Lecture Notes in Mathematics 2207

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Series of Bessel and Kummer-Type Functions

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Dedicated to Agata, Boróka, Dunja, Kati,
Koppány, Nikola, and Sanja

Preface

The summation of series of special functions (or, accepting Turán's intervention, "useful functions") is a subdiscipline of Classical Analysis. Functional series built from members of the, so-called, Bessel function family play a particularly important role in this field. The Bessel function family includes a vast range of functions: Bessel functions of the first and second kind, modified Bessel functions of the first and second kind, Hankel functions, Struve and modified Struve functions, von Lommel functions, for instance. There is also an extensive literature, including the monumental monograph [333], concerned with important properties and the vast range of applications of such functions and various functional series built from them. An important topic within the theory of Bessel functions is the study of functional series of Bessel and related functions, whose role in mathematical physics, science, astronomy, and engineering is immense.

The classes of infinite series explored in this monograph are Neumann, Kapteyn, Schlömilch, and Dini series, whose terms contain certain members of the Bessel function family or special functions that arise from the class of hypergeometric functions (Kummer function). The building blocks of these series depend on certain parameters. So, in short, the main difference between these series is that in terms of the Neumann series the summation index is the order (parameter) of the Bessel function; in terms of the Kapteyn series the summation indices are the order and the argument, while in terms of the Schlömilch series the argument is the summation index. Also, using similar criteria, one can define general Neumann, Schlömilch, and Kapteyn series of hypergeometric or other special functions, guided by the above classification principle. On the other hand, the coefficient of the argument in a Dini series involves the zeros of the initial function from the Bessel family, or those of the related Dini function.

Functions in the Bessel family and the Kummer function have either power series or definite integral representations or they are particular solutions of ordinary differential equations. Thus we shall adopt a three-pronged approach in our study and will explore summations of sums, summations of integrals, and summations of functions that are solutions of Bessel, Struve, Kummer, or certain other classical ordinary differential equations. While we are addressing mainly the same problems as some of the great forefathers of the field of Fourier–Bessel series, including Carl Gottfried Neumann (1832, Königsberg–1925, Leipzig); Willem Kapteyn (Kapteijn) (1849, Barneveld–1927, Utrecht); Oscar Xavier Schlömilch (1823, Weimar–1901, Dresden); and in parallel Ulisse Dini (1845, Pisa–1918, Pisa), our approach to these considerations is significantly different.

Baricz and Pogány in [20, p. 815, Theorem 3.2] introduced a method, which completely reorganizes the classification "Fourier–Bessel series of the first type" (where one input Bessel family member function occurs in terms of the series) versus "Fourier–Bessel series of the second type" (where products of two or more Bessel-like functions appear in terms of the series). More precisely, Baricz and Pogány have incorporated all input functions in the products *except a chosen one*, which is included into the coefficient, and they consider the initial Fourier–Bessel series as the "series of the first type" with the newly constituted coefficients. The importance of these results is further seen by bearing in mind various new findings concerning derivatives of the Bessel function family with respect to the order posted on the Wolfram Functions website (http://blog.wolfram.com/2016/05/16/new-derivatives-of-the-bessel-functions-have-been-discovered-with-the-help-of-the-wolfram-language/).

We appreciate that the title of a monograph should be concise and informative, and not "too long." To cover the phrase "Neumann, Kapteyn, Schlömilch and Dini Series of Bessel Functions or Hypergeometric Type Functions," which is a precise but excessively long title for a book, we adopted "Fourier–Bessel Series" as a working title, influenced by the title of the article [145], and, e.g., by the title of section XVIII, "Series of Fourier–Bessel and Dini" in the monograph [333] by Watson. His presentation significantly differs from ours; we will briefly present this treatment of functions by Fourier–Bessel series, which actually belongs to the class of Schlömilch series, in the related subsection of the introductory chapter, emphasizing that we treat Fourier–Bessel series in a weaker sense than Kapteyn and Watson. We also note the fact that Bessel functions are linked to hypergeometric functions; see, also, [314, Chapter 8]. So, the title "Series of Bessel and Kummer-Type Functions" interpolates the previously mentioned two descriptions of the contents of this monograph.

The starting point for our research was the study [249] by Pogány and Süli in 2009 on Neumann series of Bessel functions of the first kind J_ν and von Lommel functions in which an integral expression was derived for Neumann series. There, the cornerstones of the study were Dirichlet series associated with the input Fourier–Bessel series and the Laplace integral of this Dirichlet series. While proceeding with our research on mathematical tools associated with appropriate Bessel-type homogeneous and nonhomogeneous ordinary differential equations, we extended our study, which then resulted, among others, in the Ph.D. thesis of Jankov Maširević [130] in 2011 and the habilitation thesis [244] of Pogány in 2015. Those two theses arose from several joint or separate publications and constitute a major part of this monograph.

Our main objective in this monograph is to give a systematic overview of our results concerning such series; textual material is gathered from diverse sources including journal articles, theses, and conference papers, which had not appeared before in the form of a book.

The book is aimed at a mathematical audience, graduate students, and those in the scientific community with interest in a new perspective on Fourier–Bessel series, and their manifold and polyvalent applications, mainly in general classical analysis, applied mathematics, or mathematical physics.

A general introduction to the subject will be found in Chap. 1, together with a necessarily short overview of special functions, Dirichlet series, Cahen's formula, and the Euler–Maclaurin summation formula, among others, as it is assumed that readers have a general background in real and complex analysis, and possess some familiarity with functional analysis. Then, results on Neumann–Bessel series are collected in the identically entitled Chap. 2, followed by Chap. 3, Kapteyn series, where, in addition to Kapteyn–Bessel series, also Kapteyn–Kummer series are presented. Chapter 4 focuses on Schlömilch–Bessel series and Schlömilch series of the p-extended Mathieu series, which represents a transition to Chap. 5, entitled Miscellanea, where Dini–Bessel series, Neumann and Kapteyn series of Struve and modified Struve functions, and Neumann series of Jacobi polynomials are considered. The main body of the book ends with a short overview of Neumann series of Meijer G functions, which is followed by an exhaustive list of references and an Index. We note that a detailed overview of diverse applications, with links to further relevant sources, is given in the introductory part of each chapter.

Besides the pure mathematical aspects of the obtained results, many potential application items exist, e.g., the Kapteyn series' applications in various problems of mathematical physics, e.g., Kepler's equation, pulsar physics, and electromagnetic radiation; Neumann series' use in infinite dielectric wedge problem, description of internal gravity waves in a Boussinesq fluid, propagation properties of diffracted light beams, the orbital angular momentum quantum number, the wave functions that describe the states of motion of charged particles in a Coulomb field, inversion probability of a large spin, evaluation of the capacitance matrix of a system of finite-length conductors, modeling of the free vibrations of a wooden pole, and analysis of an isotropic medium containing a cylindrical borehole are routine procedures. These numerical calculations mainly take into account truncation of infinite series. Instead, the derived integral expressions may lead to numerical quadrature implementation for which numerous in-built software routines are widely developed.

The authors take great pleasure in thanking Endre Süli (Oxford) for taking part in the research endeavor, which initiated and now finally encompasses this manuscript. We are also very grateful to Paul Leo Butzer (Aachen), Diego Dominici (New Paltz), Saminathan Ponnusamy (Chennai), and Sanjeev Singh (Chennai) for numerous valuable suggestions, remarks, and discussions, which resulted in crucial improvements of the exposition.

Székelyudvarhely — Árpád Baricz
Trpinja — Dragana Jankov Maširević
Sušak — Tibor K. Pogány
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Survey

The aim of this brief survey is to present a short overview of the topics discussed in this book.

Bessel Functions Bessel functions are solutions to the second-order linear homogeneous Bessel differential equation. Discovered by the mathematician Daniel Bernoulli and studied systematically by the astronomer Friedrich Bessel, Bessel functions appear frequently in problems of applied mathematics. They are particularly important in problems associated with wave propagation and static potentials. Bessel functions of integer order are also known as cylinder functions or cylindrical harmonics, because they arise in the solution of Laplace's equation in cylindrical coordinates. Although the study of Bessel functions is part of classical analysis, their beautiful properties are continually explored by numerous researchers, and several new properties are reported each year. G.N. Watson's book *A treatise on the theory of Bessel functions* [333], written almost one hundred years ago, is an important book in the theory of special functions, especially on topics associated with asymptotic expansions, series, zeros, and integrals of Bessel functions. Nowadays, Watson's book is a classic, and because of their remarkable properties, special functions, such as Bessel functions, are frequently used also in probability theory, statistics, mathematical physics, and in the engineering sciences. See, for example, the interesting book by B.G. Korenev *Bessel functions and their applications*, [156].

Series of Bessel Functions Infinite series involving different kinds of Bessel functions occur quite frequently in both mathematical and physical analysis. Watson's treatise contains four chapters on different kinds of series of Bessel functions, such as Neumann, Kapteyn, Fourier–Bessel, Dini, and Schlömilch series. Because of the range of applications in concrete problems of applied mathematics, series of Bessel functions have been considered frequently by researchers.

The Topics Discussed in This Book In this book our aim is to establish certain integral representations for Neumann, Kapteyn, Schlömilch, Dini, and Fourier series of Bessel and other special functions, such as Struve and von Lommel functions. Our objective is also to find the coefficients of the Neumann and Kapteyn series, as well as closed-form expressions, and summation formulae for the series of Bessel functions considered. In the study the so-called Euler–Maclaurin summation formula (which is a beautiful bridge between continuous and discrete), the Laplace–Stieltjes integral representation of Dirichlet series, and various bounds for Bessel and Bessel-type functions (Struve, modified Struve, modified Bessel functions of the first and second kind, von Lommel functions, and Bessel function of the second kind) play an important role. Some integral representations are also deduced by using techniques from the theory of differential equations.

Contents

Notations

$\mathbb{N}, \mathbb{Z}, \mathbb{R}, \mathbb{R}_+, \mathbb{C}$	Set of natural, integer, real, non-negative real and complex numbers, respectively
$\mathbb{N}_p$	$\{p, p+1, \cdots\}$, $p \in \mathbb{N}$
$\mathbb{Z}_0^-$	Set of non-positive integers $\{\ldots, -2, -1, 0\}$
$\mathscr{A}_\alpha(t)$	Counting function of (α_n) in Cahen's formula
$\mathscr{D}_\beta(x)$	Dirichlet series associated with sequence $(\beta_n)_{n\geq 1}$
$\mathfrak{d}_x f(x)$	$f(x) + \{x\} f'(x)$
$\Gamma(z)$	Gamma function, Euler function of the second kind
$\mathrm{B}(p, q)$	Beta function, Euler function of the first kind
$\psi(z)$	Psi (or Digamma) function, logarithmic derivative of the Gamma function
$(a)_n$	Pochhammer symbol
$(\alpha)_\mu$	Generalized Pochhammer symbol
$B_m(x)$	mth Bernoulli polynomial
$B_m = B_m(0)$	mth Bernoulli number
$\zeta(s), \eta(s)$	Riemann Zeta and Dirichlet Eta functions
$S(x), \widetilde{S}(x)$	Mathieu, alternating Mathieu series
$\mathsf{S}_\mu^{(\alpha,\beta)}(r; \mathbf{a})$	Generalized Mathieu series
$\mathfrak{M}_s(\boldsymbol{a}, \boldsymbol{\lambda}; r)$	Mathieu $(\boldsymbol{a}, \boldsymbol{\lambda})$-series
$S_{p,\nu}(x)$	p-extended Mathieu series
${}_2F_1\left[\begin{matrix} a, b \\ c \end{matrix} \middle\vert z\right]$	Gaussian hypergeometric function
${}_1F_1(a; c; z)$	Kummer (confluent hypergeometric) function

${}_pF_q\left[\begin{matrix} a_1,\cdots,a_p \\ b_1,\cdots,b_q \end{matrix}\,\middle\vert\, z\right]$	Generalized hypergeometric function
${}_p\Psi_q^*\left[\begin{matrix} (a_1,\rho_1),\cdots,(a_p,\rho_p) \\ (b_1,\sigma_1),\cdots,(b_q,\sigma_q) \end{matrix}\,\middle\vert\, z\right]$	Fox–Wright generalized hypergeometric function
$F^{p:q;k}_{l:m;n}\left[\begin{matrix} (a_p):\ (b_q);\ (c_k) \\ (\alpha_l):(\beta_m);(\gamma_n) \end{matrix}\,\middle\vert\, x,y\right]$	Kampé de Fériet generalized hypergeometric function of two variables
$\Phi_3(\beta,\gamma;x,y)$	Horn confluent hypergeometric function of two variables
$F^{A:B;B'}_{C:D;D'}\left[\begin{matrix} [(a):\nu,\varphi]:\ [(b):\psi];\ [(b'):\psi'] \\ [(c):\xi,\eta]:\ [(d):\zeta];\ [(d'):\zeta'] \end{matrix}\,\middle\vert\, x,y\right]$	Srivastava–Daoust extended generalized hypergeometric function of two variables
$G^{m,n}_{p,q}\left(z\,\middle\vert\,\begin{matrix} a_1,\cdots,a_p \\ b_1,\cdots,b_q \end{matrix}\right)$	Meijer G function
$J_\nu(z), I_\nu(z)$	Bessel and modified Bessel function of the first kind of order ν
$Y_\nu(z), K_\nu(z)$	Bessel and modified Bessel function of the second kind of order ν
$H^{(1)}_\nu(z), H^{(1)}_\nu(z)$	Hankel functions
$P_\nu(z)$	$I_\nu(z)\cdot K_\nu(z)$
$j_{\nu,n}$	nth positive zero of the Bessel function J_ν
$j_\nu(z), i_\nu(z)$	Spherical Bessel, modified spherical Bessel function of the first kind of order ν
$\omega_\nu(z)$	Generalized Bessel function of the first kind of order ν
$J^{(m)}_{\nu_1,\cdots,\nu_m}(z), I^{(m)}_{\nu_1,\cdots,\nu_m}(z)$	Delerue hyper-Bessel and modified Delerue hyper-Bessel function
${}_hJ_\nu(z), {}_hI_\nu(z)$	Wright generalized and modified Wright generalized Bessel function
b_L, c_L, d_O	First, second Landau's and Olenko's constants
$\mathrm{Ai}(x)$	Airy function
$\mathbf{H}_\nu(z), \mathbf{L}_\nu(z)$	Struve, modified Struve function of order ν
$\mathfrak{N}_\mu(z)$	Neumann series
$\mathfrak{K}_\mu(z)$	Kapteyn series
$\mathscr{K}_\kappa(z)$	Kapteyn–Kummer series
Θ_n	nth Kapteyn polynomial
$\mathfrak{S}_\mu(z), \mathfrak{T}_\mu(z)$	Schlömilch series
$\mathfrak{D}_\mu(z)$	Dini series
$d_{\nu,\alpha}(z)$	Dini function
$\lambda_{\nu,n}$	nth positive zero of the Dini function

$U_\nu(x, y), V_\nu(x, y)$	von Lommel functions of two variables
$\Omega(z)$	Butzer–Flocke–Hauss complete Omega function
$Q_\nu(a, b)$	Generalized Marcum Q-function
$\Gamma(\alpha, x; \beta)$	Generalized incomplete Gamma function
$\mathscr{W}(x, y)$	Leaky aquifer function
$\vartheta_3(z, q)$	Jacobi third Theta function
$\zeta_k(s)$	Epstein Zeta function
$r_k(n)$	Number of integer lattice points inside k-dimensional sphere of radius $\sqrt{n}$
$\Phi(z, s, a)$	Hurwitz–Lerch Zeta function
$\mathrm{Li}_s(z)$	Polylogarithm, de Jonquère's function
$\mathbb{D}_x^{-\alpha}[f]$	Grünwald–Letnikov fractional derivative
$M[y], M_\mu^\alpha[y]$	Bessel-type differential operator
$P_n^{(\alpha,\beta)}(z)$	Jacobi polynomial
$s_{\mu,\nu}(x), \mathfrak{s}_{\nu,\mu}(x)$	von Lommel functions
$\boldsymbol{C}$	Euler–Mascheroni constant
H_n	nth harmonic number
$\propto$	Proportional to; $x \propto y$ means that there is a constant C independent of x, y that $x = Cy$
$\sim$	Asymptotic to; $f(x) \sim g(x),\ x \to a$ means that $\lim_{x\to a} f(x)/g(x) = 1$
$\mathscr{O}(\cdot)$	Landau (or Big O) notation
$\mathscr{F}_c(f; x)$	Fourier cosine transform of function f
$\mathscr{L}_p[f]$	Laplace transform of function f
$\mathscr{M}_p(f)$	Mellin transform of function f
$\Re(z), \Im(z)$	Real, imaginary part of a complex number z
$\{x\}, [x]$	Fractional, largest integer part of some real x
i	imaginary unit, $\mathrm{i}^2 = -1$
$\square$	Ending Proof
$\blacksquare$	Ending Remark

Chapter 1
Introduction and Preliminaries

Abstract We begin with a brief outline of special functions and methods, which will be needed in the next chapters. We recall here briefly the Gamma, Beta, Digamma functions, Pochhammer symbol, Bernoulli polynomials and numbers, Bessel, modified Bessel, generalized hypergeometric, Fox–Wright generalized hypergeometric, Hurwitz–Lerch Zeta functions, the Euler–Maclaurin summation formula together with Dirichlet series and Cahen's formula, Mathieu series, Bessel and Struve differential equations, Fourier-Bessel and Dini series of Bessel functions and fractional differintegral.

Special functions have their roots back to the eighteenth century when it became clear that the existing elementary functions are not sufficient to describe a number of unsolved problems in various branches of mathematics and physics. Functions, appropriate to describe the new results were generally presented in the form of infinite series, integrals, or as solutions of differential equations and some of them, which appeared more frequently, were named, for example, Gamma, Beta function, etc. One of the first issues about special functions is the set of four books, published between 1893 and 1902 by Tannery and Molk [302–305]. Among others, his contribution to the theory of special functions gave F.W. Bessel who systematically investigated the Bessel functions in 1824 [72] already considered in eighteenth century by Bernoulli, Euler, Lagrange, Fourier and others in theirs researches in mechanics, astronomy and the conduction of heat. The monumental monograph *A treatise on the theory of Bessel functions*, written by Watson in 1922 [333] contains a wide range of results about Bessel functions. Nowadays, Bessel functions family counts a huge spectrum of functions: Bessel functions of the first and second kind, modified Bessel functions of the first and second kind, Struve functions, modified Struve functions, von Lommel functions *et coetera* and there are numerous literature dealing with some properties and unfailing applications of such functions. Also, an interesting and important topic in the theory of Bessel functions are functional series of mathematical physics, having great importance in engineering and technique (compare [18]), the Fourier–Bessel family of infinite series consisting of Neumann-, Kapteyn-, Schlömilch and Dini series with members containing Bessel functions of

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the first kind or some other functions which are a member of the Bessel functions family and/or belong to the hypergeometric representation. The main difference between those series is that the Neumann series constituting terms contain the summation index in the order of the Bessel function; the Kapteyn series terms have index in both order and argument, while the Schlömilch series have argument summation indices. Precisely:

- *Neumann series*, i.e. the series in which the order of the Bessel function of the first kind contains the current index of summation

$$\mathfrak{N}_\mu(z) := \sum_{n\geq 1} \alpha_n J_{\mu+n}(z), \qquad z \in \mathbb{C}$$

  are named after the German mathematician Carl Gottfried Neumann [209], who in his book *Theorie der Besselschen Funktionen*, in 1867, studied only their special cases, namely those of integer order. A few years later, in 1887, Leopold Bernhard Gegenbauer [88, 89] expanded these series, having order the whole real line.
- *Kapteyn series* are the series where the order of the Bessel function, and also the argument contains index of summation:

$$\mathfrak{K}_\mu(z) := \sum_{n\geq 1} \alpha_n J_n\left((\mu+n)z\right), \qquad z \in \mathbb{C}\,.$$

  Such series were introduced in 1893, by Willem Kapteyn [145], in his article *Recherches sur les functions de Fourier-Bessel*. These series have great applications in problems of mathematical physics. For example, a solution of famous Kepler's equation can be explicitly expressed by Kapteyn series. Their application can be found in problems of pulsar physics, electromagnetic radiation, etc. In 1906 Kapteyn [146] proved that every analytic function can be developed in such series.
- *Schlömilch series* appear when the argument contains the current index of summation, i.e. the series of the form:

$$\mathfrak{S}_\mu(z) := \sum_{n\geq 1} \alpha_n J_\mu\left((\mu+n)z\right), \quad z \in \mathbb{C}\,.$$

  Oscar Xavier Schlömilch [279] was the first who defined that series, in 1857, in the article *Über die Bessel'schen Function*, but he looked only at cases when the series contains of Bessel functions of the first kind of order $\mu = 0, 1$. Their use is so widespread in the field of physics, such as the use of Kapteyn series. Rayleigh [266] in 1911 pointed out that in the case $\mu = 0$ these series are useful in the study of periodic transverse vibrations of two-dimensional membranes. Generalized Schlömilch series appeared in the Nielsen's memoirs [212–218] from 1899, 1900 and 1901. Filon [83] in 1906 first studied the possibility of development of arbitrary function in generalized Schlömilch series.

There are also Kapteyn series of the second type, which have been studied, in details, by Nielsen [216, 217], in 1901, and that series consist of the product of two Bessel functions of the first kind, of different orders.

Neumann series are widely used. Especially interesting are the Neumann series of the zero-order Bessel function, i.e. series $\mathfrak{N}_0$, which appears as a relevant technical tool to solve the problem of infinite dielectric wedge through the Kontorovich–Lebedev transformation. They also occur in the description of internal gravity waves in Bussinesq fluid, and in defining the properties of diffracted light beams. Wilkins [334] discussed the question of existence of an integral representation for a special Neumann series; Maximon [188] in 1956 represented a simple Neumann series $\mathfrak{N}_\mu$ appearing in the literature in connection with physical problems and Luke [178] studied, in 1962, integral representation of Neumann series for $\mu = 0$.

Finally, it is worth to mention that regarding the orthogonality property of Bessel and alike functions the advanced subject of interest can be a study of orthogonality of Neumann, Kapteyn, Schlömilch and Dini series. In the orthogonal series' respect we draw the attention to the classical 1935 monograph by Kaczmarz and Steinhaus [144] and suggest to consult Alexits's book [3] concerning convergence and summability questions. Also the main source for everywhere existing and used analytic inequalities can serve among others Mitrinović's celebrated inequality collection [202].

1.1 The Gamma Function

The Gamma function has caught the interest of some of the most prominent mathematicians of all times. Its history, notably documented by Philip J. Davis in an article that won him the Chauvenet Prize, in 1963, reflects many of the major developments within mathematics since the eighteenth century. In his article [59] Davis wrote:

"Each generation has found something of interest to say about the Gamma function. Perhaps the next generation will also".

In this section, we recall some properties of the Gamma function and introduce some other functions which can be expressed in terms of the Gamma function, namely Psi and Beta function and also the Pochhammer symbol.

The Gamma function is defined by a definite integral due to Leonhard Euler

$$\Gamma(z) = \int_0^\infty \mathrm{e}^{-t}\, t^{z-1} \mathrm{d}t\,, \qquad \Re(z) > 0\,. \tag{1.1}$$

The notation $\Gamma(z)$ is due to French mathematician Adrien–Marie Legendre. Using integration by parts, from (1.1) we easily get [1, p. 256, Eq. 6.1.15]

$$z\,\Gamma(z) = \Gamma(z+1)\,, \qquad \Re(z) > 0\,. \tag{1.2}$$

That relation is called the *recurrence formula* or *recurrence relation* of the Gamma function. For $z = n \in \mathbb{N}$, from (1.2) it follows that [1, p. 255, Eq. 6.1.6]

$$\Gamma(n) = (n-1)!\,.$$

The recurrence relation is not the only functional equation satisfied by the Γ. Another important property is the *Euler's reflection formula* [1, p. 256, Eq. 6.1.17]

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)},$$

which gives relation between the Gamma function of positive and negative numbers. For $z = \frac{1}{2}$, from the previous equation, it follows that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

In what follows, we would also need *Legendre's duplication formula* [1, p. 256, Eq. 6.1.18]

$$\Gamma(2z) = \frac{2^{2z-1}}{\sqrt{\pi}}\,\Gamma(z)\,\Gamma\left(z+\tfrac{1}{2}\right). \tag{1.3}$$

In examining the convergence conditions of corresponding series of Bessel functions of the first kind, we would need the formula for asymptotic behavior of the Gamma function (in other words a Stirling's formula) [1, p. 257, Eq. 6.1.37]

$$\Gamma(z) = \sqrt{2\pi}\, z^{z-\frac{1}{2}} \mathrm{e}^{-z}\left(1 + \mathscr{O}(z^{-1})\right), \qquad |\arg z| < \pi,\ |z| \to \infty, \tag{1.4}$$

which usually one writes

$$\Gamma(z) \sim \sqrt{2\pi}\, \mathrm{e}^{-z} z^{z-\frac{1}{2}}, \qquad |z| \to \infty\,.$$

Gamma function also has the following properties (see [263, p. 9]):

- $\Gamma(z)$ is analytic except at nonpositive integers, and when $z = \infty$;
- $\Gamma(z)$ has a simple pole at each nonpositive integer, $z \in \mathbb{Z}_0^-$;
- $\Gamma(z)$ has an essential singularity at $z = \infty$, a point of condensation of poles;
- $\Gamma(z)$ is never zero, because $1/\Gamma(z)$ has no poles.

1.1.1 Psi (or Digamma) Function

The Psi (or Digamma) function $\psi(z)$ is defined as the logarithmic derivative of the Gamma function:

$$\psi(z) := \frac{\mathrm{d}}{\mathrm{d}z}\{\log\,\Gamma(z)\} = \frac{\Gamma'(z)}{\Gamma(z)} \qquad \text{or} \qquad \log\,\Gamma(z) = \int_1^z \psi(t)\,\mathrm{d}t\,.$$

We express $\psi(z)$ [1, Eq. (6.3.16)] as follows (see also [287, p. 14, Eq. 1.2(3)]:

$$\psi(z) = \sum_{k\geq 1}\left(\frac{1}{k} - \frac{1}{z+k-1}\right) - \boldsymbol{C}, \qquad z \in \mathbb{C} \setminus \mathbb{Z}_0^-,$$

where $\boldsymbol{C}$ denotes the celebrated Euler-Mascheroni constant given by

$$\boldsymbol{C} := \lim_{n\to\infty} (H_n - \log n) \approx 0.5772,$$

where H_n are called the harmonic numbers defined by

$$H_n = \sum_{k=1}^{n} \frac{1}{k}, \qquad n \in \mathbb{N}.$$

Finally, let us remark that the Digamma function $\psi(z)$ increases on its entire range and possesses the unique positive nil $\alpha_0 = \psi^{-1}(0) \approx 1.4616$. One of the useful properties of the Digamma function is that [1, p. 258, Eq. 6.3.5]

$$\psi(z+1) = \frac{1}{z} + \psi(z), \qquad z > 0.$$

1.1.2 The Beta Function

The Beta function, also called the *Euler integral of the first kind*, is a special function defined by Abramowitz and Stegun [1, p. 258, Eq. 6.2.1]

$$\mathrm{B}(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,\mathrm{d}t, \qquad \min\{\Re(x), \Re(y)\} > 0.$$

The Beta function is intimately related to the Gamma function, which is described in [263, p. 18]:

$$\mathrm{B}(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \qquad \Re(x), \Re(y) > 0. \tag{1.5}$$

Accordingly, by (1.5) it follows that Beta function is invariant with respect to parameter permutation, meaning that $\mathrm{B}(x, y) = \mathrm{B}(y, x)$.

1.1.3 The Pochhammer Symbol

The Pochhammer symbol (or the *shifted factorial*), introduced by Leo August Pochhammer, is defined, in terms of Euler's Gamma function, by

$$(\lambda)_\mu := \frac{\Gamma(\lambda+\mu)}{\Gamma(\lambda)} = \begin{cases} 1, & \text{if } \mu = 0;\ \lambda \in \mathbb{C}\setminus\{0\} \\ \lambda(\lambda+1)\cdots(\lambda+n-1), & \text{if } \mu = n \in \mathbb{N};\ \lambda \in \mathbb{C} \end{cases},$$

it being understood *conventionally* that $(0)_0 := 1$.

The Pochhammer symbol also satisfies

$$(-\lambda)_\mu = (-1)^\mu\,(\lambda-\mu+1)_\mu\,, \qquad \mu \in \mathbb{N}_0\,.$$

Excellent source for these Gamma-type functions is the monograph by Andrews et al. [7].

1.2 Bernoulli Polynomials and Numbers

The Bernoulli polynomials $B_m(x)$ [1, p. 804] satisfy [58, p. 899, Eq. (3.4b)]

$$B_m(x) = \sum_{k=0}^{m} B_k \cdot \binom{m}{k} x^{m-k}\,, \tag{1.6}$$

being $B_m = B_m(0)$, $m \in \mathbb{N}_0$ the Bernoulli numbers for which hold [93, p. 1041]

$$B_0 = 1,\ B_1 = -\tfrac{1}{2},\ B_{2m+1} = 0,\ B_{2m} = \frac{(-1)^{m-1}\Gamma(2m+1)\zeta(2m)}{2^{2m-1}\pi^{2m}}, \qquad m \in \mathbb{N}, \tag{1.7}$$

where ζ stands for the Riemann's Zeta function (see e.g. [207]).

In turn, the Bernoulli polynomials of odd degree possess definition [1, p. 805]

$$B_{2m-1}(x) = (-1)^m \frac{2(2m-1)!}{(2\pi)^{2m-1}} \sum_{n\ge 1} \frac{\sin(2n\pi x)}{n^{2m-1}}, \qquad m \in \mathbb{N} \tag{1.8}$$

where $x \in (0,1)$ for $m = 1$ and $x \in [0,1]$ if $m \in \mathbb{N}_2 = \{2,3,4,\dots\}$ and for the Bernoulli polynomials of even degree there hold

$$B_{2m}(x) = (-1)^{m-1} \frac{2(2m)!}{(2\pi)^{2m}} \sum_{n\ge 1} \frac{\cos(2n\pi x)}{n^{2m}}, \qquad m \in \mathbb{N},\ x \in [0,1]\,.$$

1.3 Euler-Maclaurin Summation Formula

Euler–Maclaurin formula provides a powerful connection between integrals and sums. It can be used to approximate integrals by finite sums, or conversely to evaluate finite sums using integrals and the machinery of calculus. The formula was discovered independently by Leonhard Euler and Colin Maclaurin around 1735. Euler needed it to compute slowly converging infinite series, while Maclaurin used it to calculate integrals. Their famous summation formula of the first degree is

$$\sum_{n=k}^{\ell} a_n = \int_k^{\ell} a(x)\mathrm{d}x + \frac{1}{2}(a_\ell + a_k) + \int_k^{\ell} a'(x)B_1(x)\,\mathrm{d}x,$$

where $B_1(x) = \{x\} - \frac{1}{2}$ is the first degree Bernoulli polynomial.

It generally holds [7, p. 619, Theorem D.2.1]

$$\sum_{n=k}^{\ell} f(n) = \int_k^{\ell} f(x)\mathrm{d}x + \tfrac{1}{2}\big(f(k) + f(\ell)\big) + \sum_{j=1}^{m} \frac{B_{2j}}{(2j)!}\left(f^{(2j-1)}(\ell) - f^{(2j-1)}(k)\right)$$
$$- \int_k^{\ell} \frac{B_{2m}(x)}{(2m)!} f^{(2m)}(x)\,\mathrm{d}x\,, \qquad m \in \mathbb{N},$$

where $B_p(x) = (x+B)^p$, $0 \le x < 1$ represents Bernoulli polynomial of order $p \in \mathbb{N}$, while B_k are appropriate Bernoulli numbers. On $[\ell, \ell+1)$, $\ell \in \mathbb{N}$, $B_p(x)$ are periodic with period 1.

Summation formulae, of the first kind ($p = 1$) we will use in condensed form, under the condition $a \in C^1[k, l]$, $k, l \in \mathbb{Z}$, $k < l$:

$$\sum_{n=k+1}^{\ell} a_n = \int_k^{\ell} (a(x) + \{x\}a'(x))\,\mathrm{d}x \equiv \int_k^{\ell} \mathfrak{d}_x a(x)\,\mathrm{d}x\,, \tag{1.9}$$

where

$$\mathfrak{d}_x := 1 + \{x\}\frac{\partial}{\partial x}\,,$$

see [249, 252].

The articles [239–241] contain certain special cases of (1.9) specifying among others $a_n = 1$.

The multiple Euler–Maclaurin summation formulae are used e.g. in [70, 250] and discussed in detail in [129, 205, 241].

1.4 Dirichlet Series and Cahen's Formula

One of our main mathematical tools is the series

$$\mathscr{D}_{\boldsymbol{a}}(s) := \sum_{n\geq 1} a_n\, \mathrm{e}^{-\lambda_n s}\,, \qquad s > 0\,, \tag{1.10}$$

where

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n \to \infty \qquad \text{as} \qquad n \to \infty\,.$$

This is called *Dirichlet series on the λ_n-type*. For $\lambda_n = n$, (1.10) becomes power series

$$\mathscr{D}_{\boldsymbol{a}}(s) := \sum_{n\geq 1} a_n\, \mathrm{e}^{-ns}\,, \qquad s > 0$$

and for $\lambda_n = \ln n$, we have series of the form

$$\mathscr{D}_{\boldsymbol{a}}(s) := \sum_{n\geq 1} a_n\, n^{-s}\,, \qquad s > 0\,,$$

which is called *ordinary Dirichlet series*.

In this monograph we mostly deal with series of the form (1.10), where s is real variable. We also need a variant of closed integral form representation of Dirichlet series, which is derived below, following mainly [104], [147, **C. §V**]. The heart of the matter is the Stieltjes integral formula

$$\int_a^b f(x)\, \mathrm{d}\mathscr{A}_s(x) = \sum_{n\geq 1} f(\lambda_n)(\mathscr{A}_s(\lambda_n+) - \mathscr{A}_s(\lambda_n-)), \tag{1.11}$$

such that is valid for $\mathscr{A}_s$-integrable f, where the *step function*

$$\mathscr{A}_s(x) := \sum_{n\colon\, \lambda_n \leq x} (\mathscr{A}_s(\lambda_n+) - \mathscr{A}_s(\lambda_n-)) \tag{1.12}$$

possesses the discontinuity set (λ_n) which forms a monotone increasing sequence of positive reals diverging to the infinity. Assuming that $\lambda(x)$ is monotone increasing positive function such that runs to the infinity with growing x and it is $(\lambda_n)_{n\geq 1} = \lambda(x)\big|_{\mathbb{N}}$, we deduce that λ is invertible with the unique inverse λ^{-1}. Now, putting $a_n =: \mathscr{A}_s(\lambda_n+) - \mathscr{A}_s(\lambda_n-)$ into (1.12) we get

$$\mathscr{A}_s(x) = \sum_{n\colon\, \lambda_n \leq x} a_n = \sum_{n=1}^{[\lambda^{-1}(x)]} a_n.$$

Here $\mathscr{A}_s(x)$ is the function such that has jump of magnitude a_n at λ_n, $n \in \mathbb{N}$. So, taking $f(x) = \mathrm{e}^{-sx}$ and having in mind that $[a, b] = [0, x]$, by (1.11) we deduce

$$\sum_{n:\ \lambda_n \le x} a_n\,\mathrm{e}^{-\lambda_n s} = \int_0^x \mathrm{e}^{-s\,t}\,\mathrm{d}\mathscr{A}_s(t). \tag{1.13}$$

Letting $x \to \infty$ in (1.13) we infer an integral which is equiconvergent with $\mathscr{D}_{\mathbf{a}}(s)$, so

$$\sum_{n\ge 1} a_n\,\mathrm{e}^{-\lambda_n s} = \int_0^\infty \mathrm{e}^{-s\,t}\,\mathrm{d}\mathscr{A}_s(t)\,, \qquad s > 0\,. \tag{1.14}$$

Now, the integration by parts results in a Laplace integral instead of the Laplace–Stieltjes integral (1.14). Indeed, as e^{-sx} decreases in x being s positive, taking $a(0) = 0$, the convergence of the Laplace–Stieltjes integral (1.14) ensures the validity of the famous Cahen's formula [47, p. 97], [104]

$$\mathscr{D}_{\boldsymbol{a}}(s) = s\int_0^\infty \mathrm{e}^{-s\,t}\mathscr{A}_s(t)\,\mathrm{d}t\,, \qquad s > 0\,. \tag{1.15}$$

However, the so-called counting sum

$$\mathscr{A}_{\boldsymbol{a}}(t) = \sum_{n:\lambda_n \le t} a_n$$

we find by the Euler–Maclaurin summation formula (see [239, 240, 249]), assuming that $\mathbf{a} := a(x)\big|_{\mathbb{N}}$, $a \in C^1[0, \infty)$ we sum up $\mathscr{A}_s(t)$ completing the desired closed form integral representation of Dirichlet series $\mathscr{D}_{\mathbf{a}}(s)$ without any sums. Namely

$$\mathscr{A}_{\boldsymbol{a}}(t) = \sum_{n=1}^{[\lambda^{-1}(t)]} a_n = \int_0^{[\lambda^{-1}(t)]} \mathfrak{d}_u a(u)\,\mathrm{d}u\,, \tag{1.16}$$

since by assumption λ is monotone with an unique inverse λ^{-1} being $\lambda|_{\mathbb{N}} = (\lambda_n)$.

1.5 Mathieu $(\boldsymbol{a}, \boldsymbol{\lambda})$-Series

The series of the form

$$S(r) = \sum_{n\ge 1} \frac{2n}{(n^2 + r^2)^2}, \qquad r > 0$$

is known in literature as Mathieu series. Émile Leonard Mathieu was the first who investigated such series in 1890 in his book [187]. There is a wide range of various generalizations of the Mathieu series for which integral representations [53, 98, 248, 251, 252, 260], related summations results and bilateral bounding inequalities are obtained (see also [245]); certain new estimates upon $S(r)$ are given also in [204], see also the related references therein.

The so-called Mathieu $(\boldsymbol{a}, \boldsymbol{\lambda})$-series

$$\mathfrak{M}_s(\boldsymbol{a}, \boldsymbol{\lambda}; r) = \sum_{n\geq 0} \frac{a_n}{(\lambda_n + r)^s}, \qquad r, s > 0, \tag{1.17}$$

has been introduced by Pogány [239], giving an exhaustive answer to an Open Problem posed by Qi [259], deriving closed form integral representation and bilateral bounding inequalities for $\mathfrak{M}_s(\boldsymbol{a}, \boldsymbol{\lambda}; r)$, generalizing at the same time some earlier results by Cerone and Lenard [53], Qi [259], Srivastava and Tomovski [293] and others.

The mentioned Pogány's integral representation formula for Mathieu $(\boldsymbol{a}, \boldsymbol{\lambda})$-series reads [240, Theorem 1]:

$$\mathfrak{M}_s(\boldsymbol{a}, \boldsymbol{\lambda}; r) = \frac{a_0}{r^s} + s \int_{\lambda_1}^{\infty} \int_0^{[\lambda^{-1}(x)]} \frac{a(u) + a'(u)\{u\}}{(r+x)^{s+1}} \, \mathrm{d}x \, \mathrm{d}u,$$

where $a \in C^1[0,\infty)$ and $a\big|_{\mathbb{N}_0} \equiv \boldsymbol{a}$, λ^{-1} stands for the inverse of λ and the series (1.17) converges. The series (1.17) is assumed to be convergent and the sequences $\boldsymbol{a} := (a_n)_{n\geq 0}$, $\boldsymbol{\lambda} := (\lambda_n)_{n\geq 0}$ are positive. Following the convention that (λ_n) is monotone increasing divergent, we have

$$\boldsymbol{\lambda}: \quad 0 \leq \lambda_0 < \lambda_1 < \cdots < \lambda_n \xrightarrow[n\to\infty]{} \infty.$$

1.6 Bessel Differential Equation

The Bessel differential equation is the linear second-order ordinary differential equation given by Olver et al. [227, §10.2.(i)]

$$x^2 y'' + x y' + (x^2 - \nu^2) y = 0, \qquad \nu \in \mathbb{C}. \tag{1.18}$$

The solutions to this equation define the Bessel function of the first kind J_ν and the Bessel function of the second kind Y_ν. The equation has a regular singularity at zero, and an irregular singularity at infinity.

The function $J_\nu(x)$ is defined by the equation

$$J_\nu(x) = \sum_{m\geq 0} \frac{(-1)^m}{m!\,\Gamma(m+\nu+1)} \left(\frac{x}{2}\right)^{2m+\nu}. \tag{1.19}$$

For $\nu \notin \mathbb{Z}$, functions $J_\nu(x)$ and $J_{-\nu}(x)$ are linearly independent, thus the solutions of the differential equation (1.18) are independent, while for $\nu \in \mathbb{Z}$ it holds

$$J_{-\nu}(x) = (-1)^\nu J_\nu(x).$$

Bessel functions of first kind, which were introduced by Daniel Bernoulli in his article [36] represent the general solution of the homogeneous Bessel differential equation of the second degree. Alexandre S. Chessin [51, 52] was a first who gave an explicit solution of Bessel differential equation with general nonhomogeneous part, in 1902.

1.7 Bounds Upon $J_\nu(x)$

We are also interested in estimates for Bessel function of the first kind. Landau [167] gave the following bounds for Bessel function $J_\nu(x)$:

$$|J_\nu(x)| \le b_L \nu^{-\frac{1}{3}}, \qquad b_L := \sqrt[3]{2} \sup_{x \in \mathbb{R}_+} \big(\mathrm{Ai}(x)\big) \tag{1.20}$$

and

$$|J_\nu(x)| \le c_L |x|^{-\frac{1}{3}}, \qquad c_L := \sup_{x \in \mathbb{R}_+} \sqrt[3]{x}\big(J_0(x)\big), \tag{1.21}$$

where $\mathrm{Ai}(x)$ stands for the familiar Airy function, which is solution of differential equation

$$y'' - xy = 0, \qquad y = \mathrm{Ai}(x)$$

and can be expressed as

$$\mathrm{Ai}(x) := \frac{\pi}{3}\sqrt{\frac{x}{3}}\left\{ J_{-\frac{1}{3}}\left(2\left(\frac{x}{3}\right)^{\frac{3}{2}}\right) + J_{\frac{1}{3}}\left(2\left(\frac{x}{3}\right)^{\frac{3}{2}}\right)\right\} .$$

Olenko [226] also gave sharp upper bound for Bessel function:

$$\sup_{x \ge 0} \sqrt{x}\,|J_\nu(x)| \le b_L \sqrt{\nu^{\frac{1}{3}} + \frac{\alpha_1}{\nu^{\frac{1}{3}}} + \frac{3\alpha_1^2}{10\nu}} =: d_O\,, \qquad \nu > 0, \tag{1.22}$$

where α_1 is the smallest positive zero of Airy's function $\mathrm{Ai}(x)$, and b_L is the first Landau's constant.

There is also Krasikov's bound [159]

$$J_\nu^2(x) \le \frac{4(4x^2-(2\nu+1)(2\nu+5))}{\pi((4x^2-\mu)^{\frac{3}{2}}-\mu)}\,, \qquad x > \sqrt{\mu+\mu^{\frac{2}{3}}}\,,\ \nu > -\tfrac{1}{2}, \tag{1.23}$$

where $\mu = (2\nu+1)(2\nu+3)$. This bound is sharp in the sense that

$$J_\nu^2(x) \ge \frac{4(4x^2-(2\nu+1)(2\nu+5))}{\pi((4x^2-\mu)^{\frac{3}{2}}-\mu)}$$

in all points between two consecutive zeros of Bessel function $J_\nu(x)$ [159, Theorem 2]. Krasikov also pointed out that the estimates (1.20) and (1.21) are sharp only for values that are in the neighborhood of the smallest positive zero $j_{\nu,1}$ of the Bessel function $J_\nu(x)$, while his estimate (1.23) gives sharp upper bound in whole area.

In turn, Krasikov's recently published a set of more precise and simpler bounds [160, 161]. Precisely, for $\nu \ge \frac{1}{2}$ and all $t \ge 0$ there holds [160, p. 210, Theorem 3]

$$\left|t^2 - \left|\nu^2-\tfrac{1}{4}\right|\right|^{\frac{1}{4}} |J_\nu(t)| \le \sqrt{\frac{2}{\pi}}, \tag{1.24}$$

where the right-hand-side constant is sharp. Next, his result [160, p. 210, Theorem 4] imply

$$|J_\nu(t)| \le \sqrt{\frac{2}{\pi t}} + \rho\, c \left|\nu^2-\tfrac{1}{4}\right| t^{-\frac{3}{2}}\,, \qquad t > 0,\ |\rho| < 1\,, \tag{1.25}$$

where

$$c = \begin{cases} \left(\dfrac{2}{\pi}\right)^{\frac{3}{2}}, & x \ge 0,\ |\nu| \le \frac{1}{2} \\[2ex] \dfrac{4}{5}, & 0 < x < \sqrt{|\nu^2-\frac{1}{4}|},\ \nu > \frac{1}{2} \\[2ex] \dfrac{2}{\pi}, & x \ge \sqrt{|\nu^2-\frac{1}{4}|},\ \nu > \frac{1}{2} \end{cases} .$$

Here c cannot be less then $\sqrt{\frac{\pi}{2}}$. For another kind bounds upon $J_\nu(t)$ consult [160, Theorems 2, 5, 6] and [161, Theorems 2, 4].

Srivastava and Pogány [292, p. 199, Eq. (19)] proposed the following hybrid estimator[1]

$$|J_\nu(x)| \le \mathfrak{W}_\nu(x) := \frac{d_O}{\sqrt{x}}\, \chi_{(0,A_\lambda]}(x) + \sqrt{\mathfrak{K}_\nu(x)}\left(1-\chi_{(0,A_\lambda]}(x)\right), \tag{1.26}$$

[1] Here, and in what follows $\chi_A(x)$ denotes the *indicator function of the set* A which equals 1, when $x \in A$, and zero else.

where

$$\mathfrak{K}_\nu(x) := \frac{4(4x^2 - (2\nu + 1)(2\nu + 5))}{\pi((4x^2 - \mu)^{\frac{3}{2}} - \mu)},$$

while

$$A_\lambda = \tfrac{1}{2}\left(\lambda + (\lambda + 1)^{\frac{2}{3}}\right).$$

Here we shall mainly use Landau's bounds, because of their simplicity. Derived results one can expand using hybrid estimator $\mathfrak{W}_\nu$ as well.

However, combining (1.24), (1.25) in $\mathfrak{W}_\nu(t)$ replacing Olenko's result and/or $\mathfrak{K}_\nu(t)$ in (1.26), we could define a set of further bounding functions for $|J_\nu|$.

Further, exponential bounding inequalities for $J_\nu(x)$ are published by Pogány [243] and Sitnik [283].

1.8 Bessel Functions Family

The Bessel and the modified Bessel function of the first kind J_ν, I_ν, Bessel and modified Bessel function of the second kind Y_ν, K_ν and the Struve and modified Struve function $\mathbf{H}_\nu, \mathbf{L}_\nu$ all of the order ν possess power series representations of the form [333], respectively:

$$J_\nu(z) = \sum_{n\geq 0} \frac{(-1)^n \left(\frac{z}{2}\right)^{2n+\nu}}{\Gamma(n+\nu+1)\, n!}, \qquad I_\nu(z) = \sum_{n\geq 0} \frac{\left(\frac{z}{2}\right)^{2n+\nu}}{\Gamma(n+\nu+1)\, n!},$$

$$Y_\nu(z) = \begin{cases} \cot(\pi\nu)\, J_\nu(z) - \csc(\pi\nu)\, J_{-\nu}(z), & \nu \notin \mathbb{Z} \\ \lim_{\mu\to\nu} Y_\mu(z), & \nu \in \mathbb{Z} \end{cases},$$

$$K_\nu(z) = \begin{cases} \frac{\pi}{2}\csc(\pi\nu)\ \left(I_{-\nu}(z) - I_\nu(z)\right), & \nu \notin \mathbb{Z} \\ \lim_{\mu\to\nu} K_\mu(z), & \nu \in \mathbb{Z} \end{cases},$$

$$\mathbf{H}_\nu(z) = \sum_{n\geq 0} \frac{(-1)^n \left(\frac{z}{2}\right)^{2n+\nu+1}}{\Gamma\left(n+\frac{3}{2}\right)\Gamma\left(n+\nu+\frac{3}{2}\right)},$$

$$\mathbf{L}_\nu(z) = \sum_{n\geq 0} \frac{\left(\frac{z}{2}\right)^{2n+\nu+1}}{\Gamma\left(n+\frac{3}{2}\right)\Gamma\left(n+\nu+\frac{3}{2}\right)},$$

where $\nu \in \mathbb{R}$ and $z \in \mathbb{C}$. Bessel functions are well-known however, the function $\mathbf{H}_\nu$ was introduced in [299] as the series solution of the non-homogeneous second order Bessel type differential equation (which carries his name). Applications of Struve functions are manyfold and an exhaustive overview is given in [18] accompanied with the long list of devoted references therein.

The family of Bessel functions also contains the spherical Bessel function of the first kind of order ν defined by the formula [1, p. 437] (also see [15, p. 9, Eq. (1.9)])

$$j_\nu(z) = \sqrt{\frac{\pi}{2z}}\, J_{\nu+\frac{1}{2}}(z)\,, \qquad z \in \mathbb{C}\,,$$

and its modified variant [1, p. 443], [15, p. 9, Eq. (1.11)]

$$i_\nu(z) = \sqrt{\frac{\pi}{2z}}\, I_{\nu+\frac{1}{2}}(z)\,, \qquad z \in \mathbb{C}\,.$$

The ultraspherical Bessel function (initiated by Ashbaugh and Benguria [10, p. 562] and studied by Lorch and Szego [176, p. 549]) and the companion modified ultraspherical Bessel function of the first kind read as follows

$$\sqrt{\frac{\pi}{2}}\, z^{-\nu+1} J_{\nu+\ell-1}(z), \qquad \sqrt{\frac{\pi}{2}}\, z^{-\nu+1} I_{\nu+\ell-1}(z)\,, \qquad \nu+\ell>0,\, z \in \mathbb{C}\,.$$

For $\nu = \frac{3}{2}$, $\ell \in \mathbb{N}_0$ the ultraspherical functions reduce to the classical spherical Bessel functions.

The generalized Bessel function $\omega_\nu(x)$ of the order ν, introduced by Baricz [15, p. 10, Eq. (1.15)], which generalizes and unifies all the classical Bessel, modified Bessel, spherical Bessel, modified spherical Bessel, ultraspherical Bessel and modified ultraspherical Bessel functions, reads as follows

$$\omega_\nu(z) = \sum_{n\geq 0} \frac{(-c)^n}{n!\,\Gamma(\nu+n+\frac{b+1}{2})} \left(\frac{z}{2}\right)^{2n+\nu}\,, \qquad \nu, b, c, z \in \mathbb{C}\,.$$

It is worth to mention the Delerue hyper-Bessel function [61, 65, 153]

$$J^{(m)}_{\nu_1,\cdots,\nu_m}(z) = \left(\frac{z}{m+1}\right)^{\sum\limits_{j=1}^m \nu_j} \sum_{n\geq 0} \frac{(-1)^n \left(\frac{z}{m+1}\right)^{n(m+1)}}{\Gamma(n+\nu_1+1)\cdots\Gamma(n+\nu_m+1)\,n!}\,,$$

and its modified variant

$$I^{(m)}_{\nu_1,\cdots,\nu_m}(z) = \left(\frac{z}{m+1}\right)^{\sum\limits_{j=1}^m \nu_j} \sum_{n\geq 0} \frac{\left(\frac{z}{m+1}\right)^{n(m+1)}}{\Gamma(n+\nu_1+1)\cdots\Gamma(n+\nu_m+1)\,n!}\,,$$

which are multi-index analogues of the Bessel J and the modified Bessel I. Here $z, \nu_k \in \mathbb{C}$ and $\Re(\nu_k) > -1$, $k = \overline{1, m}$. For $m = 1$ we arrive at the classical Bessel and modified Bessel functions, while for $m = 2$ we deduce the so-called Bessel–Clifford functions [150, 151]

$$C_{\nu,\mu}(z) = z^{-\frac{\nu+\mu}{3}} J^{(2)}_{\nu,\mu}(3\sqrt[3]{z})\,.$$

The Wright generalized Bessel function [338] (discovered also by Galué [86], and misnamed as Maitland or Bessel–Maitland function) [12, p. 184, Eq. (5)] is given by

$${}_hJ_\nu(z) = \sum_{n\geq 0} \frac{(-1)^n\, \left(\frac{z}{2}\right)^{2n+\nu}}{n!\, \Gamma(\nu + hn + 1)}, \qquad z \in \mathbb{C}\,;$$

its modified variant was introduced by Baricz [12, p. 184, Eq. (6)]

$${}_hI_\nu(z) = \sum_{n\geq 0} \frac{\left(\frac{z}{2}\right)^{2n+\nu}}{n!\, \Gamma(\nu + hn + 1)}, \qquad z \in \mathbb{C},$$

being $h \in \mathbb{N}$ in both cases. We point out that ${}_1J_\nu \equiv J_\nu$ and so does ${}_1I_\nu \equiv I_\nu$. Here we mention a h-fold definite integral expression form of ${}_hJ_\nu$ on the rectangle $[0, 1]^h$, appearing in [86, p. 398], which is not of some substantial help in our considerations. The case when h is not a non-negative integer both ${}_hJ_\nu$, ${}_hI_\nu$ are Fox–Wright generalized hypergeometric (or Wright) functions, see e.g. [297].

Now, we expose the freshly obtained result that Wright generalized Bessel function is in fact a weighted variant of Delerue hyper-Bessel function. Indeed, Jankov Maširević, Parmar and Pogány have shown that [141, Theorem 6.1]

$$\begin{aligned}
{}_hJ_\nu(z) &= \frac{2^{\frac{h+1}{2}-\nu}\pi^{\frac{h-1}{2}}}{h^{\frac{1+\nu^2-(h+\nu)^2}{2(h+1)}}}\, z^{\frac{h-1}{h+1}\nu-1} \cdot J^{(h)}_{\frac{\nu+1}{h},\cdots,\frac{\nu}{h}+1}(\zeta) \\
{}_hI_\nu(z) &= \frac{2^{\frac{h+1}{2}-\nu}\pi^{\frac{h-1}{2}}}{h^{\frac{1+\nu^2-(h+\nu)^2}{2(h+1)}}}\, z^{\frac{h-1}{h+1}\nu-1} \cdot I^{(h)}_{\frac{\nu+1}{h},\cdots,\frac{\nu}{h}+1}(\zeta)\,,
\end{aligned}$$

where

$$\zeta = (1 + h)\left(\frac{z^2}{4h^h}\right)^{\frac{1}{h+1}}$$

for all $\Re(\nu) + 1 > 0$, $h \in \mathbb{N}$ and for all $z \in \mathbb{C}$.

1.9 Struve Differential Equation

The Struve function is related to the non-homogeneous Bessel type ordinary differential equation of special type called *Struve differential equation* [1, p. 496, Eq. 12.1.1]

$$z^2\, y'' + z\, y' + (z^2 - \nu^2)\, y = \frac{4}{\sqrt{\pi}\ \Gamma(\nu + \frac{1}{2})} \left(\frac{z}{2}\right)^{\nu+1},$$

whose general solution turns out to be

$$y = C_1\, J_\nu(z) + C_2\, Y_\nu(z) + \mathbf{H}_\nu(z)\,,$$

where C_1, C_2 are the integration constants and $z^{-\nu}\, \mathbf{H}_\nu(z)$ is an entire function of z. Similarly, the *modified Struve differential equation* reads

$$z^2\, y'' + z\, y' - (z^2 + \nu^2)\, y = \frac{4}{\sqrt{\pi}\ \Gamma(\nu + \frac{1}{2})} \left(\frac{z}{2}\right)^{\nu+1},$$

whose general solution is of the form

$$y = C_3\, I_\nu(z) + C_4\, K_\nu(z) + \mathbf{L}_\nu(z)\,.$$

There are further notations for another kind Struve type functions, namely [227, §11-2 (i)]

$$\mathbf{K}_\nu(z) = \mathbf{H}_\nu(z) - Y_\nu(z), \qquad \mathbf{M}_\nu(z) = \mathbf{L}_\nu(z) - K_\nu(z)\,,$$

which correspond to the principal values of the functions occurring on the right-hand-sides of these defining equalities. Obviously, $\mathbf{K}_\nu(z)$ and $\mathbf{M}_\nu(z)$ are particular solutions of the Struve, that is, of modified Struve differential equations, respectively. In determining the integral representation of the second type Neumann series $\mathfrak{X}_\nu(z)$, which will be introduced in Chap. 2, (2.33), we need the Struve function $\mathbf{H}_\nu(z)$ of order ν whose series definition reads [333, p. 328]

$$\begin{aligned}
\mathbf{H}_\nu(z) &= \frac{2\left(\frac{z}{2}\right)^\nu}{\sqrt{\pi}\,\Gamma(\nu + \frac{1}{2})} \int_0^1 (1 - t^2)^{\nu - \frac{1}{2}} \sin(zt)\ \mathrm{d}t \\
&= \frac{2\left(\frac{z}{2}\right)^\nu}{\sqrt{\pi}\,\Gamma(\nu + \frac{1}{2})} \int_0^{\frac{1}{2}\pi} \sin(z\cos\theta)\sin^{2\nu}\theta\ \mathrm{d}\theta\,,
\end{aligned}$$

provided that $\Re(\nu) > -\frac{1}{2}$.

1.10 Series Built by Bessel Functions Family Members

Series of Bessel and/or Struve functions in which summation indices appear in the order of the considered function and/or twist arguments of the constituting functions, can be unified in a double lacunary form:

$$\mathfrak{B}_{\ell_1,\ell_2}(z) = \sum_{n\geq 1} \alpha_n \mathscr{B}_{\ell_1(n)}(\ell_2(n)z), \tag{1.27}$$

where $\ell_j(x) = \mu_j x + \nu_j$, $j \in \{1, 2\}$, $x \in \mathbb{N}_0$, $z \in \mathbb{C}$ and $\mathscr{B}_\nu$ is one of the functions $J_\nu, I_\nu, Y_\nu, K_\nu, \mathbf{H}_\nu$ and $\mathbf{L}_\nu$.

The classical theory of the Fourier–Bessel series of the first type is based on the case when $\mathscr{B}_\nu = J_\nu$, see the celebrated monograph [333]. However, varying the coefficients of ℓ_1 and ℓ_2, we get three different cases which have not only deep roles in describing physical models and have physical interpretations in numerous topics of natural sciences and technology, but are also of vital mathematical interest, like e.g. zero function series [333].

We differ Neumann series (when $\mu_1 \neq 0, \mu_2 = 0$), Kapteyn series (when $\mu_1 \cdot \mu_2 \neq 0$) and Schlömilch series (when $\mu_1 = 0, \mu_2 \neq 0$). Here, all three series are of the first type (the series' terms contain only one constituting function $\mathscr{B}_\nu$); the second type series contain product terms of two (or more) members—not necessarily different ones—chosen from $J_\nu, I_\nu, Y_\nu, K_\nu, \mathbf{H}_\nu$ and $\mathbf{L}_\nu$.

We also point out that the Neumann series (of the first type) of Bessel function of the second kind Y_ν, modified Bessel function of the second kind K_ν and Hankel functions [100, 101] (Bessel functions of the third kind) $H_\nu^{(1)}, H_\nu^{(2)}$ have been studied in [24], while Neumann series of the second type were considered by Baricz and Pogány in somewhat different purposes in [20, 21]; see also [134].

Thus, under extended Neumann series (of Bessel J_ν see [333]) we mean the following

$$\mathfrak{N}_{\mu,\eta}^{\mathscr{B}}(x) = \sum_{n\geq 1} \beta_n \mathscr{B}_{\mu n+\eta}(ax),$$

where $\mathscr{B}_\nu$ is one of the functions I_ν, Y_ν, K_ν and $\mathbf{L}_\nu$. Integral representation discussions began very recently with the introductory article by Pogány and Süli [249], which gives an detailed references list concerning physical applications too, see [24].

Also, we will concentrate to the Neumann series [18]

$$\mathfrak{B}_{\mu,\eta}^{I}(x) = \sum_{n\geq 1} \beta_n I_{\mu n+\eta}(ax)\,.$$

Kapteyn series of the first type [145, 146, 217] are of the form

$$\mathfrak{K}_{\nu,\mu}^{\mathscr{B}}(z) = \sum_{n\geq 1} \alpha_n \mathscr{B}_{\rho+\mu n}\left((\sigma + \nu n)z\right);$$

more details about Kapteyn and Kapteyn-type series for Bessel function can be found also in [21, 23, 69, 308] and the references therein.

Under Schlömilch series [279, p. 155 *et seq.*] (Schlömilch considered only cases $\mu \in \{0, 1\}$), we count the functions series

$$\mathfrak{S}_\nu^{\mu,\mathscr{B}}(z) = \sum_{n\geq 1} \alpha_n \mathscr{B}_\mu\left((\mu n + \nu)z\right).$$

Integral representation are recently obtained for this series in [131], summations are given in [316].

Finally we point out that we do not consider Neumann, Kapteyn and Schlömilch series built by another members belonging to the Bessel function family such as spherical, modified spherical, ultraspherical, modified ultraspherical, Wright generalized Bessel and Delerue generalized Bessel functions listed in the previous section.

1.11 Fourier–Bessel and Dini Series

Denote $j_{\nu,n}$ the positive zeros of Bessel function of the first kind $J_\nu(x), \nu > -1$ written in ascending order of magnitude. Then the functions $J_\nu(j_{\nu,n}x),\ n \in \mathbb{N}$ form an orthogonal system [314, p. 220] with the linear weight x. The result of expanding an arbitrary suitable function $f(x)$ into a series form [333, p. 576, Eqs. (3), (4)], [39]

$$f(x) = \sum_{n\geq 1} a_n J_\nu(j_{\nu,n}x), \tag{1.28}$$

where

$$a_n = \frac{2}{J_{\nu+1}^2(j_{\nu,n})} \int_0^1 x f(x) J_\nu(j_{\nu,n}x)\, \mathrm{d}x, \qquad n \in \mathbb{N},$$

was published firstly by von Lommel in [322, pp. 69–73] (the case $\nu = 0$ has been explored in the famous Fourier's monograph [84, §§316–319]). Naturally, the expansion today holds the name *Fourier–Bessel series*, while a_n are the *Fourier–Bessel coefficients* associated with the input function f. Watson's book draw the attention [333, p. 577] to the further efforts by Hankel [100, pp. 471–491], Schläfli [277] and Harnack [106] in giving a rigorous proof of the expansion (1.28).

Few years after than Hermann Hankel and Ludwig Schläfli published their findings, Ulisse Dini [66] has considered a more general expansion in the form [66]

$$f(z) = \sum_{n\geq 1} b_n J_\nu(\lambda_{\nu,n}z), \qquad \nu \geq -\tfrac{1}{2},\ z \in \mathbb{C} \tag{1.29}$$

where $\lambda_{\nu,n}$ stands for the nth positive zero of the so-called *Dini function*

$$d_{\nu,\alpha}(z) := z^{-\nu}(zJ'_\nu(z) + \alpha J_\nu(z)) , \tag{1.30}$$

arranged in increasing order of magnitude and the coefficients are given by Pathak and Singh [231, p. 440], Watson [333, p. 577, Eq. (6)]

$$b_n = \frac{(\nu^2 - \lambda_{\nu,n}^2)\, J_\nu^2(\lambda_{\nu,n})}{\lambda_{\nu,n}^2\, \left[J'_\nu(\lambda_{\nu,n})\right]^2} + \frac{2}{\left[J'_\nu(\lambda_{\nu,n})\right]^2} \int_0^1 x f(x)\, J_\nu(\lambda_{\nu,n}x)\, \mathrm{d}x\,. \tag{1.31}$$

The expansion (1.29), where the coefficients set is given by (1.31) we call *Dini series*. However, the Sect. 5.7 is completely devoted to the integral form representations of such series with *general* coefficients b_n.

Finally, our approach in studying our 'Fourier–Bessel' and 'Dini series' is significantly flexible, since we take series alike to the right-hand-side expressions in (1.28) and (1.29) respectively, with *general* unknown coefficients constraining exclusively the fact that the considered series should converge in a widest possible sub-region of reals or $\mathbb{C}$. Assuming this property related integral representations, functional and uniform bounding inequalities are established.

Remark 1.1 We mention here Einer Hille's work on Fourier–Laguerre series [109–111], which are in fact Neumann series of the first type. ■

1.12 Hypergeometric and Generalized Hypergeometric Functions

Hypergeometric functions form an important class of special functions (see e.g. [87, 310]). They were introduced in 1866, by Carl Friedrich Gauss and after that have proved to be of enormous significance in mathematics and the mathematical sciences elsewhere. Here, we recall some properties of hypergeometric functions which are useful for us to derive some of our main results. We suggest to the reader to examine for the hypergeometric type functions either [7] or Rainville's classical book [263].

1.12.1 Gaussian Hypergeometric Function

Gaussian hypergeometric function is the power series

$${}_2F_1\left[\begin{matrix} a, b \\ c \end{matrix} \,\middle|\, z\right] = \sum_{k\ge 0} \frac{(a)_k\,(b)_k}{(c)_k} \frac{z^k}{k!} = 1 + \frac{ab}{c}\, z + \frac{a(a+1)\, b(b+1)}{c(c+1)} \frac{z^2}{2} + \ldots, \tag{1.32}$$

where z is a complex variable, a, b and c are real or complex parameters and $(a)_k$ is the Pochhammer symbol.

The series is not defined for $c = -m$, $m \in \mathbb{N}_0$, provided that a or b is not the negative integer n such that $n < m$. Furthermore, if the series (1.32) is defined but at least one of a, b is equal to $(-n)$, $n \in \mathbb{N}_0$, then it terminates in a finite number of terms and it reduces to a polynomial of degree n in variable z. Except for this case, in which the series is absolutely convergent for $|z| < \infty$, the domain of absolute convergence of the series (1.32) is the unit disc, i.e. $|z| < 1$. In this case it is said that the series (1.32) defines the Gaussian or hypergeometric function

$$y := y(z) = {}_2F_1\left[\begin{matrix} a, b \\ c \end{matrix} \middle| z\right]. \tag{1.33}$$

Also, on the unit circle $|z| = 1$, the series in (1.32) converges absolutely when $\Re(c-a-b) > 0$, converges conditionally when $-1 < \Re(c-a-b) \leq 0$ apart from at $z = 1$, and does not converge if $\Re(c-a-b) \leq -1$.

It can be verified [284, p. 6] that the function $y(z)$ is the solution of the second order differential equation [1, p. 562 *et seq.*]

$$z(1-z)y'' + (c - (a+b+1)z)\, y' - aby = 0, \tag{1.34}$$

in the region $|z| < 1$. However, the function (1.33) can be analytically continued to the other parts of the complex plane, i.e. solutions of Eq. (1.34) are also defined outside the unit circle. These solutions are provided by following substitutions in Eq. (1.34):

- substitution $1 - z \mapsto z$ yields solutions valid in the region $|1-z| < 1$,
- substitution $z^{-1} \mapsto z$ yields solutions valid in the region $|z| > 1$.

1.12.2 Generalized Hypergeometric Function

For b_i $(i = 1, 2, \ldots, q)$ different from non-positive integers the series

$$\sum_{n\geq 0} \frac{(a_1)_n (a_2)_n \cdots (a_p)_n}{(b_1)_n (b_2)_n \cdots (b_q)_n} \frac{z^n}{n!} = \sum_{n\geq 0} \frac{\prod_{j=1}^{p} (a_j)_n}{\prod_{j=1}^{q} (b_j)_n} \frac{z^n}{n!}$$

is called a *generalized hypergeometric series* (see [199]) and is denoted by

$${}_pF_q[z] = {}_pF_q[(a_p); (b_q)|z] = {}_pF_q\left[\begin{matrix} a_1, \cdots, a_p \\ b_1, \cdots, b_q \end{matrix} \middle| z\right].$$

When $p \leq q$, the generalized hypergeometric function converges for all complex values of z; that is, ${}_pF_q[z]$ is an entire function. When $p > q+1$, the series converges only for $z = 0$, unless it terminates (as when one of the parameters $a_j, j = \overline{1,p}$ is a negative integer) in which case it is just a polynomial in z. When $p = q+1$, the series converges in the unit disk $|z| < 1$, and also for $|z| = 1$ provided that

$$\Re\left(\sum_{j=1}^{q} b_j - \sum_{j=1}^{p} a_j\right) > 0\,.$$

The complex members of the sequences (a_p), (b_q) are called *parameters* and z is the *argument* of the function.

1.12.3 Fox–Wright Generalized Hypergeometric Function

In this monograph we also need the Fox-Wright generalized hypergeometric function ${}_p\Psi_q^*[\cdot]$ with p numerator parameters $a_1, \cdots, a_p$ and q denominator parameters $b_1, \cdots, b_q$, which is defined by Kilbas et al. [149, p. 56]

$${}_p\Psi_q^*\left[\begin{matrix}(a_1,\rho_1),\cdots,(a_p,\rho_p)\\(b_1,\sigma_1),\cdots,(b_q,\sigma_q)\end{matrix}\,\Big|\,z\right] = \sum_{n\geq 0}\frac{\prod_{j=1}^{p}(a_j)_{\rho_j n}}{\prod_{j=1}^{q}(b_j)_{\sigma_j n}}\,\frac{z^n}{n!}, \tag{1.35}$$

where $a_j \in \mathbb{C}; b_k \in \mathbb{C}\setminus\mathbb{Z}_0^-$ and $\rho_j, \sigma_k \in \mathbb{R}_+, j = 1, \cdots, p; k = 1, \cdots, q$. The defining series in (1.35) converges in the whole complex z-plane when

$$\Delta := \sum_{j=1}^{q}\sigma_j - \sum_{j=1}^{p}\rho_j > -1\,;$$

when $\Delta = 0$, then the series in (1.35) converges for $|z| < \nabla$, where

$$\nabla := \frac{\prod_{j=1}^{q}\sigma_j^{\sigma_j}}{\prod_{j=1}^{p}\rho_j^{\rho_j}}\,.$$

Setting in the definition (1.35) $\rho_1 = \cdots = \rho_p = 1$ and $\sigma_1 = \cdots = \sigma_q = 1$, we get the generalized hypergeometric function ${}_pF_q[\cdot]$.

1.13 Further Hypergeometric Type Functions

The *regularized generalized hypergeometric function* ${}_p\widetilde{F}_q[z]$ defined as the series [120]

$$ {}_p\widetilde{F}_q\left[\begin{matrix} a_1, \cdots, a_p \\ b_1, \cdots, b_q \end{matrix}\middle| z\right] = \sum_{n\geq 0} \frac{\prod\limits_{j=1}^{p} (a_j)_n}{\prod\limits_{j=1}^{q} \Gamma(b_j + n)} \frac{z^n}{n!}, $$

where $q \geq p$ (in which cases it is entire function in all variables [121]); $q = p - 1$ and $|z| < 1$; $q = p - 1$ and $|z| = 1$, $\Re\left(\sum\limits_{n=1}^{p-1} b_n - \sum\limits_{n=1}^{p} a_n\right) > 0$.

The *Kampé de Fériet generalized hypergeometric function of two variables* defined by the double-series [8] in a notation given by Srivastava and Panda [291, p. 423, Eq. (26)]

$$ F^{p:q;k}_{l:m;n}\left[\begin{matrix} (a_p) : (b_q); (c_k) \\ (\alpha_l) : (\beta_m); (\gamma_n) \end{matrix}\middle| x, y\right] = \sum_{r,s\geq 0} \frac{\prod\limits_{j=1}^{p} (a_j)_{r+s} \prod\limits_{j=1}^{q} (b_j)_r \prod\limits_{j=1}^{k} (c_j)_s}{\prod\limits_{j=1}^{l} (\alpha_j)_{r+s} \prod\limits_{j=1}^{m} (\beta_j)_r \prod\limits_{j=1}^{n} (\gamma_j)_s} \frac{x^r}{r!} \frac{y^s}{s!}, $$

which converges [289] when

1. $p + q < l + m + 1, p + k < l + n + 1, \max\{|x|, |y|\} < \infty,$ or
2. $p + q = l + m + 1, p + k = l + n + 1$ and

$$ \begin{cases} |x|^{\frac{1}{p-l}} + |y|^{\frac{1}{p-l}} < 1, & l < p \\ \max\{|x|, |y|\} < 1, & l > p \end{cases}. $$

The member Φ_3 of the Horn's list, a *confluent hypergeometric function of two variables*, is defined by Srivastava and Karlsson [290, p. 26, Eq. (20)]

$$ \Phi_3(\beta, \gamma; x, y) = \sum_{n,m\geq 0} \frac{(\beta)_n}{(\gamma)_{n+m}} \frac{x^n}{n!} \frac{y^m}{m!}, \qquad x, y \in \mathbb{C}. $$

Srivastava and Daoust [288] considered a two-variable series extension of a multiple generalized hypergeometric type function which reads as follows

$$ F^{A:B;B'}_{C:D;D'}\left[\begin{matrix} [(a) : \nu, \varphi] : [(b) : \psi]; [(b') : \psi'] \\ [(c) : \xi, \eta] : [(d) : \zeta]; [(d') : \zeta'] \end{matrix}\middle| x, y\right] $$

$$ = \sum_{m,n\geq 0} \frac{\prod\limits_{j=1}^{A} (a_j)_{m\nu_j+n\varphi_j} \prod\limits_{j=1}^{B} (b_j)_{m\psi_j} \prod\limits_{j=1}^{B'} (b'_j)_{n\psi'_j}}{\prod\limits_{j=1}^{C} (c_j)_{m\xi_j+n\eta_j} \prod\limits_{j=1}^{D} (d_j)_{m\zeta_j} \prod\limits_{j=1}^{D'} (d'_j)_{n\zeta'_j}} \frac{x^m}{m!} \frac{y^n}{n!}, $$

where, for convergence of the double series,

$$1+\sum_{j=1}^{C}\xi_j+\sum_{j=1}^{D}\zeta_j-\sum_{j=1}^{A}\nu_j-\sum_{j=1}^{B}\psi_j \geq 0; \qquad 1+\sum_{j=1}^{C}\eta_j+\sum_{j=1}^{D'}\zeta_j'-\sum_{j=1}^{A}\varphi_j-\sum_{j=1}^{B'}\psi_j' \geq 0,$$

with equality only when $|x|$ and $|y|$ are constrained appropriately (see e.g., for details, [288]). Here, for the sake of convenience, (a) abbreviates the array of A parameters $a_1, \ldots, a_A$ with similar interpretations for (b), (b'), (c), (d) and (d').

