

For instance, whereas the examination process at the United States Patent and Trademark Office (USPTO) takes nearly three years on average [4], a patent examiner spends an average of only 18 hours per application comparing the application and prior art, writing provisional rejections, reviewing responses and amendments, conducting an interview with the applicant's attorney, and eventually writing a notice of allowance [191].

Despite comparatively superficial examination, the PTO has accumulated an impressive backlog of more than 750,000 patent applications. Moreover, the patent prosecution process suffers from structural deficiencies encouraging the PTO to grant patents of questionable quality. Apart from a high examiner turnover, the incentive system rewards examiners only for allowing, not for rejecting applications [234, 323]. In sum, around 85 percent of patent applications in the US lead to an issued patent. Success rates in Europe and Japan are substantially lower [280].

The European patent opposition system in particular has shown to be effective in selecting valuable patents. Patents that survive an opposition proceeding in Germany are more valuable than any other type of patent [134]. Taking advantage of the superior information of industry participants to identify patents worthy of receiving more intensive scrutiny, would enable the PTO to focus its resources on the patents that are both questionable and commercially relevant.[8]

However, such differences by no means imply that all patents issued in Europe or Japan are of high quality. Firms must confront the reality that the majority of patents is not transferable into assets for use in financial transactions [222, p. 5]

In addition, following the reinterpretation of the *inventive-step* criterion of patentability in 1952, the patent system has developed from a mechanism suitable for protecting the results of individual ingenu-

[8] Nevertheless, any opposition system requiring the active participation of third parties is also subject to a free-riding problem [193]. Furthermore, an opposition process tends to increase the cost of conflict resolution [198]. For detailed discussions of post-grant opposition see Hall et al [131], Harhoff et al [136].

ity to one supporting purposeful, routine corporate R&D [168]. Routine R&D typically takes the form of entire portfolios of risky projects. R&D portfolios, in turn, yield patent portfolios, which primarily serve as "bargaining chips" in the negotiation of cooperative arrangements, designed to prevent lockout from state-of-the-art technologies developed by competitors.

With the share of products and processes involving advanced technologies growing, patent pools gain in importance. Simply put, such agreements give all members of the pool access to the technologies of all other members, thus helping to avoid costly negotiation or litigation. If, however, the use of patent pools is forbidden by law, firms tend to engage in widespread patenting, as a safeguard against competing patents that might hinder incremental improvement in the future.[9] Eventually, the fear of being locked out leads to an "arms' race," patent strategies aimed at preventing competitors from exploiting alternative technological trajectories altogether [132, p. 4]. The resulting IP "mine field," or *patent thicket,* develops into a serious impediment to innovation [308].

For an excellent, purely qualitative account of economic problems related to imperfect patent protection and *patent thickets,* for example *royalty stacking,* the reader is referred to the fairly recent discussion of *probabilistic patents* by Lemley and Shapiro [193]. Part III of this text addresses several of the issues raised using a formal model.

Regardless of problems surrounding the patent system, knowledge undoubtedly is the primary economic resource, turning its management and protection into cornerstones of corporate strategy. This insight is also reflected in the economic literature. For instance, the number of publications on patents indexed in ECONLIT, increased from around 40 publications between 1981 and 1984 to more than 250 publications between 1999 and 2002 [132, p. 1]

Before elaborating on the various valuation methods used in practice, the following section therefore briefly discusses patents from an economist's perspective.

[9] Nevertheless, as pointed out by Shapiro [308], the strategic accumulation of patents in patent pools may also result in high barriers to entry. For instance, a new firm in the semiconductor industry typically spends around 150 million dollars in licensing fees for basic technologies that, in the end, might turn out to be only of limited practical use [130].

2.2 Patents as an Incentive Mechanism

Observation 2.2. From an economist's perspective, patents constitute an incentive mechanism, which can be designed to maximize welfare.

Although much has been written on the economics of patenting, researchers have yet to agree on an answer to the challenging question of which patent system provides optimal incentives and achieves the goal of balancing innovation and competition. David admits:

> "There is no settled body of economic theory on the subject [of IP protection] that can be stated briefly without doing serious injustice to the sophisticated insights that have emerged over many decades of debate. Instead, the relevant economic literature is extensive, convoluted, and characterized by subtle points of inconclusive controversy." [84, p. 23]

Put aside these difficulties, a useful categorization of patent research is provided by Nelson and Mazzoleni [255], who propose classifying contributions into four broad theories about the benefits and costs of patents: (1) the prospect of obtaining patents motivates invention ("invention" theory); (2) while invention takes place regardless of the level of IP protection, patents induce inventors to disclose their findings to the public ("disclosure" theory); (3) patents are required to justify investments leading to the development and commercialization of inventions ("development and commercialization" theory); (4) patents facilitate the orderly exploration of broad prospects for derivative inventions ("exploration" theory). These theories, briefly described in the following, are not necessarily mutually exclusive.

The "invention" theory in its basic form presumes that, without a patent system, incentives for invention are too weak to reflect the public interest [12, 259, 293]. According to the "invention" theory, stronger patent protection thus always increases inventive activity. Conversely, if patents are not required to induce invention, granting a patent leads to a reduction in welfare.

Executives in the pharmaceutical industry agree that 60 percent of their new drugs would not have been developed without patent protection. In addition, most studies suggesting that invention does not crucially depend on patent protection focus on large and medium-sized firms, which typically are in a position to exploit inventions by using them in their own production process. Inventors who depend on sale or licensing to reap returns are sure to consider patents highly important [255, pp. 18–21] Furthermore, a lot of what was learned in studies motivated by the invention-inducement theory might not be relevant to research tools, for instance in biotechnology.[10]

In contrast to the "invention" theory, the "disclosure" theory presumes that secrecy is possible and sufficient to induce invention. Nevertheless, society is better off granting IPRs and getting disclosure in return. Disclosure makes the invention available for uses that the inventor was not aware or in a position to take advantage of. In other words, a patent advertises the presence of an invention and facilitates licensing. This mechanism seems to play a key role in pharmaceuticals [255, pp. 21–22].

Simply put, the "development and commercialization" theory is a variant of the "invention" theory, but with patents granted early in the inventive process. Early-stage patents provide the assurance that the rewards of technologically successful development are capturable, thus motivating the decision to engage in development in the first place. What is more important is that ownership of a patent enables the holder, possibly a small firm faced with large development costs, to raise capital for development financing. Alternatively, the original inventor's possession of a patent facilitates handing off the task to a large-scale organization better situated for development and commercialization. Such scenarios are in fact fairly common in research-incentive industries, again including pharmaceuticals [255, pp. 22–23].

Similar to the "development and commercialization" theory, the "exploration" theory proposes that the social benefits of patents accrue after the initial invention [171]. Presumably, making an early-stage invention freely available leads to a development process that is chaotic, duplicative, and wasteful, whereas granting a broad patent on such an invention enables orderly development of broad technological prospects. Consequently, the main difference between the "exploration" and the "development and commercialization" theory is a wider range of subsequent development projects or inventions depending on the initial invention as an input. Many research tools invented at universities, for example protein and DNA sequencing instruments, fall in this category [255, pp. 23–25].[11]

[10] A detailed analysis of patenting in pharmaceutical biotechnology is beyond the scope of this text. For more information on IPRs in biotechnology see Adler [2], Eisenberg [99], Forman and Diner [106], Gold et al [118], Jackson [153]. For a discussion of moral issues in connection with biotechnology patents see also Crespi [80].

[11] For a thorough analysis of reach-through licensing in pharmaceuticals see Eisenberg [100].

The basic tradeoff between welfare losses and gains described so far is summarized in table 2.1. Particularly if one follows the "exploration" theory, but more generally whenever an invention is viewed as a platform for subsequent inventions, the costs and benefits of patents are no longer fully captured by a static tradeoff between positive effects on inventive activity on the one hand and negative effects on competition on the other. Rather, in contrast to the very simplistic "invention" theory, welfare losses and gains due to strong patents are determined by their long-term impact on innovation [255, p. 25].

This insight calls for formal models of the patent system accounting not only for the length and breadth of IP protection, but also for the quality of patents granted, which, as shown above, has deteriorated noticeably in recent years. The following section serves to clarify how recent developments in accounting and patent law have contributed to the evolving economic role of patents, forcing firms to continuously reassess their patent strategies.

Table 2.1. Intellectual property as an incentive for innovation (Source: table adapted from Harhoff [133, p. 5]). The effect of IP on innovation and competition requires a tradeoff between welfare losses and gains.

Welfare	Effect of patents	
	Innovation	Competition
Losses	Patent thickets, hold-up problems	Market power
Gains	Invention and diffusion	Market entry

2.3 Recent Developments

Over the last decades, legal changes in both patenting and accounting have repeatedly altered the rules of competition in research-intensive industries.

2.3.1 Patenting

The introduction of the CAFC in 1982 marked the beginning of an era of strong IPRs in the US, accompanied by a number of important changes in the legal environment [93].

Most importantly, courts became more and more "patent-friendly." Before 1980, a district court ruling that a patent was valid and infringed was upheld on appeal 62 percent of the time. Between 1980 and 1990, the share of rulings upheld increased to 90 percent. Conversely, appeals overturned only 12 percent of district court rulings on invalidity or non-infringement before 1980. The share of rulings overturned during the later period rose to 28 percent. As a result, the overall probability of a litigated patent being held valid has now reached roughly 54 percent [155]. In addition, patent holders asserting infringement are also much more likely to successfully seek a preliminary injunction barring the competitor from selling the product in question for the duration of the litigation process [180, 181].[12]

The Drug Price Competition and Patent Term Restoration Act (Hatch–Waxman Act) of 1984 aimed at promoting both pharmaceutical innovation and diffusion by compensating research-based firms for time lost during the FDA approval process, while facilitating generic entry in the US.[13]

The Act provides incentives to contest the validity of a patent by granting successful challengers the exclusive right to market a generic equivalent for 180 days [66]. Although manufacturers of branded drugs may request a 30 months postponement of the FDA approval of generic drugs [111], the Act has in fact intensified competition. In the mid-1980s, generic products accounted for a mere 19 percent of all prescriptions. By 1999, this figure had risen to 47 percent [122].

Furthermore, the scope of patentable subject matter was expanded to also include genetically engineered bacteria and animals, genetic sequences, and the like. The share of biotechnological patents grew from about 3 percent of total patents in 1961 to about 6 percent in 1991. This increase was even stronger in the nineties [111].[14] Nevertheless, the expansion of patentable subject matter alone does not justify the recent surge in patenting [155].

Consequently, Kortum and Lerner [175] empirically test a variety of hypothesis trying to explain this remarkable trend: increased patenting takes place in response to stronger IP protection; increased patenting is due to a sharp rise in a few new fields benefiting from extraordinary scientific and technological progress; increased patenting results from

[12] For a formal analysis of time factors in patent litigation see Aoki and Hu [9].

[13] Generic firms need only demonstrate that their product is bio-equivalent to the pioneering brand to receive market registration and may file an abbreviated new drug application (ANDA) [122, p. 90].

[14] For additional data see Lerner [195].

large firms reacting to a perceived relaxation of antitrust vigilance by accumulating patent portfolios. Although their analysis seems to indicate that more productive R&D is the driving force, other studies hint at completely different motives [130].

More specifically, Cohen et al [67] argue that the reconciliation of increased patenting and a lack of perceived effectiveness may lie in a steadily rising number of firms that adopt aggressive IP strategies. Patents not only protect specific inventions. As explained earlier, firms frequently employ patents to block competing products, as a "currency" in cross-licensing, and to prevent or defend against infringement lawsuits. In many cases, the use of patents turns out to be a zero-sum game with only marginal impact on innovation incentives [130].

The type of *patent portfolio race* observed in many industries is thus consistent with rising rates of patenting as well as rising patent-per-R&D ratios, which are not attributable to a significant gain in net patent value. Still, patents might increase overall efficiency by supporting the ongoing deconstruction of the pharmaceutical value chain.

There are good reasons to believe that a combination of technological opportunities, the buildup in government R&D spending and defense procurement, international competition, and other factors increasing the returns to R&D would have resulted in a rising number of patents, even without legal reform strengthening the patent system. Stronger IP protection presumably only served to reinforce existing tendencies and did little to stimulate innovation [155]. In summary, patent regime changes in the 1980s possibly caused a substantial part of resources to be diverted from innovation towards the acquisition and defense of IPRs.

Weak patent protection does not necessarily hinder inventive activity; and strong patent protection does not necessarily slow down diffusion. An optimal configuration of patent and antitrust law may ensure sufficient incentives for R&D and, at the same time, promote interfirm knowledge transfer by inducing the diffusion of results through cooperative arrangements, for example licensing [260]. An overly broad interpretation of patent scope, however, may be harmful to economic growth, leading to lengthy battles in court instead of fostering genuine rivalry in technical progress [235].

Introduced in August 2005, the Patent Reform Act clearly is the most comprehensive change to US patent law since Congress passed the 1952 Patent Act. It proposes a dramatic change in the patent application process, again granting rights to the inventor (redundantly referred to as the first to invent), as apposed to the first to file a patent application. In essence, the reform addresses several issues in connection with excessive litigation mentioned earlier, including, for example, treble damages in the case of willful infringement, invalidity due to inequitable conduct, and access to injunctions [236]. Nevertheless, it is comparatively safe to assume that imperfect patent protection will continue to trouble decision makers and researchers in the foreseeable future.

2.3.2 Accounting

Although accounting principles are not the focus of this text, recent reforms in this important regulatory field should not be neglected.

The last decade has brought a number of changes in the rules for IP valuation. Above all, the Sarbanes–Oxley (SOX) Act has played a major role in the transformation of reporting standards. Together with the new SEC regulations SFAS 141, 142, and 144,[15] it has dramatically altered the way firms value and account for IP in mergers or other business combinations. Not only US-based firms, but firms operating internationally in general are affected. The new rules differ with respect to scope, level of detail, definition of asset classes, due diligence requirements, and frequency of valuation. Unsurprisingly, the potential impact on earnings following an acquisition is substantial.

The FASB released its new regulations in 2001, after a long period of study, hearings, private inputs, and public testimony. SFAS 141 and 142 primarily cover valuation issues. They require firms to use the purchase method of accounting for business combinations and to provide the *fair value* of all intangibles in a detailed purchase price allocation (PPA) report.[16] SFAS 144 is concerned with *impairment tests.* If the value of an intangible asset has been impaired and the firm chooses to dispose of it, SFAS 144 also sets standards for disposal value.

Taken everything together, the increasing use of patents as instruments of strategic management, in combination with the rising importance of *fair value accounting,* necessitates novel techniques for IP val-

[15] Written out in full, the relevant statements are "Business Combinations" (SFAS 141), "Goodwill and Other Intangible Assets" (SFAS 142), "Accounting for the Impairment or Disposal of Long-Lived Assets" (SFAS 144). Correspondingly, IAS 22, 38, and 36 address "Business Combinations," "Intangible Assets," and "Impairment of Assets."

[16] US-GAAP themselves provide no exact definition of *fair value.* IAS 22, however, defines *fair value* as "the amount for which an asset could be exchanged or a liability settled between knowledgeable, willing parties in an arm's length transaction."

uation that reflect and integrate all relevant risk factors, including imperfect patent protection. Existing techniques, by and large, are characterized by an isolated treatment of risk factors affecting future cash flows, possibly leading to an overly optimistic or pessimistic assessment of exposure.

Effective methods for valuing IP are also likely to support its role in facilitating economic growth through innovation. The following section therefore compares traditional and option-based approaches to capital budgeting, hinting at methodological challenges to be discussed in later chapters.

3

Uncertainty, Irreversibility, and Flexibility

The main characteristics of investment decisions in real-world situations are uncertainty, flexibility, and irreversibility. This chapter serves to highlight differences between the traditional and the option-based approach to capital budgeting. In addition, a detailed review of option pricing techniques employed in further analyses is provided.

3.1 Capital Budgeting

Capital budgeting is about making optimal investment decisions. However, no approach is guaranteed to yield optimal decisions under all circumstances. Hence, a variety of alternative approaches coexist. For the simple reason of clarity, techniques that do not involve option pricing are referred to as "traditional" in the following.

3.1.1 Traditional Approach

The traditional approach to capital budgeting is based on the net present value (NPV) criterion. Before presenting some of its variants geared specifically towards intangible asset valuation in more detail, a brief look at the underlying economic rationale is advisable.

3.1.1.1 Economic Rationale

Abstracting from agency problems, the separation theorem in financial markets states that all investors will accept or reject the same investment projects, regardless of their personal preferences. The investor may reduce C_0 and give up consumption today, in exchange for an increase in C_1, that is additional consumption tomorrow. The slope of the market opportunity line is determined by the interest earned on investment today.

Figure 3.1 illustrates why it is optimal to choose E_1 on the production opportunity curve and move along the market opportunity line, rather than to choose E_2, which—judging by the utility curves U_2 and U_3—is clearly inferior to E_3. Consequently, rational investment decisions are based on the NPV rule: an investment is advantageous if and only if the present value of cash flows resulting from it is positive. Personal preferences for consumption can be factored in by lending and borrowing.

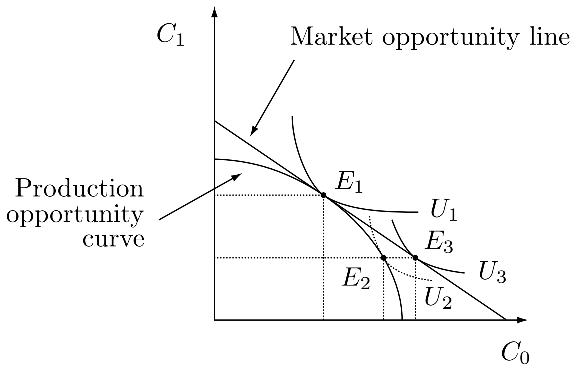

Fig. 3.1. Separation of financing and investment decisions (Source: figure adapted from Trigeorgis [328, p. 30]).

In intangible asset valuation, the NPV rule is referred to as the "income approach," contrasting it to the *market* and the *cost* approach.[1]

3.1.1.2 Variants

Practitioners in accounting have developed variants of the income approach, designed to address specific issues in the valuation of intangible assets [3]. These include the relief-from-royalty method, the Multi-Period-Excess-Earnings Method (MEEM), and the incremental-cash-flow method [282]. A certain degree of subjectivity is unavoidable, since the same asset can be valued differently, depending on the valuation context [222, p. 10].

[1] Alternative methods not discussed here include the use of value indicators such as patent renewal data [72, 186, 305].

3.1.1.2.1 Relief from Royalty

The relief-from-royalty method is based on an estimate of hypothetical after-tax royalty payments one would have to pay to a third party to reap the benefits of the intangible asset in question. It therefore requires representative data on recent licensing deals and is commonly applied in patent, franchise, and brand valuation.

3.1.1.2.2 Multi-Period Excess Earnings

When using the MEEM, incremental after-tax cash flows attributable only to the subject intangible asset are quantified to obtain its value. Following AICPA guidelines, the net cash flows attributable to the subject asset are those in excess of fair returns on all other assets that are necessary to their realization. The latter are referred to as "contributory assets." Typically, the MEEM is applied in the valuation of customer relationships and R&D processes.

3.1.1.2.3 Incremental Cash Flow

The incremental-cash-flow method aims at estimating the PV of additional cash flows resulting from the possession of a particular intangible asset and is almost exclusively applied in brand valuation.

3.1.1.3 Limitations

Put aside fundamental criticism related to the incompleteness and inefficiency of markets, NPV fails to account for managerial flexibility [140, 141]. This deficiency has lead to an increased interest in novel, option-based methods.

3.1.2 Option-Based Approach

Before introducing the approach on a non-technical level, the following section briefly addresses the origins of option-based capital budgeting.

3.1.2.1 Origins and Connections

In the early 1970s, Black and Scholes [35], Merton [238], and others developed a theory of rational option pricing. Even before a thorough methodological foundation was laid almost a decade later [76, 137, 138], Myers [253] recognized the applicability of option pricing in the wider context of corporate finance. A multitude of authors has since followed in his footsteps, analyzing almost anything from natural resource investments to real estate.[2]

3.1.2.2 Non-Technical Introduction

Roughly speaking, the approach is based on a straightforward analogy between flexibility in financial and real investment decisions.

3.1.2.2.1 From Financial to Real Options

The classic Black–Scholes equation for a European call option on stock is

$$c(S_t, t) = S_t \mathrm{N}(d_1) - K\mathrm{e}^{-r(T-t)}\mathrm{N}(d_2),$$

with

$$\begin{aligned} d_1 &= \frac{\ln\left(S_t/K\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}, \\ d_2 &= d_1 - \sigma\sqrt{T-t}, \end{aligned}$$

where S_t is the stock price at time t, K the strike price, σ volatility, r the risk-free rate, and T maturity [35, 238].

Each of the its parameters corresponds to one of the six value drivers in the *real option hexagon* (see fig. 3.2) [196, p. 9]. The use of option theory in the evaluation of real investment opportunities enables an assessment of the value of flexibility that is consistent with financial markets. Similar to the NPV rule suitable for analyzing static investment policies, option pricing is thus independent from individual preferences. Later chapters will serve to demonstrate this mechanism as well as underlying assumptions in detail.

A illustrated by fig. 3.3, flexibility creates value, because it enables firms to wait for uncertainty to resolve and revise their original action plans accordingly. Effectively, the lower part of the profit distribution is cut off, while preserving full upside potential. Of course, in realistic models of investment under uncertainty, the additional value generated by an option might not always be as obvious.

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[2] For an extensive review of the relevant literature see Baecker and Hommel [17]. See also Dixit and Pindyck [91], Grenadier [127], Trigeorgis [328].

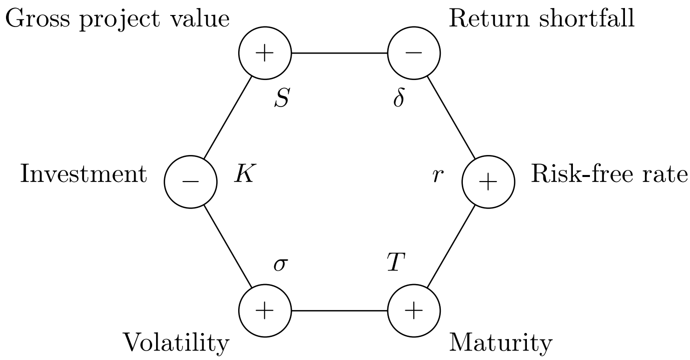
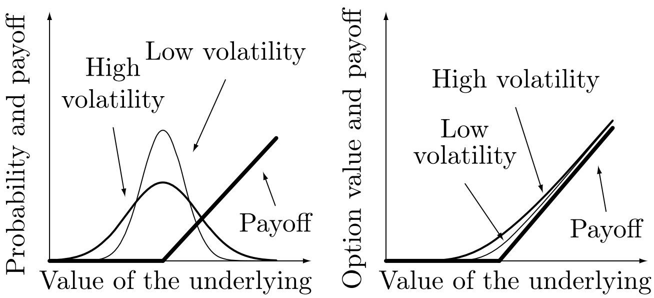


Fig. 3.2. The real option hexagon (Source: figure adapted from Leslie and Michaels [196, p. 9]).



(a) Distribution of outcomes and payoff function

(b) Option value and payoff function

Fig. 3.3. Uncertainty as a value driver (Source: Amram and Kulatilaka [6, p. 35]).

3.1.2.2.2 Limitations

Highly innovative technologies with novel uses and unknown risk profiles pose severe difficulties, since it is no longer safe to assume that markets for such assets are arbitrage-free [147]. In fact, the very nature of innovation lies in a sort of arbitrage. Hence, traditional capital budgeting techniques yield indicative results at best. The option-based approach is no different in this regard, but provides the distinct advantage of properly capturing important decisions during later stages of the investment process. In particular part II will serve to examine such issues from a more technical perspective.

3.1.2.3 Applications to Research and Development

While most of the early contributions were in the field of natural resources, there has been a growing interest in applications to R&D lately, leading to new insights about the relationship between finance and strategy. Apart from very abstract formal models, a number of conceptual papers have examined the role of options in real-world settings. One example is the opportunity portfolio developed by McGrath and MacMillan [230].

For obvious reasons, successful R&D management requires a portfolio perspective. Employing the option analogy on the conceptual level, McGrath and MacMillan propose a framework, which assists in categorizing opportunities available to the firm based on the perceived level of technological and market uncertainty (see fig. 3.4). In addition to comparatively safe enhancement launches, firms should invest in positioning and scouting options as well as stepping stones, enabling them to learn and take advantage of new opportunities as they arise.

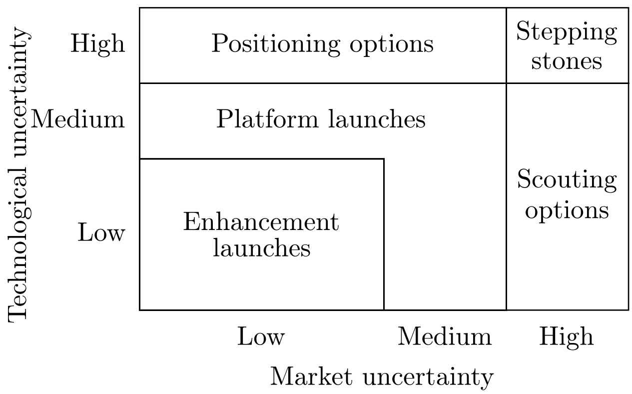


Fig. 3.4. A portfolio for categorizing opportunities available to the firm (Source: McGrath and MacMillan [231, p. 176]).

Although such frameworks are extremely helpful in practice, thorough economic analyses frequently call for sophisticated numerical procedures, to be discussed in the following section.

3.2 Numerical Methods

This section serves to give an overview of extant numerical procedures commonly employed to price financial derivatives. Furthermore, it motivates the application of similar approaches to the analysis of real investment problems for which closed-form solutions are not readily available.

3.2.1 Stochastic Processes

Uncertainty concerning the future value of assets—be it market or technical uncertainty—can be accounted for by modelling asset prices as stochastic processes. Based on the pioneering work of Bachelier [14], Einstein [98], Osborne [261], and Samuelson [290] it is now common to employ such processes and stochastic calculus when valuing flexibility in financial and real investments.

While the seminal contributions of Black and Scholes [35] and Merton [239] assume (continous) diffusion processes, asset prices often exhibit jumps [35, 239]. Closed-form solutions for european options under pure jump processes and jump-diffusion processes are given by Cox and Ross [75] and Merton [239], respectively. Current contributions to the analysis of jumps and their impact on option pricing include the works of Gukhal [128, 129] and Eraker et al [101]. Recently, there has been a rising interest in models driven by more general Lévy processes [43, 296].

Later sections will demonstrate how alternative stochastic processes may be used to better capture certain aspects distinguishing real investment opportunities from derivative securities. For now, the discussion will be limited to the standard stock price model.

3.2.2 Finite Differences

If it is safe to assume that the value of financial and real assets follows some kind of stochastic process, it follows that the value dynamics of contingent claims—including, of course, real investment opportunities—can be described in terms of partial differential equations (PDEs) subject to appropriate initial and boundary conditions.

Strictly speaking, this assumption is often violated but may nevertheless prove to be sufficiently accurate to allow conclusions as to a value-maximizing investment strategy.[3]

[3] This assumption by no means implies that the value of investment projects fluctuates randomly. Rather, the value of an investment project is determined by how the firm and its competitors respond to developments in a number of highly uncertain, seemingly random factors. Uncertainty related to these factors is resolved in the manner of a stochastic process.

3.2.2.1 Origins and Connections

While there are known closed-form solutions to many (comparatively) simple option pricing problems,[4] more complicated formulae typically require numerical methods. One type of numerical procedure is based on approximating differentials, or *infinite* differences, with *finite* differences. It is very popular in practice, since it greatly facilitates the calculation of "greeks" indicating an option's sensitivity to parameter changes—an aspect which has not been in the explicit focus of real option researchers so far.

The basic finite difference (FD) procedure for the valuation of derivatives, first discussed by Schwartz [300], Brennan and Schwartz [53], and Courtadon [73], serves as a benchmark for any novel technique introduced.[5] A more recent and rather extensive overview of the method is provided by Wilmott et al [337]. Moreover, one can draw upon a large body of literature in the areas of computational physics and other disciplines, as exemplified by numerous reference works and introductory texts [277, 332] At the risk of getting overly historical, it also seems worth noting that the FD method mirrors the arguments of Newton and Leibniz at the inception of calculus in the seventeenth century.

A related method, where only the time derivative is discretized, is the method of lines.[6] One very flexible technique for the numerical solution of PDEs is the finite element (FE) method. For the purposes of this study, it is comparable in accuracy and efficiency to the FD approach. As the FE method is less common and slightly more difficult to implement, no detailed description is given. Instead, the reader is referred to Topper [326] for a good overview with plenty examples and additional references. To the author's best knowledge, neither of these two methods has so far been applied in a real options context.

Of course, there are the various pricing trees, which are in many ways similar to the lattice approaches presented here. Applications of binomial and trinomial pricing models in the realm of investment under uncertainty are abundant [112, 327]. Although very intuitive tools, their often-claimed universal usefulness is somewhat questionable, largely due

[4] Well-known and often-cited contributions include the articles by Black and Scholes [35], Geske [115], Johnson [159], Margrabe [221], Merton [238], Stulz [317]. In an analogical sense, these formulae are often considered generic building blocks for demanding investment problems.

[5] See the seminal contributions of Brennan and Schwartz [53, 54], Courtadon [73, 74], Schwartz [300].

[6] For a brief introduction, the interested reader is referred to Meyer and van der Hoek [241]. The method of lines is occasionally referred to as "Rothe's method." [289]

to the impossibility of optimizing a multi-dimensional function by performing a brute-force search over an exponentially increasing number of grid points.[7] The problem boils down to what Bellmann calls "the curse of dimensionality." Elaborating on more advanced models, the discussion will return to this key point in later sections [28, p. 97].

Similar shortcomings, however, are shared by most techniques including (direct-sampling) Monte Carlo simulation. This is where the need for creative model formulation and suitable heuristics arises.

3.2.2.2 Preliminaries

Before proceeding to the discussion of the actual FD method, a few definitions are in order.

3.2.2.2.1 Difference Formulae

There are several ways in which changes in asset prices can be calculated. Consider the variable x denoting the value of some investment project. Given $f(x)$, an investment opportunity somehow dependent on the value of this variable, project value is examined at discrete time intervals $x_i \equiv i\Delta x$, where $i \in \mathbf{N}^+$.

Let $f_i \equiv f(x_i)$ represent equidistant table values. Each of the following expressions constitutes a different way of describing changes in $f(x)$ around $x = x_i$:

$$\begin{aligned}
\overrightarrow{\Delta} f_i &\equiv f_{i+1} - f_i \quad (\textit{forward} \text{ difference}), \\
\overleftarrow{\Delta} f_i &\equiv f_i - f_{i-1} \quad (\textit{backward} \text{ difference}), \\
\overleftrightarrow{\Delta} f_i &\equiv f_{i+1/2} - f_{i-1/2} \quad (\textit{central} \text{ difference}).
\end{aligned}$$

In addition, let

$$\overline{\Delta} f_i \equiv \tfrac{1}{2}\left(f_{i+1} - f_{i-1}\right)$$

denote the *central mean.* It is useful to furthermore establish these recursive definitions:

$$\begin{aligned}
\overrightarrow{\Delta}^2 f_i &\equiv \overrightarrow{\Delta} f_{i+1} - \overrightarrow{\Delta} f_i, \\
\overleftarrow{\Delta}^2 f_i &\equiv \overleftarrow{\Delta} f_i - \overleftarrow{\Delta} f_{i-1}, \\
\overleftrightarrow{\Delta}^2 f_i &\equiv \overleftrightarrow{\Delta} f_{i+1/2} - \overleftrightarrow{\Delta} f_{i-1/2},
\end{aligned}$$

and

[7] For a brief description of trees for option pricing see sect. B.1 in the appendix.

$$\overline{\Delta}^2 f_i \equiv \tfrac{1}{2}\left(\overline{\Delta} f_{i+1} - \overline{\Delta} f_{i-1}\right).$$

The purpose of higher order approximations in particular becomes immediately evident in the following subsection.

3.2.2.2.2 Interpolation Formulae

Based on the difference formulae introduced in the previous subsection, it is possible to thread a polynomial through the points (x_i, f_i). A differentiable interpolated function is obtained that can be used to arrive at suitable approximations for derivatives.

The normalized distance between x and x_i is $u \equiv (x - x_i)/\Delta x$. Employing the forward equation, the interpolated polynomial becomes

$$F_k(x) = f_i + \sum_{l=1}^{k} \binom{u}{l} \overrightarrow{\Delta}^l f_i + \mathrm{O}\left(\Delta x^{k+1}\right),$$

where $\overrightarrow{\Delta}^l f_i$ denotes the difference approximation of order l and $\mathrm{O}(\cdot)$ signifies the order of the truncation error. This procedure is known as Newton–Gregory *forward* interpolation. The simplest possible polynomial is derived by setting $k = 1$:

$$F_1(x) = f_i + \frac{\overrightarrow{\Delta} f_i}{\Delta x}(x - x_i) + \mathrm{O}\left(\Delta x^2\right).$$

Correspondingly, applying the backward difference formula yields the Newton–Gregory *backward* interpolation:

$$F_k(x) = f_i + \sum_{l=1}^{k} \binom{u + l - 1}{l} \overleftarrow{\Delta}^l f_i + \mathrm{O}\left(\Delta x^{k+1}\right).$$

The polynomial of first order is

$$F_1(x) = f_i + \frac{\overleftarrow{\Delta} f_i}{\Delta x}(x - x_i) + \mathrm{O}\left(\Delta x^2\right).$$

Due to the difficulty of evaluating central differences of odd order, terms of the form $\overleftrightarrow{\Delta}^{2l+1}$ have to be replaced by the corresponding central *mean:*

$$F_k(x) = f_i + \sum_{l=1}^{k/2} \binom{u + l - 1}{2l - 1}\left(\overline{\Delta}^{2l-1} f_i + \overleftrightarrow{\Delta}^{2l} f_i\right) + \mathrm{O}\left(\Delta x^{k+1}\right).$$

This equation represents the slightly more complicated *Stirling* interpolation. Substitute the smallest even positive integer for k and it becomes clear that

$$F_2(x) = f_i + \frac{\overline{\Delta} f_i}{\Delta x}(x - x_i) + \tfrac{1}{2}\frac{\overleftrightarrow{\Delta}^2 f_i}{\Delta x^2}(x - x_i)^2 + \mathrm{O}\big(\Delta x^3\big)\,.$$

3.2.2.2.3 Difference Quotients

Derivatives at x_i are easily obtained by differentiating the polynomials with respect to x and setting $x = x_i$.

First derivatives can be approximated based on interpolations of first order using either of the following formulae:

$$F_1'(x) = \frac{\overrightarrow{\Delta} f_i}{\Delta x} = \frac{f_{i+1} - f_i}{\Delta x} + \mathrm{O}(\Delta x) \quad (\textit{forward} \text{ differences}),$$
$$F_1'(x) = \frac{\overleftarrow{\Delta} f_i}{\Delta x} = \frac{f_i - f_{i-1}}{\Delta x} + \mathrm{O}(\Delta x) \quad (\textit{backward} \text{ differences}).$$

Using central differences and a polynomial of second order, one obtains

$$F_2'(x) = \frac{\overline{\Delta} f_i}{\Delta x} = \frac{f_{i+1} - f_{i-1}}{2\Delta x} + \mathrm{O}\big(\Delta x^2\big)\,.$$

The Stirling formula thus yields the most accurate approximation with the smallest truncation error. Approximations of higher order are possible, but rarely used, due to the additional complexity they introduce into the following calculations.

Second derivatives are approximated in a similar manner. As can be easily verified,

$$F_2''(x) = \frac{\overleftrightarrow{\Delta}^2 f_i}{\Delta x^2} = \frac{f_{i+1} + f_{i-1} - 2f_i}{\Delta x^2} + \mathrm{O}\big(\Delta x^2\big)\,.$$

Refer to the overviews provided by Vesely [332] and also Press et al [277] for a more detailed account of the concepts presented.

3.2.2.3 Differencing Schemes

The choice of FD approximations for various derivatives constitutes a *differencing scheme.* The following subsections serve as a brief summary of the explicit, the implicit, and the Crank–Nicolson scheme as well as solution procedures for European and, in particular, American options. Figure 3.5 illustrates the grid points used in the calculation of unknown function values for the schemes mentioned above.

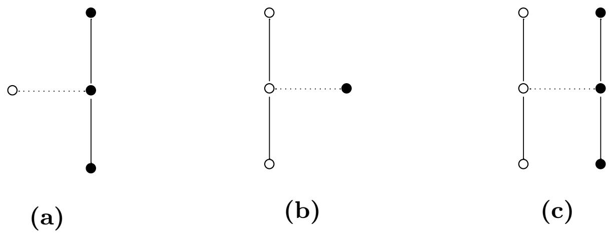


Fig. 3.5. Finite difference schemes. The diagrams illustrate (a) the explicit, (b) the implicit, and (c) the Crank–Nicolson scheme. Open circles are new points at which a solution is desired. Filled circles are known points whose function values are used in calculating the new point. Solid lines connect points that are used to calculate spatial derivatives. Dashed lines connect points that are used to calculate time derivatives (Source: Press et al [277]).

3.2.2.3.1 Fully Explicit Scheme

In this section the explicit method is introduced through an examination of the general case laid out by Hull and White [151]. The discussion then proceeds to the Black–Scholes equation which also serves as a model problem for further analysis.

3.2.2.3.1.1 General Case

Consider a real (abandonment) option, with value f, that depends on a single stochastic variable X_t, usually the gross present value of some investment. Let the stochastic Itô process followed by X_t be described by the autonomous stochastic differential equation (SDE)

$$\mathrm{d}X_t = a(X_t)\,\mathrm{d}t + b(X_t)\,\mathrm{d}W_t, \tag{3.4}$$

where $\mathrm{d}W_t$ is the increment of a Wiener process.[8] The assumption of constant instantaneous (proportional) drift α and volatility σ together with a slightly simplified notation leads to

$$\mathrm{d}x = \alpha x\,\mathrm{d}t + \sigma x\,\mathrm{d}W.$$

If λ is the market price of risk for x and r is the risk-free interest rate, f then satisfies the partial differential equation

$$\frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}\,(\alpha - \lambda\sigma)\,x + \tfrac{1}{2}x^2\sigma^2\frac{\partial^2 f}{\partial x^2} = rf, \tag{3.5}$$

[8] Equation (3.4) is considered autonomous, since neither a nor b depend explicitly upon t.

which is to be solved numerically here [77]. For convenience, let $\eta \equiv \lambda\sigma$ denote the risk premium. An $(m+1)$-by-$(n+1)$ grid is constructed considering values of f when time equals

$$t_0, t_0 + \Delta t, t_0 + 2\Delta t, \ldots, T,$$

and x equals

$$x_0, x_0 + \Delta x, x_0 + 2\Delta x, \ldots, x_{\max}.$$

Define $t_j \equiv t_0 + j\Delta t$ and $x_i \equiv x_0 + i\Delta x$. In line with the common notation, denote $f(x_i, t_j)$ by $f_{i,j}$. Partial derivatives are approximated as follows:

$$\left.\frac{\partial f}{\partial x}\right|_{i,j} \approx \frac{f_{i+1,j+1} - f_{i-1,j+1}}{2\Delta x}, \tag{3.6a}$$

$$\left.\frac{\partial^2 f}{\partial x^2}\right|_{i,j} \approx \frac{f_{i+1,j+1} + f_{i-1,j+1} - 2f_{i,j+1}}{\Delta x^2}, \tag{3.6b}$$

and

$$\left.\frac{\partial f}{\partial t}\right|_{i,j} \approx \frac{f_{i,j+1} - f_{i,j}}{\Delta t}. \tag{3.6c}$$

Derivatives at the current node are approximated by looking at nodes that succeed it in time. This scheme is only conditionally stable.

Substituting (3.6a), (3.6b), and (3.6c) into (3.5) gives

$$\frac{f_{i,j+1} - f_{i,j}}{\Delta t} + \frac{f_{i+1,j+1} - f_{i-1,j+1}}{2\Delta x}(\alpha - \eta)\, x_i + \tfrac{1}{2}x^2\sigma^2 \frac{f_{i+1,j+1} + f_{i-1,j+1} - 2f_{i,j+1}}{\Delta x^2} = r f_{i,j}.$$

Solving for $f_{i,j}$ leads to

$$f_{i,j} = a_i f_{i-1,j+1} + b_i f_{i,j+1} + c_i f_{i+1,j+1}, \tag{3.7}$$

where $i \in \{0, \ldots, m-1\}$, $j \in \{1, \ldots, n-1\}$, and

$$a_i = \frac{1}{1 + r\Delta t}\left(-\frac{(\alpha - \eta)\, x_i \Delta t}{2\Delta x} + \frac{1}{2}\frac{x_i^2 \sigma^2 \Delta t}{\Delta x^2}\right),$$

$$b_i = \frac{1}{1 + r\Delta t}\left(1 - \frac{x_i^2 \sigma^2 \Delta t}{\Delta x^2}\right),$$

$$c_i = \frac{1}{1 + r\Delta t}\left(+\frac{(\alpha - \eta)\, x_i \Delta t}{2\Delta x} + \frac{1}{2}\frac{x_i^2 \sigma^2 \Delta t}{\Delta x^2}\right).$$

This system of equations forms the basis of the *explicit* FD procedure. Since $f_{i,n}$, that is the value of the contingent claim at maturity T, is known for all relevant values of x, it can be solved by simple recursion.

Define

$$\begin{aligned} p_{i,i-1} &= -x_i\,(\alpha-\eta)\,\frac{\Delta t}{2\Delta x} + \tfrac{1}{2}\frac{x_i^2\sigma^2\Delta t}{\Delta x^2}, \\ p_{i,i} &= 1 - x_i^2\sigma^2\frac{\Delta t}{\Delta x^2}, \\ p_{i,i+1} &= +x_i\,(\alpha-\eta)\,\frac{\Delta t}{2\Delta x} + \tfrac{1}{2}\frac{x_i^2\sigma^2\Delta t}{\Delta x^2}, \end{aligned}$$

so that (3.7) becomes

$$f_{i,j} = \frac{1}{1+r\Delta t}\left(p_{i,i-1}f_{i-1,j+1} + p_{i,i}f_{i,j+1} + p_{i,i+1}f_{i+1,j+1}\right). \tag{3.10}$$

The scheme is also termed *fully explicit,* because $f_{i,j}$ for each j can be calculated explicitly from the quantities that are already known. Moreover, it is even a *single-level* scheme, because only values at time-level $j+1$ have to be stored to find values at time level j. The solution is computed on a narrowing mesh of triangular shape. All this makes the explicit FD methods quite similar to dynamic programming—and easier to implement in practice.

As a matter of fact, the resulting lattice can be regarded as a type of trinomial event tree. Obviously, $p_{i,i-1}$, $p_{i,i}$, and $p_{i,i+1}$ then represent the probabilities of moving from x_i to x_{i-1}, x_i, and x_{i+1}, respectively.

If $x_0 = t_0 = 0$, then the probabilities reduce to

$$\begin{aligned} p_{i,i-1} &= \tfrac{1}{2}\Delta t\big(i^2\sigma^2 - i\,(\alpha-\eta)\big), \\ p_{i,i} &= 1 - i^2\sigma^2\Delta t, \\ p_{i,i+1} &= \tfrac{1}{2}\Delta t\big(i^2\sigma^2 + i\,(\alpha-\eta)\big). \end{aligned}$$

It is always appropriate—though not without pitfalls—to define a new state variable $u(x,t)$ that has a constant (instantaneous) standard deviation. From Itô's Lemma the process followed by u is

$$\mathrm{d}u = q(x,t)\,\mathrm{d}t + \frac{\partial u}{\partial x}\sigma x\,\mathrm{d}W,$$

where the drift is given by

$$q(x,t) = \frac{\partial u}{\partial t} + (\alpha-\eta)\,x\frac{\partial u}{\partial x} + \tfrac{1}{2}\sigma^2 x^2\frac{\partial^2 u}{\partial x^2}.$$

The transformation is chosen so that

$$\sigma x \frac{\partial u}{\partial x} = v$$

for some constant v.[9] The probabilities in (3.10) then become

$$p_{i,i-1} = -q\frac{\Delta t}{2\Delta u} + \tfrac{1}{2}v^2\frac{\Delta t}{\Delta u^2}, \tag{3.11a}$$

$$p_{i,i} = 1 - v^2\frac{\Delta t}{\Delta u^2}, \tag{3.11b}$$

$$p_{i,i+1} = +q\frac{\Delta t}{2\Delta u} + \tfrac{1}{2}v^2\frac{\Delta t}{\Delta u^2}. \tag{3.11c}$$

To ensure convergence it is sufficient that $p_{i,i-1}$, $p_{i,i}$, and $p_{i,i+1}$ be positive as Δt and $\Delta u \to 0$. It follows from (3.11a), (3.11b), and (3.11c) that

$$v^2\frac{\Delta t}{\Delta u^2} < 1 \tag{3.12}$$

and

$$q < \frac{v^2}{\Delta u} \tag{3.13}$$

as Δt, $\Delta u \to 0$. If q is bounded, convergence is ensured as long as the ratio $\Delta t/\Delta u$ is kept constant and less than $1/v^2$.[10] Otherwise, it is possible to adjust the branching process as described by Hull and White [151].

3.2.2.3.1.2 Black–Scholes

For reasons of simplicity, consider the case analogous to a non-dividend paying stock discussed by Brennan and Schwartz [54], where $\alpha - \eta = r$. Equation (3.5) becomes the Black–Scholes equation

$$\frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}rx + \tfrac{1}{2}x^2\sigma^2\frac{\partial^2 f}{\partial x^2} = rf. \tag{3.14}$$

---

[9] Wilmott et al [337] take this idea further by transforming the Black–Scholes PDE into the *diffusion* PDE. This alternative approach is not discussed in more detail here, although it is frequently used for pricing financial options.

[10] More generally speaking, Lax's *equivalence theorem* states that "[g]iven a properly posed linear initial-value problem and an FD approximation to it that is consistent, stability is the necessary and sufficient condition for convergence." [188, 283] The scheme presented is clearly consistent, as the FDs converge to the respective derivatives as $\Delta u \to 0$ and $\Delta t \to 0$. Stability follows from a von Neumann analysis as described for example by Press et al [277] and Vesely [332]. The discussion of convergence is limited to heuristic arguments in the following sections.

Define $u \equiv \ln x$ and $g(u,t) \equiv f(x,t)$, so that

$$\begin{aligned}
\frac{\partial f}{\partial x} &= \frac{\partial g}{\partial u}\mathrm{e}^{-u}, \\
\frac{\partial^2 f}{\partial x^2} &= \left(\frac{\partial^2 g}{\partial u^2} - \frac{\partial g}{\partial u}\right)\mathrm{e}^{-2u}, \\
\frac{\partial f}{\partial t} &= \frac{\partial g}{\partial t}.
\end{aligned}$$

Making the appropriate substitutions in (3.14) gives the transformed equation

$$\frac{\partial g}{\partial t} + \frac{\partial g}{\partial u}\left(r - \tfrac{1}{2}\sigma^2\right) + \tfrac{1}{2}\sigma^2 \frac{\partial^2 g}{\partial u^2} = rg.$$

with constant coefficients. Analogous to $f_{i,j}$ denote $g(u_i, t_j)$ by $g_{i,j}$. From

$$\mathrm{d}u = \left(r - \tfrac{1}{2}\sigma^2\right)\mathrm{d}t + \sigma\,\mathrm{d}W \tag{3.16}$$

it can be seen that $q = r - \frac{1}{2}\sigma^2$ and $v = \sigma$. Consequently, (3.10) becomes

$$f_{i,j} = g_{i,j} = \frac{1}{r + \Delta t}\left(p_{i,i-1}g_{i-1,j+1} + p_{i,i}g_{i,j+1} + p_{i,i+1}g_{i+1,j+1}\right),$$

where

$$\begin{aligned}
p_{i,i-1} &= \left(\tfrac{1}{2}\left(\frac{\sigma}{\Delta u}\right)^2 - \tfrac{1}{2}\frac{r - \frac{1}{2}\sigma^2}{\Delta u}\right)\Delta t, \\
p_{i,i} &= 1 - \left(\frac{\sigma}{\Delta u}\right)^2 \Delta t, \\
p_{i,i+1} &= \left(\tfrac{1}{2}\left(\frac{\sigma}{\Delta u}\right)^2 + \tfrac{1}{2}\frac{r - \frac{1}{2}\sigma^2}{\Delta u}\right)\Delta t.
\end{aligned}$$

The probabilities stay constant throughout the grid, as they no longer depend on the value of x. Since q is clearly bounded, there is no need to modify the branching process. Substituting for q and v in (3.12) and (3.13) leads to

$$\begin{aligned}
\sigma^2 \frac{\Delta t}{\Delta u^2} &< 1, \\
r - \tfrac{1}{2}\sigma^2 &< \frac{\sigma^2}{\Delta u},
\end{aligned}$$

which implies that

$$\Delta t < \frac{\sigma^2}{\left(r - \frac{1}{2}\sigma^2\right)^2}, \tag{3.18}$$
$$\Delta u < \frac{\sigma^2}{\left|r - \frac{1}{2}\sigma^2\right|}.$$

By looking at

$$\begin{aligned}\mathbf{E}[\Delta u] &= \Delta u \left(p_{i,i+1} - p_{i,i-1}\right) \\ &= \left(r - \tfrac{1}{2}\sigma^2\right) \Delta u\end{aligned}$$

and

$$\begin{aligned}\mathbf{Var}[\Delta u] &= \Delta u^2 \left(p_{i,i+1} - p_{i,i-1}\right) - \left(\mathbf{E}[\Delta u]\right)^2 \\ &= \sigma^2 \Delta t - \left(r - \tfrac{1}{2}\sigma^2\right)^2 \Delta t^2\end{aligned}$$

one can verify that the diffusion limit of the jump process indeed corresponds to (3.16) and the diffusion limit of Δx is (3.4). The approximation suffers from a downward bias of size $\left(r - \frac{1}{2}\sigma^2\right) \Delta t$. Given (3.18), an upper bound for this bias is σ^4.

The explicit method is easily extended to deal with two-factor models [337]. Likely applications include, for instance, convertible bonds and long-term real investment opportunities, where both underlying and interest rate can be considered stochastic.

To facilitate comparison, the following sections focus on the Black–Scholes equation. The more general case discussed in the previous subsection is not examined further.

3.2.2.3.2 Fully Implicit Scheme

Due to several reasons to be addressed in detail later in this and the following section, the implicit scheme is often considered superior to the explicit scheme in option pricing. While tools and techniques evolve rapidly, it is also widely used in the literature. For example, the implicit method is used as a benchmark in a comparative study of numerical procedures by Trigeorgis [327]. Furthermore, Barone-Adesi and Whaley [25] describe the implicit FD scheme as being the most accurate.[11]

Compared to the explicit scheme, the implicit scheme uses the approximations

[11] For a more careful weighing of the particular advantages and disadvantages of various approximation techniques see also Geske and Shastri [116].

$$\frac{\partial g}{\partial u} \approx \frac{g_{i+1,j} - g_{i-1,j}}{2\Delta u},$$
$$\frac{\partial^2 g}{\partial u^2} \approx \frac{g_{i+1,j} + g_{i-1,j} - 2g_{i,j}}{\Delta u^2},$$

and

$$\frac{\partial g}{\partial t} \approx \frac{g_{i,j+1} - g_{i,j}}{\Delta t}.$$

Derivatives at the current node are approximated by looking at nodes that precede it in time. This scheme has the distinct advantage that it is unconditionally stable without further adjustments. As it couples the $g_{i,j}$ for various i and requires matrix inversion, it is also somewhat more difficult to implement.

Substitute to obtain an alternative FD representation of (3.14):

$$\frac{g_{i,j+1} - g_{i,j}}{\Delta t} + \frac{g_{i+1,j} - g_{i-1,j}}{2\Delta u}\left(r - \frac{\sigma^2}{2}\right) + \tfrac{1}{2}\sigma^2 \frac{g_{i+1,j} + g_{i-1,j} - 2g_{i,j}}{\Delta u^2} = r g_{i,j+1}.$$

Solving for $g_{i,j+1}(1 - r\Delta t)$ and collecting terms yields

$$a g_{i-1,j} + b g_{i,j} + c g_{i+1,j} = g_{i,j+1}(1 - r\Delta t),$$

where $i \in \{1, \ldots, m-1\}$, $j \in \{0, \ldots, n-1\}$, and

$$a = \left(-\tfrac{1}{2}\left(\frac{\sigma}{\Delta u}\right)^2 + \tfrac{1}{2}\frac{r - \frac{1}{2}\sigma^2}{\Delta u}\right)\Delta t,$$
$$b = 1 + \left(\frac{\sigma}{\Delta u}\right)^2 \Delta t,$$
$$c = \left(-\tfrac{1}{2}\left(\frac{\sigma}{\Delta u}\right)^2 - \tfrac{1}{2}\frac{r - \frac{1}{2}\sigma^2}{\Delta u}\right)\Delta t.$$

Note that, since the coefficients do not depend on the value of u, subscripts may be dropped.

The boundary conditions for the put option are

$$g_{0,j} = K,$$
$$g_{m,j} = 0 \quad \text{for } j \in \{0, 1, \ldots, n-1\},$$

and

$$g_{i,n} = \max\{0, K - \mathrm{e}^{u_i}\} \quad \text{for } i \in \{0, 1, \ldots, m\}, \tag{3.21a}$$

where K is the strike price. Equation (3.21a) corresponds to the *inner value,* which, at maturity, is equal to the total value of the option. The interior points and one boundary, the value of $g_{i,0}$, are left to be determined. Again going backward in time, it is possible to solve the following matrix equation for all $i \in \{1, \ldots, m-1\}$:

$$\begin{pmatrix} b\ c\ 0 \ldots \\ a\ b\ c\ 0\ . \\ \ldots\ldots\ldots \\ \ldots 0\ a\ b \end{pmatrix} \cdot \begin{pmatrix} g_{1,j} \\ g_{2,j} \\ \cdots \\ g_{m-1,j} \end{pmatrix} + \begin{pmatrix} a g_{0,j} \\ 0 \\ \cdots \\ c g_{m,j} \end{pmatrix} = \begin{pmatrix} g_{1,j+1} \\ g_{2,j+1} \\ \cdots \\ g_{m-1,j+1} \end{pmatrix} (1 - r\Delta t)\,.$$

Other types of boundary conditions—for more complex derivatives—could be absorbed into the matrix equation in a similar manner.

As the coefficient matrix is independent of j, it suffices to invert it once. For each time step, the current price vector is then to be multiplied by this constant matrix inverse, eventually giving the value of the derivative at t_0.

The solution of such *tridiagonal* systems of equations using *LU* decomposition is also straightforward and described by Press et al [277, sect. 2.4] and others. An alternative method, the more generally applicable and widely used successive overrelaxation (SOR) algorithm, is discussed in the following section [81]. Section 3.2.2.5 briefly hints at some of the more advanced techniques not addressed in this text.

As shown by Brennan and Schwartz [54], the elements of the coefficient matrix can again be regarded as probabilities discounted at the risk-free rate. In contrast to the previously discussed explicit scheme, the underlying asset price may jump to an infinite number of possible future values instead of only three.

The implicit method is biased upward by the square of the expected size of these jumps, a shortcoming overcome by the improved scheme first proposed by Courtadon [73] and briefly summarized in the following section.

3.2.2.3.3 Crank–Nicolson Scheme

A closer look at the fully implicit method shows it to be only first-order accurate in time. In contrast, the Crank–Nicolson scheme, obtained by forming the average of the implicit and explicit schemes, is also unconditionally stable, but second-order accurate in both space and time [79]. In the following a basic Crank–Nicolson model for the plain-vanilla put option is presented.

Consider the slightly different transformation of the Black–Scholes equation chosen by Courtadon [73], where $\tau \equiv T - t$ and $h(x,\tau) \equiv \mathrm{e}^{r\tau} f(x,\tau)$. This transformation makes the procedure somewhat more intuitive, since the known payoffs at maturity become genuine *initial* conditions. The resulting PDE is

$$\tfrac{1}{2}\sigma^2 x^2 \frac{\partial^2 h}{\partial x^2} + rx\frac{\partial h}{\partial x} - \frac{\partial h}{\partial \tau} = 0. \tag{3.22}$$

Taking the average of the implicit and the explicit schemes leads to the following FD approximation:

$$\frac{\partial h}{\partial x} \approx \tfrac{1}{2}\left(\frac{h_{i+1,j+1} - h_{i-1,j+1}}{2\Delta x} + \frac{h_{i+1,j} - h_{i-1,j}}{2\Delta x}\right), \tag{3.23a}$$

$$\begin{aligned}\frac{\partial^2 h}{\partial x^2} \approx \tfrac{1}{2}\Bigg(&\frac{h_{i+1,j+1} + h_{i-1,j+1} - 2h_{i,j+1}}{\Delta x^2} \\ &+\frac{h_{i+1,j} + h_{i-1,j} - 2h_{i,j}}{\Delta x^2}\Bigg),\end{aligned} \tag{3.23b}$$

and

$$\frac{\partial h}{\partial t} \approx \frac{h_{i,j+1} - h_{i,j}}{\Delta \tau}. \tag{3.23c}$$

It is also possible calculate a *weighted* average, where the weights are chosen to reduce the local truncation error.[12]

Substituting (3.23a), (3.23b), and (3.23c) in (3.22) yields

$$\begin{aligned}&\tfrac{1}{2}\sigma^2 x_i^2 \left(\frac{h_{i+1,j} - 2h_{i,j} + h_{i-1,j} + h_{i+1,j+1} - 2h_{i,j+1} + h_{i-1,j+1}}{2\Delta x^2}\right) \\ &+ rx_i\left(\frac{h_{i+1,j} - h_{i-1,j} + h_{i+1,j+1} - h_{i-1,j+1}}{4\Delta x}\right) - \frac{h_{i,j+1} - h_{i,j}}{\Delta \tau} = 0.\end{aligned}$$

Collect terms and rearrange, so that all points at timestep $j+1$ appear on the left-hand side and all points at timestep j on the right-hand side:

$$\begin{aligned}a_i h_{i-1,j+1} + (1+b_i)\, h_{i,j+1} + c_i h_{i+1,j+1} = \\ - a_i h_{i-1,j} + (1-b_i)\, h_{i,j} - c_i h_{i+1,j},\end{aligned} \tag{3.24}$$

[12] This approach is known as the "Douglas scheme." [337] A detailed discussion of three-time-level Douglas FD schemes for American options is provided by Shaw [310], who also presents suitable SOR and PSOR solvers. Three-time-level schemes offer advantages when the initial data are discontinuous. A two-time-level scheme of the same accuracy is required to get the procedure started.

where

$$
\begin{aligned}
a_i &= -\tfrac{1}{4}\left(x_i^2\sigma^2 v_1 - r x_i v_2\right),\\
b_i &= \tfrac{1}{2}x_i^2\sigma^2 v_1,\\
c_i &= -\tfrac{1}{4}\left(x_i^2\sigma^2 v_1 + r x_i v_2\right),
\end{aligned}
$$

and $v_1 = \Delta\tau/\Delta x^2$, $v_2 = \Delta\tau/\Delta x$. With x_i replaced by its FD equivalent, these coefficients reduce to those given by Courtadon [73]:

$$
\begin{aligned}
a_i &= -\tfrac{1}{4}i\Delta\tau\left(i\sigma^2 - r\right),\\
b_i &= \tfrac{1}{2}i^2\Delta\tau\sigma^2,\\
c_i &= -\tfrac{1}{4}i\Delta\tau\left(i\sigma^2 + r\right).
\end{aligned}
$$

The boundary conditions for the put option are

$$h_{0,j+1} = \mathrm{e}^{r(j+1)\Delta\tau}K, \tag{3.27a}$$

$$h_{m,j+1} = 0 \quad \text{for } j \in \{0,1,\ldots,n-1\}, \tag{3.27b}$$

and

$$h_{i,0} = \max\{0, K - i\Delta x\} \quad \text{for } i \in \{0,1,\ldots,m\}.$$

Equation (3.24) with boundary conditions (3.27a) and (3.27b) incorporated can be written in matrix form as

$$
\begin{pmatrix}
1+b_1 & c_1 & 0 & \ldots\ldots\ldots\ldots & \\
a_2 & 1+b_2 & c_2 & 0 & \ldots\ldots\ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
\ldots\ldots\ldots\ldots & 0 & a_{m-1} & 1+b_{m-1} &
\end{pmatrix}
\cdot
\begin{pmatrix}
h_{1,j+1}\\ h_{2,j+1}\\ \ldots\\ h_{m-1,j+1}
\end{pmatrix}
+
\begin{pmatrix}
a_1\mathrm{e}^{r(j+1)\Delta\tau}K\\ 0\\ \ldots\\ 0
\end{pmatrix}
$$

$$
=
\begin{pmatrix}
-a_1 & 1-b_1 & -c_1 & 0 & \ldots\ldots\ldots\ldots & \\
0 & -a_2 & 1-b_2 & -c_2 & 0 & \ldots\ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\ldots\ldots\ldots & 0 & -a_{m-1} & 1-b_{m-1} & -c_{m-1} &
\end{pmatrix}
\cdot
\begin{pmatrix}
h_{0,j}\\ h_{1,j}\\ \ldots\\ h_{m,j}
\end{pmatrix},
$$

or, more concisely,

$$\mathbf{A}_{j+1}\cdot\mathbf{h}_{j+1} = \mathbf{A}_j\cdot\mathbf{h}_j - \mathbf{r}_{j+1}.$$

Defining $\mathbf{b} \equiv \mathbf{A}_j\cdot\mathbf{h}_j - \mathbf{r}_{j+1}$, $\mathbf{x} \equiv \mathbf{h}_{j+1}$, and dropping the remaining subscript leads to the generic matrix equation

$$\mathbf{A}\cdot\mathbf{x} = \mathbf{b}. \tag{3.28}$$

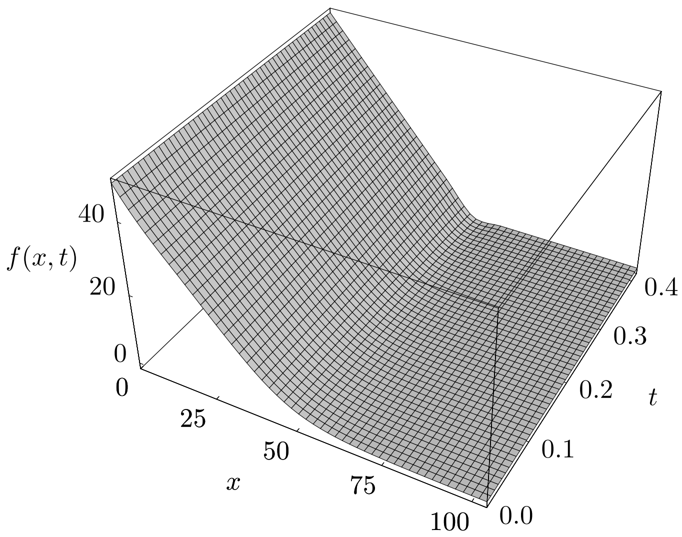


Fig. 3.6. Value of a put option p as a function of asset price X_t and time t ($X_0 = 50.0$, $K = 50.0$, $T = 5/12$, $\sigma = 0.3$, $r = 0.1$). The Black–Scholes PDE was discretized on a 500-by-500 grid using Courtadon's Crank–Nicolson scheme.

