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*Editors*

# Commutative Algebra

Noetherian and  
Non-Noetherian Perspectives

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 Springer

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# Preface

This volume contains a collection of invited survey articles by some of the leading experts in commutative algebra carefully selected for their impact on the field. Commutative algebra is growing very rapidly in many directions. The intent of this volume is to feature a wide range of these directions rather than focus on a narrow research trend. The articles represent various significant developments in both Noetherian and non-Noetherian commutative algebra, including such topics as generalizations of cyclic modules, zero divisor graphs, class semigroups, forcing algebras, syzygy bundles, tight closure, Gorenstein dimensions, tensor products of algebras over fields,  $v$ -domains, multiplicative ideal theory, direct-sum decompositions, defect, almost perfect domains, defects of field extensions, ultrafilters, ultraproducts, Rees valuations, overrings of Noetherian domains, weak normality, and seminormality.

The papers give a cross-section of what is happening and of what is influential in commutative algebra now. The target audience is the researchers in the area, with the aim that the papers serve both as a reference and as a source for further investigations.

We thank the contributors for their wonderful papers. We have learned much from their expertise, and we hope that these papers are as inspirational for the readers as they have been for us. We also thank the referees for their constructive criticism, and the Springer editorial staff, especially Elizabeth Loew and Nathan Brothers, for their patience and assistance in getting this volume into print.

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# Principal-like ideals and related polynomial content conditions\*

D.D. Anderson

**Abstract** We discuss several classes of ideals (resp., modules) having properties shared by principal ideals (resp., cyclic modules). These include multiplication ideals and modules and cancellation ideals and modules. We also discuss polynomial content conditions including Gaussian ideals and rings and Armendariz rings.

## 1 Introduction

Of all ideals in a commutative ring certainly principal ideals are the simplest. Now, principal ideals have many useful properties. We concentrate on three of these properties. First, if  $Ra$  is a principal ideal of a commutative ring  $R$  and  $A \subseteq Ra$  is an ideal, then  $A = BRa$  for some ideal  $B$  of  $R$ , namely  $B = A : Ra$ . An ideal  $I$  of  $R$  sharing this property that for any ideal  $A \subseteq I$ , we have  $A = BI$  for some ideal  $B$  is called a *multiplication ideal*. Second, if further  $a \in R$  is not a zero divisor, then for ideals  $A$  and  $B$  of  $R$ ,  $RaA = RaB$  implies  $A = B$ . An ideal  $I$  of  $R$  with the property that  $IA = IB$  for ideals  $A$  and  $B$  of  $R$  implies  $A = B$  is called a *cancellation ideal*. More generally,  $I$  is a *weak cancellation ideal* if  $IA = IB$  implies  $A + 0 : I = B + 0 : I$ . Any principal ideal is a weak cancellation ideal. Third, if  $f = a_0 + a_1X + \cdots + a_nX^n \in R[X]$  is a polynomial with content  $c(f) = Ra_0 + \cdots + Ra_n$  principal, then  $c(fg) = c(f)c(g)$  for all  $g \in R[X]$ . A polynomial  $f \in R[X]$  is called *Gaussian* if  $c(fg) = c(f)c(g)$  for all  $g \in R[X]$ . And  $R$  is said to be *Gaussian* (resp., *Armendariz*) if  $c(fg) = c(f)c(g)$  for all  $f, g \in R[X]$  (resp., with  $c(fg) = 0$ ).

We view a finitely generated locally principal ideal as the appropriate generalization of a principal ideal. It turns out that a finitely generated locally principal

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\* Dedicated to the memory of my teacher, Irving Kaplansky, who piqued my interest in these topics.

ideal  $I$  is a multiplication ideal and a weak cancellation ideal and if  $c(f) = I$ , then  $f$  is Gaussian. We will be particularly interested in how close the converses of these results are true.

The purpose of this paper is to survey principal-like ideals, especially multiplication ideals and cancellation ideals, and polynomial content conditions, especially Gaussian polynomials and rings, and Armendariz rings. We also discuss the natural extension of these concepts to modules. This paper consists of five sections besides the introduction. In the second section, we look at principal-like elements in a multiplicative lattice and lattice module and what these elements are in the case of the lattice of ideals of a commutative ring or lattice of submodules. The third section surveys multiplication ideals and modules and multiplication rings (rings in which every ideal is a multiplication ideal). The fourth section discusses cancellation ideals and modules and their various generalizations. In Section 5, we survey the recent characterizations of Gaussian polynomials and Gaussian rings. In the last (Section 6) we cover Armendariz rings. Two topics that we do not discuss are invertible ideals and  $*$ -invertible ideals. Excellent surveys already exist. See for instance [1.2]. We also give an extensive (but not exhaustive) bibliography arranged by sections.

Except for several fleeting instances, all rings will be commutative with identity and all modules unitary. For any undefined terms or notation, the reader is referred to [1.1].

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## 2 Principal elements in multiplicative lattices

In this section, we discuss principal elements in multiplicative lattices. I begin with this section as it was through multiplicative lattices that I became interested in principal-like ideals in commutative rings. By a multiplicative lattice  $\mathcal{L}$  we mean a complete lattice  $\mathcal{L}$  with greatest element  $I$  and least element  $O$  having a commutative, associative product that distributes over arbitrary joins and has  $I$  as a multiplicative identity. We do not assume that  $\mathcal{L}$  is modular. Of course, here the most important example is  $\mathcal{L} = \mathcal{L}(R)$ , the lattice of ideals of a commutative ring  $R$  with identity. We mention only two other examples. If  $R$  is a graded ring, then the set  $\mathcal{L}_h(R)$  of homogeneous ideals of  $R$  is a multiplicative sublattice of  $\mathcal{L}(R)$ , and if  $S$  is a commutative monoid with zero, the set  $\mathcal{L}(S)$  of ideals of  $S$  is a quasilocal distributive multiplicative lattice with  $A \vee B = A \cup B$ ,  $A \wedge B = A \cap B$ , and  $AB = \{ab \mid a \in A, b \in B\}$ . All three

of these multiplicative lattices are modular. A multiplicative lattice  $\mathcal{L}$  has a natural residuation  $A : B = \bigvee \{X \in \mathcal{L} \mid XB \leq A\}$ . So  $(A : B)B \leq A \wedge B \leq A$  and  $A : B$  is the greatest element  $C$  of  $\mathcal{L}$  with  $CB \leq A$ .

Early work in multiplicative lattices is due to M. Ward and R. P. Dilworth, especially see [2.13]. For a brief history of this early work see Dilworth's comments [2.5, pp. 305–307] and [2.3]. A number of these papers are reprinted in [2.5]. In [2.13] Ward and Dilworth defined an element  $M$  of a multiplicative lattice  $\mathcal{L}$  to be “principal” if for  $A \leq M$ , there exists  $B \in \mathcal{L}$  with  $A = BM$ . They showed that a modular multiplicative lattice satisfying ACC in which every element is a join of “principal” elements satisfied the usual Noether normal decomposition theory. However, this notion of “principal” element was too weak to prove deeper results such as the Krull Intersection Theorem or the Principal Ideal Theorem. Twenty years later Dilworth [2.6] returned to multiplicative lattice theory with a strengthened definition of a “principal” element (see [2.8] and the next paragraph). He defined a *Noether lattice* to be a modular multiplicative lattice satisfying ACC in which every element is a (finite) join of principal elements. He proved lattice versions of the Krull Intersection Theorem and the Principal Ideal Theorem.

Let  $\mathcal{L}$  be a multiplicative lattice and let  $M \in \mathcal{L}$ . Then  $M$  is *meet* (resp., *join*) *principal* if  $AM \wedge B = (A \wedge (B : M))M$  (resp.,  $(A : M) \vee B = (A \vee BM) : M$ ) for all  $A, B \in \mathcal{L}$ . And  $M$  is *weak meet* (resp., *weak join*) *principal* if these respective identities hold for  $A = I$  (resp.,  $A = 0$ ) and arbitrary  $B$ . So  $M$  is weak meet (resp., weak join) principal if  $M \wedge B = (B : M)M$  (resp.,  $(0 : M) \vee B = MB : M$ ) for all  $B \in \mathcal{L}$ . Finally,  $M$  is *(weak) principal* if  $M$  is (weak) meet principal and (weak) join principal. Note that (weak) meet principal and (weak) join principal elements are dual if we interchange multiplication and residuation and interchange meet and join. We will discuss this duality later in this section. Dilworth also observed that the product of two meet (join) principal elements is again meet (join) principal.

It is easily checked that a principal ideal of a commutative ring  $R$  is a principal element of  $\mathcal{L}(R)$ . McCarthy [2.11] has shown that an ideal  $M$  of  $R$  is a principal element of  $\mathcal{L}(R)$  if and only if  $M$  is finitely generated and locally principal. In particular, an invertible ideal is a principal element. Thus, a principal element of  $\mathcal{L}(R)$  need not be a principal ideal. In fact, as pointed out by Subramanian [2.12], since  $\mathbb{Z}$  and  $\mathbb{Z}[\sqrt{-5}]$  have isomorphic ideal lattices, there is no way to define a “principal element” in the ideal lattice  $\mathcal{L}(R)$  so that an element of  $\mathcal{L}(R)$  is a principal element of  $\mathcal{L}(R)$  if and only if it is a principal ideal of  $R$ . Thus, we view a finitely generated locally principal ideal as the appropriate generalization of a principal ideal.

The following proposition gives another point of view of principal elements and weak principal elements.

**Proposition 2.1.** [2.4, Lemma 1] *Let  $\mathcal{L}$  be a multiplicative lattice.*

- (1) *An element  $e \in \mathcal{L}$  is weak meet principal if and only if  $a \leq e$  implies  $a = qe$  for some  $q \in \mathcal{L}$ .*
- (2) *An element  $e \in \mathcal{L}$  is meet principal if and only if  $a \leq re$  implies  $a = qe$  for some  $q \leq r$ .*

- (3) If  $\mathcal{L}$  is a domain, then a nonzero element  $e \in \mathcal{L}$  is weak join principal if and only if  $e$  is a cancellation element (i.e.,  $ae = be$  implies  $a = b$ ).
- (4) An element  $e \in \mathcal{L}$  is weak join principal if and only if  $ae = be$  implies  $a \vee (0 : e) = b \vee (0 : e)$  (or equivalently,  $ae \leq be$  implies  $a \leq b \vee (0 : e)$ ).
- (5) An element  $e \in \mathcal{L}$  is join principal if and only if  $e$  is weak join principal in  $\mathcal{L}/a$  for all  $a \in \mathcal{L}$ .

Let  $M$  be an ideal of a commutative ring  $R$ . It follows from the previous proposition that  $M$  is a weak meet principal element of  $\mathcal{L}(R)$  if and only if  $M$  is a multiplication ideal and  $M$  is a weak join principal element of  $\mathcal{L}(R)$  (resp., with  $0 : M = 0$ ) if and only if  $M$  is a weak cancellation ideal (resp., cancellation ideal).

In a modular multiplicative lattice, it is not hard to show that an element is principal if and only if it is weak principal. However, in a nonmodular multiplicative lattice, a weak meet principal element need not be meet principal, and a meet principal element that is weak join principal need not be join principal. And even in a Noether lattice a weak join principal element need not be join principal. See [2.4] for details.

The non-Noetherian analog of a Noether lattice is the  $r$ -lattice [2.1]. A modular multiplicative lattice  $\mathcal{L}$  is an  $r$ -lattice if (1) every element of  $\mathcal{L}$  is a join of principal elements (i.e.,  $\mathcal{L}$  is principally generated), (2) every element of  $\mathcal{L}$  is a join of compact elements (i.e.,  $\mathcal{L}$  is compactly generated) (recall that  $A \in \mathcal{L}$  is *compact* if  $A \leq \bigvee B_\alpha$  implies  $A \leq B_{\alpha_1} \vee \cdots \vee B_{\alpha_n}$  for some finite subset  $\{B_{\alpha_1}, \dots, B_{\alpha_n}\} \subseteq \{B_\alpha\}$ ; an ideal of a ring is compact if and only if it is finitely generated), and (3)  $I$  is compact. If  $R$  is a (graded) commutative ring,  $(\mathcal{L}_h(R)) \mathcal{L}(R)$  is an  $r$ -lattice. Also, if  $S$  is a cancellation monoid with zero,  $\mathcal{L}(S)$  is an  $r$ -lattice. If  $\mathcal{L}$  is an  $r$ -lattice and  $a \in \mathcal{L}$ , then  $\mathcal{L}/a = \{b \in \mathcal{L} | b \geq a\}$  is an  $r$ -lattice with product  $b \circ c = bc \vee a$ . If  $S$  is a multiplicatively closed subset of  $\mathcal{L}$ , then there is a localization theory for  $\mathcal{L}$  and the localization  $\mathcal{L}_S$  is again an  $r$ -lattice; see [2.1] for details. If  $A \in \mathcal{L}$  is principal, then  $A/a$  is principal in  $\mathcal{L}/a$  and  $A_S$  is principal in  $\mathcal{L}_S$ . We have the following results concerning principal elements in  $r$ -lattices.

**Theorem 2.2.** *Let  $\mathcal{L}$  be an  $r$ -lattice and  $A \in \mathcal{L}$ .*

- (1)  $A$  is a principal element if and only if  $A$  is compact and  $A_M$  is a principal element of  $\mathcal{L}_M$  for each maximal element  $M$  of  $\mathcal{L}$  ( $\mathcal{L}_M = \mathcal{L}_S$  where  $S = \{B \in \mathcal{L} | B \not\leq M\}$ ).
- (2) For a quasilocal  $r$ -lattice  $\mathcal{L}$ , the following are equivalent: (a)  $A$  is principal, (b)  $A$  is weak meet principal, and (c)  $A$  is completely join irreducible.
- (3)  $A$  is principal if and only if  $A$  is compact and weak meet principal.
- (4)  $A$  is weak meet principal if and only if  $A$  is meet principal.
- (5) If  $\mathcal{L}$  is a domain, a compact join principal element is principal.

*Proof.* (1), (2), and (3) may be found in [2.1] while (4) and (5) are given in [2.4].

□

Weak join principal and join principal elements are much less understood. See [2.4, 2.7], and [2.9] for some results on (weak) join principal elements. We mention only the following results.

**Theorem 2.3.** (1) Let  $\mathcal{L}$  be a quasilocal  $r$ -lattice and let  $e$  be a compact, join principal element of  $\mathcal{L}$ . There exist principal elements  $e_1, \dots, e_n \in \mathcal{L}$  with  $e = e_1 \vee \dots \vee e_n$  and  $e_i(\bigvee_{j \neq i} e_j) = 0$ , for all  $i = 1, \dots, n$ . (2) Let  $\mathcal{L}$  be a local Noether lattice satisfying the weak union condition (if  $a, b, c \in \mathcal{L}$ ,  $a \not\leq b$  and  $a \not\leq c$ , then there is a principal element  $e \leq a$  with  $e \not\leq b$  and  $e \not\leq c$ ;  $\mathcal{L}(R)$ ,  $R$  a commutative ring, satisfies this condition). If  $a \in \mathcal{L}$  is join principal, then  $a = e \vee ((0 : a) \wedge a)$  for some principal element  $e \in \mathcal{L}$ . Thus,  $a^2 = e^2$  is principal and if  $0 : a = 0$ ,  $a$  is principal.

*Proof.* (1) is given in [2.4] and (2) in [2.7].  $\square$

We end this section with the promised duality of principal elements with respect to a lattice module. Let  $\mathcal{L}$  be a multiplicative lattice. An  $\mathcal{L}$ -module  $\mathcal{M}$  is a complete lattice  $\mathcal{M}$  with a scalar product  $AN \in \mathcal{M}$  for  $A \in \mathcal{L}$  and  $N \in \mathcal{M}$  satisfying (1)  $(\bigvee_{\alpha} A_{\alpha})N = \bigvee_{\alpha} A_{\alpha}N$ , (2)  $A(\bigvee_{\alpha} N_{\alpha}) = \bigvee_{\alpha} AN_{\alpha}$ , (3)  $(JK)N = J(KN)$ , (4)  $IN = N$ , and (5)  $0N = 0_{\mathcal{M}}$  for all elements  $A, A_{\alpha}, J, K \in \mathcal{L}$  and  $N, N_{\alpha} \in \mathcal{M}$ . For the rather well developed theory of lattice modules, see [2.10] and other papers by E. W. Johnson and/or J. A. Johnson.

Let  $\mathcal{M}$  be an  $\mathcal{L}$ -module. Now  $\mathcal{M}^*$ , the lattice dual of  $\mathcal{M}$ , is a complete lattice with  $\bigvee^* N_{\alpha} = \bigwedge N_{\alpha}$ ,  $\bigwedge^* N_{\alpha} = \bigvee N_{\alpha}$ ,  $0_{\mathcal{M}^*} = I_{\mathcal{M}}$ , and  $I_{\mathcal{M}^*} = 0_{\mathcal{M}}$ . Moreover,  $\mathcal{M}^*$  is an  $\mathcal{L}$ -module with the new scalar product  $J * N = N : J = \bigvee \{X \in \mathcal{M} \mid JX \leq N\}$ . An element  $M \in \mathcal{L}$  is  $\mathcal{M}$ -meet (-join) principal if  $M(A \wedge (B : M)) = MA \wedge B$  ( $(A \vee MB) : M = (A : M) \vee B$ ) for all  $A, B \in \mathcal{M}$ . As expected, we define  $M \in \mathcal{L}$  to be  $\mathcal{M}$ -principal if  $M$  is both  $\mathcal{M}$ -meet principal and  $\mathcal{M}$ -join principal. Analogous definitions are given for the “weak” case. Generalizing the notion of a cyclic submodule of an  $R$ -module, there are also the notions of (weak) meet principal, (weak) join principal, and (weak) principal elements of a lattice module; see [2.10]. The next theorem exhibits the promised duality between meet principal and join principal elements.

**Theorem 2.4.** [2.2] Let  $\mathcal{L}$  be a multiplicative lattice,  $\mathcal{M}$  an  $\mathcal{L}$ -module, and  $\mathcal{M}^*$  the  $\mathcal{L}$ -module dual of  $\mathcal{M}$ . An element  $M \in \mathcal{L}$  is  $\mathcal{M}$ -meet (-join) principal if and only if  $M$  is  $\mathcal{M}^*$ -join (-meet) principal. An analogous result holds for the “weak” case. In particular,  $M$  is  $\mathcal{M}$ -principal if and only if  $M$  is  $\mathcal{M}^*$ -principal.

This duality was used in [2.2] to develop a theory of co-primary decomposition and co-grade for Artinian  $R$ -modules.

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### 3 Multiplication ideals, rings, and modules

Let  $R$  be a commutative ring. An ideal  $I$  is a *multiplication ideal* if for each ideal  $A \subseteq I$ , there is an ideal  $C$  with  $A = CI$ ; we can take  $C = A : I$ . The ring  $R$  is a *multiplication ring* if every ideal of  $R$  is a multiplication ideal. And an  $R$ -module  $M$  is a *multiplication module* if for each submodule  $N$  of  $M$ ,  $N = AM$  for some ideal  $A$  of  $R$ ; we can take  $A = N : M$ . Clearly a principal ideal (resp., cyclic  $R$ -module) is a multiplication ideal (resp., multiplication module). Also, when working with a multiplication module  $M$  we can usually assume that  $M$  is faithful by passing to  $R/(0 : M)$ .

Early work focused mostly on multiplication rings which will be discussed at the end of this section. Perhaps the first paper to focus on multiplication ideals in their own right was [3.18] where it was shown that a finitely generated multiplication ideal in a quasilocal ring is principal and that if  $J$  is a finitely generated multiplication ideal, then  $J_P$  is a principal ideal for each prime  $P$ . In [3.6] it was shown that a finitely generated multiplication ideal  $I$  with  $0 : I$  contained in only finitely many maximal ideals is principal. In [3.1], we have the result that a multiplication ideal in a quasilocal ring is principal. We give the simple proof. The result carries over to multiplication modules over quasilocal rings, mutatis mutandis [3.5]. Theorem 3.1 easily extends to the result that a multiplication ideal (resp., module)  $I$  with  $0 : I$  contained in only finitely many maximal ideals is principal, (resp., cyclic).

**Theorem 3.1.** [3.1] *In a quasilocal ring every multiplication ideal is principal.*

*Proof.* Let  $(R, M)$  be a quasilocal ring and  $A$  a multiplication ideal in  $R$ . Suppose that  $A = \Sigma(x_\alpha)$ . Then,  $(x_\alpha) = I_\alpha A$  for some ideal  $I_\alpha$  since  $A$  is a multiplication ideal. Hence,  $A = \Sigma(x_\alpha) = \Sigma I_\alpha A = (\Sigma I_\alpha)A$ . If  $\Sigma I_\alpha = R$ , then  $I_{\alpha_0} = R$  for some index  $\alpha_0$  because  $R$  is quasilocal. In this case,  $A = I_{\alpha_0} A = (x_{\alpha_0})$ . If  $\Sigma I_\alpha \neq R$ , then  $A = MA$ . Suppose that  $x \in A$ . Then, there exists an ideal  $C$  with  $(x) = CA = C(MA) = M(CA) = M(x)$ ; so  $x = 0$  by Nakayama's Lemma. Thus,  $A = 0$  is principal.  $\square$

It is easily seen that if  $I$  is a multiplication ideal and  $S$  is a multiplicatively closed subset of  $R$ , then  $I_S$  is a multiplication ideal of  $R_S$  (a similar result holds for multiplication modules). Hence, a multiplication ideal is locally principal. Moreover, for  $I$  finitely generated,  $I$  is a multiplication ideal if and only if  $I$  is finitely generated (since for  $I$  finitely generated, the equation  $I \cap B = (B : I)I$  holds if and only if it holds locally). See Theorem 4.4. However, a locally principal ideal need not be a multiplication ideal (e.g., a nonfinitely generated ideal in an almost Dedekind domain). Conditions needed for a locally principal ideal to be a multiplication ideal are given in Theorem 3.3.

However, the serious study of multiplication ideals was inaugurated in [3.2]. The principal tool introduced was a variant of the trace ideal. Let  $I$  be an ideal of the commutative ring  $R$ . Define  $\theta(I) = \sum_{x \in I} (Rx : I)$ . Then,  $\theta(I)$  is an ideal with  $I \subseteq \theta(I) \subseteq R$ .

The following theorem is a sample from [3.2].

**Theorem 3.2.** (1) *For an ideal  $I$  in a commutative ring, the following three conditions are equivalent: (a)  $I$  is meet principal, i.e.,  $AI \cap B = (A \cap (B : I))I$  for ideals  $A$  and  $B$  of  $R$ , (b)  $I$  is a multiplication ideal, and (c) if  $M \supseteq \theta(I)$  is a maximal ideal, then  $I_M = 0_M$ .*

- (2) *An ideal  $I$  is finitely generated and locally principal if and only if  $\theta(I) = R$ .*
- (3) *If  $I$  is a multiplication ideal with  $\text{ht} I > 0$ , then  $I$  is finitely generated.*
- (4) *For a multiplication ideal  $I$  and  $i \in I$ ,  $iI$  is finitely generated.*
- (5) *Let  $M$  be a maximal ideal of  $R$ . Then  $M$  is a multiplication ideal if and only if either (a)  $M$  is finitely generated and locally principal or (b)  $R_M$  is a field. If  $M$  is finitely generated and  $\text{ht} M = 0$ , then  $M$  is principal and there exists a positive integer  $n$  such that  $M^n$  is generated by an idempotent, and  $R_M \approx R/M^n$  is a direct summand of  $R$ .*

Theorems 3.1 and 3.2 carry over to multiplication modules, mutatis mutandis, where now for a module  $M$ ,  $\theta(M) = \sum_{m \in M} (Rm : M)$ . Alternatively, one can reduce the study of multiplication modules to multiplication ideals via idealization.

Let  $R$  be a commutative ring and  $M$  an  $R$ -module. The *idealization* of  $R$  and  $M$  is the commutative ring  $R(+M)$  with addition defined as  $(r_1, m_1) + (r_2, m_2) = (r_1 + r_2, m_1 + m_2)$  and multiplication as  $(r_1, m_1)(r_2, m_2) = (r_1 r_2, r_1 m_2 + r_2 m_1)$ . Thus,  $R(+M) = R \oplus M$  as abelian groups, but we use the notation  $(+)$  to indicate we are taking the idealization. Idealization was introduced by Nagata [3.15]; for a recent survey of idealization see [3.4]. Here,  $0 \oplus M$  is an ideal of  $R(+M)$  with  $(0 \oplus M)^2 = 0$ . For an ideal  $I$  of  $R$ ,  $(I \oplus M)(0 \oplus M) = 0 \oplus IM$ . Suppose that  $N$  is a submodule of  $M$ . Then,  $N$  is a multiplication module if and only if  $0 \oplus N$  is a multiplication ideal of  $R(+M)$ . See [3.3] for details.

A. A. El-Bast and P. F. Smith [3.8] introduced an alternative, useful method for studying multiplication modules. While their method does not make explicit use of  $\theta(M)$ , it is essentially equivalent to the  $\theta(M)$  approach. Let  $M$  be an  $R$ -module and  $\mathcal{M}$  a maximal ideal of  $R$ . They defined  $T_{\mathcal{M}}(M) = \{m \in M \mid (1 - p)m = 0 \text{ for some } p \in \mathcal{M}\}$  which is easily seen to be a submodule of  $M$ . They called  $M$   $\mathcal{M}$ -torsion if  $T_{\mathcal{M}}(M) = M$ . Since  $R - \mathcal{M}$  is the saturation of  $1 + \mathcal{M}$ ,  $m \in T_{\mathcal{M}}(M) \Leftrightarrow fm = 0$  for



some  $f \in R - \mathcal{M}$ . Hence  $T_{\mathcal{M}}(M)$  is the kernel of the natural map  $M \rightarrow M_{\mathcal{M}}$  and  $M$  is  $\mathcal{M}$ -torsion  $\Leftrightarrow M_{\mathcal{M}} = 0_{\mathcal{M}}$ . They defined  $M$  to be  $\mathcal{M}$ -cyclic if there exists  $q \in \mathcal{M}$  and  $m \in M$  such that  $(1 - q)M \subseteq Rm$ . Again,  $M$  is  $\mathcal{M}$ -cyclic  $\Leftrightarrow$  there exists  $f \in R - \mathcal{M}$  and  $m \in M$  such that  $fM \subseteq Rm \Leftrightarrow \mathcal{M} \not\supseteq \theta(M)$ . They showed that  $M$  is a multiplication module if and only if for each maximal ideal  $\mathcal{M}$  of  $R$ ,  $M$  is either  $\mathcal{M}$ -torsion or  $\mathcal{M}$ -cyclic. Before we discuss the work of P. F. Smith and his co-authors, we list some equivalent characterizations for multiplication modules taken from [3.3].

**Theorem 3.3.** *Let  $M$  be an  $R$ -module and  $A$  a submodule of  $M$ . Then, the following conditions on  $A$  are equivalent.*

- (1)  $A$  is a multiplication module.
- (2) If  $\mathcal{M}$  is a maximal ideal of  $R$  with  $\mathcal{M} \supseteq \theta(A)$ , then  $A_{\mathcal{M}} = 0_{\mathcal{M}}$ .
- (3) If  $B$  is a (cyclic) submodule of  $A$ , then  $\theta(A)B = B$ .
- (4) For each maximal ideal of  $\mathcal{M}$  of  $R$ , one of the following holds:
  - (a) For  $a \in A$ , there exists  $m \in \mathcal{M}$  with  $(1 - m)a = 0$ , i.e.,  $A$  is  $\mathcal{M}$ -torsion, or
  - (b) There exists  $a_0 \in A$  and  $m \in \mathcal{M}$  with  $(1 - m)A \subseteq Ra_0$ , i.e.,  $A$  is  $\mathcal{M}$ -cyclic.
- (5) For each maximal ideal  $\mathcal{M}$  of  $R$  with  $A_{\mathcal{M}} \neq 0_{\mathcal{M}}$ , there exists  $f \in R - \mathcal{M}$  and  $a_0 \in A$  with  $fA \subseteq Ra_0$ .
- (6) For each maximal ideal  $\mathcal{M}$  of  $R$  with  $A_{\mathcal{M}} \neq 0_{\mathcal{M}}$ ,  $A_{\mathcal{M}}$  is cyclic and  $(N : A)_{\mathcal{M}} = (N_{\mathcal{M}} : A_{\mathcal{M}})$  for each submodule  $N$  of  $M$  (of  $A$ ).
- (7)  $A$  is a meet principal submodule of  $M$ , i.e.,  $IA \cap N = (I \cap (N : A))A$  for all ideals  $I$  of  $R$  and submodules  $N$  of  $M$ .
- (8) If  $I$  is an ideal of  $R$  and  $N$  is a submodule of  $M$  with  $N \subseteq IA$ , then  $N = JA$  for some ideal  $J \subseteq I$ .

We next discuss the seminal work of P. F. Smith and his co-authors on multiplication modules. In [3.8], the useful notions of  $\mathcal{M}$ -torsion and  $\mathcal{M}$ -cyclic modules were introduced. It was shown that an  $R$ -module  $M$  is a multiplication module if and only if  $\bigcap I_{\lambda}M = (\bigcap (I_{\lambda} + \text{ann}(M)))M$  for every collection  $\{I_{\lambda}\}$  of ideals of  $R$  and that a direct sum  $\bigoplus M_{\lambda}$  of  $R$ -modules is a multiplication  $R$ -module if and only if each  $M_{\lambda}$  is a multiplication module and for each  $\lambda$ , there exists an ideal  $A_{\lambda}$  with  $A_{\lambda}M_{\lambda} = M_{\lambda}$  but  $A_{\lambda}(\sum_{\mu \neq \lambda} \oplus M_{\mu}) = 0$ . A proper submodule  $N$  of a multiplication module  $M$  is maximal (resp., prime, essential) if and only if  $N = \mathcal{M}M$  for some maximal (resp., prime, essential) ideal  $\mathcal{M}$  of  $R$ . It follows that every proper submodule of a multiplication module is contained in a proper maximal submodule. Moreover, a multiplication module with only finitely many maximal submodules is cyclic. Thus an Artinian multiplication module is cyclic.

Perhaps the neatest result of [3.8] is the following. Let  $M$  be a nonzero multiplication module with  $Z(M) = P_1 \cup \cdots \cup P_n$  and  $\text{ann}(M) \subseteq P_1 \cap \cdots \cap P_n$  for some finite set of prime ideals of  $R$  (e.g.,  $M$  is Noetherian). Then,  $M$  is isomorphic to  $B/A$  where  $A \subset B$  are ideals of  $R$  with  $B/A$  invertible in  $R/A$ . Call such a multiplication module *trivial*. Since  $Z(M) = Z(R)$  for a faithful multiplication  $R$ -module, it follows that if  $R$  is Noetherian, then every nonzero multiplication  $R$ -module is trivial. In particular, a faithful multiplication  $R$ -module over a Noetherian ring is isomorphic to an

invertible ideal and hence is finitely generated. The question of which rings  $R$  have the property that each finitely generated multiplication  $R$ -module is trivial is considered in [3.13]. For example, it is shown that every finitely generated  $R$ -module is trivial if and only if for every finitely generated multiplication  $R$ -module  $M$  we have  $0 : M = 0 : m$  for some  $m \in M$ . The somewhat surprising result is given that for any commutative ring  $S$  and nonempty set of indeterminates  $\{X_\lambda\}$  over  $S$ , every finitely generated faithful multiplication module over  $R = S[\{X_\lambda\}]$  is trivial.

The paper [3.9] considers generalizations of multiplication modules to rings without identity or modules which need not be unitary. Generalizing the case for an ideal, an  $R$ -module  $M$  is called an *AM-module* if for each proper submodule  $N$  of  $M$ ,  $N = IM$  for some ideal  $I$  of  $R$ . Whether or not  $R$  has an identity, let  $R' = R \oplus \mathbb{Z}$  be the Dorroh extension of  $R$ , so  $R'$  has an identity and every  $R$ -module is naturally a unitary  $R'$ -module. If  $M$  is an *AM-module*, then  $M$  is a multiplication  $R'$ -module. Call  $M$  is *weak AM-module* if  $M$  is a multiplication  $R'$ -module; so an *AM-module* is a *weak AM-module*, but not conversely. Also, an *AM-module*  $M$  is *almost unitary* in the sense that for each proper submodule  $N$  of  $M$ ,  $N \subseteq RM$ . The relationship between these three properties is thoroughly investigated. Also, Mott's Theorem for multiplication rings (discussed later in this section) is generalized to *AM-modules*: Every prime submodule of an  $R$ -module  $M$  is an *AM-module* implies every submodule of  $M$  is an *AM-module*.

The literature on multiplication modules is quite extensive; consult *Math Reviews*. Space does not permit us to discuss the many interesting results on finitely generated multiplication modules obtained by A. G. Naoum and M. A. K. Hasan using matrix methods. Some of their results are generalized in [3.16]. Finally, [3.17] relates multiplication modules to projective modules. Now W. W. Smith [3.18] showed that a projective ideal is a multiplication ideal and clearly a free  $R$ -module is multiplication if and only if it has rank 1. In [3.17], it is shown that  $M$  being a multiplication  $R$ -module is equivalent to  $M$  being a finitely projective  $R/(0 : M)$ -module (i.e., for every finitely generated submodule  $N$  of  $M$ , there exists  $n \geq 1$  and  $m_i \in M$  and  $R$ -homomorphisms  $\theta_i : M \rightarrow R$  so that  $x = \theta_1(x)m_1 + \cdots + \theta_n(x)m_n$  for all  $x \in N$ ) and any one of the following conditions holding (1) every submodule of  $M$  is fully invariant, (2)  $\text{End}(M)$  is commutative, or (3)  $M$  is locally cyclic.

We end this section with a brief report on multiplication rings, that is, rings in which every ideal is a multiplication ideal. For simplicity, we assume that our rings have an identity. Multiplication rings were introduced by Krull in 1936 and the early theory is mostly due to Mori. See [3.10] for references. We remark that [3.10] and [3.11] treat rings satisfying conditions weaker than the existence of an identity. See [3.12] for a very readable account of multiplication rings. Let  $R$  be a multiplication ring. For a maximal ideal  $M$  of  $R$ , each ideal of  $R_M$  is the localization of a multiplication ideal and thus is a multiplication ideal of  $R_M$ . So every ideal of  $R_M$  is principal; thus  $R_M$  is either a DVR or a SPIR. Call a ring  $R$  with property that each  $R_M$  is a DVR or SPIR an *almost multiplication ring* [3.7]. Thus, a multiplication ring is an almost multiplication, but the converse is false as an almost Dedekind domain is an almost multiplication ring but need not be a multiplication ring.

For an ideal  $A$  of a commutative ring  $R$ , the kernel of  $A$  is  $\ker A = \bigcap \{A_P \cap R \mid P \text{ is a minimal prime of } A\} = \bigcap \{Q \mid Q \supseteq A \text{ is } P\text{-primary where } P \text{ is a minimal prime of } A\}$ . In [3.7] it is shown that  $R$  is an almost multiplication ring if and only if each ideal with prime radical is a prime power and that in an almost multiplication ring every ideal is equal to its kernel. In [3.10] it is shown that every ideal with prime radical is primary if and only if every ideal is equal to its kernel. The following theorem gives a number of characterizations of multiplication rings.

**Theorem 3.4.** *For a commutative ring  $R$  the following conditions are equivalent.*

- (1)  $R$  is a multiplication ring.
- (2) Each prime ideal of  $R$  is a multiplication ideal.
- (3)
  - (a) Every ideal is equal to its kernel,
  - (b) Every primary ideal is a power of its radical,
  - (c) If  $P$  is a minimal prime of an ideal  $B$  and  $n$  is the least positive integer such that  $P^n$  is an isolated component of  $B$  and if  $P^n \neq P^{n+1}$ , then  $P$  does not contain the intersection of the remaining isolated primary components of  $B$  (or equivalently, if  $B \subseteq P^n$  but  $B \not\subseteq P^{n+1}$ , then  $P^n = B : (y)$  for some  $y \in R - P$ ).
- (4)  $T(R)$  is a multiplication ring, every regular ideal of  $R$  is invertible, and any nonmaximal prime ideals of  $R$  are idempotent.
- (5) For each prime ideal of  $R$ ,  $P$  is invertible, or  $R_P$  is a field, or  $P$  is maximal and  $R_P$  is an SPIR and there exists an idempotent contained in all prime ideals of  $R$  except  $P$ .

*Proof.* The equivalence of (1)–(3) for rings with identity is given in [3.14]. This is generalized to conditions weaker than having an identity in [3.10]. For the equivalence of (1) and (2) also see [3.12]. The equivalence of (1), (4), and (5) is given in [3.11], again in a context more general than rings with an identity. Finally, we remark that [3.2] contains a simplified proof of the equivalence of (1), (2), and (5).  $\square$

The paper by Griffin [3.11] contains many more interesting results on multiplication rings and is a “must read” for anyone contemplating research on multiplication rings.

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## 4 Cancellation ideals and modules

Let  $R$  be a commutative ring. An ideal  $I$  of  $R$  is a *cancellation ideal* if whenever  $IA = IB$  for ideals  $A$  and  $B$  of  $R$ , we have  $A = B$ . A principal ideal of  $R$  is a cancellation ideal if and only if it is regular, clearly a cancellation ideal is faithful, and an invertible ideal is a cancellation ideal. Unlike the case for multiplication ideals, it is not at all clear that the localization of a cancellation ideal is again a cancellation ideal. However, if  $I_M$  is a cancellation ideal for each maximal ideal  $M$  of  $R$ , then  $IA = IB$  gives  $I_M A_M = I_M B_M$  for every maximal ideal  $M$ , so  $A_M = B_M$ , and hence  $A = B$ ; so  $I$  is a cancellation ideal. Thus, an ideal that is locally a regular principal ideal is a cancellation ideal and as we shall see, the converse is true. Note that if  $a, b \in R$ , then  $(a, b)(a, b)^2 = (a, b)^3 = (a, b)(a^2, b^2)$ , but in general  $(a, b)^2 \neq (a^2, b^2)$ . For a good introduction to cancellation ideals, see [1.1].

An integral domain  $R$  is *almost Dedekind* if  $R_M$  is a DVR for each maximal ideal  $M$  of  $R$ . Gilmer [4.6] and Jensen [4.10] independently showed that a domain  $R$  has every nonzero ideal a cancellation ideal if and only if  $R$  is almost Dedekind.

The first progress in characterizing cancellation ideals was made by Kaplansky [4.11] and is given in [1.1, Exercise 7, page 67].

**Proposition 4.1.** *Let  $(R, M)$  be a quasilocal ring,  $A$  an ideal of  $R$ ,  $x_1, \dots, x_n \in R$  and  $B = A + (x_1, \dots, x_n)$ . If  $B$  is a cancellation ideal, then  $B = A + (x_i)$  for some  $i$ . In particular, if  $B$  is a finitely generated cancellation ideal, then  $B$  is principal and generated by a regular element.*

*Proof.* It suffices to do the case  $n = 2$ ; let  $B = A + (x, y)$ . Let  $J = (x^2 + y^2, xy, xA, yA, A^2)$ . Then,  $BJ = BB^2$ ; so  $J = B^2$ . So  $x^2 = \lambda(x^2 + y^2) +$  terms

from  $(xy, xA, yA, A^2)$ . If  $\lambda \in M$ ,  $1 - \lambda$  is a unit and  $(1 - \lambda)x^2 = y^2 + \dots$ ; so  $x^2 \in (y^2, xy, xA, yA, A^2)$ . Let  $K = (y) + A$ ; so  $B^2 = BK$ , and hence  $B = K$ . Next, suppose that  $\lambda \notin M$ , so  $\lambda$  is a unit. Then,  $y^2 \in (x^2, xy, xA, yA, A^2)$  and with a proof similar to the case  $\lambda \in M$ , we get  $B = (x) + A$ .  $\square$

**Theorem 4.2.** [4.4] *Let  $R$  be a commutative ring with identity. An ideal  $I$  of  $R$  is a cancellation ideal if and only if  $I$  is locally a regular principal ideal.*

*Proof.* We have already remarked that  $\Leftarrow$  holds. ( $\Rightarrow$ ) Let  $M$  be a maximal ideal of  $R$ . We show that  $I_M$  is a regular principal ideal. We can assume that  $I \subseteq M$ . Choose a subset  $\{b_\alpha\}_{\alpha \in \Lambda}$  of  $I$  so that  $\{\bar{b}_\alpha\}$  is an  $R/M$ -basis for  $I/MI$ . Suppose  $|\Lambda| > 1$ . Then, for distinct  $\alpha_1, \alpha_2 \in \Lambda$ , we get  $I = (b_{\alpha_1}, b_{\alpha_2}) + (\{b_\alpha | \alpha \in \Lambda - \{\alpha_1, \alpha_2\}\}) + MI$ . Now a modification of the proof of Proposition 4.1 by replacing “ $R$  is quasilocal” by “ $A \supseteq MB$ ” gives that say  $I = (b_{\alpha_1}) + (\{b_\alpha | \alpha \in \Lambda - \{\alpha_1, \alpha_2\}\}) + MI$ . But then  $\{\bar{b}_\alpha | \alpha \in \Lambda - \{\alpha_2\}\}$  is an  $R/M$ -basis for  $I/MI$ , a contradiction. Hence,  $I = (a) + MI$  for some  $a \in I$ . Let  $b \in I$ . Then,  $(b)I = (b)((a) + MI) = (a)(b) + M(b)I \subseteq (a)I + M(b)I = ((a) + M(b))I$ . Hence,  $(b) \subseteq (a) + M(b)$ . So  $(b)_M \subseteq (a)_M$  and hence  $I_M = (a)_M$ . Suppose  $ca = 0$  in  $R_M$ . Then,  $(cI)_M = (ca)_M = 0_M$  so  $(cI)_M = (cMI)_M$ . Since  $(cI)_N = (cMI)_N$  for the other maximal ideals  $N$ ,  $cI = cMI$  and hence  $(c) = (c)M$ . Thus,  $c = 0$  in  $R_M$ ; so  $I_M$  is regular.  $\square$

It should be noted that while a cancellation ideal  $I$  is locally a regular principal ideal,  $I$  need not be regular, even if  $I$  is finitely generated [1.1, Exercise 10, page 456]. We have the following immediate corollary to Theorem 4.2.

**Corollary 4.3.** (1) *Let  $R$  be a commutative ring,  $I$  a cancellation ideal of  $R$ , and  $S$  a multiplicatively closed subset of  $I$ . Then,  $I_S$  is a cancellation ideal of  $R_S$ .* (2) *Let  $R$  be a subring of the integral domain  $T$ . If  $I$  is a cancellation ideal of  $R$ , then  $IT$  is a cancellation ideal of  $T$ .*

In [4.5], nonzero locally principal ideals in an integral domain are investigated with an emphasis on when they are invertible (or equivalently, finitely generated). It is shown that for a nonzero-ideal  $I$  in an integral domain  $D$ , the following conditions are equivalent: (1)  $I$  is locally principal, (2)  $I$  is a cancellation ideal, and (3)  $I$  is a faithfully flat  $D$ -module. The proof shows that (2)  $\Leftrightarrow$  (3) for any commutative ring. A domain  $D$  is called an *LPI-domain* if each nonzero locally principal ideal is invertible. It is shown that a finite character intersection of LPI-domains is again an LPI-domain.

An ideal  $I$  is called a *quasi-cancellation ideal* [4.3] if  $IB = IC$  for finitely generated ideals  $B$  and  $C$  of  $R$  implies  $B = C$ . While a finitely generated quasi-cancellation ideal is a cancellation ideal, for any valuation domain  $(V, M)$  and  $0 \neq x \in M$ ,  $Mx$  is a quasi-cancellation ideal.

The notion of a cancellation ideal can be generalized to modules in several ways. Let  $R$  be a commutative ring and  $M$  an  $R$ -module. Following [4.11], we say that  $M$  is a (weak) *cancellation module* if for ideals  $I$  and  $J$  of  $R$ ,  $IM = JM$  implies  $I = J$  ( $I + 0 : M = J + 0 : M$ ). And  $M$  is a *restricted cancellation module* [4.2] if  $IM = JM \neq 0$  implies  $I = J$ . So a weak cancellation module  $M$  is a cancellation

module if and only if it is faithful and an  $R$ -module  $M$  is a weak cancellation  $R$ -module if and only if  $M$  is a cancellation  $R/(0:M)$ -module. Less obvious is that  $M$  is a restricted cancellation  $R$ -module if and only if  $M$  is a weak cancellation module and  $0 : M$  is comparable to each ideal of  $R$ . In terms of the lattice of submodules, a submodule  $N$  of an  $R$ -module  $M$  is a weak cancellation module if and only if  $N$  is a weak join principal element of  $\mathcal{L}(M)$ . If  $M$  is a cancellation  $R$ -module, then  $M \oplus N$  is a cancellation  $R$ -module for any  $R$ -module  $N$ ; hence,  $R \oplus N$  is a cancellation  $R$ -module.

Perhaps the appropriate cyclic-like generalization of a cyclic module is a finitely generated module that is locally cyclic. Our next theorem gives several characterizations of such modules.

**Theorem 4.4.** *For an  $R$ -module  $M$  the following conditions are equivalent.*

- (1)  $M$  is a finitely generated multiplication module.
- (2)  $M$  is finitely generated and locally cyclic.
- (3)  $M$  is a multiplication module and a weak cancellation module.
- (4)  $M$  is a (weak) principal element of  $\mathcal{L}(M)$ , the lattice of submodules of  $M$ .

*Proof.* (1) $\Rightarrow$ (2) This follows from Theorem 3.1 and the fact that a localization of a multiplication module is a multiplication module. (2) $\Rightarrow$ (3) For a finitely generated module the properties of being a multiplication module or a weak cancellation module hold if and only if they hold locally. (3) $\Rightarrow$ (4) This follows from the definitions and the fact that a weak principal element is a principal element in  $\mathcal{L}(M)$ . (4) $\Rightarrow$ (2) This is the previously mentioned result of McCarthy generalized to modules. (2) $\Rightarrow$ (1) This follows from (2) $\Rightarrow$ (3).  $\square$

As with multiplication modules, the study of the various types of cancellation modules can be reduced to the ideal case via idealization. Let  $M$  be an  $R$ -module and  $N$  a submodule of  $M$ . In [4.2] it was shown that (1)  $N$  is a weak cancellation submodule of  $M$  if and only if  $0 \oplus N$  is a weak cancellation ideal of  $R(+)M$ , (2)  $N$  is a cancellation submodule if and only if  $0 \oplus N$  is a weak cancellation ideal of  $R(+)M$  and  $0 : (0 \oplus N) = 0 \oplus M$ , and (3)  $0 \oplus N$  is a restricted cancellation ideal of  $R(+)M$  if and only if  $N$  is a restricted cancellation submodule and for  $r \in R$ ,  $rN \neq 0$  implies  $rM = M$ .

Using the previous results concerning idealization, we can give an example of a weak cancellation ideal  $P$  that is not a join principal ideal, i.e., some homomorphic image of  $P$  is not a weak cancellation ideal.

*Example 4.5.* Let  $(R, \mathcal{M})$  be an  $n$ -dimensional local domain that is not a DVR and let  $M = R \oplus \mathcal{M}$ . Hence  $M$  is a cancellation  $R$ -module. Then  $R(+)M$  is an  $n$ -dimensional local ring with unique minimal prime  $P = 0 \oplus M$  and  $P^2 = 0$ . Since  $M$  is a cancellation  $R$ -module,  $P$  is a weak cancellation ideal of  $R(+)M$ . Since  $\mathcal{M}$  is not a cancellation ideal of  $R$ ,  $0 \oplus \mathcal{M}$  is not a weak cancellation submodule of  $M = R \oplus \mathcal{M}$ . So  $(0 \oplus M)/(0 \oplus (R \oplus 0)) \approx 0 \oplus (0 \oplus \mathcal{M})$  is not a weak cancellation ideal of  $(R \oplus M)/(0 \oplus (R \oplus 0)) \approx R(+) (0 \oplus \mathcal{M})$ . So  $P$  is not a join principal ideal of  $R(+)M$ .

If  $M$  is an  $R$ -module that is locally a cancellation module (i.e.,  $M_{\mathcal{M}}$  is a cancellation  $R_{\mathcal{M}}$ -module for each maximal ideal  $\mathcal{M}$  of  $R$ ), then  $M$  is a cancellation module. It is shown in [4.2] that the converse is true for a one-dimensional domain. The general case remains open.

Our next result characterizes cancellation modules over a principal ideal ring  $R$ . Since a PIR is a finite direct product of PIDs and SPIRs, Theorem 4.6(1) reduces the question to the case where  $R$  is a SPIR or PID.

**Theorem 4.6.** [4.2] (1) *Let  $R = R_1 \times \cdots \times R_n$  where each  $R_i$  is a commutative ring with identity. Let  $M = M_1 \times \cdots \times M_n$  where  $M_i$  is an  $R_i$ -module; so  $M$  is naturally an  $R$ -module. Then  $M$  is a (weak) cancellation  $R$ -module if and only if each  $M_i$  is a (weak) cancellation  $R_i$ -module. However,  $M$  is a restricted cancellation  $R$ -module if and only if either  $n = 1$  and  $M = M_1$  is a restricted cancellation  $R_1$ -module or  $n > 1$  and either  $M = 0$  or  $M$  is a cancellation  $R$ -module.*

(2) *Suppose that  $R$  is an SPIR and  $M$  an  $R$ -module. Then every  $R$ -module is a weak cancellation  $R$ -module and a restricted cancellation  $R$ -module. But  $M$  is a cancellation  $R$ -module if and only if  $M$  is faithful.*

(3) *Let  $R$  be a PID and  $M$  an  $R$ -module.*

(a)  *$M$  is a weak cancellation module if and only if  $M$  is a cancellation module or  $M$  is not faithful.*

(b) *If  $R$  has a unique maximal ideal,  $M$  is a restricted cancellation module if and only if  $M$  is a weak cancellation module. If  $R$  has more than one maximal ideal, then  $M$  is a restricted cancellation module if and only if  $M = 0$  or  $M$  is a cancellation module.*

(c)  *$M$  is a cancellation  $R$ -module if and only if for each maximal ideal  $\mathcal{M}$  of  $R$ , if  $M_{\mathcal{M}} = A \oplus B$  where  $A$  is a divisible  $R_{\mathcal{M}}$ -module and  $B$  is a reduced  $R_{\mathcal{M}}$ -module, then  $B$  is faithful.*

Space does not permit us to discuss the work of M. Ali (see [4.1] for example) and especially A. G. Naoum (see [4.12] for example). One topic covered is the notion of a 1/2 (weak) cancellation module:  $M = IM$  implies  $I = R$  ( $I + 0 : M = R$ ).

We end this section with a brief discussion of an alternative definition of a cancellation  $R$ -submodule of  $K$ , the quotient field of  $R$ , due to Goeters and Olberding [4.7, 4.8, 4.9]. They defined an  $R$ -submodule  $X$  of  $K$  to be a ‘‘cancellation module’’ if  $XW = XY$  for  $R$ -submodules  $W$  and  $Y$  of  $K$  implies  $W = Y$ . Here  $XW$  is the  $R$ -submodule generated by  $\{xw | x \in X, w \in W\}$ . To avoid confusion we call such an  $R$ -module  $X$  a *GO-cancellation module*. They showed [4.7] that for a submodule  $X$  of  $K$ , the following are equivalent: (1)  $X$  is a GO-cancellation module for  $R$ , (2)  $X$  is locally a free  $R$ -module, (3)  $X$  is a faithfully flat  $R$ -module. Certainly a GO-cancellation module is a cancellation module.

Goeters and Olberding [4.8] defined an ideal  $I$  of a domain  $R$  to have *restricted cancellation* if  $IJ = IK$  implies  $J = K$  for nonzero ideals  $J$  and  $K$  of  $R$  with  $(I : I) \subseteq (J : J) \cap (K : K)$ . They showed that this is equivalent to  $I$  being a cancellation ideal of  $(I : I)$ . The domain  $R$  is said to have *restricted cancellation* if each nonzero ideal of  $R$  has restricted cancellation. In [4.9] they showed that  $R$  has restricted cancellation if

and only if (a)  $R_M$  is stable (each nonzero ideal of  $R_M$  is invertible in  $(R_M : R_M)$ ) for each maximal ideal  $M$  of  $R$  and (b)  $\text{Spec}(R/P)$  is Noetherian for each nonzero prime ideal  $P$  of  $R$ .

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## 5 Gaussian polynomials and rings

This section is an update of Section 8 Content Formulas and Gaussian Polynomials of the author’s survey article [5.2].

Let  $R$  be a commutative ring with identity. For  $f = a_0 + a_1X + \cdots + a_nX^n$ , the *content* of  $f$  is  $c(f) = (a_0, \dots, a_n)$ . For  $g \in R[X]$ , it is clear that  $c(fg) \subseteq c(f)c(g)$ , but we may have strict containment ( $R = \mathbb{Z} + 2i\mathbb{Z}$ ,  $f = 2i + 2X = g$ ; so  $c(fg) = (4) \subsetneq (4, 4i) = c(f)c(g)$ ). The polynomial  $f \in R[X]$  is said to be *Gaussian* if  $c(fg) = c(f)c(g)$  for all  $g \in R[X]$  and  $R$  is *Gaussian* if each  $f \in R[X]$  is Gaussian; i.e., the “content formula”  $c(fg) = c(f)c(g)$  holds for all  $f, g \in R[X]$ . Since  $f \in R[X]$  is Gaussian if and only if  $f/1 \in R_M[X]$  is Gaussian for each maximal ideal  $M$  of  $R$ , most questions concerning Gaussian polynomials can be reduced to the quasilocal case. In particular,  $R$  is Gaussian if and only if each localization  $R_M$  is Gaussian.

For any commutative ring  $R$  and  $f, g \in R[X]$  we have the Dedekind–Mertens Lemma:  $c(fg)c(g)^m = c(f)c(g)^{m+1}$  where  $m+1$  is the number of elements needed to generate  $c(f)$  locally. Hence, if  $c(g)$  is a cancellation ideal (e.g., invertible),  $g$  is



Gaussian. Thus, a Prüfer domain is Gaussian. For more on the Dedekind–Mertens Lemma the reader is referred to [5.2].

Gaussian polynomials and rings were first considered by H. Tsang [5.10] (a.k.a. H. T. Tang) who showed that if  $c(f)$  is locally principal, then  $f$  is Gaussian. The converse is of course false for if  $(R, M)$  is a quasilocal ring with  $M^2 = 0$ , then every  $f \in R[X]$  is Gaussian. This leads to the following question first asked by Kaplansky. Let  $R$  be a (quasilocal) ring and let  $f \in R[X]$  be Gaussian. Suppose that  $c(f)$  is a regular ideal, is  $c(f)$  (principal) invertible?

For more on the Dedekind–Mertens Lemma, its history and generalizations and for results on the “content formula” for power series, monoid rings and graded rings and involving star operations, see [5.2]. Concerning material from [5.2] we content ourselves to a brief review of Kaplansky’s question.

It is not hard to show that if  $(R, M)$  is a quasilocal domain and  $f \in R[X]$  is Gaussian with  $c(f)$  doubly generated, then  $c(f)$  is principal. The first real progress on Kaplansky’s question was made by Glaz and Vasconcelos [5.5] via Hilbert polynomials and prestable ideals [5.3]. For example, they showed that if  $R$  is a Noetherian integrally closed domain and  $f \in R[X]$  is Gaussian, then  $c(f)$  is invertible. Then Heinzer and Huneke [5.6] using techniques from approximately Gorenstein rings showed that for  $R$  locally Noetherian and  $f \in R[X]$  Gaussian (or more generally,  $c(fg) = c(f)c(g)$  for all  $g \in R[X]$  with  $\deg g \leq \deg f$ ) with  $c(f)$  regular, then  $c(f)$  is invertible. We now begin where we left off in [5.2].

Kaplansky’s question for  $R$  a quasilocal domain (and hence for  $R$  locally a domain) was answered in the affirmative by Loper and Roitman [5.7].

**Theorem 5.1.** *Let  $R$  be a ring which is locally a domain. Then a nonzero polynomial over  $R$  is Gaussian if and only if its content is locally principal.*

We outline their approach which they state is inspired by [5.5] and in particular its use of prestable ideals [5.3]. We can reduce to the case where  $R$  is a quasilocal domain.

They first show that if  $f = f(X) \in R[X]$  is Gaussian, then  $v(c(f)^n) \leq \deg f + 1$  for sufficiently large  $n$ ; here  $v(c(f))$  is the minimal number of generators for  $(c(f))^n$ . It is enough to show that  $v(c(f)^{2^m}) \leq \deg f + 1$  for all  $m \geq 0$ . Let  $f(X) = g_0(X^2) + Xg_1(X^2)$  where  $g_0(X), g_1(X) \in R[X]$ . Since  $c(f(-X)) = c(f(X))$ ;  $(c(f))^2 = c(f(X))c(f(-X)) = c(f(X)f(-X)) = c(g_0(X^2)^2 - X^2g_1(X^2)^2) = c(g_0(X)^2 - Xg_1(X)^2)$ . Since  $\deg(g_0(X)^2 - Xg_1(X)^2) = \deg(f)$ , we get  $v(c(f)^2) \leq \deg f + 1$ . They next observe that  $\ell(X) = g_0(X)^2 - Xg_1(X)^2$  is Gaussian. To see this note that if  $h(X^2)$  is Gaussian, so is  $h(X)$ . But  $g_0(X^2)^2 - X^2g_1(X^2)^2 = f(X)f(-X)$  being the product of two Gaussian polynomials is Gaussian. Thus,  $(c(f))^2 = c(\ell(X))$  where  $\ell(X)$  is Gaussian. Thus we may proceed by induction on  $m$  to get  $v(c(f)^{2^m}) \leq \deg f + 1$  for all  $m \geq 0$ .

Next, let  $\bar{R}$  be the integral closure of  $R$ . Now  $c_{\bar{R}}(f^n) = \bar{R}c(f^n) = \bar{R}c(f)^n$ ; so by the previous paragraph  $v(c_{\bar{R}}(f^n))$  is bounded. So by [5.3] the ideal  $c_{\bar{R}}(f) = \bar{R}c(f)$  is prestable and hence invertible in  $\bar{R}$ .

To descend from  $\bar{R}$  to  $R$ , “take conjugates”. Let  $f(X) = a_0 + a_1X + \cdots + a_nX^n$ . Now  $\bar{R}c(f)$  is invertible, so  $1 = \sum_{i=0}^n z_i a_i$  where  $z_i \in (\bar{R}c(f))^{-1}$ . Let  $g(X) = f(X)$

$\Sigma_{i=0}^n z_{n-i} X^i = (\Sigma_{i=0}^n \alpha_i X^i)(\Sigma_{i=0}^n z_i X^{n-i})$ . So  $g(X) = \Sigma_{i=0}^{2n} \alpha_i X^i \in \bar{R}[X]$  has  $\alpha_n = 1$  and  $f(X)|g(X)$  in  $K(X)$ ,  $K$  the quotient field of  $R$ . For each  $i \neq n$ , there is a monic  $h_i \in R[X]$  with  $h_i(\alpha_i) = 0$ . Decompose all the  $h_i(X)$  into linear factors over some integral extension  $D$  of  $R$  containing  $\bar{R}$ :  $h_i(X) = \prod_{j=1}^{m_i} (X - \beta_{ij})$ . Let  $\varphi(X)$  be the product of all possible polynomials  $\Sigma_{i=0}^{2n} \beta_{ij} X^i$  where  $0 \leq j_i \leq m_i$  for  $i \neq n$ , and  $j_n = 0, \beta_{n0} = 1$ . Now  $\varphi(X) \in R[X]$  since the coefficients of  $\varphi(X)$  can be expressed as polynomials in the elements  $\beta_{ij}$  that are symmetric in each sequence of indeterminates  $X_{i1}, \dots, X_{im_i}$  for  $i \neq n$ . Also  $c(\varphi(X)) = R$ . Now  $\varphi = f\psi$  for some  $\psi \in K[X]$ . Since  $f$  is Gaussian  $R = c(\varphi) = c(f)c(\psi)$ ; so  $c(f)$  is invertible.

Shortly afterwards, Lucas [5.8] extended Loper and Roitman's result by replacing the hypothesis that  $R$  is a domain by "the Gaussian polynomial  $f$  is a nonzero divisor in  $R[X]$ ; that is,  $\text{ann}(c(f)) = 0$ ". More precisely, he proved the following.

**Theorem 5.2.** *Let  $R$  be a commutative ring and let  $f \in R[X]$  with  $\text{ann}(c(f)) = 0$ . Then the following are equivalent.*

- (1)  $f$  is Gaussian.
- (2)  $c(f) \text{Hom}_R(c(f), R) = R$ .
- (3)  $c(f)$  is  $\mathcal{Q}_0$ -invertible where  $\mathcal{Q}_0$  is the ring of finite fractions over  $R$ .
- (4) For each maximal ideal  $M$ ,  $c(f)_M$  is an invertible ideal of  $R_M$ .
- (5)  $c(f)_M$  is principal for each maximal ideal  $M$  of  $R$ .

Here, (3) $\Rightarrow$ (1) and the equivalence (2) $\Leftrightarrow$ (5) are relatively straightforward. Lucas proceeds by showing (1) $\Rightarrow$ (4). The proofs use ideas from [5.7], but not Theorem 5.1 itself. He first shows that for any commutative ring  $R$ , if  $f \in R[X]$  is Gaussian, then  $(c(f(X)))^{2^m}$  can be generated by  $\deg f + 1$  elements. Thus, there is an integer  $k$  such that  $c(f)^{k+1}$  can be generated by  $k + 1$  element. It is then shown that  $c(f)^k R_M$  is a stable ideal of  $R_M$  and hence is a principal ideal of  $(c(f)_M^k : c(f)_M^k)$ . Hence  $c(f)_M^k$  generates a regular principal ideal of  $\bar{R}_M$ ; so  $c(f)\bar{R}_M$  is invertible. Write  $f = f_0 + f_1 X + \dots + f_n X^n$  and let  $h_0, h_1, \dots, h_n \in c(f)\bar{R}_M$  with  $\Sigma h_{n-i} f_i = 1$ . Then for  $h = \Sigma h_j X^j$ ,  $fh = u \in \bar{R}_M$  with  $c(f)\bar{R}_M$ . Thus, by [5.4], there exist  $v \in R_M$  and  $w \in \bar{R}_M$  with  $u = vw$  where  $c(w)\bar{R}_M = \bar{R}_M$ . So  $f(hw) = v$  as polynomials in the total quotient ring  $T(R_M)$ . Thus,  $R_M = c(v) = c(f(hw)) = (c(f)R_M)c(hw)$ ; so  $c(f)R_M$  is invertible.

But what happens if  $\text{ann}(c(f)) \neq 0$ ? In [5.9], Lucas gives the following.

**Theorem 5.3.** *Let  $f \in R[X]$  be a nonzero polynomial over a reduced ring  $R$  and let  $\bar{R} = R/\text{ann}(c(f))$ . Then the following are equivalent.*

- (1)  $f$  is Gaussian.
- (2)  $\bar{f} \in \bar{R}[X]$  is Gaussian.
- (3)  $c(f)\bar{R}$  is a  $\mathcal{Q}_0$ -invertible ideal of  $\bar{R}$ .
- (4)  $c(f)\bar{R}$  is locally principal.
- (5)  $c(f)$  is locally principal.