1.14 Hurwitz–Lerch Zeta Function

The general Hurwitz–Lerch Zeta function $\Phi(z,s,a)$ is defined by (see e.g. [77, p. 27, Eq. 1.11(1)]; see also [287, p. 121, *et seq.*]):

$$\Phi(z,s,a) = \sum_{n\geq 1}\frac{z^n}{(n+a)^s},$$

where $a \in \mathbb{C}\setminus\mathbb{Z}_0^-; s \in \mathbb{C}$ when $|z| < 1$; while $\Re(s) > 1$ when $|z| = 1$. The Hurwitz–Lerch Zeta function $\Phi(z,s,a)$ can indeed be continued meromorphically to the whole complex s-plane, except for a simple pole at $s = 1$ with its residue 1. The general Hurwitz–Lerch Zeta function $\Phi(z,s,a)$ contains, as its special cases, not only the Riemann Zeta function $\zeta(s)$, the Hurwitz Zeta function $\zeta(s,a)$ and the Lerch Zeta function $\ell_\xi(z)$ defined by (see [77, Chapter I] and [287, Chapter 2]). For novel results regarding generalizations and unifications of $\Phi(z,s,a)$ the interested reader is referred also to the articles [172, 275, 295, 296].

1.15 Fractional Differintegral

In order to solve the nonhomogeneous Bessel differential equation, we will also use *fractional derivation* and *fractional integration*, i.e. *fractional differintegration*.

So, let us first introduce, according to [174, p. 1488, Definition], the fractional derivative and the fractional integral of order ν of some suitable function f, see also [149, 173, 328–330, 340].

If the function $f(z)$ is analytic (regular) inside and on $\mathscr{C} := \{\mathscr{C}^-, \mathscr{C}^+\}$, where $\mathscr{C}^-$ is a contour along the cut joining the points z and $-\infty + \mathrm{i}\Im\{z\}$, which starts from the point at $-\infty$, encircles the point z once counter-clockwise, and returns to the point at $-\infty$, $\mathscr{C}^+$ is a contour along the cut joining the points z and $\infty + \mathrm{i}\Im\{z\}$, which starts from the point at ∞, encircles the point z once counter-clockwise, and returns to the point at ∞,

$$f_\mu(z) = (f(z))_\mu := \frac{\Gamma(\mu+1)}{2\pi \mathrm{i}} \int_{\mathscr{C}} \frac{f(\zeta)}{(\zeta - z)^{\mu+1}} \, \mathrm{d}\zeta$$

for all $\mu \in \mathbb{R} \setminus \mathbb{Z}^-$; $\mathbb{Z}^- := \{-1, -2, -3, \cdots\}$ and

$$f_{-n}(z) := \lim_{\mu \to -n} f_\mu(z) \,, \qquad n \in \mathbb{N},$$

where $\zeta \neq z$,

$$-\pi \leq \arg(\zeta - z) \leq \pi \,, \qquad \text{for } \mathscr{C}^-,$$

and

$$0 \leq \arg(\zeta - z) \leq 2\pi \,, \qquad \text{for } \mathscr{C}^+,$$

then $f_\mu(z)$, $\mu > 0$ is said to be the fractional derivative of $f(z)$ of order μ and $f_\mu(z)$, $\mu < 0$ is said to be the fractional integral of $f(z)$ of order $-\mu$, provided that

$$|f_\mu(z)| < \infty, \qquad \mu \in \mathbb{R}.$$

At this point let us recall that the fractional differintegral operator (see e.g. [174, 220, 221])

- is linear, i.e. if the functions $f(z)$ and $g(z)$ are single-valued and analytic in some domain $\Omega \subseteq \mathbb{C}$, then for any constants k_1 and k_2

$$(k_1 f(z) + k_2 g(z))_\nu = k_1 f_\nu(z) + k_2 g_\nu(z) \,, \qquad \nu \in \mathbb{R}, z \in \Omega;$$

- preserves the index law: if the function $f(z)$ is single-valued and analytic in some domain $\Omega \subseteq \mathbb{C}$, then

$$\left(f_\mu(z)\right)_\nu = f_{\mu+\nu}(z) = (f_\nu(z))_\mu \,,$$

  where $f_\mu(z) \neq 0, f_\nu(z) \neq 0$, $\mu, \nu \in \mathbb{R}$, $z \in \Omega$;
- permits the generalized Leibniz rule [174, p. 1489, Lemma 3]: if the functions $f(z)$ and $g(z)$ are single-valued and analytic in some domain $\Omega \subseteq \mathbb{C}$, then

$$(f(z) \cdot g(z))_\nu = \sum_{n \geq 0} \binom{\nu}{n} f_{\nu-n}(z) \cdot g_n(z) \,, \qquad \nu \in \mathbb{R},\ z \in \Omega, \tag{1.36}$$

  where $g_n(z)$ is the ordinary derivative of $g(z)$ of order $n \in \mathbb{N}_0$, it being tacitly assumed that $g(z)$ is the polynomial part (if any) of the product $f(z) \cdot g(z)$.

The fractional differintegral operator also possesses the following properties:

- for a constant λ,

$$\left(\mathrm{e}^{\lambda z}\right)_\nu = \lambda^\nu \mathrm{e}^{\lambda z}\,, \qquad \lambda \neq 0,\ \nu \in \mathbb{R},\ z \in \mathbb{C}\,;$$

- for a constant λ,

$$\left(\mathrm{e}^{-\lambda z}\right)_\nu = \mathrm{e}^{-i\pi\nu} \lambda^\nu \mathrm{e}^{\lambda z}\,, \qquad \lambda \neq 0,\ \nu \in \mathbb{R},\ z \in \mathbb{C}\,;$$

- for a constant λ,

$$\left(z^{\lambda}\right)_\nu = \mathrm{e}^{-i\pi\nu}\, \frac{\Gamma(\nu-\lambda)}{\Gamma(-\lambda)}\, z^{\lambda-\nu}\,, \qquad \nu \in \mathbb{R},\ z \in \mathbb{C}, \quad \left|\frac{\Gamma(\nu-\lambda)}{\Gamma(-\lambda)}\right| < \infty\,.$$

Chapter 2
Neumann Series

Abstract The goal of present chapter is to study in details the integral representations of the Neumann series (of the first and second type) of Bessel and modified Bessel functions of the first and second kind. In order to achieve our goal we use several methods: the Euler–Maclaurin summation technique, differential equation technique, fractional integration technique. Moreover, we present some interesting results on the coefficients of Neumann series, product of modified Bessel functions of the first and second kind and the cumulative distribution function of the non-central χ^2-distribution.

The series

$$\mathfrak{N}_\nu(z) := \sum_{n\geq 1} \alpha_n J_{\nu+n}(z), \qquad z \in \mathbb{C}, \tag{2.1}$$

where ν, α_n are constants and J_μ stands for the Bessel function of the first kind of order μ, is called a *Neumann series* [333, Chapter XVI]. Such series owe their name to the fact that they were first systematically considered (for integer μ) by Carl Gottfried Neumann in his important book [209] in 1867; subsequently, in 1877, Leopold Bernhard Gegenbauer extended such series to $\mu \in \mathbb{R}$ (see [333, p. 522]).

Neumann series of Bessel functions arise in a number of application areas. For example, in connection with random noise, Rice [268, Eqs. (3.10–17)] applied Bennett's result

$$\sum_{n\geq 1} \left(\frac{v}{a}\right)^n J_n(a\mathrm{i}\, v) = \mathrm{e}^{\frac{v^2}{2}} \int_0^v x\mathrm{e}^{-\frac{x^2}{2}} J_0(a\mathrm{i}\, x)\,\mathrm{d}x\,. \tag{2.2}$$

Luke [178, pp. 271–288] proved that

$$1 - \int_0^v \mathrm{e}^{-(u+x)} J_0\big(2\mathrm{i}\sqrt{ux}\big)\mathrm{d}x = \begin{cases} \mathrm{e}^{-(u+v)} \sum\limits_{n\geq 0} \left(\dfrac{u}{v}\right)^{\frac{n}{2}} J_n\big(2\mathrm{i}\sqrt{uv}\big), & u < v \\ 1 - \mathrm{e}^{-(u+v)} \sum\limits_{n\geq 1} \left(\dfrac{v}{u}\right)^{\frac{n}{2}} J_n\big(2\mathrm{i}\sqrt{uv}\big), & u > v \end{cases};$$

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cf. also [223, Eq. (2a)]. In both of these applications $\mathfrak{N}_0$ plays a key role. The function $\mathfrak{N}_0$ also appears as a relevant technical tool in the solution of the infinite dielectric wedge problem by Kontorovich–Lebedev transforms [272, §4, 5]. It also arises in the description of internal gravity waves in a Boussinesq fluid [208], as well as in the study of the propagation properties of diffracted light beams; see, for example, [189, Eqs. (6a,b), (7b), (10a,b)]. Recent investigations by Kravchenko, Torba and co-workers show the role of Neumann series in Schrödinger equations' solution representation [164], perturbed Bessel equation [163] and connect them to the Strum-Liouville equation [162].

Expanding a given function f, say, into a Neumann series of the form

$$\mathfrak{N}_\nu^{\mathsf{W}}(x) = \sum_{n\geq 0} a_{n\nu} J_{\nu+2n+1}(x), \qquad \nu \geq -\tfrac{1}{2},$$

where

$$a_{n\nu} = 2(\nu+2n+1)\int_0^\infty t^{-1} f(t) J_{\nu+2n+1}(t)\,\mathrm{d}t,$$

Wilkins discussed the question of existence of an integral representation for $\mathfrak{N}_\nu^{\mathsf{W}}(x)$, as well as the conditions under which the Neumann series $\mathfrak{N}_\nu^{\mathsf{W}}(x)$ converges uniformly in x to the 'input' function f [334, §11–13], [336].

By modifying a result of Watson [333, p. 23, footnote], Maximon represented a simple Neumann series $\mathfrak{N}_\nu$ appearing in the literature in connection with physical problems [188, Eq. (4)] as an indefinite integral expression containing Bessel functions. Meligy expanded into a Neumann series $\mathfrak{N}_{L+\frac{1}{2}}$ of arbitrary argument, containing Bessel functions of order $L + 1/2 + n/2$ where L is the orbital angular momentum quantum number, the wave functions that describe the states of motion of charged particles in a Coulomb field [191, Eqs. (8), (9)]. The inversion probability of a large spin is found *via* modified Neumann series of Bessel functions $J_{(2N+1)(2n-1)\pm 1}$ for integer $N \geq 2$; see [148, Theorem].

The evaluation of the capacitance matrix of a system of finite-length conductors [62] uses $\mathfrak{N}_p$, with p integer; in [183, 184], free vibrations of a wooden pole were modeled by a coupled system of ordinary differential equations and solved by Neumann series; we note in passing that the analysis of an isotropic medium containing a cylindrical borehole by Love's auxiliary function [270] and the analytical and numerical study of Neumann series of Bessel functions [268] are two further areas in which the unknown coefficients of $\mathfrak{N}_\nu$ are derived and computed from boundary and initial conditions of the problem under consideration.

Our main aims in this chapter are to establish several closed integral representation formulae for those series and also for the modified Neumann series of the first and second type, and to derive the coefficients of Neumann series when certain integral representation formulae there hold. In the considered Neumann series the building blocks turn out to be either Bessel and/or alike functions (Struve, modified Bessel of the first and second kind, Kummer functions etc).

2.1 Integral Representation for Neumann Series of Bessel Functions

In this section our main goal is to establish a closed integral representation formula for the series $\mathfrak{N}_\nu(z)$. This will be achieved by using the Laplace integral representation of the associated Dirichlet series. Thus, we replace $z \in \mathbb{C}$ with $x \in \mathbb{R}_+$ and assume in the sequel that the behaviour of $(\alpha_n)_{n\ge 1}$ ensures the convergence of the series (2.1) over $\mathbb{R}_+$.

Theorem 2.1 (Pogány and Süli [249]) *Let* $\alpha \in \mathrm{C}^1(\mathbb{R}_+)$ *and let* $\alpha\big|_{\mathbb{N}} = (\alpha_n)_{n\ge 1}$. *Then, for all* x, ν *such that*

$$x \in \mathscr{I}_\alpha = \left(0, 2\min\left\{1, \left(\mathrm{e}\,\overline{\lim_{n\to\infty}}\frac{\sqrt[n]{|\alpha_n|}}{n}\right)^{-1}\right\}\right), \qquad \nu > -\tfrac{1}{2},$$

we have that

$$\mathfrak{N}_\nu(x) = -\int_1^\infty \frac{\partial}{\partial\omega}\Big(\Gamma(\nu+\omega+\tfrac{1}{2})\,J_{\nu+\omega}(x)\Big)\int_0^{[\omega]} \mathfrak{d}_\eta\Big(\frac{\alpha(\eta)}{\Gamma(\nu+\eta+\tfrac{1}{2})}\Big)\,\mathrm{d}\eta\,\mathrm{d}\omega. \tag{2.3}$$

Proof Consider the integral representation formula [93, **8.411** Eq. (10)]

$$J_\nu(z) = \frac{\big(z/2\big)^\nu}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})}\int_{-1}^1 \cos(zt)(1-t^2)^{\nu-\frac{1}{2}}\,\mathrm{d}t, \qquad z\in\mathbb{C},\ \Re\{\nu\} > -\tfrac{1}{2}. \tag{2.4}$$

Applying (2.4) to (2.1) and taking $x > 0$, we get

$$\mathfrak{N}_\nu(x) = \sqrt{\frac{2x}{\pi}}\int_0^1 \cos(xt)\Big(\frac{x(1-t^2)}{2}\Big)^{\nu-\frac{1}{2}}\mathscr{D}_\alpha(t)\,\mathrm{d}t \tag{2.5}$$

with the Dirichlet series

$$\mathscr{D}_\alpha(t) := \sum_{n\ge 1}\frac{\alpha_n\big(x(1-t^2)/2\big)^n}{\Gamma(n+\nu+\frac{1}{2})} = \sum_{n\ge 1}\frac{\alpha_n\,\exp\big\{-n\log\frac{2}{x(1-t^2)}\big\}}{\Gamma(n+\nu+\frac{1}{2})}.$$

Recalling that $\Gamma(s) = \sqrt{2\pi}\,s^{s-\frac{1}{2}}\mathrm{e}^{-s}\big(1+\mathscr{O}(s^{-1})\big)$, $|s|\to\infty$, we see that the Dirichlet series $\mathscr{D}_\alpha(t)$ is absolutely convergent for all $x\in\mathbb{R}_+$ and $t\in(-1,1)$ such that

$$|x|(1-t^2) \le |x| < \tfrac{2}{\mathrm{e}}\left(\overline{\lim_{n\to\infty}}\frac{\sqrt[n]{|\alpha_n|}}{n}\right)^{-1}.$$

Furthermore, $\mathscr{D}_\alpha(t)$ has a Laplace integral representation when $\log 2/(x(1-t^2)) > 0$. In this case we can take $x \in (0,2)$ and $t \in (-1,1)$, since the required positivity condition is satisfied when

$$\frac{2}{x(1-t^2)} \geq \frac{2}{x} > 1.$$

Hence, the x-domain becomes

$$0 < x < 2\min\left\{1, \left(\mathrm{e}\,\overline{\lim_{n\to\infty}}\frac{\sqrt[n]{|\alpha_n|}}{n}\right)^{-1}\right\}.$$

Thus, for all such x we deduce that

$$\mathscr{D}_\alpha(t) = \log\frac{2}{x(1-t^2)}\int_0^\infty \left(\frac{x(1-t^2)}{2}\right)^\omega \left(\sum_{j=1}^{[\omega]}\frac{\alpha_j}{\Gamma(j+\nu+\frac{1}{2})}\right)\mathrm{d}\omega; \tag{2.6}$$

see, for example, [147, **V**] or [252, §4, §6]. The counting function

$$\mathscr{A}_\alpha(\omega) := \sum_{j=1}^{[w]}\frac{\alpha_j}{\Gamma(j+\nu+\frac{1}{2})}\,.$$

The Euler–Maclaurin summation formula gives us [252, cf. Lemma 1]

$$\mathscr{A}_\alpha(\omega) = \int_0^{[\omega]}\mathfrak{d}_\eta\left(\frac{\alpha(\eta)}{\Gamma(\nu+\eta+\frac{1}{2})}\right)\mathrm{d}\eta. \tag{2.7}$$

Substituting $\mathscr{A}_\alpha(\omega)$ and $\mathscr{D}_\alpha(t)$ from (2.7) and (2.6) into (2.5), we get

$$\begin{aligned}\mathfrak{N}_\nu(x) = -\sqrt{\frac{x}{2\pi}}\int_0^\infty\int_0^{[\omega]}\mathfrak{d}_\eta\left(\frac{\alpha(\eta)}{\Gamma(\nu+\eta+\frac{1}{2})}\right)\\ \times\left(2\int_0^1\cos(xt)\left(\frac{x(1-t^2)}{2}\right)^{\nu+\omega-\frac{1}{2}}\log\left(\frac{x(1-t^2)}{2}\right)\mathrm{d}t\right)\mathrm{d}\omega\,\mathrm{d}\eta.\end{aligned} \tag{2.8}$$

However, the inner-most (t-integral) in (2.8),

$$\mathscr{I}_x(\kappa) := 2\int_0^1\cos(xt)\left(\frac{x(1-t^2)}{2}\right)^\kappa\log\left(\frac{x(1-t^2)}{2}\right)\mathrm{d}t, \qquad \kappa := \nu+\omega-\tfrac{1}{2},$$

can be expressed in terms of the Gamma function and the Bessel function of the first kind by legitimate indefinite integration with respect to κ, as follows.

To begin, we define the Fourier cosine transform of a certain function f by

$$\mathscr{F}_c(f;x) := 2\int_0^\infty \cos(xt) f(t)\,\mathrm{d}t.$$

Now, we have that

$$\begin{aligned}\int \mathscr{I}_x(\kappa)\,\mathrm{d}\kappa &= 2\Big(\frac{x}{2}\Big)^\kappa \int_0^1 \cos(xt)(1-t^2)^\kappa \mathrm{d}t \\ &= \Big(\frac{x}{2}\Big)^\kappa \mathscr{F}_c\big((1-t^2)^\kappa \chi_{[0,1)}(t);\, x\big) = \sqrt{\frac{2\pi}{x}} \cdot \Gamma(\kappa+1)\, J_{\kappa+\frac{1}{2}}(x),\end{aligned}$$

where we applied the Fourier cosine transform table [93, **17.34** Eq. (10)]. On observing that $\mathrm{d}\kappa = \mathrm{d}\omega$, we deduce that

$$\mathscr{I}_x\left(\nu+\omega-\tfrac{1}{2}\right) = \sqrt{\frac{2\pi}{x}} \cdot \frac{\partial}{\partial\omega}\Big(\Gamma\left(\nu+\omega+\tfrac{1}{2}\right) J_{\nu+\omega}(x)\Big). \tag{2.9}$$

Substituting (2.9) into (2.8) we arrive at the asserted integral expression (2.3), remarking that the integration domain $\mathbb{R}_+$ changes into $[1,\infty)$ because $[\omega]$ equals zero for all $\omega \in [0,1)$. □

2.1.1 *Bivariate von Lommel Functions as Neumann Series*

To conclude the results in Sect. 2.1, we mention some related integral representation formulae for Neumann-type series, corresponding to special α's. Bivariate von Lommel functions of order ν are defined by Neumann-type series [333, **16.5** Eqs. (5), (6)] as follows:

$$\begin{aligned}U_\nu(y,x) &:= \sum_{m\geq 0}(-1)^m\Big(\frac{y}{x}\Big)^{\nu+2m} J_{\nu+2m}(x),\\ V_\nu(y,x) &:= \cos\Big(\frac{y}{2}+\frac{x^2}{2y}+\frac{\nu\pi}{2}\Big) + U_{-\nu+2}(y,x), \qquad x,y\in\mathbb{R}.\end{aligned}$$

These series converge for unrestricted values of ν.

Now, assuming that $\Re(\nu) > 0$, by the formulae [333, **16.53** Eqs. (1), (2)] we easily deduce that

$$\begin{aligned}U_{\nu,c}(x) := U_\nu(cx,x) &= c^\nu x\int_0^1 t^\nu J_{\nu-1}(xt)\cos\big(\tfrac{c}{2}\,x(1-t^2)\big)\,\mathrm{d}t,\\ U_{\nu+1,c}(x) &= c^\nu x\int_0^1 t^\nu J_{\nu-1}(xt)\sin\big(\tfrac{c}{2}\,x(1-t^2)\big)\,\mathrm{d}t.\end{aligned}$$

Similarly, by Watson [333, **16.53** Eqs. (11), (12)][1] we also have that

$$V_{\nu,c}(x) := V_\nu(cx, x) = -c^{2-\nu} x \int_1^\infty t^{2-\nu} J_{1-\nu}(xt) \cos\left(\tfrac{c}{2}\, x(1-t^2)\right) \mathrm{d}t,$$

$$V_{\nu-1,c}(x) = -c^{2-\nu} x \int_1^\infty t^{2-\nu} J_{1-\nu}(xt) \sin\left(\tfrac{c}{2}\, x(1-t^2)\right) \mathrm{d}t,$$

provided $x, c > 0,\ \Re(\nu) > \frac{1}{2}$.

The integral expressions developed above can be easily adapted to Neumann-type series of the form

$$\sum_{m \geq 0} \gamma^m J_{\nu+2m}(x), \qquad x > 0,\ \gamma < 0.$$

Here we mention the recent articles by Fejzullahu [80] about integral form of Neumann series connected with von Lommel functions in which complex integration technique has used and the fresh manuscript by De Micheli [60] which concerns a Fourier-type integral representation for Bessel's function of the first kind and complex order *via* Gegenbauer polynomials.

An interesting open problem is the construction of examples with specific coefficients α_n, with known explicit forms of Neumann-type series, that can be derived directly from the representation formula (2.3) and such results will be presented in the next section.

2.2 On Coefficients of Neumann–Bessel Series

The problem of computing the coefficients of the Neumann series of Bessel functions has been considered in a number of publications in the mathematical literature.

For example, Watson [333] showed that, given a function f that is analytic inside and on a circle of radius R, with center at the origin, and if C denotes the integration contour formed by that circle, then f can be expanded into a Neumann series [333, Eq. (16.1), p. 523]

$$\mathfrak{N}_0(z) = \sum_{n \geq 0} \alpha_n J_n(z).$$

The corresponding coefficients are given by Watson [333, Eq. (16.2), p. 523]

$$\alpha_n = \frac{\varepsilon_n}{2\pi \mathrm{i}} \int_C f(t)\, O_n(t) \mathrm{d}t,$$

[1] Watson remarked that all four formulae that were cited by him [333, **16.53** Eqs. (1), (2), (11), (12)] had been derived by von Lommel (cf. von Lommel's memoirs [324, 325] for further details).

where the functions $O_n(t)$, $n = 0, 1, \ldots$, are the Neumann polynomials, and can be obtained from

$$\frac{1}{t-z} = \sum_{n\geq 0} \varepsilon_n\, O_n(t)\, J_n(z)\,,$$

where

$$\varepsilon_n = \begin{cases} 1, & n = 0 \\ 2, & n \in \mathbb{N} \end{cases}$$

is the so-called Neumann factor.

Wilkins [334] showed that a function $f(x)$ can be represented on $\mathbb{R}_+$ by a Neumann series of the form

$$\mathfrak{N}_\nu^{\mathsf{W}}(x) = \sum_{n\geq 0} a_{n\nu}\, J_{\nu+2n+1}(x), \qquad \nu \geq -\tfrac{1}{2}\,, \tag{2.10}$$

where the coefficients $a_{n\nu}$ are

$$a_{n\nu} = 2(\nu + 2n + 1)\int_0^\infty t^{-1} f(t)\, J_{\nu+2n+1}(t)\mathrm{d}t.$$

The problem of integral representation of Neumann series of Bessel functions occurs not so frequently. Besides the already mentioned Rice's result (2.2), there is also Wilkins who considered the possibility of integral representation for even-indexed Neumann series (2.10). Finally, let us mention Luke's integral expression for $\mathfrak{N}_0(x)$ [178, pp. 271–288] and [223, Eq. (2a)].

Also, in the previous section, we presented completely different kind of integral representation for (2.1) given by Pogány and Süli in [249] in Theorem 2.1. As we already mentioned, in that article the authors posed the problem of constructing a function α, with $\alpha\big|_{\mathbb{N}} \equiv \big(\alpha_n\big)$, such that the integral representation (2.3) holds. The purpose of this section is to answer this open question and the results exposed below concern to the paper by Jankov et al. [134].

We will describe the class $\Lambda = \{\alpha\}$ of functions that generate the integral representation (2.3) of the corresponding Neumann series, in the sense that the restriction $\alpha\big|_{\mathbb{N}} = \big(\alpha_n\big)$ forms the coefficient array of the series (2.1). Knowing only the set of nodes $\mathbf{N} := \big\{(n, \alpha_n)\big\}_{n\geq 1}$ this question cannot be answered merely by examining the convergence of the series $\mathfrak{N}_\nu(x)$ and then interpolating the set $\mathbf{N}$. We formulate an answer to this question so that the resulting class of functions α depends on a suitable, integrable (on $\mathbb{R}_+$), scaling-function h.

Theorem 2.2 (Jankov et al. [134]) *Let* Theorem 2.1 *hold for a given convergent Neumann series of Bessel functions, and suppose that the integrand in* (2.3) *is such that*

$$\frac{\partial}{\partial\omega}\Big(\Gamma\big(\nu+\omega+\tfrac{1}{2}\big)J_{\nu+\omega}(x)\Big)\int_0^{[\omega]}\mathfrak{d}_\eta\Big(\frac{\alpha(\eta)}{\Gamma(\nu+\eta+\frac{1}{2})}\Big)\mathrm{d}\eta\ \in\ \mathrm{L}^1(\mathbb{R}_+)\,,$$

and let

$$h(\omega):=\frac{\partial}{\partial\omega}\Big(\Gamma\big(\nu+\omega+\tfrac{1}{2}\big)J_{\nu+\omega}(x)\Big)\int_0^{\omega}\mathfrak{d}_\eta\Big(\frac{\alpha(\eta)}{\Gamma(\nu+\eta+\frac{1}{2})}\Big)\mathrm{d}\eta\,.$$

Then we have that

$$\alpha(\omega)=\begin{cases}\Gamma(\nu+k+\frac{1}{2})\ \dfrac{\mathrm{d}}{\mathrm{d}\omega}\dfrac{h(\omega)}{\mathscr{B}(\omega)}\Big|_{\omega=k+}, & \omega=k\in\mathbb{N}\\ \dfrac{\Gamma(\nu+\omega+\frac{1}{2})}{\{\omega\}}\Big(\dfrac{h(\omega)}{\mathscr{B}(\omega)}-\dfrac{h(k+)}{\mathscr{B}(k)}\Big), & 1<\omega\neq k\in\mathbb{N}\end{cases}, \tag{2.11}$$

where

$$\mathscr{B}(\omega):=\frac{\partial}{\partial\omega}\Big(\Gamma(\nu+\omega+\tfrac{1}{2})\,J_{\nu+\omega}(x)\Big).$$

Proof Assume that the integral representation (2.3) holds for some class Λ of functions α whose restriction $\alpha\big|_{\mathbb{N}}$ forms the coefficient array employed in $\mathfrak{N}_\nu(x)$. Suppose that $\widetilde{h}\in\mathrm{L}^1(\mathbb{R}_+)$ is defined by

$$\widetilde{h}(\omega):=\frac{\partial}{\partial\omega}\Big(\Gamma\big(\nu+\omega+\tfrac{1}{2}\big)J_{\nu+\omega}(x)\Big)\cdot\int_0^{[\omega]}\mathfrak{d}_\eta\Big(\frac{\alpha(\eta)}{\Gamma(\nu+\eta+\frac{1}{2})}\Big)\mathrm{d}\eta; \tag{2.12}$$

in other words, $\widetilde{h}$ converges to zero sufficiently fast as $\omega\to+\infty$ so as to ensure that the integral (2.3) converges. Because $\omega\sim[\omega]$ for large ω, by (2.12) we deduce that

$$\int_0^{\omega}\mathfrak{d}_\eta\Big(\frac{\alpha(\eta)}{\Gamma(\nu+\eta+\frac{1}{2})}\Big)\mathrm{d}\eta=\frac{h(\omega)}{\mathscr{B}(\omega)}, \tag{2.13}$$

where

$$h(\omega)=\frac{\widetilde{h}(\omega)\displaystyle\int_0^{\omega}\mathfrak{d}_\eta\Big(\frac{\alpha(\eta)}{\Gamma(\nu+\eta+\frac{1}{2})}\Big)\mathrm{d}\eta}{\displaystyle\int_0^{[\omega]}\mathfrak{d}_\eta\Big(\frac{\alpha(\eta)}{\Gamma(\nu+\eta+\frac{1}{2})}\Big)\mathrm{d}\eta}\sim\widetilde{h}(\omega),\qquad\omega\to\infty.$$

Differentiating (2.13) with respect to ω we get

$$\{\omega\}\alpha'(\omega) + \big(1 - \{\omega\}\psi(\nu+\omega+\tfrac{1}{2})\big)\alpha(\omega) = \Gamma(\nu+\omega+\tfrac{1}{2}) \cdot \frac{\partial}{\partial\omega}\frac{h(\omega)}{\mathscr{B}(\omega)}. \tag{2.14}$$

For integer $\omega \equiv k \in \mathbb{N}$ we know the coefficient set $\Lambda = \{\alpha_k\}$. Therefore, let $\omega \in (k, k+1)$, where k is a fixed positive integer. By this specification (2.14) becomes a linear ordinary differential equation in the unknown α:

$$\alpha'(\omega) + \Big(\frac{1}{\omega - k} - \psi(\nu+\omega+\tfrac{1}{2})\Big)\,\alpha(\omega) = \frac{\Gamma(\nu+\omega+\frac{1}{2})}{\omega - k} \cdot \frac{\partial}{\partial\omega}\frac{h(\omega)}{\mathscr{B}(\omega)}.$$

After some routine calculations we get

$$\alpha(\omega) = \frac{\Gamma(\nu+\omega+\frac{1}{2})}{\{\omega\}}\Big(C_k + \frac{h(\omega)}{\mathscr{B}(\omega)}\Big),$$

where C_k denotes the integration constant. Thus we deduce that, for $\omega \ge 1$, we have

$$\alpha(\omega) = \begin{cases} \alpha_k, & \omega = k \in \mathbb{N} \\ \dfrac{\Gamma(\nu+\omega+\frac{1}{2})}{\{\omega\}}\Big(C_k + \dfrac{h(\omega)}{\mathscr{B}(\omega)}\Big), & 1 < \omega \ne k \in \mathbb{N} \end{cases}.$$

It remains to find the numerical value of C_k. By the assumed convergence of $\mathfrak{N}_\nu(x)$, $\alpha(\omega)$ has to decay to zero as $k \to \infty$. Indeed, Landau's bound (1.21) clarifies this claim. Since k is not a pole of $\Gamma(\nu+\omega+\frac{1}{2})$, by L'Hospital's rule we deduce that

$$\begin{aligned}\alpha_k &= \lim_{\omega\to k+}\alpha(\omega) = \lim_{\omega\to k+}\Gamma\big(\nu+\omega+\tfrac{1}{2}\big)\lim_{\omega\to k+}\frac{C_k + \dfrac{h(\omega)}{\mathscr{B}(\omega)}}{\omega - k} \\ &= \Gamma\big(\nu+k+\tfrac{1}{2}\big)\lim_{\omega\to k+}\frac{\mathrm{d}}{\mathrm{d}\omega}\frac{h(\omega)}{\mathscr{B}(\omega)} = \Gamma\big(\nu+k+\tfrac{1}{2}\big)\,\frac{\mathrm{d}}{\mathrm{d}\omega}\frac{h(\omega)}{\mathscr{B}(\omega)}\bigg|_{\omega=k+},\end{aligned}$$

such that makes sense only for

$$C_k = -\frac{h(k+)}{\mathscr{B}(k)}.$$

Hence

$$\alpha(\omega) = \begin{cases} \Gamma\big(\nu+k+\tfrac{1}{2}\big)\,\dfrac{\mathrm{d}}{\mathrm{d}\omega}\dfrac{h(\omega)}{\mathscr{B}(\omega)}\bigg|_{\omega=k+}, & \omega = k \in \mathbb{N} \\ \dfrac{\Gamma\big(\nu+\omega+\frac{1}{2}\big)}{\{\omega\}}\Big(\dfrac{h(\omega)}{\mathscr{B}(\omega)} - \dfrac{h(k+)}{\mathscr{B}(k)}\Big), & 1 < \omega \ne k \in \mathbb{N} \end{cases}.$$

This proves the assertion of Theorem 2.2. □

2.2.1 Examples

Now, we will consider some examples of the function $\widetilde{h} \in \mathrm{L}^1(\mathbb{R}_+)$, which describes the convergence rate to zero of the integrand in (2.12) at infinity, and $h(\omega) \sim \widetilde{h}(\omega)$, $\omega \to \infty$, where h is function from the Theorem 2.2.

Example 2.1 Let $\widetilde{h}(\omega) = \mathrm{e}^{-[\omega]}$. Since $\int_0^\infty \mathrm{e}^{-[\omega]}\,\mathrm{d}\omega = \mathrm{e}/(\mathrm{e}-1)$, we have that $\widetilde{h} \in \mathrm{L}^1(\mathbb{R}_+)$. As $\mathrm{e}^{-[\omega]} \sim \mathrm{e}^{-\omega} = h(\omega)$ when $\omega \to \infty$, by (2.11) we conclude

$$\alpha(\omega) = \begin{cases} \Gamma\big(\nu + k + \frac{1}{2}\big)\, \dfrac{\mathrm{d}}{\mathrm{d}\omega} \dfrac{\mathrm{e}^{-\omega}}{\mathscr{B}(\omega)}\Big|_{\omega = k+}, & \omega = k \in \mathbb{N} \\ \dfrac{\Gamma\big(\nu + \omega + \frac{1}{2}\big)}{\{\omega\}} \Big(\dfrac{\mathrm{e}^{-\omega}}{\mathscr{B}(\omega)} - \dfrac{\mathrm{e}^{-k}}{\mathscr{B}(k)}\Big), & 1 < \omega \neq k \in \mathbb{N} \end{cases}.$$

Example 2.2 Let $\widetilde{h}(\omega) = \dfrac{[\omega]^{\beta-1}}{\mathrm{e}^{[\omega]} - 1}$, $\beta > 1$; then

$$\int_0^\infty \widetilde{h}(\omega)\,\mathrm{d}\omega = \sum_{n \geq 1} \frac{(n-1)^{\beta-1}}{\mathrm{e}^{n-1} - 1}\,,$$

which is a convergent series, so $\widetilde{h} \in \mathrm{L}^1(\mathbb{R}_+)$. As $\omega \to \infty$ we have that

$$[\omega]^{\beta-1}\big(\mathrm{e}^{[\omega]} - 1\big)^{-1} \sim \omega^{\beta-1}\big(\mathrm{e}^{\omega} - 1\big)^{-1} = h(\omega)\,.$$

Hence $\int_0^\infty h(\omega)\,\mathrm{d}\omega = \Gamma(\beta)\zeta(\beta)$, where ζ is Riemann's ζ function. Then, for such β, (2.11) gives

$$\alpha(\omega) = \begin{cases} \Gamma\big(\nu + k + \frac{1}{2}\big)\, \dfrac{\mathrm{d}}{\mathrm{d}\omega} \dfrac{\omega^{\beta-1}}{(\mathrm{e}^{\omega} - 1)\mathscr{B}(\omega)}\Big|_{\omega = k+}, & \omega = k \in \mathbb{N} \\ \dfrac{\Gamma\big(\nu + \omega + \frac{1}{2}\big)}{\{\omega\}} \Big(\dfrac{\omega^{\beta-1}}{\mathscr{B}(\omega)\,(\mathrm{e}^{\omega} - 1)} & \\ \qquad - \dfrac{k^{\beta-1}}{\mathscr{B}(k)\,(\mathrm{e}^{k} - 1)}\Big), & 1 < \omega \neq k \in \mathbb{N} \end{cases}.$$

Example 2.3 Let $\widetilde{h}(\omega) = \mathrm{e}^{-s[\omega]} J_0([\omega])$, where $s > 1$ and J_0 is the Bessel function of the first kind of order zero. Since

$$\int_0^\infty \mathrm{e}^{-s[\omega]} J_0([\omega])\,\mathrm{d}\omega = \sum_{n \geq 1} \mathrm{e}^{-s(n-1)} J_0(n-1),$$

we see that $\widetilde{h} \in \mathrm{L}^1(\mathbb{R}_+)$. Because $\mathrm{e}^{-s[\omega]} J_0([\omega]) \sim \mathrm{e}^{-s\omega} J_0(\omega) = h(\omega)$ as $\omega \to \infty$, and $\int_0^\infty h(\omega)\,\mathrm{d}\omega = (s^2+1)^{-\frac{1}{2}}$, from (2.11) we deduce

$$\alpha(\omega) = \begin{cases} \Gamma\big(\nu+k+\frac{1}{2}\big) \dfrac{\mathrm{d}}{\mathrm{d}\omega} \dfrac{\mathrm{e}^{-s\omega} J_0(\omega)}{\mathscr{B}(\omega)}\bigg|_{\omega=k+}, & \omega = k \in \mathbb{N} \\ \dfrac{\Gamma(\nu+\omega+\frac{1}{2})}{\{\omega\}} \left(\dfrac{\mathrm{e}^{-s\omega} J_0(\omega)}{\mathscr{B}(\omega)} - \dfrac{\mathrm{e}^{-sk} J_0(k)}{\mathscr{B}(k)} \right), & 1 < \omega \neq k \in \mathbb{N} \end{cases}.$$

2.3 Integral Representations for $\mathfrak{N}_\nu(x)$ *via* Bessel Differential Equation

Previously, we introduced an integral representation (2.3) of Neumann series (2.1), compare Theorem 2.1. The purpose of this section is to establish another (indefinite) integral representations for Neumann series of Bessel functions by means of Chessin's results [51, 52] and by applying the variation of parameters method. Finally, by using fractional differintegral approach in solving the nonhomogeneous Bessel ordinary differential equation [173, 174, 328–330] we derive integral expression formulae for $\mathfrak{N}_\nu(x)$.

The listed results are taken from the paper of Baricz et al. [25].

2.3.1 The Approach by Chessin

One of the crucial arguments used in the proof of our main results is the simple fact that the Bessel functions of the first kind are actually particular solutions of the second-order homogeneous Bessel differential equation. We note that this approach in the study of the Neumann series of Bessel functions is much simpler than the previous methods which we have found in the literature. In the geometric theory of univalent functions the idea to use Bessel's differential equation is also useful in the study of geometric properties (like univalence, convexity, starlikeness, close-to-convexity) of Bessel functions of the first kind. For more details we refer to the monograph [15].

In the sequel we shall need the *Bessel functions of the second kind of order* ν (or MacDonald functions) $Y_\nu(z)$ which satisfy [224, p. 217, Eq. 10.2.3]

$$Y_\nu(z) = \mathrm{cosec}(\pi\nu)\big(J_\nu(z)\cos(\pi\nu) - J_{-\nu}(z)\big), \qquad \nu \notin \mathbb{Z},\ |\arg(z)| < \pi\,, \tag{2.15}$$

and which have the following differentiability properties [224, p. 222, Eqs. 10.5.1-2]

$$W[J_\nu, Y_\nu](z) = \frac{2}{\pi z}, \quad W[J_{-\nu}, J_\nu](z) = \frac{2\sin(\nu\pi)}{\pi z}, \qquad \nu \in \mathbb{R}, z \neq 0\,, \tag{2.16}$$

valid for the related Wronskians $W[\cdot,\cdot](z)$.

Explicit solution of Bessel differential equation with general nonhomogeneous part

$$y'' + \frac{1}{x}y' + \Big(1 - \frac{\nu^2}{x^2}\Big)y = f(x)\,, \tag{2.17}$$

has been derived for the first time in a set of articles by Chessin more than a century ago, see for example [51, 52]. In [51, p. 678] Chessin differs the cases:

- For $\nu = n \in \mathbb{Z}$ the solution is given by

$$y(x) = A(x)J_n(x) + B(x)Y_n(x), \tag{2.18}$$

and

$$A'(x) = \frac{Y_n(x)f(x)}{W[Y_n, J_n](x)} = -\frac{\pi x Y_n(x)f(x)}{2},$$
$$B'(x) = -\frac{J_n(x)f(x)}{W[Y_n, J_n](x)} = \frac{\pi x J_n(x)f(x)}{2}\,.$$

- If $\nu \notin \mathbb{Z}$, we have

$$y(x) = A_1(x)J_\nu(x) + B_1(x)J_{-\nu}(x), \tag{2.19}$$

where

$$A_1'(x) = \frac{J_{-\nu}(x)f(x)}{W[J_{-\nu}, J_\nu](x)} = \frac{\pi x J_{-\nu}(x)f(x)}{2\sin(\nu\pi)},$$
$$B_1'(x) = -\frac{J_\nu(x)f(x)}{W[J_{-\nu}, J_\nu](x)} = -\frac{\pi x J_\nu(x)f(x)}{2\sin(\nu\pi)}\,.$$

Consider the homogeneous Bessel differential equation of $(n+\nu)$-th index

$$x^2y'' + xy' + (x^2 - (n+\nu)^2)y = 0\,, \qquad n \in \mathbb{N}\,,\ 2\nu + 3 > 0\,,$$

of which particular solution is $J_{n+\nu}(x)$, that is

$$x^2J''_{n+\nu}(x) + xJ'_{n+\nu}(x) + (x^2 - (n+\nu)^2)J_{n+\nu}(x) = 0\,. \tag{2.20}$$

Multiplying (2.20) by α_n, then summing up this expression with respect to $n \in \mathbb{N}$ we arrive at

$$\begin{aligned} x^2\mathfrak{N}''_\nu(x) + x\mathfrak{N}'_\nu(x) + (x^2 - \nu^2)\mathfrak{N}_\nu(x) \\ = \sum_{n\geq 1} n(n+2\nu)\alpha_n J_{n+\nu}(x) =: \mathfrak{P}_\nu(x); \end{aligned} \tag{2.21}$$

the right side expression $\mathfrak{P}_\nu(x)$ defines the so-called *Neumann series of Bessel functions associated to* $\mathfrak{N}_\nu(x)$. Obviously (2.21) turns out to be a nonhomogeneous Bessel differential equation in unknown function $\mathfrak{N}_\nu(x)$, while by virtue of substitution $\alpha_n \mapsto n(n+2\nu)\alpha_n$, Theorem 2.1 gives

$$\mathfrak{P}_\nu(x) = -\int_1^\infty \int_0^{[u]} \frac{\partial}{\partial u}\Big(\Gamma\big(\nu+u+\tfrac{1}{2}\big)J_{\nu+u}(x)\Big)\cdot \mathfrak{d}_s\Big(\frac{s\,(s+2\nu)\,\alpha(s)}{\Gamma(\nu+s+\frac{1}{2})}\Big)\,\mathrm{d}u\,\mathrm{d}s. \tag{2.22}$$

Let us find the domain of associated Neumann series $\mathfrak{P}_\nu(x)$. Theorem 2.1 gives the same range of validity $x \in \mathscr{I}_\alpha$ by means of the estimate

$$\big|\mathfrak{P}_\nu(x)\big| \le \sum_{n\ge 1} n(n+2\nu)|\alpha_n|\,\big|J_{n+\nu}(x)\big|\,,$$

since

$$\limsup_{n\to\infty}\{n(n+2\nu)\}^{\frac{1}{n}} = 1\,.$$

Using the Landau's bound (1.20) we see that $\mathfrak{P}_\nu(x)$ is defined for all $x \in \mathscr{I}_\alpha$ when series $\sum_{n\ge 1} n^{\frac{5}{3}}\alpha_n$ absolutely converges such that clearly follows from

$$\big|\mathfrak{P}_\nu(x)\big| \le b_L \sum_{n\ge 1} \frac{n\,(n+2\nu)}{(n+\nu)^{\frac{1}{3}}}|\alpha_n|\,.$$

Now, we are ready to formulate our first main result in this section.

Theorem 2.3 (Baricz et al. [25]) *Let* $\alpha \in \mathrm{C}^1(\mathbb{R}_+)$, $\alpha|_{\mathbb{N}} = (\alpha_n)_{n\ge 1}$ *and assume that* $\sum_{n\ge 1} n^{\frac{5}{3}}\alpha_n$ *absolutely converges. Then for all* $x \in \mathscr{I}_\alpha$, $\nu > -\frac{1}{2}$ *we have*

$$\mathfrak{N}_\nu(x) = \begin{cases} \dfrac{\pi}{2}\left(J_n(x)\displaystyle\int \frac{Y_n(x)\mathfrak{P}_\nu(x)}{x}\mathrm{d}x - Y_n(x)\int \frac{J_n(x)\mathfrak{P}_\nu(x)}{x}\mathrm{d}x\right), & \nu = n \in \mathbb{Z} \\ \dfrac{\pi}{2\sin(\nu\pi)}\left(J_\nu(x)\displaystyle\int \frac{J_{-\nu}(x)\mathfrak{P}_\nu(x)}{x^2}\mathrm{d}x \right. & \\ \qquad\qquad \left. -J_{-\nu}(x)\displaystyle\int \frac{J_\nu(x)\mathfrak{P}_\nu(x)}{x^2}\mathrm{d}x\right), & \nu \notin \mathbb{Z} \end{cases}. \tag{2.23}$$

Proof It is enough to substitute $f(x) \equiv x^{-2}\mathfrak{P}_\nu(x)$ in nonhomogeneous Bessel differential equation (2.17) and calculate integrals in (2.18) and (2.19), using into account the differentiability properties (2.16). Then, by Chessin's procedure we arrive at the asserted expressions (2.23). □

Remark 2.1 Chessin's derivation procedure is in fact the variation of parameters method; here we mention that some credits in this respect should be given also to Siemon [282]. Repeating the calculations by variation of parameters method we will arrive at

$$\mathfrak{N}_\nu(x) = \frac{\pi}{2}\left(J_\nu(x) \int \frac{Y_\nu(x)\mathfrak{P}_\nu(x)}{x}\,\mathrm{d}x - Y_\nu(x) \int \frac{J_\nu(x)\mathfrak{P}_\nu(x)}{x}\,\mathrm{d}x \right),$$

where $\nu > -\frac{1}{2}$, $x \in \mathscr{I}_\alpha$. ■

Theorem 2.4 (Baricz et al. [25]) *Let the situation be the same as in* Theorem 2.3. *Then for* $\sum_{n\geq 1} n^{\frac{5}{3}} |\alpha_n| < \infty$, *we have*

$$\begin{aligned}\mathfrak{N}_\nu(x) &= \frac{J_\nu(x)}{2} \int \frac{1}{xJ_\nu^2(x)} \left(\int \frac{\mathfrak{P}_\nu(x)\cdot J_\nu(x)}{x}\,\mathrm{d}x \right) \mathrm{d}x \\ &\quad + \frac{Y_\nu(x)}{2} \int \frac{1}{xY_\nu^2(x)} \left(\int \frac{\mathfrak{P}_\nu(x)\cdot Y_\nu(x)}{x}\,\mathrm{d}x \right) \mathrm{d}x,\end{aligned}$$

where $\mathfrak{P}_\nu$ *stands for the Neumann series* (2.22) *associated with the initial Neumann series of Bessel functions* $\mathfrak{N}_\nu(x), x \in \mathscr{I}_\alpha$.

Proof We apply now the reduction of order method in solving the Bessel equation. Solution of

$$x^2 y''(x) + xy'(x) + (x^2 - \nu^2)y(x) = 0 \tag{2.24}$$

in $\mathscr{I}_\alpha$ is given by

$$y_h(x) = C_1 Y_\nu(x) + C_2 J_\nu(x).$$

It is well known that J_ν and Y_ν are independent solutions of the homogeneous Bessel differential equation (2.24), since the Wronskian $W(x) = W[J_\nu(x), Y_\nu(x)] = 2\,(\pi x)^{-1} \neq 0$, $x \in \mathscr{I}_\alpha$.

Since $J_\nu(x)$ is a solution of the homogeneous ordinary differential equation, a guess of the particular solution is $\mathfrak{N}_\nu(x) = J_\nu(x)w(x)$. Substituting this form into (2.20) we get

$$x^2(J_\nu''w + 2J_\nu'w' + J_\nu w'') + x(J_\nu'w + J_\nu w') + (x^2 - \nu^2)J_\nu w = \mathfrak{P}_\nu(x).$$

Rewriting the equation as

$$w\big(x^2 J_\nu'' + xJ_\nu' + (x^2 - \nu^2)J_\nu\big) + w'(2x^2 J_\nu' + xJ_\nu) + w''\,x^2 J_\nu = \mathfrak{P}_\nu(x),$$

the first term vanishes being J_ν solution of (2.24). So the following linear ordinary differential equation in w':

$$(w')' + \frac{2xJ_\nu' + J_\nu}{xJ_\nu}\, w' = \frac{\mathfrak{P}_\nu(x)}{x^2 J_\nu}\,.$$

Hence

$$w' = \frac{1}{xJ_\nu^2}\int \frac{\mathfrak{P}_\nu \cdot J_\nu}{x}\,\mathrm{d}x + \frac{C_3}{xJ_\nu^2}\,,$$

i.e.

$$w = \int \frac{1}{xJ_\nu^2}\Big(\int \frac{\mathfrak{P}_\nu \cdot J_\nu}{x}\,\mathrm{d}x\Big)\,\mathrm{d}x + C_3\frac{\pi}{2}\frac{Y_\nu}{J_\nu} + C_4\,,$$

because

$$\int \frac{1}{xJ_\nu^2}\,\mathrm{d}x = \frac{\pi}{2}\frac{Y_\nu}{J_\nu}\,.$$

Being J_ν, Y_ν independent, that make up the homogeneous solution, they do not contribute to the particular solution and the constants C_3, C_4 can be set to be zero.

Now, we can take particular solution in the form $\mathfrak{N}_\nu(x) = Y_\nu(x)w(x)$, and analogously as above, we get

$$\mathfrak{N}_\nu(x) = Y_\nu(x)\int \frac{1}{xY_\nu^2}\left(\int \frac{\mathfrak{P}_\nu \cdot Y_\nu}{x}\,\mathrm{d}x\right)\mathrm{d}x - C_5\frac{\pi}{2}J_\nu(x) + C_6 Y_\nu(x)\,,$$

having in mind that

$$\int \frac{1}{xY_\nu^2}\,\mathrm{d}x = -\frac{\pi}{2}\frac{J_\nu}{Y_\nu}.$$

Choosing $C_5 = C_6 = 0$, we complete the proof of the asserted result. □

2.3.2 *Solving Bessel Differential Equation by Fractional Integration*

In this section we will give the solution of nonhomogeneous Bessel differential equation, using properties associated with the fractional differintegration which was introduced in Chap. 1.

Below, we shall need the result given as the part of e.g. [174, p. 1492, Theorem 3], [329, p. 109, Theorem 3]). We recall the mentioned result in our setting. Thus, if [25, Eq. 12] $\left|\mathfrak{P}_\nu(x)\right| < \infty$, $x \in \mathscr{I}_\alpha$, $\nu \in \mathbb{R}$ and $\big(\mathfrak{P}_\nu(x)\big)_{-\mu} \neq 0$, then the nonhomogeneous linear ordinary differential equation (2.21) has a particular solution $y_p = y_p(x)$ in the form

$$y_p(x) = x^\nu \mathrm{e}^{\lambda x}\left(\left(x^{\nu-\frac{1}{2}}\mathrm{e}^{2\lambda x}\big(x^{-\nu-1}\mathrm{e}^{-\lambda x}\mathfrak{P}_\nu(x)\big)_{-\nu-\frac{1}{2}}\right)_{-1}\frac{\mathrm{e}^{-2\lambda x}}{x^{\nu+\frac{1}{2}}}\right)_{\nu-\frac{1}{2}} \tag{2.25}$$

where $\nu \in \mathbb{R}$; $\lambda = \pm\mathrm{i}$; $x \in (\mathbb{C}\setminus\mathbb{R})\cup\mathscr{I}_\alpha$, provided that $\mathfrak{P}_\nu(x)$ exists. Let us simplify (2.25), using the generalized Leibniz rule (1.36):

$$\begin{aligned}
&\big(x^{-\nu-1}\mathrm{e}^{-\lambda x}\mathfrak{P}_\nu(x)\big)_{-\nu-\frac{1}{2}}\\
&\quad= \sum_{n\geq 0}\binom{-\nu-\frac{1}{2}}{n}\big(x^{-\nu-1}\mathrm{e}^{-\lambda x}\big)_{-\nu-\frac{1}{2}-n}\big(\mathfrak{P}_\nu(x)\big)_n\\
&\quad= \sum_{n,k\geq 0}\binom{-\nu-\frac{1}{2}}{n}\binom{-\nu-\frac{1}{2}-n}{k}\big(x^{-\nu-1}\big)_{-\nu-\frac{1}{2}-n-k}\big(\mathrm{e}^{-\lambda x}\big)_k\big(\mathfrak{P}_\nu(x)\big)_n\\
&\quad= \frac{\Lambda_\nu(x)}{\pi}\sum_{n,k\geq 0}\binom{-\nu-\frac{1}{2}}{n}\binom{-\nu-\frac{1}{2}-n}{k}(-x)^n(\lambda x)^k\Gamma\big(-n-k+\tfrac{1}{2}\big)\big(\mathfrak{P}_\nu(x)\big)_n,
\end{aligned}$$

where

$$\Lambda_\nu(x) = \frac{\pi\mathrm{e}^{\mathrm{i}\pi(\nu+\frac{1}{2})-\lambda x}}{\Gamma(\nu+1)\sqrt{x}}\,.$$

By Euler's reflection formula we get

$$\begin{aligned}
&\big(x^{-\nu-1}\mathrm{e}^{-\lambda x}\mathfrak{P}_\nu(x)\big)_{-\nu-\frac{1}{2}}\\
&\quad= \Lambda_\nu(x)\sum_{n,k\geq 0}\binom{-\nu-\frac{1}{2}}{n}\binom{-\nu-\frac{1}{2}-n}{k}\frac{x^n(-\lambda x)^k\big(\mathfrak{P}_\nu(x)\big)_n}{\Gamma(n+k+\frac{1}{2})}\,.
\end{aligned}$$

Now we have

$$\begin{aligned}
&\left(x^{\nu-\frac{1}{2}}\mathrm{e}^{2\lambda x}\big(x^{-\nu-1}\mathrm{e}^{-\lambda x}\mathfrak{P}_\nu(x)\big)_{-\nu-\frac{1}{2}}\right)_{-1}\\
&\quad= \frac{\pi\mathrm{e}^{\mathrm{i}\imath(\nu+\frac{1}{2})}}{\Gamma(\nu+1)}\sum_{n,k\geq 0}\binom{-\nu-\frac{1}{2}}{n}\binom{-\nu-\frac{1}{2}-n}{k}\\
&\qquad\times\frac{(-\lambda)^k\Big(x^{\nu+n+k-1}\mathrm{e}^{\lambda x}\big(\mathfrak{P}_\nu(x)\big)_n\Big)_{-1}}{\Gamma(n+k+\frac{1}{2})}
\end{aligned}$$

$$= \frac{\pi x^\nu \, \mathrm{e}^{\mathrm{i}\pi(\nu+\frac12)+\lambda x}}{\Gamma(\nu+1)} \sum_{n,k\ge 0} \binom{-\nu-\frac12}{n}\binom{-\nu-\frac12-n}{k}$$
$$\times \frac{(-\lambda)^k x^{n+k} \left(\mathfrak{P}_\nu(x)\right)_n}{(\nu+n+k)\Gamma(n+k+\frac12)}\,.$$

Finally, after some simplification (again by Euler's reflection formula) we get

$$y_p(x) = \frac{-1}{\Gamma(\nu+1)} \sum_{n,k,\ell,m\ge 0} \binom{-\nu-\frac12}{n}\binom{-\nu-\frac12-n}{k}\binom{-\nu-\frac12}{\ell}(-x)^{\ell+n}$$
$$\times \binom{-\nu-\frac12-\ell}{m} \frac{\Gamma(\nu-\ell-m-n-k)}{\nu+n+k} (\lambda x)^{k+m} \left(\mathfrak{P}_\nu(x)\right)_{n+\ell}\,. \tag{2.26}$$

These in turn imply the following result.

Theorem 2.5 (Baricz et al. [25]) *Let $\alpha \in \mathrm{C}^1(\mathbb{R}_+)$, $\alpha|_{\mathbb{N}} = (\alpha_n)_{n\ge1}$ and assume that $\sum_{n\ge1} n^{\frac53}\alpha_n$ absolutely converges. Then for all $x \in (\mathbb{C}\setminus\mathbb{R}) \cup \mathscr{I}_\alpha$, $\nu > -\frac12$ there holds*

$$\mathfrak{N}_\nu(x) = y_p(x)\,,$$

where y_p is given by (2.26).

Remark 2.2 In [174, p. 1492, Theorem 3] it is given solution of the homogeneous differential equation

$$x^2y'' + xy' + (x^2-\nu^2)y = 0$$

in the form

$$y_h(x) = Kx^\nu \mathrm{e}^{\lambda x}\left(x^{-\nu-\frac12}\mathrm{e}^{-2\lambda x}\right)_{\nu-\frac12} \tag{2.27}$$

for all $\nu \in \mathbb{R}$; $\lambda = \pm i$; $x \in (\mathbb{C}\setminus\mathbb{R}) \cup \mathscr{I}_\alpha$ and where K is an arbitrary real constant. Then, summing (2.26) and (2.27) we can get another solution of non-homogeneous linear ordinary differential equation (2.21). ■

2.3.3 Fractional Integral Representation

Recently Lin, Srivastava and coworkers devoted articles to explicit fractional solutions of nonhomogeneous Bessel differential equation, such that turn out to be a special case of the Tricomi equation [173, 174, 329, 330]. In this section we will exploit their results to obtain further integral representation formulae for the Neumann series $\mathfrak{N}_\nu(x)$.

Using the fractional-calculus approach we obtain the following solutions of the homogeneous Bessel differential equation, depending on the parameter ν, which can be found in [328]:

- For $\nu = n + \frac{1}{2}$, $n \in \mathbb{N}_0$, the solution is given by

$$y_h(x) = K_1 J_{-n-\frac{1}{2}}(x) + K_2 J_{n+\frac{1}{2}}(x),$$

where K_1 and K_2 are arbitrary constants, and

$$J_{-n-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\Bigg(\cos\left(x+\tfrac{\pi}{2}n\right)\sum_{k=0}^{[n/2]}(-1)^k\frac{(n+2k)!}{(2k)!(n-2k)!}(2x)^{-2k} - \sin\left(x+\tfrac{\pi}{2}n\right)\sum_{k=0}^{[(n-1)/2]}(-1)^k\frac{(n+2k+1)!}{(2k+1)!(n-2k-1)!}(2x)^{-2k-1}\Bigg), \tag{2.28}$$

$$J_{n+\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\Bigg(\sin\left(x-\tfrac{\pi}{2}n\right)\sum_{k=0}^{[n/2]}(-1)^k\frac{(n+2k)!}{(2k)!(n-2k)!}(2x)^{-2k} + \cos\left(x-\tfrac{\pi}{2}n\right)\sum_{k=0}^{[(n-1)/2]}(-1)^k\frac{(n+2k+1)!}{(2k+1)!(n-2k-1)!}(2x)^{-2k-1}\Bigg). \tag{2.29}$$

- For $\nu \notin \mathbb{Z}$ the solution is

$$y_h(x) = K_1 J_{-\nu}(x) + K_2 J_\nu(x),$$

where K_1 and K_2 are arbitrary constants, and asymptotic estimates for $J_{-\nu}$ and J_ν follows from Eqs. (2.28) and (2.29), respectively, i.e.

$$J_{-\nu}(x) \sim \sqrt{\frac{2}{\pi x}}\Bigg(\cos\left(x+\tfrac{1}{2}\nu\pi-\tfrac{\pi}{4}\right)\sum_{k\ge 0}(-1)^k\frac{\Gamma(\nu+2k+\frac{1}{2})}{(2k)!\Gamma(\nu-2k+\frac{1}{2})}(2x)^{-2k} - \sin\left(x+\tfrac{1}{2}\nu\pi-\tfrac{\pi}{4}\right)\sum_{k\ge 0}(-1)^k\frac{\Gamma(\nu+2k+\frac{3}{2})}{(2k+1)!\Gamma(\nu-2k-\frac{1}{2})}(2x)^{-2k-1}\Bigg),$$

$$J_\nu(x) \sim \sqrt{\frac{2}{\pi x}}\Bigg(\cos\left(x-\tfrac{1}{2}\nu\pi-\tfrac{\pi}{4}\right)\sum_{k\ge 0}(-1)^k\frac{\Gamma(\nu+2k+\frac{1}{2})}{(2k)!\Gamma(\nu-2k+\frac{1}{2})}(2x)^{-2k} - \sin\left(x-\tfrac{1}{2}\nu\pi-\tfrac{\pi}{4}\right)\sum_{k\ge 0}(-1)^k\frac{\Gamma(\nu+2k+\frac{3}{2})}{(2k+1)!\Gamma(\nu-2k-\frac{1}{2})}(2x)^{-2k-1}\Bigg)$$

each of which is valid for large values of $|x|$ provided that $|\arg(x)| \leq \pi - \epsilon$, $0 < \epsilon < \pi$.

- In the case when $\nu = n \in \mathbb{Z}$, two linearly independent solutions which make a general solution of Bessel differential equation, are J_n and

$$Y_n(x) \underset{n\to\infty}{\sim} \sqrt{\frac{2}{\pi x}}\Bigg(\sin\left(x - \tfrac{\pi}{2}n - \tfrac{\pi}{4}\right) \sum_{k\geq 0}(-1)^k \frac{\Gamma(n+2k+\frac{1}{2})}{(2k)!\,\Gamma(n-2k+\frac{1}{2})}(2x)^{-2k}$$
$$+ \cos\left(x - \tfrac{\pi}{2}n - \tfrac{\pi}{4}\right) \sum_{k\geq 0}(-1)^k \frac{\Gamma(n+2k+\frac{3}{2})}{(2k+1)!\,\Gamma(n-2k-\frac{1}{2})}(2x)^{-2k-1}\Bigg).$$

Using the previous findings we deduce the following

Theorem 2.6 (Baricz et al. [25]) *Let the conditions from* Theorem 2.3 *hold. Then, the integral representation formulae for the function* $\mathfrak{N}_\nu(x)$ *reads as follows:*

- *for* $\nu = n + \frac{1}{2}$, $n \in \mathbb{N}_0$, *we have*

$$\mathfrak{N}_{n+\frac{1}{2}}(x) = \frac{(-1)^n\pi}{2}\Bigg(J_{n+\frac{1}{2}}(x)\int \frac{J_{-n-\frac{1}{2}}(x)\mathfrak{P}_\nu(x)}{x^2}\,\mathrm{d}x - J_{-n-\frac{1}{2}}(x)\int \frac{J_{n+\frac{1}{2}}(x)\mathfrak{P}_\nu(x)}{x^2}\,\mathrm{d}x\Bigg); \tag{2.30}$$

- *for* $\nu \notin \mathbb{Z}$, *it is*

$$\mathfrak{N}_\nu(x) = \frac{\pi}{2\sin(\nu\pi)}\left(J_\nu(x)\int \frac{J_{-\nu}(x)\mathfrak{P}_\nu(x)}{x^2}\,\mathrm{d}x - J_{-\nu}(x)\int \frac{J_\nu(x)\mathfrak{P}_\nu(x)}{x^2}\,\mathrm{d}x\right). \tag{2.31}$$

Here $J_{\mp n\mp\frac{1}{2}}(x)$ *are given in* (2.28) *and* (2.29) *respectively and* $\mathfrak{P}_\nu$ *stands for the Neumann series* (2.22) *associated with the initial Neumann series* $\mathfrak{N}_\nu(x), x \in \mathscr{I}_\alpha$.

Proof By the variation of parameters method and by virtue of (2.16) we get the representations (2.30) and (2.31). □

2.4 Integral Representations for Neumann–Bessel Type Series

In this section we cite the results from the paper by Baricz et al. [24].

Here we pose the problem of integral representation for another Neumann-type series of Bessel functions when J_ν is replaced in (2.1) by modified Bessel function of the first kind I_ν, Bessel functions and modified Bessel functions of the second kind Y_ν, K_ν (called Basset–Neumann and MacDonald functions respectively), Hankel functions $H_\nu^{(1)}, H_\nu^{(2)}$ (or Bessel functions of the third kind) of which precise descriptions can be found in [333].

According to the established nomenclatures in the sequel we will distinguish Neumann series of first and second type number of building Bessel functions, where in the second type series more then one building function occurs. So, the *first type Neumann series* are

$$\mathfrak{N}_\nu(z) := \sum_{n\geq 1} \alpha_n J_{\nu+n}(z), \qquad \mathfrak{M}_\nu(z) := \sum_{n\geq 1} \beta_n I_{\nu+n}(z) . \tag{2.32}$$

The first type Neumann series built by Bessel functions of the second kind we introduce as

$$\mathfrak{J}_\nu(z) := \sum_{n\geq 1} \delta_n K_{\nu+n}(z), \qquad \mathfrak{X}_\nu(z) := \sum_{n\geq 1} \gamma_n Y_{\nu+n}(z) . \tag{2.33}$$

In the next two sections our aim is to present closed form expressions for these Neumann series occurring in (2.32) and (2.33). Our main tools include Cahen's formula (1.15), the condensed form of Euler–Maclaurin summation formula (1.9) and certain bounding inequalities for I_ν and K_ν, see [14].

2.4.1 Integral Form of the First Type Neumann Series $\mathfrak{M}_\nu(x)$

First, we present an integral representation for the first type Neumann series $\mathfrak{M}_\nu(x)$.

Theorem 2.7 (Baricz et al. [24]) *Let $\beta \in \mathrm{C}^1(\mathbb{R}_+)$, $\beta|_{\mathbb{N}} = (\beta_n)_{n\geq 1}$ and assume that $\sum_{n\geq 1} \beta_n$ is absolutely convergent. Then, for all*

$$x \in \left(0, 2\min\left\{1, \left(\mathrm{e}\limsup_{n\to\infty} n^{-1}|\beta_n|^{\frac{1}{n}}\right)^{-1}\right\}\right) =: \mathscr{I}_\beta, \qquad \nu > -\tfrac{3}{2},$$

we have the integral representation

$$\mathfrak{M}_\nu(x) = -\int_1^\infty \int_0^{[u]} \frac{\partial}{\partial u}\left(\Gamma(\nu+u+\tfrac{1}{2})\, I_{\nu+u}(x)\right) \cdot \mathfrak{d}_s\left(\frac{\beta(s)}{\Gamma(\nu+s+\frac{1}{2})}\right) \mathrm{d}u\, \mathrm{d}s .$$

Proof First, we establish the convergence conditions of the first type Neumann series $\mathfrak{M}_\nu(x)$. By virtue of the bounding inequality [14, p. 583]:

$$I_\mu(x) < \frac{\left(\frac{x}{2}\right)^\mu}{\Gamma(\mu+1)}\, \mathrm{e}^{\frac{x^2}{4(\mu+1)}}, \qquad x>0, \mu+1>0 ,$$

and having in mind that $\mathscr{I}_\beta \subseteq (0, 2)$, we conclude that

$$|\mathfrak{M}_\nu(x)| < \max_{n\in\mathbb{N}} \frac{\left(\frac{x}{2}\right)^{\nu+n}}{\Gamma(\nu+n+1)} \mathrm{e}^{\frac{x^2}{4(\nu+n+1)}} \sum_{n\geq 1} |\beta_n| = \frac{\left(\frac{x}{2}\right)^{\nu+1}}{\Gamma(\nu+2)} \mathrm{e}^{\frac{x^2}{4(\nu+2)}} \sum_{n\geq 1} |\beta_n| \,,$$

so, the absolute convergence of $\sum_{n\geq 1} \beta_n$ suffices for the finiteness of $\mathfrak{M}_\nu(x)$ on $\mathscr{I}_\beta$. Here we used tacitly that for $x \in \mathscr{I}_\beta$ and $\nu > -1$ fixed, the function

$$\alpha \mapsto f(\alpha) = \frac{\left(\frac{x}{2}\right)^{\nu+\alpha}}{\Gamma(\nu+\alpha+1)} \mathrm{e}^{\frac{x^2}{4(\nu+\alpha+1)}}$$

is decreasing on $[\alpha_0, \infty)$, where $\alpha_0 \approx 1.4616$ denotes the abscissa of the minimum of Γ, because Γ is increasing on $[\alpha_0, \infty)$ and then

$$\frac{f'(\alpha)}{f(\alpha)} = \log\left(\frac{x}{2}\right) - \frac{x^2}{4(\nu+\alpha+1)^2} - \frac{\Gamma'(\nu+\alpha+1)}{\Gamma(\nu+\alpha+1)} \leq 0.$$

Consequently, for all $n \in \{2, 3, \dots\}$ we have $f(n) \leq f(2)$. Moreover, by using the inequality $\mathrm{e}^x \geq 1 + x$, it can be shown easily that $f(1) \geq f(2)$ for all $x > 0$ and $\nu > -1$. These in turn imply that indeed $\max_{n\in\mathbb{N}} f(n) = f(1)$, i.e.

$$\max_{n\in\mathbb{N}} \frac{\left(\frac{x}{2}\right)^{\nu+n}}{\Gamma(\nu+n+1)} \mathrm{e}^{\frac{x^2}{4(\nu+n+1)}} = \frac{\left(\frac{x}{2}\right)^{\nu+1}}{\Gamma(\nu+2)} \mathrm{e}^{\frac{x^2}{4(\nu+2)}},$$

as we required.

Now, recall the following integral representation [333, p. 79]

$$I_\nu(z) = \frac{2^{1-\nu} z^\nu}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})} \int_0^1 (1-t^2)^{\nu-\frac{1}{2}} \cosh(zt)\mathrm{d}t, \qquad z \in \mathbb{C},\ \Re(\nu) > -\tfrac{1}{2}, \tag{2.34}$$

which will be used in the sequel. Since (2.34) is valid only for $\nu > -\frac{1}{2}$, in what follows for the Neumann series $\mathfrak{M}_\nu(x)$ we suppose that $\nu > -\frac{3}{2}$. Setting (2.34) into right-hand series in (2.32) we have

$$\mathfrak{M}_\nu(x) = \sqrt{\frac{2x}{\pi}} \int_0^1 \cosh(xt) \left(\frac{x(1-t^2)}{2}\right)^{\nu-\frac{1}{2}} \mathscr{D}_\beta(t)\mathrm{d}t\,, \qquad x > 0, \tag{2.35}$$

with the Dirichlet series

$$\mathscr{D}_\beta(t) := \sum_{n\geq 1} \frac{\beta_n}{\Gamma(n+\nu+\frac{1}{2})} \exp\left(-n \log \frac{2}{x(1-t^2)}\right). \tag{2.36}$$

Following the lines of the proof of [249, Theorem] we deduce that the x-domain is

$$0 < x < 2 \min \left\{ 1, \left(\mathrm{e} \limsup_{n\to\infty} n^{-1} \sqrt[n]{|\beta_n|} \right)^{-1} \right\} .$$

For such x, the convergent Dirichlet series (2.36) possesses a Laplace integral form

$$\mathscr{D}_\beta(t) = \log \frac{2}{x(1-t^2)} \int_0^\infty \left(\frac{x(1-t^2)}{2} \right)^u \left(\sum_{j=1}^{[u]} \frac{\beta_j}{\Gamma(j+\nu+\frac{1}{2})} \right) \mathrm{d}u. \tag{2.37}$$

Expressing (2.37) *via* the condensed Euler–Maclaurin summation formula (1.9), we get

$$\mathscr{D}_\beta(t) = \log \frac{2}{x(1-t^2)} \int_0^\infty \int_0^{[u]} \left(\frac{x(1-t^2)}{2} \right)^u \cdot \mathfrak{d}_s \left(\frac{\beta(s)}{\Gamma(\nu+s+\frac{1}{2})} \right) \mathrm{d}u\, \mathrm{d}s. \tag{2.38}$$

Substituting (2.38) into (2.35) we get

$$\mathfrak{M}_\nu(x) = -\sqrt{\frac{2x}{\pi}} \int_0^\infty \int_0^{[u]} \mathfrak{d}_s \left(\frac{\beta(s)}{\Gamma(\nu+s+\frac{1}{2})} \right) \times \left(\int_0^1 \cosh(xt) \left(\frac{x(1-t^2)}{2} \right)^{\nu+u-\frac{1}{2}} \log \frac{x(1-t^2)}{2} \mathrm{d}t \right) \mathrm{d}u\, \mathrm{d}s. \tag{2.39}$$

Now, let us simplify the t-integral in (2.39)

$$\mathscr{I}_x(w) := \int_0^1 \cosh(xt) \cdot \left(\frac{x(1-t^2)}{2} \right)^w \log \frac{x(1-t^2)}{2} \, \mathrm{d}t, \qquad w := \nu + u - \tfrac{1}{2}. \tag{2.40}$$

Indefinite integration under the sign of integral in (2.40) results in

$$\int \mathscr{I}_x(w)\, \mathrm{d}w = \left(\frac{x}{2} \right)^w \int_0^1 \cosh(xt)(1-t^2)^w \, \mathrm{d}t = \sqrt{\frac{\pi}{2x}}\, \Gamma(w+1) I_{w+\frac{1}{2}}(x) .$$

Now, observing that $\mathrm{d}w = \mathrm{d}u$, we get

$$\mathscr{I}_x(\nu + u - \tfrac{1}{2}) = \sqrt{\frac{\pi}{2x}} \frac{\partial}{\partial u} \left(\Gamma(\nu + u + \tfrac{1}{2})\, I_{\nu+u}(x) \right) .$$

From (2.39) and (2.40), we immediately get the proof of the theorem, with the assertion that the integration domain $\mathbb{R}_+$ changes to $[1, \infty)$ because $[u]$ is equal to zero for all $u \in [0, 1)$. □

2.4.2 Integral Form of Second Type Neumann Series $\mathfrak{J}_\nu(x), \mathfrak{X}_\nu(x)$

Below, we present an integral representation for the Neumann-type series $\mathfrak{J}_\nu(x)$.