All "known" information is included in $\mathbf{b}$. Note that most *boundary* value problems reduce to solving large sparse linear systems of this form.

Though, in principle, (3.28) could be solved directly by calculating $\mathbf{A}^{-1}$, this task can be fairly time-consuming. Consequently, iterative approaches are sometimes preferable. Three such approaches are—in order of increasing sophistication—the Jacobi, the Gauss–Seidel, and the SOR methods.

All three methods can be expressed in terms of the matrix splitting concept [277]. Split $\mathbf{A}$ into a sub-diagonal, a diagonal, and a super-diagonal part:

$$\mathbf{A} = \mathbf{L} + \mathbf{D} + \mathbf{U}.$$

The r^{th} step of the Jacobi iteration is

$$\mathbf{D} \cdot \mathbf{x}^{(r)} = -\left(\mathbf{L} + \mathbf{U}\right) \cdot \mathbf{x}^{(r-1)} + \mathbf{b}.$$

The Gauss–Seidel procedure makes use of updated values as soon as they become available:

$$\left(\mathbf{L} + \mathbf{D}\right) \cdot \mathbf{x}^{(r)} = -\mathbf{U} \cdot \mathbf{x}^{(r-1)} + \mathbf{b}. \tag{3.29}$$

This equation can be written as

$$\mathbf{x}^{(r)} = \mathbf{x}^{(r-1)} - (\mathbf{L} + \mathbf{D})^{-1} \cdot \left((\mathbf{L} + \mathbf{D} + \mathbf{U}) \cdot \mathbf{x}^{(r-1)} - \mathbf{b} \right),$$

where the term in parentheses is the *residual vector* $\xi^{(r-1)}$. Overcorrecting by ω leads to the SOR algorithm

$$\mathbf{x}^{(r)} = \mathbf{x}^{(r-1)} - \omega\,(\mathbf{L} + \mathbf{D})^{-1} \cdot \xi^{(r-1)}.$$

Generally, SOR converges for linear systems arising from finite differencing only if $\omega \in (0, 2)$ [162]. The optimal value for the over-relaxation parameter ω depends on the specific problem examined and is best determined by trial and error.[13]

Written out in components, set

$$x_i^{(r)} = \omega \bar{x}_i^{(r)} + (1 - \omega)\, x_i^{(r-1)}$$

and solve the scalar equation

$$\bar{x}_i^{(r)} = \left(b_i - \sum_{j=1}^{i-1} A_{ij} x_j^{(r)} - \sum_{j=i+1}^{m-1} A_{ij} x_j^{(r-1)} \right) \frac{1}{A_{ii}}. \tag{3.30}$$

For $\omega = 1$, the formula reduces to the Gauss–Seidel algorithm, where $x_i^{(r)} = \bar{x}_i^{(r)}$. This identity can be verified by rewriting (3.29) as

$$\mathbf{x}^{(r)} = \mathbf{D}^{-1} \cdot \left(\mathbf{b} - \mathbf{L} \cdot \mathbf{x}^{(r)} - \mathbf{U} \cdot \mathbf{x}^{(r-1)} \right).$$

Figure 3.7 shows how the SOR algorithm can be implemented in practice. One way to speed up convergence is odd-even ordering with Chebyshev acceleration. The reader is referred to Press et al [277, sect. 19.5] for further details.

Figure 3.8 illustrates how the results converge towards the exact solution as the number of time and space steps increases. Option values obtained using common interpolation techniques are very close to those suggested by the Black–Scholes formula.

[13] As far as the simple model problem is concerned, a suitable value for ω lies in the interval $(1.5, 1.8)$.

Choose an initial guess $x^{(0)}$ to the solution x.
for $r \leftarrow 1, \ldots$ **do**
    for $i \leftarrow 1, m-1$ **do**
        $\bar{x}_i^{(r)} \leftarrow 0$
        for $j \leftarrow 1, i-1$ **do**
            $\bar{x}_i^{(r)} \leftarrow \bar{x}_i^{(r)} + a_{i,j} x_j^{(r)}$
        end for
        for $j \leftarrow i+1, m-1$ **do**
            $\bar{x}_i^{(r)} \leftarrow \bar{x}_i^{(r)} + a_{i,j} x_j^{(r-1)}$
        end for
        $\bar{x}_i^{(r)} \leftarrow \left(b_i - \bar{x}_i^{(r)}\right) / a_{i,i}$
        $x_i^{(r)} \leftarrow x_i^{(r-1)} + \omega \left(\bar{x}_i^{(r)} - x_i^{(r-1)}\right)$
    end for
    Check convergence; continue if necessary.
end for

Fig. 3.7. Successive overrelaxation pseudo code. The algorithm shown here is suitable for any type of coefficient matrix. If the matrix is tridiagonal—as it is generally the case in option pricing—the number of operations and thus the computing time can be reduced substantially if the summation is limited to non-zero coefficients thereby eliminating the inner loops. One possible convergence criterion is the ℓ_2-norm of the error vector (Source: figure adapted from Barrett et al [26, p. 11]).

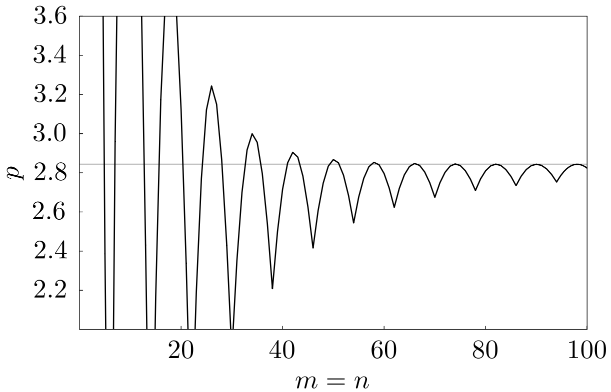


Fig. 3.8. Convergence of the Crank–Nicolson model ($X_0 = 50.0$, $K = 50.0$, $T = 5/12$, $\sigma = 0.3$, $r = 0.1$). The horizontal line indicates the Black–Scholes option value ($p = 2.84458$). While the approximation becomes acceptable comparatively quickly, a fine grid is required to obtain accurate solutions.

3.2.2.4 Early Exercise

So far, the discussion has been limited to the European put option. While European options may only be exercised at maturity, American options may be exercised early, that is prior to maturity. Real options are typically American options.

Although the SOR procedure is slightly slower than **LU** decomposition for European options, it is very easy to incorporate an early exercise feature and, at the same time, preserve the accuracy of the Crank–Nicolson scheme. The projected successive overrelaxation (PSOR) algorithm is

$$x_i^{(r)} = \max\left\{x_i^{(r-1)} + \omega\left(\bar{x}_i^{(r)} - x_i^{(r-1)}\right), \text{inner value}\right\},$$

where $\bar{x}_i^{(r)}$ is defined as in (3.30) and "inner value" reflects the specific claim to be analyzed. A much simpler, albeit less accurate, possibility is to solve for $x^{(r)}$ and then post-process the result after each iteration to account for early exercise.

The so-called free boundary is obtained as a by-product of this procedure. Only in the *continuation region,* where the price of the asset lies above a critical value, the option has a positive time value. In the *stopping region* immediate exercise is optimal: the value of an option is equal to its inner value. Correspondingly, the model problem is an optimal stopping problem, the most basic case of stochastic optimal control. Figure 3.9 depicts the free boundary resulting from an application of the Crank–Nicolson FD scheme. Explicit approximations to the early exercise boundary are proposed by Carr [58], Bunch and Johnson [56], and others.

Not seldomly, pricing American options also involves a reformulation of the optimization problem. A very popular approach is to treat the valuation of an American put option as a *linear complementarity problem* and solve the resulting system of partial differential inequalities using the procedures described above [40, 337]. From a non-technical point of view, such models offer little additional value for the problem at hand and are thus neglected.

3.2.2.5 Advanced Methods

Until the 1970s, SOR was the standard algorithm for solving matrix equations like the ones arising in option pricing. It is very easy to program, but inefficient on large problems. Press et al [277] therefore recommend SOR for problems where ease of programming outweighs expense

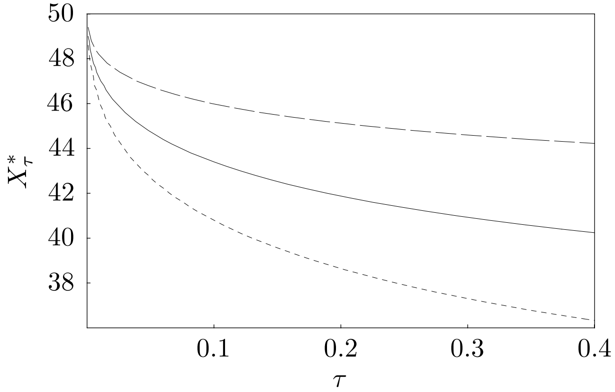


Fig. 3.9. Approximate free boundary for the American put ($X_0 = 50.0$, $K = 50.0$, $T = 5/12$, $\sigma \in \{0.2, 0.3, 0.4\}$, $r = 0.1$). Critical values X^*_τ were obtained using Crank–Nicolson FDs on a 500-by-500 grid. The inner value required to trigger (early) exercise decreases as the expiry date approaches and becomes zero at maturity. Increasing volatility is associated with lower values of X^*_τ (dotted line), while decreasing volatility moves the boundary upwards (dashed line).

of computer time. Real options models usually require solving a large problem only once, so that efficiency is rarely an issue. In addition, grids like the one shown in fig. 3.6 can be computed in fractions of a second, even using high-level languages like Mathematica. Nevertheless, it is worth pointing out that there are more advanced—and sometimes more efficient—methods available.

For example, multigrid methods, among them the projected full approximation storage (PFAS) method for American options [50], are extremely efficient for a wide variety of problems, but also more difficult to implement. There is no general-purpose solver. Rather, the components of the algorithm have to be adjusted to solve a specific problem, for example the Black–Scholes equation.[14] In more detail, the concepts are presented by Bramble [49] and Trottenberg et al [329].

As far as two-factor models are concerned, there are quite a few (partly) implicit methods to choose from, many of which are based on some type of alternating-direction implicit (ADI) scheme. Combining explicit and implicit differences, the typical ADI algorithm shows performance superior to that of a Crank–Nicolson discretization. Consequently, the ADI method is widely used for pricing interest rate and

[14] For a general description of the multigrid framework see Press et al [277, sect. 19.6].

foreign exchange products. At least to the author's knowledge, there are no examples of real option applications.

In summary, FD pricing and related methods are powerful, but depend on the existence of a pricing PDE. Often, the mathematical formalism is far from intuitive and—with the exception of the fully explicit scheme—seems only remotely related to the original economic problem.

Observation 3.1. FD pricing is based on a discretization of the pricing PDE, rendering it less intuitive and comparatively difficult to adapt to problems substantially different from conventional option contracts.

Models presented in part II will make extensive use of the fully explicit and Crank–Nicolson schemes to analyze optionalities in patenting.

What follows is an examination of alternative approaches, with particular attention to simulation models for option pricing. Unlike FD procedures, they do not require a pricing PDE. In principle, it suffices to discretize the underlying process and specify a payoff function. As a consequence, adapting simulation models to more advanced problems is usually not difficult.

3.2.3 Monte Carlo Simulation

This section serves to present Monte Carlo simulation as a slightly more intuitive means of pricing derivatives. Also, some techniques for improving accuracy and efficiency are briefly hinted at. In addition, extant algorithms for valuing American options are discussed, before the analysis proceeds to novel concepts which are the subject of further discussion in later chapters.

3.2.3.1 Itô–Taylor Expansion

In a first step, asset prices are simulated by numerically integrating the SDEs describing the value dynamics. The mathematical tool frequently used in obtaining numerical integration schemes for stochastic diffusion processes is Itô–Taylor expansion [172]. More recently, similar methods have also been applied to Poisson jump-diffusion processes [297].

Take again the autonomous SDE (3.4), restated here for convenience:

$$\mathrm{d}X_t = a(X_t)\,\mathrm{d}t + b(X_t)\,\mathrm{d}W_t.$$

Recall that it describes how an asset's price X_t evolves over time. As before, assume that $f(X_t)$ denotes an investment opportunity contingent on this asset. By Itô's Lemma

$$\mathrm{d}f(X_t) = \mathcal{L}^0 f(X_t)\,\mathrm{d}t + \mathcal{L}^1 f(X_t)\,\mathrm{d}W_t, \tag{3.31}$$

where $\mathcal{L}^0$ and $\mathcal{L}^1$ are differential operators defined as

$$\begin{aligned}
\mathcal{L}^0 f(X_t) &= a(X_t)\frac{\partial f(X_t)}{\partial X_t} + \tfrac{1}{2}b(X_t)^2\frac{\partial^2 f(X_t)}{\partial X_t^2},\\
\mathcal{L}^1 f(X_t) &= b(X_t)\frac{\partial f(X_t)}{\partial X_t}.
\end{aligned}$$

Assuming $t_0 = 0$, (3.31) is equivalent to

$$f(X_t) = f(X_0) + \int_0^t \mathcal{L}^0 f(X_s)\,\mathrm{d}s + \int_0^t \mathcal{L}^1 f(X_s)\,\mathrm{d}W_s. \tag{3.33}$$

Setting $f(X_t) \equiv X_t$ gives

$$X_t = X_0 + \int_0^t a(X_s)\,\mathrm{d}s + \int_0^t b(X_s)\,\mathrm{d}W_s.$$

Analogously to deterministic Taylor expansion, substitute $f(X_t) = a(X_t)$ and $f(X_t) = b(X_t)$ into (3.33) and get

$$\begin{aligned}
X_t = X_0 &+ \int_0^t \left(a(X_0) + \int_0^s \mathcal{L}^0 a(X_u)\,\mathrm{d}u + \int_0^s \mathcal{L}^1 a(X_u)\,\mathrm{d}W_u \right) \mathrm{d}s\\
&+ \int_0^t \left(b(X_0) + \int_0^s \mathcal{L}^0 b(X_u)\,\mathrm{d}u + \int_0^s \mathcal{L}^1 b(X_u)\,\mathrm{d}W_u \right) \mathrm{d}W_s.
\end{aligned}$$

Rearranging leads to the approximate solution

$$X_t = X_0 + a(X_0)\int_0^t \mathrm{d}s + b(X_0)\int_0^t \mathrm{d}W_s + R.$$

The residual term R contains the double integrals. Neglecting R, one obtains the most basic Monte Carlo scheme known as Euler–Maruyama discretization [224]:

$$X_{j+1} \approx X_j + a(X_j)\Delta t + b(X_j)\Delta W_j.$$

The discrete equivalent of the Wiener increment $\mathrm{d}W_t$ is $\Delta W_j \equiv \sqrt{\Delta t}\,\varepsilon_j$, where ε_j is a random number drawn from a standard normal distribution. Figure 3.10 illustrates how asset values evolve over time.

While the deterministic variant of the Euler method converges with order 1, the Euler–Maruyama scheme for SDEs converges only with order 1/2. This issue is remedied by including additional terms otherwise

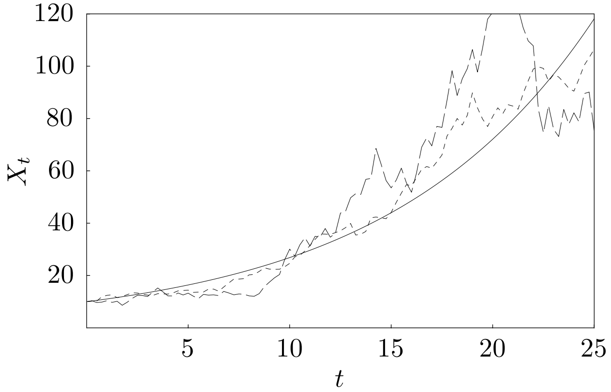


Fig. 3.10. Geometric Brownian motion sample paths. The dotted and dashed lines represent realizations of the value process for $X_0 = 10$, $\alpha = 0.1$, $\sigma \in \{0.1, 0.2\}$, and $T = 25$. The solid line illustrates the value process under certainty, that is for $\sigma = 0.0$.

contained in R. Specifically, within the well-known scheme due to Milstein [243], the additional terms arising from Itô–Taylor expansion are corrected for by setting

$$X_t = X_0 + a(X_0)\int_0^t \mathrm{d}s + b(X_0)\int_0^t \mathrm{d}W_s \\ + b(X_0)b'(X_0)\int_0^t\int_0^s \mathrm{d}W_u\,\mathrm{d}W_s + \widetilde{R}.$$

Given that

$$\int_0^t\int_0^s \mathrm{d}W_u\,\mathrm{d}W_s = \tfrac{1}{2}\Delta W_t^2 - \tfrac{1}{2}\Delta t,$$

one obtains

$$X_{j+1} \approx X_j + a(X_j)\Delta t + b(X_j)\Delta W_j + \tfrac{1}{2}b(X_j)b'(X_j)\left(\Delta W_j^2 - \Delta t\right)$$

as the Milstein scheme.

Maghsoodi [211] generalizes the Euler and Milstein schemes to scalar jump-diffusion SDEs. The vectorial case is examined by Cyganowski et al [82]. Methods of higher order are of primary theoretical value, but play an important role as a starting point for Runge–Kutta-type schemes [57].[15] Although, in principal, such schemes are not difficult to

[15] For implementional issues in the simulation of SDEs see Cyganowski et al [82], Higham [143], Higham and Kloeden [144]. For additional references see Milstein [244], Platen [271].

implement, they and even the Milstein discretization are rarely found in economic applications. In many cases, the simple Euler scheme is sufficiently accurate. After all, far more important than the scheme is the choice of asset price model on which the discretization is based.

Once the paths have been simulated, it is often straightforward to value the derivative. In a second step, option payoffs are determined for each realization of the stochastic process. A fair price for the claim is given by the average present value of payoffs. Commonly, the expectation is taken under an equivalent risk-neutral, or martingale, measure, that is the risk premium η is deducted from the drift α and the risk-free rate is used for discounting [137, 138]. For a non-dividend-paying asset, it thus suffices to simulate using the adjusted drift $\alpha^* = r$.

Reliable results require a large number of simulation runs, generally at least several thousand. For obvious reasons, a particular strength of Monte Carlo simulation lies in the valuation of path-dependent derivatives.

3.2.3.2 Variance Reduction

There are several techniques commonly used to improve convergence: antithetic variates, control variates, moment matching, importance sampling, stratified sampling, and the use of quasi-random sequences [47]. Some of these techniques have also been successfully applied to the valuation of American options discussed in the following subsection [324].[16]

Variance reduction is probably the single most important aspect in Monte Carlo simulation, since the computational effort of this method can be considerable. It is less of an issue with rainbow options, because Monte Carlo simulation scales almost linearly as the number of underlying instruments increases. Consequently, Monte Carlo methods are ideal for very complex derivatives such as realistic models of investment under uncertainty.

3.2.3.3 Early Exercise

Until recently, Monte Carlo methods were by and large limited to European options, with only few exemptions from this rule [47, 325]. As a result, almost exclusively FD models were used to account for early exercise features. A novel and comparatively simple approach developed by

[16] Another often-used technique which increases the efficiency of Monte Carlo simulation is the Brownian bridge. For an application to jump-diffusion processes see Metwally and Atiya [240].

Carrière [60], Tsitsiklis and Van Roy [330], and Longstaff and Schwartz [207] makes Monte Carlo simulation at least equally attractive for the valuation of real options.[17]

3.2.3.3.1 Least-Squares Monte Carlo

For each period, the continuation value is approximated by a linear combination of orthogonal polynomials [295] and estimated using a cross-sectional least squares regression. If, at any point in time, the inner value exceeds this conditional expected continuation value, the option is exercised. Using polynomials in several variables, the method is also applicable to multi-factor pricing problems. LSM simulation is the method of choice for a wide variety of alternative stochastic processes such as a jump-diffusion models.[18]

More formally, assume a complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$ with an equivalent martingale measure $\mathbf{P}^*$, where Ω is the set of all possible realizations of the underlying stochastic process and $\{\mathcal{F}_t\}_{t\geq 0}$ is the standard Brownian filtration.[19] Let $\omega \in \Omega$ represent a sample path. Let $\Pi(\omega, s; t, T)$ denote the path of cash flows generated by the option, conditional on the option not being exercised at or prior to time t and on the optimal stopping strategy being followed for all $s \in (t, T]$. The expected continuation value under the risk-neutral measure, conditional on the information known at time t_j, is

$$F(\omega; t_j) = \mathbf{E}_{\mathbf{P}^*}\left[\sum_{k=j+1}^{n} \mathrm{e}^{-r(t_k - t_j)} \Pi(\omega, t_k; t_j, t_n) \,\middle|\, \mathcal{F}_{t_j}\right],$$

where $\mathbf{E}_{\mathbf{P}^*}[\cdot]$ is the expectation operator under the martingale measure and $0 \leq t_j < t_n = T$.

Define an approximation

$$\widehat{F}_M(\omega; t_j) = \sum_{k=0}^{M} a_k L_k(X_{t_j}),$$

[17] For a detailed assessment of the least-squares Monte Carlo (LSM) approach see Moreno and Navas [249].

[18] For example, Longstaff and Schwartz [207] illustrate LSM by analyzing the simple jump-to-ruin model first proposed by Merton [239].

[19] This formalism seems slightly out of place at this point, but greatly facilitates notation. The significance of risk-neutral valuation will become clearer in following chapters. For a discussion of the mathematical foundation for probability theory laid by Kolmogorov in the early 1930s see Steele [314, chap. 4].

where L_k is the Laguerre polynomial of k^{th} degree, a_k are coefficients to be determined through regression analysis for each t_j, and X_{t_j} is the value of the underlying asset.

It is possible to show that the approximation indeed converges for very large M. In practice, however, M can be comparatively small. The weighted Laguerre polynomials can be replaced by any linear combination of orthonormal basis functions such as Hermite, Legendre, or Jacobi polynomials. Starting at time $t_n \equiv T$ the option value is then determined by rolling backward and making exercise decision based on the estimated continuation value $\widehat{F}_M(\omega; t_j)$.[20]

Although, in principal, the LSM procedure is applicable to option games, examples in the literature are rare [245]. Given a wide variety of simulation techniques conceivable, new methods for the are likely to appear in the future.[21]

Simulation will be employed to analyze some of the more advanced problems presented in part II, in particular complicated multi-factor models or models involving alternative stochastic processes, which are difficult to solve using conventional FDs.

3.2.3.3.2 Evolutionary Monte Carlo

This subsection focuses on a particular type of Monte Carlo simulation—in the following referred to as "evolutionary Monte Carlo (EMC)"—where the early exercise boundary (see sect. 3.2.2.4) is determined using evolutionary computation (EC), or, more specifically, GAs.[22] It is still in its infancy, but holds promise for a variety of current topics in real options resarch.

3.2.3.3.2.1 Evolutionary Computation and Evolutionary Game Theory

GAs are a computational tools for global optimization. Similar to simulated annealing, GAs represent techniques for searching large, poorly-understood spaces of possible solutions.[23] In option pricing, this search involves the identification of value-maximizing investment programs under uncertainty [65]. As already a comparatively crude approximation of

[20] For a convergence analysis see Stentoft [315].

[21] For a more recent overview see also Mußhoff et al [251].

[22] The discipline of evolutionary computation (EC) encompasses the three subdisciplines of genetic programming (GP), GAs, and evolutionary strategy (ES).

[23] Simulated annealing was originally conceived as a Monte Carlo method for examining the equations of state and frozen states of n-body systems [288]. By analogy, this approach can be applied to combinatorial problems [62, 169].

the free boundary provides option values with sufficient precision [161], various types of heuristics could be devised. In the realm of real investment under uncertainty, genetic algorithms were pioneered by Dias [86] and Balmann and Mußhoff [19].

Additional impetus comes from an increasing interest in game-theoretic option models [127]. Further development in this area has so far been hindered by a lack of pricing tools. It is therefore a distinct advantage of the EMC approach that it is readily extensible to strategic settings, although this application will not be discussed in detail.[24] More precisely, EMC is a suitable framework for numerically analyzing certain Markov perfect equilibria in differential stochastic (option) games [152].

Even in partial equilibrium analysis there exists a close connection between evolutionary Monte Carlo (EMC) and the relatively young economic subdiscipline of evolutionary game theory initially developed by Maynard and Price [228] and Maynard [227].[25] As argued by Riechmann [284], economic learning via genetic algorithms can be described as a type of evolutionary game. The author points out that GA learning results in a series of near-Nash equilibria approaching an evolutionary stable state. According to an informal definition of evolutionary stability given by Gintis "[a] strategy is *evolutionary stable* if a whole population using that strategy cannot be invaded by a small group with a mutant genotype." [117, p. 148, emphasis added] Obviously, when determining the optimal exercise policy for an option contract, only the policy that, on average, generates the highest payoff fulfills this requirement.[26]

3.2.3.3.2.2 Evolving an Optimal Investment Program

Figure 3.11 illustrates EMC simulation applied to some simple contingent claim, for example the model problem, the American put option. Without going into the details of GAs, the procedure is as follows [119, 176, 242].

Each individual in the population represents a specific investment program. The free boundary assumed by an individual (the phenotype)

[24] For a discussion of co-evolving investment strategies [15].

[25] For surveys of the literature see Mailath [213], Weibull [334].

[26] Relating evolutionary stable strategy (ESS) to the widely-known notion of Nash equilibria, it is important to stress that, while all ESS are Nash, not all Nash equilibria are necessarily evolutionary stable. An algorithm only determining ESS therefore fails to discover all Nash equilibria. These "non-obvious" Nash equilibria may, however, be very close to evolutionary stability in terms of replicator dynamics [299].

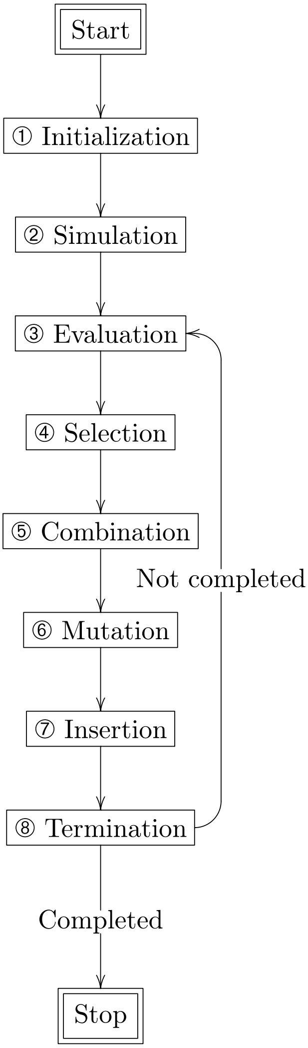


Fig. 3.11. Evolution of an optimal investment program. (1) *Start:* Specify parameters of objective funtion. (2) *Initialization:* Choose random anchor points for free boundary. (3) *Simulation:* Simulate random process. (4) *Evaluation:* Interpolate boundaries and determine option values. (5) *Selection:* Select boundaries resulting in highest option values. (6) *Combination:* Combine anchor points creating new boundaries. (7) *Mutation:* Randomly move individual anchor points modifying boundaries. (8) *Insertion:* Replace unsuitable boundaries. (9) *Termination:* Check if convergence criteria are fulfilled. (10) *Stop:* Present best boundary and resulting option value.

is described by a sequence of real numbers (the genotype) used to construct a smooth interpolation based on cubic splines or some other suitable technique. The critical values may be scaled to represent, say, a multiple of the strike price. Each individual in the population randomly chooses a boundary from the feasible set. Then, a large number of price processes is simulated.[27] Based on this information, each individual is then assigned an objective score reflecting the mean option value resulting from following her investment program. This score can be scaled leading to a relative fitness value.

The algorithm continues by selecting the individuals with the highest fitness scores, for instance scaled option values. The free boundaries of these individuals are then cut into segments and randomly recombined. Alternatively, the average of two boundaries is formed. With a certain low probability, anchor points are randomly adjusted. The population is replaced by the new generation in which the offspring with high scores is in the majority. This procedure is repeated until the maximum option value no longer increases, the population converges to homogeneity, or the maximum number of generations is exceeded. The algorithm returns the option value along with the optimal program given by the free boundary.[28]

3.2.3.3.2.3 Noisy Fitness Functions

Although, in principle, genetic algorithms are also applicable to noisy objective functions, noise may negatively affect the performance of evolutionary heuristics [11, 126, 170].[29] Noise in the objective function can be reduced by minimizing the variance of payoffs resulting from Monte Carlo simulation. While several methods have been hinted at previously, (deterministic) quasi-random sequences are particularly useful.[30]

The use of quasi-random sequences is not common among finance practitioners, but sufficiently well understood in theory [156, 338]. Among the great variety of sequences available, the Sobol sequence in particular possesses some very desirable properties [313]. Hybrid quasi–Monte Carlo (QMC) methods have been proposed to resolve accuracy

[27] To avoid an upward bias, it is important to not use the same set of realizations for each generation.

[28] By analogy to the analysis carried out by Longstaff and Schwartz, it can be beneficial to compare this result to the out-of-sample performance of other individuals with slightly lower option values [207].

[29] See also the overview by Arnold [10].

[30] The method was introduced to finance in an article by Paskov and Traub [266]. For a recent technical overview see Niederreiter [258].

issues with low-dicrepancy sequences of higher dimensionality [13, 189], although there is an obvious trade-off with efficiency gains in EMC.

Other techniques known to improve the performance of QMC simulation for high-dimensional problems include the Brownian bridge, the partial principle components, and the subsequence methods [166]. Numerical experiments by the author show that correlation effects are in fact often negligible in this context. Also note that the standard Box–Muller procedure for generating normally distributed numbers is not suitable for QMC, because the algorithm fails to preserve the low-discrepancy property [41, 250]

EMC will be used to obtain approximate solutions to a fairly complicated optimal control problem arising in the valuation of R&D and patents in part II.

Table 3.1 compares some of the option pricing methods presented. Looking at the model problem one finds that the FD method clearly outperforms the LSM and the EMC techniques in terms of accuracy and speed.[31] However, the mathematical formalism is difficult to adapt to more advanced investment problems and far less intuitive. To improve the accuracy of EMC, it is possible to combine GAs, which deliver acceptable performance globally, with other techniques better suited for local search.

Observation 3.2. EMC is a very flexible technique for valuing real options. Its flexibility, however, comes at the price of inefficiency.

This chapter served to provide an overview of investment under uncertainty, including alternative methods for numerical (real) option pricing. Among other tools, these methods will be employed in parts II and III to examine the value of R&D in the presence of patent risk.

[31] Of course, at the price of computational time, each method may be employed to generate almost infinitely accurate results.

Table 3.1. Comparison of various approaches to option pricing. All methods were used to price put options with $X_0 = 40.0$, $r = 0.05$, and $T = 7.0/12$. As far as this model problem is concerned, least-squares Monte Carlo (LSM) simulation is clearly inferior to the finite difference (FD) method both in terms of accuracy and efficiency. Its advantages lie in more advanced valuation problems. Obviously, the computational effort is even higher with evolutionary Monte Carlo (EMC) simulation. FD prices where calculated using a Courtadon scheme with 500 timesteps on a square grid. Monte Carlo prices are based on an Euler scheme, 100,000 simulation runs, Sobol quasi-random numbers, and Acklam's approximation for transforming uniformly into normally distributed numbers. The American variants employ a PSOR solver and a 40-dimensional Sobol sequence in combination with LSM, respectively. Binomial option prices reflect benchmarks reported by Bunch and Johnson [56, p. 2346]. EMC prices result from a steady state GA with 10 gray-coded real numbers, 40 individuals, evolved over 30 generations.

Volatility (σ)	Method	Option value		
		$K = 35.0$	$K = 40.0$	$K = 45.0$
0.2	Black–Scholes–Merton	0.4132	1.8688	4.8173
	FD European	0.4131	1.8686	4.8172
	Monte Carlo European	0.4131	1.8688	4.8173
	Binomial American	0.4328	1.9904	5.2670
	FD American	0.4329	1.9905	5.2671
	LSM	0.4344	1.9855	5.2627
	EMC	0.4252	1.9840	5.2208
0.3	Black–Scholes–Merton	1.1823	3.0500	5.9512
	FD European	1.1823	3.0500	5.9512
	Monte Carlo European	1.1823	3.0500	5.9512
	Binomial American	1.2198	3.1696	6.2436
	FD American	1.2000	3.1698	6.2439
	LSM	1.2213	3.1705	6.2301
	EMC	1.2177	3.1653	6.2277
0.4	Black–Scholes–Merton	2.1044	4.2332	7.1449
	FD European	2.1044	4.2331	7.1449
	Monte Carlo European	2.1044	4.2332	7.1450
	Binomial American	2.1549	4.3526	7.3830
	FD American	2.1549	4.3527	7.3829
	LSM	2.1544	4.3573	7.3759
	EMC	2.1589	4.3545	7.3868

4

Patent Protection in the Pharmaceutical Industry

Returning to the issue of patenting and the specific situation of research-driven firms already discussed in chapter 2 on the strategic and economic significance of patents, the following sections serve to present the specific financial and strategic challenges faced by firms in pharmaceutical biotechnology. In addition, a brief description of the types of risk encountered in this highly competitive industry is provided, motivating formal analysis in later chapters.

4.1 Financial and Strategic Challenges

Most experts consider new drug development as the textbook example of patent protection [267]. Although the initial cost of a pharmaceutical invention is extremely high, imitation and production is possible at comparatively low cost once the drug has been developed. In the absence of patent protection, this discrepancy would result in *free riders* capturing a significant portion of economic benefits. As a consequence, the amount of investment in pharmaceutical R&D would almost surely drop below the socially optimal level. Due to the specific nature of pharmaceutical inventions, firms are in a position to obtain IPRs on clearly-defined products or processes. Each product or process is likely to result in exceptional market power or even exclusivity in a particular field, enabling the patent holder to generate monopoly profits. Furthermore, highly codified and packaged technology facilitates licensing and sale [217].

Over the last 30 years, the pharmaceutical industry has been dominated by giant research-based multinationals generating phenomenal profits. This post-war success story reflects the fairly simple strategy of investing in drug discovery and development, following through with effective marketing, and leveraging cash flows to establish a continuing R&D pipeline that yields a steady stream of innovative products. However, as the 1980s progressed, pharmaceutical profitability began to slide, partially due to numerous government and healthcare payers' cost-containment measures implemented in key national markets. Moreover, total R&D investment in the US has grown almost exponentially, from approximately 600 million dollars in 1970 to an estimated 24 billion dollars in 1999. At the same time, the number of new chemical entity launches has fallen [29, p. 2].

A combination of cost-containment, lackluster R&D productivity, and patent expiration has brought about a situation in which the achievable growth rates in revenues and earnings fall short of the double-digit level investors have become accustomed to (see fig. 4.1). This phenomenon is commonly referred to as the "earnings gap."

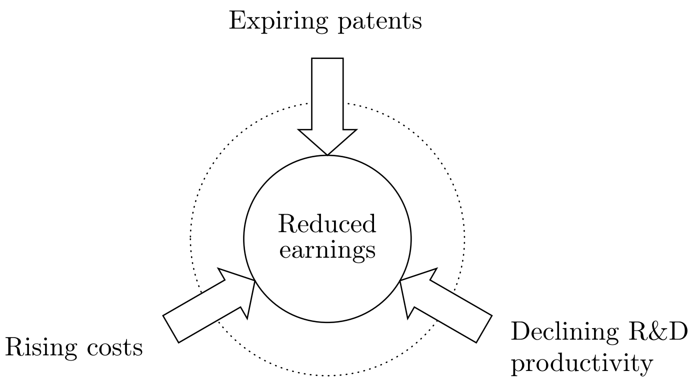


Fig. 4.1. Pressures creating the earnings gap (Source: figure adapted from Bennett [29, p. 27]).

According to Bennett, a number of key strategies have been identified to combat the earnings gap: effective in-licensing, speedier development of new products, enhanced market penetration, improved lifecycle management, strategic mergers and acquisitions (M&A) activity, improved inventory management and manufacturing asset utilization [29, pp. 2–3]. Since all major players are equally motivated to unleash the full potential of these measures, market participants are bound to face unprecedented competitive pressures. For instance, the need to maintain earnings growth has lead to cut-throat competition in the field of licensing. Correspondingly, licensing agreements have evolved to play a key role in pharmaceutical portfolio development. Although the broad outline and structure of many agreements have remained unchanged, there has been a noticeable increase in the sophistication and complexity of deals [29, p. 3].

More generally, research-based firms need to capture enough of the economic returns to make their investment worthwhile. This goal is accomplished by defending IP through formal patents and an evolving set of legal strategies. Established patent practices, however, may not be suitable for the fast-changing landscape of pharmaceutical biotechnology, because the law has yet to catch up with science [94, p. 5].

Oftentimes, the economic reward to innovators no longer lies in commercializing therapeutic or diagnostic end products, but in the use of inventions during subsequent phases of R&D by others. Unfortunately, how to use patents to capture the value of these *research tools* is less than obvious. Many firms pursue reach-through strategies, trying to claim a share of the value of derivative inventions. More precisely, these strategies entail upstream firms reaching into future revenues from end products developed using their technologies. Unsurprisingly, established pharmaceutical firms downstream strongly oppose such strategies, for example because upstream research in the biomedical field is relatively cheap and heavily subsidized with public funding. Heightened litigation activity is the unavoidable result [100, pp. 107 and 112]

4.2 Risk in Pharmaceutical Patents

Apart from the obvious market risk, research-based pharmaceutical firms face a number of risks, three of which are discussed in the following section, summarized under the headings "completion," "expiration," and "litigation." As will become clear in the course of this text, it is not difficult to incorporate them into option-based models of patents and R&D.

4.2.1 Completion

Pharmaceutical R&D very closely follows a pre-defined stage-gate process [70]. As a consequence, there is abundant statistical evidence on success probabilities for the various phases (see fig. 4.2). These probabilities vary considerably, for example across different therapeutical areas. Careful analysis may therefore lead to significant improvements in decision quality during portfolio and project management. More broadly

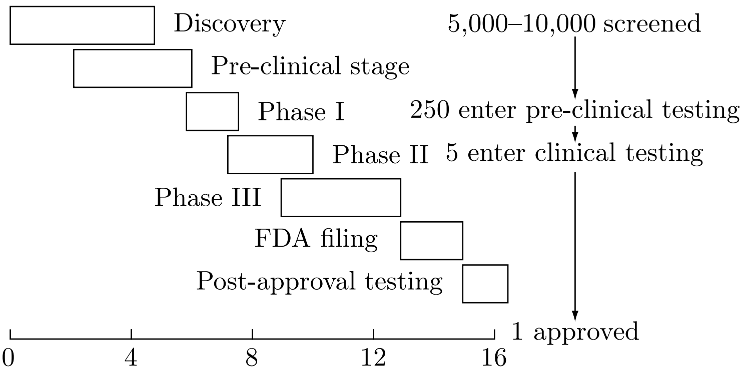


Fig. 4.2. Probability of success in pharmaceutical R&D. Regulatory requirements result in a typical stage-gate process, lending itself to DTA (Source: Gosse et al [121]).

speaking, only a very small percentage of candidates screened eventually results in a new drug approved by the FDA [123, 124].

Moreover, commercial providers collect data on the time spent in each stage as well as the total costs incurred (see table 4.1). Since the risk resulting from a positive probability of not completing R&D has been analyzed extensively elsewhere and, more importantly, poses no serious methodological challenges relevant to this discussion, the interested reader is referred to the literature for additional information [34, 316].

Table 4.1. Pre-tax costs, durations, and conditional probabilities of success in R&D. Commercial data providers have gathered very detailed information on failure rates across various stages and therapeutic areas, enabling fairly robust estimates of success probabilities (Source: Kellogg and Charnes [165, p. 79]).

R&D stage	Total cost (in EUR thousands)	Years in stage	Conditional probability of success
Discovery	2,200	1	0.60
Pre-clinical stage	13,800	3	0.90
Clinical stages			
Phase I	2,800	1	0.75
Phase II	6,400	2	0.50
Phase III	18,100	3	0.85
FDA filing	3,300	3	0.75
Post-approval testing	31,200	9	1.00

4.2.2 Expiration

Between 2000 and 2005, the pharmaceutical industry has seen unprecedented levels of patent expiration, with a total of over 20 blockbuster drugs losing patent protection. Total sales of these products were approximately 40 billon dollars in 1998, roughly equivalent to the entire Japanese ethical pharmaceutical market in 1998 [29, p. 5].

While the CAGR for global blockbuster revenues was around 23.6 percent between 1994 and 2000, strong revenue growth cannot be expected to continue into the future. Between 2001 and 2008, the blockbuster market is expected to increase with a CAGR of only 4.3 percent [83, p. 21].

This outlook poses a severe threat to the established players, because blockbuster products—that is products with annual global sales in excess of 1 billion dollars—typically are the driving force behind growth and profitability in large pharmaceutical firms. Conversely, any loss of blockbuster sales has a serious impact on both total sales and earnings. In particular, blockbuster drugs tend to attract generic manufacturers, leading to a rapid erosion in market share, once the branded product is off-patent [29, p. 28].[1]

While increasing efforts to establish a common international legal ground on patent practice have contributed to the wide application of patent extension strategies common in the US—including Supplementary Protection Certificates (SPCs), orphan drug status, and pediatric extensions—these strategies do not pertain to all products. Above all, they fail to address the fundamental problem of low R&D productivity [83, p. 37].

Major pharmaceutical firms are currently evaluating the potential for "patent protection" insurance to recover at least some of the costs associated with IP challenges from generic manufacturers [83, p. 22].

4.2.3 Litigation

Regardless of the enormous economic significance of legal risk, the vast majority of patents is actually never asserted in litigation. Only 1.5 percent of all patents are ever litigated, with a mere 0.1 percent litigated to trial [184, 191]. Although litigation rates vary by industry and reach as high as 6 percent in biotechnology [194], data from any sample of litigated patent cases must therefore be interpreted with great care.

[1] For detailed analyses of the impact of market entry by generic manufacturers Caves et al [61], Ferrándiz [105], Frank and Salkever [109], Hudson [149].

Obviously, the patents involved in litigation are those that are valuable enough to justify litigation costs and for which the parties failed to reach a mutually acceptable settlement [193, p. 79].

Some empirical data on patent litigation, categorized by international patent classification (IPC), is shown in table 4.2. Patents falling into the category "drugs and health" are clearly those most frequently litigated. The total litigation rate is almost twice as high as the average.[2]

Table 4.2. Litigation rates by technology group and ownership (Source: Lanjouw and Schankerman [184]). Numbers shown are filed cases per 1,000 patents between 1980–84. The following IPC categories are included in each group: A61 and A01N (drugs and health); A62, B31, C01–C20, and D (chemical); G01–G21 and H (electronic); B21–B30, B32–B68, C21–C30, and E01–F40 (mechanical); A not included in drug and health, B01–B20, F41–F42, and G21 (other).

Technology group	Ownership		
	Domestic	Foreign	Total
Drugs and health	26.6	6.5	20.1
Chemical	6.1	1.4	5.4
Electronic	12.7	3.3	9.6
Mechanical	20.1	3.4	11.8
Other	23.4	9.9	15.2
Total	16.4	3.5	10.7

Inevitably, the increasing importance and number of patents has been accompanied by a higher frequency of IP disputes involving patent holders and alleged infringers. Between 1978 and 1995, the number of patent suits filed rose by almost tenfold. Much of this increase occurred during the 1990s [185, p. 1], with the trend continuing between 1995 and 2000 [275]. As mentioned earlier, the current situation is aggravated by the emergence of *patent thickets* [308], requiring more and more firms to obtain multiple licenses to avoid infringement and bring their products safely to market [309, p. 391].

For some firms, notably start-ups and smaller biotechnology firms, litigation represents a necessary evil. In contrast, established pharmaceutical firms use litigation aggressively to protect their heavy investment in R&D [83, p. 78].

[2] For details see Lanjouw and Schankerman [183, p. 28] and Harhoff et al [135, p. 17].

In 2003, the median patent case with less than 25 million dollars at stake cost 2 million dollars per side to litigate to trial [192, p. 13]. According to Berman [30], patent suits filed in 2000 alone will generate around 4.2 billion dollars in legal fees. While such outrageous expenses are mainly due to the complexity of the subject matter and the large amounts of money involved, the fact that lawyers benefit more from protracted litigation than early settlement does little to improve the situation.

Concluding their review of the empirical literature on the enforcement of IPRs, Lanjouw and Lerner [180] find that the perceived danger of patent disputes indeed does reduce and distort R&D incentives, depending on the ability of firms to engage in litigation.[3] The threat of costly litigation alone may be sufficient to force a relatively weak firm to out-license its technology [83, p. 78]. As a consequence, firms with high litigation costs are less likely to patent in IPCs with many previous awards by rival firms. They also tend to avoid IPCs occupied by rivals with low litigation costs [194].

Lanjouw and Schankerman [182] show that patents that are litigated tend to have more claims and more citations per claim. The more valuable a patent, the higher the expected legal cost of enforcing it [186].

4.3 Implications for Capital Budgeting

The primary objective of capital budgeting is the efficient allocation of resources under uncertainty. Therefore, at least in theory, all types of risk described should enter into the analysis. Nevertheless, models used in practice are usually limited to selected risk factors. This text, of course, places particular emphasis on the legal risk resulting from imperfect patent protection.

In order to put discussions to come into perspective and as a general introduction to formal option-based models of pharmaceutical R&D, consider the pricing approach proposed by Kellogg and Charnes [165], an often-cited contribution from the applied literature.

A drug needs to successfully pass a number of stages before commercialization finally results in substantial cash inflows: (1) discovery, (2) pre-clinical trials, (3) clinical trials, (4) new drug application (NDA), (5) post-approval. Employing a decision tree and basic option pricing, the expected present value (EPV) of a project can be determined in the

[3] For more on litigation and infringement damage see Grandstrand [125], Kingston [167, 168], Moore [246], Parr [265].

manner originally suggested by Smith and Nau [312] and further developed by Loch and Bode-Greuel [206], Copeland and Antikarov [71], and others.

Kellogg and Charnes commence by laying out a conventional DTA. The seemingly complicated decision problem shown in fig. 4.3 can be summarized in a single formula. More specifically, the expected net present value becomes

$$\mathrm{ENPV} = \sum_{k=1}^{7} \left(p_k \sum_{j=1}^{n} \frac{\mathrm{CF}^{\mathrm{D}}_{k,j}}{(1+r^{\mathrm{D}})^j} \right) + p_7 \sum_{k=1}^{5} \left(q_k \sum_{j=1}^{n} \frac{\mathrm{CF}^{\mathrm{C}}_{k,j}}{(1+r^{\mathrm{C}})^j} \right),$$

where $\mathrm{CF}^{\mathrm{D}}_{k,j}$ is the cash flow in period j under the condition that k is the final stage and p_k is the probability of this being the case.[4] Correspondingly, $\mathrm{CF}^{\mathrm{D}}_{k,j}$ denotes the cash flow in period j, assuming that the final stage is completed successfully and the product is of quality k. Moreover, the discount rates r^{D} and r^{C} reflect the specific risk profile of the R&D stages and the commercialization stage, respectively. Probabilities q_k are associated with different categories of product quality, ranging from "dog" to "breakthrough."

Looking back at quantitative information on success probabilities given in table 4.2, it becomes immediately obvious why this data is extremely useful when estimating the option value of abandonment.[5] While a policy of blindly adopting statistical data suffers from certain drawbacks, it also avoids agency problems and cognitive biases typically encountered when interviewing experts directly involved in the project to be evaluated.

[4] Kellogg and Charnes incorrectly describe p_k as the "conditional probability that stage k is the end stage for a drug that has reached stage $k-1$" [165, p. 79]. If this were the case

$$\mathrm{ENPV} = \sum_{k=1}^{7} \left(p_k \prod_{j=1}^{k-1} (1-p_j) \sum_{j=1}^{n} \frac{\mathrm{CF}^{\mathrm{D}}_{k,j}}{(1+r^{\mathrm{D}})^j} \right) + p_7 \prod_{j=1}^{6} (1-p_j) \sum_{k=1}^{5} \left(q_k \sum_{j=1}^{n} \frac{\mathrm{CF}^{\mathrm{C}}_{k,j}}{(1+r^{\mathrm{C}})^j} \right).$$

[5] For further details concerning parameters see Kellogg and Charnes [165, p. 79].

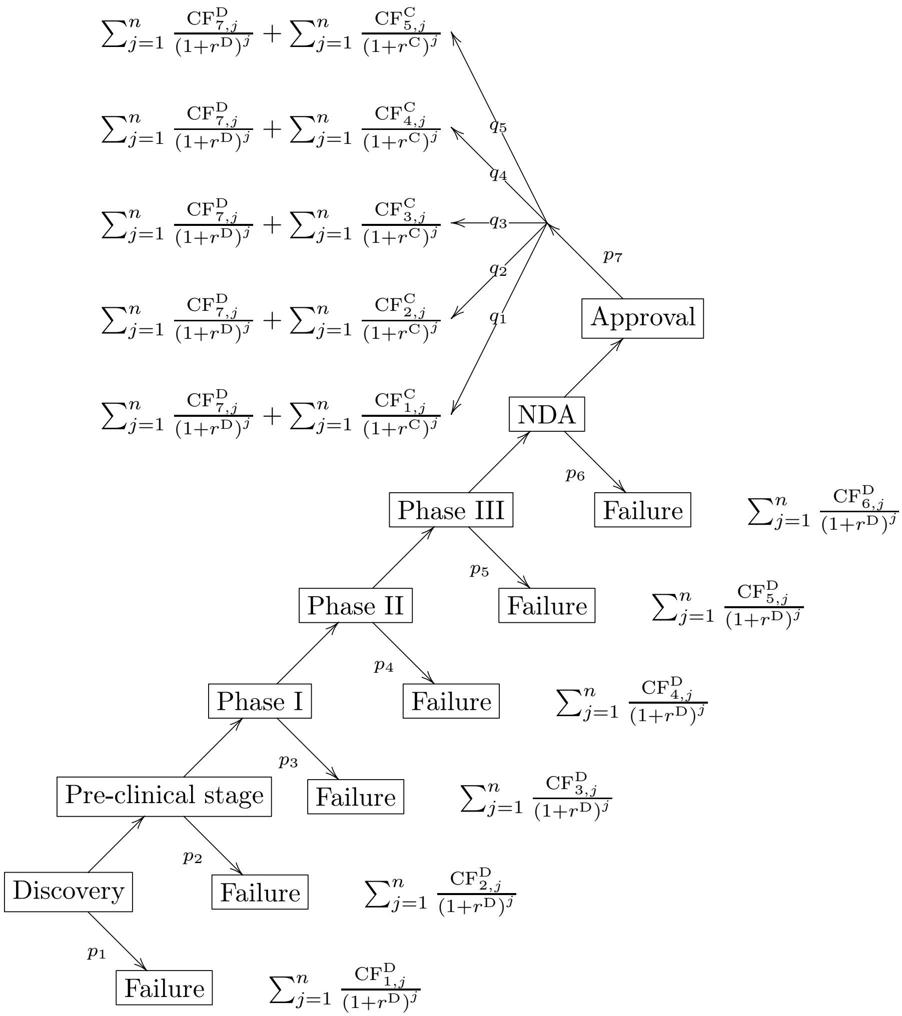


Fig. 4.3. Decision tree for pharmaceutical R&D (Source: figure adapted from Kellogg and Charnes [165, p. 80]). Conventional DTA provides little information on the appropriate choice of discount rates during R&D and commercialization.

The equivalent option-based model is similar [165, pp. 80–82], but still exhibits some subtle differences, the most important of which is a slight increase in consistency with respect to discount rates.[6]

A first step towards obtaining option values is to calculate the expected gross present value of cash flows from commercialization, which can be expressed as

$$V_0 = \sum_{k=1}^{5} \left(q_k \sum_{j=1}^{n} \frac{\mathrm{CF}_{k,j}^{\mathrm{C}}}{(1 + r^{\mathrm{C}})^j} \right),$$

where r^{C} is a risk-adjusted discount rate reflecting the systematic component of volatility, that is the standard deviation of returns on V_t. In option terms, V_t represents the underlying asset of a contingent claim.[7]

Following Cox et al [76], a discrete-time binomial approximation of V_t can be employed to perform risk-neutral valuation. Hence, the present value of cash flows from commercialization increases from $V_{i,j}$ to $V_{i+1,j+1} = uV_{i,j}$ with risk-neutral probability p. With probability $(1 - p)$, it decreases to $V_{i,j+1} = dV_{i,j}$. A recombining tree requires $u = 1/d$. Specifically,

$$u = \mathrm{e}^{\sigma\sqrt{\Delta t}},$$

where $\Delta t = T/n$ is the length of a single period, in this case one year, and σ denotes volatility.[8] The risk-neutral probability of an upward movement is

$$p = \frac{\mathrm{e}^{r\Delta t} - d}{u - d}.$$

Once R&D has been completed, the payoff is

$$F_{i,n} = \max \{V_{i,n}, 0\}.$$

Assuming positive $\mathrm{CF}_{k,j}^{\mathrm{C}}$, which implies positive $V_{i,n}$, one obtains $F_{i,n} = V_{i,n}$. The current value of R&D is then determined by dynamic programming:

---

[6] Strictly speaking, standard option valuation circumvents the issue of determining appropriate discount rates altogether, because it relies on an adjustment of the probability measure instead (see sect. 6.2.2).

[7] In the terminology of later chapters, this approach corresponds to a project-level model.

[8] For details, also see sect. B.1.1 in the appendix. Choosing such a very crude approximation of the continuous-time process may lead to substantial mispricing—an issue unfortunately often neglected in practical applications.

$$F_{i,j} = \max\left\{p_j \mathrm{e}^{-r\Delta t}\left(pF_{i+1,j+1} + (1-p)\,F_{i,j+1}\right) - \mathrm{CF}_j^{\mathrm{D}}, 0\right\},$$

where p_j is the probability of successfully completing stage j.[9]

Obviously, the risk of litigation plays no explicit role in the Kellogg and Charnes formalization. Cash flows accrue regardless of the quality of patent protection and, in addition, seem to be unaffected by competitive action.

Hence, Patent valuation calls for a careful re-examination by policy makers and practitioners alike. Before financial institutions are in a position to adequately characterize patent value in capital budgeting, M&A, licensing, or pro-forma revenue forecasts, a vigorous review of IP is mandatory. When selecting valuation methods for IP, accounting for the risk of PTO quality concerns as well as the cost of enforcement to actually extract value should be sufficient to attenuate overly optimistic predictions of profit potential [222, p. 13].

Discrete-time models like the one described are commonly employed by practitioners in the field of decision analysis.[10] The following chapters serve to develop a more comprehensive picture, also discussing more advanced models in continuous-time. More importantly, the interaction of different types of risk will be analyzed in detail.

In summary, it appears intuitively evident that patent risk should play a key role for in capital budgeting decisions, because imperfect patent protection may have a substantial impact on the valuation and, thereby, the optimal management of IPRs from a shareholder's perspective. In principal, if one chooses to follow the option-based view, there are two ways in which patent risk can be incorporated into common formal models of such rights.[11]

One way is to take patent risk as an exogenous parameter that, in addition to other parameters like drift and volatility, determines the evolution of underlying asset values over time. The modified process has an immediate impact on the value of claims contingent on these assets, such as patents granting the exclusive right to reap the benefits of commercialization. In technical terms, events negatively affecting the value of the patent take the form of discontinuities, or jumps, in the value process.

---

[9] Note the slight inconsistency in employing continuous-time as well as discrete-time compounding and discounting. In addition equation (3) provided by Kellogg and Charnes [165, p. 82] erroneously contains the discount factor $D = \mathrm{e}^{-r\sqrt{\Delta t}}$ instead of $D = \mathrm{e}^{-r\Delta t}$.

[10] For further examples see Copeland and Antikarov [71].

[11] For a recent discussion of patent risk and its implications for formal modeling see Marco [218].

Correspondingly, models incorporating both features, continuous as well as discontinuous changes, are referred to as jump-diffusion models, many of which have been successfully applied in the context of credit risk.[12] Analytic and numerical analyses serve to translate them into recommendations for value-maximizing investment policies.

Another way is to try to capture the strategic nature of patent risk by recognizing risk itself as the outcome of value-maximizing behavior of current competitors and potential new entrants. Legal action, in this framework, is just an additional strategic option open to all market participants. In fact it is a *real option,* because the antagonist will choose to litigate or to accommodate, depending on new information gathered about the expected payoff, which changes with the value of the asset underlying the challenged patent. Since the owner of a patent is likely to anticipate the behavior of other parties and act accordingly, game-theoretic arguments are, at least in theory, required to justify capital budgeting decisions concerning the creation, acquisition, and subsequent monetization of patents.

Parts II and III consider both approaches in turn, highlighting connections where appropriate.

[12] For an overview of credit risk modeling, valuation, and hedging see Bielecki and Rutkowski [33].

Part II

Exogenous Patent Risk

5

Introduction and Related Work

As pointed out by Schwartz [302], no discussion of the R&D process is complete, unless it accounts for the eventuality of patent litigation [184, 292]. This part therefore serves to present option-based models of exogenous patent risk. Following the common approach to investment analysis under uncertainty [90, 91], a formal analogy between patents and financial options is established and described in detail. Also under imperfect patent protection, the analogy proves to be as intuitive as it is fruitful, both from a theoretical and a practical standpoint.

In order to put into perspective analyses carried out, a brief overview of related work on option-based capital budgeting in R&D is provided in the following.

The application of option-based methods to real-world investment problems, especially in the pharmaceutical industry, is the subject of an interview with the CFO of Merck recorded by Nichols [257]. Smith and Nau [312] are the first to present a rigorous discussion of how to integrate option pricing theory and decision analysis in deriving optimal policies under combined market and technology risk. Simple discrete-time representations of the typical stage-gate process mirroring their approach are now widely accepted tools in R&D intensive settings (see sect. 4.3) [70].

Moreover, formal investment analysis under uncertainty has attracted the attention of researchers in strategic innovation management, exemplified by Baldwin and Clark [18], Kogut and Kulatilaka [173]. Other contributions addressing investment under uncertainty from a more strategic perspective include the papers by Lint [201], Lint and Pennings [202, 203, 204], Pennings and Lint [268]. In addition, Huchzermeier and Loch [148], Lee and Paxson [190], Ziedonis [340] focus on various issues in connection with option-based R&D management.

Game-theoretic considerations come into play in Weeds's model of R&D competition [333].

Early applications in the realm of pharmaceuticals include the articles by Jägle [157], Kellogg and Charnes [165], Ottoo [262]. Lavoie and Sheldon [187] examine macroecomic aspects. More recently, Brach and Paxson [48], Loch and Bode-Greuel [206] consider implications for valuation and value-based management. Furthermore, Robinson and Stuart [285] analyze financial contracting in strategic biotech alliances, while MacMillan and McGrath [208], McGrath and Nerkar [232] look at strategic portfolio decisions. An innovative use of extreme value theory (EVT) to study R&D options is proposed by Koh and Paxson [174].