As pointed out by Lucas, while there appears to be a relationship between  $f$  being Gaussian and  $c(f)/(c(f) \cap \text{ann}(c(f)))$  being locally principal, the relationship is not clear. A similar situation holds for join principal ideals (see Section 2).

**Question 1.** Let  $0 \neq f \in R[X]$  where  $R$  is a commutative ring. What is the relationship between the following conditions.

(1)  $f$  is Gaussian, (2)  $c(f)/(c(f) \cap \text{ann}(c(f)))$  is locally principal, (3)  $c(f)$  is join principal?

We end this section by briefly discussing Gaussian rings.

As previously mentioned, Gilmer and Tsang independently showed that an integral domain is Gaussian if and only if it is Prüfer. More generally, for  $R$  reduced,  $R$  is Gaussian if and only if  $R$  is arithmetical [5.9, 5.10], and hence if  $R$  is Gaussian  $R/\text{nil}(R)$  is arithmetical. More generally, Lucas [5.9] has shown that a ring  $R$  with  $\text{nil}(R) \neq 0$ , but  $\text{nil}(R)^2 = 0$ , is Gaussian if and only if  $I^2$  is locally principal for each finitely generated ideal  $I$  of  $R$ . For  $R$  quasilocal we have the following result [5.9, 5.10].

**Theorem 5.4.** Let  $R$  be a quasilocal ring. Then  $R$  is Gaussian if and only if (i) for  $a, b \in R$ ,  $(a, b)^2$  is principal and generated by either  $a^2$  or  $b^2$  and (ii) for all  $a, b \in R$  with  $(a, b)^2 = (a^2)$  and  $ab = 0$ , we have  $b^2 = 0$ .

In the case that  $(R, M)$  is local (= Noetherian plus quasilocal), Tsang [5.10] has shown that  $R$  is Gaussian if and only if  $M/(0 : M)$  is principal. Using this, it was shown [5.1] that a Noetherian ring  $R$  is Gaussian if and only if  $R$  is a finite direct product of indecomposable Gaussian rings of the following two types (i) a zero-dimensional local ring and (ii) a ring  $S$  in which every maximal ideal has height one and all but a finite number of its maximal ideals are invertible,  $S$  has a unique minimal prime  $P$ ,  $S/P$  is a Dedekind domain, and  $PM_1 \cdots M_n = 0$  where  $\{M_1, \dots, M_n\}$  is the set of maximal ideals of  $S$  that are not invertible. (Conversely, a ring of type (ii) is Gaussian.)

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## 6 Armendariz rings

For this section, a ring will be an associative ring with identity, not necessarily commutative unless explicitly so stated. M. B. Rege and S. Chhawchharia [6.12] introduced the notion of an Armendariz ring. They defined a ring  $R$  to be an *Armendariz ring* if whenever polynomials  $f(X) = a_0 + a_1X + \cdots + a_nX^n$ ,  $g(X) = b_0 + b_1X + \cdots + b_nX^n \in R[X]$  satisfy  $f(X)g(X) = 0$ , then  $a_i b_j = 0$  for each  $i, j$ . (So in the commutative case this amounts to saying the  $c(fg) = c(f)c(g)$  in the case where  $c(fg) = 0$ .) They chose the name “Armendariz ring” because E. Armendariz [6.2] had noted that a reduced ring satisfies this condition. They showed that a homomorphic image of a PID is Armendariz and used the method of idealization to give examples of Armendariz and non-Armendariz rings.

It is easily seen that if  $R$  is Armendariz and  $f_1, \dots, f_n \in R[X]$  with  $f_1 \cdots f_n = 0$ , then  $a_1 \cdots a_n = 0$  where  $a_i$  is a coefficient of  $f_i$ . Clearly a subring of an Armendariz ring is again Armendariz. Rege and Chhawchharia raised the question of whether  $R$  Armendariz implies  $R[X]$  is Armendariz. This question was soon answered in the affirmative by the next paper to consider Armendariz rings.

### Theorem 6.1. [6.1]

- (1) *A ring  $R$  is Armendariz if and only if  $R[X]$  is Armendariz.*
- (2) *Let  $R$  be an Armendariz ring and let  $\{X_\alpha\}$  be any set of commuting indeterminates over  $R$ . Then any subring of  $R[\{X_\alpha\}]$  is Armendariz.*
- (3) *For a ring  $R$ , the following conditions are equivalent.*
  - (a)  *$R$  is Armendariz.*
  - (b) *Let  $\{X_\alpha\}$  be any nonempty set of commuting indeterminates over  $R$  and let  $f_1, \dots, f_n \in R[\{X_\alpha\}]$  with  $f_1 \cdots f_n = 0$ . If  $a_i$  is any coefficient of  $f_i$ , then  $a_1 \cdots a_n = 0$ .*

We next briefly discuss some examples of Armendariz rings and stability properties of the Armendariz property given in [6.1]. Certainly, a direct product of rings  $\prod R_\alpha$  is Armendariz if and only if each ring  $R_\alpha$  is. A von Neumann regular ring is Armendariz if and only if it is reduced (which of course is the case for  $R$  commutative). Thus, the ring of  $n \times n$  matrices over an Armendariz ring need not be Armendariz. While a polynomial ring over an Armendariz ring is Armendariz and a subring of an Armendariz ring is Armendariz, the homomorphic image of an Armendariz ring need not be Armendariz. In fact, for  $R$  commutative, each homomorphic image of  $R$  is Armendariz if and only if  $R$  is Gaussian. Now any arithmetical

ring is Gaussian and hence Armendariz, so if  $R$  is Gaussian,  $R[X]$  is Armendariz. However, for any ring  $R$ , commutative or not,  $R[X]/(X^n)$ ,  $n \geq 2$ , is Armendariz if and only if  $R$  is reduced. Thus, if  $R$  is a nonreduced arithmetical ring (e.g.,  $\mathbb{Z}/4\mathbb{Z}$ ), then  $R[X]$  is Armendariz, but  $R[X]/(X^n)$  is not Armendariz for any  $n \geq 2$ . Suppose that  $R$  is commutative and  $S$  is an overring of  $R$ . Then  $R$  is Armendariz if and only if  $S$  is; hence  $R$  is Armendariz if and only if its total quotient ring  $T(R)$  is. Thus a commutative ring  $R$  is Armendariz if and only if  $R_P$  is Armendariz for each maximal prime  $P$  of zero divisors.

Rege and Chhawchhari showed that if  $k$  is a field and  $V$  is a vector space over  $k$ , then the idealization  $k \oplus V$  is Armendariz. More generally, we have the following example.

*Example 6.2. [6.1]* Let  $R$  be an integral domain and  $M$  an  $R$ -module. Then the idealization  $R \oplus M$  is Armendariz if and only if  $M$  is an Armendariz  $R$ -module in the sense that if  $f \in R[X]$  and  $g \in M[X]$  with  $fg = 0$ , then  $a_i b_j = 0$  for each coefficient  $a_i$  of  $f$  and  $b_j$  of  $g$ . In particular, if  $R$  is an integral domain and  $M$  is a torsion-free  $R$ -module, then  $R \oplus M$  is Armendariz.

At this point we remark that it was well known to commutative ring theorists that a reduced commutative ring satisfies the Armendariz property. For example, this easily follows from the Dedekind–Mertens Lemma. Moreover, Gilmer, Grams, and Parker [6.3] (in a paper submitted before [6.2]) had proved the stronger result that if  $R$  is a reduced commutative ring and  $f, g \in R[[X]]$  with  $fg = 0$ , then  $a_i b_j = 0$  for each coefficient  $a_i$  of  $f$  and  $b_j$  of  $g$ .

Since the appearance of [6.12] *Math Reviews* lists over fifty papers concerning Armendariz rings; almost all of them with a noncommutative flavor. We cite only a few of them and give a brief overview of some of the topics considered. The interested reader should consult *Math Reviews*.

We have remarked that for  $R$  commutative,  $R$  is Armendariz if and only if  $T(R)$  is. Several authors investigate the relationship between a noncommutative ring  $R$  and various classical quotient rings of  $R$  being Armendariz; particularly see [6.5] and [6.7]. In [6.7] it is shown that a right and left Goldie ring is Armendariz if and only if it is reduced. In [6.5] it is shown that a right Ore ring  $R$  with right quotient ring  $Q$  is Armendariz if and only if  $Q$  is. Also, a semiprime Goldie ring  $R$  is Armendariz if and only if it is semicommutative (i.e., for every  $a \in R$ ,  $\{b \in R \mid ab = 0\}$  is an ideal). However, an example of an Armendariz ring that is not semicommutative is given. Recall that a ring  $R$  is *reversible* if  $ab = 0$  implies  $ba = 0$ . The relationship between being reversible and Armendariz is investigated in [6.8]. For example, a semiprime right Goldie ring is Armendariz if and only if it is reversible.

Other topics that have been considered are graded Armendariz rings, rings Armendariz to a monoid  $M$  [6.10] (i.e., if  $f, g \in R[X; M]$  with  $fg = 0$ , then  $ab = 0$  for each coefficient  $a$  of  $f$  and  $b$  of  $g$ ). A ring  $R$  is *power series Armendariz* [6.6] if for  $f = \sum_{i=0}^{\infty} a_i X^i, g = \sum_{i=0}^{\infty} b_i X^i \in R[[X]]$ ,  $X$  a commuting indeterminate, with  $fg = 0$ , then each  $a_i b_j = 0$ . Thus, by [6.3] a reduced commutative ring is power series Armendariz. A number of papers discuss “skew Armendariz rings”. Let  $\alpha$  be an endomorphism on  $R$ . Then  $R$  is said to be  $\alpha$ -Armendariz (resp.,  $\alpha$ -skew Armendariz)

if for  $f = \sum_{i=0}^n a_i X^i, g = \sum_{i=0}^m b_i X^i$  in the skew polynomial ring  $R[X; \alpha]$  with  $fg = 0$ , then  $a_i b_j = 0$  (resp.,  $a_i \alpha^i(b_j) = 0$ ) for each  $i, j$ . See, for example [6.4].

Two other generalizations, unfortunately with the same name, are as follows. In [6.9], a ring  $R$  is said to be a *weak Armendariz ring* if for  $a_0 + a_1 X, b_0 + b_1 X \in R[X]$  with  $(a_0 + a_1 X)(b_0 + b_1 X) = 0$ , then  $a_i b_j = 0$  for  $i, j \in \{0, 1\}$ . As in the case of Gaussian polynomials, we could define  $f(X) = \sum_{i=0}^n a_i X^i \in R[X]$  to be *left Armendariz* (resp., *right Armendariz*) if for each  $g = \sum_{i=0}^m b_i X^i \in R[X]$  with  $fg = 0$  (resp.,  $gf = 0$ ) we have each  $a_i b_j = 0$  (resp.,  $b_j a_i = 0$ ). We could of course restrict  $g$  to have degree less than or equal to some natural number  $m$ . Finally, in [6.11] a ring  $R$  is said to be *weak Armendariz* if whenever  $fg = 0$ , then  $a_i b_j$  is nilpotent and to be  $\pi$ -*Armendariz* if  $fg \in \text{nil}(R[X])$  implies each  $a_i b_j \in \text{nil}(R)$ .

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# Zero-divisor graphs in commutative rings

David F. Anderson, Michael C. Axtell, and Joe A. Stickles, Jr.

**Abstract** This article surveys the recent and active area of zero-divisor graphs of commutative rings. Notable algebraic and graphical results are given, followed by a historical overview and an extensive bibliography.

## 1 Introduction

Let  $R$  be a commutative ring with nonzero identity, and let  $Z(R)$  be its set of zero-divisors. The *zero-divisor graph* of  $R$ , denoted by  $\Gamma(R)$ , is the (undirected) graph with vertices  $Z(R)^* = Z(R) \setminus \{0\}$ , the nonzero zero-divisors of  $R$ , and for distinct  $x, y \in Z(R)^*$ , the vertices  $x$  and  $y$  are adjacent if and only if  $xy = 0$ . Thus,  $\Gamma(R)$  is the empty graph if and only if  $R$  is an integral domain. Moreover,  $\Gamma(R)$  is finite and nonempty if and only if  $R$  is finite and not a field.

This article is a survey of recent results on zero-divisor graphs of commutative rings and the interplay between zero-divisors and graph theory. We are interested in how ring-theoretic properties of  $R$  determine graph-theoretic properties of  $\Gamma(R)$ , and conversely, how graph-theoretic properties of  $\Gamma(R)$  determine ring-theoretic properties of  $R$ . This subject is particularly appealing since techniques can vary from simple computations to quite sophisticated ring theory, and in many cases, all the rings or graphs satisfying a certain property can be explicitly listed. Moreover, significant results have been obtained by graduate students in their masters or doctoral theses and by undergraduates in REU programs.

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The concept of a zero-divisor graph was introduced by I. Beck [23] in 1988, and then further studied by D. D. Anderson and M. Naseer [8]. However, they let all the elements of  $R$  be vertices of the graph, and they were mainly interested in colorings. Our definition of  $\Gamma(R)$  and the emphasis on the interplay between the graph-theoretic properties of  $\Gamma(R)$  and the ring-theoretic properties of  $R$  are due to D. F. Anderson and P. S. Livingston [14] in 1999. The origins and early history of zero-divisor graphs will be discussed in more detail in Section 7.

The second section begins with the paper [14] that demonstrated the surprising amount of structure present in  $\Gamma(R)$ . It was this structure that attracted ring theorists to the area in the hopes that the graph-theoretic structure could reveal underlying algebraic structure in  $Z(R)$ . The next several sections focus on some important graph theory results concerning  $\Gamma(R)$ . Planar and toroidal zero-divisor graphs are completely characterized in Section 6. The final section gives a brief history of  $\Gamma(R)$  emphasizing the original questions that motivated the area and mentions several generalizations of  $\Gamma(R)$ . Most proofs are omitted in the interest of brevity, and we do not claim to provide all noteworthy results in this field. The bibliography is our attempt at providing an extensive list of publications in this area, although many of the papers are not explicitly cited in this survey.

We next recall some concepts from graph theory. Let  $G$  be a (undirected) graph. We say that  $G$  is *connected* if there is a path between any two distinct vertices. For distinct vertices  $x$  and  $y$  in  $G$ , the *distance* between  $x$  and  $y$ , denoted by  $d(x, y)$ , is the length of a shortest path connecting  $x$  and  $y$  ( $d(x, x) = 0$  and  $d(x, y) = \infty$  if no such path exists). The *diameter* of  $G$  is  $\text{diam}(G) = \sup\{d(x, y) \mid x \text{ and } y \text{ are vertices of } G\}$ . A *cycle* of length  $n$  in  $G$  is a path of the form  $x_1 - x_2 - \cdots - x_n - x_1$ , where  $x_i \neq x_j$  when  $i \neq j$ . We define the *girth* of  $G$ , denoted by  $\text{gr}(G)$ , as the length of a shortest cycle in  $G$ , provided  $G$  contains a cycle; otherwise,  $\text{gr}(G) = \infty$ . Finally, a vertex of  $G$  is an *end* if it is adjacent to exactly one other vertex.

A graph  $G$  is *complete* if any two distinct vertices are adjacent. The complete graph with  $n$  vertices will be denoted by  $K^n$  (we allow  $n$  to be an infinite cardinal). A *complete bipartite graph* is a graph  $G$  which may be partitioned into two disjoint nonempty vertex sets  $A$  and  $B$  such that two distinct vertices are adjacent if and only if they are in distinct vertex sets. If one of the vertex sets is a singleton, then we call  $G$  a *star graph*. We denote the complete bipartite graph by  $K^{m,n}$ , where  $|A| = m$  and  $|B| = n$  (again, we allow  $m$  and  $n$  to be infinite cardinals); so a star graph is a  $K^{1,n}$ . More generally,  $G$  is *complete  $r$ -partite* if  $G$  is the disjoint union of  $r$  nonempty vertex sets and two distinct vertices are adjacent if and only if they are in distinct vertex sets. Finally, let  $\overline{K}^{m,3}$  be the graph formed by joining  $G_1 = K^{m,3}$  ( $= A \cup B$  with  $|A| = m$  and  $|B| = 3$ ) to the star graph  $G_2 = K^{1,m}$  by identifying the center of  $G_2$  and a point of  $B$ .

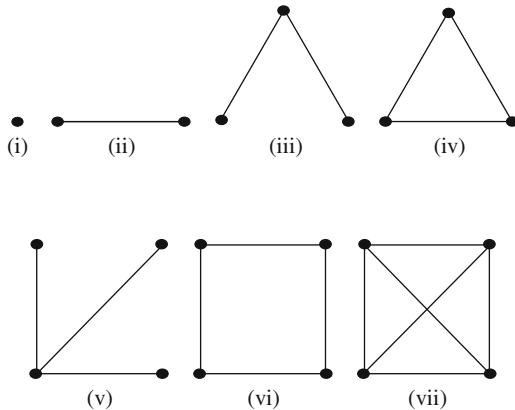
A subgraph  $G'$  of a graph  $G$  is an *induced subgraph* of  $G$  if two vertices of  $G'$  are adjacent in  $G'$  if and only if they are adjacent in  $G$ . Clearly,  $\text{gr}(G') \geq \text{gr}(G)$  when  $G'$  is an induced subgraph of  $G$ , but there is no relationship between  $\text{diam}(G')$  and  $\text{diam}(G)$ . A complete subgraph of  $G$  is called a *clique*. The *clique number* of  $G$ , denoted by  $cl(G)$ , is the greatest integer  $r \geq 1$  such that  $K^r \subseteq G$  (if  $K^r \subseteq G$  for all integers  $r \geq 1$ , then we write  $cl(G) = \infty$ ). The *chromatic number* of  $G$ , denoted



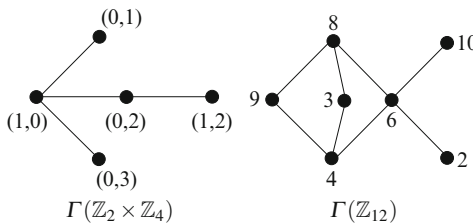
by  $\chi(G)$ , is the minimum number of colors needed to color the vertices of  $G$  so that no two adjacent vertices have the same color. Clearly  $cl(G) \leq \chi(G)$ .

Below we provide some examples of zero-divisor graphs. We will not distinguish between isomorphic graphs (two graphs  $G$  and  $G'$  are *isomorphic* if there is a bijection  $f$  between the vertices of  $G$  and the vertices of  $G'$  such that  $x$  and  $y$  are adjacent in  $G$  if and only if  $f(x)$  and  $f(y)$  are adjacent in  $G'$ ). As usual,  $\mathbb{Z}$ ,  $\mathbb{Z}_n$ ,  $\mathbb{Q}$ ,  $\mathbb{R}$ ,  $\mathbb{C}$ , and  $\mathbb{F}_q$  will denote the integers, integers modulo  $n$ , rational numbers, real numbers, complex numbers, and the finite field with  $q$  elements, respectively. In Section 5, loops will sometimes be added to vertices of  $\Gamma(R)$  corresponding to zero-divisors  $x$  with  $x^2 = 0$ .

*Example 1.1.* (a) ([12, Example 2.1]) We first give all possible nonempty zero-divisor graphs  $\Gamma(R)$  with  $|\Gamma(R)| \leq 4$ . Up to isomorphism, each graph may be realized as  $\Gamma(R)$  by precisely the following rings: (i)  $\mathbb{Z}_4$ ,  $\mathbb{Z}_2[X]/(X^2)$ ; (ii)  $\mathbb{Z}_9$ ,  $\mathbb{Z}_2 \times \mathbb{Z}_2$ ,  $\mathbb{Z}_3[X]/(X^2)$ ; (iii)  $\mathbb{Z}_6$ ,  $\mathbb{Z}_8$ ,  $\mathbb{Z}_2[X]/(X^3)$ ,  $\mathbb{Z}_4[X]/(2X, X^2 - 2)$ ; (iv)  $\mathbb{Z}_4[X, Y]/(X, Y)^2$ ,  $\mathbb{Z}_4[X]/(2, X)^2$ ,  $\mathbb{Z}_4[X]/(X^2 + X + 1)$ ,  $\mathbb{F}_4[X]/(X^2)$ ; (v)  $\mathbb{Z}_2 \times \mathbb{F}_4$ ; (vi)  $\mathbb{Z}_3 \times \mathbb{Z}_3$ ; and (vii)  $\mathbb{Z}_{25}$ ,  $\mathbb{Z}_5[X]/(X^2)$ . These examples show that a zero-divisor graph may be realized by more than one ring and that  $\Gamma(R)$  does not detect nilpotent elements of  $R$ .



(b) Up to isomorphism, the following  $\bar{K}^{1,3}$  graph may be realized as  $\Gamma(R)$  by only  $\mathbb{Z}_2 \times \mathbb{Z}_4$  or  $\mathbb{Z}_2 \times \mathbb{Z}_2[X]/(X^2)$  (Theorem 2.4, [14, p. 439], [36, Lemma 1.5], or [64, (2.0)]). The second graph may be realized as  $\Gamma(R)$  by only  $\mathbb{Z}_{12} \cong \mathbb{Z}_3 \times \mathbb{Z}_4$  or  $\mathbb{Z}_3 \times \mathbb{Z}_2[X]/(X^2)$ .



Throughout,  $R$  will be a commutative ring with nonzero identity, set of prime (resp., maximal, minimal prime, associated prime) ideals  $\text{Spec}(R)$  (resp.,  $\text{Max}(R)$ ,  $\text{Min}(R)$ ,  $\text{Ass}(R)$ ), ideal of nilpotent elements  $\text{nil}(R)$ , total quotient ring  $T(R) = R_S$ , where  $S = R \setminus Z(R)$ , and  $A^* = A \setminus \{0\}$  for  $A \subseteq R$ . Recall that  $R$  is *reduced* if  $\text{nil}(R) = \{0\}$ . We assume that a subring of a ring has the same identity element as the ring, and an *overring* of  $R$  is a subring of  $T(R)$  containing  $R$ . The Krull dimension of  $R$  will be denoted by  $\dim(R)$ , and  $\subset$  will denote proper inclusion. To avoid trivialities when  $\Gamma(R)$  is the empty graph, we will implicitly assume when necessary that  $R$  is not an integral domain. By [16, Theorem 8.7], an Artinian (e.g., finite) commutative ring is a finite direct product of local Artinian rings. Moreover,  $Z(R) = \text{nil}(R)$  is the unique prime ideal in an Artinian local ring. Thus, a finite reduced commutative ring is a finite direct product of fields. For undefined notation or terminology, see [38] for graph theory, and [16] or [45] for ring theory.

## 2 Diameter and girth

In this section, we study the girth and diameter of  $\Gamma(R)$ . However, we begin with the comforting result from [14] that (nonempty) finite zero-divisor graphs come from finite rings. This is really a result about zero-divisors and is due to N. Ganesan [43].

**Theorem 2.1.** ([43, Theorem 1], [14, Theorem 2.2]) *Let  $R$  be a commutative ring. Then  $\Gamma(R)$  is finite if and only if either  $R$  is finite or  $R$  is an integral domain. In particular, if  $1 \leq |\Gamma(R)| < \infty$ , then  $R$  is finite and not a field. Moreover,  $|R| \leq |Z(R)|^2$  if  $R$  is not an integral domain.*

*Proof.* It is sufficient to prove the “moreover” statement. Let  $x \in Z(R)^*$ . Then the  $R$ -module homomorphism  $f : R \rightarrow R$  given by  $f(r) = rx$  has kernel  $\text{ann}_R(x)$  and image  $xR$ . Thus  $|R| = |\text{ann}_R(x)||xR| \leq |Z(R)|^2$ .

The first “big” result in [14] showed that  $\Gamma(R)$  is always connected and relatively “compact.”

**Theorem 2.2.** ([14, Theorem 2.3]) *Let  $R$  be a commutative ring. Then  $\Gamma(R)$  is connected with  $\text{diam}(\Gamma(R)) \leq 3$ .*

*Proof.* Let  $x, y \in Z(R)^*$  be distinct. We will show that  $d(x, y) \leq 3$ . If  $xy = 0$ , then  $d(x, y) = 1$ . So suppose that  $xy$  is nonzero. There are  $z, w \in Z(R)^*$  such that  $xz = wy = 0$ . If  $zw \neq 0$ , then  $x - zw - y$  is a path of length 2; so  $d(x, y) = 2$ . If  $zw = 0$ , then  $x - z - w - y$  is a path of length at most 3 (we could have  $x = z$  or  $w = y$ ). Thus,  $d(x, y) \leq 3$ , and hence  $\Gamma(R)$  is connected and  $\text{diam}(\Gamma(R)) \leq 3$ .

If  $G$  contains a cycle, then  $\text{gr}(G) \leq 2 \cdot \text{diam}(G) + 1$  [38, Proposition 1.3.2]. So, if  $\Gamma(R)$  contains a cycle, then  $\text{gr}(\Gamma(R)) \leq 7$  by Theorem 2.2. Anderson and Livingston, however, noticed that all of the examples they considered had girths of

3, 4, or  $\infty$ . Based on this, they conjectured that if a zero-divisor graph has a cycle, then its girth is 3 or 4. They were able to prove this if the ring was Artinian (e.g., finite) [14, Theorem 2.4]. The conjecture was proven independently by S. B. Mulay [64] and F. DeMeyer and K. Schneider [36]. Additionally, short proofs have been given by M. Axtell, J. Coykendall, and J. Stickles [17] and S. Wright [84].

**Theorem 2.3.** ([14, Theorem 2.4], [64, (1.4)], [36, Theorem 1.6]) *Let  $R$  be a commutative ring. If  $\Gamma(R)$  contains a cycle, then  $gr(\Gamma(R)) \leq 4$ .*

*Proof.* Assume by way of contradiction that  $n = gr(\Gamma(R))$  is 5, 6, or 7. Let  $x_1 - x_2 - \cdots - x_n - x_1$  be a cycle of minimum length. So,  $x_1x_3 \neq 0$ . If  $x_1x_3 \neq x_i$  for  $1 \leq i \leq n$ , then  $x_2 - x_3 - x_4 - x_1x_3 - x_2$  is a 4-cycle, a contradiction. Thus,  $x_1x_3 = x_i$  for some  $1 \leq i \leq n$ . If  $x_1x_3 = x_1$ , then  $x_1 - x_2 - x_3 - x_4 - x_1$  is a 4-cycle. If  $x_1x_3 = x_2$ , then  $x_2 - x_3 - x_4 - x_2$  is a 3-cycle. If  $x_1x_3 = x_n$ , then  $x_1 - x_2 - x_n - x_1$  is a 3-cycle. Hence,  $x_1x_3 \neq x_1, x_2$ , or  $x_n$ . However,  $x_1 - x_2 - x_1x_3 - x_n - x_1$  is then a 4-cycle, a contradiction. Therefore, there must be a shorter cycle in  $\Gamma(R)$ , and  $gr(\Gamma(R)) \leq 4$ .

Thus,  $diam(\Gamma(R)) \in \{0, 1, 2, 3\}$  and  $gr(\Gamma(R)) \in \{3, 4, \infty\}$ . The examples given in the Introduction show that all these possible values may occur. The next result expands on Theorem 2.3.

**Theorem 2.4.** *Let  $R$  be a commutative ring which is not an integral domain. Then exactly one of the following holds:*

- (a)  $\Gamma(R)$  has a cycle of length 3 or 4 (i.e.,  $gr(\Gamma(R)) \leq 4$ );
- (b)  $\Gamma(R)$  is a singleton or a star graph; or
- (c)  $\Gamma(R) = \overline{K}^{1,3}$  (i.e.,  $R \cong \mathbb{Z}_2 \times \mathbb{Z}_4$  or  $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2[X]/(X^2)$ ).

*Moreover, if  $\Gamma(R)$  contains a cycle, then every vertex of  $\Gamma(R)$  is either an end or part of a 3-cycle or a 4-cycle.*

*Proof.* The finite case was observed in [14, p. 349], while the general case is independently given in [36, Theorem 1.6] and [64, (1.4), (2.0), and (2.1)]. The “moreover” statement is from [64, (1.4) and (2.1)].

Another characterization of girth was given in [15] using the fact that  $R$  and  $T(R)$  have isomorphic zero-divisor graphs (Theorem 4.4). The following two theorems explicitly characterize when the girth of a zero-divisor graph is 4 or  $\infty$ , and thus implicitly when the girth is 3.

**Theorem 2.5.** ([15, Theorems 2.2 and 2.4]) *Let  $R$  be a reduced commutative ring.*

(a) *The following statements are equivalent.*

- (1)  $gr(\Gamma(R)) = 4$ .
- (2)  $T(R) = K_1 \times K_2$ , where each  $K_i$  is a field with  $|K_i| \geq 3$ .
- (3)  $\Gamma(R) = K^{m,n}$  with  $m, n \geq 2$ .

(b) *The following statements are equivalent.*

- (1)  $\Gamma(R)$  is nonempty with  $gr(\Gamma(R)) = \infty$ .
- (2)  $T(R) = \mathbb{Z}_2 \times K$ , where  $K$  is a field.
- (3)  $\Gamma(R) = K^{1,n}$  for some  $n \geq 1$ .

**Theorem 2.6.** ([15, Theorems 2.3 and 2.5]) *Let  $R$  be a commutative ring with  $\text{nil}(R)$  nonzero.*

(a) *The following statements are equivalent.*

- (1)  $\text{gr}(\Gamma(R)) = 4$ .
- (2)  $R \cong D \times B$ , where  $D$  is an integral domain with  $|D| \geq 3$  and  $B = \mathbb{Z}_4$  or  $\mathbb{Z}_2[X]/(X^2)$ . (Thus  $T(R) \cong T(D) \times B$ .)
- (3)  $\Gamma(R) = \overline{K}^{m,3}$  with  $m \geq 2$ .

(b) *The following statements are equivalent.*

- (1)  $\text{gr}(\Gamma(R)) = \infty$ .
- (2)  $R \cong B$  or  $R \cong \mathbb{Z}_2 \times B$ , where  $B = \mathbb{Z}_4$  or  $\mathbb{Z}_2[X]/(X^2)$ , or  $\Gamma(R)$  is a star graph.
- (3)  $\Gamma(R)$  is a singleton, a  $\overline{K}^{1,3}$ , or a  $K^{1,n}$  for some  $n \geq 1$ .

Much of the research on zero-divisor graphs has focused on the girth and diameter for certain classes of rings. For example,  $\text{gr}(\Gamma(R))$  is studied in terms of the number of associated prime ideals of  $R$  in [3], and properties of  $\Gamma(R)$  for a reduced ring  $R$  are related to topological properties of  $\text{Spec}(R)$  in [74]. The girth and diameter of the zero-divisor graph of the direct product of two commutative rings (not necessarily with identity) are characterized in [21], and for diameter these ideas are extended to finite direct products in [41]. Also, the girth and diameter of the zero-divisor graph of an idealization are characterized in [18] and [15], and the girth and diameter of  $\Gamma(R \rtimes I)$  (the amalgamated duplication of a ring  $R$  along an ideal  $I$  [33]) are studied in [62]. The girth and diameter of  $\Gamma(R)$  for a commutative ring  $R$  which satisfies certain divisibility conditions on elements or comparability conditions on ideals or prime ideals are investigated in [10].

We next give a more detailed discussion of the zero-divisor graphs for polynomial rings and power series rings. First, we consider the easier case for girth.

**Theorem 2.7.** ([17, Theorem 4.3], [15, Theorem 3.2]) *Let  $R$  be a commutative ring.*

(a) *Suppose that  $\Gamma(R)$  is nonempty with  $\text{gr}(\Gamma(R)) = \infty$ .*

- (1) *If  $R$  is reduced, then  $\text{gr}(\Gamma(R[X])) = \text{gr}(\Gamma(R[[X]])) = 4$ .*
- (2) *If  $R$  is not reduced, then  $\text{gr}(\Gamma(R[X])) = \text{gr}(\Gamma(R[[X]])) = 3$ .*

(b) *If  $\text{gr}(\Gamma(R)) = 3$ , then  $\text{gr}(\Gamma(R[X])) = \text{gr}(\Gamma(R[[X]])) = 3$ .*

(c) *Suppose that  $\text{gr}(\Gamma(R)) = 4$ .*

- (1) *If  $R$  is reduced, then  $\text{gr}(\Gamma(R[X])) = \text{gr}(\Gamma(R[[X]])) = 4$ .*
- (2) *If  $R$  is not reduced, then  $\text{gr}(\Gamma(R[X])) = \text{gr}(\Gamma(R[[X]])) = 3$ .*

*Proof.* From [17, Theorem 4.3], we have  $\text{gr}(\Gamma(R)) \leq \text{gr}(\Gamma(R[X])) = \text{gr}(\Gamma(R[[X]]))$ , and equality holds if  $R$  is reduced and  $\Gamma(R)$  contains a cycle. The remaining cases and the result as stated above are from [15, Theorem 3.2].

The “diameter” case is not so easy. This was first studied in [17], and some cases for non-Noetherian commutative rings left open in [17] were resolved by T. G. Lucas in [59]. However, we are content here to just mention the reduced case; the interested reader should refer to [17, 59], and [15] for related results. In particular, see [59,

Theorems 3.4 and 3.6] for polynomial rings and [59, Section 5] for power series rings. Recall that a ring  $R$  is a *McCoy ring* if each finitely generated ideal contained in  $Z(R)$  has a nonzero annihilator.

**Theorem 2.8.** ([59, Theorem 4.9]) *Let  $R$  be a reduced commutative ring that is not an integral domain. Then*

$$1 \leq \text{diam}(\Gamma(R)) \leq \text{diam}(\Gamma(R[X])) \leq \text{diam}(\Gamma(R[[X]])) \leq 3.$$

Moreover, here are all possible sequences for these dimensions.

(1)  $\text{diam}(\Gamma(R)) = 1$  and  $\text{diam}(\Gamma(R[X])) = \text{diam}(\Gamma(R[[X]])) = 2$  if and only if  $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ .

(2)  $\text{diam}(\Gamma(R)) = \text{diam}(\Gamma(R[X])) = \text{diam}(\Gamma(R[[X]])) = 2$  if and only if either  $R$  has exactly two minimal primes and is not isomorphic to  $\mathbb{Z}_2 \times \mathbb{Z}_2$  or for each pair of countably generated ideals  $I$  and  $J$  with nonzero annihilators, the sum  $I + J$  has a nonzero annihilator (and  $R$  is a McCoy ring with  $Z(R)$  an ideal).

(3)  $\text{diam}(\Gamma(R)) = \text{diam}(\Gamma(R[X])) = 2$  and  $\text{diam}(\Gamma(R[[X]])) = 3$  if and only if  $R$  is a McCoy ring with  $Z(R)$  an ideal but there exists countably generated ideals  $I$  and  $J$  with nonzero annihilators such that  $I + J$  does not have a nonzero annihilator.

(4)  $\text{diam}(\Gamma(R)) = 2$  and  $\text{diam}(\Gamma(R[X])) = \text{diam}(\Gamma(R[[X]])) = 3$  if and only if  $Z(R)$  is an ideal and each two generated ideal contained in  $Z(R)$  has a nonzero annihilator but  $R$  is not a McCoy ring.

(5)  $\text{diam}(\Gamma(R)) = \text{diam}(\Gamma(R[X])) = \text{diam}(\Gamma(R[[X]])) = 3$  if and only if  $R$  has more than two minimal primes and there is a pair of zero-divisors  $a$  and  $b$  such that  $(a, b)$  does not have a nonzero annihilator.

Let  $A \subseteq B$  be an extension of commutative rings with identity. In this case,  $\Gamma(A)$  is an induced subgraph of  $\Gamma(B)$ . It may happen that  $\Gamma(A) = \Gamma(B)$  for  $A \subset B$  (this happens if and only if  $A$  is a pullback of a finite local ring [12, Theorem 4.3]). It is clear that  $\text{gr}(\Gamma(B)) \leq \text{gr}(\Gamma(A))$ . Moreover, for all  $m, n \in \{3, 4, \infty\}$  with  $m \leq n$ , there is a proper extension  $A \subset B$  of reduced finite commutative rings such that  $\text{gr}(\Gamma(B)) = m$  and  $\text{gr}(\Gamma(A)) = n$  [9, Example 2.1]. Again, the case for the diameter is not so clear since although  $Z(A) \subseteq Z(B)$ , it need not be the case that  $Z(A) = Z(B) \cap A$ . In fact, for  $m, n \in \{0, 1, 2, 3\}$ , there is a proper extension  $A \subset B$  of commutative rings with  $\text{diam}(\Gamma(A)) = m$  and  $\text{diam}(\Gamma(B)) = n$  unless  $(m, n) \in \{(0, 0), (1, 0), (2, 0), (2, 1), (3, 0), (3, 1)\}$  [9, Proposition 3.2]. Thus,  $\text{diam}(\Gamma(A)) \leq \text{diam}(\Gamma(B))$  unless  $\text{diam}(\Gamma(A)) = 3$  and  $\text{diam}(\Gamma(B)) = 2$ ; specific examples with  $\text{diam}(\Gamma(A)) = 3$  and  $\text{diam}(\Gamma(B)) = 2$  are given in [18, Example 3.7] and [9, Example 3.7]. The next theorem gives conditions when this can happen.

**Theorem 2.9.** (a) ([9, Theorem 3.8]) *Let  $A$  be a commutative ring with  $\text{diam}(\Gamma(A)) = 3$ . Then there is a commutative extension ring  $B$  of  $A$  such that  $\text{diam}(\Gamma(B)) = 2$  if and only if  $Z(A) \subseteq M$  for some maximal ideal  $M$  of  $A$ . Moreover, if  $A$  is reduced, then  $B$  can also be chosen to be reduced.*

(b) ([9, Corollary 3.12]) *Let  $A \subseteq B$  be an extension of commutative rings with  $\dim(A) = 0$ . Then  $\text{diam}(\Gamma(A)) \leq \text{diam}(\Gamma(B))$ . In particular, this holds if  $A$  is Artinian or a finite commutative ring.*

Part (b) essentially follows from part (a) since  $\text{diam}(\Gamma(R)) \leq 2$  when  $Z(R) = \text{nil}(R)$  [9, Lemma 3.11]. Theorem 2.9 illustrates a case where the zero-divisor graph of an infinite ring may behave rather differently from that of a finite ring. Also note that if  $B$  is an overring of  $A$ , then  $\text{diam}(\Gamma(A)) = \text{diam}(\Gamma(B))$  by Corollary 4.5(a).

The above results demonstrate that the zero-divisor graph of a commutative ring exhibits a remarkable amount of graphical structure that could perhaps provide some insight into the algebraic structure of  $Z(R)$ . The next several sections show some of the results in which  $\Gamma(R)$  provides information about  $R$  and  $Z(R)$ .

### 3 What the size and shape of $\Gamma(R)$ implies

Theorem 2.1 can be generalized to only require that every vertex of  $\Gamma(R)$  has finite degree (i.e., every vertex is adjacent to only finitely many other vertices).

**Theorem 3.1. ([4, Theorem 6])** *If  $R$  is a commutative ring such that  $R$  is not an integral domain and every vertex of  $\Gamma(R)$  has finite degree, then  $R$  is a finite ring.*

*Proof.* Suppose  $R$  is infinite and  $x, y \in R^*$  with  $xy = 0$ . Then  $yR^* \subseteq \text{ann}(x)$ . If  $yR^*$  is infinite, then  $x$  has infinite degree in  $\Gamma(R)$ , a contradiction. If  $yR^*$  is finite, then there exists an infinite  $A \subseteq R^*$  such that  $ya_1 = ya_2$  for all  $a_1, a_2 \in A$ . If  $a_0$  is a fixed element of  $A$ , then  $\{a_0 - a \mid a \in A\}$  is an infinite subset of  $\text{ann}(y)$ , and thus  $y$  has infinite degree in  $\Gamma(R)$ , a contradiction. Hence,  $R$  is finite.

Thus, if  $R$  is not an integral domain, we have  $|Z(R)| < \infty \Leftrightarrow |R| < \infty \Leftrightarrow$  every vertex of  $\Gamma(R)$  has finite degree. When  $R$  is Noetherian, an upper bound on  $|R|$  sometimes exists in terms of the degree of each vertex.

**Theorem 3.2. ([69, Theorem 6.1])** *Let  $R$  be a commutative Noetherian ring with identity that is not an integral domain. Suppose that there exists a positive integer  $k$  such that for all nonzero  $x \in R$ ,  $|\text{ann}(x)| \leq k$ . Then  $|R| \leq (k^2 - 2k + 2)^2$ .*

Another way to study zero-divisor graphs is to approach the structures from the opposite direction. In other words, given a graph  $G$ , is it possible to know when there is a commutative ring  $R$  such that  $\Gamma(R) \cong G$ ? One series of results has provided a list of all rings (up to isomorphism) whose zero-divisor graphs consist of  $n$  elements. The graphs on  $n = 1, 2, 3$ , or 4 vertices which can be realized as  $\Gamma(R)$ , and a complete list of rings (up to isomorphism) producing those graphs, was given in [12, Example 2.1] (Example 1.1(a)). S. P. Redmond showed in [69, Theorem 6.4] that for  $n = 5$ , there were three non-isomorphic graphs that could be realized as  $\Gamma(R)$ , while there were four non-isomorphic rings creating said graphs. Redmond continued this work in [72], where he provided all graphs on  $n = 6, 7, \dots, 14$  vertices that can be realized as the zero-divisor graph of a commutative ring with identity, and lists all rings (up to isomorphism) which produce these graphs. In addition, Redmond gave an algorithm to find all commutative reduced rings with identity (up to isomorphism) which give rise to a zero-divisor graph on  $n$  vertices for any  $n \geq 1$ .

In a similar vein, J. D. LaGrange [47] developed an algorithm for constructing the zero-divisor graph of a direct product of integral domains, as well as classified which graphs are realizable as zero-divisor graphs of direct products of integral domains or zero-divisor graphs of Boolean rings (also see [51]).

One can also ask for which positive integers  $n$  is there a commutative ring  $R$  with  $|\Gamma(R)| = n$ , equivalently, when is there a commutative ring  $R$  with  $|Z(R)| = n + 1$ ? Using a formula for the number of zero-divisors in a direct product of rings by R. Gilmer [44], S. P. Redmond [73] used computer calculations to show that there are no reduced commutative rings with 1206, 1210, 1806, 3342, 5466, 6462, 6534, 6546, or 7430 zero-divisors. Additional work showed that there are no commutative rings with 1210, 3342, or 5466 zero-divisors. Thus, there is no commutative ring  $R$  (with identity) such that  $\Gamma(R)$  has 1209, 3341, or 5465 vertices (for rings without identity, see the comments after Theorem 3.3).

Two of the most elementary forms that a connected graph can take are being complete or complete bipartite. Whenever the zero-divisor graph of a ring assumes one of these two shapes, we gain a remarkable amount of information about the ring. We first handle the “complete” case. By definition,  $xy = 0$  for distinct  $x, y \in Z(R)$  if  $\Gamma(R)$  is complete. Except in the case  $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ , we must also have  $x^2 = 0$  for all  $x \in Z(R)$ , and thus  $Z(R)$  is an (prime) ideal of  $R$ . So, except for that case, a finite ring  $R$  has  $\Gamma(R)$  complete if and only if  $(R, M)$  is local and  $M^2 = 0$ .

**Theorem 3.3.** (a) ([14, Theorem 2.8]) *Let  $R$  be a commutative ring. Then  $\Gamma(R)$  is complete if and only if either  $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$  or  $xy = 0$  for all  $x, y \in Z(R)$ .*

(b) ([14, Theorem 2.10]) *Let  $R$  be a finite commutative ring. If  $\Gamma(R)$  is complete, then either  $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$  or  $(R, M)$  is local with  $M^2 = 0$ ,  $\text{char}R = p$  or  $p^2$ , and  $|\Gamma(R)| = p^n - 1$ , where  $p$  is prime and  $n \geq 1$ .*

Conversely, for each prime  $p$  and integer  $n \geq 1$ , the ring  $R = \mathbb{F}_{p^n}[X]/(X^2)$  has  $\Gamma(R)$  complete with  $|\Gamma(R)| = p^n - 1$  [14, Example 2.11]. For a partial characterization of which finite rings  $R$  have  $\Gamma(R)$  complete, see [58, Section 3]. More generally, for each infinite cardinal number  $\alpha$ , let  $K$  be a field with  $|K| = \alpha$  and  $R = K[X]/(X^2)$ . Then  $\Gamma(R)$  is complete and  $|\Gamma(R)| = \alpha$ . However, for infinite rings,  $\Gamma(R)$  may be complete when  $R$  is not quasilocal. For example,  $R = \mathbb{Z}_4[X]$  is not quasilocal, but  $\Gamma(R)$  is complete since  $Z(R) = 2R$ . Theorem 3.3(b) not only illustrates the difference between finite and infinite rings, but also the necessity of an identity. Let  $R$  be the additive group  $\mathbb{Z}_{n+1}$  with multiplication defined by  $xy = 0$  for all  $x, y \in R$ ; then  $\Gamma(R)$  is a complete graph on  $n$  vertices.

When  $\Gamma(R)$  is a complete bipartite graph, we can say even more about  $R$ . Note that for integral domains  $R_1$  and  $R_2$ ,  $\Gamma(R_2 \times R_2) = K^{m,n}$ , where  $m = |R_1| - 1$  and  $n = |R_2| - 1$ . The converse holds for finite rings except when the graph is a  $K^{1,1}$  or  $K^{1,2}$ . Thus,  $K^{m,n} = \Gamma(R)$  for a finite commutative ring  $R$  if and only if  $m = p_1^{k_1} - 1$  and  $n = p_2^{k_2} - 1$  for primes  $p_1, p_2$  and integers  $k_1, k_2 \geq 1$ .

**Theorem 3.4.** ([14, p. 439], [36, Theorem 1.14, Corollary 1.11]) *Let  $R$  be a finite commutative ring.*

(a)  $\Gamma(R)$  is complete bipartite if and only if either  $R \cong F_1 \times F_2$ , where  $F_1$  and  $F_2$  are finite fields, or  $R$  is isomorphic to  $\mathbb{Z}_9$ ,  $\mathbb{Z}_3[X]/(X^2)$ ,  $\mathbb{Z}_8$ ,  $\mathbb{Z}_2[X]/(X^3)$ , or  $\mathbb{Z}_4/(2X, X^2 - 2)$ .

(b)  $\Gamma(R)$  is a star graph if and only if either  $R \cong \mathbb{Z}_2 \times F$ , where  $F$  is a finite field, or  $R$  is isomorphic to  $\mathbb{Z}_9$ ,  $\mathbb{Z}_3[X]/(X^2)$ ,  $\mathbb{Z}_8$ ,  $\mathbb{Z}_2[X]/(X^3)$ , or  $\mathbb{Z}_4/(2X, X^2 - 2)$ .

For infinite complete bipartite zero-divisor graphs, see Theorems 2.5, 2.6, 3.6, 3.7, and [36, Theorems 1.12 and 1.14].

Complete  $r$ -partite graphs have been described by S. Akbari, H. R. Maimani, and S. Yassemi [3]; some of their results are listed below. Of course,  $K^n$  is a complete  $n$ -partite graph (cf. Theorem 3.3).

**Theorem 3.5.** *Let  $R$  be a commutative ring such that  $\Gamma(R)$  is a complete  $r$ -partite graph for  $r \geq 3$  with vertex sets  $V_1, \dots, V_r$ .*

(a) ([3, Theorem 3.1]) *At most one vertex set has more than one element. If  $V_i = \{x\}$ , then  $x^2 = 0$ . Further,  $Z(R) \in \text{Max}(R) \cap \text{Ass}(R)$ .*

(b) ([3, Theorem 3.2]) *If  $R$  is finite, then  $R$  is local. Moreover, if  $|V_r| \geq 2$ , then there is a prime  $p$  and positive integers  $t$  and  $k$  such that  $r = p^t$  and  $|R| = p^k$ .*

(c) ([3, Theorem 3.4]) *If  $r = p$  for  $p$  prime, then  $|Z(R)| = p^2$ ,  $|R| = p^3$ , and  $R$  is isomorphic to exactly one of the rings  $\mathbb{Z}_{p^3}$ ,  $\mathbb{Z}_p[X, Y]/(XY, Y^2 - X)$ , or  $\mathbb{Z}_{p^2}[Y]/(pY, Y^2 - ps)$ , where  $1 \leq s < p$ .*

We next turn to characterizing the zero-divisor graphs of commutative rings with von Neumann regular total quotient rings. Recall that a commutative ring  $R$  is *von Neumann regular* if for each  $x \in R$ , there is a  $y \in R$  with  $x^2y = x$  (equivalently,  $R$  is reduced and  $\dim(R) = 0$  [45, Remark, p. 5]). The simplest examples of von Neumann regular rings are direct products of fields.

Let  $G$  be a (undirected) graph. As in [54], for vertices  $a$  and  $b$  of  $G$ , we define  $a \leq b$  if  $a$  and  $b$  are not adjacent and each vertex of  $G$  adjacent to  $a$  is also adjacent to  $b$ ; and define  $a \sim b$  if  $a \leq b$  and  $b \leq a$ . Thus,  $a \sim b$  if and only if  $a$  and  $b$  are adjacent to exactly the same vertices. Clearly  $\sim$  is an equivalence relation on  $G$ . For distinct vertices  $a$  and  $b$  of  $G$ , we say that  $a$  and  $b$  are *orthogonal*, written  $a \perp b$ , if  $a$  and  $b$  are adjacent and there is no vertex  $c$  of  $G$  which is adjacent to both  $a$  and  $b$ , i.e., the edge  $a - b$  is not part of any triangle in  $G$ . We say that  $G$  is *complemented* if for each vertex  $a$  of  $G$ , there is a vertex  $b$  of  $G$  (called a *complement* of  $a$ ) such that  $a \perp b$ , and that  $G$  is *uniquely complemented* if  $G$  is complemented and whenever  $a \perp b$  and  $a \perp c$ , then  $b \sim c$ . For  $a, b \in Z(R)^*$ , we have  $a \sim b$  in  $\Gamma(R)$  if and only if  $\text{ann}(a) \setminus \{a\} = \text{ann}(b) \setminus \{b\}$ .

We next determine when  $\Gamma(R)$  is complemented or uniquely complemented. Since  $\Gamma(R)$  and  $\Gamma(T(R))$  are isomorphic by Theorem 4.4, we can only characterize when  $T(R)$  is von Neumann regular. Work on the zero-divisor graph of a von Neumann regular ring was initiated by R. Levy and J. Shapiro [54] and then continued in [13, 46–48], and [49].



**Theorem 3.6. ([13, Theorem 3.5])** *The following statements are equivalent for a reduced commutative ring  $R$ .*

- (1)  $T(R)$  is von Neumann regular.
- (2)  $\Gamma(R)$  is uniquely complemented.
- (3)  $\Gamma(R)$  is complemented.

*Moreover, a nonempty  $\Gamma(R)$  is a star graph if and only if  $R \cong D \times \mathbb{Z}_2$  for some integral domain  $D$ .*

For nonreduced rings, we have the following characterizations.

**Theorem 3.7. (a) ([13, Theorem 3.9])** *Let  $R$  be a commutative ring with  $\text{nil}(R)$  nonzero. If  $\Gamma(R)$  is uniquely complemented, then either  $\Gamma(R)$  is a star graph with at most two edges or  $\Gamma(R)$  is an infinite star graph with center  $x$ , where  $\text{nil}(R) = \{0, x\}$ .*

*(b) ([13, Theorem 3.14]) Let  $R$  be a commutative ring. Then  $\Gamma(R)$  is complemented, but not uniquely complemented, if and only if  $R$  is isomorphic to  $D \times B$ , where  $D$  is an integral domain and  $B$  is either  $\mathbb{Z}_4$  or  $\mathbb{Z}_2[X]/(X^2)$ .*

A commutative ring  $R$  is a *Boolean ring* if  $x^2 = x$  for all  $x \in R$ . Clearly, a Boolean ring is von Neumann regular. The simplest example of a Boolean ring is the power set of a set with symmetric difference as addition and intersection as multiplication (i.e., a direct product of  $\mathbb{Z}_2$ 's). Note that  $R$  is Boolean if and only if  $T(R)$  is Boolean, and in this case  $T(R) = R$ . In [46], J. D. LaGrange used these ideas to characterize Boolean rings in terms of zero-divisor graphs (also see [51]).

**Theorem 3.8. ([46, Theorem 2.5])** *A commutative ring  $R$  is a Boolean ring if and only if either  $R \cong \mathbb{Z}_2$  or  $\Gamma(R)$  is not the empty graph,  $R \notin \{\mathbb{Z}_9, \mathbb{Z}_3[X]/(X^2)\}$ , and  $\Gamma(R)$  has the property that every vertex has a unique complement. In particular, if  $|\Gamma(R)| \geq 3$ , then  $R$  is Boolean if and only if every vertex of  $\Gamma(R)$  has a unique complement.*

**Theorem 3.9. ([46, Theorem 4.3])** *Let  $R$  be a commutative ring with the property that every element of  $\Gamma(R)$  is either an end or is adjacent to an end. Then exactly one of the following holds:*

- (1)  $R \cong \mathbb{Z}_2 \times \mathbb{Z}_4$  or  $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2[X]/(X^2)$  (i.e.,  $\Gamma(R) = \overline{K}^{1,3}$ ).
- (2)  $\Gamma(R)$  is a star graph.
- (3)  $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ .

## 4 When does $\Gamma(R) \cong \Gamma(S)$ imply that $R \cong S$ ?

A very natural question when studying zero-divisor graphs is whether they are unique; i.e., is  $\Gamma(R) \cong \Gamma(S)$  if and only if  $R \cong S$ ? Clearly, one direction holds, but Example 1.1(a) shows that non-isomorphic rings may have isomorphic zero-divisor graphs. Specifically, the zero-divisor graphs of  $\mathbb{Z}_2 \times \mathbb{Z}_2$  and  $\mathbb{Z}_9$  are isomorphic, yet the two rings are clearly not isomorphic. This question has a positive answer when the rings are finite products of finite fields.

**Theorem 4.1. ([12, Theorem 4.1])** *Let  $R$  and  $S$  be finite reduced commutative rings which are not fields. Then  $\Gamma(R) \cong \Gamma(S)$  if and only if  $R \cong S$ .*

Earlier mentioned examples show that “reduced” is a necessary condition. “Finite” is also a necessary condition since the rings  $R = \mathbb{Z}_2 \times \mathbb{Z}$  and  $S = \mathbb{Z}_2 \times \mathbb{Q}$  are not isomorphic, but  $\Gamma(R)$  and  $\Gamma(S)$  are each a  $K^{1,\omega}$ . As becomes clear in Theorem 4.3, Theorem 4.1 is really a cardinality result.

Theorem 4.1 is a remarkable result which says that for certain rings, the behavior of the zero-divisors uniquely determines the entire ring. In fact, this result has been generalized as shown in the two results below and Corollary 4.5(b). The special case of Theorem 4.3 for finite fields is from [54, Corollary 2.4].

**Theorem 4.2. ([4, Theorem 5])** *Let  $R$  be a finite reduced commutative ring and  $S$  not an integral domain. If  $\Gamma(R) \cong \Gamma(S)$ , then  $R \cong S$ , unless  $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$  or  $\mathbb{Z}_6$ , and  $S$  is a local ring.*

**Theorem 4.3. ([13, Theorem 2.1])** *Let  $\{R_i\}_{i \in I}$  ( $|I| \geq 2$ ) and  $\{S_j\}_{j \in J}$  be two families of integral domains, and let  $R = \prod_{i \in I} R_i$  and  $S = \prod_{j \in J} S_j$ . Then  $\Gamma(R) \cong \Gamma(S)$  if and only if there is a bijection  $\varphi : I \rightarrow J$  such that  $|R_i| = |S_{\varphi(i)}|$  for each  $i \in I$ . In particular, if  $\Gamma(R) \cong \Gamma(S)$  and each  $R_i$  is a finite field, then each  $S_j$  is also a finite field and  $R_i \cong S_{\varphi(i)}$  for each  $i \in I$ , and thus  $R \cong S$ .*

The next theorem dashes any hope of completely characterizing many classes of commutative rings solely in terms of zero-divisor graphs. This “problem” never arises for finite rings since  $\Gamma(R) = R$  when  $R$  is finite.

**Theorem 4.4. ([13, Theorem 2.2])** *Let  $R$  be a commutative ring with total quotient ring  $T(R)$ . Then the graphs  $\Gamma(T(R))$  and  $\Gamma(R)$  are isomorphic.*

**Corollary 4.5. (a)** *([13, Corollary 2.3]) Let  $A$  and  $B$  be commutative rings. If  $T(A) \cong T(B)$ , then  $\Gamma(A) \cong \Gamma(B)$ . In particular,  $\Gamma(A) \cong \Gamma(B)$  if  $B$  is an overring of  $A$ .*

*(b)* *([13, Corollaries 2.4 and 2.5]) Let  $A$  and  $B$  be reduced commutative Noetherian rings which are not integral domains. Then  $\Gamma(A) \cong \Gamma(B)$  if and only if there is a bijection  $\varphi : \text{Min}(A) \rightarrow \text{Min}(B)$  such that  $|A/P| = |B/\varphi(P)|$  for each  $P \in \text{Min}(A)$ . In particular, if  $\text{Min}(A) = \{P_1, \dots, P_n\}$ , then  $\Gamma(A) \cong \Gamma(K_1 \times \dots \times K_n)$ , where each  $K_i = T(A/P_i)$  is a field.*

J. D. LaGrange has investigated the zero-divisor graph of the complete ring of quotients  $Q(R)$  of  $R$  (see [52] for the definition of  $Q(R)$ ) in [46, 48], and [49]. In this case, we may have  $\Gamma(R) \not\cong \Gamma(Q(R))$  [46, p. 606].

Note that the two von Neumann regular rings  $\mathbb{Z}_2 \times \mathbb{R}$  and  $\mathbb{Z}_2 \times \mathbb{C}$  have isomorphic zero-divisor graphs by Theorem 4.3, but are not isomorphic. Also see [54] for several related results and examples. However, a Boolean ring is determined by its zero-divisor graph (cf. [13, Theorem 4.1], [51], [57, Theorem 2.1], and [58, Section 4]).

**Theorem 4.6. ([46, Theorem 4.1])** *Let  $R$  be a commutative ring with nonzero zero-divisors, not isomorphic to  $\mathbb{Z}_9$  or  $\mathbb{Z}_3[X]/(X^2)$ . If  $S$  is a Boolean ring such that  $\Gamma(R) \cong \Gamma(S)$ , then  $R \cong S$ . In particular, if  $R$  and  $S$  are Boolean rings, then  $\Gamma(R) \cong \Gamma(S)$  if and only if  $R \cong S$ .*

Several authors have also studied (graph) automorphisms of  $\Gamma(R)$ . They were first investigated in [14], and in more detail in [36, 64], and [66]. There is a natural homomorphism  $\varphi : \text{Aut}(R) \longrightarrow \text{Aut}(\Gamma(R))$  which is injective when  $R$  is finite [14, Theorem 3.1]. In general,  $\ker(\varphi)$  is nonzero, but is abelian and may be identified with a group of derivations (see [64, Section 3] and [36, Section 2]).

## 5 Ideals and $Z(R)$

The zero-divisors of a commutative ring typically exhibit little additive structure, and it is this lack of closure under addition that prevents them from forming an ideal. Thus, it is natural to ask when  $Z(R)$  is an ideal of  $R$ . In particular, can we determine conditions on  $\Gamma(R)$  that will ensure  $Z(R)$  is an ideal? One approach has been to create cases based on the diameter of  $\Gamma(R)$ . For example, if  $R$  is a commutative ring with  $\text{diam}(\Gamma(R)) = 0$ , then either  $Z(R) = \{0\}$  or  $Z(R) = \{0, x\}$  with  $x^2 = 0$ . In the latter case, we must have  $x + x = 0$  since  $x(x + x) = x^2 + x^2 = 0$ . So in both instances  $Z(R)$  is an ideal. The case when  $\text{diam}(\Gamma(R)) = 1$  is also quickly solved. By Theorem 3.3(a),  $\Gamma(R)$  is complete if and only if  $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$  or  $xy = 0$  for all  $x, y \in Z(R)$ . If  $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ , then  $Z(R)$  is clearly not an ideal. Otherwise,  $Z(R)$  can quickly be shown to be closed under addition and hence an ideal. So, we turn our attention to the case when  $\text{diam}(\Gamma(R)) = 2$ . For this section, it will sometimes be useful to consider any vertex corresponding to a zero-divisor  $x$  with  $x^2 = 0$  as having a loop attached; i.e., an edge connecting the vertex  $x$  to itself. We call such a vertex *looped*. We denote this extension of  $\Gamma(R)$  by  $\Gamma^*(R)$ . Let  $R = R_1 \times \cdots \times R_n$  be a product of finite local commutative rings ( $n \geq 2$ ). Given  $\Gamma^*(R)$ , M. Taylor [80] has given an algorithm to determine  $n$ ,  $|R_i|$ , and  $\Gamma(R_i)$  for each  $1 \leq i \leq n$ .

**Lemma 5.1.** ([20, Lemma 3.1]) *Let  $R$  be a commutative ring with  $\text{diam}(\Gamma(R)) = 2$ . Then  $Z(R)$  is an ideal if and only if for all pairs  $x, y \in Z(R)$  there exists a nonzero (not necessarily distinct)  $z$  such that  $xz = yz = 0$ .*

**Definition 5.2.** A graph  $G$  is said to be *star-shaped reducible* if there exists a looped vertex  $g \in G$  such that  $g$  is connected to all other points in  $G$ .

In terms of  $\Gamma^*(R)$  where  $\text{diam}(\Gamma^*(R)) = 2$ , the condition that for all pairs  $x, y \in Z(R)$  there exists a nonzero (not necessarily distinct)  $z$  such that  $xz = yz = 0$  has a graph-theoretic description. Namely, if  $x - y$ , then either  $x$  or  $y$  has a loop ( $x^2 = 0$  or  $y^2 = 0$ ), or  $x - y$  is part of a cycle of length 3.

**Theorem 5.3.** ([20, Theorem 2.3], [9, Lemma 3.11]) *Let  $R$  be a finite commutative ring with identity. Then  $Z(R)$  is an ideal if and only if  $\Gamma^*(R)$  is star-shaped reducible. In this case,  $\text{diam}(\Gamma(R)) \leq 2$ .*

The case where  $R$  is infinite with  $\text{diam}(\Gamma(R)) = 3$  was settled in [59] and [9]. A method for producing a commutative ring  $R$  with  $\text{diam}(\Gamma(R)) = 3$  and  $Z(R)$  an ideal was outlined in [59, Example 5.1], and this was actually accomplished in [9, Example 3.13]. A simpler example is given in [20]

As an interesting side note to the above results, the information revealed helped to determine which finite graphs are realizable as zero-divisor graphs of commutative rings (such rings must be finite by Theorem 2.1).

**Theorem 5.4. ([20, Theorem 4.5])** *If  $G$  is realizable as a zero-divisor graph of a finite commutative ring with identity, then it is star-shaped reducible, complete bipartite, or  $\text{diam}(G) = 3$ .*

It is well known that  $Z(R)$  is the set-theoretic union of prime ideals [16, Exercise 14, p. 12]. By placing some modest restrictions on  $\Gamma(R)$  and  $R$ , the union is over a surprisingly small number of prime ideals.