Theorem 2.8 (Baricz et al. [24]) *Let* $\delta \in \mathrm{C}^1(\mathbb{R}_+)$ *and let* $\delta|_{\mathbb{N}} = \big(\delta_n\big)_{n\geq 1}$. *Then for all* $\nu > 0$ *and*

$$x \in \mathscr{I}_\delta := \left(\tfrac{2}{\mathrm{e}} \limsup_{n\to\infty}\, n|\delta_n|^{\frac{1}{n}}, +\infty\right),$$

we have the integral representation

$$\mathfrak{J}_\nu(x) = -\int_1^\infty \int_0^{[u]} \frac{\partial}{\partial u} K_{\nu+u}(x)\cdot \mathfrak{d}_s\, \delta(s)\, \mathrm{d}u\, \mathrm{d}s\,.$$

Proof We begin by establishing first the convergence conditions for $\mathfrak{J}_\nu(x)$. To this aim let us consider the integral representation referred to Basset [333, p. 172]:

$$K_\nu(x) = \frac{2^\nu\, \Gamma(\nu+\frac{1}{2})}{x^\nu\sqrt{\pi}} \int_0^\infty \frac{\cos(xt)}{(1+t^2)^{\nu+\frac{1}{2}}}\,\mathrm{d}t, \qquad \Re(\nu) > -\tfrac{1}{2},\ \Re(x) > 0\,. \tag{2.41}$$

Consequently, for all $\Re(\nu) > 0$, $x > 0$ there holds

$$K_\nu(x) \leq \frac{2^\nu\, \Gamma(\nu+\frac{1}{2})}{x^\nu\sqrt{\pi}} \int_0^\infty \frac{\mathrm{d}t}{(1+t^2)^{\nu+\frac{1}{2}}} = \frac{1}{2}\left(\frac{2}{x}\right)^\nu \Gamma(\nu)\,. \tag{2.42}$$

Now, recalling that $\Gamma(s) = \sqrt{2\pi}\, s^{s-\frac{1}{2}}\mathrm{e}^{-s}\big(1+\mathscr{O}(s^{-1})\big)$, $|s| \to \infty$, we have

$$\begin{aligned}|\mathfrak{J}_\nu(x)| &\leq \frac{1}{2}\left(\frac{2}{x}\right)^\nu \sum_{n\geq 1} |\delta_n|\Gamma(\nu+n)\left(\frac{2}{x}\right)^n \\ &\sim \sqrt{\frac{\pi}{2}}\left(\frac{2}{\mathrm{e}x}\right)^\nu \sum_{n\geq 1}(\nu+n)^{\nu+n-\frac{1}{2}}|\delta_n|\left(\frac{2}{\mathrm{e}x}\right)^n,\end{aligned}$$

where the last series converges uniformly for all $\nu > 0$ and $x \in \mathscr{I}_\delta$. Note that more convenient integral representation for the modified Bessel function of the second kind is [333, p. 183]

$$K_\nu(x) = \frac{1}{2}\left(\frac{x}{2}\right)^\nu \int_0^\infty t^{-\nu-1}\mathrm{e}^{-t-\frac{x^2}{4t}}\,\mathrm{d}t, \qquad |\arg(x)| < \tfrac{\pi}{2},\ \Re(\nu) > 0. \tag{2.43}$$

Thus, combining the right-hand equality in (2.33) and (2.43) we get

$$\mathfrak{J}_\nu(x) = \frac{1}{2}\left(\frac{x}{2}\right)^\nu \int_0^\infty t^{-\nu-1} \mathrm{e}^{-t-\frac{x^2}{4t}} \cdot \mathscr{D}_\delta(t)\,\mathrm{d}t, \qquad x \in \mathscr{I}_\delta, \tag{2.44}$$

where $\mathscr{D}_\delta(t)$ is the Dirichlet series

$$\mathscr{D}_\delta(t) = \sum_{n\geq 1} \delta_n \left(\frac{x}{2t}\right)^n = \sum_{n\geq 1} \delta_n \exp\left(-n\log\frac{2t}{x}\right). \tag{2.45}$$

The Dirichlet series' parameter is necessarily positive, therefore (2.45) converges for all $x \in \mathscr{I}_\delta$. Now, the related Laplace integral and the Euler–Maclaurin summation formula give us:

$$\mathscr{D}_\delta(t) = \log\frac{2t}{x} \int_0^\infty \int_0^{[u]} \left(\frac{x}{2t}\right)^u \cdot \mathfrak{d}_s\, \delta(s)\,\mathrm{d}u\,\mathrm{d}s. \tag{2.46}$$

Substituting (2.46) into (2.44) we get

$$\mathfrak{J}_\nu(x) = -\frac{x^\nu}{2^{\nu+1}} \int_0^\infty \int_0^{[u]} \mathfrak{d}_s\delta(s) \left(\int_0^\infty \left(\frac{x}{2t}\right)^u \log\left(\frac{x}{2t}\right) t^{-\nu-1} \mathrm{e}^{-t-\frac{x^2}{4t}}\,\mathrm{d}t\right) \mathrm{d}u\,\mathrm{d}s. \tag{2.47}$$

Denoting

$$\mathscr{I}_x(u) := \int_0^\infty \left(\frac{x}{2t}\right)^u \log\left(\frac{x}{2t}\right) t^{-\nu-1} \mathrm{e}^{-t-\frac{x^2}{4t}}\,\mathrm{d}t,$$

we obtain

$$\int \mathscr{I}_x(u)\mathrm{d}u = \left(\frac{x}{2}\right)^u \int_0^\infty t^{-(\nu+u)-1} \mathrm{e}^{-t-\frac{x^2}{4t}}\,\mathrm{d}t = 2\left(\frac{2}{x}\right)^\nu K_{\nu+u}(x)\,.$$

Therefore

$$\mathscr{I}_x(u) = 2\left(\frac{2}{x}\right)^\nu \frac{\partial}{\partial u} K_{\nu+u}(x)\,. \tag{2.48}$$

Finally, by using (2.47) and (2.48) the proof of this theorem is done. □

Remark 2.3 It is worthwhile to note that, since $[x^\nu K_\nu(x)]' = -x^\nu K_{\nu-1}(x)$, the function $x \mapsto x^\nu K_\nu(x)$ is decreasing on $\mathbb{R}_+$ for all $\nu \in \mathbb{R}$, and because of the asymptotic relation $x^\nu K_\nu(x) \sim 2^{\nu-1}\Gamma(\nu)$, where $\nu > 0$ and $x \to 0$, we obtain again the inequality (2.42). This inequality is actually the counterpart of the inequality (see [125, 165])

$$x^\nu \mathrm{e}^x K_\nu(x) > 2^{\nu-1}\Gamma(\nu),$$

valid for all $\nu > \frac{1}{2}$ and $x > 0$. Moreover, by using the classical Čebyšev integral inequality, it can be shown that (see [26]) the above lower bound can be improved as follows

$$x^{\nu-1}K_\nu(x) \geq 2^{\nu-1}\Gamma(\nu)K_1(x), \tag{2.49}$$

where $\nu \geq 1$ and $x > 0$. Summarizing, for all $x > 0$ and $\nu \geq 1$, we have the following chain of inequalities

$$\frac{1}{x}\left(\frac{2}{x}\right)^{\nu-1}\Gamma(\nu)\mathrm{e}^{-x} < \left(\frac{2}{x}\right)^{\nu-1}\Gamma(\nu)K_1(x) \leq K_\nu(x) \leq \frac{1}{2}\left(\frac{2}{x}\right)^{\nu}\Gamma(\nu)\,.$$

Finally, observe that (see [26]) the inequality (2.49) is reversed when $0 < \nu \leq 1$, and this reversed inequality is actually better than (2.42) for $0 < \nu \leq 1$, that is, we have

$$x^\nu K_\nu(x) \leq 2^{\nu-1}\Gamma(\nu)xK_1(x) \leq 2^{\nu-1}\Gamma(\nu),$$

where in the last inequality we used (2.42) for $\nu = 1$. ■

Now, we deduce a closed integral expression for the Neumann series $\mathfrak{X}_\nu(x)$, by using the Struve function $\mathbf{H}_\nu$.

Theorem 2.9 (Baricz et al. [24]) *Let* $\gamma \in \mathrm{C}^1(\mathbb{R}_+)$ *and let* $\gamma|_{\mathbb{N}} = \big(\gamma_n\big)_{n\geq 1}$. *Then for all*

$$x \in \mathscr{I}_\gamma = \begin{cases} \big(0, 2(\mathrm{e}\,\ell)^{-1}\big)\,, & -\frac{1}{2} < \nu \leq \frac{1}{2} \\ \big(2L\mathrm{e}^{-1}, 2(\mathrm{e}\,\ell)^{-1}\big)\,, & \frac{1}{2} < \nu \leq \frac{3}{2} \\ \big(4L\mathrm{e}^{-1}, (\mathrm{e}\,\ell)^{-1}\big)\,, & \nu > \frac{3}{2} \end{cases}, \tag{2.50}$$

where

$$\ell := \limsup_{n\to\infty} n^{-1}|\gamma_n|^{\frac{1}{n}}, \qquad L := \limsup_{n\to\infty} n|\gamma_n|^{\frac{1}{n}}\,,$$

there holds

$$\mathfrak{X}_\nu(x) = \int_1^\infty \int_0^{[u]} \frac{\partial}{\partial u}\Big(\big(\Gamma(\nu+u+\tfrac{1}{2}) - \Gamma(\nu+u-\tfrac{1}{2})\big)\,\mathbf{H}_{\nu+u}(x)$$
$$+\Gamma(\nu+u-\tfrac{1}{2})Y_{\nu+u}(x)\Big)\cdot \mathfrak{d}_s\left(\frac{\gamma(s)}{\Gamma(\nu+s+\frac{1}{2})}\right)\mathrm{d}u\,\mathrm{d}s \tag{2.51}$$

for Neumann series of the second kind $\mathfrak{X}_\nu(x)$ *with coefficients* $\big(\gamma_n\big)_{n\geq 1}$ *satisfying*

$$\ell > \begin{cases} \mathrm{e}^{-1}, & \nu \in (-\frac{1}{2}, \frac{3}{2}] \\ (2\mathrm{e})^{-1}, & \nu > \frac{3}{2} \end{cases}, \qquad L \in \begin{cases} \big(\mathrm{e}^{-1}, 1\big), & \nu \in (-\frac{1}{2}, \frac{3}{2}] \\ \big((2e)^{-1}, \frac{1}{2}\big), & \nu > \frac{3}{2} \end{cases}. \tag{2.52}$$

Proof First we establish the convergence region and related parameter constraints upon ν for $\mathfrak{X}_\nu(x)$. The Gubler–Weber formula [333, p. 165]

$$Y_\nu(z) = \frac{2\left(\frac{z}{2}\right)^\nu}{\Gamma(\nu+\frac{1}{2})\sqrt{\pi}} \left(\int_0^1 \sin(zt)(1-t^2)^{\nu-\frac{1}{2}}\mathrm{d}t + \int_0^\infty \mathrm{e}^{-zt}(1+t^2)^{\nu-\frac{1}{2}}\,\mathrm{d}t \right), \tag{2.53}$$

where $\Re(z) > 0$ and $\nu > -\frac{1}{2}$, enables the derivation of integral expression for the Neumann series of the second type $\mathfrak{X}_\nu(x)$, by following the lines of derivation for $\mathfrak{J}_\nu(x)$. From (2.53), by means of the well-known moment inequality

$$(1+t^2)^{\nu-\frac{1}{2}} \leq C_\nu(1+t^{2\nu-1}), \qquad \text{where} \qquad C_\nu = \begin{cases} 1, & \frac{1}{2} < \nu \leq \frac{3}{2} \\ 2^{\nu-\frac{3}{2}}, & \nu > \frac{3}{2} \end{cases},$$

we distinguish the following two cases.

Assuming $\nu \in \left(\frac{1}{2}, \frac{3}{2}\right]$ we have

$$\begin{aligned} Y_\nu(x) &\leq \frac{2\left(\frac{x}{2}\right)^\nu}{\Gamma(\nu+\frac{1}{2})\sqrt{\pi}} \left(\int_0^1 (1-t^2)^{\nu-\frac{1}{2}}\mathrm{d}t + \int_0^\infty \mathrm{e}^{-xt}\left(1+t^{2\nu-1}\right)\mathrm{d}t \right) \\ &= \frac{2\left(\frac{x}{2}\right)^\nu}{\Gamma(\nu+\frac{1}{2})\sqrt{\pi}} \left(\frac{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})}{2\Gamma(\nu+1)} + x^{-1} + \frac{\Gamma(2\nu)}{x^{2\nu}} \right) \\ &= \frac{1}{\Gamma(\nu+1)}\left(\frac{x}{2}\right)^\nu + \frac{1}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})}\left(\frac{x}{2}\right)^{\nu-1} + \frac{\Gamma(\nu)}{\pi}\left(\frac{2}{x}\right)^\nu . \end{aligned}$$

Hence

$$\begin{aligned} |\mathfrak{X}_\nu(x)| \leq \left(\frac{x}{2}\right)^\nu \sum_{n\geq 1} \frac{|\gamma_n|}{\Gamma(\nu+n+1)}\left(\frac{x}{2}\right)^n + \frac{1}{\sqrt{\pi}}\left(\frac{x}{2}\right)^{\nu-1} \sum_{n\geq 1} \frac{|\gamma_n|}{\Gamma(\nu+n+\frac{1}{2})}\left(\frac{x}{2}\right)^n \\ + \frac{1}{\pi}\left(\frac{2}{x}\right)^\nu \sum_{n\geq 1} |\gamma_n|\Gamma(\nu+n)\left(\frac{2}{x}\right)^n . \end{aligned}$$

The first two series converge uniformly in $\big(0, 2(\mathrm{e}\,\ell)^{-1}\big)$, and the third one is uniformly convergent in $\big(2L\mathrm{e}^{-1}, \infty\big)$. Consequently the interval of convergence becomes $\mathscr{I}_\gamma = \big(2L\mathrm{e}^{-1}, 2(\mathrm{e}\,\ell)^{-1}\big)$, and then the coefficients γ_n satisfy the condition

$\ell \cdot L < 1$. This implies that the necessary condition for convergence of $\mathfrak{X}_\nu(x)$ is $\limsup\limits_{n\to\infty} |\gamma_n|^{\frac{1}{n}} < 1$.

In the case $\nu > \frac{3}{2}$ we have

$$\begin{aligned} Y_\nu(x) &\le \frac{2\left(\frac{x}{2}\right)^\nu}{\Gamma(\nu+\frac{1}{2})\sqrt{\pi}} \left(\int_0^1 (1-t^2)^{\nu-\frac{1}{2}} \mathrm{d}t + 2^{\nu-\frac{3}{2}} \int_0^\infty \mathrm{e}^{-xt}\left(1+t^{2\nu-1}\right) \mathrm{d}t \right) \\ &= \frac{2\left(\frac{x}{2}\right)^\nu}{\Gamma(\nu+\frac{1}{2})\sqrt{\pi}} \left(\frac{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})}{2\Gamma(\nu+1)} + 2^{\nu-\frac{3}{2}}\left(x^{-1} + \frac{\Gamma(2\nu)}{x^{2\nu}}\right)\right) \\ &= \frac{1}{\Gamma(\nu+1)}\left(\frac{x}{2}\right)^\nu + \frac{x^{\nu-1}}{\sqrt{2\pi}\,\Gamma(\nu+\frac{1}{2})} + \frac{2^{2\nu-\frac{3}{2}}\Gamma(\nu)}{\pi x^\nu}. \end{aligned}$$

Therefore

$$\begin{aligned} |\mathfrak{X}_\nu(x)| \le \left(\frac{x}{2}\right)^\nu \sum_{n\ge1} \frac{|\gamma_n|}{\Gamma(\nu+n+1)}\left(\frac{x}{2}\right)^n + \frac{x^{\nu-1}}{\sqrt{2\pi}} \sum_{n\ge1} \frac{|\gamma_n| x^n}{\Gamma(\nu+n+\frac{1}{2})} \\ + \frac{1}{2\pi\sqrt{2}}\left(\frac{4}{x}\right)^\nu \sum_{n\ge1} |\gamma_n|\Gamma(\nu+n)\left(\frac{4}{x}\right)^n. \end{aligned}$$

The first two series converge in $\left(0, 2(\mathrm{e}\,\ell)^{-1}\right)$, $\left(0, (\mathrm{e}\,\ell)^{-1}\right)$ respectively, while the third series converges uniformly for all $x > 4L\mathrm{e}^{-1}$. This yields the interval of convergence $\mathscr{I}_\gamma = \left(4L\mathrm{e}^{-1}, (\mathrm{e}\,\ell)^{-1}\right)$. In this case the coefficients γ_n satisfy the constraint $4\ell L < 1$, and then the necessary condition for convergence of $\mathfrak{X}_\nu(x)$ is $\limsup\limits_{n\to\infty} |\gamma_n|^{\frac{1}{n}} < \frac{1}{2}$.

It remains the case $-\frac{1}{2} < \nu \le \frac{1}{2}$. Then, because of $(1+t^2)^{\nu-\frac{1}{2}} \le 1$, we conclude

$$Y_\nu(x) \le \frac{1}{\Gamma(\nu+1)}\left(\frac{x}{2}\right)^\nu + \frac{1}{\Gamma(\nu+\frac{1}{2})\sqrt{\pi}}\left(\frac{x}{2}\right)^{\nu-1},$$

and consequently $\mathscr{I}_\gamma = \left(0, 2(\mathrm{e}\,\ell)^{-1}\right)$. Collecting these cases we get (2.50) and (2.52).

Now, let us focus on the integral representation for $\mathfrak{X}_\nu(x)$, where $x \in \mathscr{I}_\gamma$. By the Gubler–Weber formula (2.53) we have

$$\begin{aligned} \mathfrak{X}_\nu(x) = \frac{2}{\sqrt{\pi}}\left(\frac{x}{2}\right)^\nu \sum_{n\ge1} \frac{\gamma_n}{\Gamma(\nu+n+\frac{1}{2})}\left(\frac{x}{2}\right)^n \\ \times \left(\int_0^1 \sin(xt)(1-t^2)^{\nu+n-\frac{1}{2}}\mathrm{d}t + \int_0^\infty \mathrm{e}^{-xt}(1+t^2)^{\nu+n-\frac{1}{2}}\,\mathrm{d}t \right). \end{aligned} \tag{2.54}$$

The first expression in (2.54) we rewrite as

$$\begin{aligned}\Sigma_1(x) &= \frac{2}{\sqrt{\pi}}\left(\frac{x}{2}\right)^{\nu}\sum_{n\geq 1}\frac{\gamma_n\left(\frac{x}{2}\right)^n}{\Gamma(\nu+n+\frac{1}{2})}\int_0^1 \sin(xt)(1-t^2)^{\nu+n-\frac{1}{2}}\,\mathrm{d}t \\ &= \sqrt{\frac{2x}{\pi}}\int_0^1 \sin(xt)\left(\frac{x(1-t^2)}{2}\right)^{\nu-\frac{1}{2}}\mathscr{D}_\gamma(t)\,\mathrm{d}t,\end{aligned}$$

where

$$\mathscr{D}_\gamma(t) := \sum_{n\geq 1}\frac{\gamma_n}{\Gamma(n+\nu+\frac{1}{2})}\exp\left(-n\log\frac{2}{x(1-t^2)}\right)$$

is the Dirichlet series analogous to one in (2.36). It is easy to see that in view of (2.52) for all $x \in \mathscr{I}_\gamma$ and $t \in (0,1)$ we have

$$\log\frac{2}{x(1-t^2)} > 0.$$

More precisely, if $-\frac{1}{2} < \nu \leq \frac{3}{2}$, then $x < 2(\mathrm{e}\,\ell)^{-1}$, and

$$\frac{2}{x(1-t^2)} > \frac{\mathrm{e}\,\ell}{1-t^2} > \mathrm{e}\,\ell > 1\,.$$

Similarly, if $\nu > \frac{3}{2}$, then $x < (\mathrm{e}\,\ell)^{-1}$, and

$$\frac{2}{x(1-t^2)} > \frac{2\mathrm{e}\,\ell}{1-t^2} > 2\mathrm{e}\,\ell > 1\,.$$

Thus, the Dirichlet series' parameter is necessarily positive, and therefore $\mathscr{D}_\gamma(t)$ converges for all $x \in \mathscr{I}_\gamma$.

Following the same lines as in the proof of Theorem 2.7 we deduce that

$$\Sigma_1(x) = -\int_0^\infty\int_0^{[u]}\mathfrak{d}_s\left(\frac{\gamma(s)}{\Gamma(\nu+s+\frac{1}{2})}\right)\frac{\partial}{\partial u}\left(\Gamma\left(\nu+u+\tfrac{1}{2}\right)\mathbf{H}_{\nu+u}(x)\right)\mathrm{d}u\,\mathrm{d}s, \tag{2.55}$$

where $\mathbf{H}_\nu$ stands for the familiar Struve function.

Below, we will simplify the second expression in (2.54):

$$\begin{aligned}\Sigma_2(x) &= \frac{2}{\sqrt{\pi}}\left(\frac{x}{2}\right)^{\nu}\sum_{n\geq 1}\frac{\gamma_n\left(\frac{x}{2}\right)^n}{\Gamma(\nu+n+\frac{1}{2})}\int_0^\infty \mathrm{e}^{-xt}(1+t^2)^{\nu+n-\frac{1}{2}}\,\mathrm{d}t \\ &= \sqrt{\frac{2x}{\pi}}\int_0^\infty \mathrm{e}^{-xt}\left(\frac{x(1+t^2)}{2}\right)^{\nu-\frac{1}{2}}\widetilde{\mathscr{D}}_\gamma(t)\,\mathrm{d}t,\end{aligned}$$

where $\widetilde{\mathscr{D}}_\gamma(t) = \mathscr{D}_\gamma(\mathrm{i}t)$. Thus,

$$\begin{aligned}
\Sigma_2(x) &= -\sqrt{\frac{2x}{\pi}}\int_0^\infty\int_0^{[u]} \mathfrak{d}_s\left(\frac{\gamma(s)}{\Gamma(\nu+s+\frac12)}\right)\\
&\quad\times\left(\int_0^\infty \mathrm{e}^{-xt}\left(\frac{x(1+t^2)}{2}\right)^{\nu+u-\frac12}\log\frac{x(1+t^2)}{2}\mathrm{d}t\right)\mathrm{d}u\,\mathrm{d}s\\
&= -\pi\int_0^\infty\int_0^{[u]} \mathfrak{d}_s\left(\frac{\gamma(s)}{\Gamma(\nu+s+\frac12)}\right)\cdot\frac{\partial}{\partial u}\frac{1}{\Gamma(\frac12-\nu-u)}\\
&\quad\times\left(\frac{2J_{-\nu-u}(x)}{\sin 2\pi(\nu+u)}-\frac{J_{\nu+u}(x)}{\sin\pi(\nu+u)}+\frac{\mathbf{H}_{\nu+u}(x)}{\cos\pi(\nu+u)}\right)\mathrm{d}u\,\mathrm{d}s\\
&= \int_1^\infty\int_0^{[u]}\frac{\partial}{\partial u}\Gamma\left(\nu+u-\tfrac12\right)\left(Y_{\nu+u}(x)-\mathbf{H}_{\nu+u}(x)\right)\mathfrak{d}_s\left(\frac{\gamma(s)}{\Gamma(\nu+s+\frac12)}\right)\mathrm{d}u\,\mathrm{d}s\,.
\end{aligned} \tag{2.56}$$

Here we applied the Euler's reflection formula and the well-known property of the Bessel functions which was noted in Eq. (2.15). Summing (2.55) and (2.56) we have the desired integral representation (2.51). □

Remark 2.4 Another two linearly independent solutions of the Bessel homogeneous differential equation are *the Hankel functions* $H_\nu^{(1)}$ and $H_\nu^{(2)}$ which can be expressed as [333, p. 73]

$$H_\nu^{(1)}(x) = \frac{J_{-\nu}(x)-\mathrm{e}^{-\nu\pi\mathrm{i}}J_\nu(x)}{\mathrm{i}\sin(\nu\pi)}\,, \tag{2.57}$$

$$H_\nu^{(2)}(x) = \frac{J_{-\nu}(x)-\mathrm{e}^{\nu\pi\mathrm{i}}J_\nu(x)}{-\mathrm{i}\sin(\nu\pi)}\,, \tag{2.58}$$

which build the third type Neumann series:

$$\mathfrak{N}_{1,\nu}^{H^{(\kappa)}}(z) = \sum_{n\ge1}\alpha_n H_{\nu+n}^{(\kappa)}(z), \qquad \kappa = 1,2\,.$$

Using formulae (2.57), (2.58) we see that integral expressions for third type Neumann series are linear combinations of similar fashion integrals achieved for $\mathfrak{N}_\nu(x)$ in Theorem 2.1. ■

2.5 Integral Form of Neumann Series $\mathfrak{N}^{a,b}_{\mu,\nu}(x)$

In our investigations regarding Turán type determinants of Bessel functions we also aimed to establish integral formula for the second type Neumann type series of Bessel J as

$$\mathfrak{N}^{a,b}_{\mu,\nu}(x) := \sum_{n\geq 1} \alpha_n J_{\mu+an}(x) J_{\nu+bn}(x), \qquad \mu, \nu, a, b \in \mathbb{R}\,. \tag{2.59}$$

This was motivated by the fact that $\mathfrak{N}^{2,2}_{\nu,\nu}(x)$ constitutes the right-hand side series in von Lommel's expression for all $x \in \mathbb{R}$, $\nu > -1$ [333, p. 152]

$$x^2 \left[J_\nu^2(x) - J_{\nu-1}(x) J_{\nu+1}(x) \right] = 4 \sum_{n\geq 0} (\nu + 1 + 2n) J^2_{\nu+1+2n}(x), \tag{2.60}$$

and for the Al-Salam series [5]

$$\begin{aligned} &\frac{4^m (2m)!}{x^{2m} m! (m-1)!} \sum_{k\geq 0} (\nu + m + 2k)(k+1)_{m-1} (\nu + k + 1)_{m-1} J^2_{\nu+m+2k}(x) \\ &\quad = \sum_{n=-m}^{m} (-1)^n \binom{2m}{m-n} J_{\nu-n}(x) J_{\nu+n}(x), \end{aligned} \tag{2.61}$$

while $\mathfrak{N}^{1,1}_{\nu,\nu}(x)$ covers the series considered in [311]. Also, $\mathfrak{N}^{1,-1}_{\nu,\nu}(x)$ appears in (2.61), and $\mathfrak{N}^{1,-1}_{n,n+2k}(x)$ occurs in our study [21]. In order to obtain the integral representation formula for (2.59) we shall use the main idea from [249], that is, Cahen's Laplace integral representation of the associated Dirichlet series. Thus, we take $x \in \mathbb{R}_+$ and assume in the sequel that the behavior of $(\alpha_n)_{n\geq 1}$ ensures the convergence of the series (2.59) over $\mathbb{R}_+$.

Theorem 2.10 (Baricz and Pogány [21]) *Let $\alpha \in \mathrm{C}^1(\mathbb{R}_+)$, $\alpha\big|_{\mathbb{N}} = (\alpha_n)_{n\geq 1}$ and assume that series $\sum_{n\geq 1} \alpha_n n^{-\frac{2}{3}}$ is absolutely convergent. Then, for all $a, b > 0$ such that*

$$0 < x < 2 \min \left\{ 1, \frac{1}{\mathrm{e}} \left(a^a b^b / \rho^{a,b}_{\mathfrak{N}} \right)^{\frac{1}{a+b}} \right\} = \mathscr{I}_{\mathfrak{N}}, \quad \min\{\mu + a, \nu + b\} > 0, \tag{2.62}$$

where

$$\rho^{a,b}_{\mathfrak{N}} = \limsup_{n\to\infty} n^{-(a+b)} |\alpha_n|^{\frac{1}{n}}\,,$$

we have that

$$\mathfrak{N}_{\mu,\nu}^{a,b}(x) = \int_1^\infty \int_0^{[u]} \frac{\partial}{\partial u}\big(\Gamma(\mu+au+1)\Gamma(\nu+bu+1)\, J_{\mu+au}(x)J_{\nu+bu}(x)\big) \times \mathfrak{d}_v \left(\frac{-\alpha(v)}{\Gamma(\mu+av)\Gamma(\nu+bv)}\right) \mathrm{d}u\, \mathrm{d}v\,. \tag{2.63}$$

Proof Landau's bound (1.20) gives the estimate upon $\mathfrak{N}_{\mu,\nu}^{a,b}(x)$

$$\big|\mathfrak{N}_{\mu,\nu}^{a,b}(x)\big| \le b_L^2 \sum_{n\ge1} \frac{|\alpha_n|}{\sqrt[3]{(\mu+an)(\nu+bn)}} \sim \frac{b_L^2}{\sqrt[3]{ab}} \sum_{n\ge1} \frac{|\alpha_n|}{n^{\frac{2}{3}}}\,,$$

therefore $\mathfrak{N}_{\mu,\nu}^{a,b}(x)$ absolutely and uniformly converges for $x > 0$. Taking the integral expression (2.4) (listed also in [333, p. 48]),

$$J_\nu(x) = \frac{2\left(\frac{x}{2}\right)^\nu}{\sqrt{\pi}\,\Gamma(\nu+\frac12)} \int_0^1 \cos(xt)(1-t^2)^{\nu-\frac12}\mathrm{d}t, \qquad x\in\mathbb{R},\ \nu > -\tfrac12,$$

in (2.59) we get

$$\mathfrak{G}_{\mu,\nu}^{a,b}(x) = \frac{4}{\pi}\left(\frac{x}{2}\right)^{\mu+\nu} \int_0^1\int_0^1 \frac{\cos(xt)\cos(xs)}{(1-t^2)^{\frac12-\mu}(1-s^2)^{\frac12-\nu}}\,\mathscr{D}_\alpha(t,s)\,\mathrm{d}t\mathrm{d}s\,, \tag{2.64}$$

where we should obtain the Dirichlet series'

$$\mathscr{D}_\alpha(t,s) = \sum_{n\ge1} \frac{\alpha_n\left((x/2)^{a+b}(1-t^2)^a(1-s^2)^b\right)^n}{\Gamma(\mu+an)\Gamma(\nu+bn)}$$

x-domain of convergence. Expressing $\mathscr{D}_\alpha(t,s)$ by the Cahen's formula (1.15) it is necessary to have positive Dirichlet parameter, that is,

$$-\log\left(x/2\right)^{a+b}(1-t^2)^a(1-s^2)^b > 0,$$

which holds for all $|x| < 2$ when $a+b > 0$. Also $\mathscr{D}_\alpha(t,s)$ is equi-convergent to the auxiliary power series

$$\sum_{n\ge1} \frac{\alpha_n}{n^{(a+b)n}} \left(\frac{(\mathrm{e}x)^{a+b}(1-t^2)^a(1-s^2)^b}{2^{a+b}a^ab^b}\right)^n$$

with radius of convergence

$$\rho_{\mathfrak{N}}^{a,b} = \left(\limsup_{n\to\infty} \frac{|\alpha_n|^{\frac{1}{n}}}{n^{a+b}} \right)^{-1} .$$

This yields the convergence region $\mathscr{I}_{\mathfrak{N}}$ described in (2.62).

Next, by the Cahen's formula (1.15), $\mathscr{D}_\alpha(t, s)$ becomes

$$\begin{aligned}
\mathscr{D}_\alpha(t,s) &= \sum_{n\ge1} \frac{\alpha_n}{\Gamma(\mu+an)\Gamma(\nu+bn)} \exp\left\{ -n \log \frac{2^{a+b}}{x^{a+b}(1-t^2)^a(1-s^2)^b} \right\} \\
&= \log \frac{2^{a+b}}{x^{a+b}(1-t^2)^a(1-s^2)^b} \int_0^\infty \left(\frac{x^{a+b}(1-t^2)^a(1-s^2)^b}{2^{a+b}} \right)^u \\
&\quad \times \left(\sum_{n=1}^{[u]} \frac{\alpha_n}{\Gamma(\nu+an)\Gamma(\mu+bn)} \right) \mathrm{d}u \\
&= \log \frac{2^{a+b}}{x^{a+b}(1-t^2)^a(1-s^2)^b} \int_0^\infty \left(\frac{x^{a+b}(1-t^2)^a(1-s^2)^b}{2^{a+b}} \right)^u \\
&\quad \times \int_0^{[u]} \mathfrak{d}_v \left(\frac{\alpha(v)\,\mathrm{d}v}{\Gamma(\nu+an)\Gamma(\mu+bn)} \right) \mathrm{d}u ,
\end{aligned}$$

where the last equality we deduced by virtue of condensed Euler–Maclaurin summation formula (1.16). The last formula in conjunction with (2.64) gives

$$\mathfrak{N}_{\mu,\nu}^{a,b}(x) = \frac{4}{\pi} \left(\frac{x}{2}\right)^{\mu+\nu} \int_0^\infty \int_0^{[u]} \mathscr{J}_{t,s}(u)\, \mathfrak{d}_v \left(\frac{\alpha(v)}{\Gamma(\nu+an)\Gamma(\mu+bn)} \right) \mathrm{d}u\mathrm{d}v ,$$

where

$$\begin{aligned}
\mathscr{J}_{t,s}(u) &= -\left(\frac{x}{2}\right)^{(a+b)u} \int_0^1 \int_0^1 \cos(xt)\cos(xs)\,(1-t^2)^{\mu+au-\frac12}(1-s^2)^{\nu+bv-\frac12} \\
&\quad \times \log\left(\frac{x^{a+b}(1-t^2)^a(1-s^2)^b}{2^{a+b}} \right) \mathrm{d}t\mathrm{d}s .
\end{aligned}$$

Because

$$\int \mathscr{J}_{t,s}(u)\,\mathrm{d}u = -\left(\frac{x}{2}\right)^{(a+b)u} \mathscr{I}_{\mu,a}(u)\mathscr{I}_{\nu,b}(u) ,$$

where

$$\mathscr{I}_{\mu,a}(u) = \int_0^1 \cos(xt)(1-t)^{\mu+au-\frac12}\mathrm{d}t = \frac{\sqrt{\pi}}{2}\left(\frac{x}{2}\right)^{-\mu-au} \Gamma(\mu+au+1)J_{\mu+au}(x),$$

there holds

$$\mathscr{I}_{t,s}(u) = -\frac{\pi}{4}\left(\frac{x}{2}\right)^{-\mu-\nu} \frac{\partial}{\partial u}\big(\Gamma(\mu+au+1)\Gamma(\nu+bu+1)J_{\mu+au}(x)J_{\nu+bu}(x)\big)\,,$$

which immediately implies the asserted formula (2.63). □

In the preliminary part of this section we mentioned the equalities by von Lommel, Thiruvenkatachar and Nanjundiah and Al-Salam. By particular choice of a and b we conclude

Corollary 2.1 (Baricz and Pogány [21]) *If* $\nu > 2$ *and*

$$x \in \left(0, 2\min\left\{1, 2\left(\mathrm{e}^4\rho_{\mathfrak{N}}^{2,2}\right)^{-\frac{1}{4}}\right\}\right),$$

then we have that

$$\mathfrak{N}_{\nu-2,\nu-2}^{2,2}(x) = \int_1^\infty \int_0^{[u]} \frac{\partial}{\partial u}\big(\Gamma^2(\nu-1+2u)J_{\nu-2+2u}^2(x)\big) \times \mathfrak{d}_v\left(-\frac{(\nu-2+2v)\Gamma(-1+v)\Gamma(\nu-1+v)}{\Gamma^2(\nu-2-2v)\Gamma(v)\Gamma(\nu+v)}\right)\mathrm{d}u\mathrm{d}v\,. \qquad (2.65)$$

Moreover, for $\nu > 0$ *and*

$$x \in \left(0, 2\min\left\{1, \left(\mathrm{e}^2\rho_{\mathfrak{N}}^{1,1}\right)^{-\frac{1}{2}}\right\}\right)$$

there holds true

$$\mathfrak{N}_{\nu+1,\nu+1}^{1,1}(x) = \int_1^\infty \int_0^{[u]} \frac{\partial}{\partial u}\big(\Gamma^2(\nu+u+2)J_{\nu+u+2}^2(x)\big) \times \mathfrak{d}_v\left(-\frac{(\nu+v)^{-1}(\nu+v+2)^{-1}}{\Gamma^2(\nu+v+1)}\right)\mathrm{d}u\mathrm{d}v. \qquad (2.66)$$

Remark 2.5 We note that the second type Neumann series in von Lommel's formula (2.60) possesses divergent auxiliary series $\sum_{n\ge1}\alpha_n n^{-\frac{2}{3}}$, therefore it is not covered by Theorem 2.10. Also, Al-Salam's series (2.61), converges only when $m < \frac{1}{3}$, so the formula (2.65). The auxiliary series associated with the Neumann series by Thiruvenkatachar and Nanjundiah is equiconvergent to the Riemannian $\zeta(\frac{8}{3})$, thus this case meet Theorem 2.10, see (2.66). Finally, the second type finite sum (2.61) has to be studied separately, being the parameter space of $\mathfrak{N}_{\mu,\nu}^{a,b}(x)$ restricted by the required positivity of upper parameters a, b in Theorem 2.10. ■

2.5.1 Second Type Neumann Series $\mathfrak{N}_{\mu,\nu}^{\alpha I_\eta}(x)$ and $\mathfrak{N}_{\mu,\nu}^{\alpha K_\nu}(x)$

Both modified Bessel functions of the first and second kind I_ν, K_ν are frequently considered in physics, applied mathematics and engineering applications. The product $I_\nu K_\nu$ is also used in some application items, see e.g. [261, 262] which concern the hydrodynamic and hydromagnetic instability of certain cylindrical models, in which the monotonicity of $x \mapsto I_\nu(x)K_\nu(x)$ for $\nu > 1$ is used. Also, different kind proofs on the monotonicity of $I_\nu(x)K_\nu(x)$ can be found in the recent article [13]. We focus here on integral representations for second type Neumann series of modified Bessel functions I_ν and K_μ in the manner of previous results exposed in a set of articles [21, 24, 25, 249] for the first type Neumann series.

We introduce a second type Neumann-series

$$\mathfrak{N}_{\nu,\mu}^{\alpha}(z) := \sum_{n\geq 1} \alpha_n \, I_{\nu+n}(z) K_{\mu+n}(z) \, .$$

Our main derivation tools include Cahen's Laplace integral form of the Dirichlet series (1.15), the condensed form of Euler–Maclaurin summation formula (1.9) and certain bounding inequalities for I_ν and K_ν, see [14]. Our goal is to give integral representations for the second type Neumann series $\mathfrak{N}_{\nu,\mu}^{\alpha}(x)$, $\mathfrak{N}_{\nu,\mu}^{\alpha I_\eta}(x)$ and $\mathfrak{N}_{\nu,\mu}^{\alpha K_\eta}(x)$. Obviously in the last two series $\alpha I_\eta \mapsto \alpha$, that is $\alpha K_\eta \mapsto \alpha$ was used, which means that both underlying second type Neumann series consist from products of three modified Bessel functions of the first and second kind:

$$\mathfrak{N}_{\nu,\mu}^{\alpha I_\eta}(x) = \sum_{n\geq 1} \alpha_n I_{\eta+n}(x) I_{\nu+n}(x) K_{\mu+n}(x)$$

$$\mathfrak{N}_{\nu,\mu}^{\alpha K_\eta}(x) = \sum_{n\geq 1} \alpha_n I_{\nu+n}(x) K_{\mu+n}(x) K_{\eta+n}(x) \, .$$

Finally, our aim is to establish indefinite integral representation formulae for the one-parameter second type Neumann series of the product of two modified Bessel functions of the first kind $P_\nu = I_\nu \, K_\nu$, observing that it is a particular solution of the homogeneous third order ordinary differential equation [20, p. 816, Eq. (17)]

$$x^2 y'''(x) + 3xy''(x) - (4\nu^2 + 4x^2 - 1)y'(x) - 4xy(x) = 0.$$

Also, let us define

$$\mathfrak{H}_{\nu,\nu}^{\alpha}(x) =: 4 \sum_{n\geq 1} n(n+2\nu)\alpha_n I_{n+\nu}(x) K_{n+\nu}(x) \, ,$$

the *second type Neumann series of modified Bessel functions associated with the Neumann series* $\mathfrak{N}_{\nu,\nu}^{\mu}(x)$.

By these considerations we finish the first essay, the one in which another view to Fourier–Bessel Neumann series was exposed.

2.6 Properties of Product of Modified Bessel Functions

In Sect. 2.5.1 we already mentioned some applications of the product $x \mapsto P_\nu(x) := I_\nu(x)K_\nu(x)$ [261, 262]; see also the paper of Hasan [107], where the electrogravitational instability of non-oscillating streaming fluid cylinder under the action of the selfgravitating, capillary and electrodynamic forces has been discussed. In these papers the authors use (without proof) the inequality

$$P_\nu(x) < \tfrac{1}{2}$$

for all $\nu > 1$ and $x > 0$. We note that the above inequality readily follows from the fact that $x \mapsto P_\nu(x)$ is decreasing on $\mathbb{R}_+$ for all $\nu > -1$. More precisely, for all $x > 0$ and $\nu > 1$ we have

$$P_\nu(x) < \lim_{x\to 0} P_\nu(x) = \tfrac{1}{2\nu} < \tfrac{1}{2}.$$

For different proofs on the monotonicity of the function $x \mapsto P_\nu(x)$ we refer to the papers [13, 234, 237]. It is worth to mention that the above monotonicity property has been used also in a problem in biophysics (see [95]). Moreover, recently Klimek and McBride [152] used this monotonicity to prove that a Dirac operator (subject to Atiyah–Patodi–Singer-like boundary conditions on the solid torus) has a bounded inverse, which is actually a compact operator. In [319, 320] van Heijster et al. investigated the existence, stability and interaction of localized structures in a one-dimensional generalized FitzHugh–Nagumo type model. Recently, van Heijster and Sandstede [318] started to analyze the existence and stability of radially symmetric solutions in the planar variant of this model. The product of modified Bessel functions P_ν arises naturally in their stability analysis, and the monotonicity (see [22, 318]) of $\nu \mapsto P_\nu(x)$ is important to conclude (in)stability of these radially symmetric solutions.

In this section, motivated by the above applications, we focus on Chebyshev-type discrete inequalities for Neumann series of modified Bessel functions I_ν and K_μ of the first and the second kind, respectively. Moreover, we deduce integral representations formulae for these Neumann series appearing in newly derived discrete Chebyshev inequalities in the manner of such results given recently by Baricz, Jankov, Pogány and Süli in a set of articles [21, 24, 25, 249] for the first type Neumann series.

In the sequel we will consider *first type Neumann series* introduced in (2.32) and (2.33) as

$$\mathfrak{M}_\nu^\mu(z) := \sum_{n\geq 1} \mu_n\, I_{\nu+n}(z) \quad \text{and} \quad \mathfrak{J}_\nu^\mu(z) := \sum_{n\geq 1} \mu_n\, K_{\nu+n}(z)\,. \tag{2.67}$$

In this section our aim is to present the Chebyshev-type discrete inequality in the terminology of Neumann-series (2.67) and its closed form integral representation. In this goal we consider the (in Sect. 2.5.1) introduced second type Neumann-series

$$\mathfrak{N}^{\mu}_{\nu,\eta}(z) := \sum_{n\geq 1} \mu_n \, I_{\nu+n}(z) K_{\eta+n}(z) \, .$$

Our main derivation tools include Cahen's Laplace integral form of a Dirichlet series (1.15) (see the exact proof in Perron's article [235]), the condensed form of Euler–Maclaurin summation formula (1.9) and certain bounding inequalities for I_ν and K_ν, see [14].

2.6.1 Discrete Chebyshev Inequalities

We begin with the discrete form of the celebrated Chebyshev inequality reported (in part) by Graham [94, p. 116]. Here, and in what follows let μ be a nonnegative discrete measure, $\mu(n) \equiv \mu_n$, $n \in \mathbb{N}$. Assuming f, g are both nonnegative and same (opposite) kind monotone, then

$$\sum_{n\geq 1} \mu_n f(n) \sum_{n\geq 1} \mu_n g(n) \;\leq\; (\geq)\; \|\mu\|_1 \sum_{n\geq 1} \mu_n f(n) g(n) \, , \tag{2.68}$$

where $\|\cdot\|_1$ stands for the appropriate ℓ_1-norm. Let us signify throughout

$$\|\mathbb{N}^\alpha \mu\|_1 := \sum_{n\geq 1} n^\alpha \mu_n \, , \qquad \alpha \in \mathbb{R} \, .$$

Now, let us recall some monotonicity properties of modified Bessel functions. Jones [143] proved that $I_{\nu_1}(x) < I_{\nu_2}(x)$ holds for all $x > 0$ and $\nu_1 > \nu_2 \geq 0$, while Cochran [56] and Reudink [267] established the inequality $\partial I_\nu(x)/\partial \nu < 0$ for all $x, \nu > 0$. In other words, the function $\nu \mapsto I_\nu(x)$ is strictly decreasing on $\mathbb{R}_+$ for all $x > 0$ fixed. Moreover, as it was pointed out by Laforgia [165], the function $\nu \mapsto K_\nu(x)$ is strictly increasing on $\mathbb{R}_+$ for all $x > 0$ fixed. Finally, recall that recently in [22, 318] it was proved the function $\nu \mapsto P_\nu(x)$ is strictly decreasing on $\mathbb{R}_+$ for all $x > 0$ fixed.

Having in mind these properties we can see that modified Bessel functions of the first and second kind I_ν, K_η and also their equal order product P_ν are ideal candidates to establish discrete Chebyshev inequalities of the type (2.68).

Our first main result is the following theorem.

Theorem 2.11 (Baricz and Pogány [20]) *Let $\nu, \eta > 0$ and let μ be a positive discrete measure on $\mathbb{N}$ such that $\|\mu\|_1 < \infty$, not necessarily the same in different* *occasions. Then the following assertions are true:*

(a) *For all fixed* $x \in \mathscr{I}_0 := \big(2\mathrm{e}^{-1}\,\limsup_{n\to\infty} n\mu_n^{\frac{1}{n}}, \infty\big)$ *we have*

$$\mathfrak{M}_\nu^\mu(x)\,\mathfrak{I}_\eta^\mu(x) \geq \|\mu\|_1\,\mathfrak{N}_{\nu,\eta}^\mu(x). \tag{2.69}$$

(b) *For all fixed* $x \in \mathscr{I}_1 := \big(0, 2\mathrm{e}^{-1}/\limsup_{n\to\infty} n^{-1}\mu_n^{\frac{1}{n}}\big)$ *, it holds*

$$\|\mu\|_1\,\mathfrak{N}_{\nu,\eta}^{\mu I_\eta}(x) \geq \mathfrak{M}_\nu^\mu(x)\,\mathfrak{N}_{\eta,\eta}^\mu(x), \tag{2.70}$$

whenever $\|\mathbb{N}^{(\eta-\nu-1)_+}\mu\|_1 < \infty$, *where* $(a)_+ = \max\{0, a\}$.

(c) *Moreover, for all fixed* $x \in \mathscr{I}_0$ *and* $\|\mathbb{N}^{(\eta-\nu-1)_+}\,\mu\|_1 < \infty$ *we have*

$$\mathfrak{I}_\nu^\mu(x)\,\mathfrak{N}_{\nu,\eta}^\mu(x) \geq \|\mu\|_1\,\mathfrak{N}_{\nu,\eta}^{\mu K_\nu}(x). \tag{2.71}$$

Proof We apply the Chebyshev inequality (2.68) by choosing **(a)** $f \equiv I_\nu, g \equiv K_\eta$, **(b)** $f \equiv I_\nu, g \equiv I_\eta K_\eta$ and **(c)** $f \equiv K_\nu, g \equiv I_\nu K_\eta$. In the cases **(a)** and **(c)** the functions f and g are opposite kind monotone, and thus we immediately conclude (2.69) and (2.71), respectively. Moreover, in the case **(b)** both f and g decrease, which imply the derived inequality (2.70).

It remains only to find the x-domains of the inequalities.

Observe that $\|\mu\|_1 < \infty$ suffices for the absolute and uniform convergence of the Neumann series $\mathfrak{M}_\nu^\mu(x)$. This has been established by Baricz et al. in the proof of [24, Theorem 2.1] for all $x > 0$ and $\nu > -1$. Moreover, in the same paper [24] the authors proved that $\mathfrak{I}_\eta^\mu(x)$ converges absolutely and uniformly when $\eta > 0$ and $x \in \mathscr{I}_0$. Now, by the inequalities [14, p. 583]

$$I_\nu(x) < \frac{\left(\frac{x}{2}\right)^\nu}{\Gamma(\nu+1)}\,\mathrm{e}^{\frac{x^2}{4(\nu+1)}}, \qquad \nu > -1,\ x > 0,$$

and [24]

$$K_\eta(x) \leq \frac{2^{\eta-1}}{x^\eta}\,\Gamma(\eta), \qquad \eta > 0,\ x > 0\,,$$

applied to the summands of $\mathfrak{N}_{\nu,\eta}^\mu(x)$, we obtain

$$\big|\mathfrak{N}_{\nu,\eta}^\mu(x)\big| \leq \frac{1}{2}\left(\frac{x}{2}\right)^{\eta-\nu} \mathrm{e}^{\frac{x^2}{4(\nu+2)}} \sum_{n\geq 1} n^{\eta-\nu-1}\,\mu_n\,.$$

Observe that the convergence of the right-hand-side series, that is $\|\mathbb{N}^{\eta-\nu-1}\,\mu\|_1 < \infty$, ensures the convergence of the second type Neumann series $\mathfrak{G}_{\nu,\eta}^\mu(x)$ for all $\nu, \eta, x > 0$. This yields together with the additional requirement $\|\mu\|_1 < \infty$

$$\max\big\{\|\mu\|_1, \|\mathbb{N}^{\eta-\nu-1}\,\mu\|_1\big\} = \|\mathbb{N}^{(\eta-\nu-1)_+}\,\mu\|_1 < \infty.$$

Finally, consider the series $\mathfrak{N}_{\nu,\eta}^{\mu I_\eta}(x)$ which ensures the convergence of both left-hand-side Neumann series in (2.70). By virtue of the above listed upper bounds for I_ν, I_η and K_η, we have

$$\begin{aligned}\mathfrak{N}_{\nu,\eta}^{\mu I_\eta}(x) &= \sum_{n\geq 1} \mu_n I_{\eta+n}(x) I_{\nu+n}(x) K_{\eta+n}(x)\\ &\leq \frac{1}{2\sqrt{2\pi}} \left(\frac{x}{2}\right)^\nu \mathrm{e}^{\frac{x^2}{4}\left(\frac{1}{\nu+2}+\frac{1}{\eta+2}\right)} \sum_{n\geq 1} \frac{\mu_n}{n^{n+\nu+\frac{3}{2}}} \left(\frac{x\mathrm{e}}{2}\right)^n,\end{aligned}$$

where the bounding power series converges for all $x \in \mathscr{I}_1$.

Combining all these estimates we arrive at the asserted inequality domains. □

2.6.2 Integral Form of Related Second Type Neumann Series

Our next goal is to give integral representations for the second type Neumann series

$$\mathfrak{N}_{\nu,\eta}^{\mu}(x),\ \mathfrak{N}_{\nu,\eta}^{\mu I_\eta}(x) \quad \text{and} \quad \mathfrak{N}_{\nu,\eta}^{\mu K_\nu}(x),$$

which appeared in Theorem 2.12. This will be realized on the account of procedure introduced by Pogány and Süli in [249] and further developed and promoted by Baricz et al. [21, 24, 25].

Theorem 2.12 (Baricz and Pogány [20]) *Let* $\mu \in \mathrm{C}^1(\mathbb{R}_+)$, $\mu|_{\mathbb{N}} = (\mu_n)_{n\geq 1}$ *such that* $\limsup_{n\to\infty} |\mu_n|^{\frac{1}{n}} \leq 1$. *Then, for all* $x > 0$ *and* $\nu, \eta > -\frac{3}{2}$ *we have the integral representation*

$$\begin{aligned}\mathfrak{N}_{\nu,\eta}^{\mu}(x) = -\frac{x^{\nu-\eta}}{4} \int_1^\infty \int_0^{[t]} \frac{\partial}{\partial t} &\left(\frac{\Gamma(t+\nu+\frac{1}{2})}{\Gamma(t+\eta+\frac{1}{2})} I_{t+\nu}(x)\, K_{t+\eta}(x) \right)\\ &\times \mathfrak{d}_s \left(\frac{\mu(s)\Gamma(s+\eta+\frac{1}{2})}{\Gamma(s+\nu+\frac{1}{2})} \right) \mathrm{d}t\, \mathrm{d}s\,, \end{aligned} \tag{2.72}$$

where

$$\mathfrak{d}_x := 1 + \{x\} \frac{\mathrm{d}}{\mathrm{d}x}.$$

Proof By the Basset formula (2.41), applying (2.34) (see e.g. [333, p. 79]) to $\mathfrak{N}_{\nu,\eta}^{\mu}(x)$, we conclude

$$\begin{aligned}\mathfrak{N}_{\nu,\eta}^{\mu}(x) = \frac{x^{\nu-\eta}}{2\pi} \int_0^1 \int_0^\infty &\frac{(1-t^2)^{\nu-\frac{1}{2}} \cosh(xt)\cos(xs)}{(1+s^2)^{\eta+\frac{1}{2}}}\\ &\times \sum_{n\geq 1} \frac{\mu_n \Gamma(n+\eta+\frac{1}{2})}{\Gamma(n+\nu+\frac{1}{2})} \left(\frac{1-t^2}{1+s^2}\right)^n \mathrm{d}t\mathrm{d}s\,. \end{aligned} \tag{2.73}$$

The inner sum we recognize as the Dirichlet series

$$\mathscr{D}_0(t,s) = \sum_{n\geq 1} \frac{\mu_n \Gamma(n+\eta+\frac{1}{2})}{\Gamma(n+\nu+\frac{1}{2})} \exp\Big(-n\log\frac{1+s^2}{1-t^2}\Big), \tag{2.74}$$

which parameter $\log(1+s^2)(1-t^2)^{-1}$ is obviously positive on $(t,s)\in(0,1)\times\mathbb{R}_+$ independently of x. Also, the power series (2.74) has the radius of convergence

$$\rho_{\mathscr{D}_0} = \frac{1}{\limsup\limits_{n\to\infty} |\mu_n|^{\frac{1}{n}}},$$

and then $\mathscr{D}_0(t,s)$ is convergent for all $(t,s)\in(0,1)\times\mathbb{R}_+$, being $\rho_{\mathscr{D}_0}\geq 1$ according to the assumption of the theorem.

Thus, by Cahen's Laplace integral formula for the Dirichlet series (1.15) and by the condensed Euler–Maclaurin summation formula (1.9), we get

$$\mathscr{D}_0(t,s) = \log\frac{1+s^2}{1-t^2}\int_0^\infty\int_0^{[w]}\Big(\frac{1-t^2}{1+s^2}\Big)^w \mathfrak{d}_z\Big(\frac{\mu(z)\Gamma(z+\eta+\frac{1}{2})}{\Gamma(z+\nu+\frac{1}{2})}\Big)\,\mathrm{d}w\mathrm{d}z\,. \tag{2.75}$$

Substituting (2.75) into (2.73) we get

$$\begin{aligned}\mathfrak{N}^{\mu}_{\nu,\eta}(x) &= -\frac{x^{\nu-\eta}}{2\pi}\int_0^\infty\int_0^{[w]}\mathfrak{d}_z\Big(\frac{\mu(z)\Gamma(z+\eta+\frac{1}{2})}{\Gamma(z+\nu+\frac{1}{2})}\Big)\\ &\times\left(\int_0^1\int_0^\infty\Big(\frac{1-t^2}{1+s^2}\Big)^{w+\nu-\frac{1}{2}}\log\frac{1-t^2}{1+s^2}\cdot\frac{\cosh(xt)\cos(xs)}{(1+s^2)^{\eta-\nu+1}}\,\mathrm{d}t\mathrm{d}s\right)\mathrm{d}w\mathrm{d}z\,.\end{aligned}$$

Denote

$$\mathscr{I}(\alpha) := \int_0^1\int_0^\infty\Big(\frac{1-t^2}{1+s^2}\Big)^{\alpha}\log\frac{1-t^2}{1+s^2}\cdot\frac{\cosh(xt)\cos(xs)}{(1+s^2)^{\eta-\nu+1}}\,\mathrm{d}t\mathrm{d}s\,.$$

Now, having in mind (2.34) and (2.41), we deduce

$$\begin{aligned}\int\mathscr{I}(\alpha)\,\mathrm{d}\alpha &= \int_0^1\int_0^\infty\Big(\frac{1-t^2}{1+s^2}\Big)^{\alpha}\frac{\cosh(xt)\cos(xs)}{(1+s^2)^{\eta-\nu+1}}\,\mathrm{d}t\mathrm{d}s\\ &= \int_0^1\int_0^\infty\frac{(1-t^2)^\alpha\cosh(xt)\cos(xs)}{(1+s^2)^{\alpha+\eta-\nu+1}}\,\mathrm{d}t\mathrm{d}s\\ &= \frac{\pi}{2}\,\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+\eta-\nu+1)}\,I_{\alpha+\frac{1}{2}}(x)K_{\alpha+\eta-\nu+\frac{1}{2}}(x)\,,\end{aligned}$$

that is, choosing $\alpha \mapsto w + \nu - \frac{1}{2}$ we have

$$\mathscr{I}(w + \nu - \tfrac{1}{2}) = \frac{\pi}{2}\,\frac{\partial}{\partial w}\,\frac{\Gamma(w + \nu + \frac{1}{2})}{\Gamma(w + \eta + \frac{1}{2})}\, I_{w+\nu}(x) K_{w+\eta}(x)\,.$$

Hence

$$\begin{aligned}\mathfrak{N}_{\nu,\eta}^{\mu}(x) = -\frac{x^{\nu-\eta}}{4}\int_0^\infty\int_0^{[t]}\frac{\partial}{\partial t}\left(\frac{\Gamma(t + \nu + \frac{1}{2})}{\Gamma(t + \eta + \frac{1}{2})}\, I_{t+\nu}(x)\, K_{t+\eta}(x)\right)\\ \times\, \mathfrak{d}_s\left(\frac{\mu(s)\Gamma(s + \eta + \frac{1}{2})}{\Gamma(s + \nu + \frac{1}{2})}\right)\mathrm{d}t\,\mathrm{d}s\,,\end{aligned}$$

which is equivalent to the asserted double integral expression (2.72). □

Theorem 2.13 (Baricz and Pogány [20]) *Let* $\mu \in \mathrm{C}^1(\mathbb{R}_+)$, $\mu|_{\mathbb{N}} = (\mu_n)_{n\geq 1}$. *Then, for all* $x \in \mathscr{I}_1,\ \nu, \eta > -\frac{3}{2}$ *there holds*

$$\begin{aligned}\mathfrak{N}_{\nu,\eta}^{\mu I_\eta}(x) = -\frac{x^{\nu-\eta}}{4}\int_1^\infty\int_0^{[t]}\frac{\partial}{\partial t}\left(\frac{\Gamma(t + \nu + \frac{1}{2})}{\Gamma(t + \eta + \frac{1}{2})}\, I_{t+\nu}(x)\, K_{t+\eta}(x)\right)\\ \times\, \mathfrak{d}_s\left(\frac{\mu(s) I_{s+\eta}(x)\Gamma(s + \eta + \frac{1}{2})}{\Gamma(s + \nu + \frac{1}{2})}\right)\mathrm{d}t\,\mathrm{d}s\,.\end{aligned}$$

Moreover, for $x \in \mathscr{I}_0,\ \nu > -1, \eta > -\frac{3}{2}$ *we have*

$$\begin{aligned}\mathfrak{N}_{\nu,\eta}^{\mu K_\nu}(x) = -\frac{x^{\nu-\eta}}{4}\int_1^\infty\int_0^{[t]}\frac{\partial}{\partial t}\left(\frac{\Gamma(t + \nu + \frac{1}{2})}{\Gamma(t + \eta + \frac{1}{2})}\, I_{t+\nu}(x)\, K_{t+\eta}(x)\right)\\ \times\, \mathfrak{d}_s\left(\frac{\mu(s) K_{s+\nu}(x)\Gamma(s + \eta + \frac{1}{2})}{\Gamma(s + \nu + \frac{1}{2})}\right)\mathrm{d}t\,\mathrm{d}s\,.\end{aligned}$$

Proof We follow the proof of (2.72) to get the integral representations. It remains only to remark that the Dirichlet series $\mathscr{D}_1(t, s)$ associated with $\mathfrak{N}_{\nu,\eta}^{\mu I_\eta}(x)$ satisfies

$$\begin{aligned}|\mathscr{D}_1(t, s)| &\leq \sum_{n\geq 1}\frac{|\mu_n||I_{n+\eta}(x)|\Gamma(n + \eta + \frac{1}{2})}{\Gamma(n + \nu + \frac{1}{2})}\left(\frac{1 - t^2}{1 + s^2}\right)^n\\ &\leq \frac{1}{\sqrt{2\pi}}\left(\frac{|x|}{2}\right)^\eta \mathrm{e}^{\frac{x^2}{2(\eta+2)}}\sum_{n\geq 1}\frac{\mu_n}{n^{n+\nu+\frac{1}{2}}}\left(\frac{|x|e}{2}\,\frac{1 - t^2}{1 + s^2}\right)^n,\end{aligned}$$

so x has to be from $\mathscr{I}_1$. Similarly can be concluded that for the Dirichlet series $\mathscr{D}_2(t,s)$ associated with $\mathfrak{N}_{\nu,\eta}^{\mu I_\eta}(x)$ holds the estimate

$$\begin{aligned}|\mathscr{D}_2(t,s)| &\leq \sum_{n\geq 1} \frac{|\mu_n||K_{n+\nu}(x)|\Gamma(n+\nu)\Gamma(n+\eta+\frac{1}{2})}{\Gamma(n+\nu+\frac{1}{2})}\Big(\frac{1-t^2}{1+s^2}\Big)^n \\ &\leq \sqrt{\frac{\pi}{2}}\Big(\frac{2}{|x|}\Big)^\nu \sum_{n\geq 1} n^{n+\eta-\frac{1}{2}}\,\mu_n\Big(\frac{2}{|x|e}\,\frac{1-t^2}{1+s^2}\Big)^n,\end{aligned}$$

of which convergence requirement causes $x \in \mathscr{I}_0$. □

2.6.3 *Indefinite Integral Expressions for Second Type Neumann Series* $\mathfrak{N}_{\nu,\nu}^{\mu}(x)$

In this section our aim is to establish indefinite integral representation formulae for the one-parameter second type Neumann series of the product of two modified Bessel functions of the first kind P_ν. First of all, observe that P_ν is a particular solution of the homogeneous third order linear differential equation

$$x^2y'''(x) + 3xy''(x) - (4\nu^2 + 4x^2 - 1)y'(x) - 4xy(x) = 0. \tag{2.76}$$

To see this, let us recall that I_ν and K_ν both satisfy the differential equation

$$x^2y''(x) + xy'(x) - (x^2 + \nu^2)y(x) = 0$$

and consequently

$$x^2I_\nu''(x) = (x^2 + \nu^2)I_\nu(x) - xI_\nu'(x) \tag{2.77}$$

and

$$x^2K_\nu''(x) = (x^2 + \nu^2)K_\nu(x) - xK_\nu'(x). \tag{2.78}$$

Applying these relations we obtain

$$x^2P_\nu''(x) = 2(x^2 + \nu^2)P_\nu(x) - xP_\nu'(x) + 2x^2I_\nu'(x)K_\nu'(x).$$

Now, differentiating both sides of this equation and applying again the previous relations we arrive at

$$x^2P_\nu'''(x) + 3xP_\nu''(x) - (4\nu^2 + 4x^2 - 1)P_\nu'(x) - 4xP_\nu(x) = 0.$$

Repeating this procedure twice in view of (2.77) and (2.78), we can show[2] that actually I_ν^2 and K_ν^2 are also particular solutions of the third order linear differential equation (2.76).

Now, let us show that I_ν^2, $I_\nu K_\nu$ and K_ν^2 are independent being the Wronskian $W[I_\nu^2, I_\nu K_\nu, K_\nu^2] \neq 0$ on $\mathbb{R}$. After some computation we get

$$\begin{aligned} W[I_\nu^2, I_\nu K_\nu, K_\nu^2](x) &= \begin{vmatrix} I_\nu^2(x) & I_\nu(x)K_\nu(x) & K_\nu^2(x) \\ \big(I_\nu^2(x)\big)' & (I_\nu(x)K_\nu(x))' & \big(K_\nu^2(x)\big)' \\ \big(I_\nu^2(x)\big)'' & (I_\nu(x)K_\nu(x))'' & \big(K_\nu^2(x)\big)'' \end{vmatrix} \\ &= -\frac{1}{4}\big(I_\nu(x)K_{\nu-1}(x) + I_{\nu-1}(x)K_\nu(x) \\ &\quad + I_{\nu+1}(x)K_\nu(x) + I_\nu(x)K_{\nu+1}(x)\big)^3 \\ &= 2\big(I_\nu(x)K_\nu'(x) - I_\nu'(x)K_\nu(x)\big)^3 = 2W^3[I_\nu, K_\nu](x) = -\frac{2}{x^3} \neq 0, \end{aligned}$$

where we used the fact that $W[I_\nu, K_\nu](x) = -x^{-1}$.

Thus, by the variation of constants method we get the desired particular solution of the non-homogeneous variant of (2.76), that is,

$$x^2 y'''(x) + 3xy''(x) - (4\nu^2 + 4x^2 - 1)y'(x) - 4xy(x) = f(x)\,, \tag{2.79}$$

where f is a suitable real function. Hence, bearing in mind (2.77), the general solution reads as follows

$$y(x) = c_1 I_\nu^2(x) + c_2 I_\nu(x)K_\nu(x) + c_3 K_\nu^2(x) - 4\int_1^x tf(t)\big(I_\nu(x)K_\nu(t) - I_\nu(t)K_\nu(x)\big)^2\,\mathrm{d}t\,.$$

Choosing the constants c_1, c_2 and c_3 to be zero, the particular solution y_p of the non-homogeneous ordinary differential equation (2.79) becomes

$$y_p(x) = -4\int_1^x tf(t)\big(I_\nu(x)K_\nu(t) - I_\nu(t)K_\nu(x)\big)^2\,\mathrm{d}t\,. \tag{2.80}$$

---

[2] It is worth to mention here that the above procedure for modified Bessel functions is similar to the method for Bessel functions applied by Wilkins [335]. See also Andrews et al. [7] for more details. More precisely, Wilkins proved that the Hankel functions $\big(H_\nu^{(1)}\big)^2$ and $\big(H_\nu^{(2)}\big)^2$, as well as $J_\nu^2 + Y_\nu^2$, where J_ν and Y_ν stand for the Bessel functions of the first and second kind, are particular solutions of the third order homogeneous differential equation [7, p. 225]

$$x^2 y'''(x) + 3xy''(x) + (1 + 4x^2 - 4\nu^2)y'(x) + 4xy(x) = 0.$$

The above result was used to prove the celebrated Nicholson formula [7, p. 224]

$$J_\nu^2(x) + Y_\nu^2(x) = \frac{8}{\pi^2}\int_0^\infty K_0(2x\sinh t)\cosh(2\nu t)\mathrm{d}t,$$

which generalizes the trigonometric identity $\sin^2 x + \cos^2 x = 1$.

Now, by using (2.76) we have

$$x^2\, P'''_{n+\nu}(x) + 3x\, P''_{n+\nu}(x) - (4(n+\nu)^2 + 4x^2 - 1)\, P'_{n+\nu}(x) - 4x\, P_{n+\nu}(x) = 0$$

and multiplying with the weight μ_n and summing up on the set of positive integers $\mathbb{N}$, transformations lead to the non-homogeneous third order linear differential equation

$$\begin{aligned} &x^2\left(\mathfrak{N}_{\nu,\nu}^{\mu}(x)\right)''' + 3x\left(\mathfrak{N}_{\nu,\nu}^{\mu}(x)\right)'' - (4\nu^2 + 4x^2 - 1)\left(\mathfrak{N}_{\nu,\nu}^{\mu}(x)\right)' - 4x\,\mathfrak{N}_{\nu,\nu}^{\mu}(x) \\ &\quad = 4\sum_{n\ge 1} n(n+2\nu)\mu_n I_{n+\nu}(x) K_{n+\nu}(x) := \mathfrak{H}_{\nu,\nu}^{\mu}(x)\,, \end{aligned}$$

where $\mathfrak{H}_{\nu,\nu}^{\mu}(x)$ stands for the *second kind equal parameter Neumann series of modified Bessel functions associated with the Neumann series* $\mathfrak{N}_{\nu,\nu}^{\mu}(x)$.

Theorem 2.14 (Baricz and Pogány [20]) *Let* $\mu \in \mathrm{C}^1(\mathbb{R}_+)$, $\mu|_{\mathbb{N}} = (\mu_n)_{n\ge 1}$ *such that* $\limsup_{n\to\infty} |\mu_n|^{\frac{1}{n}} \le 1$ *and* $\|\mathbb{N}^{-1}\,\mu\|_1 < \infty$. *Then for all* $\nu > -\frac{3}{2}$ *and* $x > 0$ *we have*

$$\mathfrak{N}_{\nu,\nu}^{\mu}(x) = -4\int_1^x u\mathfrak{H}_{\nu,\nu}^{\mu}(u)\big(I_\nu(x)K_\nu(u) - I_\nu(u)K_\nu(x)\big)^2\,\mathrm{d}u\,, \tag{2.81}$$

where $\mathfrak{H}_{\nu}^{\mu}(x)$ *possesses the integral representation*

$$\mathfrak{H}_{\nu,\nu}^{\mu}(x) = -\int_1^\infty \int_0^{[t]} \frac{\partial}{\partial t}\big(I_{t+\nu}(x)\, K_{t+\nu}(x)\big)\, \mathfrak{d}_s\big(s(s+2\nu)\mu(s)\big)\mathrm{d}t\,\mathrm{d}s\,. \tag{2.82}$$

Proof The integral representation (2.82) of the associated second type Neumann series of Bessel function $\mathfrak{H}_{\nu,\nu}^{\mu}(x)$ can be obtained by using the integral expression (2.72) in Theorem 2.12, just putting $\eta \equiv \nu$ for the weight function $\mu_n \mapsto 4n(n+2\nu)\mu_n$, when

$$\limsup_{n\to\infty} |4n(n+2\nu)\mu_n|^{\frac{1}{n}} = \limsup_{n\to\infty} |\mu_n|^{\frac{1}{n}} \le 1\,.$$

After that by straightforward application of (2.80) with $f(x) \equiv \mathfrak{H}_{\nu,\nu}^{\mu}(x)$, we deduce the desired integral expression (2.81). □

2.7 Summation Formulae for the First and Second Type Neumann Series

The problem of summing the Neumann series of Bessel and modified Bessel functions of the first kind i.e. $\mathfrak{N}_\nu$ and $\mathfrak{M}_\nu \equiv \mathfrak{N}_{1,\nu}^{I}$, has been widely considered in the mathematical literature [1, 24, 257]. Also, the NIST project, the Wolfram virtual formula collection and Hansen's classical monograph [102] contain an exhaustive list of summations for alike series.

Quite recently, a summation formula for Neumann series of modified Bessel functions of the first kind was given by Al-Jarrah et al. [4, p. 3, Theorem 1]:

$$\sum_{n\geq 0} I_{kn}(x) = \frac{1}{2}I_0(x) + \frac{1}{2k}\sum_{n=0}^{k-1} \mathrm{e}^{x\cos(2\pi n/k)}, \qquad k \in \mathbb{N}\,. \tag{2.83}$$

This summation possesses numerous already known special cases. For instance, see [1, p. 376, Eqs. **9.6.37**, **9.6.39**], [102, p. 411–412, Eqs. (58.1.2), (58.1.12)]

$$\sum_{n\geq 0} I_n(x) = \tfrac{1}{2}\left(I_0(x) + \mathrm{e}^x\right), \qquad \sum_{n\geq 0} I_{2n}(x) = \tfrac{1}{2}\left(I_0(x) + \cosh x\right)\,,$$

respectively and [102, p. 412, Eq. (58.1.17)]

$$\sum_{n\geq 0} (-)^n\, I_{\nu+2n}(x) = \tfrac{1}{2}\int I_{\nu-1}(x)\,\mathrm{d}x\,;$$

also inspect [102, §§58, 74.6, 79.2] together with suggested links to further results.