In contrast to more complicated analyses of patents and R&D, some of which have been mentioned so far, formalizations chosen throughout the following chapters are deliberately stylized, drawing a clear picture of the impact of various levels of patent protection on value-maximizing strategies. For example, choosing a setup closely related to the classic model by Majd and Pindyck [214], the author clarifies the signifcance of limited patent duration [263]. Generally speaking, modifying such models to better reflect real-world investment problems should be straightforward.

Similarly, imperfect patent protection is studied employing classic models from the option pricing literature. Based on seminal contributions by Black and Scholes [35], McKean [233], Merton [238] who only consider the basic stock price model devised by Samuelson [14, 261, 290], Merton [239] extends the analysis of options prices to conventional jump-diffusion processes. This extended option pricing model constitutes the framework for introductory analyses of patent risk as jumps of *deterministic* size. Proceeding with an obvious generalization of Merton's setup, the discussion of patent risk as jumps of *stochastic* size then draws upon fairly recent literature in the field of option pricing under alternative price dynamics, more specifically, Lévy processes.

Although not in widespread use among practitioners, more general Lévy processes in financial applications have been the subject of research for a long time. Fama [102, 103], Mandelbrot [215, 216] are among the first to examine pure jump Lévy processes with a *stable* Pareto–Lévy measure to address deviations of stock returns from normality. In contrast, Press [276] analyzes an exponential Lévy process model with a *non-stable* distribution. His log price process combines Brownian motion and an independent compound Poisson process with normally distributed jumps. The use of generalized hyperbolic (GH) distributions is proposed by Barndorff-Nielsen and Halgreen [20], Eberlein and Keller [97]. These distributions encompass the important subclasses of hyperbolic and normal inverse Gaussian (NIG) distributions. Other possibilities are explored by Carr et al [59], Madan et al [210], analyzing variance gamma distributions and the more general case of so-called Carr–Geman–Madan–Yor (CGMY) distributions. In addition, Cont et al [69], Matacz [226] consider truncated Lévy processes. To simplify matters, the models proposed here are limited to the case of exponential jump-diffusion.

Several authors discuss the valuation of American options under Lévy processes, including, for example Gerber and Landry [113], Gerber and Shiu [114], Mordecki [247], Mulinacci [252], Pham [269], Zhang [339]. Analytical solutions presented in this part of the text are largely based on the pricing approach developed by Boyarchenko and Levendorskiĭ [44, 46], Mordecki [248]. Similar results concerning real investment options are obtained in a series of papers by Boyarchenko [42], Boyarchenko and Levendorskiĭ [45, 46] For a more general introduction to Lévy processes and applications, the reader is referred to the literature Bertoin and Doney [32], Boyarchenko and Levendorskiĭ [43], Jacod and Shiryaev [154], Raible [281], Sato [291], Schoutens [296], Skorokhod [311].

Again drawing upon a classic paper, the basic model of investment under market uncertainty with time-to-build due to Majd and Pindyck [214] is extended to the case of imperfect patent protection. Originally developed to capture the threat of radical innovation faced by a large producer of consumer electronics, it leads to several findings on the nature of risky R&D and serves as a point of reference for the more advanced models that follow.

Building on fairly recent contributions to the field of option-based R&D management by Hsu and Schwartz [146], Schwartz [302], Schwartz and Moon [303], Schwartz and Zozaya-Gorostiza [304], these advanced models are extensions of Pindyck's original work on investment of uncertain costs [270]. In contrast to Miltersen and Schwartz [245], the explicit analysis of strategic interaction is postponed to part III.

Spanning a comparatively wide range of analytical as well as numerical methods, the discussion is structured as follows. To lay the foundation for a detailed description of option-based models of exogenous patent risk, chap. 6 introduces the general concept of option-based patent valuation, starting with deterministic analyses before proceeding to a stochastic setting. Examples are provided to illustrate the application of project-level, profit-level, and demand-level models. Particular attention is devoted to the impact of finite patent protection, a type of "hard" portfolio-level patent risk on optimal policies under uncertainty.

Chapter 7 then addresses the implications of "soft" patent risk, modeled as jumps in the underlying processes, covering patent-related events of deterministic as well as stochastic severity. Moving from simple single-stage, single-factor models to more advanced two-stage, two-factor models, a novel approach for determining value-maximizing policies in R&D is presented, followed by an extensive numerical analysis of the effects of patent risk.

Shedding light on some of the issues raised from a slightly more practical perspective, chap. 8 serves to point out connections between the advanced jump-diffusion models of R&D described and the business shift approach to R&D valuation proposed by Lint and Pennings [203].

In chap. 9, the author draws preliminary conclusions and identifies opportunities for future research.

6

Patents as Investment Opportunities

On a very abstract level, patents represent investment opportunities or, to use a technical term borrowed from financial markets, *options.* Depending on whether decisions are made upfront or over time as new information is obtained, investment policies in connection with patents are termed "static" or "dynamic." Of course, truly option-based methods require the policy to be dynamic, whereas traditional techniques based on the NPV criterion are static in nature. A brief description of static investment policy sets the scene for extensions to come.

6.1 Static Investment Policy

A patent protects its holder from competition, enabling him or her to generate additional cash flows from commercializing certain goods or services. These cash flows accrue in the form of monopoly rents, that is profits in excess of socially-optimal levels. The resulting price increase leads to a loss in welfare, but creates the incentive to obtain the patent in the first place, usually by investing into R&D and completing a lengthy and expensive application process. In other words, patent protection compensates the innovator for accepting potentially significant technological and market-related risk. Patent systems thus aim at maximizing economic growth by ensuring the optimal level of patent protection that balances social costs and benefits.[1]

[1] Despite the widespread assumption that patents in general have a positive impact on innovation and economic growth, there is hardly any empirical evidence to support this claim. In fact, many authors assume that the existence of patents is by and large owed to path dependency. Innovation activity in the software and financial industry seems to support their standpoint. See the references in sect. 2.2 for a more detailed account of patenting from an economist's perspective.

Leaving aside macroeconomic considerations, R&D and patenting decisions are obviously firm-level, that is microeconomic, investment problems. Practitioners have developed a variety of income-based, market-based, and cost-based valuation models to determine the value of a patent, including, for example, the popular relief-from-royalty method. Here, a more stylized approach is adopted, which aims at outlining the general mechanisms driving investment policy.

Simply put, the firm weighs individual costs and benefits of innovation by comparing the gross present value of cash flows from commercialization to the investment required to enter the market. What would be the value of the patent in a one-period model with no flexibility beyond the immediate investment decision?

For reasons of simplicity, let V denote the gross present value of future cash flows and I the cost of commercialization. It is straightforward to determine the so-called *static value* of the patent.

Proposition 1 (Static patent value). *Abstracting from the possibility to postpone commercialization, the (static) value of a patent is*

$$F(V) = \begin{cases} V - I & \textit{if } V^* < V, \\ 0 & \textit{otherwise}, \end{cases} \tag{6.1}$$

where

$$V^* = I \tag{6.2}$$

denotes the critical gross payoff from commercialization.

Proof (Proposition 1). Equation (6.1) and (6.2) simply describe the standard NPV rule, or Marshallian investment trigger. It is optimal to invest if and only if the overall payoff is positive.

Since the corresponding investment policy implies a now-or-never decision, it is termed "static." Once the patent has been obtained, commercialization takes place immediately. Strictly speaking, according to the static view, a patent represents an investment *project,* rather than an investment *opportunity.* In contrast, dynamic policies are contingent on future events and thereby typically involve a number of subsequent decisions. In the most basic setup, investment into R&D is followed by an additional investment required to commercialize the patent.

Using a slightly different notation, similarities to financial contracts become obvious. Equation (6.1) is equivalent to

$$F(V) = \max\{V - I, 0\} \equiv (V - I)^+,$$

which is just the inner value of a plain-vanilla call option.[2] It is useful however, to express the rule in terms of an upper threshold V^*, since this makes static investment policies readily comparable to dynamic ones.

The length of the protection period granted to the holder of the patent only comes into play if V is broken down into a series of cash flows. These are then discounted and summed up to arrive at the gross payoff from commercialization. In a continuous-time setting with finite time horizon T,

$$V(\Pi_0) = \int_0^T \mathrm{e}^{-\mu t} \left(\Pi_0 + \int_0^t \alpha \Pi_s \, \mathrm{d}s \right) \mathrm{d}t,$$

where Π_t is the (expected) cash flow rate at time t, α is the instantaneous growth rate (drift), and μ is a risk-adjusted discount rate. Assuming $0 < \alpha < \mu$, integration yields

$$V(\Pi_0) = \left(1 - \mathrm{e}^{-(\mu-\alpha)T}\right) \frac{\Pi_0}{\mu - \alpha}. \tag{6.3}$$

The longer the protection period, the higher the value of the patent. If T is infinitely large, (6.3) reduces to the standard perpetuity formula

$$V(\Pi_0) = \frac{\Pi_0}{\mu - \alpha},$$

so that

$$F(\Pi_0) = \begin{cases} \frac{\Pi_0}{\mu-\alpha} - I & \text{if } \Pi^* < \Pi_0, \\ 0 & \text{otherwise,} \end{cases}$$

where

$$\Pi^* = I\,(\mu - \alpha)$$

is an upper investment threshold.

To overcome the obvious limitations of such a static model, a number of extension are introduced in the following section.

6.2 Dynamic Investment Policy

Although uncertainty is the single most important element in R&D-related investments, the deterministic case provides a suitable benchmark for further analysis. Therefore, sect. 6.2.1 serves to present basic

[2] Recall that the total value of a financial option consists of an *inner* value and a *time* value. The latter captures the possibility to respond flexibly as uncertainty about asset prices resolves over time.

models of dynamic investment policy under certainty. The stochastic case is discussed in sect. 6.2.2.

6.2.1 Deterministic Payoff

Depending on the level of detail, investment policies may be analyzed on the project, profit or demand level. Despite the fact that demand-level analysis is most comprehensive and, as a consequence, perhaps also most realistic, many important results can already be derived in a very simplified setting.

6.2.1.1 Project-Level Analysis

As closed-form expressions for the patent value under uncertainty are only available for the subclass of problems with infinite patent protection, this particular case is also chosen as the starting point for the following discussion of investment under certainty.

6.2.1.1.1 Infinite Protection Period

The optimal policy is derived analytically and illustrated graphically using a numerical example.

6.2.1.1.1.1 Analytical Derivation

Similar to the time-dependent profit rate specified in sect. 6.1, assume a gross present value that grows at a constant rate α:

$$V_t = V_0 + \int_0^t \alpha V_s \, \mathrm{d}s, \qquad V_0 = v. \tag{6.4}$$

This specification is similar to the widely-used standard model of stock prices [290], except for the random component, which is omitted at this point, but re-introduced in sect. 6.2.2.[3]

Assumption 6.1. Changes in commercialization payoff are analogous to stock price movements.

[3] Although, for example, typical drug lifecycles observed in the market suggest that a constant growth rate might not properly reflect important stylized facts of the industry, the model is a good starting point for further analysis.

Equation (6.4) implies

$$V_t = e^{\alpha t} v.$$

With the initial project value and growth rate known, the value of the project at any point in time $t \in [0, \infty)$ is easily calculated. The current value of the patent $F(v)$ is then obtained by solving the optimization problem

$$\begin{aligned} F(v) &= \max_{\tau \in [0,\infty)} e^{-r\tau} \left(e^{\alpha\tau} v - I \right)^+ \\ &= e^{-r\tau^*} \left(e^{\alpha\tau^*} v - I \right)^+ , \end{aligned} \tag{6.5}$$

where τ^* is the optimal time to commercialize. Future project values are known with certainty, so that the risk-free rate r may be used for discounting. As commercialization effectively "stops" the process describing the evolution of V_t over time, τ^* is also referred to as the "optimal stopping time." The investment amount is still considered to be constant.

Assumption 6.2. The holder of a patent maximizes value by choosing an optimal time to commercialize.

As explained in sect. 6.1, the dynamic value of a patent can be expressed in terms of a threshold, or *critical value.* The set $\{v \in [0, \infty) : v^* < v\}$ is known as the *stopping region,* whereas $\{v \in [0, \infty) : v \leq v^*\}$ represents the *continuation region* [177, pp. 193–221].

Proposition 2. *Assuming a moderate growth rate $\alpha \in (0, r)$, the dynamic value of a patent under certainty and infinite patent protection is*

$$F(v) = \begin{cases} v - I & \text{if } v^* < v, \\ \frac{I\alpha}{r-\alpha} \left(\frac{v(r-\alpha)}{Ir} \right)^{r/\alpha} & \text{otherwise.} \end{cases} \tag{6.6}$$

where

$$v^* = \frac{Ir}{r - \alpha} \tag{6.7}$$

denotes the critical gross payoff from commercialization at time $t = 0$.

Proof (Proposition 2). The optimal investment policy depends on the growth rate α and the discount rate r. The value of the project may (1) decrease over time, (2) increase over time, but decrease in present-value terms, (3) stay constant in present-value terms, or (4) increase in present-value terms. These cases are now examined in turn.

Case 1 ($\alpha \leq 0$). Simply by looking at (6.5) it becomes obvious that, as long as I remains constant, postponing commercialization will always decrease the investor's payoff. Consequently, it is optimal to commercialize immediately, or never. This situation is similar to the static case.

Case 2 ($0 < \alpha < r$). An interior solution to the optimization problem exists only for moderately positive growth rates. Set

$$G(v, \tau) = \mathrm{e}^{-r\tau} \left(\mathrm{e}^{\alpha\tau} v - I\right). \tag{6.8}$$

A necessary condition for an optimum is

$$\left.\frac{\partial G(v, \tau)}{\partial \tau}\right|_{\tau=\tau^*} = \mathrm{e}^{-r\tau^*} \left(Ir - \mathrm{e}^{\alpha\tau^*} v \,(r - \alpha)\right) = 0. \tag{6.9}$$

Solving for τ^* leads to a solution candidate:

$$\tau^* = \frac{1}{\alpha} \ln \frac{Ir}{v\,(r - \alpha)}. \tag{6.10}$$

Since

$$\left.\frac{\partial^2 G(v, \tau)}{\partial \tau^2}\right|_{\tau=\tau^*} = -Ir\alpha \left(\frac{v\,(r-\alpha)}{Ir}\right)^{r/\alpha} < 0,$$

this candidate indeed yields a maximum. However, τ^* is required to be non-negative. The critical project value v^* given by (6.7) is derived as the lowest project value for which τ^* is actually positive. Solving (6.9) for $V_t = \mathrm{e}^{\alpha\tau^*} v$ leads to

$$\mathrm{e}^{\alpha\tau^*} v = \frac{Ir}{r - \alpha}.$$

Setting $\tau^* = 0$ yields the threshold v^*.[4] Inserting (6.10) in (6.8) shows that

$$\begin{aligned} F(v) &= G(v, \tau^*) \\ &= \mathrm{e}^{-r\tau^*} \left(\mathrm{e}^{\alpha\tau^*} v - I\right) = \frac{I\alpha}{r - \alpha} \left(\frac{v\,(r-\alpha)}{Ir}\right)^{r/\alpha} > 0 \end{aligned}$$

for project values lower than or equal to v^*. Conversely, for all project values above the threshold, commercializing immediately is optimal. As evident from (6.7), immediate commercialization also results in a positive payoff, because $0 < v^* - I = \alpha / (r - \alpha)$.

[4] At the critical threshold v^*, it optimal to invest immediately.

Case 3 ($r = \alpha$). If the growth rate equals the discount rate, timing has no effect on the investor's payoff. He or she is indifferent with respect to the time of commercialization.

Case 4 ($r < \alpha$). If the project value grows at a rate in excess of the risk-free rate, it is never optimal to commercialize. The rational manager postpones the investment indefinitely.[5]

Patent value for moderate growth rates is thus given by the piecewise function (6.6).

Comparing proposition 2 to proposition 1 shows that timing flexibility indeed increases the value of the patent. In this setting, option value is not driven by uncertainty, but by growth. The higher the growth rate, the higher the value of the patent.

Occasionally, v^* is also referred to as a "deterministic investment trigger." Compared to the Marshallian trigger presented in sect. 6.1, the trigger calculated here is substantially higher, because

$$I < v^* = I\left(1 + \frac{\alpha}{r - \alpha}\right).$$

The investment required to commercialize is equivalent to a perpetuity of Ir. Therefore, the critical project value v^* corresponds to a growing perpetuity of the same amount.[6]

Furthermore, it is important to emphasize that, if payoffs are deterministic, the optimal time to commercialize can be calculated explicitly.

Corollary 1. *Assuming a moderate growth rate $\alpha \in (0, r)$, the optimal commercialization time of a patent under certainty and infinite patent protection is*

$$\tau^* = \begin{cases} 0 & \textit{if } v^* < v, \\ \frac{1}{\alpha} \ln \frac{Ir}{v(r-\alpha)} & \textit{otherwise.} \end{cases} \tag{6.11}$$

Proof (Corollary 1). Equation 6.11 simply restates a result obtained in the derivation of proposition 2.

The lower the initial project value, the more attractive postponement becomes. More formally,

[5] This case seems like a mere technicality. However, a similar condition involving the rate of return shortfall δ will appear under uncertainty.

[6] Correspondingly, Dixit and Pindyck [91, p. 184] coin the term "flow-equivalent cost of investment."

$$\left.\frac{\partial \tau^*}{\partial v}\right|_{v \leq v^*} = -\frac{1}{v\alpha} < 0. \tag{6.12}$$

This straightforward relationship could provide a somewhat naïve explanation for the existence of sleeping patents, which are not commercialized despite obvious market opportunities. However, the setup is still too simplistic to permit such far-reaching conclusions.

6.2.1.1.1.2 Numerical Illustration

Figure 6.1 shows how patent values change depending on the initial value of the project for various growth rates ($\alpha \in \{0.000, 0.015, 0.030\}$). Assuming an investment amount of $I = 1.0$ and a risk-free rate of $r = 0.05$, critical project values become

$$v_0^* = \frac{1.000 \times 0.050}{0.050 - 0.000} = 1.000,$$
$$v_1^* = \frac{1.000 \times 0.050}{0.050 - 0.015} = 1.429,$$
$$v_2^* = \frac{1.000 \times 0.050}{0.050 - 0.030} = 2.500.$$

Already under certainty, the typical option-like shape is clearly recognizable. Obviously, the impact of growth on patent value is most significant for initial project values close to the investment amount.

6.2.1.1.2 Finite Protection Period

Under finite patent protection, the value of the patent is not necessarily positive, even for positive growth rates. Specifically, if the patent expires at time $t = T$ and $\mathrm{e}^{\alpha T} v \leq I$, it is never optimal to commercialize. More importantly, postponing in real-world situations decreases the payoff from commercialization, because the window of opportunity for accumulating cash flows closes. In order to investigate this effect and draw appropriate conclusions, it is indispensable to examine cash flows instead of aggregate project values.

6.2.1.2 Profit-Level Analysis

Building on previous analyses, extensions introduced in this section explicitly recognize the fact that project values represent the present value of future cash flows. This relationship was hinted at in sect. 6.1 and is fleshed out here in detail.

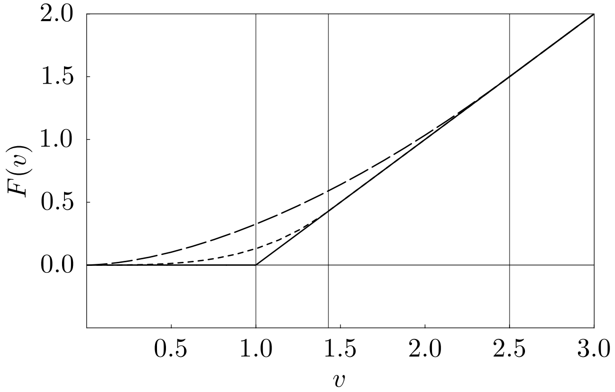


Fig. 6.1. Dynamic patent value under certainty and infinite patent protection according to project-level analysis ($I = 1.0$, $r = 0.05$, and $\alpha \in \{0.000, 0.015, 0.030\}$). The impact of growth on patent value is most significant for initial project values close to the investment amount. Vertical lines indicate critical initial project values at $v \in \{1.00, 1.43, 2.50\}$.

6.2.1.2.1 Infinite Protection Period

For reasons of clarity and comparability, consider first the case of an infinite protection period. Again, the optimal policy is first derived analytically, before resorting to numerical methods to accentuate noticeable results.

6.2.1.2.1.1 Analytical Derivation

By analogy, the time-dependent value of commercialization becomes

$$V(\varpi, t) = \int_t^{\infty} \mathrm{e}^{-r(s-t)} \mathit{\Pi}_s \, \mathrm{d}s,$$

where

$$\mathit{\Pi}_t = \mathit{\Pi}_0 + \int_0^t \alpha \mathit{\Pi}_s \, \mathrm{d}s, \qquad \mathit{\Pi}_0 = \varpi,$$

and $\mathit{\Pi}_t$ again denotes the (commercialization) cash flow rate at time t. For moderate growth rates, V_t is bounded. Integration leads to

$$V(\varpi, t) = \frac{\mathrm{e}^{\alpha t} \varpi}{r - \alpha}, \qquad 0 < \alpha < r. \tag{6.13}$$

The rational manager's objective function is thus

$$F(\varpi) = \max_{\tau \in [0,\infty)} \mathrm{e}^{-r\tau} \left(\frac{\mathrm{e}^{\alpha\tau}\varpi}{r-\alpha} - I \right)^{+}$$
$$= \mathrm{e}^{-r\tau^*} \left(\frac{\mathrm{e}^{\alpha\tau^*}\varpi}{r-\alpha} - I \right)^{+} . \tag{6.14}$$

Analogous to the project-level case, τ is chosen to maximize patent value.

Proposition 3. *Assuming a moderate growth rate $\alpha \in (0, r)$, the dynamic value of a patent under certainty and infinite patent protection is*

$$F(\varpi) = \begin{cases} \frac{\varpi}{r-\alpha} - I & \text{if } \varpi^* < \varpi, \\ \frac{I\alpha}{r-\alpha} \left(\frac{\varpi}{Ir}\right)^{r/\alpha} & \text{otherwise,} \end{cases} \tag{6.15}$$

where

$$\varpi^* = Ir \tag{6.16}$$

denotes the critical cash flow rate at time $t = 0$.

The simple proof follows the steps outlined in sect. 6.2.1.1.

Proof (Proposition 3). Analogous to project-level analysis, set

$$G(\varpi, \tau) = \mathrm{e}^{-r\tau} \left(\frac{\mathrm{e}^{\alpha\tau}\varpi}{r-\alpha} - I \right) . \tag{6.17}$$

Given that $0 < \alpha < r$, a necessary condition for optimal commercialization is

$$\left. \frac{\partial G(\varpi, \tau)}{\partial \tau} \right|_{\tau=\tau^*} = \mathrm{e}^{-r\tau^*} \left(Ir - \mathrm{e}^{\alpha\tau^*}\varpi \right) = 0. \tag{6.18}$$

Solving for τ^* leads to an optimal commercialization time,

$$\tau^* = \frac{1}{\alpha} \ln \frac{Ir}{\varpi}, \tag{6.19}$$

which, because of

$$\left. \frac{\partial^2 G(\varpi, \tau)}{\partial \tau^2} \right|_{\tau=\tau^*} = -Ir\alpha \left(\frac{\varpi}{Ir} \right)^{r/\alpha} < 0, \tag{6.20}$$

in fact represents a partial solution to the maximization problem. Again, the critical cash flow rate ϖ^* is obtained by requiring the optimal commercialization time to equal zero. Cash flow rates in excess of ϖ^* trigger immediate investment. Simply inserting (6.19) in (6.17) and verifying that the non-negativity constraint is fulfilled as in sect. 6.2.1.1 produces (6.15).

Alternatively, proposition 3 can be derived directly from proposition 6.2 by substituting

$$v = V(\varpi, 0) = \frac{\varpi}{r - \alpha}, \tag{6.21}$$

which follows from (6.13), in (6.11).

Since proposition 3 describes the Jorgensonian trigger known from neoclassical investment theory, it is of particular interest when comparing the option-based approach to traditional capital budgeting techniques. By (6.16), it is optimal to invest as soon as the marginal profit from employing an additional unit of capital exceeds the user cost of capital [160].

The main difference between propositions 2 and 3 is that, according to the latter, the payoff from immediate commercialization depends explicitly on α. As a consequence, patent values vary substantially for different growth rates, even for very high values of ϖ.

As before, the optimal time to commercialize can be given explicitly.

Corollary 2. *Assuming a moderate growth rate $\alpha \in (0, r)$, the optimal commercialization time of a patent under certainty and infinite patent protection is*

$$\tau^* = \begin{cases} 0 & \textit{if } \varpi^* < \varpi \\ \frac{1}{\alpha} \ln \frac{Ir}{\varpi} & \textit{otherwise}, \end{cases} \tag{6.22}$$

Proof (Corollary 2). Proposition 3 requires (6.22) to hold.

Decreasing initial cash flow rates will cause the rational investor to postpone commercialization further. Since, by (6.21), v is a multiple of the initial cash flow rate, the value of waiting behaves completely analogous to (6.12), that is τ^* is inversely proportional to $-\varpi$, or

$$\left.\frac{\partial \tau^*}{\partial \varpi}\right|_{\varpi \leq \varpi^*} = -\frac{1}{\alpha \varpi}.$$

6.2.1.2.1.2 Numerical Illustration

Figure 6.2 illustrates how the value of the patent changes with the initial cash flow rate ϖ and the growth rate α. In contrast to project-level analysis, high growth rates result in patent values being particularly sensitive to changes in the initial cash flow rate, whereas the trigger is obviously unaffected by such changes. Again assuming $I = 1.0$ and $r = 0.05$, the rational investor commercializes if the initial cash flow rate exceeds $\varpi^* = 1.0 \times 0.05 = 0.05$, no matter what the growth rate α.

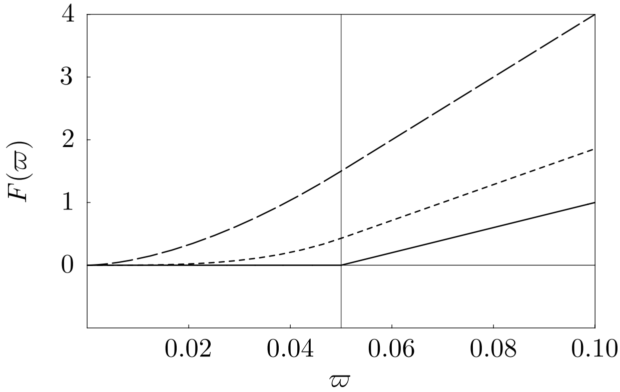


Fig. 6.2. Dynamic patent value under certainty and infinite patent protection according to profit-level analysis ($I = 1.0$, $r = 0.05$, and $\alpha \in \{0.000, 0.015, 0.030\}$). High initial cash flow rates result in patent values being particularly sensitive to changes in growth. The critical cash flow rate, which is unaffected by such changes, is at $\varpi = 0.05$.

6.2.1.2.2 Finite Protection Period

Consider the case of a finite protection period of length T, which, as opposed to project-level analysis (see sect. 6.2.1.1), is presented here in some detail, employing both analytical and numerical methods.

6.2.1.2.2.1 Analytical Derivation

The gross present value of cash flows from commercialization is

$$V(\varpi, t) = \int_t^T \mathrm{e}^{-r(s-t)} \mathrm{e}^{\alpha s} \varpi \, \mathrm{d}s.$$

If cash flows grow at a moderate rate, that is $0 < \alpha < r$,

$$V(\varpi, t) = \left(\mathrm{e}^{\alpha t} - \mathrm{e}^{-r(T-t)+\alpha T}\right) \frac{\varpi}{r - \alpha}.$$

The resulting objective function is

$$\begin{aligned} F(\varpi, 0) &= \max_{\tau \in [0,T]} \mathrm{e}^{-r\tau} \left(\left(\mathrm{e}^{\alpha\tau} - \mathrm{e}^{-r(T-\tau)+\alpha T}\right) \frac{\varpi}{r-\alpha} - I \right)^+ \\ &= \mathrm{e}^{-r\tau^*} \left(\left(\mathrm{e}^{\alpha\tau^*} - \mathrm{e}^{-r(T-\tau^*)+\alpha T}\right) \frac{\varpi}{r-\alpha} - I \right)^+ . \qquad (6.23) \end{aligned}$$

In contrast to the case of an infinite protection period, commercialization has to take place before the patent expires, so that $\tau^* \in [0, T]$.

Proposition 4. *Assuming a moderate growth rate $\alpha \in (0, r)$, the dynamic value of a patent under certainty and finite patent protection is*

$$F(\varpi, 0) = \big(G(\varpi, \tau^*)\big)^+, \tag{6.24}$$

where

$$G(\varpi, \tau^*) = \begin{cases} \left(1 - \mathrm{e}^{-(r-\alpha)T}\right) \frac{\varpi}{r-\alpha} - I & \text{if } \varpi^* < \varpi, \\ \frac{I\alpha}{r-\alpha} \left(\frac{\varpi}{Ir}\right)^{r/\alpha} - \mathrm{e}^{-(r-\alpha)T} \frac{\varpi}{r-\alpha} & \text{if } \mathrm{e}^{-\alpha T}\varpi^* < \varpi \leq \varpi^*, \\ -\mathrm{e}^{-rT} I & \text{otherwise}, \end{cases}$$

and

$$\varpi^* = Ir$$

denotes the critical cash flow rate at time $t = 0$.

The most noticeable difference consists in an additional, *lower* threshold for ϖ, which marks the highest level of profitability that is not yet sufficient to justify commercialization within the protection period.

Proof (Proposition 4). Again, the optimal commercialization time is found by introducing an auxiliary function. Define

$$G(\varpi, \tau) = \mathrm{e}^{-r\tau} \left(\left(\mathrm{e}^{\alpha\tau} - \mathrm{e}^{-r(T-\tau)+\alpha T} \right) \frac{\varpi}{r - \alpha} - I \right). \tag{6.25}$$

As can be verified easily, the derivatives of (6.25) with respect to τ are identical to (6.18) and (6.20), so that the Jorgensonian trigger also applies under finite patent protection.[7] The corresponding lower threshold is determined by solving

$$\tau^* = \frac{1}{\alpha} \ln \frac{Ir}{\varpi} = T$$

for ϖ. Consequently, for all cash flow rates lower than or equal to $\mathrm{e}^{-\alpha T}\varpi^*$, postponing commercialization to the end of the protection period is, in principal, optimal.

For certain parameter values, however, it is optimal to forgo the opportunity to invest, because even optimal commercialization timing results in a loss. Assuming $\varpi^* < \varpi$ and solving for the minimum initial cash flow rate that provides a positive payoff leads to

[7] The detailed derivation is omitted, because it is completely identical to the case of infinite patent protection discussed above.

$$\frac{I\,(r-\alpha)}{1-\mathrm{e}^{-(r-\alpha)T}} < \varpi.$$

If $\varpi \leq \mathrm{e}^{\alpha T}\varpi^*$ the optimal commercialization payoff is clearly negative. Finally, for all initial cash flow rates $\mathrm{e}^{\alpha T}\varpi^* < \varpi \leq \varpi^*$, a threshold below which commercializing is disadvantageous can be determined numerically. The rational investor, of course, requires the payoff to be positive, which implies (6.24).

Similar to the case described in sect. 6.2.1.1, a limited protection period gives rise to situations in which patents are in fact worthless. For the sake of completeness, the optimal commercialization time is also provided.

Corollary 3. *Assuming a moderate growth rate $\alpha \in (0, r)$, the optimal commercialization time of a patent under certainty and finite patent protection is*

$$\tau^* = \begin{cases} 0 & \textit{if } \varpi^* < \varpi, \\ \frac{1}{\alpha}\ln\frac{Ir}{\varpi} & \textit{if } \mathrm{e}^{-\alpha T}\varpi^* < \varpi \leq \varpi^*, \\ T & \textit{otherwise.} \end{cases} \tag{6.26}$$

Proof (Corollary 3). Proposition 4 implies (6.26).

Decreases in ϖ still lead to a higher value of waiting. Nevertheless, falling initial profit rates eventually result in a negative payoff, causing the investor to refrain from commercializing the patent. Thereafter, τ^* becomes completely insensitive to variations in profitability.

6.2.1.2.2.2 Numerical Illustration

As before, a numerical example is chosen to illustrate the sensitivity of patent value to changes in cash flow rates and growth. Figure 6.3 shows how $G(\varpi, \tau^*)$ changes with ϖ under various growth assumptions. Compared to the graphs in fig. 6.2, finite patent protection leads to a downward shift. Patent value in each case corresponds to the non-negative part of the function.

6.2.1.3 Demand-Level Analysis

An obvious extension to the basic model is to analyze a simple monopoly. In the interest of brevity, the discussion is limited to the case of infinite patent protection. To this purpose, an inverse demand function P_t is introduced. Set

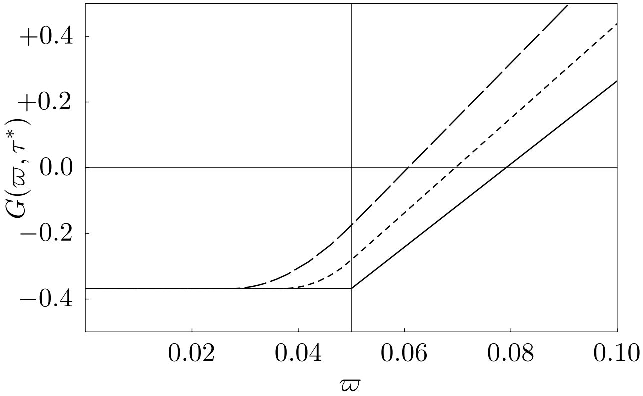


Fig. 6.3. Dynamic patent value under certainty and finite patent protection according to profit-level analysis ($I = 1.0$, $r = 0.05$, $\alpha \in \{0.000, 0.015, 0.030\}$, and $T = 20.0$). The thresholds above which the patent carries positive value are at $\varpi \in \{0.061, 0.070, 0.079\}$.

$$P(y, t, Q_t) = Y_t D(Q_t),$$

where

$$Y_t = Y_0 + \int_0^t \alpha Y_s \,\mathrm{d}s, \qquad Y_0 = y \tag{6.27}$$

represents a time-dependent demand-scaling parameter. Furthermore,

$$D(Q_t) = a\mathrm{e}^{-bQ_t^2},$$

where a and b are positive constants. Admittedly, the inverse demand function employed is non-standard, but has the distinct advantage of providing optimal interior solutions in the absence of a cost component [245]. At each point in time, the rational monopolist chooses a profit-maximizing output level Q_t^*:

$$\begin{aligned} \Pi(y, t, Q_t) &= P(y, t, Q_t^*)Q_t^* \\ &= \max_{Q_t} P(y, t, Q_t)Q_t. \end{aligned} \tag{6.28}$$

Solving

$$\left.\frac{\mathrm{d}\Pi(y, t, Q_t)}{\mathrm{d}Q_t}\right|_{Q_t = Q_t^*} = a\mathrm{e}^{-b(Q_t^*)^2}\mathrm{e}^{\alpha t} y \left(1 - 2b\,(Q_t^*)^2\right) = 0$$

for the optimal output level yields

$$Q_t^* = \frac{1}{\sqrt{2b}}. \tag{6.29}$$

The candidate represents a maximum, because

$$\left.\frac{\mathrm{d}^2 \Pi(y,t,Q_t)}{\mathrm{d}Q_t^2}\right|_{Q_t=Q_t^*} = -2a\sqrt{\frac{2b}{\mathrm{e}}}\mathrm{e}^{\alpha t} y < 0.$$

Substituting (6.29) in (6.28) leads to

$$\Pi(y,t,Q_t^*) = \frac{a}{\sqrt{2b\mathrm{e}}}\mathrm{e}^{\alpha t} y. \tag{6.30}$$

Consequently, the gross present value of cash flows from commercialization is

$$\begin{aligned} V(y,t) &= \int_t^\infty \mathrm{e}^{-r(s-t)} \Pi(y,s,Q_s^*)\,\mathrm{d}s \\ &= \frac{a}{\sqrt{2b\mathrm{e}}} \frac{\mathrm{e}^{\alpha t} y}{r-\alpha}, \qquad 0<\alpha<r. \end{aligned}$$

The objective function, which is analogous to (6.5), (6.14), and (6.23), becomes

$$F(y) = \max_{\tau \in [0,\infty)} \mathrm{e}^{-r\tau} \left(\frac{a}{\sqrt{2b\mathrm{e}}} \frac{\mathrm{e}^{\alpha\tau} y}{r-\alpha} - I \right)^+.$$

Both, patent value and optimal commercialization time, are easily derived.

Proposition 5. *Assuming a moderate growth rate $\alpha \in (0,r)$, the dynamic value of a patent under certainty and infinite patent protection is*

$$F(y) = \begin{cases} \frac{a}{\sqrt{2b\mathrm{e}}} \frac{y}{r-\alpha} - I & \text{if } y^* < y, \\ \frac{I\alpha}{r-\alpha} \left(\frac{a}{\sqrt{2b\mathrm{e}}} \frac{y}{Ir} \right)^{r/\alpha} & \text{otherwise,} \end{cases} \tag{6.31}$$

where

$$y^* = \frac{\sqrt{2b\mathrm{e}}}{a} Ir \tag{6.32}$$

denotes the critical value of the demand-scaling parameter at time $t=0$.

Proof (Proposition 5). According to (6.30),

$$\begin{aligned} \varpi &= \Pi(y,0,Q_0^*) \\ &= \frac{a}{\sqrt{2b\mathrm{e}}} y. \end{aligned} \tag{6.33}$$

Inserting (6.33) into (6.15) yields (6.31). Inserting (6.33) in (6.19) and solving for the critical value of the demand-scaling parameters leads to (6.32).

Of course, the assumption underlying the demand-level model and all previous analyses is that the gross present value of cash flows from commercialization is zero unless exclusivity is granted to the holder of a patent. Consequently, the value of the patent equals total monopoly profits. This simplification is not always appropriate. For example, a duopoly may prove to be a benchmark better reflecting industry structure in the absence of patent protection. To introduce such aspects into the model, however, is a non-trivial task, mainly because strategic interaction during commercialization is likely to affect investment behavior during R&D. Corresponding extensions are therefore postponed to later sections.

Furthermore, the investor is interested in the optimal time to commercialize.

Corollary 4. *Assuming a moderate growth rate $\alpha \in (0, r)$, the optimal commercialization time of a patent under certainty and infinite patent protection is*

$$\tau^* = \begin{cases} 0 & \textit{if } y^* < y \\ \frac{1}{\alpha} \ln\left(\frac{\sqrt{2b\mathrm{e}}}{a}\frac{Ir}{y}\right) & \textit{otherwise.} \end{cases} \tag{6.34}$$

Proof (Corollary 4). Equation (6.34) is completely analogous to (6.22).

As demonstrated in this section, the value of a patent and the time of commercialization that maximizes total profit may be determined based on project-level, profit-level, or demand-level analysis. Project-level analysis fails to fully capture the effects of finite patent protection. Demand-level analysis represents an important step towards a better understanding of competitive interaction during commercialization and R&D, but, under the simplifying assumptions used, provides hardly any new insights into real-world strategic investment policies.

Moreover, deterministic analyses obviously neglect the element of uncertainty, arguably the most influential factor in R&D management. In order to address this important issue, market risk is introduced at this point. In the following, a project-level, profit-level, or demand-level approach is adopted, depending on the respective investment problem at hand.

6.2.2 Stochastic Payoff

In this section, the contingent-claims method is used to derive patent value and optimal timing under uncertainty.[8] Again it is important to distinguish infinite-horizon from finite-horizon analysis.

6.2.2.1 Infinite Protection Period

While the assumption of an infinite protection period may seem fairly restrictive, it greatly facilitates analytical treatment. Of course, the infinite-horizon setting represents an important limiting case of the more realistic model analyzed numerically in sect. 6.2.2.2.

6.2.2.1.1 Analytical Derivation

Quite similar to investment analysis under certainty, the derivation proceeds in two steps: (1) the value of the investment *project,* that is the gross present value of cash flows from commercialization, is calculated; (2) the value of the investment *opportunity,* that is the value of the patent itself, is determined.

6.2.2.1.1.1 Investment Project

Changes in demand have a random component. The obvious way to introduce randomness into (6.27) is to specify

$$Y_t = Y_0 + \int_0^t \alpha Y_s \,\mathrm{d}s + \int_0^t \sigma Y_s \,\mathrm{d}W_s, \qquad Y_0 = y, \tag{6.35}$$

where σ denotes volatility and $W = \{W_t\}_{t \geq 0}$ is a Wiener process. Since expected changes are proportional to the current level of Y_t, (6.35) is an example of geometric Brownian motion (GBM). As demonstrated above, the profit rate Π_t is a multiple of Y_t. Consequently, in the absence of variable costs, Π_t itself can be taken as the stochastic variable. Without loss of generality, Π_t is assumed to follow the process

$$\Pi_t = \Pi_0 + \int_0^t \alpha \Pi_s \,\mathrm{d}s + \int_0^t \sigma \Pi_s \,\mathrm{d}W_s, \qquad \Pi_0 = \varpi. \tag{6.36}$$

[8] The dynamic-programming approach yields identical results and is therefore perfectly equivalent. However, as opposed to contingent-claims analysis, it may also be employed if *spanning* does not hold, but requires an exogneously-specified discount rate under these circumstances. For a brief description of dynamic programming see sect. A.1.1 in the appendix.

Changes in the profit rate are therefore given by the SDE

$$\mathrm{d}\Pi_t = \alpha \Pi_t \,\mathrm{d}t + \sigma \Pi_t \,\mathrm{d}W_t. \tag{6.37}$$

Using Itô's Lemma (see lemma 1 below), it can be solved for the cash flow rate at time t, which is

$$\Pi_t = \Pi_0 \mathrm{e}^{\left(\alpha - \frac{1}{2}\sigma^2\right)t + \sigma W_t}.$$

Together with the balance of the risk-free savings account B_t, Π_t constitutes a complete Black–Scholes market:

$$\begin{cases} \mathrm{d}B_t = rB_t \,\mathrm{d}t, \\ \mathrm{d}\Pi_t = \alpha \Pi_t \,\mathrm{d}t + \sigma \Pi_t \,\mathrm{d}W_t. \end{cases}$$

Assuming infinite patent protection, one obtains the conditional expectation

$$\begin{aligned} V(\Pi_t) &= \int_t^\infty \mathrm{e}^{-\mu(s-t)} \mathbf{E}[\Pi_s \,|\, \mathcal{F}_t] \,\mathrm{d}s \\ &= \int_t^\infty \mathrm{e}^{-\mu(s-t)} \mathrm{e}^{\alpha(s-t)} \Pi_s \,\mathrm{d}s, \end{aligned}$$

where $\mathbf{E}[\cdot]$ is the expectation operator and $\mathcal{F}_t$, loosely speaking, represents the information known at time t [91, p. 72].[9] The required risk-adjusted return according to the capital asset pricing model (CAPM) is

$$\mu = r + \frac{r_\mathrm{m} - r}{\sigma_\mathrm{m}} \sigma\rho, \tag{6.38}$$

where r_m is the rate of return on the market portfolio, σ_m the corresponding standard deviation, and ρ denotes the correlation coefficient.

Note that $\lambda = (r_\mathrm{m} - r)/\sigma_\mathrm{m}$ is the *market price of risk,* and $\beta = \sigma/\sigma_\mathrm{m}\rho$ is the widely-used measure of systematic risk. In summary, project values are determined by discounting expected profits under the real-world probability measures at a risk-adjusted rate. A suitable alternative lies in risk-neutral valuation, where an adjusted probability measure enables the investor to employ the risk-free rate and that is formally introduced below.

Returning to (6.38), project values are obviously bounded for $\alpha < \mu$. Direct integration leads to

---

[9] For a formal definition of the standard Brownian filtration in particular see Steele [314, pp. 50–51].

$$V(\Pi_t) = \frac{\Pi_t}{\mu - \alpha}. \tag{6.39}$$

The same result follows from contingent-claims analysis. In this context, V represents a contingent claim, because it is a function of Π_t. Therefore, the profit rate is also referred to as an "underlying."[10]

To clarify the notion of *spanning,* assume an asset X_t that is perfectly correlated with Π_t [91, pp. 117–119]. In contrast to Π_t, the replicating asset is directly tradeable and can be used to hedge changes in $V(\Pi_t)$. Uncertainty in Π_t is thus tracked, or *spanned,* by financial markets. For reasons of simplicity, X_t is replaced by Π_t in the following calculations. Nevertheless, the reader should keep in mind that a perfect hedge might not always be available.[11]

Assumption 6.3. A single replicating asset or a portfolio of assets, perfectly correlated with the rate of cash flows from commercialization, allows the investor to hedge against demand fluctuations.

Without spanning, the risk-free rate r is replaced by some exogenous discount rate. Determining this discount rate requires restrictive assumptions about the investor's utility function; the CAPM no longer applies [91, p. 152]. Unfortunately, R&D ventures in particular are often unrelated to existing traded assets [91, pp. 147–148]. This deficiency is a fundamental issue shared by all known capital budgeting techniques, rather than a shortcoming specific to option-based analysis.[12]

For valuation purposes, assume the investor sets up a portfolio consisting of the investment project and a short position in the traded asset. Since total project return must equal the required rate of return given by (6.38), beyond capital gains, holders of the spanning asset earn a dividend or convenience yield. Let

$$\delta = \mu - \alpha \tag{6.40}$$

denote this rate of return shortfall. Then (6.39) is just

$$V(\Pi_t) = \Pi_t/\delta. \tag{6.41}$$

Assuming the portfolio includes $-n$ units of Π_t, total portfolio return over a very small time interval $\mathrm{d}t$ becomes

[10] While the notion of an *underlying asset* is still comparatively vague at this point, it will become clearer quickly in the following discussion.

[11] Strictly speaking, a perfect hedge is the rare exception. Nevertheless, existence is a common assumption to facilitate analysis.

[12] In the words of Harrison, the assumption of a replicating asset could rightfully be referred to as a *custom of our tribe* [314, p. 291].

$$\mathrm{d}\Phi(\Pi_t) = \mathrm{d}V(\Pi_t) - n\,\mathrm{d}\Pi_t + (\Pi_t - n\delta\Pi_t)\,\mathrm{d}t. \tag{6.42}$$

Itô's Lemma can be used to expand this expression further. For convenience, it is restated here somewhat informally.

Lemma 1 (Itô's Lemma for scalar processes). *If x_t follows the (scalar) Itô process described by*

$$\mathrm{d}x_t = a(x_t, t)\,\mathrm{d}t + b(x_t, t)\,\mathrm{d}W_t,$$

the contingent claim $f(x_t, t)$ also follows an Itô process, namely

$$\mathrm{d}f(x_t, t) = \frac{\partial f(x_t, t)}{\partial t}\,\mathrm{d}t + \frac{\partial f(x_t, t)}{\partial x_t}\,\mathrm{d}x_t + \tfrac{1}{2}\frac{\partial^2 f(x_t, t)}{\partial x_t^2}(\mathrm{d}x_t)^2$$

or, in expanded form,

$$\begin{aligned}\mathrm{d}f(x_t, t) = \Bigg(\frac{\partial f(x_t, t)}{\partial t} &+ a(x_t, t)\frac{\partial f(x_t, t)}{\partial x_t} \\ &+ \tfrac{1}{2}b^2(x_t, t)\frac{\partial^2 f(x_t, t)}{\partial x_t^2}\Bigg)\,\mathrm{d}t \\ &\qquad + b(x_t, t)\frac{\partial f(x_t, t)}{\partial x_t}\,\mathrm{d}W_t.\end{aligned}$$

Equation (6.37) is thus equivalent to an Itô process with time-invariant parameters. In this case, $x_t \equiv \Pi_t$, $a(\Pi_t, t) \equiv \alpha\Pi_t$, and $b(\Pi_t, t) \equiv \sigma\Pi_t$, and $f(\Pi_t, t) \equiv V(\Pi_t)$. Consequently, combining (6.36) and (6.42) yields

$$\begin{aligned}\mathrm{d}\Phi(\Pi_t) = \left(\alpha\Pi_t\frac{\mathrm{d}V(\Pi_t)}{\mathrm{d}\Pi_t} + \tfrac{1}{2}\sigma^2\Pi_t^2\frac{\mathrm{d}^2V(\Pi_t)}{\mathrm{d}\Pi_t^2}\right)\mathrm{d}t \\ + \sigma\Pi_t\frac{\mathrm{d}V(\Pi_t)}{\mathrm{d}\Pi_t}\,\mathrm{d}W_t - n\left(\alpha\Pi_t\,\mathrm{d}t + \sigma\Pi_t\,\mathrm{d}W_t\right) \\ + (\Pi_t - n\delta\Pi_t)\,\mathrm{d}t.\end{aligned}$$

Setting $n = \mathrm{d}V(\Pi_t)/\,\mathrm{d}\Pi_t$ eliminates $\mathrm{d}W_t$ and thereby all randomness. The portfolio is risk-free and should therefore earn the risk-free rate:

$$\left(\tfrac{1}{2}\sigma^2\Pi_t^2\frac{\mathrm{d}^2V(\Pi_t)}{\mathrm{d}\Pi_t^2} + \Pi_t - n\delta\Pi_t\right)\mathrm{d}t = r\big(V(\Pi_t) - n\Pi_t\big)\,\mathrm{d}t.$$

Dividing by $\mathrm{d}t$ and rearranging terms leads to the differential equation

$$\tfrac{1}{2}\sigma^2 \mathit{\Pi}_t^2 \frac{\mathrm{d}^2 V(\mathit{\Pi}_t)}{\mathrm{d}\mathit{\Pi}_t^2} + (r-\delta)\,\mathit{\Pi}_t \frac{\mathrm{d}V(\mathit{\Pi}_t)}{\mathrm{d}\mathit{\Pi}_t} - rV(\mathit{\Pi}_t) + \mathit{\Pi}_t = 0. \tag{6.43}$$

It applies to any claim with time-invariant payoff and needs to be solved subject to appropriate boundary conditions. Fortunately, the function $V(\mathit{\Pi}_t)$ is already known. Simply inserting (6.41) verifies that the present value of future cash flows indeed happens to be one solution to (6.43). It can be shown that (6.41) is in fact the only solution that is not due to speculative bubbles, but represents a "fundamental component" of value [91, pp. 181–182].

As demonstrated by this basic example, contingent-claims analysis is a suitable tool for capital budgeting if payoffs are stochastic. Furthermore, there is a way of obtaining the same result that explicitly involves the market price of risk. From a technical perspective, choosing a market price of risk is the same as defining a *probability measure.* The market price of risk under the real-world measure $\mathbf{P}$ is $\lambda = (r_{\mathrm{m}} - r)/\sigma_{\mathrm{m}}$, and $\mathit{\Pi}_t$ grows at the rate $\alpha = \mu - \delta$. In contrast, the dynamics of $\mathit{\Pi}_t$ in a risk-neutral world are described by

$$\mathrm{d}\mathit{\Pi}_t = \alpha^* \mathit{\Pi}_t \,\mathrm{d}t + \sigma \mathit{\Pi}_t \,\mathrm{d}W_t,$$

where $\alpha^* = r - \delta$.

An alternative procedure leading to the gross present value of cash flows from commercialization thus lies in taking expectations under the transformed measure $\mathbf{P}^*$ and then simply discounting at the risk-free rate [91, pp. 123–124]:

$$\begin{aligned} V(\mathit{\Pi}_t) &= \int_t^\infty \mathrm{e}^{-r(s-t)} \mathbf{E}_{\mathbf{P}^*}[\mathit{\Pi}_s \,|\, \mathcal{F}_t]\,\mathrm{d}s \\ &= \int_t^\infty \mathrm{e}^{-r(s-t)} \mathrm{e}^{(r-\delta)(s-t)} \mathit{\Pi}_s \,\mathrm{d}s = \mathit{\Pi}_t/\delta. \end{aligned}$$

This technique is known as risk-neutral valuation [137, 138]: Under $\mathbf{P}^*$, $\{\mathrm{e}^{-(r-\delta)t}\mathit{\Pi}_t\}_{t\geq 0}$ is a martingale, that is a zero-drift stochastic process. Therefore, $\mathbf{P}^*$ is an equivalent martingale probability measure.[13] In complete markets, the absence of arbitrage implies the existence of a unique equivalent risk-neutral measure. Identifying an appropriate measure if markets are incomplete, an issue to be addressed in chapter 7, is non-trivial.

[13] For an accessible and entertaining introduction to martingales see [314, pp. 11–60].

6.2.2.1.1.2 Investment Opportunity

The optimization problem faced by the patentholder can be expressed in terms of the risk-neutral valuation technique, namely as

$$\begin{aligned} F(\Pi_t) &= \sup_{\tau\in[t,\infty)} \mathrm{e}^{-r(\tau-t)}\mathbf{E}_{\mathbf{P}^*}\left[\left(V(\Pi_\tau)-I\right)^+ \,\middle|\, \mathcal{F}_t\right] \\ &= \mathrm{e}^{-r(\tau^*-t)}\mathbf{E}_{\mathbf{P}^*}\left[\left(V(\Pi_{\tau^*})-I\right)^+ \,\middle|\, \mathcal{F}_t\right]. \end{aligned}$$

The same procedure employed in determining the gross present value of cash flows from commercialization also makes it possible to calculate patent value under uncertainty.

Proposition 6. *Assuming a positive rate of return shortfall δ, the dynamic value of a patent under uncertainty and infinite patent protection is*

$$F(\Pi_t) = \begin{cases} \Pi_t/\delta - I & \textit{if } \Pi^* < \Pi_t, \\ A^+\Pi_t^{\gamma^+} & \textit{otherwise}, \end{cases} \tag{6.44}$$

where

$$\Pi^* = \frac{\gamma^+}{\gamma^+-1} I\delta \tag{6.45}$$

denotes the critical cash flow rate,

$$\begin{aligned} A^+ &= \frac{I}{\gamma^+-1}\left(\frac{1}{\Pi^*}\right)^{\gamma^+} \\ &= \left(\frac{1}{\gamma^+\delta}\right)^{\gamma^+}\left(\frac{\gamma^+-1}{I}\right)^{\gamma^+-1}, \end{aligned} \tag{6.46}$$

and

$$\gamma^+ = \tfrac{1}{2} - \frac{r-\delta}{\sigma^2} + \sqrt{\left(\frac{r-\delta}{\sigma^2}-\tfrac{1}{2}\right)^2 + 2\frac{r}{\sigma^2}}. \tag{6.47}$$

The proof follows standard arguments from the option pricing literature [233, 238, 290].

Proof (Proposition 6). Again, a risk-free portfolio is constructed, this time from the investment opportunity and a short position of n units of a suitable spanning asset. Changes in portfolio value are given by

$$\mathrm{d}\Phi(\Pi_t) = \mathrm{d}F(\Pi_t) - n\,\mathrm{d}\Pi_t - n\delta\Pi_t\,\mathrm{d}t.$$

Of course, in contrast to (6.42), there is no profit flow $\Pi_t \, \mathrm{d}t$, because commercialization has yet to take place. Following the steps of contingent-claims analysis outlined above, one obtains

$$\tfrac{1}{2}\sigma^2 \Pi_t^2 \frac{\mathrm{d}^2 F(\Pi_t)}{\mathrm{d}\Pi_t^2} + (r-\delta)\,\Pi_t \frac{\mathrm{d}F(\Pi_t)}{\mathrm{d}\Pi_t} - rF(\Pi_t) = 0, \tag{6.48}$$

which is a homogeneous linear equation of second order. It holds in the continuation region, where the cash flow rate is below the threshold and postponement is optimal. In the stopping region, where the cash flow rate exceeds the critical value, the rational investor commercializes immediately, so that $F(\Pi_t) = \Pi_t/\delta - I$. A general solution to (6.48) takes the form

$$F(\Pi_t) = A^+ \Pi_t^{\gamma^+} + A^- \Pi_t^{\gamma^-}, \tag{6.49}$$

where A^+, A^- are constants to be determined and γ^+, γ^- are roots of the quadratic equation

$$\tfrac{1}{2}\sigma^2 \gamma\,(\gamma-1) + (r-\delta)\,\gamma - r = 0, \tag{6.50}$$

sometimes referred to as the "fundamental quadratic" [91, pp. 142–143]. Assuming γ^+ is the positive root, $A^- = 0$, because patent value should become zero if the investment project itself is worthless. Imposing C^1-continuity at $\Pi_t = \Pi^*$ leads to the so-called *value-matching* and *smooth-pasting* conditions,[14] namely

$$A^+ \left(\Pi^*\right)^{\gamma^+} = \Pi^*/\delta - I$$

and

$$\gamma^+ A^+ \left(\Pi^*\right)^{\gamma^+ - 1} = 1/\delta.$$

Solving this system of equations leads to (6.46) and (6.47), which proves that (6.44) in fact represents the value of a patent under uncertainty.

If the value of the investment project, as in this case, is a constant multiple of Π_t, patent value can be obtained by working directly in terms of V_t, which follows a stochastic process with identical drift and volatility [91, p. 184]. Of course, this procedure requires the protection period to be infinite.

[14] For in-depth discussions of such conditions see Brekke and Øksendal [52], Dumas [96], Shackleton and Sødal [306, 307].

There is a close connection between (6.45) and the Jorgensonian trigger presented in sect. 6.2.1.2. For moderate growth rates under certainty, by (6.38) and (6.40), $\delta = r - \alpha < r$. Since

$$\lim_{\sigma \to 0} = \frac{\gamma^+}{\gamma^+ - 1} = \begin{cases} r/\delta & \text{if } \delta < r, \\ 1 & \text{otherwise,} \end{cases}$$

the critical value at time $t = 0$ becomes $\Pi^* = Ir$, which is identical to (6.16). Therefore, the new investment rule represents a generalization that converges to the deterministic case for very small σ [91, pp. 144 and 184]. The fraction $\gamma^+/(\gamma^+ - 1)$ is an "option value multiple," which happens to be identical to Tobin's q in conventional capital budgeting [91, pp. 146–147].

Remarkably, in order to calculate patent value, it is not essential to know μ and α, but it suffices to know the rate of return shortfall δ, which reflects the opportunity cost of keeping the option alive. If this opportunity cost is zero, commercialization never takes place, regardless of expected payoff [91, pp. 147–150].

Although, due to an infinite protection period, the critical value is constant, τ^* is obviously random. The optimal time to commercialize is the time Π_t exits the continuation region given by (6.45):

$$\tau^* = \inf\{t \geq 0 : \Pi^* < \Pi_t\}.$$

Since this specification suffices as an investment policy, the characteristics of τ^* are not discussed further at this point.[15] Instead, consider the following numerical example.

6.2.2.1.2 Numerical Illustration

All other things equal, additional demand uncertainty is value-creating, because commercialization takes place only under favorable market conditions: on the one hand, the patent has significant upside potential; on the other hand, the holder bears little downside risk. This relationship, well-known in the field of financial option pricing, is depicted in fig. 6.4. Assuming an investment amount of $I = 1.0$ and a risk-free rate of $r = 0.05$, the critical project value rises from 0.050 to 0.093 as σ increases from 0.0 to 0.2.

When examining the value of the investment opportunity as a function of the underlying asset value, graphs turn out to be perfectly similar

[15] The expected exit time in this particular case is $\mathbf{E}[\tau^* \mid \mathcal{F}_t] = 1/\left(\alpha - \frac{1}{2}\sigma^2\right) \ln\left(\Pi^*/\Pi_t\right)$, where $\frac{1}{2}\sigma^2 < \alpha$ [337, p. 371]. For a brief introduction to the mathematical procedure see Dixit [90, pp. 54–57].

to those of options on stock, demonstrating that patents are, in a way, similar to financial instruments and thereby can be thought of as *real options.* An infinite protection period makes the investment opportunity analogous to a perpetual American call [238].

Observation 6.1. Patents, under certain assumptions, exhibit characteristics typically found in financial derivatives. This analogy can be used for valuation purposes.

This observation is neither revolutionary, nor does it come at a surprise, because assumptions 6.1 through 6.3 were deliberately chosen to demonstrate a similarity between real and financial investments. It is important to note, however, that the widely-known similarity is not limited to *tangible* assets, but also extends to *intangible* assets, including, of course, intellectual property in the form of patents.

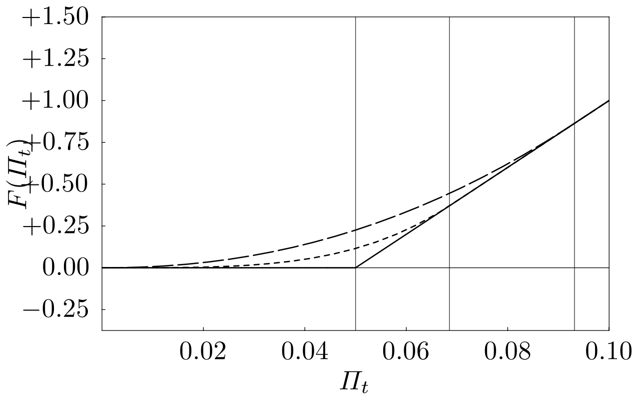


Fig. 6.4. Dynamic patent value under uncertainty and infinite patent protection as a function of profitability ($I = 1.0$ and $r = \delta = 0.05$). Vertical lines indicate critical profit rates at $\Pi_t \in \{0.050, 0.069, 0.093\}$ for various levels of uncertainty ($\sigma \in \{0.0, 0.1, 0.2\}$). The higher the volatility of cash flow rates, the higher the investment threshold [figure adapted from 91, 154].