**Theorem 5.5. ([17, Proposition 3.4])** *Let  $R$  be a commutative ring with  $\text{diam}(\Gamma(R)) \leq 2$  and let  $Z(R) = \bigcup_{i \in \Lambda} P_i$  for prime ideals  $P_i$  of  $R$ . If there is an element in  $Z(R)$  that is contained in a unique maximal  $P_i$ , then  $|\Lambda| \leq 2$ . In particular, if  $\Lambda$  is a finite set (e.g., if  $R$  is Noetherian), then  $|\Lambda| \leq 2$ .*

The following theorem is in the spirit of Theorems 2.5 and 2.6.

**Theorem 5.6. ([15, Theorem 2.7])** *Let  $R$  be a commutative ring with  $\text{diam}(\Gamma(R)) \leq 2$ . Then exactly one of the following holds.*

- (1)  $Z(R)$  is an (prime) ideal of  $R$ .
- (2)  $T(R) = K_1 \times K_2$ , where each  $K_i$  is a field.

We close this section with three results from [59] that provide further links between the ideal structure of  $R$  and  $Z(R)$  and  $\text{diam}(\Gamma(R))$ .

**Theorem 5.7. ([59, Theorem 2.1])** *Let  $R$  be a reduced commutative ring. If  $R$  has more than two minimal prime ideals and there are nonzero elements  $a, b \in Z(R)$  such that  $(a, b)$  has no nonzero annihilator, then  $\text{diam}(\Gamma(R)) = 3$ .*

It is of interest to note that Theorem 5.7 generalizes to the nonreduced case, and in this case we do not need the assumption that  $R$  has more than two minimal primes (see [59, Theorem 2.4]). A corollary to this appears below.

**Theorem 5.8. ([59, Theorem 2.2])** *Let  $R$  be a reduced commutative ring with  $Z(R)$  not an ideal. Then  $\text{diam}(\Gamma(R)) \leq 2$  if and only if  $R$  has exactly two minimal prime ideals.*

**Corollary 5.9. ([59, Corollary 2.5])** *If  $R$  is a non-reduced commutative ring such that  $Z(R)$  is not an ideal, then  $\text{diam}(\Gamma(R)) = 3$ .*

## 6 Planar and toroidal graphs

A graph  $G$  is *planar* if it can be embedded (i.e., drawn with no crossings) in the plane and is *toroidal* if it is not planar, but can be embedded in a torus. More generally,  $G$  has *genus*  $g$  if it can be embedded in a surface of genus  $g$ , but not in

one of genus  $g - 1$ . Let  $\gamma(G)$  denote the genus of  $G$ ; so  $G$  is planar (resp., toroidal) when  $\gamma(G) = 0$  (resp.,  $\gamma(G) = 1$ ). In this section, we determine all finite commutative rings with planar or toroidal zero-divisor graphs.

For the planar case, the proof uses the theorem of Kuratowski [38, Theorem 4.4.6] that a graph  $G$  is planar if and only if it contains no subdivisions homeomorphic to  $K^5$  or  $K^{3,3}$  and the fact that a finite commutative ring is the direct product of a finite number of local rings. The idea is to get a bound on the number of local ring factors, and then handle the local case. If  $\Gamma(R)$  is planar, then  $R$  has at most 3 local ring factors since otherwise it would contain a  $K^{3,3}$ . The cases for 2 or 3 local ring factors are then handled separately. For a finite local ring  $(R, M)$ ,  $M^n = 0$  for some positive integer  $n$ . If  $n \geq 5$ , then  $\Gamma(R)$  would contain a  $K^5$ , and thus not be planar. The toroidal case is similar, but uses the facts that  $K^m$  is toroidal if and only if  $m = 5, 6, 7$ , while  $K^{4,4}$  is toroidal and  $K^{3,n}$  is toroidal if and only if  $n = 3, 4, 5, 6$ .

The first work in this direction was given in [12], where they asked which finite commutative rings  $R$  have  $\Gamma(R)$  planar and gave the following partial answer.

**Theorem 6.1. ([12, Theorem 5.1])** (a) *Let  $R = \mathbb{Z}_n$ , where  $n \geq 2$  is not prime. Then  $\Gamma(R)$  is planar if and only if  $n \in \{8, 12, 16, 18, 25, 27\} \cup \{2p, 3p \mid p \text{ is prime}\}$ .*

(b) *Let  $R = \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_r}$ , where  $r \geq 2$  and  $2 \leq n_1 \leq \cdots \leq n_r$ . Then  $\Gamma(R)$  is planar if and only if  $R$  is one of  $\mathbb{Z}_2 \times \mathbb{Z}_4$ ,  $\mathbb{Z}_2 \times \mathbb{Z}_6$ ,  $\mathbb{Z}_2 \times \mathbb{Z}_8$ ,  $\mathbb{Z}_2 \times \mathbb{Z}_9$ ,  $\mathbb{Z}_2 \times \mathbb{Z}_p$ ,  $\mathbb{Z}_3 \times \mathbb{Z}_4$ ,  $\mathbb{Z}_3 \times \mathbb{Z}_9$ ,  $\mathbb{Z}_3 \times \mathbb{Z}_q$ ,  $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ ,  $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$ , where  $p \geq 2$  and  $q \geq 3$  are primes.*

**Theorem 6.2. ([12, Theorem 5.2])** *Let  $R_{n,m} = \mathbb{Z}_n[X]/(X^m)$ , where  $m, n \geq 2$ .*

- (a)  $\Gamma(R_{n,2})$  is planar if and only if  $n \leq 5$ .
- (b)  $\Gamma(R_{n,3})$  is planar if and only if  $n \leq 3$ .
- (c)  $\Gamma(R_{n,4})$  is planar if and only if  $n = 5$ .
- (d)  $\Gamma(R_{n,m})$  is never planar if  $m \geq 5$ .

S. Akbari, H. R. Maimani, and S. Yassemi [3] gave a partial answer by showing that if  $\Gamma(R)$  is planar, then  $R$  has at most 3 local ring factors, describing those local ring factors, and giving the following theorem for when  $R$  is local.

**Theorem 6.3. ([3, Theorems 1.2 and 1.4])** *Let  $(R, M)$  be a finite local commutative ring. Then  $\Gamma(R)$  is not planar if one of the following holds.*

- (1)  $|R/M| \geq 4$  and  $|R| \geq 26$ .
- (2)  $|R/M| = 3$  and  $|R| \geq 28$ .
- (3)  $|R/M| = 2$  and  $|R| \geq 33$ .

In [3, Remark 1.5], they also asked the following question: “Is it true that, for any local ring  $R$  of cardinality 32, which is not a field,  $\Gamma(R)$  is not planar?” N. O. Smith [76] answered their question affirmatively and explicitly gave all the finite commutative rings with planar zero-divisor graphs (their zero-divisor graphs are given in [77]). Their question has also been answered independently by H.-J. Wang [82] and R. Belshoff and J. Chapman [25] ([76] was not reviewed in Math Reviews). Belshoff and Chapman also give all finite local rings with planar zero-divisor graphs using somewhat different techniques than [76].

**Theorem 6.4. ([76, Theorem 3.7])** *Let  $R$  be a finite commutative ring (not a field), and  $k$  a (finite) field. Then  $\Gamma(R)$  is planar if and only if  $R$  is isomorphic to one of the following 44 types of rings:  $\mathbb{Z}_2 \times k$ ,  $\mathbb{Z}_3 \times k$ ,  $\mathbb{Z}_2 \times \mathbb{Z}_4$ ,  $\mathbb{Z}_2 \times \mathbb{Z}_2[X]/(X^2)$ ,  $\mathbb{Z}_3 \times \mathbb{Z}_4$ ,  $\mathbb{Z}_3 \times \mathbb{Z}_2[X]/(X^2)$ ,  $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ ,  $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$ ,  $\mathbb{Z}_2 \times \mathbb{Z}_8$ ,  $\mathbb{Z}_2 \times \mathbb{Z}_2[X]/(X^3)$ ,  $\mathbb{Z}_2 \times \mathbb{Z}_4[X]/(2X, X^2 - 2)$ ,  $\mathbb{Z}_2 \times \mathbb{Z}_9$ ,  $\mathbb{Z}_2 \times \mathbb{Z}_3[X]/(X^2)$ ,  $\mathbb{Z}_3 \times \mathbb{Z}_9$ ,  $\mathbb{Z}_3 \times \mathbb{Z}_3[X]/(X^2)$ ,  $\mathbb{Z}_4$ ,  $\mathbb{Z}_2[X]/(X^2)$ ,  $\mathbb{Z}_9$ ,  $\mathbb{Z}_3[X]/(X^2)$ ,  $\mathbb{Z}_8$ ,  $\mathbb{Z}_2[X]/(X^3)$ ,  $\mathbb{Z}_4[X]/(2X, X^2 - 2)$ ,  $\mathbb{Z}_{16}$ ,  $\mathbb{Z}_2[X]/(X^4)$ ,  $\mathbb{Z}_4[X]/(2X, X^3 - 2)$ ,  $\mathbb{Z}_4[X]/(X^2 - 2)$ ,  $\mathbb{Z}_4[X]/(X^2 + 2X + 2)$ ,  $\mathbb{F}_4[X]/(X^2)$ ,  $\mathbb{Z}_4[X]/(X^2 + X + 1)$ ,  $\mathbb{Z}_2[X, Y]/(X, Y)^2$ ,  $\mathbb{Z}_4[X]/(2, X)^2$ ,  $\mathbb{Z}_{27}$ ,  $\mathbb{Z}_3[X]/(X^3)$ ,  $\mathbb{Z}_3[X]/(X^2 - 3, 3X)$ ,  $\mathbb{Z}_9[X]/(X^2 - 6, 3X)$ ,  $\mathbb{Z}_2[X, Y]/(X^2, Y^2 - XY)$ ,  $\mathbb{Z}_2[X, Y]/(X^2, Y^2)$ ,  $\mathbb{Z}_8[X]/(2X - 4, X^2)$ ,  $\mathbb{Z}_4[X]/(X^2)$ ,  $\mathbb{Z}_4[X]/(X^2 - 2X)$ ,  $\mathbb{Z}_4[X, Y]/(X^2, XY - 2, Y^2 - XY, 2X, 2Y)$ ,  $\mathbb{Z}_4[X, Y]/(X^2, XY - 2, Y^2, 2X, 2Y)$ ,  $\mathbb{Z}_{25}$ , or  $\mathbb{Z}_5[X]/(X^2)$ .*

**Corollary 6.5. ([76, Corollary 3.8], [82, Theorem 3.2], [25, Proposition 5])** *Let  $R$  be a finite commutative local ring (not a field) with either  $|R| \geq 28$  or  $|Z(R)| \geq 10$ . Then  $\Gamma(R)$  is not planar.*

Note that the above corollary is best possible since  $R = \mathbb{Z}_{27}$  is a local ring with  $|R| = 27$ ,  $|Z(R)| = 9$ , and planar zero-divisor graph.

Smith [79] has also characterized the infinite planar zero-divisor graphs. Note that if  $\Gamma(R)$  is planar, then necessarily  $|\Gamma(R)| \leq c$ ; so a  $K^{1,\alpha}$  is planar if and only if  $\alpha \leq c$ . Moreover, the graphs given in the following theorem can all be realized as the zero-divisor graphs of commutative rings [79, Remark 2.20].

**Theorem 6.6. ([79, Theorem 2.19])** *Let  $R$  be an infinite commutative ring (not an integral domain) such that  $\Gamma(R)$  is planar. Then  $\Gamma(R)$  is isomorphic to either a star graph, a  $K^{2,\alpha}$ , where  $\alpha \leq c$ , or the graph obtained by taking such a  $K^{2,\alpha}$  and adding an edge between the two vertices of infinite degree.*

We next proceed to zero-divisor graphs of genus one. This research was initiated by H.-J. Wang in [82], where he determined which finite commutative rings of the type in Theorems 6.1 and 6.2 have genus at most one, answered the question about planar local rings raised in [3], and gave bounds on the cardinality of local rings of genus one. The complete genus-one solution was achieved independently by C. Wickham [83] and H.-J. Chiang-Hsieh, H.-J. Wang, and N. O. Smith [31]. The planar case is also redone in [31]. All three papers also give partial results for zero-divisor graphs of higher genus. The next theorem lists all the finite commutative rings with  $\gamma(\Gamma(R)) = 1$ .

**Theorem 6.7. ([83, Theorems 3.1 and 4.1], [31, Theorems 3.5.2 and 3.6.2])** *Let  $R$  be a finite commutative ring which is not a field.*

(a) *If  $R$  is local, then  $\gamma(\Gamma(R)) = 1$  if and only if  $R$  is isomorphic to one of the following 17 rings:  $\mathbb{Z}_{32}$ ,  $\mathbb{Z}_{49}$ ,  $\mathbb{Z}_2[X]/(X^5)$ ,  $\mathbb{F}_8[X]/(X^2)$ ,  $\mathbb{Z}_2[X]/(X^3, XY, Y^2)$ ,  $\mathbb{Z}_2[X, Y, Z]/(X, Y, Z)^2$ ,  $\mathbb{Z}_4[X]/(X^3 + X + 1)$ ,  $\mathbb{Z}_4[X]/(X^3 - 2, X^5)$ ,  $\mathbb{Z}_4[X]/(X^4 - 2, X^5)$ ,  $\mathbb{Z}_4[X, Y]/(X^3, X^2 - 2, XY, Y^2)$ ,  $\mathbb{Z}_4[X]/(X^3, 2X)$ ,  $\mathbb{Z}_4[X, Y]/(2X, 2Y, X^2, XY, Y^2)$ ,  $\mathbb{Z}_7[X]/(X^2)$ ,  $\mathbb{Z}_8[X]/(X^2, 2X)$ ,  $\mathbb{Z}_8[X]/(X^2 - 2, X^5)$ ,  $\mathbb{Z}_8[X]/(X^2 + 2X - 2, X^5)$ , or  $\mathbb{Z}_8[X]/(X^2 - 2x + 2, X^5)$ .*

(b) If  $R$  is not local, then  $\gamma(\Gamma(R)) = 1$  if and only if  $R$  is isomorphic to one of the following 29 rings:  $\mathbb{F}_4 \times \mathbb{F}_4$ ,  $\mathbb{F}_4 \times \mathbb{Z}_5$ ,  $\mathbb{F}_4 \times \mathbb{Z}_7$ ,  $\mathbb{Z}_5 \times \mathbb{Z}_5$ ,  $\mathbb{Z}_2 \times \mathbb{F}_4[X]/(X^2)$ ,  $\mathbb{Z}_2 \times \mathbb{Z}_4[X]/(X^2 + X + 1)$ ,  $\mathbb{Z}_2 \times \mathbb{Z}_2[X, Y]/(X^2, XY, Y^2)$ ,  $\mathbb{Z}_2 \times \mathbb{Z}_4[X]/(2X, X^2)$ ,  $\mathbb{Z}_3 \times \mathbb{Z}_2[X]/(X^3)$ ,  $\mathbb{Z}_3 \times \mathbb{Z}_4[X]/(X^2 - 2, X^3)$ ,  $\mathbb{Z}_3 \times \mathbb{Z}_8$ ,  $\mathbb{Z}_4 \times \mathbb{F}_4$ ,  $\mathbb{F}_4 \times \mathbb{Z}_2[X]/(X^2)$ ,  $\mathbb{Z}_4 \times \mathbb{Z}_4$ ,  $\mathbb{Z}_4 \times \mathbb{Z}_2[X]/(X^2)$ ,  $\mathbb{Z}_2[X]/(X^2) \times \mathbb{Z}_2[X]/(X^2)$ ,  $\mathbb{Z}_4 \times \mathbb{Z}_5$ ,  $\mathbb{Z}_5 \times \mathbb{Z}_2[X]/(X^2)$ ,  $\mathbb{Z}_4 \times \mathbb{Z}_7$ ,  $\mathbb{Z}_7 \times \mathbb{Z}_2[X]/(X^2)$ ,  $\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ ,  $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ ,  $\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{F}_4$ ,  $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_5$ ,  $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_7$ ,  $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{F}_4$ ,  $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4$ ,  $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2[X]/(X^2)$ , or  $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ .

We conclude this section by mentioning that H.-J. Chiang-Hsieh [30] has recently determined all the finite commutative rings  $R$  such that  $\Gamma(R)$  is projective (a nonplanar graph is said to be *projective* if it can be embedded in the projective plane). Such a ring has at most 4 prime ideals, and up to isomorphism, there are 36 finite commutative rings with projective zero-divisor graph.

## 7 Origins and generalizations

In this final section, we first trace the origins and early history of  $\Gamma(R)$ , and then briefly mention several other directions this research has taken. It all started in 1988 when I. Beck [23] presented the idea of associating a “zero-divisor” graph with a commutative ring. However, Beck used a slightly different definition for  $\Gamma(R)$  and was mainly interested in colorings. He let all elements of  $R$  be vertices; so 0 is adjacent to every other vertex. We denote Beck’s zero-divisor graph of  $R$  by  $\Gamma_0(R)$  (Beck just used  $R$ ); so  $\Gamma(R)$  is an induced subgraph of  $\Gamma_0(R)$ .

If either  $R$  is an integral domain,  $R \cong \mathbb{Z}_4$ , or  $R \cong \mathbb{Z}_2[X]/(X^2)$ , then  $\Gamma_0(R)$  is the star graph  $K^{1,\alpha}$ , where  $\alpha = |R^*|$ ; so  $\text{gr}(\Gamma_0(R)) = \infty$ . Otherwise,  $\text{gr}(\Gamma_0(R)) = 3$  since 0 and any two distinct zero-divisors  $x$  and  $y$  of  $R$  with  $xy = 0$  determine a triangle. Clearly,  $\Gamma_0(R)$  is always connected. We have  $\text{diam}(\Gamma_0(R)) = 1$  for  $R \cong \mathbb{Z}_2$ , and  $\text{diam}(\Gamma_0(R)) = 2$  for all other rings  $R$  since  $x - 0 - y$  is a path of length two between any two distinct nonzero elements  $x$  and  $y$  of  $R$ . Also, it is easily verified that  $\Gamma_0(R)$  is complete if and only if  $R \cong \mathbb{Z}_2$ , and  $\Gamma_0(R)$  is complete bipartite if and only if it is one of the star graphs mentioned above. We can always recover  $\Gamma(R)$  from  $\Gamma_0(R)$  except when  $\Gamma_0(R)$  is a  $K^{1,3}$ , since in this case  $R$  could be either  $\mathbb{F}_4$ ,  $\mathbb{Z}_4$ , or  $\mathbb{Z}_2[X]/(X^2)$ , and thus  $\Gamma(R)$  could either be empty or a singleton. Based on the above comments and what we have seen in the earlier sections, it is clear that  $\Gamma(R)$  has a much richer and more appealing structure than  $\Gamma_0(R)$  and better reflects properties of  $Z(R)$ .

Beck’s focus was on rings that could be finitely colored, i.e.,  $\chi(\Gamma_0(R)) < \infty$ . He called such rings *Colorings*, and characterized these rings in the following theorem.

**Theorem 7.1.** ([23, Theorem 3.9]) *The following conditions are equivalent for a commutative ring  $R$ .*

- (1)  $\chi(\Gamma_0(R))$  is finite (i.e.,  $R$  is a *Coloring*).
- (2)  $cl(\Gamma_0(R))$  is finite.
- (3)  $\text{nil}(R)$  is finite and equals a finite intersection of prime ideals.
- (4)  $\Gamma_0(R)$  does not contain an infinite clique.

Beck proved many other results about Colorings. For example, he showed that if  $R$  is a Coloring, then  $Ass(R)$  is finite [23, Theorem 4.3] and for  $P \in Ass(R)$ , either  $R_P$  is a field or  $P$  is a maximal ideal [23, Theorem 4.4]. He also investigated the stability of the family of Colorings, and showed that a finite direct product of Colorings is a Coloring [23, Theorem 5.5], a localization of a Coloring is a Coloring [23, Theorem 5.8], and certain factor rings of Colorings are Colorings [23, Theorems 5.2, 5.4, and 5.6]. Moreover, Beck determined all the finite commutative rings  $R$  with  $\chi(\Gamma_0(R)) \leq 3$  [23, p. 226](see Theorem 7.4(a)-(b)).

It is easy to see that  $cl(\Gamma_0(R)) \leq \chi(\Gamma_0(R))$  for any commutative ring  $R$ . Based on the evidence given in the next theorem, Beck conjectured that  $\chi(\Gamma_0(R)) = cl(\Gamma_0(R))$  for any Coloring  $R$ .

**Theorem 7.2.** *Let  $R$  be a commutative ring with  $\chi(\Gamma_0(R)) < \infty$ .*

(a) ([23, Theorem 3.8])  $\chi(\Gamma_0(R)) = cl(\Gamma_0(R)) = |Min(R)| + 1$  if  $R$  is reduced.

(b) ([23, Theorem 6.13]) *If  $R$  is a finite direct product of reduced rings and principal ideal rings, then  $\chi(\Gamma_0(R)) = cl(\Gamma_0(R))$ .*

(c) ([23, Theorem 7.3]) *Let  $n \leq 4$  be a positive integer. Then  $\chi(\Gamma_0(R)) = n$  if and only if  $cl(\Gamma_0(R)) = n$ . Moreover,  $\chi(\Gamma_0(R)) = 5$  implies  $cl(\Gamma_0(R)) = 5$ .*

In 1993, five years after Beck's paper appeared, D. D. Anderson and M. Naseer [8] provided a counterexample to Beck's conjecture (also see [27] and [39]).

*Example 7.3.* ([8, Theorem 2.1]) Let  $R$  be the commutative ring  $\mathbb{Z}_4[X, Y, Z]/(X^2 - 2, Y^2 - 2, Z^2, 2X, 2Y, 2Z, XY, XZ, YZ - 2)$ . In this example,  $cl(\Gamma_0(R)) = 5$ , but  $\chi(\Gamma_0(R)) = 6$ . This ring is a local ring  $(R, M)$  with 32 elements,  $R/M \cong \mathbb{Z}_2$ ,  $M^2 \neq 0$ , but  $M^3 = 0$ . This counterexample is minimal in several senses. Firstly, it has the smallest possible clique or chromatic number for a counterexample by Theorem 7.2(c). Secondly, it is minimal in the sense that a Coloring  $S$  with  $nil(S)^2 = 0$  has  $\chi(\Gamma_0(S)) = cl(\Gamma_0(S)) = |nil(S)| + 1$  by [8, Theorem 3.1]. Finally, it has the smallest number of elements possible since if  $S$  is a finite commutative ring with  $|S| \leq 31$ , then  $\chi(\Gamma_0(S)) = cl(\Gamma_0(S))$ .

In addition, their paper contained several positive results. For example, they showed that a Noetherian ring  $R$  is a Coloring if and only if it is a subring of a finite direct product of fields and a finite ring [8, Theorem 3.6] and that a ring of the form  $A/M_1^{m_1} \cdots M_n^{m_n}$ , where  $A$  is a regular Noetherian ring,  $M_1, \dots, M_n$  are maximal ideals of  $A$  with each  $A/M_i$  finite, and  $m_1, \dots, m_n$  are positive integers is a Coloring and determined its chromatic number [8, Corollary 3.3]. They also determined the finite commutative rings  $R$  with  $\chi(\Gamma_0(R)) = 4$  (see Theorem 7.4(c)).

Although recent directions in zero-divisor graph theory have not involved colorings, the papers of Beck and Anderson–Naseer did determine the chromatic number of certain rings. It is easy to see that  $cl(\Gamma_0(R)) = cl(\Gamma(R)) + 1$  and  $\chi(\Gamma_0(R)) = \chi(\Gamma(R)) + 1$ . Using those two facts and Theorem 7.2(c), the next theorem translates their results about  $\chi(\Gamma_0(R))$  to  $cl(\Gamma(R))$ . The finite nonlocal commutative rings  $R$  with  $cl(\Gamma(R)) = 4$  have been computed by N. O. Smith [76].



**Theorem 7.4.** *Let  $R$  be a finite commutative ring, and  $K_1$ ,  $K_2$ , and  $K_3$  be finite fields.*

(a) ([23, Proposition 2.2])  *$cl(\Gamma(R)) = 1$  if and only if  $R$  is isomorphic to  $\mathbb{Z}_4$  or  $\mathbb{Z}_2[X]/(X^2)$ .*

(b) ([23, p. 226])  *$cl(\Gamma(R)) = 2$  if and only if  $R$  is isomorphic to one of the following 8 types of rings:  $K_1 \times K_2$ ,  $K_1 \times \mathbb{Z}_4$ ,  $K_1 \times \mathbb{Z}_2[X]/(X^2)$ ,  $\mathbb{Z}_8$ ,  $\mathbb{Z}_9$ ,  $\mathbb{Z}_3[X]/(X^2)$ ,  $\mathbb{Z}_2[X]/(X^3)$ , or  $\mathbb{Z}_4[X]/(2X, X^2 - 2)$ .*

(c) ([8, Theorem 4.4])  *$cl(\Gamma(R)) = 3$  if and only if  $R$  is isomorphic to one of the following 31 types of rings:  $\mathbb{Z}_4 \times \mathbb{Z}_4$ ,  $\mathbb{Z}_4 \times \mathbb{Z}_2[X]/(X^2)$ ,  $\mathbb{Z}_2[X]/(X^2) \times \mathbb{Z}_2[X]/(X^2)$ ,  $K_1 \times K_2 \times K_3$ ,  $K_1 \times K_2 \times \mathbb{Z}_4$ ,  $K_1 \times K_2 \times \mathbb{Z}_2[X]/(X^2)$ ,  $K_1 \times \mathbb{Z}_8$ ,  $K_1 \times \mathbb{Z}_9$ ,  $K_1 \times \mathbb{Z}_3[X]/(X^2)$ ,  $K_1 \times \mathbb{Z}_2[X]/(X^3)$ ,  $K_1 \times \mathbb{Z}_4[X]/(2X, X^2 - 2)$ ,  $\mathbb{Z}_{16}$ ,  $\mathbb{Z}_2[X]/(X^4)$ ,  $\mathbb{Z}_4[X]/(2X, X^3 - 2)$ ,  $\mathbb{Z}_4[X]/(X^2 - 2)$ ,  $\mathbb{Z}_4[X]/(X^2 + 2X + 2)$ ,  $\mathbb{F}_4[X]/(X^2)$ ,  $\mathbb{Z}_4[X]/(X^2 + X + 1)$ ,  $\mathbb{Z}_2[X, Y]/(X, Y)^2$ ,  $\mathbb{Z}_4[X]/(2, X)^2$ ,  $\mathbb{Z}_{27}$ ,  $\mathbb{Z}_3[X]/(X^3)$ ,  $\mathbb{Z}_9[X]/(3X, X^2 - 3)$ ,  $\mathbb{Z}_9[X]/(3X, X^2 - 6)$ ,  $\mathbb{Z}_2[X, Y]/(X^2, Y^2 - XY)$ ,  $\mathbb{Z}_2[X, Y]/(X^2, Y^2)$ ,  $\mathbb{Z}_8[X]/(2X - 4, X^2)$ ,  $\mathbb{Z}_4[X]/(X^2)$ ,  $\mathbb{Z}_4[X]/(X^2 - 2X)$ ,  $\mathbb{Z}_4[X, Y]/(X^2, XY - 2, Y^2, 2X, 2Y)$ , or  $\mathbb{Z}_4[X, Y]/(X^2, XY - 2, X^2 - XY, 2X, 2Y)$ .*

Another 6 years passed before the 1999 article by D. F. Anderson and P. S. Livingston [14], which was based on Livingston's 1997 Master's Thesis [55]. This paper introduced our present definition of  $\Gamma(R)$  and emphasized the interplay between ring-theoretic properties of  $R$  and graph-theoretic properties of  $\Gamma(R)$ . Besides the basic results on diameter and girth given in Section 2 and characterizations of complete and complete bipartite zero-divisor graphs given in Section 3, they also studied graph automorphisms of  $\Gamma(R)$ . Then in 2002, the papers by S. B. Mulay [64] and F. DeMeyer and K. Schneider [36] were published. These two papers built on the work in [14], independently answered the girth conjecture, and gave a more detailed study of automorphisms of  $\Gamma(R)$  (see Section 4).

Finally, the article by D. F. Anderson, A. Frazier, A. Lauve, and P. S. Livingston [12] in 2001 reviewed and consolidated earlier work on diameter and girth, did additional work on clique numbers, and initiated work on planar zero-divisor graphs (see Section 6) and isomorphisms of zero-divisor graphs (see Section 4). They also gave explicit formulas to compute the number of complete subgraphs (cliques) of  $\Gamma(R)$  of order  $n$  for  $R$  a finite reduced commutative ring or  $\mathbb{Z}_{p^m}$  with  $p$  prime.

The two papers on Colorings ([23] and [8]) and the four papers on  $\Gamma(R)$  [12, 14, 36, 64] discussed in this section contain many more results. The interested reader should consult them to get a flavor for the formative work on zero-divisor graphs of commutative rings. These papers either contain or motivate much of the work discussed in earlier sections and the many generalizations discussed below.

This survey has concentrated on zero-divisor graphs of commutative rings with identity. This idea has been extended in many different directions, usually to either zero-divisor graphs for different algebraic structures or to different types of graphs for commutative rings.

Zero-divisor graphs for noncommutative rings were first studied in [67]. In this case, there are several possible definitions and the graph may be either directed or undirected. Besides rings, the same definition makes sense for any algebraic structure with a zero element. For semigroups, this was first studied in [35]. A second

direction would be to start with a commutative ring  $R$  and use a different set of vertices or adjacency relation. For example, see [61, 75], or [11]. Again, these ideas can be extended to other algebraic structures.

A survey paper of this nature cannot hope to provide a complete picture of all the avenues of research being pursued within the study of zero-divisor graphs. The bibliography is our attempt to provide the reader with as complete as possible listing of works in this area. Other avenues being explored include ideal-based zero-divisor graphs, zero-divisor graphs of ideals, homology of zero-divisor graphs, and applying a graph structure to other algebraic constructs such as factorizations into irreducibles and to commuting elements in matrix algebras.

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# Class semigroups and $t$ -class semigroups of integral domains

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**Abstract** The class (resp.,  $t$ -class) semigroup of an integral domain is the semigroup of the isomorphism classes of the nonzero fractional ideals (resp.,  $t$ -ideals) with the operation induced by ideal ( $t$ -) multiplication. This paper surveys recent literature which studies ring-theoretic conditions that reflect reciprocally in the Clifford property of the class (resp.,  $t$ -class) semigroup. Precisely, it examines integral domains with Clifford class (resp.,  $t$ -class) semigroup and describes their idempotent elements and the structure of their associated constituent groups.

## 1 Introduction

All rings considered in this paper are integral domains. The notion of ideal class group of a domain is classical in commutative algebra and is also one of major objects of investigation in algebraic number theory. Let  $R$  be a domain. The ideal class group  $\mathcal{C}(R)$  (also called Picard group) of  $R$  consists of the isomorphism classes of the invertible ideals of  $R$ , that is, the factor group  $\mathcal{J}(R)/\mathcal{P}(R)$ , where  $\mathcal{J}(R)$  is the group of invertible fractional ideals and  $\mathcal{P}(R)$  is the subgroup of nonzero principal fractional ideals of  $R$ . A famous result by Claiborne states that every Abelian group can be regarded as the ideal class group of a Dedekind domain.

If  $R$  is Dedekind, then  $\mathcal{J}(R)$  coincides with the semigroup  $\mathcal{F}(R)$  of nonzero fractional ideals of  $R$ . Thus, a natural generalization of the ideal class group is the semigroup  $\mathcal{F}(R)/\mathcal{P}(R)$  of the isomorphism classes of nonzero fractional ideals of  $R$ . The factor semigroup  $\mathcal{F}(R)/\mathcal{P}(R)$  is denoted by  $\mathcal{S}(R)$  and called the *class*

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*semigroup* of  $R$ . The class semigroup of an order in an algebraic number field was first investigated by Dade, Taussky and Zassenhaus [18] and later by Zanardo and Zannier [59]. Halter-Koch [34] considered the case of the class semigroup of lattices over Dedekind domains.

The investigation of the structure of a semigroup is not as attractive as the study of a group. This is the reason why it is convenient to restrict attention to the case of a particular type of semigroups, namely, the Clifford semigroups. A commutative semigroup  $S$  (with 1) is said to be Clifford if every element  $x$  of  $S$  is (von Neumann) regular, i.e., there exists  $a \in S$  such that  $x^2a = x$ . The importance of a Clifford semigroup  $S$  resides in its ability to stand as a disjoint union of groups  $G_e$ , each one associated to an idempotent element  $e$  of the semigroup and connected by *bonding homomorphisms* induced by multiplications by idempotent elements [16]. The semigroup  $S$  is said to be Boolean if for each  $x \in S$ ,  $x = x^2$ .

Let  $R$  be a domain with quotient field  $K$ . For a nonzero fractional ideal  $I$  of  $R$ , let  $I^{-1} := (R : I) = \{x \in K \mid xI \subseteq R\}$ . The  $v$ - and  $t$ -closures of  $I$  are defined, respectively, by  $I_v := (I^{-1})^{-1}$  and  $I_t := \bigcup J_v$  where  $J$  ranges over the set of finitely generated subideals of  $I$ . The ideal  $I$  is said to be divisorial or a  $v$ -ideal if  $I_v = I$ , and  $I$  is said to be a  $t$ -ideal if  $I_t = I$ . Under the ideal  $t$ -multiplication  $(I, J) \mapsto (IJ)_t$ , the set  $\mathcal{F}_t(R)$  of fractional  $t$ -ideals of  $R$  is a semigroup with unit  $R$ . An invertible element for this operation is called a  $t$ -invertible  $t$ -ideal of  $R$ .

The  $t$ -operation in integral domains is considered as one of the keystones of multiplicative ideal theory. It originated in Jaffard's 1960 book "Les Systèmes d'Idéaux" [37] and was investigated by many authors in the 1980s. From the  $t$ -operation stemmed the notion of ( $t$ -)class group of an arbitrary domain, extending both notions of divisor class group (in Krull domains) and ideal class group (in Prüfer domains). Class groups were introduced and developed by Bouvier and Zafrullah [12, 13], and have been extensively studied in the literature. The ( $t$ -)class group of  $R$ , denoted  $\text{Cl}(R)$ , is the group under  $t$ -multiplication of fractional  $t$ -invertible  $t$ -ideals modulo its subgroup of nonzero principal fractional ideals. The  $t$ -class semigroup of  $R$ , denoted  $\mathcal{S}_t(R)$ , is the semigroup under  $t$ -multiplication of fractional  $t$ -ideals modulo its subsemigroup of nonzero principal fractional ideals. One may view  $\mathcal{S}_t(R)$  as the  $t$ -analogue of  $\mathcal{S}(R)$ , similarly as the ( $t$ -)class group  $\text{Cl}(R)$  is the  $t$ -analogue of the ideal class group  $\mathcal{C}(R)$ . We have the set-theoretic inclusions

$$\mathcal{C}(R) \subseteq \text{Cl}(R) \subseteq \mathcal{S}_t(R) \subseteq \mathcal{S}(R).$$

The properties of the class group or class semigroup of a domain can be translated into ideal-theoretic information on the domain and conversely. If  $R$  is a Prüfer domain,  $\mathcal{C}(R) = \text{Cl}(R)$  and  $\mathcal{S}_t(R) = \mathcal{S}(R)$ ; and then  $R$  is a Bézout domain if and only if  $\text{Cl}(R) = 0$ . If  $R$  is a Krull domain,  $\text{Cl}(R) = \mathcal{S}_t(R)$  equals its usual divisor class group, and then  $R$  is a UFD if and only if  $\text{Cl}(R) = 0$  (so that  $R$  is a UFD if and only if every  $t$ -ideal of  $R$  is principal). Trivially, Dedekind domains (resp., PIDs) have Clifford (resp., Boolean) class semigroup. In 1994, Zanardo and Zannier proved that all orders in quadratic fields have Clifford class semigroup, whereas the ring of all entire functions in the complex plane (which is Bézout) fails to have this property [59].

Thus, the natural question arising is to characterize the domains with Clifford class (resp.,  $t$ -class) semigroup and, moreover, to describe their idempotent elements and the structure of their associated constituent groups.

## 2 Class semigroups of integral domains

A domain is said to be *Clifford regular* if its class semigroup is a Clifford semigroup. The first significant example of a Clifford regular domain is a valuation domain. In fact, in [9], Salce and the first named author proved that the class semigroup of any valuation domain is a Clifford semigroup whose constituent groups are either trivial or groups associated to the idempotent prime ideals of  $R$ . Next, the investigation was carried over for the class of Prüfer domains of finite character, that is, the Prüfer domains such that every nonzero ideal is contained in only finitely many maximal ideals. In [5], the first named author proved that if  $R$  is a Prüfer domain of finite character, then  $R$  is a Clifford regular domain and moreover, in [6] and [7] a description of the idempotent elements of  $\mathcal{S}(R)$  and of their associated groups was given.

A complete characterization of the class of integrally closed Clifford regular domains was achieved in [8] where it is proved that it coincides with the class of the Prüfer domains of finite character. Moreover, [8] explores the relation between Clifford regularity, stability and finite stability. Recall that an ideal of a commutative ring is said to be *stable* if it is projective over its endomorphism ring and a ring  $R$  is said to be *stable* if every ideal of  $R$  is stable. The notion of stability was first introduced in the Noetherian case with various different definitions which turned out to be equivalent in the case of a local Noetherian ring (cf. [51]). Olberding has described the structural properties of an arbitrary stable domain. In [51] and [50] he proves that a domain is stable if and only if it is of finite character and locally stable. Rush, in [52] considered the class of *finitely stable* rings, that is, rings with the property that every finitely generated ideal is stable and proved that the integral closure of such rings is a Prüfer ring.

In [8], it is shown that the class of Clifford regular domains is properly intermediate between the class of finitely stable domains and the class of stable domains. In particular, the integral closure of a Clifford regular domain is a Prüfer domain. Moreover, this implies that a Noetherian domain is Clifford regular if and only if it is a stable domain. Thus, [8] provides for a characterization of the class of Clifford regular domains in the classical cases of Noetherian and of integrally closed domains. In the general case, the question of determining whether Clifford regularity always implies finite character is still open.

In [8], was also outlined a relation between Clifford regularity and the *local invertibility property*. A domain is said to have the local invertibility property if every locally invertible ideal is invertible. In [5] and again in [8] the question of deciding if a Prüfer domain with the local invertibility property is necessarily of finite character was proposed as a conjecture. The question was of an interest on its own



independently of Clifford regularity and it attracted the interest of many authors. Recently the validity of the conjecture has been proved by Holland, Martinez, McGovern and Tesemma [36]. They translated the problem into a statement on the lattice ordered group of the invertible fractional ideals of a Prüfer domain and then used classical results by Conrad [17] on lattice ordered groups.

## 2.1 Preliminaries and notations

Let  $\mathcal{S}$  be a commutative multiplicative semigroup. The subsemigroup  $\mathcal{E}$  of the idempotent elements of  $\mathcal{S}$  has a natural partial order defined by  $e \leq f$  if and only if  $ef = e$ , for every  $e, f \in \mathcal{E}$ . Clearly,  $e \wedge f = ef$  and thus  $\mathcal{E}$  is a  $\wedge$ -semilattice under this order. An element  $a$  of a semigroup  $\mathcal{S}$  is *von Neumann regular* if  $a = a^2x$  for some  $x \in \mathcal{S}$ .

**Definition 2.1.** A commutative semigroup  $\mathcal{S}$  is a *Clifford semigroup* if every element of  $\mathcal{S}$  is regular.

By [16] a Clifford semigroup  $\mathcal{S}$  is the disjoint union of the family of groups  $\{G_e \mid e \in \mathcal{E}\}$ , where  $G_e$  is the largest subgroup of  $\mathcal{S}$  containing the idempotent element  $e$ , that is:

$$G_e = \{ae \mid abe = e \text{ for some } b \in \mathcal{S}\}.$$

In fact, if  $a \in \mathcal{S}$  and  $a = a^2x$ ,  $x \in \mathcal{S}$ , then  $e = ax$  is the unique idempotent element such that  $a \in G_e$ . We say that  $e = ax$  is the idempotent associated to  $a$ . The groups  $G_e$  are called the constituent groups of  $\mathcal{S}$ . If  $e \leq f$  are idempotent elements, that is  $fe = e$ , the multiplication by  $e$  induces a group homomorphism  $\phi_e^f : G_f \rightarrow G_e$  called the bonding homomorphism between  $G_f$  and  $G_e$ . Moreover, the set  $\mathcal{S}^*$  of the regular elements of a commutative semigroup  $\mathcal{S}$  is a Clifford subsemigroup of  $\mathcal{S}$ . In fact, if  $a^2x = a$  and  $e = ax$ , then also  $a^2xe = a$  and  $xe$  is a regular element of  $\mathcal{S}$ , since  $(xe)^2a = xe$ .

Throughout this section  $R$  will denote a domain and  $Q$  its field of quotients. For  $R$ -submodules  $A$  and  $B$  of  $Q$ ,  $(A : B)$  is defined as follows:

$$(A : B) = \{q \in Q \mid qB \subseteq A\}.$$

A fractional ideal  $F$  of  $R$  is an  $R$ -submodule of  $Q$  such that  $(R : F) \neq 0$ . By an overring of  $R$  is meant any ring between  $R$  and  $Q$ . We say that a domain  $R$  is of finite character if every nonzero ideal of  $R$  is contained only in a finite number of maximal ideals. If (P) is any property, we say that a fractional ideal  $F$  of  $R$  satisfies (P) locally if each localization  $FR_m$  of  $F$  at a maximal ideal  $m$  of  $R$  satisfies (P).

Let  $\mathcal{F}(R)$  be the semigroup of the nonzero fractional ideals of  $R$  and let  $\mathcal{P}(R)$  be the subsemigroup of the nonzero principal fractional ideals of the domain  $R$ . The factor semigroup  $\mathcal{F}(R)/\mathcal{P}(R)$  is denoted by  $\mathcal{S}(R)$  and called the *class semigroup of  $R$* . For every nonzero ideal  $I$  of  $R$ ,  $[I]$  will denote the isomorphism class of  $I$ .

**Definition 2.2.** A domain  $R$  is said to be Clifford regular if the class semigroup  $\mathcal{S}(R)$  of  $R$  is a Clifford semigroup.

## 2.2 Basic properties of regular elements of $\mathcal{S}(R)$ and of Clifford regular domains

If  $R$  is a domain and  $I$  is a nonzero ideal of  $R$ ,  $[I]$  is a regular element of  $\mathcal{S}(R)$  if and only if  $I = I^2X$  for some fractional ideal  $X$  of  $R$ . Let  $E(I) = (I : I)$  be the endomorphism ring of the ideal  $I$  of  $R$ . The homomorphisms from  $I$  to  $E(I)$  are multiplication by elements of  $(E(I) : I) = (I : I^2)$ . The trace ideal of  $I$  in  $E(I)$  is the sum of the images of the homomorphisms of  $I$  into  $E(I)$ , namely  $I(I : I^2)$ . Thus, we have the following basic properties of regular elements of  $\mathcal{S}(R)$ .

**Proposition 2.3** ([8, Lemma 1.1, Proposition 1.2]). *Let  $I$  be a nonzero ideal of a domain  $R$  with endomorphism ring  $E = (I : I)$  and let  $T = I(E : I)$  be the trace ideal of  $I$  in  $E$ . Assume that  $[I]$  is a regular element of  $\mathcal{S}(R)$ , that is,  $I = I^2X$  for some fractional ideal  $X$  of  $R$ . The following hold:*

- (1)  $I = I^2(I : I^2)$ .
- (2)  $IX = T$  and  $[T]$  is an idempotent of  $\mathcal{S}(R)$  associated to  $[I]$ .
- (3)  $T$  is an idempotent ideal of  $E$  and  $IT = I$ .
- (4)  $E = (T : T) = (E : T)$

*Proof.* (1) By assumption  $X \subseteq (I : I^2)$  and so  $I = I^2X \subseteq I^2(I : I^2) \subseteq I$  implies  $I = I^2(I : I^2)$ .

(2) and (3). Since  $(I : I^2) = (E : I)$ , part (1) implies  $IX = I^2(E : I)X = I(E : I)$ , hence  $T = IX$  is an idempotent ideal of  $E$  and  $IT = I$ .

(4) We have  $E \subseteq (E : T) = (I : IT) = E$  and  $E \subseteq (T : T) \subseteq (E : T)$ .

Recall that a nonzero ideal of a domain is said to be *stable* if it is projective, or equivalently invertible, as an ideal of its endomorphism ring and  $R$  is said to be *(finitely) stable* if every nonzero (finitely generated) ideal of  $R$  is stable.

An ideal  $I$  of a domain  $R$  is said to be  *$L$ -stable* (here  $L$  stands for Lipman) if  $R^I := \bigcup_{n \geq 1} (I^n : I^n) = (I : I)$ , and  $R$  is called  *$L$ -stable* if every nonzero ideal is  $L$ -stable. Lipman introduced the notion of stability in the specific setting of one-dimensional commutative semi-local Noetherian rings in order to give a characterization of Arf rings; in this context,  $L$ -stability coincides with Boole regularity [46].

The next proposition illustrates the relation between the notions of (finite) stability,  $L$ -stability and Clifford regularity. A preliminary key observation is furnished by the following lemma.

**Lemma 2.4** ([8, Lemma 2.1]). *Let  $I$  be a nonzero finitely generated ideal of a domain  $R$ . Then  $[I]$  is a regular element of  $\mathcal{S}(R)$  if and only if  $I$  is a stable ideal.*

**Proposition 2.5** ([8, Propositions 2.2 and 2.3, Lemma 2.6]).

- (1) *A stable domain is Clifford regular.*
- (2) *A Clifford regular domain is finitely stable.*
- (3) *A Clifford regular domain is  $L$ -stable.*

In order to better understand the situation, it is convenient to recall some properties of finitely stable and stable domains.

**Theorem 2.6** ([52, Proposition 2.1] and [51, Theorem 3.3]).

- (1) *The integral closure of a finitely stable domain is a Prüfer domain.*
- (2) *A domain is stable if and only if it has finite character and every localization at a maximal ideal is a stable domain.*

It is also useful to state properties of Clifford regular domains relative to localization and overrings. To this end we can state:

**Lemma 2.7** ([8, Lemmas 2.14 and 2.5]).

- (1) *A fractional overring of a Clifford regular domain is Clifford regular.*
- (2) *If  $R$  is a Clifford regular domain and  $S$  is a multiplicatively closed subset of  $R$ , then  $R_S$  is a Clifford regular domain.*

Recall that an overring  $T$  of a domain  $R$  is fractional if  $T$  is a fractional ideal of  $R$ . The next result is useful in reducing the problem of the characterization of a Clifford regular domain to the local case: it states that a domain is Clifford regular if and only if it is locally Clifford regular and the trace of any ideal in its endomorphism ring localizes. In this vein, recall that [58] contributes to the classification of Clifford regular local domains.

**Proposition 2.8** ([8, Proposition 2.8]). *Let  $R$  be a domain. The following are equivalent:*

- (1)  *$R$  is a Clifford regular domain;*
- (2) *For every maximal ideal  $m$  of  $R$ ,  $R_m$  is a Clifford regular domain and for every ideal  $I$  of  $R$ ,  $(I : I^2)_m = I_m(I_m : I_m^2)$ , i.e., the trace of the localization  $I_m$  in its endomorphism ring coincides with the localization at  $m$  of the trace of  $I$  in its endomorphism ring.*

In case the Clifford regular domain  $R$  is stable or integrally closed, a better result can be proved.

**Lemma 2.9.** *Let  $R$  be a stable or an integrally closed Clifford regular domain. If  $I$  is any ideal of  $R$  and  $m$  is any maximal ideal of  $R$ , then the following hold:*

- (1)  $(I : I)_m = (I_m : I_m)$ .
- (2)  $(I : I^2)_m = (I_m : I_m^2)$ .

The connection between Clifford regularity and stability stated by Proposition 2.5 is better illustrated by the concepts of local stability and local invertibility in the way that we are going to indicate.

**Definition 2.10.** A domain  $R$  is said to have the *local invertibility property* (resp., *local stability property*) if every locally invertible (resp., locally stable) ideal is invertible (resp., stable).

The next result is a consequence of Proposition 2.8 and the fact that a locally invertible ideal of a domain is cancellative.

**Proposition 2.11** ([8, Lemmas 4.2 and 5.7]). *A Clifford regular domain has the local invertibility property and the local stability property.*

The preceding result together with the observation that stable domains are of finite character, prompts one to ask if a Clifford regular domain is necessarily of finite type. The question has a positive answer if the Clifford regular domain is Noetherian or integrally closed as we are going to show in the next two sections.

### 2.3 The Noetherian case

From Proposition 2.5, the characterization of the Clifford regular Noetherian domains is immediate.

**Theorem 2.12** ([8, Theorem 3.1]). *A Noetherian domain is Clifford regular if and only if it is stable.*

The Noetherian stable rings have been extensively studied by Sally and Vasconcelos in the two papers [53] and [54]. We list some of their results.

- (a) A stable Noetherian ring has Krull dimension at most 1.
- (b) If every ideal of a domain  $R$  is two-generated (i.e., generated by at most two elements), then  $R$  is stable.
- (c) If  $R$  is a Noetherian domain and the integral closure  $\bar{R}$  of  $R$  is a finitely generated  $R$ -module, then  $R$  is stable if and only if every ideal of  $R$  is two-generated.
- (d) Ferrand and Raynaud [24, Proposition 3.1] constructed an example of a local Noetherian stable domain admitting non two-generated ideals. This domain is not Gorenstein.
- (e) A local Noetherian Gorenstein domain is Clifford regular if and only if every ideal is two-generated. ([8, Theorem 3.2])

It is not difficult to describe the idempotent elements of the class semigroup of a Noetherian domain and the groups associated to them.

**Proposition 2.13** ([8, Proposition 3.4 and Corollary 3.5]). *Let  $R$  be a Noetherian domain. The following hold:*

- (1) *The idempotent elements of  $\mathcal{S}(R)$  are the isomorphism classes of the fractional overrings of  $R$  and the groups associated to them are the ideal class groups of the fractional overrings of  $R$ .*
- (2) *If  $R$  is also a Clifford regular domain, then the class semigroup  $\mathcal{S}(R)$  of  $R$  is the disjoint union of the ideal class groups of the fractional overrings of  $R$  and the bonding homomorphisms between the groups are induced by extending ideals to overrings.*

## 2.4 The integrally closed case

The starting point for the study of integrally closed Clifford regular domains is the following fact.

**Proposition 2.14** ([59, Proposition 3]). *An integrally closed Clifford regular domain is a Prüfer domain.*

In [9], it was proved that any valuation domain is Clifford regular and in [5] the result was extended by proving that a Prüfer domain of finite character is a Clifford regular domain. Finally, in [8] it was proved that an integrally closed Clifford regular domain is of finite character.

While trying to prove the finite character property for a Clifford regular Prüfer domain, a more general problem arose and in the papers [7] and [8] the following conjecture was posed. Its interest goes beyond the Clifford regularity of Prüfer domains.

*Conjecture.* If  $R$  is a Prüfer domain with the local invertibility property, then  $R$  is of finite character.

In [8], the conjecture was established in the affirmative for the class of Prüfer domains satisfying a particular condition. To state the condition we need to recall a notion on prime ideals: a prime ideal  $P$  of a Prüfer domain is *branched* if there exists a prime ideal  $Q$  properly contained in  $P$  and such that there are no other prime ideals properly between  $Q$  and  $P$ .

**Theorem 2.15** ([8, Theorem 4.4]). *Let  $R$  be a Prüfer domain with the local invertibility property. If the endomorphism ring of every branched prime ideal of  $R$  satisfies the local invertibility property, then  $R$  is of finite character.*

Theorem 2.15 together with Proposition 2.11 and the fact that every fractional overring of a Clifford regular domain is again Clifford regular, imply the characterization of integrally closed Clifford regular domains.

**Theorem 2.16** ([8, Theorem 4.5]). *An integrally closed domain is Clifford regular if and only if it is a Prüfer domain of finite character.*

We wish to talk a little about the conjecture mentioned above. It attracted the interest of many authors and its validity has been proved recently. In [36], Holland, Martinez, McGovern, and Tesemma proved that the conjecture is true by translating the problem into a statement on lattice ordered groups. In fact, as shown by Brewer and Klingler in [14], the group  $G$  of invertible fractional ideals of a Prüfer domain endowed with the reverse inclusion, is a latticed ordered group and the four authors noticed that both the property of finite character and the local invertibility property of a Prüfer domain can be translated into statements on prime subgroups of the group  $G$  and filters on the positive cone of  $G$ .

Then, they used a crucial result by Conrad [17] on lattice ordered groups with finite basis to prove that the two statements translating the finite character and the local invertibility property are equivalent, so that the validity of the conjecture follows.

Subsequently, McGovern [47] has provided a ring theoretic proof of the conjecture by translating from the language of lattice ordered groups to the language of ring theory the techniques used in [36]. At one point it was necessary to introduce a suitable localization of the domain in order to translate the notion of the kernel of a lattice homomorphism on the lattice ordered group.

Independently, almost at the same time, Halter-Koch [35] proved the validity of the conjecture by using the language of ideal systems on cancellative commutative monoids and he proved that an  $r$ -Prüfer monoid with the local invertibility property is a monoid of Krull type (see [33, Theorem 22.4]).

## 2.5 The structure of the class semigroup of an integrally closed Clifford regular domain

In order to understand the structure of the class semigroup  $\mathcal{S}(R)$  of a Clifford regular domain it is necessary to describe the idempotent elements, the constituent groups associated to them and the bonding homomorphisms between those groups. Complete information is available for the case of integrally closed Clifford regular domains, that is, the class of Prüfer domains of finite character.

In [9], Salce and the first named author proved that the class semigroup of a valuation domain  $R$  is a Clifford semigroup with idempotent elements of two types: they are represented either by fractional overrings of  $R$ , that is, localizations  $R_P$  at prime ideals  $P$ , or by nonzero idempotent prime ideals. The groups corresponding to localizations are trivial and the group associated to a nonzero idempotent prime ideal  $P$  is described as a quotient of the form  $\overline{\Gamma}/\Gamma$ , where  $\Gamma$  is the value group of the localization  $R_P$  and  $\overline{\Gamma}$  is the completion of  $\Gamma$  in the order topology. This group is also called the *archimedean group* of the localizations  $R_P$  and denoted by  $\text{Arch}R_P$ .

If  $I$  is a nonzero ideal of  $R$ ,  $[I]$  belongs to  $\text{Arch}R_P$  if and only if  $R_P$  is the endomorphism ring of  $I$  and  $I$  is not principal as an  $R_P$ -ideal. Note that the endomorphism ring of an ideal  $I$  of a valuation domain  $R$  is the localization of  $R$  at the prime ideal  $P$  associated to  $I$  defined by  $P = \{r \in R \mid rI \subsetneq I\}$  (cf. [29, II p. 69]).

The idempotent elements, the constituent groups and the bonding homomorphisms of the class semigroup of a Prüfer domain of finite character have been characterized by the first named author in [6] and [7].

If  $\mathcal{S}(R)$  is a Clifford semigroup and  $I$  is a nonzero ideal of  $R$ , then by Proposition 2.3, the unique idempotent of  $\mathcal{S}(R)$  associated to  $[I]$  is the trace ideal  $T$  of  $I$  in its endomorphism ring, that is,  $T = I(I : I^2)$ . Moreover, every idempotent of  $\mathcal{S}(R)$  is of this form. The next two propositions describe the subsemigroup  $\mathcal{E}(R)$  of the idempotent elements of  $\mathcal{S}(R)$

**Proposition 2.17** ([6, Theorem 3.1 and Proposition 3.2]). *Assume that  $R$  is a Prüfer domain of finite character. Let  $I$  be a nonzero ideal of  $R$  such that  $[I]$  is*

an idempotent element of  $\mathcal{S}(R)$ . Then there exists a unique nonzero idempotent fractional ideal  $L$  isomorphic to  $I$  such that

$$L = P_1 \cdot P_2 \cdots P_n D \quad n \geq 0$$

with uniquely determined factors satisfying the following conditions:

- (1)  $D = (L : L)$  is a fractional overring of  $R$ ;
- (2) The  $P_i$  are pairwise incomparable idempotent prime ideals of  $R$ ;
- (3) Each  $P_i D$  is a maximal ideal of  $D$ ;
- (4)  $D \supseteq \text{End}(P_i)$ .

The preceding result shows that the semigroup  $\mathcal{E}(R)$  of the idempotent elements of  $\mathcal{S}(R)$  is generated by the classes  $[P]$  and  $[D]$  where  $P$  vary among the nonzero idempotent prime ideals of  $R$  and  $D$  are arbitrary overrings of  $R$ . Moreover, every element of  $\mathcal{E}(R)$  has a unique representation as a finite product of these classes provided they satisfy the conditions of Proposition 2.17.

For each nonzero idempotent fractional ideal  $L$ , denote by  $G_L$  the constituent group of  $\mathcal{S}(R)$  associated to the idempotent element  $[L]$  of  $\mathcal{E}(R)$ , as defined in Section 2.1. The properties and the structure of the groups  $G_L$  have been investigated in [7].

We recall some useful information on ideals of a Prüfer domain of finite character.

**Lemma 2.18 ([7, Lemma 3.1]).** *Let  $I$  and  $J$  be locally isomorphic ideals of a Prüfer domain of finite character  $R$ . Then there exists a finitely generated fractional ideal  $B$  of  $D = \text{End}(I)$  such that  $I = BJ$ . In particular, if  $R$  is also a Bézout domain, then  $I \cong J$ .*

A key observation in order to describe the constituent groups of the class semigroup of a Prüfer domain of finite character  $R$  is to note that, for each nonzero idempotent prime ideal  $P$  of  $R$ , there is a relation between  $G_P$  and the archimedean group  $\text{Arch}R_P$  of the valuation domain  $R_P$  (cf. [7, Proposition 3.3]). In fact, the correspondence

$$[I] \mapsto [IR_P], \quad [I] \in G_P$$

induces an epimorphism of Abelian groups

$$\psi: G_P \rightarrow \text{Arch}R_P$$

such that  $\text{Ker } \psi = \{[CP] \mid C \text{ is a finitely generated ideal of } \text{End}(P)\}$ . In particular,  $\text{Ker } \psi \cong \mathcal{C}(\text{End}(P))$  and  $\psi$  is injective if and only if  $\text{End}(P)$  is a Bézout domain.

The preceding remark can be extended to each group  $G_L$  in the class semigroup  $\mathcal{S}(R)$ .

**Theorem 2.19 ([7, Theorem 3.5]).** *Assume that  $R$  is a Prüfer domain of finite character. Let  $L = P_1 \cdot P_2 \cdots P_n D$  be a nonzero idempotent fractional ideal of  $R$  satisfying the conditions of Proposition 2.17. For every nonzero ideal  $I$  of  $R$  such that  $[I] \in G_L$ ,*

consider the diagonal map  $\pi([I]) = ([IR_{P_1}], \dots, [IR_{P_n}])$ . Then the group  $G_L$  fits in the short exact sequence:

$$1 \rightarrow \mathcal{C}(D) \rightarrow G_L \xrightarrow{\pi} \text{Arch}R_{P_1} \times \cdots \times \text{Arch}R_{P_n} \rightarrow 1.$$

If  $R$  is a Bézout domain, then so is every overring  $D$  of  $R$ , hence the ideal class groups  $\mathcal{C}(D)$  are all trivial. The constituent groups are then built up by means of the groups associated to the idempotent prime ideals of  $R$  and the structure of the class semigroup  $\mathcal{S}(R)$  is simpler, precisely we can state the following:

**Proposition 2.20 ([7, Proposition 4.4]).** *If  $R$  is a Bézout domain of finite character, then the constituent groups associated to every idempotent element of  $\mathcal{S}(R)$  are isomorphic to a finite direct product of archimedean groups  $\text{Arch}R_P$  of the valuation domain  $R_P$ , where  $P$  is a nonzero idempotent prime ideal of  $R$ .*

It remains to describe the partial order on the semigroup  $\mathcal{E}(R)$  of the idempotent elements and the bonding homomorphisms between the constituent groups of  $\mathcal{S}(R)$ .

Recalling that if  $P$  and  $Q$  are two idempotent prime ideals of a domain  $R$ ,  $P \subseteq Q$  if and only if  $PQ = P$  and if  $D$  and  $S$  are overrings of  $R$ , then  $S \subseteq D$  if and only if  $SD = D$ , then the partial order on  $\mathcal{E}(R)$  is induced by the inclusion between prime ideals and the reverse inclusion between fractional overrings. Moreover, we have:

**Proposition 2.21 ([7, Proposition 4.1]).** *Assume that  $R$  is a Prüfer domain of finite character. Let  $L = P_1 \cdot P_2 \cdots P_n D$ ,  $H = Q_1 \cdot Q_2 \cdots Q_k S$  be nonzero idempotent fractional ideals of  $R$  satisfying the conditions of Proposition 2.17. Then  $[L] \leq [H]$  if and only if*

- (1)  $S \subseteq D$ ,
- (2) For every  $1 \leq j \leq k$  either  $Q_j D = D$  or there exists  $1 \leq i \leq n$  such that  $Q_j = P_i$ .

To describe the bonding homomorphisms between the constituent groups of the class semigroup  $\mathcal{S}(R)$  it is convenient to consider the properties of two special types of such homomorphisms, that is, those induced by multiplication by a fractional overring of  $R$  or by an idempotent prime ideal of  $R$ .

**Lemma 2.22 ([7, Lemma 4.2]).** *Let  $P, Q$  be nonzero idempotent prime ideals of the Prüfer domain of finite character  $R$  and let  $D$  and  $S$  be overrings of  $R$  such that  $S \subseteq D$ . Then:*

- (1) The maps

$$\phi_D^S: G_S \rightarrow G_D \quad \text{and} \quad \phi_{PD}^P: G_P \rightarrow G_{PD}$$

are surjective homomorphisms induced by multiplication by  $D$ .

- (2) Assume that  $D \supseteq \text{End}(QP)$  and that  $P, Q$  are non-comparable, then:

$$\phi_{PD}^D: G_D \rightarrow G_{PD} \quad \text{and} \quad \phi_{QPD}^{QD}: G_{QD} \rightarrow G_{QPD}$$

are injective homomorphisms induced by multiplication by  $P$ .



The bonding homomorphisms are then described by the following proposition.

**Proposition 2.23** ([7, Proposition 4.3]). *Assume that  $R$  is a Prüfer domain of finite character. Let  $L = P_1 \cdot P_2 \cdots P_n D$ ,  $H = Q_1 \cdot Q_2 \cdots Q_k S$  be nonzero idempotent fractional ideals of  $R$  satisfying the conditions of Proposition 2.17 and such that  $[L] \leq [H]$ . Let  $K = Q_1 \cdot Q_2 \cdots Q_k D$ , then the bonding homomorphism  $\phi_L^H: G_H \rightarrow G_L$  is the composition of the bonding epimorphism  $\phi_K^H$  and the bonding monomorphism  $\phi_L^K$ , namely  $\phi_L^H = \phi_L^K \circ \phi_K^H$ .*

The results on the structure of the Clifford semigroup of a Prüfer domain of finite character have been generalized by Fuchs [28] by considering an arbitrary Prüfer domain  $R$  and restricting considerations to the subsemigroup  $\mathcal{S}'(R)$  of  $\mathcal{S}(R)$  consisting of the isomorphy classes of ideals containing at least one element of finite character.