Our first main purpose is to extend (2.83) to a new summation formulae for

$$\mathfrak{N}^{I,\pm}_{\mu,\nu}(x) = \sum_{n\geq 0} (\pm)^n\, I_{\nu+\mu n}(x), \qquad x \in \mathbb{R}\,, \tag{2.84}$$

for the widest possible parameter space upon μ, ν and such results will be presented in the next section.

Also, motivated by the fact that the modified Bessel function of the first kind frequently occurs in probability and statistics [269], mostly as a part of the distributions of spherical and directional random variables such as, for instance, the probability density function of the non-central χ^2 distribution with non-centrality parameter $a > 0$ and $n \in \mathbb{N}$ degrees of freedom [142, p. 436, Eq. (29.4)]

$$f_{n,a}(x) = \tfrac{1}{2}\mathrm{e}^{-\frac{a+x}{2}}\left(\sqrt{\frac{x}{a}}\right)^{\frac{n}{2}-1} I_{\frac{n}{2}-1}(\sqrt{ax})\,, \qquad x > 0 \tag{2.85}$$

in Sect. 2.8 we will also present a new summation formula for the special kind of Neumann series $\mathfrak{N}^I_{1,\nu}$ which is connected to the cumulative distribution function (CDF) of the non-central χ^2 distribution (usually denoted by $\chi'^2_n(a)$ [142, p. 433]).

Section 2.9 is devoted to new summation formulae for the second type Neumann series which members contain product of two modified Bessel functions of the first kind and also to new results which connect Neumann series of the first and second type.

Further, as a by-product of the mentioned results new summation formulae will be established for the Neumann series which members contain Bessel functions of the first kind J_ν and also their products.

2.7.1 Closed Form of the First Type Neumann Series $\mathfrak{N}_{\mu,\nu}^{I,\pm}$

Here, we present new summation formulae for the Neumann series given by (2.84), together with some consequences and generalizations.

Theorem 2.15 (Jankov Maširević and Pogány [139]) *For all* $\min\{x,\nu\} > 0$ *there hold*

$$\mathfrak{N}_{1,\nu}^{I,\pm}(x) = \frac{1}{2(1-\nu)}\Big(\frac{2\,\mathrm{e}^{\pm x}}{\Gamma(\nu+1)}\Big(\frac{x}{2}\Big)^\nu {}_2F_2\Big[\begin{matrix}\nu, \nu-\frac{1}{2}\\ \nu+1, 2\nu-1\end{matrix}\Big| \mp 2x\Big] - xI_{\nu-1}(x) \mp xI_\nu(x)\Big)\,. \tag{2.86}$$

Proof First, let us establish the absolute convergence of the series $\mathfrak{N}_{1,\nu}^{I,\pm}(x)$. By virtue of asymptotic behavior [227, p. 256, Eq. 10.41.1]

$$I_\nu(x) \sim \frac{1}{\sqrt{2\pi\nu}}\Big(\frac{\mathrm{e}\,x}{2\nu}\Big)^\nu,$$

valid for all $x > 0$ fixed and $\nu \to \infty$, we have

$$\big|\mathfrak{N}_{1,\nu}^{\pm}(x)\big| \le \sum_{n\ge 0}\big|I_{\nu+n}(x)\big| \sim \frac{1}{\sqrt{2\pi}}\Big(\frac{\mathrm{e}\,x}{2}\Big)^\nu \sum_{n\ge 0}\frac{\Big(\frac{\mathrm{e}\,x}{2}\Big)^n}{(\nu+n)^{\nu+n+\frac{1}{2}}}$$

which obviously converges for all $x > 0$.

The appropriate integral representation [257, p. 694] yields

$$\mathfrak{N}_{1,\nu}^{I,\pm}(x) = \frac{1}{2(1-\nu)}\Big(\mathrm{e}^{\pm x}\int_0^x \mathrm{e}^{\mp t} I_{\nu-1}(t)\,\mathrm{d}t - xI_{\nu-1}(x) \mp xI_\nu(x)\Big), \qquad \nu > 0\,.$$

On the other hand, expanding the exponential term in the integrand into Maclaurin series, by the legitimate change of order of summation and integration, we get

$$\int_0^x \mathrm{e}^{\mp t} I_{\nu-1}(t)\,\mathrm{d}t = \frac{2^{1-\nu}x^\nu}{\Gamma(\nu+1)}\,{}_2F_2\Big[\begin{matrix}\nu, \nu-\frac{1}{2}\\ \nu+1, 2\nu-1\end{matrix}\Big| \mp 2x\Big].$$

This evidently leads to the asserted formula (2.86). □

Theorem 2.16 (Jankov Maširević and Pogány [139]) *For all* $\min\{x, \nu\} > 0$ *there holds*

$$\mathfrak{N}_{2,\nu}^{I,+}(x) = \frac{1}{2(1-\nu)}\left\{\frac{1}{\Gamma(\nu+1)}\left(\frac{x}{2}\right)^{\nu}\left(\mathrm{e}^{x}\,{}_2F_2\left[\begin{matrix}\nu, \nu-\frac{1}{2}\\ \nu+1, 2\nu-1\end{matrix}\,\Big|\,-2x\right]\right.\right.$$
$$\left.\left. + \mathrm{e}^{-x}\,{}_2F_2\left[\begin{matrix}\nu, \nu-\frac{1}{2}\\ \nu+1, 2\nu-1\end{matrix}\,\Big|\,2x\right]\right) - xI_{\nu-1}(x)\right\} \tag{2.87}$$

and

$$\mathfrak{N}_{2,\nu}^{I,-}(x) = \frac{1}{\Gamma(\nu+1)}\left(\frac{x}{2}\right)^{\nu}\,{}_1\Psi_2^{*}\left[\begin{matrix}(\nu,2)\\ (\nu,1),(\nu+1,2)\end{matrix}\,\Big|\,\frac{x^2}{4}\right]. \tag{2.88}$$

Proof We establish the convergence conditions of $\mathfrak{N}_{2,\nu}^{I,\pm}(x)$ analogously as previously in Theorem 2.15, being

$$\left|\mathfrak{N}_{2,\nu}^{I,\pm}(x)\right| \le \sum_{n\ge 0}\left|I_{\nu+2n}(x)\right| \sim \frac{1}{\sqrt{2\pi}}\left(\frac{\mathrm{e}\,x}{2}\right)^{\nu}\sum_{n\ge 0}\frac{1}{(\nu+2n)^{\nu+2n+\frac{1}{2}}}\left(\frac{\mathrm{e}\,x}{2}\right)^{2n},$$

which converges for all $x > 0$.

As to (2.87), we use the elementary transformation

$$\sum_{n\ge 0} a_n = \sum_{n\ge 0} a_{2n} + \sum_{n\ge 0} a_{2n+1}\,. \tag{2.89}$$

Now, it is easy to see that $\mathfrak{N}_{1,\nu}^{I,\pm}(x) = \mathfrak{N}_{2,\nu}^{I,+}(x) \pm \mathfrak{N}_{2,\nu+1}^{I,+}(x)$. Summing these expressions we get

$$\mathfrak{N}_{2,\nu}^{I,+}(x) = \tfrac{1}{2}\left\{\mathfrak{N}_{1,\nu}^{I,+}(x) + \mathfrak{N}_{1,\nu}^{I,-}(x)\right\}, \tag{2.90}$$

so (2.87) follows from Theorem 2.15.

Next, by the identity [1, p. 377, Eq. (9.6.47)]

$$I_\nu(x) = \frac{1}{\Gamma(\nu+1)}\left(\frac{x}{2}\right)^{\nu}\,{}_0F_1\left[\begin{matrix}-\\ \nu+1\end{matrix}\,\Big|\,\frac{x^2}{4}\right], \qquad -\nu \notin \mathbb{N}, \tag{2.91}$$

the Bailey-transform technique in summing up double infinite series [11]

$$\sum_{n,m\ge 0} a_{m,n} = \sum_{n\ge 0}\sum_{m=0}^{n} a_{m,n-m} \tag{2.92}$$

and the transformation formula $(m-n)!(-m)_n = (-1)^n m!$, we get

$$\begin{aligned}
\mathfrak{N}_{2,\nu}^{I,-}(x) &= \left(\frac{x}{2}\right)^\nu \sum_{n,m\geq 0} \frac{(-1)^n}{\Gamma(\nu+2n+m+1)\, m!} \left(\frac{x^2}{4}\right)^{n+m} \\
&= \frac{1}{\Gamma(\nu+1)} \left(\frac{x}{2}\right)^\nu \sum_{m\geq 0} \frac{1}{(\nu+1)_m\, m!} \left(\frac{x^2}{4}\right)^m \sum_{n=0}^{m} \frac{(-m)_n (1)_n}{(\nu+m+1)_n\, n!} \\
&= \frac{1}{\Gamma(\nu+1)} \left(\frac{x}{2}\right)^\nu \sum_{m\geq 0} \frac{1}{(\nu+1)_m\, m!} \left(\frac{x^2}{4}\right)^m {}_2F_1\left[\begin{matrix} -m, 1 \\ \nu+m+1 \end{matrix}\,\middle|\, 1\right] \\
&= \frac{1}{\Gamma(\nu+1)} \left(\frac{x}{2}\right)^\nu \sum_{m\geq 0} \frac{(\nu+m)_m}{(\nu+1)_m\, (\nu+m+1)_m\, m!} \left(\frac{x^2}{4}\right)^m,
\end{aligned}$$

where in the last equality the Chu–Vandermonde identity [227, p. 387, Eq. 15.4.24] has been used. Now, with the help of the well-known identity $(a)_{m+n} = (a)_m(a+m)_n$ we conclude (2.88). □

Formula (2.88) can be also written in a slightly different form:

Corollary 2.2 (Jankov Maširević and Pogány [139]) *For all* $\min\{x,\nu\} > 0$ *there holds*

$$\mathfrak{N}_{2,\nu}^{I,-}(x) = \frac{1}{\Gamma(\nu+1)} \left(\frac{x}{2}\right)^\nu \left({}_1F_2\left[\begin{matrix} \frac{\nu}{2} \\ \frac{\nu}{2}+1, \nu+1 \end{matrix}\,\middle|\, \frac{x^2}{4}\right] \right. \tag{2.93}$$

$$\left. + \frac{x^2}{4(\nu+1)(\nu+2)}\, {}_1F_2\left[\begin{matrix} \frac{\nu}{2}+1 \\ \frac{\nu}{2}+2, \nu+2 \end{matrix}\,\middle|\, \frac{x^2}{4}\right] \right). \tag{2.94}$$

Proof Transforming

$$h_\nu^-(x) := \Gamma(\nu+1) 2^\nu x^{-\nu} \mathfrak{N}_{2,\nu}^{I,-}(x)$$

into

$$\begin{aligned}
h_\nu^-(x) &= \sum_{m\geq 0} \frac{\nu}{(\nu+1)_m\, (\nu+2m)\, m!} \left(\frac{x^2}{4}\right)^m + \sum_{m\geq 0} \frac{m}{(\nu+1)_m\, (\nu+2m)\, m!} \left(\frac{x^2}{4}\right)^m \\
&= \sum_{m\geq 0} \frac{\left(\frac{\nu}{2}\right)_m \left(\frac{x^2}{4}\right)^m}{(\nu+1)_m \left(\frac{\nu}{2}+1\right)_m m!} + \frac{x^2}{8} \sum_{n\geq 0} \frac{\Gamma(\nu+1) \left(\frac{x^2}{4}\right)^n}{\left(1+\frac{\nu}{2}+n\right) \Gamma(\nu+2+n)\, n!} \\
&= {}_1F_2\left[\begin{matrix} \frac{\nu}{2} \\ \frac{\nu}{2}+1, \nu+1 \end{matrix}\,\middle|\, \frac{x^2}{4}\right] + \frac{x^2}{4(\nu+1)(\nu+2)} \sum_{n\geq 0} \frac{\left(\frac{\nu}{2}+1\right)_n}{\left(\frac{\nu}{2}+2\right)_n (\nu+2)_n\, n!} \left(\frac{x^2}{4}\right)^n;
\end{aligned}$$

the stated formula follows. □

The next result is an interesting by-product of Theorem 2.16 and Corollary 2.2. Namely equating the right-hand-side expressions in (2.88) and (2.93) we get

Corollary 2.3 (Jankov Maširević and Pogány [139]) *For all* $\min\{x, \nu\} > 0$ *there holds*

$$ {}_1\Psi_2^*\left[\begin{matrix}(\nu,2)\\(\nu,1),(\nu+1,2)\end{matrix}\,\middle|\,\frac{x^2}{4}\right] = {}_1F_2\left[\begin{matrix}\frac{\nu}{2}\\\frac{\nu}{2}+1,\nu+1\end{matrix}\,\middle|\,\frac{x^2}{4}\right] + \frac{x^2}{4(\nu+1)(\nu+2)}\,{}_1F_2\left[\begin{matrix}\frac{\nu}{2}+1\\\frac{\nu}{2}+2,\nu+2\end{matrix}\,\middle|\,\frac{x^2}{4}\right]. $$

Using (2.83) a double Neumann series result was given in [4]; in our setting:

$$ \sum_{n\ge 0}\mathfrak{N}_{2,kn+1}^{I,-}(x) = \frac{1}{2}\mathfrak{N}_{2,1}^{I,-}(x) + \frac{1}{4k}\left\{\sum_{\substack{n=0\\4n\neq k,\,3k}}^{k-1}\frac{\mathrm{e}^{x\cos(2\pi n/k)}-1}{\cos(2\pi n/k)} + Nx\right\}, \tag{2.95} $$

where $N = 2$ if $k \equiv 0 \pmod 4$, otherwise $N = 0$.

Corollary 2.4 (Jankov Maširević and Pogány [139]) *For all* $x > 0$ *and* $k \in \mathbb{N}$ *there holds*

$$ \sum_{n\ge 0}\mathfrak{N}_{2,kn+1}^{I,-}(x) = \frac{x}{8}\left\{2I_0(x) + \pi\big(I_0(x)\boldsymbol{L}_1(x) - I_1(x)\boldsymbol{L}_0(x)\big)\right\} + \frac{1}{4k}\left\{\sum_{\substack{n=0\\4n\neq k,\,3k}}^{k-1}\frac{\mathrm{e}^{x\cos(2\pi n/k)}-1}{\cos(2\pi n/k)} + Nx\right\}, $$

where $N = 2$ $k \equiv 0$ *(mod 4), otherwise* $N = 0$.

Proof From (2.88) and (2.95) we have

$$ \sum_{n\ge 0}\mathfrak{N}_{2,kn+1}^{-}(x) = \frac{x}{4}\,{}_1\Psi_2^*\left[\begin{matrix}(1,2)\\(1,1),(2,2)\end{matrix}\,\middle|\,\frac{x^2}{4}\right] + \frac{1}{4k}\left\{\sum_{\substack{n=0\\4n\neq k,\,3k}}^{k-1}\frac{\mathrm{e}^{x\cos(2\pi n/k)}-1}{\cos(2\pi n/k)} + Nx\right\}. $$

However, Corollary 2.3 enables to rewrite the Fox–Wright function as a weighted sum of two hypergeometric ${}_1F_2$ functions. Here $\nu = 1$, and having in mind the formulae [113, 114] respectively

$$ {}_1F_2\left[\begin{matrix}\frac{1}{2}\\\frac{3}{2},2\end{matrix}\,\middle|\,\frac{x^2}{4}\right] = 2I_0(x) - \frac{2}{x}I_1(x) + \pi\left\{I_0(x)\mathbf{L}_1(x) - I_1(x)\mathbf{L}_0(x)\right\} $$

$$ {}_1F_2\left[\begin{matrix} \frac{3}{2} \\ \frac{5}{2}, 3 \end{matrix} \,\Big|\, \frac{x^2}{4}\right] = -\frac{12}{x^3}\left\{2xI_0(x) - 4I_1(x) + \pi x\left(I_0(x)\mathbf{L}_1(x) - I_1(x)\mathbf{L}_0(x)\right)\right\}, $$

after some routine calculation we arrive at the statement. □

Ending this section, motivated by the summation [257, p. 694, Eq. (4)]

$$ \sum_{n\ge 0} \frac{t^n}{n!} I_{\nu+n}(x) = \left(\frac{2t}{x} + 1\right)^{-\frac{\nu}{2}} I_\nu(\sqrt{x^2 + 2tx}), \tag{2.96} $$

valid for all t, x which satisfy $2|t| < |x|$, we present some associated closed form result for the Neumann series

$$ \widetilde{\mathfrak{N}}_\nu(x) = \sum_{n\ge 0} \frac{t^{2n} I_{\nu+2n}(x)}{(2n)!}, \qquad t, x \in \mathbb{R}. $$

Theorem 2.17 (Jankov Maširević and Pogány [139]) *For all* $\nu, t, x > 0$ *such that* $2t < x$ *we have*

$$ \widetilde{\mathfrak{N}}_\nu(x) = \tfrac{1}{2} x^{\frac{\nu}{2}} \left\{ (x+2t)^{-\frac{\nu}{2}} I_\nu(\sqrt{x^2+2tx}) + (x^2-2tx)^{-\frac{\nu}{2}} I_\nu(\sqrt{x^2-2tx}) \right\}. $$

Proof Following the lines of previous theorems by the asymptotic behavior of the modified Bessel function we can show that $\widetilde{\mathfrak{N}}_\nu(x)$ converges for all $\min\{\nu, x\} > 0$.

Using the identity (2.89) and repeating the proving procedure of (2.90) we infer

$$ \begin{aligned} \widetilde{\mathfrak{N}}_\nu(x) &= \frac{1}{2}\sum_{n\ge 0} \frac{t^n I_{\nu+n}(x)}{n!} + \frac{1}{2}\sum_{n\ge 0} \frac{(-t)^n I_{\nu+n}(x)}{n!} \\ &= \frac{1}{2}\left(\frac{2t}{x} + 1\right)^{-\frac{\nu}{2}} I_\nu(\sqrt{x^2+2tx}) + \frac{1}{2}\sum_{n\ge 0} \frac{(-t)^n I_{\nu+n}(x)}{n!}, \end{aligned} $$

where in the last equality (2.96) was used. By (2.91), (2.92) and $(m-n)!(-m)_n = (-1)^n m!$ we get

$$ \begin{aligned} \sum_{n\ge 0} \frac{(-t)^n I_{\nu+n}(x)}{n!} &= \left(\frac{x}{2}\right)^\nu \sum_{n\ge 0} \frac{1}{\Gamma(\nu+n+1)\, n!} \left(\frac{-tx}{2}\right)^n {}_0F_1\left[\begin{matrix} - \\ \nu+n+1 \end{matrix} \,\Big|\, \frac{x^2}{4}\right] \\ &= \frac{1}{\Gamma(\nu+1)}\left(\frac{x}{2}\right)^\nu \sum_{n\ge 0} \frac{1}{(\nu+1)_n\, n!}\left(-\frac{tx}{2}\right)^n \sum_{m=0}^n \frac{(-n)_m}{m!}\left(\frac{x}{2t}\right)^m \\ &= \frac{1}{\Gamma(\nu+1)}\left(\frac{x}{2}\right)^\nu \sum_{n\ge 0} \frac{1}{(\nu+1)_n\, n!}\left(-\frac{tx}{2}\right)^n \left(1 - \frac{x}{2t}\right)^n \end{aligned} $$

$$= \frac{1}{\Gamma(\nu+1)} \left(\frac{x}{2}\right)^\nu {}_0F_1\left[\begin{matrix} - \\ \nu+1 \end{matrix} \,\middle|\, \frac{tx}{2}\left(\frac{x}{2t}-1\right)\right]$$

$$= x^\nu\,(x^2-2tx)^{-\frac{\nu}{2}}\, I_\nu(\sqrt{x^2-2tx})\,, \tag{2.97}$$

which implies the asserted expression. □

Remark 2.6 It is worth to mention that instead of the independently derived (2.97) we can apply the formula [102, p. 414, Eq. (58.8.1)] (also see [1, p. 377, Eq. **9.6.51**])

$$\sum_{n\geq 0} \frac{c^n}{n!} \mathscr{I}_{\nu+n}(x) = w^{-\nu}\, \mathscr{I}_\nu(wx), \qquad 2c = x(w^2-1)\,,$$

where $\mathscr{I}_\mu(x) = C_1 I_\mu(x) + C_2 \mathrm{e}^{\mathrm{i}\pi\mu} K_\mu(x)$ (C_1, C_2 arbitrary constants) denotes the general modified Bessel function, being $K_\mu(x)$ the modified Bessel function of the second kind of the order μ. Moreover, setting $t = x$ in (2.97) we get summation formula

$$\sum_{n\geq 0} \frac{(-x)^n I_{\nu+n}(x)}{n!} = J_\nu(x),$$

compare [1, p. 377, Eq. **9.6.51**] and [257, p. 694, Eq. (5)] for instance. ■

2.7.2 *Confluent Hypergeometric Functions and Srivastava–Daoust Function*

In this section we establish reduction formulae for the Horn's Φ_3 function and generalized Srivastava–Daoust $F^{A:B;B'}_{C:D;D'}$ function of two variables in some special cases of their parameters by virtue of newly established summations from Theorems 2.15 and 2.16.

Theorem 2.18 (Jankov Maširević and Pogány [139]) *For all* $x, \nu > 0$ *it is*

$$\Phi_3\left(1, \nu+1; \pm\frac{x}{2}, \frac{x^2}{4}\right) = \frac{\mathrm{e}^{\pm x}}{1-\nu}\, {}_2F_2\left[\begin{matrix} \nu, \nu-\frac{1}{2} \\ \nu+1, 2\nu-1 \end{matrix} \,\middle|\, \mp 2x\right]$$
$$- \frac{\Gamma(\nu+1)}{2(1-\nu)} \left(\frac{2}{x}\right)^\nu (xI_{\nu-1}(x) \pm xI_\nu(x))\,.$$

Proof Using the formula (2.91), by means of Theorem 2.15, we conclude

$$\mathfrak{N}^{I,\pm}_{1,\nu}(x) = \frac{1}{\Gamma(\nu+1)} \left(\frac{x}{2}\right)^\nu \sum_{n,m\geq 0} \frac{(1)_n}{(\nu+1)_{n+m}\, n!\, m!} \left(\pm\frac{x}{2}\right)^n \left(\frac{x^2}{4}\right)^m$$
$$= \frac{1}{\Gamma(\nu+1)} \left(\frac{x}{2}\right)^\nu \Phi_3\left(1, \nu+1; \pm\frac{x}{2}, \frac{x^2}{4}\right),$$

which completes the proof. □

Theorem 2.19 (Jankov Maširević and Pogány [139]) *For all* $x, \nu > 0$ *we have*

$$F^{0:1;0}_{1:0;0}\left[\begin{matrix} -: & [1:1]; - \\ [\nu+1:2,1]: & -; \quad - \end{matrix}\,\middle|\, \frac{x^2}{4}, \frac{x^2}{4}\right]$$
$$= \frac{1}{2(1-\nu)}\left[\left(\mathrm{e}^x\, {}_2F_2\left[\begin{matrix} \nu, \nu-\frac{1}{2} \\ \nu+1, 2\nu-1 \end{matrix}\,\middle|\, -2x\right]\right.\right.$$
$$\left.\left. + \; \mathrm{e}^{-x}\, {}_2F_2\left[\begin{matrix} \nu, \nu-\frac{1}{2} \\ \nu+1, 2\nu-1 \end{matrix}\,\middle|\, 2x\right]\right) - 2^\nu \Gamma(\nu+1) x^{1-\nu} I_{\nu-1}(x)\right] \tag{2.98}$$

and

$$F^{0:1;0}_{1:0;0}\left[\begin{matrix} - : [1:1] ; - \\ [\nu+1:2,1]:-;- \end{matrix}\,\middle|\, -\frac{x^2}{4}, \frac{x^2}{4}\right] = {}_1\Psi_2^*\left[\begin{matrix} (\nu, 2) \\ (\nu,1), (\nu+1, 2) \end{matrix}\,\middle|\, \frac{x^2}{4}\right]. \tag{2.99}$$

Proof Bearing in mind again the hypergeometric representation of the modified Bessel function identity (2.91) and the definition of Srivastava–Daoust function, we can write

$$\mathfrak{N}^{I,+}_{1,\nu}(x) = \left(\frac{x}{2}\right)^\nu \sum_{n,m\ge 0} \frac{\left(\frac{x^2}{4}\right)^{n+m}}{\Gamma(2n+m+\nu+1)\, m!}$$
$$= \frac{1}{\Gamma(\nu+1)} \left(\frac{x}{2}\right)^\nu \sum_{n,m\ge 0} \frac{(1)_n \left(\frac{x^2}{4}\right)^{n+m}}{(\nu+1)_{2n+m}\, n!\, m!}$$
$$= \frac{1}{\Gamma(\nu+1)} \left(\frac{x}{2}\right)^\nu F^{0:1;0}_{1:0;0}\left[\begin{matrix} -: & [1:1]; - \\ [\nu+1:2,1]: & -; \quad - \end{matrix}\,\middle|\, \frac{x^2}{4}, \frac{x^2}{4}\right],$$

which, in combination with (2.87) gives the desired formula (2.98). Analogously, using (2.88), we obtain (2.99). □

Theorem 2.20 (Jankov Maširević and Pogány [139]) *For all* $\min\{x,\nu\} > 0$ *and* $k \in \mathbb{N}$ *there holds*

$$F^{0:1;0}_{1:0;0}\left[\begin{matrix} - : [1:1] ; - \\ [1:k,1]:-;- \end{matrix}\,\middle|\, \left(\frac{x}{2}\right)^k, \frac{x^2}{4}\right] = \frac{1}{2} I_0(x) + \frac{1}{2k} \sum_{n=0}^{k-1} \mathrm{e}^{x\cos(2\pi n/k)}. \tag{2.100}$$

When k is odd, we have

$$F^{0:1;0}_{1:0;0}\left[\begin{matrix} - : [1:1] ; - \\ [1:k,1]:-;- \end{matrix}\,\middle|\, -\left(\frac{x}{2}\right)^k, \frac{x^2}{4}\right] = \frac{1}{2} I_0(x) + \frac{1}{2k} \sum_{n=0}^{k-1} \mathrm{e}^{-x\cos(2\pi n/k)}. \tag{2.101}$$

Proof Following the same lines as in the proof of Theorem 2.19 we deduce that

$$\mathfrak{N}_{k,0}^{I,+}(x) = \sum_{n,m\geq 0} \frac{(1)_n}{(1)_{kn+m}\, n!\, m!} \left(\frac{x}{2}\right)^{kn+2m}$$
$$= F_{1:0;0}^{0:1;0}\left[\begin{matrix} - \ : \ [1:1]\ ;\ - \\ [1:k,1] : -; - \end{matrix} \,\middle|\, \left(\frac{x}{2}\right)^k, \frac{x^2}{4}\right],$$

which, in combination with (2.83) gives the desired formula (2.100). Analogously, using the following result by Al-Jarrah et al. [4, p. 3, Corollary 2, Eq. (17)]

$$\sum_{n\geq 0}(-)^n I_{nk}(x) = \frac{1}{2}I_0(x) + \frac{1}{2k}\sum_{n=0}^{k-1} \mathrm{e}^{-x\cos(2\pi n/k)}\,, \tag{2.102}$$

valid for odd positive integer numbers k, we deduce (2.101). □

2.8 Neumann Series Regarding the $\chi_n'^2(a)$ Distribution

As we already stated, the probability density function of the non-central χ^2 random variable is given in terms of modified Bessel function of the first kind (2.85).

Considering the definition of the generalized Marcum Q-function of order $\nu > 0$ defined by Marcum [181], András et al. [6]

$$Q_\nu(a,b) = \frac{1}{a^{\nu-1}} \int_b^\infty t^\nu \mathrm{e}^{-\frac{t^2+a^2}{2}} I_{\nu-1}(at)\,\mathrm{d}t,$$

where $a, \nu > 0$ and $b \geq 0$ it is obvious that the explicit formula for the cumulative distribution function of the non-central χ^2 random variable can be represented in terms of generalized Marcum Q-function as

$$F_{n,a}(x) = 1 - Q_{\frac{n}{2}}(\sqrt{a}, \sqrt{x}), \qquad x > 0. \tag{2.103}$$

Such cumulative distribution function has been widely considered in mathematical literature (see e.g. Johnson et al. [142], Patnaik [232], Pearson [233], Sankaran [274], Wilson and Hilfetry [337]) and one of the recently derived formulae for such cumulative distribution function, which are claimed to have some computational advantages, was given in 1993 by Temme [309].

The non-central χ^2 distribution is frequently used in communication theory and in that context it is called *the generalized Marcum Q-function* and the non-centrality parameter is interpreted as a signal-to-noise ratio [142].

Motivated by closed-form formula for the generalized Marcum Q-function, derived by Brychkov [43, p. 178, Eq. (7)]

$$Q_{n+\frac{1}{2}}(a,b) = \frac{1}{2}\left(\mathrm{Erfc}\left(\frac{b-a}{\sqrt{2}}\right) + \mathrm{Erfc}\left(\frac{b+a}{\sqrt{2}}\right)\right) + \mathrm{e}^{-\frac{a^2+b^2}{2}} \sum_{m=1}^{n} \left(\frac{b}{a}\right)^{m-\frac{1}{2}} I_{m-\frac{1}{2}}(ab),$$

where Erfc is the complementary error function, which also implies new formula for cumulative distribution function (2.103) in the case of odd $n \in \mathbb{N}$, our main aim is to derive new closed form formula for such cumulative distribution function in the case of even number of the degrees of freedom. In that case, having in mind that there holds [203]

$$Q_n(a,b) = 1 - Q_{1-n}(b,a), \qquad n \in \mathbb{Z}$$

and [43, p. 178, Eq. (3)]

$$Q_{\nu-n}(a,b) = Q_\nu(a,b) - \left(\frac{b}{a}\right)^\nu \mathrm{e}^{-\frac{a^2+b^2}{2}} \sum_{m=1}^{n} \left(\frac{a}{b}\right)^m I_{\nu-m}(ab), \qquad \nu > 0$$

it follows from (2.103)

$$\begin{aligned} F_{2n,a}(x) &= Q_1(\sqrt{x}, \sqrt{a}) - \sqrt{\frac{a}{x}}\,\mathrm{e}^{-\frac{a+x}{2}} \sum_{m=1}^{n} \left(\sqrt{\frac{x}{a}}\right)^m I_{1-m}(\sqrt{ax}) \\ &= \mathrm{e}^{-\frac{a+x}{2}} \left(\sum_{n\geq 0} \left(\sqrt{\frac{x}{a}}\right)^n I_n(\sqrt{ax}) - \sqrt{\frac{a}{x}} \sum_{m=1}^{n} \left(\sqrt{\frac{x}{a}}\right)^m I_{m-1}(\sqrt{ax}) \right), \end{aligned}$$

where the last equality is a direct consequence of a parity of modified Bessel function of the first kind i.e. $I_{-n}(x) = I_n(x)$, when $n \in \mathbb{Z}$ and the relation between generalized Marcum Q-function (which is known in literature, for $\nu = 1$ as the (first order) Marcum Q-function) and a Neumann series with members containing modified Bessel functions of the first kind (see e.g. [123]):

$$Q_m(a,b) = \mathrm{e}^{-\frac{a^2+b^2}{2}} \sum_{n\geq 1-m} \left(\frac{a}{b}\right)^n I_n(ab), \qquad m \in \mathbb{Z}.$$

So, the problem of deriving new closed form expression for cumulative distribution function $F_{2n,a}$ is equivalent to a problem of deriving a closed-form expression for a Neumann series given above.

Below, we will need the definition of the so-called incomplete MacDonald function [2, p. 26, Eq. (1.30)]

$$K_\nu(w,z) = \frac{\sqrt{\pi}}{\Gamma\left(\nu+\frac{1}{2}\right)}\left(\frac{z}{2}\right)^\nu \int_0^w \mathrm{e}^{-z\cosh t}\sinh^{2\nu} t\,\mathrm{d}t, \quad \Re(\nu) > -\tfrac{1}{2},$$

which reduces to the MacDonald function K_ν when $w \mapsto \infty$ and $\Re(z) > 0$.

Now, we are ready to state and prove our first main result in this section.

Theorem 2.21 (Jankov Maširević [138]) *For all* $\min\{a,x\} > 0$ *there holds*

$$\sum_{n\ge 0}\left(\sqrt{\frac{x}{a}}\right)^n I_n(\sqrt{ax}) = \mathrm{e}^{\frac{a+x}{2}}\left[1 - \frac{\sqrt{ax}}{2} I_1(\sqrt{ax})\left(K_0(\sqrt{ax}) - K_0\left(\log\sqrt{\frac{x}{a}}, \sqrt{ax}\right)\right)\right.$$
$$\left. + a\, I_0(\sqrt{ax})\frac{\partial}{\partial a}\left(K_0(\sqrt{ax}) - K_0\left(\log\sqrt{\frac{x}{a}}, \sqrt{ax}\right)\right)\right].$$

Proof Considering the identity [6, p. 63]

$$Q_\nu(\sqrt{2a}, \sqrt{2b}) = 1 - b^\nu \mathrm{e}^{-a}\sum_{n\ge 0}\frac{(-b)^n L_n^{(\nu-1)}(a)}{\Gamma(\nu+n+1)},$$

where [6, p. 62]

$$L_n^{(\alpha)}(x) = \sum_{k=0}^n \frac{(-x)^k\Gamma(n+\alpha+1)}{\Gamma(k+\alpha+1)\,\Gamma(n-k+1)\,k!}$$

is the generalized Laguerre polynomial of degree $n \in \mathbb{N}$ and order $\alpha > -1$ we get

$$\sum_{n\ge 0}\left(\sqrt{\frac{x}{a}}\right)^n I_n(\sqrt{ax}) = \mathrm{e}^{\frac{a+x}{2}} - \frac{a}{2}\mathrm{e}^{\frac{a}{2}}\sum_{n\ge 0}\frac{\left(\frac{-a}{2}\right)^n}{n+1}\sum_{k=0}^n\frac{\left(\frac{-x}{2}\right)^k}{\Gamma(n-k+1)\,k!\,k!}.$$

Now, using the Bailey transform technique (2.92) and the definition of the Kampé de Fériet function we arrive at

$$\begin{aligned}\sum_{n\ge 0}\left(\sqrt{\frac{x}{a}}\right)^n I_n(\sqrt{ax}) &= \mathrm{e}^{\frac{a+x}{2}} - \frac{a}{2}\mathrm{e}^{\frac{a}{2}}\sum_{n,k\ge 0}\frac{(1)_{k+n}\left(\frac{ax}{4}\right)^k\left(\frac{-a}{2}\right)^n}{(2)_{k+n}(1)_k\,k!\,n!}\\ &= \mathrm{e}^{\frac{a+x}{2}} - \frac{a}{2}\mathrm{e}^{\frac{a}{2}}\,F^{1:0;0}_{1:1;0}\left[\begin{matrix}1 : -; -\\ 2 : 1; -\end{matrix}\,\middle|\,\frac{ax}{4}, \frac{-a}{2}\right]\\ &= \mathrm{e}^{\frac{a+x}{2}} - \frac{a}{2}\mathrm{e}^{\frac{a}{2}}\sum_{n\ge 0}\frac{(1)_n\left(\frac{-a^2x}{8}\right)^n}{(1)_{2n}(2)_{2n}}\,{}_0F_1\left[\begin{matrix}-\\ 2+2n\end{matrix}\,\middle|\,\frac{ax}{4}\right]{}_1F_1\left[\begin{matrix}1+n\\ 2+2n\end{matrix}\,\middle|\,-\frac{a}{2}\right],\end{aligned} \tag{2.104}$$

where in the last equality we used the transformation [290, p. 337, Eq. (242)]

$$F^{1:r;u}_{1:s;v}\left[\begin{matrix}\alpha\,:\,(a_r);\,(c_u)\\ \gamma\,:\,(b_s);\,(d_v)\end{matrix}\Bigg|\,x,y\right] = \sum_{n\geq 0}\frac{(\alpha)_n(\gamma-\alpha)_n\prod_{j=1}^{r}(a_j)_n\prod_{j=1}^{u}(c_j)_n}{(\gamma+n-1)_n(\gamma)_{2n}\prod_{j=1}^{s}(b_j)_n\prod_{j=1}^{v}(d_j)_n}\frac{(xy)^n}{n!}$$
$$\times\,{}_{r+1}F_{s+1}\left[\begin{matrix}(a_r)+n,\alpha+n\\(b_s)+n,\gamma+2n\end{matrix}\Bigg|\,x\right]$$
$$\times\,{}_{u+1}F_{v+1}\left[\begin{matrix}(c_u)+n,\alpha+n\\(d_v)+n,\gamma+2n\end{matrix}\Bigg|\,y\right].$$

Further, by the identity $I_n(\mathrm{i}\,z) = \mathrm{i}^n J_n(z)$, formula [1, p. 377, Eq. (9.6.47)]

$$I_\nu(x) = \frac{1}{\Gamma(\nu+1)}\left(\frac{x}{2}\right)^\nu {}_0F_1\left[\begin{matrix}-\\ \nu+1\end{matrix}\Bigg|\,\frac{x^2}{4}\right], \qquad -\nu\notin\mathbb{N} \tag{2.105}$$

and the integral representation [116]

$${}_1F_1(a;b;z) = \frac{1}{\Gamma(a)}\int_0^\infty \mathrm{e}^{-t}t^{a-1}{}_0F_1\left[\begin{matrix}-\\ b\end{matrix}\Bigg|\,zt\right]\mathrm{d}t, \qquad \Re(a)>0$$

we get

$$\sum_{n\geq 0}\left(\sqrt{\frac{x}{a}}\right)^n I_n(\sqrt{ax}) = \mathrm{e}^{\frac{a+x}{2}} - \sqrt{\frac{2}{x}}\,\mathrm{e}^{\frac{a}{2}}\sum_{n\geq 0}(-1)^n(2n+1)$$
$$\times\int_0^\infty \frac{\mathrm{e}^{-t}}{\sqrt{t}}I_{2n+1}(\sqrt{ax})J_{2n+1}(\sqrt{2at})\,\mathrm{d}t$$
$$= \mathrm{e}^{\frac{a+x}{2}} - \mathrm{e}^{\frac{a}{2}}\left(\sqrt{ax}\,I_1(\sqrt{ax})\int_0^\infty\frac{\mathrm{e}^{-t}J_0(\sqrt{2at})}{x+2t}\,\mathrm{d}t\right.$$
$$\left. + \sqrt{2a}\,I_0(\sqrt{ax})\int_0^\infty\frac{\mathrm{e}^{-t}\sqrt{t}\,J_1(\sqrt{2at})}{x+2t}\,\mathrm{d}t\right), \tag{2.106}$$

where we also used the summation formula [102, p. 396, Eq. (57.18.19)]

$$\sum_{n\geq 0}(2n+1)J_{2n+1}(z)J_{2n+1}(t) = \frac{zt}{2(z^2-t^2)}\left(zJ_1(z)J_0(t) - tJ_0(z)J_1(t)\right)$$

with an appropriate substitution $z \mapsto \mathrm{i}\sqrt{ax}$, $t \mapsto \sqrt{2at}$ in order to get the last expression. Finally, having in mind that [118] $J_0'(z) = -J_1(z)$ and also using the identity [2, p. 154, Eq. (4.17)]

$$\int_0^\infty \frac{\mathrm{e}^{-pt^2} J_0(bt)}{t^2+a}\, t\,\mathrm{d}t = \frac{\mathrm{e}^{ap}}{2}\left(K_0(\sqrt{a}\,b) - K_0\left(\log\frac{2p\sqrt{a}}{b}, \sqrt{a}\,b\right)\right),$$

where $\min\{a,b,p\} > 0$, the desired formula immediately follows. □

In order to state and prove the second set of results in this section we introduce the *generalized incomplete Gamma function* [321, p. 4107, Eq. (1)]

$$\Gamma(\alpha, x; b) = \int_x^\infty t^{\alpha-1}\mathrm{e}^{-t-\frac{b}{t}}\,\mathrm{d}t,$$

where $\alpha \in \mathbb{R}$, $x, b \geq 0$, but not both $x = b = 0$ if $\alpha \leq 0$, defined by Chaudhry and Zubair [49] in order to present closed-form solutions to several problems in heat conduction and the *leaky aquifer function*

$$\mathscr{W}(x,y) = \int_1^\infty \frac{\mathrm{e}^{-xt-\frac{y}{t}}}{t}\,\mathrm{d}t,$$

valid for $x, y \geq 0$ and introduced by Hantush and Jacob [103] who showed that water levels in pumped aquifer systems with finite transmissivity and leakage could be analyzed in terms of such integral, consult also [253].

Corollary 2.5 (Jankov Maširević [138]) *For all* $\min\{a,x\} > 0$ *there holds*

$$\sum_{n\geq 0}\left(\sqrt{\frac{x}{a}}\right)^n I_n(\sqrt{ax}) = \mathrm{e}^{\frac{a+x}{2}}\left[1 - \frac{\sqrt{ax}}{2} I_1(\sqrt{ax})\Gamma\left(0, \frac{x}{2}; \frac{ax}{4}\right)\right. \tag{2.107}$$

$$\left. + aI_0(\sqrt{ax})\frac{\partial}{\partial a}\left(\Gamma\left(0, \frac{x}{2}; \frac{ax}{4}\right)\right)\right]$$

$$= \mathrm{e}^{\frac{a+x}{2}}\left[1 - \frac{\sqrt{ax}}{2} I_1(\sqrt{ax})\mathscr{W}\left(\frac{x}{2}, \frac{a}{2}\right)\right. \tag{2.108}$$

$$\left. + aI_0(\sqrt{ax})\frac{\partial}{\partial a}\left(\mathscr{W}\left(\frac{x}{2}, \frac{a}{2}\right)\right)\right].$$

Proof Using the definition of the Bessel function of the first kind (1.19) and the integral representation of the incomplete Gamma function [78, p. 137]

$$\int_0^\infty \frac{\mathrm{e}^{-t}t^{-a}}{x+t}\,\mathrm{d}t = \frac{\Gamma(1-a)\,\Gamma(a,x)}{x^a}\mathrm{e}^x$$

we get

$$\int_0^\infty \frac{\mathrm{e}^{-t} J_0(\sqrt{2at})}{x+2t}\,\mathrm{d}t = \frac{1}{2}\sum_{n\geq 0}\frac{(-1)^n}{n!\,n!}\left(\frac{a}{2}\right)^n \int_0^\infty \frac{\mathrm{e}^{-t} t^n}{\frac{x}{2}+t}\,\mathrm{d}t = \tfrac{1}{2}\mathrm{e}^{\frac{x}{2}} \sum_{n\geq 0}\frac{\left(\frac{-ax}{4}\right)^n}{n!}\Gamma\left(-n, \tfrac{x}{2}\right).$$

Now, from the identity [49, p. 9, Eq. (66)]

$$\Gamma(\alpha, x; b) = \sum_{n\geq 0}\frac{(-b)^n}{n!}\Gamma(\alpha-n, x),$$

and (2.106) the formula (2.107) immediately follows. Further, using the obvious connection between leaky aquifer function and generalized incomplete Gamma function $\mathscr{W}(x, y) = \Gamma(0, x; xy)$ we deduce (2.108). □

The results stated in the next corollary are direct consequence of the closed-form summation formulae stated in Corollary 2.5 and the substitution $a \mapsto -a$:

Corollary 2.6 *For all* $\min\{-a, x\} > 0$ *there holds*

$$\begin{aligned}\sum_{n\geq 0}\left(-\sqrt{\frac{x}{a}}\right)^n J_n(\sqrt{ax}) &= \mathrm{e}^{\frac{x-a}{2}}\left[1 + \frac{\sqrt{ax}}{2} J_1(\sqrt{ax})\Gamma\left(0, \frac{x}{2}; \frac{-ax}{4}\right)\right.\\ &\quad \left. -aJ_0(\sqrt{ax})\frac{\partial}{\partial a}\left(\Gamma\left(0, \frac{x}{2}; \frac{-ax}{4}\right)\right)\right]\\ &= \mathrm{e}^{\frac{x-a}{2}}\left[1 + \frac{\sqrt{ax}}{2} J_1(\sqrt{ax})\mathscr{W}\left(\frac{x}{2}, \frac{-a}{2}\right)\right.\\ &\quad \left. -aJ_0(\sqrt{ax})\frac{\partial}{\partial a}\left(\mathscr{W}\left(\frac{x}{2}, \frac{-a}{2}\right)\right)\right].\end{aligned}$$

2.9 Connecting First and Second Type Neumann Series

In this section we establish connection formulae between first and second type Neumann series. After that, closed-form expressions for some second type Neumann series will be presented.

Theorem 2.22 *For all* $\min\{a, x\} > 0$ *the following connection formulae hold*

$$\begin{aligned}&\sum_{n\geq 0}(-1)^n(2n+1)I_{2n+1}(\sqrt{ax})I_{n+\frac{1}{2}}\left(\frac{a}{4}\right) \qquad (2.109)\\ &\quad = \sqrt{\frac{x}{2\pi}}\mathrm{e}^{-\frac{a}{4}}\left(\mathrm{e}^{\frac{a+x}{2}} - \sum_{n\geq 0}\left(\sqrt{\frac{x}{a}}\right)^n I_n(\sqrt{ax})\right)\end{aligned}$$

and

$$\sum_{n\ge 0}(-1)^n I_{2n+1}(\sqrt{ax})\,\left(I_{n-\frac{1}{2}}\left(\frac{a}{4}\right)-I_{n+\frac{3}{2}}\left(\frac{a}{4}\right)\right) \tag{2.110}$$
$$= \frac{4}{a}\sqrt{\frac{x}{2\pi}}\mathrm{e}^{-\frac{a}{4}}\left(\mathrm{e}^{\frac{a+x}{2}}-\sum_{n\ge 0}\left(\sqrt{\frac{x}{a}}\right)^n I_n(\sqrt{ax})\right).$$

Proof Applying the formulae (2.105) and [227, p. 255, Eq. 10.39.5]

$$I_\nu(x)=\frac{\mathrm{e}^x}{\Gamma(\nu+1)}\left(\frac{x}{2}\right)^\nu {}_1F_1(\nu+\frac{1}{2},\,2\nu+1;-2x),\qquad -\nu\notin\mathbb{N},$$

to equality (2.104) derived in the proof of Theorem 2.21 and also the identity

$$(a)_{2n}=2^{2n}\left(\frac{a}{2}\right)_n\left(\frac{a+1}{2}\right)_n,$$

we immediately get

$$\sum_{n\ge 0}\left(\sqrt{\frac{x}{a}}\right)^n I_n(\sqrt{ax})=\mathrm{e}^{\frac{a+x}{2}}-\sqrt{\frac{2\pi}{x}}\mathrm{e}^{\frac{a}{4}}\sum_{n\ge 0}(-1)^n(2n+1)I_{2n+1}(\sqrt{ax})I_{n+\frac{1}{2}}\left(\frac{a}{4}\right),$$

which is exactly equal to (2.109). Further, using the identity

$$2\nu I_\nu(x)=x\,(I_{\nu-1}(x)-I_{\nu+1}(x))$$

Eq. (2.110) immediately follows. □

Remark 2.7 Combination of results achieved in Theorem 2.21 and Corollary 2.5 with those derived in Theorem 2.22 yields the closed-form expressions for the second type Neumann series (2.109) and (2.110). ■

Ending this section, let us derive a generalization of the following formulae [257, p. 665, Eq. 5.7.11.8]

$$\sum_{n\ge 1}(\pm 1)^n J_{n+\nu}(x)J_{n-\nu}(x)=\frac{1}{2\pi}\begin{Bmatrix}\nu^{-1}\sin(\nu\pi)\\ -4\nu\sin(\nu\pi)s_{-1,2\nu}(2x)\end{Bmatrix}-\frac{1}{2}J_\nu(x)J_{-\nu}(x)$$

and [257, p. 666, Eqs. 20–21]

$$\sum_{n\ge 0}(\pm 1)^n J_{2n+\nu}(x)J_{2n-\nu}(x)=\frac{1}{4\nu\pi}\sin(\nu\pi)\begin{Bmatrix}1\\0\end{Bmatrix}-\frac{\nu}{\pi}\sin(\nu\pi)\begin{Bmatrix}s_{-1,2\nu}(2x)\\ 2s_{-1,2\nu}(\sqrt{2}x)\end{Bmatrix}$$
$$+\frac{1}{2}J_\nu(x)J_{-\nu}(x).$$

First, we derive new summation formulae for the second type Neumann series of I_ν.

Theorem 2.23 *For all $b \in \mathbb{R}$, $2a \in \mathbb{N}$ and $x > 0$ the following summation formulae hold*

$$\sum_{n\geq 0} I_{an+b}(x)I_{an-b}(x) = \frac{1}{2}I_b(x)I_{-b}(x) + \frac{1}{4a}\sum_{n=0}^{2a-1}\Big({}_1\widetilde{F}_2\Big[\begin{matrix}1\\ 1+b, 1-b\end{matrix}\Big| x^2\cos^2\Big(\frac{\pi n}{a}\Big)\Big] \pm x\cos\Big(\frac{\pi n}{a}\Big)\, {}_1\widetilde{F}_2\Big[\begin{matrix}1\\ \frac{3}{2}+b, \frac{3}{2}-b\end{matrix}\Big| x^2\cos^2\Big(\frac{\pi n}{a}\Big)\Big]\Big). \tag{2.111}$$

Proof It is not hard to show that our series absolutely converges for all $x > 0$, using the same proving procedure as in Theorem 2.15.

Now, from the integral representation [227, p. 253, Eq. **10.32.15**]

$$I_\mu(x)I_\nu(x) = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} I_{\mu+\nu}(2x\cos t)\cos((\mu-\nu)t)\, \mathrm{d}t,$$

valid for $\Re(\mu+\nu) > -1$ and with the help of the formula (2.83) we infer that

$$\begin{aligned}\sum_{n\geq 0} I_{an+b}(x)I_{an-b}(x) &= \frac{2}{\pi}\int_0^{\frac{\pi}{2}}\cos(2bt)\sum_{n\geq 0} I_{2an}(2x\cos t)\,\mathrm{d}t\\ &= \frac{1}{\pi}\int_0^{\frac{\pi}{2}}\cos(2bt)\left(I_0(2x\cos t) + \frac{1}{2a}\sum_{n=0}^{2a-1}\mathrm{e}^{2x\cos(\pi n/a)\cos t}\right)\mathrm{d}t\\ &= \frac{1}{2}I_b(x)I_{-b}(x) + \frac{1}{2a\pi}\sum_{n=0}^{2a-1}\int_0^{\frac{\pi}{2}}\cos(2bt)\,\mathrm{e}^{2x\cos(\pi n/a)\cos t}\,\mathrm{d}t.\end{aligned} \tag{2.112}$$

To solve this integral we apply [96, p. 108, Eq. 9c)]

$$\int_0^{\frac{\pi}{2}}\cos^{\gamma-1}t\cos(\beta t)\,\mathrm{d}t = \frac{\pi\Gamma(\gamma)}{2^\gamma\Gamma\left(\frac{\gamma+\beta+1}{2}\right)\Gamma\left(\frac{\gamma-\beta+1}{2}\right)}, \qquad \gamma > 0.$$

Expanding the exponential term in our main integral into Maclaurin series, by the legitimate change of order of summation and integration, we get

$$\begin{aligned}\int_0^{\frac{\pi}{2}}\mathrm{e}^{\alpha\cos t}\cos(\beta t)\,\mathrm{d}t &= \sum_{n\geq 0}\frac{\alpha^n}{n!}\int_0^{\frac{\pi}{2}}\cos^n t\cos(\beta t)\,\mathrm{d}t\\ &= \frac{\pi}{2}\sum_{n\geq 0}\frac{\alpha^n}{2^n\Gamma\left(1+\frac{\beta+n}{2}\right)\Gamma\left(1+\frac{-\beta+n}{2}\right)}\end{aligned}$$

$$= \frac{\pi}{2}\left(\sum_{n\geq 0} \frac{\left(\frac{\alpha}{2}\right)^{2n}(1)_n}{\Gamma\left(1+\frac{\beta}{2}+n\right)\Gamma\left(1-\frac{\beta}{2}+n\right)n!}\right.$$

$$\left. + \sum_{n\geq 0} \frac{\left(\frac{\alpha}{2}\right)^{2n+1}(1)_n}{\Gamma\left(\frac{3+\beta}{2}+n\right)\Gamma\left(\frac{3-\beta}{2}+n\right)n!}\right)$$

$$= \frac{\pi}{2}\left({}_1\widetilde{F}_2\left[\begin{matrix}1\\ 1+\frac{\beta}{2}, 1-\frac{\beta}{2}\end{matrix}\middle| \frac{\alpha^2}{4}\right] + \frac{\alpha}{2}\,{}_1\widetilde{F}_2\left[\begin{matrix}1\\ \frac{3+\beta}{2}, \frac{3-\beta}{2}\end{matrix}\middle| \frac{\alpha^2}{4}\right]\right),$$

where (2.89) implicates the third equality. Substituting $\alpha = 2x\cos\left(\frac{\pi n}{a}\right)$ and $\beta = 2b$ in the previous expression from (2.112) we obtain the desired formula. Another formula follows analogously, using (2.102) instead of (2.83). □

Substituting $\mathrm{i}\,x$ instead of x in (2.111), we immediately arrive at the following particular result.

Corollary 2.7 *For all $a, b \in \mathbb{N}$ and $x > 0$ the following summation formulae hold*

$$\sum_{n\geq 0}(-1)^{an} J_{an+b}(x)J_{an-b}(x)$$

$$= \frac{1}{2}J_b(x)J_{-b}(x) + \frac{1}{4a}\sum_{n=0}^{2a-1}\left({}_1\widetilde{F}_2\left[\begin{matrix}1\\ 1+b, 1-b\end{matrix}\middle| -x^2\cos^2\left(\frac{\pi n}{a}\right)\right]\right.$$

$$\left. \pm \mathrm{i}\,x\cos\left(\frac{\pi n}{a}\right)\,{}_1\widetilde{F}_2\left[\begin{matrix}1\\ \frac{3}{2}+b, \frac{3}{2}-b\end{matrix}\middle| -x^2\cos^2\left(\frac{\pi n}{a}\right)\right]\right).$$

Remark 2.8 It should be mention that Newberger derived a set of summation results close to second type Neumann–Bessel series when the summation set is $\mathbb{Z}$, see [210]. ■

Chapter 3
Kapteyn Series

Abstract In this chapter we deduce several results for Kapteyn series of Bessel and Kummer hypergeometric functions. We present some integral representations and results on coefficients by using the Euler-Maclaurin summation technique and the differential equation technique.

Series of the type

$$\mathfrak{K}_\nu(z) := \sum_{n\geq 1} \alpha_n J_{\nu+n}\left((\nu+n)z\right), \qquad z \in \mathbb{C}, \tag{3.1}$$

where ν, α_n are constants and J_ν stands for the Bessel function of the first kind of order ν, are called *Kapteyn–Bessel series of the first type*. Willem Kapteyn was the first who investigated such series in 1893, in his important memoir [145]. Kapteyn series have been considered in a number of mathematical physics problems. For example, the solution of the famous *Kepler equation* [73, 183, 238]

$$E - \epsilon \sin E = M,$$

where $M \in (0, \pi)$, $\epsilon \in (0, 1]$, can be expressed *via* a Kapteyn series of the first type:

$$E = M + 2\sum_{n\geq 1} \frac{\sin(nM)}{n} J_n(n\epsilon)\,.$$

There is also an integral expression for E obtained in [81, p. 133]. *Kepler's problem* was for the first time analytically solved by Lagrange [166], and the solution was rediscovered half a century later by Bessel in [37], in which he introduced the famous functions named after him. See also [57] for more details.

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There are also *Kapteyn series of the second type*, studied in detail e.g. by Nielsen [217]. Such series are defined by terms consisting of a product of two Bessel functions of the first kind:

$$\sum_{n\geq 1} \beta_n J_{\mu+n}\left(\left(\frac{\mu+\nu}{2}+n\right)z\right) J_{\nu+n}\left(\left(\frac{\mu+\nu}{2}+n\right)z\right), \qquad z, \nu, \mu \in \mathbb{C}.$$

Summations for second type Kapteyn series were obtained in [168, 169, 171]. More about Kapteyn series of the first and second type can be found in [306]. Also, in [182, 228, 229] we can find some asymptotic formulae and estimates for sums of special kind of Kapteyn series.

The importance of Kapteyn series extends from pulsar physics [168] through radiation from rings of discrete charges [169, 312], electromagnetic radiation [280], quantum modulated systems [54, 171], traffic queueing problems [67, 68] and plasma physics problems in ambient magnetic fields [170, 281]. For more details see also the paper [307].

One of the most representative result concerning this type of series is Kapteyn's own expansion [145, p. 103, Eq. (17)] of integral powers z^n:

$$\left(\frac{z}{2}\right)^n = n^2 \sum_{m\geq 1} \frac{(n+m-1)!}{(n+2m)^{n+1}\, m!} J_{n+2m}\big\{(n+2m)z\big\}, \qquad z \in K \tag{3.2}$$

where

$$K := \left\{z \in \mathbb{C}\colon \left|\frac{z \exp\sqrt{1-z^2}}{1+\sqrt{1-z^2}}\right| < 1\right\}.$$

A few years later, in 1906 (see [146]) Kapteyn obtained a generalization of the previous result, i.e. he concluded that it is possible to expand an arbitrary analytic function into a series of Bessel functions of the first kind (3.1), see for example [69, 146, 333]. Namely, let f be a function which is analytic throughout the region

$$D_a = \left\{z \in \mathbb{C}\colon \Omega(z) = \left|\frac{z \exp\{\sqrt{1-z^2}\}}{1+\sqrt{1-z^2}}\right| \leq a\right\},$$

with $a < 1$. Then,

$$f(z) = \alpha_0 + 2\sum_{n\geq 1} \alpha_n J_n(nz), \qquad z \in D_a,$$

where

$$\alpha_n = \frac{1}{2\pi i} \oint \Theta_n(z) f(z) \mathrm{d}z$$

and the path of integration is the curve on which $\Omega(z) = a$. Here the function Θ_n is the so-called Kapteyn polynomial defined by

$$\Theta_0(z) = \frac{1}{z}, \quad \Theta_n(z) = \frac{1}{4}\sum_{k=0}^{[\frac{n}{2}]} \frac{(n-2k)^2(n-k-1)!}{k!}\left(\frac{nz}{2}\right)^{2k-n}, \qquad n \in \mathbb{N}.$$

Before we state our results on Kapteyn series let us first consider the results given by Exton [79] which are the starting point for our first set of main results.

By certain formal manipulations Exton [79, Eqs. (1.1), (4.1)] generalize (3.2), getting

$$\frac{\Gamma(a_1+\frac{\nu}{2})\cdots\Gamma(a_n+\frac{\nu}{2})}{\Gamma(b_1+\frac{\nu}{2})\cdots\Gamma(b_r+\frac{\nu}{2})}\left(\frac{z}{2}\right)^\nu = \sum_{k\geq 0}\frac{\nu^2\,\Gamma(\nu+k)}{(\nu+2k)^{\nu+1}\,k!}\,{}_n\mathbb{X}_{\nu+2k}\left[\begin{matrix}(a)\\(b)\end{matrix}\,\middle|\,(\nu+2k)z\right] \tag{3.3}$$

where

$${}_n\mathbb{X}_{\nu+2k}\left[\begin{matrix}(a)\\(b)\end{matrix}\,\middle|\,z\right] := {}_n\mathbb{X}_{\nu+2k}\left[\begin{matrix}a_1,\cdots,a_n\\b_1,\cdots,b_n\end{matrix}\,\middle|\,z\right]$$
$$\equiv \sum_{k\geq 0}\frac{(-1)^k\left(\frac{z}{2}\right)^{\nu+2k}}{k!\,\Gamma(\nu+1+k)}\prod_{r=1}^n\frac{\Gamma(a_r+\frac{\nu}{2}+k)}{\Gamma(b_r+\nu/2+k)}\,. \tag{3.4}$$

Here the case $n = 0$ corresponds to (3.2) in which $J_\mu \equiv {}_0\mathbb{X}_\mu$. Then Exton examined the convergence of the series on the right of (3.4) but only for integer values of the order parameter μ in ${}_n\mathbb{X}_\mu$, for real z using Hansen's bounding inequalities [333, p. 31][1]

$$|J_0(z)| \leq 1,\ |J_r(z)| \leq \tfrac{1}{\sqrt{2}}, \qquad r \in \mathbb{N},\ z \in \mathbb{R}\,. \tag{3.5}$$

Hence, his conclusion was that the series in (3.3) converges absolutely and uniformly for all real z and for $\Re(d) < 1$, where

$$d := \sum_{j=1}^n (a_j - b_j)\,.$$

Using more sophisticated bounding inequalities than (3.5), precisely the ones by Landau [167] and then Olenko's [226], we will show that the range of d can be extended to $\Re(d) < \frac{4}{3}$ in the case of absolute and uniform convergence and this

[1] In fact Exton applied the inequality $J_{N+2k}\{(N+2k)z\} \leq 1$, $N+2k \in \mathbb{N}_0$ such that didn't appear in [333], but which one readily follows by Hansen's bounds (3.5).

value is optimal when $z \in \mathbb{R}$. Those results, which concern the paper by Pogány [242], will be presented in the next section.

In order to state and prove the second set of results in this chapter, let us mention the well-known fact that the series $\mathfrak{K}_\nu(z)$ is convergent and represents an analytic function (see [333, p. 559]) throughout the domain

$$\Omega(z) < \liminf_{n\to\infty} |\alpha_n|^{-\frac{1}{\nu+n}}.$$

But, when $z = x \in \mathbb{R}$, the convergence region depends on the nature of the sequence $(\alpha_n)_{n\geq 1}$. This question will be tested by using Landau's bounds (1.20), (1.21) for J_ν in the proof of Theorem 3.2, in Sect. 3.2 which also contains a new double definite integral representation of $\mathfrak{K}_\nu$.

Motivated by the above applications in mathematical physics, the main objective of Sect. 3.3 is to establish two different types integral representations for the Kapteyn series of the first type. The first one is a definite integral representation, while the second is an indefinite integral representation formula. Also, in Sect. 3.4 we establish an integral representation for the special kind of Kapteyn series which generalize an integral representation given in Sect. 3.2. Finally, the last section is devoted to new results concerning coefficients of Kapteyn series.

Let us mention, that our main findings are associated with the published papers [23, 131] and [133].

3.1 On Convergence of Generalized Kapteyn Expansion

In the sequel, we begin with Exton's research procedure steps, specifying here and in what follows the parameter sequences $(a) := (a_1, \cdots, a_n)$, $(b) := (b_1, \cdots, b_n)$ by assuming

$$-(\Re(a_j) + h),\ -(\Re(b_j) + k) \notin \mathbb{N}_0 \qquad j = \overline{1,n},\ h,k \in \mathbb{N}_0\,. \tag{3.6}$$

Recalling the property (1.4) of the Gamma function:

$$\Gamma(z) = \sqrt{2\pi}\, z^{z-\frac{1}{2}} \mathrm{e}^{-z} \left(1 + \mathscr{O}(z^{-1})\right) \qquad |\arg z| < \pi,\ |z| \to \infty,$$

we obviously have for k enough large

$${}_n\mathbb{X}_{\nu+2k}\left[\begin{matrix}(a)\\(b)\end{matrix}\,\middle|\,(\nu+2k)z\right] \sim k^d\, J_{\nu+2k}\big\{(\nu+2k)z\big\}\,.$$

Therefore, we examine the convergence of auxiliary Kapteyn-series

$$\sum_{k\geq 0} \frac{\Gamma(\nu+k)\, k^d}{(\nu+2k)^{\nu+1}\, k!}\, J_{\nu+2k}\big\{(\nu+2k)z\big\}\,. \tag{3.7}$$

We already introduced the Landau's bounds (1.20) and (1.21) for the Bessel function of the first kind $J_\nu(x)$ with respect to ν and x. The bounds are uniform, (1.20) for $x \in \mathbb{R}$, and (1.21) for $\nu \in \mathbb{R}_+$, and the exponents $\frac{1}{3}$ are the best possible [167]. Thus, for

$$\Big|\sum_{k\geq 0} \frac{\Gamma(\nu+k)\,k^d}{(\nu+2k)^{\nu+1}\,k!} J_{\nu+2k}\{(\nu+2k)z\}\Big| \leq \sum_{k\geq 0} \frac{\Gamma(\nu+k)\,k^{\Re(d)}}{(\nu+2k)^{\nu+1}\,k!} \big|J_{\nu+2k}\{(\nu+2k)z\}\big|\,.$$

Hence, by means of (1.20) we conclude that the absolute convergence (and consequently the uniform as well!) of (3.7) follows from the convergence of the depending M-series (Weierstraß):

$$\sum_{k\geq 0} \frac{\Gamma(\nu+k)\,k^{\Re(d)}}{(\nu+2k)^{\nu+\frac{4}{3}}\,k!} \Big(= \sum_{k\geq 0} u_k\Big)\,. \tag{3.8}$$

But, since

$$\frac{u_{k+1}}{u_k} = 1 - \frac{\frac{7}{3} - \Re(d)}{k} + \frac{\theta_{\nu,d}(k)}{k^2}\,,$$

where

$$\theta_{\nu,d}(k) := \frac{\nu-1}{3} + \Big(\Re(d) - \tfrac{7}{3}\Big)^2 + \mathscr{O}(k^{-1})\,,$$

is bounded in $k \in \mathbb{N}$, by the Gaußian convergence-test [154, §172, p. 297], the series (3.8) converges for $\frac{7}{3} - \Re(d) > 1$ and diverges elsewhere. That means, the convergence of (3.3) is *a fortiori* established with $\Re(d) < \frac{4}{3}$. Finally, being Landau's bound (1.20) uniform, the constant $\frac{4}{3}$ is optimal for all real z-values so, cannot be improved. The estimate (1.21) gives the same result.

Olenko [226, Theorem 1] has established the following sharp upper bound:

$$\sup_{x\geq 0} \sqrt{x}\,|J_\nu(x)| \leq b_L \sqrt{\nu^{\frac{1}{3}} + \frac{\alpha_1}{\nu^{\frac{1}{3}}} + \frac{3\alpha_1^2}{10\nu}} = \gamma_\nu(\alpha_1), \qquad \nu > 0\,, \tag{3.9}$$

where α_1 is the smallest positive zero of the Airy-function $\mathrm{Ai}(x)$ and b_L is the Landau's constant in (1.20), see also [226, §3]. With the aid of (3.9) it is obvious that

$$\gamma_\nu(\alpha_1) \asymp \nu^{\frac{1}{6}} \qquad \nu \to \infty\,,$$

so, for some absolute constant C_γ we easily deduce that, for fixed x

$$\begin{aligned}
&\sum_{k\geq 0} \frac{\Gamma(\nu+k)\, k^{\Re(d)}}{(\nu+2k)^{\nu+1}\, k!} \left| J_{\nu+2k}\{(\nu+2k)z\}\right| \\
&\qquad \leq \frac{1}{\sqrt{x}} \sum_{k\geq 0} \frac{\Gamma(\nu+k)\, k^{\Re(d)}}{(\nu+2k)^{\nu+\frac{3}{2}}\, k!} \sup_{x>0} \sqrt{(\nu+2k)x}\left| J_{\nu+2k}\{(\nu+2k)z\}\right| \\
&\qquad \leq \frac{1}{\sqrt{x}} \sum_{k\geq 0} \frac{\Gamma(\nu+k)\, k^{\Re(d)} \gamma_{\nu+2k}(\alpha_1)}{(\nu+2k)^{\nu+\frac{3}{2}}\, k!} \\
&\qquad \leq \frac{C_\gamma}{\sqrt{x}} \sum_{k\geq 0} \frac{\Gamma(\nu+k)\, k^{\Re(d)}}{(\nu+2k)^{\nu+\frac{4}{3}}\, k!} .
\end{aligned}$$

The series in the last expression we recognize as $\sum_{k\geq 0} u_k$, considered earlier in display (3.8). Collecting all these considerations, we prove the following

Theorem 3.1 (Pogány [242]) *Consider Extons's Kapteyn-type expansion*

$$\frac{\Gamma(a_1+\frac{\nu}{2})\cdots\Gamma(a_n+\frac{\nu}{2})}{\Gamma(b_1+\frac{\nu}{2})\cdots\Gamma(b_r+\frac{\nu}{2})}\left(\frac{z}{2}\right)^\nu = \sum_{k\geq 0} \frac{\nu^2\,\Gamma(\nu+k)}{(\nu+2k)^{\nu+1}\, k!}\, {}_n\mathbb{X}_{\nu+2k}\left[\begin{matrix}(a)\\(b)\end{matrix}\,\middle|\, (\nu+2k)z\right],$$

where

$${}_n\mathbb{X}_{\nu+2k}\left[\begin{matrix}(a)\\(b)\end{matrix}\,\middle|\, z\right] = \sum_{k\geq 0} \frac{(-1)^k \left(\frac{z}{2}\right)^{\nu+2k}}{k!\,\Gamma(\nu+1+k)} \prod_{r=1}^n \frac{\Gamma(a_r+\frac{\nu}{2}+k)}{\Gamma(b_r+\frac{\nu}{2}+k)}, \qquad \nu>0$$

and the sequences $(a), (b)$ *satisfy* (3.6)*. Then the series* (3.3) *converges absolutely and uniformly for all* $z\in\mathbb{R}$*, when* $\Re(d)<\frac{4}{3}$ *and the bound* $\frac{4}{3}$ *is sharp.*

Remark 3.1 Exton pointed out (hypergeometric display) [79, Eq. (1.2)] that

$$\begin{aligned}
{}_n\mathbb{X}_{\nu}\left[\begin{matrix}(a)\\(b)\end{matrix}\,\middle|\, z\right] &= \left(\prod_{r=1}^n \frac{\Gamma(a_r+\frac{\nu}{2}+k)}{\Gamma(b_r+\frac{\nu}{2}+k)}\right) \frac{\left(\frac{z}{2}\right)^\nu}{\Gamma(\nu+1)}\, {}_nF_{n+1}\left[\begin{matrix}(a)_n+\frac{\nu}{2}\\(b)_n+\frac{\nu}{2},\nu+1\end{matrix}\,\middle|\, -\frac{z^2}{4}\right] \\
&= \left(\frac{z}{2}\right)^\nu {}_n\Psi_{n+1}\left[\begin{matrix}((a)_n+\frac{\nu}{2},1)\\((b)_n+\frac{\nu}{2},1),(\nu+1,1)\end{matrix}\,\middle|\, -\frac{z^2}{4}\right],
\end{aligned}$$

where $(u)_n+v$ is a shorthand for the sequence of parameters $u_1+v,\cdots,u_n+v$, while ${}_nF_{n+1}[\cdot]$ is the generalized hypergeometric and ${}_n\Psi_{n+1}$ the Fox–Wright Psi function, both with n numerator and $n+1$ denominator parameters. Therefore the

Kapteyn-type expansions and Theorem 3.1 can be translated into hypergeometric framework too. The case $n = 0$ corresponds to well-known result

$$ {}_0\mathbb{X}_\nu\big[-\,\big|z\big] = J_\nu(z) = \frac{(z/2)^\nu}{\Gamma(\nu+1)}\,{}_0F_1\Big[{-\atop \nu+1}\Big|-\frac{z^2}{4}\Big]. $$

For a comprehensive treatment of generalized hypergeometric functions the reader can consult the classical monographs [185, 294]. ■

3.2 Integral Representation of Kapteyn Series

In this section our aim is to deduce the double definite integral representation of the Kapteyn series $\mathfrak{K}_\nu(z)$. We shall replace $z \in \mathbb{C}$ with $x > 0$ and assume that the behavior of $(\alpha_n)_{n\geq 1}$ ensures the convergence of the series (3.1) over a proper subset of $\mathbb{R}_+$.

Theorem 3.2 (Baricz et al. [23]) *Let* $\alpha \in \mathrm{C}^1(\mathbb{R}_+)$, $\alpha|_{\mathbb{N}} = (\alpha_n)_{n\geq 1}$ *and assume that* $\sum\limits_{n\geq 1} n^{-\frac13}\alpha_n < \infty$. *Then, for all* $2\nu > -3$ *and*

$$ x \in \left(0, 2\min\left\{1, \mathrm{e}^{-1}\big(\limsup_{n\to\infty}|\alpha_n|^{\frac1n}\big)^{-1}\right\}\right) =: \mathscr{I}_\alpha $$

we have the integral representation

$$ \mathfrak{K}_\nu(x) = -\int_{1+\nu}^\infty\int_\nu^{[u-\nu]+\nu} \frac{\partial}{\partial u}\Big(u^{-u}\,\Gamma\left(u+\tfrac12\right)\,J_u\big(u\,x\big)\Big)\,\mathfrak{d}_s\left(\frac{s^s\,\alpha(s-\nu)}{\Gamma(s+\frac12)}\right)\mathrm{d}u\,\mathrm{d}s\,. \tag{3.10} $$

Proof Let us first establish the convergence conditions for the Kapteyn series of the first type $\mathfrak{K}_\nu(x)$. For this purpose we use Landau's bounds (1.20), (1.21) for the first kind Bessel function introduced in Chap. 1. It is easy to see that there holds the estimate

$$ |\mathfrak{K}_\nu(x)| \leq \max\left\{b_L, \frac{c_L}{x^{\frac13}}\right\}\sum_{n\geq 1}\frac{|\alpha_n|}{(n+\nu)^{\frac13}}\,, $$

and thus the series (3.1) converges for all $x > 0$ when $\sum_{n\geq 1} n^{-\frac13}\alpha_n$ absolutely converges.