Another key aspect concerns the role of demand uncertainty in patent valuation. To be precise, the sensitivity of patent value to the volatility of cash flow rates depends on how the rate of return shortfall δ is affected by changes in the uncertainty parameter σ [91, p. 155]. According to (6.38) and (6.40), higher levels of uncertainty tend to increase δ, as long as cash flow rates are positively correlated with market returns, and the instantaneous growth rate of the replicating asset remains constant. Consequently, the gross payoff given by (6.41) drops, thereby counterbalancing a rising value of flexibility:

$$\Pi_t/\delta = \Pi_t \left(\frac{r_{\mathrm{m}} - r}{\sigma_{\mathrm{m}}} \sigma - \alpha \right)^{-1} .$$

Figure 6.5 depicts the suggested relationship: patent value declines rapidly as a result of rising volatility. This observation reveals a common misconception in connection with option-based capital budgeting, namely that increasing demand uncertainty is value-enhancing under all circumstances. In many cases, however, it is more convenient to assume δ remains fixed, for example because rising uncertainty also goes along with a higher growth rate α. Unless stated otherwise, this view is adopted in further analyses [91, pp. 178–179].

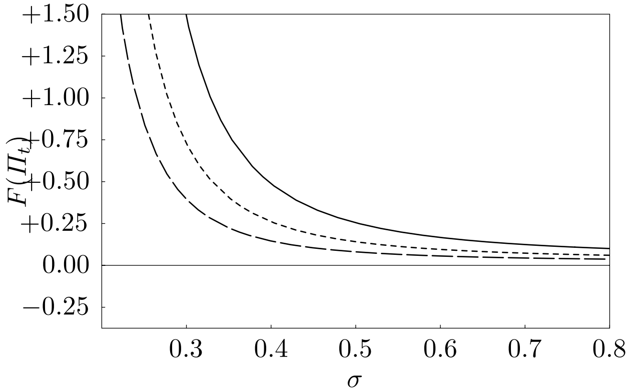


Fig. 6.5. Dynamic patent value under uncertainty and infinite patent protection as a function of cash flow rate volatility ($\Pi_t = 0.05$, $I = 1.0$, $r = 0.05$, $\alpha = 0.08$, $(r_{\mathrm{m}} - r)/\sigma_{\mathrm{m}} = 0.4$, $\rho \in \{0.6, 0.7, 0.8\}$). Contrary to common economic knowledge, but in line with intuition, adverse effects of rising uncertainty on patent value can be observed if δ is assumed to depend on σ. Lower correlation coefficients are associated with less systematic risk and, as a consequence, higher patent values.

Figure 6.6 illustrates how the value of a patent changes randomly over time. A simple discretization is used to generate a sample path with $\varpi = 0.05$, $\alpha = 0.04$, $\sigma = 0.2$, and $r = \delta = 0.05$. The net payoff from commercialization, that is the value of the investment *project,* may even drop below zero, whereas the value of the investment *opportunity* before commercialization stays positive. Equations (6.47) and (6.45) lead to

$$\Pi^* = \frac{1.86}{1.86 - 1} \times 1.0 \times 0.05 = 0.09,$$

which is the constant investment threshold under uncertainty. As soon as the cash flow rate exceeds this critical value, the investor should prefer immediate commercialization over postponement.

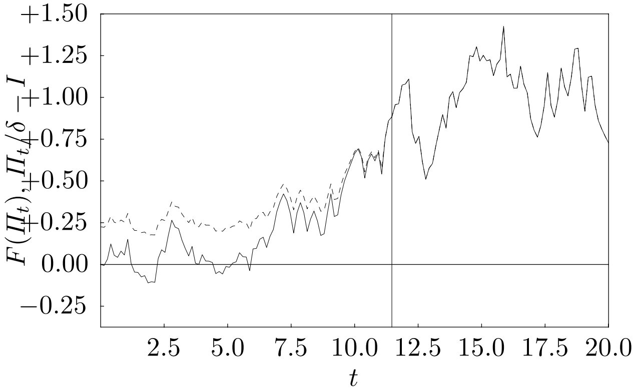


Fig. 6.6. Changes in dynamic patent value under uncertainty and infinite patent protection over time ($I = 1.0$, $r = \delta = 0.05$, $\varpi = 0.05$, $\alpha = 0.04$, and $\sigma = 0.2$). A single vertical line marks the optimal commercialization time $\tau^* = 11.47$ for the particular realization shown. Below the threshold, a solid line represents the payoff from immediate commercialization, while the line representing actual patent value is dashed [figure adapted from 91, 160].

6.2.2.2 Finite Protection Period

The impact of a finite protection period on patent value crucially depends on the investor's ability to maintain profitability beyond the expiration date. While the profit rate is assumed to drop to zero in sect. 6.2.1, this section also examines the opposite case of an infinite commercialization phase, which is computationally similar, but, as expected, yields fundamentally different results.

6.2.2.2.1 Infinite Commercialization Phase

First assume that, even under finite patent protection, the gross payoff from commercialization stays constant until expiration, so that (6.41) continues to hold. Nevertheless, market entry is impossible once the patent has expired.[16] Equation (6.48) becomes

[16] It is important to carefully distinguish the length of the protection period from the length of the commercialization phase.

$$\tfrac{1}{2}\sigma^2 \Pi_t^2 \frac{\partial^2 F(\Pi_t,t)}{\partial \Pi_t^2} + (r-\delta)\,\Pi_t \frac{\partial F(\Pi_t,t)}{\partial \Pi_t} - rF(\Pi_t,t) + \frac{\partial F(\Pi_t,t)}{\partial t} = 0, \quad (6.52)$$

because patent value depends on the current time t. In addition, a valid solution has fulfill certain constraints. Specifically,

$$F(\Pi_T,T) = (\Pi_T/\delta - I)^+\,, \quad (6.53a)$$

$$F(0,t) = 0, \quad (6.53b)$$

and

$$\lim_{\Pi_t\to\infty} \frac{\partial^2 F(\Pi_t,t)}{\partial \Pi_t^2} = 0, \quad (6.53c)$$

At the end of the protection period, the investor chooses between commercializing immediately and letting the patent expire. Again, in the absence of profits, the patent is worthless. Equation (6.53c) represents a generic boundary condition that is independent of the type of contract and applies as long as the payoff is at most linear in the underlying [336, p. 642]. Furthermore, the usual value-matching and smooth-pasting conditions apply.

Although there is no closed-form solution to this problem, numerical procedures may be employed to derive patent value.[17] Since the PDE shown above is identical to the one developed by Black and Scholes [35], Merton [238] for financial options, suitable methods are readily available, for example the Crank–Nicolson variant of the FD method [336, pp. 639–640].[18] However, the numerical scheme is usually not applied directly. To facilitate calculations, consider the transformation

$$u \equiv T - t \quad (6.54a)$$

and

$$F(\Pi_u,u) \equiv \mathrm{e}^{-ru} G(\Pi_u,u). \quad (6.54b)$$

Equation (6.54a) reverses time, transforming (6.52) into a genuine intial value problem. Equation (6.54b) eliminates the "reaction term"

[17] For details see Paddock et al [263], Dixit and Pindyck [91, pp. 396–405].

[18] Originally, the method was devised by Courtadon [73], Crank and Nicolson [79].

$-rF(\Pi_t, t)$ by changing the problem from present value to future value terms.[19]

Hence,

$$\frac{\partial F(\Pi_t,t)}{\partial \Pi_t} = \mathrm{e}^{-ru}\frac{\partial G(\Pi_u,u)}{\partial \Pi_u},$$
$$\frac{\partial^2 F(\Pi_t,t)}{\partial \Pi_t^2} = \mathrm{e}^{-ru}\frac{\partial^2 G(\Pi_u,u)}{\partial \Pi_u^2},$$

and

$$\frac{\partial F(\Pi_t,t)}{\partial t} = \mathrm{e}^{-ru}\left(rG(\Pi_u,u) - \frac{\partial G(\Pi_u,u)}{\partial u}\right),$$

so that (6.52) becomes

$$\tfrac{1}{2}\sigma^2\Pi_u^2\frac{\partial^2 G(\Pi_u,u)}{\partial \Pi_u^2} + (r-\delta)\,\Pi_u\frac{\partial G(\Pi_u,u)}{\partial \Pi_u} - \frac{\partial G(\Pi_u,u)}{\partial u} = 0.$$

An $(m+1)$-by-$(n+1)$ grid is constructed, where $m \equiv \Pi_{\max}/\Delta\Pi$ and $n \equiv T/\Delta u$. Nodes correspond to $G(\Pi_u, u)$ with Π_u and u assuming discrete values, that is

$$\Pi_u \in \{0, \Delta\Pi, \ldots, i\Delta\Pi, \ldots, \Pi_{\max} - \Delta\Pi, \Pi_{\max}\},$$

and

$$u \in \{0, \Delta u, \ldots, j\Delta u, \ldots, T - \Delta u, T\}.$$

Furthermore, denote $G(\Pi_i, u_j)$ by $G_{i,j}$. Using the approximations

$$\left.\frac{\partial G(\Pi_u,u)}{\partial \Pi_u}\right|_{u=u_j}^{\Pi_u=\Pi_i} \approx \tfrac{1}{2}\frac{G_{i+1,j+1} - G_{i-1,j+1}}{2\Delta\Pi} + \tfrac{1}{2}\frac{G_{i+1,j} - G_{i-1,j}}{2\Delta\Pi},$$
$$\left.\frac{\partial^2 G(\Pi_u,u)}{\partial \Pi_u^2}\right|_{u=u_j}^{\Pi_u=\Pi_i} \approx \tfrac{1}{2}\frac{G_{i+1,j+1} + G_{i-1,j+1} - 2G_{i,j+1}}{\Delta\Pi^2} + \tfrac{1}{2}\frac{G_{i+1,j} + G_{i-1,j} - 2G_{i,j}}{\Delta\Pi^2},$$

and

[19] This manipulation could of course be taken further, for example by adopting a log-transformed model, where $u \equiv T - t$, $\Psi_u \equiv \ln(\Pi_u)$, and $F(\Pi_u, u) \equiv \mathrm{e}^{-ru}G(\Psi_u, u)$, leading to constant coefficients. The slight computational advantage achieved is irrelevant in this context [336, pp. 82 and 92–93].

$$\left.\frac{\partial G(\Pi_u, u)}{\partial u}\right|_{u=u_j}^{\Pi_u=\Pi_i} \approx \frac{G_{i,j+1} - G_{i,j}}{\Delta u}$$

one obtains

$$aG_{i-1,j} + (1+b)G_{i,j} + cG_{i+1,j} = \\ - aG_{i-1,j+1} + (1-b)G_{i,j+1} - cG_{i+1,j+1}, \quad (6.57)$$

where

$$a_i = -\tfrac{1}{4} i\Delta u\big(i\sigma^2 - (r-\delta)\big),$$
$$b_i = \tfrac{1}{2} i^2 \Delta u \sigma^2,$$
$$c_i = -\tfrac{1}{4} i\Delta u\big(i\sigma^2 + (r-\delta)\big).$$

Equation (6.57) relates three adjacent nodes on time level $j+1$ to three nodes on the preceding level j. To obtain a solution, initial and boundary conditions must be translated into their FD equivalent. A central difference approximation is used to calculate the second derivative around $G_{m-1,j+1}$:

$$\frac{G_{m,j+1} + G_{m-2,j+1} - 2G_{m-1,j+1}}{\Delta \Pi^2} = 0.$$

Consequently, for all $j \in \{0, 1, \dots, n-1\}$,

$$G_{m,j+1} = 2G_{m-1,j+1} - G_{m-2,j+1},$$

and

$$G_{0,j+1} = 0.$$

In addition,

$$G_{i,0} = \left(\frac{i\Delta \Pi}{\delta} - I\right)^+$$

for all $i \in \{0, 1, \dots, m\}$. Using this information as a starting point and proceeding backwards in time, approximate patent values are obtained at each step. However, due to timing flexibility, it is not sufficient to simply solve the resulting matrix equations. As the net payoff from immediate commercialization may never exceed the value of the patent, the following constraint has to be fulfilled at all times:

$$V(\Pi_t) - I = \Pi_t/\delta - I \le F(\Pi_t, t), \quad (6.60)$$

where $V(\Pi_t) - I$, in analogy to financial options, is referred to as the "inner value" of the real option. The time-dependent threshold, beyond which the rational investor prefers to commercialize immediately and which thus separates the continuation from the stopping region, constitutes a *free boundary* that is found together with the solution. To preserve the accuracy of the Crank–Nicolson scheme, options of this type are usually priced using PSOR, instead of resorting to LU decomposition and simply post-processing the column vectors to account for timing flexibility [81].

The resulting patent value as a function of profit rate and time is depicted in fig. 6.7(a). Due to the implicit assumption of an infinite commercialization phase, the diagram still closely resembles that of a plain-vanilla (American) call option—not of a perpetual one, but one with finite maturity T. The free boundary is depicted in fig. 6.7(b). The longer the time to expiration, the higher the profit rate required to trigger commercialization. At the end of the protection period, the value of waiting is zero. Therefore, the critical profit rate becomes $\Pi_T^* = I\delta$. In contrast, the trigger at time $t = 0$ quickly converges to (6.45) as T grows larger [91, p. 401]. However, investment thresholds differ substantially when the commercialization phase is finite.

Observation 6.2. Patents with a finite lifetime are analogous to plain-vanilla call options as long as the commercialization phase following investment is infinite.

6.2.2.2.2 Finite Commercialization Phase

When the commercialization phase is finite, that is the period during which cash flows are accumulated ends with the expiration of the patent, the gross present value of cash flows from commercialization under uncertainty decreases to

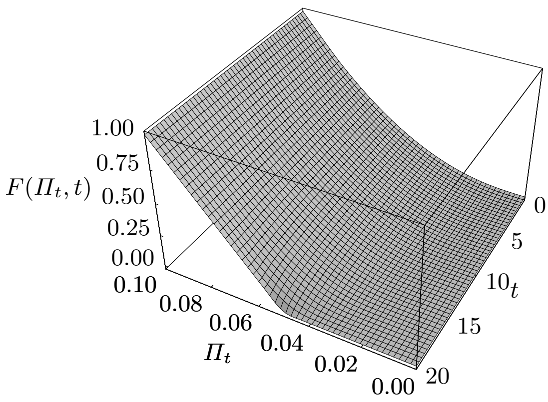
$$V(\Pi_t, t) = \mathbf{E}\left[\int_t^T \mathrm{e}^{-\mu(s-t)} \Pi_s \, \mathrm{d}s \,\middle|\, \mathcal{F}_t\right]. \tag{6.61}$$

Since $\mathbf{E}[\Pi_s \,|\, \mathcal{F}_t] = \mathrm{e}^{\alpha(s-t)} \Pi_t$ and $\delta = \mu - \alpha$, integration yields

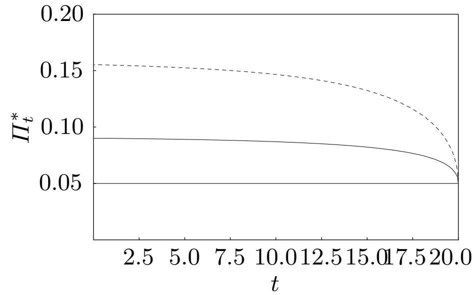
$$V(\Pi_t, t) = \left(1 - \mathrm{e}^{-\delta(T-t)}\right) \Pi_t / \delta. \tag{6.62}$$

Equations (6.60) and (6.53a) become

$$F(\Pi_t, T) = 0$$



(a) Patent value



(b) Critical profit rates

Fig. 6.7. Dynamic patent value under uncertainty and finite patent protection when the commercialization phase is infinite ($I = 1.0$, $r = \delta = 0.05$, and $T = 20.0$). Panel (a) shows approximate patent values as a function of profit rate and time ($\sigma = 0.2$). Critical profit rates for various levels of uncertainty ($\sigma \in \{0.0, 0.2, 0.4\}$) are depicted in (b). Under certainty, the boundary becomes a horizontal line.

and

$$V(\Pi_t, t) - I = \left(1 - \mathrm{e}^{-\delta(T-t)}\right) \Pi_t/\delta - I \leq F(\Pi_t, t).$$

Boundary conditions, (6.53b) and (6.53c), still apply.

In essence, a finite commercialization phase represents a form depreciation by sudden death [91, p. 205]. The "profit generator," in this case a patent and not a machine, ceases to function at the end of the protection period. If the present value integral, unlike (6.61), is difficult to evaluate, it is most convenient to employ contingent-claims analysis as demonstrated in sect. 6.2.2.1 [91, pp. 205–206].

Relying on the numerical procedures outlined above, it is straightforward to obtain approximate patent values.[20] Figure 6.8(a) illustrates how the value of a patent depends on profit rate and time. Compared to the model analyzed in the previous subsection, patent values are substantially lower and eventually drop to zero. The time-dependent investment threshold is shown in fig. 6.8(b). It is quite different from the boundary calculated above, as the critical cash flow rate in fact is not a decreasing, but an increasing function of time. More importantly, infinite patent protection no longer is a suitable approximation for analyzing the investment opportunity.

Observation 6.3. Assuming an infinite protection period in the valuation of patents as real options leads to substantial overpricing and faulty investment policy.

While this result seems obvious from a practical perspective, formal option-based analyses of patenting in continuous time are typically based on the assumption of an infinite protection period, enabling the derivation of closed-form solutions.

Regardless of these differences, uncertainty, measured by the volatility of cash flow rates, continues to be an important parameter. The longer the protection period, the higher the sensitivity of the investment trigger to variations in σ. Again, the impact of uncertainty is determined by the relationship between volatility and the rate of return shortfall. Since the arguments brought forward in sect. 6.2.2.1 are also applicable in this context, no detailed analysis is provided at this point.

The option-based view has far-reaching implications. For example, consider a situation in which the investor has to decide on the acquisition of a patent with a lifetime of 20 years. The expected present value of future profits is around 0.63 million euros, which seems too low to justify the upfront investment of 1.00 million euros that is required to turn the patent into a marketable product. A closer look at the decision problem reveals additional option value, because the investor is not obliged to commercialize immediately.

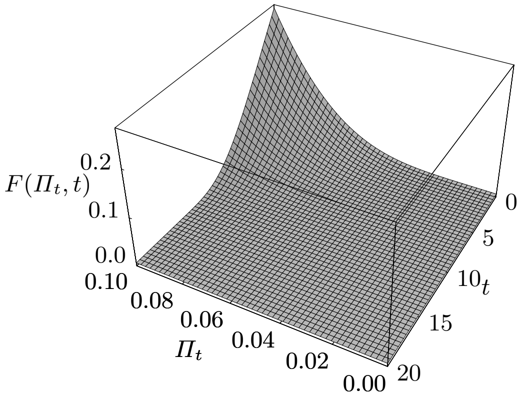
Cash flow rates are perfectly correlated with a stock portfolio.[21] The standard deviation of returns on this portfolio is 0.02. The rate of return

[20] Alternatively, this option can be priced using a simple binomial tree. See sect. B.1.1 in the appendix for details.

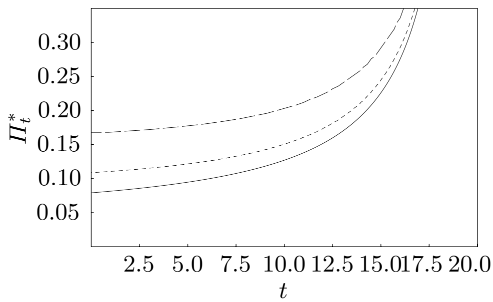
[21] If this assumption is dropped, a precise value cannot be determined [147].

Table 6.1. Dynamic patent value under uncertainty and finite patent protection when the commercialization phase is finite ($I = 1.0$ and $r = \delta = 0.05$). Longer protection periods, higher volatilities, and higher cash flow rates are associated with higher patent values. Note that this relationship hinges on the assumption of a fixed rate of return shortfall.

Volatility (σ)	Protection period (T)	Patent value		
		$\Pi_t = 0.06$	$\Pi_t = 0.08$	$\Pi_t = 0.10$
0.20	10	0.00	0.00	0.02
	12	0.00	0.01	0.04
	14	0.00	0.03	0.07
	16	0.01	0.05	0.12
	18	0.02	0.09	0.19
	20	0.04	0.15	0.28
0.24	10	0.00	0.01	0.03
	12	0.00	0.03	0.06
	14	0.01	0.05	0.10
	16	0.02	0.08	0.15
	18	0.04	0.12	0.22
	20	0.06	0.18	0.30
0.28	10	0.00	0.02	0.05
	12	0.01	0.04	0.08
	14	0.02	0.07	0.13
	16	0.04	0.10	0.18
	18	0.06	0.15	0.25
	20	0.08	0.20	0.33
0.32	10	0.01	0.03	0.07
	12	0.02	0.06	0.11
	14	0.03	0.09	0.16
	16	0.05	0.13	0.21
	18	0.08	0.18	0.28
	20	0.11	0.23	0.36
0.36	10	0.02	0.05	0.09
	12	0.03	0.08	0.13
	14	0.05	0.11	0.18
	16	0.07	0.16	0.25
	18	0.10	0.21	0.32
	20	0.13	0.26	0.39
0.40	10	0.02	0.06	0.11
	12	0.04	0.10	0.16
	14	0.06	0.14	0.21
	16	0.09	0.18	0.28
	18	0.12	0.23	0.35
	20	0.16	0.29	0.43



(a) Patent value



(b) Critical profit rates

Fig. 6.8. Dynamic patent value under uncertainty and finite patent protection when the commercialization phase is finite ($I = 1.0$, $r = \delta = 0.05$, and $T = 20.0$). Panel (a) shows how patent value declines over time ($\sigma = 0.2$). Panel (b) illustrates how the time-dependent stochastic trigger Π_t^* converges to the Marshallian one, indicated by the solid line, as volatility decreases ($\sigma \in \{0.0, 0.2, 0.4\}$).

shortfall is 0.05. According to (6.62), the current cash flow rate is close to 0.05 million euros per unit of time:

$$\begin{aligned}\Pi_0 &= \frac{e^{\delta T}}{e^{\delta T} - 1} V(\Pi_0, 0)\delta \\ &= \frac{2.71}{2.71 - 1} \times 0.63 \times 0.05 = 0.05.\end{aligned}$$

Based on the very stylized model presented, patent value amounts to almost 0.02 million euros (see table 6.1). Hence, the rational investor should be willing to pay as much as 20,000 euros for the opportunity to invest and enter the market in later years, although commercialization is unattractive under current circumstances. The net present value rule and the option-based approach yield fundamentally different results.

In addition, it is worth pointing out similarities between the proposed setup and the classic model by Paddock et al, in which offshore petroleum leases are treated as real options [263]. Similar to the valuation of patents, it is important to account for a finite time horizon, which, for development reserves, is due to a relinquishment requirement. Also, GBM might be a poor approximation of the true value process in both cases. An alternative specification is presented in chapter 7.

If one chooses to adopt a project-level approach, instead of resorting to numerical methods, the value of a patent can be obtained via analytical approximations developed for the American call option [25, 209].[22] The resulting equations are formally similar to those for perpetual options and can be solved iteratively. For obvious reasons, the impact of a finite commercialization period is not properly captured under these circumstances.

In summary, a patent under uncertainty represents a real option on commercialization. Consequently, it can be analyzed in analogy to financial derivatives. However, although the framework presented is intuitively appealing, patents, like all real investment opportunities, differ substantially from standardized financial contracts. In the following, the discussion focuses on one particular aspect, namely imperfect patent protection due to the risk of litigation.

[22] For a more thorrough discussion of analytical approximations see Hull [150, pp. 425 and 432–434].

7

Patent Risk as Jumps in the Underlying Process

Realistic option-based models account explicitly for at least two stages with distinct characteristics, namely R&D and commercialization. Properly capturing the pecularities of each stage usually means introducing additional sources of risk, including, for example, cost uncertainty or a random time to completion. Before proceeding to this level of sophistication, however, it is worth studying patent risk using much simpler single-stage, single-factor models.

7.1 Single-Stage and Single-Factor Models

Not only the frequency, but also the size of jumps in project value caused by patent-related events may be stochastic. The case of deterministic jump size is discussed first, followed by an extension to randomly distributed jumps.

7.1.1 Deterministic Jump Size

Introducing the concept of patent risk as jumps in an otherwise continuous price process, a project-level model serves to illustrate how patent risk, in principal, affects patent value. Then, a profit-level model is employed to examine the effect of finite patent protection.

7.1.1.1 Project-Level Analysis

Following the derivation of general valuation formulae, comparative statics are presented to examine the dependence of patent value on patent quality.

7.1.1.1.1 Analytical Derivation

The jump-diffusion model used in this section extends the Itô process describing project value dynamics by adding an additional term $\mathrm{d}J_t$ representing the increment of a Poisson process with intensity λ:

$$\mathrm{d}V_t = \alpha V_t\,\mathrm{d}t + \sigma V_t\,\mathrm{d}W_t - \phi V_t\,\mathrm{d}J_t, \qquad 0 \le \phi \le 1. \tag{7.1}$$

Hence, changes in J are given by

$$\mathrm{d}J_t = \begin{cases} 1 & \text{with probability } \lambda\,\mathrm{d}t, \\ 0 & \text{with probability } 1 - \lambda\,\mathrm{d}t. \end{cases} \tag{7.2}$$

Within an infinitesimally small time interval $\mathrm{d}t$, a jump occurs with probability $\lambda\,\mathrm{d}t$, that is λ denotes the mean arrival rate of a value-relevant event [91, p. 115]. Upon the occurrence of such an event, V_t instantaneously drops by ϕV_t. The relative size of jumps is thus fixed.[1]

A variety of scenarios are conceivable, which serve as practical examples of events reducing the payoff from commercialization: (1) Should competitors succeed in obtaining a patent enabling them to market a substitute, decreased profits will be the result. (2) Also, if a court ruling finds the original patent to be invalid or to unlawfully restrict the rights of another patent holder, profits will drop.

The value of a patent is tied to some underlying stochastic process and is subject to a kind of default risk. In addition, the investor may choose to give up flexibility in exchange for volatile cash flows at any point in time, making the patent somewhat similar to a defaultable claim or, more specifically, convertible debt [33, p. 33]. Although this similarity justifies an investigation into the applicability of credit risk models in patent valuation, these models are usually quite complex and, as a consequence, often require numerical methods. In any case, the analogy can be exploited to gain insights into the nature of patent risk from a quantitative perspective. In particular, the formalization of patent risk presented here is conceptually related to the subclass of instantaneous-risk-of-default models [336, pp. 565–567 and 580–582].

Observation 7.1. Patents are defaultable contingent claims comparable to convertible bonds.

Put aside conceptual issues, difficulties in connection with discrete jumps in the value process arise due to a pivotal assumption underlying

[1] For a graphical illustration of this type of model see also fig. 7.3 in sect. 7.1.1.2.

contingent-claims analysis, namely the possibility to continuously rebalance a replicating portfolio, which creates a position free of risk (see sect. 6.2.2). Whereas continuous diffusion processes allow for dynamic hedging strategies, these techniques can no longer be applied once a jump term is added to the corresponding PDE.

A pragmatic solution to this issue is offered by common equilibrium models of financial markets. If discontinuous changes in the value of the underlying asset are unsystematic, that is uncorrelated with the market portfolio, the diversified investor is not rewarded for accepting jump risk. Consequently, there is no need to adjust the discount factor, and the risk-free rate applies [91, p. 120].

While this procedure may appear as an oversimplification, possibly the only suitable alternative lies in exogenous discount rates, which are arbitrarily chosen or at least require specific assumptions about the investor's risk tolerance. Since the initial motivation behind option-based methods for capital budgeting was to obtain values independent from individual preferences, patent risk is assumed to be unsystematic for the moment. An alternative approach based on the so-called Esscher transform is briefly discussed in the following section (sect. 7.1.2).

In analogy to previous discussions of dynamic patent value, one obtains a modified investment rule reflecting the possibility of sudden drops in the gross payoff from commercialization.

Proposition 7. *The dynamic value of a patent under uncertainty and infinite, but imperfect, patent protection is*

$$F(V_t) = \begin{cases} V_t - I & \text{if } V^* < V_t, \\ A^+ V_t^{\gamma^+} & \text{otherwise,} \end{cases} \tag{7.3}$$

where

$$V^* = \frac{\gamma^+}{\gamma^+ - 1} I$$

denotes the critical project value,

$$\begin{aligned} A^+ &= \frac{I}{\gamma^+ - 1} \left(\frac{1}{V^*} \right)^{\gamma^+} \\ &= \left(\frac{1}{\gamma^+} \right)^{\gamma^+} \left(\frac{\gamma^+ - 1}{I} \right)^{\gamma^+ - 1}, \end{aligned}$$

γ^+ *is the positive root of the exponential equation*

$$\tfrac{1}{2}\sigma^2 \gamma (\gamma - 1) + (r - \delta)\gamma - (r + \lambda) + \lambda (1 - \phi)^\gamma = 0, \tag{7.4}$$

and λ *is the mean arrival rate of a value-decreasing event.*

Proof (Proposition 7). Similar to sect. 6.2.2, a type of Itô–Taylor expansion for jump-diffusion processes makes it possible to express changes in the value of a patent as a function of the underlying variate V_t [91, pp. 85–87, 112–114, and 171]. Since, by assumption, the investor is risk-neutral with respect to discontinuous changes, expected values can be used in all terms related to the jump process.

Changes in the value of the patent thus become

$$\mathrm{d}F(V_t) = \left(\alpha V_t \frac{\mathrm{d}F(V_t)}{\mathrm{d}V_t} + \tfrac{1}{2}\sigma^2 V_t^2 \frac{\mathrm{d}^2 F(V_t)}{\mathrm{d}V_t^2}\right)\mathrm{d}t + \sigma V_t \frac{\mathrm{d}F(V_t)}{\mathrm{d}V_t}\,\mathrm{d}W_t \\ + \lambda\Big(F\big((1-\phi)V_t\big) - F(V_t)\Big)\,\mathrm{d}t.$$

The hedge portfolio is constructed to eliminate demand uncertainty:[2]

$$\mathrm{d}\Phi(V_t) = \mathrm{d}F(V_t) - \frac{\mathrm{d}F(V_t)}{\mathrm{d}V_t}\left(\alpha V_t\,\mathrm{d}t + \sigma V_t\,\mathrm{d}W_t\right) - \delta V_t \frac{\mathrm{d}F(V_t)}{\mathrm{d}V_t}\,\mathrm{d}t.$$

Consequently, it yields the risk-free return:

$$\tfrac{1}{2}\sigma^2 V_t^2 \frac{\mathrm{d}^2 F(V_t)}{V_t^2}\,\mathrm{d}t + \lambda\Big(F\big((1-\phi)\,V_t\big) - F(V_t)\Big)\,\mathrm{d}t \\ - \delta V_t \frac{\mathrm{d}F(V_t)}{\mathrm{d}V_t}\,\mathrm{d}t = r\left(F(V_t) - V_t \frac{\mathrm{d}F(V_t)}{\mathrm{d}V_t}\right)\mathrm{d}t,$$

which leads to

$$\tfrac{1}{2}\sigma^2 V_t^2 \frac{\mathrm{d}^2 F(V_t)}{\mathrm{d}V_t^2} + (r-\delta)\,V_t \frac{\mathrm{d}F(V_t)}{\mathrm{d}V_t} \\ - (r+\lambda)\,F(V_t) + \lambda F\big((1-\phi)\,V_t\big) = 0. \quad (7.5)$$

Solutions again take the general form given by (6.49), only that γ^+ is now the positive root of the slightly more complicated equation (7.4).

Employing numerical methods, it is possible to successively determine the root γ^+ that ensures $F(0) = 0$, the critical gross present value of cash flows from commercialization V^*, the constant factor A^+, and, eventually, the patent value $F(V_t)$.

Note that (7.5) is in fact a partial integro-differential equation (PIDE), which becomes obvious as soon as the expected value is kept unevaluated. Let $h(\phi)$ denote the probability density of ϕ, and it can be rewritten as

[2] Another possibility lies in constructing a minimum-variance hedge [336, p. 331].

$$\tfrac{1}{2}\sigma^2 V_t^2 \frac{\mathrm{d}^2 F(V_t)}{\mathrm{d}V_t^2} + (r-\delta) V_t \frac{\mathrm{d}F(V_t)}{\mathrm{d}V_t} \\ - (r+\lambda) F(V_t) + \lambda \int_{\mathbf{R}} F\big((1-\phi)\, V_t\big) h(\phi)\, \mathrm{d}\phi = 0. \quad (7.6)$$

In this case, assuming a Poisson process given by (7.2) and a fixed jump size, the PIDE simplifies considerably, resulting in the PDE shown above. The general case of stochastic jump size is discussed in the following subsection.

If ϕ is deterministic and equals unity, the occurrence of a jump decreases the cash flow rate to zero and, by (7.3), destroys the entire patent value ("sudden death"). As a pleasant side effect, setting $\phi = 1$ reduces (7.4) to the quadratic equation

$$\tfrac{1}{2}\sigma^2 \gamma\,(\gamma - 1) + (r - \delta)\,\gamma - (r + \lambda) = 0, \quad (7.7)$$

which is almost identical to (6.50), with r increased by λ in the constant term. Consequently, there is a closed-form expression for the positive root [91, p. 171]:

$$\gamma^+ = \tfrac{1}{2} - \frac{r-\delta}{\sigma^2} + \sqrt{\left(\frac{r-\delta}{\sigma^2} - \tfrac{1}{2}\right)^2 + 2\frac{r+\lambda}{\sigma^2}}. \quad (7.8)$$

While this finding is closely related to the well-known pricing formula for jump-diffusion processes developed by Merton [239], there is a crucial difference between his results and the relationship suggested by proposition 7. Merton chooses to increase α to compensate rising jump risk and maintain the rate of return required by investors. Rational investors, so the argument, would not be willing to invest in a project yielding inadequate returns. Specifically, $r - \delta$ in (7.7) and (7.8) is replaced by $r - \delta + \lambda$.[3] As a consequence, a higher frequency of jumps leads to higher patent value [91, pp. 172–173].

In a real options setting, α is given by the expected return on the underlying project and, as such, less likely to adapt to rising patent risk. Strictly speaking, the impact of increases in λ is indeterminate.

[3] The characteristic equation becomes

$$\tfrac{1}{2}\sigma^2 \gamma\,(\gamma - 1) + (r - \delta + \lambda)\,\gamma - (r + \lambda) = 0$$

and

$$\gamma^+ = \tfrac{1}{2} - \frac{r-\delta+\lambda}{\sigma^2} + \sqrt{\left(\frac{r-\delta+\lambda}{\sigma^2} - \tfrac{1}{2}\right)^2 + 2\frac{r+\lambda}{\sigma^2}}.$$

Similar to the sensitivity of patent value to changes in volatility (see sect. 6.2.2.1) results crucially depend on the assumptions made about interdependencies between key parameters—and on whether patent values are assumed to be at least approximately arbitrage-free.[4]

Another interesting observation concerns the aggregate impact of uncertainty, measured by the variance of changes in commercialization payoff, on patent value [91, pp. 168–170]. Employing a discrete-time approximation of the jump-diffusion process, one obtains

$$\mathrm{d}V_t = \begin{cases} -\phi V_t & \text{with probability } \lambda\,\mathrm{d}t, \\ \alpha V_t\,\mathrm{d}t + \sigma V_t\sqrt{\mathrm{d}t} & \text{with probability } \frac{1}{2}\,(1-\lambda\,\mathrm{d}t), \\ \alpha V_t\,\mathrm{d}t - \sigma V_t\sqrt{\mathrm{d}t} & \text{with probability } \frac{1}{2}\,(1-\lambda\,\mathrm{d}t). \end{cases}$$

Neglecting all terms of order $\mathrm{d}t^2$ and above leads to

$$\begin{aligned} \mathbf{E}[\mathrm{d}V_t] &= \alpha V_t\,\mathrm{d}t - \lambda\phi V_t\,\mathrm{d}t, \\ \mathbf{E}\left[(\mathrm{d}V_t)^2\right] &= \sigma^2 V_t^2\,\mathrm{d}t + \lambda\phi^2 V_t^2\,\mathrm{d}t, \end{aligned}$$

and thus

$$\begin{aligned} \mathbf{V}\,[\mathrm{d}V_t] &= \mathbf{E}\left[(\mathrm{d}V_t)^2\right] - (\mathbf{E}[\mathrm{d}V_t])^2 \\ &= \sigma^2 V_t^2\,\mathrm{d}t + \lambda\phi^2 V_t^2\,\mathrm{d}t. \end{aligned}$$

Hence, total variance consists of an instantaneous, or local, component $\sigma^2 V_t^2\,\mathrm{d}t$ as well as a jump component $\lambda\phi^2 V_t^2\,\mathrm{d}t$. Consequently, if total variance is to be kept constant, an increase in patent risk requires a downward adjustment of volatility [207, pp. 135–137]. Economically speaking, however, market and patent risk are not equivalent. After all, patent risk is assumed to be unsystematic in the derivation of proposition 7, whereas market risk, by definition, is correlated with the market portfolio.

7.1.1.1.2 Numerical Illustration

A numerical example, where $V_t = 1.0$, $I = 1.0$, $r = \delta = 0.05$, and $\phi = 0.5$ may serve as an illustration. Consider a real options model under standard assumptions. As evident from fig. 7.1, different types of risk affect the value of the patent in fundamentally different ways. While higher market risk tends to increase patent value, the opposite is true of patent risk, which clearly diminishes the attractiveness of the

[4] This insight is illustrated by fig. 7.2 discussed in the following section.

investment opportunity. An increase in patent risk is thus compensated by higher demand uncertainty.

This relationship comes at no surprise, because uncertainty related to patent protection lacks the upside potential characteristic of random changes in demand. Although flexibility enables the investor to react to a sudden drop in the value of the underlying project by further postponing commercialization, he or she is worse off compared to a situation that is characterized by perfect patent protection.

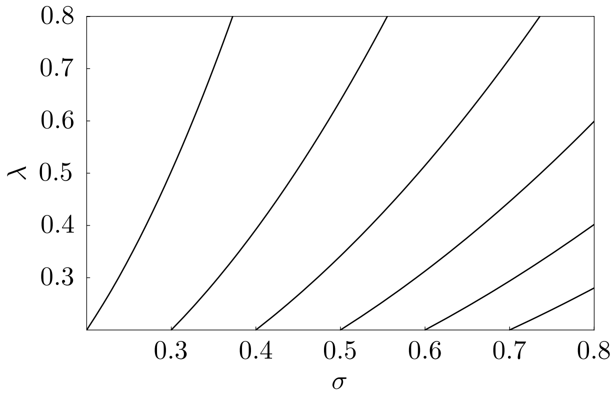


Fig. 7.1. Dynamic patent value under certainty and infinite, but imperfect, patent protection according to project-level analysis ($V_t = 1.0$, $I = 1.0$, $r = \delta = 0.05$, and $\phi = 0.5$). The plot shows isolines for $F(V_t) \in \{0.11, 0.16, 0.22, 0.28, 0.34, 0.39\}$, corresponding to $\sigma \in \{0.2, 0.3, 0.3, 0.5, 0.6, 0.7\}$ for $\lambda = 0.2$.

A slightly different picture is drawn by fig. 7.2, which demonstrates how possible adjustments drive valuation. As expected, depending on whether drift or volatility are held constant or allowed to vary, increases in patent risk bring about rising or falling patent values. When examining the short-term impact of sudden changes in patent quality, other parameters are very likely to remain unchanged, perhaps resulting in below-equilibrium rates of return on the investment.

From a more practical perspective, it is important to stress the fact that it is not a good idea to estimate the jump-diffusion models of the proposed type as one would normally estimate a stock price model with fat-tailed return distributions.[5] Rather, since value-decreasing patent events can typically be identified clearly in historical time series, these events are to be excluded explicitly from a dataset used only for es-

[5] For example, see Honoré [145] and the references therein.

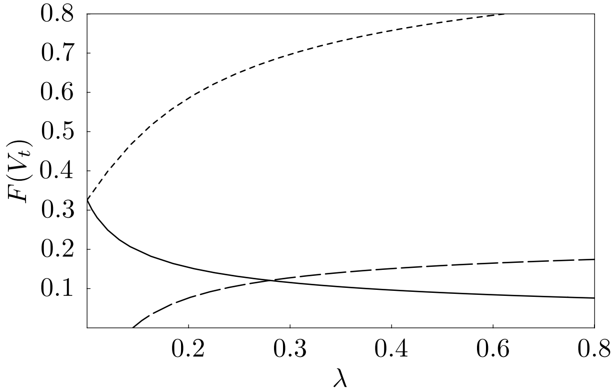


Fig. 7.2. Drift and volatility adjustments for patent risk ($V_t = 1.0$, $I = 1.0$, $r = \delta = 0.05$, $\sigma = 0.3$, $\phi = 1.0$). All other things equal, patent risk leads to lower patent values (solid line). Patent value becomes an increasing function of patent risk if returns are held constant (short dashes). If total variance of changes in patent value is held constant by reducing volatility, patent value rises as well, but, for obvious reasons, is substantially lower (long dashes).

timating the diffusive part of the process. Data on changes in patent value immediately following a patent event may then be employed to fit a suitable probability distribution or, in the simplest of cases treated in this section, to simply determine the average size of jumps. Such empirical issues will be re-addressed in chapter 8.

7.1.1.2 Profit-Level Analysis

Since profit-level models were shown to better reflect typical patent characteristics, including finite patent protection, the discussions that follow focus on profit rates instead of aggregate project values. As in sect. 6.2.2, the analysis proceeds in two steps, that is by first finding an expression for the value of the investment project and then calculating the value of the investment opportunity.

7.1.1.2.1 Investment Project

In analogy to (7.1) set

$$\mathrm{d}\Pi_t = \alpha\Pi_t\,\mathrm{d}t + \sigma\Pi_t\,\mathrm{d}W_t - \phi\Pi_t\,\mathrm{d}J_t, \qquad 0 \le \phi \le 1.$$

When the commercialization phase is finite, the expected present value of cash flows from commercialization, determined via direct integration, is

$$
\begin{aligned}
V(\Pi_t, t) &= \Pi_t \int_t^T \mathrm{e}^{-\mu(s-t)} \mathrm{e}^{\alpha(s-t)} \left(\prod_{i=1}^{J_s} (1-\phi) \right) \mathrm{d}s \\
&= \Pi_t \int_t^T \mathrm{e}^{-\mu(s-t)} \mathrm{e}^{\alpha(s-t)} \sum_{i=0}^{\infty} \Big((1-\phi)^i \, \mathbf{P}\{J_s = i\} \Big) \mathrm{d}s \\
&= \Pi_t \int_t^T \mathrm{e}^{-\mu(s-t)} \mathrm{e}^{\alpha(s-t)} \\
&\qquad \sum_{i=0}^{\infty} \left((1-\phi)^i \, \mathrm{e}^{-\lambda(s-t)} \frac{\big(\lambda (s-t)\big)^i}{i!} \right) \mathrm{d}s \\
&= \left(1 - \mathrm{e}^{-(\delta+\lambda\phi)(T-t)} \right) \frac{\Pi_t}{\delta + \lambda\phi},
\end{aligned} \tag{7.9}
$$

where i is the number of value-relevant events in the interval $[t, s]$. The summation term represents the expected multiplicative decrease in the cash flow rate during that period, assuming J is again a Poisson process of intensity λ and deterministic amplitude specified by ϕ. The product $\lambda\phi$ can be interpreted as a type of hazard score measuring the overall patent risk associated with a commercialization opportunity. For $\phi = 0$, jumps have no effect, and (7.9) is identical to (6.62).

When the commercialization phase is infinite, one obtains

$$
\begin{aligned}
V(\Pi_t) &= \lim_{T \to \infty} V(\Pi_t, t) \\
&= \frac{\Pi_t}{\delta + \lambda\phi}.
\end{aligned} \tag{7.10}
$$

Following Dixit and Pindyck [91, p. 200], the value of a project under the assumption of exponentially decaying cash flows is

$$
\begin{aligned}
V(\Pi_t) &= \Pi_t \int_t^{\infty} \mathrm{e}^{-\mu(s-t)} \mathrm{e}^{\alpha(s-t)} \mathbf{P}\{J_s = 0\} \, \mathrm{d}s \\
&= \Pi_t \int_t^{\infty} \mathrm{e}^{-\mu(s-t)} \mathrm{e}^{\alpha(s-t)} \mathrm{e}^{-\lambda(s-t)} \, \mathrm{d}s \\
&= \frac{\Pi_t}{\delta + \lambda}.
\end{aligned}
$$

This expression is identical to (7.10) with $\phi = 1$. Consequently, apart from the bond analogy pointed out in sect. 7.1.1.1, patent risk is closely related to the concept of economic depreciation.

Observation 7.2. Patent risk can be regarded as a form of depreciation.

If jump risk is again assumed to be unsystematic, and the drift is adjusted as in Merton's model,

$$\mathrm{d}\mathit{\Pi}_t = (\alpha + \lambda\phi)\,\mathit{\Pi}_t\,\mathrm{d}t + \sigma\mathit{\Pi}_t\,\mathrm{d}W_t - \phi\mathit{\Pi}_t\,\mathrm{d}J_t. \tag{7.11}$$

Consequently, $\mathit{\Pi}_t$ grows at a risk-neutral rate $\alpha^* + \lambda\phi = r - \delta + \lambda\phi$, which implies

$$\begin{aligned} V(\mathit{\Pi}_t, t) = {} & \mathit{\Pi}_t \int_t^T \mathrm{e}^{-r(s-t)}\mathrm{e}^{(r-\delta+\lambda\phi)(s-t)} \\ & \sum_{i=0}^{\infty}\left((1-\phi)^i\,\mathrm{e}^{-\lambda(s-t)}\frac{(\lambda\,(s-t))^i}{i!}\right)\mathrm{d}s \\ = {} & \left(1-\mathrm{e}^{-\delta(T-t)}\right)\mathit{\Pi}_t/\delta. \end{aligned}$$

Jump risk is exactly offset by the drift correction and (6.62) is obtained.

The extension of proposition 6 to the case of imperfect patent protection is straightforward. Analyzing the investment problem under the assumption of a finite protection period, however, is far more informative.

7.1.1.2.2 Investment Opportunity

Patent valuation boils down to the optimization problem

$$\begin{aligned} F(\mathit{\Pi}_t, t) &= \sup_{\tau\in[t,T)} \mathrm{e}^{-r(\tau-t)}\mathbf{E}_{\mathbf{P}^*}\left[\left(V(\mathit{\Pi}_\tau,\tau) - I\right)^+ \,\middle|\, \mathcal{F}_t\right], \\ &= \mathrm{e}^{-r(\tau^*-t)}\mathbf{E}_{\mathbf{P}^*}\left[\left(V(\mathit{\Pi}_{\tau^*},\tau) - I\right)^+ \,\middle|\, \mathcal{F}_t\right], \end{aligned}$$

with $V(\mathit{\Pi}_t, t)$ given by (7.9).

As explained in sect. 6.2.2.2, numerical methods are required to obtain a solution. In principle, also jump–diffusion models may be solved using variants of Crank–Nicolson FD discretization. Incorporating the integral part, however, proves to be more difficult than apparent at first glance. Direct numerical integration via Gaussian quadrature or Simpson's method is computationally expensive. Alternatively, PIDEs like (7.6) lend themselves to the application of fast Fourier transform (FFT) methods [336, p. 331], which involves a variety of prepatory steps [55, 85].

A completely different approch is adopted here, namely the extremely simple LSM algorithm proposed by Longstaff and Schwartz [207]. LSM has the distinct advantage of providing a solution without requiring a PDE, or PIDE, for the value of the patent in the first place. Due to its versatility, the technique is also employed to analyze more advanced models in sect. 7.2.

Expectations under $\mathbf{P}^*$ can be computed by generating many realizations of the risk-adjusted stochastic process, discounting the optimal payoff for each realization at the risk-free rate, and then taking the average. To this purpose, time is divided into steps of equal size $\Delta t = T/n$. Using the simple Euler scheme for the logarithm of the profit rate process, a discretization of (7.11) under the risk-neutral measure is obtained:[6]

$$\Pi_{j+1} = \begin{cases} \Pi_j \,(1-\phi) & \text{with probability } \lambda \Delta t, \\ \Pi_j \mathrm{e}^{\left(r-\delta+\lambda\phi-\frac{1}{2}\sigma^2\right)\Delta t + \sigma\sqrt{\Delta t}\,\varepsilon_j} & \text{with probability } 1-\lambda\Delta t, \end{cases}$$

where $\varepsilon_j \sim \mathrm{N}(0,1)$ are random numbers picked from the standard normal distribution, that is a normal distribution with zero mean and variance of one.

If the proposed discretization is employed to generate sample paths, the resulting trajectories are similar to the one depicted in fig. 7.3. Profit rates evolve continuously until a downward jump occurs, leading to a sudden (discontinuous) change. Immediately following an event, $\Pi = \{\Pi_t\}_{t\geq 0}$ again behaves just like a regular diffusion process.

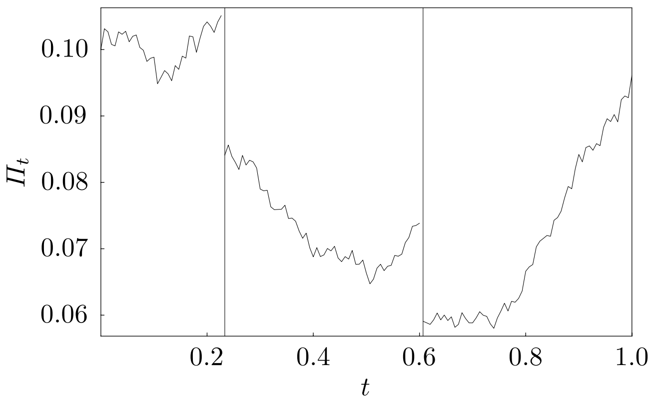


Fig. 7.3. Evolution of profit rates under patent risk ($\Pi_0 = 0.1$, $\alpha = 0.1$, $\sigma = 0.2$, $\lambda = 1.0$, $\phi = 0.2$, $T = 1.0$, $n = 150$). For the realization shown, jumps at $t \in \{0.23, 0.61\}$ are indicated by vertical lines. Since the relative jump size is assumed to be fixed, the profit rate drops exactly by one fifth of its original value each time a patent event occurs.

[6] For alternative methods see Kloeden and Platen [172], Milstein [243, 244].

Of course, timing flexibility makes it necessary to account for the possibility of early commercialization. For each timestep, the payoff from immediate investment is compared to the expected presented value of cash flows from continuation, which is estimated using least-squares regression.[7] The optimal policy derived then leads to the value of the patent. Although LSM, in principle, is very reliable, the accuracy of results depends on the quality of the random numbers used. Compared to conventional methods, convergence is improved substantially by employing a high-dimensional Sobol QRNG.[8]

Specifically, the conditional expected value of continuation on each time level is estimated by regressing previously determined option values for the following period on a set of basis functions f_k in Π_t [207, pp. 122–123]:[9]

$$\mathbf{E}_{\mathbf{P}^*}\left[\mathrm{e}^{-r\Delta t}F\big(\Pi_{j+1},(j+1)\,\Delta t\big)\,\big|\,\mathcal{F}_j\right]=\sum_{k=0}^{\infty}a_k f_k\left(\Pi_j\right).$$

While it is possible to employ a variety of orthogonal polynomials, for example weighted Laguerre polynomials, simple powers of the state variable provide sufficiently accurate results.

If the expected continuation value is estimated to exceed the present value of the project $V(\Pi_j, j\Delta t)$ less the investment amount I, commercialization is postponed for an additional period. Otherwise, immediate exercise of the real option is optimal. Project values are easily obtained from (6.62).

Table 7.1 shows results for a patent with $\Pi_0 = 0.10$, $I = 1.0$, $\sigma = 0.40$, $T = 20.0$, and $r = \delta = 0.05$, based on $n = 32$ timesteps and $m = 50{,}000$ simulation runs. The observed increase in patent value as a consequence of additional patent risk is due to the drift adjustment.

The same technique applied in this section may of course be used to reproduce results obtained in sect. 6.2.2.2.

Observation 7.3. LSM is suitable for valuing both simple and advanced option-based models, specifically those based on jump-diffusions.

---

[7] The best-fit is found by singular value decomposition (SVD), more specifically, the modified Golub–Reinsch algorithm with column scaling implemented in the GNU Scientific Library (GSL). Any components which have zero singular value (to machine precision) are discarded from the fit [110, pp. 373–374]. See also Chan [63].

[8] See the appendix for an implementation of such a QRNG see Baecker [16]. See also Bratley and Fox [51], Joe and Kuo [158], Sobol [313]. For discussions of Brownian bridge techniques see Metwally and Atiya [240], Papageorgiou [264].

[9] See sect. 3.2 for a more detailed description of LSM simulation for option pricing and benchmark results for the plain-vanilla put.

Table 7.1. Impact of frequency and severity of patent-related events on patent value ($\Pi_0 = 0.10$, $I = 1.0$, $\sigma = 0.40$, $T = 20.0$, $r = \delta = 0.05$, $n = 32$, $m = 50{,}000$). Results illustrate how increases in patent risk, measured by the frequency and severity of value-decreasing events, lead to higher patent values in a Merton-type model with drift adjustment.

Frequency (λ)	Patent value					
	$\phi = 0.0$	$\phi = 0.2$	$\phi = 0.4$	$\phi = 0.6$	$\phi = 0.8$	$\phi = 1.0$
0.00	0.43	0.42	0.42	0.42	0.41	0.42
0.20	0.41	0.43	0.44	0.49	0.52	0.62
0.40	0.42	0.44	0.48	0.55	0.61	0.78
0.60	0.42	0.44	0.49	0.53	0.67	1.04
0.80	0.42	0.43	0.50	0.59	0.63	1.00
1.00	0.42	0.45	0.51	0.59	0.76	1.07
1.20	0.42	0.45	0.54	0.60	0.75	0.96
1.40	0.42	0.44	0.51	0.54	0.70	1.13
1.60	0.43	0.46	0.48	0.59	0.70	0.81
1.80	0.43	0.45	0.52	0.59	0.65	0.88
2.00	0.43	0.44	0.50	0.71	0.60	0.84

7.1.2 Stochastic Jump Size

As seen in the previous section, patent risk can be modeled as jumps in the process underlying a real option on commercialization. Assuming jumps of a fixed size known in advance, however, only poorly reflects real-world investment problems. Rather, the impact of patent-related events is random with large jumps occurring less frequently. Closed-form solutions to investment problems with jumps of stochastic size in the underlying project value, only mentioned briefly by Dixit and Pindyck [91, p. 173], can be obtained based on the works of Boyarchenko and Levendorskiĭ [43], Mordecki [247], and others.

7.1.2.1 Project-Level Analysis

If the size of jumps is stochastic, deriving a clear-cut solution becomes challenging. The somewhat elaborate mathematics involved in obtaining closed-form results for non-Gaussian processes tends to obscure economic implications. Nevertheless, analyzing this generalization is a worthwhile exercise. Consequently, for the sake of clarity, the model is again presented in several steps, and mathematical formalism is kept to a necessary minimum. Previous results, as shown later in this section, can be obtained by analyzing limiting cases of the extended models presented.

7.1.2.1.1 Analytical Derivation

On a very abstract level, the optimization problem stays the same, only the gross present value of cash flows from commercialization follows a slightly less simplistic stochastic process. To better emphasize the connection between the results obtained in this section and previous findings, the discussion focuses on the optimal investment rule in the presence of exclusively value-decreasing events before proceeding to a more widely applicable jump-diffusion model. Both analyses are based on the same general formula [248] and a common pricing approach [46].

7.1.2.1.1.1 General Pricing Formula

Very broadly speaking and restating a basic insight from earlier sections, the value of a patent according to the risk-neutral valuation approach is

$$\begin{aligned} F(V_t) &= \sup_{\tau \in [t,\infty)} \mathrm{e}^{-r(\tau - t)} \mathbf{E}_{\mathbf{P}^*} \left[(V_\tau - I)^+ \,\middle|\, \mathcal{F}_t \right] \\ &= \mathrm{e}^{-r(\tau^* - t)} \mathbf{E}_{\mathbf{P}^*} \left[(V_{\tau^*} - I)^+ \,\middle|\, \mathcal{F}_t \right], \end{aligned}$$

where $\mathbf{P}^*$ is an appropriate risk-neutral measure. As suggested by earlier discussions of hedging issues in the presence of jumps, choosing this measure constitutes a non-trivial task if price processes are partly discontinuous.

An equally important choice concerns the specific stochastic model for the underlying asset price. Jump-diffusion models found in the finance literature are often based on price processes belonging to the wider class of Lévy processes, that is processes with stationary independent increments, encompassing Itô processes (see sect. 6.2.2) as a special case [95]. It therefore seems natural to express the extended model in terms of a Lévy process. Pricing techniques introduced recently as a generalization of Merton [238]'s approach then yield the value of the investment opportunity under imperfect patent protection.[10]

[10] For relatively recent discussions of perpetual option pricing in Lévy markets see for example Mordecki [247, 248]. For rare examples of applications to real investments see Boyarchenko [42], Boyarchenko and Levendorskiĭ [45]. For overviews of Lévy processes in general see Sato [291]. For financial applications see also Boyarchenko and Levendorskiĭ [43], Raible [281], Schoutens [296].

A Lévy process is fully characterized by the triple $\big(b, \sigma^2, H(\mathrm{d}y)\big)$, where b is the drift, σ^2 is the variance of the continuous component, and $H(\mathrm{d}y)$ the so-called Lévy measure determining jump frequency and magnitude. The characteristic function of a Lévy process is given by the widely-known Lévy–Kintchine formula, which states that, if $q \in \mathbf{R}$,

$$\mathbf{E}\left[\mathrm{e}^{iqX_t}\right] = \exp\left(t \left(ibq - \tfrac{1}{2}\sigma^2 q^2 + \int_{\mathbf{R}} \big(\mathrm{e}^{iqy} - 1 - iqy\mathbf{1}_{\{|y|<1\}}\big) H(\mathrm{d}y) \right) \right)$$

with

$$\int_{\mathbf{R}} (y^2 \wedge 1)\, \mathrm{d}H(\mathrm{d}y) < \infty.$$

Furthermore, define the convex set

$$\mathbf{C}_0 = \left\{ c \in \mathbf{R} : \int_{\{1<|y|\}} \mathrm{e}^{cy} H(\mathrm{d}y) < \infty \right\}.$$

For all $z \in \mathbf{C}$ with $\Re(z) \in \mathbf{C}_0$, the Lévy exponent

$$\psi(z) = bz + \tfrac{1}{2}\sigma^2 z^2 + \int_{\mathbf{R}} \big(\mathrm{e}^{zy} - 1 - zy\mathbf{1}_{\{|y|<1\}}\big) H(\mathrm{d}y) \tag{7.12}$$

can be determined from $\mathbf{E}\left[\mathrm{e}^{zX_t}\right] = \mathrm{e}^{t\psi(z)}$. It contains all relevant process parameters and is thus equivalent to the triple $\big(b, \sigma^2, H(\mathrm{d}y)\big)$.

The random process associated with this triple can be interpreted as the superposition of a GBM and an infinite superposition of independent compensated Poisson processes. As long as $\int H(\mathrm{d}y) = \lambda$ is finite, it is possible to derive a compound Poisson representation of the Lévy process, which leads to the type of jump-diffusion model analyzed in previous sections [298]. Specifically, a Lévy measure of the form $H(\mathrm{d}y) = \lambda h(y)\,\mathrm{d}y$ implies a Poisson process of intensity λ with $h(y)$ being the distribution of jump sizes. For example, assume the Lévy measure

$$H(\mathrm{d}y) = \begin{cases} \lambda^+ \left(+c^+\right) \mathrm{e}^{-c^+ y}\,\mathrm{d}y & \text{if } 0 < y, \\ \lambda^- \left(-c^-\right) \mathrm{e}^{-c^- y}\,\mathrm{d}y & \text{if } y < 0, \end{cases}$$

equivalent to the Lévy exponent

$$\begin{aligned} \psi(z) =\ & bz + \tfrac{1}{2}\sigma^2 z^2 + \int_0^{+\infty} \left(\mathrm{e}^{zy} - 1\right) \lambda^+ \left(+c^+\right) \mathrm{e}^{-c^+ y}\,\mathrm{d}y \\ & \qquad + \int_{-\infty}^{0} \left(\mathrm{e}^{zy} - 1\right) \lambda^- \left(-c^-\right) \mathrm{e}^{-c^- y}\,\mathrm{d}y, \\ =\ & bz + \tfrac{1}{2}\sigma^2 z^2 + \lambda^+ \frac{z}{c^+ - z} + \lambda^- \frac{z}{c^- - z}. \end{aligned} \tag{7.13}$$

Note that, in this case, the truncation of small jumps found in (7.12) can be safely omitted [68].