## 2.6 Boole regular domains

Recall that a semigroup  $S$  (with 1) is said to be Boolean if for each  $x \in S$ ,  $x = x^2$ . This subsection seeks ring-theoretic conditions of a domain  $R$  that reflects in the Boolean property of its class semigroup  $\mathcal{S}(R)$ . Precisely, it characterizes integrally closed domains with Boolean class semigroup; in this case,  $\mathcal{S}(R)$  happens to identify with the Boolean semigroup formed of all fractional overrings of  $R$ . It also treats Noetherian-like settings where the Clifford and Boolean properties of  $\mathcal{S}(R)$  coincide with stability conditions; a main feature is that the Clifford property forces  $t$ -locally Noetherian domains to be one-dimensional Noetherian domains. It closes with a study of the transfer of the Clifford and Boolean properties to various pullback constructions. These results lead to new families of domains with Clifford or Boolean class semigroup, moving therefore beyond the contexts of integrally closed domains or Noetherian domains.

By analogy with Clifford regularity, we define Boole regularity as follows:

**Definition 2.24** ([38]). A domain  $R$  is Boole regular if  $\mathcal{S}(R)$  is a Boolean semigroup.

Clearly, a PID is Boole regular and a Boole regular domain is Clifford regular. The integral closure of a Clifford regular domain is Prüfer [8, 59]. The next result is an analogue for Boole regularity.

**Proposition 2.25** ([38, Proposition 2.3]). *The integral closure of a Boole regular domain is Bézout.*

A first application characterizes almost Krull domains subject to Clifford or Boole regularity as shown below:

**Corollary 2.26** ([38, Corollary 2.4]). *A domain  $R$  is almost Krull and Boole (resp., Clifford) regular if and only if  $R$  is a PID (resp., Dedekind).*

A second application handles the transfer to polynomial rings:

**Corollary 2.27** ([38, Corollary 2.5]). *Let  $R$  be a domain and  $X$  an indeterminate over  $R$ . Then:*

$$R \text{ is a field} \iff R[X] \text{ is Boole regular} \iff R[X] \text{ is Clifford regular}$$

One of the aims is to establish sufficient conditions for Boole regularity in integrally closed domains. One needs first to examine the valuation case. For this purpose, recall first a stability condition that best suits Boole regularity:

**Definition 2.28.** A domain  $R$  is strongly stable if each nonzero ideal  $I$  of  $R$  is principal in its endomorphism ring  $(I : I)$ .

Note that for a domain  $R$ , the set  $\mathcal{F}_{OV}(R)$  of fractional overrings of  $R$  is a Boolean semigroup with identity equal to  $R$ . Recall that a domain  $R$  is said to be strongly discrete if  $P^2 \not\subseteq P$  for every nonzero prime ideal  $P$  of  $R$  [26].

**Theorem 2.29** ([38, 39, Theorem 3.2]). *Let  $R$  be an integrally closed domain. Then  $R$  is a strongly discrete Bézout domain of finite character if and only if  $R$  is strongly stable. Moreover, when any one condition holds,  $R$  is Boole regular with  $\mathcal{S}(R) \cong \mathcal{F}_{OV}(R)$ .*

The proof lies partially on the following lemmas.

**Lemma 2.30.** *Let  $R$  be a domain. Then:*

$$R \text{ is stable Boole regular} \iff R \text{ is strongly stable.}$$

**Lemma 2.31.** *Let  $R$  be an integrally closed domain. Then:*

$$R \text{ is strongly discrete Clifford regular} \iff R \text{ is stable.}$$

**Lemma 2.32.** *Let  $V$  be a valuation domain. The following are equivalent:*

- (1)  $V_P$  is a divisorial domain, for each nonzero prime ideal  $P$  of  $R$ ;
- (2)  $V$  is a stable domain;
- (3)  $V$  is a strongly discrete valuation domain.

*Moreover, when any one condition holds,  $V$  is Boole regular.*

This lemma gives rise to a large class of Boole regular domains that are not PIDs. For example, any strongly discrete valuation domain of dimension  $\geq 2$  (cf. [27]) is a Boole regular domain which is not Noetherian. The rest of this subsection studies the class semigroups for two large classes of Noetherian-like domains, that is,  $t$ -locally Noetherian domains and Mori domains. Precisely, it examines conditions under which stability and strong stability characterize Clifford regularity and Boole regularity, respectively.

Next, we review some terminology related to the  $w$ -operation. For a nonzero fractional ideal  $I$  of  $R$ ,  $I_w := \bigcup(I : J)$  where the union is taken over all finitely generated ideals  $J$  of  $R$  with  $J^{-1} = R$ . We say that  $I$  is a  $w$ -ideal if  $I_w = I$ . The domain  $R$  is

said to be *Mori* if it satisfies the ascending chain condition on divisorial ideals [3] and *strong Mori* if it satisfies the ascending chain condition on  $w$ -ideals [23, 48]. Trivially, a Noetherian domain is strong Mori and a strong Mori domain is Mori. Finally, we say that  $R$  is  *$t$ -locally Noetherian* if  $R_M$  is Noetherian for each  $t$ -maximal ideal  $M$  of  $R$  [43]. Recall that strong Mori domains are  *$t$ -locally Noetherian* [23, Theorem 1.9].

The next result handles the  *$t$ -locally Noetherian* setting.

**Theorem 2.33 ([38, Theorem 4.2]).** *Let  $R$  be a  $t$ -locally Noetherian domain. Then  $R$  is Clifford (resp., Boole) regular if and only if  $R$  is stable (resp., strongly stable). Moreover, when any one condition holds,  $R$  is either a field or a one-dimensional Noetherian domain.*

The proof relies partially on the next lemma.

**Lemma 2.34.** *Let  $R$  be a Clifford regular domain. Then  $I_t \subsetneq R$  for each nonzero proper ideal  $I$  of  $R$ . In particular, every maximal ideal of  $R$  is a  $t$ -ideal.*

The above theorem asserts that a strong Mori Clifford regular domain is necessarily Noetherian. Here, Clifford regularity forces the  $w$ -operation to be trivial (see also [48, Proposition 1.3]). Also noteworthy is that while a  *$t$ -locally Noetherian stable domain is necessarily a one-dimensional  $L$ -stable domain*, the converse does not hold in general. For instance, consider an almost Dedekind domain which is not Dedekind and appeal to Corollary 2.26. However, the equivalence holds for Noetherian domains [8, Theorem 2.1] and [1, Proposition 2.4].

**Corollary 2.35 ([38, Corollary 4.4]).** *Let  $R$  be a local Noetherian domain such that the extension  $R \subseteq \bar{R}$  is maximal, where  $\bar{R}$  denotes the integral closure of  $R$ . The following are equivalent:*

- (1)  $R$  is Boole regular;
- (2)  $R$  is strongly stable;
- (3)  $R$  is stable and  $\bar{R}$  is a PID.

This result generates new families of Boole regular domains beyond the class of integrally closed domains.

*Example 2.36.* Let  $R := k[X^2, X^3]_{(X^2, X^3)}$  where  $k$  is a field and  $X$  an indeterminate over  $k$ . Clearly,  $\bar{R} = k[X]_{R \setminus (X^2, X^3)}$  is a PID and the extension  $R \subseteq \bar{R}$  is maximal. Further,  $R$  is a Noetherian Warfield domain, hence stable (cf. [10]). Consequently,  $R$  is a one-dimensional non-integrally closed local Noetherian domain that is Boole regular.

The next results handle the Mori setting. In what follows, we shall use  $\bar{R}$  and  $R^*$  to denote the integral closure and complete integral closure, respectively, of a domain  $R$ .

**Theorem 2.37 ([38, Theorem 4.7]).** *Let  $R$  be a Mori domain. Then the following are equivalent:*

- (1)  $R$  is one-dimensional Clifford (resp., Boole) regular and  $R^*$  is Mori;
- (2)  $R$  is stable (resp., strongly stable).

It is worth recalling that for a Noetherian domain  $R$  we have  $\dim(R) = 1 \Leftrightarrow \dim(R^*) = 1 \Leftrightarrow R^*$  is Dedekind since here  $R^* = \overline{R}$ . The same result holds if  $R$  is a Mori domain such that  $(R : R^*) \neq 0$  [4, Corollary 3.4(1) and Corollary 3.5(1)]. Also, it was stated that the “only if” assertion holds for seminormal Mori domains [4, Corollary 3.4(2)]. However, beyond these contexts, the problem remains open. This explains the cohabitation of “ $\dim(R) = 1$ ” and “ $R^*$  is Mori” assumptions in the above theorem. In this vein, we set the following open question:

“Let  $R$  be a local Mori Clifford regular domain is it true that:  
 $\dim(R) = 1 \iff R^*$  is Dedekind?”

The next result partly draws on the above theorem and treats two well-studied large classes of Mori domains [3]. Recall that a domain  $R$  is seminormal if  $x \in R$  whenever  $x \in K$  and  $x^2, x^3 \in R$ .

**Theorem 2.38 ([38, Theorem 4.9]).** *Let  $R$  be a Mori domain. Consider the following statements:*

- (1) The conductor  $(R : R^*) \neq 0$ ,
- (2)  $R$  is seminormal,
- (3) The extension  $R \subseteq R^*$  has at most one proper intermediate ring.

Assume that either (1), (2), or (3) holds. Then  $R$  is Clifford (resp., Boole) regular if and only if  $R$  is stable (resp., strongly stable).

## 2.7 Pullbacks

The purpose here is to examine Clifford regularity and Boole regularity in pullback constructions. This allows for the construction of new families of domains with Clifford or Boolean class semigroup, beyond the contexts of integrally closed or Noetherian domains.

Let us fix the notation for the rest of this subsection. Let  $T$  be a domain,  $M$  a maximal ideal of  $T$ ,  $K$  its residue field,  $\phi : T \rightarrow K$  the canonical surjection,  $D$  a proper subring of  $K$  with quotient field  $k$ . Let  $R := \phi^{-1}(D)$  be the pullback issued from the following diagram of canonical homomorphisms:

$$\begin{array}{ccc} R & \longrightarrow & D \\ \downarrow & & \downarrow \\ T & \xrightarrow{\phi} & K = T/M \end{array}$$

Next, we announce the first theorem which provides a necessary and sufficient condition for a pseudo-valuation domain (i.e., PVD) to inherit Clifford or Boole regularity.

**Theorem 2.39 ([38, Theorem 5.1]).**

- (1) If  $R$  is Clifford (resp., Boole) regular, then so are  $T$  and  $D$ , and  $[K : k] \leq 2$ .
- (2) Assume  $D = k$  and  $T$  is a valuation (resp., strongly discrete valuation) domain. Then  $R$  is Clifford (resp., Boole) regular if and only if  $[K : k] = 2$ .

The following example shows that this theorem does not hold in general, and hence nor does the converse of (1).

*Example 2.40.* Let  $\mathbb{Z}$  and  $\mathbb{Q}$  denote the ring of integers and field of rational numbers, respectively, and let  $X$  and  $Y$  be indeterminates over  $\mathbb{Q}$ . Set  $V := \mathbb{Q}(\sqrt{2}, \sqrt{3})[[X]]$ ,  $M := X\mathbb{Q}(\sqrt{2}, \sqrt{3})[[X]]$ ,  $T := \mathbb{Q}(\sqrt{2}) + M$ , and  $R := \mathbb{Q} + M$ . Both  $T$  and  $R$  are one-dimensional local Noetherian domains arising from the DVR  $V$ , with  $\bar{T} = V$  and  $\bar{R} = T$ . By the above theorem,  $T$  is Clifford (actually, Boole) regular, whereas  $R$  is not. More specifically, the isomorphism class of the ideal  $I := X(\mathbb{Q} + \sqrt{2}\mathbb{Q} + \sqrt{3}\mathbb{Q} + M)$  is not regular in  $\mathcal{S}(R)$ .

Now, one can build original example using the above theorem as follows:

*Example 2.41.* Let  $n$  be an integer  $\geq 1$ . Let  $R$  be a PVD associated with a non-Noetherian  $n$ -dimensional valuation (resp., strongly discrete valuation) domain  $(V, M)$  with  $[V/M : R/M] = 2$ . Then  $R$  is an  $n$ -dimensional local Clifford (resp., Boole) regular domain that is neither integrally closed nor Noetherian.

Recall that a domain  $A$  is said to be *conductive* if the conductor  $(A : B)$  is nonzero for each overring  $B$  of  $A$  other than its quotient field. Examples of conductive domains include arbitrary pullbacks of the form  $R := D + M$  arising from a valuation domain  $V := K + M$  [19, Propositions 2.1 and 2.2]. We are now able to announce the last theorem of this subsection. It treats Clifford regularity, for the remaining case “ $k = K$ ”, for pullbacks  $R := \phi^{-1}(D)$  where  $D$  is a conductive domain.

**Theorem 2.42 ([38, Theorem 5.6]).** *Under the same notation as above, consider the following statements:*

- (1)  $T$  is a valuation domain and  $R := \phi^{-1}(D)$ ,
- (2)  $T := K[X]$  and  $R := D + XK[X]$ , where  $X$  is an indeterminate over  $K$ .

*Assume that  $D$  is a semilocal conductive domain with quotient field  $k = K$  and either (1) or (2) holds. Then  $R$  is Clifford regular if and only if so is  $D$ .*

Now a combination of Theorems 2.39 and 2.42 generates new families of examples of Clifford regular domains, as shown by the following construction [38, Example 5.8]:

*Example 2.43.* For every positive integer  $n \geq 2$ , there exists an example of a domain  $R$  satisfying the following conditions:

- (1)  $\dim(R) = n$ ,
- (2)  $R$  is neither integrally closed nor Noetherian,
- (3)  $R$  is Clifford regular,
- (4) Each overring of  $R$  is Clifford regular,
- (5)  $R$  has infinitely many maximal ideals.

## 2.8 Open problems

By Proposition 2.5 the class of Clifford regular domains contains the class of stable domains and is contained in the class of finitely stable domains. Both inclusions are proper. In fact, every Prüfer domain is finitely stable, but only the Prüfer domains of finite character are Clifford regular. Moreover, a Prüfer domain is stable if and only if it is of finite character and strongly discrete, that is, every nonzero prime ideal is not idempotent (cf. [49, Theorem 4.6]), hence there exists a large class of nonstable integrally closed Clifford regular domains. The classification of stable domains obtained by B. Olberding in [50], shows that there are stable domains which are neither Noetherian nor integrally closed. Furthermore, there is an example of a non-coherent stable domain ([50, Section 5]), hence there exist non-coherent Clifford regular domains.

There are also examples of Clifford regular domains which are neither stable nor integrally closed, as illustrated by [8, Example 6.1].

*Example 2.44.* Let  $k_0$  be a field and let  $K$  be an extension field of  $k_0$  such that  $[K : k_0] = 2$ . Consider a valuation domain  $V$  of the form  $K + M$  where  $M$  is the maximal ideal of  $V$  and assume  $M^2 = M$ . Let  $R$  be the domain  $k_0 + M$ . The ideals of  $R$  can be easily described: they are either ideals of  $V$  or principal ideals of  $R$ . Thus,  $R$  is Clifford regular, but it is not stable, since  $M$  is an idempotent ideal of  $R$ ; moreover the integral closure of  $R$  is  $V$ .

There are still many questions related to the problem of characterizing the class of Clifford regular domains in general. Note that if a domain  $R$  is stable, then  $R$  is of finite character and every overring of  $R$  is again stable ([51, Theorems 3.3 and 5.1]). If  $R$  is an integrally closed Clifford regular domain, then  $R$  is a Prüfer domain of finite character (Theorem 2.16) and thus the same holds for every overring of  $R$ . Hence, the two subclasses of Clifford regular domains consisting of the stable domains and of the integrally closed domains are closed for overrings and their members are domains of finite character. We may ask the following major questions concerning Clifford regular domains:

**Question 2.45** *Is every Clifford regular domain of finite character?*

**Question 2.46** (a) *Is every overring of a Clifford regular domain again Clifford regular?*

(b) *In particular, is the integral closure of a Clifford regular domain a Clifford regular domain?*

In [56], Sega gives partial answers to part (a) of this question. In particular, he proves that if  $R$  is a Clifford regular domain such that the integral closure of  $R$  is a fractional overring, then every overring of  $R$  is Clifford regular. An affirmative answer to part (b) would imply that a Clifford regular domain is necessarily of finite character, since the integral closure of a Clifford regular domain is a Prüfer domain.

In view of the validity of the conjecture about the finite character of Prüfer domains with the local invertibility property proved in [36], Question 2.46 (b) may

be weakened by asking if the integral closure of a Clifford regular domain satisfies the local invertibility property. More generally we may ask:

**Question 2.47** *If a finitely stable domain satisfies the local invertibility property, is it true that its integral closure satisfies the same property?*

A positive answer to the above question would imply that a finitely stable domain satisfying the local invertibility property has finite character.

Another interesting problem is to characterize the local Clifford regular domains. The next example shows that not every finitely stable local domain is Clifford regular.

*Example 2.48.* Let  $A$  be a DVR with quotient field  $Q$  and let  $B$  be the ring  $Q[[X^2, X^3]]$ . Denote by  $P$  the maximal ideal of  $B$  and let  $R = A + P$ . By [50, Proposition 3.6],  $R$  is finitely stable but it is not  $L$ -stable. In fact,  $J = Q + AX + P$  is a fractional ideal of  $R$ , since  $JP \subseteq P \subseteq R$  and  $(J : J) = R$ , but  $J^2 = Q[[X]]$ . Thus, by Proposition 2.5,  $R$  is not Clifford regular.

However, the following result holds.

**Proposition 2.49 ([8, Corollary 5.6]).** *Let  $R$  be a local Clifford regular domain with principal maximal ideal. Then  $R$  is a valuation domain.*

In the case of a Clifford regular domain  $R$  of finite character a description of the idempotent elements of  $\mathcal{S}(R)$  is available. It generalizes the situation illustrated in Proposition 2.17 for Clifford regular Prüfer domains.

**Lemma 2.50.** *Let  $R$  be a Clifford regular domain of finite character and let  $T$  be a nonzero idempotent fractional ideal of  $R$ . If  $E = \text{End}(T)$ , then either  $T = E$  or  $T$  is a product of idempotent maximal ideals of  $E$ .*

We end this subsection by recalling a partial result regarding the finite character of Clifford regular domains. We denote by  $\mathcal{T}(R)$  the set of maximal ideals  $m$  of  $R$  for which there exists a finitely generated ideal with the property that  $m$  is the only maximal ideal containing it.

**Proposition 2.51.** *Let  $R$  be a finitely stable domain satisfying the local stability property. Then every nonzero element of  $R$  is contained in at most a finite number of maximal ideals of  $\mathcal{T}(R)$ . In particular the result holds for every Clifford regular domain.*

### 3 $t$ -Class semigroups of integral domains

A domain  $R$  is called a PVMD (for Prüfer  $\nu$ -multiplication domain) if the  $\nu$ -finite  $\nu$ -ideals form a group under the  $t$ -multiplication; equivalently, if  $R_M$  is a valuation domain for each  $t$ -maximal ideal  $M$  of  $R$ . Ideal  $t$ -multiplication converts ring notions

such as PID, Dedekind, Bézout, Prüfer, and integrality to UFD, Krull, GCD, PVMD, and pseudo-integrality, respectively. The pseudo-integrality (i.e.,  $t$ -integrality) was introduced and studied in 1991 by D. F. Anderson, Houston, and Zafrullah [2].

The  $t$ -class semigroup of  $R$  is defined by

$$\mathcal{S}_t(R) := \mathcal{F}_t(R)/\mathcal{P}(R)$$

where  $\mathcal{P}(R)$  is the subsemigroup of  $\mathcal{F}_t(R)$  consisting of nonzero principal fractional ideals of  $R$ . Thus,  $\mathcal{S}_t(R)$  stands as the  $t$ -analogue of  $\mathcal{S}(R)$ , the class semigroup of  $R$ . For the reader's convenience we recall from the introduction the set-theoretic inclusions:

$$\mathcal{C}(R) \subseteq \text{Cl}(R) \subseteq \mathcal{S}_t(R) \subseteq \mathcal{S}(R).$$

By analogy with Clifford regularity and Boole regularity (Section 2), we define  $t$ -regularity as follows:

**Definition 3.1 ([40]).** A domain  $R$  is Clifford (resp., Boole)  $t$ -regular if  $\mathcal{S}_t(R)$  is a Clifford (resp., Boolean) semigroup.

This section reviews recent works that examine ring-theoretic conditions of a domain  $R$  that reflect reciprocally in semigroup-theoretic properties of its  $t$ -class semigroup  $\mathcal{S}_t(R)$ . Contexts that suit best  $t$ -regularity are studied in [40–42] in an attempt to parallel analogous developments and generalize the results on class semigroups (reviewed in Section 2).

Namely, [40] treats the case of PVMDs extending Bazzoni's results on Prüfer domains [5, 8]; [41] describes the idempotents of  $\mathcal{S}_t(R)$  and the structure of their associated groups recovering well-known results on class semigroups of valuation domains [9] and Prüfer domains [6, 7]; and [42] studies the  $t$ -class semigroup of a Noetherian domain. All results are illustrated by original examples distinguishing between the two concepts of class semigroup and  $t$ -class semigroup. Notice that in Prüfer domains, the  $t$ - and trivial operations (and hence the  $t$ -class and class semigroups) coincide.

### 3.1 Basic results on $t$ -regularity

Here, we discuss  $t$ -analogues of basic results on  $t$ -regularity. First we notice that Krull domains and UFDs are Clifford and Boole  $t$ -regular, respectively. These two classes of domains serve as a starting ground for  $t$ -regularity as Dedekind domains and PIDs do for regularity. Also, we will see that  $t$ -regularity stands as a default measure for some classes of Krull-like domains, e.g., “UFD = Krull + Boole  $t$ -regular.” Moreover, while an integrally closed Clifford regular domain is Prüfer (Proposition 2.14), an integrally closed Clifford  $t$ -regular domain need not be a PVMD. An example is built to this end, as an application of the main theorem of this subsection, which examines the transfer of  $t$ -regularity to pseudo-valuation domains.



The first result displays necessary and/or sufficient ideal-theoretic conditions for the isomorphism class of an ideal to be regular in the  $t$ -class semigroup.

**Lemma 3.2** ([40, Lemma 2.1]). *Let  $I$  be a  $t$ -ideal of a domain  $R$ . Then*

- (1)  $[I]$  is regular in  $\mathcal{S}_t(R)$  if and only if  $I = (I^2(I : I^2))_t$ .
- (2) If  $I$  is  $t$ -invertible, then  $[I]$  is regular in  $\mathcal{S}_t(R)$ .

A domain  $R$  is Krull if every  $t$ -ideal of  $R$  is  $t$ -invertible. From the lemma one can obviously see that a Krull domain is Clifford  $t$ -regular. Recall that a domain  $R$  is  $t$ -almost Dedekind if  $R_M$  is a rank-one DVR for each  $t$ -maximal ideal  $M$  of  $R$ ;  $t$ -almost Dedekind domains lie strictly between Krull domains and general PVMDs [43]. A domain  $R$  is said to be strongly  $t$ -discrete if it has no  $t$ -idempotent  $t$ -prime ideals (i.e., for every  $t$ -prime ideal  $P$ ,  $(P^2)_t \not\subseteq P$ ) [22]. The next results (cf. [40, Proposition 2.3]) show that  $t$ -regularity measures how far some Krull-like domains are from being Krull or UFDs.

**Proposition 3.3.** *Let  $R$  be a domain. The following are equivalent:*

- (1)  $R$  is Krull;
- (2)  $R$  is  $t$ -almost Dedekind and Clifford  $t$ -regular;
- (3)  $R$  is strongly  $t$ -discrete, completely integrally closed, and Clifford  $t$ -regular.

**Proposition 3.4.** *Let  $R$  be a domain. The following are equivalent:*

- (1)  $R$  is a UFD;
- (2)  $R$  is Krull and Boole  $t$ -regular;
- (3)  $R$  is  $t$ -almost Dedekind and Boole  $t$ -regular;
- (4)  $R$  is strongly  $t$ -discrete, completely integrally closed, and Boole  $t$ -regular.

Note that the assumptions in the previous results are not superfluous. For, the (Bézout) ring of all entire functions in the complex plane is strongly ( $t$ -)discrete [26, Corollary 8.1.6] and completely integrally closed, but it is not ( $t$ -)almost Dedekind (since it has an infinite Krull dimension). Also, a non-discrete rank-one valuation domain is completely integrally closed and Clifford ( $t$ -)regular [9], but it is not Krull.

The  $t$ -regularity transfers to polynomial rings and factor rings providing more examples of Clifford or Boole  $t$ -regular domains, as shown in the next result. Recall that Clifford regularity of  $R[X]$  forces  $R$  to be a field (Corollary 2.27).

**Proposition 3.5** ([40, Propositions 2.4 and 2.5]). *Let  $R$  be a domain,  $X$  an indeterminate over  $R$ , and  $S$  a multiplicative subset of  $R$ .*

- (1) Assume  $R$  is integrally closed. Then  $R$  is Clifford (resp., Boole)  $t$ -regular if and only if so is  $R[X]$ .
- (2) If  $R$  is Clifford (resp., Boole)  $t$ -regular, then so is  $R_S$ .

Now, one needs to examine the integrally closed setting. At this point, recall that an integrally closed Clifford (resp., Boole) regular domain is necessarily Prüfer (resp., Bézout) [38, 59]. This fact does not hold for  $t$ -regularity; namely, an integrally closed Clifford (or Boole)  $t$ -regular domain need not be a PVMD (i.e.,  $t$ -Prüfer). Examples stem from the following theorem on the inheritance of  $t$ -regularity by PVDs (for pseudo-valuation domains).

**Theorem 3.6** ([40, Theorem 2.7]). *Let  $R$  be a PVD issued from a valuation domain  $V$ . Then:*

- (1)  $R$  is Clifford  $t$ -regular.
- (2)  $R$  is Boole  $t$ -regular if and only if  $V$  is Boole regular.

Contrast this result with Theorem 2.39 about regularity; which asserts that if  $R$  is a PVD issued from a valuation (resp., strongly discrete valuation) domain  $(V, M)$ , then  $R$  is a Clifford (resp., Boole) regular domain if and only if  $[V/M : R/M] = 2$ .

Now, using Theorem 3.6, one can build integrally closed Boole (hence Clifford)  $t$ -regular domains which are not PVMDs. For instance, let  $k$  be a field and  $X, Y$  two indeterminates over  $k$ . Let  $R := k + M$  be the PVD associated to the rank-one DVR  $V := k(X)[[Y]] = k(X) + M$ , where  $M = YV$ . Clearly,  $R$  is an integrally closed Boole  $t$ -regular domain but not a PVMD [25, Theorem 4.1].

### 3.2 The PVMD case

A domain  $R$  is of finite  $t$ -character if each proper  $t$ -ideal is contained in only finitely many  $t$ -maximal ideals. It is worthwhile recalling that the PVMDs of finite  $t$ -character are exactly the Krull-type domains introduced and studied by Griffin in 1967–1968 [31, 32]. This subsection discusses the  $t$ -analogue for Bazzoni’s result that “an integrally closed domain is Clifford regular if and only if it is a Prüfer domain of finite character” (Theorem 2.16).

Recall from [2] that the pseudo-integral closure of a domain  $R$  is defined as  $\tilde{R} = \bigcup (I_t : I_t)$ , where  $I$  ranges over the set of finitely generated ideals of  $R$ ; and  $R$  is said to be pseudo-integrally closed if  $R = \tilde{R}$ . This is equivalent to saying that  $R$  is a  $v$ -domain, i.e. a domain such that  $(I_v : I_v) = R$  for each nonzero finitely generated ideal  $I$  of  $R$ . A domain with this property is called in Bourbaki’s language regularly integrally closed [11, Chap. VII, Exercise 30]. Clearly  $\bar{R} \subseteq \tilde{R} \subseteq R^*$ , where  $\bar{R}$  and  $R^*$  are respectively the integral closure and the complete integral closure of  $R$ . In view of the example provided in the previous subsection, one has to elevate the “integrally closed” assumption in Bazzoni’s result to “pseudo-integrally closed.” Accordingly, in [40, Conjecture 3.1], the authors sustained the following:

*Conjecture 3.7.* A pseudo-integrally closed domain (i.e.,  $v$ -domain) is Clifford  $t$ -regular if and only if it is a PVMD of finite  $t$ -character.

The next result presented a crucial step towards a satisfactory  $t$ -analogue.

**Theorem 3.8** ([40, Theorem 3.2]). *A PVMD is Clifford  $t$ -regular if and only if it is a Krull-type domain.*

Since in Prüfer domains the  $t$ - and trivial operations coincide, this theorem recovers Bazzoni’s result (mentioned above) and also uncovers the fact that in the class of PVMDs, Clifford  $t$ -regularity coincides with the finite  $t$ -character condition.

The proof involves several preliminary lemmas, some of which are of independent interest and their proofs differ in form from their respective analogues – if any – for the trivial operation. These lemmas are listed below.

**Lemma 3.9.** *Let  $R$  be a PVMD and  $I$  a nonzero fractional ideal of  $R$ . Then for every  $t$ -prime ideal  $P$  of  $R$ ,  $I_t R_P = IR_P$ .*

**Lemma 3.10.** *Let  $R$  be a PVMD which is Clifford  $t$ -regular and  $I$  a nonzero fractional ideal of  $R$ . Then  $I$  is  $t$ -invertible if and only if  $I$  is  $t$ -locally principal.*

**Lemma 3.11.** *Let  $R$  be a PVMD which is Clifford  $t$ -regular and let  $P \subsetneq Q$  be two  $t$ -prime ideals of  $R$ . Then there exists a finitely generated ideal  $I$  of  $R$  such that  $P \subsetneq I_t \subseteq Q$ .*

**Lemma 3.12.** *Let  $R$  be a PVMD which is Clifford  $t$ -regular and  $P$  a  $t$ -prime ideal of  $R$ . Then  $(P : P)$  is a PVMD which is Clifford  $t$ -regular and  $P$  is a  $t$ -maximal ideal of  $(P : P)$ .*

**Lemma 3.13.** *Let  $R$  be a PVMD which is Clifford  $t$ -regular and  $Q$  a  $t$ -prime ideal of  $R$ . Suppose there is a nonzero prime ideal  $P$  of  $R$  such that  $P \subsetneq Q$  and  $\text{ht}(Q/P) = 1$ . Then there exists a finitely generated subideal  $I$  of  $Q$  such that  $\text{Max}_t(R, I) = \text{Max}_t(R, Q)$ , where  $\text{Max}_t(R, I)$  consists of  $t$ -maximal ideals containing  $I$ .*

As a consequence of Theorem 3.8, the next result handles the context of strongly  $t$ -discrete domains.

**Corollary 3.14 ([40, Corollary 3.12]).** *Assume  $R$  is a strongly  $t$ -discrete domain. Then  $R$  is a pseudo-integrally closed Clifford  $t$ -regular domain if and only if  $R$  is a PVMD of finite  $t$ -character.*

Recently, Halter-Koch solved Conjecture 3.7 by using the language of ideal systems on cancellative commutative monoids. Precisely, he proved that “every  $t$ -Clifford regular  $v$ -domain is a Krull-type domain” [35, Propositions 6.11 and 6.12]. This result combined with the “if” statement of Theorem 3.8 provides a  $t$ -analogue for Bazzoni’s result (mentioned above):

**Theorem 3.15.** *A  $v$ -domain is Clifford  $t$ -regular if and only if it is a Krull-type domain.*

The rest of this subsection is devoted to generating examples. For this purpose, two results will handle the possible transfer of the PVMD notion endowed with the finite  $t$ -character condition to pullbacks and polynomial rings, respectively. This will allow for the construction of original families of Clifford  $t$ -regular domains via PVMDs.

**Proposition 3.16 ([40, Proposition 4.1]).** *Let  $T$  be a domain,  $M$  a maximal ideal of  $T$ ,  $K$  its residue field,  $\phi : T \rightarrow K$  the canonical surjection, and  $D$  a proper*

subring of  $K$ . Let  $R = \phi^{-1}(D)$  be the pullback issued from the following diagram of canonical homomorphisms:

$$\begin{array}{ccc} R & \longrightarrow & D \\ \downarrow & & \downarrow \\ T & \xrightarrow{\phi} & K = T/M \end{array}$$

Then  $R$  is a PVMD of finite  $t$ -character if and only if  $D$  is a semilocal Bézout domain with quotient field  $K$  and  $T$  is a Krull-type domain such that  $T_M$  is a valuation domain.

**Proposition 3.17** ([40, Proposition 4.2]). *Let  $R$  be an integrally closed domain and  $X$  an indeterminate over  $R$ . Then  $R$  has finite  $t$ -character if and only if so does  $R[X]$ .*

Note that the “integrally closed” condition is unnecessary in the above result, as pointed out recently in [30]. Now one can build new families of Clifford  $t$ -regular domains originating from the class of PVMDs via a combination of the two previous results and Theorem 3.8 (cf. [40, Example 4.3]).

*Example 3.18.* For each integer  $n \geq 2$ , there exists a PVMD  $R_n$  subject to the following conditions:

- (1)  $\dim(R_n) = n$ .
- (2)  $R_n$  is Clifford  $t$ -regular.
- (3)  $R_n$  is not Clifford regular.
- (4)  $R_n$  is not Krull.

Here are two ways to realize this. Let  $V_0$  be a rank-one valuation domain with quotient field  $K$ . Let  $V = K + N$  be a rank-one non-strongly discrete valuation domain (cf. [21, Remark 6(b)]). Take  $R_n = V[X_1, \dots, X_{n-1}]$ .

For  $n \geq 4$ , the classical  $D + M$  construction provides more examples. Indeed, consider an increasing sequence of valuation domains  $V = V_1 \subset V_2 \subset \dots \subset V_{n-2}$  such that, for each  $i \in \{2, \dots, n-2\}$ ,  $\dim(V_i) = i$  and  $V_i/M_i = V/N = K$ , where  $M_i$  denotes the maximal ideal of  $V_i$ . Set  $T = V_{n-2}[X]$  and  $M = (M_{n-2}, X)$ . Therefore  $R_n = V_0 + M$  is the desired example.

### 3.3 The structure of the $t$ -class semigroup of a Krull-type domain

This subsection extends Bazzoni and Salce’s study of groups in the class semigroup of a valuation domain [9] and recovers Bazzoni’s results on the constituents groups of the class semigroup of a Prüfer domain of finite character [6, 7] to a larger class of domains. Precisely, it describes the idempotents of  $\mathcal{S}_t(R)$  and the structure of their associated groups when  $R$  is a Krull-type domain (i.e., PVMD of finite  $t$ -character). Indeed, it states that there are two types of idempotents in  $\mathcal{S}_t(R)$ : those represented by fractional overrings of  $R$  and those represented by finite intersections

of  $t$ -maximal ideals of fractional overrings of  $R$ . Further, it shows that the group associated with an idempotent of the first type equals the class group of the fractional overring, and characterizes the elements of the group associated with an idempotent of the second type in terms of their localizations at  $t$ -prime ideals.

In any attempt to extend classical results on Prüfer domains to PVMDs (via  $t$ -closure), the  $t$ -linked notion plays a crucial role in order to make the  $t$ -move possible. An overring  $T$  of a domain  $R$  is  $t$ -linked over  $R$  if, for each finitely generated ideal  $I$  of  $R$ ,  $I^{-1} = R \Rightarrow (T : IT) = T$  [2,45]. In Prüfer domains, the  $t$ -linked property coincides with the notion of overring (since every finitely generated proper ideal is invertible and then its inverse is a fortiori different from  $R$ ). Recall also that an overring  $T$  of  $R$  is fractional if  $T$  is a fractional ideal of  $R$ . Of significant importance too for the study of  $t$ -class semigroups is the notion of  $t$ -idempotence; namely, a  $t$ -ideal  $I$  is  $t$ -idempotent if  $(I^2)_t = I$ .

Let  $R$  be a PVMD. Note that  $T$  is a  $t$ -linked overring of  $R$  if and only if  $T$  is a subintersection of  $R$ , that is,  $T = \bigcap R_P$ , where  $P$  ranges over some set of  $t$ -prime ideals of  $R$  [44, Theorem 3.8] or [15, p. 206]. Further, every  $t$ -linked overring of  $R$  is a PVMD [44, Corollary 3.9]; in fact, this condition characterizes the notion of PVMD [20, Theorem 2.10]. Finally, let  $I$  be a  $t$ -ideal of  $R$ . Then  $(I : I)$  is a fractional  $t$ -linked overring of  $R$  and hence a PVMD.

Theorem 3.8 asserts that if  $R$  is a Krull-type domain, then  $\mathcal{S}_t(R)$  is Clifford and hence a disjoint union of subgroups  $G_{[J]}$ , where  $[J]$  ranges over the set of idempotents of  $\mathcal{S}_t(R)$  and  $G_{[J]}$  is the largest subgroup of  $\mathcal{S}_t(R)$  with unit  $[J]$ . At this point, it is worthwhile recalling Bazzoni-Salce's result that valuation domains have Clifford class semigroup [9]. To the main result of this subsection:

**Theorem 3.19 ([41, Theorem 2.1]).** *Let  $R$  be a Krull-type domain and  $I$  a  $t$ -ideal of  $R$ . Set  $T := (I : I)$  and  $\Gamma(I) := \{\text{finite intersections of } t\text{-idempotent } t\text{-maximal ideals of } T\}$ . Then  $[I]$  is an idempotent of  $\mathcal{S}_t(R)$  if and only if there exists a unique  $J \in \{T\} \cup \Gamma(I)$  such that  $[I] = [J]$ . Moreover,*

- (1) *If  $J = T$ , then  $G_{[J]} \cong \text{Cl}(T)$ ;*
- (2) *If  $J = \bigcap_{1 \leq i \leq r} Q_i \in \Gamma(I)$ , then the sequence*

$$0 \longrightarrow \text{Cl}(T) \xrightarrow{\phi} G_{[J]} \xrightarrow{\psi} \prod_{1 \leq i \leq r} G_{[Q_i T_{Q_i}]} \longrightarrow 0$$

*of natural group homomorphisms is exact, where  $G_{[Q_i T_{Q_i}]}$  denotes the constituent group of the Clifford semigroup  $\mathcal{S}(T_{Q_i})$  associated with  $[Q_i T_{Q_i}]$ .*

The proof of the theorem draws partially on the following lemmas, which are of independent interest.

**Lemma 3.20.** *Let  $R$  be a PVMD. Let  $T$  be a  $t$ -linked overring of  $R$  and  $Q$  a  $t$ -prime ideal of  $T$ . Then  $P := Q \cap R$  is a  $t$ -prime ideal of  $R$  with  $R_P = T_Q$ . If, in addition,  $Q$  is  $t$ -idempotent in  $T$ , then so is  $P$  in  $R$ .*

**Lemma 3.21.** *Let  $R$  be a PVMD and  $T$  a  $t$ -linked overring of  $R$ . Let  $J$  be a common (fractional) ideal of  $R$  and  $T$ . Then:*

- (1)  $J_{t_1} = J_t$ , where  $t_1$  denotes the  $t$ -operation with respect to  $T$ .
- (2)  $J$  is a  $t$ -idempotent  $t$ -ideal of  $R \iff J$  is a  $t$ -idempotent  $t$ -ideal of  $T$ .

**Lemma 3.22.** *Let  $R$  be a PVMD,  $I$  a  $t$ -ideal of  $R$ , and  $T := (I : I)$ . Let  $J := \bigcap_{1 \leq i \leq r} Q_i$ , where each  $Q_i$  is a  $t$ -idempotent  $t$ -maximal ideal of  $T$ . Then  $J$  is a fractional  $t$ -idempotent  $t$ -ideal of  $R$ .*

**Lemma 3.23.** *Let  $R$  be a PVMD,  $I$  a  $t$ -idempotent  $t$ -ideal of  $R$ , and  $M \supseteq I$  a  $t$ -maximal ideal of  $R$ . Then  $IR_M$  is an idempotent (prime) ideal of  $R_M$ .*

**Lemma 3.24.** *Let  $R$  be a Krull-type domain,  $L$  a  $t$ -ideal of  $R$ , and  $J$  a  $t$ -idempotent  $t$ -ideal of  $R$ . Then:*

$$[L] \in G_{[J]} \iff (L : L) = (J : J) \text{ and } (JL(L : L^2))_t = (L(L : L^2))_t = J.$$

**Lemma 3.25.** *Let  $R$  be a PVMD and  $I$  a  $t$ -ideal of  $R$ . Then:*

- (1)  $I$  is a  $t$ -ideal of  $(I : I)$ .
- (2) If  $R$  is Clifford  $t$ -regular, then so is  $(I : I)$ .

Since in a Prüfer domain the  $t$ -operation collapses to the trivial operation, Theorem 3.19 recovers Bazzoni's results on Prüfer domains of finite character (Proposition 2.17 and Theorem 2.19). Moreover, there is the following consequence:

**Corollary 3.26 ([41, Corollary 2.9]).** *Let  $R$  be a Krull-type domain which is strongly  $t$ -discrete. Then  $\mathcal{S}_t(R)$  is a disjoint union of subgroups  $\text{Cl}(T)$ , where  $T$  ranges over the set of fractional  $t$ -linked overrings of  $R$ .*

Now one can develop numerous illustrative examples via Theorem 3.19 and Corollary 3.26. Two families of such examples can be provided by means of polynomial rings over valuation domains. First, the following lemma investigates this setting:

**Lemma 3.27 ([41, Lemma 3.1]).** *Let  $V$  be a nontrivial valuation domain and  $X$  an indeterminate over  $V$ . Then:*

- (1)  $R := V[X]$  is a Krull-type domain which is not Prüfer.
- (2) Every fractional  $t$ -linked overring of  $R$  has the form  $V_p[X]$  for some nonzero prime ideal  $p$  of  $V$ .
- (3) Every  $t$ -idempotent  $t$ -prime ideal of  $R$  has the form  $p[X]$  for some idempotent prime ideal  $p$  of  $V$ .

*Example 3.28.* Let  $n$  be an integer  $\geq 1$ . Let  $V$  be an  $n$ -dimensional strongly discrete valuation domain and let  $(0) \subset p_1 \subset p_2 \subset \dots \subset p_n$  denote the chain of its prime ideals. Let  $R := V[X]$ , a Krull-type domain. A combination of Lemma 3.27 and Corollary 3.26 yields

$$\mathcal{S}_t(R) = \{V_{p_1}[X], V_{p_2}[X], \dots, V_{p_n}[X]\}$$

where, for each  $i$ , the class  $[V_{p_i}[X]]$  in  $\mathcal{S}_t(R)$  is identified with  $V_{p_i}[X]$  (due to the uniqueness stated by the main theorem).

*Example 3.29.* Let  $V$  be a one-dimensional valuation domain with idempotent maximal ideal  $M$  and  $R := V[X]$ , a Krull-type domain. By Theorem 3.19 and Lemma 3.27, we have:

$$\mathcal{S}_t(R) = \{[R]\} \cup \{[I] \mid I \text{ } t\text{-ideal of } R \text{ with } (II^{-1})_t = M[X]\}.$$

### 3.4 The Noetherian case

A domain  $R$  is called *strong Mori* if  $R$  satisfies the ascending chain condition on  $w$ -ideals (cf. Section 2.6). Recall that the  $t$ -dimension of  $R$ , abbreviated  $t\text{-dim}(R)$ , is by definition equal to the length of the longest chain of  $t$ -prime ideals of  $R$ .

This subsection discusses  $t$ -regularity in Noetherian and Noetherian-like domains. Precisely, it studies conditions under which  $t$ -stability (see definition below) characterizes  $t$ -regularity. Unlike regularity,  $t$ -regularity over Noetherian domains does not force the  $t$ -dimension to be one. However, Noetherian strong  $t$ -stable domains happen to have  $t$ -dimension 1.

Recall that an ideal  $I$  of a domain  $R$  is said to be  $L$ -stable if  $R^I := \bigcup_{n \geq 1} (I^n : I^n) = (I : I)$ .

The next result compares Clifford  $t$ -regularity to two forms of stability.

**Theorem 3.30 ([42, Theorem 2.2]).** *Let  $R$  be a Noetherian domain and consider the following:*

- (1)  $R$  is Clifford  $t$ -regular,
- (2) Each  $t$ -ideal  $I$  of  $R$  is  $t$ -invertible in  $(I : I)$ ,
- (3) Each  $t$ -ideal is  $L$ -stable.

Then (1)  $\Rightarrow$  (2)  $\Rightarrow$  (3). If  $t\text{-dim}(R) = 1$ , then the 3 conditions are equivalent.

Recall that an ideal  $I$  of a domain  $R$  is said to be *stable* (resp., *strongly stable*) if  $I$  is invertible (resp., principal) in  $(I : I)$ , and  $R$  is called a *stable* (resp., *strongly stable*) domain provided each nonzero ideal of  $R$  is stable (resp., strongly stable). A stable domain is  $L$ -stable [1, Lemma 2.1]. By analogy,  $t$ -stability is defined in [42] as follows:

**Definition 3.31.** A domain  $R$  is  *$t$ -stable* if each  $t$ -ideal of  $R$  is stable, and  $R$  is *strongly  $t$ -stable* if each  $t$ -ideal of  $R$  is strongly stable.

Recall that a Noetherian domain  $R$  is Clifford regular if and only if  $R$  is stable if and only if  $R$  is  $L$ -stable and  $\dim(R) = 1$  [8, Theorem 2.1] and [38, Corollary 4.3]. Unlike Clifford regularity, Clifford (or even Boole)  $t$ -regularity does not force a Noetherian domain  $R$  to be of  $t$ -dimension one. In order to illustrate this fact with an example, a result first establishes the transfer of Boole  $t$ -regularity to pullbacks issued from local Noetherian domains.

**Proposition 3.32** ([42, Proposition 2.3]). *Let  $(T, M)$  be a local Noetherian domain with residue field  $K$  and  $\phi : T \rightarrow K$  the canonical surjection. Let  $k$  be a proper subfield of  $K$  and  $R := \phi^{-1}(k)$  the pullback issued from the following diagram of canonical homomorphisms:*

$$\begin{array}{ccc} R & \longrightarrow & k \\ \downarrow & & \downarrow \\ T & \xrightarrow{\phi} & K = T/M \end{array}$$

*Then  $R$  is Boole  $t$ -regular if and only if  $T$  is Boole  $t$ -regular.*

Now the next example provides a Boole  $t$ -regular Noetherian domain with  $t$ -dimension  $\geq 1$ .

*Example 3.33.* Let  $K$  be a field,  $X$  and  $Y$  two indeterminates over  $K$ , and  $k$  a proper subfield of  $K$ . Let  $T := K[[X, Y]] = K + M$  and  $R := k + M$  where  $M := (X, Y)$ . Since  $T$  is a UFD, then  $T$  is Boole  $t$ -regular (Proposition 3.4). Further,  $R$  is a Boole  $t$ -regular Noetherian domain by the above proposition. Further  $M$  is a  $\nu$ -ideal of  $R$ , so that  $t\text{-dim}(R) = \dim(R) = 2$ , as desired.

Next, the main result of this subsection presents a  $t$ -analogue for Boole regularity as stated in Theorem 2.33.

**Theorem 3.34** ([42, Theorem 2.6]). *Let  $R$  be a Noetherian domain. Then:*

$$R \text{ is strongly } t\text{-stable} \iff R \text{ is Boole } t\text{-regular and } t\text{-dim}(R) = 1.$$

An analogue of this result does not hold for Clifford  $t$ -regularity. For, there exists a Noetherian Clifford  $t$ -regular domain with  $t\text{-dim}(R) = 1$  such that  $R$  is not  $t$ -stable. Indeed, recall first that a domain  $R$  is said to be pseudo-Dedekind [43] (or generalized Dedekind [57]) if every  $\nu$ -ideal is invertible. In [55], P. Samuel gave an example of a Noetherian UFD  $R$  for which  $R[[X]]$  is not a UFD. In [43], Kang noted that  $R[[X]]$  is a Noetherian Krull domain which is not pseudo-Dedekind (otherwise,  $\text{Cl}(R[[X]]) = \text{Cl}(R) = 0$  forces  $R[[X]]$  to be a UFD, absurd). Moreover,  $R[[X]]$  is a Clifford  $t$ -regular domain with  $t$ -dimension 1 (since Krull). But  $R[[X]]$  not being a UFD translates into the existence of a  $\nu$ -ideal of  $R[[X]]$  that is not invertible, as desired.

The next result extends the above theorem to the larger class of strong Mori domains.

**Theorem 3.35** ([42, Theorem 2.10]). *Let  $R$  be a strong Mori domain. Then:*

$$R \text{ is strongly } t\text{-stable} \iff R \text{ is Boole } t\text{-regular and } t\text{-dim}(R) = 1.$$

Unlike Clifford regularity, Clifford (or even Boole)  $t$ -regularity does not force a strong Mori domain to be Noetherian. Indeed, it suffices to consider a UFD which is not Noetherian. We close with the following discussion about the limits and possible extensions of the above results.



*Remark 3.36.* (1) It is not known whether the assumption “ $t\text{-dim}(R) = 1$ ” in Theorem 3.30 can be omitted.

(2) Following Proposition 2.25, the integral closure  $\bar{R}$  of a Noetherian Boole regular domain  $R$  is a PID. By analogy, it is not known if  $\bar{R}$  is a UFD in the case of Boole  $t$ -regularity. (It is the case if the conductor  $(R : \bar{R}) \neq 0$ .)

(3) It is not known if the assumption “ $R$  strongly  $t$ -discrete, i.e.,  $R$  has no  $t$ -idempotent  $t$ -prime ideals” forces a Clifford  $t$ -regular Noetherian domain to be of  $t$ -dimension one.

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# Forcing algebras, syzygy bundles, and tight closure

Holger Brenner

**Abstract** We give a survey about some recent work on tight closure and Hilbert-Kunz theory from the viewpoint of vector bundles. This work is based in understanding tight closure in terms of forcing algebras and the cohomological dimension of torsors of syzygy bundles. These geometric methods allowed to answer some fundamental questions of tight closure, in particular the equality between tight closure and plus closure in graded dimension two over a finite field and the rationality of the Hilbert-Kunz multiplicity in graded dimension two. Moreover, this approach showed that tight closure may behave weirdly under arithmetic and geometric deformations, and provided a negative answer to the localization problem.

## 1 Introduction

In this survey article, we describe some developments which led to a detailed geometric understanding of tight closure in dimension two in terms of vector bundles and torsors. Tight closure is a technique in positive characteristic introduced by M. Hochster and C. Huneke 20 years ago [21–23, 40]. We recall its definition. Let  $R$  be a commutative ring of positive characteristic  $p$  with  $e$ th Frobenius homomorphism  $F^e : R \rightarrow R$ ,  $f \mapsto f^q$ ,  $q = p^e$ . For an ideal  $I$  let  $I^{[q]} := F^e(I)$  be the extended ideal under the  $e$ th Frobenius. Then the *tight closure* of  $I$  is given by

$$I^* = \{f \in R : \text{there exists } t, \text{ not in any minimal prime,} \\ \text{such that } tf^q \in I^{[q]} \text{ for } q \gg 0\}$$

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(in the domain case this means just  $t \neq 0$ , and for all  $q$ ). In this paper, we will not deal with the applications of tight closure in commutative algebra, homological algebra and algebraic geometry, but with some of its intrinsic problems. One of them is whether tight closure commutes with localization, that is, whether for a multiplicative system  $S \subseteq R$  the equality

$$(I^*)R_S = (IR_S)^*$$

holds (the inclusion  $\subseteq$  is always true). A directly related question is whether tight closure is the same as plus closure. The *plus closure* of an ideal  $I$  in a domain  $R$  is defined to be

$$I^+ = \{f \in R : \text{there exists } R \subseteq S \text{ finite domain extension such that } f \in IS\}.$$

This question is known as the *tantalizing question* of tight closure theory. The inclusion  $I^+ \subseteq I^*$  always holds. Since the plus closure commutes with localization, a positive answer to the tantalizing question for a ring and all its localizations implies a positive answer for the localization problem for this ring. The tantalizing question is a problem already in dimension two, the localization problem starts to get interesting in dimension three.

What makes these problems difficult is that there are no exact criteria for tight closure. There exist many important inclusion criteria for tight closure, and in all these cases the criteria also hold for plus closure (in general, with much more difficult proofs). The situation is that the heartlands of “tight closure country” and of “non tight closure country” have been well exploited, but not much is known about the thin line which separates them. This paper is about approaching this thin line geometrically.

The original definition of tight closure, where one has to check infinitely many conditions, is difficult to apply. The starting point of the work we are going to present here is another description of tight closure due to Hochster as *solid closure* based on the concept of *forcing algebras*. Forcing algebras were introduced by Hochster in [19] in an attempt to build up a characteristic-free closure operation with similar properties as tight closure. This approach rests on the fact that  $f \in (f_1, \dots, f_n)^*$  holds in  $R$  if and only if  $H_m^{\dim R}(A) \neq 0$ , where  $A = R[T_1, \dots, T_n]/(f_1T_1 + \dots + f_nT_n - f)$  is the forcing algebra for these data (see Theorem 5.1 for the exact statement). This gives a new interpretation for tight closure, where, at least at first glance, not infinitely many conditions are involved. This cohomological interpretation can be refined geometrically, and the goal of this paper is to describe how this is done and where it leads to. We give an overview.

We will describe the basic properties of forcing algebras in Section 2. A special feature of the cohomological condition for tight closure is that it depends only on the open subset  $D(\mathfrak{m}A) \subseteq \text{Spec}A$ . This open subset is a “torsor” over  $D(\mathfrak{m}) \subseteq \text{Spec}R$ , on which the *syzygy bundle*  $\text{Syz}(f_1, \dots, f_n)$  acts. This allows a more geometric view of the situation (Section 3). In general, closure operations for ideals can be expressed with suitable properties of forcing algebras. We mention some examples of this correspondence in Section 4 and come back to tight closure and solid closure in Section 5.

To obtain a detailed understanding, we restrict in Section 6 to the situation of a two-dimensional standard-graded normal domain  $R$  over an algebraically closed field and homogeneous  $R_+$ -primary ideals. In this setting, the question about the cohomological dimension is the question whether a torsor coming from forcing data is an affine scheme. Moreover, to answer this question we can look at the corresponding torsor over the smooth projective curve  $\text{Proj}R$ . This translates the question into a projective situation. In particular, we can then use concepts from algebraic geometry like *semistable bundles* and the *strong Harder–Narasimhan filtration* to prove results. We obtain an exact numerical criterion for tight closure in this setting (Theorems 6.2 and 6.3). The containment in the plus closure translates to a geometric condition for the torsors on the curve, and in the case where the base field is the algebraic closure of a finite field we obtain the same criterion. This implies that under all these assumptions, tight closure and plus closure coincide (Theorem 6.4).

With this geometric approach also some problems in Hilbert–Kunz theory could be solved, in particular it was shown that the *Hilbert–Kunz multiplicity* is a rational number in graded dimension two (Theorem 7.3). In fact, there is a formula for it in terms of the strong Harder–Narasimhan filtration of the syzygy bundle. In Section 8, we change the setting and look at families of two-dimensional rings parametrized by a one-dimensional base. Typical bases are  $\text{Spec} \mathbb{Z}$  (*arithmetic deformations*) or  $\mathbb{A}_K^1$  (*geometric deformations*). Natural questions are how tight closure, Hilbert–Kunz multiplicity and strong semistability of bundles vary in such a family. Examples of P. Monsky already showed that the Hilbert–Kunz multiplicity behaves “weirdly” in the sense that it is not almost constant. It follows from the geometric interpretation that also strong semistability behaves wildly. Further extra work is needed to show that tight closure also behaves wildly under such a deformation. We present the example of Brenner–Katzman in the arithmetic case and of Brenner–Monsky in the geometric case (Examples 8.4 and 8.7). The latter example shows also that tight closure does not commute with localization and that even in the two-dimensional situation, the tantalizing question has a negative answer, if the base field contains a transcendental element. We close the paper with some open problems (Section 9).

As this is a survey article, we usually omit the proofs and refer to the original research papers and to [9]. I thank Helena Fischbacher-Weitz, Almar Kaid, Axel Stähler, and the referees for useful comments.

## 2 Forcing algebras

Let  $R$  be a commutative ring, let  $M$  be a finitely generated  $R$ -module and  $N \subseteq M$  a finitely generated  $R$ -submodule. Let  $s \in M$  be an element. The *forcing algebra* for these data is constructed as follows: choose generators  $y_1, \dots, y_m$  for  $M$  and generators  $x_1, \dots, x_n$  for  $N$ . This gives rise to a surjective homomorphism  $\varphi : R^m \rightarrow M$ , a submodule  $N' = \varphi^{-1}(N)$  and a morphism  $R^n \rightarrow R^m$  which sends  $e_i$  to a preimage  $x'_i$  of  $x_i$ . Altogether we get the commutative diagram with exact rows

$$\begin{array}{ccccccc}
 R^n & \xrightarrow{\alpha} & R^m & \longrightarrow & M/N & \longrightarrow & 0 & (*) \\
 & & & & & & & \\
 & & & & & & & \\
 & & & & & & & \\
 & & & & & & & \\
 & & & & & & & \\
 & & & & & & & \\
 & & & & & & & \\
 0 & \longrightarrow & N & \longrightarrow & M & \longrightarrow & M/N & \longrightarrow & 0
 \end{array}$$

( $\alpha$  is a matrix). The element  $s$  is represented by  $(s_1, \dots, s_m) \in R^m$ , and  $s$  belongs to  $N$  if and only if the linear equation

$$\alpha \begin{pmatrix} t_1 \\ \cdot \\ \cdot \\ \cdot \\ t_n \end{pmatrix} = \begin{pmatrix} s_1 \\ \cdot \\ \cdot \\ \cdot \\ s_m \end{pmatrix}$$

has a solution. An important insight due to Hochster is that this equation can be formulated with new variables  $T_1, \dots, T_n$ , and then the “distance of  $s$  to  $N$ ” – in particular, whether  $s$  belongs to a certain closure of  $N$  – is reflected by properties of the resulting (generic) forcing algebra. Explicitly, if  $\alpha$  is the matrix describing the submodule  $N$  as above and if  $(s_1, \dots, s_m)$  represents  $s$ , then the forcing algebra is defined by

$$A = R[T_1, \dots, T_n]/(\alpha T - s),$$

where  $\alpha T - s$  stands for

$$\alpha \begin{pmatrix} T_1 \\ \cdot \\ \cdot \\ \cdot \\ T_n \end{pmatrix} = \begin{pmatrix} s_1 \\ \cdot \\ \cdot \\ \cdot \\ s_m \end{pmatrix}$$

or, in other words, for the system of *inhomogeneous linear equations*

$$\begin{aligned}
 a_{11}T_1 + \dots + a_{1n}T_n &= s_1 \\
 a_{21}T_1 + \dots + a_{2n}T_n &= s_2 \\
 &\vdots \\
 a_{m1}T_1 + \dots + a_{mn}T_n &= s_m
 \end{aligned}$$

In the case of an ideal  $I = (f_1, \dots, f_n)$  and  $f \in R$  the forcing algebra is just  $A = R[T_1, \dots, T_n]/(f_1T_1 + \dots + f_nT_n - f)$ . Forcing algebras are given by the easiest algebraic equations at all, namely linear equations. Yet we will see that forcing algebras already have a rich geometry. Of course, starting from the data  $(M, N, s)$  we had to make some choices in order to write down a forcing algebra, hence only properties which are independent of these choices are interesting.

The following lemma expresses the *universal property* of a forcing algebra.

**Lemma 2.1.** *Let the situation be as above, and let  $R \rightarrow R'$  be a ring homomorphism. Then there exists an  $R$ -algebra homomorphism  $A \rightarrow R'$  if and only if  $s \otimes 1$  lies in  $\text{im}(N \otimes R' \rightarrow M \otimes R')$ .*

*Proof.* This follows from the right exactness of tensor products applied to the sequence (\*) above.

The lemma implies in particular that for two forcing algebras  $A$  and  $A'$  we have (not uniquely determined)  $R$ -algebra homomorphisms  $A \rightarrow A'$  and  $A' \rightarrow A$ . It also implies that  $s \in N$  if and only if there exists an  $R$ -algebra homomorphism  $A \rightarrow R$  (equivalently,  $\text{Spec} A \rightarrow \text{Spec} R$  has a section).

We continue with some easy geometric properties of the mapping  $\text{Spec} A \rightarrow \text{Spec} R$ . The formation of forcing algebras commutes with arbitrary base change  $R \rightarrow R'$ . Therefore, for every point  $\mathfrak{p} \in \text{Spec} R$  the fiber ring  $A \otimes_R \kappa(\mathfrak{p})$  is the forcing algebra given by

$$\alpha(\mathfrak{p})T = s(\mathfrak{p}),$$

which is an inhomogeneous linear equation over the field  $\kappa(\mathfrak{p})$ . Hence, the fiber of  $\text{Spec} A \rightarrow \text{Spec} R$  over  $\mathfrak{p}$  is the *solution set* to a system of linear inhomogeneous equations.

We know from linear algebra that the solution set to such a system might be empty, or it is an affine space (in the sense of linear algebra) of dimension at least  $n - m$ . Hence, one should think of  $\text{Spec} A \rightarrow \text{Spec} R$  as a family of affine-linear spaces varying with the base. Also, from linear algebra we know that such a solution set is given by adding to one particular solution a solution of the corresponding system of homogeneous of linear equations. The solution set to  $\alpha(\mathfrak{p})T = 0$  is a vector space over  $\kappa(\mathfrak{p})$ , and this solution set is the fiber over  $\mathfrak{p}$  of the forcing algebra of the zero element, namely

$$B = R[T_1, \dots, T_n]/(\alpha T) = R[T_1, \dots, T_n]/\left(\sum_{i=1}^n a_{ij}T_i, j = 1, \dots, m\right).$$

As just remarked, the fibers of  $V = \text{Spec} B$  over a point  $\mathfrak{p}$  are vector spaces of possibly varying dimensions. Therefore,  $V$  is in general not a vector bundle. It is, however, a commutative *group scheme* over  $\text{Spec} R$ , where the addition is given by

$$V \times V \rightarrow V, (s_1, \dots, s_n), (s'_1, \dots, s'_n) \mapsto (s_1 + s'_1, \dots, s_n + s'_n)$$

(written on the level of sections) and the coaddition by

$$R[T_1, \dots, T_n]/(\alpha T) \rightarrow R[T_1, \dots, T_n]/(\alpha T) \otimes R[\tilde{T}_1, \dots, \tilde{T}_n]/(\alpha \tilde{T}), T_i \mapsto T_i + \tilde{T}_i.$$

This group scheme is the kernel group scheme of the group scheme homomorphism

$$\alpha : \mathbb{A}_R^n \longrightarrow \mathbb{A}_R^m$$

between the trivial additive group schemes of rank  $n$  and  $m$ . We call it the *syzygy group scheme* for the given generators of  $N$ .



The syzygy group scheme acts on the spectrum of a forcing algebra  $\text{Spec}A$ ,  $A = R[T]/(\alpha T - s)$  for every  $s \in M$ . The action is exactly as in linear algebra, by adding a solution of the system of homogeneous equations to a solution of the system of inhomogeneous equations. An understanding of the syzygy group scheme is necessary before we can understand the forcing algebras.