Now, recall the following integral representation for the Bessel function [93, p. 902, 8.411, Eq. (10)]

$$ J_\nu(z) = \frac{(z/2)^\nu}{\sqrt{\pi}\,\Gamma(\nu+\frac12)}\int_{-1}^1 \mathrm{e}^{\mathrm{i}zt}(1-t^2)^{\nu-\frac12}\,\mathrm{d}t\,, \qquad z\in\mathbb{C},\ \Re(\nu) > -\tfrac12, \tag{3.11} $$

and thus, having in mind the definition of $\mathfrak{K}_\nu(x)$ in what follows we suppose that $\nu > -\frac{3}{2}$. Replacing (3.11) into (3.1) we have

$$\mathfrak{K}_\nu(x) = \sqrt{\frac{x}{2\pi}} \int_{-1}^{1} \mathrm{e}^{\mathrm{i}\nu x t} \left(\frac{x(1-t^2)}{2}\right)^{\nu-\frac{1}{2}} \mathscr{D}_\alpha(t)\,\mathrm{d}t, \qquad x > 0, \tag{3.12}$$

where $\mathscr{D}_\alpha(t)$ is the Dirichlet series

$$\mathscr{D}_\alpha(t) := \sum_{n\ge 1} \frac{\alpha_n(\nu+n)^{\nu+n}}{\Gamma(n+\nu+\frac{1}{2})} \exp\left(-n \log \frac{2}{\mathrm{e}^{\mathrm{i}xt}x(1-t^2)}\right). \tag{3.13}$$

For the convergence of (3.13) we find that the related radius of convergence equals

$$\rho_{\mathfrak{K}}^{-1} = \mathrm{e} \limsup_{n\to\infty} |\alpha_n|^{\frac{1}{n}}.$$

So, the convergence domain of $\mathscr{D}_\alpha(t)$ is $x \in (0, 2\rho_{\mathfrak{K}})$. Moreover, the Dirichlet series' parameter needs to have positive real part [147, 249], i.e.

$$\Re\left(\log \frac{2}{\mathrm{e}^{\mathrm{i}tx}x(1-t^2)}\right) = \ln \frac{2}{x(1-t^2)} > \log \frac{2}{x} > 0, \qquad |t| < 1,$$

and hence the additional convergence range is $x \in (0, 2)$. Collecting all these estimates, we deduce that the asserted integral expression exists for $x \in \mathscr{I}_\alpha$.

Expressing (3.13) first by virtue of (1.15) as the Laplace integral, then using the Euler–Maclaurin formula (1.9), we get

$$\begin{aligned}\mathscr{D}_\alpha(t) &= \log \frac{2}{\mathrm{e}^{\mathrm{i}xt}x(1-t^2)} \int_0^\infty \left(\frac{\mathrm{e}^{\mathrm{i}xt}x(1-t^2)}{2}\right)^u \sum_{n=1}^{[u]} \frac{\alpha_n(\nu+n)^{\nu+n}}{\Gamma(\nu+n+\frac{1}{2})}\,\mathrm{d}u \\ &= -\int_0^\infty \int_0^{[u]} \left(\frac{\mathrm{e}^{\mathrm{i}xt}x(1-t^2)}{2}\right)^u \log \frac{\mathrm{e}^{\mathrm{i}xt}x(1-t^2)}{2}\, \mathfrak{d}_s\left(\frac{\alpha(s)(\nu+s)^{\nu+s}}{\Gamma(\nu+s+\frac{1}{2})}\right) \mathrm{d}u\,\mathrm{d}s.\end{aligned} \tag{3.14}$$

Combination of (3.12) and (3.14) yields

$$\begin{aligned}\mathfrak{K}_\nu(x) = -\sqrt{\frac{x}{2\pi}} \int_0^\infty \int_0^{[u]} \mathfrak{d}_s\left(\frac{\alpha(s)(\nu+s)^{\nu+s}}{\Gamma(\nu+s+\frac{1}{2})}\right) \\ \times\left(\int_{-1}^{1} \mathrm{e}^{\mathrm{i}x(\nu+u)t}\Big(\frac{x(1-t^2)}{2}\Big)^{\nu+u-\frac{1}{2}} \log \frac{\mathrm{e}^{\mathrm{i}xt}x(1-t^2)}{2}\,\mathrm{d}t\right) \mathrm{d}u\,\mathrm{d}s.\end{aligned} \tag{3.15}$$

Denoting

$$\mathscr{I}_x(u) := \int_{-1}^{1} \mathrm{e}^{\mathrm{i}(\nu+u)xt} \left(\frac{x(1-t^2)}{2}\right)^{\nu+u-\frac{1}{2}} \log \frac{\mathrm{e}^{\mathrm{i}xt} x(1-t^2)}{2}\, \mathrm{d}t\,,$$

we have

$$\int \mathscr{I}_x(u)\, \mathrm{d}u = \sqrt{\frac{2\pi}{x}} \frac{\Gamma(\nu+u+\frac{1}{2})}{(\nu+u)^{\nu+u}} J_{\nu+u}\left((\nu+u)x\right)\,,$$

that is

$$\mathscr{I}_x(u) = \sqrt{\frac{2\pi}{x}} \frac{\partial}{\partial u} \left(\frac{\Gamma(\nu+u+\frac{1}{2})}{(\nu+u)^{\nu+u}} J_{\nu+u}\left((\nu+u)x\right)\right)\,. \tag{3.16}$$

Now, by virtue of (3.15) and (3.16) we conclude that

$$\mathfrak{K}_\nu(x) = -\int_1^\infty \int_0^{[u]} \frac{\partial}{\partial u} \left(\frac{\Gamma(\nu+u+\frac{1}{2})}{(\nu+u)^{\nu+u}} J_{\nu+u}\big((\nu+u)x\big)\right) \mathfrak{d}_s \left(\frac{\alpha(s)(\nu+s)^{(\nu+s)}}{\Gamma(\nu+s+\frac{1}{2})}\right) \mathrm{d}u\mathrm{d}s,$$

and the change of variables $\nu + t \mapsto t,\ t \in \{u, s\}$ completes the proof of (3.10). □

3.3 Another Integral Form of Kapteyn Series Through Bessel Differential Equation

In the following, we deduce another integral representation for the Kapteyn series (3.1), by using the already mentioned fact (2.20) that $J_{\nu+n}$ satisfies

$$x^2 J''_{n+\nu}(x) + x J'_{n+\nu}(x) + (x^2 - (n+\nu)^2) J_{n+\nu}(x) = 0.$$

Now, taking $x \mapsto (\nu+n)x$ we obtain

$$\begin{aligned} &x^2(\nu+n)^2 J''_{\nu+n}\left((\nu+n)x\right) + x(\nu+n) J'_{\nu+n}\left((\nu+n)x\right) \\ &\quad + (\nu+n)^2(x^2-1) J_{\nu+n}\left((\nu+n)x\right) = 0\,. \end{aligned} \tag{3.17}$$

Multiplying (3.17) by α_n, then summing up that expression for $n \in \mathbb{N}$ we arrive at

$$\begin{aligned} &x^2 \mathfrak{K}''_\nu(x) + x\mathfrak{K}'_\nu(x) + (x^2-\nu^2)\mathfrak{K}_\nu(x) \\ &\quad = \sum_{n\ge1} \big(x^2 - \nu^2 + (1-x^2)(\nu+n)^2\big) \alpha_n J_{n+\nu}\big((\nu+n)x\big) =: \mathfrak{L}_\nu(x); \end{aligned} \tag{3.18}$$

the right-hand side expression $\mathfrak{L}_\nu(x)$ defines the *Kapteyn series of Bessel functions associated with* $\mathfrak{K}_\nu(x)$.

Our main results in this section follows.

Theorem 3.3 (Baricz et al. [23]) *For all* $\nu > -\frac{3}{2}$ *the particular solution of the nonhomogeneous Bessel-type differential equation*

$$x^2y'' + xy' + (x^2 - \nu^2)y = \mathfrak{L}_\nu(x)\,, \tag{3.19}$$

with nonhomogeneous part (3.18)*, represents a Kapteyn series* $y = \mathfrak{K}_\nu(x)$ *of order* ν*. Moreover, let* $\alpha \in \mathrm{C}^1(\mathbb{R}_+)$*,* $\alpha|_{\mathbb{N}} = (\alpha_n)_{n\geq 1}$ *and assume that series* $\sum_{n\geq 1} n^{\frac{5}{3}}\alpha_n$ *absolutely converges. Then, for all* $x \in \mathscr{I}_\alpha$ *we have the integral representation*

$$\begin{aligned}\mathfrak{L}_\nu(x) = &-\int_{1+\nu}^{\infty}\int_{\nu}^{[u-\nu]+\nu} \frac{\partial}{\partial u}\big(u^{-u}\,\Gamma(u+\tfrac{1}{2})\,J_u(u\,x)\big) \\ &\times \mathfrak{d}_s\left(\frac{s^s\,\big((1-x^2)s^2 + x^2 - \nu^2\big)\,\alpha(s-\nu)}{\Gamma(s+\frac{1}{2})}\right)\mathrm{d}u\,\mathrm{d}s\,.\end{aligned} \tag{3.20}$$

Proof Equation (3.19) was established already in the beginning of this section. Further, since the associated Kapteyn series $\mathfrak{L}_\nu(x)$ is a linear combination of two Kapteyn-series, reads as follows

$$\mathfrak{L}_\nu(x) = (x^2 - \nu^2)\mathfrak{K}_\nu(x) + (1 - x^2)\sum_{n\geq 1}(\nu + n)^2\alpha_n J_{\nu+n}\big((\nu + n)x\big)\,,$$

the uniform convergence of the second series can be easily recognized (by Landau's bounds) to be such that $\sum\limits_{n\geq 1} n^{\frac{5}{3}}|\alpha_n| < \infty$. Making use of Theorem 3.2 with

$$\alpha_n \mapsto \big((1 - x^2)(\nu + n)^2 + x^2 - \nu^2\big)\alpha_n,$$

we get the statement, the x-range for the integral expression (3.20) remains unchanged. □

Below, we shall need the Bessel functions of the second kind of order ν (or MacDonald functions) Y_ν which are defined by Eq. (2.15), in Chap. 2:

$$Y_\nu(x) = \operatorname{cosec}(\pi\nu)\,(J_\nu(x)\cos(\pi\nu) - J_{-\nu}(x))\,, \qquad \nu \notin \mathbb{Z},\ |\arg(z)| < \pi.$$

Remember that linear combination of J_ν and Y_ν gives the particular solutions of homogeneous Bessel differential equation (1.18), when $\nu \in \mathbb{Z}$. On the other hand, when $\nu \notin \mathbb{Z}$, the particular solution is given as the linear combination of the Bessel functions of the first kind, J_ν and $J_{-\nu}$.

Theorem 3.4 (Baricz et al. [23]) *Let the situation be the same as in* Theorem 3.3. *Then we have*

$$\mathfrak{K}_\nu(x) = \frac{J_\nu(x)}{2}\int \frac{1}{xJ_\nu^2(x)}\left(\int \frac{J_\nu(x)\mathfrak{L}_\nu(x)}{x}\,\mathrm{d}x\right)\mathrm{d}x + \frac{Y_\nu(x)}{2}\int \frac{1}{xY_\nu^2(x)}\left(\int \frac{Y_\nu(x)\mathfrak{L}_\nu(x)}{x}\,\mathrm{d}x\right)\mathrm{d}x\,, \qquad (3.21)$$

where $\mathfrak{L}_\nu$ is the Kapteyn series associated with the initial Kapteyn series of Bessel functions.

Proof It is a well-known fact that J_ν and Y_ν are independent solutions of the homogeneous Bessel differential equation. Thus, the solution of the homogeneous ordinary differential equation is

$$y_h(x) = C_1 Y_\nu(x) + C_2 J_\nu(x)\,.$$

Since J_ν is a solution of Bessel's differential equation, a guess of the particular solution is $\mathfrak{K}_\nu(x) = J_\nu(x)w(x)$. Substituting this form into non-homogeneous Bessel differential equation (3.18) we get

$$x^2(J_\nu''w + 2J_\nu'w' + J_\nu w'') + x(J_\nu'w + J_\nu w') + (x^2-\nu^2)J_\nu w = \mathfrak{L}_\nu(x).$$

Rewriting the equation as

$$w\big(x^2J_\nu'' + xJ_\nu' + (x^2-\nu^2)J_\nu\big) + w'(2x^2J_\nu' + xJ_\nu) + w''(x^2J_\nu) = \mathfrak{L}_\nu(x),$$

and using again the fact that J_ν is a solution of the homogeneous Bessel differential equation, leads to the solution

$$w = \int \frac{1}{xJ_\nu^2}\left(\int \frac{\mathfrak{L}_\nu J_\nu}{x}\,\mathrm{d}x\right)\mathrm{d}x + C_3\frac{\pi}{2}\frac{Y_\nu}{J_\nu} + C_4\,,$$

because

$$\int \frac{1}{xJ_\nu^2}\,\mathrm{d}x = \frac{\pi}{2}\frac{Y_\nu}{J_\nu}.$$

Therefore, the desired particular solution is

$$\mathfrak{K}_\nu(x) = J_\nu(x)w(x) = J_\nu(x)\int \frac{1}{xJ_\nu^2}\left(\int \frac{\mathfrak{L}_\nu J_\nu}{x}\,\mathrm{d}x\right)\mathrm{d}x + C_3\frac{\pi}{2}Y_\nu(x) + C_4J_\nu(x)\,.$$

Finally, as J_ν and Y_ν are independent functions that build up the solution y_h, they do not contribute to the particular solution y_p and the constants C_3, C_4 can be taken to be zero.

On the other hand, taking a particular solution in the form $\mathfrak{K}_\nu(x) = Y_\nu(x)w(x)$ and repeating the procedure, we arrive at

$$\mathfrak{K}_\nu(x) = Y_\nu(x)w(x) = Y_\nu(x)\int \frac{1}{xY_\nu^2}\left(\int \frac{\mathfrak{L}_\nu\, Y_\nu}{x}\,\mathrm{d}x\right)\mathrm{d}x - C_5\frac{\pi}{2}J_\nu(x) + C_6Y_\nu(x)\,,$$

bearing in mind that

$$\int \frac{1}{xY_\nu^2}\,\mathrm{d}x = -\frac{\pi}{2}\frac{J_\nu}{Y_\nu}.$$

Choosing $C_5 = C_6 = 0$, we obtain the integral representation (3.21). □

3.4 Integral Expression of Special Kind Kapteyn Series

Here we derive an integral expression for the special Kapteyn–Bessel series

$$\widetilde{K}^\mu_{\nu,\beta}(z) := \sum_{n\geq 1}\alpha_n J_{\nu+\beta n}\big((\mu+n)z\big), \qquad z\in\mathbb{C} \tag{3.22}$$

where ν, α_n are constants, $\mu\in\mathbb{C}$ and $\beta > 0$. This integral will be useful to us in the next chapter, devoted to similar questions concerning Schlömilch series.

Theorem 3.5 (Jankov and Pogány [131]) *Let* $\alpha \in \mathrm{C}^1(\mathbb{R}_+)$, $\alpha|_{\mathbb{N}} = (\alpha_n)_{n\geq 1}$ *and assume* $\mathscr{C} = \limsup_{n\to\infty}|\alpha_n|^{\frac{1}{n}} < 1$. *Then, for all* $\beta > 0$, $2(\nu+\beta)+1 > 0$ *and*

$$x \in \mathscr{I}_{\alpha,\beta} := \Big(0, 2\min\Big\{1, \beta\,(\mathrm{e}\,\mathscr{C}^{\frac{1}{\beta}})^{-1}\Big\}\Big)$$

we have

$$\widetilde{K}^\mu_{\nu,\beta}(x) = -\int_1^\infty\int_0^{[u]}\frac{\partial}{\partial u}\Big(\frac{\Gamma(\beta u+\nu+\frac{1}{2})}{(\mu+u)^{\beta u+\nu}}J_{\beta u+\nu}\big((\mu+u)\,x\big)\Big)$$
$$\times\, \mathfrak{d}_s\left(\frac{\alpha(s)(\mu+s)^{\nu+\beta s}}{\Gamma(\nu+\beta s+\frac{1}{2})}\right)\mathrm{d}u\,\mathrm{d}s. \tag{3.23}$$

Proof By virtue of the Landau's bounds (1.20), (1.21) there holds

$$\Big|\widetilde{K}^\mu_{\nu,\beta}(x)\Big| \leq \sum_{n\geq 1}|\alpha_n|\max\left\{\frac{b_L}{(\nu+\beta n)^{\frac{1}{3}}}, \frac{c_L}{((\mu+n)|x|)^{\frac{1}{3}}}\right\}$$
$$\leq \max\left\{\frac{b_L}{(\nu+\beta)^{\frac{1}{3}}}, \frac{c_L}{((\mu+1)|x|)^{\frac{1}{3}}}\right\}\sum_{n\geq 1}|\alpha_n|\,,$$

therefore the series (3.22) absolutely converges being $\mathscr{C} < 1$.

In the following, again we need the integral representation of the Bessel function (3.11), it has to be $2(\nu + \beta) + 1 > 0$. Substituting (3.11) into (3.22) we get

$$\widetilde{K}^{\mu}_{\nu,\beta}(x) = \sqrt{\frac{x}{2\pi}} \int_{-1}^{1} \mathrm{e}^{\mathrm{i}\mu x t} \left(\frac{x(1-t^2)}{2}\right)^{\nu-\frac{1}{2}} \mathscr{D}_\alpha(t)\,\mathrm{d}t, \qquad x > 0, \tag{3.24}$$

where $\mathscr{D}_\alpha(t)$ is the Dirichlet series

$$\mathscr{D}_\alpha(t) := \sum_{n\ge 1} \frac{\alpha_n (\mu+n)^{\nu+\beta n}}{\Gamma(\nu+\beta n+\frac{1}{2})} \exp\left\{-n \log\left(\frac{2}{\mathrm{e}^{\mathrm{i}xt/\beta} x(1-t^2)}\right)^{\beta}\right\}. \tag{3.25}$$

For the convergence of (3.25) we find that the related radius of convergence equals

$$\rho = \left(\frac{\beta}{\mathrm{e}}\right)^{\beta} \left(\limsup_{n\to\infty} |\alpha_n|^{\frac{1}{n}}\right)^{-1} = \frac{\beta^\beta}{\mathrm{e}^\beta\,\mathscr{C}}.$$

Now, because of $|t| < 1$, there holds

$$\left|\mathrm{e}^{\mathrm{i}xt}\left(\frac{x(1-t^2)}{2}\right)^{\beta}\right| \le \left|\frac{x}{2}\right|^{\beta} < \rho\,,$$

hence the convergence domain of $\mathscr{D}_\alpha(t)$ is

$$|x| < 2\rho^{\frac{1}{\beta}} = \frac{2\beta}{\mathrm{e}}\,\mathscr{C}^{-\frac{1}{\beta}}.$$

Moreover, the Dirichlet series' parameter needs to have positive real part [147, 249]:

$$\Re\left(\log \frac{2^\beta}{\mathrm{e}^{\mathrm{i}tx} x^\beta (1-t^2)^\beta}\right) = \beta \ln \frac{2}{x(1-t^2)} > \beta \log\frac{2}{x} > 0, \qquad |t| < 1\,,$$

so, this additional convergence range is $x \in (0, 2)$. Collecting all these estimates, we deduce that the desired integral expression exists for $x \in \mathscr{I}_{\alpha,\beta}$. Expressing (3.25) as the Laplace integral we get

$$\begin{aligned}
\mathscr{D}_\alpha(t) &= \log \frac{2^\beta}{\mathrm{e}^{\mathrm{i}xt}\,(x(1-t^2))^\beta} \int_0^\infty \left(\mathrm{e}^{\mathrm{i}xt}\left(\frac{x(1-t^2)}{2}\right)^{\beta}\right)^{u} \sum_{n=1}^{[u]} \frac{\alpha_n(\mu+n)^{\nu+\beta n}}{\Gamma\left(\nu+\beta n+\frac{1}{2}\right)}\,\mathrm{d}u \\
&= -\int_0^\infty \int_0^{[u]} \left(\mathrm{e}^{\mathrm{i}xt}\left(\frac{x(1-t^2)}{2}\right)^{\beta}\right)^{u} \log \frac{\mathrm{e}^{\mathrm{i}xt}\left(x(1-t^2)\right)^\beta}{2^\beta} \\
&\quad \times \mathfrak{d}_s \left(\frac{\alpha(s)(\mu+s)^{\nu+\beta s}}{\Gamma\left(\nu+\beta s+\frac{1}{2}\right)}\right) \mathrm{d}u\mathrm{d}s\,.
\end{aligned} \tag{3.26}$$

Combination of (3.24) and (3.26) yields

$$\widetilde{K}^{\mu}_{\nu,\beta}(x) = -\sqrt{\frac{x}{2\pi}} \int_0^\infty \int_0^{[u]} \mathfrak{d}_s \left(\frac{\alpha(s)(\mu+s)^{\nu+\beta s}}{\Gamma\left(\nu+\beta s+\frac{1}{2}\right)} \right) \times \left(\int_{-1}^1 \mathrm{e}^{\mathrm{i}x(\mu+u)t} \left(\frac{x(1-t^2)}{2}\right)^{\nu+\beta u-\frac{1}{2}} \log \frac{\mathrm{e}^{\mathrm{i}xt}\left(x(1-t^2)\right)^\beta}{2^\beta}\,\mathrm{d}t \right) \mathrm{d}u\mathrm{d}s. \tag{3.27}$$

In the following, we will simplify the t-integral

$$\mathscr{J}_x(u) := \int_{-1}^1 \mathrm{e}^{\mathrm{i}(\mu+u)xt} \left(\frac{x(1-t^2)}{2}\right)^{\nu+\beta u-\frac{1}{2}} \log \frac{\mathrm{e}^{\mathrm{i}xt}\left(x(1-t^2)\right)^\beta}{2^\beta}\,\mathrm{d}t\,.$$

We have

$$\begin{aligned}\int \mathscr{J}_x(u)\mathrm{d}u &= \int_{-1}^1 \mathrm{e}^{\mathrm{i}(\mu+u)xt} \left(\frac{x(1-t^2)}{2}\right)^{\nu+\beta u-\frac{1}{2}} \mathrm{d}t \\ &= \sqrt{\frac{2\pi}{x}} \frac{\Gamma\left(\nu+\beta u+\frac{1}{2}\right)}{(\mu+u)^{\nu+\beta u}} J_{\beta u+\nu}\left((\mu+u)x\right)\,,\end{aligned}$$

that is

$$\mathscr{J}_x(u) = \sqrt{\frac{2\pi}{x}} \frac{\partial}{\partial u} \left(\frac{\Gamma\left(\nu+\beta u+\frac{1}{2}\right)}{(\mu+u)^{\nu+\beta u}} J_{\beta u+\nu}\left((\mu+u)x\right) \right). \tag{3.28}$$

Now, by virtue of (3.27) and (3.28) we immediately get the integral representation (3.23). □

3.5 On Coefficients of Kapteyn Series

Here we describe the class of functions $\Lambda = \{\alpha\}$ which generate an integral representation like (3.23) for the corresponding Kapteyn-type series, in the sense that $\alpha\big|_{\mathbb{N}} = \left(\alpha_n\right)_{n\ge1}$ generates the coefficients of the series (3.22). For the fixed set of nodes $\left\{(n,\alpha_n)\right\}_{n\ge1}$, we derive the class of functions α which depends on a certain integrable (on $\mathbb{R}_+$), scaling-function h, say. It is important to note that Jankov et al. [134] applied a similar way of concluding the coefficient-function class for Neumann series $\mathfrak{N}^J_{\mu,\nu}(x)$ (compare Sect. 2.2).

Theorem 3.6 (Jankov and Pogány [133]) *Let $\beta > 0, 1 + \min\{\mu, \nu/\beta\} > 0$ and assume that* Theorem 3.5 *holds for a given Kapteyn-type series of Bessel functions. Suppose that the integrand in* (3.23)*, is such that*

$$\frac{\partial}{\partial u}\Big(\frac{\Gamma(\beta u+\nu+\frac{1}{2})}{(\mu+u)^{\beta u+\nu}}\,J_{\beta u+\nu}\big((\mu+u)\,x\big)\Big)\int_0^{[u]}\mathfrak{d}_s\left(\frac{\alpha(s)(\mu+s)^{\nu+\beta s}}{\Gamma(\nu+\beta s+\frac{1}{2})}\right)\mathrm{d}s,$$

is $\mathrm{L}^1(\mathbb{R}_+)$*-integrable and let*

$$h(u) := \frac{\partial}{\partial u}\Big(\frac{\Gamma(\beta u+\nu+\frac{1}{2})}{(\mu+u)^{\beta u+\nu}}\,J_{\beta u+\nu}\big((\mu+u)\,x\big)\Big)\int_0^{u}\mathfrak{d}_s\left(\frac{\alpha(s)(\mu+s)^{\nu+\beta s}}{\Gamma(\nu+\beta s+\frac{1}{2})}\right)\mathrm{d}s\,.$$

Then the following formula holds

$$\alpha(u) = \begin{cases} \dfrac{\Gamma(\nu+\beta k+\frac{1}{2})}{(\mu+k)^{\nu+\beta k}}\,\dfrac{\mathrm{d}}{\mathrm{d}u}\dfrac{h(u)}{\mathscr{H}(u)}\Big|_{u=k+}, & u=k,\ k\in\mathbb{N} \\ \dfrac{\Gamma(\nu+\beta u+\frac{1}{2})}{\{u\}\,(\mu+u)^{\nu+\beta u}}\Big(\dfrac{h(u)}{\mathscr{H}(u)}-\dfrac{h(k+)}{\mathscr{H}(k)}\Big), & 1<u\neq k,\ k\in\mathbb{N} \end{cases}, \tag{3.29}$$

where

$$\mathscr{H}(u) := \frac{\partial}{\partial u}\Big(\frac{\Gamma(\beta u+\nu+\frac{1}{2})}{(\mu+u)^{\beta u+\nu}}\,J_{\beta u+\nu}\big((\mu+u)\,x\big)\Big).$$

Proof Assume that the integral representation (3.23) holds for some class of functions $\Lambda = \{\alpha\}$ such that $\alpha|_{\mathbb{N}}$ represents the coefficient array appearing in $\widetilde{K}^{\mu}_{\nu,\beta}(x)$. Suppose that the function $\widetilde{h}\in \mathrm{L}^1(\mathbb{R}_+)$ is defined by

$$\widetilde{h}(u) := \mathscr{H}(u)\int_0^{[u]}\mathfrak{d}_s\left(\frac{\alpha(s)(\mu+s)^{\nu+\beta s}}{\Gamma(\nu+\beta s+\frac{1}{2})}\right)\mathrm{d}s\,. \tag{3.30}$$

Because $u\sim[u]$ for large u, using (3.30) we conclude that

$$\int_0^{u}\mathfrak{d}_s\left(\frac{\alpha(s)(\mu+s)^{\nu+\beta s}}{\Gamma(\nu+\beta s+\frac{1}{2})}\right)\mathrm{d}s = \frac{h(u)}{\mathscr{H}(u)}, \tag{3.31}$$

where

$$h(u) := \frac{\widetilde{h}(u)\displaystyle\int_0^{u}\mathfrak{d}_s\left(\frac{\alpha(s)(\mu+s)^{\nu+\beta s}}{\Gamma(\nu+\beta s+\frac{1}{2})}\right)\mathrm{d}s}{\displaystyle\int_0^{[u]}\mathfrak{d}_s\left(\frac{\alpha(s)(\mu+s)^{\nu+\beta s}}{\Gamma(\nu+\beta s+\frac{1}{2})}\right)\mathrm{d}s}\sim\widetilde{h}(u), \qquad u\to\infty.$$

If we differentiate (3.31) with respect to u, we get

$$\{u\}\alpha'(u) + \Big(1 + \{u\}\Big(\log(\mu+u)^\beta + \frac{\nu+\beta u}{\mu+u} - \beta\,\psi\left(\nu+\beta u+\tfrac{1}{2}\right)\Big)\Big)\alpha(u)$$
$$= \frac{\Gamma(\nu+\beta u+\frac{1}{2})}{(\mu+u)^{\nu+\beta u}}\cdot\frac{\partial}{\partial u}\,\frac{h(u)}{\mathscr{H}(u)}\,. \tag{3.32}$$

For $u \equiv k \in \mathbb{N}$, we have the coefficient-set (α_k). When $u \in (k, k+1)$, where k is a fixed positive integer, from (3.32) we deduce

$$\alpha'(u) + \Big(\frac{1}{u-k} + \log(\mu+u)^\beta + \frac{\nu+\beta u}{\mu+u} - \beta\,\psi\left(\nu+\beta u+\tfrac{1}{2}\right)\Big)\alpha(u)$$
$$= \frac{\Gamma(\nu+\beta u+\frac{1}{2})}{(u-k)\,(\mu+u)^{\nu+\beta u}}\cdot\frac{\partial}{\partial u}\,\frac{h(u)}{\mathscr{H}(u)}\,.$$

Now it is easy to find the solution of the previous linear ordinary differential equation in the form

$$\alpha(u) = \frac{\Gamma(\nu+\beta u+\frac{1}{2})}{\{u\}\,(\mu+u)^{\nu+\beta u}}\Big(C_k + \frac{h(u)}{\mathscr{H}(u)}\Big),$$

where C_k denotes the integration constant. Thus we conclude that for $u \geq 1$ it holds $\alpha(u) = \alpha_k$ for $u = k$, $k \in \mathbb{N}$ and

$$\alpha(u) = \frac{\Gamma(\nu+\beta u+\frac{1}{2})}{\{u\}\,(\mu+u)^{\nu+\beta u}}\Big(C_k + \frac{h(u)}{\mathscr{H}(u)}\Big), \qquad 1 < u \neq k, \quad k \in \mathbb{N}.$$

Using Landau's bounds (1.20) and (1.21) we have the estimate

$$\big|\widetilde{K}^{\mu}_{\nu,\beta}(x)\big| \leq \max\left\{\frac{b_L}{\sqrt[3]{\beta}}, \frac{c_L}{\sqrt[3]{|x|}}\right\}\sum_{n\geq 1}\frac{|\alpha_n|}{\big(n+\min\{\mu,\nu/\beta\}\big)^{\frac{1}{3}}} \sim \sum_{n\geq 1}\frac{|\alpha_n|}{n^{\frac{1}{3}}}\,,$$

which converges by assumption. So, it is sufficient to take $\alpha(u) \to 0$, as $k \to \infty$.

Let us find the constant C_k. Because

$$\alpha_k = \lim_{u\to k+}\alpha(u) = \lim_{u\to k+}\Gamma\left(\nu+\beta u+\tfrac{1}{2}\right)\lim_{u\to k+}\frac{C_k + \dfrac{h(u)}{\mathscr{H}(u)}}{(u-k)\,(\mu+u)^{\nu+\beta u}}$$

becomes an indeterminate form for

$$C_k = -\frac{h(k+)}{\mathscr{H}(k)},$$

by L'Hospital's rule we conclude

$$\alpha_k = \frac{\Gamma(\nu + \beta k + \frac{1}{2})}{(\mu + k)^{\nu+\beta k}} \cdot \frac{\mathrm{d}}{\mathrm{d}u} \frac{h(u)}{\mathscr{H}(u)}\Big|_{u=k+} .$$

Finally, we get the desired formula

$$\alpha(u) = \begin{cases} \dfrac{\Gamma(\nu + \beta k + \frac{1}{2})}{(\mu + k)^{\nu+\beta k}} \dfrac{\mathrm{d}}{\mathrm{d}u} \dfrac{h(u)}{\mathscr{H}(u)}\Big|_{u=k+}, & u = k, \quad k \in \mathbb{N} \\ \dfrac{\Gamma(\nu + \beta u + \frac{1}{2})}{\{u\}\,(\mu + u)^{\nu+\beta u}} \Big(\dfrac{h(u)}{\mathscr{H}(u)} - \dfrac{h(k+)}{\mathscr{H}(k)} \Big), & 1 < u \neq k, \quad k \in \mathbb{N} \end{cases},$$

such that finishes the proof of the Theorem 3.6. □

Remark 3.2 Specifying $\beta = 1, \mu = \nu$ in Theorem 3.6, we deduce the coefficient function class Λ for the Kapteyn-series $\mathfrak{K}_\nu(x)$ associated with the integral representation result by Baricz et al. [23, Theorem 1]. ■

3.5.1 Examples

In the introduction we pointed out a wide range of applications of Kapteyn-series, while in this section, we present two illustrative examples for functions $\widetilde{h} \in \mathrm{L}^1(\mathbb{R}_+)$, which describes the convergence rate to zero of the integrand in (3.30) at infinity. Simultaneously, the associated function $h(u) \sim \widetilde{h}(u)$, $u \to \infty$ and finally related coefficient-functions α are obtained by (3.29). We remark that in both examples $\mathscr{H}(u)$ remains the same as in Theorem 3.6.

Example 3.1 Let $\widetilde{h}(u) = [u]^s(\mathrm{e}^{[u]} + 1)^{-1}$, $s > 0$; then

$$\int_0^\infty \widetilde{h}(u)\,\mathrm{d}u = \sum_{n\geq 1} \frac{n^s}{\mathrm{e}^n + 1} ,$$

which is a convergent series, so $\widetilde{h} \in \mathrm{L}^1(\mathbb{R}_+)$. Next, we have

$$[u]^s\big(\mathrm{e}^{[u]} + 1\big)^{-1} \sim u^s\big(\mathrm{e}^u + 1\big)^{-1} = h(u) \qquad u \to \infty .$$

On the other side

$$\int_0^\infty h(u)\,\mathrm{d}u = (1 - 2^{-s})\Gamma(1 + s)\zeta(1 + s),$$

where ζ stands for the Riemann Zeta function. For $s > 0$, from (3.29) it follows that the coefficient-function is of the form

$$\alpha(u) = \begin{cases} \dfrac{\Gamma(\nu+\beta k+\frac{1}{2})}{(\mu+k)^{\nu+\beta k}} \dfrac{\mathrm{d}}{\mathrm{d}u} \dfrac{u^s}{(\mathrm{e}^u+1)\mathscr{H}(u)}\Bigg|_{u=k+}, & u = k \in \mathbb{N} \\ \dfrac{\Gamma(\nu+\beta u+\frac{1}{2})}{\{u\}(\mu+u)^{\nu+\beta u}} \Big(\dfrac{u^s}{\mathscr{H}(u)\,(\mathrm{e}^u+1)} \\ \qquad -\dfrac{k^s}{\mathscr{H}(k)\,(\mathrm{e}^k+1)}\Big), & 1 < u \neq k \in \mathbb{N} \end{cases}.$$

Example 3.2 Take $\widetilde{h}(u) = \mathrm{e}^{-s[u]}J_0(a\,[u])$, where $s \geq 0$, $a \in \mathbb{R}$. Since

$$\int_0^\infty \mathrm{e}^{-s[u]}J_0(a\,[u])\,\mathrm{d}u = \sum_{n\geq 0}\mathrm{e}^{-sn}J_0(an),$$

the auxiliary function $\widetilde{h}$ belongs to $\mathrm{L}^1(\mathbb{R}_+)$. Further

$$\mathrm{e}^{-s[u]}J_0(a\,[u]) \sim \mathrm{e}^{-su}J_0(au) = h(u) \qquad u \to \infty\,,$$

and

$$\int_0^\infty h(u)\,\mathrm{d}u = \frac{1}{\sqrt{s^2+a^2}}\,,$$

being the integral the Laplace transform of $J_0(au)$. Thus, by (3.29) we conclude

$$\alpha(u) = \begin{cases} \dfrac{\Gamma(\nu+\beta k+\frac{1}{2})}{(\mu+k)^{\nu+\beta k}} \dfrac{\mathrm{d}}{\mathrm{d}u} \dfrac{\mathrm{e}^{-su}J_0(au)}{\mathscr{H}(u)}\Bigg|_{u=k+}, & u = k \in \mathbb{N} \\ \dfrac{\Gamma(\nu+\beta u+\frac{1}{2})}{\{u\}(\mu+u)^{\nu+\beta u}} \Big(\dfrac{\mathrm{e}^{-su}J_0(au)}{\mathscr{H}(u)} \\ \qquad -\dfrac{\mathrm{e}^{-sk}J_0(ak)}{\mathscr{H}(k)}\Big), & 1 < u \neq k \in \mathbb{N} \end{cases}.$$

3.6 On Kapteyn–Kummer Series' Integral Form

As in this section our main goal concerns the Kapteyn series we will focus our exposition to this kind of series, pointing out that a set of problems associated with Kapteyn type series are solved in [23, 133].

The Kummer's differential equation [227, §13.2]

$$z\,y'' + (b-z)\,y' - ay = 0, \qquad y \equiv \Phi(a,b,z)$$

is the limiting form of the hypergeometric differential equation with the first standard series solution

$$ {}_1F_1(a;b;z) = \sum_{n\geq 0} \frac{(a)_n}{(b)_n}\frac{z^n}{n!}, \qquad a\in\mathbb{C},\ b\in\mathbb{C}\setminus\mathbb{Z}_0^- . $$

The series converges for all $z\in\mathbb{C}$. Another notations which occur for Kummer's function are: $M(a;b;z)$, $\Phi(a,b,z)$.

Having in mind the structure of Fourier–Bessel series (1.27), let us introduce the *Kapteyn–Kummer series* as

$$ \mathscr{K}_\kappa(z) =: \mathscr{K}_\kappa\left(\begin{matrix} a,b \\ \alpha,\beta,\zeta \end{matrix}; z\right) = \sum_{n\geq 0} \kappa_n\, {}_1F_1(a+\alpha n; b+\beta n; z(1+\zeta n)), \qquad (3.33) $$

where $\kappa_n\in\mathbb{C}$; the parameter range and the z-domain will be described in the sequel. We point out that for at least one non-zero α,β, and $\zeta=0$, this series becomes a Neumann—while in the case $\alpha=\beta=0, \zeta\neq 0$ we are faced with the Schlömilch–Kummer series.

We are motivated by the fact that Kummer's function ${}_1F_1(a;b;z)$ generate diverse special functions such as [1, pp. 509–10, §13.6. Special Cases]

$$ {}_1F_1(\nu+\tfrac12; 2\nu+1; 2\mathrm{i}z) = \Gamma(1+\nu)\,\mathrm{e}^{\mathrm{i}z}\left(\tfrac12 z\right)^{-\nu} J_\nu(z) $$

$$ {}_1F_1(-\nu+\tfrac12; -2\nu+1; 2\mathrm{i}z) = \Gamma(1-\nu)\,\mathrm{e}^{\mathrm{i}z}\left(\tfrac12 z\right)^{\nu}\left[\cos(\nu\pi)J_\nu(z) - \sin(\nu\pi)Y_\nu(z)\right] $$

$$ {}_1F_1(\nu+\tfrac12; 2\nu+1; 2z) = \Gamma(1+\nu)\,\mathrm{e}^{z}\left(\tfrac12 z\right)^{-\nu} I_\nu(z) $$

where $J_\nu(I_\nu), Y_\nu(K_\nu)$ stand for the Bessel (modified Bessel) functions of the first and second kind of the order ν respectively, for which their Fourier–Bessel series have been studied in [24, 130, 131, 134, 244, 249] and [133], among others. Further special cases of Φ listed in [1, pp. 509–510, §13.6.] are: Hankel, spherical Bessel, Coulomb wave [17], Laguerre, incomplete Gamma, Poisson–Charlier, Weber, Hermite, Airy, Kelvin, error function and elementary functions like trigonometric, exponential and hyperbolic ones. These links from Kummer's Φ to the above mentioned special functions and then *a fortiori* to their Schlömilch-, Neumann- and Kapteyn-series obviously justify the definition of the Kapteyn–Kummer $\mathscr{K}_\kappa$-series (3.33).

Our main aim here is to establish integral representation for the Kapteyn–Kummer series $\mathscr{K}_\kappa$. The main derivation tools will be the associated Dirichlet series, the famous Cahen's formula (1.15) and the Euler–Maclaurin summation formula firstly used for similar purposes in [239] and in [249].

3.6.1 The Master Integral Representation Formula

The derivation of the integral representation formula we split into few crucial steps assuming that all auxiliary parameters a, b, α, β *mutatis mutandis* are non-negative, and ζ real. Having in mind the integral expression of Kummer's function [1, p. 505, Eq. 13.2.1]

$$ {}_1F_1(a;b;z) = \frac{\Gamma(b)}{\Gamma(b-a)\Gamma(a)} \int_0^1 \mathrm{e}^{zt}\, t^{a-1}(1-t)^{b-a-1}\, \mathrm{d}t, \tag{3.34} $$

valid for all $\Re(b) > \Re(a) > 0$, we transform the Kapteyn–Kummer series into

$$ \begin{aligned} \mathscr{K}_\kappa(z) = &\sum_{n\geq 0} \frac{\kappa_n \Gamma(b+\beta n)}{\Gamma(b-a+(\beta-\alpha)n)\Gamma(a+\alpha n)} \\ &\times \int_0^1 \mathrm{e}^{z(1+\zeta n)t}\, t^{a+\alpha n-1}(1-t)^{b-a+(\beta-\alpha)n-1}\, \mathrm{d}t. \end{aligned} \tag{3.35} $$

Hence, for all $\beta \geq \alpha \geq 0$ using (3.35) we get

$$ \begin{aligned} \left|\mathscr{K}_\kappa(z)\right| &\leq \sum_{n\geq 0} \frac{|\kappa_n|\Gamma(b+\beta n)}{\Gamma(b-a+(\beta-\alpha)n)\Gamma(a+\alpha n)} \\ &\quad\times \int_0^1 \left|\mathrm{e}^{\Re(z)(1+\zeta n)t}\right|\, t^{a+\alpha n-1}(1-t)^{b-a+(\beta-\alpha)n-1}\, \mathrm{d}t \\ &\leq \sum_{n\geq 0} \frac{|\kappa_n|\Gamma(b+\beta n)}{\Gamma(b-a+(\beta-\alpha)n)\Gamma(a+\alpha n)} \\ &\quad\times \int_0^1 \mathrm{e}^{|\Re(z)|(1+|\zeta|n)t}\, t^{a+\alpha n-1}(1-t)^{b-a+(\beta-\alpha)n-1}\, \mathrm{d}t \\ &\leq \mathrm{e}^{|\Re(z)|} \sum_{n\geq 0} \frac{|\kappa_n|\Gamma(b+\beta n)\, \mathrm{e}^{|\zeta\Re(z)|n}}{\Gamma(b-a+(\beta-\alpha)n)\Gamma(a+\alpha n)} \\ &\quad\times \int_0^1 t^{a+\alpha n-1}(1-t)^{b-a+(\beta-\alpha)n-1}\, \mathrm{d}t \\ &= \mathrm{e}^{|\Re(z)|} \sum_{n\geq 0} |\kappa_n|\, \mathrm{e}^{|\zeta\Re(z)|n}. \end{aligned} \tag{3.36} $$

Here we employ the Euler Beta function's integral form and its connection to the Gamma function:

$$ \mathrm{B}(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1}\, \mathrm{d}t = \frac{\Gamma(p)\,\Gamma(q)}{\Gamma(p+q)}, $$

where $\min\{\Re(p), \Re(q)\} > 0$. Indeed, specifying $p = a + \alpha n$, $q = b - a + (\beta - \alpha)n$ (3.36) immediately follows. Finally, by virtue of e.g. Cauchy's convergence test we get the convergence region of $\mathscr{K}_\kappa(z)$:

$$\mathsf{R}'_\kappa(\zeta) = \left\{ z \in \mathbb{C} \colon |\zeta \Re(z)| < -\log \lim_{n\to\infty} \sqrt[n]{|\kappa_n|} \right\} ,$$

for any fixed real ζ.

The integral representation formula (3.34) of Kummer's function enable us to re-formulate the series (3.35) into the following form

$$\begin{aligned} \mathscr{K}_\kappa(z) &= \sum_{n\geq 0} \frac{\kappa_n \Gamma(b+\beta n)}{\Gamma(b-a+(\beta-\alpha)n)\Gamma(a+\alpha n)} \\ &\quad \times \int_0^1 \mathrm{e}^{z(1+\zeta n)t}\, t^{a+\alpha n-1}(1-t)^{b-a+(\beta-\alpha)n-1}\,\mathrm{d}t \\ &= \int_0^1 \mathrm{e}^{zt}\, t^{a-1}(1-t)^{b-a-1}\, \mathscr{D}_\kappa(t)\,\mathrm{d}t\,, \end{aligned} \tag{3.37}$$

where the Dirichlet series

$$\mathscr{D}_\kappa(t) = \sum_{n\geq 0} \frac{\kappa_n \Gamma(b+\beta n)\,\mathrm{e}^{-\mathfrak{p}_t n}}{\Gamma(b-a+(\beta-\alpha)n)\Gamma(a+\alpha n)} \,.$$

Here the parameter

$$\mathfrak{p}_t = \log\left(t^{-\alpha}(1-t)^{\alpha-\beta}\right) - z\zeta t$$

should have positive real part. In turn, bearing in mind that for $\zeta\Re(z) < 0$ for all $t \in (0,1)$ it is

$$\Re(\mathfrak{p}_t) = -\alpha\log t - (\beta-\alpha)\log(1-t) - \zeta\Re(z)t > 0\,,$$

we have to take into account the following subset of $\mathsf{R}'_\kappa(\zeta)$:

$$\mathsf{R}_\kappa(\zeta) = \left\{ z \in \mathbb{C} : \log \lim_{n\to\infty} \sqrt[n]{|\kappa_n|} < \zeta\Re(z) < 0 \right\} .$$

Using $z \in \mathsf{R}_\kappa(\zeta)$ where ζ is a fixed real number, applying Cahen's formula (1.15) and the consequent Euler–Maclaurin summation's condensed writing developed in [239], we arrive at

Theorem 3.7 (Pogány et al. [254]) *Let* $\kappa \in \mathrm{C}^1(\mathbb{R}_+)$ *be the function which restriction into* $\mathbb{N}_0$ *is the sequence* (κ_n)*. For all* $b > a > 0$*;* $\beta \geq \alpha \geq 0$*;* $\zeta \in \mathbb{R}$ *and for all* $z \in \mathsf{R}_\kappa(\zeta)$*, we have*

$$\mathscr{D}_\kappa(t) = \frac{\kappa_0\Gamma(b)}{\Gamma(b-a)\Gamma(a)} + \mathfrak{p}_t \int_0^\infty \mathrm{e}^{-\mathfrak{p}_t s}\mathscr{A}_\kappa(s)\,\mathrm{d}s\,, \tag{3.38}$$

where $\mathfrak{p}_t = \log\big(t^{-\alpha}(1-t)^{\alpha-\beta}\mathrm{e}^{-z\zeta t}\big)$ *and*

$$\mathscr{A}_\kappa(s) = \int_0^{[s]} \mathfrak{d}_u\Big(\frac{\kappa(u)\,\Gamma(b+\beta u)}{\Gamma(b-a+(\beta-\alpha)u)\Gamma(a+\alpha u)}\Big)\,\mathrm{d}u\,.$$

Proof It only remains to explain the sum-structure of (3.38). As to the use of Cahen's formula for the Dirichlet series, which involves summation over $n \in \mathbb{N}$, we rewrite

$$\mathscr{D}_\kappa(t) = \frac{\kappa_0\Gamma(b)}{\Gamma(b-a)\Gamma(a)} + \sum_{n\ge 1} \frac{\kappa_n\,\Gamma(b+\beta n)\,\mathrm{e}^{-\mathfrak{p}_t n}}{\Gamma(b-a+(\beta-\alpha)n)\Gamma(a+\alpha n)}\,.$$

The rest is straightforward. □

Remark 3.3 Obviously the constituting addend constant term

$$\frac{\kappa_0\Gamma(b)}{\Gamma(b-a)\Gamma(a)}$$

can be avoided in the Dirichlet series' integral expression (3.38) by using without any loss of generality $\kappa_0 = 0$. ■

To derive the master integral representation formula for the Kapteyn–Kummer series we need further special functions and auxiliary results. Putting now the integral expression (3.38) of the Dirichlet series $\mathscr{D}_\kappa(t)$ into the integral form (3.37) of the Kapteyn–Kummer series $\mathscr{K}_\kappa(z)$, by (3.34), we deduce

$$\mathscr{K}_\kappa(z) = \kappa_0\,{}_1F_1(a;b;z) + \int_0^1\int_0^\infty \mathrm{e}^{zt}t^{a-1}(1-t)^{b-a-1}\mathfrak{p}_t\mathscr{A}_\kappa(s)\,\mathrm{d}t\mathrm{d}s\,. \tag{3.39}$$

Let us concentrate on the double integral $\mathscr{I}_\kappa(z)$ appearing above. By the legitimate change of integration order we have

$$\begin{aligned}\mathscr{I}_\kappa(z) &= -\int_0^\infty \mathscr{A}_\kappa(s)\Bigg(\int_0^1 \mathrm{e}^{z(1+\zeta s)t}\,t^{a+\alpha s-1}(1-t)^{b-a+(\beta-\alpha)s-1}\\ &\qquad\times\big(\zeta zt+\alpha\log t+(\beta-\alpha)\log(1-t)\big)\,\mathrm{d}t\Bigg)\mathrm{d}s\\ &=: -\int_0^\infty \mathscr{A}_\kappa(s)\Big(\zeta z\mathscr{J}_\kappa(z,1)+\alpha\frac{\partial}{\partial a}\mathscr{J}_\kappa(z,0)+\beta\frac{\partial}{\partial b}\mathscr{J}_\kappa(z,0)\Big)\mathrm{d}s\,,\end{aligned} \tag{3.40}$$

where for $\rho \in \{0,1\}$ the following auxiliary integral occurs:

$$\mathscr{J}_\kappa(z,\rho) = \int_0^1 \mathrm{e}^{z(1+\zeta s)t}t^{a+\alpha s-1+\rho}(1-t)^{b-a+(\beta-\alpha)s-1}\mathrm{d}s.$$

In turn, by (3.34) it is explicitly

$$\mathscr{J}_\kappa(z,\rho) = \boldsymbol{\Gamma}_\rho(s)\, \Phi\big(a+\alpha s+\rho, b+\beta s+\rho, z(1+\zeta s)\big)\,,$$

where we use the notation

$$\boldsymbol{\Gamma}_\rho(s) = \frac{\Gamma(b-a+(\beta-\alpha)s)\Gamma(a+\alpha s+\rho)}{\Gamma(b+\beta s+\rho)}\,.$$

Theorem 3.8 (Pogány et al. [254]) *Let* $\kappa \in \mathrm{C}^1(\mathbb{R}_+)$ *be the function for which* $\kappa\big|_{\mathbb{N}_0} = (\kappa_n)$*. For all* $b > a > 0$*;* $\beta \geq \alpha > 0$*;* $\zeta \in \mathbb{R}$ *and for all* $z \in \mathsf{R}_\kappa(\zeta)$*, we have*

$$\begin{aligned}\mathscr{K}_\kappa(z) &= \kappa_0\, {}_1F_1(a;b;z) - \int_0^\infty \int_0^{[s]} \mathfrak{d}_u\Big(\frac{\kappa(u)\,\Gamma(b+\beta u)}{\Gamma(b-a+(\beta-\alpha)u)\Gamma(a+\alpha u)}\Big)\\ &\times \Big\{\zeta z\boldsymbol{\Gamma}_1(s)\,\Phi\big(a+\alpha s+1, b+\beta s+1, z(1+\zeta s)\big)\\ &+ \Phi^*\big(\beta\frac{\partial}{\partial b}\boldsymbol{\Gamma}_0(s) + \alpha\frac{\partial}{\partial a}\boldsymbol{\Gamma}_0(s)\big)\\ &+ \boldsymbol{\Gamma}_0(s)\big(\beta\frac{\partial\Phi^*}{\partial b} + \alpha\,\frac{\partial\Phi^*}{\partial a}\big)\Big\}\mathrm{d}s\,\mathrm{d}u.\end{aligned}$$

Here $\mathscr{A}_\kappa(s)$ *and* $\Gamma_\rho(s), \rho = 0, 1$ *are described previously, while*

$$\Phi^* := {}_1F_1\big(a+\alpha s; b+\beta s; z(1+\zeta s)\big).$$

Accordingly, writing $\mathfrak{u} = z(1+\zeta s)$*:*

$$\begin{aligned}\frac{\partial\Phi^*}{\partial a} &= \frac{z(1+\zeta s)}{b+\beta s}\,F^{1:1;2}_{2:0;1}\Big[\begin{matrix}a+\alpha s+1 : 1; 1, a+\alpha s\\ 2, b+\beta s+1 : -; a+\alpha s+1\end{matrix}\Big|\,\mathfrak{u},\mathfrak{u}\Big]\\ \frac{\partial\Phi^*}{\partial b} &= -\frac{(a+\alpha s)z(1+\zeta s)}{(b+\beta s)^2}\,F^{1:1;2}_{2:0;1}\Big[\begin{matrix}a+\alpha s+1 : 1; 1, b+\beta s\\ 2, b+\beta s+1 : -; b+\beta s+1\end{matrix}\Big|\,\mathfrak{u},\mathfrak{u}\Big].\end{aligned}$$

Proof Collecting all these expressions, that is (3.39) and (3.40), we finish the proof. So, from

$$\begin{aligned}\mathscr{K}_\kappa(z) &= \kappa_0\, {}_1F_1(a;b;z) - \int_0^\infty \int_0^{[s]} \mathfrak{d}_u\Big(\frac{\kappa(u)\,\Gamma(b+\beta u)}{\Gamma(b-a+(\beta-\alpha)u)\Gamma(a+\alpha u)}\Big)\\ &\times \Big\{\zeta z\boldsymbol{\Gamma}_1(s)\,\Phi\big(a+\alpha s+1, b+\beta s+1, z(1+\zeta s)\big)\\ &+ \beta\frac{\partial}{\partial b}\boldsymbol{\Gamma}_0(s)\,\Phi\big(a+\alpha s, b+\beta s, z(1+\zeta s)\big)\\ &+ \alpha\,\frac{\partial}{\partial a}\boldsymbol{\Gamma}_0(s)\Phi\big(a+\alpha s, b+\beta s, z(1+\zeta s)\big)\Big\}\mathrm{d}s\,\mathrm{d}u,\end{aligned}$$

with some algebra the double integral will take the form

$$\int_0^\infty \int_0^{[s]} \mathfrak{d}_u\Big(\frac{\kappa(u)\,\Gamma(b+\beta u)}{\Gamma(b-a+(\beta-\alpha)u)\Gamma(a+\alpha u)}\Big) \times \Big\{\zeta z \boldsymbol{\Gamma}_1(s)\,\Phi\big(a+\alpha s+1, b+\beta s+1, z(1+\zeta s)\big) + \Phi^*\Big(\beta\frac{\partial}{\partial b}\boldsymbol{\Gamma}_0(s)+\alpha\frac{\partial}{\partial a}\boldsymbol{\Gamma}_0(s)\Big)+\boldsymbol{\Gamma}_0(s)\Big(\beta\frac{\partial\Phi^*}{\partial b}+\alpha\,\frac{\partial\Phi^*}{\partial a}\Big)\Big\}\mathrm{d}s\,\mathrm{d}u.$$

Applying the formulae [115]

$$\frac{\partial}{\partial a}{}_1F_1(a;b;z) = \frac{z}{b}F^{1:1;2}_{2:0;1}\Big[\begin{matrix} a+1 : 1; 1, a \\ 2, b+1 : -; a+1\end{matrix}\Big| z, z\Big]$$

$$\frac{\partial}{\partial b}{}_1F_1(a;b;z) = -\frac{az}{b^2}F^{1:1;2}_{2:0;1}\Big[\begin{matrix} a+1 : 1; 1, b \\ 2, b+1 : -; b+1\end{matrix}\Big| z, z\Big]$$

for getting the partial derivatives of Φ^*, in which should be specified $a \to a+\alpha s$, $b \to b+\beta s$ and $z \to z(1+\zeta s)$, we arrive at the assertion of the Theorem 3.8. □

3.6.2 *The Neumann–Kummer and Schlömilch–Kummer Series*

Consider finally the limiting cases: **(i)** $\alpha \to 0$, which implies a *two-parameter Kapteyn–Kummer series*; when either **(ii)** $\zeta \to 0$ or **(iii)** $\alpha, \zeta \to 0$ that infer *Neumann–Kummer series*. In the last possible common-sense case **(iv)** $\beta \to 0$ we earn a *Schlömilch–Kummer series*—all from $\mathscr{K}_\kappa(z)$, provided the conditions of Theorem 3.8 hold.

We point out that for the sake of simplicity in this section we take vanishing κ_0.

(i) $\alpha \to 0$. Since $\alpha \to 0$ independently of β, in this case we have a Kapteyn–Kummer series:

$$\mathscr{K}_\kappa\Big(\begin{matrix} a, b \\ 0, \beta, \zeta\end{matrix}; z\Big) = \int_0^\infty \int_0^{[s]} \mathfrak{d}_u\Big(\frac{-\kappa(u)\,\Gamma(b+\beta u)}{\Gamma(b-a+\beta u)}\Big) \times \Bigg(\zeta z a \boldsymbol{\Gamma}_1(s)\,{}_1F_1\big(a+1; b+\beta s+1; z(1+\zeta s)\big) + \beta\Big(\Phi^*\big|_{\alpha=0}\frac{\partial}{\partial b}\boldsymbol{\Gamma}_0(s)+\boldsymbol{\Gamma}_0(s)\frac{\partial \Phi^*|_{\alpha=0}}{\partial b}\Big)\Bigg)\mathrm{d}s\,\mathrm{d}u.$$

(ii) $\zeta \to 0$. This case results in a two-parameter Neumann–Kummer series

$$\mathscr{K}_\kappa\binom{a, b}{\alpha, \beta, 0}; z\Big) = \int_0^\infty \int_0^{[s]} \mathfrak{d}_u\Big(\frac{-\kappa(u)\Gamma(b+\beta u)/\Gamma(a+\alpha u)}{\Gamma(b-a+(\beta-\alpha)u)}\Big) \times \left(\Phi^*|_{\zeta=0}\Big(\beta\frac{\partial}{\partial b}\boldsymbol{\Gamma}_0(s) + \alpha\frac{\partial}{\partial a}\boldsymbol{\Gamma}_0(s)\Big) + \boldsymbol{\Gamma}_0(s)\Big(\beta\,\frac{\partial\Phi^*|_{\zeta=0}}{\partial b} + \alpha\,\frac{\partial\Phi^*|_{\zeta=0}}{\partial a}\Big)\right) \mathrm{d}s\,\mathrm{d}u.$$

(iii) $\alpha,\ \zeta \to 0$. Further simplification of the previous integral gives one-parameter Neumann–Kummer series, reads as follows:

$$\mathscr{K}_\kappa\Big(\begin{matrix} a, b \\ 0, \beta, 0\end{matrix}; z\Big) = -\frac{\beta}{\Gamma(a)}\int_0^\infty \int_0^{[s]} \mathfrak{d}_u\Big(\frac{\kappa(u)\Gamma(b+\beta u)}{\Gamma(b-a+\beta u)}\Big) \times \left(\Phi^*|_{\alpha,\zeta=0}\frac{\partial}{\partial b}\boldsymbol{\Gamma}_0(s) + \boldsymbol{\Gamma}_0(s)\frac{\partial\Phi^*|_{\alpha,\zeta=0}}{\partial b}\right)\mathrm{d}s\,\mathrm{d}u.$$

(iv) $\beta \to 0$. We end this overview of special cases of Master Theorem 3.8 with the Schlömilch–Kummer series integral representation formula

$$\mathscr{K}_\kappa\Big(\begin{matrix} a, b \\ 0, 0, \zeta\end{matrix}; z\Big) = -\frac{a\zeta z}{b}\int_0^\infty \int_0^{[s]} \mathfrak{d}_u \kappa(u)\,{}_1F_1\big(a+1; b+1; z(1+\zeta s)\big)\,\mathrm{d}s\,\mathrm{d}u.$$

Chapter 4
Schlömilch Series

Abstract This chapter is devoted to the study of integral representations of Schlömilch series built by Bessel functions of the first kind and modified Bessel functions of the second kind. Closed expressions for some special Schlömilch series together with their connection to Mathieu series are also investigated. The chapter ends with an integral representation formula for number theoretical summation by Popov, which also covers the theta-transform identity coming from functional equation for the Epstein Zeta function.

Oscar Xavier Schlömilch introduced in 1857 in his article [279, pp. 155–158] the series of the form

$$\mathfrak{S}_\nu(z) := \sum_{n\geq 1} \alpha_n \, J_\nu\left((\nu+n)z\right), \qquad z \in \mathbb{C}, \tag{4.1}$$

where ν, α_n are constants and J_ν stands for the Bessel function of the first kind of order ν. So, this kind series are known as *Schlömilch series* (of the order ν). Schlömilch considered only the cases $\nu = 0, 1$. Rayleigh [266] has showed that such series play important roles in physics, because for $\nu = 0$ they are useful in investigation of a periodic transverse vibrations uniformly distributed in direction through the two dimensions of the membrane. Also, Schlömilch series present various features [278] of purely mathematical interest and it is remarkable that a null-function can be represented by such series in which the coefficients are not all zero [333, p. 634].

It is worth to mention, that Schlömilch [279] proved that there exists a series $\mathfrak{S}_0^f(x)$ associated with any analytic function f. Namely, according to Watson (in renewed formulation) [333, p. 619]: *let $f(x)$ be an arbitrary function, with a derivative $f'(x)$ which is continuous in the interval $(0,\pi)$ and which has limited total fluctuation in this interval. Then $f(x)$ admits of the expansion*

$$f(x) = \frac{a_0}{2} + \sum_{m\geq 1} a_m J_0(mx) =: \mathfrak{S}_0^f(x), \tag{4.2}$$

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where

$$a_0 = 2f(0) + \tfrac{2}{\pi}\int_0^{\pi}\int_0^{\frac{1}{2}\pi} uf'(u\sin\phi)\,\mathrm{d}\phi\,\mathrm{d}u\,,$$

$$a_m = \tfrac{2}{\pi}\int_0^{\pi}\int_0^{\frac{1}{2}\pi} uf'(u\sin\phi)\cos(mu)\,\mathrm{d}\phi\,\mathrm{d}u, \qquad m\in\mathbb{N}$$

and this expansion is valid, and the series converges in $(0,\pi)$.

We point out that this Schlömilch's result may be generalized by replacing the expansion (4.2) of order zero by $\mathfrak{S}_\nu^f(x)$ of arbitrary order ν, see [38, 216, 317] and [333, Ch. XIX.].

The next generalization is suggested by the theory of Fourier series. The functions which naturally extend $\mathfrak{S}_0^f(x)$ are Bessel functions of the second kind and Struve functions. The types of series to be considered may be written in the forms:

$$\frac{a_0}{2\,\Gamma(\nu+1)} + \sum_{m\geq 1}\frac{a_m J_\nu(mx) + b_m Y_\nu(mx)}{\left(\frac{1}{2}mx\right)^\nu}\,,$$

$$\frac{a_0}{2\,\Gamma(\nu+1)} + \sum_{m\geq 1}\frac{a_m J_\nu(mx) + b_m \mathbf{H}_\nu(mx)}{\left(\frac{1}{2}mx\right)^\nu}\,.$$

Such series, with $\nu = 0$ have been considered in 1886 by Coates [55], but his proof of expanding an arbitrary functions $f(x)$ into this kind of series seems to be invalid except in some trivial case in which $f(x)$ is defined to be periodic (with period 2π) and to tend to zero as $x\to\infty$. Also for further subsequent generalizations consult e.g. Bondarenko's recent article [38] and the references therein and Miller's multidimensional expansion [197].

The series of much greater interest are direct generalization of trigonometrical series and they are called *generalized Schlömilch series*. Nielsen studied such kind of series in his memoirs consecutively in 1899 [212–214], in 1900 [215] and finally in 1901 [216, 218]. He has given the forms for the coefficients in the generalized Schlömilch expansion of arbitrary function and he has investigated the construction of Schlömlich series which represents null-functions [219, p. 348]. Filon also investigated the possibility of expanding an arbitrary function into a generalized Schlömilch series for $\nu = 0$ [83]. Using Filon's method for finding coefficients in the generalized Schlömlich expansion, Watson proved a similar fashion expansion result.

Theorem A (Watson [333]) *Let* ν *be a number such that* $2|\nu| < 1$ *and let* $f(x)$ *be defined arbitrarily in the interval* $(-\pi,\pi)$ *subject to the following conditions: (i) the function* $h(x) = 2\nu f(x) + xf'(x)\in \mathrm{C}^1(-\pi,\pi)$ *and it has limited total fluctuation in the interval* $(-\pi,\pi)$*, and (ii) the integral*

$$\int_0^{\Delta}\frac{\mathrm{d}}{\mathrm{d}x}\left(|x|^{2\nu}\{f(x)-f(0)\}\right)\mathrm{d}x\,,\qquad \nu\in\left(-\tfrac{1}{2},0\right)$$

is absolutely convergent when Δ is a (small) number either positive or negative. Then $f(x)$ admits of the expansion

$$f(x) = \frac{a_0}{2\,\Gamma(\nu+1)} + \sum_{m\geq 1} \frac{a_m J_\nu(mx) + b_m \mathbf{H}_\nu(mx)}{\left(\frac12 mx\right)^\nu}\,,$$

where

$$a_m = \int_{-\pi}^{\pi}\int_0^{\frac12\pi} \frac{\mathrm{d}}{\mathrm{d}\phi}\left(\{f(u\sin\phi)-f(0)\}\sin^{2\nu}\phi\right) \frac{\cos(mu)}{\cos^{2\nu+1}\phi}\,\frac{\mathrm{d}\phi\,\mathrm{d}u}{\sqrt{\pi}\,\Gamma\left(\frac12-\nu\right)}\,, \tag{4.3}$$

$$b_m = \int_{-\pi}^{\pi}\int_0^{\frac12\pi} \frac{\mathrm{d}}{\mathrm{d}\phi}\left(\{f(u\sin\phi)-f(0)\}\sin^{2\nu}\phi\right) \frac{\sin(mu)}{\cos^{2\nu+1}\phi}\,\frac{\mathrm{d}\phi\,\mathrm{d}u}{\sqrt{\pi}\,\Gamma\left(\frac12-\nu\right)}\,,$$

when $m > 0$; a_0 is obtained by inserting an additional term $2\Gamma(\nu+1)f(0)$ on the right in (4.3).

Now we refer about few Schlömilch–Bessel type series, which are closely connected to certain number theoretical functions. First, we define $r_k(n)$ as the number of representations of n by k squares, allowing zeros and distinguishing signs and order. We mention that by convention $r_2(n) \equiv r(n)$ [105], this number is connected to the famous Gauss' circle problem [108, pp. 33 *et seq.*]. In fact, $r_k(n)$ denotes the number of integer-coordinate lattice points in the k-dimensional sphere of radius $\sqrt{n}$

$$x_1^2 + \cdots + x_k^2 = n\,.$$

The generating function of $r_k(n)$ turns out to be [255, p. 801]

$$\sum_{n\geq 0} r_k(n)x^n = \vartheta_3^k(0;x)\,,$$

where $\vartheta_3(z;q)$ stands for the Jacobi third Theta function.

Next, it can be given another type generating function *via* the Epstein Zeta function [76]

$$\zeta_k(s) = \sum_{n\geq 0} \frac{r_k(n)}{n^s}, \qquad k \geq 2;\ 2\Re(s) > k\,,$$

where $\zeta_k(s)$ has analytical continuation to the punctured complex s-plane $\mathbb{C}\setminus\{\frac{k}{2}\}$, with residue $\mathrm{Res}\big[\zeta_k;\frac{k}{2}\big] = \pi^{\frac{k}{2}}\,\Gamma\left(\frac{k}{2}\right)$, [48, p. 18, Example 3]. This function satisfies the functional equation [48, p. 18, Example 3]

$$\pi^{-s}\,\Gamma(s)\,\zeta_k(s) = \pi^{s-\frac{k}{2}}\,\Gamma\left(\tfrac{k}{2}-s\right)\zeta_k\left(\tfrac{k}{2}-s\right). \tag{4.4}$$

Popov obtained the formula [255, p. 801, Eq. (1)] (also see [157])

$$\frac{(\pi\sqrt{z})^{\frac{k}{2}-1}}{\Gamma(\frac{k}{2})} + \sum_{n\ge1}\frac{r_k(n)}{\sqrt{n}^{\frac{k}{2}-1}}\, J_{\frac{k}{2}-1}\big(2\pi\sqrt{nz}\big)\, \mathrm{e}^{-\pi n t}$$
$$= t^{-1}\,\mathrm{e}^{-\frac{\pi z}{t}}\left\{\frac{(\pi\sqrt{z})^{\frac{k}{2}-1}}{t^{\frac{k}{2}-1}\,\Gamma(\frac{k}{2})} + \sum_{n\ge0}\frac{r_k(n)}{\sqrt{n}^{\frac{k}{2}-1}}\, I_{\frac{k}{2}-1}\Big(\frac{2\pi\sqrt{nz}}{t}\Big)\,\mathrm{e}^{-\frac{\pi n}{t}}\right\} \tag{4.5}$$

which holds for all $\Re(t) > 0$. In turn, Chandrasekharan and Narasimhan (compare both references [35, 48]) have proved that the functional equation (4.4), the theta-transform identity [48, p. 19]

$$\sum_{n\ge0} r_k(n)\,\mathrm{e}^{-\pi n y} = y^{-\frac{k}{2}}\sum_{n\ge0} r_k(n)\,\mathrm{e}^{-\frac{\pi n}{y}} \qquad \Re(y) > 0, \tag{4.6}$$

and the Bessel sum formula [255], [48, p. 19]

$$\frac{1}{\Gamma(q+1)}\sum_{0\le n\le[x]}{}' r_k(n)(x-n)^q$$
$$= \frac{\pi^{\frac{k}{2}}\,x^{\frac{k}{2}+q}}{\Gamma(\frac{k}{2}+q+1)} + \pi^{-q}\sum_{n\ge1} r_k(n)\left(\frac{x}{n}\right)^{\frac{k}{4}+\frac{q}{2}} J_{\frac{k}{2}+q}\big(2\pi\sqrt{nx}\big) \tag{4.7}$$

are equivalent if $x > 0, 2q > k-1$, see [48, p. 10, Theorem I]. The dashed sum indicates that if $q = 0$ and x equals to an integer N, the left-hand-side reduces only to $\frac{1}{2}\, r_k(N)$, see [255, p. 801] and also [35].[1]

However, Popov's formula (4.5) covers *inter alia* both results (4.6) and (4.7), consult [35].

The double Schlömilch–Bessel series analogue of (4.7), when $k = 2, q = 0$ was established by Ramanujan. Namely, denoting

$$F(x) = \begin{cases} [x], & x \quad \text{not an integer} \\ x - \frac{1}{2}, & x \quad \text{integer} \end{cases},$$

for all $\theta \in (0, 1)$, $x > 0$, we have [264, p. 335, Entry 1.1]

$$\sum_{n\ge0} F\Big(\frac{x}{n}\Big)\sin(2\pi n\theta) = \pi x\Big(\frac{1}{2}-\theta\Big) - \frac{1}{4}\cot(\pi\theta)$$
$$+ \frac{\sqrt{x}}{2}\sum_{\substack{m\ge1\\ n\ge0}}\left\{\frac{J_1(4\pi\sqrt{m(n+\theta)x})}{\sqrt{m(n+\theta)}} - \frac{J_1(4\pi\sqrt{m(n+1-\theta)x})}{\sqrt{m(n+1-\theta)}}\right\}.$$

[1] Moreover, Berndt et al. extended (4.7) to all $2q > k-3$, compare [35, Theorem 2.1].

Further reading about Bessel functions summations formulating Schlömilch series with respect to another kind number theoretical functions the interested reader can find in papers by Berndt with different coauthors, see [28, 30–35].

In the first two sections of this chapter we derive several new integral representations for Schlömilch-type series, while the last section is devoted to new closed form formulae for the Schlömilch series built by members which contain modified Bessel function of the second kind. Let us also mention that results presented in this chapter concern to the papers by Jankov Maširević [135] and Jankov et al. [131].

Finally, we will close this section with establishing integral expressions for the right-hand-side Schlömilch–Bessel series in (4.7) which also concerns a finite sum evaluation formula for an associated triple integral.

4.1 Integral Representation of Schlömilch Series

In this section we will derive the double definite integral representation of the special kind of Schlömilch series

$$\mathfrak{S}_\nu^\mu(z) := \sum_{n\geq 1} \alpha_n J_\nu\left((\mu+n)z\right), \qquad z \in \mathbb{C}, \tag{4.8}$$

using an integral representation of Kapteyn-type series

$$\widetilde{K}_{\nu,\beta}^\mu(z) := \sum_{n\geq 1} \alpha_n J_{\nu+\beta n}\big((\mu+n)z\big), \qquad z\in\mathbb{C},\ \beta>0, \tag{4.9}$$

which has been proven in the previous chapter by Theorem 3.5. Now, we can establish a connection between Schlömilch series (4.8) and Kapteyn-type series (4.9) by

$$\mathfrak{S}_\nu^\mu(x) = \lim_{\beta\to 0} \widetilde{K}_{\nu,\beta}^\mu(x)\,. \tag{4.10}$$

Using this equality, we have the following result.