Two positive intensity parameters λ^+ and λ^- characterize the intensity of upward and downward jumps, respectively, while $c^- < 0 < c^+$ determine the relative intensity of large jumps in each direction. Smaller values for c^+ result in a higher probability of large upward jumps as opposed to small ones. Conversely, the larger c^-, the larger the probability of large downward jumps. If $\lambda^+ = 0$ or $\lambda^- = 0$ there are no jumps in the corresponding direction [45, p. 4]. It can be assumed that λ^+ and λ^- are, by and large, determined by the intensity of competition in a particular industry, whereas c^+ and c^- reflect patent quality.

An equation for the project value process analogous to the specification used in sect. 6.2.2, a compound Poisson representation of (7.42), is $V_t = V_0 \mathrm{e}^{X_t}$, where

$$X_t = bt + \sigma W_t + \sum_{i=1}^{J_t} Y_i,$$

and $b = \alpha - \frac{1}{2}\sigma^2$. Since $\int H(\mathrm{d}y) = \lambda^+ + \lambda^-$, $J = \{J_t\}_{t\geq 0}$ is a Poisson process of intensity $\lambda = \lambda^+ + \lambda^-$ and $Y = \{Y_i\}_{i\in J}$ a sequence of independent and identically distributed upward and downward jumps described by the density

$$h(y) = \mathbf{1}_{\{0<y\}} \frac{\lambda^+}{\lambda^+ + \lambda^-} \left(+c^+\right) \mathrm{e}^{-c^+ y} + \mathbf{1}_{\{y<0\}} \frac{\lambda^-}{\lambda^+ + \lambda^-} \left(-c^-\right) \mathrm{e}^{-c^- y}.$$

Of course, a risk-neutral measure remains to be chosen. One possibility is to apply the Esscher transform known from actuarial sciences [114]. A transformation of this type results in another Lévy process and makes $\left\{\mathrm{e}^{-(r-\delta)t} V_t\right\}_{t\geq 0}$ a martingale.[11] More specifically,

$$\frac{\mathrm{d}\mathbf{P}_t^*}{\mathrm{d}\mathbf{P}_t} = \frac{\mathrm{e}^{\zeta X_t}}{\mathbf{E}[\mathrm{e}^{\zeta X_t}]} \Leftrightarrow \mathrm{d}\mathbf{P}_t^* = \mathrm{e}^{\zeta X_t - t\psi(\zeta)}\, \mathrm{d}\mathbf{P}_t,$$

where ζ is the unique real root of $\psi^*(1) = \psi(1+\zeta) - \psi(\zeta) = r - \delta < r$.[12] The Lévy exponent under $\mathbf{P}^*$ is $\psi^*(z) = \psi(z+\zeta) - \psi(\zeta)$, or

[11] For related transformations see Goovaerts and Laeven [120].

[12] For details see Mordecki [248, p. 483], Boyarchenko and Levendorskiĭ [46, p. 22]. Equivalent martingale measures are usually calculated via the Radon–Nikodým derivative $\mathrm{d}\mathbf{P}_t^* / \mathrm{d}\mathbf{P}_t$ used in Girsanov's theorem. For a detailed discussion of the procedure see Gerber and Shiu [114, pp. 168–175], Karatzas and Shreve [163].

$$\psi^*(z) = \left(b + \zeta\sigma^2\right) z + \tfrac{1}{2}\sigma^2 z^2 \\ + \lambda^+ \frac{c^+}{c^+ - \zeta} \frac{z}{(c^+ - \zeta) - z} + \lambda^- \frac{c^-}{c^- - \zeta} \frac{z}{(c^- - \zeta) - z},$$

which leads to the risk-neutral parameters

$$\begin{aligned} b^* &= b + \zeta\sigma^2, & \sigma^* &= \sigma, \\ \left(\lambda^+\right)^* &= \lambda^+ \frac{c^+}{c^+ - \zeta}, & \left(\lambda^-\right)^* &= \lambda^- \frac{c^-}{c^- - \zeta}, \\ \left(c^+\right)^* &= c^+ - \zeta, & \left(c^-\right)^* &= c^- - \zeta. \end{aligned}$$

Evidently, instead of employing the Esscher transform, one may also leave the jump component unchanged, and adjust the Gaussian component only. Recall that this choice implies entirely unsystematic jump risk.

As demonstrated in previous sections, the investment problem to be analyzed is analogous to a perpetual call. A general formula for the value of a perpetual call if asset prices follow a Lévy process is available and can be applied in the context of patent valuation.

To clarify this point, consider the supremum and infimum of X killed at the random time $\tau(r)$, which is exponentially distributed with parameter r.[13] More specifically, define

$$\overline{X} = \sup_{t \in [0, \tau(r)]} X_t$$

to be the *supremum* and

$$\underline{X} = \inf_{t \in [0, \tau(r)]} X_t$$

to be the *infimum*. Both processes and their dependency on X are illustrated by fig. 7.4. In a way, the supremum reflects the most cheerful outlook, while the infimum corresponds to the most gloomy one [45, p. 5].

As long as the payoff from commercialization is instantaneous it suffices to focus on the supremum when calculating the value of a (real) option.

Theorem 1. *If* $\mathbf{E}\left[\mathrm{e}^{X_1}\right] < \mathrm{e}^r$, *and the payoff is instantaneous, the value of a call option is*

[13] For an introduction to the concept of *killed* processes and the Feynman–Kac formula see Steele [314, pp. 264–265], Øksendal [177, pp. 135–137].

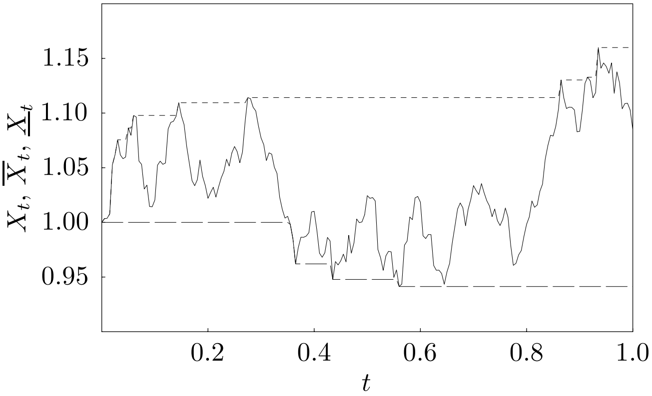


Fig. 7.4. A process, its supremum and infimum. Increases in the supremum $\overline{X}$ (short dashes) require record-setting good news. Conversely, only record-setting bad news cause the infimum $\underline{X}$ (long dashes) to decrease. Note that the specific X shown is a pure diffusion process, while the "record-setting good news principle" applies to *any* kind of Lévy process, including processes with jumps.

$$F(V_0) = \frac{\mathbf{E}\left[\left(V_0 e^{\overline{X}} - \mathbf{E}\left[e^{\overline{X}}\right] I\right)^+\right]}{\mathbf{E}\left[e^{\overline{X}}\right]},$$

where

$$\tau^* = \inf\{t \geq 0 : V^* < V_t\},$$

and

$$V^* = \mathbf{E}\left[e^{\overline{X}}\right] I$$

is the critical asset price.

Proof (Proof of theorem 1). The option pricing formula corresponds to theorem 1 in Mordecki [248].[14]

The investment rule looks somewhat involved at first glance, but has an intuitive interpretation related to the "record-setting good news principle," a term coined by Boyarchenko [42].

In the project-level scenario, the patentholder obtains an instantaneous payoff upon commercialization. Since, at any point in time, V_t reflects all future cash flows, he or she is not concerned with any price movements once the investment has been undertaken. The process X

[14] For a complete proof see Mordecki [248, pp. 476 and 486–488], and the references therein.

first reaches a constant upper threshold together with its supremum. Consequently, only record-setting *good* news, which can never be due to downward jumps, matter to the patentholder.

In contrast, it can be shown that profit-level analysis is tied to the infimum of the price process. Roughly speaking, when making the investment decision, the patentholder ignores all temporary increases in the gross payoff from commercialization, because he or she remains exposed to declining profit rates. Delaying commercialization therefore enables the patentholder to avoid record-setting *bad* news.[15]

Before elaborating on this point in more detail, a few definitions are in order, which make it possible to derive option prices, or patent values, for specific processes. The Laplace transform [1] in t of the distribution of X_t is given by

$$r\mathbf{E}\left[\int_0^\infty \mathrm{e}^{-rt}\mathrm{e}^{zX_t}\,\mathrm{d}t\right] = \frac{r}{r-\psi(z)}. \tag{7.14}$$

Employing the Wiener–Hopf factorization for Lévy processes,[16] the transform can be related to the supremum and infimum processes defined above.[17] Specifically,

$$\frac{r}{r-\psi(z)} = \psi_r^+(z)\psi_r^-(z),$$

where

$$\psi_r^+(z) = \mathbf{E}\left[\mathrm{e}^{z\overline{X}}\right] = r\mathbf{E}\left[\int_0^\infty \mathrm{e}^{-rt}\mathrm{e}^{z\overline{X}_t}\,\mathrm{d}t\right],$$
$$\psi_r^-(z) = \mathbf{E}\left[\mathrm{e}^{z\underline{X}}\right] = r\mathbf{E}\left[\int_0^\infty \mathrm{e}^{-rt}\mathrm{e}^{z\underline{X}_t}\,\mathrm{d}t\right]$$

are closely tied to EPVs, namely of the supremum and infimum processes. The procedure employed to derive option pricing formulae in terms of these and other EPVs—quite similar to theorem 1—is now known as the EPV pricing model [46].

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15 For a careful analysis see Boyarchenko [42, pp. 558 and 565]. The original "bad news" principle is due to Bernanke [31].

16 For a description of this procedure see Rogers and Williams [286, p. 89], Rogozin [287].

17 For details see also Boyarchenko and Levendorskiĭ [45, pp. 5–6], Levendorskiĭ [197].

7.1.2.1.1.2 Value-Decreasing Patent Events Only

Assume the gross present value of cash flows from commercialization follows a process described by the Lévy exponent

$$\psi(z) = bz + \tfrac{1}{2}\sigma^2 z^2 + \lambda^- \frac{z}{c^- - z}, \tag{7.16}$$

corresponding to a special case of the jump-diffusion (7.13) with no upward jumps. In line with previous analyses, the risk-neutral, or risk-adjusted, drift is assumed to be

$$b^* = \alpha^* - \tfrac{1}{2}\sigma^2 = (r - \delta) - \tfrac{1}{2}\sigma^2. \tag{7.17}$$

This choice allows for a straightforward extension of proposition 7.

It can be shown that,[18] if $\gamma_1^- < c^- < \gamma_2^- < 0 < \gamma^+$ are the roots of the characteristic equation,

$$\psi(\gamma)^* = r, \tag{7.18}$$

the supremum and infimum processes are given by

$$\begin{aligned} \psi_r^+(z) &= \frac{\gamma^+}{\gamma^+ - z}, \qquad (7.19a)\\ \psi_r^-(z) &= \left(\prod_{i=1}^{2} \frac{\gamma_i^-}{\gamma_i^- - z}\right) \frac{c^- - z}{c^-}. \end{aligned}$$

Evidently, all operations are performed not under the true, but under the equivalent martingale measure.

Armed with this information, theorem 1 can be used to derive patent values for the Lévy process with exponentially distributed downward jumps described by (7.16).

Proposition 8. *The dynamic value of a patent under uncertainty and infinite, but imperfect, patent protection is*

$$F(V_t) = \begin{cases} V_t - I & \text{if } V^* < V_t, \\ A^+ V_t^{\gamma^+} & \text{otherwise,} \end{cases}$$

where

[18] Only if the Lévy exponent is a rational function, there is a simple analytical formula [42, p. 567]. For a more detailed discussion see Mordecki [248], Boyarchenko and Levendorskiĭ [45, pp. 4–7], Levendorskiĭ [197, pp. 310–311 and 319], Boyarchenko and Levendorskiĭ [46, p. 14].

$$V^* = \frac{\gamma^+}{\gamma^+ - 1} I$$

denotes the critical project value,

$$\begin{aligned} A^+ &= \frac{I}{\gamma^+ - 1} \left(\frac{1}{V^*} \right)^{\gamma^+} \\ &= \left(\frac{1}{\gamma^+} \right)^{\gamma^+} \left(\frac{\gamma^+ - 1}{I} \right)^{\gamma^+ - 1}, \end{aligned}$$

and γ^+ *is the positive root of*

$$\tfrac{1}{2}\sigma^2\gamma(\gamma - 1) + (r - \delta)\gamma - \left(r - \lambda^- \frac{\gamma}{c^- - \gamma} \right) = 0. \tag{7.20}$$

Proof (Proposition 8). Loosely speaking, when dealing with a call option, downward jumps affect the characteristic equation, but not the option pricing formula itself [45, p. 9].[19] Subtituting (7.17) in (7.16) yields

$$\psi^*(\gamma) = \left((r - \delta) - \tfrac{1}{2}\sigma^2 \right)\gamma + \tfrac{1}{2}\sigma^2\gamma^2 + \lambda^- \frac{\gamma}{c^- - \gamma} = r.$$

and, after rearranging, (7.20). The trigger follows directly from (7.19a) and theorem 1.

As a sidenote, it is interesting to observe that the standard model of investment under uncertainty is easily derived as a special case. Simply omitting the jump component in (7.13) altogether leads to

$$\psi(z) = \left(\alpha - \tfrac{1}{2}\sigma^2 \right) z + \tfrac{1}{2}\sigma^2 z^2.$$

Consequently, if there are no jumps, (7.17) implies

$$\psi^*(\gamma) = \left((r - \delta) - \tfrac{1}{2}\sigma^2 \right)\gamma + \tfrac{1}{2}\sigma^2\gamma^2 = r,$$

which is identical to the fundamental quadratic (6.50):

$$\tfrac{1}{2}\sigma^2\gamma(\gamma - 1) + (r - \delta)\gamma - r = 0.$$

In addition, note that $\psi^*(1) = r - \delta$, which means that $\{\mathrm{e}^{-(r-\delta)t} V_t\}_{t \geq 0}$ again is a martingale [248, p. 483].

[19] See also proposition 7.

7.1.2.1.1.3 Value-Decreasing and Value-Increasing Patent Events

Although it was assumed that only downward jumps occur with positive probability, it is not difficult to extend the analysis to also include value-increasing events.[20] This subsection serves to outline necessary modifications to the EPV formulae and investment rule.

For a change, consider the Esscher transform discussed earlier. Given the true Lévy exponent (7.13), the characteristic equation under the martingale measure has four roots, namely $\gamma_1^- < (c^-)^* < \gamma_2^- < 0 < \gamma_1^+ < (c^+)^* < \gamma_2^+$, and

$$\psi_r^+(z) = \left(\prod_{i=1}^{2} \frac{\gamma_i^+}{\gamma_i^+ - z}\right) \frac{(c^+)^* - z}{(c^+)^*},$$

$$\psi_r^-(z) = \left(\prod_{i=1}^{2} \frac{\gamma_i^-}{\gamma_i^- - z}\right) \frac{(c^-)^* - z}{(c^-)^*}.$$

Nevertheless, starting with a valuation formula applicable under perfect patent protection, it is no longer sufficient to simply adapt the characteristic equation, because jumps may now result in the gross payoff from commercialization crossing the upper boundary V^*. As before, set $b = \alpha - \frac{1}{2}\sigma^2$. Preserving formal similarity to investment rules presented up to this point leads to the following proposition.

Proposition 9. *The dynamic value of a patent under uncertainty and infinite, but imperfect, patent protection is*

$$F(V_t) = \begin{cases} V_t - I & \text{if } V^* < V_t, \\ A^+ V_t^{\gamma_1^+} + A^- V_t^{\gamma_2^+} & \text{otherwise,} \end{cases}$$

where

$$V^* = \left(\prod_{i=1}^{2} \frac{\gamma_i^+}{\gamma_i^+ - 1}\right) \frac{(c^+)^* - 1}{(c^+)^*} I$$

denotes the critical project value,

$$A^+ = \frac{1}{\gamma_1^+ - 1} \frac{\gamma_2^+}{\gamma_2^+ - \gamma_1^+} \frac{(c^+)^* - \gamma_1^+}{(c^+)^*} \left(\frac{1}{V^*}\right)^{\gamma_1^+},$$

$$A^- = \frac{1}{\gamma_2^+ - 1} \frac{\gamma_1^+}{\gamma_1^+ - \gamma_2^+} \frac{(c^+)^* - \gamma_2^+}{(c^+)^*} \left(\frac{1}{V^*}\right)^{\gamma_2^+},$$

[20] See the appendix and the references provided in footnote 18 above.

and γ_1^+, γ_2^+ *are the positive roots of*

$$\tfrac{1}{2}\sigma^2\gamma^2 + b^*\gamma - \left(r - (\lambda^+)^* \frac{\gamma}{(c^+)^* - \gamma} - (\lambda^-)^* \frac{\gamma}{(c^-)^* - \gamma}\right) = 0.$$

Proof (Proposition 9). The option pricing formula follows directly from corollary 1 in Mordecki [248, pp. 480 and 490]. Risk-neutral parameters are obtained by applying the Esscher transform as laid out in the previous subsection.[21]

Taking everything together, patent values can be obtained based on a limitless variety of probability measures, some of which are easier to justify than others, but none of which may be regarded as the "most accurate" martingale measure, which comes closest to a hypothetical fair value. This dilemma arises, because it is very difficult to determine the appropriate market price of patent risk.

7.1.2.1.2 Numerical Illustration

Connections between both models presented in this section are underlined by similarities between the characteristic equations appearing in the investment rule. While, under certain circumstances, the roots of such equations can be given explicitly, the resulting expressions are extremely complicated and, as a consequence, not very informative. Therefore, the sensitivity to key parameters is perhaps best analyzed numerically.

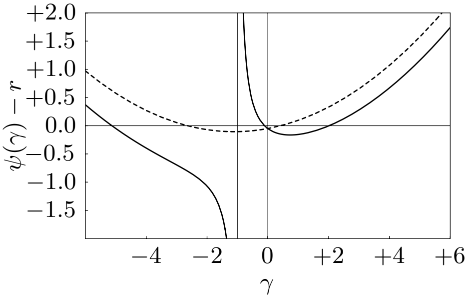
Panel (a) in fig. 7.5 illustrates how downward jumps affect (7.18) and thereby the value of patents. As shown in panel (b), once uncertainty surrounding patent quality is symmetric—implying upward as well as downward jumps—an additional root appears, leading to the slightly more complicated investment rule of proposition 9.

As argued previously on more than one occasion, a more realistic picture of patent value is painted if profit flows instead of aggregate project values are considered. Therefore, instead of engaging into a more detailed analysis of the investment rule at hand, the following section deals with a straightforward extension to the profit level.

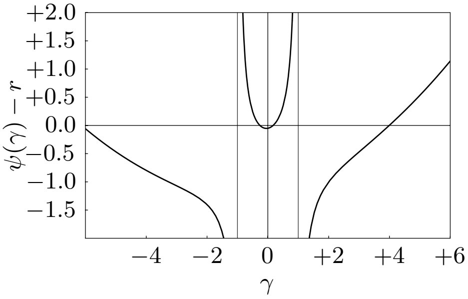
7.1.2.2 Profit-Level Analysis

As before, a closed-form expression for the dynamic value of a patent will be developed and then analyzed numerically.

[21] Note that $\sigma^* \equiv \sigma$.



(a) No patent events and value-decreasing patent events only



(b) Value-decreasing and value-increasing patent events

Fig. 7.5. Fundamental equation in the presence and the absence of patent risk ($b = 0.1$, $\sigma = 0.3$, and $r = 0.05$). The dashed line in panel (a) illustrates the case of perfect patent protection, which leads to the fundamental quadratic discussed previously. For the given choice of parameters, the roots are at $\gamma^- = -2.64$ and $\gamma^+ = +0.42$. If patent protection is imperfect (solid line), the characteristic equation exhibits a singularity at $\gamma = c^-$ and there are three roots. For $c^- = -1.0$ and $\lambda^- = 0.5$ the roots are at $\gamma_1^- = -5.13$, $\gamma_2^- = -0.11$, and $\gamma^+ = +2.01$. Furthermore, panel (b) shows how the possibility of value-increasing events, that is upward jumps in the underlying process, gives rise to a second singularity and an additional (positive) root ($c^+ = +1.0$, $c^- = -1.0$, $\lambda^+ = +0.5$, and $\lambda^- = -0.5$). Specifically, $\gamma_1^- = -5.92$, $\gamma_2^- = -0.16$, $\gamma_1^+ = +0.14$, and $\gamma_2^+ = +4.22$.

7.1.2.2.1 Analytical Derivation

Recognizing that

$$V(\Pi_t) = \mathbf{E}_{\mathbf{P}^*}\left[\int_t^\infty \mathrm{e}^{-rs}\Pi_s \,\mathrm{d}s\right]$$

one can use the Laplace transform (7.14) to obtain

$$\begin{aligned} V(\Pi_t) &= \frac{\Pi_t}{r-\psi^*(1)} \\ &= \frac{\Pi_t}{r-\left(b^*+\frac{1}{2}\sigma^2+\lambda^-\frac{1}{c^--1}\right)}. \end{aligned}$$

Substituting (7.17) then leads to

$$V(\Pi_t) = \frac{\Pi_t}{\delta-\lambda^-\frac{1}{c^--1}}. \tag{7.23}$$

Essentially, this is all that is needed to calculate patent value.

Proposition 10. *The dynamic value of a patent under uncertainty and infinite, but imperfect, patent protection is*

$$V(\Pi_t) = \begin{cases} \frac{\Pi_t}{\delta-\lambda^-\frac{1}{c^--1}} - I & \text{if } \Pi^* < \Pi_t, \\ A^+\Pi_t^{\gamma^+} & \text{otherwise.} \end{cases}$$

where

$$\Pi^* = \frac{\gamma^+}{\gamma^+-1} I\left(\delta-\lambda^-\frac{1}{c^--1}\right), \tag{7.24}$$

denotes the critical project value,

$$\begin{aligned} A^+ &= \frac{I}{\gamma^+-1}\left(\frac{1}{\Pi^*}\right)^{\gamma^+} \\ &= \left(\frac{1}{\gamma^+\left(\delta-\lambda^-\frac{1}{c^--1}\right)}\right)^{\gamma^+}\left(\frac{\gamma^+-1}{I}\right)^{\gamma^+-1}, \end{aligned}$$

and γ^+ is the positive root of

$$\tfrac{1}{2}\sigma^2\gamma(\gamma-1) + (r-\delta)\gamma - \left(r-\lambda^-\frac{\gamma}{c^--\gamma}\right) = 0.$$

Proof (Proposition 10). In analogy to proposition 8, it remains to be shown that (7.24) indeed represents the critical project value. By theorem 1, (7.16) and (7.19a),

$$\psi_r^+(1)I = \frac{\Pi^*}{r - \psi^*(1)}$$
$$\Leftrightarrow \frac{\gamma^+}{\gamma^+ - 1} I = \frac{\Pi^*}{\delta - \lambda^- \frac{1}{c^- - 1}}.$$

Solving for the critical profit rate yields (7.24). The characteristic equation is the same as in proposition 8.

A closer look at the proof reveals how the Wiener–Hopf factorization

$$\Pi^* \frac{\psi_r^+(1)\psi_r^-(1)}{r} = \psi_r^+(1)I$$

ties the critical profit rate to the infimum of the underlying process, resulting in

$$\Pi^* = \frac{Ir}{\psi_r^-(1)}.$$

This finding is in contrast to the project-level model, where

$$V^* = \psi_r^+(1)I.$$

As hinted at previously, value-decreasing events do not show up in the investment rule if the payoff from commercialization is instantaneous. Nevertheless, they do appear in the investment rule for profit-level models, demonstrating the validity of the record-setting *bad* news principle proposed by Boyarchenko [42].

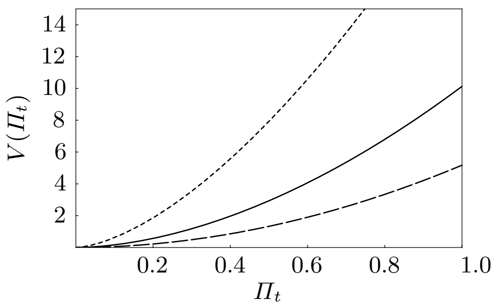
If, in addition, there is a positive probability of jumps that increase the cash flow rate, (7.23) becomes

$$V(\Pi_t) = \frac{\Pi_t}{\delta - \lambda^+ \frac{1}{c^+ - 1} - \lambda^- \frac{1}{c^- - 1}}.$$

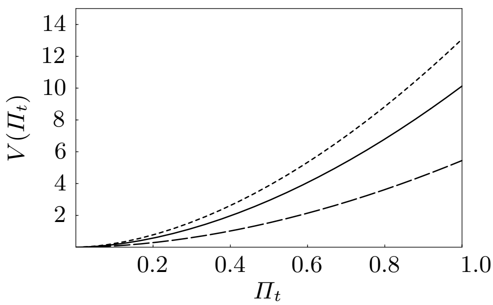
Focusing on the adverse effects of patent-related events, the following section provides a brief numerical analysis of the project-level model with exponentially distributed jumps.

7.1.2.2.2 Numerical Illustration

Recall that, in option-based models under Lévy processes, a higher frequency of downward jumps could be attributed to increased intensity of competition, while larger jumps may reflect decreasing patent quality. As expected, patent value is highly sensitive to changes in either of these two parameters. Fig. 7.6 illustrates this dependency for $\sigma = 0.2$, $r = 0.05$, and $\delta = 0.03$.



(a) Competitive intensity (λ^-)



(b) Patent quality (c^-)

Fig. 7.6. Impact of competitive intensity and patent quality on patent value in a model with exponentially distributed jumps ($\sigma = 0.2$, $r = 0.05$, and $\delta = 0.03$). Panel (a) shows patent value for various jump frequencies ($\lambda^- \in \{0.00, 0.01, 0.02\}$ and $c^- = 0.50$), whereas panel (b) reflects changes in the expected size of jumps ($\lambda^- = 0.01$ and $c^- \in \{0.25, 0.50, 0.75\}$).

Probably the most important insight to be gained from the formal analysis of Lévy markets is the applicability of the record-setting news principle in the context of patent risk. Only two-stage, two-factor models, however, are suited to capture the process of R&D and commercialization in its entirety. For reasons of simplicity, the discussion will focus on the case of deterministic jump size in the following.

7.2 Two-Stage and Two-Factor Models

In order to make the following discussion more accessible and to put the effects of patent risk into perspective, the case of perfect patent protection is analyzed first. The models presented are then extended in analogy to previous sections.

7.2.1 Perfect Patent Protection

In contrast to the very stylized formalization of the investment process proposed in previous sections, analyses in this subsection will serve to illustrate the significance of "time-to-build" and sequentiality, more accurately reflecting several stylized facts about capital budgeting in R&D-intensive industries.

7.2.1.1 Time-to-Build

Depending on whether the impact of investment is deterministic or stochastic, cost to completion is certain or uncertain. Both cases will be analyzed in turn, employing FD or Monte Carlo methods, repectively.

7.2.1.1.1 Certain Cost

The model presented in this subsection is similar to one developed by Majd and Pindyck [214]. Obtaining a solution requires numerical methods, not too different from those employed in sect. 6.2.2.2 [91, pp. 328–336 and 353–356].

Consider an aggregate gross payoff from commercialization that develops stochastically according to the following specification:

$$\mathrm{d}V_t = \alpha V_t \,\mathrm{d}t + \sigma V_t \,\mathrm{d}W_t.$$

In contrast to the more simple case of instantaneous payoff, it takes time to complete the investment. This feature found in some real option models is known as "time to build" [91, pp. 329–339].

The length of the investment phase is determined by a time-dependent investment rate I_t and a known cost to completion C_t. Changes in C_t are given by

$$\mathrm{d}C_t = -I_t\,\mathrm{d}t. \tag{7.25}$$

Although deterministic, (7.25) describes a controlled Itô process. Consequently, in order to calculate the value of a claim contingent on V_t and C_t, for example a patent, a multi-dimensional variant of Itô's Lemma is required.[22]

Lemma 2 (Itô's Lemma for vector-valued processes). *If $x_1, \ldots, x_m$ follow the (vector-valued) Itô process described by*

$$\mathrm{d}x_i = a_i(x_1, \ldots, x_m, t)\,\mathrm{d}t + b_i(x_1, \ldots, x_m, t)\,\mathrm{d}W_i, \quad i = 1, \ldots, m,$$

and $\mathbf{E}[\mathrm{d}W_i\,\mathrm{d}W_j] = \rho_{i,j}\,\mathrm{d}t$, the contingent claim $f(x_1, \ldots, x_m, t)$ also follows an Itô process, namely

$$\mathrm{d}f(\cdot) = \frac{\partial f(\cdot)}{\partial t} + \sum_i \frac{\partial f(\cdot)}{\partial x_i}\,\mathrm{d}x_i + \tfrac{1}{2}\sum_i \sum_j \frac{\partial^2 f(\cdot)}{\partial x_i \partial x_j}\,\mathrm{d}x_i\,\mathrm{d}x_j$$

or, in expanded form,

$$\begin{aligned}\mathrm{d}f(\cdot) = \Bigg(&\frac{\partial f(\cdot)}{\partial t} + \sum_i a_i(\cdot)\frac{\partial f(\cdot)}{\partial x_i} + \tfrac{1}{2}\sum_i b_i^2(\cdot)\frac{\partial^2 f(\cdot)}{\partial x_i^2} \\ &+ \tfrac{1}{2}\sum_{i \neq j} \rho_{i,j} b_i(\cdot) b_j(\cdot)\frac{\partial^2 f(\cdot)}{\partial x_i \partial x_j}\Bigg)\,\mathrm{d}t + \sum_i b_i(\cdot)\frac{\partial f(\cdot)}{\partial x_i}\,\mathrm{d}W_i.\end{aligned}$$

In this case, $m = 2$, $x_1 \equiv V_t$, $x_2 \equiv C_t$, $a_1(V_t, C_t, t) \equiv \alpha V_t$, $a_2(V_t, C_t, t) \equiv -I_t$, $b_1(V_t, C_t, t) \equiv \sigma V_t$, $b_2(V_t, C_t, t) \equiv 0$, $f(V_t, C_t, t) \equiv F(V_t, C_t)$, $\rho_{1,2} = 0$, and $\mathrm{d}W_1 \equiv \mathrm{d}W_t$. Hence, changes in patent value become

$$\begin{aligned}\mathrm{d}F(V_t, C_t) = {}&\alpha V_t \frac{\partial F(V_t, C_t)}{\partial V_t}\,\mathrm{d}t - I_t \frac{\partial F(V_t, C_t)}{\partial C_t}\,\mathrm{d}t \\ &+ \tfrac{1}{2}\sigma^2 V_t^2 \frac{\partial^2 F(V_t, C_t)}{\partial V_t^2}\,\mathrm{d}t + \sigma V_t \frac{\partial F(V_t, C_t)}{\partial V_t}\,\mathrm{d}W_t.\end{aligned}$$

As before, a portfolio is constructed to eliminate all terms involving $\mathrm{d}W_t$:

[22] Note that time indices have been omitted.

$$\Phi(V_t, C_t) = F(V_t, C_t) - V_t \frac{\partial F(V_t, C_t)}{\partial V_t}. \tag{7.26}$$

Changes in portfolio value are given by

$$\begin{aligned} \mathrm{d}\Phi(V_t, C_t) &= \mathrm{d}F(V_t, C_t) - \frac{\partial F(V_t, C_t)}{\partial V_t}\,\mathrm{d}V_t \\ &= -I_t \frac{\partial F(V_t, C_t)}{\partial C_t}\,\mathrm{d}t + \tfrac{1}{2}\sigma^2 V_t^2 \frac{\partial^2 F(V_t, C_t)}{\partial V_t^2}\,\mathrm{d}t. \end{aligned}$$

Holding the short position again requires a "dividend" payment. In addition, investment results in a cash outflow. Taking everything together, the portfolio yields the risk-free rate, that is

$$\begin{aligned} r\Phi(V_t, C_t)\,\mathrm{d}t = -I_t \frac{\partial F(V_t, C_t)}{\partial C_t}\,\mathrm{d}t &+ \tfrac{1}{2}\sigma^2 V_t^2 \frac{\partial^2 F(V_t, C_t)}{\partial V_t^2}\,\mathrm{d}t \\ &- \delta V_t \frac{\partial F(V_t, C_t)}{\partial V_t}\,\mathrm{d}t - I_t\,\mathrm{d}t. \end{aligned}$$

Substituting (7.26) and subsequent rearranging leads to a variant of the option pricing PDE deduced previously:

$$\begin{aligned} \tfrac{1}{2}\sigma^2 V_t^2 \frac{\partial^2 F(V_t, C_t)}{\partial V_t^2} &+ (r - \delta)\, V_t \frac{\partial F(V_t, C_t)}{\partial V_t} \\ &- rF(V_t, C_t) - I_t \frac{\partial F(V_t, C_t)}{\partial C_t} - I_t = 0. \end{aligned} \tag{7.27}$$

It is solved subject to the boundary conditions

$$F(V_t, 0) = V_t, \tag{7.28a}$$

$$F(0, C_t) = 0, \tag{7.28b}$$

$$\begin{aligned} \lim_{V_t \to \infty} \frac{\partial F(V_t, C_t)}{\partial V_t} &= 1 - \int_0^{C_t/I_{\max}} \delta \mathrm{e}^{-\mu t} \mathrm{e}^{\alpha t}\,\mathrm{d}t \\ &= \mathrm{e}^{-\delta C_t / I_{\max}}. \end{aligned} \tag{7.28c}$$

While the first two are more or less obvious, the last one requires some explanation. For very high gross payoffs it is safe to assume that commercialization takes place and the investment rate reaches a maximum ($I_t = I_{\max}$). Nevertheless, the "dividend" outflow continues until project completion, which implies (7.28c).

Similar to the basic model (see sect. 6.2.2), it is possible to specifiy a threshold V^*, obtain patent values above and below the boundary and then paste both solutions together, producing a smooth value function.

In the absence of adjustment costs, it is either optimal to invest at the maximum rate or not to invest at all [91, p. 330]:

$$I_t = \begin{cases} I_{\max} & \text{for } V^* < V_t, \\ 0 & \text{otherwise.} \end{cases}$$

Note that V^* is not constant, but varies with C_t. Hence, it is a *free* boundary that is found together with the solution. In the "continuation region," where $V_t \leq V^*$ and thus $I_t = 0$, the PDE to be solved simplifies considerably and becomes

$$\tfrac{1}{2}\sigma^2 V_t^2 \frac{\partial^2 F(V_t, C_t)}{\partial V_t^2} + (r - \delta) V_t \frac{\partial F(V_t, C_t)}{\partial V_t} - rF(V_t, C_t) = 0.$$

As demonstrated earlier, a solution to this equation is

$$F(V_t, C_t) = A^+ V_t^{\gamma^+}, \tag{7.29}$$

where

$$\gamma^+ = \tfrac{1}{2} - \frac{r - \delta}{\sigma^2} + \sqrt{\left(\frac{r - \delta}{\sigma^2} - \tfrac{1}{2}\right)^2 + 2\frac{r}{\sigma^2}}.$$

Just like V^*, the constant A^+ depends on C_t.

In the "investment region," where $I_t = I_{\max}$, the following PDE holds:

$$\begin{aligned} \tfrac{1}{2}\sigma^2 V_t^2 \frac{\partial^2 F(V_t, C_t)}{\partial V_t^2} &+ (r - \delta)\, V_t \frac{\partial F(V_t, C_t)}{\partial V_t} \\ &- rF(V_t, C_t) - I_{\max} \frac{\partial F(V_t, C_t)}{\partial C_t} - I_{\max} = 0. \end{aligned} \tag{7.30}$$

Unfortunately, (7.30) is of the parabolic type and a closed-form solution does not exist [91, p. 331]. Taking advantage of boundary conditions (7.28a), (7.28b), and (7.28c), however, FD procedures can be employed to obtain a numerical approximation of $F(V_t, C_t)$ in the investment region. As opposed to the discretization described in sect. 6.2.2.2, the grid constructed represents patent values along the dimensions V_t and C_t. Since the procedure is slightly more involved, the steps are described in detail below.

Value-matching and smooth-pasting conditions can be used to check for the free boundary and calculate A^+ as a function of C_t. Specifically, C^1-continuity at the free boundary implies

$$F\big(V^*, C_t\big) = A^+ \left(V^*\right)^{\gamma^+}, \tag{7.31a}$$

$$\frac{\partial F\big(V^*, C_t\big)}{\partial V_t} = A^+ \gamma^+ \left(V^*\right)^{\gamma^+ - 1}. \tag{7.31b}$$

Solving (7.31b) for A^+ and substituting the resulting expression in (7.31a) yields the free-boundary condition

$$F(V^*, C_t) = \frac{V^*}{\gamma^+} \frac{\partial F(V^*, C_t)}{\partial V_t}.$$

Although following discussions will focus on a more sophisticated model, a brief look at the numerical scheme proposed by Dixit and Pindyck [91, pp. 353–356] may serve to motivate the use of alternative pricing techniques.

A complete solution to the problem could be determined based on the familiar Crank–Nicolson scheme presented in sect. 6.2.2.2. To simplify calculations, however, Dixit and Pindyck consider an explicit FD scheme applied to a log-transform of (7.30), where $X_t \equiv \ln V_t$ and $F(V_t, C_t) \equiv \mathrm{e}^{-rC_t/I_{\max}} G(X_t, C_t)$. Derivatives,

$$\begin{aligned}
\frac{\partial F(V_t, C_t)}{\partial V_t} &\equiv \mathrm{e}^{-rC_t/I_{\max} - X_t} \frac{\partial G(X_t, C_t)}{\partial X_t}, \\
\frac{\partial^2 F(V_t, C_t)}{\partial V_t^2} &\equiv \mathrm{e}^{-rC_t/I_{\max} - 2X_t} \left(\frac{\partial^2 G(X_t, C_t)}{\partial X_t^2} - \frac{\partial G(X_t, C_t)}{\partial X_t} \right), \\
\frac{\partial F(V_t, C_t)}{\partial C_t} &\equiv \mathrm{e}^{-rC_t/I_{\max}} \left(\frac{\partial G(X_t, C_t)}{\partial C_t} - \frac{r}{I_{\max}} G(X_t, C_t) \right).
\end{aligned}$$

The transformed PDE becomes

$$\begin{aligned}
\tfrac{1}{2}\sigma^2 \frac{\partial^2 G(X_t, C_t)}{\partial X_t^2} &+ \left(r - \delta - \tfrac{1}{2}\sigma^2\right) \frac{\partial G(X_t, C_t)}{\partial X_t} \\
&- I_{\max} \frac{\partial F(V_t, C_t)}{\partial C_t} - \mathrm{e}^{rC_t/I_{\max}} I_{\max} = 0
\end{aligned} \tag{7.33}$$

with boundary conditions

$$\begin{aligned}
G(X_t, 0) &= \mathrm{e}^{X_t}, \\
G(0, C_t) &= 0, \\
\lim_{X_t \to \infty} \left(\mathrm{e}^{-rC_t/I_{\max} - X_t} \frac{\partial G(X_t, C_t)}{\partial X_t} \right) &= \mathrm{e}^{-\delta C_t/I_{\max}}.
\end{aligned}$$

Similarly, the free boundary condition can be expressed as

$$
\begin{aligned}
\mathrm{e}^{-rC_t/I_{\max}} G(X^*, C_t) &= \frac{\mathrm{e}^{X^*}}{\gamma^+} \mathrm{e}^{-rC_t/I_{\max} - X^*} \frac{\partial G(X^*, C_t)}{\partial X_t} \\
\Leftrightarrow G(X^*, C_t) &= \frac{1}{\gamma^+} \frac{\partial G(X^*, C_t)}{\partial X_t}.
\end{aligned} \tag{7.35}
$$

Transformation facilitates the FD procedure considerably, because coefficients no longer depend on X_t (or V_t). Set $X_{\max} \equiv m\Delta X$ and $C_{\max} \equiv n\Delta C$. Assume a rectangular grid, where

$$
X_t \in \{0, \Delta X, \ldots, i\Delta X, \ldots, X_{\max} - \Delta X, X_{\max}\}
$$

and

$$
C_t \in \{0, \Delta C, \ldots, j\Delta C, \ldots, C_{\max} - \Delta C, C_{\max}\}.
$$

Furthermore, let $G_{i,j}$ denote $G(X_i, C_j)$. Derivatives are approximated as follows:

$$
\begin{aligned}
\left.\frac{\partial G(X_t, C_t)}{\partial X_t}\right|_{C_t = C_j}^{X_t = X_i} &\approx \frac{G_{i+1,j} - G_{i-1,j}}{2\Delta X}, \\
\left.\frac{\partial^2 G(X_t, C_t)}{\partial X_t^2}\right|_{C_t = C_j}^{X_t = X_i} &\approx \frac{G_{i+1,j} + G_{i-1,j} - 2G_{i,j}}{\Delta X^2},
\end{aligned}
$$

and

$$
\left.\frac{\partial G(X_t, C_t)}{\partial C_t}\right|_{C_t = C_j}^{X_t = X_i} \approx \frac{G_{i,j+1} - G_{i,j}}{\Delta C}.
$$

Applying this discretization to (7.33) and solving for $G_{i,j+1}$ yields

$$
G_{i,j+1} = aG_{i+1,j} + (1 - b)\, G_{i,j} + cG_{i-1,j} + d_j, \tag{7.37}
$$

where

$$
\begin{aligned}
a &= \tfrac{1}{2} \frac{\Delta C}{I_{\max} \Delta X} \left(\frac{\sigma^2}{\Delta X} + \left(r - \delta - \tfrac{1}{2}\sigma^2\right) \right), \\
b &= \frac{\Delta C}{I_{\max} \Delta X} \frac{\sigma^2}{\Delta X}, \\
c &= \tfrac{1}{2} \frac{\Delta C}{I_{\max} \Delta X} \left(\frac{\sigma^2}{\Delta X} - \left(r - \delta - \tfrac{1}{2}\sigma^2\right) \right), \\
d_j &= -\mathrm{e}^{rj\Delta C/I_{\max}} \Delta C.
\end{aligned}
$$

Note that $a + (1 - b) + c \equiv 1$. The resulting expressions are very similar to those appearing in the standard model outlined in sect. 3.2.2.3.1, with the noteworthy exception of an additional constant term resulting from the time-to-build feature. Furthermore,

$$G_{i,0} = \mathrm{e}^{i\Delta X}, \tag{7.39a}$$

$$G_{0,j} = 0, \tag{7.39b}$$

$$\mathrm{e}^{-rj\Delta C/I_{\max} - m\Delta X} \frac{G_{m+1,j} - G_{m-1,j}}{2\Delta X} = \mathrm{e}^{-\delta j \Delta C/I_{\max}}. \tag{7.39c}$$

Equation (7.39c) is equivalent to

$$G_{m+1,j} = \mathrm{e}^{(r-\delta)j\Delta C/I_{\max} + m\Delta X} 2\Delta X + G_{m-1,j}.$$

Inserting this expression into (7.37) and setting $i = m$ leads to

$$\begin{aligned} G_{m,j+1} = a\mathrm{e}^{(r-\delta)j\Delta C/I_{\max} + m\Delta X} & 2\Delta X \\ & + (1-b)\, G_{m,j} + (a+c)\, G_{m-1,j} + d_j. \end{aligned}$$

The free boundary condition (7.35) is approximately fulfilled if

$$G_{i^*,j+1} \approx \frac{1}{\gamma^+} \frac{G_{i^*+1,j+1} - G_{i^*,j+1}}{\Delta X},$$

that is

$$G_{i^*,j+1} - \frac{G_{i^*+1,j+1}}{1 + \Delta X \gamma^+} \leq \epsilon,$$

where ϵ is some small number [91, p. 356].[23] Equation (7.29), after transformation, becomes

$$\mathrm{e}^{-rC_t/I_{\max}} G(X_t, C_t) = A^+ \mathrm{e}^{X_t \gamma^+}.$$

Consequently, A^+ can be determined from

$$A^+ = \mathrm{e}^{-r(j+1)\Delta C/I_{\max} - i^* \Delta X \gamma^+} G_{i^*,j+1}.$$

Using

$$G_{i,j+1} = A^+ \mathrm{e}^{r(j+1)\Delta C/I_{\max} + i\Delta X \gamma^+},$$

it is then possible to calculate patent values below the threshold.

[23] Note that, deviating from the original scheme, a forward difference approximation is used to express the first derivative.

Assuming $\sigma = 0.15$, $r = \delta = 0.05$, and a maximum investment rate $I_{\max} = 1.00$, one obtains the surface plot shown in fig. 7.7. Dynamic patent value increases as the project nears completion and eventually equals gross payoff from commercialization. Furthermore, the procedure yields an estimate of the free boundary to be discussed later in this chapter.

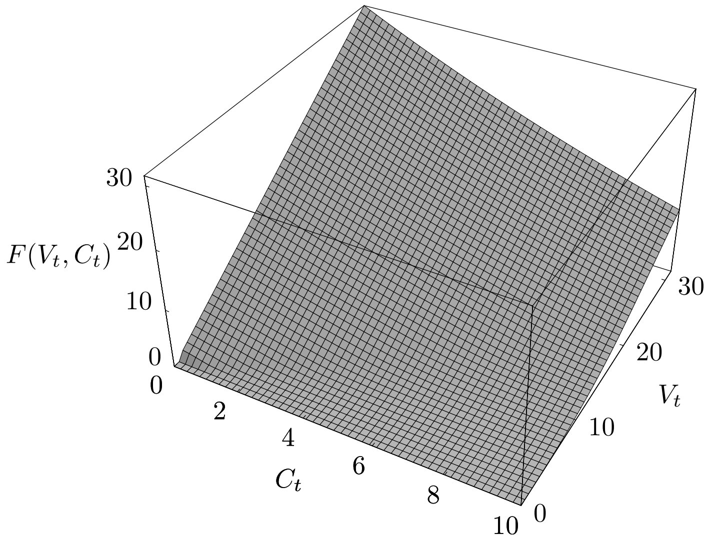


Fig. 7.7. Dynamic patent value with time-to-build under cost certainty ($\sigma = 0.15$, $r = \delta = 0.05$, $I_{\max} = 1.00$, $m = 50$ and $n = 25$). The plot shows patent value, which does not depend explicitly on time, as a function of gross payoff from commercialization and cost to completion. However, a positive investment rate will drive down cost to completion and thus increase patent value over time.

As illustrated by this comparatively simple example, there are limitations to common methods of analyzing option-based models of the R&D process, because FD pricing obviously lacks the intuitive appeal of much simpler approaches, for example multinomial trees. Nevertheless, building on the work of Majd and Pindyck [214], Pindyck [270] takes the FD approach further by introducing cost uncertainty. Similarly, Schwartz and Moon [303], Schwartz and Zozaya-Gorostiza [304] present analyses based on FD procedures.

In contrast, Hsu and Schwartz [146], Miltersen and Schwartz [245], Schwartz [302] resort to alternative optimization techniques, some of which were already successfully employed to study single-factor models in earlier chapters. These techniques are more promising, mainly due to the additional flexibility they provide in a jump-diffusion setting. Instead of proceeding along the path of more complicated FD calculations [85, pp. 25–57], analyses to come therefore highlight the implications of multiple sources of risk, drawing upon relatively recent advances in Monte Carlo simulation [207, 249].

Nevertheless, the discussion will return to the project-level model under certain cost in sect. 7.2.2.1, extending it to a setting of imperfect patent protection.

7.2.1.1.2 Uncertain Cost

At the heart of analyses carried out so far were uncertainty, flexibility, and irreversibility. Consequently, the following two subsections will serve to examine the impact of an additional source of uncertainty, namely one related to cost, presenting solutions to a model of the now familiar optimal-stopping type, but also considering the more general case of optimal control. The next section will then develop these results further, extending the discussion to a setting of imperfect patent protection.

7.2.1.1.2.1 Optimal Stopping

As shown in earlier sections, a straightforward extension of any project-level model—like the one analyzed in sect. 7.2.1.1.1—can be obtained by decomposing the gross payoff from commercialization into cash flows.

While Majd and Pindyck [214] and Pindyck [270] consider the case of instantaneous payoff upon completion and present solutions assuming both certain and uncertain cost to completion, cash flows during the commercialization phase can and should be modeled explicitly, for example to capture finite patent protection.

As usual, the profit rate process follows GBM:

$$\mathrm{d}\Pi_t = \alpha \Pi_t \,\mathrm{d}t + \sigma \Pi_t \,\mathrm{d}W_t^{\Pi}.$$

Replacing the real-world drift by the risk-neutral drift leads to

$$\mathrm{d}\Pi_t = \alpha^* \Pi_t \,\mathrm{d}t + \sigma \Pi_t \,\mathrm{d}W_t^{\Pi}, \tag{7.40}$$

where $\alpha^* = r - \delta$ and $\delta = \mu - \alpha$. To put it differently, $\alpha^* = \alpha - \eta$, where $\eta = \beta\,(r_{\mathrm{m}} - r)$ is the risk premium over the risk-free rate, such that $\mu = r + \eta$.

Furthermore, if $I_t \in [0, I_{\max}]$ denotes the investment rate at time t, and ς represents cost uncertainty, changes in the expected cost to completion of R&D are

$$\mathrm{d}C_t = -I_t \,\mathrm{d}t + \varsigma\sqrt{I_t C_t}\,\mathrm{d}W_t^C. \tag{7.41}$$

The underlying Wiener processes $W^{\Pi} = \{W_t^{\Pi}\}_{t\geq 0}$ and $W^C = \{W_t^C\}_{t\geq 0}$ may be correlated.[24]

Project value upon completion of R&D then fulfills

$$\begin{aligned}\tfrac{1}{2}\sigma^2 \Pi_t^2 \frac{\partial^2 V(\Pi_t,t)}{\partial \Pi_t^2} + \alpha^* \Pi_t \frac{\partial V(\Pi_t,t)}{\partial \Pi_t}& \\ + \frac{\partial V(\Pi_t,t)}{\partial t} - rV(\Pi_t,t) + \Pi_t &= 0,\end{aligned}$$

subject to the terminal condition

$$V(\Pi_T, T) = M\Pi_T, \tag{7.42}$$

where M is a kind of "exit multiple." It can be shown that gross payoff from commercialization is

$$\begin{aligned}V(\Pi_t,t) &= \mathbf{E}_{\mathbf{P}^*}\left[\int_t^T \mathrm{e}^{-r(s-t)}\Pi_s\,\mathrm{d}s + \mathrm{e}^{-r(T-t)}M\Pi_T \,\middle|\, \mathcal{F}_t\right] \\ &= \left(1-\mathrm{e}^{-(r-\alpha^*)(T-t)}\right)\frac{\Pi_t}{r-\alpha^*} \\ &\qquad + \mathrm{e}^{-(r-\alpha^*)(T-t)}M\Pi_t.\end{aligned} \tag{7.43}$$

Patent value is maximized by continuously choosing an optimal investment rate I_t^*, leading to

$$\begin{aligned}F(\Pi_t, C_t, t) = \mathbf{E}_{\mathbf{P}^*}\Bigg[&-\int_t^T \mathrm{e}^{-r(s-t)} I_s^*\,\mathrm{d}s \\ &+\int_t^T \mathrm{e}^{-r(s-t)}\mathbf{1}_{\{C_s=0\}}\Pi_s\,\mathrm{d}s \\ &+\mathrm{e}^{-r(T-t)}\mathbf{1}_{\{C_T=0\}}M\Pi_T \,\Bigg|\, \mathcal{F}_t\Bigg].\end{aligned}$$

Cash flows are accumulated once the expected cost to completion reaches zero.

[24] Equation (7.41) belongs to the class of Feller processes [104].

Since the protection period is finite, it seems unlikely that the project is resumed once it has been abandoned.[25] In analogy to the deterministic case, the absence of adjustment costs and low correlation between both processes imply investment at the maximum rate ("bang-bang policy"). Technically speaking, these simplifications turn the more general optimal *control* problem into a much more easily tractable optimal *stopping* problem,[26] which can be analyzed using the LSM approach introduced in sect. 7.1.1.2 and described in detail by Schwartz [302].

Developing a numerical procedure for the original problem is more challenging and could involve some type of Markov chain approximation (MCA) [178, pp. 144–145]. Simply employing a trinomial model and fitting it to appropriate drift and volatility surfaces—as commonly done for option pricing models with stochastic parameters—is likely to result in severe computational issues due to the non-recombining nature of the trees created. In any case, with the state space encompassing two stochastic variables, the problem unfortunately represents a fine example of Bellman's "curse of dimensionality." Nevertheless, the concept of a "bang-bang policy" may be used as a simplifying assumption, rendering the problem tractable by means of EMC simulation.

Turning again to the simplified model, the optimization problem can be summarized as follows:

$$F(\Pi_t, C_t, t) = \sup_{\tau \in [t,T]} \mathbf{E}_{\mathbf{P}^*}\left[- \int_t^{\tau} \mathrm{e}^{-r(s-t)} \mathbf{1}_{\{0<C_s\}} I_{\max}\, \mathrm{d}s \right. \\ \left. + \int_{\tau}^{T} \mathrm{e}^{-r(s-t)} \mathbf{1}_{\{C_s=0\}} \Pi_s\, \mathrm{d}s + \mathrm{e}^{-r(T-t)} \mathbf{1}_{\{C_T=0\}} M \Pi_T \,\middle|\, \mathcal{F}_t \right].$$

Figure 7.8 illustrates the development of expected cost to completion and profit rates over time. Although investment continues at the maximum rate until completion, cost uncertainty causes random fluctuations, that is C_t may even *increase* despite positive I_t.

Employing methods introduced in sect. 7.1.1, it is possible to account for catastophic risk, that is the possibility of project failure. Let κ denote the probability of project failure per unit of time. As demonstrated earlier, "jump-to-ruin" risk simply increases the discount rate for uncompleted projects, thus leading to

[25] A detailed assessment of this presumption is provided in the following subsection.

[26] In financial terms, one might refer to the contingent claim as a "down-and-in call option."

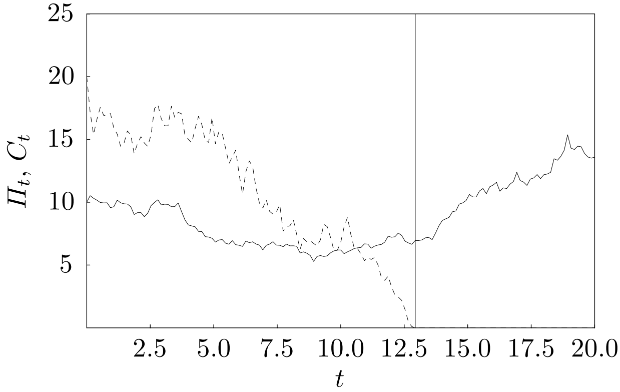


Fig. 7.8. Cash flow rates and expected cost to completion ($C_0 = 20$, $\Pi_0 = 10$, $\alpha^* = 0.02$, $I_{\max} = 2$, $\sigma = 0.1$, $\varsigma = 0.5$, $T = 20$, and $n = 150$). Investment continues at the maximum rate until the expected cost to completion C_t becomes zero (dashed line). The cash flow rate Π_t evolves separately (solid line), but obviously increases the investor's wealth only beyond $\tau^* = 12.93$ (vertical line).

$$\begin{aligned} F(\Pi_t, C_t, t) = \sup_{\tau \in [t,T]} \mathbf{E}_{\mathbf{P}^*} \Bigg[& - \int_t^{\tau} \mathrm{e}^{-(r+\kappa)(s-t)} \mathbf{1}_{\{0<C_s\}} I_{\max} \,\mathrm{d}s \\ & + \int_{\tau}^{T} \exp\left(- \int_t^s r + \mathbf{1}_{\{0<C_u\}} \kappa \,\mathrm{d}u \right) \mathbf{1}_{\{C_s = 0\}} \Pi_s \,\mathrm{d}s \\ & + \exp\left(- \int_t^T r + \mathbf{1}_{\{0<C_s\}} \kappa \,\mathrm{d}s \right) \mathbf{1}_{\{C_T = 0\}} M \Pi_T \,\Bigg|\, \mathcal{F}_t \Bigg] . \end{aligned} \quad (7.44)$$

Although this expression looks involved, it actually represents a fairly concise summary of the valuation procedure. Cash flows are discounted at the increased rate $r+\kappa$ during the investment phase. Once the project has been completed, the risk-free rate applies.

For the sake of completeness, note that the value of the ongoing project $F(\Pi_t, C_t, t)$ is then described by

$$\begin{aligned} & \tfrac{1}{2}\sigma^2 \Pi_t^2 \frac{\partial^2 F(\Pi_t, C_t, t)}{\partial \Pi_t^2} + \tfrac{1}{2}\varsigma^2 \sqrt{I_{\max} C_t} \frac{\partial^2 F(\Pi_t, C_t, t)}{\partial C_t^2} \\ & \quad + \sigma\varsigma\rho \Pi_t \sqrt{I_{\max} C_t} \frac{\partial^2 F(\Pi_t, C_t, t)}{\partial \Pi_t \partial C_t} + \alpha^* \Pi_t \frac{\partial F(\Pi_t, C_t, t)}{\partial \Pi_t} \\ & \quad - I_{\max} \frac{\partial F(\Pi_t, C_t, t)}{\partial C_t} + \frac{\partial F(\Pi_t, C_t, t)}{\partial t} \\ & \quad - (r+\kappa) F(\Pi_t, C_t, t) - I_{\max} = 0. \end{aligned} \quad (7.45)$$

Strictly speaking, however, this PDE is not required, because Monte Carlo simulation—not FD pricing—can be used to determine patent value.

In addition to project failure, patent events may also give rise to jump risk not included in κ. Section 7.2.2 will serve to examine this point in more detail. In this section, patent protection is assumed to be perfect.

The NPV solution may serve as a benchmark for future analyses. In the absence of cost uncertainty ($\varsigma = 0$), time to completion is known in advance:

$$\tau = C_0 / I_{\max}.$$

For the moment, assume a protection period which is sufficiently long to neglect projects not completed on time. Hence, one might argue that the NPV is given by

$$F(\Pi_0, C_0, 0) = \mathrm{e}^{-r\tau} V(\Pi_\tau, \tau) - \left(1 - \mathrm{e}^{-(r+\kappa)\tau}\right) \frac{I_{\max}}{r+\kappa},$$

where $V(\Pi_\tau, \tau)$ follows from (7.43) with $\mathbf{E}_{\mathbf{P}^*}[\Pi_\tau] = \mathrm{e}^{(\alpha^*-\kappa)\tau}\Pi_0$ [302]. More specifically,

$$V(\Pi_\tau, \tau) = \mathrm{e}^{(\alpha^*-\kappa)\tau} \left(\left(1 - \mathrm{e}^{-(r-\alpha^*)(T-\tau)}\right) \frac{\Pi_0}{r-\alpha^*} + \mathrm{e}^{-(r-\alpha^*)(T-\tau)} M \Pi_0 \right).$$

A closer look, however, reveals that this approach—discounting investments at the same rate used for cash flows—implicitly assumes partial flexibility, namely that investment stops if a catastrophic event causes the cash flow rate to drop to zero. Compared to

$$F(\Pi_0, C_0, 0) = \mathrm{e}^{-r\tau} V(\Pi_\tau, \tau) - \left(1 - \mathrm{e}^{-r\tau}\right) I_{\max}/r,$$

substantial mispricing may be the result. Admittedly, this insight is of primarily theoretical value, because only under very unusual circumstances a sensible person would continue to invest into an obviously worthless project.[27]

Applying Itô's Lemma as in the derivation of (7.45) [303, pp. 93–94], results in

[27] Note, however, that such irrational behavior is one of the basic assumptions of NPV analysis.

$$
\begin{aligned}
\mathrm{d}F(\Pi_t, C_t, t) = \Bigg(&\alpha^* \Pi_t \frac{\partial F(\Pi_t, C_t, t)}{\partial \Pi_t} - I_t \frac{\partial F(\Pi_t, C_t, t)}{\partial C_t} \\
&+ \tfrac{1}{2}\sigma^2 \Pi^2 \frac{\partial^2 F(\Pi_t, C_t, t)}{\partial \Pi_t^2} + \tfrac{1}{2}\varsigma^2 I_t C_t \frac{\partial^2 F(\Pi_t, C_t, t)}{\partial C_t^2} \\
&+ \sigma\varsigma\rho \Pi_t \sqrt{I_t C_t} \frac{\partial^2 F(\Pi_t, C_t, t)}{\partial \Pi_t \partial C_t}\Bigg) \mathrm{d}t \\
&+ \sigma \Pi_t \frac{\partial F(\Pi_t, C_t, t)}{\partial \Pi_t} \mathrm{d}W_t^{\Pi} + \varsigma \sqrt{I_t C_t} \frac{\partial F(\Pi_t, C_t, t)}{\partial C_t} \mathrm{d}W_t^{C}.
\end{aligned}
$$

The volatility $\left(\sigma^F\right)^2$ of the incomplete project thus becomes

$$
\begin{aligned}
\left(\sigma^F\right)^2 = &\left(\sigma \Pi_t \frac{\partial F(\Pi_t, C_t, t)}{\partial \Pi_t} / F(\Pi_t, C_t, t)\right)^2 \\
&+ I_t C_t \left(\varsigma \frac{\partial F(\Pi_t, C_t, t)}{\partial C_t} / F(\Pi_t, C_t, t)\right)^2 \\
&+ 2\sigma\varsigma\rho \Pi_t \sqrt{I_t C_t} \frac{\partial F(\Pi_t, C_t, t)}{\partial \Pi_t} \frac{\partial F(\Pi_t, C_t, t)}{\partial C_t} / F(\Pi_t, C_t, t)^2.
\end{aligned}
$$

Similarly, based on Merton's intertemporal asset pricing model [237], its beta is quickly derived as

$$
\beta^F = \beta \Pi_t \frac{\partial F(\Pi_t, C_t, t)}{\partial \Pi_t} / F(\Pi_t, C_t, t).
$$

A detailed analysis of patent value requires numerical methods.