Although  $V$  is not a vector bundle in general, it is not too far away. Let  $U \subseteq \text{Spec}R$  be the open subset of points  $\mathfrak{p}$  where the mapping  $\alpha(\mathfrak{p})$  is surjective. Then the restricted group scheme  $V|_U$  is a vector bundle of rank  $n - m$ . If  $M/N$  has its support in a maximal ideal  $\mathfrak{m}$ , then the syzygy group scheme induces a vector bundle on the *punctured spectrum*  $\text{Spec}R - \{\mathfrak{m}\}$ , which we call the *syzygy bundle*. Hence on  $U$  we have a short exact sequence

$$0 \longrightarrow \text{Syz} \longrightarrow \mathcal{O}_U^n \longrightarrow \mathcal{O}_U^m \longrightarrow 0$$

of vector bundles on  $U$ .

We will mostly be interested in the situation where the submodule is an ideal  $I \subseteq R$  in the ring. We usually fix ideal generators  $I = (f_1, \dots, f_n)$  and  $(*)$  becomes

$$R^n \xrightarrow{f_1, \dots, f_n} R \longrightarrow R/I \longrightarrow 0.$$

The ideal generators and an element  $f \in R$  defines then the forcing equation  $f_1 T_1 + \dots + f_n T_n - f = 0$ . Moreover, if the ideal is primary to a maximal ideal  $\mathfrak{m}$ , then we have a syzygy bundle  $\text{Syz} = \text{Syz}(f_1, \dots, f_n)$  defined on  $D(\mathfrak{m})$ .

### 3 Forcing algebras and torsors

Let  $Z \subset \text{Spec}R$  be the support of  $M/N$  and let  $U = \text{Spec}R - Z$  be the open complement where  $\alpha$  is surjective. Let  $s \in M$  with forcing algebra  $A$ . We set  $T = \text{Spec}A|_U$  and we assume that the fibers are non-empty (in the ideal case this means that  $f$  is not a unit). Then the action of the group scheme  $V$  on  $\text{Spec}A$  restricts to an action of the syzygy bundle  $\text{Syz} = V|_U$  on  $T$ , and this action is *simply transitive*. This means that locally the actions looks like the action of  $\text{Syz}$  on itself by addition.

In general, if a vector bundle  $\mathcal{S}$  on a separated scheme  $U$  acts simply transitively on a scheme  $T \rightarrow U$  – such a scheme is called a *geometric  $\mathcal{S}$ -torsor* or an *affine-linear bundle* –, then this corresponds to a cohomology class  $c \in H^1(U, \mathcal{S})$  (where  $\mathcal{S}$  is now also the sheaf of sections in the vector bundle  $\mathcal{S}$ ). This follows from the Čech cohomology by taking an open covering where the action can be trivialized. Hence, forcing data define, by restricting the forcing algebra, a torsor  $T$  over  $U$ .

On the other hand, the forcing data define the short exact sequence  $0 \rightarrow \text{Syz} \rightarrow \mathcal{O}_U^n \rightarrow \mathcal{O}_U^m \rightarrow 0$  and  $s$  is represented by an element  $s' \in R^m \rightarrow \Gamma(U, \mathcal{O}_U^m)$ . By the connecting homomorphism  $s'$  defines a cohomology class

$$c = \delta(s') \in H^1(U, \text{Syz}).$$

An explicit computation of Čech cohomology shows that this class corresponds to the torsor given by the forcing algebra.

Starting from a cohomology class  $c \in H^1(U, \mathcal{S})$ , one may construct a geometric model for the torsor classified by  $c$  in the following way: because of  $H^1(U, \mathcal{S}) \cong \text{Ext}^1(\mathcal{O}_U, \mathcal{S})$  we have an extension

$$0 \longrightarrow \mathcal{S} \longrightarrow \mathcal{S}' \longrightarrow \mathcal{O}_U \longrightarrow 0.$$

This sequence induces projective bundles  $\mathbb{P}(\mathcal{S}^\vee) \hookrightarrow \mathbb{P}(\mathcal{S}'^\vee)$  and  $T(c) \cong \mathbb{P}(\mathcal{S}'^\vee) - \mathbb{P}(\mathcal{S}^\vee)$ . If  $\mathcal{S} = \text{Syz}(f_1, \dots, f_n)$  is the syzygy bundle for ideal generators, then the extension given by the cohomology class  $\delta(f)$  coming from another element  $f$  is easy to describe: it is just

$$0 \longrightarrow \text{Syz}(f_1, \dots, f_n) \longrightarrow \text{Syz}(f_1, \dots, f_n, f) \longrightarrow \mathcal{O}_U \longrightarrow 0$$

with the natural embedding (extending a syzygy by zero in the last component). This follows again from an explicit computation in Čech cohomology.

If the base  $U$  is projective, a situation in which we will work starting with Section 6, then  $\mathbb{P}(\mathcal{S}'^\vee)$  is also a projective variety and  $\mathbb{P}(\mathcal{S}^\vee)$  is a subvariety of codimension one, a divisor. Then properties of the torsor are reflected by properties of the divisor and vice versa.

## 4 Forcing algebras and closure operations

A *closure operation* for ideals or for submodules is an assignment

$$N \longmapsto N^c$$

for submodules  $N \subseteq M$  of  $R$ -modules  $M$  such that  $N \subseteq N^c = (N^c)^c$  holds. One often requires further nice properties of a closure operation, like *monotony* or the *independence of representation* (meaning that  $s \in N^c$  if and only if  $\bar{s} \in 0^c$  in  $M/N$ ). Forcing algebras are very natural objects to study such closure operations. The underlying philosophy is that  $s \in N^c$  holds if and only if the forcing morphism  $\text{Spec} A \rightarrow \text{Spec} R$  fulfills a certain property (depending on and characterizing the closure operation). The property is in general not uniquely determined; for the identical closure operation one can take the properties to be faithfully flat, to be (cyclic) pure, or to have a (module- or ring-) section.

Let us consider some easy closure operations to get a feeling for this philosophy. In Section 5 we will see how tight closure can be characterized with forcing algebras.

*Example 4.1.* For the *radical*  $\text{rad}(I)$  the corresponding property is that  $\text{Spec} A \rightarrow \text{Spec} R$  is surjective. It is not surprising that a rough closure operation corresponds to a rough property of a morphism. An immediate consequence of this viewpoint is that we get at once a hint of what the radical of a submodule should be: namely  $s \in \sqrt{N}$  if and only if the forcing algebra is Spec-surjective. This is equivalent to the property that  $s \otimes 1 \in \text{im}(N \otimes_R K \rightarrow M \otimes_R K)$  for all homomorphism  $R \rightarrow K$  to fields (or just for all  $\kappa(\mathfrak{p})$ ,  $\mathfrak{p} \in \text{Spec} R$ ).

*Example 4.2.* We now look at the *integral closure* of an ideal, which is defined by

$$\bar{I} = \{f \in R : \text{there exists } f^k + a_1 f^{k-1} + \cdots + a_{k-1} f + a_k = 0, a_i \in I^i\}.$$

The integral closure was first used by Zariski as it describes the normalization of blow-ups. What is the corresponding property of a morphism?

We look at an example. For  $R = K[X, Y]$  we have  $X^2 Y \in \overline{(X^3, Y^3)}$  and  $XY \notin \overline{(X^3, Y^3)}$ . The inclusion follows from  $(X^2 Y)^3 = X^6 Y^3 \in (X^3, Y^3)^3$ . The non-inclusion follows from the *valuation criterion* for integral closure: This says for a noetherian domain  $R$  that  $f \in \bar{I}$  if and only if for all mappings to discrete valuation rings  $\varphi : R \rightarrow V$  we have  $\varphi(f) \in IV$ . In the example, the mapping  $K[X, Y] \rightarrow K[X], Y \mapsto X$ , yields  $X^2 \notin (X^3)$ , so it can not belong to the integral closure. In both cases the mapping  $\text{Spec} A \rightarrow \text{Spec} R$  is surjective. In the second case, the forcing algebra over the line  $V(Y - X)$  is given by the equation  $T_1 X^3 + T_2 X^3 + X^2 = X^2((T_1 + T_2)X + 1)$ . The fiber over the zero point is a plane and is an affine line over a hyperbola for every other point of the line. The topologies above and below are not much related: The inverse image of the non-closed punctured line is closed, hence the topology downstairs does not carry the *image topology* from upstairs. In fact, the relationship in general is

$$f \in \bar{I} \text{ if and only if } \text{Spec} A \rightarrow \text{Spec} R \text{ is universally a } \textit{submersion}$$

(a submersion in the topological sense). This relies on the fact that both properties can be checked with (in the noetherian case discrete) valuations (for this criterion for submersions, see [15] and [1]).

Let us consider the forcing algebras for  $(X, Y)$  and  $1$  and for  $(X^2, Y^2)$  and  $XY$  in  $K[X, Y]$ . The restricted spectra of the forcing algebras over the punctured plane for these two forcing data are isomorphic, because both represent the torsor given by the cohomology class  $\frac{1}{XY} = \frac{XY}{X^2 Y^2} \in H^1(D(X, Y), \mathcal{O})$ . However,  $XY \in \overline{(X^2, Y^2)}$ , but  $1 \notin \overline{(X, Y)}$  (not even in the radical). Hence, integral closure can be characterized by the forcing algebra, but not by the induced torsor. An interesting feature of tight closure is that it only depends on the cohomology class in the syzygy bundle and the torsor induced by the forcing algebra, respectively.

*Example 4.3.* In the case of finitely generated algebras over the complex numbers, there is another interesting closure operation, called *continuous closure*. An element  $s$  belongs to the continuous closure of  $N$  if the forcing algebra  $A$  is such that the morphism  $\mathbb{C} - \text{Spec} A \rightarrow \mathbb{C} - \text{Spec} R$  has a continuous section in the *complex topology*. For an ideal  $I = (f_1, \dots, f_n)$  this is equivalent to the existence of complex-continuous functions  $g_1, \dots, g_n : \mathbb{C} - \text{Spec} R \rightarrow \mathbb{C}$  such that  $\sum_{i=1}^n g_i f_i = f$  (as an identity on  $\mathbb{C} - \text{Spec} R$ ).

*Remark 4.4.* One can go one step further with the understanding of closure operations in terms of forcing algebras. For this we take the forcing algebras which are allowed by the closure operation (i.e., forcing algebras for  $s, N, M, s \in N^c$ ) and declare them to be the defining covers of a (non-flat) *Grothendieck topology*.

This works basically for all closure operations fulfilling certain natural conditions. This embeds closure operations into the much broader context of Grothendieck topologies, see [6].

## 5 Tight closure as solid closure

We come back to tight closure, and its interpretation in terms of forcing algebras and *solid closure*.

**Theorem 5.1.** *Let  $(R, \mathfrak{m})$  be a local excellent normal domain of positive characteristic and let  $I$  denote an  $\mathfrak{m}$ -primary ideal. Then  $f \in I^*$  if and only if  $H_{\mathfrak{m}}^{\dim R}(A) \neq 0$ , where  $A$  denotes the forcing algebra.*

*Proof.* We indicate the proof of the direction that the cohomological property implies the tight closure inclusion. By the assumptions we may assume that  $R$  is complete. Because of  $H_{\mathfrak{m}}^{\dim R}(A) \neq 0$  there exists by Matlis-duality a non-trivial  $R$ -module homomorphism  $\psi : A \rightarrow R$  and we may assume  $\psi(1) =: c \neq 0$ . In  $A$  we have the equality  $f = \sum_{i=1}^n f_i T_i$  and hence

$$f^q = \sum_{i=1}^n f_i^q T_i^q \text{ for all } q = p^e.$$

Applying the  $R$ -homomorphism  $\psi$  to these equations gives

$$c f^q = \sum_{i=1}^n f_i^q \psi(T_i^q),$$

which is exactly the tight closure condition (the  $\psi(T_i^q)$  are the coefficients in  $R$ ). For the other direction see [19].

This theorem provides the bridge between tight closure and cohomological properties of forcing algebras. The first observation is that the property about *local cohomology* on the right hand side does not refer to positive characteristic. The closure operation defined by this property is called *solid closure*, and the theorem says that in positive characteristic and under the given further assumptions solid closure and tight closure coincide. The hope was that this would provide a closure operation in all (even mixed) characteristics with similar properties as tight closure. This hope was however destroyed by the following example of Paul Roberts (see [36])

*Example 5.2.* (Roberts) Let  $K$  be a field of characteristic zero and consider

$$A = K[X, Y, Z]/(X^3 T_1 + Y^3 T_2 + Z^3 T_3 - X^2 Y^2 Z^2).$$

Then  $H_{(X, Y, Z)}^3(A) \neq 0$ . Therefore  $X^2 Y^2 Z^2 \in (X^3, Y^3, Z^3)^{\text{sc}}$  in the regular ring  $K[X, Y, Z]$ . This means that in a three-dimensional regular ring an ideal needs

not be solidly-closed. It is, however, an important property of tight closure that every regular ring is *F-regular*, namely that every ideal is tightly closed. Hence, solid closure is not a good replacement for tight closure (for a variant called *parasolid closure* with all good properties in equal characteristic zero, see [2]).

Despite this drawback, solid closure provides an important interpretation of tight closure. First of all we have for  $d = \dim(R) \geq 2$  (the one-dimensional case is trivial) the identities

$$H_{\mathfrak{m}}^d(A) \cong H_{\mathfrak{m}A}^d(A) \cong H^{d-1}(D(\mathfrak{m}A), \mathcal{O}).$$

This easy observation is quite important. The open subset  $D(\mathfrak{m}A) \subseteq \text{Spec}A$  is exactly the torsor induced by the forcing algebra over the punctured spectrum  $D(\mathfrak{m}) \subset \text{Spec}R$ . Hence, we derive an important particularity of tight closure: tight closure of primary ideals in a normal excellent local domain depends only on the torsor (or, what is the same, only on the cohomology class of the syzygy bundle). We recall from the last section that this property does not hold for integral closure.

By Theorem 5.1, tight closure can be understood by studying the global sheaf cohomology of the torsor given by a first cohomology class of the syzygy bundle. The forcing algebra provides a geometric model for this torsor. An element  $f$  belongs to the tight closure if and only if the *cohomological dimension* of the torsor  $T$  is  $d - 1$  (which is the cohomological dimension of  $D(\mathfrak{m})$ ), and  $f \notin I^*$  if and only if the cohomological dimension drops. Recall that the cohomological dimension of a scheme  $U$  is the largest number  $i$  such that  $H^i(U, \mathcal{F}) \neq 0$  for some (quasi-)coherent sheaf  $\mathcal{F}$  on  $U$ . In the quasiaffine case, where  $U \subseteq \text{Spec}B$  (as in the case of torsors inside the spectrum of the forcing algebra), one only has to look at the structure sheaf  $\mathcal{F} = \mathcal{O}$ .

In dimension two, this means that  $f \in I^*$  if and only if the cohomological dimension of the torsor is one, and  $f \notin I^*$  if and only if this is zero. By a theorem of Serre ([18, Theorem III.3.7]) cohomological dimension zero means that  $U$  is an *affine scheme*, i.e., isomorphic as a scheme to the spectrum of a ring (do not confuse the “affine” in affine scheme with the “affine” in affine-linear bundle).

It is, in general, a difficult question to decide whether a quasiaffine scheme is an affine scheme. Even in the special case of torsors there is no general machinery to answer it. A necessary condition is that the complement has pure codimension one (which is fulfilled in the case of torsors). So far we have not gained any criterion from our geometric interpretation.

## 6 Tight closure in graded dimension two

From now on we deal with the following situation:  $R$  is a two-dimensional normal standard-graded domain over an algebraically closed field of any characteristic,  $I = (f_1, \dots, f_n)$  is a homogeneous  $R_+$ -primary ideal with homogeneous generators of degree  $d_i = \deg(f_i)$ . Let  $C = \text{Proj}R$  be the corresponding smooth projective curve. The ideal generators define the homogeneous resolution

$$0 \longrightarrow \mathrm{Syz}(f_1, \dots, f_n) \longrightarrow \bigoplus_{i=1}^n R(-d_i) \xrightarrow{f_1, \dots, f_n} R \longrightarrow R/I \longrightarrow 0,$$

and the short exact sequence of vector bundles on  $C$

$$0 \longrightarrow \mathrm{Syz}(f_1, \dots, f_n) \longrightarrow \bigoplus_{i=1}^n \mathcal{O}_C(-d_i) \xrightarrow{f_1, \dots, f_n} \mathcal{O}_C \longrightarrow 0.$$

We also need the  $m$ -twists of this sequence for every  $m \in \mathbb{Z}$ ,

$$0 \longrightarrow \mathrm{Syz}(f_1, \dots, f_n)(m) \longrightarrow \bigoplus_{i=1}^n \mathcal{O}_C(m - d_i) \xrightarrow{f_1, \dots, f_n} \mathcal{O}_C(m) \longrightarrow 0.$$

It follows from this *presenting sequence* by the additivity of rank and degree that the vector bundle  $\mathrm{Syz}(f_1, \dots, f_n)(m)$  has rank  $n - 1$  and degree

$$(m(n-1) - \sum_{i=1}^n d_i) \deg C$$

(where  $\deg C = \deg \mathcal{O}_C(1)$  is the degree of the curve).

A homogeneous element  $f \in R_m = \Gamma(C, \mathcal{O}_C(m))$  defines again a cohomology class  $c \in H^1(C, \mathcal{O}_C(m))$  as well as a torsor  $T(c) \rightarrow C$ . This torsor is a homogeneous version of the torsor induced by the forcing algebra on  $D(m) \subset \mathrm{Spec} R$ . This can be made more precise by endowing the forcing algebra  $A = R[T_1, \dots, T_n]/(f_1 T_1 + \dots + f_n T_n - f)$  with a (not necessarily positive)  $\mathbb{Z}$ -grading and taking  $T = D_+(R_+) \subseteq \mathrm{Proj} A$ . From this it follows that the affineness of this torsor on  $C$  is decisive for tight closure. The translation of the tight closure problem via forcing algebras into torsors over projective curves has the following advantages:

- (1) We can work over a smooth projective curve, i.e., we have reduced the dimension of the base and we have removed the singularity.
- (2) We can work in a projective setting and use intersection theory.
- (3) We can use the theory of vector bundles, in particular the notion of semistable bundles and their moduli spaces.

We will give a criterion when such a torsor is affine and hence when a homogeneous element belongs to the tight closure of a graded  $R_+$ -primary ideal. For this we need the following definition (for background on semistable bundles we refer to [25]).

**Definition 6.1.** Let  $\mathcal{S}$  be a locally free sheaf on a smooth projective curve  $C$ . Then  $\mathcal{S}$  is called *semistable*, if  $\deg(\mathcal{T})/\mathrm{rk}(\mathcal{T}) \leq \deg(\mathcal{S})/\mathrm{rk}(\mathcal{S})$  holds for all subbundles  $\mathcal{T} \neq 0$ . In positive characteristic,  $\mathcal{S}$  is called *strongly semistable*, if all *Frobenius pull-backs*  $F^{e*}(\mathcal{S})$ ,  $e \geq 0$ , are semistable (here  $F : C \rightarrow C$  denotes the *absolute Frobenius* morphism).

Note that for the syzygy bundle we have the natural isomorphism (by pulling back the presenting sequence)

$$F^{e*}(\mathrm{Syz}(f_1, \dots, f_n)) \cong \mathrm{Syz}(f_1^q, \dots, f_n^q).$$

Therefore, the Frobenius pull-back of the cohomology class  $\delta(f) \in H^1(C, \text{Syz}(f_1, \dots, f_n)(m))$  is

$$F^{e*}(\delta(f)) = \delta(f^q) \in H^1(C, \text{Syz}(f_1^q, \dots, f_n^q)(qm)).$$

The following two results establish an exact numerical *degree bound* for tight closure under the condition that the syzygy bundle is strongly semistable.

**Theorem 6.2.** *Suppose that  $\text{Syz}(f_1, \dots, f_n)$  is strongly semistable. Then we have  $R_m \subseteq I^*$  for  $m \geq (\sum_{i=1}^n d_i)/(n-1)$ .*

*Proof.* Note that the degree condition implies that  $\mathcal{S} := \text{Syz}(f_1, \dots, f_n)(m)$  has non-negative degree. Let  $c \in H^1(C, \mathcal{S})$  be any cohomology class (it might be  $\delta(f)$  for some  $f$  of degree  $m$ ). The pull-back  $F^{e*}(c)$  lives in  $H^1(C, F^{e*}(\mathcal{S}))$ . Let now  $k$  be such that  $\mathcal{O}_C(-k) \otimes \omega_C$  has negative degree, where  $\omega_C$  is the canonical sheaf on the curve. Let  $z \in \Gamma(C, \mathcal{O}_C(k)) = R_k, z \neq 0$ . Then  $zF^{e*}(c) \in H^1(C, F^{e*}(\mathcal{S}) \otimes \mathcal{O}_C(k))$ . However, by degree considerations, these cohomology groups are zero: by Serre duality they are dual to  $H^0(C, F^{e*}(\mathcal{S}^\vee) \otimes \mathcal{O}_C(-k) \otimes \omega_C)$ , and this bundle is semistable of negative degree, hence it can not have global sections. Because of  $zF^{e*}(c) = 0$  it follows that  $zf^q$  is in the image of the mapping given by  $f_1^q, \dots, f_n^q$ , so  $zf^q \in I^{[q]}$  and  $f \in I^*$ .

**Theorem 6.3.** *Suppose that  $\text{Syz}(f_1, \dots, f_n)$  is strongly semistable. Let  $m < (\sum_{i=1}^n d_i)/(n-1)$  and let  $f$  be a homogeneous element of degree  $m$ . Suppose that  $f^{p^a} \notin I^{[p^a]}$  for a such that  $p^a > gn(n-1)$  (where  $g$  is the genus of  $C$ ). Then  $f \notin I^*$ .*

*Proof.* Here, the proof works with the torsor  $T$  defined by  $c = \delta(f)$ . The syzygy bundle  $\mathcal{S} = \text{Syz}(f_1, \dots, f_n)(m)$  has now negative degree, hence its dual bundle  $\mathcal{F} = \mathcal{S}^\vee$  is an *ample* vector bundle (as it is strongly semistable of positive degree). The class defines a non-trivial dual extension  $0 \rightarrow \mathcal{O}_C \rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow 0$ . By the assumption also a certain Frobenius pull-back of this extension is still non-trivial. Hence,  $\mathcal{F}'$  is also ample and therefore  $\mathbb{P}(\mathcal{F}) \subset \mathbb{P}(\mathcal{F}')$  is an ample divisor and its complement  $T = \mathbb{P}(\mathcal{F}') - \mathbb{P}(\mathcal{F})$  is affine. Hence,  $f \notin I^*$ .

It is in general not easy to establish whether a bundle is strongly semistable or not. However, even if we do not know whether the syzygy bundle is strongly semistable, we can work with its *strong Harder–Narasimhan filtration*. The Harder–Narasimhan filtration of a vector bundle  $\mathcal{S}$  on a smooth projective curve is a filtration

$$0 = \mathcal{S}_0 \subset \mathcal{S}_1 \subset \mathcal{S}_2 \subset \dots \subset \mathcal{S}_{t-1} \subset \mathcal{S}_t = \mathcal{S}$$

with  $\mathcal{S}_i/\mathcal{S}_{i-1}$  semistable and descending slopes

$$\mu(\mathcal{S}_1) > \mu(\mathcal{S}_2/\mathcal{S}_1) > \dots > \mu(\mathcal{S}/\mathcal{S}_{t-1}).$$

Since the Frobenius pull-back of a semistable bundle need not be semistable anymore, the Harder–Narasimhan filtration of  $F^*(\mathcal{S})$  is quite unrelated to the

Harder–Narasimhan filtration of  $\mathcal{S}$ . However, by a result of A. Langer [29, Theorem 2.7], there exists a certain number  $e$  such that the quotients in the Harder–Narasimhan filtration of  $F^{e*}(\mathcal{S})$  are strongly semistable. Such a filtration is called *strong*. With a strong Harder–Narasimhan filtration one can now formulate an exact numerical criterion for tight closure inclusion building on Theorems 6.2 and 6.3.

The criterion basically says that a torsor is affine (equivalently,  $f \notin I^*$ ), if and only if the cohomology class is non-zero in some strongly semistable quotient of negative degree of the strong Harder–Narasimhan filtration. One should remark here that even if we start with a syzygy bundle, the bundles in the filtration are no syzygy bundles, hence it is important to develop the theory of torsors of vector bundles in full generality. From this numerical criterion one can deduce an answer to the tantalizing question.

**Theorem 6.4.** *Let  $K = \overline{\mathbb{F}}_p$  be the algebraic closure of a finite field and let  $R$  be a normal standard-graded  $K$ -algebra of dimension two. Then  $I^* = I^+$  for every  $R_+$ -primary homogeneous ideal.*

*Proof.* This follows from the numerical criterion for the affineness of torsors mentioned above. The point is that the same criterion holds for the non-existence of projective curves inside the torsor. One reduces to the situation of a strongly semistable bundle  $\mathcal{S}$  of degree 0. Every cohomology class of such a bundle defines a non-affine torsor and hence we have to show that there exists a projective curve inside, or equivalently, that the cohomology class can be annihilated by a finite cover of the curve. Here, is where the finiteness assumption about the field enters.  $\mathcal{S}$  is defined over a finite subfield  $\mathbb{F}_q \subseteq K$ , and it is represented (or rather, its  $S$ -equivalence class) by a point in the moduli space of semistable bundles of that rank and degree 0. The Frobenius pull-backs  $F^{e*}(\mathcal{S})$  are again semistable (by strong semistability) and they are defined over the same finite field. Because semistable bundles form a bounded family (itself the reason for the existence of the moduli space), there exist only finitely many semistable bundles defined over  $\mathbb{F}_q$  of degree zero. Hence there exists a repetition, i.e., there exists  $e' > e$  such that we have an isomorphism  $F^{e'*}(\mathcal{S}) \cong F^{e*}(\mathcal{S})$ . By a result of H. Lange and U. Stuhler [28] there exists a finite mapping  $C' \xrightarrow{\varphi} C \xrightarrow{F^e} C$  (with  $\varphi$  étale) such that the pull-back of the bundle is trivial. Then one is left with the problem of annihilating a cohomology class  $c \in H^1(C, \mathcal{O}_C)$ , which is possible using Artin–Schreier theory (or the graded version of K. Smith’s parameter theorem, [38, 39]).

*Remark 6.5.* This theorem was extended by G. Dietz for  $R_+$ -primary ideals which are not necessarily homogeneous [14]. The above proof shows how important the assumption is that the base field is finite or the algebraic closure of a finite field. Indeed, we will see in the last section that the statement is not true when the base field contains transcendental elements. Also some results on Hilbert–Kunz functions depend on the property that the base field is finite.



## 7 Applications to Hilbert–Kunz theory

The geometric approach to tight closure was also successful in Hilbert–Kunz theory. This theory originates in the work of E. Kunz [26, 27] and was largely extended by P. Monsky [17, 32].

Let  $R$  be a commutative ring of positive characteristic and let  $I$  be an ideal which is primary to a maximal ideal. Then all  $R/I^{[q]}$ ,  $q = p^e$ , have finite length, and the *Hilbert–Kunz function* of the ideal is defined to be

$$e \longmapsto \varphi(e) = \lg(R/I^{[p^e]}).$$

Monsky proved the following fundamental theorem of Hilbert–Kunz theory ([32], [22, Theorem 6.7]).

**Theorem 7.1.** *The limit*

$$\lim_{e \rightarrow \infty} \frac{\varphi(e)}{p^{e \dim(R)}}$$

*exists (as a positive real number) and is called the Hilbert–Kunz multiplicity of  $I$ , denoted by  $e_{\text{HK}}(I)$ .*

The Hilbert–Kunz multiplicity of the maximal ideal in a local ring is usually denoted by  $e_{\text{HK}}(R)$  and is called the Hilbert–Kunz multiplicity of  $R$ . It is an open question whether this number is always rational. Strong numerical evidence suggests that this is probably not true in dimension  $\geq 4$ , see [35]. We will deal with the two-dimensional situation in a minute, but first we relate Hilbert–Kunz theory to tight closure (see [22, Theorem 5.4]).

**Theorem 7.2.** *Let  $R$  be an analytically unramified and formally equidimensional local ring of positive characteristic and let  $I$  be an  $\mathfrak{m}$ -primary ideal. Let  $f \in R$ . Then  $f \in I^*$  if and only if*

$$e_{\text{HK}}(I) = e_{\text{HK}}((I, f)).$$

This theorem means that the Hilbert–Kunz multiplicity is related to tight closure in the same way as the Hilbert–Samuel multiplicity is related to integral closure.

We restrict now again to the case of an  $R_+$ -primary homogeneous ideal in a standard-graded normal domain  $R$  of dimension two over an algebraically closed field  $K$  of positive characteristic  $p$ . In this situation, Hilbert–Kunz theory is directly related to global sections of the Frobenius pull-backs of the syzygy bundle on  $\text{Proj } R$  (see Section 6). We shall see that it is possible to express the Hilbert–Kunz multiplicity in terms of the strong Harder–Narasimhan filtration of this bundle.

For homogeneous ideal generators  $f_1, \dots, f_n$  of degrees  $d_1, \dots, d_n$  we write down again the presenting sequence on  $C = \text{Proj } R$ ,

$$0 \longrightarrow \text{Syz}(f_1, \dots, f_n) \longrightarrow \bigoplus_{i=1}^n \mathcal{O}_C(-d_i) \xrightarrow{f_1, \dots, f_n} \mathcal{O}_C \longrightarrow 0.$$

The  $m$ -twists of the Frobenius pull-backs of this sequence are

$$0 \longrightarrow \text{Syz}(f_1^q, \dots, f_n^q)(m) \longrightarrow \bigoplus_{i=1}^n \mathcal{O}_C(m - qd_i) \xrightarrow{f_1^q, \dots, f_n^q} \mathcal{O}_C(m) \longrightarrow 0.$$

The global evaluation of the last short exact sequence is

$$0 \longrightarrow \Gamma(C, \text{Syz}(f_1^q, \dots, f_n^q)(m)) \longrightarrow \bigoplus_{i=1}^n R_{m-qi} \xrightarrow{f_1^q, \dots, f_n^q} R_m,$$

and the cokernel of the map on the right is

$$R_m / (f_1^q, \dots, f_n^q) = (R/I^{[q]})_m.$$

Because of  $R/I^{[q]} = \bigoplus_m (R/I^{[q]})_m$ , the length of  $R/I^{[q]}$  is the sum over the degrees  $m$  of the  $K$ -dimensions of these cokernels. The sum is in fact finite because the ideals  $I^{[q]}$  are primary (or because  $H^1(C, \text{Syz}(f_1^q, \dots, f_n^q)(m)) = 0$  for  $m \gg 0$ ), but the bound for the summation grows with  $q$ . Anyway, we have

$$\begin{aligned} \dim(R/I^{[q]})_m &= \dim(\Gamma(C, \mathcal{O}_C(m))) - \sum_{i=1}^n \dim(\Gamma(C, \mathcal{O}_C(m-qi))) \\ &\quad + \dim(\Gamma(C, \text{Syz}(f_1^q, \dots, f_n^q)(m))). \end{aligned}$$

The computation of the dimensions  $\dim(\Gamma(C, \mathcal{O}_C(\ell)))$  is easy, hence the problem is to control the global sections of  $\text{Syz}(f_1^q, \dots, f_n^q)(m)$ , more precisely, its behavior for large  $q$ , and its sum over a suitable range of  $m$ . This behavior is encoded in the strong Harder–Narasimhan filtration of the syzygy bundle. Let  $e$  be fixed and large enough such that the Harder–Narasimhan filtration of the pull-back  $\mathcal{H} = F^{e*}(\text{Syz}(f_1, \dots, f_n)) = \text{Syz}(f_1^q, \dots, f_n^q)$  is strong. Let  $\mathcal{H}_j \subseteq \mathcal{H}$ ,  $j = 1, \dots, t$ , be the subsheaves occurring in the Harder–Narasimhan filtration and set

$$v_j := \frac{-\mu(\mathcal{H}_j/\mathcal{H}_{j-1})}{p^e \deg(C)} \text{ and } r_j = \text{rk}(\mathcal{H}_j/\mathcal{H}_{j-1}).$$

Because the Harder–Narasimhan filtration of  $\mathcal{H}$  and of all its pull-backs is strong, these numbers are independent of  $e$ . The following theorem was shown by Brenner and Trivedi independently [7, 41].

**Theorem 7.3.** *Let  $R$  be a normal two-dimensional standard-graded domain over an algebraically closed field and let  $I = (f_1, \dots, f_n)$  be a homogeneous  $R_+$ -primary ideal,  $d_i = \deg(f_i)$ . Then the Hilbert–Kunz multiplicity of  $I$  is given by the formula*

$$e_{\text{HK}}(I) = \frac{\deg(C)}{2} \left( \sum_{j=1}^t r_j v_j^2 - \sum_{i=1}^n d_i^2 \right).$$

*In particular, it is a rational number.*

We can also say something about the behavior of the Hilbert–Kunz function under the additional condition that the base field is the algebraic closure of a finite field (see [8]).

**Theorem 7.4.** *Let  $R$  and  $I$  be as before and suppose that the base field is the algebraic closure of a finite field. Then the Hilbert–Kunz function has the form*

$$\varphi(e) = e_{\text{HK}}(I)p^{2e} + \gamma(e),$$

where  $\gamma$  is eventually periodic.

This theorem also shows that here the “linear term” in the Hilbert–Kunz function exists and that it is zero. It was proved in [24] that for normal excellent  $R$  the Hilbert–Kunz function looks like

$$e_{\text{HK}}q^{\dim(R)} + \beta q^{\dim(R)-1} + \text{smaller terms}.$$

For possible behavior of the second term in the non-normal case in dimension two see [34]. See also Remark 8.2.

## 8 Arithmetic and geometric deformations of tight closure

The geometric interpretation of tight closure theory led to a fairly detailed understanding of tight closure in graded dimension two. The next easiest case is to study how tight closure behaves in families of two-dimensional rings, parametrized by a one dimensional ring. Depending on whether the base has mixed characteristic (like  $\text{Spec } \mathbb{Z}$ ) or equal positive characteristic  $p$  (like  $\text{Spec } K[T] = \mathbb{A}_K^1$ ) we talk about *arithmetic* or *geometric deformations*.

More precisely, let  $D$  be a one-dimensional domain and let  $S$  be a  $D$ -standard-graded domain of dimension three, such that for every point  $\mathfrak{p} \in \text{Spec } D$  the fiber rings  $S_{\kappa(\mathfrak{p})} = S \otimes_D \kappa(\mathfrak{p})$  are normal standard-graded domains over  $\kappa(\mathfrak{p})$  of dimension two. The data  $I = (f_1, \dots, f_n)$  in  $S$  and  $f \in S$  determine corresponding data in these fiber rings, and the syzygy bundle  $\text{Syz}(f_1, \dots, f_n)$  on  $\text{Proj } S \rightarrow \text{Spec } D$  determines syzygy bundles on the curves  $C_{\kappa(\mathfrak{p})} = \text{Proj } S_{\kappa(\mathfrak{p})}$ . The natural questions here are: how does the property  $f \in I^*$  (in  $S_{\kappa(\mathfrak{p})}$ ) depend on  $\mathfrak{p}$ , how does  $e_{\text{HK}}(I)$  depend on  $\mathfrak{p}$ , how does strong semistability depend on  $\mathfrak{p}$ , how does the affineness of torsors depend on  $\mathfrak{p}$ ?

Semistability itself is an open property and behaves nicely in a family in the sense that if the syzygy bundle is semistable on the curve over the generic point, then it is semistable over almost all closed points. D. Gieseker gave in [16] an example of a collection of bundles such that, depending on the parameter, the  $e$ th pull-back is semistable, but the  $(e + 1)$ th is not semistable anymore (for every  $e$ ). The problem how strong semistability behaves under arithmetic deformations was explicitly formulated by Y. Miyaoka and by N. Shepherd-Barron [31, 37].

In the context of Hilbert–Kunz theory, this question has been studied by P. Monsky [17, 33], both in the arithmetic and in the geometric case. Monsky (and Han) gave examples that the Hilbert–Kunz multiplicity may vary in a family.

*Example 8.1.* Let  $R_p = \mathbb{Z}/(p)[X, Y, Z]/(X^4 + Y^4 + Z^4)$ . Then the Hilbert–Kunz multiplicity of the maximal ideal is

$$e_{\text{HK}}(R_p) = \begin{cases} 3 & \text{for } p \equiv \pm 1 \pmod{8} \\ 3 + 1/p^2 & \text{for } p \equiv \pm 3 \pmod{8} \end{cases} .$$

Note that by the theorem on prime numbers in arithmetic progressions there exist infinitely many prime numbers of all these congruence types.

*Remark 8.2.* In the previous example, there occur infinitely many different values for  $e_{\text{HK}}(R_p)$  depending on the characteristic, the limit as  $p \mapsto \infty$  is, however, well defined. Trivedi showed [42] that in the graded two-dimensional situation this limit always exists, and that this limit can be computed by the Harder–Narasimhan filtration of the syzygy bundle in characteristic zero. Brenner showed that one can define, using this Harder–Narasimhan filtration, a Hilbert–Kunz multiplicity directly in characteristic zero, and that this Hilbert–Kunz multiplicity characterizes solid closure [3] in the same way as Hilbert–Kunz multiplicity characterizes tight closure in positive characteristic (Theorem 7.2 above). Combining these results one can say that “solid closure is the limit of tight closure” in graded dimension two, in the sense that  $f \in I^{\text{sc}}$  in characteristic zero if and only if the Hilbert–Kunz difference  $e_{\text{HK}}((I, f)) - e_{\text{HK}}(I)$  tends to 0 for  $p \mapsto \infty$ .

It is an open question whether in all dimensions the Hilbert–Kunz multiplicity has always a limit as  $p$  goes to infinity, whether this limit, if it exists, has an interpretation in characteristic zero alone (independent of reduction to positive characteristic) and what closure operation it would correspond to. See also [12].

In the geometric case, Monsky gave the following example [33].

*Example 8.3.* Let  $K = \mathbb{Z}/(2)$  and let

$$S = \mathbb{Z}/(2)[T][X, Y, Z]/(Z^4 + Z^2XY + Z(X^3 + Y^3) + (T + T^2)X^2Y^2) .$$

We consider  $S$  as an algebra over  $\mathbb{Z}/(2)[T]$  ( $T$  has degree 0). Then the Hilbert–Kunz multiplicity of the maximal ideal is

$$e_{\text{HK}}(S_{\kappa(\mathfrak{p})}) = \begin{cases} 3 & \text{if } \kappa(\mathfrak{p}) = K(T) \text{ (generic case)} \\ 3 + 1/4^m & \text{if } \kappa(\mathfrak{p}) = \mathbb{Z}/(2)(t) \text{ is finite over } \mathbb{Z}/(2) \text{ of degree } m. \end{cases}$$

By the work of Brenner and Trivedi (see Section 7) these examples can be translated immediately into examples where strong semistability behaves weirdly. From the first example we get an example of a vector bundle of rank two over a projective relative curve over  $\text{Spec } \mathbb{Z}$  such that the bundle is semistable on the generic curve (in characteristic zero), and is strongly semistable for infinitely many prime reductions, but also not strongly semistable for infinitely many prime reductions.

From the second example we get an example of a vector bundle of rank two over a projective relative curve over the affine line  $\mathbb{A}_{\mathbb{Z}/(2)}^1$ , such that the bundle is strongly

semistable on the generic curve (over the function field  $\mathbb{Z}/(2)(T)$ ), but not strongly semistable for the curve over any finite field (and the degree of the field extension determines which Frobenius pull-back destabilizes).

To get examples where tight closure varies with the base one has to go one step further (in short, weird behavior of Hilbert–Kunz multiplicity is a necessary condition for weird behavior of tight closure). Interesting behavior can only happen for elements of degree  $(\sum d_i)/(n-1)$  (the degree bound, see Theorems 6.2 and 6.3).

In [11], Brenner and M. Katzman showed that tight closure does not behave uniformly under an arithmetic deformation, thus answering negatively a question in [19].

*Example 8.4.* Let

$$R = \mathbb{Z}/(p)[X, Y, Z]/(X^7 + Y^7 + Z^7)$$

and  $I = (X^4, Y^4, Z^4)$ ,  $f = X^3Y^3$ . Then  $f \in I^*$  for  $p = 3 \pmod{7}$  and  $f \notin I^*$  for  $p = 2 \pmod{7}$  (see [11, Propositions 2.4 and 3.1]). Hence, we have infinitely many prime reductions where the element belongs to the tight closure and infinitely many prime reductions where it does not.

*Remark 8.5.* Arithmetic deformations are closely related to the question “what is tight closure over a field of characteristic zero”. The general philosophy is that characteristic zero behavior of tight closure should reflect the behavior of tight closure for almost all primes, after expressing the relevant data over an arithmetic base. By declaring  $f \in I^*$ , if this holds for almost all primes, one obtains a satisfactory theory of tight closure in characteristic zero with the same impact as in positive characteristic. This is a systematic way to do reduction to positive characteristic (see [22, Appendix 1] and [20]). However, the example above shows that there is not always a uniform behavior in positive characteristic. A consequence is also that solid closure in characteristic zero is not the same as tight closure (but see Remark 8.2). From the example we can deduce that  $f \in I^{\text{sc}}$ , but  $f \notin I^*$  in  $\mathbb{Q}[X, Y, Z]/(X^7 + Y^7 + Z^7)$ . Hence, the search for a good tight closure operation in characteristic zero remains.

We now look at geometric deformations. They are directly related to the localization problem and to the tantalizing problem which we have mentioned in the introduction.

**Lemma 8.6.** *Let  $D$  be a one-dimensional domain of finite type over  $\mathbb{Z}/(p)$  and let  $S$  be a  $D$ -domain of finite type. Let  $f \in S$  and  $I \subseteq S$  be an ideal. Suppose that localization holds for  $S$ . If then  $f \in I^*$  in the generic fiber ring  $S_{Q(D)}$ , then also  $f \in I^*$  in  $S_{\kappa(\mathfrak{p})} = S \otimes_D \kappa(\mathfrak{p})$  for almost all closed points  $\mathfrak{p} \in \text{Spec } D$ .*

*Proof.* The generic fiber ring is the localization of  $S$  at the multiplicative system  $M = D - \{0\}$  (considered in  $S$ ). So if  $f \in I^*$  holds in  $S_{Q(D)} = S_M$ , and if localization holds, then there exists  $h \in M$  such that  $hf \in I^*$  in  $S$  (the global ring of the deformation). By the persistence of tight closure ([22, Theorem 2.3] applied to  $S \rightarrow S_{\kappa(\mathfrak{p})}$ ) it follows that  $hf \in I^*$  in  $S_{\kappa(\mathfrak{p})}$  for all closed points  $\mathfrak{p} \in \text{Spec } D$ . But  $h$  is a unit in almost all residue class fields  $\kappa(\mathfrak{p})$ , so the result follows.

*Example 8.7.* Let

$$S = \mathbb{Z}/(2)[T][X, Y, Z]/(Z^4 + Z^2XY + Z(X^3 + Y^3) + (T + T^2)X^2Y^2)$$

as in Example 8.3 and let  $I = (X^4, Y^4, Z^4)$ ,  $f = Y^3Z^3$  ( $X^3Y^3$  would not work). Then  $f \in I^*$ , as is shown in [13], in the generic fiber ring  $S_{\mathbb{Z}/(2)(T)}$ , but  $f \notin I^*$  in  $S_{\kappa(\mathfrak{p})}$  for all closed points  $\mathfrak{p} \in \text{Spec } D$ . Hence, tight closure does not commute with localization.

*Example 8.8.* Let  $K = \mathbb{Z}/(2)(T)$  and  $R = K[X, Y, Z]/(Z^4 + Z^2XY + Z(X^3 + Y^3) + (T + T^2)X^2Y^2)$ . This is the generic fiber ring of the previous example. It is a normal, standard-graded domain of dimension two and it is defined over the function field. In this ring we have  $Y^3Z^3 \in (X^4, Y^4, Z^4)^*$ , but  $Y^3Z^3 \notin (X^4, Y^4, Z^4)^+$ . Hence, tight closure is not the same as plus closure, not even in dimension two.

## 9 Some open problems

We collect some open questions and problems, together with some comments of what is known and some guesses. We first list problems which are weaker variants of the localization problem.

*Question 9.1.* Is F-regular the same as weakly F-regular?

Recall that a ring is called *weakly F-regular* if every ideal is tightly closed, and *F-regular* if this is true for all localizations. A positive answer would have followed from a positive answer to the localization problem. This path is not possible anymore, but there are many positive results on this: it is true in the Gorenstein case, in the graded case [30], it is true over an uncountable field (proved by Murthy, see [22, Theorem 12.2]). All this shows that a positive result is likely, at least under some additional assumptions.

*Question 9.2.* Does tight closure commute with the localization at one element?

There is no evidence why this should be true. It would be nice to see a counterexample, and it would also be nice to have examples of bad behavior of tight closure under geometric deformations in all characteristics.

*Question 9.3.* Suppose  $R$  is of finite type over a finite field. Is tight closure the same as plus closure?

This is known in graded dimension two for  $R_+$ -primary ideals by Theorem 6.4 and the extension for non-homogeneous ideals (but still graded ring) by Dietz (see [14]). To attack this problem one probably needs first to establish new exact criteria of what tight closure is. Even in dimension two, but not graded, the best way to establish results is probably to develop a theory of strongly semistable modules on a local ring.

Can one characterize the rings where tight closure is plus closure? Are rings, where every ideal coincides with its plus closure, F-regular? This is true for Gorenstein rings.

For a two-dimensional standard-graded domain and the corresponding projective curve, the following problems remain.

*Question 9.4.* Let  $C$  be a smooth projective curve over a field of positive characteristic, and let  $\mathcal{L}$  be an invertible sheaf of degree zero. Let  $c \in H^1(C, \mathcal{L})$  be a cohomology class. Does there exist a finite mapping  $C' \rightarrow C$ ,  $C'$  another projective curve, such that the pull-back annihilates  $c$ .

This is known for the structure sheaf  $\mathcal{O}_C$  and holds in general over (the algebraic closure of) a finite field. It is probably not true over a field with transcendental elements, the heuristic being that otherwise there would be a uniform way to annihilate the class over every finite field (an analogue is that every invertible sheaf of degree zero over a finite field has finite order in  $\text{Pic}^0(C)$ , but the orders do not have much in common as the field varies, and the order over larger fields might be infinite).

*Question 9.5.* Let  $R$  be a two-dimensional normal standard-graded domain and let  $I$  be an  $R_+$ -primary homogeneous ideal. Write  $\varphi(e) = e_{HK}p^2 + \gamma(e)$ . Is  $\gamma(e)$  eventually periodic?

By Theorem 7.4 this is true if the base field is finite, but this question is open if the base field contains transcendental elements. How does (the lowest term of) the Hilbert–Kunz function behave under a geometric deformation?

*Question 9.6.* Let  $C \rightarrow \text{Spec} D$  be a relative projective curve over an arithmetic base like  $\text{Spec} \mathbb{Z}$ , and let  $\mathcal{S}$  be a vector bundle over  $C$ . Suppose that the generic bundle  $\mathcal{S}_0$  over the generic curve of characteristic zero is semistable. Is then  $\mathcal{S}_p$  over  $C_p$  strongly semistable for infinitely many prime numbers  $p$ ?

This question was first asked by Y. Miyaoka [31]. Corresponding questions for an arithmetic family of two-dimensional rings are: Does there exist always infinitely many prime numbers where the Hilbert–Kunz multiplicity coincides with the characteristic zero limit? If an element belongs to the solid closure in characteristic zero, does it belong to the tight closure for infinitely many prime reductions? In [5], there is a series of examples where the number of primes with not strongly semistable reduction has an arbitrary small density under the assumption that there exist infinitely many Sophie Germain prime numbers (a prime number  $p$  such that also  $2p + 1$  is prime).

We come back to arbitrary dimension.

*Question 9.7.* Understand tight closure geometrically, say for standard-graded normal domains with an isolated singularity. The same for Hilbert–Kunz theory.

Some progress in this direction has been made in [4] and in [10], but much more has to be done. What is apparent from this work is that positivity properties of the top-dimensional syzygy bundle coming from a resolution are important. A problem is that strong semistability controls global sections and by Serre duality also top-dimensional cohomology, but one problem is to control the intermediate cohomology.

*Question 9.8.* Find a good closure operation in equal characteristic zero, with tight closure like properties, with no reduction to positive characteristic.

The notion of *parasolid closure* gives a first answer to this [2]. However, not much is known about it beside that it fulfills the basic properties one expects from tight closure, and many proofs depend on positive characteristic (though the notion itself does not). Is there a more workable notion?

One should definitely try to understand here several candidates with the help of forcing algebras and the corresponding Grothendieck topologies. A promising approach is to allow the forcing algebras as coverings which do not annihilate (top-dimensional) local cohomology unless it is annihilated by a resolution of singularities.

Is there a closure operation which commutes with localization (this is also not known for characteristic zero tight closure, but probably false)?

*Question 9.9.* Find a good closure operation in mixed characteristic and prove the remaining homological conjectures.

In Hilbert–Kunz theory, the following questions are still open.

*Question 9.10.* Is the Hilbert–Kunz multiplicity always a rational number? Is it at least an algebraic number?

The answer to the first question is probably no, as the numerical material in [35] suggests. However, this still has to be established.

*Question 9.11.* Prove or disprove that the Hilbert–Kunz multiplicity has always a limit as the characteristic tends to  $\infty$ .

If it has, or in the cases where it has, one should also find a direct interpretation in characteristic zero and study the corresponding closure operation.

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# Beyond totally reflexive modules and back

## A survey on gorenstein dimensions

Lars Winther Christensen, Hans-Bjørn Foxby, and Henrik Holm

**Abstract** Starting from the notion of totally reflexive modules, we survey the theory of Gorenstein homological dimensions for modules over commutative rings. The account includes the theory's connections with relative homological algebra and with studies of local ring homomorphisms. It ends close to the starting point: with a characterization of Gorenstein rings in terms of total acyclicity of complexes.

### Introduction

An important motivation for the study of homological dimensions dates back to 1956, when Auslander and Buchsbaum [7] and Serre [98] proved:

*Theorem A.* Let  $R$  be a commutative Noetherian local ring with residue field  $k$ . Then the following conditions are equivalent.

- (i)  $R$  is regular.
- (ii)  $k$  has finite projective dimension.
- (iii) Every  $R$ -module has finite projective dimension.

This result opened to the solution of two long-standing conjectures of Krull. Moreover, it introduced the theme that finiteness of a homological dimension for all

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modules characterizes rings with special properties. Later work has shown that modules of finite projective dimension over a general ring share many properties with modules over a regular ring. This is an incitement to study homological dimensions of individual modules.

In line with these ideas, Auslander and Bridger [6] introduced in 1969 the G-dimension. It is a homological dimension for finitely generated modules over a Noetherian ring, and it gives a characterization of Gorenstein local rings (Theorem 1.27), which is similar to Theorem A. Indeed,  $R$  is Gorenstein if  $k$  has finite G-dimension, and only if every finitely generated  $R$ -module has finite G-dimension.

In the early 1990s, the G-dimension was extended beyond the realm of finitely generated modules over a Noetherian ring. This was done by Enochs and Jenda who introduced the notion of Gorenstein projective modules [41]. With the Gorenstein projective dimension at hand, a perfect parallel to Theorem A becomes available (Theorem 2.19). Subsequent work has shown that modules of finite Gorenstein projective dimension over a general ring share many properties with modules over a Gorenstein ring.

## Classical homological algebra as precedent

The notions of injective dimension and flat dimension for modules also have Gorenstein counterparts. It was Enochs and Jenda who introduced Gorenstein injective modules [41] and, in collaboration with Torrecillas, Gorenstein flat modules [47]. The study of Gorenstein dimensions is often called *Gorenstein homological algebra*; it has taken directions from the following:

*Meta Question.* Given a result in classical homological algebra, does it have a counterpart in Gorenstein homological algebra?

To make this concrete, we review some classical results on homological dimensions and point to their Gorenstein counterparts within the main text. In the balance of this introduction,  $R$  is assumed to be a commutative Noetherian local ring with maximal ideal  $\mathfrak{m}$  and residue field  $k = R/\mathfrak{m}$ .

### DEPTH AND FINITELY GENERATED MODULES

The projective dimension of a finitely generated  $R$ -module is closely related to its depth. This is captured by the *Auslander–Buchsbaum Formula* [8]:

*Theorem B.* For every finitely generated  $R$ -module  $M$  of finite projective dimension there is an equality  $\text{pd}_R M = \text{depth } R - \text{depth}_R M$ .

The Gorenstein counterpart (Theorem 1.25) actually strengthens the classical result; this is a recurring theme in Gorenstein homological algebra.

The injective dimension of a non-zero finitely generated  $R$ -module is either infinite or it takes a fixed value:

*Theorem C.* For every non-zero finitely generated  $R$ -module  $M$  of finite injective dimension there is an equality  $\text{id}_R M = \text{depth} R$ .

This result of Bass [20] has its Gorenstein counterpart in Theorem 3.24.

## CHARACTERIZATIONS OF COHEN–MACAULAY RINGS

Existence of special modules of finite homological dimension characterizes Cohen–Macaulay rings. The equivalence of (i) and (iii) in the theorem below is still referred to as *Bass’ conjecture*, even though it was settled more than 20 years ago. Indeed, Peskine and Szpiro proved in [86] that it follows from the New Intersection Theorem, which they proved *ibid.* for equicharacteristic rings. In 1987, Roberts [87] settled the New Intersection Theorem completely.

*Theorem D.* The following conditions on  $R$  are equivalent.

- (i)  $R$  is Cohen–Macaulay.
- (ii) There is a non-zero  $R$ -module of finite length and finite projective dimension.
- (iii) There is a non-zero finitely generated  $R$ -module of finite injective dimension.

A Gorenstein counterpart to this characterization is yet to be established; see Questions 1.31 and 3.26.

Gorenstein rings are also characterized by existence of special modules of finite homological dimension. The equivalence of (i) and (ii) below is due to Peskine and Szpiro [86]. The equivalence of (i) and (iii) was conjectured by Vasconcelos [108] and proved by Foxby [56]. The Gorenstein counterparts are given in Theorems 3.22 and 4.28; see also Question 4.29.

*Theorem E.* The following conditions on  $R$  are equivalent.

- (i)  $R$  is Gorenstein.
- (ii) There is a non-zero cyclic  $R$ -module of finite injective dimension.
- (iii) There is a non-zero finitely generated  $R$ -module of finite projective dimension and finite injective dimension.

## LOCAL RING HOMOMORPHISMS

The Frobenius endomorphism detects regularity of a local ring of positive prime characteristic. The next theorem collects results of Avramov, Iyengar and Miller [17], Kunz [82], and Rodicio [89]. The counterparts in Gorenstein homological algebra to these results are given in Theorems 6.4 and 6.5.

*Theorem F.* Let  $R$  be of positive prime characteristic, and let  $\phi$  denote its Frobenius endomorphism. Then the following conditions are equivalent.

- (i)  $R$  is regular.
- (ii)  $R$  has finite flat dimension as an  $R$ -module via  $\phi^n$  for some  $n \geq 1$ .
- (iii)  $R$  is flat as an  $R$ -module via  $\phi^n$  for every  $n \geq 1$ .
- (iv)  $R$  has finite injective dimension as an  $R$ -module via  $\phi^n$  for some  $n \geq 1$ .
- (v)  $R$  has injective dimension equal to  $\dim R$  as an  $R$ -module via  $\phi^n$  for every  $n \geq 1$ .

Let  $(S, \mathfrak{n})$  be yet a commutative Noetherian local ring. A ring homomorphism  $\varphi: R \rightarrow S$  is called *local* if there is an inclusion  $\varphi(\mathfrak{m}) \subseteq \mathfrak{n}$ . A classical chapter of local algebra, initiated by Grothendieck, studies transfer of ring theoretic properties along such homomorphisms. If  $\varphi$  is flat, then it is called *Cohen–Macaulay* or *Gorenstein* if its closed fiber  $S/\mathfrak{m}S$  is, respectively, a Cohen–Macaulay ring or a Gorenstein ring. These definitions have been extended to homomorphisms of finite flat dimension. The theorem below collects results of Avramov and Foxby from [12] and [14]; the Gorenstein counterparts are given in Theorems 7.8 and 7.11.

*Theorem G.* Let  $\varphi: R \rightarrow S$  be a local homomorphism and assume that  $S$  has finite flat dimension as an  $R$ -module via  $\varphi$ . Then the following hold:

- (a)  $S$  is Cohen–Macaulay if and only if  $R$  and  $\varphi$  are Cohen–Macaulay.
- (b)  $S$  is Gorenstein if and only if  $R$  and  $\varphi$  are Gorenstein.

## VANISHING OF COHOMOLOGY

The projective dimension of a module  $M$  is at most  $n$  if and only if the absolute cohomology functor  $\text{Ext}^{n+1}(M, -)$  vanishes. Similarly (Theorem 5.25),  $M$  has Gorenstein projective dimension at most  $n$  if and only if the relative cohomology functor  $\text{Ext}_{\text{GP}}^{n+1}(M, -)$  vanishes. Unfortunately, the similarity between the two situations does not run too deep. We give a couple of examples:

The absolute Ext is balanced, that is, it can be computed from a projective resolution of  $M$  or from an injective resolution of the second argument. In general, however, the only known way to compute the relative Ext is from a (so-called) proper Gorenstein projective resolution of  $M$ .

Secondly, if  $M$  is finitely generated, then the absolute Ext commutes with localization, but the relative Ext is not known to do so, unless  $M$  has finite Gorenstein projective dimension.

Such considerations motivate the search for an alternative characterization of modules of finite Gorenstein projective dimension, and this has been a driving force behind much research on Gorenstein dimensions within the past 15 years. What follows is a brief review.

## EQUIVALENCE OF MODULE CATEGORIES

For a finitely generated  $R$ -module, Foxby [57] gave a “resolution-free” criterion for finiteness of the Gorenstein projective dimension; that is, one that does not

involve construction of a Gorenstein projective resolution. This result from 1994 is Theorem 8.2. In 1996, Enochs, Jenda, and Xu [49] extended Foxby’s criterion to non-finitely generated  $R$ -modules, provided that  $R$  is Cohen–Macaulay with a dualizing module  $D$ . Their work is related to a 1972 generalization by Foxby [54] of a theorem of Sharp [100]. Foxby’s version reads:

*Theorem H.* Let  $R$  be Cohen–Macaulay with a dualizing module  $D$ . Then the horizontal arrows below are equivalences of categories of  $R$ -modules.

$$\begin{array}{ccc}
 \mathcal{A}_D(R) & \begin{array}{c} \xrightarrow{D \otimes_R -} \\ \xleftarrow{\text{Hom}_R(D, -)} \end{array} & \mathcal{B}_D(R) \\
 \uparrow \downarrow & & \uparrow \downarrow \\
 \{A \mid \text{pd}_R A \text{ is finite}\} & \begin{array}{c} \xrightarrow{D \otimes_R -} \\ \xleftarrow{\text{Hom}_R(D, -)} \end{array} & \{B \mid \text{id}_R B \text{ is finite}\}
 \end{array}$$

Here,  $\mathcal{A}_D(R)$  is the *Auslander class* (Definition 9.1) with respect to  $D$  and  $\mathcal{B}_D(R)$  is the *Bass class* (Definition 9.4). What Enochs, Jenda, and Xu prove in [49] is that the  $R$ -modules of finite Gorenstein projective dimension are exactly those in  $\mathcal{A}_D(R)$ , and the modules in  $\mathcal{B}_D(R)$  are exactly those of finite Gorenstein injective dimension. Thus, the upper level equivalence in Theorem H is the Gorenstein counterpart of the lower level equivalence.

A commutative Noetherian ring has a dualizing complex  $D$  if and only if it is a homomorphic image of a Gorenstein ring of finite Krull dimension; see Kawasaki [79]. For such rings, a result similar to Theorem H was proved by Avramov and Foxby [13] in 1997. An interpretation in terms of Gorenstein dimensions (Theorems 9.2 and 9.5) of the objects in  $\mathcal{A}_D(R)$  and  $\mathcal{B}_D(R)$  was established by Christensen, Frankild, and Holm [31] in 2006. Testimony to the utility of these results is the frequent occurrence—e.g., in Theorems 3.16, 4.13, 4.25, 4.30, 6.5, 6.8, 7.3, and 7.7—of the assumption that the ground ring is a homomorphic image of a Gorenstein ring of finite Krull dimension. Recall that every complete local ring satisfies this assumption.

Recent results, Theorems 2.20 and 4.27, by Esmkhani and Tousi [52] and Theorem 9.11 by Christensen and Sather-Wagstaff [35] combine with Theorems 9.2 and 9.5 to provide resolution-free criteria for finiteness of Gorenstein dimensions over general local rings; see Remarks 9.3 and 9.12.

### Scope and organization

A survey of this modest length is a portrait painted with broad pen strokes. Inevitably, many details are omitted, and some generality has been traded in for simplicity. We have chosen to focus on modules over commutative, and often Noetherian, rings. Much of Gorenstein homological algebra, though, works flawlessly over non-commutative rings, and there are statements in this survey about

Noetherian rings that remain valid for coherent rings. Furthermore, most statements about modules remain valid for complexes of modules. The reader will have to consult the references to qualify these claims.

In most sections, the opening paragraph introduces the main references on the topic. We strive to cite the strongest results available and, outside of this introduction, we do not attempt to trace the history of individual results. In notes, placed at the end of sections, we give pointers to the literature on directions of research—often new ones—that are not included in the survey. Even within the scope of this paper, there are open ends, and more than a dozen questions and problems are found throughout the text.

From this point on,  $R$  denotes a commutative ring. Any extra assumptions on  $R$  are explicitly stated. We say that  $R$  is *local* if it is Noetherian and has a unique maximal ideal. We use the shorthand  $(R, \mathfrak{m}, k)$  for a local ring  $R$  with maximal ideal  $\mathfrak{m}$  and residue field  $k = R/\mathfrak{m}$ .

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## 1 G-dimension of finitely generated modules

The topic of this section is Auslander and Bridger's notion of G-dimension for finitely generated modules over a Noetherian ring. The notes [5] from a seminar by Auslander outline the theory of G-dimension over commutative Noetherian rings. In [6] Auslander and Bridger treat the G-dimension within a more abstract framework. Later expositions are given by Christensen [28] and by Mašek [84].

A *complex*  $\mathbf{M}$  of modules is (in homological notation) an infinite sequence of homomorphisms of  $R$ -modules

$$\mathbf{M} = \cdots \xrightarrow{\partial_{i+1}} M_i \xrightarrow{\partial_i} M_{i-1} \xrightarrow{\partial_{i-1}} \cdots$$



such that  $\partial_i \partial_{i+1} = 0$  for every  $i$  in  $\mathbb{Z}$ . The  $i$ th homology module of  $\mathbf{M}$  is  $H_i(\mathbf{M}) = \text{Ker } \partial_i / \text{Im } \partial_{i+1}$ . We call  $\mathbf{M}$  acyclic if  $H_i(\mathbf{M}) = 0$  for all  $i \in \mathbb{Z}$ .