Corollary 4.1 (Jankov and Pogány [131]) *Let* $\alpha \in \mathrm{C}^1(\mathbb{R}_+)$ *for which the function*

$$\kappa(u,s) := \frac{\partial}{\partial u}\left(\frac{\Gamma\left(\beta u+\nu+\frac{1}{2}\right)}{(\mu+u)^{\beta u+\nu}} J_{\beta u+\nu}\big((\mu+u)\,x\big)\right)\mathfrak{d}_s\left(\frac{\alpha(s)(\mu+s)^{\nu+\beta s}}{\Gamma\left(\nu+\beta s+\frac{1}{2}\right)}\right)$$

is integrable for all $\beta>0$. *Let* $\alpha|_{\mathbb{N}} = (\alpha_n)_{n\geq 1}$ *and let*

$$\mathscr{C} = \limsup_{n\to\infty} |\alpha_n|^{\frac{1}{n}} < 1\,.$$

Then, for all $\nu > -\frac{1}{2}$ *and* $x \in (0,2) =: \mathscr{I}_{\alpha,0}$ *we have the integral representation*

$$\mathfrak{S}_\nu^\mu(x) = -\int_1^\infty \int_0^{[u]} \frac{\partial}{\partial u}\Big(\frac{J_\nu\big((\mu+u)\,x\big)}{(\mu+u)^\nu}\Big)\,\mathfrak{d}_s\,(\alpha(s)(\mu+s)^\nu)\,\mathrm{d}u\,\mathrm{d}s. \tag{4.11}$$

Proof It is enough to establish the behavior of the convergence domain $\mathscr{I}_{\alpha,\beta}$ when β vanishes. Having in mind that by assumption $\mathscr{C} < 1$ and this implies

$$\lim_{\beta\to 0_+} \beta\mathscr{C}^{-\frac{1}{\beta}} = -\log\mathscr{C}\,\lim_{\beta\to 0_+} \mathscr{C}^{-\frac{1}{\beta}} = +\infty\,,$$

and $\mathscr{I}_{\alpha,0} = (0,2)$. Thus the statement (4.11) immediately follows from Theorem 3.5, relation (4.10) and Lebesgue dominated convergence theorem. □

4.2 Another Integral Representation of Schlömilch Series

In this section our aim is to derive integral representations for Schlömlich series (4.1), using Bessel differential equation. First, applying the same procedure exploiting the non-homogeneous Bessel ordinary differential equation as in Sect. 3.3, we have the auxiliary result:

Theorem 4.1 (Jankov and Pogány [131]) *Schlömilch series* (4.1) *is the solution of the nonhomogeneous Bessel-type differential equation*

$$x^2y'' + xy' + (x^2-\nu^2)y = \mathfrak{T}_\nu(x)\,, \tag{4.12}$$

where

$$\mathfrak{T}_\nu(x) = \sum_{n\ge 1}\big(1-(\nu+n)^2\big)x^2\alpha_n J_\nu\big((\nu+n)x\big)\,. \tag{4.13}$$

Moreover, if we assume that $\alpha \in \mathrm{C}^1(\mathbb{R}_+)$*,* $\alpha|_{\mathbb{N}} = (\alpha_n)_{n\ge1}$ *and the series* $\sum_{n\ge1} n^{\frac{5}{3}}\alpha_n$ *absolutely converges, then for all* $x \in \mathscr{I}_{\alpha,0}$ *we have the integral representation*

$$\mathfrak{T}_\nu(x) = x^2\int_1^\infty \int_0^{[u]} \frac{\partial}{\partial u}\left(\frac{J_\nu\big((\nu+u)x\big)}{(\nu+u)^\nu}\right)\mathfrak{d}_s\Big(-\alpha(s)\left(1-(\nu+s)^2\right)(\nu+s)^\nu\Big)\,\mathrm{d}u\mathrm{d}s\,.$$

Below, we will derive a new integral representation of the Schlömilch series (4.1), using the Bessel differential equation (1.18).

Theorem 4.2 (Jankov and Pogány [131]) *Let* $\alpha \in \mathrm{C}^1(\mathbb{R}_+)$, $\alpha|_{\mathbb{N}} = \big(\alpha_n\big)_{n\ge 1}$ *and assume that series* $\sum_{n\ge1} n^{\frac{5}{3}}\alpha_n$ *absolutely converges. Then, for all* $\nu > -\frac{1}{2}$ *and* $x \in \mathscr{I}_{\alpha,0}$ *we have*

$$\mathfrak{S}_\nu(x) = \frac{J_\nu(x)}{2}\int \frac{1}{xJ_\nu^2(x)}\left(\int \frac{J_\nu(x)\mathfrak{T}_\nu(x)}{x}\,\mathrm{d}x\right)\mathrm{d}x + \frac{Y_\nu(x)}{2}\int \frac{1}{xY_\nu^2(x)}\left(\int \frac{Y_\nu(x)\mathfrak{T}_\nu(x)}{x}\,\mathrm{d}x\right)\mathrm{d}x\,, \tag{4.14}$$

where $\mathfrak{T}_\nu$ *is the Schlömilch series, given with* (4.13).

Proof The homogeneous solution of the Bessel differential equation is given with

$$y_h(x) = C_1 Y_\nu(x) + C_2 J_\nu(x)\,,$$

where J_ν and Y_ν are independent solutions of the Bessel differential equation.

As J_ν is a solution, we search for the particular solution y_p in the form $y_p(x) = J_\nu(x)w(x)$. Substituting this form into (4.12), we have

$$x^2(J_\nu''w + 2J_\nu'w' + J_\nu w'') + x(J_\nu'w + J_\nu w') + (x^2-\nu^2)J_\nu w = \mathfrak{T}_\nu(x).$$

If we write the previous equation in the following form

$$w\big(x^2J_\nu'' + xJ_\nu' + (x^2-\nu^2)J_\nu\big) + w'(2x^2J_\nu' + xJ_\nu) + w''(x^2J_\nu) = \mathfrak{T}_\nu(x),$$

using the fact that J_ν is a solution of the homogeneous Bessel differential equation, we get the solution

$$w = \int \frac{1}{xJ_\nu^2}\left(\int \frac{\mathfrak{T}_\nu J_\nu}{x}\,\mathrm{d}x\right)\mathrm{d}x + C_3\frac{\pi}{2}\frac{Y_\nu}{J_\nu} + C_4\,,$$

because

$$\int \frac{1}{xJ_\nu^2}\mathrm{d}x = \frac{\pi}{2}\frac{Y_\nu}{J_\nu}.$$

So, we have the particular solution

$$\mathfrak{S}_\nu(x) = J_\nu(x)w(x) = J_\nu(x)\int \frac{1}{xJ_\nu^2}\left(\int \frac{\mathfrak{T}_\nu J_\nu}{x}\,\mathrm{d}x\right)\mathrm{d}x + C_3\frac{\pi}{2}Y_\nu(x) + C_4 J_\nu(x)\,.$$

Using the fact that y_h is formed by independent functions J_ν and Y_ν, that functions do not contribute to the particular solution y_p and the constants C_3, C_4 can be taken to be zero.

Analogously, taking particular solution in the form $\mathfrak{y}_p(x) = Y_\nu(x)w(x)$ and using the equality

$$\int \frac{1}{xY_\nu^2}\,\mathrm{d}x = -\frac{\pi}{2}\frac{J_\nu}{Y_\nu}$$

we get

$$\mathfrak{S}_\nu(x) = Y_\nu(x)w(x) = Y_\nu(x)\int \frac{1}{xY_\nu^2}\left(\int \frac{\mathfrak{T}_\nu\,Y_\nu}{x}\,\mathrm{d}x\right)\mathrm{d}x - C_5\frac{\pi}{2}J_\nu(x) + C_6Y_\nu(x)\,.$$

Again, choosing $C_5 = C_6 = 0$, we get the integral representation (4.14). □

4.3 Schlömilch Series Built by Modified Bessel K_ν

The problem of summing up special kind Schlömilch series built by members which contain modified Bessel function of the second kind K_ν in the form

$$\sum_{n\geq 1}\alpha_n K_\nu(nz), \qquad z\in\mathbb{C}, \tag{4.15}$$

where ν, α_n are constants, has not been considered in mathematical literature so often. More general results about this kind series, with members containing Bessel function of the first kind J_ν are recently studied in [131] and can be also found in e.g. [198, 265, 315].

According to our knowledge, summations of the series (4.15) has been studied only in the article [316] where the authors proved that [316, p. 217, Example 4]

$$\begin{aligned}\sum_{n\geq 1}\frac{K_{\nu-\rho+1}(nz)}{n^{2m+\nu-\rho+1}} &= \frac{(-1)^m\pi|z|^{2m+2\nu+1}\Gamma\left(\rho-m-\nu-\frac12\right)}{2^{2m+\nu-\rho+2}z^{\nu+\rho+1}\Gamma\left(m+\frac12\right)}\\ &\quad+\sum_{j=0}^{m}\frac{(-1)^j\zeta(2m-2j)\,\Gamma(\rho-\nu-j-1)}{2^{\nu+2j-\rho+2}z^{\nu+\rho+1}\,j!\,|z|^{-2\nu-2j-2}},\end{aligned}$$

which holds for $m\in\mathbb{N}$, $z>0$, $-1<2\Re(\nu)<4\Re(\rho)-1$ and [316, p. 218, Example 7]

$$\begin{aligned}\sum_{n\geq 1}\frac{K_{\frac12}(nx)}{n^{\alpha-\frac12}} &= (-1)^{\alpha/2}\frac{\pi\,x^{\alpha-\frac32}\,\Gamma\left(1-\frac{\alpha}{2}\right)}{2^{\alpha+\frac12}\,\Gamma\left(\frac{\alpha+1}{2}\right)}\\ &\quad+\sum_{j=0}^{M}\frac{(-1)^j\zeta(\alpha-2j)\,x^{2j-\frac12}\,\Gamma\left(\frac12-j\right)}{2^{2j+\frac12}\,j!},\end{aligned}$$

where $\alpha\in\mathbb{N}$, $M=\frac{\alpha-1}{2}$ for α odd and $M=\frac{\alpha}{2}$ for α even.

Also, knowing that [93, p. 925, Eq. (8.469.3)]

$$K_{\frac{1}{2}}(z) = \sqrt{\frac{\pi}{2z}}\, \mathrm{e}^{-z},$$

some summation formulae for Schlömilch series with members containing $K_{\frac{1}{2}}$ can be derived by using collection of 17 infinite summation formulae for exponential function given on the widely known website [122].

In this section, we are interested in summing the series of the form

$$\mathfrak{S}_\mu(a, x, \nu) = \sum_{n \ge 1} \frac{\varepsilon_n\, n^\nu K_\nu(nx)}{(n^2 - a^2)\, n^\mu}, \qquad x \in \mathbb{R}, \tag{4.16}$$

where a, ν, μ are constants and $\varepsilon_n = 1$ or $\varepsilon_n = (-1)^n$.

First, we can observe that

$$\mathfrak{S}_{\mu-2}(a, x, \nu) - a^2 \mathfrak{S}_\mu(a, x, \nu) = \mathfrak{S}_{\mu-2}(0, x, \nu),$$

from which, setting $\mu = 2k$, $k \in \mathbb{N}$, it follows by mathematical induction

$$a^{2r} \mathfrak{S}_{2r}(a, x, \nu) = \mathfrak{S}_0(a, x, \nu) - \sum_{k=1}^{r} a^{2k-2}\, \mathfrak{S}_{2k-2}(0, x, \nu), \qquad r \in \mathbb{N}\,. \tag{4.17}$$

So, the previous problem, in the case when $\mu = 2k$, $k \in \mathbb{N}$ one reduces to summing the series $\mathfrak{S}_{2k-2}(0, x, \nu)$, for all $k \in \mathbb{N}$ and also to derivation of a summation formula for $\mathfrak{S}_0(a, x, \nu)$.

In what follows we will use notation $s_\mu(a, x, \nu)$ for the series (4.16), when $\varepsilon_n = 1$ and $S_\mu(a, x, \nu)$ in the case when $\varepsilon_n = (-1)^{n-1}$.

First, in Sect. 4.3.1 we prove our main results on summation formulae for the above mentioned series, when $\mu = 2k$, $k \in \mathbb{N}$, together with a set of corresponding results which concern special kind Schlömilch series with members containing products of K_ν and modified Bessel function of the first kind I_ν.

In Sect. 4.4, the connection between special case of the series (4.16) and a generalized Mathieu series (see e.g. [293]) will be established.

4.3.1 Closed Expressions for $\mathfrak{S}_{2k-2}(0, x, \nu)$ and $\mathfrak{S}_0(a, x, \nu)$

Let us establish our first main result in this section.

Theorem 4.3 (Jankov Maširević [135]) *For all $\nu > k$, $k \in \mathbb{N}$ there holds*

$$s_{2k-2}(0,x,\nu) = \frac{(-1)^{k-1}}{2}\left(\frac{x}{2}\right)^{2k-\nu}\left(-\frac{\pi\,\Gamma\left(\nu-k+\frac{1}{2}\right)}{x\,\Gamma\left(k+\frac{1}{2}\right)} + \sum_{n=0}^{k}\frac{(-1)^{n-1}\Gamma(n+\nu-k)\,\zeta(2n)}{(k-n)!}\left(\frac{2}{x}\right)^{2n}\right), \tag{4.18}$$

where $x \in (0, 2\pi]$. Moreover, for all $x \in (0, \pi]$ it is

$$S_{2k-2}(0,x,\nu) = \frac{(-1)^k}{2}\left(\frac{x}{2}\right)^{2k-\nu}\sum_{n=0}^{k}\frac{(-1)^n\Gamma(n-k+\nu)\,\Phi(-1,2n,1)}{(k-n)!}\left(\frac{2}{x}\right)^{2n}. \tag{4.19}$$

Proof In order to establish the convergence conditions of the series

$$\mathfrak{S}_{2k-2}(0,x,\nu) = \sum_{n\ge 1}\frac{\varepsilon_n\, n^\nu K_\nu(nx)}{n^{2k}}, \qquad k \in \mathbb{N}$$

let us consider already derived bound (2.42)

$$|K_\nu(x)| \le \frac{1}{2}\left(\frac{2}{x}\right)^{\nu}\Gamma(\nu)\,, \qquad \min\{\Re(\nu), x\} > 0,$$

which follows from the integral representation given by Basset (2.41) and gives

$$|\mathfrak{S}_{2k-2}(0,x,\nu)| \le \sum_{n\ge 1}\frac{|K_\nu(nx)|}{n^{2k-\nu}} \le \frac{\Gamma(\nu)}{2}\left(\frac{2}{x}\right)^{\nu}\sum_{n\ge 1}\frac{1}{n^{2k}} = \frac{\Gamma(\nu)}{2}\left(\frac{2}{x}\right)^{\nu}\zeta(2k),$$

so the convergence holds for any positive integer k.

Now, by the integral representation (2.41), the formulae (1.6)–(1.8) and Legendre's duplication formula (1.3) we get

$$\begin{aligned}
s_{2k-2}(0,x,\nu) &= \frac{(-1)^{k-1}\,\Gamma\left(\nu+\frac{1}{2}\right)(2\pi)^{2k}}{2\sqrt{\pi}\,\Gamma(2k+1)}\left(\frac{2}{x}\right)^{\nu}\int_0^\infty (t^2+1)^{-\nu-\frac{1}{2}}B_{2k}\left(\frac{xt}{2\pi}\right)\mathrm{d}t \\
&= \frac{(-1)^{k-1}}{4}\left(\frac{x}{2}\right)^{2k-\nu}\sum_{n=0}^{2k}\frac{\Gamma\left(\nu-k+\frac{n}{2}\right)B_n}{\Gamma(n+1)\,\Gamma\left(k-\frac{n}{2}+1\right)}\left(\frac{4\pi}{x}\right)^{n} \\
&= \frac{(-1)^{k-1}}{4}\left(\frac{x}{2}\right)^{2k-\nu}\left(\frac{\Gamma(\nu-k)}{\Gamma(k+1)} - \frac{2\pi}{x}\frac{\Gamma\left(\nu-k+\frac{1}{2}\right)}{\Gamma\left(k+\frac{1}{2}\right)}\right. \\
&\quad \left. + 2\sum_{n=1}^{k}\frac{(-1)^{n-1}\Gamma(n+\nu-k)\,\zeta(2n)}{\Gamma(k-n+1)}\left(\frac{2}{x}\right)^{2n}\right),
\end{aligned}$$

which is equal to (4.18), knowing that $\zeta(0) = -\frac{1}{2}$. Because the conditions given in (1.8) and (2.42), we see that the range of x is $(0, 2\pi]$. Similarly, by the integral (2.41) we infer

$$S_{2k-2}(0,x,\nu) = \frac{(-1)^k(2\pi)^{2k}\,\Gamma\left(\nu+\frac{1}{2}\right)}{2\sqrt{\pi}\,\Gamma(2k+1)}\left(\frac{2}{x}\right)^\nu \int_0^\infty (t^2+1)^{-\nu-\frac{1}{2}} B_{2k}\left(\frac{xt}{2\pi}+\frac{1}{2}\right)\mathrm{d}t,$$

where $x \in (0, \pi]$ for all $k \in \mathbb{N}$. Now, since [1, p. 804, Eq. (23.1.10)]

$$B_n(mt) = m^{n-1}\sum_{k=0}^{m-1} B_n\left(t+\frac{k}{m}\right), \qquad n \in \mathbb{N}_0,\ m \in \mathbb{N}, \tag{4.20}$$

for $m = 2$, we deduce

$$\begin{aligned} S_{2k-2}(0,x,\nu) &= \frac{(-1)^k(2\pi)^{2k}\,\Gamma\left(\nu+\frac{1}{2}\right)}{2\sqrt{\pi}\,\Gamma(2k+1)}\left(\frac{2}{x}\right)^\nu \\ &\quad\times \int_0^\infty (t^2+1)^{-\nu-\frac{1}{2}}\left(2^{1-2k}B_{2k}\left(\frac{xt}{\pi}\right) - B_{2k}\left(\frac{xt}{2\pi}\right)\right)\mathrm{d}t, \end{aligned}$$

and using (1.6), (1.7) and the Legendre's duplication formula (1.3) we obtain

$$\begin{aligned} S_{2k-2}(0,x,\nu) &= \frac{(-1)^k}{2}\left(\frac{x}{2}\right)^{2k-\nu}\sum_{n=0}^{2k}\frac{(1-2^{n-1})\,\Gamma\left(-k+\frac{n}{2}+\nu\right)B_n}{n!\,\Gamma\left(k-\frac{n}{2}+1\right)}\left(\frac{2\pi}{x}\right)^n \\ &= \frac{(-1)^k}{4}\left(\frac{x}{2}\right)^{2k-\nu}\left(\frac{\Gamma(\nu-k)}{k!}\right. \\ &\qquad\left. +2\sum_{n=1}^{k}\frac{(-1)^{n-1}(2-2^{2n})\,\Gamma(-k+n+\nu)\,\zeta(2n)}{x^{2n}\,\Gamma(k-n+1)}\right) \end{aligned}$$

which becomes (4.19), knowing that the Hurwitz–Lerch Zeta function $\Phi(z,s,a)$ (see p. 22, **1.14** and [77, p. 27, Eq. (1)]) has the properties [77, p. 32], [175, p. 54]

$$\begin{aligned} &\Phi(-1,0,1) = \tfrac{1}{2} \\ &\Phi(-1,s,1) = (1-2^{1-s})\,\zeta(s), \qquad \Re(s) > 0, \end{aligned} \tag{4.21}$$

by which we end the proof. □

Theorem 4.4 (Jankov Maširević [135]) *For all* $\min\{x, a, \nu\} > 0$ *there holds*

$$s_0(a, x, \nu) = \frac{\Gamma(\nu)}{(2a)^2}\left(\frac{2}{x}\right)^\nu - \frac{\pi a^{\nu-1}}{2\sin(a\pi)}\Bigg(\cos(a\pi)K_\nu(ax) + \frac{\pi \sin(a\pi)}{2\cos(\nu\pi)}\left(I_\nu(ax) - \mathbf{L}_{-\nu}(ax)\right)\Bigg). \tag{4.22}$$

Moreover

$$S_0(a, x, \nu) = -\frac{\Gamma(\nu)}{(2a)^2}\left(\frac{2}{x}\right)^\nu + \frac{\pi a^{\nu-1}}{2\sin(a\pi)} K_\nu(ax) . \tag{4.23}$$

Proof In order to establish the convergence conditions of $\mathfrak{S}_0(a, x, \nu)$, analogously as we did in the previous theorem, we can deduce that for all $\min\{x, a, \nu\} > 0$ there holds

$$|\mathfrak{S}_0(a, x, \nu)| \le \frac{\Gamma(\nu)}{2}\left(\frac{2}{x}\right)^\nu \sum_{n\ge1} \frac{1}{n^2 - a^2} = \frac{\Gamma(\nu)}{2}\left(\frac{2}{x}\right)^\nu \left(\frac{1}{2a^2} - \frac{\pi}{2a}\cot(a\pi)\right),$$

where in the last equality we used the formula [256, p. 685, Eq. (5.1.25.4)].

Now, using the integral (2.41) and the summation formula [256, p. 730, Eq. (5.4.5.1)]

$$\sum_{n\ge1} \frac{\cos(nx)}{n^2 - a^2} = \frac{1}{2a^2} - \frac{\pi\cos\left(a(\pi - x)\right)}{2a\sin(a\pi)}, \qquad 0 \le x \le 2\pi,$$

we conclude that

$$s_0(a, x, \nu) = \frac{\Gamma\left(\nu + \frac{1}{2}\right)}{\sqrt{\pi}}\left(\frac{2}{x}\right)^\nu \int_0^\infty (t^2 + 1)^{-\nu-\frac{1}{2}} \times \left(\frac{1}{2a^2} - \frac{\pi\left(\cos(a\pi)\cos(axt) + \sin(a\pi)\sin(axt)\right)}{2a\sin(a\pi)}\right).$$

Combination of the previous expression, formula (2.41), the integral [1, p. 498, Eq. (12.2.3)]

$$I_\nu(x) - \mathbf{L}_{-\nu}(x) = \frac{2\left(\frac{2}{x}\right)^\nu}{\sqrt{\pi}\,\Gamma\left(-\nu + \frac{1}{2}\right)} \int_0^\infty (t^2 + 1)^{-\nu-\frac{1}{2}} \sin(xt)\,\mathrm{d}t,$$

where $\Re(\nu) > -\frac{1}{2}$, $x > 0$ and Euler's reflection formula yields (4.22).

Analogously, combining the summation formula [256, p. 730, Eq. (5.4.5.2)]

$$\sum_{n\geq 1} \frac{(-1)^{n-1}\cos(nx)}{n^2-a^2} = -\frac{1}{2a^2} + \frac{\pi\cos(ax)}{2a\sin(a\pi)}, \qquad |x|\leq \pi$$

with (2.41) we arrive at (4.23). □

Corollary 4.2 (Jankov Maširević [135]) *For all* $\max\{a,x\}>0$, $\nu>k$, $k\in\mathbb{N}$ *and* $r\in\mathbb{N}$ *there holds*

$$\begin{aligned} a^{2r}s_{2r}(a,x,\nu) &= \frac{\Gamma(\nu)}{(2a)^2}\left(\frac{2}{x}\right)^\nu - \frac{\pi a^{\nu-1}}{2\sin(a\pi)}\Bigg(\cos(a\pi)K_\nu(ax) \\ &\quad + \frac{\pi\sin(a\pi)}{2\cos(\nu\pi)}\left(I_\nu(ax)-\mathbf{L}_{-\nu}(ax)\right)\Bigg) \\ &\quad + \sum_{k=1}^{r}\frac{(-1)^k}{2\,a^{2-2k}}\left(\frac{x}{2}\right)^{2k-\nu}\Bigg(\frac{-\pi\,\Gamma\left(\nu-k+\frac12\right)}{x\,\Gamma\left(k+\frac12\right)} \\ &\quad + \sum_{n=0}^{k}\frac{(-1)^{n-1}\Gamma(n+\nu-k)\,\zeta(2n)}{(k-n)!}\left(\frac{2}{x}\right)^{2n}\Bigg) \end{aligned}$$

and

$$\begin{aligned} a^{2r}S_{2r}(a,x,\nu) &= \frac{-\Gamma(\nu)}{(2a)^2}\left(\frac{2}{x}\right)^\nu + \frac{\pi a^{\nu-1}}{2\sin(a\pi)}K_\nu(ax) \\ &\quad - \sum_{k=1}^{r}\frac{(-1)^k a^{2k-2}}{2}\left(\frac{x}{2}\right)^{2k-\nu} \\ &\quad \times \sum_{n=0}^{k}\frac{(-1)^n\Gamma(n-k+\nu)\,\Phi(-1,2n,1)}{(k-n)!}\left(\frac{2}{x}\right)^{2n}. \end{aligned}$$

Proof The desired formulae follow immediately from the equality (4.17), Theorems 4.3 and 4.4. □

Also, using the results obtained in Theorem 4.3, we derive summation formulae for the following special kind Schlömilch series

$$\mathfrak{S}^{I,K}_{\nu,2k}(x) = \sum_{n\geq 1}\frac{I_\nu(nx)\,K_\nu(nx)}{n^{2k}}; \qquad \widetilde{\mathfrak{S}}^{I,K}_{\nu,2k}(x) = \sum_{n\geq 1}\frac{(-1)^{n-1}I_\nu(nx)\,K_\nu(nx)}{n^{2k}},$$

where $x\in\mathbb{R}$, ν is constant and $k\in\mathbb{N}$.

Corollary 4.3 (Jankov Maširević [135]) *Let the conditions from* Theorem 4.3 *hold. Then, for all* $x \in (0, \pi]$

$$\mathfrak{S}_{\nu,2k}^{I,K}(x) = \frac{(-1)^{k-1}\, x^{2k}}{2\sqrt{\pi}} \left(-\frac{\pi}{2x}\, \frac{\Gamma\left(\nu - k + \frac{1}{2}\right)\, \Gamma(k)}{\Gamma\left(k + \frac{1}{2}\right)\, \Gamma\left(k + \nu + \frac{1}{2}\right)} + \sum_{n=0}^{k} \frac{(-1)^{n-1}\Gamma\,(n + \nu - k)\;\Gamma\left(k - n + \frac{1}{2}\right)\, \zeta(2n)}{\Gamma(k - n + 1)\, \Gamma\,(1 + k - n + \nu)\; x^{2n}} \right)$$

and

$$\widetilde{\mathfrak{S}}_{\nu,2k}^{I,K}(x) = \frac{(-1)^k\, x^{2k}}{2\sqrt{\pi}} \sum_{n=0}^{k} \frac{(-1)^n\, \Gamma\,(n - k + \nu)\;\Gamma\left(k - n + \frac{1}{2}\right)\, \Phi(-1, 2n, 1)}{\Gamma(k - n + 1)\, \Gamma(k - n + \nu + 1)\, x^{2n}},$$

where $x \in \left(0, \frac{\pi}{2}\right]$.

Proof It is well known [1, p. 378, Eq. (9.7.5)] that

$$I_\nu(z)\, K_\nu(z) = \frac{1}{2z}\left(1 + \mathscr{O}(z^{-2})\right), \qquad |z| \to \infty\,,$$

i.e., for n enough large it holds that

$$\mathfrak{S}_{\nu,2k}^{I,K}(x) \sim \frac{1}{2x}\,\zeta(2k+1), \qquad \widetilde{\mathfrak{S}}_{\nu,\mu}^{I,K}(x) \sim \frac{1}{2x}\,\eta(2k+1),$$

so the convergence of $\mathfrak{S}_{\nu,2k}^{I,K}(x)$ is ensured being $k \geq 1$ and the same holds for $\widetilde{\mathfrak{S}}_{\nu,2k}^{I,K}(x)$, because the fulfilled convergence condition for Dirichlet Eta function η [1, p. 807, Eq. (23.2.19)].

Using the integral representation [93, p. 680, Eq. (6.567.11)]

$$2^{\nu-1}x^{-\nu}\sqrt{\pi}\,\Gamma\left(\nu + \frac{1}{2}\right) I_\nu\left(\frac{x}{2}\right) K_\nu\left(\frac{x}{2}\right) = \int_0^1 t^\nu (1-t^2)^{\nu-\frac{1}{2}} K_\nu(xt)\,\mathrm{d}t,$$

valid for $\nu > -\frac{1}{2}$ and Theorem 4.3 we deduce the desired results. □

4.4 Connection Between $s_{-2}(0, x, \nu)$ and Generalized Mathieu Series

In this section, the main tools we refer to are generalized Mathieu series and a Poisson formula due to Titchmarsh. Namely, a family of generalized Mathieu series was introduced in [293] by Srivastava and Tomovski:

$$\mathsf{S}_\mu^{(\alpha,\beta)}(r; \mathbf{a}) = \mathsf{S}_\mu^{(\alpha,\beta)}(r; (a_k)_{k\geq 1}) := \sum_{n\geq 1} \frac{2a_n^\beta}{(a_n^\alpha + r^2)^\mu}, \tag{4.24}$$

where $r, \alpha, \beta, \mu \in \mathbb{R}^+$ and it is assumed that the positive sequence

$$\mathbf{a} := (a_k)_{k\geq 1} = (a_1, a_2, a_3, \ldots, a_k, \ldots), \qquad \lim_{k\to\infty} a_k = \infty$$

is so chosen (and then the positive parameters α, β and μ are so constrained) that the infinite series in the definition (4.24) converges, that is, that the auxiliary series $\sum_{n\geq 1} a_n^{\beta-\mu\alpha}$ is convergent. Srivastava and Tomovski also showed that when $\beta \to 0$ it holds that

$$\mathsf{S}_\mu^{(\alpha,0)}(r; (k^{\frac{2}{\alpha}})_{k\geq 1}) := \sum_{n\geq 1} \frac{2}{(n^2 + r^2)^\mu},$$

where $r \in \mathbb{R}^+$, $\mu > \frac{1}{2}$ and the right-hand side series can be found in literature, more often, in the following form (see e.g. [53, 93, p. 5, Eq. (2.12)])

$$\frac{(2r)^{\mu-\frac{1}{2}}\Gamma(\mu)}{\sqrt{\pi}} \sum_{n\geq 1} \frac{1}{(n^2 + r^2)^\mu} = \int_0^\infty \frac{x^{\mu-\frac{1}{2}}}{\mathrm{e}^x - 1} J_{\mu-\frac{1}{2}}(rx)\, \mathrm{d}x,$$

which is also attributed by Watson [333] to a 1906 result by Kapteyn.

In order to state a Poisson formula due to Titchmarsh, i.e. the *Titchmarsh theorem* [4, p. 2], let us mention that here and in what follows the function f and its Fourier cosine transform $\mathscr{F}_c$ are related by

$$\mathscr{F}_c(f; x) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \cos(xt)\, \mathrm{d}t, \qquad f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \mathscr{F}_c(f; t) \cos(xt)\, \mathrm{d}t.$$

Theorem B (Titchmarsh theorem [313, p. 61, Theorem 45]) *Let* $\alpha\beta = 2\pi$, $\alpha > 0$, *and let* f *be of bounded variation on* $(0, \infty)$, *and tend to* 0 *as* $x \to \infty$. *Then*

$$\sqrt{\beta} \sum_{n\geq 1} \mathscr{F}_c(f; n\beta) = \sqrt{\alpha} \lim_{N\to\infty} \left(\frac{1}{2} f(0+) + \sum_{n=1}^{N} \frac{f(n\alpha-) + f(n\alpha+)}{2} \right.$$
$$\left. - \frac{1}{\alpha} \int_0^{(N+\frac{1}{2})\alpha} f(t)\, \mathrm{d}t \right).$$

Also, if $\int_0^\infty f(t)\, \mathrm{d}t$ *exists as an improper Riemann integral, then*

$$\sqrt{\frac{\beta}{\alpha}} \left(\frac{1}{2} \mathscr{F}_c(f; 0) + \sum_{n\geq 1} \mathscr{F}_c(f; n\beta) \right) = \frac{1}{2} f(0+) + \sum_{n\geq 1} \frac{f(n\alpha-) + f(n\alpha+)}{2}. \tag{4.25}$$

Also, if f is continuous, then (4.25) *reduces to Poisson's formula*

$$\sqrt{\frac{\beta}{\alpha}}\left(\frac{1}{2}\mathscr{F}_c(f;0)+\sum_{n\geq 1}\mathscr{F}_c(f;n\beta)\right)=\frac{1}{2}f(0)+\sum_{n\geq 1}f(n\alpha)\,. \tag{4.26}$$

Now, we are ready to state and prove the main result of this section.

Theorem 4.5 (Jankov Maširević [135]) *For all* $\min\{x,\nu\}>0$ *there holds*

$$\begin{aligned} s_{-2}(0,x,\nu) &= \frac{\sqrt{\pi}}{4}\left(\frac{2}{x}\right)^{\nu+1}\Gamma\left(\nu+\frac{1}{2}\right)-\frac{\Gamma(\nu)}{4}\left(\frac{2}{x}\right)^{\nu} \\ &\quad+\frac{\Gamma\left(\nu+\frac{1}{2}\right)}{4\,\pi^{2\nu+\frac{1}{2}}}\left(\frac{x}{2}\right)^{\nu}\,\mathsf{S}^{(2\pi/x,0)}_{\nu+\frac{1}{2}}\left(\frac{x}{2\pi};(k^{x/\pi})_{k\geq 1}\right)\,. \end{aligned} \tag{4.27}$$

Proof First, we establish the convergence conditions of the series $s_{-2}(0,x,\nu)$. By virtue of the well-known formula [333, p. 202]

$$K_\nu(z)=\mathrm{e}^{-z}\sqrt{\frac{\pi}{2z}}\left(1+\mathscr{O}(z^{-1})\right),\qquad |z|\to\infty\,,$$

and the definition of the polylogarithm (or *de Jonquère's*) function

$$\mathrm{Li}_s(z):=\sum_{n\geq 1}\frac{z^n}{n^s}=z\,\Phi(z,s,1),$$

defined for $s\in\mathbb{C}$, when $|z|<1$; $\Re(s)>1$ when $|z|=1$, we conclude that

$$|s_{-2}(0,x,\nu)|\leq\sum_{n\geq 1}n^\nu|K_\nu(nx)|\sim\sqrt{\frac{\pi}{2x}}\sum_{n\geq 1}\frac{\mathrm{e}^{-nx}}{n^{\frac{1}{2}-\nu}}=\sqrt{\frac{\pi}{2x}}\,\mathrm{e}^{-x}\,\Phi\left(\mathrm{e}^{-x},\tfrac{1}{2}-\nu,1\right),$$

where the convergence is ensured for $x>0$, $\nu\in\mathbb{R}$.

Now, from the integral representation (2.41) we infer that $\mathscr{F}_c(x)=2^{\frac{1}{2}-\nu}x^\nu K_\nu(x)/\Gamma\left(\nu+\frac{1}{2}\right)$ and the continuous function $f(t)=(t^2+1)^{-\nu-\frac{1}{2}}$ tends to zero as $t\to\infty$; also

$$\lim_{x\to 0}x^\nu K_\nu(x)=\frac{2^\nu}{\sqrt{\pi}}\Gamma\left(\nu+\frac{1}{2}\right)\int_0^\infty\frac{1}{(t^2+1)^{\nu+\frac{1}{2}}}\,\mathrm{d}t=2^{\nu-1}\Gamma(\nu),\;\;\nu>0.$$

Making use of (4.26) and choosing $\beta = x$, $\alpha = 2\pi/x$ we get

$$\frac{\sqrt{x}}{\Gamma\left(\nu+\frac{1}{2}\right)}\left(\frac{\Gamma(\nu)}{2^{\frac{3}{2}}}+\frac{x^\nu}{2^{\nu-\frac{1}{2}}}\sum_{n\geq 1} n^\nu K_\nu(nx)\right)$$
$$=\sqrt{\frac{\pi}{2x}}+\frac{1}{2}\left(\frac{x}{2\pi}\right)^{2\nu+\frac{1}{2}}\mathsf{S}^{(2\pi/x,0)}_{\nu+\frac{1}{2}}\left(\frac{x}{2\pi};(k^{x/\pi})_{k\geq 1}\right),$$

which is equal to (4.27). □

Finally, setting $\nu = m - \frac{1}{2}$, $m \in \mathbb{N}$, in the previous theorem, in turn implies the following result.

Corollary 4.4 (Jankov Maširević [135]) *For all* $\nu = m - \frac{1}{2}$, $m \in \mathbb{N}$ *and* $x > 0$ *there holds*

$$s_{-2}(0,x,\nu) = \frac{\sqrt{\pi}\,\Gamma(m)}{4}\left(\frac{2}{x}\right)^{m+\frac{1}{2}} - \frac{\Gamma\left(m-\frac{1}{2}\right)}{4}\left(\frac{2}{x}\right)^{m-\frac{1}{2}}$$
$$+\frac{\Gamma(m)}{4\,\pi^{2m-\frac{1}{2}}}\left(\frac{x}{2}\right)^{m-\frac{1}{2}}\sum_{n\geq 1}\frac{1}{\left(n^2+\left(\frac{x}{2\pi}\right)^2\right)^m}.$$

Remark 4.1 Using Mathematica 8.0 we have calculated the following special cases for $m = 1, 2, 3$

$$M_m = \sum_{n\geq 1}\frac{1}{\left(n^2+\left(\frac{x}{2\pi}\right)^2\right)^m} = \begin{cases} \dfrac{\pi^2}{x^2}\left(x\coth\left(\dfrac{x}{2}\right)-2\right), & m=1 \\[2ex] \dfrac{\pi^4}{x^4\sinh^2\left(\frac{x}{2}\right)}\left(x\sinh x - 4\cosh x + x^2 + 4\right), & m=2 \\[2ex] \dfrac{\pi^6}{2x^6}\left(\dfrac{x^3\sinh x}{\sinh^4\left(\frac{x}{2}\right)}+\dfrac{6x^2}{\sinh^2\left(\frac{x}{2}\right)}\right. \\ \qquad\left. +12x\coth\left(\dfrac{x}{2}\right)-64\right), & m=3 \end{cases}.$$

Further illustrative examples show the structure of connection between $s_{-2}(0, x, \nu)$ and the Mathieu type sum M_m. ■

4.5 p-Extended Mathieu Series as Schlömilch Series

One of the actual generalizations of Mathieu series, defined in Sect. 1.5, is the so-called generalized Mathieu series with a fractional power reads [53, p. 2, Eq. (1.6)] (and also consult [201, p. 181])

$$S_\mu(r) = \sum_{n\ge 1} \frac{2n}{(n^2+r^2)^{\mu+1}}, \qquad r>0,\ \mu>0;$$

which can also be presented in terms of the Riemann Zeta function [53, p. 3, Eq. (2.1)]

$$S_\mu(r) = 2\sum_{n\ge 0} r^{2n}(-1)^n \binom{\mu+n}{n} \zeta(2\mu+2n+1), \qquad |r|<1. \tag{4.28}$$

Having in mind (4.28) Pogány and Parmar [245] recently introduced the *p-extended Mathieu series*

$$S_{\mu,p}(r) = 2\sum_{n\ge 0} r^{2n}(-1)^n \binom{\mu+n}{n} \zeta_p(2\mu+2n+1), \tag{4.29}$$

where $\Re(p)>0$ or $p=0$, $\mu>0$. Here and in what follows ζ_p stands for the p-extension of the Riemann ζ function [50]:

$$\zeta_p(\alpha) = \frac{1}{\Gamma(\alpha)} \int_0^\infty \frac{t^{\alpha-1}\mathrm{e}^{-\frac{p}{t}}}{\mathrm{e}^t-1}\,\mathrm{d}t \tag{4.30}$$

defined for $\Re(p)>0$ or $p=0$ and $\Re(\alpha)>0$ and it reduces to the Riemann Zeta function when $p=0$. Also, (4.29) one reduces to (4.28) when $p=0$.

Pogány and Parmar [245] obtained an integral form of such series, which reads

$$S_{\mu,p}(r) = \frac{\sqrt{\pi}}{(2r)^{\mu-\frac12}\Gamma(\mu+1)} \int_0^\infty \frac{t^{\mu+\frac12}\mathrm{e}^{-\frac{p}{t}}}{\mathrm{e}^t-1} J_{\mu-\frac12}(rt)\,\mathrm{d}t; \tag{4.31}$$

here $\Re(p)>0$ or $p=0$, $\mu>0$.

The whole set of recent extensions of Gamma and Beta function [50] were motivated by the wide applications of the generalization of these special function's family. In [50] the same extension method has been used also in the kernel of the integral expression of the Zeta function, compare (4.30). On the other hand, one of the important properties of the Mathieu series $S_\mu(r)$ turns out to be (4.29). What we require is that the results be naturally and simply extended. This approach is met by (4.31). It is expected that such natural extensions would be found useful in answering some of the classical problems.

Motivated by that newly introduced p-extended Mathieu series which members contain the extension of the Riemann Zeta ζ_p and also the fact that ζ_p can be presented as Schlömilch series of modified Bessel functions of the second kind i.e. as [50, p. 1240]

$$\zeta_p(\alpha) = \frac{2p^{\frac{\alpha}{2}}}{\Gamma(\alpha)} \sum_{n\geq 1} \frac{K_\alpha(2\sqrt{np})}{n^{\frac{\alpha}{2}}}, \qquad \alpha, p > 0$$

our main aim in this section is to derive new representations of our series in terms of the various Schlömilch series. This new, deeper insight into the Schlömilch series' structure of p-extended Mathieu series will give an important bridge to the fractional calculus considerations, approach and background of further understanding the Mathieu series studies. After necessary preliminaries in the next section we derive new expressions of (4.29) in terms of Schlömilch series which members contain derivation (ordinary or fractional) of a combination of Bessel function of the first kind J_ν and modified Bessel function of the second kind K_ν. In Sect. 4.5.2 we would also derive some connection formulae between our Mathieu series and Schlömilch series but this time with members containing only modified Bessel functions of the second kind.

4.5.1 Connection Between $S_{\mu,p}(r)$ and Schlömilch Series of $J_\nu \cdot K_\mu$

In this section, our main aim is to derive connection formulae between p-extended Mathieu series $S_{\mu,p}(r)$ and Schlömilch series which members contain combination of Bessel functions of the first kind J_ν and modified Bessel functions of the second kind K_ν of the order ν.

Our derivation procedure requires the Grünwald–Letnikov fractional derivative of order $-\alpha$, $\alpha > 0$ with respect to an argument x of a suitable function f defined by Samko et al. [273]

$$\mathbb{D}_x^{-\alpha}[f] = \lim_{n\to\infty} \left(\frac{n}{x-a}\right)^\alpha \sum_{m=0}^{n} \frac{\Gamma(\alpha+m)}{m!\,\Gamma(\alpha)} f\left(x - m\frac{x-a}{h}\right), \qquad a < x. \tag{4.32}$$

Several numerical algorithms are available for the direct computation of (4.32); see e.g. [63, 206, 285].

Theorem 4.6 (Jankov Maširević and Pogány [140]) *For all* $\min\{\Re(p) > 0, \Re(q), \gamma\} > 0$ *and* $\alpha > \frac{1}{2}$ *there holds*

$$\begin{aligned} S_{\alpha-\frac{3}{2},p}(\gamma) = \frac{2(-1)^\alpha\sqrt{\pi}}{(2\gamma)^{\alpha-2}\Gamma(\alpha-\frac{1}{2})} \sum_{k\geq 1} \mathbb{D}_q^\alpha \Big(J_{\alpha-2}\left(\sqrt{2p}\,[\sqrt{q^2+\gamma^2}-q]^{\frac{1}{2}}\right) \\ \times K_{\alpha-2}\left(\sqrt{2p}\,[\sqrt{q^2+\gamma^2}+q]^{\frac{1}{2}}\right)\Big)\Big|_{q=k}. \end{aligned} \tag{4.33}$$

Further, for $\alpha = n \in \mathbb{N}$ *we have*

$$S_{n-\frac{3}{2},p}(\gamma) = \frac{2(-1)^n\sqrt{\pi}}{(2\gamma)^{n-2}\Gamma(n-\frac{1}{2})}\sum_{k\geq 1}\frac{\partial^n}{\partial q^n}\Big[J_{n-2}\left(\sqrt{2p}\,[\sqrt{q^2+\gamma^2}-q]^{\frac{1}{2}}\right) \times K_{n-2}\left(\sqrt{2p}\,[\sqrt{q^2+\gamma^2}+q]^{\frac{1}{2}}\right)\Big]\Big|_{q=k}.$$

Proof In order to prove the desired results, let us first consider the integral [93, p. 708, Eq. 6.635.3]

$$A_{p,q}(\gamma) = \int_0^\infty x^{-1}\mathrm{e}^{-qx-p/x}J_\nu(\gamma x)\mathrm{d}x = 2\,J_\nu\left(\sqrt{2p}\,[\sqrt{q^2+\gamma^2}-q]^{\frac{1}{2}}\right)K_\nu\left(\sqrt{2p}\,[\sqrt{q^2+\gamma^2}+q]^{\frac{1}{2}}\right), \tag{4.34}$$

where $\min\{\Re(p), \Re(q), \gamma\} > 0$.[2]

Now, using the Grünwald-Letnikov fractional derivative

$$\mathbb{D}_q^\alpha \mathrm{e}^{-qx} = (-x)^\alpha\,\mathrm{e}^{-qx}$$

valid for every real $\alpha > -\nu$ we get

$$\mathbb{D}_q^\alpha A_{p,q}(\gamma) = (-1)^\alpha\int_0^\infty x^{\alpha-1}\mathrm{e}^{-qx-p/x}J_\nu(\gamma x)\mathrm{d}x.$$

Further, specifying $q = k+1$ and summing up the previous equality for $k \in \mathbb{N}_0$ we have

$$\sum_{k\geq 0}\mathbb{D}_q^\alpha A_{p,q}(\gamma)\big|_{q=k+1} = (-1)^\alpha\int_0^\infty \frac{x^{\alpha-1}\mathrm{e}^{-p/x}}{\mathrm{e}^x-1}J_\nu(\gamma x)\mathrm{d}x\,.$$

Setting $\nu = \alpha - 2$ with the help of the integral representation (4.31) we get

$$\int_0^\infty \frac{x^{\alpha-1}\mathrm{e}^{-p/x}}{\mathrm{e}^x-1}J_{\alpha-2}(\gamma x)\mathrm{d}x = \frac{(2\gamma)^{\alpha-2}\Gamma(\alpha-\frac{1}{2})}{\sqrt{\pi}}S_{\alpha-\frac{3}{2},p}(\gamma).$$

Now, from the previous calculations, using also (4.34), we have

$$\begin{aligned} S_{\alpha-\frac{3}{2},p}(\gamma) &= \frac{(-1)^\alpha\sqrt{\pi}}{(2\gamma)^{\alpha-2}\Gamma(\alpha-\frac{1}{2})}\sum_{k\geq 1}\mathbb{D}_q^\alpha A_{p,q}(\gamma)\Big|_{q=k} \\ &= \frac{2(-1)^\alpha\sqrt{\pi}}{(2\gamma)^{\alpha-2}\Gamma(\alpha-\frac{1}{2})}\sum_{k\geq 1}\mathbb{D}_q^\alpha\Big(J_{\alpha-2}\left(\sqrt{2p}[\sqrt{q^2+\gamma^2}-q]^{\frac{1}{2}}\right) \\ &\quad\times K_{\alpha-2}\left(\sqrt{2p}[\sqrt{q^2+\gamma^2}+q]^{\frac{1}{2}}\right)\Big)\Big|_{q=k} \end{aligned}$$

which is equal to (4.33).

[2]Actually, $A_{p,q}(\gamma)$ is the Laplace transform of $x \mapsto x^{-1}\mathrm{e}^{-\frac{p}{x}}J_\nu(\gamma x)$ at the argument q.

Next, for a positive integer $\alpha = n$ (in fact $A_{p,q}(\gamma)$ converges for all $n + \nu > 0$) consider

$$\frac{\partial^n}{\partial q^n} A_{p,q}(\gamma) = (-1)^n \int_0^\infty x^{n-1} \mathrm{e}^{-qx-p/x} J_\nu(\gamma x) \mathrm{d}x.$$

The same procedure as above yields

$$\int_0^\infty \frac{x^{n-1} \mathrm{e}^{-p/x}}{\mathrm{e}^x - 1} J_\nu(\gamma x) \mathrm{d}x = (-1)^n \sum_{k\geq 0} \frac{\partial^n}{\partial q^n} A_{p,q}(\gamma)\Big|_{q=k+1}.$$

Again with the help of (4.31) and (4.34) and substituting $\nu = n - 2$ we have

$$\begin{aligned} S_{n-\frac{3}{2},p}(\gamma) &= \frac{(-1)^n \sqrt{\pi}}{(2\gamma)^{n-2}\Gamma(n-\frac{1}{2})} \sum_{k\geq 1} \frac{\partial^n}{\partial q^n} A_{p,q}(\gamma)\Big|_{q=k} = \frac{2(-1)^n \sqrt{\pi}}{(2\gamma)^{n-2}\Gamma(n-\frac{1}{2})} \\ &\quad \times \sum_{k\geq 1} \frac{\partial^n}{\partial q^n} \Big[J_{n-2}\Big(\sqrt{2p}[\sqrt{q^2+\gamma^2} - q]^{\frac{1}{2}}\Big) \\ &\quad \times K_{n-2}\Big(\sqrt{2p}[\sqrt{q^2+\gamma^2} + q]^{\frac{1}{2}}\Big)\Big]\Big|_{q=k}, \end{aligned}$$

which completes the proof. □

Theorem 4.7 (Jankov Maširević and Pogány [140]) *For all* $\Re(p) > 0$ *we have*

$$\begin{aligned} S_{\frac{1}{2},p}(\gamma) = -4\gamma \sum_{k\geq 1} \frac{\partial^3}{\partial q^3} \Bigg(&\frac{J_1(\sqrt{2p}[\sqrt{q^2+\gamma^2} - q]^{\frac{1}{2}}) K_0(\sqrt{2p}[\sqrt{q^2+\gamma^2} + q]^{\frac{1}{2}})}{\sqrt{2p}[\sqrt{q^2+\gamma^2} + q]^{\frac{1}{2}}} \\ &+ \frac{J_0(\sqrt{2p}[\sqrt{q^2+\gamma^2} - q]^{\frac{1}{2}}) K_1(\sqrt{2p}[\sqrt{q^2+\gamma^2} + q]^{\frac{1}{2}})}{\sqrt{2p}[\sqrt{q^2+\gamma^2} - q]^{\frac{1}{2}}} \Bigg)\Bigg|_{q=k}. \end{aligned} \tag{4.35}$$

Moreover, it is

$$S_{-\frac{1}{2},p}(\gamma) = 4\gamma \sum_{k\geq 1} \frac{\partial}{\partial q} \Big(J_1(\sqrt{2p}[\sqrt{q^2+\gamma^2} - q]^{\frac{1}{2}}) K_1(\sqrt{2p}[\sqrt{q^2+\gamma^2} + q]^{\frac{1}{2}}) \Big)\Big|_{q=k}. \tag{4.36}$$

Proof With the help of the integral [257, p. 188, Eq. **2.12.10.2**]

$$B_{p,q}(\gamma) = \int_0^\infty x^{-2} \mathrm{e}^{-qx-p/x} J_0(\gamma x)\, \mathrm{d}x = 2\gamma \left(z_+^{-1} J_1(z_-) K_0(z_+) + z_-^{-1} J_0(z_-) K_1(z_+) \right),$$

where $z_\pm = \sqrt{2p}[\sqrt{q^2+\gamma^2} \pm q]^{\frac{1}{2}}$, $\min\{\Re(q), \Re(p)\} > 0$, we conclude

$$\frac{\partial^3}{\partial q^3} B_{p,q}(\gamma) = -\int_0^\infty x\mathrm{e}^{-qx-p/x} J_0(\gamma x)\mathrm{d}x$$

which, with the help of (4.31), gives us

$$\sum_{k\geq 0} \frac{\partial^3}{\partial q^3} B_{p,q}(\gamma)\Big|_{q=k+1} = -\int_0^\infty \frac{x\mathrm{e}^{-p/x}}{\mathrm{e}^x-1} J_0(\gamma x)\mathrm{d}x = -\frac{1}{2} S_{\frac{1}{2},p}(\gamma)$$

which coincides with (4.35).

In the same way, but this time using [257, p. 188, Eq. **2.12.10.1**]

$$C_{p,q}(\gamma) = \int_0^\infty x^{-1}\mathrm{e}^{-qx-p/x} J_\nu(\gamma x)\,\mathrm{d}x = 2J_\nu(z_-)K_\nu(z_+),$$

where $\min\{\Re(p), \Re(q)\} > 0$, and $z_\pm$ has the same meaning as above, with the aid of parity of Bessel and modified Bessel function $J_{-1}(x) = -J_1(x); K_{-1}(x) = K_1(x)$ we deduce (4.36). □

Remark 4.2 From (4.36), bearing in mind [117, 119]:

$$2\,(J_1(x)\,K_1(x))' = (J_0(x) - J_2(x))\,K_1(x) - J_1(x)\,(K_0(x) + K_2(x))\,,$$

we can infer a new representation for $S_{-\frac{1}{2},p}(\gamma)$. ■

4.5.2 $S_{\mu,p}(r)$ and the Schlömilch Series of K_ν Terms

Considering now specialized p-extended Mathieu series, that is in which $\mu = 0, 1, 2$, we report on their Schlömilch-series expansion *via* modified Bessel functions of the second kind $K_{\mu+1}$.

Theorem 4.8 (Jankov Maširević and Pogány [140]) *For all $\Re(p) > 0$, $\gamma > 0$ there hold*

$$S_{0,p}(\gamma) = 2\sqrt{p} \sum_{k\geq 1} \left(\frac{K_1\left(2\sqrt{p(k+\mathrm{i}\,\gamma)}\right)}{\sqrt{k+\mathrm{i}\,\gamma}} + \frac{K_1\left(2\sqrt{p(k-\mathrm{i}\,\gamma)}\right)}{\sqrt{k-\mathrm{i}\,\gamma}} \right), \tag{4.37}$$

$$S_{1,p}(\gamma) = \frac{\mathrm{i}p}{\gamma} \sum_{k\geq 1} \left(\frac{K_2\left(2\sqrt{p(k+\mathrm{i}\,\gamma)}\right)}{\sqrt{k+\mathrm{i}\,\gamma}} - \frac{K_2\left(2\sqrt{p(k-\mathrm{i}\,\gamma)}\right)}{\sqrt{k-\mathrm{i}\,\gamma}} \right). \tag{4.38}$$

Proof In order to prove the desired results we will need the following formula [258]

$$E^{\mp}_{p,q}(\gamma) = \int_0^\infty x^\nu \mathrm{e}^{-qx-p/x} \begin{Bmatrix} \sin(\gamma x) \\ \cos(\gamma x) \end{Bmatrix} \,\mathrm{d}x \tag{4.39}$$
$$= \mathrm{i}^{\frac{1\pm 1}{2}} p^{\frac{\nu+1}{2}} \left(\frac{K_{\nu+1}\left(2\sqrt{p(q+\mathrm{i}\,\gamma)}\right)}{(q+\mathrm{i}\,\gamma)^{\frac{\nu+1}{2}}} \mp \frac{K_{\nu+1}\left(2\sqrt{p(q-\mathrm{i}\,\gamma)}\right)}{(q-\mathrm{i}\,\gamma)^{\frac{\nu+1}{2}}} \right),$$

which holds for $\min\{\Re(p), \Re(q)\} > 0$.

Now, since

$$J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \cos x,$$

by virtue of (4.31) and (4.39) setting $q = k + 1$, $k \in \mathbb{N}_0$ and $\nu = 0$ it follows

$$\sum_{k \ge 0} E^{+}_{p,k+1}(\gamma) = \sqrt{\frac{\pi\gamma}{2}} \int_0^\infty \frac{\sqrt{x}\,\mathrm{e}^{-p/x}}{\mathrm{e}^x - 1} J_{-\frac{1}{2}}(\gamma x)\,\mathrm{d}x = \frac{1}{2} S_{0,p}(\gamma),$$

which results in (4.37).

Analogously, from (4.31) for $\nu = 1$, applying (4.39) for $E^{-}_{p,k+1}(\gamma)$ and $J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \sin x$, one implies the second statement (4.38). □

Theorem 4.9 (Jankov Maširević and Pogány [140]) *For all* $\Re(p) > 0$, $r > 0$ *there holds*

$$S_{2,p}(r) = \frac{S_{1,p}(r)}{(2r)^2} - \frac{p^{\frac{3}{2}}}{(2r)^2} \sum_{n \ge 1} \left(\frac{K_3\left(2\sqrt{p(n+\mathrm{i}\,r)}\right)}{(n+\mathrm{i}\,r)^{\frac{3}{2}}} + \frac{K_3\left(2\sqrt{p(n-\mathrm{i}\,r)}\right)}{(n-\mathrm{i}\,r)^{\frac{3}{2}}} \right). \tag{4.40}$$

Proof From the integral representation (4.31) of $S_{\mu,p}(r)$, for $\mu = 2$ it is

$$S_{2,p}(r) = \frac{\sqrt{\pi}}{2(2r)^{\frac{3}{2}}} \int_0^\infty \frac{x^{\frac{5}{2}} \mathrm{e}^{-p/x} J_{\frac{3}{2}}(rx)}{\mathrm{e}^x - 1}\,\mathrm{d}x$$
$$= \frac{\sqrt{\pi}}{2(2r)^{\frac{3}{2}}} \sum_{k \ge 1} \int_0^\infty x^{\frac{5}{2}} \mathrm{e}^{-kx-p/x} J_{\frac{3}{2}}(rx)\,\mathrm{d}x$$
$$= \frac{1}{(2r)^2} \sum_{k \ge 1} \int_0^\infty x^2 \mathrm{e}^{-kx-p/x} \left(\frac{\sin(rx)}{rx} - \cos(rx) \right) \mathrm{d}x,$$

where in the last equality we used the well-known formula

$$J_{\frac{3}{2}}(x) = \sqrt{\frac{2}{\pi x}}\left(\frac{\sin x}{x} - \cos x\right).$$

Further, with the help of (4.31) and $J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\,\sin x$ the previous expression can be rewritten into

$$S_{2,p}(r) = \frac{1}{(2r)^2}\,S_{1,p}(r) - \frac{1}{(2r)^2}\sum_{k\geq 1}\int_0^\infty x^2 \mathrm{e}^{-kx-p/x}\cos(rx)\,\mathrm{d}x.$$

Finally, using the Laplace transform of the function $x \mapsto x^2\mathrm{e}^{-p/x}\cos(rx)$, in the argument k given by (4.39) we get the display (4.40). □

Remark 4.3 Using the formula (4.38) derived in Theorem 4.8 and the formula (4.40) which connects $S_{2,p}(r)$ and $S_{1,p}(r)$ new representation for $S_{2,p}(r)$ can be derived. ■

4.6 Integral Form of Popov's Formula (4.7)

Recall the relation (4.7)

$$\frac{1}{\Gamma(q+1)}\sum_{0\leq n\leq [x]}{}' r_k(n)(x-n)^q = \frac{\pi^{\frac{k}{2}}\,x^{\frac{k}{2}+q}}{\Gamma(\frac{k}{2}+q+1)} + \pi^{-q}\,\mathfrak{S}_{k,q}(x)\,,$$

where the following notation is introduced

$$\mathfrak{S}_{k,q}(x) = \sum_{n\geq 1} r_k(n)\left(\frac{x}{n}\right)^{\frac{k}{4}+\frac{q}{2}} J_{\frac{k}{2}+q}\big(2\pi\sqrt{nx}\big), \qquad \Re(x) > 0. \tag{4.41}$$

Our task is to derive the integral expression for the Schlömilch–Bessel type series $\mathfrak{S}_{k,q}(x)$ when $2q > k-3$ and x belongs to the widest possible sub-domain of the positive reals.

The x-convergence domain we determine with the help of Olenko's bound (1.22), which is more efficient then the Landau's bounds (1.20) and (1.21). Indeed, applying Olenko's estimate we get the upper bound

$$\big|\mathfrak{S}_{k,q}(x)\big| \leq \frac{d_O}{\sqrt{2\pi}}\,x^{\frac{k}{4}+\frac{q}{2}-\frac{1}{4}}\sum_{n\geq 1}\frac{r_k(n)}{n^{\frac{k}{2}+\frac{2q-k+1}{4}}}\,,$$

which shows that the bound is enough sensitive to give upper bound which finiteness do not depend on x, but upon the convergence of associated Epstein Zeta function.

On the other hand Walfisz precised that [326, p. 417]

$$\sum_{n\geq 1} r_k(n)\, n^{-\frac{k}{2}-\varepsilon} < \infty, \qquad \varepsilon > 0\,,$$

which yields convergence of the Schlömilch–Bessel type series $\mathfrak{S}_{k,q}(x)$ on the whole $x \in \mathbb{R}_+$, only for $2q > k-1$. We point out the Landau's bounds are inferior with respect Olenko's in this question, giving constraints $2q > k$ and $2q > k - \frac{2}{3}$, respectively.

The next step is to apply the integral representation by Schläfli [276, p. 204][3]

$$J_\nu(z) = \frac{1}{\pi}\int_0^\pi \cos(\nu\theta - z\sin\theta)\,\mathrm{d}\theta - \frac{\sin\nu\pi}{\pi}\int_0^\infty \mathrm{e}^{-\nu t - z\sinh t}\,\mathrm{d}t, \qquad |\arg(z)| < \frac{\pi}{2}\,.$$

The concluded form of the Schlömilch series becomes

$$\begin{aligned}\mathfrak{S}_{k,q}(x) = \frac{1}{\pi}\, x^{\frac{k}{4}+\frac{q}{2}} \Bigg\{ \int_0^\pi \sum_{n\geq 1} \frac{r_k(n)}{n^{\frac{k}{4}+\frac{q}{2}}}\, \cos\Big(\big(\tfrac{k}{2}+q\big)\theta - 2\pi\,\sqrt{nx}\,\sin\theta\Big)\,\mathrm{d}\theta \\ - \sin\pi\big(\tfrac{k}{2}+q\big)\int_0^\infty \mathrm{e}^{-(\frac{k}{2}+q)t} \sum_{n\geq 1}\frac{r_k(n)}{n^{\frac{k}{4}+\frac{q}{2}}}\,\mathrm{e}^{-2\pi\sqrt{nx}\,\sinh t}\,\mathrm{d}t\Bigg\}\,. \qquad (4.42)\end{aligned}$$

Consider a function $\mathfrak{r}_k(t)$ which restriction $\mathfrak{r}_k(t)\big|_{\mathbb{N}} = (r_k(n))_{n\geq 1}$, using by convention the value $\mathfrak{r}_k(0) \equiv 1$ which holds for any positive integer $k \in \mathbb{N}$. Obviously such function there exists—take for instance an interpolation polynomial of suitably high degree—and it is differentiable.

Both inner sums in (4.42) are in fact Dirichlet series of the form

$$\mathscr{D}_h(x) = \sum_{n\geq 1} r_k(n) h_n(x) \mathrm{e}^{-(\frac{k}{4}+\frac{q}{2})\,\log n}\,,$$

---

[3]The usually used integral expression (2.4) for the Bessel function in the summands of (4.41) results in

$$\mathfrak{S}_{k,q}(x) = \frac{2(\pi x)^{\frac{k}{2}+q}}{\sqrt{\pi}\,\Gamma\left(\frac{k+1}{2}+q\right)}\int_0^1 (1-t^2)^{\frac{k-1}{2}+q} \sum_{n\geq 1} r_k(n)\,\cos\big(2\pi t\sqrt{nx}\big)\,\mathrm{d}t.$$

On the other hand, also by Walfisz was found that [326, p. 40]

$$\sum_{j=1}^n r_k(n) = cn^{\frac{k}{2}} + \mathscr{O}\big(n^{\frac{k-1}{2}}\big)\,,$$

being c an absolute constant. All together imply that the inner sum diverges in a neighborhood of $t = 0$, therefore the integral diverges too.

where $h_n(x)$ takes either the cosine or the exponential form, respectively. Being the Dirichlet series' parameter positive by the Cahen's formula (1.15) we deduce

$$\begin{aligned}\mathscr{D}_h(x) &= \frac{k+2q}{4}\int_0^\infty \mathrm{e}^{-(\frac{k}{4}+\frac{q}{2})y}\left\{\sum_{n=1}^{[\mathrm{e}^y]} r_k(n)h_n(x)\right\}\mathrm{d}y\\ &= \frac{k+2q}{4}\int_0^\infty \mathrm{e}^{-(\frac{k}{4}+\frac{q}{2})y}\mathscr{A}_h(y)\,\mathrm{d}y\,,\end{aligned}$$

where the finite counting sum

$$\mathscr{A}_h(y) = \sum_{n=1}^{[\mathrm{e}^y]} r_k(n)h_n(x) = \sum_{n=1}^{[\mathrm{e}^y]} r_k(n)\left\{\begin{matrix}\cos\Big((\frac{k}{2}+q)\theta - 2\pi\sqrt{nx}\,\sin\theta\Big)\\ \mathrm{e}^{-2\pi\sqrt{nx}\,\sinh t}\end{matrix}\right\}\,,$$

we sum up using the Euler–Maclaurin summation formula (1.9). The result reads

$$\mathscr{A}_h(y) = \int_0^{[\mathrm{e}^y]} \mathfrak{d}_u\,\mathfrak{r}_k(u)\left\{\begin{matrix}\cos\Big((\frac{k}{2}+q)\theta - 2\pi\sqrt{ux}\,\sin\theta\Big)\\ \mathrm{e}^{-2\pi\sqrt{ux}\,\sinh t}\end{matrix}\right\}\mathrm{d}u\,.$$

Collecting all these expressions in multiple replacing procedure, we arrive at

Theorem 4.10 *For all $k \in \mathbb{N}, x > 0$ and $q > \frac{1}{2}(k-1)$ we have the integral representation*

$$\begin{aligned}\mathfrak{S}_{k,q}(x) = \frac{k+2q}{4\pi}\,x^{\frac{k}{4}+\frac{q}{2}}\Bigg\{&\int_0^\pi\int_0^\infty\int_0^{[\mathrm{e}^y]} \mathrm{e}^{-(\frac{k}{4}+\frac{q}{2})y}\,\mathfrak{d}_u\,\mathfrak{r}_k(u)\,\cos\Big((\tfrac{k}{2}+q)\theta\\ &- 2\pi\sqrt{ux}\,\sin\theta\Big)\,\mathrm{d}\theta\,\mathrm{d}y\,\mathrm{d}u - \sin\pi\big(\tfrac{k}{2}+q\big)\\ &\times\int_0^\infty\int_0^\infty\int_0^{[\mathrm{e}^y]} \mathrm{e}^{-(\frac{k}{4}+\frac{q}{2})y}\,\mathfrak{d}_u\,\mathfrak{r}_k(u)\,\mathrm{e}^{-2\pi\sqrt{ux}\,\sinh t}\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}u\Bigg\}\,,\end{aligned}$$

where $\mathfrak{r}_k(t)$ is a differentiable function which restriction to the set of positive integers coincides with the sequence $\mathfrak{r}_k\big|_{\mathbb{N}} = (r_k(n))_{n\geq 1}$ and by convention $\mathfrak{r}_k(0) = 1$.

The conjunction of (4.7) and Theorem 4.10 leads to the following form of a complicated triple integral.

Corollary 4.5 *For all $k \in \mathbb{N}, x > 0$ and $q > \frac{1}{2}(k-1)$ there holds*

$$\mathfrak{S}_{k,q}(x) = \frac{\pi^q}{\Gamma(q+1)}\sum_{0\leq n\leq [x]}{}' r_k(n)(x-n)^q - \frac{(\pi x)^{\frac{k}{2}+q}}{\Gamma(\frac{k}{2}+q+1)}\,.$$

Chapter 5
Miscellanea

Abstract In this chapter we will present various results concerning Neumann, Kapteyn and Schlömilch series with members containing functions from the Bessel functions family (Bessel functions of the first and second kind, modified Bessel functions of the first and second kind, Struve functions, modified Struve functions etc.). In Sects. 5.7–5.9 we consider Dini series and Jacobi polynomials, respectively. Next section is devoted to summations of Schlömilch series which members contain some von Lommel functions of the first kind. Section 5.11 finishes this chapter with Neumann–Meijer G series results.

5.1 The Fourier–Bessel Series Associated with Struve Functions

The function $\mathbf{H}_\nu$ [299] was introduced by Struve and today this function is carrying his name. However, the modified Struve function $\mathbf{L}_\nu$ appeared into mathematical literature by Nicholson [211, p. 218], *viz.*

$$\mathbf{H}_\nu(z) = \sum_{n\geq 0} \frac{(-1)^n \left(\frac{z}{2}\right)^{2n+\nu+1}}{\Gamma\left(n+\frac{3}{2}\right)\Gamma\left(n+\nu+\frac{3}{2}\right)}, \qquad \mathbf{L}_\nu(z) = \sum_{n\geq 0} \frac{\left(\frac{z}{2}\right)^{2n+\nu+1}}{\Gamma\left(n+\frac{3}{2}\right)\Gamma\left(n+\nu+\frac{3}{2}\right)}.$$

These functions are related to the non-homogeneous Bessel type ordinary differential equation of special type called Struve differential equation, see Sect. 1.6.

Here we consider the series

$$\mathfrak{B}_{\ell_1,\ell_2}(z) = \sum_{n\geq 1} \alpha_n \mathscr{B}_{\ell_1(n)}(\ell_2(n)z),$$

where $\mathscr{B}_\nu$ is one of the functions $\mathbf{H}_\nu$ and/or $\mathbf{L}_\nu$. The Sonin–Gubler formula which connects modified Bessel function of the first kind I_ν, modified Struve function $\mathbf{L}_\nu$

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and a definite integral of the Bessel function of the first kind J_ν [97, p. 424] (actually a special case of a Sonin-formula [333, p. 434]) reads:

$$\int_0^\infty \frac{J_\nu(ax)}{x^2+n^2}\,\frac{\mathrm{d}x}{x^\nu} = \frac{\pi}{2n^{\nu+1}}\left(I_\nu(an) - \mathbf{L}_\nu(an)\right), \tag{5.1}$$

where $\Re(\nu) > -\frac{1}{2}$, $a > 0$ and $\Re(n) > 0$; also see [333, p. 426].

The main results exposed in this section have been recently obtained by Baricz and Pogány in [18]. So, the section is devoted *inter alia* to the study of specific Kapteyn-type series of the following form:

$$\mathfrak{K}^{\alpha}_{\nu,\mu}(x) := \sum_{n\geq 1} \frac{\alpha_n}{n^\mu}\left(I_{\nu n}(xn) - \mathbf{L}_{\nu n}(xn)\right);$$

to the Schlömilch series'

$$\mathfrak{S}^{I,\mathbf{L}}_{\mu,\nu}(z) := \sum_{n\geq 1} \frac{\alpha_n}{n^\mu}\left(I_\nu(xn) - \mathbf{L}_\nu(xn)\right),$$

its special case $\alpha_n \equiv 1$, see [18, p. 257, Eqs. (1.5), (1.6)]

$$\mathfrak{T}^{I,\mathbf{L}}_{\nu,\mu}(x) := \sum_{n\geq 1} \frac{I_\nu(nx) - \mathbf{L}_\nu(nx)}{n^\mu},$$

and its alternating variant $\widetilde{\mathfrak{T}}^{I,\mathbf{L}}_{\nu,\mu}(x)$.

The links to the Butzer–Flocke–Hauss complete Ω function [45, 46] and the generalized alternating Mathieu series [252] are also given there, see [18, p. 266, Theorem 7 *et seq.*].

Finally ordinary differential equation approach was involved in the considerations and novel contour integral expressions were derived for $\mathfrak{T}^{I,\mathbf{L}}_{\nu,\mu}(x)$ *via* Mellin transform technique, applying the associated first kind Fredholm type convolutional integral equation with degenerate kernel, compare [18, p. 276 *et seq.*, Theorems 13, 14, 15].

5.2 Summations of Series Built by Modified Struve Function

We will consider in this section the power series representations of Struve and modified Struve functions $\mathbf{H}_\nu(z)$ and $\mathbf{L}_\nu(z)$ listed above respectively, according to Watson [333] for $\nu \in \mathbb{R}$ and $z \in \mathbb{C}$. Struve [299] introduced $\mathbf{H}_\nu$ as the series solution of the nonhomogeneous second order Bessel type differential equation, which carries also his name. However, the modified Struve function $\mathbf{L}_\nu$ appeared into mathematical literature by Nicholson [211, p. 218]. Applications of Struve functions are manyfold and include among others optical investigations [327, pp. 392–395]; general expression of the power carried by a transverse magnetic or electric beam, is given in terms of $\mathbf{L}_{n+\frac{1}{2}}$ [9]; triplet phase shifts of the scattering by the singular nucleon-nucleon potentials $\propto \exp(-x)/x^n$ [85]; leakage inductance in transformer windings [124]; boundary element solutions of the two-dimensional multi-energy-group neutron diffusion equation which governs the neutronic phenomena in nuclear reactors [127]; effective isotropic potential for a pair of dipoles [192]; perturbation approximations of lee-waves in a stratified flow [200]; quantum-statistical distribution functions of a hard-sphere system [222]; scattering of plane waves by circular cylinders for the general case of oblique incidence and for both real and complex values of particle refractive index [298]; aerodynamic sensitivities for subsonic, sonic, and supersonic unsteady, non-planar lifting-surface theory [339]; stress concentration around broken filaments [82] and lift and downwash distributions of oscillating wings in subsonic and supersonic flow [331, 332].