A first step in obtaining a numerical solution is to specify approximations of the original processes, dividing time into small intervals of equal length $\Delta t = T/n$. The Euler scheme leads to

$$
\begin{aligned}
\Pi_{j+1} - \Pi_j &= \alpha^* \Pi_j \Delta t + \sigma \Pi_j \sqrt{\Delta t}\, \varepsilon_j^{\Pi}, \\
C_{j+1} - C_j &= -I_j \Delta t + \varsigma \sqrt{I_j C_j} \sqrt{\Delta t}\, \varepsilon_j^{C},
\end{aligned}
$$

where ε^{Π} and ε^{C} are correlated sequences of random numbers drawn from the standard normal distribution.[28] Patent value is then calculated in a dynamic programming fashion.

[28] At the price of a slight inconsistency, one could apply the Euler scheme to the logarithm of Π:

$$
\Pi_{t+\Delta t} = \Pi_t \exp\left(\left(\alpha^* - \tfrac{1}{2}\sigma^2\right) \Delta t + \sigma \sqrt{\Delta t}\, \varepsilon_t^{\Pi}\right).
$$

Since the process describing expected cost to completion is required to reach zero, such a transformation is not an option for C. Nevertheless, introducing a lower

At the end of the protection period, set

$$F(\Pi_m, C_m, m\Delta t) = \begin{cases} M\Pi_m & \text{if } C_m = 0, \\ 0 & \text{otherwise.} \end{cases} \tag{7.47}$$

For all previous periods, if expected cost to completion is zero,

$$F(\Pi_j, C_j, j\Delta t) = \Pi_j \Delta t + \mathrm{e}^{-r\Delta t} F\big(\Pi_{j+1}, C_{j+1}, (j+1)\,\Delta t\big). \tag{7.48}$$

If expected cost to completion is positive, the investor chooses between continuation and abandonment. Continuation implies

$$\begin{aligned} F(\Pi_j, C_j, j\Delta t) = {} & -I_{\max}\Delta t \\ & + \mathrm{e}^{-(r+\kappa)\Delta t} F\big(\Pi_{j+1}, C_{j+1}, (j+1)\,\Delta t\big), \end{aligned} \tag{7.49}$$

whereas abandonment leads to $F(\Pi_j, C_j, j\Delta t) = 0$. Note that, by (7.44), the discount rates used in (7.48) and (7.49) differ.

According to the LSM approach to option pricing, a suitable estimate of future project value—conditional on the project not being terminated during the next small time interval—can be obtained by performing a least-squares regression [207],[29] that is

$$\begin{aligned} \mathbf{E}_{\mathbf{P}^*}\left[\mathrm{e}^{-(r+\kappa)\Delta t} F\left(\Pi_{j+1}, C_{j+1}, (j+1)\,\Delta t\right) \,\middle|\, \mathcal{F}_j\right] = {} & \\ a_0 + a_1\Pi_j + a_2 C_j + a_3 \Pi_j C_j & \\ + a_4 \Pi_j^2 + a_5 C_j^2 + a_6 \Pi_j^2 C_j + a_7 \Pi_j C_j^2 + a_8 \Pi_j^2 C_j^2 &, \end{aligned}$$

where a_k are the regression coefficients. If this estimate, less the marginal investment required to keep the project alive ($I_{\max}\Delta t$) becomes negative, rational investors will choose to abandon the project. Continuation is optimal otherwise.

Unsurprisingly, Monte Carlo techniques run into difficulties in determining the boundary around points that are rarely encountered during simulation, in this case low expected cost to completion at the beginning of the protection period. Since the policy in these areas has no significant impact on overall patent value, the issue is negligible.

---

barrier $C_{\min} > 0$ makes it possible to also use a log-transformed model, namely

$$C_{t+\Delta t} = C_t \exp\left(-I_t/C_t \left(1 + \tfrac{1}{2}\varsigma^2\right)\Delta t + \varsigma\sqrt{I_t/C_t}\sqrt{\Delta t}\,\varepsilon_t^C\right).$$

[29] See also sections 3.2.3.3.1 and 7.1.2.2.2.

As far as LSM is concerned, little guidance is provided by the estimates of continuation value calculated at each timestep, because R^2 is typically low and the functional form not very intuitive. In general, recovering a smooth boundary from the data can be challenging [207, p. 137]. In order to facilitate comparison, a more detailed analysis of results is postponed to sect. 7.2.2, which also covers the case of imperfect patent protection.

Parametrization of the model requires a number of simplifying assumptions. Loosely speaking, one has to look at parameters related to volatility and correlation, cost and cash flow, compounding and discounting, as well as time to expiration. In the following, these parameters are discussed in turn, with the aim of justifying the choice of values for the base case described by Schwartz [301, pp. 16–18].

An approximate number for total out-of-the-pocket expenditures per project is 100 million euros. Given an expected time to completion of around 10 years, an investment rate of approximately 10 million euros per year is obtained. This view of course abstracts from different stages in the R&D process, which are typically characterized by specific investment rates.

Most statistical data available are on projects not completed, encompassing failed as well as abandoned endeavors, whereas λ is supposed to include project failure only. Consequently, to be precise, the probability of project failure is best determined in an iterative manner, starting with an initial guess that is refined by repeatedly deducing the number of projects abandoned implied and making necessary adjustments. Following Schwartz, assume for now that roughly half of all projects started do not reach commercialization due to catastrophic events. This assumption implies

$$\mathrm{e}^{-\lambda \times 10} = 0.5 \Leftrightarrow \lambda = 0.07.$$

Based on simulation results, a better choice would be $\lambda = 0.09$, more closely approximating the fraction of projects reaching commercialization according to DiMasi et al, namely 23 percent. In contrast, setting $\lambda = 0.07$ results in more than 30 percent of projects completed [89].

By (7.43), the value of the complete project is proportional to Π_t and thus exhibits the same volatility. Consequently, traded pure play benchmark firms may be used to calculate cash flow drift, risk premium, and volatility ($\alpha = 0.02$, $\eta = 0.036$, and $\sigma = 0.35$). Correlation between cash flow and cost processes is very difficult to observe. Frankly speaking, Schwartz's choice ($\rho = -0.1$) is a mere ad-hoc assumption. Values for the remaining parameters are $T = 20$, $M = 5$, and $r = 0.05$.[30]

When examining valuation results, it is important to keep in mind that LSM valuation of such complicated claims is subject to numerical inaccuracies, which can only be reduced at the price of considerable computational effort. Figure 7.2 gives an impression of the ranges to expect for the base case.

Table 7.2. Inaccuracies in LSM patent valuation ($C_0 = 100$, $I_{\max} = 10$, $\varsigma = 0.5$, $\Pi_0 = 20$, $\sigma = 0.35$, $\alpha = 0.02$, $M = 5$, $\kappa = 0.07$, $T = 20$, $\rho = -0.1$, $\eta = 0.036$, $r = 0.05$, $\Delta t = 0.25$, and $m = 100{,}000$). Even with 50,000 antithetic pairs, results for different seeds still vary. Nevertheless, the procedure is sufficiently accurate to allow for sensitivity analyses. Static patent value is provided for different assumptions concerning investments (see sect. 7.2.1.1.2). Calculations for the number of paths optimally abandoned are based on the risk-neutral measure and thereby do not reflect true probabilities.

Dynamic value	Percentage of projects abandoned	Static value	
		Partial flexibility	No flexibility
13.24	0.47	6.05	−14.46
13.27	0.46	5.79	−14.67
13.66	0.48	5.99	−14.78
14.26	0.51	5.94	−14.69
13.67	0.53	5.00	−13.84

Looking at the original analysis of Schwartz [302], employing high-dimensional quasi-random instead of pseudo-random numbers brings about only marginal improvements in terms of accuracy and computational speed. However, it can be shown how trinomial trees, under certain assumptions, provide equivalent results.[31]

In addition, it is worth pointing out the more pragmatic approach to option-based models of R&D taken by Hsu and Schwartz [146]. More specifically, the authors replace the controlled diffusion (7.41) by arithmetic Brownian motion (ABM):

$$\mathrm{d}C_t = -I_t\,\mathrm{d}t + \varsigma\,\mathrm{d}W_t^C.$$

The most attractive feature of the modified model is a simple expression for the expected first time the process hits zero. However, assuming that the diffusion part neither depends on the currently expected cost to

[30] For a more detailed account of parametrization see Schwartz [301, 302].

[31] For a detailed description of the algorithm, see sect. B.1.2 in the appendix.

completion nor on the investment rate seems quite unrealistic. Moreover, because analytical tractability is not of vital importance for the kind of numerical experiments that follow, implications are not examined further at this point.

More interesting, because directly comparable in terms of value maximization, is the extended problem outlined previously, which enables the investor to costlessly pause and resume investment. In particular, it is unclear at this point, whether, in line with the argumentation of Schwartz [302], a finite protection period really renders the option of "mothballing" an incomplete project entirely unattractive.

7.2.1.1.2.2 Optimal Control

Instead of considering policies allowing for any positive investment rate, the analysis will again focus on a "bang-bang policy." The claim to be analyzed, originally an *option to abandon,* becomes an *option to switch.* If the adjustment costs of switching between the two states of investing at the full rate and not investing at all are zero, then there will be a critical profit rate, which is a function of time and expected cost to completion.

As long as the current profit rate exceeds the threshold, it will be optimal to invest. Below the threshold, the investor is better off waiting for profits to increase again. Neglecting the possibility of projects not being completed within the protection period, the optimal stopping and optimal control problems can be contrasted by examining their state diagrams, shown in fig. 7.9.

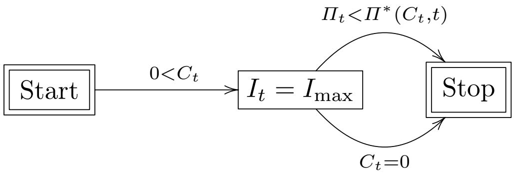
Determining the critical profit rate, however, is a non-trivial task. Almost needless to say, there is no known closed-form solution. Consequently, a numerical procedure is required to arrive at the value-maximizing program. The method proposed here, a variant of EMC simulation described in sect. 3.2.3.3.2, is highly effective, but also computationally expensive.

Recall that the generic EMC algorithm involves directly approximating the threshold by some suitable interpolation, which, for the problem at hand, leads to splines fitted to two-dimensional gridded data.[32] For

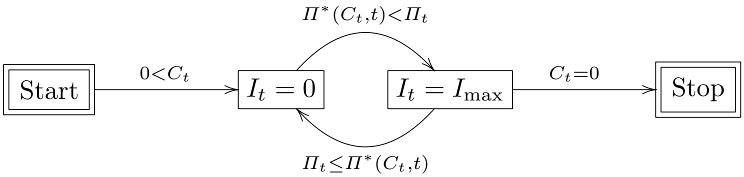
[32] A polynomial spline of order k can be described in terms of its breaks $\xi_1, \ldots, \xi_{l+1}$ and the local polynomial coefficients $c_{j,i}$ of its l pieces:

$$p_j(x) = \sum_{i=1}^{k} (x - \xi_j)^{k-i} c_{j,i}.$$

Analogously, a bivariate spline comprises a number of break sequences, a multidimensional coefficient matrix, a vector of number pieces, and a vector of polynomial



(a) Option to abandon



(b) Option to switch

Fig. 7.9. State diagram of the optimal stopping and control problems. Panel (b) can be regarded as a generalization of panel (a). Due to the absence of adjustment costs, a single boundary $\Pi^*(C_t, t)$ is sufficient to control transitions between states in both cases.

practical reasons, the optimization procedure is based on a relatively coarse three-by-three grid,[33] producing nine nodes in total. Each node corresponds to one data point. Taken together, these real numbers form the genotype of an individual in the population. Every time the fitness function is evaluated, the phenotype, that is the free boundary, is constructed from this input using piecewise polynomials.[34]

Numerical experiments show that explicitly introducing a jump component into the simulation yields more accurate results than simply increasing the discount rate. Equation (7.40) becomes

$$\mathrm{d}\Pi_t = \alpha^* \Pi_t \,\mathrm{d}t + \sigma \Pi_t \,\mathrm{d}W_t^{\Pi} - \phi \Pi_t \,\mathrm{d}J_t,$$

orders [39, pp. 2–16 and 2–36]. The computer program used in this analysis employs the cubic spline implementation ($k = 4$) described by de Boor [38]. See also de Boor [38, chap. 17], de Boor [37].

[33] Robustness checks were performed using a finer mesh. Significant changes in patent value could not be observed.

[34] For an overview of splines and applications see de Boor [38].

where $\mathrm{d}J_t$, in analogy to sect. 7.1.1.2.1, represents the increment of a Poisson process. As a side effect, this approach also facilitates the remaining steps.

A distinct advantage of the optimal stopping problem lies in the possibility to simply generate realizations of an uncontrolled diffusion process, whereas the optimal control problem requires a genuine controlled process. Fortunately, because, in contrast to LSM, an estimate of the free boundary is available from the outset in EMC simulation, this requirement poses no serious difficulties.

Discretization of the controlled vector-valued process produces two coupled difference equations, namely

$$\Pi_{j+1} - \Pi_j = \begin{cases} \alpha^* \Pi_j \Delta t + \sigma \Pi_j \sqrt{\Delta t}\, \varepsilon_j^{\Pi} - \Pi_j \,\mathrm{d}J_j & \text{if } 0 < C_j, \\ \alpha^* \Pi_j \Delta t + \sigma \Pi_j \sqrt{\Delta t}\, \varepsilon_j^{\Pi} & \text{otherwise} \end{cases}$$

and

$$C_{j+1} - C_j = \begin{cases} -I_j \Delta t + \varsigma \sqrt{I_j C_j} \sqrt{\Delta t}\, \varepsilon_j^{C} & \text{if } \Pi_j^* < \Pi_j, \\ 0 & \text{otherwise.} \end{cases}$$

Once a large number of realizations has been generated, it is straightforward to derive patent values in the usual manner, starting at the end of the protection period.

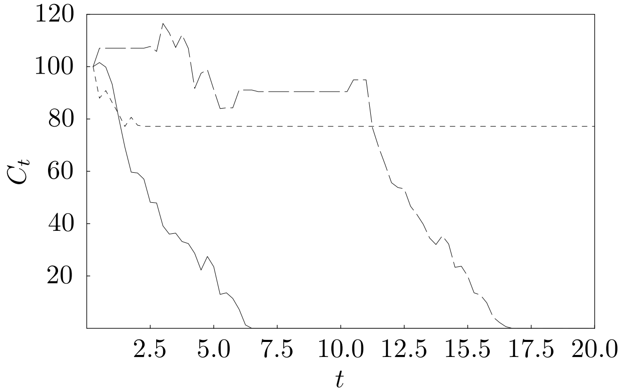


Fig. 7.10. Realizations of the controlled diffusion process ($C_0 = 100$, $I_{\max} = 10$, $\Pi_0 = 20$, $\alpha = 0.02$, $M = 5$, $\kappa = 0.07$, $T = 20$, $\eta = 0.036$, $r = 0.05$, $\Delta t = 0.25$, and $m = 100{,}000$). Long dashes illustrate how investment may be paused and later resumed. Other projects are abandoned altogether (short dashes) or continue at the maximum investment rate until completion (solid line).

The terminal condition given by (7.47) remains unchanged. Patent values are then calculated as follows:

$$F(\Pi_j, C_j, j\Delta t) = \begin{cases} A_1 & \text{if } 0 < C_j \wedge \Pi_j^* < \Pi_j, \\ A_2 & \text{if } 0 < C_j \wedge \Pi_j \leq \Pi_j^*, \\ A_3 & \text{if } C_j = 0, \end{cases}$$

where

$$\begin{aligned} A_1 &= -I_{\max}\Delta t + e^{-r\Delta t} F\big(\Pi_{j+1}, C_{j+1}, (j+1)\,\Delta t\big), \\ A_2 &= e^{-r\Delta t} F\big(\Pi_{j+1}, C_{j+1}, (j+1)\,\Delta t\big), \\ A_3 &= \Pi_j \Delta t + e^{-r\Delta t} F\big(\Pi_{j+1}, C_{j+1}, (j+1)\,\Delta t\big). \end{aligned}$$

Table 7.3 compares patent values for the base case according to the optimal stopping and the optimal control policies. As outlined previously, high correlation in particular may cause results to diverge. Nevertheless, already moderate negative correlation is sufficient to yield substantial differences in terms of dynamic patent value and the average cost to completion.

Table 7.3. Optimal stopping versus optimal control policies in patent valuation ($C_0 = 100$, $I_{\max} = 10$, $\Pi_0 = 20$, $\alpha = 0.02$, $M = 5$, $\kappa = 0.07$, $T = 20$, $\eta = 0.036$, $r = 0.05$, $\Delta t = 0.25$, and $m = 100{,}000$). Conditional expected cost to completion reflects average values for projects successfully completed. Static patent values obviously do not depend on the policy chosen and are provided as benchmarks only.

Policy	Patent risk (λ)	Static value	Dynamic value	Conditional expected cost to completion
Optimal stopping	0.00	−14.87	13.09	80.49
	0.01	−22.67	9.94	78.16
Optimal control	0.00	−14.87	15.33	77.38
	0.01	−22.67	11.45	73.41

In summary, the simplified optimal stopping problem is a quite poor approximation of the original model, subject to substantial underpricing. Although the protection period is limited, it is often advantageous to resume projects abandoned earlier. Furthermore, even the optimal control problem described in this section assumes a “bang-bang policy,” potentially underestimating the true value of flexibility.

Conceptually speaking, the proposed algorithm is closely related to the project-level model by Schwartz and Moon [303] and offers a pragmatic approach to optimal control in project-level analyses of R&D investments under cost uncertainty.

The EMC approach—with obvious modifications—can also be employed to replicate results obtained in the previous subsection. However, such a procedure is not advisable, due to a substantial increase in computational effort. The following analyses of patent risk will therefore be limited to the simplified optimal stopping case. Note that, for the sake of completeness, fig. 7.3 also presents indicative results under imperfect patent protection.

7.2.1.2 Sequential Investment

R&D in the pharmaceutical industry very closely resembles the stereotypical stage-gate model, each stage roughly corresponding to one of the the different phases prescribed by the FDA. In economic terms, sequential investments are compound options and, as such, can be priced using familiar option pricing techniques. Although breaking down the process into its parts may be advantageous in practical applications, particularly because stage-specific success probabilities and costs can be accounted for, the added value is limited from a conceptual perspective.

The reader should keep in mind, however, that the formal models presented could be extended to better capture special types of patents, most importantly platform technologies, which form the basis of subsequent innovations [173].

7.2.2 Imperfect Patent Protection

Based on the models presented so far, this section examines the impact of imperfect patent protection on optimal investment policies. As before, the cases of certain and uncertain cost are considered in turn.

7.2.2.1 Certain Cost

The extended FD model proposed here is readily comparable to the problem studied by Majd and Pindyck [214] and summarized earlier (see sect. 7.2.1.1.1). While the setup is quite simplistic, more sophisticated formalizations will be explored towards the end of this chapter.

Again, patent risk takes the form of jumps in the underlying process. If the size of jumps is stochastic,[35] advanced methods for solving PIDEs are required. For example, a generally applicable FD scheme for option pricing under Lévy processes has been developed recently by Cont and Voltchkova [68]. Such models will not be discussed in detail, because Monte Carlo techniques may be used to obtain similar results with less implementational effort.

However, as pointed out earlier, the task simplifies considerably if one limits the analysis to the case of proportional deterministic jumps. Gross payoff from commercialization then follows the familiar jump-diffusion process

$$\mathrm{d}V_t = \alpha V_t \,\mathrm{d}t + \sigma V_t \,\mathrm{d}W_t - \phi V_t \,\mathrm{d}J_t.$$

The steps to be carried out are almost completely analogous a setting with perfect patent protection. Nevertheless, to illustrate some subtle differences, the procedure is described in detail below.

According to Itô's Lemma for jump-diffusion processes, changes in patent value are given by

$$\begin{aligned}\mathrm{d}F(V_t, C_t) = \alpha V_t \frac{\partial F(V_t, C_t)}{\partial V_t} \,\mathrm{d}t - I_t \frac{\partial F(V_t, C_t)}{\partial C_t} \,\mathrm{d}t \\ + \tfrac{1}{2}\sigma^2 V_t^2 \frac{\partial^2 F(V_t, C_t)}{\partial V_t^2} \,\mathrm{d}t + \sigma V_t \frac{\partial F(V_t, C_t)}{\partial V_t} \,\mathrm{d}W_t \\ + \lambda\Big(F\big((1-\phi)V_t, C_t\big) - F(V_t, C_t)\Big) \,\mathrm{d}t.\end{aligned}$$

Constructing a portfolio free from market risk by setting

$$\Phi(V_t, C_t) = F(V_t, C_t) - V_t \frac{\partial F(V_t, C_t)}{\partial V_t}$$

leads to

$$\begin{aligned}- I_t \frac{\partial F(V_t, C_t)}{\partial C_t} \,\mathrm{d}t + \tfrac{1}{2}\sigma^2 V_t^2 \frac{\partial^2 F(V_t, C_t)}{\partial V_t^2} \,\mathrm{d}t \\ + \lambda\Big(F\big((1-\phi)V_t, C_t\big) - F(V_t, C_t)\Big) \,\mathrm{d}t - \delta V_t \frac{\partial F(V_t, C_t)}{\partial V_t} \,\mathrm{d}t \\ - I_t \,\mathrm{d}t = r\left(F(V_t, C_t) - V_t \frac{\partial F(V_t, C_t)}{\partial V_t}\right) \mathrm{d}t.\end{aligned}$$

Consequently, for the special case $\varphi = 1$, (7.27) becomes

[35] See the discussion of stochastic jump size in sect. 7.1.2.

$$\tfrac{1}{2}\sigma^2 V_t^2 \frac{\partial^2 F(V_t, C_t)}{\partial V_t^2} + (r-\delta)\, V_t \frac{\partial F(V_t, C_t)}{\partial V_t} \\ - (r+\lambda)\, F(V_t, C_t) - I_t \frac{\partial F(V_t, C_t)}{\partial C_t} - I_t = 0.$$

Rewriting the original upper boundary condition in terms of risk-neutral valuation yields

$$\lim_{V_t \to \infty} \frac{\partial F(V_t, C_t)}{\partial V_t} = 1 - \int_0^{C_t/I_{\max}} \delta \mathrm{e}^{-rt} \mathrm{e}^{(r-\delta)t}\, \mathrm{d}t.$$

Since the probability of no jump occuring up to time t is $\mathrm{e}^{\lambda t}$ (see sect. 7.1.1.1), (7.28c) becomes

$$\begin{aligned}\lim_{V_t \to \infty} \frac{\partial F(V_t, C_t)}{\partial V_t} &= 1 - \int_0^{C_t/I_{\max}} \delta \mathrm{e}^{-rt} \mathrm{e}^{(r-\delta)t} \mathrm{e}^{-\lambda t}\, \mathrm{d}t \\ &= 1 - \left(1 - \mathrm{e}^{-(\delta+\lambda)C_t/I_{\max}}\right) \frac{\delta}{\delta + \lambda}.\end{aligned}$$

Above the critical project value $I_t = I_{\max}$, whereas

$$\tfrac{1}{2}\sigma^2 V_t^2 \frac{\partial^2 F(V_t, C_t)}{\partial V_t^2} + (r-\delta)\, V_t \frac{\partial F(V_t, C_t)}{\partial V_t} - (r+\lambda)\, F(V_t, C_t) = 0$$

holds below the threshold. In analogy to perfect patent protection, solutions in the "continuation region" take the form $F(V_t, C_t) = A^+ V_t^{\gamma^+}$, where

$$\gamma^+ = \tfrac{1}{2} - \frac{r-\delta}{\sigma^2} + \sqrt{\left(\frac{r-\delta}{\sigma^2} - \tfrac{1}{2}\right)^2 + 2\frac{r+\lambda}{\sigma^2}}.$$

Applying the transformation

$$\begin{aligned} X_t &\equiv \ln V_t, \\ F(V_t, C_t) &\equiv \mathrm{e}^{-(r+\lambda)C_t/I_{\max}} G(X_t, C_t) \end{aligned}$$

results in the following identities:

$$\begin{aligned}\frac{\partial F(V_t, C_t)}{\partial V_t} &\equiv \mathrm{e}^{-(r+\lambda)C_t/I_{\max} - X_t} \frac{\partial G(X_t, C_t)}{\partial X_t}, \\ \frac{\partial^2 F(V_t, C_t)}{\partial V_t^2} &\equiv \mathrm{e}^{-(r+\lambda)C_t/I_{\max} - 2X_t} \left(\frac{\partial^2 G(X_t, C_t)}{\partial X_t^2} - \frac{\partial G(X_t, C_t)}{\partial X_t}\right), \\ \frac{\partial F(V_t, C_t)}{\partial V_t} &\equiv \mathrm{e}^{-(r+\lambda)C_t/I_{\max}} \left(\frac{\partial G(X_t, C_t)}{\partial C_t} - \frac{r+\lambda}{I_{\max}} G(V_t, C_t)\right).\end{aligned}$$

The transformed PDE is easily derived as

$$\tfrac{1}{2}\sigma^2 \frac{\partial^2 G(X_t, C_t)}{\partial X_t^2} + \left(r - \delta - \tfrac{1}{2}\sigma^2\right) \frac{\partial G(X_t, C_t)}{\partial X_t} \\ - I_{\max} \frac{\partial F(V_t, C_t)}{\partial C_t} - \mathrm{e}^{(r+\lambda)C_t/I_{\max}} I_{\max} = 0.$$

The upper boundary condition is

$$\lim_{X_t \to \infty} \left(\mathrm{e}^{-(r+\lambda)C_t/I_{\max} - X_t} \frac{\partial G(X_t, C_t)}{\partial X_t} \right) = \\ 1 - \left(1 - \mathrm{e}^{-(\delta+\lambda)C_t/I_{\max}}\right) \frac{\delta}{\delta + \lambda}.$$

In contrast, the free boundary condition remains unchanged, because

$$\mathrm{e}^{-(r+\lambda)C_t/I_{\max}} G(X^*, C_t) = \frac{\mathrm{e}^{X^*}}{\gamma^+} \mathrm{e}^{-(r+\lambda)C_t/I_{\max} - X_t} \frac{\partial G(X^*, C_t)}{\partial X_t}$$
$$\Leftrightarrow G(X^*, C_t) = \frac{1}{\gamma^+} \frac{\partial G(X^*, C_t)}{\partial X_t}.$$

Again, in order to obtain a solution, all equations must be translated into their discrete-time equivalents. The coefficients a, b, and c are those derived previously. However,

$$d_j = -\mathrm{e}^{(r+\lambda)j\Delta C/I_{\max}} \Delta C.$$

Armed with this information, it is straightforward to show that, for very high project values,

$$\mathrm{e}^{-(r+\lambda)j\Delta C/I_{\max} - m\Delta X} \frac{G_{m+1,j} - G_{m-1,j}}{2\Delta X} = \\ 1 - \left(1 - \mathrm{e}^{-(\delta+\lambda)j\Delta C/I_{\max}}\right) \frac{\delta}{\delta + \lambda},$$

which implies

$$G_{m,j+1} = a\mathrm{e}^{(r-\delta)j\Delta C/I_{\max} + m\Delta X} 2\Delta X \frac{1}{\delta + \lambda} \left(\delta + \mathrm{e}^{(\delta+\lambda)j\Delta C/I_{\max}} \lambda\right) \\ + (1 - b)\, G_{m,j} + (a + c)\, G_{m-1,j} + d_j.$$

For low project values,

$$G_{i,j+1} = A^+ \mathrm{e}^{(r+\lambda)(j+1)\Delta C/I_{\max} + i\Delta X \gamma^+},$$

where A^+ follows from

$$A^+ = \mathrm{e}^{-(r+\lambda)(j+1)\Delta C/I_{\max}-i^*\Delta X\gamma^+} G_{i^*,j+1}.$$

Setting $\lambda = 0$ shows that the extended model encompasses perfect patent protection as a limiting case.

Table 7.4. Dynamic patent value with time-to-build under perfect and imperfect patent protection ($\sigma = 0.15$, $r = \delta = 0.05$, $I_{\max} = 1.00$, $m = 50$, and $n = 25$). Due to the project-level specification, patent risk has no impact once the expected cost to completion reaches zero.

Patent risk (λ)	Payoff (V_t)	Patent value				
		$C_t = 10.0$	$C_t = 7.5$	$C_t = 5.0$	$C_t = 2.5$	$C_t = 0.0$
0.00	0.00	0.00	0.00	0.00	0.00	0.00
	2.12	0.01	0.02	0.06	0.22	2.12
	5.21	0.12	0.25	0.61	2.27	5.21
	12.81	1.33	2.71	5.60	8.98	12.81
	31.50	11.31	15.46	20.15	25.47	31.50
0.05	0.00	0.00	0.00	0.00	0.00	0.00
	2.12	0.00	0.00	0.01	0.11	2.12
	5.21	0.02	0.07	0.29	2.14	5.21
	12.81	0.48	1.60	5.03	8.85	12.81
	31.50	8.90	14.13	19.58	25.34	31.50
0.10	0.00	0.00	0.00	0.00	0.00	0.00
	2.12	0.00	0.00	0.00	0.07	2.12
	5.21	0.00	0.02	0.14	2.00	5.21
	12.81	0.13	0.76	4.35	8.71	12.81
	31.50	5.46	12.41	18.91	25.20	31.50

Table 7.4 shows numerical results for the choice of parameters discussed earlier, namely $\sigma = 0.15$, $r = \delta = 0.05$, and $I_{\max} = 1.00$. Judging from the example, patent value is particularly sensitive to changes in expected cost to completion if patent risk is high.

7.2.2.2 Uncertain Cost

A more realistic picture of the R&D process is drawn if costs and thus the time to completion are uncertain. Following an extension of the project-level analysis developed by Majd and Pindyck [214] in the previous section, this section will now turn to Schwartz's profit-level model [302].

7.2.2.2.1 Formalization

Under imperfect patent protection, the cash flow process is still governed by the discrete-time jump-diffusion equivalent

$$\Pi_{j+1} - \Pi_j = \alpha^* \Pi_j \Delta t + \sigma \Pi_j \sqrt{\Delta t}\, \varepsilon_j^{\Pi} - \Pi_j \, \mathrm{d}J_j .$$

Nevertheless, the intensity of the Poisson process is now $\kappa + \lambda$ in the R&D phase, dropping to λ during commercialization. The first parameter (κ) captures project-related risk, while the second parameter (λ), in analogy to previous chapters, represents patent-related risk. Alternatively, $J = \{J_t\}_{t \geq 0}$ could be considered as the superposition of two independent Poisson processes with intensities κ and λ. In any case, the occurrence of a jump causes the cash flow rate to drop to zero.

In order to develop a better understanding of the impact of patent risk, the following discussion will focus on comparative statics. Again, the familiar base case is examined, greatly facilitating comparison with the original setup described by Schwartz [301, 302].

7.2.2.2.2 Analysis

According to subtable 7.5(a), the adverse effects of patent risk are less pronounced if the investment policy is dynamic. This result is expected, because an abandonment option allows managers to discontinue projects following value-reducing patent events, thereby taking advantage of flexibility to "cut off" the lower end of the probability distribution. Furthermore, especially in the absence of patent risk, even static value increases sharply in cost uncertainty, which, by assumption, is symmetric. Sensitivity to cost uncertainty is substantially lower under patent risk.

As evident from subtable 7.5(b), rising cash flow uncertainty leads to higher dynamic project values, while static patent value is virtually unaffected. Negative correlation is value-enhancing regardless of patent risk, because more profitable projects are also completed earlier.

Of particular interest is the relationship between patent risk and expected cost to completion. On average, costly projects take longer to complete, rendering dynamic patent value more sensitive to patent risk. Looking at subtable 7.6(a), however, one finds that the opposite is true of static patent value, which is in fact less sensitive to patent risk

Table 7.5. Comparative statics (volatilities and correlation parameters) for two-factor R&D valuation ($C_0 = 100$, $I_{\max} = 10$, $\Pi_0 = 20$, $\alpha = 0.02$, $M = 5$, $\kappa = 0.07$, $T = 20$, $\eta = 0.036$, $r = 0.05$, $\Delta t = 0.25$, and $m = 100{,}000$). Subcolumns on the left-hand side represent absolute values, subcolumns on the right-hand side percentage changes relative to the middle row.

(a) Cost uncertainty ($\sigma = 0.35$, $\rho = -0.1$)

Cost uncertainty (ς)	Dynamic value		Static value	
	$\lambda = 0.00$	$\lambda = 0.01$	$\lambda = 0.00$	$\lambda = 0.01$
0.40	10.43	6.05	−18.20	−26.23
0.45	11.28	7.09	−17.53	−25.82
0.50	14.12	9.42	−14.33	−22.67
0.55	16.32	10.65	−11.82	−20.98
0.60	18.33	12.82	−9.90	−18.27

(b) Cash flow uncertainty ($\varsigma = 0.5$ and $\rho = -0.1$)

Cash flow uncertainty (σ)	Dynamic value		Static value	
	$\lambda = 0.00$	$\lambda = 0.01$	$\lambda = 0.00$	$\lambda = 0.01$
0.25	11.69	7.52	−15.33	−23.16
0.30	13.26	8.41	−14.60	−22.97
0.35	14.12	8.96	−14.33	−22.67
0.40	16.19	10.43	−13.11	−22.29
0.45	16.74	13.48	−14.43	−19.88

(c) Correlation ($\varsigma = 0.5$ and $\sigma = 0.35$)

Correlation (ρ)	Dynamic value		Static value	
	$\lambda = 0.00$	$\lambda = 0.01$	$\lambda = 0.00$	$\lambda = 0.01$
−0.10	14.12	9.42	−14.33	−22.67
−0.05	13.06	8.56	−15.45	−23.25
0.00	12.01	7.49	−16.90	−25.02
+0.05	10.03	6.60	−19.06	−26.11
+0.10	8.61	5.07	−20.33	−27.67

if expected cost to completion is high. Intuitively, this finding can be explained by the increased significance of cash outflows as opposed to cash inflows resulting from a static policy. Table 7.7 shows clearly that similar arguments apply as far as the initial cash flow rate and the cash flow multiple are concerned. More specifically, higher cash flows lead to higher reductions in static value due to additional patent risk.

Correspondingly, as illustrated by subtable 7.6(b), high investment rates in the absence of flexibility turn patent risk into a key value driver, whereas rising investment rates slightly reduce sensitivity to patent risk if there is an abandonment option.

Comparative statics with respect to compounding and discounting parameters are provided in table 7.8. Subtable 7.8(b) yields insights into the relationship between different types of risk. Flexible projects exhibiting high technology-related risk happen to be slightly more sensitive to changes in patent-related risk. In the absence of optionality, however, the opposite is the case, that is sensitivity to patent risk is high for technologically safe projects. Since the kind of jump risk considered essentially increases the discount rate, corresponding changes in sensitivity can also be observed in table 7.8(c).

Finally, table 7.9 contains information on the role of limited patent duration. Small changes in the time to expiration have little impact on how different levels of patent risk translate into dynamic value. In contrast, longer patent duration makes static value substantially more sensitive to variations in patent risk.

Figure 7.11 illustrates total investments for completed projects, based on a Kaplan–Meier estimate of the corresponding cumulative density function (CDF). Rising patent risk reduces the conditional probability of completed projects being costly. Nevertheless, the expected percentage of projects that are optimally abandoned before completion increases more than twofold, resulting in much higher expected investment outlays overall.

Consequently, rounding off the numerical analysis, fig. 7.12 depicts the effect of patent risk on abandonment by comparing survivor functions for various intensities of the patent-related jump process.[36] Higher patent risk greatly increases the probability of a project being abandoned early, creating substantial value compared to the static policy benchmark.

[36] For another example of survival analysis in the context of patent valuation see Barney [21].

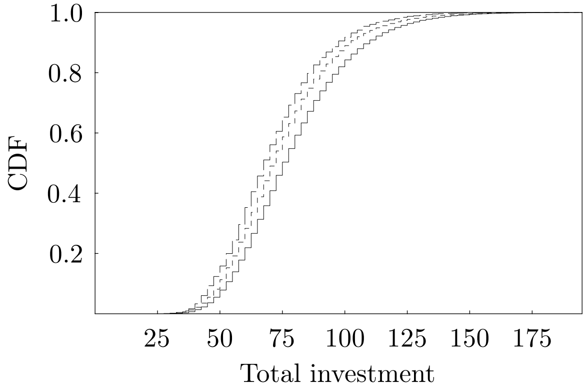


Fig. 7.11. Effect of patent risk on the CDF of total investment for successfully completed projects ($C_0 = 100$, $I_{\max} = 10$, $\varsigma = 0.5$, $\Pi_0 = 20$, $\sigma = 0.35$, $\alpha = 0.02$, $M = 5$, $\kappa = 0.07$, $T = 20$, $\rho = -0.1$, $\eta = 0.036$, $r = 0.05$, $\Delta t = 0.25$, and $m = 100{,}000$). Graphs shown reflect Kaplan–Meier estimates of the CDF. Compared to the case of perfect patent protection (solid line), additional patent risk tends to lower the conditional probability of completed projects being costly ($\lambda \in \{0.000, 0.025, 0.050\}$).

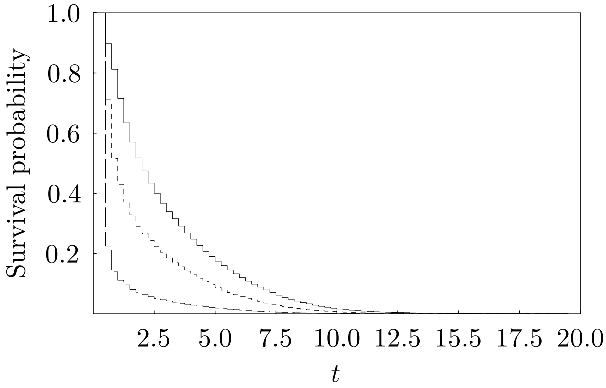


Fig. 7.12. Effect of patent risk on abandonment ($C_0 = 100$, $I_{\max} = 10$, $\varsigma = 0.5$, $\Pi_0 = 20$, $\sigma = 0.35$, $\alpha = 0.02$, $M = 5$, $\kappa = 0.07$, $T = 20$, $\rho = -0.1$, $\eta = 0.036$, $r = 0.05$, $\Delta t = 0.25$, and $m = 100{,}000$). Probabilities of project not having been abandoned at time t decrease significantly as patent risk rises ($\lambda \in \{0.000, 0.025, 0.050\}$). Again, results are based on Kaplan–Meier estimates.

Table 7.6. Comparative statics (cost parameters) for two-factor R&D valuation ($\varsigma = 0.5$, $\Pi_0 = 20$, $\sigma = 0.35$, $\alpha = 0.02$, $M = 5$, $\kappa = 0.07$, $T = 20$, $\rho = -0.1$, $\eta = 0.036$, $r = 0.05$, $\Delta t = 0.25$, and $m = 100{,}000$). Subcolumns on the left-hand side represent absolute values, subcolumns on the right-hand side percentage changes relative to the middle row.

(a) Expected cost to completion ($I_{\max} = 10$)

Cost to completion (C_0)	Dynamic value		Static value	
	$\lambda = 0.00$	$\lambda = 0.01$	$\lambda = 0.00$	$\lambda = 0.01$
80.00	34.91	26.38	17.02	7.04
90.00	22.64	16.51	−0.26	−9.27
100.00	14.12	9.42	−14.33	−22.67
110.00	7.66	4.44	−27.95	−35.10
120.00	3.41	1.28	−40.07	−45.84

(b) Investment rate ($C_0 = 100$)

Investment rate ($I_{\max}$)	Dynamic value		Static value	
	$\lambda = 0.00$	$\lambda = 0.01$	$\lambda = 0.00$	$\lambda = 0.01$
8.00	5.93	3.68	−27.58	−33.07
9.00	9.76	6.13	−21.01	−27.83
10.00	14.12	9.42	−14.33	−22.67
11.00	17.85	13.05	−8.94	−16.81
12.00	23.73	16.79	−1.09	−11.34

7.2.2.2.3 Variations and Extensions

More complicated jump-diffusion models can be implemented by a modification of the underlying processes themselves. Among the many specifications possible, consider the comparably basic variant of deterministic proportional jumps, or

$$\begin{aligned} \mathrm{d}\Pi_t &= \alpha \Pi_t \,\mathrm{d}t + \sigma \Pi_t \,\mathrm{d}W_t^{\Pi} - \phi \Pi_t \,\mathrm{d}J_t^{\Pi}, \\ \mathrm{d}C_t &= -I_t \,\mathrm{d}t + \varsigma \sqrt{I_t C_t} \,\mathrm{d}W_t^{C} + \varphi C_t \,\mathrm{d}J_t^{C}, \end{aligned}$$

where $\mathrm{d}J_t^{\Pi}$ and $\mathrm{d}J_t^{C}$ are increments of Poisson processes with intensities λ and κ, respectively. Intuitively, it makes sense to associate project-related risk with upward jumps in the expected cost to completion,

Table 7.7. Comparative statics (cash flow parameters) for two-factor R&D valuation ($C_0 = 100$, $I_{\max} = 10$, $\varsigma = 0.5$, $\sigma = 0.35$, $\alpha = 0.02$, $\kappa = 0.07$, $T = 20$, $\rho = -0.1$, $\eta = 0.036$, $r = 0.05$, $\Delta t = 0.25$, and $m = 100{,}000$). Subcolumns on the left-hand side represent absolute values, subcolumns on the right-hand side percentage changes relative to the middle row.

(a) Cash flow rate ($M = 5$)

Cash flow rate (Π_0)	Dynamic value		Static value	
	$\lambda = 0.00$	$\lambda = 0.01$	$\lambda = 0.00$	$\lambda = 0.01$
16.00	5.54	3.30	−27.15	−33.28
18.00	9.51	6.46	−20.25	−27.04
20.00	14.12	9.42	−14.33	−22.67
22.00	18.97	13.43	−8.65	−16.88
24.00	24.14	17.13	−2.50	−12.05

(b) Cash flow multiple ($\Pi_0 = 20$)

Cash flow multiple (M)	Dynamic value		Static value	
	$\lambda = 0.00$	$\lambda = 0.01$	$\lambda = 0.00$	$\lambda = 0.01$
3.00	10.33	6.67	−20.05	−26.70
4.00	12.99	7.83	−16.75	−25.02
5.00	14.12	9.42	−14.33	−22.67
6.00	15.80	10.46	−11.89	−20.43
7.00	18.41	11.75	−8.70	−18.05

whereas patent-related risk corresponds to downward jumps in the cash flow rate. For example, if $\varphi = 1.0$, project events cause the expected cost to completion to double.

Figure 7.13 shows how simultaneous project-related and patent-related risk affects patent value. Project events do not mean complete failure; and a proportional increase in cost to completion has little impact towards the end of the project. The opposite is true of patent events, because the cash flow rate tends to increase over time. As a consequence, patent value is less sensitive to changes in φ than to variations in ϕ.

In order to determine suitable parameters for the modified model, it obviously is no longer sufficient to simply estimate the percentage

Table 7.8. Comparative statics (compounding and discounting parameters) for two-factor R&D valuation ($C_0 = 100$, $I_{\max} = 10$, $\varsigma = 0.5$, $\Pi_0 = 20$, $\sigma = 0.35$, $M = 5$, $T = 20$, $\rho = -0.1$, $\eta = 0.036$, $\Delta t = 0.25$, $m = 100{,}000$). Subcolumns on the left-hand side represent absolute values, subcolumns on the right-hand side percentage changes relative to the middle row.

(a) Cash flow drift ($\kappa = 0.07$ and $r = 0.05$)

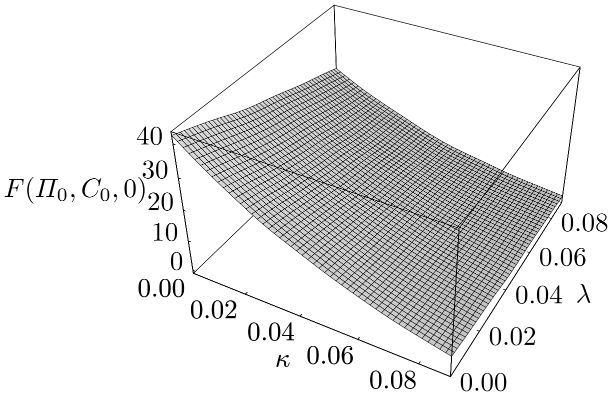
Cash flow drift (Π_0)	Dynamic value		Static value	
	$\lambda = 0.00$	$\lambda = 0.01$	$\lambda = 0.00$	$\lambda = 0.01$
0.00	4.00	2.42	−29.87	−35.92
0.01	7.87	5.28	−23.21	−29.56
0.02	14.12	9.42	−14.33	−22.67
0.03	21.59	14.82	−4.90	−14.82
0.04	31.07	22.30	6.47	−5.57

(b) Technological risk ($\alpha = 0.02$ and $r = 0.05$)

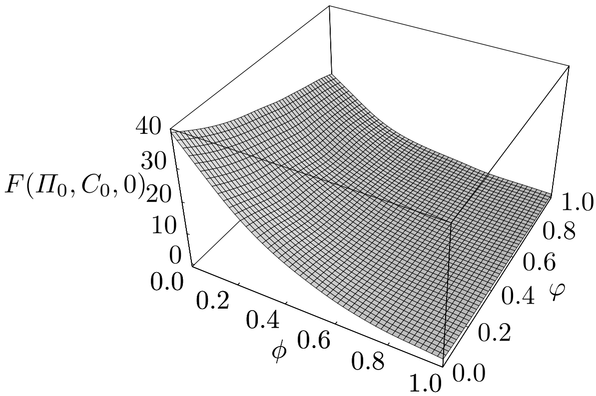
Technological risk (κ)	Dynamic value		Static value	
	$\lambda = 0.00$	$\lambda = 0.01$	$\lambda = 0.00$	$\lambda = 0.01$
0.05	20.58	13.40	−2.91	−13.60
0.06	17.68	11.20	−9.12	−18.40
0.07	14.12	9.42	−14.33	−22.67
0.08	12.03	8.14	−18.91	−26.02
0.09	9.94	5.66	−23.72	−30.89

(c) Risk-free rate ($\alpha = 0.02$ and $\kappa = 0.07$)

Risk-free rate (r)	Dynamic value		Static value	
	$\lambda = 0.00$	$\lambda = 0.01$	$\lambda = 0.00$	$\lambda = 0.01$
0.03	28.25	19.59	−1.93	−13.34
0.04	20.69	13.46	−8.56	−18.85
0.05	14.12	9.42	−14.33	−22.67
0.06	9.46	5.55	−18.60	−26.68
0.07	6.02	3.38	−22.27	−28.28



(a) Frequency ($\phi = \varphi = 0.5$)



(b) Severity ($\kappa = \lambda = 0.07$)

Fig. 7.13. Impact of simultaneous project-related and patent-related risk ($C_0 = 100$, $I_{\max} = 10$, $\varsigma = 0.5$, $\Pi_0 = 20$, $\sigma = 0.35$, $\alpha = 0.02$, $M = 5$, $T = 20$, $\rho = -0.1$, $\eta = 0.036$, $r = 0.05$, $\Delta t = 0.25$, and $m = 100{,}000$). Randomness in Monte-Carlo simulation leads to slight inaccuracy. Surfaces shown are based on a smoothing spline interpolation fitted to an 11-by-11 grid.

Table 7.9. Comparative statics (time to expiration) for two-factor R&D valuation ($C_0 = 100$, $I_{\max} = 10$, $\varsigma = 0.5$, $\mathit{\Pi}_0 = 20$, $\sigma = 0.35$, $\alpha = 0.02$, $M = 5$, $\kappa = 0.07$, $\rho = -0.1$, $\eta = 0.036$, $r = 0.05$, $\Delta t = 0.25$, and $m = 100{,}000$). Subcolumns on the left-hand side represent absolute values, subcolumns on the right-hand side percentage changes relative to the middle row.

Time to expiration (T)	Dynamic value		Static value	
	$\lambda = 0.00$	$\lambda = 0.01$	$\lambda = 0.00$	$\lambda = 0.01$
18.00	10.79	7.30	−19.19	−25.64
19.00	12.73	8.50	−16.01	−23.78
20.00	14.12	9.42	−14.33	−22.67
21.00	15.47	9.95	−12.79	−21.20
22.00	16.77	10.87	−10.69	−20.30

of projects successfully completed. The following chapter thus aims at pointing out some of the issues to be addressed in practical applications.

8

From Business Shifts to Jump Processes

This chapter serves to demonstrate the applicability of the models presented in real-world settings. In particular, the discussion will focus on the relationship between advanced jump-diffusion models of R&D covered up to this point and the business shift approach to R&D valuation described by Lint and Pennings [203].

Recall from sect. 7.1.1.1.1 that, if

$$\mathrm{d}V_t = \alpha V_t \,\mathrm{d}t + \sigma V_t \,\mathrm{d}W_t - \phi V_t \,\mathrm{d}J_t,$$

where $J = \{J_t\}_{t\geq 0}$ is a Poisson process of intensity λ. The variance of changes in commercialization payoff is

$$\mathbf{Var}[\mathrm{d}V_t] = \sigma^2 V^2 \,\mathrm{d}t + \lambda\phi^2 V_t^2 \,\mathrm{d}t.$$

Consequently, in terms of variance, σ^2 and $\lambda\phi^2$ are equivalent. This insight motivates the pragmatic approach proposed by Lint and Pennings, who replace σ by $\sqrt{\lambda}\,\phi$ in the Black–Scholes equation for European options [268].

The dynamic value of a patent, modeled as a European call option becomes

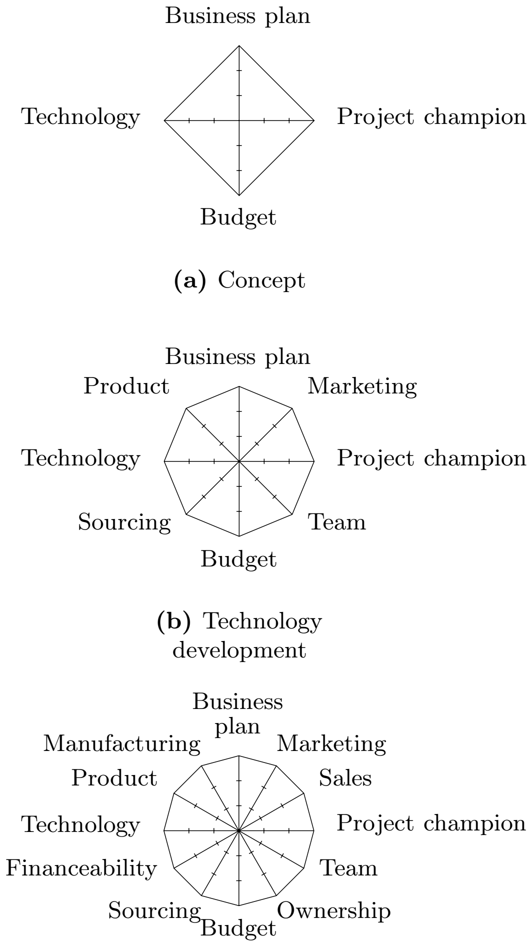
$$F(V_t, t) = V_t \mathrm{N}(d_1) - I\mathrm{e}^{-r(T-t)}\mathrm{N}(d_2),$$

where

$$d_1 = \frac{\ln\left(V_t/I\right) + \left(r + \frac{1}{2}\lambda\phi^2\right)(T-t)}{\sqrt{\lambda}\,\phi\sqrt{T-t}},$$
$$d_2 = d_1 - \sqrt{\lambda}\,\phi\sqrt{T-t}.$$

Obviously, the resulting equation is primarily based on practical considerations, ignoring the necessity to devise effective hedging strategies.

Nevertheless, it may be used to obtain approximate option values. Reliability is improved by embedding the formula into a comprehensive assessment tool (see fig. 8.1).



(a) Concept

(b) Technology development

(c) Market development

Fig. 8.1. Business development graphs generated by the PAT (Source: Lint and Pennings [203, p. 130]).

As shown in fig. 8.2, preliminary analyses carried out by Lint and Pennings seem to indicate that each industry is characterized by a distinct frequency and severity of value-relevant events. Similarly, impact-arrival portfolios could be employed to assess the patent risk profile of certain patent classes or even therapeutical areas. The formal models described in previous chapters constitute a comprehensive toolkit for translating such portfolios into optimal investment decisions.

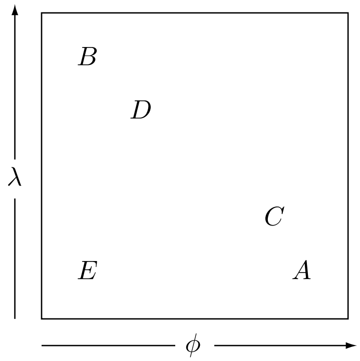


Fig. 8.2. Impact-arrival portfolio (Source: Lint and Pennings [203, p. 130]). Different industries typically exhibit different risk profiles (A = pharmaceutical, B = financial, C = multimedia, D = transport, and E = natural resources).

9

Preliminary Conclusion

The aim of this part of the text was to develop a formal, option-based framework for analyzing the impact of imperfect patent protection on capital budgeting in pharmaceuticals and other research-intensive industries, treating patent risk as an exogenous parameter. Chapter 6 provided an overview of patent valuation under uncertainty. Developing a model in the spirit of Paddock et al [263], an FD approximation of the free boundary was obtained, thereby clarifying the significance of finite patent protection when determining value-maximizing policies.

A variety of single-stage, single-factor models in sect. 7.1 served to introduce the notion of patent risk as jumps in the underlying process. Taking advantage of recent developments in the valuation of options under Lévy processes [60, 207, 249, 315] as well as advanced Monte Carlo techniques [42, 46], variants of these models were shown to capture both the frequency and severity of patent-related events.

Section 7.2 then offered more realistic two-stage, two-factor formalizations of patents and R&D, extending work by Schwartz [302] to allow for additional flexibility. In contrast to presumptions underlying this earlier analysis, the possibility to pause and later resume investment was shown to be a significant value driver, regardless of finite patent duration. The genetic algorithm employed represents, to the author's best knowledge, the first application of such optimization procedures to multi-factor option valuation. In addition, the models presented, also including the classic problem with time-to-build originally analyzed by Majd and Pindyck [214], were extended to account for exogenous patent risk.

Finally, chapter 8 added a more applied perspective, briefly hinting at connections between advanced jump-diffusion models of R&D and the business shift approach to R&D valuation [203, 268].

In summary, the quality of capital budgeting decisions in research-intensive industries is increased by employing comprehensive models, encompassing not only market-related and project-related, but also patent-related risk factors. Future research should therefore aim at developing an integrated, option-based view of imperfect patent protection.

Part III

Endogenous Patent Risk

10

Introduction and Related Work

In previous analyses, patent risk appeared as an exogenous parameter. In reality, the litigation is the result of value-maximizing behavior on the part of potential challengers. Consequently, the discussion proceeds by endogenizing patent risk, which, as will become clear, is best treated as an option to litigate.

Generally speaking, the aim of this dicussion is to illucidate the applicability of option pricing in the wider context of uncertain property rights and flexible managerial decisions surrounding them.

Lemley [191] points out a noticeable degradation of patent examination quality at the USPTO in recent years. However, because the vast majority of patents are of no appreciable business value, the incremental cost associated with marginally improving patent examination would not be justified by a substantial reduction in litigation costs.

In light of such serious deficiencies and heightened levels of competition, patenting has come to resemble the purchase of a lottery ticket, admittedly complicated by interdependencies between individual patents. Lemley and Shapiro conclude:

> "Under patent law, a patent is no guarantee of exclusion but more precisely a legal right to try to exclude. ... [M]ost patents represent highly uncertain or probabilistic property rights. By this we mean that patents are a mixture of a property right and a lottery." [192, p. 2]

Translating the vague notion of a *lottery* into a consistent valuation approach, the author will demonstrate how patents can be described as a mixture of a property right and a short option to litigate.

The analysis of patent risk as an endogenous parameter in option-based models of intellectual property is still in its infancy. To the author's best knowledge, the only detailed discussion of the option value of litigation is due to Marco [219]. His paper, however, has a strong empirical focus. Furthermore, the approach to formalizing patent risk differs from the one adopted here. Aoki and Hu [9] discuss time factors of patent litigation and licensing in a deterministic setting, also examining the role of settlement.

Roughly speaking, the discussion is structured as follows. The intuition behind the formal model and some basic definitions are provided in section 11.1, before section 11.2 lays out the details. Section 11.3 hints at a a number of variations and extensions of the original setup. Section 12 concludes and contains suggestions for future research.

11

Patent Risk as an Option to Litigate

Over the years, the literature on investment under uncertainty has seen a variety of generic (real) options, covering investment as well as disinvestment decisions. Since litigation, in a way, represents an investment with uncertain outcome, it seems natural to examine more closely the option value of litigation and its impact on capital budgeting decisions.

11.1 Formalization

The incumbent innovator owns a patent expiring at time T allowing him or her to commercialize some pharmaceutical product. Commercialization is associated with some expected revenue, which fluctuates randomly. This randomness is captured by specifying the revenue rate as a stochastic process.

While a variety of specifications are possible, a common choice in line with models discussed earlier is to let such variables evolve in analogy to the standard stock price model [290]. As demonstrated previously, abstracting from operating costs, the dynamics of the associated profit rate or net cash flow Π_t under the martingale measure $\mathbf{P}^*$ are then described by

$$\mathrm{d}\Pi_t = \alpha^* \Pi_t \,\mathrm{d}t + \sigma \Pi_t \,\mathrm{d}W_t, \qquad 0 < \Pi_0 = \varpi, \tag{11.1}$$

or, in integral notation,

$$\Pi_t = \Pi_0 + \int_0^t \alpha^* \Pi_s \,\mathrm{d}s + \int_0^t \sigma \Pi_s \,\mathrm{d}W_s, \tag{11.2}$$

where $\alpha^* = \alpha - \eta = r - \delta$ is the risk-adjusted drift,[1] σ the corresponding volatility, that is standard deviation of returns, and $W = \{W_t\}_{t \geq 0}$ is

[1] See section 7.2.1.1.2 for a more thorough account of the risk premium η.

one-dimensional Brownian motion. Following the risk-neutral pricing approach [137, 138], (11.1) and (11.2) describe the profit rate process in an equivalent risk-neutral world, making it possible to discount cash flows at the risk-free rate.

Without further emphasis, risk-neutral pricing is adopted for the rest of this analysis. While, in practice, profit rate and beta are difficult to determine, similar shortcomings are shared by all capital budgeting techniques (see sect. 7.1.2.1). Whoever accepts the validity of the CAPM will also accept the existence and uniqueness of the risk-neutral measure $\mathbf{P}^*$.

Another assumption worth pointing out is the non-negativity of net cash flows resulting from (11.1). It seems restrictive at first, but is sensible in many practical applications, including, in particular, pharmaceutical patents. Commercialization itself is almost always value-enhancing, because the lion's share of costs is incurred during R&D.

Due to the limited life of patents, however, cash flows will not continue indefinitely. The profit rate usually drops sharply upon expiration of the patent. In this model, the patent is taken to have a terminal value of $M\varPi_T$, where M is some multiple. The fiercer competition by imitators, or generics manufacturers, the lower M.

Let $\mathbf{E}_{\mathbf{P}^*}[\cdot]$ denote the expectation operator under the risk-neutral measure. In the absence of additional costs, the value of the project to the incumbent at time t, conditional on the information available to him or her at that time, is then given by

$$V_{\mathrm{I}}(\varPi_t, t) = \mathbf{E}_{\mathbf{P}^*}\left[\int_t^T \varPi_t \mathrm{e}^{-r(s-t)}\,\mathrm{d}s + M\varPi_T \mathrm{e}^{-r(T-t)} \,\middle|\, \mathcal{F}_t \right].$$

Note that, throughout this analysis and in line with the notation familiar from previous sections, V is used to refer to the project, whereas F signifies the option.

Following arguments laid out in detail in part II, the commercialization value must satisfy the PDE

$$\tfrac{1}{2}\sigma^2 \varPi_t^2 \frac{\partial^2 V_{\mathrm{I}}(\varPi_t, t)}{\partial \varPi_t^2} + (r-\delta)\,\varPi_t \frac{\partial V_{\mathrm{I}}(\varPi_t, t)}{\partial \varPi_t} - rV_{\mathrm{I}}(\varPi_t, t) + \varPi_t + \frac{\partial V_{\mathrm{I}}(\varPi_t, t)}{\partial t} = 0$$

with boundary condition

$$V_{\mathrm{I}}(\varPi_T, T) = M\varPi_T.$$

Again ruling out speculative bubbles and assuming perfect patent protection, the complete solution to this problem can be derived as

$$V_{\mathrm{I}}(\Pi_t, t) = \left(1 - \mathrm{e}^{-\delta(T-t)}\right) \Pi_t/\delta + \mathrm{e}^{-\delta(T-t)} M \Pi_t. \tag{11.3}$$

This formula corresponds to (6.62) plus a terminal value.