**Lemma 1.1.** *Let  $\mathbf{L}$  be an acyclic complex of finitely generated projective  $R$ -modules. The following conditions on  $\mathbf{L}$  are equivalent.*

- (i) *The complex  $\text{Hom}_R(\mathbf{L}, R)$  is acyclic.*
- (ii) *The complex  $\text{Hom}_R(\mathbf{L}, F)$  is acyclic for every flat  $R$ -module  $F$ .*
- (iii) *The complex  $E \otimes_R \mathbf{L}$  is acyclic for every injective  $R$ -module  $E$ .*

*Proof.* The Lemma is proved in [28], but here is a cleaner argument: Let  $F$  be a flat module and  $E$  be an injective module. As  $\mathbf{L}$  consists of finitely generated projective modules, there is an isomorphism of complexes

$$\text{Hom}_R(\text{Hom}_R(\mathbf{L}, F), E) \cong \text{Hom}_R(F, E) \otimes_R \mathbf{L}.$$

It follows from this isomorphism, applied to  $F = R$ , that (i) implies (iii). Applied to a faithfully injective module  $E$ , it shows that (iii) implies (ii), as  $\text{Hom}_R(F, E)$  is an injective module. It is evident that (ii) implies (i).  $\square$

The following nomenclature is due to Avramov and Martsinkovsky [19]; Lemma 1.6 clarifies the rationale behind it.

**Definition 1.2.** A complex  $\mathbf{L}$  that satisfies the conditions in Lemma 1.1 is called *totally acyclic*. An  $R$ -module  $M$  is called *totally reflexive* if there exists a totally acyclic complex  $\mathbf{L}$  such that  $M$  is isomorphic to  $\text{Coker}(\mathbf{L}_1 \rightarrow \mathbf{L}_0)$ .

Note that every finitely generated projective module  $L$  is totally reflexive; indeed, the complex  $0 \rightarrow L \xrightarrow{\cong} L \rightarrow 0$ , with  $L$  in homological degrees 0 and  $-1$ , is totally acyclic.

*Example 1.3.* If there exist elements  $x$  and  $y$  in  $R$  such that  $\text{Ann}_R(x) = (y)$  and  $\text{Ann}_R(y) = (x)$ , then the complex

$$\dots \xrightarrow{x} R \xrightarrow{y} R \xrightarrow{x} R \xrightarrow{y} \dots$$

is totally acyclic. Thus,  $(x)$  and  $(y)$  are totally reflexive  $R$ -modules. For instance, if  $X$  and  $Y$  are non-zero non-units in an integral domain  $D$ , then their residue classes  $x$  and  $y$  in  $R = D/(XY)$  generate totally reflexive  $R$ -modules.

An elementary construction of rings of this kind—Example 1.4 below—shows that non-projective totally reflexive modules may exist over a variety of rings; see also Problem 1.30.

*Example 1.4.* Let  $S$  be a commutative ring, and let  $m > 1$  be an integer. Set  $R = S[X]/(X^m)$ , and denote by  $x$  the residue class of  $X$  in  $R$ . Then for every integer  $n$  between 1 and  $m - 1$ , the module  $(x^n)$  is totally reflexive.

From Lemma 1.1 it is straightforward to deduce:

**Proposition 1.5.** *Let  $S$  be an  $R$ -algebra of finite flat dimension. For every totally reflexive  $R$ -module  $G$ , the module  $S \otimes_R G$  is totally reflexive over  $S$ .*

Proposition 1.5 applies to  $S = R/(\mathbf{x})$ , where  $\mathbf{x}$  is an  $R$ -regular sequence. If  $(R, \mathfrak{m})$  is local, then it also applies to the  $\mathfrak{m}$ -adic completion  $S = \widehat{R}$ .

## Noetherian rings

Recall that a finitely generated  $R$ -module  $M$  is called *reflexive* if the canonical map from  $M$  to  $\text{Hom}_R(\text{Hom}_R(M, R), R)$  is an isomorphism. The following characterization of totally reflexive modules goes back to [6, 4.11].

**Lemma 1.6.** *Let  $R$  be Noetherian. A finitely generated  $R$ -module  $G$  is totally reflexive if and only if it is reflexive and for every  $i \geq 1$  one has*

$$\text{Ext}_R^i(G, R) = 0 = \text{Ext}_R^i(\text{Hom}_R(G, R), R).$$

**Definition 1.7.** An (augmented)  $G$ -resolution of a finitely generated module  $M$  is an exact sequence  $\cdots \rightarrow G_i \rightarrow G_{i-1} \rightarrow \cdots \rightarrow G_0 \rightarrow M \rightarrow 0$ , where each module  $G_i$  is totally reflexive.

Note that if  $R$  is Noetherian, then every finitely generated  $R$ -module has a  $G$ -resolution, indeed it has a resolution by finitely generated free modules.

**Definition 1.8.** Let  $R$  be Noetherian. For a finitely generated  $R$ -module  $M \neq 0$  the  $G$ -dimension, denoted by  $\text{G-dim}_R M$ , is the least integer  $n \geq 0$  such that there exists a  $G$ -resolution of  $M$  with  $G_i = 0$  for all  $i > n$ . If no such  $n$  exists, then  $\text{G-dim}_R M$  is infinite. By convention, set  $\text{G-dim}_R 0 = -\infty$ .

The ‘ $G$ ’ in the definition above is short for Gorenstein.

In [6, Chap. 3] one finds the next theorem and its corollary; see also [28, 1.2.7].

**Theorem 1.9.** *Let  $R$  be Noetherian and  $M$  be a finitely generated  $R$ -module of finite  $G$ -dimension. For every  $n \geq 0$  the next conditions are equivalent.*

- (i)  $\text{G-dim}_R M \leq n$ .
- (ii)  $\text{Ext}_R^i(M, R) = 0$  for all  $i > n$ .
- (iii)  $\text{Ext}_R^i(M, N) = 0$  for all  $i > n$  and all  $R$ -modules  $N$  with  $\text{fd}_R N$  finite.
- (iv) In every augmented  $G$ -resolution

$$\cdots \rightarrow G_i \rightarrow G_{i-1} \rightarrow \cdots \rightarrow G_0 \rightarrow M \rightarrow 0$$

the module  $\text{Coker}(G_{n+1} \rightarrow G_n)$  is totally reflexive.

**Corollary 1.10.** *Let  $R$  be Noetherian. For every finitely generated  $R$ -module  $M$  of finite  $G$ -dimension there is an equality*

$$\text{G-dim}_R M = \sup\{i \in \mathbb{Z} \mid \text{Ext}_R^i(M, R) \neq 0\}.$$

*Remark 1.11.* Examples due to Jorgensen and Şega [77] show that in Corollary 1.10 one cannot avoid the *a priori* condition that  $\text{G-dim}_R M$  is finite.

*Remark 1.12.* For a module  $M$  as in Corollary 1.10, the small finitistic projective dimension of  $R$  is an upper bound for  $\text{G-dim}_R M$ ; cf. Christensen and Iyengar [33, 3.1(a)].

A standard argument, see [6, 3.16] or [19, 3.4], yields:

**Proposition 1.13.** *Let  $R$  be Noetherian. If any two of the modules in an exact sequence  $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$  of finitely generated  $R$ -modules have finite  $G$ -dimension, then so has the third.*

The following quantitative comparison establishes the  $G$ -dimension as a refinement of the projective dimension for finitely generated modules. It is easily deduced from Corollary 1.10; see [28, 1.2.10].

**Proposition 1.14.** *Let  $R$  be Noetherian. For every finitely generated  $R$ -module  $M$  one has  $G\text{-dim}_R M \leq \text{pd}_R M$ , and equality holds if  $\text{pd}_R M$  is finite.*

By [6, 4.15] the  $G$ -dimension of a module can be measured locally:

**Proposition 1.15.** *Let  $R$  be Noetherian. For every finitely generated  $R$ -module  $M$  there is an equality  $G\text{-dim}_R M = \sup\{G\text{-dim}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} \mid \mathfrak{p} \in \text{Spec } R\}$ .*

For the projective dimension even more is known: Bass and Murthy [21, 4.5] prove that if a finitely generated module over a Noetherian ring has finite projective dimension locally, then it has finite projective dimension globally—even if the ring has infinite Krull dimension. A Gorenstein counterpart has recently been established by Avramov, Iyengar, and Lipman [18, 6.3.4].

**Theorem 1.16.** *Let  $R$  be Noetherian and let  $M$  be a finitely generated  $R$ -module. If  $G\text{-dim}_{R_{\mathfrak{m}}} M_{\mathfrak{m}}$  is finite for every maximal ideal  $\mathfrak{m}$  in  $R$ , then  $G\text{-dim}_R M$  is finite.*

Recall that a local ring is called *Gorenstein* if it has finite self-injective dimension. A Noetherian ring is Gorenstein if its localization at each prime ideal is a Gorenstein local ring, that is,  $\text{id}_{R_{\mathfrak{p}}} R_{\mathfrak{p}}$  is finite for every prime ideal  $\mathfrak{p}$  in  $R$ . Consequently, the self-injective dimension of a Gorenstein ring equals its Krull dimension; that is  $\text{id}_R R = \dim R$ . The next result follows from [6, 4.20] in combination with Proposition 1.15.

**Theorem 1.17.** *Let  $R$  be Noetherian and  $n \geq 0$  be an integer. Then  $R$  is Gorenstein with  $\dim R \leq n$  if and only if one has  $G\text{-dim}_R M \leq n$  for every finitely generated  $R$ -module  $M$ .*

A corollary to Theorem 1.16 was established by Goto [63] already in 1982; it asserts that also Gorenstein rings of infinite Krull dimension are characterized by finiteness of  $G$ -dimension.

**Theorem 1.18.** *Let  $R$  be Noetherian. Then  $R$  is Gorenstein if and only if every finitely generated  $R$ -module has finite  $G$ -dimension.*

Recall that the *grade* of a finitely generated module  $M$  over a Noetherian ring  $R$  can be defined as follows:

$$\text{grade}_R M = \inf\{i \in \mathbb{Z} \mid \text{Ext}_R^i(M, R) \neq 0\}.$$

Foxby [55] makes the following:

**Definition 1.19.** Let  $R$  be Noetherian. A finitely generated  $R$ -module  $M$  is called *quasi-perfect* if it has finite G-dimension equal to  $\text{grade}_R M$ .

The next theorem applies to  $S = R/(x)$ , where  $x$  is an  $R$ -regular sequence. Special (local) cases of the theorem are due to Golod [62] and to Avramov and Foxby [13, 7.11]. Christensen's proof [29, 6.5] establishes the general case.

**Theorem 1.20.** *Let  $R$  be Noetherian and  $S$  be a commutative Noetherian module-finite  $R$ -algebra. If  $S$  is a quasi-perfect  $R$ -module of grade  $g$  such that  $\text{Ext}_R^g(S, R) \cong S$ , then the next equality holds for every finitely generated  $S$ -module  $N$ ,*

$$\text{G-dim}_R N = \text{G-dim}_S N + \text{G-dim}_R S.$$

Note that an  $S$ -module has finite G-dimension over  $R$  if and only if it has finite G-dimension over  $S$ ; see also Theorem 7.7. The next question is raised in [13]; it asks if the assumption of quasi-perfectness in Theorem 1.20 is necessary.

*Question 1.21.* Let  $R$  be Noetherian, let  $S$  be a commutative Noetherian module-finite  $R$ -algebra, and let  $N$  be a finitely generated  $S$ -module. If  $\text{G-dim}_S N$  and  $\text{G-dim}_R S$  are finite, is then  $\text{G-dim}_R N$  finite?

This is known as the *Transitivity Question*. By [13, 4.7] and [29, 3.15 and 6.5] it has an affirmative answer if  $\text{pd}_S N$  is finite; see also Theorem 7.4.

## Local rings

Before we proceed with results on G-dimension of modules over local rings, we make a qualitative comparison to the projective dimension. Theorem 1.20 reveals a remarkable property of the G-dimension, one that has almost no counterpart for the projective dimension. Here is an example:

*Example 1.22.* Let  $(R, \mathfrak{m}, k)$  be local of positive depth. Pick a regular element  $x$  in  $\mathfrak{m}$  and set  $S = R/(x)$ . Then one has  $\text{grade}_R S = 1 = \text{pd}_R S$  and  $\text{Ext}_R^1(S, R) \cong S$ , but  $\text{pd}_S N$  is infinite for every  $S$ -module  $N$  such that  $x$  is in  $\mathfrak{m} \text{Ann}_R N$ ; see Shamash [99, Section 3]. In particular, if  $R$  is regular and  $x$  is in  $\mathfrak{m}^2$ , then  $S$  is not regular, so  $\text{pd}_S k$  is infinite while  $\text{pd}_R k$  is finite; see Theorem A.

If  $G$  is a totally reflexive  $R$ -module, then every  $R$ -regular element is  $G$ -regular. A strong converse holds for modules of finite projective dimension; it is (still) referred to as *Auslander's zero-divisor conjecture*: let  $R$  be local and  $M \neq 0$  be a finitely generated  $R$ -module with  $\text{pd}_R M$  finite. Then every  $M$ -regular element is  $R$ -regular; for a proof see Roberts [88, 6.2.3]. An instance of Example 1.3 shows that one can *not* relax the condition on  $M$  to finite G-dimension:

*Example 1.23.* Let  $k$  be a field and consider the local ring  $R = k[[X, Y]]/(XY)$ . Then the residue class  $x$  of  $X$  generates a totally reflexive module. The element  $x$  is  $(x)$ -regular but nevertheless a zero-divisor in  $R$ .

While a tensor product of projective modules is projective, the next example shows that totally reflexive modules do not have an analogous property.

*Example 1.24.* Let  $R$  be as in Example 1.23. The  $R$ -modules  $(x)$  and  $(y)$  are totally reflexive, but  $(x) \otimes_R (y) \cong k$  is not. Indeed,  $k$  is not a submodule of a free  $R$ -module.

The next result [6, 4.13] is parallel to Theorem B in the Introduction; it is known as the *Auslander–Bridger Formula*.

**Theorem 1.25.** *Let  $R$  be local. For every finitely generated  $R$ -module  $M$  of finite  $G$ -dimension there is an equality*

$$G\text{-dim}_R M = \text{depth} R - \text{depth}_R M.$$

In [84] Mašek corrects the proof of [6, 4.13]. Proofs can also be found in [5] and [28].

By Lemma 1.6 the  $G$ -dimension is preserved under completion:

**Proposition 1.26.** *Let  $R$  be local. For every finitely generated  $R$ -module  $M$  there is an equality*

$$G\text{-dim}_R M = G\text{-dim}_{\widehat{R}}(\widehat{R} \otimes_R M).$$

The following main result from [5, Section 3.2] is akin to Theorem A, but it differs in that it only deals with finitely generated modules.

**Theorem 1.27.** *For a local ring  $(R, \mathfrak{m}, k)$  the next conditions are equivalent.*

- (i)  $R$  is Gorenstein.
- (ii)  $G\text{-dim}_R k$  is finite.
- (iii)  $G\text{-dim}_R M$  is finite for every finitely generated  $R$ -module  $M$ .

It follows that non-projective totally reflexive modules exist over any non-regular Gorenstein local ring. On the other hand, Example 1.4 shows that existence of such modules does not identify the ground ring as a member of one of the standard classes, say, Cohen–Macaulay rings.

A theorem of Christensen, Piepmeyer, Striuli, and Takahashi [34, 4.3] shows that fewness of totally reflexive modules comes in two distinct flavors:

**Theorem 1.28.** *Let  $R$  be local. If there are only finitely many indecomposable totally reflexive  $R$ -modules, up to isomorphism, then  $R$  is Gorenstein or every totally reflexive  $R$ -module is free.*

This dichotomy brings two problems to light:

*Problem 1.29.* Let  $R$  be a local ring that is not Gorenstein and assume that there exists a non-free totally reflexive  $R$ -module. Find constructions that produce infinite families of non-isomorphic indecomposable totally reflexive modules.

*Problem 1.30.* Describe the local rings over which every totally reflexive module is free.

While the first problem is posed in [34], the second one was already raised by Avramov and Martsinkovsky [19], and it is proved *ibid.* that over a Golod local ring that is not Gorenstein, every totally reflexive module is free. Another partial answer to Problem 1.30 is obtained by Yoshino [116], and by Christensen and Veliche [36]. The problem is also studied by Takahashi in [105].

Finally, Theorem D in the Introduction motivates:

*Question 1.31.* Let  $R$  be a local ring. If there exists a non-zero  $R$ -module of finite length and finite G-dimension, is then  $R$  Cohen–Macaulay?

A partial answer to this question is obtained by Takahashi [101, 2.3].

### Notes

A topic that was only treated briefly above is constructions of totally reflexive modules. Such constructions are found in [16] by Avramov, Gasharov and Peeva, in work of Takahashi and Watanabe [106], and in Yoshino’s [116].

Hummel and Marley [73] extend the notion of G-dimension to finitely presented modules over coherent rings and use it to define and study coherent Gorenstein rings.

Gerko [61, Section 2] studies a dimension—the PCI-dimension or  $CI_*$ -dimension—based on a subclass of the totally reflexive modules. Golod [62] studies a generalized notion of G-dimension: the  $G_C$ -dimension, based on total reflexivity with respect to a semidualizing module  $C$ . These studies are continued by, among others, Gerko [61, Section 1] and Salarian, Sather-Wagstaff, and Yassemi [91]; see also the notes in Section 8.

An approach to homological dimensions that is not treated in this survey is based on so-called quasi-deformations. Several authors—among them Avramov, Gasharov, and Peeva [16] and Veliche [109]—take this approach to define homological dimensions that are intermediate between the projective dimension and the G-dimension for finitely generated modules. Gerko [61, Section 3] defines a Cohen–Macaulay dimension, which is a refinement of the G-dimension. Avramov [10, Section 8] surveys these dimensions.

## 2 Gorenstein projective dimension

To extend the G-dimension beyond the realm of finitely generated modules over Noetherian rings, Enochs and Jenda [41] introduced the notion of Gorenstein projective modules. The same authors, and their collaborators, studied these modules in several subsequent papers. The associated dimension, which is the focus of this section, was studied by Christensen [28] and Holm [66].

In organization, this section is parallel to Section 1.

**Definition 2.1.** An  $R$ -module  $A$  is called *Gorenstein projective* if there exists an acyclic complex  $\mathbf{P}$  of projective  $R$ -modules such that  $\text{Coker}(P_1 \rightarrow P_0) \cong A$  and such that  $\text{Hom}_R(\mathbf{P}, Q)$  is acyclic for every projective  $R$ -module  $Q$ .

It is evident that every projective module is Gorenstein projective.

*Example 2.2.* Every totally reflexive module is Gorenstein projective; this follows from Definition 1.2 and Lemma 1.1.

Basic categorical properties are recorded in [66, Section 2]:

**Proposition 2.3.** *The class of Gorenstein projective  $R$ -modules is closed under direct sums and summands.*

Every projective module is a direct summand of a free one. A parallel result for Gorenstein projective modules, Theorem 2.5 below, is due to Bennis and Mahdou [24, Section 2]; as substitute for free modules they define:

**Definition 2.4.** An  $R$ -module  $A$  is called *strongly Gorenstein projective* if there exists an acyclic complex  $\mathbf{P}$  of projective  $R$ -modules, in which all the differentials are identical, such that  $\text{Coker}(P_1 \rightarrow P_0) \cong A$ , and such that  $\text{Hom}_R(\mathbf{P}, Q)$  is acyclic for every projective  $R$ -module  $Q$ .

**Theorem 2.5.** *An  $R$ -module is Gorenstein projective if and only if it is a direct summand of a strongly Gorenstein projective  $R$ -module.*

**Definition 2.6.** An (augmented) *Gorenstein projective resolution* of a module  $M$  is an exact sequence  $\cdots \rightarrow A_i \rightarrow A_{i-1} \rightarrow \cdots \rightarrow A_0 \rightarrow M \rightarrow 0$ , where each module  $A_i$  is Gorenstein projective.

Note that every module has a Gorenstein projective resolution, as a free resolution is trivially a Gorenstein projective one.

**Definition 2.7.** The *Gorenstein projective dimension* of a module  $M \neq 0$ , denoted by  $\text{Gpd}_R M$ , is the least integer  $n \geq 0$  such that there exists a Gorenstein projective resolution of  $M$  with  $A_i = 0$  for all  $i > n$ . If no such  $n$  exists, then  $\text{Gpd}_R M$  is infinite. By convention, set  $\text{Gpd}_R 0 = -\infty$ .

In [66, Section 2] one finds the next standard theorem and corollary.

**Theorem 2.8.** *Let  $M$  be an  $R$ -module of finite Gorenstein projective dimension. For every integer  $n \geq 0$  the following conditions are equivalent.*

- (i)  $\text{Gpd}_R M \leq n$ .
- (ii)  $\text{Ext}_R^i(M, Q) = 0$  for all  $i > n$  and all projective  $R$ -modules  $Q$ .
- (iii)  $\text{Ext}_R^i(M, N) = 0$  for all  $i > n$  and all  $R$ -modules  $N$  with  $\text{pd}_R N$  finite.
- (iv) *In every augmented Gorenstein projective resolution*

$$\cdots \rightarrow A_i \rightarrow A_{i-1} \rightarrow \cdots \rightarrow A_0 \rightarrow M \rightarrow 0$$

*the module  $\text{Coker}(A_{n+1} \rightarrow A_n)$  is Gorenstein projective.*

**Corollary 2.9.** *For every  $R$ -module  $M$  of finite Gorenstein projective dimension there is an equality*

$$\text{Gpd}_R M = \sup\{i \in \mathbb{Z} \mid \text{Ext}_R^i(M, Q) \neq 0 \text{ for some projective } R\text{-module } Q\}.$$

*Remark 2.10.* For every  $R$ -module  $M$  as in the corollary, the finitistic projective dimension of  $R$  is an upper bound for  $\text{Gpd}_R M$ ; see [66, 2.28].

The next result [66, 2.24] extends Proposition 1.13.

**Proposition 2.11.** *Let  $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$  be an exact sequence of  $R$ -modules. If any two of the modules have finite Gorenstein projective dimension, then so has the third.*

The Gorenstein projective dimension is a refinement of the projective dimension; this follows from Corollary 2.9:

**Proposition 2.12.** *For every  $R$ -module  $M$  one has  $\text{Gpd}_R M \leq \text{pd}_R M$ , and equality holds if  $M$  has finite projective dimension.*

Supplementary information comes from Holm [67, 2.2]:

**Proposition 2.13.** *If  $M$  is an  $R$ -module of finite injective dimension, then there is an equality  $\text{Gpd}_R M = \text{pd}_R M$ .*

The next result of Foxby is published in [32, Ascent table II(b)].

**Proposition 2.14.** *Let  $S$  be an  $R$ -algebra of finite projective dimension. For every Gorenstein projective  $R$ -module  $A$ , the module  $S \otimes_R A$  is Gorenstein projective over  $S$ .*

## Noetherian rings

Finiteness of the Gorenstein projective dimension characterizes Gorenstein rings. The next result of Enochs and Jenda [43, 12.3.1] extends Theorem 1.17.

**Theorem 2.15.** *Let  $R$  be Noetherian and  $n \geq 0$  be an integer. Then  $R$  is Gorenstein with  $\dim R \leq n$  if and only if  $\text{Gpd}_R M \leq n$  for every  $R$ -module  $M$ .*

The next result [28, 4.2.6] compares the Gorenstein projective dimension to the G-dimension.

**Proposition 2.16.** *Let  $R$  be Noetherian. For every finitely generated  $R$ -module  $M$  there is an equality  $\text{Gpd}_R M = \text{G-dim}_R M$ .*

The Gorenstein projective dimension of a module can not be measured locally; that is, Proposition 1.15 does not extend to non-finitely generated modules. As a consequence of Proposition 2.14, though, one has the following:

**Proposition 2.17.** *Let  $R$  be Noetherian of finite Krull dimension. For every  $R$ -module  $M$  and every prime ideal  $\mathfrak{p}$  in  $R$  one has  $\text{Gpd}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} \leq \text{Gpd}_R M$ .*

Theorem E and Proposition 2.13 yield:

**Theorem 2.18.** *Let  $R$  be Noetherian and  $M$  a finitely generated  $R$ -module. If  $\text{Gpd}_R M$  and  $\text{id}_R M$  are finite, then  $R_{\mathfrak{p}}$  is Gorenstein for all  $\mathfrak{p} \in \text{Supp}_R M$ .*



## Local rings

The next characterization of Gorenstein local rings—akin to Theorem A in the Introduction—follows from Theorems 1.27 and 2.15 via Proposition 2.16.

**Theorem 2.19.** *For a local ring  $(R, \mathfrak{m}, k)$  the next conditions are equivalent.*

- (i)  $R$  is Gorenstein.
- (ii)  $\text{Gpd}_R k$  is finite.
- (iii)  $\text{Gpd}_R M$  is finite for every  $R$ -module  $M$ .

The inequality in the next theorem is a consequence of Proposition 2.14. The second assertion is due to Esmkhani and Tousi [52, 3.5], cf. [31, 4.1]. The result should be compared to Proposition 1.26.

**Theorem 2.20.** *Let  $R$  be local and  $M$  be an  $R$ -module. Then one has*

$$\text{Gpd}_{\widehat{R}}(\widehat{R} \otimes_R M) \leq \text{Gpd}_R M,$$

*and if  $\text{Gpd}_{\widehat{R}}(\widehat{R} \otimes_R M)$  is finite, then so is  $\text{Gpd}_R M$ .*

### Notes

Holm and Jørgensen [69] extend Golod’s [62] notion of  $G_C$ -dimension to non-finitely generated modules in the form of a  $C$ -Gorenstein projective dimension. Further studies of this dimension are made by White [112].

## 3 Gorenstein injective dimension

The notion of Gorenstein injective modules is (categorically) dual to that of Gorenstein projective modules. The two were introduced in the same paper by Enochs and Jenda [41] and investigated in subsequent works by the same authors, by Christensen and Sather-Wagstaff [35], and by Holm [66].

This section is structured parallelly to the previous ones.

**Definition 3.1.** An  $R$ -module  $B$  is called *Gorenstein injective* if there exists an acyclic complex  $I$  of injective  $R$ -modules such that  $\text{Ker}(I^0 \rightarrow I^1) \cong B$ , and such that  $\text{Hom}_R(E, I)$  is acyclic for every injective  $R$ -module  $E$ .

It is clear that every injective module is Gorenstein injective.

*Example 3.2.* Let  $L$  be a totally acyclic complex of finitely generated projective  $R$ -modules, see Definition 1.2, and let  $I$  be an injective  $R$ -module. Then the acyclic complex  $I = \text{Hom}_R(L, I)$  consists of injective modules, and from Lemma 1.1 it follows that the complex  $\text{Hom}_R(E, I) \cong \text{Hom}_R(E \otimes_R L, I)$  is acyclic for every injective module  $E$ . Thus, if  $G$  is a totally reflexive  $R$ -module and  $I$  is injective, then the module  $\text{Hom}_R(G, I)$  is Gorenstein injective.

Basic categorical properties are established in [66, 2.6]:

**Proposition 3.3.** *The class of Gorenstein injective  $R$ -modules is closed under direct products and summands.*

Under extra assumptions on the ring, Theorem 3.16 gives more information.

**Definition 3.4.** An (augmented) *Gorenstein injective resolution* of a module  $M$  is an exact sequence  $0 \rightarrow M \rightarrow B^0 \rightarrow \dots \rightarrow B^{i-1} \rightarrow B^i \rightarrow \dots$ , where each module  $B^i$  is Gorenstein injective.

Note that every module has a Gorenstein injective resolution, as an injective resolution is trivially a Gorenstein injective one.

**Definition 3.5.** The *Gorenstein injective dimension* of an  $R$ -module  $M \neq 0$ , denoted by  $\text{Gid}_R M$ , is the least integer  $n \geq 0$  such that there exists a Gorenstein injective resolution of  $M$  with  $B^i = 0$  for all  $i > n$ . If no such  $n$  exists, then  $\text{Gid}_R M$  is infinite. By convention, set  $\text{Gid}_R 0 = -\infty$ .

The next standard theorem is [66, 2.22].

**Theorem 3.6.** *Let  $M$  be an  $R$ -module of finite Gorenstein injective dimension. For every integer  $n \geq 0$  the following conditions are equivalent.*

- (i)  $\text{Gid}_R M \leq n$ .
- (ii)  $\text{Ext}_R^i(E, M) = 0$  for all  $i > n$  and all injective  $R$ -modules  $E$ .
- (iii)  $\text{Ext}_R^i(N, M) = 0$  for all  $i > n$  and all  $R$ -modules  $N$  with  $\text{id}_R N$  finite.
- (iv) In every augmented Gorenstein injective resolution

$$0 \rightarrow M \rightarrow B^0 \rightarrow \dots \rightarrow B^{i-1} \rightarrow B^i \rightarrow \dots$$

the module  $\text{Ker}(B^n \rightarrow B^{n+1})$  is Gorenstein injective.

**Corollary 3.7.** *For every  $R$ -module  $M$  of finite Gorenstein injective dimension there is an equality*

$$\text{Gid}_R M = \sup\{i \in \mathbb{Z} \mid \text{Ext}_R^i(E, M) \neq 0 \text{ for some injective } R\text{-module } E\}.$$

*Remark 3.8.* For every  $R$ -module  $M$  as in the corollary, the finitistic injective dimension of  $R$  is an upper bound for  $\text{Gid}_R M$ ; see [66, 2.29].

The next result [66, 2.25] is similar to Proposition 2.11.

**Proposition 3.9.** *Let  $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$  be an exact sequence of  $R$ -modules. If any two of the modules have finite Gorenstein injective dimension, then so has the third.*

The Gorenstein injective dimension is a refinement of the injective dimension; this follows from Corollary 3.7:

**Proposition 3.10.** *For every  $R$ -module  $M$  one has  $\text{Gid}_R M \leq \text{id}_R M$ , and equality holds if  $M$  has finite injective dimension.*

Supplementary information comes from Holm [67, 2.1]:

**Proposition 3.11.** *If  $M$  is an  $R$ -module of finite projective dimension, then there is an equality  $\text{Gid}_R M = \text{id}_R M$ . In particular, one has  $\text{Gid}_R R = \text{id}_R R$ .*

In [32] Christensen and Holm study (co)base change of modules of finite Gorenstein homological dimension. The following is elementary to verify:

**Proposition 3.12.** *Let  $S$  be an  $R$ -algebra of finite projective dimension. For every Gorenstein injective  $R$ -module  $B$ , the module  $\text{Hom}_R(S, B)$  is Gorenstein injective over  $S$ .*

For a conditional converse see Theorems 3.27 and 9.11.

The next result of Bennis, Mahdou, and Ouarghi [25, 2.2] should be compared to characterizations of Gorenstein rings like Theorems 2.15 and 3.14, and also to Theorems 2.18 and 3.21. It is a perfect Gorenstein counterpart to a classical result due to Faith and Walker among others; see e.g. [111, 4.2.4].

**Theorem 3.13.** *The following conditions on  $R$  are equivalent.*

- (i)  $R$  is quasi-Frobenius.
- (ii) Every  $R$ -module is Gorenstein projective.
- (iii) Every  $R$ -module is Gorenstein injective.
- (iv) Every Gorenstein projective  $R$ -module is Gorenstein injective.
- (v) Every Gorenstein injective  $R$ -module is Gorenstein projective.

## Noetherian rings

Finiteness of the Gorenstein injective dimension characterizes Gorenstein rings; this result is due to Enochs and Jenda [42, 3.1]:

**Theorem 3.14.** *Let  $R$  be Noetherian and  $n \geq 0$  be an integer. Then  $R$  is Gorenstein with  $\dim R \leq n$  if and only if  $\text{Gid}_R M \leq n$  for every  $R$ -module  $M$ .*

A ring is Noetherian if every countable direct sum of injective modules is injective (and only if every direct limit of injective modules is injective). The “if” part has a perfect Gorenstein counterpart:

**Proposition 3.15.** *If every countable direct sum of Gorenstein injective  $R$ -modules is Gorenstein injective, then  $R$  is Noetherian.*

*Proof.* It is sufficient to see that every countable direct sum of injective  $R$ -modules is injective. Let  $\{E_n\}_{n \in \mathbb{N}}$  be a family of injective modules. By assumption, the module  $\bigoplus E_n$  is Gorenstein injective; in particular, there is an epimorphism  $\varphi: I \rightarrow \bigoplus E_n$  such that  $I$  is injective and  $\text{Hom}_R(E, \varphi)$  is surjective for every injective  $R$ -module  $E$ . Applying this to  $E = E_n$  it is elementary to verify that  $\varphi$  is a split epimorphism.  $\square$

Christensen, Frankild, and Holm [31, 6.9] provide a partial converse:

**Theorem 3.16.** *Assume that  $R$  is a homomorphic image of a Gorenstein ring of finite Krull dimension. Then the class of Gorenstein injective modules is closed under direct limits; in particular, it is closed under direct sums.*

As explained in the Introduction, the hypothesis on  $R$  in this theorem ensures the existence of a dualizing  $R$ -complex and an associated Bass class, cf. Section 9. These tools are essential to the known proof of Theorem 3.16.

*Question 3.17.* Let  $R$  be Noetherian. Is then every direct limit of Gorenstein injective  $R$ -modules Gorenstein injective?

Next follows a Gorenstein version of Chouinard's formula [27, 3.1]; it is proved in [35, 2.2]. Recall that the *width* of a module  $M$  over a local ring  $(R, \mathfrak{m}, k)$  is defined as

$$\text{width}_R M = \inf\{i \in \mathbb{Z} \mid \text{Tor}_i^R(k, M) \neq 0\}.$$

**Theorem 3.18.** *Let  $R$  be Noetherian. For every  $R$ -module  $M$  of finite Gorenstein injective dimension there is an equality*

$$\text{Gid}_R M = \{\text{depth } R_{\mathfrak{p}} - \text{width}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} \mid \mathfrak{p} \in \text{Spec } R\}.$$

Let  $M$  be an  $R$ -module, and let  $\mathfrak{p}$  be a prime ideal in  $R$ . Provided that  $\text{Gid}_{R_{\mathfrak{p}}} M_{\mathfrak{p}}$  is finite, the inequality  $\text{Gid}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} \leq \text{Gid}_R M$  follows immediately from the theorem. However, the next question remains open.

*Question 3.19.* Let  $R$  be Noetherian and  $B$  be a Gorenstein injective  $R$ -module. Is then  $B_{\mathfrak{p}}$  Gorenstein injective over  $R_{\mathfrak{p}}$  for every prime ideal  $\mathfrak{p}$  in  $R$ ?

A partial answer is known from [31, 5.5]:

**Proposition 3.20.** *Assume that  $R$  is a homomorphic image of a Gorenstein ring of finite Krull dimension. For every  $R$ -module  $M$  and every prime ideal  $\mathfrak{p}$  there is an inequality  $\text{Gid}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} \leq \text{Gid}_R M$ .*

Theorem E and Proposition 3.11 yield:

**Theorem 3.21.** *Let  $R$  be Noetherian and  $M$  a finitely generated  $R$ -module. If  $\text{Gid}_R M$  and  $\text{pd}_R M$  are finite, then  $R_{\mathfrak{p}}$  is Gorenstein for all  $\mathfrak{p} \in \text{Supp}_R M$ .*

## Local rings

The following theorem of Foxby and Frankild [58, 4.5] generalizes work of Peskine and Szpiro [86], cf. Theorem E.

**Theorem 3.22.** *A local ring  $R$  is Gorenstein if and only if there exists a non-zero cyclic  $R$ -module of finite Gorenstein injective dimension.*

Theorems 3.14 and 3.22 yield a parallel to Theorem 1.27, akin to Theorem A.

**Corollary 3.23.** *For a local ring  $(R, \mathfrak{m}, k)$  the next conditions are equivalent.*

- (i)  $R$  is Gorenstein.
- (ii)  $\text{Gid}_R k$  is finite.
- (iii)  $\text{Gid}_R M$  is finite for every  $R$ -module  $M$ .

The first part of the next theorem is due to Christensen, Frankild, and Iyengar, and published in [58, 3.6]. The equality in Theorem 3.24—the Gorenstein analogue of Theorem C in the Introduction—is proved by Khatami, Tousi, and Yassemi [80, 2.5]; see also [35, 2.3].

**Theorem 3.24.** *Let  $R$  be local and  $M \neq 0$  be a finitely generated  $R$ -module. Then  $\text{Gid}_R M$  and  $\text{Gid}_{\widehat{R}}(\widehat{R} \otimes_R M)$  are simultaneously finite, and when they are finite, there is an equality*

$$\text{Gid}_R M = \text{depth} R.$$

*Remark 3.25.* Let  $R$  be local and  $M \neq 0$  be an  $R$ -module. If  $M$  has finite length and finite G-dimension, then its Matlis dual has finite Gorenstein injective dimension, cf. Example 3.2. See also Takahashi [103].

This remark and Theorem D from the Introduction motivate:

*Question 3.26.* Let  $R$  be local. If there exists a non-zero finitely generated  $R$ -module of finite Gorenstein injective dimension, is then  $R$  Cohen–Macaulay?

A partial answer to this question is given by Yassemi [115, 1.3].

Esmkhani and Tousi [53, 2.5] prove the following conditional converse to Proposition 3.12. Recall that an  $R$ -module  $M$  is said to be *cotorsion* if  $\text{Ext}_R^1(F, M) = 0$  for every flat  $R$ -module  $F$ .

**Theorem 3.27.** *Let  $R$  be local. An  $R$ -module  $M$  is Gorenstein injective if and only if it is cotorsion and  $\text{Hom}_R(\widehat{R}, M)$  is Gorenstein injective over  $\widehat{R}$ .*

The example below demonstrates the necessity of the cotorsion hypothesis. Working in the derived category one obtains a stronger result; see Theorem 9.11.

*Example 3.28.* Let  $(R, \mathfrak{m})$  be a local domain which is not  $\mathfrak{m}$ -adically complete. Aldrich, Enochs, and López-Ramos [1, 3.3] show that the module  $\text{Hom}_R(\widehat{R}, R)$  is zero and hence Gorenstein injective over  $\widehat{R}$ . However,  $\text{Gid}_R R$  is infinite if  $R$  is not Gorenstein, cf. Proposition 3.11.

### Notes

Dual to the notion of strongly Gorenstein projective modules, see Definition 2.4, Bennis and Mahdou [24] also study strongly Gorenstein injective modules.

Several authors—Asadollahi, Sahandi, Salarian, Sazeedeh, Sharif, and Yassemi—have studied the relationship between Gorenstein injectivity and local cohomology; see [3], [90], [96], [97], and [115].

## 4 Gorenstein flat dimension

Another extension of the G-dimension is based on Gorenstein flat modules—a notion due to Enochs, Jenda, and Torrecillas [47]. Christensen [28] and Holm [66] are other main references for this section.

The organization of this section follows the pattern from Sections 1–3.

**Definition 4.1.** An  $R$ -module  $A$  is called *Gorenstein flat* if there exists an acyclic complex  $F$  of flat  $R$ -modules such that  $\text{Coker}(F_1 \rightarrow F_0) \cong A$ , and such that  $E \otimes_R F$  is acyclic for every injective  $R$ -module  $E$ .

It is evident that every flat module is Gorenstein flat.

*Example 4.2.* Every totally reflexive module is Gorenstein flat; this follows from Definition 1.2 and Lemma 1.1.

Here is a direct consequence of Definition 4.1:

**Proposition 4.3.** *The class of Gorenstein flat  $R$ -modules is closed under direct sums.*

See Theorems 4.13 and 4.14 for further categorical properties of Gorenstein flat modules.

**Definition 4.4.** An (augmented) *Gorenstein flat resolution* of a module  $M$  is an exact sequence  $\cdots \rightarrow A_i \rightarrow A_{i-1} \rightarrow \cdots \rightarrow A_0 \rightarrow M \rightarrow 0$ , where each module  $A_i$  is Gorenstein flat.

Note that every module has a Gorenstein flat resolution, as a free resolution is trivially a Gorenstein flat one.

**Definition 4.5.** The *Gorenstein flat dimension* of an  $R$ -module  $M \neq 0$ , denoted by  $\text{Gfd}_R M$ , is the least integer  $n \geq 0$  such that there exists a Gorenstein flat resolution of  $M$  with  $A_i = 0$  for all  $i > n$ . If no such  $n$  exists, then  $\text{Gfd}_R M$  is infinite. By convention, set  $\text{Gfd}_R 0 = -\infty$ .

The next duality result is an immediate consequence of the definitions.

**Proposition 4.6.** *Let  $M$  be an  $R$ -module. For every injective  $R$ -module  $E$  there is an inequality  $\text{Gid}_R \text{Hom}_R(M, E) \leq \text{Gfd}_R M$ .*

Recall that an  $R$ -module  $E$  is called *faithfully injective* if it is injective and  $\text{Hom}_R(M, E) = 0$  only if  $M = 0$ . The next question is inspired by the classical situation. It has an affirmative answer for Noetherian rings; see Theorem 4.16.

*Question 4.7.* Let  $M$  and  $E$  be  $R$ -modules. If  $E$  is faithfully injective and the module  $\text{Hom}_R(M, E)$  is Gorenstein injective, is then  $M$  Gorenstein flat?

A straightforward application of Proposition 4.6 shows that the Gorenstein flat dimension is a refinement of the flat dimension; cf. Bennis [23, 2.2]:

**Proposition 4.8.** *For every  $R$ -module  $M$  one has  $\text{Gfd}_R M \leq \text{fd}_R M$ , and equality holds if  $M$  has finite flat dimension.*

The following result is an immediate consequence of Definition 4.1. Over a local ring a stronger result is available; see Theorem 4.27.

**Proposition 4.9.** *Let  $S$  be a flat  $R$ -algebra. For every  $R$ -module  $M$  there is an inequality  $\text{Gfd}_S(S \otimes_R M) \leq \text{Gfd}_R M$ .*

**Corollary 4.10.** *Let  $M$  be an  $R$ -module. For every prime ideal  $\mathfrak{p}$  in  $R$  there is an inequality  $\text{Gfd}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} \leq \text{Gfd}_R M$ .*

## Noetherian rings

Finiteness of the Gorenstein flat dimension characterizes Gorenstein rings; this is a result of Enochs and Jenda [42, 3.1]:

**Theorem 4.11.** *Let  $R$  be Noetherian and  $n \geq 0$  be an integer. Then  $R$  is Gorenstein with  $\dim R \leq n$  if and only if  $\text{Gfd}_R M \leq n$  for every  $R$ -module  $M$ .*

A ring is coherent if and only if every direct product of flat modules is flat. We suggest the following problem:

*Problem 4.12.* Describe the rings over which every direct product of Gorenstein flat modules is Gorenstein flat.

Partial answers are due to Christensen, Frankild, and Holm [31, 5.7] and to Murfet and Salarian [85, 6.21].

**Theorem 4.13.** *Let  $R$  be Noetherian. The class of Gorenstein flat  $R$ -modules is closed under direct products under either of the following conditions:*

- (a)  *$R$  is homomorphic image of a Gorenstein ring of finite Krull dimension.*
- (b)  *$R_{\mathfrak{p}}$  is Gorenstein for every non-maximal prime ideal  $\mathfrak{p}$  in  $R$ .*

The next result follows from work of Enochs, Jenda, and López-Ramos [46, 2.1] and [40, 3.3].

**Theorem 4.14.** *Let  $R$  be Noetherian. Then the class of Gorenstein flat  $R$ -modules is closed under direct summands and direct limits.*

A result of Govorov [64] and Lazard [83, 1.2] asserts that a module is flat if and only if it is a direct limit of finitely generated projective modules. For Gorenstein flat modules, the situation is more complicated:

*Remark 4.15.* Let  $R$  be Noetherian. It follows from Example 4.2 and Theorem 4.14 that a direct limit of totally reflexive modules is Gorenstein flat. If  $R$  is Gorenstein of finite Krull dimension, then every Gorenstein flat  $R$ -module can be written as a direct limit of totally reflexive modules; see Enochs and Jenda [43, 10.3.8]. If  $R$  is not Gorenstein, this conclusion may fail; see Beligiannis and Krause [22, 4.2 and 4.3] and Theorem 4.30.

The next result [28, 6.4.2] gives a partial answer to Question 4.7.

**Theorem 4.16.** *Let  $R$  be Noetherian, and let  $M$  and  $E$  be  $R$ -modules. If  $E$  is faithfully injective, then there is an equality*

$$\text{Gid}_R \text{Hom}_R(M, E) = \text{Gfd}_R M.$$

Theorem 4.17 is found in [66, 3.14]. It can be obtained by application of Theorem 4.16 to Theorem 3.6.

**Theorem 4.17.** *Let  $R$  be Noetherian and  $M$  be an  $R$ -module of finite Gorenstein flat dimension. For every integer  $n \geq 0$  the following are equivalent.*

- (i)  $\text{Gfd}_R M \leq n$ .
- (ii)  $\text{Tor}_i^R(E, M) = 0$  for all  $i > n$  and all injective  $R$ -modules  $E$ .
- (iii)  $\text{Tor}_i^R(N, M) = 0$  for all  $i > n$  and all  $R$ -modules  $N$  with  $\text{id}_R N$  finite.
- (iv) In every augmented Gorenstein flat resolution

$$\cdots \rightarrow A_i \rightarrow A_{i-1} \rightarrow \cdots \rightarrow A_0 \rightarrow M \rightarrow 0$$

the module  $\text{Coker}(A_{n+1} \rightarrow A_n)$  is Gorenstein flat.

**Corollary 4.18.** *Let  $R$  be Noetherian. For every  $R$ -module  $M$  of finite Gorenstein flat dimension there is an equality*

$$\text{Gfd}_R M = \sup\{i \in \mathbb{Z} \mid \text{Tor}_i^R(E, M) \neq 0 \text{ for some injective } R\text{-module } E\}.$$

*Remark 4.19.* For every  $R$ -module  $M$  as in the corollary, the finitistic flat dimension of  $R$  is an upper bound for  $\text{Gfd}_R M$ ; see [66, 3.24].

The next result [66, 3.15] follows by Theorem 4.16 and Proposition 3.9.

**Proposition 4.20.** *Let  $R$  be Noetherian. If any two of the modules in an exact sequence  $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$  have finite Gorenstein flat dimension, then so has the third.*

A result of Holm [67, 2.6] supplements Proposition 4.8:

**Proposition 4.21.** *Let  $R$  be Noetherian of finite Krull dimension. For every  $R$ -module  $M$  of finite injective dimension one has  $\text{Gfd}_R M = \text{fd}_R M$ .*

Recall that the *depth* of a module  $M$  over a local ring  $(R, \mathfrak{m}, k)$  is given as

$$\text{depth}_R M = \inf\{i \in \mathbb{Z} \mid \text{Ext}_R^i(k, M) \neq 0\}.$$

Theorem 4.22 is a Gorenstein version of Chouinard's [27, 1.2]. It follows from [66, 3.19] and [30, 2.4(b)]; see also Iyengar and Sather-Wagstaff [76, 8.8].



**Theorem 4.22.** *Let  $R$  be Noetherian. For every  $R$ -module  $M$  of finite Gorenstein flat dimension there is an equality*

$$\text{Gfd}_R M = \{ \text{depth } R_{\mathfrak{p}} - \text{depth}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} \mid \mathfrak{p} \in \text{Spec } R \}.$$

The next two results compare the Gorenstein flat dimension to the Gorenstein projective dimension. The inequality in Theorem 4.23 is [66, 3.4], and the second assertion in this theorem is due to Esmkhani and Tousi [52, 3.4].

**Theorem 4.23.** *Let  $R$  be Noetherian of finite Krull dimension, and let  $M$  be an  $R$ -module. Then there is an inequality*

$$\text{Gfd}_R M \leq \text{Gpd}_R M,$$

and if  $\text{Gfd}_R M$  is finite, then so is  $\text{Gpd}_R M$ .

It is not known whether the inequality in Theorem 4.23 holds over every commutative ring. For finitely generated modules one has [28, 4.2.6 and 5.1.11]:

**Proposition 4.24.** *Let  $R$  be Noetherian. For every finitely generated  $R$ -module  $M$  there is an equality  $\text{Gfd}_R M = \text{Gpd}_R M = \text{G-dim}_R M$ .*

The next result [31, 5.1] is related to Theorem 4.16; the question that follows is prompted by the classical situation.

**Theorem 4.25.** *Assume that  $R$  is a homomorphic image of a Gorenstein ring of finite Krull dimension. For every  $R$ -module  $M$  and every injective  $R$ -module  $E$  there is an inequality*

$$\text{Gfd}_R \text{Hom}_R(M, E) \leq \text{Gid}_R M,$$

and equality holds if  $E$  is faithfully injective.

*Question 4.26.* Let  $R$  be Noetherian and  $M$  and  $E$  be  $R$ -modules. If  $M$  is Gorenstein injective and  $E$  is injective, is then  $\text{Hom}_R(M, E)$  Gorenstein flat?

## Local rings

Over a local ring there is a stronger version [52, 3.5] of Proposition 4.9:

**Theorem 4.27.** *Let  $R$  be local. For every  $R$ -module  $M$  there is an equality*

$$\text{Gfd}_{\widehat{R}}(\widehat{R} \otimes_R M) = \text{Gfd}_R M.$$

Combination of [67, 2.1 and 2.2] with Theorem E yields the next result. Recall that a non-zero finitely generated module has finite depth.

**Theorem 4.28.** *For a local ring  $R$  the following conditions are equivalent.*

- (i)  $R$  is Gorenstein.
- (ii) There is an  $R$ -module  $M$  with  $\text{depth}_R M$ ,  $\text{fd}_R M$ , and  $\text{Gid}_R M$  finite.
- (iii) There is an  $R$ -module  $M$  with  $\text{depth}_R M$ ,  $\text{id}_R M$ , and  $\text{Gfd}_R M$  finite.

We have for a while been interested in:

*Question 4.29.* Let  $R$  be local. If there exists an  $R$ -module  $M$  with  $\text{depth}_R M$ ,  $\text{Gfd}_R M$ , and  $\text{Gid}_R M$  finite, is then  $R$  Gorenstein?

A theorem of Jørgensen and Holm [71] brings perspective to Remark 4.15.

**Theorem 4.30.** *Assume that  $R$  is Henselian local and a homomorphic image of a Gorenstein ring. If every Gorenstein flat  $R$ -module is a direct limit of totally reflexive modules, then  $R$  is Gorenstein or every totally reflexive  $R$ -module is free.*

### Notes

Parallel to the notion of strongly Gorenstein projective modules, see Definition 2.4, Bennis and Mahdou [24] also study strongly Gorenstein flat modules. A different notion of strongly Gorenstein flat modules is studied by Ding, Li, and Mao in [38].

## 5 Relative homological algebra

Over a Gorenstein local ring, the totally reflexive modules are exactly the maximal Cohen–Macaulay modules, and their representation theory is a classical topic. Over rings that are not Gorenstein, the representation theory of totally reflexive modules was taken up by Takahashi [102] and Yoshino [116]. Conclusive results have recently been obtained by Christensen, Piepmeyer, Striuli, and Takahashi [34] and by Holm and Jørgensen [71]. These results are cast in the language of precovers and preenvelopes; see Theorem 5.4.

Relative homological algebra studies dimensions and (co)homology functors based on resolutions that are constructed via precovers or preenvelopes. Enochs and Jenda and their collaborators have made extensive studies of the precovering and preenveloping properties of the classes of Gorenstein flat and Gorenstein injective modules. Many of their results are collected in [43].

### Terminology

Let  $\mathcal{H}$  be a class of  $R$ -modules. Recall that an  $\mathcal{H}$ -precover (also called a right  $\mathcal{H}$ -approximation) of an  $R$ -module  $M$  is a homomorphism  $\varphi: H \rightarrow M$  with  $H$  in  $\mathcal{H}$  such that

$$\text{Hom}_R(H', \varphi): \text{Hom}_R(H', H) \longrightarrow \text{Hom}_R(H', M)$$

is surjective for every  $H'$  in  $\mathcal{H}$ . That is, every homomorphism from a module in  $\mathcal{H}$  to  $M$  factors through  $\varphi$ . Dually one defines  $\mathcal{H}$ -preenvelopes (also called left  $\mathcal{H}$ -approximations).

*Remark 5.1.* If  $\mathcal{H}$  contains all projective modules, then every  $\mathcal{H}$ -precover is an epimorphism. Thus, Gorenstein projective/flat precovers are epimorphisms.

If  $\mathcal{H}$  contains all injective modules, then every  $\mathcal{H}$ -preenvelope is a monomorphism. Thus, every Gorenstein injective preenvelope is a monomorphism.

Fix an  $\mathcal{H}$ -precover  $\varphi$ . It is called *special* if one has  $\text{Ext}_R^1(H', \text{Ker } \varphi) = 0$  for every module  $H'$  in  $\mathcal{H}$ . It is called a *cover* (or a minimal right approximation) if in every factorization  $\varphi = \varphi\psi$ , the map  $\psi: H \rightarrow H$  is an automorphism. If  $\mathcal{H}$  is closed under extensions, then every  $\mathcal{H}$ -cover is a special precover. This is known as Wakamatsu’s lemma; see Xu [113, 2.1.1]. Dually one defines special  $\mathcal{H}$ -preenvelopes and  $\mathcal{H}$ -envelopes.

*Remark 5.2.* Let  $I$  be a complex of injective modules as in Definition 3.1. Then every differential in  $I$  is a special injective precover of its image; this fact is used in the proof of Proposition 3.15. Similarly, in a complex  $P$  of projective modules as in Definition 2.1, every differential  $\partial_i$  is a special projective preenvelope of the cokernel of the previous differential  $\partial_{i+1}$ .

## Totally reflexive covers and envelopes

The next result of Avramov and Martsinkovsky [19, 3.1] corresponds over a Gorenstein local ring to the existence of maximal Cohen–Macaulay approximations in the sense of Auslander and Buchweitz [9].

**Proposition 5.3.** *Let  $R$  be Noetherian. For every finitely generated  $R$ -module  $M$  of finite  $G$ -dimension there is an exact sequence  $0 \rightarrow K \rightarrow G \rightarrow M \rightarrow 0$  of finitely generated modules, where  $G$  is totally reflexive and  $\text{pd}_R K = \max\{0, G\text{-dim}_R M - 1\}$ . In particular, every finitely generated  $R$ -module of finite  $G$ -dimension has a special totally reflexive precover.*

An unpublished result of Auslander states that every finitely generated module over a Gorenstein local ring has a totally reflexive cover; see Enochs, Jenda, and Xu [50] for a generalization. A strong converse is contained in the next theorem, which combines Auslander’s result with recent work of several authors; see [34] and [71].

**Theorem 5.4.** *For a local ring  $(R, \mathfrak{m}, k)$  the next conditions are equivalent.*

- (i) *Every finitely generated  $R$ -module has a totally reflexive cover.*
- (ii) *The residue field  $k$  has a special totally reflexive precover.*
- (iii) *Every finitely generated  $R$ -module has a totally reflexive envelope.*

- (iv) *Every finitely generated  $R$ -module has a special totally reflexive preenvelope.*
- (v)  *$R$  is Gorenstein or every totally reflexive  $R$ -module is free.*

If  $R$  is local and Henselian (e.g. complete), then existence of a totally reflexive precover implies existence of a totally reflexive cover; see [102, 2.5]. In that case one can drop “special” in part (ii) above. In general, though, the next question from [34] remains open.

*Question 5.5.* Let  $(R, \mathfrak{m}, k)$  be local. If  $k$  has a totally reflexive precover, is then  $R$  Gorenstein or every totally reflexive  $R$ -module free?

## Gorenstein projective precovers

The following result is proved by Holm in [66, 2.10].

**Proposition 5.6.** *For every  $R$ -module  $M$  of finite Gorenstein projective dimension there is an exact sequence  $0 \rightarrow K \rightarrow A \rightarrow M \rightarrow 0$ , where  $A$  is Gorenstein projective and  $\text{pd}_R K = \max\{0, \text{Gpd}_R M - 1\}$ . In particular, every  $R$ -module of finite Gorenstein projective dimension has a special Gorenstein projective precover.*

For an important class of rings, Jørgensen [78] and Murfet and Salarian [85] prove existence of Gorenstein projective precovers for all modules:

**Theorem 5.7.** *If  $R$  is Noetherian of finite Krull dimension, then every  $R$ -module has a Gorenstein projective precover.*

*Remark 5.8.* Actually, the argument in Krause’s proof of [81, 7.12(1)] applies to the setup in [78] and yields existence of a *special* Gorenstein projective precover for every module over a ring as in Theorem 5.7.

Over any ring, every module has a special projective precover; hence:

*Problem 5.9.* Describe the rings over which every module has a (special) Gorenstein projective precover.

## Gorenstein injective preenvelopes

In [66, 2.15] one finds:

**Proposition 5.10.** *For every  $R$ -module  $M$  of finite Gorenstein injective dimension there is an exact sequence  $0 \rightarrow M \rightarrow B \rightarrow C \rightarrow 0$ , where  $B$  is Gorenstein injective and  $\text{id}_R C = \max\{0, \text{Gid}_R M - 1\}$ . In particular, every  $R$ -module of finite Gorenstein injective dimension has a special Gorenstein injective preenvelope.*

Over Noetherian rings, existence of Gorenstein injective preenvelopes for all modules is proved by Enochs and López-Ramos in [51]. Krause [81, 7.12] proves a stronger result:

**Theorem 5.11.** *If  $R$  is Noetherian, then every  $R$ -module has a special Gorenstein injective preenvelope.*

Over Gorenstein rings, Enochs, Jenda, and Xu [48, 6.1] prove even more:

**Proposition 5.12.** *If  $R$  is Gorenstein of finite Krull dimension, then every  $R$ -module has a Gorenstein injective envelope.*

Over any ring, every module has an injective envelope; this suggests:

*Problem 5.13.* Describe the rings over which every module has a Gorenstein injective (pre)envelope.

Over a Noetherian ring, every module has an injective cover; see Enochs [39, 2.1]. A Gorenstein version of this result is recently established by Holm and Jørgensen [72, 3.3(b)]:

**Proposition 5.14.** *If  $R$  is a homomorphic image of a Gorenstein ring of finite Krull dimension, then every  $R$ -module has a Gorenstein injective cover.*

## Gorenstein flat covers

The following existence result is due to Enochs and López-Ramos [51, 2.11].

**Theorem 5.15.** *If  $R$  is Noetherian, then every  $R$ -module has a Gorenstein flat cover.*

*Remark 5.16.* Let  $R$  be Noetherian and  $M$  be a finitely generated  $R$ -module. If  $M$  has finite G-dimension, then by Proposition 5.3 it has a finitely generated Gorenstein projective/flat precover, cf. Proposition 4.24. If  $M$  has infinite G-dimension, it still has a Gorenstein projective/flat precover by Theorems 5.7 and 5.15, but by Theorem 5.4 this need *not* be finitely generated.

Over any ring, every module has a flat cover, as proved by Bican, El Bashir, and Enochs [26]. This motivates:

*Problem 5.17.* Describe the rings over which every module has a Gorenstein flat (pre)cover.

Over a Noetherian ring, every module has a flat preenvelope; cf. Enochs [39, 5.1]. A Gorenstein version of this result follows from Theorem 4.13 and [51, 2.5]:

**Proposition 5.18.** *If  $R$  is a homomorphic image of a Gorenstein ring of finite Krull dimension, then every  $R$ -module has a Gorenstein flat preenvelope.*

## Relative cohomology via Gorenstein projective modules

The notion of a proper resolution is central in relative homological algebra. Here is an instance:

**Definition 5.19.** An augmented Gorenstein projective resolution,

$$\mathbf{A}^+ = \cdots \xrightarrow{\varphi_{i+1}} A_i \xrightarrow{\varphi_i} A_{i-1} \xrightarrow{\varphi_{i-1}} \cdots \xrightarrow{\varphi_1} A_0 \xrightarrow{\varphi_0} M \longrightarrow 0$$

of an  $R$ -module  $M$  is said to be *proper* if the complex  $\mathrm{Hom}_R(A', \mathbf{A}^+)$  is acyclic for every Gorenstein projective  $R$ -module  $A'$ .

*Remark 5.20.* Assume that every  $R$ -module has a Gorenstein projective precover. Then every  $R$ -module has a proper Gorenstein projective resolution constructed by taking as  $\varphi_0$  a Gorenstein projective precover of  $M$  and as  $\varphi_i$  a Gorenstein projective precover of  $\mathrm{Ker} \varphi_{i-1}$  for  $i > 0$ .

**Definition 5.21.** The *relative Gorenstein projective dimension* of an  $R$ -module  $M \neq 0$ , denoted by  $\mathrm{rel}\text{-Gpd}_R M$ , is the least integer  $n \geq 0$  such that there exists a proper Gorenstein projective resolution of  $M$  with  $A_i = 0$  for all  $i > n$ . If no such  $n$  (or no such resolution) exists, then  $\mathrm{rel}\text{-Gpd}_R M$  is infinite. By convention, set  $\mathrm{rel}\text{-Gpd}_R 0 = -\infty$ .

The following result is a consequence of Proposition 5.6.

**Proposition 5.22.** For every  $R$ -module  $M$  one has  $\mathrm{rel}\text{-Gpd}_R M = \mathrm{Gpd}_R M$ .

It is shown in [43, Section 8.2] that the next definition makes sense.

**Definition 5.23.** Let  $M$  and  $N$  be  $R$ -modules and assume that  $M$  has a proper Gorenstein projective resolution  $\mathbf{A}$ . The  $i$ th relative cohomology module  $\mathrm{Ext}_{\mathrm{GP}}^i(M, N)$  is  $H^i(\mathrm{Hom}_R(\mathbf{A}, N))$ .

*Remark 5.24.* Let  $R$  be Noetherian, and let  $M$  and  $N$  be finitely generated  $R$ -modules. Unless  $M$  has finite G-dimension, it is not clear whether the cohomology modules  $\mathrm{Ext}_{\mathrm{GP}}^i(M, N)$  are finitely generated, cf. Remark 5.16.

Vanishing of relative cohomology  $\mathrm{Ext}_{\mathrm{GP}}^i$  characterizes modules of finite Gorenstein projective dimension. The proof is standard; see [68, 3.9].

**Theorem 5.25.** Let  $M$  be an  $R$ -module that has a proper Gorenstein projective resolution. For every integer  $n \geq 0$  the next conditions are equivalent.

- (i)  $\mathrm{Gpd}_R M \leq n$ .
- (ii)  $\mathrm{Ext}_{\mathrm{GP}}^i(M, -) = 0$  for all  $i > n$ .
- (iii)  $\mathrm{Ext}_{\mathrm{GP}}^{n+1}(M, -) = 0$ .

*Remark 5.26.* Relative cohomology based on totally reflexive modules is studied in [19]. The results that correspond to Proposition 5.22 and Theorem 5.25 in that setting are contained in [19, 4.8 and 4.2].

*Notes*

Based on a notion of coproper Gorenstein injective resolutions, one can define a relative Gorenstein injective dimension and cohomology functors  $\text{Ext}'_{\text{GI}}$  with properties analogous to those of  $\text{Ext}'_{\text{GP}}$  described above. There is, similarly, a relative Gorenstein flat dimension and a relative homology theory based on proper Gorenstein projective/flat resolutions. The relative Gorenstein injective dimension and the relative Gorenstein flat dimension were first studied by Enochs and Jenda [42] for modules over Gorenstein rings. The question of balancedness for relative (co)homology is treated by Enochs and Jenda [43], Holm [65], and Iacob [74].