Series of Bessel and/or Struve functions in which summation indices appear in the order of the considered function and/or twist arguments of the constituting functions, can be unified in a double lacunary form:

$$\mathfrak{B}_{\ell_1,\ell_2}(z) := \sum_{n\geq 1} \alpha_n \mathscr{B}_{\ell_1(n)}(\ell_2(n)z),$$

where $x \mapsto \ell_j(x) = \mu_j + a_j x$, $j \in \{1, 2\}$, $x \in \{0, 1, \dots\}$, $z \in \mathbb{C}$ and $\mathscr{B}_\nu$ is one of the functions $J_\nu, I_\nu, \mathbf{H}_\nu$ and $\mathbf{L}_\nu$. The classical theory of the Fourier–Bessel series of the first type is based on the case when $\mathscr{B}_\nu = J_\nu$, see the celebrated monograph by Watson [333]. However, varying the coefficients of ℓ_1 and ℓ_2, we get three different cases which have not only deep roles in describing physical ordinary differential equation and have physical interpretations in numerous topics of natural sciences and technology, but are also of deep mathematical interest, like e.g. zero function series [333]. Hence we differ: Neumann series (when $a_1 \neq 0, a_2 = 0$), Kapteyn series (when $a_1 \cdot a_2 \neq 0$) and Schlömilch series (when $a_1 = 0, a_2 \neq 0$). Here, all three series are of the first type (the series' terms contain only one constituting function $\mathscr{B}_\nu$); the second type series contain product terms of two (or more) members—not necessarily different ones—from $J_\nu, I_\nu, \mathbf{H}_\nu$ and $\mathbf{L}_\nu$. We also point out that the Neumann series (of the first type) of Bessel function of the second kind Y_ν, modified Bessel function of the second kind K_ν and Hankel functions (Bessel functions of the third kind) $H_\nu^{(1)}, H_\nu^{(2)}$ have been studied by Baricz et al. [24], while Neumann series of the second type were considered by Baricz and Pogány in somewhat different purposes in [20, 21]; see also [134]. An important role has throughout of this paper the Sonin–Gubler formula (5.1). Thus, under extended Neumann series (of Bessel J_ν see [333]) we mean the following

$$\mathfrak{N}_{\mu,\eta}^{\mathscr{B}}(x) := \sum_{n\geq 1} \beta_n \mathscr{B}_{\mu n+\eta}(ax),$$

where $\mathscr{B}_\nu$ is one of the functions I_ν and $\mathbf{L}_\nu$. Integral representation discussions began very recently with the introductory article by Pogány and Süli [249], which gives an exhaustive references list concerning physical applications too; see also [24]. In Sect. 5.2.1 we will concentrate to the Neumann series

$$\mathfrak{N}_{\mu,\eta}(x) := \sum_{n\geq 1} \beta_n I_{\mu n+\eta}(ax)\,. \tag{5.2}$$

Secondly, Kapteyn series of the first type [145, 146, 217] are of the form

$$\mathfrak{K}_{\nu,\mu}^{\mathscr{B}}(z) := \sum_{n\geq 1} \alpha_n \mathscr{B}_{\rho+\mu n}\left((\sigma+\nu n)z\right);$$

more details about Kapteyn and Kapteyn-type series for Bessel functions can be found also in [21, 23, 69, 308] and the references therein. Here we will consider specific Kapteyn-type series of the following form:

$$\mathfrak{K}_{\nu,\mu}^{\alpha}(x) := \sum_{n\geq 1} \frac{\alpha_n}{n^\mu}\,\left(I_{\nu n}(xn) - \mathbf{L}_{\nu n}(xn)\right); \tag{5.3}$$

this series appear as auxiliary expression in the fourth section of this chapter. Thanks to Sonin–Gubler formula (5.1) we give an alternative proof for integral representation of $\mathfrak{K}_{\nu,\mu}^{\alpha}(x)$, see Sect. 5.4. Thirdly, under Schlömilch series [279, pp. 155–158] (Schlömilch considered only cases $\mu\in\{0,1\}$), we understand the functions series

$$\mathfrak{S}_{\mu,\nu}^{\mathscr{B}}(z) := \sum_{n\geq 1} \alpha_n\, \mathscr{B}_\mu\left((\nu+n)z\right).$$

Integral representation are recently obtained for this series in [133], summations are given in [316]. Our attention is focused currently to

$$\mathfrak{S}_{\mu,\nu}^{I,\mathbf{L}}(z) := \sum_{n\geq 1} \frac{\alpha_n}{n^\mu}\left(I_\nu(xn) - \mathbf{L}_\nu(xn)\right)\,. \tag{5.4}$$

The next generalization is suggested by the theory of Fourier series, and the functions which naturally come under consideration instead of the classical sine and cosine, are the Bessel functions of the first kind and Struve's functions. The next type series considered here we call generalized Schlömilch series [333, p. 622], [128, p. 1803]

$$\frac{a_0}{2\Gamma(\nu+1)} + \left(\frac{x}{2}\right)^{-\nu} \sum_{n\geq 1} \frac{a_n J_\nu(nx) + b_n \mathbf{H}_\nu(nx)}{n^\nu}\,.$$

For further subsequent generalizations consult e.g. Bondarenko's recent article [38] and the references therein and Miller's multidimensional expansion [197]. A set of summation formulae of Schlömilch series for Bessel function of the first kind can be found in the literature, such as the Nielsen formula [333, p. 636]; further, we have [313, p. 65], also consult [90, 236, 265, 316, 341, 342]. Similar summations, for Schlömilch series of Struve function, have been given by Miller [198], consult [316] too.

Further, we are interested in a specific variant of generalized Schlömilch series in which $J_\nu, \mathbf{H}_\nu$ are exchanged by I_ν and $\mathbf{L}_\nu$ respectively, when a_n, b_n are of the form $a_0 = 0$, $a_n = 2^{-\nu} n^{\nu-\mu} x^\nu = -b_n$, $\mu \geq \nu > 0$, which results in

$$\mathfrak{T}^{I,\mathbf{L}}_{\nu,\mu}(x) := \sum_{n\geq 1} \frac{I_\nu(nx) - \mathbf{L}_\nu(nx)}{n^\mu}.$$

Its alternating variant $\widetilde{\mathfrak{T}}^{I,\mathbf{L}}_{\nu,\mu}(x)$ we perform setting $(-1)^{n-1}a_n \mapsto a_n$, where $n \in \{0, 1, \dots\}$:

$$\widetilde{\mathfrak{T}}^{I,\mathbf{L}}_{\nu,\mu}(x) := \sum_{n\geq 1} \frac{(-1)^{n-1}}{n^\mu} \left(I_\nu(nx) - \mathbf{L}_\nu(nx)\right) .$$

Summations of these series are one of tools in obtaining explicit expressions for integrals containing Butzer–Flocke–Hauss complete Omega-function $\Omega(x)$ [44–46] and Mathieu series $S(x), \widetilde{S}(x)$ [187, 252].

Let us also mention that summation results in form of a double definite integral representation for $\mathfrak{S}^J_{\mu,\nu}(z)$, achieved *via* Kapteyn-series, have been recently derived in [131] (see Sect. 4.1).

Finally, we mention that except the Sonin–Gubler formula (5.1) another main tool we refer to is the Cahen's formula on the Laplace integral representation of Dirichlet series.

5.2.1 $\mathbf{L}_\nu$ as a Neumann Series of Modified Bessel I Functions

Let us observe the well-known formulae [230, Eqs. 11.4.18–19–20]

$$\mathbf{H}_\nu(z) = \begin{cases} \dfrac{4}{\sqrt{\pi}\, \Gamma\left(\nu + \frac{1}{2}\right)} \displaystyle\sum_{n\geq 0} \dfrac{(2n+\nu+1)\Gamma(n+\nu+1)}{n!(2n+1)(2n+2\nu+1)} J_{2n+\nu+1}(z) \\[2ex] \sqrt{\dfrac{z}{2\pi}} \displaystyle\sum_{n\geq 0} \dfrac{\left(\frac{z}{2}\right)^n}{n!(n+\frac{1}{2})} J_{n+\nu+\frac{1}{2}}(z) \\[2ex] \dfrac{\left(\frac{z}{2}\right)^{\nu+\frac{1}{2}}}{\Gamma\left(\nu+\frac{1}{2}\right)} \displaystyle\sum_{n\geq 0} \dfrac{\left(\frac{z}{2}\right)^n}{n!\left(n+\nu+\frac{1}{2}\right)} J_{n+\frac{1}{2}}(z) \end{cases},$$

where the first formula is valid for $-\nu \notin \mathbb{N}$. So, having in mind that $\mathbf{L}_\nu(z) = -\mathrm{i}^{1-\nu}\mathbf{H}_\nu(\mathrm{i}z)$ and $J_\nu(\mathrm{i}z) = \mathrm{i}^\nu I_\nu(z)$, we immediately conclude that

$$\mathbf{L}_\nu(z) = \begin{cases} \dfrac{4}{\sqrt{\pi}} \displaystyle\sum_{n\geq 0} \dfrac{(-1)^n(2n+\nu+1)\Gamma(n+\nu+1)}{\Gamma\left(\nu+\frac{1}{2}\right)(2n+1)(2n+2\nu+1)\,n!} I_{2n+\nu+1}(z) \\ \sqrt{\dfrac{z}{2\pi}} \displaystyle\sum_{n\geq 0} \dfrac{\left(-\frac{z}{2}\right)^n}{n!\left(n+\frac{1}{2}\right)} I_{n+\nu+\frac{1}{2}}(z) \\ \dfrac{\left(\frac{z}{2}\right)^{\nu+\frac{1}{2}}}{\Gamma\left(\nu+\frac{1}{2}\right)} \displaystyle\sum_{n\geq 0} \dfrac{\left(-\frac{z}{2}\right)^n}{n!\left(n+\nu+\frac{1}{2}\right)} I_{n+\frac{1}{2}}(z) \end{cases} . \tag{5.5}$$

However, all three series expansions we recognize as Neumann-series built by modified Bessel functions of the first kind. This kind of series have been intensively studied very recently by the authors in [24].

Exploiting the appropriate findings, we give new integral expressions for the modified Struve function $\mathbf{L}_\nu$.

First, let us modestly generalize [24, Theorem 2.1] which concerns $\mathfrak{N}_{1,\nu}(x)$, to integral expression for $\mathfrak{N}_{\mu,\eta}$ defined by (5.2), following the same procedure as in [24].

Theorem 5.1 (Baricz and Pogány [18]) *Let* $\beta \in \mathrm{C}^1(\mathbb{R}_+)$, $\beta|_{\mathbb{N}} = \big(\beta_n\big)_{n\geq 1}$, $\mu > 0$ *and assume that*

$$\lim_{n\to\infty} \frac{|\beta_n|^{\frac{1}{\mu n}}}{n} < \frac{\mu}{\mathrm{e}} . \tag{5.6}$$

Then, for μ, η *such that*

$$\min\{\eta + \tfrac{3}{2}, \mu + \eta + 1\} > 0$$

and

$$x \in \left(0, 2\,\min\left\{1, \left((e/\mu)^\mu \limsup_{n\to\infty} n^{-\mu}|\beta_n|^{\frac{1}{n}}\right)^{-1}\right\}\right) := \mathscr{I}_\beta ,$$

we have the integral representation

$$\mathfrak{N}_{\mu,\eta}(x) = -\int_1^\infty \int_0^{[u]} \frac{\partial}{\partial u}\big(\Gamma\left(\mu u + \eta + \tfrac{1}{2}\right) I_{\mu u+\eta}(x)\big)\, \mathfrak{d}_s \left(\frac{\beta(s)}{\Gamma\left(\mu s + \eta + \frac{1}{2}\right)}\right) \mathrm{d}u\mathrm{d}s . \tag{5.7}$$

Proof The proof is a copy of the proving procedure delivered for Theorem 2.7 (i.e. [24, Theorem 2.1]). The only exception is to refine the convergence condition upon $\mathfrak{N}_{\mu,\eta}(x)$. By the bound [14, p. 583]:

$$I_\nu(x) < \frac{\left(\frac{x}{2}\right)^\nu}{\Gamma(\nu+1)}\, e^{\frac{x^2}{4(\nu+1)}},$$

where $x > 0$ and $\nu + 1 > 0$, we have

$$\left|\mathfrak{N}_{\mu,\eta}(x)\right| < \left(\frac{x}{2}\right)^{\mu+\eta} e^{\frac{x^2}{4(\mu+\eta+1)}} \sum_{n\geq 1} \frac{|\beta_n|}{\Gamma(\mu n+\eta+1)},$$

so, the absolute convergence of the right hand side series suffices for the finiteness of $\mathfrak{N}_{\mu,\eta}(x)$ on $\mathscr{I}_\beta$. However, condition (5.6) ensures the absolute convergence by the Cauchy convergence criterion.

The remaining part of the proof mimics the one performed for Theorem 2.7, having in mind that $\mu = 1$ reduces Theorem 5.1 to the ancestor result Theorem 2.7. □

Theorem 5.2 (Baricz and Pogány [18]) *If* $\nu > 0$ *and* $x \in (0, 2)$, *then we have the integral representation*

$$\begin{aligned}\mathbf{L}_\nu(x) &- \frac{2\Gamma(\nu+2)}{\sqrt{\pi}\,\Gamma\left(\nu+\frac{3}{2}\right)}\, I_{\nu+1}(x)\\ &= \int_1^\infty \int_0^{[u]} \frac{\partial}{\partial u}\left(\Gamma\left(2u+\nu+\tfrac{3}{2}\right) I_{2u+\nu+1}(x)\right) \mathfrak{d}_s \left(\frac{\beta(s)}{\Gamma\left(2s+\nu+\frac{3}{2}\right)}\right) \mathrm{d}u\, \mathrm{d}s\,,\end{aligned} \tag{5.8}$$

where

$$\beta(s) = -\frac{\mathrm{e}^{\mathrm{i}\pi s}(2s+\nu+1)\Gamma(s+\nu+1)}{\sqrt{\pi}\,\Gamma\left(\nu+\frac{1}{2}\right)\,\Gamma(s+1)\,\left(s+\frac{1}{2}\right)\left(s+\nu+\frac{1}{2}\right)}.$$

Proof Consider the first Neumann sum expansion of $\mathbf{L}_\nu(x)$ in (5.5), that is

$$\begin{aligned}\mathbf{L}_\nu(x) &= \frac{4}{\sqrt{\pi}\,\Gamma\left(\nu+\frac{1}{2}\right)} \sum_{n\geq 0} \frac{(-1)^n(2n+\nu+1)\Gamma(n+\nu+1)}{n!(2n+1)(2n+2\nu+1)}\, I_{2n+\nu+1}(x)\\ &= \frac{2\Gamma(\nu+2)}{\sqrt{\pi}\,\Gamma\left(\nu+\frac{3}{2}\right)}\, I_{\nu+1}(x) - \sum_{n\geq 1} \beta_n\, I_{2n+\nu+1}(x)\,,\end{aligned}$$

in which we specify

$$\beta_n = \frac{(-1)^{n-1}(2n+\nu+1)\Gamma(n+\nu+1)}{\sqrt{\pi}\,\Gamma\left(\nu+\frac{1}{2}\right)\,\Gamma(n+1)\,\left(n+\frac{1}{2}\right)\left(n+\nu+\frac{1}{2}\right)}\,.$$

Observe that

$$\mathbf{L}_\nu(x) = \frac{2\Gamma(\nu+2)}{\sqrt{\pi}\,\Gamma\left(\nu+\frac{3}{2}\right)}\,I_{\nu+1}(x) - \mathfrak{N}_{2,\nu+1}(x).$$

Since

$$|\beta(s)| \sim \frac{2s^{\nu-2}}{\sqrt{\pi}\,\Gamma\left(\nu+\frac{1}{2}\right)}, \qquad s\to\infty\,,$$

we deduce (by means of Theorem 5.1) that (5.8) is valid for $x \in \mathscr{I}_\beta = (0,2)$. □

Theorem 5.3 (Baricz and Pogány [18]) *For* $\nu+2>0$ *and* $x\in(0,2)$ *we have the integral representation*

$$\mathbf{L}_\nu(x) - \sqrt{\frac{2x}{\pi}}\,I_{\nu+\frac{1}{2}}(x) = \int_1^\infty\int_0^{[u]}\frac{\partial}{\partial u}\left(\Gamma(u+\nu+1)\,I_{u+\nu+\frac{1}{2}}(x)\right)\mathfrak{d}_s\left(\frac{\beta(s)}{\Gamma(s+\nu+1)}\right)\mathrm{d}u\,\mathrm{d}s\,, \tag{5.9}$$

where

$$\beta(s) = -\sqrt{\frac{x}{2\pi}}\,\frac{\mathrm{e}^{\mathrm{i}\pi s}\left(\frac{x}{2}\right)^s}{\Gamma(s+1)\,(s+\frac{1}{2})}\,.$$

Proof Let us observe now the second Neumann sum expansion of $\mathbf{L}_\nu(x)$ in (5.5):

$$\begin{aligned}\mathbf{L}_\nu(x) &= \sqrt{\frac{x}{2\pi}}\sum_{n\ge0}\frac{\left(-\frac{x}{2}\right)^n}{n!\left(n+\frac{1}{2}\right)}\,I_{n+\nu+\frac{1}{2}}(x)\\ &= \sqrt{\frac{2x}{\pi}}\,I_{\nu+\frac{1}{2}}(x) - \sqrt{\frac{x}{2\pi}}\sum_{n\ge1}\frac{(-1)^{n-1}\left(\frac{x}{2}\right)^n}{n!\left(n+\frac{1}{2}\right)}\,I_{n+\nu+\frac{1}{2}}(x)\,.\end{aligned}$$

In other words,

$$\mathbf{L}_\nu(x) = \sqrt{\frac{2x}{\pi}}\,I_{\nu+\frac{1}{2}}(x) - \mathfrak{N}_{1,\nu+\frac{1}{2}}(x)$$

in which we specify

$$\beta(s) = -\sqrt{\frac{x}{2\pi}}\,\frac{e^{i\pi s}\left(\frac{x}{2}\right)^s}{\Gamma(s+1)\left(s+\frac{1}{2}\right)}\,.$$

The convergence condition (5.6) reduces to the behavior of the auxiliary series

$$\sum_{n\geq 0}\frac{|\beta_n|}{\Gamma\left(n+\nu+\frac{1}{2}\right)} \sim \sqrt{\frac{2x}{\pi}}\;{}_1F_2\left[\begin{matrix}\frac{1}{2}\\ \frac{3}{2},\ \nu+\frac{1}{2}\end{matrix}\,\Bigg|\,\frac{|x|}{2}\right],$$

which converges for all bounded $x \in \mathbb{C}$, unconditionally upon ν.

However, for $\nu > -2$ we have the integral expression (2.34) [333, p. 79]. This was used in the proof of the ancestor result (5.7), see [24, Theorem 2.1].

Now, we apply Theorem 5.1 and immediately conclude that (5.9) is valid for $x \in \mathscr{I}_\beta = (0,2)$. □

The third formula in (5.5) one reduces to the case $\mathfrak{N}_{1,\frac{1}{2}}(x)$. Concerning this case we remark the proof is omitted because the slightly modified derivation procedure used for getting (5.9) directly implies the above asserted integral expression.

Theorem 5.4 (Baricz and Pogány [18]) *Assume that*

$$\nu + 2 > 0 \qquad \text{and} \qquad x \in (0,2)\,.$$

Then we have the integral representation

$$\mathbf{L}_\nu(x) - \frac{x^\nu\ \sinh x}{2^\nu\sqrt{\pi}\,\Gamma\left(\nu+\frac{3}{2}\right)} = \int_1^\infty\int_0^{[u]}\frac{\partial}{\partial u}\left(\Gamma\,(u+1)\ I_{u+\frac{1}{2}}(x)\right)\mathfrak{d}_s\left(\frac{\beta(s)}{\Gamma\,(s+1)}\right)\mathrm{d}u\,\mathrm{d}s,$$

where

$$\beta(s) = -\frac{\left(\frac{x}{2}\right)^{\nu+\frac{1}{2}}}{\Gamma\left(\nu+\frac{1}{2}\right)}\,\frac{e^{i\pi s}\left(\frac{x}{2}\right)^s}{\Gamma(s+1)\left(s+\nu+\frac{1}{2}\right)}\,.$$

Now, applying the integral representation (2.34) we derive another integral expression for $\mathbf{L}_\nu(x)$ in terms of hypergeometric functions in the integrand.

Theorem 5.5 (Baricz and Pogány [18]) *Let* $\nu > -\frac{1}{2}$. *Then for* $x > 0$ *we have*

$$\mathbf{L}_\nu(x) = \frac{x^{\nu+1}\Gamma(\nu+2)}{\sqrt{\pi}\,2^{2\nu-\frac{1}{2}}\Gamma\left(\nu+\frac{3}{2}\right)\Gamma\left(\frac{\nu}{2}+\frac{3}{4}\right)\Gamma\left(\frac{\nu}{2}+\frac{5}{4}\right)}\int_0^1(1-t^2)^{\nu+\frac{1}{2}}\cosh(xt)$$
$$\times\;{}_4F_5\left[\begin{matrix}\frac{1}{2},\ \frac{\nu+3}{2},\ \nu+\frac{1}{2},\ \nu+1\\ \frac{3}{2},\frac{\nu+1}{2},\frac{\nu}{2}+\frac{3}{4},\frac{\nu}{2}+\frac{5}{4},\nu+\frac{3}{4}\end{matrix}\,\Bigg|\,-\frac{x^2}{16}(1-t^2)^2\right]\mathrm{d}t\,.$$

Proof Consider the first Bessel function series expansion for $\mathbf{L}_\nu(x)$ given in (5.5). Applying *mutatis mutandis* the integral representation formula (2.34), the Pochhammer symbol technique, the familiar formula $(A)_n(n + A) = A(A + 1)_n$, $n \in \{0, 1, \dots\}$, and the Legendre's duplication formula (1.3) to the summands, we get the chain of equivalent legitimate transformations:

$$\begin{aligned}\mathbf{L}_\nu(x) &= \frac{8}{\sqrt{\pi}\,\Gamma\left(\nu+\frac{1}{2}\right)} \sum_{n\ge 0} \frac{(-1)^n(2n+\nu+1)\Gamma(n+\nu+1)}{n!(2n+1)(2n+2\nu+1)} \frac{2\left(\frac{x}{2}\right)^{2n+\nu+1}}{\sqrt{\pi}\,\Gamma\left(2n+\nu+\frac{3}{2}\right)} \\ &\quad\times \int_0^1 (1-t^2)^{2n+\nu+\frac{1}{2}} \cosh(xt)\,\mathrm{d}t \\ &= \frac{4\left(\frac{x}{2}\right)^{\nu+1}}{\pi\Gamma\left(\nu+\frac{1}{2}\right)} \int_0^1 (1-t^2)^{\nu+\frac{1}{2}} \cosh(xt) \\ &\quad\times \sum_{n\ge 0} \frac{\left(n+\frac{\nu+1}{2}\right)\Gamma(n+\nu+1)\left[-\frac{x^2}{4}(1-t^2)^2\right]^n}{\left(n+\frac{1}{2}\right)\left(n+\nu+\frac{1}{2}\right)\Gamma\left(2n+\nu+\frac{3}{2}\right)n!}\,\mathrm{d}t \\ &= \frac{4\left(\frac{x}{2}\right)^{\nu+1}(\nu+1)\Gamma(\nu+1)}{\sqrt{\pi}\left(\nu+\frac{1}{2}\right)\Gamma\left(\nu+\frac{1}{2}\right)\Gamma\left(\frac{\nu}{2}+\frac{3}{4}\right)\Gamma\left(\frac{\nu}{2}+\frac{5}{4}\right)} \int_0^1 (1-t^2)^{\nu+\frac{1}{2}} \cosh(xt) \\ &\quad\times \sum_{n\ge 0} \frac{(\frac{1}{2})_n(\frac{\nu+3}{2})_n(\nu+\frac{1}{2})_n(\nu+1)_n\left[-\frac{x^2}{16}(1-t^2)^2\right]^n}{\left(\frac{3}{2}\right)_n\left(\frac{\nu+1}{2}\right)_n\left(\frac{\nu}{2}+\frac{3}{4}\right)_n\left(\frac{\nu}{2}+\frac{5}{4}\right)_n\left(\nu+\frac{3}{4}\right)_n n!}\,\mathrm{d}t,\end{aligned}$$

which proves the assertion. □

By virtue of similar manipulations presented above, we conclude the following results.

Theorem 5.6 (Baricz and Pogány [18]) *Let* $\nu > -\frac{1}{2}$ *and* $x > 0$. *Then there holds*

$$\mathbf{L}_\nu(x) = \begin{cases} \dfrac{x^{\nu+1}}{2^{\nu-1}\pi\,\Gamma(\nu+1)} \displaystyle\int_0^1 (1-t^2)^\nu \cosh(xt)\, {}_1F_2\left[\begin{matrix}\frac{1}{2}\\ \frac{3}{2},\ \nu+1\end{matrix}\,\middle|\, -\frac{x^2}{4}(1-t^2)\right]\mathrm{d}t, \\ \dfrac{x^{\nu+1}}{\sqrt{\pi}\,2^{\nu+\frac{1}{2}}\,\Gamma\left(\nu+\frac{1}{2}\right)} \displaystyle\int_0^1 \cosh(xt)\, {}_1F_2\left[\begin{matrix}\nu+\frac{1}{2}\\ 1,\ \nu+\frac{3}{2}\end{matrix}\,\middle|\, -\frac{x^2}{4}(1-t^2)\right]\mathrm{d}t.\end{cases}$$

The proof of Theorem 5.6 follows from the same proving procedure as the previous theorem but now considering the second and third series expansion results in (5.5), so we shall omit the proofs of these integral representations.

5.3 Integrals of $\Omega(x)$-Function and Mathieu Series *via* $\mathfrak{T}_\nu^{I,\mathbf{L}}(x)$

By virtue of the Sonin–Gubler formula (5.1) we establish the convergence conditions for the generalized Schlömilch series $\mathfrak{T}_{\nu,\mu}^{I,\mathbf{L}}(x)$ and $\widetilde{\mathfrak{T}}_{\nu,\mu}^{I,\mathbf{L}}(x)$. As for n enough large we have

$$I_\nu(an) - \mathbf{L}_\nu(an) = \frac{2n^{\nu-1}}{\pi} \int_0^\infty \frac{J_\nu(ax)\,\mathrm{d}x}{(1+n^{-2}x^2)x^\nu} = \mathscr{O}\left(n^{\nu-1}\right), \tag{5.10}$$

we immediately conclude that the following equi-convergences hold true

$$\mathfrak{T}_{\nu,\mu}^{I,\mathbf{L}}(x) \sim \zeta(\mu-\nu+1), \qquad \widetilde{\mathfrak{T}}_{\nu,\mu}^{I,\mathbf{L}}(x) \sim \eta(\mu-\nu+1),$$

that is, $\mathfrak{T}_{\nu,\mu}^{I,\mathbf{L}}(x)$ converges for $\mu > \nu > 0$, while $\widetilde{\mathfrak{T}}_{\nu,\mu}^{I,\mathbf{L}}(x)$ converges for $\mu + 1 > \nu > 0$. On the other hand, we connect $\widetilde{\mathfrak{T}}_{\nu,\nu}^{I,\mathbf{L}}(x)$ and the Butzer–Flocke–Hauss (BFH) complete Omega function [44, Definition 7.1]

$$\Omega(w) = 2\int_{0+}^{\frac{1}{2}} \sinh(wu)\cot(\pi u)\,\mathrm{d}u, \qquad w \in \mathbb{C}.$$

By the Hilbert transform terminology, $\Omega(w)$ is the Hilbert transform $\mathscr{H}(\mathrm{e}^{-wx})_1(0)$ at 0 of the 1-periodic function $(\mathrm{e}^{-wx})_1$ defined by the periodic continuation of the following exponential function [44, p. 67]: e^{-xw}, $x \in \left[-\frac{1}{2}, \frac{1}{2}\right)$, $w \in \mathbb{C}$, that is,

$$\mathscr{H}(\mathrm{e}^{-xw})_1(0) := \mathrm{P.V.}\int_{-\frac{1}{2}}^{\frac{1}{2}} \mathrm{e}^{wu}\cot(\pi u)\,\mathrm{d}u \equiv \Omega(w)$$

where the integral is to be understood in the sense of Cauchy's Principal Value at zero, see e.g. [46, 247].

On the other side by differentiating once (5.1) with respect to n we get a tool to obtain Mathieu series $S(x)$ (introduced by Mathieu [187]) and its alternating variant $\widetilde{S}(x)$ (introduced by Pogány et al. [252]), which are defined as follows

$$S(x) = \sum_{n\ge1} \frac{2n}{(x^2+n^2)^2}, \qquad \widetilde{S}(x) = \sum_{n\ge1} \frac{2(-1)^{n-1}n}{(x^2+n^2)^2}.$$

Closed integral expression for $S(r)$ was considered by Emersleben [75] and subsequently by Elbert [74], while for $\widetilde{S}_\mu(x)$ integral representation has been given by Pogány et al. [252]:

$$S(x) = \frac{1}{x}\int_0^\infty \frac{t\,\sin(xt)}{\mathrm{e}^t-1}\,\mathrm{d}t, \tag{5.11}$$

$$\widetilde{S}(x) = \frac{1}{x}\int_0^\infty \frac{t\,\sin(xt)}{\mathrm{e}^t+1}\,\mathrm{d}t.$$

Another kind integral expressions for underlying Mathieu series can be found in [252].

Theorem 5.7 (Baricz and Pogány [18]) *Assume that* $\Re(\nu) > 0$ *and* $a > 0$. *Then we have*

$$\int_0^\infty \frac{J_\nu(ax)\,\Omega(2\pi x)}{x^\nu\,\sinh(\pi x)}\,\mathrm{d}x = \nu\int_0^\infty \mathrm{e}^{-\nu t}\int_0^{[\mathrm{e}^t]} \mathfrak{d}_u\left(\mathrm{e}^{\mathrm{i}\pi u}\left(\mathbf{L}_\nu(au) - I_\nu(au)\right)\right)\,\mathrm{d}t\,\mathrm{d}u\,.$$

Proof When we multiply (5.1) by $(-1)^{n-1}n$ and sum up all three series with respect to $n \in \mathbb{N}$, the following partial-fraction representation of the Omega function [44, Theorem 1.3]

$$\frac{\pi\,\Omega(2\pi w)}{\sinh(\pi w)} = \sum_{n\ge1}\frac{2(-1)^{n-1}\,n}{n^2+w^2}$$

immediately gives

$$\int_0^\infty \frac{J_\nu(ax)\,\Omega(2\pi x)}{x^\nu\,\sinh(\pi x)}\,\mathrm{d}x = \sum_{n\ge1}(-1)^{n-1}n^{-\nu}\,\left(I_\nu(an) - \mathbf{L}_\nu(an)\right) = \widetilde{\mathfrak{T}}^{I,\mathbf{L}}_{\nu,\nu}(a)\,.$$

We recognize the right-hand-side sums as Dirichlet series of I_ν and $\mathbf{L}_\nu$, respectively. Being

$$\sum_{n\ge1}(-1)^{n-1}n^{-\nu}I_\nu(an) = \sum_{n\ge1}\mathrm{e}^{\mathrm{i}\pi(n-1)}I_\nu(an)\mathrm{e}^{-\nu\log n}, \qquad \Re(\nu) > 0\,,$$

we get

$$\sum_{n\ge1}(-1)^{n-1}n^{-\nu}I_\nu(an) = \nu\int_0^\infty \mathrm{e}^{-\nu t}\sum_{n:\log n\le t}\mathrm{e}^{\mathrm{i}\pi(n-1)}I_\nu(an)\,\mathrm{d}t.$$

So, making use of the Euler–Maclaurin summation to the Cahen's formula (1.15) we deduce

$$\sum_{n\ge1}(-1)^{n-1}n^{-\nu}I_\nu(an) = -\nu\int_0^\infty\int_0^{[\mathrm{e}^t]}\mathrm{e}^{-\nu t}\,\mathfrak{d}_u\left(\mathrm{e}^{\mathrm{i}\pi u}I_\nu(au)\right)\,\mathrm{d}t\,\mathrm{d}u;$$

and repeating the procedure to the second Dirichlet series containing $\mathbf{L}_\nu(an)$, the proof is complete. □

The next result concerns a hypergeometric integral, which we integrate by means of Schlömilch series of modified Bessel and modified Struve functions.

Theorem 5.8 (Baricz and Pogány [18]) *Let* $\Re(\nu) > 0$ *and* $a > 0$. *Then we have*

$$\int_0^\infty J_\nu(ax)\,S(x)\frac{\mathrm{d}x}{x^\nu} = \frac{\sqrt{\pi}\,a^{\nu+2}}{2^{\nu+1}\Gamma(\nu+\frac{1}{2})}\int_0^1 \frac{t^2}{\mathrm{e}^{at}-1}\,{}_2F_1\left[\left.\begin{array}{c}\frac{1}{2},\frac{1}{2}-\nu\\ \frac{3}{2}\end{array}\right| t^2\right]\mathrm{d}t$$
$$+\frac{\pi a^\nu\,[\mathrm{Li}_2(\mathrm{e}^{-a})+a\,\mathrm{Li}_1(\mathrm{e}^{-a})]}{2^{\nu+1}\Gamma(\nu+1)},$$

where $\mathrm{Li}_\alpha(z)$ *stands for the* dilogarithm function.

Proof Differentiating (5.1) with respect to n, we get

$$\int_0^\infty \frac{2n\,J_\nu(ax)}{(x^2+n^2)^2}\,\frac{\mathrm{d}x}{x^\nu} = \frac{\pi(\nu+1)}{2n^{\nu+2}}\,(I_\nu(an)-\mathbf{L}_\nu(an)) - \frac{a\pi}{2n^{\nu+1}}\left(I_\nu'(an)-\mathbf{L}_\nu'(an)\right).$$

Summing up this relation with respect to positive integers $n \in \mathbb{N}$, we have

$$\kappa_\nu(a) := \int_0^\infty J_\nu(ax)\,S(x)\frac{\mathrm{d}x}{x^\nu} = \frac{\pi(\nu+1)}{2}\sum_{n\ge 1}\frac{I_\nu(an)-\mathbf{L}_\nu(an)}{n^{\nu+2}}$$
$$-\frac{a\pi}{2}\sum_{n\ge1}\frac{I_\nu'(an)-\mathbf{L}_\nu'(an)}{n^{\nu+1}} = \frac{\pi(\nu+1)}{2}\,\mathfrak{T}_{\nu,\nu+2}^{I,\mathbf{L}}(a) - \frac{a\pi}{2}\,\frac{\mathrm{d}}{\mathrm{d}a}\,\mathfrak{T}_{\nu,\nu+2}^{I,\mathbf{L}}(a)\,.$$

By the Emersleben–Elbert formula (5.11) we conclude that

$$\kappa_\nu(a) = \int_0^\infty J_\nu(ax)\,S(x)\frac{\mathrm{d}x}{x^\nu} = \int_0^\infty \frac{t}{\mathrm{e}^t-1}\left(\int_0^\infty \frac{J_\nu(ax)\,\sin(xt)}{x^{\nu+1}}\,\mathrm{d}x\right)\mathrm{d}t\,.$$

Expressing the sine *via* $J_{\frac{1}{2}}$, we get that the inner-most integral equals

$$\int_0^\infty \frac{J_\nu(ax)\,\sin(xt)}{x^{\nu+1}}\,\mathrm{d}x = \sqrt{\frac{\pi t}{2}}\int_0^\infty \frac{J_\nu(ax)J_{\frac{1}{2}}(tx)}{x^{\nu+\frac{1}{2}}}\,\mathrm{d}x\,. \tag{5.12}$$

Now, we apply the Weber–Sonin–Schafheitlin formula [333, §**13.41**] for $\lambda = \nu + \frac{1}{2}$, which reduces to

$$\int_0^\infty J_\nu(ax)J_{\frac{1}{2}}(tx)x^{-\nu-\frac{1}{2}}\,\mathrm{d}x = \begin{cases} \dfrac{a^\nu\sqrt{\pi}}{2^{\nu+\frac{1}{2}}\sqrt{t}\,\Gamma(\nu+1)}, & 0 < a \le t,\\[2ex] \dfrac{a^{\nu-1}\sqrt{t}}{2^{\nu+\frac{1}{2}}\Gamma\left(\nu+\frac{1}{2}\right)}\,{}_2F_1\left[\left.\begin{array}{c}\frac{1}{2},\frac{1}{2}-\nu\\ \frac{3}{2}\end{array}\right| \frac{t^2}{a^2}\right], & 0 < t < a.\end{cases}$$

Accordingly, (5.12) becomes

$$\begin{aligned}\kappa_\nu(a) &= \frac{\sqrt{\pi}\, a^{\nu-1}}{2^{\nu+1}\Gamma(\nu+\frac{1}{2})}\int_0^a \frac{t^2}{\mathrm{e}^t-1}\, {}_2F_1\Big[\begin{matrix}\frac{1}{2}, \frac{1}{2}-\nu\\ \frac{3}{2}\end{matrix}\Big|\frac{t^2}{a^2}\Big]\,\mathrm{d}t \\ &\quad + \frac{\pi a^\nu}{2^{\nu+1}\Gamma(\nu+1)}\int_a^\infty \frac{t}{\mathrm{e}^t-1}\,\mathrm{d}t \\ &= \frac{\sqrt{\pi}\, a^{\nu+2}}{2^{\nu+1}\Gamma\left(\nu+\frac{1}{2}\right)}\int_0^1 \frac{t^2}{\mathrm{e}^{at}-1}\, {}_2F_1\Big[\begin{matrix}\frac{1}{2}, \frac{1}{2}-\nu\\ \frac{3}{2}\end{matrix}\Big|t^2\Big]\,\mathrm{d}t \\ &\quad + \frac{\pi a^\nu}{2^{\nu+1}\Gamma(\nu+1)}\int_0^\infty \frac{t+a}{\mathrm{e}^{t+a}-1}\,\mathrm{d}t \\ &= \frac{\sqrt{\pi}\, a^{\nu+2}}{2^{\nu+1}\Gamma\left(\nu+\frac{1}{2}\right)}\int_0^1 \frac{t^2}{\mathrm{e}^{at}-1}\, {}_2F_1\Big[\begin{matrix}\frac{1}{2}, \frac{1}{2}-\nu\\ \frac{3}{2}\end{matrix}\Big|t^2\Big]\,\mathrm{d}t \\ &\quad + \frac{\pi a^\nu}{2^{\nu+1}\Gamma(\nu+1)}\left[\mathrm{Li}_2(\mathrm{e}^{-a}) + a\,\mathrm{Li}_1(\mathrm{e}^{-a})\right],\end{aligned}$$

where the dilogarithm $\mathrm{Li}_\alpha(z) = \sum_{n\ge1} z^n n^{-\alpha}$, $|z|\le 1$, has the integral representation

$$\mathrm{Li}_\alpha(z) = \frac{z}{\Gamma(\alpha)}\int_0^\infty \frac{t^{\alpha-1}}{\mathrm{e}^t - z}\,\mathrm{d}t\,, \qquad \Re(\alpha) > 0.$$

This completes the proof. □

5.4 Differential Equations for Kapteyn and Schlömilch Series of I_ν, $\mathbf{L}_\nu$

Kapteyn series of Bessel functions were introduced by Kapteyn [145, 146], and were considered and discussed in details by Nielsen [217] and Watson [333], who devoted a whole section of his celebrated monograph to this theme. Recently, Baricz, Jankov and Pogány obtained integral representation and ordinary differential equations descriptions and related results for real variable Kapteyn series [23, 133].

Now, we will consider the Kapteyn series built by modified Bessel functions of the first kind, and modified Struve functions

$$\mathfrak{K}_{\nu,\mu}^{\alpha}(x) = \sum_{n\ge1}\frac{\alpha_n}{n^\mu}\left(I_{\nu n}(xn) - \mathbf{L}_{\nu n}(xn)\right)\,,$$

where the parameter space includes positive $a > 0$, while sequence $(\alpha_n)_{n\ge1}$ ensures the convergence of $\mathfrak{K}_{\nu,\mu}^{\alpha}(x)$. Our first goal is to establish double definite integral representation formula for $\mathfrak{K}_{\nu,\mu}^{\alpha}(x)$. In this goal we recall the definition of the confluent Fox-Wright generalized hypergeometric function ${}_1\Psi_1^*$ (for the general case ${}_p\Psi_q^*$ consult Sect. 1.12.3):

$$ {}_1\Psi_1^*\left[\begin{array}{c}(a,\rho)\\(b,\sigma)\end{array}\bigg| z\right] = \sum_{n\geq 0} \frac{(a)_{\rho n}}{(b)_{\sigma n}} \frac{z^n}{n!}, \tag{5.13} $$

where $a, b \in \mathbb{C}$, $\rho, \sigma > 0$ and where, as usual, $(\lambda)_\mu$ denotes the Pochhammer symbol. The defining series in (5.13) converges in the whole complex z-plane when $\Delta = \sigma - \rho + 1 > 0$; if $\Delta = 0$, then the series converges for $|z| < \nabla$, where $\nabla := \rho^{-\rho}\sigma^\sigma$.

Theorem 5.9 (Baricz and Pogány [18]) *Let $\mu > \nu > 0$ and let $\alpha \in \mathrm{C}^2(\mathbb{R}_+)$, such that $\alpha\,|_{\mathbb{N}} = (\alpha_n)_{n\geq 1}$. Then for*

$$ x \in \left(0, 2\min\left\{1, \frac{\nu}{\mathrm{e}\,\limsup\limits_{n\to\infty} |\alpha_n|^{\frac{1}{\nu n}}}\right\}\right) := \mathscr{I}_\alpha\,, $$

we have

$$ \begin{aligned} \mathfrak{K}_{\nu,\mu}^{\alpha}(x) = &-\int_1^\infty \int_0^{[t]} \frac{\partial}{\partial t} \frac{\Gamma(\nu t + \frac12)}{\Gamma(\nu t)} \left(\frac{x}{2}\right)^{\nu t} {}_1\Psi_1^\star\left[\begin{array}{c}(\frac12, \frac12)\\(\nu t, 1)\end{array}\bigg| -xt\right] \\ &\times \mathfrak{d}_s \frac{\alpha(s)s^{\nu s-\mu}}{\Gamma\left(\nu s + \frac12\right)}\, \mathrm{d}t\mathrm{d}s\,. \end{aligned} \tag{5.14} $$

Proof The Sonin–Gubler formula enables us to transform the summands of the Kapteyn series $\mathfrak{K}_{\nu,\mu}^{\alpha}(x)$ into

$$ \mathfrak{K}_{\nu,\mu+1}^{\alpha}(x) = \frac{2}{\pi}\int_0^\infty \sum_{n\geq 1} \frac{\alpha_n}{n^{\mu-\nu n}} \frac{J_{\nu n}(xy)}{(y^2+n^2)\,y^{\nu n}}\,\mathrm{d}y\,. $$

Making use of the Gegenbauer's integral expression for J_α [7, p. 204, Eq. (4.7.5)], after some algebra we get

$$ \begin{aligned} \mathfrak{K}_{\nu,\mu+1}^{\alpha}(x) &= \frac{1}{\sqrt{\pi}}\int_0^1 \frac{1}{\sqrt{1-t^2}} \left\{\sum_{n\geq 1} \frac{\alpha_n \left(\frac{x}{2}(1-t^2)\right)^{\nu n}}{n^{\mu-\nu n}\,\Gamma\left(\nu n+\frac12\right)} \int_0^\infty \frac{\cos(xty)}{y^2+n^2}\,\mathrm{d}y\right\} \mathrm{d}t \\ &= \frac{2}{\sqrt{\pi}}\int_0^1 \frac{1}{\sqrt{1-t^2}} \left\{\sum_{n\geq 1} \frac{\alpha_n \left(\frac{x}{2}(1-t^2)\right)^{\nu n}\mathrm{e}^{-xtn}}{n^{\mu-\nu n+1}\,\Gamma\left(\nu n+\frac12\right)}\right\} \mathrm{d}t \\ &= \frac{2}{\sqrt{\pi}}\int_0^1 \frac{1}{\sqrt{1-t^2}}\,\mathscr{D}_\alpha(t)\,\mathrm{d}t\,, \end{aligned} $$

where the inner sum is evidently the following Dirichlet series

$$\mathscr{D}_\alpha(t) = \sum_{n\geq 1} \frac{\alpha_n \, \exp\left\{-n\left(xt + \nu\log\frac{2}{x(1-t^2)}\right)\right\}}{n^{\mu-\nu n+1}\,\Gamma\left(\nu n + \frac{1}{2}\right)}\,,$$

and $p(t) = xt + \nu \log \frac{2}{x(1-t^2)} > 0$ for $x \in (0, 2)$, since p is increasing on $(0, 1)$. By the Cauchy convergence test applied to $\mathscr{D}_\alpha(t)$ we deduce that

$$\left(\frac{\mathrm{e}x}{2\nu}\,(1-t^2)\right)^\nu \mathrm{e}^{-xt} \limsup_{n\to\infty} |\alpha_n|^{\frac{1}{n}} \leq \left(\frac{\mathrm{e}x}{2\nu}\right)^\nu \limsup_{n\to\infty} |\alpha_n|^{\frac{1}{n}} < 1\,,$$

that is, for all $x \in \mathscr{I}_\alpha$ the series converges absolutely and uniformly. By the Cahen's formula (1.15) we have

$$\mathscr{D}_\alpha(t) = \log \mathrm{e}^{xt}\left(\frac{2}{x(1-t^2)}\right)^\nu \int_0^\infty \int_0^{[z]} \left(\left(\frac{x}{2}(1-t^2)\right)^\nu \mathrm{e}^{-xt}\right)^z \cdot \mathfrak{d}_s \frac{\alpha(s)s^{\nu s-\mu-1}}{\Gamma\left(\nu s+\frac{1}{2}\right)}\,\mathrm{d}z\,\mathrm{d}s\,.$$

Thus

$$\mathfrak{K}_{\nu,\mu+1}^{\alpha}(x) = -\frac{1}{\sqrt{\pi}} \int_0^\infty \int_0^{[z]} \mathfrak{d}_s \frac{\alpha(s)s^{\nu s-\mu-1}}{\Gamma\left(\nu s+\frac{1}{2}\right)}\,\Phi_\nu(z)\,\mathrm{d}z\,\mathrm{d}s\,,$$

where the t-integral

$$\Phi_\nu(z) = \int_0^1 \frac{\log \mathrm{e}^{-xt}\left(\frac{x}{2}(1-t^2)\right)^\nu}{\sqrt{1-t^2}} \left(\left(\frac{x}{2}(1-t^2)\right)^\nu \mathrm{e}^{-xt}\right)^z \mathrm{d}t$$

has to be evaluated. After indefinite integration, under definite integral, expanding the exponential term into Maclaurin series, legitimate term-wise integration leads to

$$\begin{aligned}
\int \Phi_\nu(z)\,\mathrm{d}z &= \left(\frac{x}{2}\right)^{\nu z} \int_0^1 (1-t^2)^{\nu z-\frac{1}{2}}\mathrm{e}^{-xzt}\,\mathrm{d}t \\
&= \frac{\sqrt{\pi}\,\Gamma\left(\nu z+\frac{1}{2}\right)}{2\,\Gamma(\nu z)} \left(\frac{x}{2}\right)^{\nu z} \sum_{j\geq 0} \frac{\left(\frac{1}{2}\right)_{\frac{1}{2}j}}{(\nu z)_j}\,\frac{(-xz)^j}{j!} \\
&= \frac{\sqrt{\pi}\,\Gamma\left(\nu z+\frac{1}{2}\right)}{2\,\Gamma(\nu z)} \left(\frac{x}{2}\right)^{\nu z} {}_1\Psi_1^\star\left[\left.\begin{matrix}\left(\frac{1}{2},\frac{1}{2}\right)\\ (\nu z, 1)\end{matrix}\,\right|\, -xz\right].
\end{aligned}$$

Consequently

$$\Phi_\nu(z) = \frac{\sqrt{\pi}}{2}\,\frac{\partial}{\partial z}\,\frac{\Gamma\left(\nu z+\frac{1}{2}\right)}{\Gamma(\nu z)} \left(\frac{x}{2}\right)^{\nu z} {}_1\Psi_1^\star\left[\left.\begin{matrix}\left(\frac{1}{2},\frac{1}{2}\right)\\ (\nu z, 1)\end{matrix}\,\right|\, -xz\right],$$

and thus

$$\mathfrak{K}^{\alpha}_{\nu,\mu+1}(x) = -\int_1^\infty \int_0^{[t]} \frac{\partial}{\partial t}\, \frac{\Gamma(\nu t+\frac12)}{\Gamma(\nu t)} \left(\frac{x}{2}\right)^{\nu t} \\ \times {}_1\Psi_1^\star\left[\begin{array}{c} (\frac12,\frac12) \\ (\nu t,1)\end{array}\middle| -xt\right] \cdot \mathfrak{d}_s \frac{\alpha(s)s^{\nu s-\mu-1}}{\Gamma\left(\nu s+\frac12\right)}\, \mathrm{d}t\mathrm{d}s\,.$$

The proof is complete. □

Now, our goal is to establish a second order nonhomogeneous ordinary differential equation which particular solution is the above introduced special kind Kapteyn series (5.3). Firstly, we introduce the modified Bessel type differential operator

$$M[y] \equiv y'' + \frac{1}{x}\,y' - \left(1+\frac{\nu^2}{x^2}\right)y\,;$$

this operator is associated with the modified Struve differential equation, reads as follows

$$M[y] \equiv y'' + \frac{1}{x}\,y' - \left(1+\frac{\nu^2}{x^2}\right)y = \frac{\left(\frac{x}{2}\right)^{\nu-1}}{\sqrt{\pi}\,\Gamma(\nu+\frac12)}\,. \tag{5.15}$$

Theorem 5.10 (Baricz and Pogány [18]) *Let* $\min\{\nu,\mu\}>0$. *Then for* $x\in\mathscr{I}_\alpha$ *the Kapteyn series* $\mathfrak{K}=\mathfrak{K}^{\alpha}_{\nu,\mu}(x)$ *is a particular solution of the nonhomogeneous linear second order ordinary differential equation*

$$M^{\alpha}_{\mu}[\mathfrak{K}] \equiv \mathfrak{K}'' + \frac{1}{x}\,\mathfrak{K}' - \left(1+\frac{\nu^2}{x^2}\right)\mathfrak{K} = \frac{1}{x}\,\Xi^{\alpha}_{\nu,\mu}(x) + \frac{2}{x\sqrt{\pi}} \sum_{n\ge 1} \frac{\alpha_n\left(\frac{x}{2}\right)^{\nu n}}{\Gamma\left(\nu n+\frac12\right) n^{\mu-\nu n+1}}\,, \tag{5.16}$$

where

$$\Xi^{\alpha}_{\nu,\mu}(x) = \frac{\mathrm{d}}{\mathrm{d}x}\int_1^\infty \int_1^{[t]} \frac{\partial}{\partial t}\, \frac{\Gamma\left(\nu t+\frac12\right)}{\Gamma(\nu t)} \left(\frac{x}{2}\right)^{\nu t} \\ \times {}_1\Psi_1^\star\left[\begin{array}{c} (\frac12,\frac12) \\ (\nu t,1)\end{array}\middle| -xt\right] \cdot \mathfrak{d}_s \frac{\alpha(s)s^{\nu s-\mu-1}(s-1)}{\Gamma\left(\nu s+\frac12\right)}\, \mathrm{d}t\mathrm{d}s\,.$$

Proof Consider the modified Struve differential equation (5.15)

$$M[y] \equiv y''(x) + \frac{1}{x}\,y'(x) - \left(1+\frac{\nu^2}{x^2}\right)y(x) = \frac{\left(\frac{x}{2}\right)^{\nu-1}}{\sqrt{\pi}\,\Gamma(\nu+\frac12)}$$

which possesses the solution $y(x) = c_1 I_\nu(x) + c_2 \mathbf{L}_\nu(x) + c_3 K_\nu(x)$. Being I_ν and K_ν independent particular solutions (the Wronskian $W[I_\nu, K_\nu] = -x^{-1}$) of the homogeneous modified Bessel ordinary differential equation, which appears on the left side in (5.15), the choice $c_3 = 0$ is legitimate. Thus $y(x) = I_{\nu n}(x) - \mathbf{L}_{\nu n}(x)$ is also a particular solution of (5.15). Setting $\nu \mapsto \nu n$, we get

$$\begin{aligned}(I_{\nu n}(x) - \mathbf{L}_{\nu n}(x))'' &+ \frac{1}{x}\,(I_{\nu n}(x) - \mathbf{L}_{\nu n}(x))' \\ &- \left(1 + \frac{\nu^2 n^2}{x^2}\right)(I_{\nu n}(x) - \mathbf{L}_{\nu n}(x)) = \frac{\left(\frac{x}{2}\right)^{\nu n - 1}}{\sqrt{\pi}\,\Gamma\left(\nu n + \frac{1}{2}\right)}\,.\end{aligned}$$

Finally, putting $x \mapsto xn$, multiplying the above display with $n^{-\mu}\alpha_n$ and summing up in $n \in \mathbb{N}$, we obtain

$$M\Big[\mathfrak{K}^{\alpha}_{\nu,\mu}\Big] = M^{\alpha}_{\mu}[\mathfrak{K}] = \frac{1}{x}\left(\mathfrak{K}^{\alpha}_{\nu,\mu}(x) - \mathfrak{K}^{\alpha}_{\nu,\mu+1}(x)\right)' + \frac{2}{x\sqrt{\pi}} \sum_{n\ge 1} \frac{\alpha_n \left(\frac{xn}{2}\right)^{\nu n}}{\Gamma\left(\nu n + \frac{1}{2}\right) n^{\mu+1}}\,,$$

where all three right-hand side series converge uniformly inside $\mathscr{I}_\alpha$. Applying the result (5.14) of the previous theorem to the series

$$\mathfrak{K}^{\alpha}_{\nu,\mu}(x) - \mathfrak{K}^{\alpha}_{\nu,\mu+1}(x) = \sum_{n\ge 2} \frac{\alpha_n (n-1)}{n^{\mu+1}}\,(I_{\nu n}(xn) - \mathbf{L}_{\nu n}(xn))\,,$$

the summation begins with 2. So, the current lower integration limit in the Euler-Maclaurin summation formula related to (5.14) becomes 1. By this we clarify the stated relation (5.16). □

In the following we concentrate on the summation of Schlömilch series

$$\begin{aligned}\mathfrak{T}^{I,\mathbf{L}}_{\nu,\mu}(x) &:= \sum_{n\ge 1} \frac{1}{n^\mu}\,(I_\nu(nx) - \mathbf{L}_\nu(nx)) \\ \widetilde{\mathfrak{T}}^{I,\mathbf{L}}_{\nu,\mu}(x) &:= \sum_{n\ge 1} \frac{(-1)^{n-1}}{n^\mu}\,(I_\nu(nx) - \mathbf{L}_\nu(nx))\,.\end{aligned}$$

To unify these procedures, we consider the generalized Schlömilch series like (5.4)

$$\mathfrak{S}^{I,\mathbf{L}}_{\nu,\mu}(x) = \sum_{n\ge 1} \frac{\alpha_n}{n^\mu}\,(I_\nu(xn) - \mathbf{L}_\nu(xn))\,;$$

obviously $\mathfrak{T}^{I,\mathbf{L}}_{\nu,\mu}(x), \widetilde{\mathfrak{T}}^{I,\mathbf{L}}_{\nu,\mu}(x)$ are special cases of $\mathfrak{S}^{I,\mathbf{L}}_{\nu,\mu}(x)$. However, bearing in mind the asymptotics in Sonin–Gubler formula (5.10), we see that the necessary condition for the convergence of $\mathfrak{S}^{I,\mathbf{L}}_{\nu,\mu}(x)$ for a fixed $x > 0$ becomes $\alpha_n = o\left(n^{\mu-\nu+1}\right)$ as $n \to \infty$.

Theorem 5.11 (Baricz and Pogány [18]) *Let* $\min\{\nu, \mu, x\} > 0$ *and* $\alpha \in \mathrm{C}^1(\mathbb{R}_+)$ *be monotone increasing, such that* $\alpha|_{\mathbb{N}} = (\alpha_n)_{n\geq 1}$*, and* $\sum_{n\geq 1} n^{-\mu+\nu-1}\alpha_n$ *converges. Then* $\mathfrak{S} = \mathfrak{S}_{\nu,\mu}^{I,\mathbf{L}}(x)$ *is a particular solution of the nonhomogeneous linear second order ordinary differential equation*

$$M_\mu^\alpha[\mathfrak{S}] = M\left[\mathfrak{S}_{\nu,\mu}^{I,\mathbf{L}}\right] = \frac{1}{x}\left(\Upsilon_{\mu+1}^{\alpha,1}(x)\right)' - \frac{\nu^2}{x^2}\,\Upsilon_{\mu+2}^{\alpha,2}(x) + \frac{(\frac{x}{2})^{\nu-1}}{\sqrt{\pi}\,\Gamma\left(\nu+\frac{1}{2}\right)}\sum_{n\geq 1}\frac{\alpha_n}{n^{\mu-\nu+1}}\,,$$

where

$$\Upsilon_\mu^{\alpha,\beta}(x) = \mu\int_0^\infty \mathrm{e}^{-\mu t}\int_1^{[\mathrm{e}^t]} \mathfrak{d}_u\left(\alpha(u)(u^\beta-1)\,(I_\nu(xu)-\mathbf{L}_\nu(xu))\right)\,\mathrm{d}t\,\mathrm{d}u\,.$$

Proof Consider again the modified Struve differential equation (5.15), which possesses the solution $y(x) = c_1 I_\nu(x) + c_2\mathbf{L}_\nu(x) + c_3 K_\nu(x)$, and choose the particular solution associated with $c_1 = -c_2 = 1$ and $c_3 = 0$. Transforming (5.15) by putting $x \mapsto xn$, multiplying it by $\alpha_n n^{-\mu}$ and summing the equation with respect to $n \in \mathbb{N}$, we arrive at

$$\left(\sum_{n\geq 1}\frac{\alpha_n}{n^\mu}y(xn)\right)'' + \frac{1}{x}\left(\sum_{n\geq 1}\frac{\alpha_n}{n^{\mu+1}}y(xn)\right)' - \sum_{n\geq 1}\frac{\alpha_n}{n^\mu}y(xn)$$
$$-\frac{\nu^2}{x^2}\sum_{n\geq 1}\frac{\alpha_n}{n^{\mu+2}}y(xn) = \frac{\left(\frac{x}{2}\right)^{\nu-1}}{\sqrt{\pi}\,\Gamma\left(\nu+\frac{1}{2}\right)}\sum_{n\geq 1}\frac{\alpha_n}{n^{\mu-\nu+1}}\,.$$

Thus

$$M\left[\mathfrak{S}_{\nu,\mu}^{I,\mathbf{L}}\right] = \frac{1}{x}\left(\sum_{n\geq 2}\frac{\alpha_n(n-1)}{n^{\mu+1}}\,y(xn)\right)'$$
$$-\frac{\nu^2}{x^2}\sum_{n\geq 2}\frac{\alpha_n(n^2-1)}{n^{\mu+2}}\,y(xn) + \frac{\left(\frac{x}{2}\right)^{\nu-1}}{\sqrt{\pi}\,\Gamma\left(\nu+\frac{1}{2}\right)}\sum_{n\geq 1}\frac{\alpha_n}{n^{\mu-\nu+1}}\,.$$

Denote

$$\Upsilon_\mu^{\alpha,\beta}(x) = \sum_{n\geq 2}\frac{\alpha_n(n^\beta-1)}{n^\mu}\,y(xn), \qquad 0<\nu\leq\mu, x>0\,.$$

Following the same lines of the proof of Theorem 5.7, by Cahen's formula and the Euler–Maclaurin summation we immediately yield the double definite integral representation

$$\Upsilon_{\mu}^{\alpha,\beta}(x) = \mu \int_0^\infty \mathrm{e}^{-\mu t} \int_1^{[\mathrm{e}^t]} \mathfrak{d}_u \left(\alpha(u)(u^\beta - 1)\, y(xu)\right)\, \mathrm{d}t\, \mathrm{d}u\,,$$

which leads to the stated result. □

Corollary 5.1 (Baricz and Pogány [18]) *Let* $\mu - 1 > \nu > 0$ *and* $x > 0$. *Then* $\mathfrak{T} = \mathfrak{T}_{\nu,\mu}^{I,\mathbf{L}}(x)$ *is a particular solution of the nonhomogeneous linear second order ordinary differential equation*

$$M[\mathfrak{T}] = \frac{1}{x}\left(\Upsilon_{\mu+1}^{1,1}(x)\right)' - \frac{\nu^2}{x^2}\,\Upsilon_{\mu+2}^{1,2}(x) + \frac{\zeta(\mu-\nu+1)}{\sqrt{\pi}\,\Gamma\left(\nu+\frac{1}{2}\right)}\left(\frac{x}{2}\right)^{\nu-1}, \tag{5.17}$$

where

$$\Upsilon_{\mu}^{1,\beta}(x) = \mu \int_0^\infty \mathrm{e}^{-\mu t} \int_1^{[\mathrm{e}^t]} \mathfrak{d}_u \left((u^\beta - 1)\,(I_\nu(xu) - \mathbf{L}_\nu(xu))\right)\, \mathrm{d}t\, \mathrm{d}u\,.$$

Corollary 5.2 (Baricz and Pogány [18]) *Let* $\mu > \nu > 0$ *and* $x > 0$. *Then* $\widetilde{\mathfrak{T}} = \widetilde{\mathfrak{T}}_{\nu,\mu}^{I,\mathbf{L}}(x)$ *is a particular solution of the nonhomogeneous linear second order ordinary differential equation*

$$M_\mu^\alpha\left[\widetilde{\mathfrak{T}}\right] = M\left[\widetilde{\mathfrak{T}}_{\nu,\mu}^{I,\mathbf{L}}\right] = \frac{1}{x}\left(\widetilde{\Upsilon}_{\mu+1}^{1}(x)\right)' - \frac{\nu^2}{x^2}\,\widetilde{\Upsilon}_{\mu+2}^{2}(x) + \frac{\eta(\mu-\nu+1)}{\sqrt{\pi}\,\Gamma\left(\nu+\frac{1}{2}\right)}\left(\frac{x}{2}\right)^{\nu-1},$$

where

$$\widetilde{\Upsilon}_{\mu}^{\beta}(x) = \mu \int_0^\infty \mathrm{e}^{-\mu t} \int_1^{[\mathrm{e}^t]} \mathfrak{d}_u \left(\mathrm{e}^{\mathrm{i}\pi u}\,(u^\beta - 1)(\mathbf{L}_\nu(xu) - I_\nu(xu))\right)\, \mathrm{d}t\, \mathrm{d}u\,.$$

Now, a completely different type of integral representation formula will be derived for $\mathfrak{T}_{\nu,\nu+1}^{I,\mathbf{L}}(x)$ which simplifies the nonhomogeneous part of related differential equation (5.17).

Theorem 5.12 (Baricz and Pogány [18]) *If* $\nu > 0$ *and* $x > 0$, *then we have*

$$\mathfrak{T}_{\nu,\nu+1}^{I,\mathbf{L}}(x) = \int_0^\infty J_\nu(xt)\left(\coth(\pi t) - \frac{1}{\pi t}\right)\frac{\mathrm{d}t}{t^{\nu+1}}\,.$$

Proof Consider the well-known summation formula [99]

$$\sum_{n\geq 1}\frac{1}{a^2+n^2}=\frac{\pi}{2a}\,\coth(\pi a)-\frac{1}{2a^2}, \qquad a\neq \mathrm{i}n\,.$$

In conjunction with the Sonin–Gubler formula (5.1) we conclude that

$$\begin{aligned}\mathfrak{T}^{I,\mathbf{L}}_{\nu,\nu+1}(x) &= \frac{2}{\pi}\int_0^\infty J_\nu(xt)\left(\sum_{n\geq 1}\frac{1}{t^2+n^2}\right)\frac{\mathrm{d}t}{t^\nu}\\ &= \int_0^\infty J_\nu(xt)\left(\coth(\pi t)-\frac{1}{\pi t}\right)\frac{\mathrm{d}t}{t^{\nu+1}}\,,\end{aligned}$$

which confirms the assertion. □

Remark 5.1 Actually, the formula

$$\sum_{n\geq 1}\frac{1}{a^2+n^2}=\frac{\pi}{2a}\,\coth(\pi a)-\frac{1}{2a^2}, \qquad a\neq 0$$

has been considered by Hamburger [99, p. 130, Eq. (C)] in the slightly different form

$$1+2\sum_{n\geq 1}\mathrm{e}^{-2\pi n a}=\mathrm{i}\cot\pi\mathrm{i}a=\frac{1}{\pi a}+\frac{2a}{\pi}\sum_{n\geq 1}\frac{1}{a^2+n^2}, \qquad a\neq \mathrm{i}n\,. \tag{C}$$

Hamburger proved that the functional equation for the Riemann Zeta function is equivalent to (C), see also [45] for connections of the above formulae to Eisenstein series.

Also, it is worth to mention that further, complex analytical generalizations of above formula can be found in [29]. ■

5.5 Bromwich–Wagner Integral Form of $J_\nu(x)$

As a by-product of Theorem 5.12, it turns out the integral relation

$$\mathfrak{T}^{I,\mathbf{L}}_{\nu,\nu+1}(x)=\sum_{n\geq 1}\frac{1}{n^{\nu+1}}\left(I_\nu(xn)-\mathbf{L}_\nu(xn)\right), \qquad \nu>0, x>0$$

which, in the expanded form reads

$$\int_0^\infty J_\nu(xt)\left(\coth(\pi t)-\frac{1}{\pi t}\right)\frac{\mathrm{d}t}{t^{\nu+1}}=(\nu+1)\int_0^\infty\int_0^{[\mathrm{e}^s]}\mathfrak{d}_u\left(I_\nu(xu)-\mathbf{L}_\nu(xu)\right)\frac{\mathrm{d}s\,\mathrm{d}u}{\mathrm{e}^{(\nu+1)s}}\,.$$

This arises in a Fredholm type convolutional integral equation of the first kind with degenerate kernel

$$\int_0^\infty f(xt)\left(\coth(\pi t) - \frac{1}{\pi t}\right)\frac{\mathrm{d}t}{t^{\nu+1}} = F_\nu(x)\,, \tag{5.18}$$

having nonhomogeneous part

$$F_\nu(x) = (\nu+1)\int_0^\infty \int_0^{[\mathrm{e}^s]} \mathrm{e}^{-(\nu+1)s}\,\mathfrak{d}_u\left(I_\nu(xu) - \mathbf{L}_\nu(xu)\right)\,\mathrm{d}s\,\mathrm{d}u\,. \tag{5.19}$$

Obviously, J_ν is a particular solution of this equation.

Before we state our result, we say that the functions f and g are *orthogonal a.e.* with respect to the ordinary Lebesgue measure on the positive half-line when $\int_0^\infty f(x)g(x)\mathrm{d}x$ vanishes, writing this as $f \perp g$.

Theorem 5.13 (Baricz and Pogány [18]) *Let $\nu > 0$ and $x > 0$. The first kind Fredholm type convolutional integral equation with degenerate kernel (5.18) possesses particular solution $f = J_\nu + h$, where $h \in L^1(\mathbb{R}_+)$ and*

$$h(x) \perp x^{-\nu-1}\left(\coth(\pi x) - \frac{1}{\pi x}\right), \qquad x > 0$$

if and only if the nonhomogeneous part of the integral equation equals $F_\nu(x)$ given by (5.19).

We mention that h as in the above theorem has been constructed in [71, Example]. To solve the integral equation (5.19) we use the Mellin integral transform technique, following some lines of a similar procedure used by Draščić–Pogány in [71]. The Mellin transform pairs of certain suitable f we define as [301]

$$\mathscr{M}_p(f) = \int_0^\infty x^{p-1} f(x)\,\mathrm{d}x := g(p)$$

$$\mathscr{M}_x^{-1}(g) = \frac{1}{2\pi\mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} x^{-p}\mathscr{M}_p(f)\,\mathrm{d}p\,,$$

where the inverse Mellin transform is given in the form of a line integral with Bromwich–Wagner type integration path which begins at $c - \mathrm{i}\infty$ and terminates at $c + \mathrm{i}\infty$. Here the real c belongs to the fundamental strip of the inverse Mellin transform $\mathscr{M}^{-1}$.

Theorem 5.14 (Baricz and Pogány [18]) *Let $\nu > 0, x > 0$. Then the following Bromwich–Wagner type line integral representation holds true*

$$J_\nu(x) = \frac{\nu+1}{2\mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \frac{\mathscr{M}_p\left(\int_0^\infty \int_0^{[\mathrm{e}^s]} \mathrm{e}^{-(\nu+1)s}\,\mathfrak{d}_u\left(I_\nu(xu) - \mathbf{L}_\nu(xu)\right)\,\mathrm{d}s\,\mathrm{d}u\right)}{\Gamma\left(\frac{p-\nu}{2}\right)\,\Gamma\left(\frac{\nu-p}{2}+1\right)\,\zeta(\nu-p+2)}\,x^{p-1}\,\mathrm{d}p\,, \tag{5.20}$$

where $c \in (\nu, \nu+1)$.

Proof Applying $\mathscr{M}_p$ to the equation (5.18), we get

$$\mathscr{M}_p\left(\int_0^\infty J_\nu(xt)\left(\coth(\pi t)-\frac{1}{\pi t}\right)\frac{\mathrm{d}t}{t^{\nu+1}}\right)=\mathscr{M}_p(F_\nu)\,.$$

By the Mellin-convolution property

$$\mathscr{M}_p\,(f\star g)=\mathscr{M}_p\left(\int_0^\infty f(rt)\cdot g(t)\,\mathrm{d}t\right)=\mathscr{M}_p(f)\cdot\mathscr{M}_{1-p}(g)\,,$$

it follows that

$$\mathscr{M}_p\left(x^{-\nu-1}\left(\coth\pi x-(\pi x)^{-1}\right)\right)\cdot\mathscr{M}_{1-p}(J_\nu)=\mathscr{M}_p(F_\nu)\,. \tag{5.21}$$

The fundamental analytic strip contains $(\nu, 1+\nu)$, because the coth behaves like

$$\coth z=\begin{cases}\dfrac{1}{z}+\dfrac{z}{3}+\mathscr{O}(z^3), & z\to 0\\ 1+2\mathrm{e}^{-2z}\left[1+\mathscr{O}\left(\mathrm{e}^{-2z}\right)\right], & z\to\infty\end{cases};$$

in both cases $\Re(z)>0$. Now, rewriting $x^{-\nu-1}\left(\coth\pi x-(\pi x)^{-1}\right)$ by (C) and using termwise the Beta function description, we conclude that

$$\mathscr{M}_p\left(x^{-\nu-1}\left(\coth\pi x-(\pi x)^{-1}\right)\right)=\frac{1}{\pi}\,\mathrm{B}\left(\frac{p-\nu}{2},\frac{\nu-p}{2}+1\right)\zeta(\nu-p+2)\,,$$

for all $p\in(\nu,\nu+1)$. Therefore

$$\mathscr{M}_{1-p}(J_\nu)=\frac{\pi\,\mathscr{M}_p(F_\nu)}{\mathrm{B}\left(\frac{p-\nu}{2},\frac{\nu-p}{2}+1\right)\zeta(\nu-p+2)}\,,$$

which finally results in

$$J_\nu(x)=\frac{\nu+1}{2\mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty}\frac{\mathscr{M}_p\left(\int_0^\infty\int_0^{[\mathrm{e}^s]}\mathrm{e}^{-(\nu+1)s}\,\mathfrak{d}_u\left(I_\nu(xu)-\mathbf{L}_\nu(xu)\right)\,\mathrm{d}s\,\mathrm{d}u\right)}{\mathrm{B}\left(\frac{p-\nu}{2},\frac{\nu-p}{2}+1\right)\zeta(\nu-p+2)}\,x^{p-1}\,\mathrm{d}p\,,$$

where the fundamental strip contains $c=\nu+\frac{1}{2}$. So, the desired integral representation formula is established. □

We note that the formula-collection [112] does not contain (5.20).

Theorem 5.15 (Baricz and Pogány [18]) *Let* $0 < \nu < \frac{3}{2}, x > 0$. *Then*

$$\mathfrak{T}^{I,\mathbf{L}}_{\nu,\nu+1}(x) = \frac{1}{2^{p+1}\,\pi\,\mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \frac{\Gamma\left(\frac{\nu-p}{2}+\frac{1}{2}\right)\zeta(\nu-p+2)}{\sin\left[\frac{\pi}{2}(p-\nu)\right]\cdot\Gamma\left(\frac{\nu+p}{2}+\frac{1}{2}\right)}\, x^{-p}\,\mathrm{d}p, \qquad c\in(\nu,\nu+1)\,.$$

Proof Consider relation (5.21). Expressing $\mathscr{M}_{1-p}(J_\nu)$ *via* formula [225, p. 93, Eq. 10.1]

$$\mathscr{M}_p\left(J_\nu(ax)\right) = \frac{2^{p-1}}{a^p}\,\frac{\Gamma\left(\frac{\nu+p}{2}\right)}{\Gamma\left(\frac{\nu-p}{2}+1\right)}, \qquad a>0,\ -\nu<p<\tfrac{3}{2}\,,$$

equality (5.21), by virtue of the Euler's reflection formula becomes

$$\begin{aligned}\mathscr{M}_p(F_\nu) &= \frac{\Gamma\left(\frac{\nu-p}{2}+1\right)\Gamma\left(\frac{\nu-p}{2}+\frac{1}{2}\right)\Gamma\left(\frac{p-\nu}{2}\right)\zeta(\nu-p+2)}{2^p\,\pi\,\Gamma\left(\frac{\nu+p}{2}+\frac{1}{2}\right)}\\ &= \frac{\Gamma\left(\frac{\nu-p}{2}+\frac{1}{2}\right)\zeta(\nu-p+2)}{2^p\,\sin\left[\frac{\pi}{2}(p-\nu)\right]\cdot\Gamma\left(\frac{\nu+p}{2}+\frac{1}{2}\right)}\,.\end{aligned}$$

Having in mind that $F_\nu(x)$ is the integral representation of the Schlömilch series $\mathfrak{T}^{I,\mathbf{L}}_{\nu,\nu+1}(x)$, inverting the last display by $\mathscr{M}_p^{-1}$ we arrive at the asserted result. □

5.6 Summing up Schlömilch Series of Struve Functions

In 1987 Lorch and Szego [175] considered the series

$$s_{-q}(a,x,\nu) = \sum_{n\geq 1} \frac{\varepsilon_n\,\mathbf{H}_\nu(nx)}{n^q(n^2-a^2)(nx)^\nu}, \tag{5.22}$$

for positive odd integers q, where $\varepsilon_n = 1$ or $\varepsilon_n = (-1)^{n-1}$. Using mathematical induction, they proved that [175, p. 56, Eq. (22)]

$$\begin{aligned}\sigma_{-(2k+1)}(0,x,\nu) = &\sum_{n=0}^{k}(-1)^n \frac{\zeta(2k+2-2n)\,x^{2n+1}}{(2n+1)!!\,\sqrt{\pi}\,2^{\nu+n}\,\Gamma\left(\nu+n+\frac{3}{2}\right)}\\ &- \frac{(-1)^k\pi\,x^{2k+2}}{(k+1)!\,2^{\nu+2k+3}\Gamma(\nu+k+2)}\\ &+ \frac{(-1)^k x^{2k+3}}{(2k+3)!!\,\sqrt{\pi}\,\Gamma\left(\nu+k+\frac{5}{2}\right)\,2^{\nu+k+2}},\end{aligned} \tag{5.23}$$

where $k \in \mathbb{N}_{-1} = \{-1, 0, 1, \dots\}$, $\nu > -\frac{3}{2}$, $x \in (0, 2\pi)$, $\sigma_{-(2k+1)}(0, x, \nu)$ stands for the series $s_{-(2k+1)}(0, x, \nu)$ containing $\varepsilon_n = 1$, and ζ signifies the Riemann's Zeta function. We point out that for $k = -1$, the sum in $\sigma_{-(2k+1)}(0, x, \nu)$ has to be taken to be zero.