As discussed earlier and argued by Schwartz [301, 302], the associated process exhibits, in terms of risk premium and volatility, characteristics identical to those of the underlying cash flow process. It is thus possible to estimate η as well as σ from data on the drift and volatility of comparable completed projects.[2]

Proposition 11. *The dynamics of $V_{\mathrm{I}}(\Pi_t, t)$ under $\mathbf{P}$ observed in the real world are described by*

$$\mathrm{d}V_{\mathrm{I}}(\Pi_t, t) = (r + \eta)\, V_{\mathrm{I}}(\Pi_t, t)\, \mathrm{d}t + \sigma V_{\mathrm{I}}(\Pi_t, t)\, \mathrm{d}W_t,$$

where $r + \eta = \alpha + \delta$ is the required total return on Π_t according to the CAPM.

This proposition is verified by applying Itô's Lemma to (11.3) and using

$$\mathrm{d}\Pi_t = \alpha \Pi_t\, \mathrm{d}t + \sigma \Pi_t\, \mathrm{d}W_t,$$

which is just (11.3) under the true probability measure.[3]

Schwartz [302] also points out that the project value is linear in Π_t and independent of volatility. However, this conclusion hinges on the absence of flexibility once the incumbent has committed to commercialization. This not only means taking an un-realistic now-or-never view of decision-making on the side of the incumbent. It also neglects the effect of competitive action which is similarly contingent on how the revenue rate develops over time.

In the context of patent risk, which is the main focus of this analysis, it is important to note the profound impact a potential challenger has on the incumbent's optimal investment policy, as will be shown in more detail below. In the spirit of the real options paradigm, patent risk can thus be regarded as one of the many manifestation of optionality. As option value is heavily influenced by volatility, the project turns out to be sensitive to changes in this important parameter as well.

[2] Since $V_{\mathrm{I}}(\Pi_t, t)$ represents the value of a completed project under *perfect* patent protection, additional adjustments may become necessary in practice.

[3] Careful analysis draws a slightly different picture. See section A.2 in the appendix for details.

The discussion now proceeds by formalizing the above intuition. Based on the alleged infringement of a related patent, a challenger may decide to litigate at any time $\tau \in [0, T]$ and, if successful, receives a damage award, equal to a fraction $\zeta \in [0, 1]$ of the value of *past* cash flows, compounded to time τ.[4] Furthermore, the successful challenger may claim a fraction $\theta \in [0, 1]$ of *future* net cash flows. Due to improved monitoring after litigation, this fraction may very well be higher than the proportion of past net cash flows claimed.

If, on the one hand, the challenger is not willing or able to commercialize the patent, the incumbent will continue to market the product for the challenger as long as his or her participation constraint is fulfilled. Abstracting from a possible super-game, some marginally small profit is sufficient for this to be the case. Competition in other products and the threat of various forms of opportunistic behavior, however, may lead to concessions on the side of the challenger.

If the challenger, on the other hand, does not depend on the incumbent to market the product, θ becomes unity. Furthermore, the doctrine of *lashes* prevents a patent holder from obtaining damages for a time span during which he or she was aware of the alleged infringement, but did not take action. Otherwise, the challenger would be well-advised to wait for all market uncertainty to resolve, before taking the risk of a costly patent dispute. While it might prove difficult to establish the exact point in time at which the challenger took notice, the resulting damages award, expressed as a proportion of past cash flows, should be comparatively low.

Although, in principle, it might be interesting to examine the case in which the challenger is active in the market from the outset and, together with the incumbent, forms a duopoly, the challenger is assumed to be idle at time $t = 0$. Such a variation of the model would lower the challenger's expected gain over the status quo and thereby also diminish the incentive to litigate. Moreover, an alternative scenario with mutual litigation is conceivable.

According to the so-called *American* rule, both parties have to pay their lawyers out of their own pockets. For now, the American rule is applied to calculate litigation costs incurred by the incumbent and the challenger, which are denoted by L_{I} and L_{C}, respectively. In addition, let p denote the probability of successful litigation. The expected payoff from litigation becomes

[4] For reasons of simplicity, litigation has to take place within the specified time-frame and cannot be postponed beyond patent expiration.

$$\mathbf{E}_{\mathbf{P}^*}[V_{\mathrm{C}}(\Pi_\tau,\tau)-L_{\mathrm{C}}\,|\,\mathcal{F}_\tau]=$$
$$p\Bigg(\zeta\left(\int_t^\tau \mathrm{e}^{r(\tau-s)}\Pi_s\,\mathrm{d}s+\mathbf{1}_{\{\tau=T\}}M\Pi_T\right)$$
$$+\mathbf{E}_{\mathbf{P}^*}\left[\theta\left(\int_\tau^T \mathrm{e}^{-r(s-\tau)}\Pi_s\,\mathrm{d}s\right.\right.$$
$$\left.\left.+\mathbf{1}_{\{\tau<T\}}\mathrm{e}^{-r(T-\tau)}M\Pi_T\right)\,\Big|\,\mathcal{F}_\tau\right]\Bigg)-L_{\mathrm{C}}.$$

Given the information available at the time of litigation, cash flows are known for all $t\le\tau$. Cash flows beyond this point are still uncertain, making it necessary to take expectation over all possible realizations. Litigation costs are constant and known in advance.

The expected payoff is maximized by choosing an optimal litigation time. At time $t=0$, the option to litigate is worth

$$\begin{aligned} F_{\mathrm{C}}(\varpi,0) &= \sup_{\tau\in[0,T]}\mathbf{E}_{\mathbf{P}^*}\left[\mathrm{e}^{-r\tau}\left(V_{\mathrm{C}}(\Pi_\tau,\tau)-L_{\mathrm{C}}\right)^+\right] \\ &= \mathbf{E}_{\mathbf{P}^*}\left[\mathrm{e}^{-r\tau^*}\left(V_{\mathrm{C}}(\Pi_{\tau^*},\tau^*)-L_{\mathrm{C}}\right)^+\right]. \end{aligned} \tag{11.4}$$

All agents are assumed to follow a policy of value-maximization. Of course, the optimal litigation time τ^* cannot be specified in advance, but is chosen by the challenger in response to the resolution of uncertainty related to Π_t over time. For this reason, the stopping time τ^* is stochastic and can be described as the first time Π_t exceeds a critical level Π_t^*,

$$\tau^*=\inf\{t:\Pi_t^*<\Pi_t\},$$

which is sufficiently high to justify the cost of litigation. If litigation has not become optimal by the time the patent expires, no action is taken. Intuitively, the value of the project to the incumbent, including patent risk, becomes

$$\begin{aligned} \widetilde{V}_{\mathrm{I}}(\varpi,0) &= \mathbf{E}_{\mathbf{P}^*}\left[\left(1-\mathrm{e}^{-\delta T}\right)\varpi/\delta+M\Pi_T\mathrm{e}^{-\delta T}\right] \\ &\quad -F_{\mathrm{C}}(\varpi,0)-\mathbf{E}_{\mathbf{P}^*}\left[\mathbf{1}_{\{\tau^*\le T\}}\mathrm{e}^{-r\tau^*}\left(L_{\mathrm{I}}+L_{\mathrm{C}}\right)\right], \end{aligned} \tag{11.5}$$

that is the expected present value of cash flows from commercialization, less the option value of litigation, less the present value of additional litigation costs. In option terms, the incumbent is long the commercialization project and short an option to litigate. This important result is valid beyond the scope of this model and applies to any uncertain property right. For this reason, detailed discussion is deferred to a later chapter.

11.2 Analysis

Procedures employed to derive optimal policies are familiar from part II. Again, the deterministic and the stochastic case are examined in turn.

11.2.1 Deterministic Payoff

As a benchmark, this subsection considers optimal litigation under certainty. In contrast to the stochastic case, an optimal litigation time τ^* is straightforward to determine.

11.2.1.1 Finite Protection Period

Since deriving closed-form solutions in the presence of a finite patent protection period poses no difficulties under certainty, the analysis will focus on the more general model, before examining the limiting case of infinite patent duration.

11.2.1.1.1 Analytical Derivation

With $\sigma = 0$, (11.2) reduces to

$$\Pi_t = \varpi + \int_0^t \alpha \, \mathrm{d}s,$$

which implies $\Pi_t = \mathrm{e}^{\alpha t}\varpi$. Assuming intense competition after patent expiration, there will be no revenue for all $t > T$, which implies $M = 0$. Further assuming $0 < \alpha < r$ [91, p. 138], the discounted payoff from litigation at some future time τ becomes

$$\begin{aligned}
\mathrm{e}^{-r\tau}\big(V_\mathrm{C}(\varpi,\tau) - L_\mathrm{C}\big) &= p\left(\zeta \int_0^\tau \mathrm{e}^{-(r-\alpha)t}\varpi \, \mathrm{d}t\right. \\
&\qquad \left. +\theta \int_\tau^T \mathrm{e}^{-(r-\alpha)t}\varpi \, \mathrm{d}t\right) - \mathrm{e}^{-r\tau}L_\mathrm{C} \\
&= p\Bigg(\zeta\left(1 - \mathrm{e}^{-(r-\alpha)\tau}\right) \qquad (11.6) \\
&\qquad +\theta\left(\mathrm{e}^{-(r-\alpha)\tau} - \mathrm{e}^{-(r-\alpha)T}\right)\Bigg)\frac{\varpi}{r-\alpha} - \mathrm{e}^{-r\tau}L_\mathrm{C}.
\end{aligned}$$

The Marshallian rule commonly used in practice neglects timing issues altogether and simply requires positive net present value, or $0 < V_{\mathrm{C}}(\varpi, 0) - L_{\mathrm{C}}$. If, in addition, the value of waiting is accounted for, one obtains a critical revenue rate ϖ^* that triggers litigation, provided there is a positive expected payoff. This view leads to the following proposition for the deterministic case.

Proposition 12 (Deterministic trigger). *A critical cash flow rate ϖ^*, above which immediate litigation becomes optimal, exists if and only if $\zeta < \theta$, and it is given by*

$$\varpi^* = \frac{L_{\mathrm{C}} r}{p\,(\theta - \zeta)}. \tag{11.7}$$

Proof (Proposition 12). Consider the optimization problem

$$F_{\mathrm{C}}(\varpi, 0) = \max_{\tau \in [0,T]} \big(G_{\mathrm{C}}(\varpi, 0)\big)^+,$$

where

$$G_{\mathrm{C}}(\varpi, 0) = \mathrm{e}^{-r\tau}\big(V_{\mathrm{C}}(\varpi, \tau) - L_{\mathrm{C}}\big). \tag{11.8}$$

A necessary condition for a maximum is

$$\left.\frac{\partial G_{\mathrm{C}}(\varpi, 0)}{\partial \tau}\right|_{\tau=\tau^*} = \mathrm{e}^{-r\tau^*}\left(L_{\mathrm{C}} r - p\,(\theta - \zeta)\,\mathrm{e}^{\alpha\tau^*}\varpi\right) = 0.$$

The optimal policy depends on the ratio ζ/θ. If $\theta \le \zeta$, that is the successful challenger receives a larger proportion of past than of future cash flows, (11.8) is strictly increasing in τ, there is no interior solution, and it is optimal to postpone litigation as long as possible. Recall that, by assumption, $0 < \alpha < r$. Provided $\zeta < \theta$, (11.7) holds, and

$$\tau^* = \begin{cases} 0 & \text{if } \varpi^* < \varpi, \\ \frac{1}{\alpha} \ln \frac{L_{\mathrm{C}} r}{p(\theta-\zeta)\varpi} & \text{if } \mathrm{e}^{-\alpha T}\varpi^* < \varpi \le \varpi^*, \\ T & \text{otherwise.} \end{cases} \tag{11.9}$$

It is easily verified that the sufficient condition is always fulfilled, because

$$\left.\frac{\partial^2 G_{\mathrm{C}}(\varpi, 0)}{\partial \tau^2}\right|_{\tau=\tau^*} = -L_{\mathrm{C}} r \alpha \left(\frac{p\,(\theta - \zeta)\,\varpi}{L_{\mathrm{C}} r}\right)^{r/\alpha} < 0.$$

For the critical cash flow rate $\tau^* = 0$, so that, by (11.9), the deterministic trigger is in fact given by (11.7).

The lower the probability of success, the longer the optimal time to litigation. Increases in the fraction $L_{\mathrm{C}}r/\big(p\,(\theta-\zeta)\,\varpi\big)$ make postponing litigation more attractive. This result corresponds to the Jorgensonian investment rule,

$$p\,(\theta-\zeta)\,\varpi^* = L_{\mathrm{C}}r,$$

which triggers investment when the marginal revenue product equals the user cost of capital [160]. This rule applies regardless of patent duration.

Corollary 5 (Independence of patent length). *The deterministic trigger ϖ^* is independent of patent length.*

Proof (Corollary 5). Corollary 5 follows from proposition 12.

Nevertheless, patent duration does have an impact on whether it will be ever optimal to litigate at all, because optimal timing alone does not automatically lead to a positive payoff. Substituting (11.9) into (11.6) yields

$$F_{\mathrm{C}}(\varpi,0) = \left(p\zeta\left(1-\mathrm{e}^{-(r-\alpha)T}\right)\frac{\varpi}{r-\alpha} - \mathrm{e}^{-rT}L_{\mathrm{C}}\right)^+,$$

if $\varpi < \mathrm{e}^{-\alpha T}\varpi^*$, that is litigation takes place at the end of the protection period ($\tau^* = T$). Immediate litigation ($\tau^* = 0$) is optimal if $\varpi^* < \varpi$, and

$$F_{\mathrm{C}}(\varpi,0) = \left(p\theta\left(1-\mathrm{e}^{-(r-\alpha)T}\right)\frac{\varpi}{r-\alpha} - L_{\mathrm{C}}\right)^+.$$

For any profit rate that does not exceed the critical level, but is greater than $\mathrm{e}^{-\alpha T}\varpi^*$, there is an interior solution to the optimization problem ($0 < \tau^* < T$), and

$$F_{\mathrm{C}}(\varpi,0) = \left(p\left(\zeta-\theta\mathrm{e}^{-(r-\alpha)T}\right)\frac{\varpi}{r-\alpha} + \frac{L_{\mathrm{C}}\alpha}{r-\alpha}\left(\frac{p\,(\theta-\zeta)\,\varpi}{L_{\mathrm{C}}r}\right)^{r/\alpha}\right)^+. \qquad (11.10)$$

Intuitively, (11.10) decomposes the option value of litigation into two perpetuities and an option (see fig. 11.1). The latter is quite similar to the contingent claim of proposition 3. This observation comes in handy also under uncertainty (see sect. 11.2.2).

One implication of the above analysis is that, in the absence of substantial litigation costs, immediate legal action always maximizes the

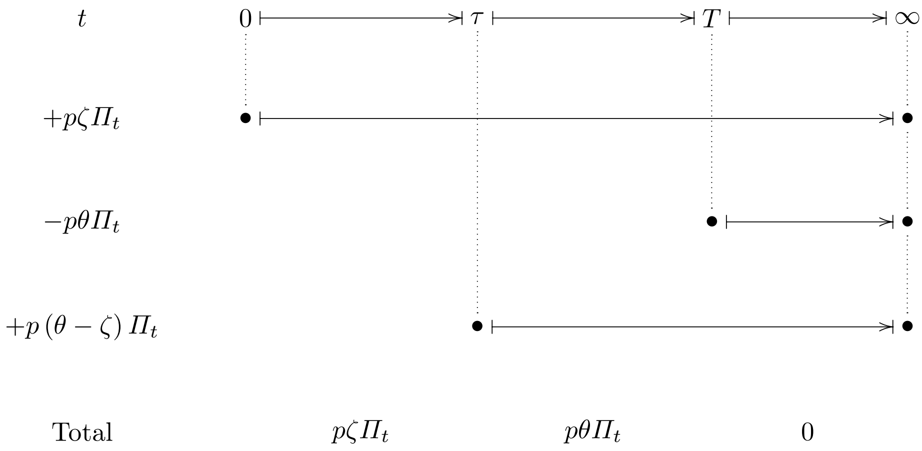


Fig. 11.1. Decomposing the payoff from litigation. Total payoff from litigation can be decomposed into two perpetuities and one option, creating two closed and one open interval with distinct profit rates. While T is pre-specified, the stopping time τ is chosen to maximize litigation payoff.

expected payoff from litigation. If litigation costs are comparatively high, however, challengers who, on the one hand, are likely to experience difficulties in claiming the full amount of their damage in court, but, on the other hand, will probably be able to negotiate participation in future cash flows benefit from immediate litigation. Firms that aim at being compensated in full and cannot participate in future increases of commercial value should postpone litigation. Since, in reality, litigation costs are usually substantial, optimal timing becomes essential. Optimal timing is determined by the ratio ζ/θ capturing a firm's relative ability to participate in past and future profits.

11.2.1.1.2 Numerical Illustration

The impact of this ratio is illustrated by fig. 11.2, which shows the discounted expected payoff from litigation as a function of litigation time for $p = 0.5$, $\varpi = 1.0$, $r = \delta = 0.05$, $\alpha = 0.1$, $\theta = 1.0$, $T = 20.0$, and $L_C = 10.0$. Simply inserting these parameters and $\zeta \in \{0.0, 0.5, 1.0\}$ into (11.7) yields the thresholds

$$\varpi_1^* = \frac{10.0 \times 0.05}{0.5\,(1.0 - 0.0)} = 1.0,$$
$$\varpi_2^* = \frac{10.0 \times 0.05}{0.5\,(1.0 - 0.5)} = 2.0.$$

Furthermore,

$$\lim_{\zeta \to 1.0} \frac{10.0 \times 0.05}{0.5\,(1.0 - \zeta)} = \infty.$$

Setting $\zeta = 0.0$ or $\zeta = 1.0$ thus produces the limiting cases of immediate litigation and litigation at the end of the protection period. An interior solution exists only for $\zeta = \zeta/\theta = 0.5$, namely

$$\tau^* = \frac{1}{0.1} \ln \frac{2.0}{1.0} = 6.93.$$

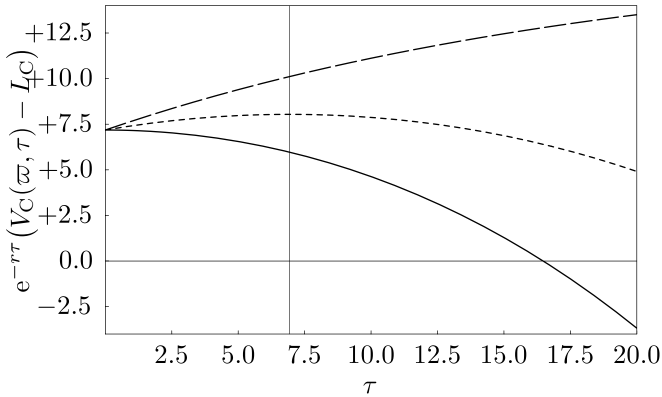


Fig. 11.2. Discounted expected payoff from litigation as a function of litigation time ($p = 0.5$, $\varpi = 1.0$, $r = \delta = 0.05$, $\alpha = 0.1$, $\theta = 1.0$, $T = 20.0$, and $L_C = 10.0$). If $\zeta = \zeta/\theta = 0.0$, immediate litigation is optimal (solid line); if $\zeta = \zeta/\theta = 1.0$, postponing litigation to the end of the protection period is the value-maximizing strategy (long dashes). For $\zeta = \zeta/\theta = 0.5$, the rational investor will litigate at time $\tau = 6.93$ (short dashes and vertical line).

11.2.1.2 Infinite Protection Period

At this point, an additional simplifying assumptions is introduced, which makes it possible to separate the effects of patent expiration and patent litigation, namely that the protection period T is infinite. This assumption also greatly facilitates the derivation of closed-form solutions for the stochastic case, analyzed in section 11.2.2. Equation (11.4) becomes

$$\begin{aligned} F_{\mathrm{C}}(\Pi_t) &= \max_{\tau \in [t,\infty)} p\left(\zeta + \mathrm{e}^{-(r-\alpha)(\tau-t)}\left(\theta - \zeta\right)\right)\frac{\Pi_t}{r-\alpha} - \mathrm{e}^{-r(\tau-t)}L_{\mathrm{C}} \\ &= p\left(\zeta + \mathrm{e}^{-(r-\alpha)(\tau^*-t)}\left(\theta - \zeta\right)\right)\frac{\Pi_t}{r-\alpha} - \mathrm{e}^{-r(\tau^*-t)}L_{\mathrm{C}}. \end{aligned}$$

By corollary 5, the trigger deduced previously applies regardless of patent length and thus continues to hold if the protection period is infinite. In addition,

$$\widetilde{V}_{\mathrm{I}}(\Pi_t) = \frac{\Pi_t}{r-\alpha} - F_{\mathrm{C}}(\Pi_t) - \mathrm{e}^{-r(\tau^*-t)}\left(L_{\mathrm{I}} + L_{\mathrm{C}}\right). \tag{11.11}$$

For example, immediate litigation of a perpetual patent implies

$$F_{\mathrm{C}}(\Pi_t) = V_{\mathrm{C}}(\Pi_t) - L_{\mathrm{C}} = p\theta\frac{\Pi_t}{r-\alpha} - L_{\mathrm{C}}$$

and thus

$$\begin{aligned} \widetilde{V}_{\mathrm{I}}(\Pi_t) &= \frac{\Pi_t}{r-\alpha} - \left(p\theta\frac{\Pi_t}{r-\alpha} - L_{\mathrm{C}}\right) - \left(L_{\mathrm{I}} + L_{\mathrm{C}}\right) \\ &= (1-p\theta)\frac{\Pi_t}{r-\alpha} - L_{\mathrm{I}}. \end{aligned}$$

If litigation is successful, there is no damage award, simply a participation in future cash flows from commercialization.

Previous discussions served to highlight timing flexibility in patent litigation under certainty, that is for the special case $\sigma = 0$. Under certainty, the value of waiting is solely driven by the ratio ζ/θ. As this ratio increases, so does the critical cash flow rate. Nevertheless, this view neglects the impact of σ, which is another important value driver. Therefore, in the following section, the effect of uncertainty on the litigation decision will be considered. The case of an infinite protection period under certainty is not examined further, because it obviously represents a limiting case of the stochastic model and is more or less analogous to the analyses carried out in section 6.2.1.1.

11.2.2 Stochastic Payoff

With the option value of litigation under certainty established, it is now possible to extend the model to a stochastic setting. The option value of litigation interacts with the option value of investing into R&D. Proceeding backwards in time, a sequential stochastic game for patent valuation will be developed.

11.2.2.1 Option to Litigate

The first step involves determining the payoff from commercialization, accounting for the short option to litigate held by a potential challenger.

Recall from section 11.1 that, under uncertainty,

$$\mathrm{d}\varPi_t = \alpha^* \varPi_t \,\mathrm{d}t + \sigma \varPi_t \,\mathrm{d}W_t, \qquad\qquad \varPi_0 = \varpi,$$

and, by assumption, $0 < \alpha^* < r$. The simplified optimization problem with no terminal value becomes

$$\begin{aligned} F_{\mathrm{C}}(\varPi_t) = \sup_{\tau \in [t,\infty)} \mathbf{E}_{\mathbf{P}^*} \Bigg[p \Bigg(\zeta \int_t^{\tau} \mathrm{e}^{-r(s-t)} \varPi_s \,\mathrm{d}s \\ + \theta \int_{\tau}^{\infty} \mathrm{e}^{-r(s-t)} \varPi_s \,\mathrm{d}s \Bigg) - \mathrm{e}^{-r(\tau-t)} L_{\mathrm{C}} \Bigg]. \end{aligned} \tag{11.12}$$

It looks challenging at first glance, but decomposes into tractable parts just like the deterministic model. Equation (11.12) can be re-written as

$$\begin{aligned} F_{\mathrm{C}}(\varPi_t) = \sup_{\tau \in [t,\infty)} \mathbf{E}_{\mathbf{P}^*} \left[-p\,(\theta - \zeta) \int_t^{\tau} \mathrm{e}^{-r(s-t)} \varPi_s \,\mathrm{d}s - \mathrm{e}^{-r(\tau-t)} L_{\mathrm{C}} \right] \\ + \mathbf{E}_{\mathbf{P}^*} \left[p\theta \int_t^{\infty} \mathrm{e}^{-r(s-t)} \varPi_s \,\mathrm{d}s \right] \end{aligned} \tag{11.13}$$

and

$$\begin{aligned} \mathbf{E}_{\mathbf{P}^*} \left[p\theta \int_t^{\infty} \mathrm{e}^{-r(s-t)} \varPi_s \,\mathrm{d}s \right] &= p\theta \int_0^{\infty} \mathrm{e}^{-(r-\alpha^*)(s-t)} \varPi_t \,\mathrm{d}s \\ &= \frac{p\theta \varPi_t}{r - \alpha^*}. \end{aligned}$$

The second term in (11.13) is thus independent of τ and can be neglected in determining an optimal stopping time.

Proposition 13. *Assuming $\zeta < \theta$, the option value of litigation is*

$$F_{\mathrm{C}}(\Pi_t) = \begin{cases} p\theta\Pi_t/\delta - L_{\mathrm{C}} & \text{if } \Pi^* < \Pi_t, \\ A^+\Pi_t^{\gamma^+} + p\zeta\Pi_t/\delta & \text{otherwise,} \end{cases} \tag{11.14}$$

where

$$\Pi^* = \frac{\gamma^+}{\gamma^+ - 1}\frac{L_{\mathrm{C}}\delta}{p\,(\theta-\zeta)} \tag{11.15}$$

denotes the critical cash flow rate,

$$\begin{aligned} A^+ &= \frac{L_{\mathrm{C}}}{\gamma^+ - 1}\left(\frac{1}{\Pi^*}\right)^{\gamma^+} \\ &= \left(\frac{p\,(\theta-\zeta)}{\gamma^+\delta}\right)^{\gamma^+}\left(\frac{\gamma^+ - 1}{L_{\mathrm{C}}}\right)^{\gamma^+ - 1}, \end{aligned}$$

and

$$\gamma^+ = \tfrac{1}{2} - \frac{r-\delta}{\sigma^2} + \sqrt{\left(\frac{r-\delta}{\sigma^2} - \tfrac{1}{2}\right)^2 + 2\frac{r}{\sigma^2}}.$$

Proof (Proposition 13). For convenience, define

$$\Psi_{\mathrm{C}}(\Pi_t) = \sup_{\tau\in[t,\infty)} \mathbf{E}_{\mathbf{P}^*}\left[-p\,(\theta-\zeta)\int_t^\tau \mathrm{e}^{-r(s-t)}\Pi_s\,\mathrm{d}s - \mathrm{e}^{-r(\tau-t)}L_{\mathrm{C}}\,\middle|\,\mathcal{F}_t\right]. \tag{11.16}$$

Under the abovementioned assumption that the risk in Π_t can be spanned by existing assets (see sect. 6.2.2), it is possible to construct a risk-free portfolio consisting of one unit of the claim $\Psi_{\mathrm{C}}(\Pi_t)$ and a short position of n units of Π_t. This feat is accomplished by choosing an appropriate quantity n. Economically speaking, the claim $\Psi_{\mathrm{C}}(\Pi_t)$ represents an abandonment (put) option on a project yielding a profit rate of $-p\,(\theta-\zeta)\,\Pi_t$. Nevertheless, the valuation procedure does not differ significantly from the approach adopted in previous chapters. Holding the portfolio yields a "dividend" of $-\big(p\,(\theta-\zeta) + n\delta\big)\Pi_t\,\mathrm{d}t$. Expanding $\mathrm{d}\Psi_{\mathrm{C}}(\Pi_t)$ using Itô's Lemma gives the "capital gain" on the portfolio, which is

$$\begin{aligned} \mathrm{d}\Psi_{\mathrm{C}}(\Pi_t) - n\,\mathrm{d}\Pi_t = \Bigg(&\alpha\Pi_t\left(\frac{\partial\Psi_{\mathrm{C}}(\Pi_t)}{\partial\Pi_t} - n\right) \\ &+ \tfrac{1}{2}\sigma^2\Pi_t^2\frac{\partial^2\Psi_{\mathrm{C}}(\Pi_t)}{\partial\Pi_t^2}\Bigg)\,\mathrm{d}t + \sigma\Pi_t\left(\frac{\partial\Psi_{\mathrm{C}}(\Pi_t)}{\partial\Pi_t} - n\right)\mathrm{d}W_t. \end{aligned}$$

For the portfolio to be risk-free, set $n = \partial\Psi_C(\Pi_t)/\partial\Pi_t$ and assume continuous rebalancing. Total return equals the risk-free return:

$$\left(\tfrac{1}{2}\sigma^2\Pi_t^2\frac{\partial^2\Psi_C(\Pi_t)}{\partial\Pi_t^2} - p\,(\theta-\zeta)\,\Pi_t - \frac{\partial\Psi_C(\Pi_t)}{\partial\Pi_t}\delta\Pi_t\right)\mathrm{d}t = r\left(\Psi_C(\Pi_t) - \frac{\partial\Psi_C(\Pi_t)}{\partial\Pi_t}\Pi_t\right)\mathrm{d}t$$

or

$$\tfrac{1}{2}\sigma^2\Pi_t^2\frac{\partial^2\Psi_C(\Pi_t)}{\partial\Pi_t^2} + (r-\delta)\,\Pi_t\frac{\partial\Psi_C(\Pi_t)}{\partial\Pi_t} - r\Psi_C(\Pi_t) - p\,(\theta-\zeta)\,\Pi_t = 0.$$

A general solution, which holds in the continuation region, is

$$\Psi_C(\Pi_t) = A^+\Pi_t^{\gamma^+} + A^-\Pi_t^{\gamma^-} - \frac{p\,(\theta-\zeta)\,\Pi_t}{r-\alpha^*},$$

where the roots of the characteristic equation $\{\gamma^+,\gamma^-\}$ are those derived in section 6.2.2. Given that $\alpha^* = r-\delta$,

$$\gamma^\pm = \tfrac{1}{2} - \frac{\alpha^*}{\sigma^2} \pm \sqrt{\left(\frac{\alpha^*}{\sigma^2} - \tfrac{1}{2}\right)^2 + 2\frac{r}{\sigma^2}}.$$

The constants A^+ and A^- are to be determined. Setting $A^- = 0$ ensures that $\Psi_C(\Pi_t)$ is bounded near $\Pi_t = 0$. Since the option grants the holder the right to exchange uncertain negative profits for a negative cash flow known with certainty, it should be worthless for small Π_t.

In the stopping region, immediate exercise is optimal and $\Psi_C(\Pi_t) = -L_C$. Hence,

$$\Psi_C(\Pi_t) = \begin{cases} -L_C & \text{if } \Pi^* < \Pi_t, \\ A^+\Pi_t^{\gamma^+} - \frac{p(\theta-\zeta)\Pi_t}{r-\alpha^*} & \text{otherwise.} \end{cases} \tag{11.17}$$

Imposing C^1-continuity at $\Pi_t = \Pi^*$ as usual leads to

$$A^+(\Pi^*)^{\gamma^+} - \frac{p\,(\theta-\zeta)\,\Pi_t}{r-\alpha^*} = -L_C \tag{11.18a}$$

and

$$\gamma^+A^+(\Pi^*)^{\gamma^+-1} - \frac{p\,(\theta-\zeta)}{r-\alpha^*} = 0. \tag{11.18b}$$

These equations are the *value-matching* and *smooth-pasting* conditions, respectively. Solving (11.18b) for A^+, substituting the result in (11.18a) and subsequently solving for Π^* leads to

$$\begin{aligned} \Pi^* &= \frac{\gamma^+}{\gamma^+ - 1} \frac{L_{\mathrm{C}} (r - \alpha^*)}{p (\theta - \zeta)} \\ \Leftrightarrow \frac{p (\theta - \zeta) \Pi^*}{r - \alpha^*} &= \frac{\gamma^+}{\gamma^+ - 1} L_{\mathrm{C}} \end{aligned} \tag{11.19}$$

and

$$A^+ = \frac{1}{\gamma^+} \frac{p (\theta - \zeta) (\Pi^*)^{1-\gamma^+}}{r - \alpha^*}. \tag{11.20}$$

Since $\alpha^* < r$ (by assumption) and $1 < \gamma^+$, Π^* will take positive values if and only if $\zeta < \theta$. Since $\Pi_t = 0$ is an absorbing barrier, litigation will never be optimal otherwise.

Summing up, provided $\zeta < \theta$, by (11.13), (11.16), and (11.17), the option value of litigation is given by (11.14).

As expected (11.15) is analogous to the deterministic case from (11.7), but, in addition, includes the well-known "option value multiple" $\gamma^+ / (\gamma^+ - 1)$. It is increasing in σ, which implies a higher value of waiting for higher levels of uncertainty. Also note that, compared to the Jorgensonian rule, $r - \alpha^*$ replaces r. As volatility approaches zero, the stochastic trigger converges to the deterministic trigger (see sect. 6.2.2):

$$\lim_{\sigma \to 0} \frac{\gamma^+}{\gamma^+ - 1} \frac{L_{\mathrm{C}} (r - \alpha^*)}{p (\theta - \zeta)} = \frac{L_{\mathrm{C}} r}{p (\theta - \zeta)}.$$

Convergence is demonstrated by fig. 11.3.

Litigation will be postponed as long as possible if $\theta \leq \zeta$ and

$$F_{\mathrm{C}}(\Pi_t) = \frac{p \zeta \Pi_t}{r - \alpha^*}.$$

The latter result obviously fundamentally relies on the assumption of infinite patent protection and is thus primarily of theoretical relevance.

If the revenue rate lies above the critical level, that is immediate litigation is optimal, the option value equals the expected share of future revenues, less litigation costs. Below the critical level, the option value has two components. One component is the expected payoff from litigation under the assumption of indefinite postponement. Continuation in this setting implies that the holder of the option acquires an (expected) claim on past cash flows. The other component is the value of flexibility.

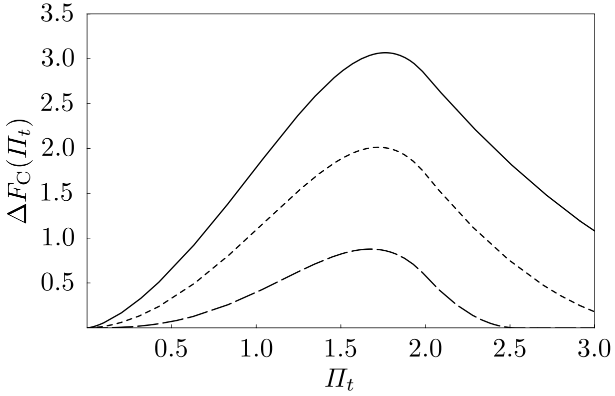


Fig. 11.3. Impact of uncertainty on the option value of litigation ($p = 0.5$, $r = 0.05$, $\alpha = r - \delta = 0.01$, $\theta = 1.0$, $\zeta = 0.5$, and $L_{\mathrm{C}} = 10.0$). As volatility decreases ($\sigma \in \{0.1, 0.2, 0.3\}$), option values converge to the deterministic solution.

Under the condition that the initial revenue rate is sufficiently high, the option holder will litigate and give up this flexibility in exchange for immediate benefits.

The net payoff from commercialization corresponds to the value a rational investor would attribute to a patent if he or she were to enter the relevant market immediately. As outlined previously, it equals the net present value of expected profits, less the option value of litigation, less the expected value of additional litigation costs. The latter component deserves more detailed analysis.

With the option value of litigation known, determining the gross payoff from commercialization to the incumbent under patent risk $\widetilde{V}_{\mathrm{I}}(\Pi_t)$ seems straightforward. However, one has to account for the fact that the cost of litigation for the incumbent and the challenger might differ, that is $L_{\mathrm{I}} \neq L_{\mathrm{C}}$. It is therefore insufficient to simply subtract the "short position." Finding the appropriate discount rate for the correction introduced in (11.5), however, is non-trivial, because the occurrence of litigation is random. Consequently, one needs to form expectations about the "first hitting time" τ^*.[5]

Theorem 2. *If $\Pi^* \geq \Pi_t$ is a fixed upper threshold, and $\tau^* \geq t$ is the first hitting time,*

$$\mathbf{E}_{\mathbf{P}^*}\left[\mathrm{e}^{-r(\tau^*-t)} \,\middle|\, \mathcal{F}_t\right] = \left(\frac{\Pi_t}{\Pi^*}\right)^{\gamma^+}. \tag{11.21}$$

[5] For details see Dixit and Pindyck [91, pp. 315–316], Dixit et al [92], Harrison [139, p. 42], Karlin and Taylor [164, p. 362].

The proof presented here is based on the dynamic-programming approach outlined in the appendix (see sect. A.1).

Proof (Theorem 2). The discount rate is a function of Π_t. Set

$$G(\Pi_t, t) = \mathbf{E}_{\mathbf{P}^*}\left[\mathrm{e}^{-r(\tau^*-t)} \,\middle|\, \mathcal{F}_t\right].$$

If Δt is sufficiently small, the threshold will not be crossed during the next time interval. The Bellmann equation thus becomes

$$G(\Pi_t, t) = \frac{1}{1+r\Delta t}\mathbf{E}_{\mathbf{P}^*}[G(\Pi_t + \Delta\Pi_t, t + \Delta t) \,|\, \mathcal{F}_t].$$

Proceeding in analogy to section A.1, that is multiplying by $1 + r\Delta t$, dividing by Δt, letting it go to zero, and expanding the right-hand side using Itô's Lemma, one obtains

$$\begin{aligned} rG(\Pi_t) &= \mathbf{E}_{\mathbf{P}^*}[\mathrm{d}G(\Pi_t) \,|\, \mathcal{F}_t] \\ &= (r-\delta)\,\Pi_t \frac{\mathrm{d}G(\Pi_t)}{\mathrm{d}\Pi_t} + \tfrac{1}{2}\sigma^2\Pi_t^2 \frac{\mathrm{d}^2 G(\Pi_t)}{\mathrm{d}\Pi_t^2}. \end{aligned}$$

As usual, a general solution to this differential equation is

$$G(\Pi_t) = A^+\Pi_t^{\gamma^+} + A^-\Pi_t^{\gamma^-}.$$

If $\Pi_t = \Pi^*$, the expected time until the process first crosses the threshold is zero, so that $G(\Pi^*) = 1$. Conversely, if $\Pi_t = 0$, the process will never cross the boundary, which implies $G(0) = 0$. From these boundary conditions it is possible to deduce $A^- = 0$ and $A^+(\Pi^*)^{\gamma^+} = 1$, resulting in

$$G(\Pi_t) = \left(\frac{\Pi_t}{\Pi^*}\right)^{\gamma^+},$$

which concludes the proof.

Similarities between (11.21) and the option pricing formula of proposition 13 are no coincidence. As outlined in the appendix, patent value under uncertainty can also be derived based on the first hitting time (see sect. A.1.2).

Theorem 2 holds in the continuation region of the litigation option. Immediate litigation obviously implies $\tau^* = 0$. Therefore, the following proposition can be derived.

Proposition 14. *The gross payoff from commercializing in the presence of imperfect patent protection is*

$$\widetilde{V}_{\mathrm{I}}(\Pi_t) = \begin{cases} (1 - p\theta)\,\Pi_t/\delta - L_{\mathrm{I}} & \text{if } \Pi^* < \Pi_t, \\ (1 - p\zeta)\,\Pi_t/\delta - B^+ \Pi_t^{\gamma^+} & \text{otherwise,} \end{cases}$$

where

$$\Pi^* = \frac{\gamma^+}{\gamma^+ - 1} \frac{L_{\mathrm{C}}\delta}{p\,(\theta - \zeta)}.$$

is the critical profit rate, and

$$B^+ = \left(L_{\mathrm{I}} + \frac{\gamma^+}{\gamma^+ - 1} L_{\mathrm{C}} \right) \left(\frac{1}{\Pi^*} \right)^{\gamma^+}.$$

Proof (Proposition 14). Since litigation risk hinges on the ratio ζ/θ, it becomes necessary to distinguish the cases $\zeta < \theta$ and $\theta \leq \zeta$.

Case 1 ($\zeta < \theta$). On the one hand, provided that $\zeta < \theta$ and Π_t is in the continuation region, combining (11.11) and proposition 13 yields

$$\begin{aligned} \widetilde{V}_{\mathrm{I}}(\Pi_t) &= \Pi_t/\delta - F_{\mathrm{C}}(\Pi_t) - \mathbf{E}_{\mathbf{P}^*}\left[\mathrm{e}^{-r(\tau^* - t)}\,(L_{\mathrm{I}} + L_{\mathrm{C}}) \right] \\ &= \Pi_t/\delta - \left(\frac{L_{\mathrm{C}}}{\gamma^+ - 1} \left(\frac{\Pi_t}{\Pi^*} \right)^{\gamma^+} + p\zeta \Pi_t/\delta \right) \\ &\qquad - \mathbf{E}_{\mathbf{P}^*}\left[\mathrm{e}^{-r(\tau^* - t)} \right] (L_{\mathrm{I}} + L_{\mathrm{C}}) \\ &= (1 - p\zeta)\,\Pi_t/\delta - \frac{L_{\mathrm{C}}}{\gamma^+ - 1} \left(\frac{\Pi_t}{\Pi^*} \right)^{\gamma^+} - (L_{\mathrm{I}} + L_{\mathrm{C}}) \left(\frac{\Pi_t}{\Pi^*} \right)^{\gamma^+} \\ &= (1 - p\zeta)\,\Pi_t/\delta - \left(L_{\mathrm{I}} + \frac{\gamma^+}{\gamma^+ - 1} L_{\mathrm{C}} \right) \left(\frac{\Pi_t}{\Pi^*} \right)^{\gamma^+}. \end{aligned}$$

If the cash flow rate exceeds the critical level, the challenger will litigate immediately, resulting in litigation costs of L_{I}. With probability p, the challenger is successful and obtains a fraction of future profits, namely $\theta \Pi_t/\delta$. The gross present value of cash flows from commercialization thus becomes

$$\begin{aligned} \widetilde{V}_{\mathrm{I}}(\Pi_t) &= \Pi_t/\delta - (p\theta \Pi_t/\delta - L_{\mathrm{C}}) - (L_{\mathrm{I}} + L_{\mathrm{C}}) \\ &= (1 - p\theta)\,\Pi_t/\delta - L_{\mathrm{I}}. \end{aligned}$$

Case 2 ($\theta \leq \zeta$). If, on the other hand, $\theta \leq \zeta$ one obtains

$$\widetilde{V}_{\mathrm{I}}(\Pi_t) = (1 - p\zeta)\,\Pi_t/\delta. \tag{11.22}$$

The cost of litigation, which takes place in the very distant future, becomes negligible in present-value terms.

Intuitively speaking, gross payoff equals the value of the project if the challenger were to litigate immediately, plus the value of waiting, less the expected present value of litigation costs.[6]

Figure 11.4 shows $\widetilde{V}_{\mathrm{I}}$ as a function of Π_t. Interestingly, rising profit rates under imperfect patent protection may result in declining patent value. This seemingly counter-intuitive result is due to litigation risk, which—under certain conditions—may over-compensate the positive effects of heightened profitability. The adverse effects of patent risk are particularly pronounced if the challenger's litigation costs are small compared to those incurred by the incumbent.

Since the payoff from commercialization includes a short position, it may also turn out to be negative.

11.2.2.2 Option to Commercialize

The analysis can be carried one step further by examining the option to invest held by the incumbent who owns the patent, but has not yet commenced commercialization. This view implies that a patent is properly valued by pricing a (compound) call on a portfolio consisting of a project and a short option to litigate.

11.2.2.2.1 Analytical Derivation

The extended decision problem is a sequential game in continuous time. Its discrete-time equivalent is depicted in fig. 11.5.

Proposition 14 gives the value of *commercialization* under litigation risk. The value of the *patent* is the result of the nested optimization problem

$$\begin{aligned}\widetilde{F}_{\mathrm{I}}(\Pi_t) &= \sup_{\tau \in [t,\infty)} \mathbf{E}_{\mathbf{P}^*}\left[\mathrm{e}^{-r(\tau-t)}\left(\widetilde{V}_{\mathrm{I}}(\Pi_\tau) - I\right)\right] \\ &= \mathbf{E}_{\mathbf{P}^*}\left[\mathrm{e}^{-r(\tau^{**}-t)}\left(\widetilde{V}_{\mathrm{I}}(\Pi_{\tau^{**}}) - I\right)\right],\end{aligned}$$

[6] While this intuition served as the starting point for the proof just presented, the appendix offers an alternative derivation of proposition 14, based on the expected first hitting time (see sect. A.4).

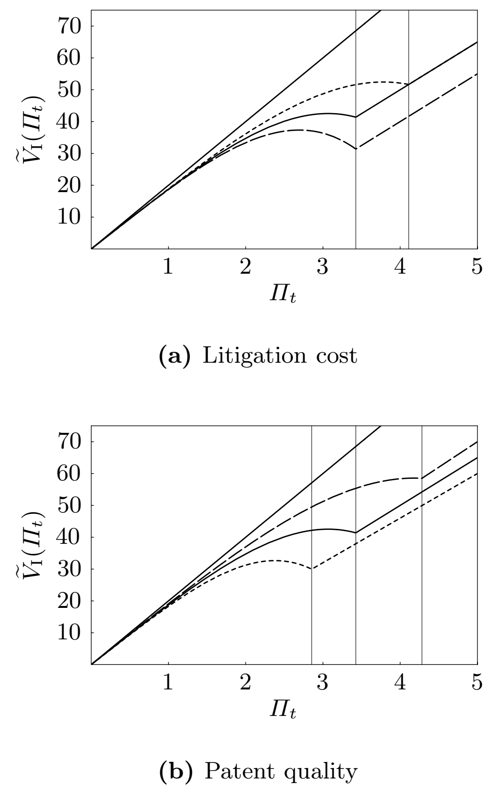


Fig. 11.4. Gross payoff from commercialization under endogenous patent risk when the protection period is infinite ($\sigma = 0.1$, $r = \delta = 0.05$, $\zeta = 0.1$, and $\theta = 0.5$). Panel (a) shows $\widetilde{V}_{\mathrm{I}}(\Pi_t)$ for $p = 0.5$. Assuming $L_{\mathrm{C}} = 10$, an increase in the incumbent's litigation cost L_{I} from 10 to 20 results in a downward shift of the corresponding graph, but does not affect the trigger (long dashes). In contrast, holding the incumbent's litigation cost constant at $L_{\mathrm{I}} = 10$, an increase in the challenger's litigation cost L_{C} from 10 to 12 leads to a higher investment threshold Π^* (vertical lines), but, for obvious reason, has no influence on $\widetilde{V}_{\mathrm{I}}(\Pi_t)$ in the stopping region (short dashes). Panel (b) illustrates the impact of patent quality, measured by the probability of litigation success p, on $\widetilde{V}_{\mathrm{I}}(\Pi_t)$. As p decreases, higher profit rates are required to trigger litigation; and $\widetilde{V}_{\mathrm{I}}(\Pi_t)$ eventually equals $V_{\mathrm{I}}(\Pi_t)$ ($L_{\mathrm{I}} = L_{\mathrm{C}} = 10$ and $p = \{0.4, 0.5, 0.6\}$).

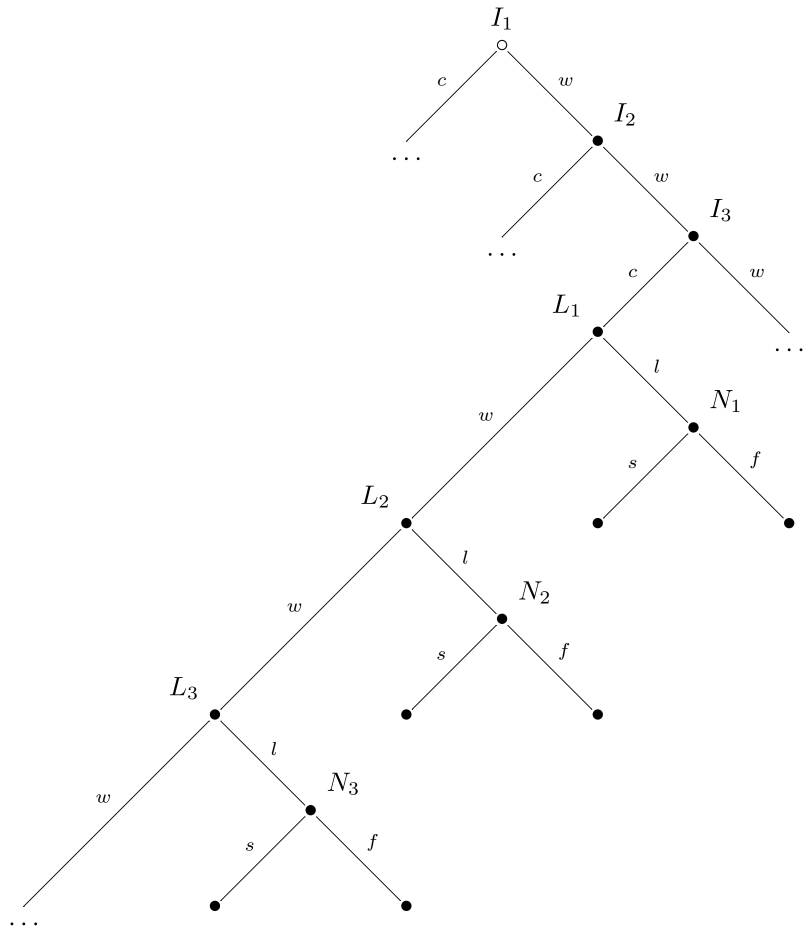


Fig. 11.5. Sequential game for patent valuation. At each node in $\{I_1, I_2, I_3, \dots\}$ the incumbent decides whether to commercialize the patent (c) or wait an additional period (w). Once the incumbent has decided to commercialize, the challenger faces a similar sequence of choices $\{L_1, L_2, L_3, \dots\}$. At each node he or she may either litigate (l) or postpone litigation to a later point in time (w). In the event of litigation, nature determines the outcome at $\{N_1, N_2, N_3, \dots\}$. The sub-tree starting at L_1 corresponds to the litigation option discussed in the previous subsection.

where I denotes the up-front investment required to commercialize the patent.

Due to the non-linear payoff function, deriving specific patent value requires careful analysis. Among other parameters, the ratio ζ/θ plays a key role.

Consider the case $\theta \leq \zeta$. As shown previously, litigation will be postponed as long as possible, and the underlying becomes linear in Π_t. Option exercise is only optimal above some threshold Π^{**}, making the claim quite similar to the type of real call option discussed earlier:

$$\widetilde{F}_{\mathrm{I}}(\Pi_t, t) = \begin{cases} \widetilde{V}_{\mathrm{I}}(\Pi_\tau) - I & \text{if } \Pi^{**} < \Pi_t, \\ C^+ \Pi_t^{\gamma+} & \text{otherwise,} \end{cases}$$

Substituting (11.22) one obtains the value-matching and smooth-pasting conditions

$$\begin{aligned} C^+ (\Pi^{**})^{\gamma^+} &= (1 - p\zeta)\, \Pi^{**}/\delta - I, \\ \gamma^+ C^+ (\Pi^{**})^{\gamma^+ - 1} &= (1 - p\zeta)\, /\delta. \end{aligned}$$

Consequently,

$$\begin{aligned} C^+ &= \frac{1}{\gamma^+} (1 - p\zeta)\, (\Pi^{**})^{1-\gamma^+}/\delta \\ &= \frac{I}{\gamma^+ - 1} \left(\frac{1}{\Pi^{**}} \right)^{\gamma^+}, \end{aligned}$$

where

$$\Pi^{**} = \frac{\gamma^+}{\gamma^+ - 1} \frac{I\delta}{1 - p\zeta}.$$

These equations correspond to (11.19) and (11.20), respectively. In summary, the dynamic value of a patent under imperfect patent protection is

$$\widetilde{F}_{\mathrm{I}}(\Pi_t) = \begin{cases} (1 - p\zeta)\, \Pi_t/\delta - I & \text{if } \Pi^{**} < \Pi_t, \\ \frac{I}{\gamma^+ - 1} \left(\frac{\Pi_t}{\Pi^{**}} \right)^{\gamma^+} & \text{otherwise.} \end{cases}$$

For obvious reasons, patent value does not dependent on θ. Patent value is almost completely analogous to the case of perfect patent protection, with the noteworthy exception of an expected payment to the challenger litigating in the very distant future.

Consider now the case $\zeta < \theta$. If Π_t is very large, both options will end up in their respective stopping regions, so that

$$\widetilde{F}_{\mathrm{I}}(\Pi_t) = (1 - p\theta)\,\Pi_t/\delta - L_{\mathrm{I}} - I.$$

The incumbent commercializes, followed by immediate litigation. Nevertheless, the option value will proof to be more complicated to determine for a wide range of moderate cash flow rates. In order to provide a more comprehensive picture under various assumptions, especially with respect to patent duration, further analysis are best carried out numerically.

11.2.2.2.2 Numerical Illustration

In the following, a numerical method for determining patent value under litigation risk is described. Taking advantage of a decomposition similar to the one depicted in fig. 11.1 and the pricing approach for profit-level models discussed in the appendix (see chap. B.1.1), it also captures the effect of a finite protection period.

In order to improve accuracy, not a standard Cox–Ross–Rubinstein (CRR) tree like the one employed in chapter 7, but the log-transformed variant proposed by Trigeorgis is constructed.[7] Based on Itô's Lemma, the discrete-time equivalent of the profit rate process under the risk-neutral measure is

$$\Pi_{t+\Delta t} = \Pi_t \exp\Big(\left(r - \delta - \tfrac{1}{2}\sigma^2\right)\Delta t + \sigma \Delta W_t\Big).$$

Furthermore, consider the transformation $X_t \equiv \ln(\Pi_t)$ and $u \equiv \sigma^2 t$ [328, p. 321], so that $X = \{X_u\}_{u \geq 0}$ becomes ABM, and time is expressed "in units of variance." Assuming the protection period is divided into intervals of equal length $\Delta t \equiv T/n$, this choice implies $\Delta u \equiv \sigma^2 \Delta t$. Over each interval, $X_{i,j} \equiv X_{i\Delta X, j\Delta u}$ increases by

$$\begin{aligned}\Delta X &= \ln\left(\frac{\Pi_{t+\Delta t}}{\Pi_t}\right)\\ &= \left(r - \delta - \tfrac{1}{2}\sigma^2\right)\Delta t + \sigma \Delta W_t\end{aligned}$$

or decreases by the same amount. The probability of an upward movement is q. Figure 11.6 shows the binomial tree representing this discrete-time process.

Parameters are chosen to mirror continuous-time dynamics. Set $\mu \equiv (r - \delta)/\sigma^2 - \frac{1}{2}$.[8] Hence,

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[7] For a more detailed account see Trigeorgis [327], Trigeorgis [328, pp. 320–322]. For an extension to multi-variate processes see Gamba and Trigeorgis [112].

[8] Deviating from the notation of previous chapters, μ here does *not* signify the required rate of return.

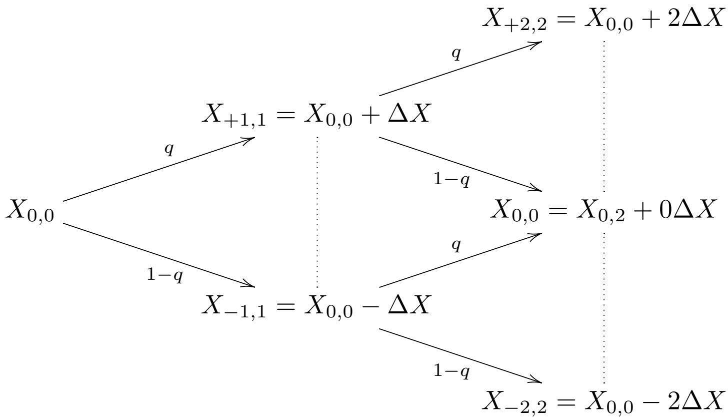


Fig. 11.6. Log-transformed binomial tree for the numerical valuation of patents under endogenous litigation risk.

$$\begin{aligned}\mathbf{E}[\Delta X] &= \mu \Delta u \\ &= q\Delta X - (1-q)\,\Delta X \\ &= 2q\Delta X - \Delta X\end{aligned}$$

and

$$\begin{aligned}\mathbf{Var}[\Delta X] &= \Delta u \\ &= \mathbf{E}\left[\Delta X^2\right] - (\mathbf{E}[\Delta X])^2 \\ &= \Delta X^2 - (\mathbf{E}[\Delta X])^2 \,.\end{aligned}$$

Solving for the risk-neutral probability leads to

$$q = \tfrac{1}{2}\left(1 + \mu \frac{\Delta u}{\Delta X}\right),$$

where

$$\Delta X = \sqrt{\Delta u + (\mu \Delta u)^2}.$$

Note that the procedure is unconditionally stable [328, p. 322].

Recall from (11.13) that the option to litigate decomposes into a call option and a perpetuity. However, in order to solve the optimization problem, it becomes necessary to choose a slightly different decomposition, namely

$$F_{\mathrm{C}}(\Pi_t, t) = \mathbf{E}_{\mathbf{P}^*}\left[p\zeta \int_t^T \mathrm{e}^{-r(s-t)} \Pi_s \,\mathrm{d}s \,\middle|\, \mathcal{F}_t \right] + \Psi_{\mathrm{C}}(\Pi_t, t), \qquad (11.24)$$

where

$$\begin{aligned}\mathit{\Psi}_{\mathrm{C}}(\mathit{\Pi}_t, t) &= \sup_{\tau\in[t,T]} \mathbf{E}_{\mathbf{P}^*}\left[p\,(\theta-\zeta)\int_{\tau}^{T} \mathrm{e}^{-r(s-t)}\mathit{\Pi}_s\,\mathrm{d}s - \mathrm{e}^{-r(\tau-t)}L_{\mathrm{C}}\,\middle|\,\mathcal{F}_t\right]\\ &= \mathbf{E}_{\mathbf{P}^*}\left[p\,(\theta-\zeta)\int_{\tau^*}^{T} \mathrm{e}^{-r(s-t)}\mathit{\Pi}_s\,\mathrm{d}s - \mathrm{e}^{-r(\tau^*-t)}L_{\mathrm{C}}\,\middle|\,\mathcal{F}_t\right].\end{aligned}$$

The stopping times derived lead to the gross present value of commercialization under patent risk, which is

$$\begin{aligned}V_{\mathrm{I}}(\mathit{\Pi}_t, t) = \mathbf{E}_{\mathbf{P}^*}\Bigg[&(1-p\zeta)\int_{t}^{\tau^*} \mathrm{e}^{-r(s-t)}\mathit{\Pi}_s\,\mathrm{d}s\\ &-\mathrm{e}^{-r(\tau^*-t)}L_{\mathrm{I}} + (1-p\theta)\int_{\tau^*}^{T} \mathrm{e}^{-r(s-t)}\mathit{\Pi}_s\,\mathrm{d}s\,\Bigg|\,\mathcal{F}_t\Bigg]\end{aligned}$$

or, after rearranging,

$$V_{\mathrm{I}}(\mathit{\Pi}_t, t) = \mathbf{E}_{\mathbf{P}^*}\left[(1-p\zeta)\int_{t}^{T} \mathrm{e}^{-r(s-t)}\mathit{\Pi}_s\,\mathrm{d}s\,\middle|\,\mathcal{F}_t\right] + \widetilde{\mathit{\Psi}}_{\mathrm{I}}(\mathit{\Pi}_t, t), \quad (11.25)$$

where

$$\widetilde{\mathit{\Psi}}_{\mathrm{I}}(\mathit{\Pi}_t, t) = \mathbf{E}_{\mathbf{P}^*}\left[-p\,(\theta-\zeta)\int_{\tau^*}^{T} \mathrm{e}^{-r(s-t)}\mathit{\Pi}_s\,\mathrm{d}s - \mathrm{e}^{-r(\tau^*-t)}L_{\mathrm{I}}\,\middle|\,\mathcal{F}_t\right].$$

For implementation purposes, the algorithm has to be translated into discrete-time formulae.