In [19] Avramov and Martsinkovsky also study the connection between relative and Tate cohomology for finitely generated modules. This study is continued by Veliche [110] for arbitrary modules, and a dual theory is developed by Asadollahi and Salarian [4]. Jørgensen [78], Krause [81], and Takahashi [104] study connections between Gorenstein relative cohomology and generalized notions of Tate cohomology.

Sather-Wagstaff and White [95] use relative cohomology to define an Euler characteristic for modules of finite G-dimension. In collaboration with Sharif, they study cohomology theories related to generalized Gorenstein dimensions [93].

## 6 Modules over local homomorphisms

In this section,  $\varphi: (R, \mathfrak{m}) \rightarrow (S, \mathfrak{n})$  is a local homomorphism, that is, there is a containment  $\varphi(\mathfrak{m}) \subseteq \mathfrak{n}$ . The topic is Gorenstein dimensions over  $R$  of finitely generated  $S$ -modules. The utility of this point of view is illustrated by a generalization, due to Christensen and Iyengar [33, 4.1], of the Auslander–Bridger Formula (Theorem 1.25):

**Theorem 6.1.** *Let  $N$  be a finitely generated  $S$ -module. If  $N$  has finite Gorenstein flat dimension as an  $R$ -module via  $\varphi$ , then there is an equality*

$$\text{Gfd}_R N = \text{depth } R - \text{depth}_R N.$$

For a finitely generated  $S$ -module  $N$  of finite flat dimension over  $R$ , this equality follows from work of André [2, II.57]. For a finitely generated  $S$ -module of finite injective dimension over  $R$ , an affirmative answer to the next question is already in [107, 5.2] by Takahashi and Yoshino.

*Question 6.2.* Let  $N$  be a non-zero finitely generated  $S$ -module. If  $N$  has finite Gorenstein injective dimension as an  $R$ -module via  $\varphi$ , does then the equality  $\text{Gid}_R N = \text{depth } R$  hold? (For  $\varphi = \text{Id}_R$  this is Theorem 3.24.)

The next result of Christensen and Iyengar [33, 4.8] should be compared to Theorem 4.27.

**Theorem 6.3.** *Let  $N$  be a finitely generated  $S$ -module. If  $N$  has finite Gorenstein flat dimension as an  $R$ -module via  $\varphi$ , there is an equality*

$$\text{Gfd}_R N = \text{Gfd}_{\widehat{R}}(\widehat{S} \otimes_S N).$$

## The Frobenius endomorphism

If  $R$  has positive prime characteristic, we denote by  $\phi$  the Frobenius endomorphism on  $R$  and by  $\phi^n$  its  $n$ -fold composition. The next two theorems are special cases of [76, 8.14 and 8.15] by Iyengar and Sather-Wagstaff and of [58, 5.5] by Foxby and Frankild.; together, they constitute the Gorenstein counterpart of Theorem F.

**Theorem 6.4.** *Let  $R$  be local of positive prime characteristic. The following conditions are equivalent.*

- (i)  $R$  is Gorenstein.
- (ii)  $R$  has finite Gorenstein flat dimension as an  $R$ -module via  $\phi^n$  for some  $n \geq 1$ .
- (iii)  $R$  is Gorenstein flat as an  $R$ -module via  $\phi^n$  for every  $n \geq 1$ .

**Theorem 6.5.** *Let  $R$  be local of positive prime characteristic, and assume that it is a homomorphic image of a Gorenstein ring. The following conditions are equivalent.*

- (i)  $R$  is Gorenstein.
- (ii)  $R$  has finite Gorenstein injective dimension as  $R$ -module via  $\phi^n$  for some  $n \geq 1$ .
- (iii)  $R$  has Gorenstein injective dimension equal to  $\dim R$  as an  $R$ -module via  $\phi^n$  for every  $n \geq 1$ .

Part (iii) in Theorem 6.5 is actually not included in [58, 5.5]. Part (iii) follows, though, from part (i) by Corollary 3.23 and Theorem 3.18.

## G-dimension over a local homomorphism

The homomorphism  $\varphi: (R, \mathfrak{m}) \rightarrow (S, \mathfrak{n})$  fits in a commutative diagram of local homomorphisms:

$$\begin{array}{ccccc}
 & & R' & & \\
 & \nearrow \phi & & \searrow \phi' & \\
 R & \xrightarrow{\varphi} & S & \xrightarrow{\iota} & \widehat{S}
 \end{array}$$

where  $\phi$  is flat with regular closed fiber  $R'/\mathfrak{m}R'$ , the ring  $R'$  is complete, and  $\phi'$  is surjective. Set  $\widehat{\varphi} = \iota\varphi$ ; a diagram as above is called a *Cohen factorization of  $\widehat{\varphi}$* . This is a construction due to Avramov, Foxby, and Herzog [15, 1.1].

The next definition is due to Iyengar and Sather-Wagstaff [76, Section 3]; it is proved *ibid.* that it is independent of the choice of Cohen factorization.

**Definition 6.6.** Choose a Cohen factorization of  $\widehat{\varphi}$  as above. For a finitely generated  $S$ -module  $N$ , the *G-dimension of  $N$  over  $\varphi$*  is given as

$$\text{G-dim}_{\varphi} N = \text{G-dim}_{R'}(\widehat{S} \otimes_S N) - \text{edim}(R'/\mathfrak{m}R').$$

*Example 6.7.* Let  $k$  be a field and let  $\varphi$  be the extension from  $k$  to the power series ring  $k[[x]]$ . Then one has  $\text{G-dim}_{\varphi} k[[x]] = -1$ .



Iyengar and Sather-Wagstaff [76, 8.2] prove:

**Theorem 6.8.** *Assume that  $R$  is a homomorphic image of a Gorenstein ring. A finitely generated  $S$ -module  $N$  has finite  $G$ -dimension over  $\varphi$  if and only if it has finite Gorenstein flat dimension as an  $R$ -module via  $\varphi$ .*

It is clear from Example 6.7 that  $\text{Gfd}_R N$  and  $\text{G-dim}_\varphi N$  need not be equal.

## 7 Local homomorphisms of finite $G$ -dimension

This section treats transfer of ring theoretic properties along a local homomorphism of finite  $G$ -dimension. Our focus is on the Gorenstein property, which was studied by Avramov and Foxby in [13], and the Cohen–Macaulay property, studied by Frankild in [59].

As in Section 6,  $\varphi: (R, \mathfrak{m}) \rightarrow (S, \mathfrak{n})$  is a local homomorphism. In view of Definition 6.6, a notion from [13, 4.3] can be defined as follows:

**Definition 7.1.** Set  $\text{G-dim } \varphi = \text{G-dim}_\varphi S$ ; the homomorphism  $\varphi$  is said to be of *finite  $G$ -dimension* if this number is finite.

*Remark 7.2.* The homomorphism  $\varphi$  has finite  $G$ -dimension if  $S$  has finite Gorenstein flat dimension as an  $R$ -module via  $\varphi$ , and the converse holds if  $R$  is a homomorphic image of a Gorenstein ring. This follows from Theorems 6.3 and 6.8, in view of [76, 3.4.1],

The next descent result is [13, 4.6].

**Theorem 7.3.** *Let  $\varphi$  be of finite  $G$ -dimension, and assume that  $R$  is a homomorphic image of a Gorenstein ring. For every  $S$ -module  $N$  one has:*

- (a) *If  $\text{fd}_S N$  is finite then  $\text{Gfd}_R N$  is finite.*
- (b) *If  $\text{id}_S N$  is finite then  $\text{Gid}_R N$  is finite.*

It is not known if the composition of two local homomorphisms of finite  $G$ -dimension has finite  $G$ -dimension, but it would follow from an affirmative answer to Question 1.21, cf. [13, 4.8]. Some insight is provided by Theorem 7.9 and the next result, which is due to Iyengar and Sather-Wagstaff [76, 5.2].

**Theorem 7.4.** *Let  $\psi: S \rightarrow T$  be a local homomorphism such that  $\text{fd}_S T$  is finite. Then  $\text{G-dim } \psi\varphi$  is finite if and only if  $\text{G-dim } \varphi$  is finite.*

## Quasi-Gorenstein homomorphisms

Let  $M$  be a finitely generated module over a local ring  $(R, \mathfrak{m}, k)$ . For every integer  $i \geq 0$  the  $i$ th Bass number  $\mu_R^i(M)$  is the dimension of the  $k$ -vector space  $\text{Ext}_R^i(k, M)$ .

**Definition 7.5.** The homomorphism  $\varphi$  is called *quasi-Gorenstein* if it has finite G-dimension and for every  $i \geq 0$  there is an equality of Bass numbers

$$\mu_R^{i+\text{depth}R}(R) = \mu_S^{i+\text{depth}S}(S).$$

*Example 7.6.* If  $R$  is Gorenstein, then the natural surjection  $R \rightarrow k$  is quasi-Gorenstein. More generally, if  $\varphi$  is surjective and  $S$  is quasi-perfect as an  $R$ -module via  $\varphi$ , then  $\varphi$  is quasi-Gorenstein if and only if there is an isomorphism  $\text{Ext}_R^g(S, R) \cong S$  where  $g = \text{G-dim}_R S$  with; see [13, 6.5, 7.1, 7.4].

Several characterizations of the quasi-Gorenstein property are given in [13, 7.4 and 7.5]. For example, it is sufficient that  $\text{G-dim } \varphi$  is finite and the equality of Bass numbers holds for some  $i > 0$ .

The next ascent–descent result is [13, 7.9].

**Theorem 7.7.** *Let  $\varphi$  be quasi-Gorenstein and assume that  $R$  is a homomorphic image of a Gorenstein ring. For every  $S$ -module  $N$  one has:*

- (a)  $\text{Gfd}_S N$  is finite if and only if  $\text{Gfd}_R N$  is finite.
- (b)  $\text{Gid}_S N$  is finite if and only if  $\text{Gid}_R N$  is finite.

Ascent and descent of the Gorenstein property is described by [13, 7.7.2]. It should be compared to part (b) in Theorem G.

**Theorem 7.8.** *The following conditions on  $\varphi$  are equivalent.*

- (i)  $R$  and  $S$  are Gorenstein.
- (ii)  $R$  is Gorenstein and  $\varphi$  is quasi-Gorenstein.
- (iii)  $S$  is Gorenstein and  $\varphi$  is of finite G-dimension.

The following (de)composition result is [13, 7.10, 8.9, and 8.10]. It should be compared to Theorem 1.20.

**Theorem 7.9.** *Assume that  $\varphi$  is quasi-Gorenstein, and let  $\psi: S \rightarrow T$  be a local homomorphism. The following assertions hold.*

- (a)  $\text{G-dim } \psi\varphi$  is finite if and only if  $\text{G-dim } \psi$  is finite.
- (b)  $\psi\varphi$  is quasi-Gorenstein if and only if  $\psi$  is quasi-Gorenstein.

## Quasi-Cohen–Macaulay homomorphisms

The next definition from [59, 5.8 and 6.2] uses terminology from Definition 1.19 and the remarks before Definition 6.6.

**Definition 7.10.** The homomorphism  $\varphi$  is *quasi-Cohen–Macaulay*, for short *quasi-CM*, if  $\varphi$  has a Cohen factorization where  $\tilde{S}$  is quasi-perfect over  $R'$ .

If  $\varphi$  is quasi-CM, then  $\widehat{S}$  is a quasi-perfect  $R'$ -module in every Cohen factorization of  $\widehat{\varphi}$ ; see [59, 5.8]. The following theorem is part of [59, 6.7]; it should be compared to part (a) in Theorem G.

**Theorem 7.11.** *The following assertions hold.*

- (a) *If  $R$  is Cohen–Macaulay and  $\varphi$  is quasi-CM, then  $S$  is Cohen–Macaulay.*
- (b) *If  $S$  is Cohen–Macaulay and  $\text{G-dim } \varphi$  is finite, then  $\varphi$  is quasi-CM.*

In view of Theorem 7.9, Frankild’s work [59, 6.4 and 6.5] yields:

**Theorem 7.12.** *Assume that  $\varphi$  is quasi-Gorenstein, and let  $\psi: S \rightarrow T$  be a local homomorphism. Then  $\psi\varphi$  is quasi-CM if and only if  $\psi$  is quasi-CM.*

*Notes*

The composition question addressed in the remarks before Theorem 7.4 is investigated further by Sather-Wagstaff [92].

## 8 Reflexivity and finite G-dimension

In this section,  $R$  is Noetherian. Let  $M$  be a finitely generated  $R$ -module. If  $M$  is totally reflexive, then the cohomology modules  $\text{Ext}_R^i(M, R)$  vanish for all  $i > 0$ . The converse is true if  $M$  is known *a priori* to have finite G-dimension, cf. Corollary 1.10. In general, though, one can not infer from such vanishing that  $M$  is totally reflexive—explicit examples to this effect are constructed by Jorgensen and Šega in [77]—and this has motivated a search for alternative criteria for finiteness of G-dimension.

### Reflexive complexes

One such criterion was given by Foxby and published in [114]. Its habitat is the derived category  $\mathcal{D}(R)$  of the category of  $R$ -modules. The objects in  $\mathcal{D}(R)$  are  $R$ -complexes, and there is a canonical functor  $F$  from the category of  $R$ -complexes to  $\mathcal{D}(R)$ . This functor is the identity on objects and it maps homology isomorphisms to isomorphisms in  $\mathcal{D}(R)$ . The restriction of  $F$  to modules is a full embedding of the module category into  $\mathcal{D}(R)$ .

The homology  $H(\mathbf{M})$  of an  $R$ -complex  $\mathbf{M}$  is a (graded)  $R$ -module, and  $\mathbf{M}$  is said to have *finitely generated homology* if this module is finitely generated. That is, if every homology module  $H_i(\mathbf{M})$  is finitely generated and only finitely many of them are non-zero.

For  $R$ -modules  $M$  and  $N$ , the (co)homology of the derived Hom and tensor product complexes gives the classical Ext and Tor modules:

$$\mathrm{Ext}_R^i(M, N) = \mathrm{H}^i(\mathbf{R}\mathrm{Hom}_R(M, N)) \quad \text{and} \quad \mathrm{Tor}_i^R(M, N) = \mathrm{H}_i(M \otimes_R^{\mathbf{L}} N).$$

**Definition 8.1.** An  $R$ -complex  $M$  is *reflexive* if  $M$  and  $\mathbf{R}\mathrm{Hom}_R(M, R)$  have finitely generated homology and the canonical morphism

$$M \longrightarrow \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(M, R), R)$$

is an isomorphism in the derived category  $\mathcal{D}(R)$ . The full subcategory of  $\mathcal{D}(R)$  whose objects are the reflexive  $R$ -complexes is denoted by  $\mathcal{R}(R)$ .

The requirement in the definition that the complex  $\mathbf{R}\mathrm{Hom}_R(M, R)$  has finitely generated homology is redundant but retained for historical reasons; see [18, 3.3].

Theorem 8.2 below is Foxby's criterion for finiteness of G-dimension of a finitely generated module [114, 2.7]. It differs significantly from Definition 1.8 as it does not involve construction of a G-resolution of the module.

**Theorem 8.2.** *Let  $R$  be Noetherian. A finitely generated  $R$ -module has finite G-dimension if and only if it belongs to  $\mathcal{R}(R)$ .*

If  $R$  is local, then the next result is [28, 2.3.14]. In the generality stated below it follows from Theorems 1.18 and 8.2: the implication (ii)  $\Rightarrow$  (iii) is the least obvious, it uses [28, 2.1.12].

**Corollary 8.3.** *Let  $R$  be Noetherian. The following conditions are equivalent.*

- (i)  $R$  is Gorenstein.
- (ii) Every  $R$ -module is in  $\mathcal{R}(R)$ .
- (iii) Every  $R$ -complex with finitely generated homology is in  $\mathcal{R}(R)$ .

## G-dimension of complexes

Having made the passage to the derived category, it is natural to consider G-dimension for complexes. For every  $R$ -complex  $M$  with finitely generated homology there exists a complex  $G$  of finitely generated free  $R$ -modules, which is isomorphic to  $M$  in  $\mathcal{D}(R)$ ; see [11, 1.7(1)]. In Christensen's [28, Chap. 2] one finds the next definition and the two theorems that follow.

**Definition 8.4.** Let  $M$  be an  $R$ -complex with finitely generated homology. If  $\mathrm{H}(M)$  is not zero, then the *G-dimension* of  $M$  is the least integer  $n$  such that there exists a complex  $G$  of totally reflexive  $R$ -modules which is isomorphic to  $M$  in  $\mathcal{D}(R)$  and has  $G_i = 0$  for all  $i > n$ . If no such integer  $n$  exists, then  $\mathrm{G-dim}_R M$  is infinite. If  $\mathrm{H}(M) = 0$ , then  $\mathrm{G-dim}_R M = -\infty$  by convention.

Note that this extends Definition 1.8. As a common generalization of Theorem 8.2 and Corollary 1.10 one has [28, 2.3.8]:

**Theorem 8.5.** *Let  $R$  be Noetherian. An  $R$ -complex  $M$  with finitely generated homology has finite  $G$ -dimension if and only if it is reflexive. Furthermore, for every reflexive  $R$ -complex  $M$  there is an equality*

$$G\text{-dim}_R M = \sup\{i \in \mathbb{Z} \mid H^i(\mathbf{R}\text{Hom}_R(M, R)) \neq 0\}.$$

In broad terms, the theory of  $G$ -dimension for finitely generated modules extends to complexes with finitely generated homology. One example is the next extension [28, 2.3.13] of the Auslander–Bridger Formula (Theorem 1.25).

**Theorem 8.6.** *Let  $R$  be local. For every complex  $M$  in  $\mathcal{R}(R)$  one has*

$$G\text{-dim}_R M = \text{depth } R - \text{depth}_R M.$$

Here, the *depth* of a complex  $M$  over a local ring  $(R, \mathfrak{m}, k)$  is defined by extension of the definition for modules, that is,

$$\text{depth}_R M = \inf\{i \in \mathbb{Z} \mid H^i(\mathbf{R}\text{Hom}_R(k, M)) \neq 0\}.$$

*Notes*

In [28, Chap. 2] the theory of  $G$ -dimension for complexes is developed in detail.

Generalized notions of  $G$ -dimension—based on reflexivity with respect to semidualizing modules and complexes—are studied by Avramov, Iyengar, and Lipman [18], Christensen [29], Frankild and Sather-Wagstaff [60], Gerko [61], Golod [62], Holm and Jørgensen [70], by Salarian, Sather-Wagstaff, and Yassemi [91], and White [112]. See also the notes in Section 1.

## 9 Detecting finiteness of Gorenstein dimensions

In the previous section, we discussed a resolution-free characterization of modules of finite  $G$ -dimension (Theorem 8.2). The topic of this section is similar characterizations of modules of finite Gorenstein projective/injective/flat dimension. By work of Christensen, Frankild, and Holm [31], appropriate criteria are available for modules over a Noetherian ring that has a dualizing complex (Theorems 9.2 and 9.5). As mentioned in the Introduction, a Noetherian ring has a dualizing complex if and only if it is a homomorphic image of a Gorenstein ring of finite Krull dimension. For example, every complete local ring has a dualizing complex by Cohen’s structure theorem.

### Auslander categories

The next definition is due to Foxby; see [13, 3.1] and [54, Section 2].

**Definition 9.1.** Let  $R$  be Noetherian and assume that it has a dualizing complex  $\mathbf{D}$ . The *Auslander class*  $\mathcal{A}(R)$  is the full subcategory of the derived category  $\mathcal{D}(R)$  whose objects  $\mathbf{M}$  satisfy the following conditions.

- (1)  $H_i(\mathbf{M}) = 0$  for  $|i| \gg 0$ .
- (2)  $H_i(\mathbf{D} \otimes_R^{\mathbf{L}} \mathbf{M}) = 0$  for  $i \gg 0$ .
- (3) The natural map  $\mathbf{M} \rightarrow \mathbf{RHom}_R(\mathbf{D}, \mathbf{D} \otimes_R^{\mathbf{L}} \mathbf{M})$  is invertible in  $\mathcal{D}(R)$ .

The relation to Gorenstein dimensions is given by [31, 4.1]:

**Theorem 9.2.** *Let  $R$  be Noetherian and assume that it has a dualizing complex. For every  $R$ -module  $M$ , the following conditions are equivalent.*

- (i)  $M$  has finite Gorenstein projective dimension.
- (ii)  $M$  has finite Gorenstein flat dimension.
- (iii)  $M$  belongs to  $\mathcal{A}(R)$ .

*Remark 9.3.* The equivalence of (i)/(ii) and (iii) in Theorem 9.2 provides a resolution-free characterization of modules of finite Gorenstein projective/flat dimension over a ring that has a dualizing complex. Every complete local ring has a dualizing complex, so in view of Theorem 2.20/4.27 there is a resolution-free characterization of modules of finite Gorenstein projective/flat dimension over any local ring.

The next definition is in [13, 3.1]; the theorem that follows is [31, 4.4].

**Definition 9.4.** Let  $R$  be Noetherian and assume that it has a dualizing complex  $\mathbf{D}$ . The *Bass class*  $\mathcal{B}(R)$  is the full subcategory of the derived category  $\mathcal{D}(R)$  whose objects  $\mathbf{M}$  satisfy the following conditions.

- (1)  $H^i(\mathbf{M}) = 0$  for  $|i| \gg 0$ .
- (2)  $H^i(\mathbf{RHom}_R(\mathbf{D}, \mathbf{M})) = 0$  for  $i \gg 0$ .
- (3) The natural map  $\mathbf{D} \otimes_R^{\mathbf{L}} \mathbf{RHom}_R(\mathbf{D}, \mathbf{M}) \rightarrow \mathbf{M}$  is invertible in  $\mathcal{D}(R)$ .

**Theorem 9.5.** *Let  $R$  be Noetherian and assume that it has a dualizing complex. For every  $R$ -module  $M$ , the following conditions are equivalent.*

- (i)  $M$  has finite Gorenstein injective dimension.
- (ii)  $M$  belongs to  $\mathcal{B}(R)$ .

From the two theorems above and from Theorems 3.14 and 4.11 one gets:

**Corollary 9.6.** *Let  $R$  be Noetherian and assume that it has a dualizing complex. The following conditions are equivalent.*

- (i)  $R$  is Gorenstein.
- (ii) Every  $R$ -complex  $\mathbf{M}$  with  $H_i(\mathbf{M}) = 0$  for  $|i| \gg 0$  belongs to  $\mathcal{A}(R)$ .
- (iii) Every  $R$ -complex  $\mathbf{M}$  with  $H_i(\mathbf{M}) = 0$  for  $|i| \gg 0$  belongs to  $\mathcal{B}(R)$ .

## Gorenstein dimensions of complexes

It turns out to be convenient to extend the Gorenstein dimensions to complexes; this is illustrated by Theorem 9.11 below.

In the following we use the notion of a semi-projective resolution. Every complex has such a resolution, by [11, 1.6], and a projective resolution of a module is semi-projective. In view of this and Theorem 2.8, the next definition, which is due to Veliche [110, 3.1 and 3.4], extends Definition 2.7.

**Definition 9.7.** Let  $\mathbf{M}$  be an  $R$ -complex. If  $H(\mathbf{M}) \neq 0$ , then the *Gorenstein projective dimension* of  $\mathbf{M}$  is the least integer  $n$  such that  $H_i(\mathbf{M}) = 0$  for all  $i > n$  and there exists a semi-projective resolution  $\mathbf{P}$  of  $\mathbf{M}$  for which the module  $\text{Coker}(P_{n+1} \rightarrow P_n)$  is Gorenstein projective. If no such  $n$  exists, then  $\text{Gpd}_R \mathbf{M}$  is infinite. If  $H(\mathbf{M}) = 0$ , then  $\text{Gpd}_R \mathbf{M} = -\infty$  by convention.

In the next theorem, which is due to Iyengar and Krause [75, 8.1], the *unbounded Auslander class*  $\widehat{\mathcal{A}}(R)$  is the full subcategory of  $\mathcal{D}(R)$  whose objects satisfy conditions (2) and (3) in Definition 9.1.

**Theorem 9.8.** *Let  $R$  be Noetherian and assume that it has a dualizing complex. For every  $R$ -complex  $\mathbf{M}$ , the following conditions are equivalent.*

- (i)  $\mathbf{M}$  has finite Gorenstein projective dimension.
- (ii)  $\mathbf{M}$  belongs to  $\widehat{\mathcal{A}}(R)$ .

One finds the next definition in [4, 2.2 and 2.3] by Asadollahi and Salarian. It uses the notion of a semi-injective resolution. Every complex has such a resolution, by [11, 1.6], and an injective resolution of a module is semi-injective. In view of Theorem 3.6, the following extends Definition 3.5.

**Definition 9.9.** Let  $\mathbf{M}$  be an  $R$ -complex. If  $H(\mathbf{M}) \neq 0$ , then the *Gorenstein injective dimension* of  $\mathbf{M}$  is the least integer  $n$  such that  $H^i(\mathbf{M}) = 0$  for all  $i > n$  and there exists a semi-injective resolution  $\mathbf{I}$  of  $\mathbf{M}$  for which the module  $\text{Ker}(I^n \rightarrow I^{n+1})$  is Gorenstein injective. If no such integer  $n$  exists, then  $\text{Gid}_R \mathbf{M}$  is infinite. If  $H(\mathbf{M}) = 0$ , then  $\text{Gid}_R \mathbf{M} = -\infty$  by convention.

In the next result, which is [75, 8.2], the *unbounded Bass class*  $\widehat{\mathcal{B}}(R)$  is the full subcategory of  $\mathcal{D}(R)$  whose objects satisfy (2) and (3) in Definition 9.4.

**Theorem 9.10.** *Let  $R$  be Noetherian and assume that it has a dualizing complex. For every  $R$ -complex  $\mathbf{M}$ , the following conditions are equivalent.*

- (i)  $\mathbf{M}$  has finite Gorenstein injective dimension.
- (ii)  $\mathbf{M}$  belongs to  $\widehat{\mathcal{B}}(R)$ .

The next result is [35, 1.7]; it should be compared to Theorem 3.27.

**Theorem 9.11.** *Let  $R$  be local. For every  $R$ -module  $M$  there is an equality*

$$\text{Gid}_R M = \text{Gid}_{\widehat{R}} \mathbf{R}\text{Hom}_R(\widehat{R}, M).$$

*Remark 9.12.* Via this result, Theorem 9.10 gives a resolution-free characterization of modules of finite Gorenstein injective dimension over any local ring.

## Acyclicity versus total acyclicity

The next results characterize Gorenstein rings in terms of the complexes that define Gorenstein projective/injective/flat modules. The first one is [75, 5.5].

**Theorem 9.13.** *Let  $R$  be Noetherian and assume that it has a dualizing complex. Then the following conditions are equivalent.*

- (i)  $R$  is Gorenstein.
- (ii) For every acyclic complex  $\mathbf{P}$  of projective  $R$ -modules and every projective  $R$ -module  $Q$ , the complex  $\text{Hom}_R(\mathbf{P}, Q)$  is acyclic.
- (iii) For every acyclic complex  $\mathbf{I}$  of injective  $R$ -modules and every injective  $R$ -module  $E$ , the complex  $\text{Hom}_R(E, \mathbf{I})$  is acyclic.

In the terminology of [75], part (ii)/(iii) above says that every acyclic complex of projective/injective modules is totally acyclic.

The final result is due to Christensen and Veliche [37]:

**Theorem 9.14.** *Let  $R$  be Noetherian and assume that it has a dualizing complex. Then there exist acyclic complexes  $\mathbf{F}$  and  $\mathbf{I}$  of flat  $R$ -modules and injective  $R$ -modules, respectively, such that the following conditions are equivalent.*

- (i)  $R$  is Gorenstein.
- (ii) For every injective  $R$ -module  $E$ , the complex  $E \otimes_R \mathbf{F}$  is acyclic.
- (iii) For every injective  $R$ -module  $E$ , the complex  $\text{Hom}_R(E, \mathbf{I})$  is acyclic.

The complexes  $\mathbf{F}$  and  $\mathbf{I}$  in the theorem have explicit constructions. It is not known, in general, if there is an explicit construction of an acyclic complex  $\mathbf{P}$  of projective  $R$ -modules such that  $R$  is Gorenstein if  $\text{Hom}_R(\mathbf{P}, Q)$  is acyclic for every projective  $R$ -module  $Q$ .

## Notes

In broad terms, the theory of Gorenstein dimensions for modules extends to complexes. It is developed in detail by Asadollahi and Salarian [4], Christensen, Frankild, and Holm [31] and [32], Christensen and Sather-Wagstaff [35], and by Veliche [110].

Objects in Auslander and Bass classes with respect to semi-dualizing complexes have interpretations in terms of generalized Gorenstein dimensions; see [44] and [45] by Enochs and Jenda, [69] by Holm and Jørgensen, and [92] by Sather-Wagstaff.



Sharif, Sather-Wagstaff, and White [94] study totally acyclic complexes of Gorenstein projective modules. They show that the cokernels of the differentials in such complexes are Gorenstein projective. That is, a “Gorenstein projective” module is Gorenstein projective.

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# On $\nu$ -domains: a survey

Marco Fontana and Muhammad Zafrullah

**Abstract** An integral domain  $D$  is a  $\nu$ -domain if, for every finitely generated nonzero (fractional) ideal  $F$  of  $D$ , we have  $(FF^{-1})^{-1} = D$ . The  $\nu$ -domains generalize Prüfer and Krull domains and have appeared in the literature with different names. This paper is the result of an effort to put together information on this useful class of integral domains. In this survey, we present old, recent and new characterizations of  $\nu$ -domains along with some historical remarks. We also discuss the relationship of  $\nu$ -domains with their various specializations and generalizations, giving suitable examples.

## 1 Preliminaries and introduction

Let  $D$  be an integral domain with quotient field  $K$ . Let  $\overline{F}(D)$  be the set of all nonzero  $D$ -submodules of  $K$  and let  $\mathbf{F}(D)$  be the set of all nonzero fractional ideals of  $D$ , i.e.,  $A \in \mathbf{F}(D)$  if  $A \in \overline{F}(D)$  and there exists an element  $0 \neq d \in D$  with  $dA \subseteq D$ . Let  $f(D)$  be the set of all nonzero finitely generated  $D$ -submodules of  $K$ . Then, obviously  $f(D) \subseteq \mathbf{F}(D) \subseteq \overline{F}(D)$ .

Recall that a *star operation* on  $D$  is a map  $*$  :  $\mathbf{F}(D) \rightarrow \mathbf{F}(D)$ ,  $A \mapsto A^*$ , such that the following properties hold for all  $0 \neq x \in K$  and all  $A, B \in \mathbf{F}(D)$ :

- (\*<sub>1</sub>)  $D = D^*$ ,  $(xA)^* = xA^*$ ;
- (\*<sub>2</sub>)  $A \subseteq B$  implies  $A^* \subseteq B^*$ ;
- (\*<sub>3</sub>)  $A \subseteq A^*$  and  $A^{**} := (A^*)^* = A^*$ .

(the reader may consult [53, Sections 32 and 34] for a quick review of star operations).

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In [107], the authors introduced a useful generalization of the notion of a star operation: a *semistar operation* on  $D$  is a map  $\star : \overline{F}(D) \rightarrow \overline{F}(D)$ ,  $E \mapsto E^\star$ , such that the following properties hold for all  $0 \neq x \in K$  and all  $E, F \in \overline{F}(D)$ :

- ( $\star_1$ )  $(xE)^\star = xE^\star$ ;
- ( $\star_2$ )  $E \subseteq F$  implies  $E^\star \subseteq F^\star$ ;
- ( $\star_3$ )  $E \subseteq E^\star$  and  $E^{\star\star} := (E^\star)^\star = E^\star$ .

Clearly, a semistar operation  $\star$  on  $D$ , restricted to  $F(D)$ , determines a star operation if and only if  $D = D^\star$ .

If  $\star$  is a star operation on  $D$ , then we can consider the map  $\star_f : F(D) \rightarrow F(D)$  defined as follows:

$$A^{\star_f} := \bigcup \{F^\star \mid F \in f(D) \text{ and } F \subseteq A\} \quad \text{for all } A \in F(D).$$

It is easy to see that  $\star_f$  is a star operation on  $D$ , called the *star operation of finite type associated to  $\star$* . Note that  $F^\star = F^{\star_f}$  for all  $F \in f(D)$ . A star operation  $\star$  is called a *star operation of finite type* (or a *star operation of finite character*) if  $\star = \star_f$ . It is easy to see that  $(\star_f)_f = \star_f$  (i.e.,  $\star_f$  is of finite type).

If  $\star_1$  and  $\star_2$  are two star operations on  $D$ , we say that  $\star_1 \leq \star_2$  if  $A^{\star_1} \subseteq A^{\star_2}$  for all  $A \in F(D)$ . This is equivalent to saying that  $(A^{\star_1})^{\star_2} = A^{\star_2} = (A^{\star_2})^{\star_1}$  for all  $A \in F(D)$ . Obviously, for any star operation  $\star$  on  $D$ , we have  $\star_f \leq \star$ , and if  $\star_1 \leq \star_2$ , then  $(\star_1)_f \leq (\star_2)_f$ .

Let  $I \subseteq D$  be a nonzero ideal of  $D$ . We say that  $I$  is a  *$\star$ -ideal* of  $D$  if  $I^\star = I$ . We call a  $\star$ -ideal of  $D$  a  *$\star$ -prime ideal* of  $D$  if it is also a prime ideal and we call a maximal element in the set of all proper  $\star$ -ideals of  $D$  a  *$\star$ -maximal ideal* of  $D$ .

It is not hard to prove that a  $\star$ -maximal ideal is a prime ideal and that each proper  $\star_f$ -ideal is contained in a  $\star_f$ -maximal ideal.

Let  $\Delta$  be a set of prime ideals of an integral domain  $D$  and set

$$E^{\star_\Delta} := \bigcap \{ED_Q \mid Q \in \Delta\} \quad \text{for all } E \in \overline{F}(D).$$

The operation  $\star_\Delta$  is a semistar operation on  $D$  called the *spectral semistar operation associated to  $\Delta$* . Clearly, it gives rise to a star operation on  $D$  if (and only if)  $\bigcap \{D_Q \mid Q \in \Delta\} = D$ .

Given a star operation  $\star$  on  $D$ , when  $\Delta$  coincides with  $\text{Max}^{\star_f}(D)$ , the (nonempty) set of all  $\star_f$ -maximal ideals of  $D$ , the operation  $\tilde{\star}$  defined as follows:

$$A^{\tilde{\star}} := \bigcap \{AD_Q \mid Q \in \text{Max}^{\star_f}(D)\} \quad \text{for all } A \in F(D)$$

determines a star operation on  $D$ , called the *stable star operation of finite type associated to  $\star$* . It is not difficult to show that  $\tilde{\star} \leq \star_f \leq \star$ .

It is easy to see that, *mutatis mutandis*, all the previous notions can be extended to the case of a semistar operation.

Let  $A, B \in F(D)$ , set  $(A : B) := \{z \in K \mid zB \subseteq A\}$ ,  $(A :_D B) := (A : B) \cap D$ ,  $A^{-1} := (D : A)$ . As usual, we let  $\nu_D$  (or just  $\nu$ ) denote the  *$\nu$ -operation* defined by

$A^v := (D : (D : A)) = (A^{-1})^{-1}$  for all  $A \in F(D)$ . Moreover, we denote  $(v_D)_f$  by  $t_D$  (or just by  $t$ ), the  $t$ -operation on  $D$ ; and we denote the stable semistar operation of finite type associated to  $v_D$  (or, equivalently, to  $t_D$ ) by  $w_D$  (or, just by  $w$ ), i.e.,  $w_D := \widetilde{v}_D = \widetilde{t}_D$ .

Clearly,  $w_D \leq t_D \leq v_D$ . Moreover, from [53, Theorem 34.1(4)], we immediately deduce that  $* \leq v_D$ , and thus  $\widetilde{*} \leq w_D$  and  $*_f \leq t_D$ , for each star operation  $*$  on  $D$ .

Integral ideals that are maximal with respect to being  $*$ -ideals, when  $*$  =  $v$  or  $t$  or  $w$  are relevant in many situations. However, maximal  $v$ -ideals are not a common sight. There are integral domains, such as a nondiscrete rank one valuation domain, that do not have any maximal  $v$ -ideal [53, Exercise 12, p. 431]. Unlike maximal  $v$ -ideals, the maximal  $t$ -ideals are everywhere, in that every  $t$ -ideal is contained in at least one maximal  $t$ -ideal, which is always a prime ideal [80, Corollaries 1 and 2, pp. 30–31] (or, [93, Proposition 3.1.2], in the integral domains setting). Note also that the set of maximal  $t$ -ideals coincides with the set of maximal  $w$ -ideals [10, Theorem 2.16].

We will denote simply by  $d_D$  (or just  $d$ ) the identity star operation on  $D$  and clearly  $d_D \leq *$ , for each star operation  $*$  on  $D$ . Another important star operation on an integrally closed domain  $D$  is the  $b_D$ -operation (or just  $b$ -operation) defined as follows:

$$A^{b_D} := \bigcap \{AV \mid V \text{ is a valuation overring of } D\} \quad \text{for all } A \in F(D).$$

Given a star operation on  $D$ , for  $A \in F(D)$ , we say that  $A$  is  $*$ -finite if there exists a  $F \in f(D)$  such that  $F^* = A^*$ . (Note that in the above definition, we do not require that  $F \subseteq A$ .) It is immediate to see that if  $*_1 \leq *_2$  are star operations and  $A$  is  $*_1$ -finite, then  $A$  is  $*_2$ -finite. In particular, if  $A$  is  $*_f$ -finite, then it is  $*$ -finite. The converse is not true in general, and one can prove that  $A$  is  $*_f$ -finite if and only if there exists  $F \in f(D)$ ,  $F \subseteq A$ , such that  $F^* = A^*$  [126, Theorem 1.1].

Given a star operation on  $D$ , for  $A \in F(D)$ , we say that  $A$  is  $*$ -invertible if  $(AA^{-1})^* = D$ . From the fact that the set of maximal  $\widetilde{*}$ -ideals,  $\text{Max}^{\widetilde{*}}(D)$ , coincides with the set of maximal  $*_f$ -ideals,  $\text{Max}^{*_f}(D)$ , [10, Theorem 2.16], it easily follows that a nonzero fractional ideal  $A$  is  $\widetilde{*}$ -invertible if and only if  $A$  is  $*_f$ -invertible (note that if  $*$  is a star operation of finite type, then  $(AA^{-1})^* = D$  if and only if  $AA^{-1} \not\subseteq Q$  for all  $Q \in \text{Max}^*(D)$ ).

An invertible ideal is a  $*$ -invertible  $*$ -ideal for any star operation  $*$  and, in fact, it is easy to establish that, if  $*_1$  and  $*_2$  are two star operations on an integral domain  $D$  with  $*_1 \leq *_2$ , then any  $*_1$ -invertible ideal is also  $*_2$ -invertible.

A classical result due to Krull [80, Théorème 8, Chap. I, § 4] shows that for a star operation  $*$  of finite type,  $*$ -invertibility implies  $*$ -finiteness. More precisely, for  $A \in F(D)$ , we have that  $A$  is  $*_f$ -invertible if and only if  $A$  and  $A^{-1}$  are  $*_f$ -finite (hence, in particular,  $*$ -finite) and  $A$  is  $*$ -invertible (see [46, Proposition 2.6] for the semistar operation case).

We recall now some notions and properties of monoid theory needed later. A nonempty set with a binary associative and commutative law of composition “ $\cdot$ ” is called a *semigroup*. A *monoid*  $\mathcal{H}$  is a semigroup that contains an identity



element  $\mathbf{1}$  (i.e., an element such that, for all  $x \in \mathcal{H}$ ,  $\mathbf{1} \cdot x = x \cdot \mathbf{1} = x$ ). If there is an element  $\mathbf{o}$  in  $\mathcal{H}$  such that, for all  $x \in \mathcal{H}$ ,  $\mathbf{o} \cdot x = x \cdot \mathbf{o} = \mathbf{o}$ , we say that  $\mathcal{H}$  has a *zero element*. Finally if, for all  $a, x, y$  in a monoid  $\mathcal{H}$  with  $a \neq \mathbf{o}$ ,  $a \cdot x = a \cdot y$  implies that  $x = y$  we say that  $\mathcal{H}$  is a *cancellative monoid*. In what follows we shall be working with commutative and cancellative monoids with or without zero. Note that, if  $D$  is an integral domain then  $D$  can be considered as a monoid under multiplication and, more precisely,  $D$  is a cancellative monoid with zero element  $0$ .

Given a monoid  $\mathcal{H}$ , we can consider the set of invertible elements in  $\mathcal{H}$ , denoted by  $\mathcal{U}(\mathcal{H})$  (or, by  $\mathcal{H}^\times$ ) and the set  $\mathcal{H}^\bullet := \mathcal{H} \setminus \{\mathbf{o}\}$ . Clearly,  $\mathcal{U}(\mathcal{H})$  is a subgroup of (the monoid)  $\mathcal{H}^\bullet$  and the monoid  $\mathcal{H}$  is called a *groupoid* if  $\mathcal{U}(\mathcal{H}) = \mathcal{H}^\bullet$ . A monoid with a unique invertible element is called *reduced*. The monoid  $\mathcal{H}/\mathcal{U}(\mathcal{H})$  is reduced. A monoid shall mean a reduced monoid unless specifically stated.

Given a monoid  $\mathcal{H}$ , we can easily develop a divisibility theory and we can introduce a GCD. A *GCD-monoid* is a monoid having a uniquely determined GCD for each finite set of elements. In a monoid  $\mathcal{H}$  an element, distinct from the unit element  $\mathbf{1}$  and the zero element  $\mathbf{o}$ , is called *irreducible* (or, *atomic*) if it is divisible only by itself and  $\mathbf{1}$ . A monoid  $\mathcal{H}$  is called *atomic* if every nonzero noninvertible element of  $\mathcal{H}$  is a product of finitely many atoms of  $\mathcal{H}$ . A nonzero noninvertible element  $p \in \mathcal{H}$  with the property that  $p \mid a \cdot b$ , with  $a, b \in \mathcal{H}$  implies  $p \mid a$  or  $p \mid b$  is called a *prime* element. It is easy to see that in a GCD-monoid, irreducible and prime elements coincide.

Given a monoid  $\mathcal{H}$ , we can also form the monoids of fractions of  $\mathcal{H}$  and, when  $\mathcal{H}$  is cancellative, the groupoid of fractions  $q(\mathcal{H})$  of  $\mathcal{H}$  in the same manner, avoiding the zero element  $\mathbf{o}$  in the denominator, as in the constructions of the rings of fractions and the field of fractions of an integral domain  $D$ .

◇ ◇ ◇

This survey paper is the result of an effort to put together information on the important class of integral domains called  $\nu$ -domains, i.e., integral domains in which every finitely generated nonzero (fractional) ideal is  $\nu$ -invertible. In the present work, we will use a ring theoretic approach. However, because in multiplicative ideal theory we are mainly interested in the multiplicative structure of the integral domains, the study of monoids came into multiplicative ideal theory at an early stage. For instance, as we shall indicate in the sequel,  $\nu$ -domains came out of a study of monoids. During the second half of the 20th century, essentially due to the work of Griffin [57], and due to Gilmer's books [53] and [54], multiplicative ideal theory from a ring theoretic point of view became a hot topic for the ring theorists. However, things appear to be changing. Halter-Koch has put together in [59], in the language of monoids, essentially all that was available at that time and essentially all that could be translated to the language of monoids. On the other hand, more recently, Matsuda, under the influence of [54], is keen on converting into the language of additive monoids and semistar operations all that is available and permits conversion [95].

Since translation of results often depends upon the interest, motivation and imagination of the "translator", it is a difficult task to indicate what (and in which way)

can be translated into the language of monoids, multiplicative or additive, or to the language of semistar operations. But, one thing is certain, as we generalize, we gain a larger playground but, at the same time, we lose the clarity and simplicity that we had become so accustomed to.

With these remarks in mind, we indicate below some of the results that may or may not carry over to the monoid treatment, and we outline some general problems that can arise when looking for generalizations, without presuming to be exhaustive. The first and foremost is any result to do with polynomial ring extensions may not carry over to the language of monoids even though some of the concepts translated to monoids do get used in the study of semigroup rings. The other trouble-spot is the results on integral domains that use the identity ( $d$ -)operation. As soon as one considers the multiplicative monoid of an integral domain, with or without zero, some things get lost. For instance, the multiplicative monoid  $R \setminus \{0\}$  of a PID  $R$ , with more than one maximal ideal, is no longer a principal ideal monoid, because a monoid has only one maximal ideal, which in this case is not principal. All you can recover is that  $R \setminus \{0\}$  is a unique factorization monoid; similarly, from a Bézout domain you can recover a GCD-monoid. Similar comments can be made for Dedekind and Prüfer domains. On the other hand, if the  $\nu$ -operation is involved then nearly every result, other than the ones involving polynomial ring extensions, can be translated to the language of monoids. So, a majority of old ring theoretic results on  $\nu$ -domains and their specializations can be found in [59] and some in [95], in one form or another. We will mention or we will provide precise references only for those results on monoids that caught our fancy for one reason or another, as indicated in the sequel.

The case of semistar operations and the possibility of generalizing results on  $\nu$ -domains, and their specializations, in this setting is somewhat difficult in that the area of research has only recently opened up [107]. Moreover, a number of results involving semistar invertibility are now available, showing a more complex situation for the invertibility in the semistar operation setting see for instance [46, 109, 110]. However, in studying semistar operations, in connection with  $\nu$ -domains, we often gain deeper insight, as recent work indicates, see [6, 14].

## 2 When and in what context did the $\nu$ -domains show up?

### 2.1 The genesis

The  $\nu$ -domains are precisely the integral domains  $D$  for which the  $\nu$ -operation is an “endlich arithmetisch brauchbar” operation, cf. [52, p. 391]. Recall that a star operation  $*$  on an integral domain  $D$  is *endlich arithmetisch brauchbar* (for short, *e.a.b.*) (respectively, *arithmetisch brauchbar* (for short, *a.b.*)) if for all  $F, G, H \in f(D)$  (respectively,  $F \in f(D)$  and  $G, H \in F(D)$ )  $(FG)^* \subseteq (FH)^*$  implies that  $G^* \subseteq H^*$ .

In [90], the author only considered the concept of “a.b.  $*$ -operation” (actually, Krull’s original notation was “ $'$ -Operation”, instead of “ $*$ -operation”). He did not consider the (weaker) concept of “e.a.b.  $*$ -operation”.

The e.a.b. concept stems from the original version of Gilmer’s book [52]. The results of Section 26 in [52] show that this (presumably) weaker concept is all that one needs to develop a complete theory of Kronecker function rings. Robert Gilmer explained to us saying that  $\ll$  I believe I was influenced to recognize this because during the 1966 calendar year in our graduate algebra seminar (Bill Heinzer, Jimmy Arnold, and Jim Brewer, among others, were in that seminar) we had covered Bourbaki’s Chaps. 5 and 7 of *Algèbre Commutative*, and the development in Chap. 7 on the  $v$ -operation indicated that e.a.b. would be sufficient.  $\gg$

Apparently there are no examples in the literature of star operations which are e.a.b. but not a.b.. A forthcoming paper [45] (see also [44]) will contain an explicit example to show that Krull’s a.b. condition is really stronger than the Gilmer’s e.a.b. condition.

We asked Robert Gilmer and Joe Mott about the origins of  $v$ -domains. They had the following to say:  $\ll$  We believe that Prüfer’s paper [111] is the first to discuss the concept in complete generality, though we still do not know who came up with the name of “ $v$ -domain”.  $\gg$

However, the basic notion of  $v$ -ideal appeared around 1929. More precisely, the notion of quasi-equality of ideals (where, for  $A, B \in \mathbf{F}(D)$ ,  $A$  is *quasi-equal* to  $B$ , if  $A^{-1} = B^{-1}$ ), special cases of  $v$ -ideals and the observation that the classes of quasi-equal ideals of a Noetherian integrally closed domain form a group first appeared in [119] (cf. also [89, p. 121]), but this material was put into a more polished form by E. Artin and in this form was published for the first time by Bartel Leendert van der Waerden in “Modern Algebra” [120]. This book originated from notes taken by the author from E. Artin’s lectures and it includes research of E. Noether and her students. Note that the “ $v$ ” of a  $v$ -ideal (or a  $v$ -operation) comes from the German “Vielfachenideale” or “ $V$ -Ideale” (“ideal of multiples”), terminology used in [111, Section 7]. It is important to recall also the papers [16] and [91] that introduce the study of  $v$ -ideals and  $t$ -ideals in semigroups.

The paper [31] provides a clue to where  $v$ -domains came out as a separate class of rings, though they were not called  $v$ -domains there. Note that [31] has been cited in [80, p. 23] and, later, in [59, p. 216], where it is mentioned that J. Dieudonné gives an example of a  $v$ -domain that is not a Prüfer  $v$ -multiplication domain (for short,  $PvMD$ , i.e., an integral domain  $D$  in which every  $F \in f(D)$  is  $t$ -invertible).

## 2.2 Prüfer domains and $v$ -domains

The  $v$ -domains generalize the Prüfer domains (i.e., the integral domains  $D$  such that  $D_M$  is a valuation domain for all  $M \in \text{Max}(D)$ ), since an integral domain  $D$  is a Prüfer domain if and only if every  $F \in f(D)$  is invertible [53, Theorem 22.1]. Clearly, an invertible ideal is  $*$ -invertible for all star operations  $*$ . In particular, a

Prüfer domain is a *Prüfer  $*$ -multiplication domain* (for short,  $P^*MD$ , i.e., an integral domain  $D$  such that, for each  $F \in f(D)$ ,  $F$  is  $*$ -invertible [75, p. 48]). It is clear from the definitions that a  $P^*MD$  is a  $P\nu MD$  (since  $*$   $\leq \nu$  for all star operations  $*$ , cf. [53, Theorem 34.1]) and a  $P\nu MD$  is a  $\nu$ -domain.

The picture can be refined. M. Griffin, a student of Ribenboim's, showed that  $D$  is a  $P\nu MD$  if and only if  $D_M$  is a valuation domain for each maximal  $t$ -ideal  $M$  of  $D$  [57, Theorem 5]. A generalization of this result is given in [75, Theorem 1.1] by showing that  $D$  is a  $P^*MD$  if and only if  $D_Q$  is a valuation domain for each maximal  $*$ -ideal  $Q$  of  $D$ .

Call a *valuation overring*  $V$  of  $D$  *essential* if  $V = D_P$  for some prime ideal  $P$  of  $D$  (which is invariably the center of  $V$  over  $D$ ) and call  $D$  an *essential domain* if  $D$  is expressible as an intersection of its essential valuation overrings. Clearly, a Prüfer domain is essential and so it is a  $P^*MD$  and, in particular, a  $P\nu MD$  (because, in the first case,  $D = \bigcap D_Q$  where  $Q$  varies over maximal  $*$ -ideals of  $D$  and  $D_Q$  is a valuation domain; in the second case,  $D = \bigcap D_M$  where  $M$  varies over maximal  $t$ -ideals of  $D$  and  $D_M$  is a valuation domain; see [57, Proposition 4] and [84, Proposition 2.9]).

From a local point of view, it is easy to see from the definitions that every integral domain  $D$  that is locally essential is essential. The converse is not true and the first example of an essential domain having a prime ideal  $P$  such that  $D_P$  is not essential was given in [67].

Now add to this information the following well known result [85, Lemma 3.1] that shows that the essential domains sit in between  $P\nu MD$ 's and  $\nu$ -domains.

**Proposition 2.1.** *An essential domain is a  $\nu$ -domain.*

*Proof.* Let  $\Delta$  be a subset of  $\text{Spec}(D)$  such that  $D = \bigcap \{D_P \mid P \in \Delta\}$ , where each  $D_P$  is a valuation domain with center  $P \in \Delta$ , let  $F$  be a nonzero finitely generated ideal of  $D$ , and let  $*_\Delta$  be the star operation induced by the family of (flat) overrings  $\{D_P \mid P \in \Delta\}$  on  $D$ . Then

$$\begin{aligned} (FF^{-1})^{*\Delta} &= \bigcap \{(FF^{-1})D_P \mid P \in \Delta\} = \bigcap \{FD_P F^{-1}D_P \mid P \in \Delta\} \\ &= \bigcap \{FD_P(FD_P)^{-1} \mid P \in \Delta\} \quad (\text{because } F \text{ is f.g.}) \\ &= \bigcap \{D_P \mid P \in \Delta\} \quad (\text{because } D_P \text{ is a valuation domain}). \end{aligned}$$

Therefore,  $(FF^{-1})^{*\Delta} = D$  and so  $(FF^{-1})^\nu = D$  (since  $*_\Delta \leq \nu$  [53, Theorem 34.1]).

For an alternate implicit proof of Proposition 2.1, and much more, the reader may consult [124, Theorem 3.1 and Corollary 3.2].

*Remark 2.2.* (a) Note that Proposition 2.1 follows also from a general result for essential monoids [59, Exercise 21.6 (i), p.244], but the result as stated above (for essential domains) was already known for instance as an application of [125, Lemma 4.5].

If we closely look at [59, Exercise 21.6, p.244], we note that part (ii) was already known for the special case of integral domains (i.e., an essential domain is a  $P\nu MD$  if and only if the intersection of two principal ideals is a  $\nu$ -finite  $\nu$ -ideal, [122, Lemma 8]) and part (iii) is related to the following fact concerning integral

domains: for  $F \in f(D)$ ,  $F$  is  $t$ -invertible if and only if  $(F^{-1} : F^{-1}) = D$  and  $F^{-1}$  is  $v$ -finite. The previous property follows immediately from the following statements:

- (a.1) Let  $F \in f(D)$ , then  $F$  is  $t$ -invertible if and only if  $F$  is  $v$ -invertible and  $F^{-1}$  is  $v$ -finite;  
 (a.2) Let  $A \in \mathbf{F}(D)$ , then  $A$  is  $v$ -invertible if and only if  $(A^{-1} : A^{-1}) = D$ .

The statement (a.1) can be found in [127] and (a.2) is posted in [128]. For reader's convenience, we next give their proofs.

For the "only if part" of (a.1), if  $F \in f(D)$  is  $t$ -invertible, then  $F$  is clearly  $v$ -invertible and  $F^{-1}$  is also  $t$ -invertible. Hence,  $F^{-1}$  is  $t$ -finite and thus  $v$ -finite.

For a "semistar version" of (a.1), see for instance [46, Lemma 2.5].

For the "if part" of (a.2), note that  $AA^{-1} \subseteq D$  and so  $(AA^{-1})^{-1} \supseteq D$ . Let  $x \in (AA^{-1})^{-1}$ , hence  $xAA^{-1} \subseteq D$  and so  $xA^{-1} \subseteq A^{-1}$ , i.e.,  $x \in (A^{-1} : A^{-1}) = D$ . For the "only if part", note that in general  $D \subseteq (A^{-1} : A^{-1})$ . For the reverse inclusion, let  $x \in (A^{-1} : A^{-1})$ , hence  $xA^{-1} \subseteq A^{-1}$ . Multiplying both sides by  $A$  and applying the  $v$ -operation, we have  $xvD = x(AA^{-1})^v \subseteq (AA^{-1})^v = D$ , i.e.,  $x \in D$  and so  $D \supseteq (A^{-1} : A^{-1})$ . A simple proof of (a.2) can also be deduced from [59, Theorem 13.4].

It is indeed remarkable that all those results known for integral domains can be interpreted and extended to monoids.

(b) We have observed in (a) that a PvMD is an essential domain such that the intersection of two principal ideals is a  $v$ -finite  $v$ -ideal. It can be also shown that  $D$  is a PvMD if and only if  $(a) \cap (b)$  is  $t$ -invertible in  $D$ , for all nonzero  $a, b \in D$  [94, Corollary 1.8].

For  $v$ -domains we have the following " $v$ -version" of the previous characterization for PvMD's:

$$D \text{ is a } v\text{-domain} \Leftrightarrow (a) \cap (b) \text{ is } v\text{-invertible in } D, \text{ for all nonzero } a, b \in D.$$

The idea of proof is simple and goes along the same lines as those of PvMD's. Recall that every  $F \in f(D)$  is invertible (respectively,  $v$ -invertible;  $t$ -invertible) if and only if every nonzero two generated ideal of  $D$  is invertible (respectively,  $v$ -invertible;  $t$ -invertible) [111, p. 7] or [53, Theorem 22.1] (respectively, for the " $v$ -invertibility case", [99, Lemma 2.6]; for the " $t$ -invertibility case", [94, Lemma 1.7]); for the general case of star operations, see the following Remark 2.5 (c). Moreover, for all nonzero  $a, b \in D$ , we have:

$$(a, b)^{-1} = \frac{1}{a}D \cap \frac{1}{b}D = \frac{1}{ab}(aD \cap bD),$$

$$(a, b)(a, b)^{-1} = \frac{1}{ab}(a, b)(aD \cap bD).$$

Therefore, in particular, the fractional ideal  $(a, b)^{-1}$  (or, equivalently,  $(a, b)$ ) is  $v$ -invertible if and only if the ideal  $aD \cap bD$  is  $v$ -invertible.

(c) Note that, by the observations contained in the previous point (b), if  $D$  is a Prüfer domain then  $(a) \cap (b)$  is invertible in  $D$ , for all nonzero  $a, b \in D$ . However, the converse is not true, as we will see in Sections 2.3 and 2.5 (Irreversibility of

$\Rightarrow_7$ ). The reason for this is that  $aD \cap bD$  invertible allows only that the ideal  $\frac{(a,b)^v}{ab}$  (or, equivalently,  $(a,b)^v$ ) is invertible and not necessarily the ideal  $(a,b)$ .

Call a *P-domain* an integral domain such that every ring of fractions is essential (or, equivalently, a *locally essential domain*, i.e., an integral domain  $D$  such that  $D_P$  is essential, for each prime ideal  $P$  of  $D$ ) [100, Proposition 1.1]. Note that every ring of fractions of a PvMD is still a PvMD (see Section 3 for more details), in particular, since a PvMD is essential, a locally PvMD is a P-domain. Examples of P-domains include Krull domains. As a matter of fact, by using Griffin’s characterization of PvMD’s [57, Theorem 5], a Krull domain is a PvMD, since in a Krull domain  $D$  the maximal  $t$ -ideals (= maximal  $v$ -ideals) coincide with the height 1 prime ideals [53, Corollary 44.3 and 44.8] and  $D = \bigcap \{D_P \mid P \text{ is an height 1 prime ideal of } D\}$ , where  $D_P$  is a discrete valuation domain for all height 1 prime ideals  $P$  of  $D$  [53, (43.1)]. Furthermore, it is well known that every ring of fractions of a Krull domain is still a Krull domain [24, BAC, Chap. 7, § 1, N. 4, Proposition 6].

With these observations at hand, we have the following picture:

$$\begin{aligned} \text{Krull domain} &\Rightarrow_0 \text{PvMD}; \\ \text{Prüfer domain} &\Rightarrow_1 \text{PvMD} \Rightarrow_2 \text{locally PvMD} \\ &\Rightarrow_3 \text{P-domain} \Rightarrow_4 \text{essential domain} \\ &\Rightarrow_5 v\text{-domain}. \end{aligned}$$

*Remark 2.3.* Note that P-domains were originally defined as the integral domains  $D$  such that  $D_Q$  is a valuation domain for every *associated prime ideal*  $Q$  of a *principal ideal of*  $D$  (i.e., for every prime ideal which is minimal over an ideal of the type  $(aD : bD)$  for some  $a \in D$  and  $b \in D \setminus aD$ ) [100, p. 2]. The P-domains were characterized in a somewhat special way in [108, Corollary 2.3]:  $D$  is a P-domain if and only if  $D$  is integrally closed and, for each  $u \in K$ ,  $D \subseteq D[u]$  satisfies INC at every associated prime ideal  $Q$  of a principal ideal of  $D$ .

### 2.3 Bézout-type domains and $v$ -domains

Recall that an integral domain  $D$  is a *Bézout domain* if every finitely generated ideal of  $D$  is principal and  $D$  is a *GCD domain* if, for all nonzero  $a, b \in D$ , a greatest common divisor of  $a$  and  $b$ ,  $\text{GCD}(a,b)$ , exists and is in  $D$ . Among the characterizations of the GCD domains we have that  $D$  is a GCD domain if and only if, for every  $F \in f(D)$ ,  $F^v$  is principal or, equivalently, if and only if the intersection of two (integral) principal ideals of  $D$  is still principal (see, for instance, [2, Theorem 4.1] and also Remark 2.2 (b)). From Remark 2.2 (b), we deduce immediately that a GCD domain is a  $v$ -domain.

However, in between GCD domains and  $v$ -domains lie several other distinguished classes of integral domains. An important generalization of the notion of GCD domain was introduced in [3] where an integral domain  $D$  is called a *Generalized GCD* (for short, *GGCD*) *domain* if the intersection of two (integral) invertible ideals of  $D$  is invertible  $D$ . It is well known that  $D$  is a GGCD domain if and only if,

for each  $F \in f(D)$ ,  $F^v$  is invertible [3, Theorem 1]. In particular, a Prüfer domain is a GGCD domain. From the fact that an invertible ideal in a local domain is principal [86, Theorem 59], we easily deduce that a GGCD domain is locally a GCD domain. On the other hand, from the definition of PvMD, we easily deduce that a GCD domain is a PvMD (see also [2, Section 3]). Therefore, we have the following addition to the existing picture:

$$\begin{aligned} \text{Bézout domain} &\Rightarrow_6 \text{GCD domain} \Rightarrow_7 \text{GGCD domain} \\ &\Rightarrow_8 \text{locally GCD domain} \Rightarrow_9 \text{locally PvMD} \\ &\Rightarrow_3 \dots \Rightarrow_4 \dots \Rightarrow_5 \text{v-domain.} \end{aligned}$$

## 2.4 Integral closures and $v$ -domains

Recall that an integral domain  $D$  with quotient field  $K$  is called a *completely integrally closed* (for short, *CIC*) *domain* if  $D = \{z \in K \mid \text{for all } n \geq 0, az^n \in D \text{ for some nonzero } a \in D\}$ . It is well known that *the following statements are equivalent*.

- (i)  $D$  is CIC;
- (ii) for all  $A \in F(D)$ ,  $(A^v : A^v) = D$ ;
- (ii') for all  $A \in F(D)$ ,  $(A : A) = D$ ;
- (ii'') for all  $A \in F(D)$ ,  $(A^{-1} : A^{-1}) = D$ ;
- (iii) for all  $A \in F(D)$ ,  $(AA^{-1})^v = D$ ;

(see [53, Theorem 34.3] and Remark 2.2 (a.2); for a general monoid version of this characterization, see [59, p. 156]).

In Bourbaki [24, BAC, Chap. 7, § 1, Exercice 30], an integral domain  $D$  is called *regularly integrally closed* if, for all  $F \in f(D)$ ,  $F^v$  is regular with respect to the  $v$ -multiplication (i.e., if  $(FG)^v = (FH)^v$  for  $G, H \in f(D)$  then  $G^v = H^v$ ).

**Theorem 2.4.** ([53, Theorem 34.6] and [24, BAC, Chap. 7, § 1, Exercice 30 (b)]) *Let  $D$  be an integral domain, then the following are equivalent.*

- (i)  $D$  is a regularly integrally closed domain.
- (ii) <sub>$f$</sub>  For all  $F \in f(D)$ ,  $(F^v : F^v) = D$ .
- (iii) <sub>$f$</sub>  For all  $F \in f(D)$   $(FF^{-1})^{-1} = D$  (or, equivalently,  $(FF^{-1})^v = D$ ).
- (iv)  $D$  is a  $v$ -domain.