Also, when $\varepsilon_n = (-1)^{n-1}$, writing $S_{-q}(a, x, \nu)$ for (5.22), the same authors obtained [175, p. 54, Eq. (18)]

$$S_{-(2k+1)}(0, x, \nu) = \sum_{n=0}^{k+1} (-1)^n \frac{\Phi(-1, 2k+2-2n, 1)\, x^{2n+1}}{(2n+1)!!\, \sqrt{\pi}\, 2^{\nu+n}\, \Gamma\left(\nu + n + \frac{3}{2}\right)}, \tag{5.24}$$

for all $k \in \mathbb{N}_{-1}$, $\nu > -\frac{3}{2}$, $x \in [0, \pi)$; here Φ denotes the Hurwitz–Lerch Zeta.

Motivated by already stated results by Lorch and Szego, Jankov Maširević [137] presented a new proof of the summations (5.23), (5.24) for $\sigma_{-(2k+1)}(0, x, \nu)$ and $S_{-(2k+1)}(0, x, \nu)$, respectively and show its validity for a significantly wider range of variable x, in the case when $\nu > -\frac{1}{2}$. We recall those results in the Theorem 5.16, below.

Theorem 5.16 (Jankov Maširević [137]) *For all $\nu > -\frac{1}{2}$ there holds*

$$\begin{aligned}\sigma_{-(2k+1)}(0, x, \nu) &= \sum_{n=0}^{k} (-1)^n \frac{\zeta(2k+2-2n)\, x^{2n+1}}{(2n+1)!!\, \sqrt{\pi}\, 2^{\nu+n}\, \Gamma\left(\nu + n + \frac{3}{2}\right)} \\ &\quad - \frac{(-1)^k \pi\, x^{2k+2}}{(k+1)!\, 2^{\nu+2k+3} \Gamma(\nu + k + 2)} \\ &\quad + \frac{(-1)^k x^{2k+3}}{(2k+3)!!\, \sqrt{\pi}\, \Gamma\left(\nu + k + \frac{5}{2}\right)\, 2^{\nu+k+2}},\end{aligned} \tag{5.25}$$

where $x \in (0, 2\pi)$ for $k = -1$ and $x \in [0, 2\pi]$ for all $k \in \mathbb{N}_0$. Moreover

$$S_{-(2k+1)}(0, x, \nu) = \sum_{n=0}^{k+1} (-1)^n \frac{\Phi(-1, 2k+2-2n, 1)\, x^{2n+1}}{(2n+1)!!\, \sqrt{\pi}\, 2^{\nu+n}\, \Gamma\left(\nu + n + \frac{3}{2}\right)}, \tag{5.26}$$

where $x \in (-\pi, \pi)$ for $k = -1$ and $x \in [-\pi, \pi]$ for all $k \in \mathbb{N}_0$.

Proof Firstly, let us establish the convergence conditions of the series

$$s_{-(2k+1)}(0, x, \nu) = \frac{1}{x^\nu} \sum_{n \ge 1} \frac{\varepsilon_n}{n^{2k+\nu+3}}\, \mathbf{H}_\nu(nx), \qquad k \in \mathbb{N}_{-1}.$$

Using the identity [1, p. 497, Eq. (12.1.21)]

$$\mathbf{H}_\nu(z) = \frac{z^{\nu+1}}{2^\nu \sqrt{\pi}\, \Gamma\left(\nu + \frac{3}{2}\right)}\, {}_1F_2\left[\begin{matrix} 1 \\ \frac{3}{2}, \nu + \frac{3}{2} \end{matrix} \,\Big|\, -\frac{z^2}{4}\right]$$

and the asymptotic expansion for the generalized hypergeometric function [199, p. 274, Eq. (2.2b)]

$$ {}_pF_{p+1}\left[\begin{matrix} a_1, \cdots, a_p \\ b_1, \cdots, b_{p+1} \end{matrix} \,\middle|\, -z^2\right] \sim \sum_{k=1}^{p} A_k z^{-2a_k} + A_{p+1} z^{\frac{1}{2}+C} \cos(2z + B)\,, \tag{5.27} $$

where $|z| \to \infty$, $|\arg z| < \frac{\pi}{2}$, the A_k, $k = 1, 2, \ldots, p+1$ and B are dependent on the parameters of the function ${}_pF_{p+1}$ and $C := \sum_{k=1}^{p} a_k - \sum_{k=1}^{p+1} b_k$, we conclude

$$ |s_{-(2k+1)}(0, x, \nu)| \le \frac{1}{|x|^{\nu}} \sum_{n\ge 1} \frac{|\mathbf{H}_\nu(nx)|}{n^{2k+\nu+3}} $$

$$ \sim \frac{|x|}{2^{\nu-2}\sqrt{\pi}\,\Gamma\left(\nu+\frac{3}{2}\right)} \left(\frac{|A_1|}{x^2}\zeta(2k+4) + \frac{2^{\nu-\frac{1}{2}}|A_2|}{|x|^{\nu+\frac{3}{2}}}\zeta\left(2k+\nu+\frac{7}{2}\right)\right), $$

where the convergence is ensured for $\nu > -\frac{1}{2}$.

Next, letting $x \to 0$ in the integral representation [1, p. 496, Eq. (12.1.6)]

$$ \mathbf{H}_\nu(x) = \frac{2\left(\frac{x}{2}\right)^{\nu}}{\sqrt{\pi}\,\Gamma\left(\nu+\frac{1}{2}\right)} \int_0^1 (1-t^2)^{\nu-\frac{1}{2}} \sin(xt)\,\mathrm{d}t, \tag{5.28} $$

valid for $|\arg x| < \frac{\pi}{2}$, $\Re(\nu) > -\frac{1}{2}$, we see that $\mathbf{H}_\nu(x) = x^{\nu+1}(1+o(1))$. So, the series (5.22) is also defined at $x = 0$.

Thus, by the previous integral representation, formulae (1.6), (1.8) and Legendre's duplication formula (1.3), we get

$$ \begin{aligned} \sigma_{-(2k+1)}(0, x, \nu) &= \frac{(-1)^k 2^{-\nu}(2\pi)^{2k+3}}{\sqrt{\pi}\,\Gamma\left(\nu+\frac{1}{2}\right)\Gamma(2k+4)} \int_0^1 (1-t^2)^{\nu-\frac{1}{2}} B_{2k+3}\left(\frac{xt}{2\pi}\right) \mathrm{d}t \\ &= \frac{(-1)^k}{2^{1+\nu}}\left(\frac{x}{2}\right)^{2k+3} \\ &\quad \times \sum_{n=0}^{2k+3} \frac{B_n}{\Gamma(n+1)\,\Gamma\left(k-\frac{n}{2}+\frac{5}{2}\right)\,\Gamma\left(k+\nu-\frac{n}{2}+\frac{5}{2}\right)} \left(\frac{4\pi}{x}\right)^{n} \\ &= \frac{(-1)^k x^{2k+3}}{2^{4+\nu+2k}\,\Gamma\left(k+\frac{5}{2}\right)\,\Gamma\left(k+\nu+\frac{5}{2}\right)} \\ &\quad - \frac{(-1)^k \pi x^{2k+2}}{\Gamma(k+2)2^{\nu+2k+3}\,\Gamma(\nu+k+2)} \\ &\quad + \frac{(-1)^k}{2^{\nu}}\left(\frac{x}{2}\right)^{2k+1} \\ &\quad \times \sum_{m=0}^{k} \frac{(-1)^m \zeta(2m+2)}{\Gamma\left(k-m+\frac{3}{2}\right)\,\Gamma\left(k+\nu-m+\frac{3}{2}\right)} \left(\frac{2}{x}\right)^{2m} \end{aligned} $$

$$= \frac{(-1)^k x^{2k+3}\,\Gamma(k+2)}{2^{1+\nu}\sqrt{\pi}\,\Gamma\left(k+\nu+\frac{5}{2}\right)\,\Gamma(2k+4)}$$
$$- \frac{(-1)^k \pi x^{2k+2}}{\Gamma(k+2)2^{\nu+2k+3}\,\Gamma(\nu+k+2)}$$
$$+ \sum_{n=0}^{k} \frac{(-1)^n x^{2n+1}\,\zeta(2k-2n+2)\,\Gamma(n+1)}{2^{\nu}\sqrt{\pi}\,\Gamma\left(n+\nu+\frac{3}{2}\right)\,\Gamma(2n+2)}\,.$$

Finally, using the identity $(2n+1)! = (2n+1)!! \cdot 2^n\, n!$, $n \in \mathbb{N}_0$ we conclude the formula (5.25), where $x \in (0, 2\pi)$, for $k = -1$ and $x \in [0, 2\pi]$, for $k \in \mathbb{N}_0$, based on the conditions under which (1.8) holds.

Analogously, by virtue of the integral representation (5.28) we infer

$$S_{-(2k+1)}(0, x, \nu) = \frac{(-1)^{k+1} 2^{-\nu} (2\pi)^{2k+3}}{\sqrt{\pi}\,\Gamma\left(\nu+\frac{1}{2}\right)\Gamma(2k+4)} \int_0^1 (1-t^2)^{\nu-\frac{1}{2}} B_{2k+3}\left(\frac{xt}{2\pi}+\frac{1}{2}\right) \mathrm{d}t,$$

where $x \in (-\pi, \pi)$ for $k = -1$ and $x \in [-\pi, \pi]$ for $k \in \mathbb{N}_0$. Now, using the same proving procedure as in Theorem 4.3, with a help of (4.20), (1.6) and the Legendre's duplication formula (1.3) we obtain

$$S_{-(2k+1)}(0, x, \nu) = \frac{(-1)^{k+1} x^{2k+3}}{2^{2+\nu+k}\sqrt{\pi}(2k+3)!!\,\Gamma\left(\frac{5}{2}+k+\nu\right)}$$
$$+ \sum_{n=0}^{k} \frac{(-1)^n (1-2^{2n-2k-1})\,\zeta(2k-2n+2)\,x^{1+2n}}{2^{\nu+n}\sqrt{\pi}\,(2n+1)!!\,\Gamma\left(\frac{3}{2}+\nu+n\right)}\,,$$

which becomes (5.26), using the properties of Hurwitz–Lerch Zeta function (4.21). □

5.7 Dini Series

The series of the form

$$\mathfrak{D}_\nu(z) := \sum_{n\geq 1} b_n\, J_\nu\left(\lambda_{\nu,n} z\right), \tag{5.29}$$

where $\nu \geq -\frac{1}{2}$, $z \in \mathbb{C}$, the coefficients b_n are constants, J_ν stands for the Bessel function of the first kind of order ν and $\lambda_{\nu,n}$ denotes the nth positive zero of $z^\nu\, d_{\nu,\alpha}(z)$, where $d_{\nu,\alpha}$ denotes the Dini-function (1.30), arranged in ascending order of magnitude, is called *Dini series of Bessel functions*, which is a generalization of the Schlömilch series, compare Chap. 4. The coefficients b_n, $n \in \mathbb{N}$ read (1.31)

$$(\lambda_{\nu,n}^2 - \nu^2)J_{\nu+1}^2(j_{\nu,n}) + b_n = \frac{(\nu^2 - \lambda_{\nu,n}^2)}{\lambda_{\nu,n}^2\,\big[J_\nu'(j_{\nu,n})\big]} + \frac{2}{\big[J_\nu'(j_{\nu,n})\big]}\int_0^1 xf(x)\,J_\nu(j_{\nu,n})x)\,\mathrm{d}x\,.$$

Observe that changing the argument $(\beta+n)x$ inside summands of $\mathfrak{S}_\nu^\beta(x) \equiv \mathfrak{S}_\nu^{\beta,J}(x)$ in (4.8) to $\lambda_{\nu,n}x$ we arrive at the Dini series $\mathfrak{D}_\nu(x)$.

It is also worth to mention that Fourier [84] considered Dini series in the case when $\nu = 0$ in solving the problem of the propagation of heat in a circular cylinder. In this problem the heat is radiated from the cylinder, where the physical significance of the constant α in the Dini's function is the ratio of the external conductivity of the cylinder to the internal conductivity. For the more detailed historical overview the interested reader is referred to [333, Section XVIII].

Our first main aim in this section is to derive the double definite integral representation of the Dini series (5.29); this result is given in [27].

Theorem 5.17 (Baricz et al. [27]) *Let $\varepsilon > 0$ and $b, \lambda \in \mathrm{C}^1(\mathbb{R}_+)$ be such that the function*

$$\kappa(u,w) = \frac{\partial}{\partial u}\left(\frac{\Gamma(\varepsilon u+\nu+\frac12)}{\lambda^{\varepsilon u+\nu}(u)}J_{\varepsilon u+\nu}\big(\lambda(u)x\big)\right)\cdot \mathfrak{d}_w\left(\frac{b(w)\lambda^{\varepsilon w+\nu}(w)}{\Gamma(\varepsilon w+\nu+\frac12)}\right)$$

is integrable. Let $b\big|_{\mathbb{N}} = (b_n)_{n\ge1}$, $\lambda\big|_{\mathbb{N}} = (\lambda_{\nu,n})_{n\ge1}$ and assume that

$$\ell_b = \limsup_{n\to\infty}|b_n|^{\frac1n} < 1\,.$$

Then for all $\nu > -\frac12$ and $x \in (0,2)$ we have

$$\mathfrak{D}_\nu(x) = -\int_1^\infty\int_0^{[u]}\frac{\partial}{\partial u}\left(\frac{J_\nu\big(\lambda(u)x\big)}{\lambda^\nu(u)}\right)\mathfrak{d}_w\Big(b(w)\lambda^\nu(w)\Big)\,\mathrm{d}u\,\mathrm{d}w\,.$$

Proof First, let us consider the integral form for the Kapteyn series $\mathfrak{K}_{\nu,\varepsilon}^\mu(x) = \sum_{n\ge1}b_nJ_{\nu+\varepsilon n}((\mu+n)x)$ which is given in Theorem 3.5. Then, consider certain suitable $\lambda \in \mathrm{C}^1(\mathbb{R}_+)$, which interpolates the set $(\lambda_{\nu,n})_{n\ge1}$ which constitutes of all positive zeros of $d_{\nu,\alpha}(x)$ taken in ascending order. Substituting instead of $\mu+n$ the expression $\lambda(n)$ in the integral representation (3.23), given in Theorem 3.5 and applying similar procedure as in that theorem, we conclude that $|x| < 2\rho^{\frac1\varepsilon}$, where

$$\rho = \left(\frac{\varepsilon}{\mathrm{e}}\right)^\varepsilon\left(\limsup_{n\to\infty}|b_n|^{\frac1n}\left(\frac{\lambda_{\nu,n}}{n}\right)^\varepsilon\right)^{-1}\,.$$

So, the argument's range changes to

$$x \in \left(0, 2\min\left\{1, \frac{\varepsilon}{\mathrm{e}}\left(\limsup_{n\to\infty} |b_n|^{\frac{1}{n}}\left(\frac{\lambda_{\nu,n}}{n}\right)^{\varepsilon}\right)^{-\frac{1}{\varepsilon}}\right\}\right) =: \mathscr{I}_{b,\varepsilon}\,.$$

In the next step we are interested in the limit of $\rho^{\frac{1}{\varepsilon}}$ as ε approaches zero from the right. Since

$$\rho^{-\frac{1}{\varepsilon}} \le \frac{\mathrm{e}}{\varepsilon}\,\ell_b^{\frac{1}{\varepsilon}} \cdot \left(\limsup_{n\to\infty}\left(\frac{\lambda_{\nu,n}}{n}\right)^{\varepsilon}\right)^{\frac{1}{\varepsilon}},$$

it is necessary to determine the behavior of $\lambda_{\nu,n}/n$ for large n. For this, first we show that for all $n \in \{1, 2, \dots\}$ we have $\lambda_{\nu,n} \in (j_{\nu,n-1}, j_{\nu,n})$, where $j_{\nu,n}$ is the nth positive zero of $J_\nu(x)$. Note that for $n = 1$ the fact that $\lambda_{\nu,1} \in (0, j_{\nu,1})$ was pointed out in [126, p. 11]. Now, suppose that $n \in \{2, 3, \dots\}$. In view of the recurrence relation

$$J'_\nu(x) = J_{\nu-1}(x) - (\nu/x)J_\nu(x)$$

we have

$$d_{\nu,\alpha}(j_{\nu,s}) = j_{\nu,s}J'_\nu(j_{\nu,s}) = j_{\nu,s}J_{\nu-1}(j_{\nu,s}),$$

where $s \in \{n-1, n\}$. On the other hand it is known that zeros of $J_{\nu-1}$ and J_ν interlace, so $d_{\nu,\alpha}(j_{\nu,n-1}) \cdot d_{\nu,\alpha}(j_{\nu,n}) < 0$, because $J_{\nu-1}(x)$ has opposite sign at the subsequent zeros $j_{\nu,n-1}$ and $j_{\nu,n}$. Thus, the root of $d_{\nu,\alpha}(x) = 0$, that is, $\lambda_{\nu,n}$ belongs to $(j_{\nu,n-1}, j_{\nu,n})$. Consequently, for all $n \in \{1, 2, \dots\}$ we obtain

$$\frac{j_{\nu,n-1}}{n} < \frac{\lambda_{\nu,n}}{n} < \frac{j_{\nu,n}}{n},$$

which in view of the MacMahon expansion [333, p. 506] (see also Schläfli's footnote [277, p. 137])

$$j_{\nu,n} = \left(n + \tfrac{\nu}{2} - \tfrac{1}{4}\right)\pi + \mathscr{O}\left(\tfrac{1}{n}\right), \qquad n \to \infty,$$

shows that $\lim_{n\to\infty} \lambda_{\nu,n}/n = \pi$. Hence

$$\left(\limsup_{n\to\infty}\left(\frac{\lambda_{\nu,n}}{n}\right)^{\varepsilon}\right)^{\frac{1}{\varepsilon}} = \lim_{n\to\infty}\frac{\lambda_{\nu,n}}{n} = \pi\,,$$

since $x \mapsto x^\varepsilon \in \mathrm{C}(\mathbb{R}^+)$, where $\varepsilon > 0$. Consequently, these show that

$$\lim_{\varepsilon\searrow 0} \rho^{-\frac{1}{\varepsilon}} \le \mathrm{e}\pi \lim_{\varepsilon\searrow 0}\frac{\ell_b^{\frac{1}{\varepsilon}}}{\varepsilon} = 0\,.$$

Now, it immediately follows that $\mathscr{I}_{b,0} = (0, 2)$, and (3.23) becomes

$$\widetilde{\mathfrak{K}}_{\nu,\varepsilon}^{\lambda}(x) = -\int_1^\infty \int_0^{[u]} \frac{\partial}{\partial u}\left(\frac{\Gamma\left(\varepsilon u + \nu + \frac{1}{2}\right)}{\lambda^{\varepsilon u+\nu}(u)}\, J_{\varepsilon u+\nu}\big(\lambda(u)\, x\big)\right) \times \mathfrak{d}_w\left(\frac{b(w)\lambda^{\nu+\varepsilon w}(w)}{\Gamma\left(\nu + \varepsilon w + \frac{1}{2}\right)}\right) \mathrm{d}u\, \mathrm{d}w\,. \tag{5.30}$$

Since the integrand in (5.30) is integrable, by the Lebesgue dominated convergence theorem we get

$$\lim_{\varepsilon \searrow 0} \widetilde{\mathfrak{K}}_{\nu,\varepsilon}^{\lambda}(x) = \mathfrak{D}_\nu(x)\,,$$

which finishes the proof of Theorem 5.17. □

We draw the reader's attention to the fact that

$$\lim_{\varepsilon \searrow 0} \mathfrak{K}_{\nu,\varepsilon}^{\mu}(x) = \mathfrak{S}_\nu^{\mu}(x),$$

so actually Kapteyn series $\mathfrak{K}_{\nu,\varepsilon}^{\mu}, \widetilde{\mathfrak{K}}_{\nu,\varepsilon}^{\lambda}$ connect Schlömilch's $\mathfrak{S}_\nu^{\mu}$ and Dini's $\mathfrak{D}_\nu$ series.

5.8 Dini Series and the Bessel Differential Equation

Analogously as we derived integral representations for Kapteyn and Schlömlich series, using Bessel differential equation, in Sects. 3.3 and 4.2, respectively, exploiting the non-homogeneous Bessel ordinary differential equation we can conclude the following result:

Theorem 5.18 (Baricz et al. [27]) *Let $b, \lambda \in \mathrm{C}^1(\mathbb{R}_+)$, and $b\big|_{\mathbb{N}} = (b_n)_{n\geq 1}$, $\lambda\big|_{\mathbb{N}} = (\lambda_{\nu,n})_{n\geq 1}$. Assume that series $\sum_{n\geq 1} b_n \lambda_{\nu,n}^{\frac{5}{3}}$ absolutely converges. Then the Dini series (5.29) is a particular solution of the nonhomogeneous Bessel-type differential equation*

$$x^2 y'' + xy' + (x^2 - \nu^2)y = \mathfrak{E}_\nu(x)\,,$$

where $\mathfrak{E}_\nu(x)$ is given with

$$\mathfrak{E}_\nu(x) := x^2 \sum_{n\geq 1} (1 - \lambda_{\nu,n}^2) b_n J_\nu(\lambda_{\nu,n} x)$$

and for all $x \in (0,2)$ *it is*

$$\mathfrak{E}_\nu(x) = -x^2 \int_1^\infty \int_0^{[u]} \frac{\partial}{\partial u}\Big(\frac{J_\nu(\lambda(u)\,x)}{\lambda^\nu(u)}\Big)\, \mathfrak{d}_w\,\big(b(w)\,(1-\lambda^2(w))\lambda^\nu(w)\big)\; \mathrm{d}u\,\mathrm{d}w. \tag{5.31}$$

At the end of this section, analogous procedure as we used to derive familiar integral representations of Neumann, Kapteyn and Schlömilch series lead us to the following integral representation of (5.29):

Theorem 5.19 (Baricz et al. [27]) *Let the situation for* b, λ *be the same as in* Theorem 5.18. *Then, for all* $\nu > -\frac{1}{2}$ *and* $x \in (0,2)$ *there holds*

$$\begin{aligned}\mathfrak{D}_\nu(x) &= \frac{J_\nu(x)}{2}\int \frac{1}{xJ_\nu^2(x)}\left(\int \frac{J_\nu(x)\mathfrak{E}_\nu(x)}{x}\,\mathrm{d}x\right)\mathrm{d}x\\ &\quad + \frac{Y_\nu(x)}{2}\int \frac{1}{xY_\nu^2(x)}\left(\int \frac{Y_\nu(x)\mathfrak{E}_\nu(x)}{x}\,\mathrm{d}x\right)\mathrm{d}x\,,\end{aligned}$$

where $\mathfrak{E}_\nu$ *is the Dini series associated with* $\mathfrak{D}_\nu(x)$, *which possesses the integral form* (5.31).

5.9 Jacobi Polynomials in Sum

The Jacobi polynomials, which are also called hypergeometric polynomials [155], can be represented with the following formula [286]

$$P_n^{(\alpha,\beta)}(z) = \frac{(1+\alpha)_n}{n!}\; {}_2F_1\left[\begin{matrix}-n,\ 1+\alpha+\beta+n\\ 1+\alpha\end{matrix}\,\middle|\,\frac{1-z}{2}\right].$$

It is worth mentioning that Luke and Wimp [180] proved that if we have continuous function $f(x)$, which has a piecewise continuous derivative for $0 \le x \le \lambda$, then $f(x)$ may be expanded into a uniformly convergent series of shifted Jacobi polynomials in the form

$$f(x) = \sum_{n\ge 0} a_n(\lambda) P_n^{(\alpha,\beta)}\left(\frac{2x}{\lambda} - 1\right),$$

where $\epsilon \le \lambda^{-1}x \le 1-\epsilon$, $\epsilon > 0$, $\alpha > -1$, $\beta > -1$. Various techniques are available for the determination of the coefficients $a_n(\lambda)$.

Let us define a functional series in the following form

$$\mathfrak{P}_{\alpha,\beta}(z) = \sum_{n\ge 1} \alpha_n P_n^{(\alpha,\beta)}(z), \qquad z \in \mathbb{C}. \tag{5.32}$$

We point out that the Bulgarian mathematician P. Rusev studied in [271] the convergence of the series $\mathfrak{P}_{\alpha,\beta}(z)$ (precisely, he considered $a_0 + \mathfrak{P}_{\alpha,\beta}(z)$).

In this section, our main aim is to derive several integral representations for the Rusev series (5.32), derived in the article [132, p. 109 *et seq.*]. The double integral representation is given in the following theorem:

Theorem 5.20 (Jankov and Pogány [132]) *Let* $a \in \mathrm{C}^1(\mathbb{R}_+)$ *and* $a|_{\mathbb{N}} = (a_n)_{n\geq 1}$. *Then for all* $\alpha > -\frac{1}{2}$, $\alpha + \beta > -1$ *and for all* x *belonging to*

$$\mathscr{I}_a = \big(\max\{0, 2\eta - 1\}, 1\big] \tag{5.33}$$

we have the integral representation

$$\mathfrak{P}_{\alpha,\beta}(x) = -\int_1^\infty \int_0^{[s]} \frac{\partial}{\partial s}\left(\frac{\Gamma(2s+1)\, P_s^{(\alpha,\beta)}(x)}{\Gamma(\alpha+s+\frac{1}{2})\Gamma(\beta+s+\frac{1}{2})}\right) \times \mathfrak{d}_w \left(\frac{a(w)\,\Gamma(\alpha+w+\frac{1}{2})\Gamma(\beta+w+\frac{1}{2})}{\Gamma(2w+1)}\right) \mathrm{d}s\, \mathrm{d}w\,.$$

Proof First, we begin by establishing the convergence conditions for the series $\mathfrak{P}_{\alpha,\beta}(x)$.

For that purpose, let us consider the integral representation given by Feldheim [81]:

$$P_n^{(\alpha,\beta)}(x) = \frac{1}{\Gamma(\alpha+\beta+n+1)} \int_0^\infty t^{\alpha+\beta+n} \mathrm{e}^{-t} L_n^{(\alpha)}\left(\frac{1}{2}(1-x)t\right) \mathrm{d}t\,, \tag{5.34}$$

valid for all $n \in \mathbb{N}_0, \alpha + \beta > -1$, where $L_n^{(\alpha)}$ is the Laguerre polynomial. We estimate (5.34) *via* the bounding inequality for Laguerre functions $L_\nu^{(\mu)}(x)$, given by Love [177, p. 295, Theorem 2]:

$$|L_\nu^{(\mu)}(x)| \leq \frac{\Gamma(\Re(\nu+\mu+1))}{|\Gamma(\nu+1)|\Gamma(\Re(\mu)+1)}\, \frac{\Gamma(\Re(\mu)+\frac{1}{2})}{|\Gamma(\mu+\frac{1}{2})|}\, \mathrm{e}^x\,, \tag{5.35}$$

where $\nu \in \mathbb{C}$, $x > 0$, $\Re(\mu) > -\frac{1}{2}$ and $\Re(\mu+\nu) > -1$, which has been generalized by Pogány and Srivastava [246]. Specifying $\mu = \alpha \in \mathbb{R}, \nu = n \in \mathbb{N}_0$ the bound (5.35) reduces to

$$|L_n^{(\alpha)}(x)| \leq \frac{\Gamma(n+\alpha+1)}{n!\,\Gamma(\alpha+1)}\, \mathrm{e}^x, \qquad x > 0\,. \tag{5.36}$$

Now, applying bound (5.36) to the integrand of (5.34), we have that

$$\big|\mathfrak{P}_{\alpha,\beta}(x)\big| \leq \frac{1}{\Gamma(\alpha+1)}\left(\frac{2}{1+x}\right)^{\alpha+\beta+1} \sum_{n\geq 1} \frac{|a_n|\Gamma(\alpha+n+1)}{n!}\left(\frac{2}{1+x}\right)^n.$$

The resulting power series converges uniformly for all x satisfying constraint (5.33).

A more convenient integral representation for the Jacobi polynomials has been given by Braaksma and Meulenbeld [41], [64, p. 191]

$$P_n^{(\alpha,\beta)}(1-2z^2) = \frac{(-1)^n 4^n (\alpha+\frac{1}{2})_n(\beta+\frac{1}{2})_n}{\pi(2n)!} \int_{-1}^{1}\int_{-1}^{1} \left(zu \pm \mathrm{i}\sqrt{1-z^2}\, v\right)^{2n} \times (1-u^2)^{\alpha-\frac{1}{2}}(1-v^2)^{\beta-\frac{1}{2}}\, \mathrm{d}u\mathrm{d}v, \qquad 0 \le z \le 1,$$

where $2\min\{\alpha,\beta\} > -1$. This expression in an obvious way one reduces to

$$P_n^{(\alpha,\beta)}(x) = \frac{2^n(\alpha+\frac{1}{2})_n(\beta+\frac{1}{2})_n}{\pi(2n)!} \int_{-1}^{1}\int_{-1}^{1} \left(\mathrm{i}\sqrt{1-x}\,u - \sqrt{1+x}\,v\right)^{2n} \times (1-u^2)^{\alpha-\frac{1}{2}}(1-v^2)^{\beta-\frac{1}{2}}\, \mathrm{d}u\mathrm{d}v, \qquad |x| \le 1. \tag{5.37}$$

Thus, combining (5.32) and (5.37) we get

$$\mathfrak{P}_{\alpha,\beta}(x) = \frac{1}{\pi}\int_{-1}^{1}\int_{-1}^{1}(1-u^2)^{\alpha-\frac{1}{2}}(1-v^2)^{\beta-\frac{1}{2}}\, \mathscr{D}_a(u,v)\,\mathrm{d}u\mathrm{d}v, \tag{5.38}$$

where $\mathscr{D}_a(u,v)$ is the Dirichlet series

$$\begin{aligned}\mathscr{D}_a(u,v) &= \sum_{n\ge 1} \frac{a_n\,(\alpha+\frac{1}{2})_n(\beta+\frac{1}{2})_n}{(2n)!}\left(2\big(\mathrm{i}\sqrt{1-x}\,u - \sqrt{1+x}\,v\big)^2\right)^n \\ &= \sum_{n\ge 1} \frac{a_n\,(\alpha+\frac{1}{2})_n(\beta+\frac{1}{2})_n}{(2n)!}\, \mathrm{e}^{-n\log\left(\sqrt{2}(\mathrm{i}\sqrt{1-x}\,u-\sqrt{1+x}\,v)\right)^{-2}}.\end{aligned}$$

The Dirichlet series possesses Laplace integral representation when its parameter has positive real part, therefore we are looking for the two-dimensional region $\mathscr{S}_{uv}(x)$ in the uv-plane where

$$\Re\left\{\log 2\big(\mathrm{i}\sqrt{1-x}\,u - \sqrt{1+x}\,v\big)^2\right\} = \log 2\big((1+x)v^2 + (1-x)u^2\big) < 0\,.$$

So, we get the ellipse

$$\mathscr{S}_{uv}(x) = \left\{(u,v)\in\mathbb{R}^2\colon (1+x)v^2 + (1-x)u^2 < \tfrac{1}{2}\right\},$$

such that is nonempty for all $x \in \mathscr{I}_a$, so $\mathscr{D}_a(u,v)$ converges in $\mathscr{I}_a$.

Now, the related Laplace-integral and the Euler–Maclaurin summation formula (see for instance [23, 24]) give us:

$$\mathscr{D}_a(u,v) = -\frac{\log\left(\sqrt{2}(\mathrm{i}\sqrt{1-x}\,u - \sqrt{1+x}\,v)\right)^2}{\Gamma(\alpha+\frac{1}{2})\Gamma(\beta+\frac{1}{2})} \times \int_0^\infty \int_0^{[s]} \left(\sqrt{2}(\mathrm{i}\sqrt{1-x}\,u - \sqrt{1+x}\,v)\right)^{2s} \times \mathfrak{d}_w \left(\frac{a(w)\,\Gamma(\alpha+w+\frac{1}{2})\Gamma(\beta+w+\frac{1}{2})}{\Gamma(2w+1)}\right) \mathrm{d}s\,\mathrm{d}w. \tag{5.39}$$

Substituting (5.39) into (5.38) we get

$$\mathfrak{P}_{\alpha,\beta}(x) = -\frac{1}{\pi\Gamma(\alpha+\frac{1}{2})\Gamma(\beta+\frac{1}{2})} \int_{-1}^1 \int_{-1}^1 \int_0^\infty \int_0^{[s]} (1-u^2)^{\alpha-\frac{1}{2}}(1-v^2)^{\beta-\frac{1}{2}} \times \log\left(\sqrt{2}(\mathrm{i}\sqrt{1-x}\,u - \sqrt{1+x}\,v)\right)^2 \cdot \left(\sqrt{2}(\mathrm{i}\sqrt{1-x}\,u - \sqrt{1+x}\,v)\right)^{2s} \times \mathfrak{d}_w \left(\frac{a(w)\,\Gamma(\alpha+w+\frac{1}{2})\Gamma(\beta+w+\frac{1}{2})}{\Gamma(2w+1)}\right) \mathrm{d}u\,\mathrm{d}v\,\mathrm{d}s\,\mathrm{d}w\,. \tag{5.40}$$

Denoting

$$\mathscr{I}_x(s) := \int_{-1}^1 \int_{-1}^1 \log\left(\sqrt{2}(\mathrm{i}\sqrt{1-x}\,u - \sqrt{1+x}\,v)\right)^2 \times \left(\sqrt{2}(\mathrm{i}\sqrt{1-x}\,u - \sqrt{1+x}\,v)\right)^{2s} (1-u^2)^{\alpha-\frac{1}{2}}(1-v^2)^{\beta-\frac{1}{2}}\,\mathrm{d}u\,\mathrm{d}v,$$

we get

$$\int \mathscr{I}_x(s)\mathrm{d}s = \int_{-1}^1 \int_{-1}^1 \left(\sqrt{2}(\mathrm{i}\sqrt{1-x}\,u - \sqrt{1+x}\,v)\right)^{2s}(1-u^2)^{\alpha-\frac{1}{2}}(1-v^2)^{\beta-\frac{1}{2}}\mathrm{d}u\mathrm{d}v = \pi\,\frac{\Gamma(\alpha+\frac{1}{2})\Gamma(\beta+\frac{1}{2})\Gamma(2s+1)\,P_s^{(\alpha,\beta)}(x)}{\Gamma(\alpha+s+\frac{1}{2})\Gamma(\beta+s+\frac{1}{2})}\,.$$

Therefore, we can easily conclude that

$$\mathscr{I}_x(s) = \pi\Gamma(\alpha+\tfrac{1}{2})\Gamma(\beta+\tfrac{1}{2})\,\frac{\partial}{\partial s}\left(\frac{\Gamma(2s+1)\,P_s^{(\alpha,\beta)}(x)}{\Gamma(\alpha+s+\frac{1}{2})\Gamma(\beta+s+\frac{1}{2})}\right). \tag{5.41}$$

Finally, by using (5.40) and (5.41), we immediately get the proof of the theorem, with the assertion that the integration domain $\mathbb{R}_+$ becomes $[1,\infty)$ because $[s]$ is equal to zero for all $s \in [0,1)$. □

There exists another, indefinite type integral representation for the functional series (5.32) which can be obtained by having in mind that the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ satisfy a linear homogeneous ordinary differential equation of the second order [263, 300]:

$$(1-x^2)y'' + (\beta - \alpha - (2+\alpha+\beta)x)\,y' + n(1+\alpha+\beta+n)y = 0\,. \qquad (5.42)$$

Now, using the analogous procedure as in the previous sections concerning Neumann, Kapteyn, Schlömlich and Dini series, but this time using the differential equation (5.42) instead of Bessel differential equation, we can conclude the following results:

Theorem 5.21 (Jankov and Pogány [132]) *For all $\alpha > -\frac{1}{2}$, $\alpha+\beta > -1$ the particular solution of the linear* ordinary differential equation*:*

$$(1-x^2)y' + (\beta - \alpha - (2+\alpha+\beta)x)\,y = \mathfrak{R}_{\alpha,\beta}(x)\,,$$

represents the first derivative $\frac{\partial}{\partial x}\mathfrak{P}_{\alpha,\beta}(x)$ of the functional series (5.32), where $\mathfrak{R}_{\alpha,\beta}(x)$ is given with

$$\begin{aligned}\mathfrak{R}_{\alpha,\beta}(x) :&= -\sum_{n\geq 1} a_n\, n(1+\alpha+\beta+n)P_n^{(\alpha,\beta)}(x)\\ &= (1-x^2)\mathfrak{P}''_{\alpha,\beta}(x) + (\beta-\alpha-(2+\alpha+\beta)x)\,\mathfrak{P}'_{\alpha,\beta}(x)\,.\end{aligned}$$

Here for $a \in \mathrm{C}^1(\mathbb{R}_+)$, $a|_{\mathbb{N}} = (a_n)_{n\geq 1}$ and letting $\sum_{n\geq 1} n^2\, a_n$ absolutely converges, for all $x \in \mathscr{I}_a$ we have the integral representation

$$\begin{aligned}\mathfrak{R}_{\alpha,\beta}(x) = \int_1^\infty \int_0^{[s]} \frac{\partial}{\partial s}\left(\frac{\Gamma(2s+1)\,P_s^{(\alpha,\beta)}(x)}{\Gamma(\alpha+s+\frac{1}{2})\Gamma(\beta+s+\frac{1}{2})}\right)&\\ \times\, \mathfrak{d}_w\left(\frac{a(w)\,w\,(1+\alpha+\beta+w)\,\Gamma(\alpha+w+\frac{1}{2})\Gamma(\beta+w+\frac{1}{2})}{\Gamma(2w+1)}\right)&\mathrm{d}s\,\mathrm{d}w\,.\end{aligned}$$

Theorem 5.22 (Jankov and Pogány [132]) *Let the situation be the same as in* Theorem 5.21*. Then we have*

$$\mathfrak{P}_{\alpha,\beta}(x) = \int \frac{1}{(1-x)^{\alpha+1}(1+x)^{\beta+1}}\left(\int \mathfrak{R}_{\alpha,\beta}(x)(1-x)^\alpha(1+x)^\beta\,\mathrm{d}x\right)\mathrm{d}x\,,$$

where $\mathfrak{R}_{\alpha,\beta}(x)$ is the series associated with the series $\mathfrak{P}_{\alpha,\beta}(x)$.

5.10 Schlömilch Series of von Lommel Functions

The von Lommel function of the first kind [93, 323]

$$s_{\mu,\nu}(z) = \frac{z^{\mu+1}}{4} \sum_{n\geq 0} \frac{(-1)^n \Gamma(\frac{1}{2}(\mu-\nu+1))\, \Gamma(\frac{1}{2}(\mu+\nu+1))}{\Gamma(\frac{1}{2}(\mu-\nu+3)+n2)\, \Gamma(\frac{1}{2}(\mu+\nu+3)+n)} \left(\frac{z}{2}\right)^{2n}$$

is defined for all $\mu, \nu \in \mathbb{C}$ such that neither $\mu-\nu$ nor $\mu+\nu$ is an odd negative integer, and for all $z \in \mathbb{C}$ which satisfy $-\pi < \arg z \leq \pi$ and it is a particular solution of the inhomogeneous Bessel differential equation

$$z^2 y'' + zy' + (z^2 - \nu^2)y = z^{\mu+1}, \qquad y = s_{\mu,\nu}(z).$$

Motivated by an importance of von Lommel functions which arise in the theory of positive trigonometric sums [158] and occurs in several places in physics and engineering (see e.g. [91]) we are interested in this section in summing up the special kind Schlömilch series built by members which contain von Lommel function of the first kind in the form

$$\mathfrak{S}_{\mu,\nu}(z) = \sum_{n\geq 1} \alpha_n s_{\mu,\nu}(nz), \qquad z \in \mathbb{C},$$

for some special cases of the constants μ, ν, α_n. According to our knowledge, such problem has not been considered in mathematical literature.

More general results about this kind series, with members containing Bessel function of the first kind J_ν are recently studied in [131]; also, in 1995 Rawn [265, p. 285, Eq. (5)] showed that

$$\sum_{n\geq 1} \frac{(-1)^{n-1} J_\nu(nx)}{n^\nu} = \frac{x^\nu}{2^{\nu+1}\, \Gamma(\nu+1)}, \tag{5.43}$$

where $\Re(\nu) > -\frac{1}{2}$, $x \in (-\pi, \pi)$ and 2 years later Miller [198, p. 91] proved

$$\sum_{n\geq 1} \frac{J_\nu(nx)}{n^\nu} = -\frac{x^\nu}{2^{\nu+1}\, \Gamma(\nu+1)} + \frac{\sqrt{\pi}\, x^{\nu-1}}{2^\nu\, \Gamma\left(\nu+\frac{1}{2}\right)}, \tag{5.44}$$

where $\Re(\nu) > -\frac{1}{2}$, $x \in (0, 2\pi)$.

Quite recently, Tričković et al. [316] proved that for all $m \in \mathbb{N}$, $\nu > -\frac{1}{2}$ there holds:

$$\sum_{n\geq 1} \frac{(-1)^{n-1} J_\nu(nx)}{n^{2m+\nu}} = \sum_{n=0}^{m} \frac{(-1)^n \eta(2m-2n)}{n!\, \Gamma(\nu+n+1)} \left(\frac{x}{2}\right)^{\nu+2n}, \tag{5.45}$$

valid for $x \in (-\pi, \pi)$, where

$$\eta(s) = \sum_{n\geq 1} \frac{(-1)^{n-1}}{n^s}, \qquad \Re(s) > 0,$$

signifies the Dirichlet Eta function and

$$\sum_{n\geq 1} \frac{J_\nu(nx)}{n^{2m+\nu}} = \frac{(-1)^m \pi}{2\Gamma\left(m+\frac{1}{2}\right)\Gamma\left(m+\nu+\frac{1}{2}\right)} \left(\frac{x}{2}\right)^{2m+\nu-1} + \sum_{n=0}^{m} \frac{(-1)^n \zeta(2m-2n)}{\Gamma(n+1)\Gamma(\nu+n+1)} \left(\frac{x}{2}\right)^{\nu+2n}, \tag{5.46}$$

where $x \in (0, 2\pi)$ and ζ stands for the Riemann's Zeta function.

It is important to mention that the authors stated the previous formulae in form of infinite sums, but knowing the property of Riemann's Zeta function that $\zeta(-2n) = 0$, for $n \in \mathbb{N}$ and also that [195, p. 4]

$$\eta(-2n) = (1 - 2^{1+2n})\,\zeta(-2n),$$

the sums in the previous expressions vanish when $n \geq m+1$. Also, substituting $m = 0$ in (5.45) and (5.46) and knowing that

$$-\eta(0) = \zeta(0) = -\tfrac{1}{2}$$

we immediately get (5.43) and (5.44), so the formulae derived by Tričković et al. are also valid for $m = 0$, which is not mentioned in theirs article.

Let us also mention that in the book of Brychkov [42, sections 6.8.6, 6.10.3, 6.17.2] one can find exhaustive list of summations for Schlömilch series containing Bessel function of the first kind and Struve function $\mathbf{H}_\nu$ and hypergeometric function ${}_pF_{q+2}$ as well which are connected with von Lommel function of the first kind (see Eqs. (5.50), (5.51)).

Our main objective is to establish closed form expressions for the Schlömilch series

$$\mathfrak{S}^{\alpha}_{\mu,\nu}(x) = \sum_{n\geq 1} \frac{\varepsilon_n\, s_{\mu,\nu}(nx)}{n^\alpha}, \qquad x \in \mathbb{R}, \tag{5.47}$$

where α, ν, μ are constants and $\varepsilon_n = 1$ or $\varepsilon_n = (-1)^{n-1}$. Also, we will derive several closed expressions for the series which members contain some trigonometric functions, as a by-product of the mentioned main results and all of those results concern to the paper by Jankov Maširević [136].

In what follows we will use notation $\mathfrak{T}^{\alpha}_{\mu,\nu}(x)$ for the series (5.47), when $\varepsilon_n = 1$ and $\widetilde{\mathfrak{T}}^{\alpha}_{\mu,\nu}(x)$ in the case when $\varepsilon_n = (-1)^{n-1}$.

5.10.1 Closed Form Expressions for $\mathfrak{S}^{\alpha}_{\mu,\nu}(x)$

Our first set of main results is based essentially upon already stated formulae due to Tričković et al. (5.45). The second set of results would make use of the Bernoulli polynomials defined in Chap. 1.

Theorem 5.23 (Jankov Maširević [136]) *For all* $m \in \mathbb{N}_0$, $\nu \in \mathbb{R}$ *and* $\mu > \max\{-\nu-1, \nu-2, -\frac{1}{2}\}$ *there holds*

$$\mathfrak{T}^{2m+\mu+1}_{\mu,\nu}(x) = \frac{x^{\mu+1}}{4}\Gamma\left(\frac{1+\mu-\nu}{2}\right)\,\Gamma\left(\frac{1+\mu+\nu}{2}\right) \times\left(\frac{(-1)^m\pi}{2\,\Gamma\left(m+1+\frac{\mu-\nu}{2}\right)\,\Gamma\left(m+1+\frac{\mu+\nu}{2}\right)}\left(\frac{x}{2}\right)^{2m-1} + \sum_{n=0}^{m}\frac{(-1)^n\zeta(2m-2n)}{\Gamma\left(n+1+\frac{1+\mu-\nu}{2}\right)\,\Gamma\left(n+1+\frac{1+\mu+\nu}{2}\right)}\left(\frac{x}{2}\right)^{2n}\right), \tag{5.48}$$

where $x \in (0, 2\pi)$. *Moreover for all* $x \in (-\pi, \pi)$

$$\widetilde{\mathfrak{T}}^{2m+\mu+1}_{\mu,\nu}(x) = \frac{x^{\mu+1}}{4}\Gamma\left(\frac{1+\mu-\nu}{2}\right)\,\Gamma\left(\frac{1+\mu+\nu}{2}\right) \times\sum_{n=0}^{m}\frac{(-1)^n\,\eta(2m-2n)}{\Gamma\left(n+1+\frac{1+\mu-\nu}{2}\right)\,\Gamma\left(n+1+\frac{1+\mu+\nu}{2}\right)}\left(\frac{x}{2}\right)^{2n}. \tag{5.49}$$

Proof First, let us establish the convergence conditions of the series

$$\mathfrak{S}^{2m+\mu+1}_{\mu,\nu}(x) = \sum_{n\ge 1}\frac{\varepsilon_n\, s_{\mu,\nu}(nx)}{n^{2m+\mu+1}}, \qquad m \in \mathbb{N}_0\,.$$

Using the fact that the von Lommel function of the first kind can be expressed in terms of a hypergeometric function [224, p. 281] as

$$s_{\mu,\nu}(z) = \frac{z^{\mu+1}}{(\mu-\nu+1)(\mu+\nu+1)}\;{}_1F_2\left[\begin{matrix}1\\ \frac{\mu-\nu+3}{2},\ \frac{\mu+\nu+3}{2}\end{matrix}\,\middle|\, -\frac{z^2}{4}\right] \tag{5.50}$$

and the asymptotic expansion for the generalized hypergeometric function (5.27) we conclude

$$|\mathfrak{S}_{\mu,\nu}^{2m+\mu+1}(x)| \leq \sum_{n\geq 1} \frac{|s_{\mu,\nu}(nx)|}{n^{2m+\mu+1}} \sim \frac{|x|^{\mu+1}}{(\mu-\nu+1)(\mu+\nu+1)}$$

$$\times \left(\frac{4\,|A_1|}{x^2} \zeta(2m+2) + |A_2| \zeta\left(2m+\mu+\tfrac{3}{2}\right) \left(\frac{2}{|x|}\right)^{\mu+\frac{3}{2}} \right),$$

where the convergence is ensured for $\mu > -\frac{1}{2}$.

Now, by virtue of an integral representation [78, p. 42, Eq. (86)]

$$s_{\mu,\nu}(x) = 2^{\mu}\,\Gamma\left(\frac{1+\mu-\nu}{2}\right)\left(\frac{x}{2}\right)^{\frac{1+\nu+\mu}{2}}$$

$$\times \int_0^{\frac{\pi}{2}} J_{(1+\mu-\nu)/2}(x \sin t)\,(\sin t)^{(1+\nu-\mu)/2} (\cos t)^{\nu+\mu}\, \mathrm{d}t,$$

valid for $\Re(\nu+\mu+1) > 0$ and summation formula (5.46) we get

$$\mathfrak{T}_{\mu,\nu}^{2m+\mu+1}(x) = 2^{\mu}\,\Gamma\left(\frac{1+\mu-\nu}{2}\right)\left(\frac{x}{2}\right)^{\frac{1+\nu+\mu}{2}} \int_0^{\frac{\pi}{2}} (\sin t)^{(1+\nu-\mu)/2} (\cos t)^{\nu+\mu}$$

$$\times \sum_{n\geq 1} \frac{1}{n^{2m+(1+\mu-\nu)/2}} J_{(1+\mu-\nu)/2}(n x \sin t)\,\mathrm{d}t$$

$$= \frac{x^{\mu+1}}{2}\Gamma\left(\frac{1+\mu-\nu}{2}\right)\left(\frac{(-1)^m \pi}{2\,\Gamma\left(m+\frac{1}{2}\right)\,\Gamma\left(m+1+\frac{\mu-\nu}{2}\right)}\right.$$

$$\times \left(\frac{x}{2}\right)^{2m-1} \int_0^{\frac{\pi}{2}} (\sin t)^{2m} (\cos t)^{\nu+\mu}\,\mathrm{d}t$$

$$+ \sum_{n=0}^{m} \frac{(-1)^n \zeta(2m-2n)}{\Gamma(n+1)\,\Gamma\left(n+1+\frac{1+\mu-\nu}{2}\right)}$$

$$\left. \times \left(\frac{x}{2}\right)^{2n} \int_0^{\frac{\pi}{2}} (\sin t)^{1+2n} (\cos t)^{\nu+\mu}\,\mathrm{d}t \right),$$

which immediately gives the desired formula.

Analogously, using the previous integral representation and (5.45) we conclude

$$\widetilde{\mathfrak{T}}_{\mu,\nu}^{2m+\mu+1}(x) = \frac{x^{\mu+1}}{2}\Gamma\left(\frac{1+\mu-\nu}{2}\right) \sum_{n=0}^{m} \frac{(-1)^n\, \eta(2m-2n)}{\Gamma(n+1)\,\Gamma\left(n+1+\frac{1+\mu-\nu}{2}\right)}$$

$$\times \left(\frac{x}{2}\right)^{2n} \int_0^{\frac{\pi}{2}} (\sin t)^{1+2n} (\cos t)^{\nu+\mu}\,\mathrm{d}t,$$

which is equal to (5.49). □

Remark 5.2 Using the well-known symmetry property [333, p. 348] $s_{\mu,-\nu}(x) = s_{\mu,\nu}(x)$ we deduce that the summation formulae for the series $\mathfrak{T}^{2m+\mu+1}_{\mu,-\nu}(x)$ and $\widetilde{\mathfrak{T}}^{2m+\mu+1}_{\mu,-\nu}(x)$ has the same form as (5.48) and (5.49), respectively.

Furthermore, knowing that the von Lommel function is related to the Struve function $\mathbf{H}_\nu$ by [78, p. 42, Eq. (84)]

$$s_{\nu,\nu}(x) = \frac{\sqrt{\pi}\,\Gamma\left(\nu+\frac{1}{2}\right)\mathbf{H}_\nu(x)}{2^{1-\nu}}, \tag{5.51}$$

and setting $m \mapsto m+1$ and $\mu = \nu$ in the formulae (5.48) and (5.49), we can conclude the summation formulae (5.25) and (5.26) for the Schlömlich series which members containing Struve functions, derived in Theorem 5.16. ■

In what follows, we will prove that, in the case when $m \in \mathbb{N}$, $\mu \mapsto \mu - \frac{3}{2}$ and $\nu = \frac{1}{2}$, the summation formulae (5.48) and (5.49) are also valid for x equal to the endpoints of a given intervals.

Theorem 5.24 (Jankov Maširević [136]) *For all $m \in \mathbb{N}$, $\mu > 0$*

$$\mathfrak{T}^{2m+\mu-\frac{1}{2}}_{\mu-\frac{3}{2},\frac{1}{2}}(x) = \frac{(-1)^{m-1}x^{2m+\mu-\frac{1}{2}}\Gamma(\mu-1)}{2} \times\left(\frac{-\pi}{x\,\Gamma(2m+\mu)} + 2\sum_{n=0}^{m}\frac{(-1)^{n-1}\zeta(2n)}{\Gamma(2m+\mu+1-2n)\,x^{2n}}\right),$$

where $x \in [0, 2\pi]$. Moreover

$$\widetilde{\mathfrak{T}}^{2m+\mu-\frac{1}{2}}_{\mu-\frac{3}{2},\frac{1}{2}}(x) = (-1)^m x^{2m+\mu-\frac{1}{2}}\Gamma(\mu-1)\sum_{n=0}^{m}\frac{(-1)^n(1-2^{1-2n})\,\zeta(2n)}{x^{2n}\,\Gamma(2m+\mu+1-2n)},$$

holds for $x \in [-\pi, \pi]$.

Proof First, let us establish the convergence conditions of the series $\mathfrak{S}^{2m+\mu-\frac{1}{2}}_{\mu-\frac{3}{2},\frac{1}{2}}(x)$. To this aim let us consider the integral representation referred to Baricz et al. [16], valid for $\mu > 0$:

$$s_{\mu-\frac{3}{2},\frac{1}{2}}(x) = \frac{x^{\mu-\frac{1}{2}}}{\mu-1}\int_0^1(1-t)^{\mu-1}\cos(xt)\,\mathrm{d}t. \tag{5.52}$$

Consequently, we get

$$|s_{\mu-\frac{3}{2},\frac{1}{2}}(x)| \le \frac{|x|^{\mu-\frac{1}{2}}}{|\mu-1|}\int_0^1(1-t)^{\mu-1}\,\mathrm{d}t = \frac{|x|^{\mu-\frac{1}{2}}}{\mu\,|\mu-1|},$$

that is

$$|\mathfrak{S}^{2m+\mu-\frac{1}{2}}_{\mu-\frac{3}{2},\frac{1}{2}}(x)| \le \sum_{n\ge 1} \frac{|s_{\mu-\frac{3}{2},\frac{1}{2}}(nx)|}{n^{2m+\mu-\frac{1}{2}}} \le \frac{|x|^{\mu-\frac{1}{2}}}{\mu\,|\mu-1|} \sum_{n\ge 1} \frac{1}{n^{2m}} = \frac{|x|^{\mu-\frac{1}{2}}}{\mu\,|\mu-1|}\,\zeta(2m),$$

where the convergence is ensured being $m \ge 1$.

Using the previous integral representation and the same proving procedure as in Theorems 4.3 and 5.16 we can conclude the desired formulae. □

Theorem 5.25 (Jankov Maširević [136]) *For all* $\mu > 0$ *there holds*

$$\mathfrak{T}^{2m+\mu-\frac{3}{2}}_{\mu-\frac{1}{2},\frac{1}{2}}(x) = \frac{(-1)^m \Gamma(\mu)\, x^{2m+\mu-\frac{3}{2}}}{2} \left(\frac{-\pi}{x\,\Gamma(2m+\mu-1)} + 2\sum_{n=0}^{m-1} \frac{(-1)^{n-1}\zeta(2n)}{\Gamma(2m+\mu-2n)\,x^{2n}} \right),$$

where $x \in (0, 2\pi)$, *for* $m = 1$ *and* $x \in [0, 2\pi]$, *for* $m \in \mathbb{N}_2$. *Moreover*

$$\widetilde{\mathfrak{T}}^{2m+\mu-\frac{3}{2}}_{\mu-\frac{1}{2},\frac{1}{2}}(x) = (-1)^{m+1} x^{2m+\mu-\frac{3}{2}}\,\Gamma(\mu) \sum_{n=0}^{m-1} \frac{(-1)^n (1-2^{1-2n})\,\zeta(2n)}{\Gamma(2m+\mu-2n)\,x^{2n}}$$

holds true for $x \in (-\pi, \pi)$ *when* $m = 1$ *and* $x \in [-\pi, \pi]$ *for* $m \in \mathbb{N}_2$.

Proof Analogously as we did in the Theorem 5.23, we can conclude that the series

$$\mathfrak{S}^{2m+\mu-\frac{3}{2}}_{\mu-\frac{1}{2},\frac{1}{2}}(x) = \sum_{n\ge 1} \frac{\varepsilon_n\, s_{\mu-\frac{1}{2},\frac{1}{2}}(nx)}{n^{2m+\mu-\frac{3}{2}}}, \qquad m \in \mathbb{N}$$

converges for all $\mu > 0$ being

$$|\mathfrak{S}^{2m+\mu-\frac{3}{2}}_{\mu-\frac{1}{2},\frac{1}{2}}(x)| \le \sum_{n\ge 1} \frac{|s_{\mu-\frac{1}{2},\frac{1}{2}}(nx)|}{n^{2m+\mu-\frac{3}{2}}} \sim \frac{|x|^{\mu+\frac{1}{2}}}{\mu\,(\mu+1)} \left(\frac{4\,|A_1|}{x^2}\zeta(2m) + |A_2|\zeta(2m+\mu-1)\left(\frac{2}{|x|}\right)^{\mu+1} \right).$$

The rest is obvious and follows by considering the same concluding process as in the previous theorem, except here we use an integral representation [16]

$$s_{\mu-\frac{1}{2},\frac{1}{2}}(x) = x^{\mu-\frac{1}{2}} \int_0^1 (1-t)^{\mu-1} \sin(xt)\,\mathrm{d}t,$$

valid for $\mu > 0$, instead of (5.52). □

5.11 Neumann Series of Meijer *G* Function

One of the most powerful tools in Mellin–Barnes type complex integrals turns out to be Meijer *G* function, which contains as special cases higher transcendental hypergeometric functions, Bessel functions family members including the Struve functions. The symbol $G^{m,n}_{p,q}(\cdot|\cdot)$ denotes Meijer's *G*-function [190] and [186] defined in terms of the Mellin–Barnes integral reads

$$G^{m,n}_{p,q}\left(z\,\middle|\,\begin{matrix}a_1,\cdots,a_p\\ b_1,\cdots,b_q\end{matrix}\right)=\frac{1}{2\pi\mathrm{i}}\int_{\mathfrak{C}}\frac{\prod\limits_{j=1}^{m}\Gamma(b_j-s)\prod\limits_{j=1}^{n}\Gamma(1-a_j+s)\cdot z^s}{\prod\limits_{j=m+1}^{q}\Gamma(1-b_j+s)\prod\limits_{j=n+1}^{p}\Gamma(a_j-s)}\,\mathrm{d}s, \tag{5.53}$$

where $0\le m\le q,\ 0\le n\le p$ and the poles a_j, b_j are such that no pole of $\Gamma(b_j-s)$, $j=\overline{1,m}$ coincides with any pole of $\Gamma(1-a_j+s), j=\overline{1,n}$; i.e. $a_j-b_j\notin\mathbb{N}$, while $z\neq 0$. $\mathfrak{C}$ is a suitable integration contour which starts at $-\mathrm{i}\infty$ and goes to $\mathrm{i}\infty$ separating the poles of $\Gamma(b_j-s), j=\overline{1,m}$ which lie to the right of the contour, from all poles of $\Gamma(1-a_j+s), j=\overline{1,n}$, which lie to the left of $\mathfrak{C}$. The integral converges if $\delta=m+n-\frac{1}{2}(p+q)>0$ and $|\arg(z)|<\delta\pi$, see [178, p. 143], [179, 186] and [190].

The asymptotic expansion results of Meijer's *G*-function derived by Braaksma [40, Section 11,Theorems 10–17] for various values and constraints upon m,n,p,q. By Braaksma these asymptotic expansions are actually rederived results given earlier by Meijer in [190]. However, the advantages of novel different kind proving procedures and formulations in [40] are that they hold uniformly in closed sectors and in the transition regions. Moreover, a recurrence method was proposed for expansion coefficients finding.

The Meijer *G* function allows expressing of all four Bessel functions for all values of the parameter by the following special cases [77, p. 219]

$$\begin{aligned}
J_\nu(z)&=\left(\frac{z}{2}\right)^\nu\ G^{1,0}_{0,2}\left(\frac{z^2}{4}\,\middle|\,0,\ -\nu\right)\\
I_\nu(z)&=\pi\ \left(\frac{z}{2}\right)^\nu\ G^{1,0}_{1,3}\left(\frac{z^2}{4}\,\middle|\,\begin{matrix}\frac{1}{2}\\ 0,\ -\nu,\ \frac{1}{2}\end{matrix}\right)\\
Y_\nu(z)&=\pi\ \left(\frac{z}{2}\right)^\nu\ G^{2,0}_{1,3}\left(\frac{z^2}{4}\,\middle|\,\begin{matrix}-\frac{1}{2}(\nu+1)\\ \frac{\nu}{2},-\frac{\nu}{2},-\frac{1}{2}(\nu+1)\end{matrix}\right), && \Re(z)>0\\
K_\nu(z)&=\frac{1}{2}\ \left(\frac{z}{2}\right)^\nu\ G^{2,0}_{0,2}\left(\frac{z^2}{4}\,\middle|\,-\frac{\nu}{2},\ -\frac{\nu}{2}\right), && \Re(z)>0.
\end{aligned}$$

Furthermore, also hold the representations for the Struve functions [77, p. 220]

$$\mathbf{H}_\nu(z) = G^{1,1}_{1,3}\Big(\frac{z^2}{4}\Big|\, \begin{matrix} \frac12(\nu+1) \\ \frac12(\nu+1),\ -\frac{\nu}{2},\ \frac{\nu}{2} \end{matrix}\Big)$$

$$\mathbf{L}_\nu(z) = -\frac{\pi}{\sin\left(\frac{\pi\nu}{2}\right)}\, G^{1,1}_{2,4}\Big(\frac{z^2}{4}\Big|\, \begin{matrix} \frac12(\nu+1),\ \frac12 \\ \frac12(\nu+1),\ \frac12,\ -\frac{\nu}{2},\ \frac{\nu}{2} \end{matrix}\Big),$$

where in both cases $\Re(z) > 0$. Finally the von Lommel function

$$\mathfrak{s}_{\nu,\mu}(z) = \frac{2^{\mu-1}}{\Gamma\left(\frac12(1-\mu-\nu)\right)\, \Gamma\left(\frac12(1-\mu+\nu)\right)}\, G^{3,1}_{1,3}\Big(\frac{z^2}{4}\Big|\, \begin{matrix} \frac12(\mu+1) \\ \frac12(\mu+1),\ -\frac{\nu}{2},\ \frac{\nu}{2} \end{matrix}\Big).$$

These and another special functions' connections can be found for instance in [93, §**9.34**]. Having in mind these formulae we see that the earlier exposed Neumann, Kapteyn and Schlömilch series result can be written in Meijer G function form too. So, we collect and discuss here the related results when the general G-function series are in the focus of our considerations, getting associated Neumann–Meijer series.

In two consecutive papers Milgram [194, 196] exposed the analysis of two integrals associated with the integral transport equation in infinitely long, annular geometry. These integrals have been expressed as sums built by Meijer G function terms. Certain further special results include the probability integrals and the generalized Bickley–Nailer function (compare [19] as well) [194, p. 2457, Eq. (2.5)]

$$\mathrm{Ki}^\tau_\nu(x) = \int_0^{\frac{\pi}{2}} \cos^{\nu-1} t\, \sin^{\tau-1} t\, \mathrm{e}^{-\frac{x}{\cos t}}\, \mathrm{d}t, \qquad \Re(\nu), \Re(\tau) > 0\,,$$

which Meijer G series are presented too. The basic model includes the probability [196, p. 417]

$$P^{oo} = \frac{4}{\pi}\int_{\arcsin\kappa}^{\frac{\pi}{2}} \mathrm{Ki}_3(2x\cos\theta)\, \cos\theta\, \mathrm{d}\theta\,,$$

where $\kappa \in [0,1], x \in \mathbb{R}_+$ and we omit the upper parameter being $\tau = 1$. The probability P^{oo} is the radial component of the outer-outer transmission probability, while

$$P^{io} = \frac{4}{\pi}\int_{\arcsin\kappa}^{\frac{\pi}{2}} \mathrm{Ki}_3(R\,x)\, \cos\theta\, \mathrm{d}\theta$$

$$R = \sqrt{1-\kappa^2\, \arcsin^2\theta} - \kappa\, cos\theta\,,$$

turns out to be the radial component of the inner-outer transmission probability [194, §2, p. 2457]. To expand both integrals into Meijer G sums and then evaluate them in [196], Milgram developed the necessary mathematical tool in the technical report [193]. In turn, these series are Neumann series of Meijer function since the parameters of the G functions depend linearly by the summation index, *viz.*

$$\sum_{k\ge 0} \frac{\Gamma(c+k)}{\Gamma(d+k)}\, G^{m,n}_{p,q}\Big(z\,\Big|\,\begin{matrix} a_1,\cdots,a_p \\ b_1,\cdots,b_q\end{matrix}\Big)\, \frac{w^k}{k!}\,,$$

where some of the G functions parameters are of the form $a_j = r_j \pm k, b_\ell = s_\ell \pm k$ and ensure the convergence either of the Mellin–Barnes type integral G and the constituting Gamma function coefficients. The basic model includes only the $|w| < 1$, while the extension in $|w| \ge 1$ we realize by analytic continuation.

Finally, it is worth to mention Milgram's Neumann–Bessel series corollaries mostly associated with the main topic of our monograph [193, p. 7, Eqs. (4.5–6)]:

$$\sum_{n\ge 0} \frac{(c)_n}{(b)_n} J_{\alpha-n}(2x)\, \frac{(-x)^n}{n!} = \frac{\Gamma(b)\,\Gamma(b+\alpha-c)\, x^\alpha}{\Gamma(b-c)\,\Gamma(\alpha+1)\,\Gamma(b+\alpha)}\; {}_1F_2\Big[\begin{matrix} b+\alpha-c \\ \alpha+1,\; b+\alpha\end{matrix}\Big|-x^2\Big]$$

$$\sum_{n\ge 0} \frac{(c)_n}{(b)_n} K_{\alpha+n}(2x)\, \frac{x^n}{n!} = \frac{\Gamma(b)}{2}\Bigg\{ \frac{\Gamma(-\alpha)\, x^\alpha}{\Gamma(d)}\; {}_1F_2\Big[\begin{matrix} b-c \\ \alpha+1,\; d\end{matrix}\Big|x^2\Big] + \frac{\Gamma(\alpha)\,\Gamma(b-\alpha-c)\, x^{-\alpha}}{\Gamma(b-c)\,\Gamma(d-\alpha)}\; {}_1F_2\Big[\begin{matrix} b-\alpha-c \\ 1-\alpha,\; d-\alpha\end{matrix}\Big|x^2\Big]\Bigg\},$$

see also [178, p. 20, Eq. (7)].

Motivated by the expansion sum results we introduce the Neumann–Meijer series in the following form:

$$\mathfrak{N}^G_{\boldsymbol{a},\boldsymbol{b}}(w,z) = \sum_{k\ge 0} \frac{g_k\, w^k}{k!}\, G^{m,n}_{p,q}\Big(z\,\Big|\,\begin{matrix} c+k, a_1,\cdots,a_{p-1} \\ d+k, b_1,\cdots,b_{q-1}\end{matrix}\Big)\,. \tag{5.54}$$

However, we point out that either the integral transforms or inverse integral transforms of Meijer G terms are expressible also *via* another numeration and/or parameters Meijer G functions; therefore our already presented mathematical tools are not suitable to establish an integral representation for $\mathfrak{N}^G_{\boldsymbol{a},\boldsymbol{b}}(w,z)$. Indeed, replacing the path integral form of the G function in (5.54) and interchanging the summation and integration order we get

$$\mathfrak{N}^G_{\boldsymbol{a},\boldsymbol{b}}(w,z) = \frac{1}{2\pi \mathrm{i}} \int_{\mathfrak{C}} z^s\, \frac{\prod\limits_{j=1}^{m-1} \Gamma(b_j - s) \prod\limits_{j=1}^{n-1} \Gamma(1-a_j+s)}{\prod\limits_{j=m}^{q-1} \Gamma(1-b_j+s) \prod\limits_{j=n}^{p-1} \Gamma(a_j - s)}\, \mathscr{E}_g(s;w)\, \mathrm{d}s\,, \tag{5.55}$$

where

$$\mathscr{E}_g(s;w) = \sum_{k\ge 0} \frac{g_k\, w^k}{k!}\, \Gamma(d+k-s)\,\Gamma(1-c-k+s)\,.$$

Employing the formula $(a)_k(1-a)_{-k} = (-1)^k$, we have

$$\mathscr{E}_g(s; w) = \Gamma(d-s)\,\Gamma(1-c+s) \sum_{k\ge 0} \frac{g_k\,(-w)^k}{k!}\,\frac{(d-s)_k}{(c-s)_k};$$

the radius of convergence $\rho_{\mathscr{E}}$ of this power series satisfies

$$\rho_{\mathscr{E}}^{-1} = \mathrm{e}^{-1}\,\limsup_{k\to\infty} \frac{|g_k|^{\frac{1}{k}}}{k}.$$

When the coefficients' behavior is polynomial in k, of degree $p \in \mathbb{N}_0$, say, that is

$$g_k = \sum_{j=0}^{p} \mathfrak{q}_j\, k^j,$$

$\mathscr{E}_g(s; w)$ becomes entire in w-plane and having in mind that

$$\sum_{k\ge 0} \frac{k^j\,(d-s)_k}{(c-s)_k}\,\frac{(-w)^k}{k!} = \frac{(d-s)\,w}{c-s}\;{}_jF_j\left[\begin{matrix}2,\cdots,2, d-s+1\\ 1,\cdots,1, c-s+1\end{matrix}\,\Big|\, -w\right], \qquad j\in\mathbb{N},$$

we conclude

$$\begin{aligned}\mathscr{E}_g(s; w) &= -\Gamma(d-s+1)\,\Gamma(s-c)\,w\,\Big\{\mathfrak{q}_0\cdot {}_1F_1\left[\begin{matrix}d-s+1\\ c-s+1\end{matrix}\,\Big|\, -w\right] \\ &\quad + \sum_{j=1}^{p} \mathfrak{q}_j \cdot {}_jF_j\left[\begin{matrix}2,\cdots,2, d-s+1\\ 1,\cdots,1, c-s+1\end{matrix}\,\Big|\, -w\right]\Big\}. \end{aligned} \tag{5.56}$$

Returning the expression (5.56) to (5.55) we infer the following $p+1$ term linear combination of integral transforms of generalized hypergeometric functions:

$$\begin{aligned}\mathfrak{N}^G_{\boldsymbol{a},\boldsymbol{b}}(w,z) &= -\frac{\mathfrak{q}_0\,w}{2\pi\mathrm{i}} \int_{\mathfrak{C}} z^s\, \frac{\prod\limits_{j=1}^{m-1}\Gamma(b_j-s)\prod\limits_{j=1}^{n-1}\Gamma(1-a_j+s)}{\prod\limits_{j=m}^{q-1}\Gamma(1-b_j+s)\prod\limits_{j=n}^{p-1}\Gamma(a_j-s)} \\ &\quad \times \Gamma(d-s+1)\,\Gamma(s-c)\;{}_1F_1\left[\begin{matrix}d-s+1\\ c-s+1\end{matrix}\,\Big|\, -w\right]\mathrm{d}s\end{aligned}$$

$$- \frac{w}{2\pi \mathrm{i}} \sum_{j=1}^{p} \mathfrak{q}_j \int_{\mathfrak{C}} z^s \frac{\prod_{j=1}^{m-1} \Gamma(b_j - s) \prod_{j=1}^{n-1} \Gamma(1 - a_j + s)}{\prod_{j=m}^{q-1} \Gamma(1 - b_j + s) \prod_{j=n}^{p-1} \Gamma(a_j - s)}$$

$$\times \Gamma(d - s + 1)\, \Gamma(s - c)\, {}_jF_j\left[\begin{matrix} 2, \cdots, 2, d - s + 1 \\ 1, \cdots, 1, c - s + 1 \end{matrix} \,\middle|\, -w \right] \mathrm{d}s\,.$$

This formula holds true for all $w \in \mathbb{C}$ and the related parameter space which is now not hard to precise by the convergence conditions given around (5.53).

Changing the coefficient sequence $(\mathfrak{g}_k)_{k \geq 0}$ we can arrive at similar subsequent set of results. However, general method for deriving integral representation formula for the Neumann–Meijer series (5.54) will be the task of a forthcoming study.

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