Employing the log-transformed model described above, a profit rate tree is constructed. Once the value of the underlying has been determined at each node, it is not difficult to calculate the present value of cash flows. Starting at the leaves of the tree, one obtains

$$V_{i,n} = \mathit{\Pi}_{i,n}\Delta t.$$

For all previous periods, the expected present value of cash flows is

$$V_{i,j} = \mathit{\Pi}_{i,j}\Delta t + \mathrm{e}^{-r\Delta t}\left(q\mathit{\Pi}_{i+1,j+1} + (1-q)\,\mathit{\Pi}_{i-1,j+1}\right).$$

Standard dynamic programming techniques lead to the flexible component of option value, namely

$$\mathit{\Psi}_{\mathrm{C}}(\mathit{\Pi}_{i,n}, n\Delta t) = \max\left\{p(\theta-\zeta)V_{i,j} - L_{\mathrm{C}}, 0\right\}$$

and

$$\begin{aligned}\Psi_{\mathrm{C}}(\Pi_{i,j}, j\Delta t) = \max\Big\{&p(\theta-\zeta)V_{i,j} - L_{\mathrm{C}},\\ &\mathrm{e}^{-r\Delta t}\Big(q\Psi_{\mathrm{C}}\left(\Pi_{i+1,j+1},(j+1)\,\Delta t\right)\\ &+(1-q)\,\Psi_{\mathrm{C}}\left(\Pi_{i-1,j+1},(j+1)\,\Delta t\right)\Big)\Big\}.\end{aligned}$$

The resulting policy is then used to arrive at the corresponding component of project value. At the end of the protection period,

$$\widetilde{\Psi}_{\mathrm{I}}(\Pi_{i,n}, n\Delta t) = \begin{cases}-p(\theta-\zeta)V_{i,n} - L_{\mathrm{I}} & \text{if } \Pi^* < \Pi_{i,n},\\ 0 & \text{otherwise.}\end{cases}$$

For all previous nodes, if option exercise is optimal ($\Pi^* < \Pi_{i,j}$),

$$\widetilde{\Psi}_{\mathrm{I}}(\Pi_{i,j}, j\Delta t) = -p(\theta-\zeta)V_{i,j} - L_{\mathrm{I}}.$$

If continuation is optimal ($\Pi_{i,j} \leq \Pi^*$),

$$\begin{aligned}\widetilde{\Psi}_{\mathrm{I}}(\Pi_{i,j}, j\Delta t) = \mathrm{e}^{-r\Delta t}\Big(&q\Psi_{\mathrm{I}}\left(\Pi_{i+1,j+1},(j+1)\,\Delta t\right)\\ &+(1-q)\,\Psi_{\mathrm{I}}\left(\Pi_{i-1,j+1},(j+1)\,\Delta t\right)\Big).\end{aligned}$$

Using (11.24) and (11.25), one obtains the option value of litigation as well as the gross payoff from commercialization.

For example, consider the illustrative example shown in table 11.1, where $\Delta t = T/n = 20.0/2 = 10.0$. Assuming an initial profit rate of $\Pi_0 = 1.00$, cash flow volatility of $\sigma = 0.1$, $r = \delta = 0.05$, $p = 0.5$, $\zeta = 0.1$, $\theta = 0.5$, $L_{\mathrm{C}} = 2$, and $L_{\mathrm{I}} = 10$, the option value of litigation is

$$\begin{aligned}F_C(\Pi_{0,0}, 0) &= p\zeta V_{0,0} + \Psi_{\mathrm{C}}(\Pi_{0,0}, 0)\\ &= 0.5 \times 0.1 \times 19.755 + 1.951 = 2.939.\end{aligned}$$

Gross payoff from commercialization under patent risk becomes

$$\begin{aligned}\widetilde{V}_{\mathrm{I}}(\Pi_{0,0}, 0) &= (1-p\zeta)\,V_{0,0} + \widetilde{\Psi}_{\mathrm{I}}(\Pi_{0,0}, 0)\\ &= (1-0.5\times 0.1)\times 19.755 - 13.951 = 4.816.\end{aligned}$$

Accurate patent and project values, however, require significantly larger trees.

Figure 11.7 presents selected numerical results graphically. Although discretization brings about visible inaccuracies around the challenger's critical threshold, the overall shape of curves is in line with analytical project values provided earlier. In addition, fig. 11.7(b) shows that

Table 11.1. Log-transformed binomial model for patent valuation under endogenous litigation risk ($\Pi_0 = 1.00$, $\sigma = 0.1$, $r = \delta = 0.05$, $p = 0.5$, $\zeta = 0.1$, $\theta = 0.5$, $L_C = 2$, $L_I = 10$, $T = 20.0$, and $n = 2$). Carrying out the steps described, it is possible to determine the gross payoff from commercialization, which is indispensable for calculating dynamic patent value. Panel (d) shows the challenger's optimal policy, ones indicating nodes at which litigation is optimal.

(a) Underlying

State	$\Pi_{i,j}$		
	$t = 0$	$t = 10$	$t = 20$
+2			1.897
+1		1.377	
0	1.000		1.000
−1		0.726	
−2			0.527

(b) Present value

State	$V_{i,j}$		
	$t = 0$	$t = 10$	$t = 20$
+2			18.971
+1		22.134	
0	19.755		10.000
−1		11.668	
−2			5.271

(c) Option

State	$\Psi_C(\Pi_{i,j}, j\Delta t)$		
	$t = 0$	$t = 10$	$t = 20$
+2			1.794
+1		2.427	
0	1.951		0.000
−1		0.334	
−2			0.000

(d) Policy

State	Policy		
	$t = 0$	$t = 10$	$t = 20$
+2			1
+1		1	
0	1		0
−1		1	
−2			0

(e) Project

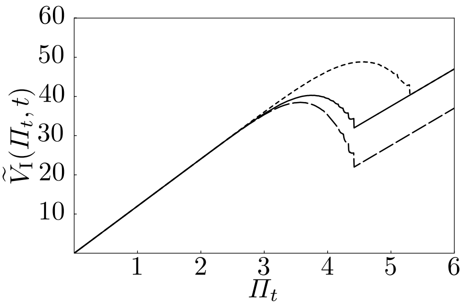
State	$\widetilde{\Psi}_I(\Pi_{i,j}, j\Delta t)$		
	$t = 0$	$t = 10$	$t = 20$
+2			−13.794
+1		−14.427	
0	−13.951		0.000
−1		−12.334	
−2			0.000

(f) Patent

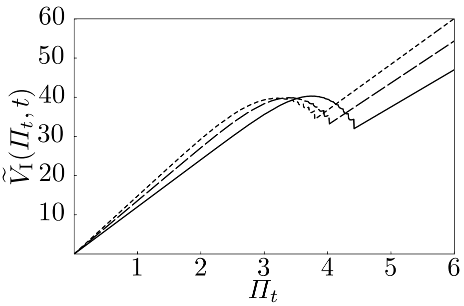
State	$\widetilde{F}_I(\Pi_{i,j}, j\Delta t)$		
	$t = 0$	$t = 10$	$t = 20$
+2			8.022
+1		11.028	
0	8.767		0.000
−1		1.084	
−2			0.000

longer protection periods are associated with higher project values, but also make litigation attractive at comparatively low levels of profitability.

Under finite patent protection, the potentially adverse effect of rising profit rates on gross payoff from commercialization are more pronounced. As evident from fig. 11.7(a) and in analogy to the case of an infinite protection period, comparatively high costs of litigation for the incumbent cause project values to drop sharply as rising profit rates approach the critical threshold.



(a) Litigation cost



(b) Protection period

Fig. 11.7. Gross payoff from commercialization under endogenous patent risk when the protection period is finite ($\sigma = 0.1$, $r = \delta = 0.05$, $p = 0.5$, $\zeta = 0.1$, $\theta = 0.5$, and $n = 500$). Panel (a) shows $\widetilde{V}_{\mathrm{I}}(\Pi_t)$ for $T = 20.0$, the base case ($L_{\mathrm{C}} = L_{\mathrm{I}} = 10.0$) represented by a solid line. In analogy to previous analyses, long and short dashes illustrate results for $L_{\mathrm{I}} = 20.0$ and $L_{\mathrm{C}} = 12.0$, respectively. Panel (b) depicts gross payoff for $T = 20.0$ (solid line), $T = 25.0$ (long dashes), and $T = 30.0$ (short dashes).

Finally, dynamic patent value can be quantified by pricing an option on the gross payoff from commercialization, that is

$$\widetilde{V}_{\mathrm{I}}(\Pi_{i,j}, j\Delta t) = (1 - p\zeta)\, V_{i,j} + \widetilde{\Psi}_{\mathrm{I}}(\Pi_{i,j}, j\Delta t).$$

Patent values at the leaves of the tree are given by

$$\widetilde{F}_{\mathrm{I}}(\Pi_{i,n}, n\Delta t) = \max\left\{\widetilde{V}_{\mathrm{I}}(\Pi_{i,n}, n\Delta t) - I, 0\right\}.$$

Proceeding backwards in time, all previous nodes are calculated as follows:

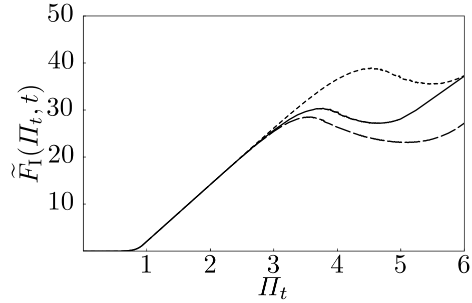
$$\begin{aligned}\widetilde{F}_{\mathrm{I}}(\Pi_{i,j}, j\Delta t) = \max\Big\{&\widetilde{V}_{\mathrm{I}}(\Pi_{i,j}, j\Delta t) - I,\\ &\mathrm{e}^{-r\Delta t}\Big(q\widetilde{F}_{\mathrm{I}}\big(\Pi_{i+1,j+1}, (j+1)\,\Delta t\big)\\ &\quad + (1-q)\,\widetilde{F}_{\mathrm{I}}\big(\Pi_{i-1,j+1}, (j+1)\,\Delta t\big)\Big)\Big\}.\end{aligned}$$

Figure 11.8 shows dynamic patent value as a function of the initial profit rate under various assumptions concerning litigation costs and the investment required to commercialize the patent. Obviously, the resulting diagram differs substantially from the familiar "hockey stick" associated with plain-vanilla call options—real or financial. Although the drop in patent value due to rising patent risk is mitigated by the value of flexibility, it is still noticeable, in particular if commercialization is inexpensive.

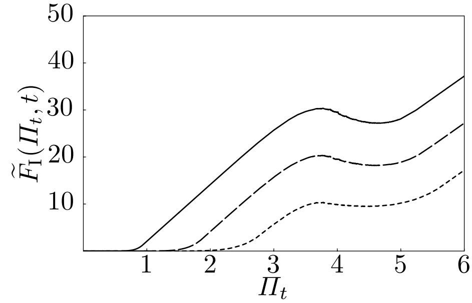
Correspondingly, the optimal policy is far more complicated than for the fairly simple litigation option. For very low profit rates, early exercise is unattractive. As profit rates rise, early exercise becomes optimal, before increasing patent risk renders it unattractive again. Eventually, profit rates are high enough to justify early exercise despite the threat of litigation. Fig. 11.9 shows numerical approximations of the resulting boundaries Π_t^*, Π_t^{**}, and Π_t^{***} as a function of time, assuming $L_{\mathrm{C}} = 1.0$, $L_{\mathrm{I}} = 10.0$, and $I = 1.0$.

11.3 Variations and Extensions

A stylized model like the one presented in this chapter can be extended in a number of ways. In the following, a selection of possible extension will be discussed in more detail.



(a) Litigation cost



(b) Investment

Fig. 11.8. Dynamic patent value under endogenous patent risk when the protection period is finite ($\sigma = 0.1$, $r = \delta = 0.05$, $p = 0.5$, $\zeta = 0.1$, $\theta = 0.5$, $T = 20.0$, and $n = 500$). Panel (a) shows patent values for $I = 10.0$. Again, the base case with $L_{\mathrm{C}} = L_{\mathrm{I}} = 10.0$ is represented by a solid line, while long and short dashes serve to illustrate the sensitivity of patent value to changes in these parameters. Moreover, panel (b) depicts how increases in the investment amount required to commercialize lower patent value ($I \in \{10.0, 20.0, 30.0\}$).

11.3.1 Alternative Litigation Systems

An important area of research is the design of the legal system, and the patent system in particular, addressing important issues such as optimal patent length and breadth. Moreover, incentives to litigate and the outcome of disputes are determined by the cost of litigation.

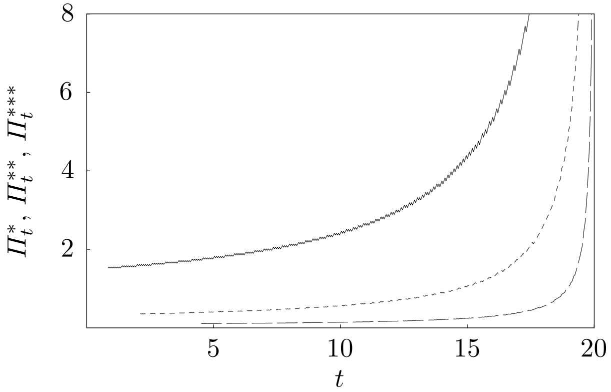


Fig. 11.9. Incumbent's critical thresholds under endogenous patent risk when the protection period is finite ($\Pi_0 = 1.0$, $\sigma = 0.1$, $r = \delta = 0.05$, $L_C = 1.0$, $L_I = 10.0$, $I = 1.0$, $p = 0.5$, $\zeta = 0.1$, $\theta = 0.5$, and $T = 20.0$). Again, approximations of the boundaries were obtained using a log-transformed binomial tree with $n = 500$ timesteps.

11.3.1.1 Settlement

Apart from litigation, settlement of patent disputes plays an important role in the value-based management of property rights.[9] Lanjouw and Schankerman [185] find that some 95 percent of patent lawsuits are settled prior to a court judgment. More importantly, as argued by Shapiro [309, p. 391], a wide range of commercial arrangements involving IP—including patent licenses, mergers, and joint ventures—can be regarded as settlements of IP disputes, effectively or even literally. Royalty rates in licensing deals, for instance, reflect the bargaining power of the contracting parties, which fundamentally depends on the likelihood of winning in court.

Technically speaking, the tradeoff between seeking a decision in court or opting to settle most likely involves the calculation of Nash bargaining solutions. The current model may be used to establish suitable threat points.

11.3.1.2 European Rule

As mentioned before, the American rule requires both parties to bear their own legal costs. However, legal systems differ in the treatment of such expenses. If the loosing party or the state covers costs of litigation,

[9] For example see Aoki and Hu [7, 8], Bebchuk [27], Crampes and Langinier [78].

different option values result. A thorough comparison of alternatives could provide insights into the impact on innovation incentives.

11.3.1.3 Variable Cost of Litigation

Almost needless to say, assuming a constant cost of litigation is a simplification of the actual process, because lawyers might claim a proportion of the damage award. In essence, variable costs of litigation correspond to a stochastic strike price, which, depending on the choice of parameters, could lead to a higher or lower option value of litigation. Again, an extensive sensitivity analysis would be required to draw meaningful conclusions.

11.3.2 Alternative Underlying Dynamics

Simulation results might change considerably, depending on the dynamics employed to capture the development of expected cost to completion and cash flow rates. Common variations of the standard stock price model include mean reversion and stochastic interest rates.

11.3.2.1 Mean Reversion

Cash flow rates usually track a product-specific lifecycle. In contrast, cash flow rates in this chapter were assumed to follow GBM with a positive drift, on average leading to an increase in profitability as the end of the protection period approaches. While a variety of alternative specifications are conceivable, mean-reversion processes probably better reflect the stylized facts [91, pp. 74–78]. One example is the Ornstein–Uhlenbeck process

$$\mathrm{d}\Pi_t = \vartheta \left(\overline{\Pi} - \Pi_t \right) \mathrm{d}t + \sigma \, \mathrm{d}W_t,$$

where ϑ is the speed of reversion and $\overline{\Pi}$ denotes the long-run average level of profitability, to which Π tends to revert [331].

Similar SDEs are very popular in option-based models of natural resource investments. One way to answer the question of whether a mean-reversion process indeed matches empirical data more closely is the application of *unit root* tests [87].

11.3.2.2 Stochastic Interest Rates

As pointed out by Schwartz [301, p. 18], analyzing patent value under stochastic interest rates is facilitated by the Monte Carlo approach. In principal, it suffices to specifiy a suitable model, generate the required number of interest rate processes and carry out all calculations employing a time-variant discount factor [207, pp. 131–135]. Similarly, stochastic interest rates can be accounted for in tree-based option pricing, for example employing the widely-used Heath–Jarrow–Morton model of interest rates.[10]

11.3.3 Exit Option

Due to the fact that the current setup abstracts from operating costs, exit options during the commercialization phase have so far been neglected. Introducing an exit option along the lines of existing analyses would complicate matters somewhat, but should not pose severe difficulties [91, 229]. It is important to note, however, that—at least in the pharmaceutical industry—firms very rarely exercise the option to stop commercializing, mainly due to the paramount importance of expenditures during R&D.

11.3.4 Industry Equilibrium

A closer look at industry equilibrium would call for a demand-level model. Roughly speaking, excess profits earned by commercializing certain patents are likely to attract challengers, thereby increasing patent risk. In equilibrium, these excess profits are exactly offset by the threat of litigation.

[10] Other possibilities include the Black–Derman–Toy model, the Hull–White model, and its Black–Karanski modification, all of which are available in commercial implementations [36, 142, 150].

12

Preliminary Conclusion

According to Lemley and Shapiro, uncertainty surrounding patent protection is an inherent part of the patent system, reflecting the hundreds of thousands of patent applications filed each year, the inability of third parties to participate effectively in determining whether a patent should issue, and the fact that for most issued patents, scope and validity are of little commercial significance. *Probabilistic patents* thus require rethinking patent granting, opposition, litigation, and settlement [193, p. 95].[1]

Consequently, the aim of this part of the dissertation was to develop a deeper understanding of patent risk, looking beyond the seemingly random occurrence of patent-related events, suggested by jump-diffusion models of the R&D process. Following introductory definitions and the model setup in section 11.1, section 11.2.1 served to discuss the option value of litigation under certainty. Building on some basic insights into the composition of cash flows, section 11.2.2 outlined a sequential stochastic game for patent valuation, which was studied using both analytical and numerical methods. Finally, section 11.3 hinted at some variations and extensions of the basic framework presented.

Among the many noteworthy findings is the non-obvious functional relationship between cash flow rates and commercialization payoff with important implications for the option value of R&D under imperfect patent protection. As outlined in detail, higher profitability not necessarily goes along with higher patent values.

This insight highlights an important stylized fact of market entry in research intensive industries. Common sense dictates that a high probability of litigation is an indicator of attractive commercial opportunities. Not only do high profits attract potential challengers; high profits are often the result of novel products and services, which due to the lim-

[1] For more on antitrust issues in the context of settlement see Shapiro [309].

ited experience of all parties involved, are typically difficult to protect through patents. This uncertainty, in turn, gives rise to increased litigation activity. Consequently, potential entrants have to trade off growth and profit potential in markets driven by innovation for a comparatively high reliability of intellectual property protection in more mature markets.

While, as a result of various barriers to entry imposed by incumbent oligopolists, this consideration appears somewhat theoretical on the level of whole industries, similar issues arise on the project level. The formal model analyzed in this part of the dissertation may be seen as a first step to more comprehensive models of R&D and commercialization, demonstrating that the impact of litigation on patent value in strategic settings can in fact be anticipated and, to some degree, even quantified.

The type of model proposed may serve as a tool for studying the optimal level of patent protection from an option-based perspective, including, for example, not only the length of the protection period, but also other aspects, such as the reliability of patent protection, which may differ substantially across countries and industries.

In addition, the discussion served to present the option-based view of patent risk as a special case of a more general reconceptualization of uncertain property rights, capturing legal risk as embedded short options to litigate.

13

Conclusion

This final chapter summarizes some important results and provides suggestions for future research.

13.1 Summary

Since many of the more technical results have already been discussed extensively in chapters 9 and 12, the aim of this section is to provide non-technical answers to the four research questions formulated in the introduction: (1) How does patent risk change patent value and investment policy? (2) What is the source of patent risk? (3) How do market, technology, and patent risk interact? (4) Are option-based models of imperfect patent protection ready for practice?

As shown in part II of this dissertation, the impact of patent risk on patent value and investment policy is essentially determined by the frequency and severity of value-relevant events. Jump-diffusion processes may serve to capture these effects in formal option-based models. Several models of this type were presented, extending previous work on capital budgeting under uncertainty to the case of imperfect patent protection.

Part II shed light on the source of patent risk, which arises as the result of strategic interdependency in the increasingly complex IP landscape. As explained in detail, this interdependency yields surprising results, calling for careful analysis in real-world settings.

While parts II and II offered a variety of explanations as to the interaction of market, technology, and patent risk, findings crucially depend on the specific setup chosen. More generally, an isolated treatment was shown to result in suboptimal decisions under most circumstances, in particular including strategic settings.

Not all option-based models are ready for practice. The author is convinced, however, that the option-based view (OBV) outlined in this dissertation will contribute to a more widespread use of advanced capital budgeting techniques when assessing the impact of risk on optimal investment policies in R&D. As demonstrated by the case of imperfect patent protection, complicated decision problems sometimes require equally complicated tools.

13.2 Suggestions for Future Research

Although a multitude of further theoretical studies—for instance integrating the exogenous and endogenous perspective—could certainly be conceived of, both academia and practice are most likely to profit from empirical investigations into the nature of imperfect patent protection, aimed at testing hypotheses derived from the exogenous and endogenous models of patent risk presented. Furthermore, extensive case studies may lower the hurdles that have so far hindered the adoption of more advanced capital budgeting techniques. Improvement potential exists also on the portfolio level, which has largely been neglected in formal analyses. Given recent challenges posed by accounting rules and patent law, the potential economic benefits are sizeable.

Part IV

Appendices

A

Proofs

In addition to the proofs presented throughout the chapters of this dissertation, the appendix contains alternative proofs of important propositions.

A.1 Proposition 6

There are at least two alternative approaches to deriving dynamic patent value if the gross payoff from commercialization is stochastic. One is based on dynamic programming, the other one on the distribution of first hitting times.

A.1.1 Dynamic Programming

Starting with a Bellman [28] equation in discrete time, it is possible to express the current value of a patent as

$$F(\mathit{\Pi}_t, t) = \frac{1}{1 + r\Delta t}\mathbf{E}_{\mathbf{P}^*}\left[F(\mathit{\Pi}_t + \Delta\mathit{\Pi}_t, t + \Delta t) \,|\, \mathcal{F}_t\right],$$

namely the expected future value of the same patent, discounted back to time t. In line with the risk-neutral pricing approach, the martingale measure is adopted here; and r is the appropriate discount rate. Again, in contrast to the project itself, there is no profit flow $\mathit{\Pi}_t\Delta t$ [91, pp. 104–108]. Multiplying by $1 + r\Delta t$ and subtracting $F(\mathit{\Pi}_t, t)$ results in

$$\begin{aligned} r\Delta t F(\mathit{\Pi}_t, t) &= \mathbf{E}_{\mathbf{P}^*}\left[F(\mathit{\Pi}_t + \Delta\mathit{\Pi}_t, t + \Delta t) - F(\mathit{\Pi}_t, t) \,|\, \mathcal{F}_t\right] \\ &= \mathbf{E}_{\mathbf{P}^*}\left[\Delta F(\mathit{\Pi}_t, t) \,|\, \mathcal{F}_t\right]. \end{aligned}$$

Dividing by Δt, letting it go to zero, and expanding the right-hand side using Itô's Lemma yields

$$rF(\Pi_t) = (r - \delta)\, \Pi_t \frac{\mathrm{d}F(\Pi_t)}{\mathrm{d}\Pi_t} + \tfrac{1}{2}\sigma^2 \Pi_t^2 \frac{\mathrm{d}^2 F(\Pi_t)}{\mathrm{d}\Pi_t^2},$$

which, after re-arranging becomes the familiar option pricing PDE. Note that, since expectations are taken under the risk-neutral measure, $\alpha = \alpha^* \equiv r - \delta$. Furthermore, in the case of an infinite protection period, patent value no longer depends explicitly on time. Similar to contingent-claims analysis, the simplified equation can be solved subject to appropriate terminal and boundary conditions, leading to a closed-form expression for the dynamic value of a patent.

A.1.2 First Hitting Time

In order to demonstrate the validity of theorem 2, consider the following derivation of dynamic patent value based on the distribution of first hitting times. A closed-form expression for $\mathbf{E}_{\mathbf{P}^*}\left[\mathrm{e}^{-r(\tau^*-t)}\right]$ as a function of the critical profit rate Π^* leads to a simplified optimization problem.[1] In the continuation region $\{\Pi_t \in [0,\infty) : \Pi_t \leq \Pi^*\}$,

$$\begin{aligned} F(\Pi_t) &= \max_{\Pi^* \in [0,\infty)} \mathbf{E}_{\mathbf{P}^*}\left[\mathrm{e}^{-r(\tau^*-t)}\,(\Pi^*/\delta - I)\,\middle|\,\mathcal{F}_t\right] \\ &= \max_{\Pi^* \in [0,\infty)} \mathbf{E}_{\mathbf{P}^*}\left[\mathrm{e}^{-r(\tau^*-t)}\,\middle|\,\mathcal{F}_t\right](\Pi^*/\delta - I) \\ &= \max_{\Pi^* \in [0,\infty)} (\Pi^*/\delta - I)\left(\frac{\Pi_t}{\Pi^*}\right)^{\gamma^+}. \end{aligned}$$

A necessary condition for Π^* to be optimal is

$$\left(\frac{\gamma^+ I}{\Pi^*} - \frac{\gamma^+ - 1}{\delta}\right)\left(\frac{\Pi_t}{\Pi^*}\right)^{\gamma^+} = 0,$$

which implies

$$\Pi^* = \frac{\gamma^+}{\gamma^+ - 1} I\delta.$$

The sufficient condition is fulfilled, because

$$-\frac{\gamma^+ I}{\Pi_t^2}\left(\frac{\Pi_t}{\Pi^*}\right)^{\gamma^+ + 2} < 0.$$

[1] Note that, in the perpetual case, one can safely examine the payoff function $(\Pi^* - I)$ instead of $(\Pi^* - I)^+$, because it is always possible to postpone option exercise indefinitely.

In the stopping region immediate exercise is optimal, hence $F(\Pi_t) = \Pi_t/\delta - I$. Consequently, in accordance with proposition 13, patent value is given by

$$F(\Pi_t) = \begin{cases} \Pi_t/\delta - I & \text{if } \Pi^* < \Pi_t, \\ \frac{I}{\gamma^+ - 1}\left(\frac{\Pi_t}{\Pi^*}\right)^{\gamma^+} & \text{otherwise,} \end{cases}$$

which is just the value calculated previously using contingent-claims analysis.

A.2 Proposition 11

Consider the cash flow rate process described by

$$\mathrm{d}\Pi_t = \alpha \Pi_t \,\mathrm{d}t + \sigma \Pi_t \,\mathrm{d}W_t.$$

In analogy to (11.3), the value of the completed project is

$$V_I(\Pi_t, t) = \left(1 - \mathrm{e}^{-\left(r-(\alpha-\eta)(T-t)\right)}\right) \frac{\Pi_t}{r-(\alpha-\eta)} + \mathrm{e}^{-\left(r-(\alpha-\eta)(T-t)\right)} M \Pi_t,$$

where η is the appropriate risk premium. Itô's Lemma leads to

$$\begin{aligned} \mathrm{d}V_I(\Pi_t, t) = (r+\eta) & \left(\left(\frac{\alpha}{r+\eta} - \mathrm{e}^{-\left(r-(\alpha-\eta)(T-t)\right)} \right) \frac{\Pi_t}{r-(\alpha-\eta)} \right. \\ & \left. + \mathrm{e}^{-\left(r-(\alpha-\eta)(T-t)\right)} M \Pi_t \right) \mathrm{d}t \\ & + \sigma \left(\left(1 - \mathrm{e}^{-\left(r-(\alpha-\eta)(T-t)\right)}\right) \frac{\Pi_t}{r-(\alpha-\eta)} \right. \\ & \left. + \mathrm{e}^{-\left(r-(\alpha-\eta)(T-t)\right)} M \Pi_t \right) \mathrm{d}W_t. \end{aligned}$$

Hence, the value of a completed project exhibits the same volatility, but does *not* exhibit the same drift as the underlying cash flow rate.

A.3 Proposition 13

Section 11.2.2 served to introduce the option value of litigation. A proof based on contingent-claims analysis was presented. The same result can be obtained in a slightly different manner, mirroring arguments brought forward in section A.1.2.

Again, the option value of litigation is decomposed into a controlled and an uncontrolled diffusion process, resulting in

$$F_{\mathrm{C}}(\Pi_t) = \max_{\Pi^* \in [0,\infty)} \mathbf{E}_{\mathbf{P}^*}\left[-p\,(\theta - \zeta) \int_t^{\tau^*} \mathrm{e}^{-r(s-t)} \Pi_s \,\mathrm{d}s \right.$$
$$\left. - \,\mathrm{e}^{-r(\tau^*-t)} L_{\mathrm{C}} + p\theta \int_t^{\infty} \mathrm{e}^{-r(s-t)} \Pi_s \,\mathrm{d}s \,\middle|\, \mathcal{F}_t \right]. \quad \text{(A.1)}$$

An expression that can be employed to discount the cost of litigation, is known from previous analyses (see theorem 2) and is restated here for convenience:

$$\mathbf{E}_{\mathbf{P}^*}\left[\mathrm{e}^{-r(\tau^*-t)} \,\middle|\, \mathcal{F}_t \right] = \left(\frac{\Pi_t}{\Pi^*} \right)^{\gamma^+}. \quad \text{(A.2)}$$

The first integral remains to be evaluated. Dixit and Pindyck [91, pp. 315–316] provide the following theorem.

Theorem 3. *If $\Pi^* \geq \Pi_t$ is a fixed upper threshold, and $\tau^* \geq t$ is the first hitting time,*

$$\mathbf{E}_{\mathbf{P}^*}\left[\int_t^{\tau^*} \mathrm{e}^{-r(s-t)} \Pi_s \,\mathrm{d}s \,\middle|\, \mathcal{F}_t \right] = \Pi_t/\delta - \Pi^*/\delta \left(\frac{\Pi_t}{\Pi^*} \right)^{\gamma^+}. \quad \text{(A.3)}$$

Proof (Proof of theorem 3). Using familiar dynamic-programming techniques [91, p. 316], it is not difficult to see that

$$G(\Pi_t) = \mathbf{E}_{\mathbf{P}^*}\left[\int_t^{\tau^*} \mathrm{e}^{-r(s-t)} \Pi_s \,\mathrm{d}s \,\middle|\, \mathcal{F}_t \right],$$

satisfies

$$\tfrac{1}{2}\sigma^2 \Pi_t^2 \frac{\mathrm{d}^2 G(\Pi_t)}{\mathrm{d}\Pi_t^2} + (r - \delta)\,\Pi_t \frac{\mathrm{d}G(\Pi_t)}{\mathrm{d}\Pi_t} - rG(\Pi_t) + G(\Pi_t) = 0.$$

General solutions take the form

$$G(\Pi_t) = A^- \Pi_t^{\gamma^-} + A^+ \Pi_t^{\gamma^+} + \Pi_t/\delta,$$

subject to $G(0) = 0$ and $G(\Pi^*) = 0$. Hence, $A^- = 0$ and $A^+ = -\Pi^*/\delta\,(1/\Pi^*)^{\gamma^+}$, leading to (A.3).

Note that theorem 3 also follows directly from theorem 2, because

$$\begin{aligned}
\mathbf{E}_{\mathbf{P}^*}\left[\int_t^{\tau^*} \mathrm{e}^{-r(s-t)} \Pi_s \,\mathrm{d}s \,\middle|\, \mathcal{F}_t\right] = \mathbf{E}_{\mathbf{P}^*}\Bigg[\int_t^{\infty} \mathrm{e}^{-r(s-t)} \Pi_s \,\mathrm{d}s& \\
-\mathrm{e}^{-r(\tau^*-t)} \int_{\tau^*}^{\infty} \mathrm{e}^{-r(s-\tau^*)} \Pi_s \,\mathrm{d}s \,\Bigg|\, \mathcal{F}_t\Bigg]& \\
= \Pi_t/\delta - \mathbf{E}_{\mathbf{P}^*}\left[\mathrm{e}^{-r(\tau^*-t)} \,\middle|\, \mathcal{F}_t\right] \Pi^*/\delta,&
\end{aligned}$$

which is equivalent to (A.3).

It is then straightforward to deduce proposition 13 by substituting (A.2) and (A.3) in (A.1), which yields

$$\begin{aligned}
F_{\mathrm{C}}(\Pi_t) = \max_{\Pi^* \in [0,\infty)} -p\,(\theta - \zeta)\left(\Pi_t/\delta - \Pi^*/\delta\left(\frac{\Pi_t}{\Pi^*}\right)^{\gamma^+}\right)& \\
- L_{\mathrm{C}}\left(\frac{\Pi_t}{\Pi^*}\right)^{\gamma^+} + p\theta\Pi_t/\delta.& \quad \text{(A.4)}
\end{aligned}$$

A necessary condition for the threshold to be optimal is

$$\left(\frac{\gamma^+ L_{\mathrm{C}}}{\Pi^*} - p\,(\theta - \zeta)\,\frac{\gamma^+ - 1}{\delta}\right)\left(\frac{\Pi_t}{\Pi^*}\right)^{\gamma^+} = 0.$$

Solving for the critical profit rate yields

$$\Pi^* = \frac{\gamma^+}{\gamma^+ - 1}\frac{L_{\mathrm{C}}\delta}{p\,(\theta - \zeta)}, \tag{A.5}$$

which is the trigger deduced earlier. As is easily verified, the sufficient condition is also fulfilled. Inserting (A.5) in (A.4) leads to

$$F_{\mathrm{C}}(\Pi_t) = \frac{L_{\mathrm{C}}}{\gamma^+ - 1}\left(\frac{\Pi_t}{\Pi^*}\right)^{\gamma^+} + p\zeta\Pi/\delta$$

for dynamic patent values in the continuation region.

A.4 Proposition 14

In section 11.2.2, gross payoff from commercialization under patent risk was derived as a portfolio of claims. Alternatively, one may arrive at the same result directly, again making use of the expected first hitting time and the threshold of proposition 13 (see sect. A.3).

Gross payoff from commercialization is

$$\begin{aligned}\widetilde{V}_{\mathrm{I}}(\Pi_t) = \mathbf{E}_{\mathbf{P}^*}\Bigg[& \int_t^{\infty} \mathrm{e}^{-r(s-t)} \Pi_s \,\mathrm{d}s \\ & - p \left(\int_t^{\tau^*} \mathrm{e}^{-r(s-t)} \zeta \Pi_s \,\mathrm{d}s + \int_{\tau^*}^{\infty} \mathrm{e}^{-r(s-t)} \theta \Pi_s \,\mathrm{d}s \right) \\ & - \mathrm{e}^{-r(\tau^*-t)} L_{\mathrm{I}} \,\Bigg|\, \mathcal{F}_t \Bigg]\end{aligned}$$

Re-write this equation to obtain

$$\begin{aligned}\widetilde{V}_{\mathrm{I}}(\Pi_t) = {} & (1 - p\theta)\, \mathbf{E}_{\mathbf{P}^*}\left[\int_t^{\infty} \mathrm{e}^{-r(s-t)} \Pi_s \,\mathrm{d}s \,\middle|\, \mathcal{F}_t \right] \\ & + p\,(\theta - \zeta)\, \mathbf{E}_{\mathbf{P}^*}\left[\int_t^{\tau^*} \mathrm{e}^{-r(s-t)} \Pi_s \,\mathrm{d}s \,\middle|\, \mathcal{F}_t \right] \\ & - \mathbf{E}_{\mathbf{P}^*}\left[\mathrm{e}^{-r(\tau^*-t)} \,\middle|\, \mathcal{F}_t \right] L_{\mathrm{I}}.\end{aligned}$$

Recall that the first and second terms represent the gross payoff from commercialization under the assumption of immediate litigation and the value of waiting, respectively. Figure A.1 illustrates this decomposition graphically.

The second and the third integral follow from (A.2) and (A.3). Applying the critical profit rate and taking expectations over first hitting times leads to

$$\begin{aligned}\widetilde{V}_{\mathrm{I}}(\Pi_t) = {} & (1 - p\theta)\, \Pi_t/\delta + p\,(\theta - \zeta) \left(\Pi_t/\delta - \Pi^*/\delta \left(\frac{\Pi_t}{\Pi^*} \right)^{\gamma^+} \right) \\ & - L_{\mathrm{I}} \left(\frac{\Pi_t}{\Pi^*} \right)^{\gamma^+} \\ = {} & (1 - p\zeta)\, \Pi_t/\delta - \left(L_{\mathrm{I}} + \frac{\gamma^+}{\gamma^+ - 1} L_{\mathrm{C}} \right) \left(\frac{\Pi_t}{\Pi^*} \right)^{\gamma^+},\end{aligned}$$

thus verifying proposition 14.

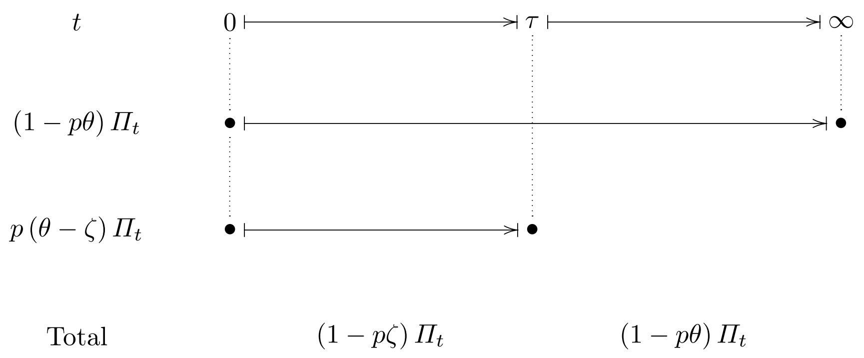


Fig. A.1. Decomposing the payoff from commercialization. Total payoff from litigation can be decomposed into one perpetuity and one option, creating one closed and one open interval with distinct profit rates. The stopping time τ is chosen by the challenger to maximize litigation payoff.

B

Numerical Methods

This appendix adds to the overview of extant numerical procedures provide in section 3.2. Furthermore, it hints at some of the particular challenges posed by the specific option-based models of imperfect patent protection presented.

B.1 Binomial and Multinomial Trees

This subsection presents variants of the CRR option pricing technique, looking a single-factor as well as multi-factor models in the context of patent valuation.

B.1.1 Single-Factor Model

The method described in this section represents an alternative way of calculating patent value in the presence of finite patent protection and thereby offers a possibility to check the accuracy of the FD techniques employed earlier (see sect. 6.2.2.2). Specifically, the proposed approach is based on the simple CRR model, which uses a binomial tree to approximate the continuous diffusion process in discrete time [76].

The protection period is divided into n periods of length $\Delta t \equiv T/n$. In each period, the discrete profit rate process exhibits an upward jump with probability p or a downward jump with probability $1 - p$. If j denotes the current period and i the number of upward jumps, the current profit rate becomes

$$\Pi_{i,j} = \Pi_{0,0} u^i d^{j-i}, \qquad 0 \leq j \leq n, \qquad 0 \leq i < j,$$

where $u > 1$ and $d = 1/u$ are chosen to properly reflect expected value and variance under the equivalent martingale measure. Figure B.1 shows an illustrative two-period example.

Equating mean and variance for the continuous-time and discrete-time process leads to

$$p = \frac{\mathrm{e}^{(r-\delta)\Delta t} - d}{u - d},$$

where $u = \mathrm{e}^{\sigma\sqrt{\Delta t}}$ [336, pp. 164 and 173]. The resulting binomial tree converges to GBM as the number of timesteps increases.[1]

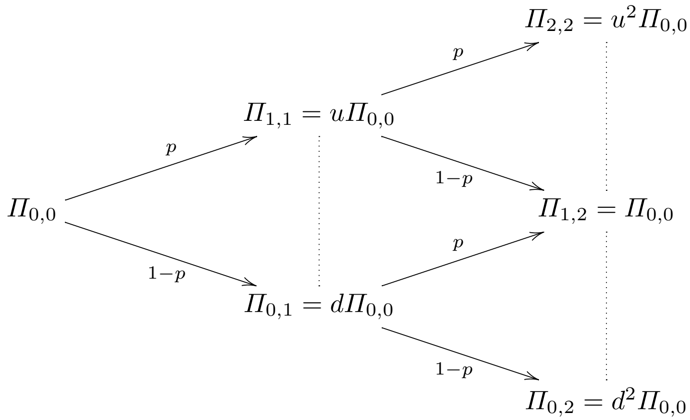


Fig. B.1. Binomial profit rate tree. Continuous dynamics are approximated by discrete upward and downward jumps. Parameters are chosen to make the tree recombining, thereby greatly facilitating calculations.

In contrast to a project-level model, which is analogous to a plain-vanilla call option, a profit-level model requires an additional step to determine the gross payoff from commercialization. In order to derive $V_{i,j} \equiv V(\Pi_{i,j}, j\Delta t)$ at each node, a second directed graph is constructed, starting at the leaves of the orginal profit rate tree. In each period the

[1] As is easily verified by applying Itô's Lemma to the logarithm of the profit rate process, instead of the standard CRR discretization, one could also set

$$u = \mathrm{e}^{\left((r-\delta)\Delta t - \frac{1}{2}\sigma^2\right) + \sigma\sqrt{\Delta t}},$$
$$d = \mathrm{e}^{\left((r-\delta)\Delta t - \frac{1}{2}\sigma^2\right) - \sigma\sqrt{\Delta t}}.$$

A similar approach is adopted in sect. B.1.2 [254, p. 495].

investor accumulates cash at the then current rate $\Pi_{i,j}$. Consequently, at the end of the protection period,

$$V_{i,n} = \Pi_{i,n}\Delta t.$$

Moving backwards in time and taking expectations, one obtains

$$V_{i,j} = \Pi_{i,j}\Delta t + \mathrm{e}^{-r\Delta t}\big(pV_{i+1,j+1} + (1-p)V_{i,j+1}\big).$$

The final step is straightforward and involves the construction of a third tree representing patent values. Again, the procedure starts in period n. A rational investor commercializes if and only if the net payoff is positive:

$$F_{i,n} = \max\{V_{i,n} - I, 0\}.$$

For each preceding period, the payoff from immediate commercialization is compared to the alternative of postponement, that is

$$F_{i,j} = \max\left\{V_{i,j} - I, \mathrm{e}^{-r\Delta t}\big(pF_{i+1,j+1} + (1-p)F_{i,j+1}\big)\right\}.$$

Eventually, the value of the patent $F_{0,0} \equiv F(\Pi_0, 0)$ is obtained.

Table B.1 shows a numerical example. Similar to section 6.2.2.2, assume an initial profit rate $\Pi_0 = 0.10$, a protection period of $T = 20$ years, $r = \delta = 0.05$, and $\sigma = 0.05$. Calculating jump sizes and risk-neutral probabilities for the two-period model discussed previously leads to

$$u = \mathrm{e}^{+0.4\times 10} = 3.54, \qquad d = 1/3.54 = 0.28,$$

and

$$p = \frac{\mathrm{e}^{(0.05-0.05)10} - 0.28}{3.54 - 0.28} = 0.22.$$

Further assuming an investment amount of $I = 1.0$, one obtains an approximate patent value of $F(\Pi_0, 0) = 0.97$. This result differs substantially from the one derived using FD techniques. However, a similar tree with $n = 50$ timesteps yields patent values very close to those reported in table 6.1.

B.1.2 Multi-Factor Model

Define $C_t \equiv \mathrm{e}^{X_t}$, so that X_t follows ABM, and use Itô's Lemma to derive the stochastic process followed by the logarithm of expected cost to completion from (7.41):

Table B.1. Dynamic patent value under uncertainty and finite patent protection if the commercialization phase is finite (binomial trees). Panel (a) shows profit rate dynamics under the risk-neutral probability measure for $\Pi_0 = 0.10$, $T = 20$, $r = \delta = 0.05$, and $\sigma = 0.05$. Panel (b) shows the gross payoff from commercialization $V(\Pi_t, t)$, and panel (c) patent value $F(\Pi_t, t)$ for $I = 1.0$. The result, which differs substantially from the more accurate benchmark, reveals that discretization errors should not be neglected.

(a) Profit rate

State	$\Pi_{i,j}$		
	$t = 0$	$t = 10$	$t = 20$
+2			1.26
+1		0.35	
0	0.10		0.10
−1		0.03	
−2			0.01

(b) Investment project

State	$V_{i,j}$		
	$t = 0$	$t = 10$	$t = 20$
+2			12.55
+1		5.69	
0	1.97		1.10
−1		0.45	
−2			0.08

(c) Investment opportunity

State	$F_{i,j}$		
	$t = 0$	$t = 10$	$t = 20$
+2			11.55
+1		4.69	
0	0.97		0.00
−1		0.00	
−2			0.00

$$\begin{aligned} \mathrm{d}X_t &= \nu(C_t, t)\,\mathrm{d}t + \theta(C_t, t)\,\mathrm{d}W_t^C \\ &= -\frac{I_t}{C_t}\left(1 + \tfrac{1}{2}\varsigma^2\right)\mathrm{d}t + \varsigma\sqrt{\frac{I_t}{C_t}}\,\mathrm{d}W_t^C, \end{aligned}$$

that is

$$\nu = -\frac{I_t}{C_t} - \tfrac{1}{2}\theta^2, \tag{B.1a}$$

$$\theta = \varsigma\sqrt{\frac{I_t}{C_t}}. \tag{B.1b}$$

This information can be used to construct a trinomial tree of the form shown in fig. B.2.

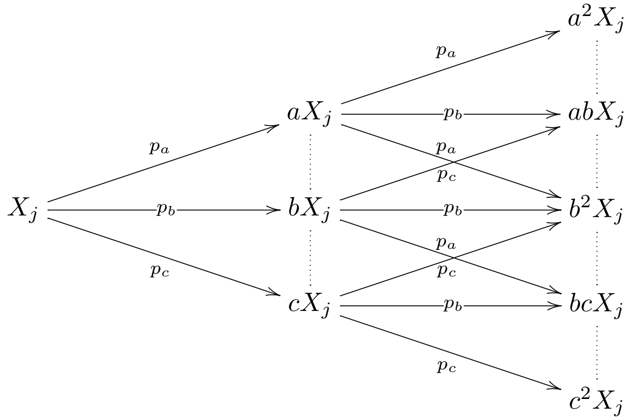


Fig. B.2. Trinomial tree for patent valuation. Compared to binomial trees, additional degrees of freedom make it possible to fit drift and volatility surfaces.

Since the tree is recombining, set

$$ac = b^2.$$

Coefficients are chosen to resemble the CRR model:

$$\begin{aligned} a &= \mathrm{e}^{\overline{\nu}\Delta t + \overline{\theta}\sqrt{\Delta t}}, \\ b &= \mathrm{e}^{\overline{\nu}\Delta t}, \\ c &= \mathrm{e}^{\overline{\nu}\Delta t - \overline{\theta}\sqrt{\Delta t}}, \end{aligned}$$

where $\overline{\nu}$ and $\overline{\theta}$ are constants to be determined. Note that this specification also implies constant a, b, and c, while probabilities (p_a, p_b, and p_c) may vary to reflect local drift and volatility. More precisely,

$$\begin{aligned} \mathbf{E}[X_t] &= \nu\Delta t = \overline{\nu}\Delta t + (p_a - p_c)\,\overline{\theta}\sqrt{\Delta t}, \\ \mathbf{V}[X_t] &= \mathbf{E}\left[X^2\right] - (\mathbf{E}[X_t])^2 \\ &= \theta^2\Delta t = (p_a + p_c)\,\overline{\theta}^2\Delta t - (\nu\Delta t)^2 . \end{aligned}$$

Neglecting terms smaller than $\mathrm{O}(\Delta t)$,

$$\theta^2\Delta t = (p_a + p_c)\,\overline{\theta}^2\Delta t. \tag{B.3}$$

Solving the resulting system of equations, one obtains

$$p_a = \tfrac{1}{2}\left(\left(\frac{\theta}{\overline{\theta}}\right)^2 + \frac{\nu - \overline{\nu}}{\overline{\theta}}\sqrt{\Delta t}\right),$$

$$p_b = 1 - \left(\frac{\theta}{\overline{\theta}}\right)^2,$$

$$p_c = \tfrac{1}{2}\left(\left(\frac{\theta}{\overline{\theta}}\right)^2 - \frac{\nu - \overline{\nu}}{\overline{\theta}}\sqrt{\Delta t}\right).$$

Define

$$p \equiv p_a + p_c = \left(\frac{\theta}{\overline{\theta}}\right)^2,$$

$$q \equiv \frac{p_a - p_c}{\overline{\theta}\sqrt{\Delta t}} = \frac{\nu - \overline{\nu}}{\overline{\theta}^2}.$$

Consequently,

$$p_a = \tfrac{1}{2}\left(p + q\overline{\theta}\sqrt{\Delta t}\right), \tag{B.6a}$$

$$p_b = 1 - p, \tag{B.6b}$$

$$p_c = \tfrac{1}{2}\left(p - q\overline{\theta}\sqrt{\Delta t}\right). \tag{B.6c}$$

Equation (B.6b) leads to

$$0 \leq p \leq 1. \tag{B.7}$$

Using (B.6a) and (B.6c), non-negative probabilities are obtained for

$$-q\overline{\theta}\sqrt{\Delta t} \leq p, \tag{B.8a}$$

$$+q\overline{\theta}\sqrt{\Delta t} \leq p. \tag{B.8b}$$

Since $p \equiv p_a + p_c$, (B.3) and (B.7) imply

$$\theta^2 \leq \overline{\theta}^2.$$

It therefore seems natural to simply choose $\overline{\theta} = \theta_{\mathrm{max}}$, where θ_{max} and θ_{min} are the maximum and minimum values of the volatility parameter within the relevant range.

Provided that $\nu_{\mathrm{min}} \leq \overline{\nu} \leq \nu_{\mathrm{max}}$, (B.8a) and (B.8b) imply

$$-\frac{\nu_{\mathrm{min}} - \overline{\nu}}{\theta_{\mathrm{max}}}\sqrt{\Delta t} \leq \left(\frac{\theta_{\mathrm{min}}}{\theta_{\mathrm{max}}}\right)^2,$$

$$+\frac{\nu_{\mathrm{max}} - \overline{\nu}}{\theta_{\mathrm{max}}}\sqrt{\Delta t} \leq \left(\frac{\theta_{\mathrm{min}}}{\theta_{\mathrm{max}}}\right)^2.$$

Consequently, $\overline{\nu} = \frac{1}{2}(\nu_{\max} + \nu_{\min})$ leads to

$$\sqrt{\Delta t} \leq \frac{2\theta_{\min}^2}{\theta_{\max}(\nu_{\max} - \nu_{\min})}.$$

Since this constraint requires the timestep size to become infinitely small as the cost of completion tends to zero, a lower boundary $C_{\min} > 0$ is introduced to ensure stability. Constructing the drift and volatility surfaces employing (B.1a) and (B.1b) is then comparatively straightforward.

Patent values are obtained by combining the resulting trinomial tree with a binomial one for the profit rate process.[2] No detailed description is given at this point, because Monte Carlo simulation has been shown to yield reliable results, while a hexanomial model suffers from the "curse of dimensionality." In general, valuing barrier options using multinomial trees is a very inefficient approach. Such limitations, however, can be overcome by making careful adjustments. Hull [150, pp. 477–482] provides an overview of suitable techniques, including the adaptive mesh model.

B.2 Monte Carlo Simulation

Some of the random numbers needed to perform Monte Carlo simulations throughout this dissertation were produced by QRNGEXTRA, an extension package for the GSL developed by the author, providing a high-dimensional Sobol sequence generator [110, 313]. The use of Sobol sequences, today considered as examples of (t, d)-sequences in base 2, represents only one of many possibilities to increase computational efficiency and accuracy by employing (deterministic) quasi-random instead of pseudo-random numbers.[3] More specifically, QRNGEXTRA is based on Algorithm 659, originally developed by Bratley and Fox [51] and later extended by Joe and Kuo [158].

Consisely, generating the jth component in a Sobol sequence requires a primitive polynomial of degree s_j in the field $\mathbf{Z}_2$:

$$x^{s_j} + a_{1,j}x^{s_j-1} + \cdots + a_{s_j-1,j}x + 1,$$

[2] For details on how to combine uncertainties in option valuation see Copeland and Antikarov [71, pp. 270–293].

[3] For a general theory of low discrepancy (t, d)-sequences in base b see Niederreiter [258].

where $\{a_{1,j}, \ldots, a^{s_j-1,j}\}$ are binary coefficients. A sequence of positive integers follows from the recursive equation

$$\begin{aligned} m_{k,j} = 2a_{1,j}m_{k-1,j} \oplus 2^2 a_2 m_{k-2,j} \oplus \ldots \\ \oplus 2^{s_j-1}a_{s_j-1,j}m_{k-s_j+1,j} \oplus 2^{s_j}m_{k-s_j,j} \oplus m_{k-s_j,j}, \end{aligned}$$

where $k \geq s_j + 1$ and $\oplus$ is the bitwise XOR operator. The quality of the sequences generated fundamentally hinges on the initial values $\{m_{1,j}, \ldots, m_{s_j,j}\}$, which can be chosen freely provided that $m_{k,j}$, $k \in [1, s_j]$, is odd and less than 2^k.

The direction numbers

$$v_{k,j} \equiv m_{k,j}/2^k$$

then make it possible to calculate the jth component of the ith point in a Sobol sequence:

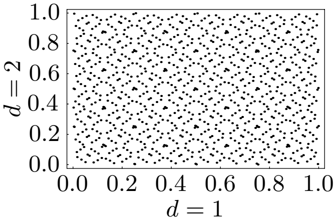
$$x_{i,j} = b_1 v_1 \oplus b_2 v_2 \oplus \ldots,$$

where b_l is the lth bit from the right in the binary representation of i, namely $(\ldots b_2 b_1)_2$ [158, p. 50].

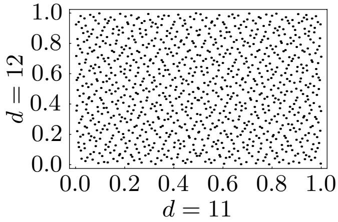
The choice of suitable primitive polynomials and direction numbers not covered here is discussed in detail by Jäckel [156], Joe and Kuo [158].

Despite some advantages over the application of pseudo-random numbers, valuing American options based on high-dimensional Sobol sequences also suffers from potential drawbacks. For example, comparatively poor two-dimensional projections may occur, reducing the accuracy of results (see fig. B.3).

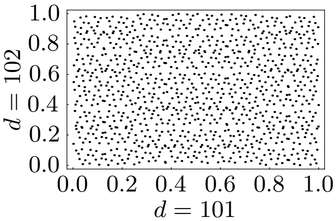
As mentioned previously, such inaccuracies can be reduced by choosing appropriate direction numbers—but not eliminated completely. The form of *scrambling* best-suited to overcome these issues continues to be the subject of scientific debate [64]. Since numerical accuracy is not of paramount importance in real option analysis, Sobol sequences are used as generated by QRNGEXTRA, without prior scrambling.



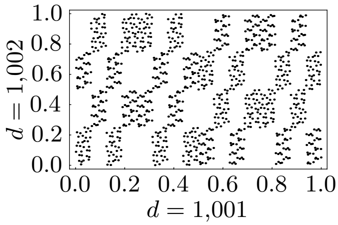
(a) Dimensions 1 and 2



(b) Dimensions 11 and 12



(c) Dimensions 101 and 102



(d) Dimensions 1,001 and 1,002

Fig. B.3. Quasi-random numbers generated by QRNGEXTRA. The quality of the numbers generated is very sensitive to the initial choice of direction numbers. Comparing the two-dimensional projections shown in panels (b) and (d) illustrates very nicely that satisfactory results are difficult to achieve for any combination of dimensions.

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List of Abbreviations

ABM	arithmetic Brownian motion
ACM	Association for Computing Machinery
ADI	alternating-direction implicit
AdW	Akademie der Wissenschaften
AICPA	American lnstitute of Certified Public Accountants
AIPLA	American Intellectual Property Law Association
ANDA	abbreviated new drug application
CAFC	Court of Appeals for the Federal Circuit
CAGR	compound annual growth rate
CAPM	capital asset pricing model
CDF	cumulative density function
CEO	chief executive officer
cf.	confer
CFO	chief financial officer
CGMY	Carr–Geman–Madan–Yor
chap.	chapter
CPA	certified public accountant
CRR	Cox–Ross–Rubinstein
DCF	discounted cash flow
DDR	Deutsche Demokratische Republik
DNA	desoxyribonucleic acid
DTA	decision tree analysis
EC	evolutionary computation
EMC	evolutionary Monte Carlo
ENPV	expected net present value
EP	evolutionary programming
EPO	European Patent Office
EPV	expected present value

ES	evolutionary strategy
ESS	evolutionary stable strategy
EUR	euros
EVT	extreme value theory
FAS	full approximation storage
FASB	Financial Accounting Standards Board
FD	finite difference
FDA	Food and Drug Administration
FE	finite element
FFT	fast Fourier transform
fig.	figure
FTC	Federal Trade Commission
GA	genetic algorithm
GAAP	Generally Accepted Accounting Principles
GBM	geometric Brownian motion
GH	generalized hyperbolic
GNU	GNU is Not UNIX
GP	genetic programming
GSL	GNU Scientific Library
HBS	Harvard Business School
IAS	International Accounting Standards
IEEE	Institute of Electrical and Electronics Engineers
IFRS	International Financial Reporting Standards
IP	intellectual property
IPC	international patent classification
IPR	intellectual property right
ITC	International Trade Commission
IUI	Industriens Utredningsinstitut (The Research Institute of Industrial Economics)
KTH	Kungliga Tekniska högskolan (Royal Institute of Technology)
LSM	least-squares Monte Carlo
M&A	mergers and acquisitions
MCA	Markov chain approximation
MEEM	Multi-Period-Excess-Earnings Method
MIT	Massachusetts Institute of Technology
MULTICS	Multiplexed Information and Computing Service
NAS	National Academies of Science
NBER	National Bureau of Economic Research
NDA	new drug application
NIG	normal inverse Gaussian

no.	number
NPD	new product development
NPV	net present value
OBV	option-based view
p., pp.	page, pages
PAT	Primary Assessment Tool
PDE	partial differential equation
PDF	probability density function
PFAS	projected full approximation storage
PIDE	partial integro-differential equation
PPA	purchase price allocation
PRNG	pseudo-random number generator
PSOR	projected successive overrelaxation
PTO	Patent and Trademark Office
PV	present value
QMC	quasi–Monte Carlo
QRNG	quasi-random number generator
RAND	research and development
RBV	resource-based view
R&D	research and development
ROI	return on investment
SDE	stochastic differential equation
SEC	Securities and Exchange Commission
sec.	section
SFAS	Statement of Financial Accounting Standards
SIAM	Society for Industrial and Applied Mathematics
SME	small and medium-sized enterprises
SOR	successive overrelaxation
SOX	Sarbanes–Oxley
SPC	Supplementary Protection Certificate
SVD	singular value decomposition
SWOT	strengths, weaknesses, opportunities, and threats
TOMAC	Transactions on Modeling and Computer Simulation
UCLA	University of California, Los Angeles
UK	United Kingdom
UNIX	UNIX is Not MULTICS
US	United States
USD	US dollars
USPTO	United States Patent and Trademark Office
VBM	value-based management
vol.	volume

VRIN	valuable, rare, inimitatable, and non-substitutable
ZIMM	Zentralinstitut für Mathematik und Mechanik

List of Symbols

α	instantaneous proportional drift
β	systematic risk according to the CAPM
$D(\cdot)$	inverse demand function excluding the demand-scaling parameter
δ	dividend yield
$\overrightarrow{\Delta}$	forward difference
$\overleftarrow{\Delta}$	backward difference
$\overleftrightarrow{\Delta}$	central difference
$\overline{\Delta}$	central mean
$\mathbf{E}[\cdot]$	expectation operator
η	risk premium
$\mathcal{F}_t$	σ-algebra generated by $\{W_s : s \leq t\}$
F	value of the investment opportunity
F_{C}	option value of litigation
$\widetilde{F}_{\mathrm{I}}$	patent value under imperfect patent protection
γ_i^+	positive root of the characteristic equation
γ_i^-	negative root of the characteristic equation
I	investment required to initiate the project
K	strike price
L_{C}	challenger's cost of litigation
L_{I}	incumbent's cost of litigation
λ	market price of risk
M	exit multiple
μ	risk-adjusted discount rate
$\mathrm{O}(\cdot)$	order
P_t	price at time t
Π^*	critical profit (cash flow) rate
Π_t	profit (cash flow) rate at time t

ϖ	initial profit (cash flow) rate
ϖ^*	critical initial profit (cash flow) rate
Q_t	supply at time t
Q_t^*	optimal supply at time t
r	risk-free rate of interest
r_{m}	market rate of return
ρ	correlation coefficient
σ	instantaneous proportional volatility
σ_{m}	standard deviation of market returns
S_t	stock price at time t
T	maturity
τ	stopping time
τ^*	optimal stopping time
θ	proportion of future cash flows claimed
V	value of the investment project
v	initial project value
V^*	critical project value
v^*	critical initial project value
V_t	project value at time t
V_{C}	payoff from litigation
V_{I}	gross payoff from commercialization under perfect patent protection
$\widetilde{V}_{\mathrm{I}}$	gross payoff from commercialization under imperfect patent protection
W_t	state of a Wiener process at time t
X_t	value of the spanning asset at time t
y	initial value of the demand-scaling parameter
y^*	critical initial value of the demand-scaling parameter
Y_t	value of the demand-scaling parameter at time t
ζ	proportion of past cash flows claimed

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