The original version of Theorem 2.4 appeared in [91, p. 538] (see also [31, p. 139] and [79, Theorem 13]). A general monoid version of the previous characterization is given in [59, Theorem 19.2].

*Remark 2.5.* (a) Note that the condition

$$(ii')_f \text{ for all } F \in f(D), (F : F) = D$$

is equivalent to say that  $D$  is integrally closed [53, Proposition 34.7] and so it is weaker than condition (ii) <sub>$f$</sub>  of the previous Theorem 2.4, since  $(F^v : F^v) = (F^v : F) \supseteq (F : F)$ .

On the other hand, by Remark 2.2 (a.2), the condition (ii')<sub>f</sub> for all  $F \in f(D)$ ,  $(F^{-1} : F^{-1}) = D$

is equivalent to the other statements of Theorem 2.4.

(b) By [99, Lemma 2.6], condition (iii')<sub>f</sub> of the previous theorem is equivalent to (iii)<sub>2</sub>) Every nonzero fractional ideal with two generators is  $\nu$ -invertible.

This characterization is a variation of Prüfer's classical result that an integral domain is Prüfer if and only if each nonzero ideal with two generators is invertible (Remark 2.2 (b)) and of the characterization of PvMD's also recalled in that remark.

(c) Note that several classes of Prüfer-like domains can be studied in a unified frame by using star and semistar operations. For instance Prüfer star-multiplication domains were introduced in [75]. Later, in [39], the authors studied Prüfer semistar-multiplication domains and gave several characterizations of these domains, that are new also for the classical case of PvMD's. Other important contributions, in general settings, were given recently in [110] and [63].

In [6, Section 2], given a star operation  $*$  on an integral domain  $D$ , the authors call  $D$  a  $*$ -Prüfer domain if every nonzero finitely generated ideal of  $D$  is  $*$ -invertible (i.e.,  $(FF^{-1})^* = D$  for all  $F \in f(D)$ ). (Note that  $*$ -Prüfer domains were previously introduced in the case of semistar operations  $\star$  under the name of  $\star$ -domains [47, Section 2].) Since a  $*$ -invertible ideal is always  $\nu$ -invertible, a  $*$ -Prüfer domain is always a  $\nu$ -domain. More precisely,  $d$ -Prüfer (respectively,  $t$ -Prüfer;  $\nu$ -Prüfer) domains coincide with Prüfer (respectively, Prüfer  $\nu$ -multiplication;  $\nu$ -) domains.

Note that, in [6, Theorem 2.2], the authors show that a star operation version of (iii)<sub>2</sub>) considered in point (b) characterizes  $*$ -Prüfer domains, i.e.,  $D$  is a  $*$ -Prüfer domain if and only if every nonzero two generated ideal of  $D$  is  $*$ -invertible. An analogous result, in the general setting of monoids, can be found in [59, Lemma 17.2].

(d) Let  $f^\nu(D) := \{F^\nu \mid F \in f(D)\}$  be the set of all divisorial ideals of finite type of an integral domain  $D$  (in [31], this set is denoted by  $\mathfrak{M}_f$ ). By Theorem 2.4, we have that a  $\nu$ -domain is an integral domain  $D$  such that each element  $F^\nu \in f^\nu(D)$  is  $\nu$ -invertible, but  $F^{-1}$  ( $= (F^\nu)^{-1}$ ) does not necessarily belong to  $f^\nu(D)$ . When (and only when), in a  $\nu$ -domain  $D$ ,  $F^{-1} \in f^\nu(D)$  for each  $F \in f^\nu(D)$ ,  $D$  is a PvMD (Remark 2.2 (a.1)).

The “regular” terminology for the elements of  $f^\nu(D)$  used by [31, p. 139] (see the above definition of  $F^\nu$  regular with respect to the  $\nu$ -multiplication) is totally different from the notion of “von Neumann regular”, usually considered for elements of a ring or of a semigroup. However, it may be instructive to record some observations showing that, in the present situation, the two notions are somehow related.

Recall that, by a Clifford semigroup, we mean a multiplicative commutative semigroup  $\mathcal{H}$ , containing a unit element, such that each element  $a$  of  $\mathcal{H}$  is von Neumann regular (this means that there is  $b \in \mathcal{H}$  such that  $a^2b = a$ ).

- ( $\alpha$ ) Let  $\mathcal{H}$  be a commutative and cancellative monoid. If  $\mathcal{H}$  is a Clifford semigroup, then  $a$  is invertible in  $\mathcal{H}$  (and conversely); in other words,  $\mathcal{H}$  is a group.
- ( $\beta$ ) Let  $D$  be a  $\nu$ -domain. If  $A \in f^\nu(D)$  is von Neumann regular in the monoid  $f^\nu(D)$  under  $\nu$ -multiplication, then  $A$  is  $t$ -invertible (or, equivalently,  $A^{-1} \in$



$f^v(D)$ ). Consequently, an integral domain  $D$  is a PvMD if and only if  $D$  is a  $v$ -domain and the monoid  $f^v(D)$  (under  $v$ -multiplication) is Clifford regular.

The proofs of  $(\alpha)$  and  $(\beta)$  are straightforward, after recalling that  $f^v(D)$  under  $v$ -multiplication is a commutative monoid and, by definition, it is cancellative if  $D$  is a  $v$ -domain.

Note that, in the “if part” of  $(\beta)$ , the assumption that  $D$  is a  $v$ -domain is essential. As a matter of fact, it is not true that an integral domain  $D$ , such that every member of the monoid  $f^v(D)$  under the  $v$ -operation is von Neumann regular, is a  $v$ -domain. For instance, in [129, Theorem 11] (see also [30]), the authors show that for every quadratic order  $D$ , each nonzero ideal  $I$  of  $D$  satisfies  $I^2J = cI$ , i.e.,  $I^2J(1/c) = I$ , for some (nonzero) ideal  $J$  of  $D$  and some nonzero  $c \in D$ . So, in particular, in this situation  $f(D) = F(D)$  and every element of the monoid  $f^v(D)$  is von Neumann regular (we do not even need to apply the  $v$ -operation in this case), however not all quadratic orders are integrally closed (e.g.,  $D := \mathbb{Z}[\sqrt{5}]$ ) and so, in general, not all elements of  $f^v(D)$  are regular with respect to the  $v$ -operation (i.e.,  $D$  is not a  $v$ -domain).

Clifford regularity for class and  $t$ -class semigroups of ideals in various types of integral domains was investigated, for instance, in [20 and 21, Bazzoni (1996), (2001)] [49], [71, 72 and 73, Kabbaj-Mimouni, (2003), (2007), (2008)], [116], and [54 and 55, Halter-Koch (2007), (2008)]. In particular, in the last paper, Halter-Koch proves a stronger and much deeper version of  $(\beta)$ , that is, a  $v$ -domain having its  $t$ -class semigroups of ideals Clifford regular is a domain of Krull-type (i.e., a PvMD with finite  $t$ -character). This result generalizes [82, Theorem 3.2] on Prüfer  $v$ -multiplication domains.

(e) In the situation of point (d,  $\beta$ ), the condition that every  $v$ -finite  $v$ -ideal is regular, in the sense of von Neumann, in the larger monoid  $F^v(D) := \{A^v \mid A \in F(D)\}$  of all  $v$ -ideals of  $D$  (under  $v$ -multiplication) is too weak to imply that  $D$  is a PvMD.

As a matter of fact, if we assume that  $D$  is a  $v$ -domain, then every  $A \in f^v(D)$  is  $v$ -invertible in the (larger) monoid  $F^v(D)$ . Therefore,  $A$  is von Neumann regular in  $F^v(D)$ , since  $(AB)^v = D$  for some  $B \in F^v(D)$  and thus, multiplying both sides by  $A$  and applying the  $v$ -operation, we get  $(A^2B)^v = A$ .

*Remark 2.6.* Regularly integrally closed integral domains make their appearance with a different terminology in the study of a weaker form of integrality, introduced in the paper [15]. Recall that, given an integral domain  $D$  with quotient field  $K$ , an element  $z \in K$  is called *pseudo-integral over  $D$*  if  $z \in (F^v : F^v)$  for some  $F \in f(D)$ . The terms *pseudo-integral closure* (i.e.,  $\tilde{D} := \bigcup \{(F^v : F^v) \mid F \in f(D)\}$ ) and *pseudo-integrally closed domain* (i.e.,  $D = \tilde{D}$ ) are coined in the obvious fashion and it is clear from the definition that pseudo-integrally closed coincides with regularly integrally closed.

From the previous observations, we have the following addition to the existing picture:

$$\text{CIC domain} \Rightarrow_{10} v\text{-domain} \Rightarrow_{11} \text{integrally closed domain.}$$

Note that in the Noetherian case, the previous three classes of domains coincide (see the following Proposition 2.8 (2) or [53, Theorem 34.3 and Proposition 34.7]). Recall also that Krull domains can be characterized by the property that, for all  $A \in \mathbf{F}(D)$ ,  $A$  is  $t$ -invertible [85, Theorem 3.6]. This property is clearly stronger than the condition (iii<sub>f</sub>) of previous Theorem 2.4 and, more precisely, it is strictly stronger than (iii<sub>f</sub>), since a Krull domain is CIC (by condition (iii) of the above characterizations of CIC domains, see also [24, BAC, Chap. 7, § 1, N. 3, Théorème 2]) and a CIC domain is a  $v$ -domain, but the converse does not hold, as we will see in the following Section 2.5.

*Remark 2.7.* Note that Okabe and Matsuda [106] generalized pseudo-integral closure to the star operation setting. Given a star operation  $*$  on an integral domain  $D$ , they call the  $*$ -integral closure of  $D$  its overring  $\bigcup\{(F^* : F^*) \mid F \in f(D)\}$  denoted by  $\text{cl}^*(D)$  in [58]. Note that, in view of this notation,  $\tilde{D} = \text{cl}^v(D)$  (Remark 2.6) and the integral closure  $\bar{D}$  of  $D$  coincides with  $\text{cl}^d(D)$  [53, Proposition 34.7]. Clearly, if  $*_1$  and  $*_2$  are two star operations on  $D$  and  $*_1 \leq *_2$ , then  $\text{cl}^{*_1}(D) \subseteq \text{cl}^{*_2}(D)$ . In particular, for each star operation  $*$  on  $D$ , we have  $\bar{D} \subseteq \text{cl}^*(D) \subseteq \tilde{D}$ .

It is not hard to see that  $\text{cl}^*(D)$  is integrally closed [106, Theorem 2.8] and is contained in the complete integral closure of  $D$ , which coincides with  $\bigcup\{(A : A) \mid A \in \mathbf{F}(D)\}$  [53, Theorem 34.3].

Recall also that, in [59, Section 3], the author introduces a star operation of finite type on the integral domain  $\text{cl}^*(D)$ , that we denote here by  $\text{cl}^*$ , defined as follows, for all  $G \in f(\text{cl}^*(D))$ :

$$G^{\text{cl}^*} := \bigcup\{((F^* : F^*)G)^* \mid F \in f(\text{cl}^*(D))\}.$$

Clearly,  $D^{\text{cl}^*} = \text{cl}^*(D)$ . Call an integral domain  $D$   $*$ -integrally closed when  $D = \text{cl}^*(D)$ . Then, from the fact that  $\text{cl}^*$  is a star operation on  $\text{cl}^*(D)$ , it follows that  $\text{cl}^*(D)$  is  $\text{cl}^*$ -integrally closed. In general, if  $D$  is not necessarily  $*$ -integrally closed, then  $\text{cl}^*$ , defined on  $f(D)$ , gives rise naturally to a semistar operation (of finite type) on  $D$  [41, Definition 4.2].

Note that the domain  $\tilde{D} (= \text{cl}^v(D))$ , even if it is  $\text{cl}^v$ -integrally closed, in general is not  $v_{\tilde{D}}$ -integrally closed; a counterexample is given in [15, Example 2.1] by using a construction due to [55]. On the other hand, since an integral domain  $D$  is a  $v$ -domain if and only if  $D = \text{cl}^v(D)$  (Theorem 2.4), from the previous observation we deduce that, in general,  $\tilde{D}$  is not a  $v$ -domain. On the other hand, using a particular “ $D + M$  construction”, in [106, Example 3.4], the authors construct an example of a non- $v$ -domain  $D$  such that  $\tilde{D}$  is a  $v$ -domain, i.e.,  $D \subsetneq \tilde{D} = \text{cl}^{v_{\tilde{D}}}(\tilde{D})$ .

### 2.5 Irreversibility of the implications “ $\Rightarrow_n$ ”

We start by observing that, under standard finiteness assumptions, several classes of domains considered above coincide. Recall that an integral domain  $D$  is called  $v$ -coherent if a finite intersection of  $v$ -finite  $v$ -ideals is a  $v$ -finite  $v$ -ideal or,

equivalently, if  $F^{-1}$  is  $v$ -finite for all  $F \in f(D)$  [35, Proposition 3.6], and it is called a  $v$ -finite conductor domain if the intersection of two principal ideals is  $v$ -finite [33]. From the definitions, it follows that a  $v$ -coherent domain is a  $v$ -finite conductor domain. From Remark 2.2 (a.1), we deduce immediately that

$$D \text{ is a PvMD} \Leftrightarrow D \text{ is a } v\text{-coherent } v\text{-domain.}$$

In case of a  $v$ -domain, the notions of  $v$ -finite conductor domain and  $v$ -coherent domain coincide. As a matter of fact, as we have observed in Remark 2.5 (c), a PvMD is exactly a  $t$ -Prüfer domain and an integral domain  $D$  is  $t$ -Prüfer if and only if every nonzero two generated ideal is  $t$ -invertible. This translates to  $D$  is a PvMD if and only if  $(a, b)$  is  $v$ -invertible and  $(a) \cap (b)$  is  $v$ -finite, for all  $a, b \in D$  (see also Remark 2.5 (b)). In other words,

$$D \text{ is a PvMD} \Leftrightarrow D \text{ is a } v\text{-finite conductor } v\text{-domain.}$$

Recall that an integral domain  $D$  is a GGCD domain if and only if  $D$  is a PvMD that is a locally GCD domain [3, Corollary 1 and p. 218] or [124, Corollary 3.4]. On the other hand, we have already observed that a locally GCD domain is essential and it is known that an essential  $v$ -finite conductor domain is a PvMD [122, Lemma 8]. The situation is summarized in the following:

**Proposition 2.8.** *Let  $D$  be an integral domain.*

- (1) *Assume that  $D$  is a  $v$ -finite conductor (e.g., Noetherian) domain. Then, the following classes of domains coincide:*
  - (a) *PvMD's;*
  - (b) *locally PvMD's;*
  - (c)  *$P$ -domains;*
  - (d) *essential domains.*
  - (e) *locally  $v$ -domains;*
  - (f)  *$v$ -domains.*
- (2) *Assume that  $D$  is a Noetherian domain. Then, the previous classes of domains (a)–(f) coincide also with the following:*
  - (g) *Krull domains;*
  - (h) *CIC domains;*
  - (i) *integrally closed domains.*
- (3) *Assume that  $D$  is a  $v$ -finite conductor (e.g., Noetherian) domain. Then, the following classes of domains coincide:*
  - (j) *GGCD domains;*
  - (k) *locally GCD domains.*

Since the notion of Noetherian Bézout (respectively, Noetherian GCD) domain coincides with the notion of PID or principal ideal domain (respectively, of Noetherian UFD (= unique factorization domain) [53, Proposition 16.4]), in the Noetherian case the picture of all classes considered above reduces to the following:

$$\begin{aligned} \text{Dedekind domain} &\Rightarrow_{1,2,3,4,5} v\text{-domain} \\ \text{PID} &\Rightarrow_6 \text{UFD} \Rightarrow_{7,8} \text{locally UFD} \Rightarrow_{9,3,4,5} v\text{-domain.} \end{aligned}$$

In general, of the implications  $\Rightarrow_n$  (with  $0 \leq n \leq 11$ ) discussed above all, except  $\Rightarrow_3$ , are known to be irreversible. We leave the case of irreversibility of  $\Rightarrow_3$  as an open question and proceed to give examples to show that all the other implications are irreversible.

- Irreversibility of  $\Rightarrow_0$ . Take any nondiscrete valuation domain or, more generally, a Prüfer non-Dedekind domain.

- Irreversibility of  $\Rightarrow_1$  (even in the Noetherian case). Let  $D$  be a Prüfer domain that is not a field and let  $X$  be an indeterminate over  $D$ . Then, as  $D[X]$  is a PvMD if and only if  $D$  is [93, Theorem 4.1.6] (see also [4, Proposition 6.5], [84, Theorem 3.7], [12, Corollary 3.3], and the following Section 4), we conclude that  $D[X]$  is a PvMD that is not Prüfer. An explicit example is  $\mathbb{Z}[X]$ , where  $\mathbb{Z}$  is the ring of integers.

- Irreversibility of  $\Rightarrow_2$ . It is well known that every ring of fractions of a PvMD is again a PvMD [69, Proposition 1.8] (see also the following Section 3). The fact that  $\Rightarrow_2$  is not reversible has been shown by producing examples of locally PvMD's that are not PvMD's. In [100, Example 2.1] an example of a non PvMD essential domain due to Heinzer and Ohm [69] was shown to have the property that it was locally PvMD and hence a P-domain.

- Irreversibility of  $\Rightarrow_3$ : Open. However, as mentioned above, [100, Example 2.1] shows the existence of a P-domain which is not a PvMD. Note that [125, Section 2] gives a general method of constructing P-domains that are not PvMD's.

- Irreversibility of  $\Rightarrow_4$ . An example of an essential domain which is not a P-domain was constructed in [67]. Recently, in [40, Example 2.3], the authors show the existence of  $n$ -dimensional essential domains which are not P-domains, for all  $n \geq 2$ .

- Irreversibility of  $\Rightarrow_5$ . Note that, by  $\Rightarrow_{10}$ , a CIC domain is a  $v$ -domain and Nagata solving with a counterexample a famous conjecture stated by Krull in 1936, has produced an example of a one dimensional quasilocal CIC domain that is not a valuation ring (cf. [101, 102, 114]). This proves that a  $v$ -domain may not be essential. It would be desirable to have an example of a nonessential  $v$ -domain that is simpler than Nagata's example.

- Irreversibility of  $\Rightarrow_6$  (even in the Noetherian case). This case can be handled in the same manner as that of  $\Rightarrow_1$ , since a polynomial domain over a GCD domain is still a GCD domain (cf. [86, Exercise 9, p. 42]).

- Irreversibility of  $\Rightarrow_7$  (even in the Noetherian case). Note that a Prüfer domain is a GGCD domain, since a GGCD domain is characterized by the fact that  $F^\vee$  is invertible for all  $F \in f(D)$  [3, Theorem 1]. Moreover, a Prüfer domain  $D$  is a Bézout domain if and only if  $D$  is GCD. In fact, according to [28] a Prüfer domain  $D$  is Bézout if and only if  $D$  is a generalization of GCD domains called a *Schreier domain* (i.e., an integrally closed integral domain whose group of divisibility is a *Riesz group*, that is a partially ordered directed group  $G$  having the following interpolation property: given  $a_1, a_2, \dots, a_m, b_1, b_2, \dots, b_n \in G$  with  $a_i \leq b_j$ , there exists  $c \in G$  with  $a_i \leq c \leq b_j$  see [28] and also [2, Section 3]). Therefore, a Prüfer non-Bézout domain (e.g., a Dedekind non principal ideal domain, like  $\mathbb{Z}[i\sqrt{5}]$ ) shows the irreversibility of  $\Rightarrow_7$ .

- Irreversibility of  $\Rightarrow_8$ . From the characterization of GGCD domains recalled in the irreversibility of  $\Rightarrow_7$  [3, Theorem 1], it follows that a GGCD domain is a PvMD. More precisely, as we have already observed just before Proposition 2.8, an integral domain  $D$  is a GGCD domain if and only if  $D$  is a PvMD that is a locally GCD domain. Finally, as noted above, there are examples in [125] of locally GCD domains that are not PvMD's. More explicitly, let  $E$  be the ring of entire functions (i.e., complex functions that are analytic in the whole plane). It is well known that  $E$  is a Bézout domain and every nonzero non unit  $x \in E$  is uniquely expressible as an associate of a “countable” product  $x = \prod p_i^{e_i}$ , where  $e_i \geq 0$  and  $p_i$  is an irreducible function (i.e., a function having a unique root) [70, Theorems 6 and 9]. Let  $S$  be the multiplicative set of  $E$  generated by the irreducible functions and let  $X$  be an indeterminate over  $E$ , then  $E + XE_S[X]$  is a locally GCD domain that is not a PvMD [125, Example 2.6 and Proposition 4.1].

- Irreversibility of  $\Rightarrow_9$  (even in the Noetherian case). This follows easily from the fact that there do exist examples of Krull domains (which we have already observed are locally PvMD's) that are not locally factorial (e.g., a non-UFD local Noetherian integrally closed domain, like the power series domain  $D[[X]]$  constructed in [115], where  $D$  is a two dimensional local Noetherian UFD). As a matter of fact, a Krull domain which is a GCD domain is a UFD, since in a GCD domain, for all  $F \in f(D)$ ,  $F^v$  is principal and so the class group  $Cl(D) = 0$  [25, Section 2]; on the other hand, a Krull domain is factorial if and only if  $Cl(D) = 0$  [48, Proposition 6.1].

- Irreversibility of  $\Rightarrow_{10}$ . Let  $R$  be an integral domain with quotient field  $L$  and let  $X$  be an indeterminate over  $L$ . By [29, Theorem 4.42]  $T := R + XL[X]$  is a  $v$ -domain if and only if  $R$  is a  $v$ -domain. Therefore, if  $R$  is not equal to  $L$ , then obviously  $T$  is an example of a  $v$ -domain that is not completely integrally closed (the complete integral closure of  $T$  is  $L[X]$  [53, Lemma 26.5]). This establishes that  $\Rightarrow_{10}$  is not reversible.

Note that, in [35, Section 4] the transfer in pullback diagrams of the PvMD property and related properties is studied. A characterization of  $v$ -domains in pullbacks is proved in [50, Theorem 4.15]. We summarize these results in the following:

**Theorem 2.9.** *Let  $R$  be an integral domain with quotient field  $k$  and let  $T$  be an integral domain with a maximal ideal  $M$  such that  $L := T/M$  is a field extension of  $k$ . Let  $\varphi : T \rightarrow L$  be the canonical projection and consider the following pullback diagram:*

$$\begin{array}{ccc}
 D := \varphi^{-1}(R) & \longrightarrow & R \\
 \downarrow & & \downarrow \\
 T_1 := \varphi^{-1}(k) & \longrightarrow & k \\
 \downarrow & & \downarrow \\
 T & \xrightarrow{\varphi} & L
 \end{array}$$

*Then,  $D$  is a  $v$ -domain (respectively, a PvMD) if and only if  $k = L$ ,  $T_M$  is a valuation domain and  $R$  and  $T$  are  $v$ -domains (respectively, PvMD's).*

*Remark 2.10.* Recently, bringing to a sort of close a lot of efforts to restate results of [29] in terms of very general pullbacks, in the paper [76], the authors use some remarkable techniques to prove a generalization of the previous theorem. Although that paper is not about  $v$ -domains in particular, it does have a few good results on  $v$ -domains. One of these results will be recalled in Proposition 3.6. Another one, with a pullback flavor, can be stated as follows: *Let  $I$  be a nonzero ideal of an integral domain  $D$  and set  $T := (I : I)$ . If  $D$  is a  $v$ -domain (respectively, a PvMD) then  $T$  is a  $v$ -domain (respectively, a PvMD)* [76, Proposition 2.5].

- Irreversibility of  $\Rightarrow_{11}$ . Recall that an integral domain  $D$  is called a *Mori domain* if  $D$  satisfies ACC on its integral divisorial ideals. According to [103, Lemma 1] or [112],  $D$  is a Mori domain if and only if for every nonzero integral ideal  $I$  of  $D$  there is a finitely generated ideal  $J \subseteq I$  such that  $J^v = I^v$  (see also [20] for an updated survey on Mori domains). Thus, if  $D$  is a Mori domain then  $D$  is CIC (i.e., every nonzero ideal is  $v$ -invertible) if and only if  $D$  is a  $v$ -domain (i.e., every nonzero finitely generated ideal is  $v$ -invertible). On the other hand, a completely integrally closed Mori domain is a Krull domain (see for example [48, Theorem 3.6]). More precisely, Mori  $v$ -domains coincide with Krull domains [104, Theorem]. Therefore, an integrally closed Mori non Krull domain provides an example of the irreversibility of  $\Rightarrow_{11}$ . An explicit example is given next.

It can be shown that, if  $k \subseteq L$  is an extension of fields and if  $X$  is an indeterminate over  $L$ , then  $k + XL[X]$  is always a Mori domain (see, for example, [50, Theorem 4.18] and references there to previous papers by V. Barucci and M. Roitman). It is easy to see that the complete integral closure of  $k + XL[X]$  is precisely  $L[X]$  [53, Lemma 26.5]. Thus if  $k \not\subseteq L$  then  $k + XL[X]$  is not completely integrally closed and, as an easy consequence of the definition of integrality, it is integrally closed if and only if  $k$  is algebraically closed in  $L$ . This shows that there do exist integrally closed Mori domains that are not Krull. A very explicit example is given by  $\overline{\mathbb{Q}} + X\mathbb{R}[X]$ , where  $\mathbb{R}$  is the field of real numbers and  $\overline{\mathbb{Q}}$  is the algebraic closure of  $\mathbb{Q}$  in  $\mathbb{R}$ .

### 3 $v$ -domains and rings of fractions

We have already mentioned that, if  $S$  is a multiplicative set of a PvMD  $D$ , then  $D_S$  is still a PvMD [69, Proposition 1.8]. The easiest proof of this fact can be given noting that, given  $F \in f(D)$ , if  $F$  is  $t$ -invertible in  $D$  then  $FD_S$  is  $t$ -invertible in  $D_S$ , where  $S$  is a multiplicative set of  $D$  [25, Lemma 2.6]. It is natural to ask if  $D_S$  is a  $v$ -domain when  $D$  is a  $v$ -domain.

The answer is no. As a matter of fact an example of an essential domain  $D$  with a prime ideal  $P$  such that  $D_P$  is not essential was given in [67]. What is interesting is that an essential domain is a  $v$ -domain by Proposition 2.1 and that, in this example,  $D_P$  is a (non essential) overring of the type  $k + XL[X]_{(X)} = (k + XL[X])_{XL[X]}$ , where  $L$  is a field and  $k$  its subfield that is algebraically closed in  $L$ . Now, a domain of type  $k + XL[X]_{(X)}$  is an integrally closed local Mori domain, see [50, Theorem 4.18]. In the irreversibility of  $\Rightarrow_{11}$ , we have also observed that if a Mori domain is a

$\nu$ -domain then it must be CIC, i.e., a Krull domain, and hence, in particular, an essential domain. Therefore, Heinzer's construction provides an example of an essential ( $\nu$ -)domain  $D$  with a prime ideal  $P$  such that  $D_P$  is not a  $\nu$ -domain.

Note that a similar situation holds for CIC domains. If  $D$  is CIC then it may be that for some multiplicative set  $S$  of  $D$  the ring of fractions  $D_S$  is not a completely integrally closed domain. A well known example in this connection is the ring  $E$  of entire functions. For  $E$  is a completely integrally closed Bézout domain that is infinite dimensional (see [61 and 62, Henriksen (1952), (1953)], [53, Examples 16–21, pp. 146–148] and [38, Section 8.1]). Localizing  $E$  at one of its prime ideals of height greater than one would give a valuation domain of dimension greater than one, which is obviously not completely integrally closed [53, Theorem 17.5]. For another example of a CIC domain that has non-CIC rings of fractions, look at the integral domain of integer-valued polynomials  $\text{Int}(\mathbb{Z})$  [7, Example 7.7 and the following paragraph at p. 127]. (This is a non-Bézout Prüfer domain, being atomic and two-dimensional.)

Note that these examples, like other well known examples of CIC domains with some overring of fractions not CIC, are all such that their overrings of fractions are at least  $\nu$ -domains (hence, they do not provide further counterexamples to the transfer of the  $\nu$ -domain property to the overrings of fractions). As a matter of fact, the examples that we have in mind are CIC Bézout domains with Krull dimension  $\geq 2$  (and polynomial domains over them), constructed using the Krull-Jaffard-Ohm-Heinzer Theorem (for the statement, a brief history and applications of this theorem see [53, Theorem 18.6, p. 214, p. 136, Example 19.12]). Therefore, it would be instructive to find an example of a CIC domain whose overrings of fractions are not all  $\nu$ -domains. Slightly more generally, we have the following.

It is well known that if  $\{D_\lambda \mid \lambda \in \Lambda\}$  is a family of overrings of  $D$  with  $D = \bigcap_{\lambda \in \Lambda} D_\lambda$  and if each  $D_\lambda$  is a completely integrally closed (respectively, integrally closed) domain then so is  $D$  (for the completely integrally closed case see for instance [53, Exercise 11, p. 145]; the integrally closed case is a straightforward consequence of the definition). It is natural to ask if in the above statement “completely integrally closed/integrally closed domain” is replaced by “ $\nu$ -domain” the statement is still true.

The answer in general is no, because by Krull's theorem every integrally closed integral domain is expressible as an intersection of a family of its valuation overrings (see e.g. [53, Theorem 19.8]) and of course a valuation domain is a  $\nu$ -domain. But, an integrally closed domain is not necessarily a  $\nu$ -domain (see the irreversibility of  $\Rightarrow_{11}$ ). If however each of  $D_\lambda$  is a ring of fractions of  $D$ , then the answer is yes. A slightly more general statement is given next.

**Proposition 3.1.** *Let  $\{D_\lambda \mid \lambda \in \Lambda\}$  be a family of flat overrings of  $D$  such that  $D = \bigcap_{\lambda \in \Lambda} D_\lambda$ . If each of  $D_\lambda$  is a  $\nu$ -domain then so is  $D$ .*

*Proof.* Let  $\nu_\lambda$  be the  $\nu$ -operation on  $D_\lambda$  and let  $* := \wedge \nu_\lambda$ , be the star operation on  $D$  defined by  $A \mapsto A^* := \bigcap_{\lambda} (AD_\lambda)^{\nu_\lambda}$ , for all  $A \in \mathbf{F}(D)$  [1, Theorem 2]. To show that  $D$  is a  $\nu$ -domain it is sufficient to show that every nonzero finitely generated ideal

is  $*$ -invertible (for  $* \leq v$  and so, if  $F \in f(D)$  and  $(FF^{-1})^* = D$ , then applying the  $v$ -operation to both sides we get  $(FF^{-1})^v = D$ ).

Now, we have

$$\begin{aligned} (FF^{-1})^* &= \bigcap_{\lambda} ((FF^{-1})D_{\lambda})^{v_{\lambda}} = \bigcap_{\lambda} ((FD_{\lambda})(F^{-1}D_{\lambda}))^{v_{\lambda}} \\ &= \bigcap_{\lambda} ((FD_{\lambda})(FD_{\lambda})^{-1})^{v_{\lambda}} \quad (\text{since } D_{\lambda} \text{ is } D\text{-flat and } F \text{ is f.g.}) \\ &= \bigcap_{\lambda} D_{\lambda} \quad (\text{since } D_{\lambda} \text{ is a } v_{\lambda}\text{-domain}) \\ &= D. \end{aligned}$$

**Corollary 3.2.** *Let  $\Delta$  be a nonempty family of prime ideals of  $D$  such that  $D = \bigcap \{D_P \mid P \in \Delta\}$ . If  $D_P$  is a  $v$ -domain for each  $P \in \Delta$ , then  $D$  is a  $v$ -domain. In particular, if  $D_M$  is a  $v$ -domain for all  $M \in \text{Max}(D)$  (for example, if  $D$  is locally a  $v$ -domain, i.e.,  $D_P$  is a  $v$ -domain for all  $P \in \text{Spec}(D)$ ), then  $D$  is a  $v$ -domain.*

Note that the previous Proposition 3.1 and Corollary 3.2 generalize Proposition 2.1, which ensures that an essential domain is a  $v$ -domain. Corollary 3.2 in turn leads to an interesting conclusion concerning the overrings of fractions of a  $v$ -domain.

**Corollary 3.3.** *Let  $S$  be a multiplicative set in  $D$ . If  $D_P$  is a  $v$ -domain for all prime ideals  $P$  of  $D$  such that  $P$  is maximal with respect to being disjoint from  $S$ , then  $D_S$  is a  $v$ -domain.*

In Corollary 3.2 we have shown that, if  $D_M$  is a  $v$ -domain for all  $M \in \text{Max}(D)$ , then  $D$  is a  $v$ -domain. However, if  $D_P$  is a  $v$ -domain for all  $P \in \text{Spec}(D)$ , we get much more in return. To indicate this, we note that, if  $S$  is a multiplicative set of  $D$ , then  $D_S = \bigcap \{D_Q \mid Q \text{ ranges over associated primes of principal ideals of } D \text{ with } Q \cap S = \emptyset\}$  [26, Proposition 4] (the definition of associated primes of principal ideals was recalled in Remark 2.3). Indeed, if we let  $S = \{1\}$ , then we have  $D = \bigcap D_Q \mid Q$  ranges over all associated primes of principal ideals of  $D$  (see also [86, Theorem 53] for a “maximal-type” version of this property). Using this terminology and the information at hand, it is easy to prove the following result.

**Proposition 3.4.** *Let  $D$  be an integral domain. Then, the following are equivalent.*

- (i)  $D$  is a  $v$ -domain such that, for every multiplicative set  $S$  of  $D$ ,  $D_S$  is a  $v$ -domain.
- (ii) For every nonzero prime ideal  $P$  of  $D$ ,  $D_P$  is a  $v$ -domain.
- (iii) For every associated prime of principal ideals of  $D$ ,  $D_Q$  is a  $v$ -domain.

From the previous considerations, we have the following addition to the existing picture:

$$\text{locally PvMD} \Rightarrow_{12} \text{locally } v\text{-domain} \Rightarrow_{13} v\text{-domain}.$$

The example discussed at the beginning of this section shows the irreversibility of  $\Rightarrow_{13}$ . Nagata’s example (given for the irreversibility of  $\Rightarrow_5$ ) of a one dimensional quasilocal CIC domain that is not a valuation ring shows also the irreversibility of  $\Rightarrow_{12}$ .



*Remark 3.5.* In the spirit of Proposition 3.4, we can make the following statement for CIC domains: *Let  $D$  be an integral domain. Then, the following are equivalent:*

- (i)  *$D$  is a CIC domain such that, for every multiplicative set  $S$  of  $D$ ,  $D_S$  is CIC.*
- (ii) *For every nonzero prime ideal  $P$  of  $D$ ,  $D_P$  is CIC.*
- (iii) *For every associated prime of a principal ideal of  $D$ ,  $D_Q$  is CIC.*

At the beginning of this section, we have mentioned the existence of examples of  $v$ -domains (respectively, CIC domains) having some localization at prime ideals which is not a  $v$ -domain (respectively, a CIC domain). Therefore, the previous equivalent properties (like the equivalent properties of Proposition 3.4) are strictly stronger than the property of being a CIC domain (respectively,  $v$ -domain).

On the other hand, for the case of integrally closed domains, the fact that, for every nonzero prime ideal  $P$  of  $D$ ,  $D_P$  is integrally closed (or, for every maximal ideal  $M$  of  $D$ ,  $D_M$  is integrally closed) returns exactly the property that  $D$  is integrally closed (i.e., the “integrally closed property” is a local property; see, for example, [17, Proposition 5.13]). Note that, more generally, the semistar integral closure is a local property (see for instance [60, Theorem 4.11]).

We have just observed that a ring of fractions of a  $v$ -domain may not be a  $v$ -domain, however there are distinguished classes of overrings for which the ascent of the  $v$ -domain property is possible.

Given an extension of integral domains  $D \subseteq T$  with the same field of quotients,  $T$  is called  $v$ -linked (respectively,  $t$ -linked) over  $D$  if whenever  $I$  is a nonzero (respectively, finitely generated) ideal of  $D$  with  $I^{-1} = D$  we have  $(IT)^{-1} = T$ . It is clear that  $v$ -linked implies  $t$ -linked and it is not hard to prove that flat overring implies  $t$ -linked [32, Proposition 2.2]. Moreover, the complete integral closure and the pseudo-integral closure of an integral domain  $D$  are  $t$ -linked over  $D$  (see [32, Proposition 2.2 and Corollary 2.3] or [58, Corollary 2]). Examples of  $v$ -linked extensions can be constructed as follows: take any nonzero ideal  $I$  of an integral domain then the overring  $T := (I^v : I^v)$  is a  $v$ -linked overring of  $D$  [76, Lemma 3.3].

The  $t$ -linked extensions were used in [32] to deepen the study of PvMD's. It is known that *an integral domain  $D$  is a PvMD if and only if each  $t$ -linked overring of  $D$  is a PvMD* (see [73, Proposition 1.6], [84, Theorem 3.8 and Corollary 3.9]). More generally, in [32, Theorem 2.10], the authors prove that *an integral domain  $D$  is a PvMD if and only if each  $t$ -linked overring is integrally closed*. On the other hand, a ring of fractions of a  $v$ -domain may not be a  $v$ -domain, so a  $t$ -linked overring of a  $v$ -domain may not be a  $v$ -domain. However, when it comes to a  $v$ -linked overring we get a different story. The following result is proven in [76, Lemma 2.4].

**Proposition 3.6.** *If  $D$  is a  $v$ -domain and  $T$  is a  $v$ -linked overring of  $D$ , then  $T$  is a  $v$ -domain.*

*Proof.* Let  $J := y_1T + y_2T + \cdots + y_nT$  be a nonzero finitely generated ideal of  $T$  and set  $F := y_1D + y_2D + \cdots + y_nD \in f(D)$ . Since  $D$  is a  $v$ -domain,  $(FF^{-1})^v = D$  and, since  $T$  is  $v$ -linked, we have  $(JF^{-1}T)^v = (FF^{-1}T)^v = T$ . We conclude easily that  $(J(T : J))^v = T$ .

## 4 $v$ -domains and polynomial extensions

### 4.1 The polynomial ring over a $v$ -domain

As for the case of integrally closed domains and of completely integrally closed domains [53, Corollary 10.8 and Theorem 13.9], we have observed in the proof of irreversibility of  $\Rightarrow_1$  that, given an integral domain  $D$  and an indeterminate  $X$  over  $D$ ,

$$D[X] \text{ is a PvMD} \Leftrightarrow D \text{ is a PvMD.}$$

A similar statement holds for  $v$ -domains. As a matter of fact, the following statements are equivalent (see part (4) of [12, Corollary 3.3]).

- (i) For every  $F \in f(D)$ ,  $F^v$  is  $v$ -invertible in  $D$ .
- (ii) For every  $G \in f(D[X])$ ,  $G^v$  is  $v$ -invertible in  $D[X]$ .

This equivalence is essentially based on a polynomial characterization of integrally closed domains given in [113], for which we need some introduction. Given an integral domain  $D$  with quotient field  $K$ , an indeterminate  $X$  over  $K$  and a polynomial  $f \in K[X]$ , we denote by  $c_D(f)$  the content of  $f$ , i.e., the (fractional) ideal of  $D$  generated by the coefficients of  $f$ . For every fractional ideal  $B$  of  $D[X]$ , set  $c_D(B) := (c_D(f) \mid f \in B)$ . The integrally closed domains are characterized by the following property: for each integral ideal  $J$  of  $D[X]$  such that  $J \cap D \neq (0)$ ,  $J^v = (c_D(J)[X])^v = c_D(J)^v[X]$  (see [113, Section 3] and [12, Theorem 3.1]). Moreover, an integrally closed domain is an agreeable domain (i.e., for each fractional ideal  $B$  of  $D[X]$ , with  $B \subseteq K[X]$ , there exists  $0 \neq s \in D$ -depending on  $B$ - with  $sB \subseteq D$ ) [12, Theorem 2.2]. (Note that agreeable domains were also studied in [65] under the name of almost principal ideal domains.)

The previous considerations show that, for an integrally closed domain  $D$ , there is a close relation between the divisorial ideals of  $D[X]$  and those of  $D$  [113, Theorem 1 and Remark 1]. The equivalence (i) $\Leftrightarrow$ (ii) will now follow easily from the fact that, given an agreeable domain, for every integral ideal  $J$  of  $D[X]$ , there exist an integral ideal  $J_1$  of  $D[X]$  with  $J_1 \cap D \neq (0)$ , a nonzero element  $d \in D$  and a polynomial  $f \in D[X]$  in such a way that  $J = d^{-1}fJ_1$  [12, Theorem 2.1].

On the other hand, using the definitions of  $v$ -invertibility and  $v$ -multiplication, one can easily show that for  $A \in F(D)$ ,  $A$  is  $v$ -invertible if and only if  $A^v$  is  $v$ -invertible. By the previous equivalence (i) $\Leftrightarrow$ (ii), we conclude that every  $F \in f(D)$  is  $v$ -invertible if and only if every  $G \in f(D[X])$  is  $v$ -invertible and this proves the following:

**Theorem 4.1.** *Given an integral domain  $D$  and an indeterminate  $X$  over  $D$ ,  $D$  is a  $v$ -domain if and only if  $D[X]$  is a  $v$ -domain.*

Note that a much more interesting and general result was proved in terms of pseudo-integral closures in [15, Theorem 1.5 and Corollary 1.6].

## 4.2 $v$ -domains and rational functions

Characterizations of  $v$ -domains can be also given in terms of rational functions, using properties of the content of polynomials.

Recall that Gauss' Lemma for the content of polynomials holds for Dedekind domains (or, more generally, for Prüfer domains). A more precise and general statement is given next.

**Lemma 4.2.** *Let  $D$  be an integral domain with quotient field  $K$  and let  $X$  be an indeterminate over  $D$ . The following are equivalent.*

- (i)  $D$  is an integrally closed domain (respectively, a PvMD; a Prüfer domain).
- (ii) For all nonzero  $f, g \in K[X]$ ,  $c_D(fg)^v = (c_D(f)c_D(g))^v$  (respectively,  $c_D(fg)^w = (c_D(f)c_D(g))^w$ ;  $c_D(fg) = c_D(f)c_D(g)$ ).

For the “Prüfer domain part” of the previous lemma, see [53, Corollary 28.5], [118], and [51]; for the “integrally closed domains part”, see [90, p. 557] and [113, Lemme 1]; for the “PvMD's part”, see [14, Corollary 1.6] and [27, Corollary 3.8]. For more on the history of Gauss' Lemma, the reader may consult [68, p. 1306] and [2, Section 8].

For general integral domains, we always have the inclusion of ideals  $c_D(fg) \subseteq c_D(f)c_D(g)$ , and, more precisely, we have the following famous lemma due to Dedekind and Mertens (for the proof, see [105] or [53, Theorem 28.1] and, for some complementary information, see [2, Section 8]):

**Lemma 4.3.** *In the situation of Lemma 4.2, let  $0 \neq f, g \in K[X]$  and let  $m := \deg(g)$ . Then*

$$c_D(f)^m c_D(fg) = c_D(f)^{m+1} c_D(g).$$

A straightforward consequence of the previous lemma is the following:

**Corollary 4.4.** *In the situation of Lemma 4.2, assume that, for a nonzero polynomial  $f \in K[X]$ ,  $c_D(f)$  is  $v$ -invertible (e.g.,  $t$ -invertible). Then  $c_D(fg)^v = (c_D(f)c_D(g))^v$  (or, equivalently,  $c_D(fg)^t = (c_D(f)c_D(g))^t$ ), for all nonzero  $g \in K[X]$ .*

From Corollary 4.4 and from the “integrally closed domain part” of Lemma 4.2, we have the following result (see [99, Theorem 2.4 and Section 3]):

**Corollary 4.5.** *In the situation of Lemma 4.2, set  $V_D := \{g \in D[X] \mid c_D(g) \text{ is } v\text{-invertible}\}$  and  $T_D := \{g \in D[X] \mid c_D(g) \text{ is } t\text{-invertible}\}$ . Then,  $T_D$  and  $V_D$  are multiplicative sets of  $D[X]$  with  $T_D \subseteq V_D$ . Furthermore,  $V_D$  (or, equivalently,  $T_D$ ) is saturated if and only if  $D$  is integrally closed.*

It can be useful to observe that, from Remark 2.2 (a.1), we have

$$T_D = \{g \in V_D \mid c_D(g)^{-1} \text{ is } t\text{-finite}\}.$$

We are now in a position to give a characterization of  $v$ -domains (and PvMD's) in terms of rational functions (see [99, Theorems 2.5 and 3.1]).

**Theorem 4.6.** *Suppose that  $D$  is an integrally closed domain, then the following are equivalent:*

- (i)  $D$  is a  $v$ -domain (respectively, a PvMD).
- (ii)  $v_D = D[X] \setminus \{0\}$  (respectively,  $T_D = D[X] \setminus \{0\}$ ).
- (iii)  $D[X]_{v_D}$  (respectively,  $D[X]_{T_D}$ ) is a field (or, equivalently,  $D[X]_{v_D} = K(X)$  (respectively,  $D[X]_{T_D} = K(X)$ )).
- (iv) Each nonzero element  $z \in K$  satisfies a polynomial  $f \in D[X]$  such that  $c_D(f)$  is  $v$ -invertible (respectively,  $t$ -invertible).

*Remark 4.7.* Note that *quasi Prüfer domains* (i.e., integral domains having integral closure Prüfer [19]) can also be characterized by using properties of the field of rational functions. In the situation of Lemma 4.2, set  $S_D := \{g \in D[X] \mid c_D(g) \text{ is invertible}\}$ . Then, by Lemma 4.4, the multiplicative set  $S_D$  of  $D[X]$  is saturated if and only if  $D$  is integrally closed. Moreover,  $D$  is quasi Prüfer if and only if  $D[X]_{S_D}$  is a field (or, equivalently,  $D[X]_{S_D} = K(X)$ ) if and only if each nonzero element  $z \in K$  satisfies a polynomial  $f \in D[X]$  such that  $c_D(f)$  is invertible [99, Theorem 1.7].

Looking more carefully at the content of polynomials, it is obvious that the set

$$N_D := \{g \in D[X] \mid c_D(g)^v = D\}$$

is a subset of  $T_D$  and it is well known that  $N_D$  is a saturated multiplicative set of  $D[X]$  [84, Proposition 2.1]. We call the *Nagata ring of  $D$  with respect to the  $v$ -operation* the ring:

$$\text{Na}(D, v) := D[X]_{N_D}.$$

We can also consider

$$\text{Kr}(D, v) := \{f/g \mid f, g \in D[X], g \neq 0, c_D(f)^v \subseteq c_D(g)^v\}.$$

When  $v$  is an e.a.b. operation on  $D$  (i.e., when  $D$  is a  $v$ -domain)  $\text{Kr}(D, v)$  is a ring called the *Kronecker function ring of  $D$  with respect to the  $v$ -operation* [53, Theorem 32.7]. Clearly, in general,  $\text{Na}(D, v) \subseteq \text{Kr}(D, v)$ . It is proven in [39, Theorem 3.1 and Remark 3.1] that  $\text{Na}(D, v) = \text{Kr}(D, v)$  if and only if  $D$  is a PvMD.

*Remark 4.8.* (a) Concerning Nagata and Kronecker function rings, note that a unified general treatment and semistar analogs of several results were obtained in the recent years, see for instance [41–43].

(b) A general version of Lemma 4.2, in case of semistar operations, was recently proved in [14, Corollary 1.2].

### 4.3 $v$ -domains and uppers to zero

Recall that if  $X$  is an indeterminate over an integral domain  $D$  and if  $Q$  is a nonzero prime ideal of  $D[X]$  such that  $Q \cap D = (0)$  then  $Q$  is called an *upper to zero*. The

“upper” terminology in polynomial rings is due to S. McAdam and was introduced in the early 1970s. In a recent paper, Houston and Zafrullah introduce the *UM $v$ -domains* as the integral domains such that the uppers to zero are maximal  $v$ -ideals and they prove the following result [78, Theorem 3.3].

**Theorem 4.9.** *Let  $D$  be an integral domain with quotient field  $K$  and let  $X$  be an indeterminate over  $K$ . The following are equivalent.*

- (i)  $D$  is a  $v$ -domain.
- (ii)  $D$  is an integrally closed *UM $v$ -domain*.
- (iii)  $D$  is integrally closed and every upper to zero in  $D[X]$  is  $v$ -invertible.
- (iii $_{\ell}$ )  $D$  is integrally closed and every upper to zero of the type  $Q_{\ell} := \ell K[X] \cap D[X]$  with  $\ell \in D[X]$  a linear polynomial is  $v$ -invertible.

It would be unfair to end the section with this characterization of  $v$ -domains without giving a hint about where the idea came from.

Gilmer and Hoffmann in 1975 gave a characterization of Prüfer domains using uppers to zero. This result is based on the following characterization of essential valuation overrings of an integrally closed domain  $D$ : let  $P$  be a prime ideal of  $D$ , then  $D_P$  is a valuation domain if and only if, for each upper to zero  $Q$  of  $D[X]$ ,  $Q \not\subseteq P[X]$ , [53, Theorem 19.15].

A globalization of the previous statement leads to the following result that can be easily deduced from [56, Theorem 2].

**Proposition 4.10.** *In the situation of Theorem 4.9, the following are equivalent:*

- (i)  $D$  is a Prüfer domain.
- (ii)  $D$  is integrally closed and if  $Q$  is an upper to zero of  $D[X]$ , then  $Q \not\subseteq M[X]$ , for all  $M \in \text{Max}(D)$  (i.e.,  $c_D(Q) = D$ ).

In [123, Proposition 4], the author proves a “ $t$ -version” of the previous result.

**Proposition 4.11.** *In the situation of Theorem 4.9, the following are equivalent:*

- (i)  $D$  is a *P $v$ MD*.
- (ii)  $D$  is integrally closed and if  $Q$  upper to zero of  $D[X]$ , then  $Q \not\subseteq M[X]$ , for all maximal  $t$ -ideal  $M$  of  $D$  (i.e.,  $c_D(Q)^t = D$ ).

The proof of the previous proposition relies on very basic properties of polynomial rings.

Note that in [123, Lemma 7] it is also shown that, if  $D$  is a *P $v$ MD*, then every upper to zero in  $D[X]$  is a maximal  $t$ -ideal. As we observed in Section 1, unlike maximal  $v$ -ideals, the maximal  $t$ -ideals are ubiquitous.

Around the same time, in [75, Proposition 2.6], the authors came up with a much better result, using the  $*$ -operations much more efficiently. Briefly, this result said that the converse holds, i.e.,  $D$  is a *P $v$ MD* if and only if  $D$  is an integrally closed integral domain and every upper to zero in  $D[X]$  is a maximal  $t$ -ideal.

It turns out that integral domains  $D$  such that their uppers to zero in  $D[X]$  are maximal  $t$ -ideals (called *UMt-domains* in [77, Section 3]; see also [36] and, for a survey on the subject, [74]) and domains such that, for each upper to zero  $Q$  of  $D[X]$ ,  $c_D(Q)^t = D$  had an independent life. In [77, Theorem 1.4], studying  $t$ -invertibility, the authors prove the following result.

**Proposition 4.12.** *In the situation of Theorem 4.9, let  $Q$  be an upper to zero in  $D[X]$ . The following statements are equivalent.*

- (i)  $Q$  is a maximal  $t$ -ideal of  $D[X]$ .
- (ii)  $Q$  is a  $t$ -invertible ideal of  $D[X]$ .
- (iii)  $c_D(Q)^t = D$ .

Based on this result, one can see that the following statement is a precursor to Theorem 4.9.

**Proposition 4.13.** *Let  $D$  be an integral domain with quotient field  $K$  and let  $X$  be an indeterminate over  $K$ . The following are equivalent.*

- (i)  $D$  is a PvMD.
- (ii)  $D$  is an integrally closed UMt-domain.
- (iii)  $D$  is integrally closed and every upper to zero in  $D[X]$  is  $t$ -invertible.
- (iii $_{\ell}$ )  $D$  is integrally closed and every upper to zero of the type  $Q_{\ell} := \ell K[X] \cap D[X]$ , with  $\ell \in D[X]$  a linear polynomial, is  $t$ -invertible.

Note that the equivalence (i) $\Leftrightarrow$ (ii) is in [77, Proposition 3.2]. (ii) $\Leftrightarrow$ (iii) is a consequence of previous Proposition 4.12. Obviously, (iii) $\Rightarrow$ (iii $_{\ell}$ ). (iii $_{\ell}$ ) $\Rightarrow$ (i) is a consequence of the characterization already cited that an integral domain  $D$  is a PvMD if and only if each nonzero two generated ideal is  $t$ -invertible [94, Lemma 1.7]. As a matter of fact, consider a nonzero two generated ideal  $I := (a, b)$  in  $D$ , set  $\ell := a + bX$  and  $Q_{\ell} := \ell K[X] \cap D[X]$ . Since  $D$  is integrally closed, then  $Q_{\ell} = \ell c_D(\ell)^{-1} D[X]$  by [113, Lemme 1, p. 282]. If  $Q_{\ell}$  is  $t$ -invertible (in  $D[X]$ ), then it is easy to conclude that  $c_D(\ell) = I$  is  $t$ -invertible (in  $D$ ).

*Remark 4.14.* Note that Prüfer domains may not be characterized by straight modifications of conditions (ii) and (iii) of Proposition 4.13. As a matter of fact, if there exists in  $D[X]$  an upper to zero which is also a maximal ideal, then the domain  $D$  is a G(oldman)-domain (i.e., its quotient field is finitely generated over  $D$ ), and conversely [86, Theorems 18 and 24]. Moreover, every upper to zero in  $D[X]$  is invertible if and only if  $D$  is a GGCD domain [11, Theorem 15].

On the other hand, a variation of condition (iii $_{\ell}$ ) of Proposition 4.13 does characterize Prüfer domains:  $D$  is a Prüfer domain if and only if  $D$  is integrally closed and every upper to zero of the type  $Q_{\ell} := \ell K[X] \cap D[X]$  with  $\ell \in D[X]$  a linear polynomial is such that  $c_D(Q_{\ell}) = D$  [75, Theorem 1.1].

## 5 $\nu$ -domains and GCD-theories

In [23, p. 170], a *factorial monoid*  $\mathcal{D}$  is a commutative semigroup with a unit element  $1$  (and without zero element) such that every element  $a \in \mathcal{D}$  can be uniquely represented as a finite product of atomic (= irreducible) elements  $q_i$  of  $\mathcal{D}$ , i.e.,  $a = q_1 q_2 \cdots q_r$ , with  $r \geq 0$  and this factorization is unique up to the order of factors; for  $r = 0$  this product is set equal to  $1$ . As a consequence, it is easy to see that this kind of uniqueness of factorization implies that  $1$  is the only invertible element in  $\mathcal{D}$ , i.e.,  $\mathcal{U}(\mathcal{D}) = \{1\}$ . Moreover, it is not hard to see that, in a factorial monoid, any two elements have GCD and every atom is a prime element [59, Theorem 10.7].

Let  $D$  be an integral domain and set  $D^\bullet := D \setminus \{0\}$ . In [23, p. 171] an integral domain  $D$  is said to have a *divisor theory* if there is a factorial monoid  $\mathcal{D}$  and a semigroup homomorphism, denoted by  $(-): D^\bullet \rightarrow \mathcal{D}$ , given by  $a \mapsto (a)$ , such that:

- (D1)  $(a) \mid (b)$  in  $\mathcal{D}$  if and only if  $a \mid b$  in  $D$  for  $a, b \in D^\bullet$ .
- (D2) If  $\mathfrak{g} \mid (a)$  and  $\mathfrak{g} \mid (b)$  then  $\mathfrak{g} \mid (a \pm b)$  for  $a, b \in D^\bullet$  with  $a \pm b \neq 0$  and  $\mathfrak{g} \in \mathcal{D}$ .
- (D3) Let  $\mathfrak{g} \in \mathcal{D}$  and set

$$\bar{\mathfrak{g}} := \{x \in D^\bullet \text{ such that } \mathfrak{g} \mid (x)\} \cup \{0\}.$$

Then  $\bar{a} = \bar{b}$  if and only if  $a = b$  for all  $a, b \in \mathcal{D}$ .

Given a divisor theory, the elements of the factorial monoid  $\mathcal{D}$  are called *divisors of the integral domain*  $D$  and the divisors of the type  $(a)$ , for  $a \in D$  are called *principal divisors of*  $D$ .

Note that, in [117, p. 119], the author shows that the axiom (D2), which guarantees that  $\bar{\mathfrak{g}}$  is an ideal of  $D$ , for each divisor  $\mathfrak{g} \in \mathcal{D}$ , is unnecessary. Furthermore, note that divisor theories were also considered in [98, Chap. 10], written in the spirit of Jaffard's volume [80].

Borevich and Shafarevich introduced domains with a divisor theory in order to generalize Dedekind domains and unique factorization domains, along the lines of Kronecker's classical theory of "algebraic divisors" (cf. [88] and also [121] and [34]). As a matter of fact, they proved that

- (a) If an integral domain  $D$  has a divisor theory  $(-): D^\bullet \rightarrow \mathcal{D}$  then it has only one (i.e., if  $((-)): D^\bullet \rightarrow \mathcal{D}'$  is another divisor theory then there is an isomorphism  $\mathcal{D} \cong \mathcal{D}'$  under which the principal divisors in  $\mathcal{D}$  and  $\mathcal{D}'$ , which correspond to a given nonzero element  $a \in D$ , are identified) [23, Theorem 1, p. 172];
- (b) An integral domain  $D$  is a unique factorization domain if and only if  $D$  has a divisor theory  $(-): D^\bullet \rightarrow \mathcal{D}$  in which every divisor of  $\mathcal{D}$  is principal [23, Theorem 2, p. 174];
- (c) An integral domain  $D$  is a Dedekind domain if and only if  $D$  has a divisor theory  $(-): D^\bullet \rightarrow \mathcal{D}$  such that, for every prime element  $p$  of  $\mathcal{D}$ ,  $D/\bar{p}$  is a field [23, Chap. 3, Section 6.2].

Note that Borevich and Shafarevich do not enter into the details of the determination of those integral domains for which a theory of divisors can be constructed

[23, p. 178], but it is known that they coincide with the Krull domains (see [120, Section 105], [18, Theorem 5], [92, Section 5], and [87] for the monoid case). In particular, note that, for a Krull domain, the group of non-zero fractional divisorial ideals provides a divisor theory.

Taking the above definition as a starting point and recalling that **(D2)** is unnecessary, in [92], the author introduces a more general class of domains, called the domain with a GCD–theory.

An integral domain  $D$  is said to have a *GCD–theory* if there is a GCD–monoid  $\mathcal{G}$  and a semigroup homomorphism, denoted by  $(-): D^\bullet \rightarrow \mathcal{G}$ , given by  $a \mapsto (a)$ , such that:

- (G1)**  $(a) \mid (b)$  in  $\mathcal{G}$  if and only if  $a \mid b$  in  $D$  for  $a, b \in D^\bullet$ .
- (G2)** Let  $\mathfrak{g} \in \mathcal{G}$  and set  $\bar{\mathfrak{g}} := \{x \in D^\bullet \text{ such that } \mathfrak{g} \mid (x)\} \cup \{0\}$ . Then  $\bar{\mathfrak{a}} = \bar{\mathfrak{b}}$  if and only if  $\mathfrak{a} = \mathfrak{b}$  for all  $\mathfrak{a}, \mathfrak{b} \in \mathcal{G}$ .

Let  $\mathcal{Q} := q(\mathcal{G})$  be the group of quotients of the GCD–monoid  $\mathcal{G}$ . It is not hard to prove that the natural extension a GCD–theory  $(-): D^\bullet \rightarrow \mathcal{G}$  to a group homomorphism  $(-)' : K^\bullet \rightarrow \mathcal{Q}$  has the following properties:

- (qG1)**  $(\alpha)' \mid (\beta)'$  with respect to  $\mathcal{G}$  if and only if  $\alpha \mid \beta$  with respect to  $D$  for  $\alpha, \beta \in K^\bullet$ .
- (qG2)** Let  $\mathfrak{h} \in \mathcal{Q}$  and set  $\bar{\mathfrak{h}} := \{\gamma \in K^\bullet \text{ such that } \mathfrak{h} \mid (\gamma)'\} \cup \{0\}$  (the division in  $\mathcal{Q}$  is with respect to  $\mathcal{G}$ ). Then  $\bar{\mathfrak{a}} = \bar{\mathfrak{b}}$  if and only if  $\mathfrak{a} = \mathfrak{b}$  for all  $\mathfrak{a}, \mathfrak{b} \in \mathcal{Q}$ .

In [92, Theorem 2.5], the author proves the following key result, that clarifies the role of the ideal  $\bar{\mathfrak{a}}$ . (Call, as before, *divisors of  $D$*  the elements of the GCD–monoid  $\mathcal{G}$  and *principal divisors of  $D$*  the divisors of the type  $(a)$ , for  $a \in D^\bullet$ .)

**Proposition 5.1.** *Let  $D$  be an integral domain with GCD–theory  $(-): D^\bullet \rightarrow \mathcal{G}$ , let  $\mathfrak{a}$  be any divisor of  $\mathcal{G}$  and  $\{(a_i)\}_{i \in I}$  a family of principal divisors with  $\mathfrak{a} = \text{GCD}(\{(a_i)\}_{i \in I})$ . Then  $\bar{\mathfrak{a}} = (\{a_i\}_{i \in I})^v = \bar{\mathfrak{a}}^v$ .*

Partly as a consequence of Proposition 5.1, we have a characterization of a  $v$ -domain as a domain with GCD–theory [92, Theorem and Definition 2.9].

**Theorem 5.2.** *Given an integral domain  $D$ ,  $D$  is a ring with GCD–theory if and only if  $D$  is a  $v$ -domain.*

The “only if part” is a consequence of Proposition 5.1 (for details see [92, Corollary 2.8]).

The proof of the “if part” is constructive and provides explicitly the GCD–theory. The GCD–monoid is constructed via Kronecker function rings. Recall that, when  $v$  is an e.a.b. operation (i.e., when  $D$  is a  $v$ -domain (Theorem 2.4)), the Kronecker function ring with respect to  $v$ ,  $\text{Kr}(D, v)$ , is well defined and is a Bézout domain [53, Lemma 32.6 and Theorem 32.7]. Let  $\mathcal{K}$  be the monoid  $\text{Kr}(D, v)^\bullet$ , let  $\mathcal{U} := \mathcal{U}(\text{Kr}(D, v))$  be the group of invertible elements in  $\text{Kr}(D, v)$  and set  $\mathcal{G} := \mathcal{K}/\mathcal{U}$ . The canonical map:



$$[-] : D^\bullet \rightarrow \mathfrak{G} = \frac{\text{Kr}(D, \nu)^\bullet}{\mathfrak{U}}, \quad a \mapsto [a] \quad (= \text{the equivalence class of } a \text{ in } \mathfrak{G})$$

defines a GCD–theory for  $D$ , called the *Kroneckerian GCD–theory* for the  $\nu$ -domain  $D$ . In particular, the GCD of elements in  $D$  is realized by the equivalence class of a polynomial; more precisely, under this GCD–theory, given  $a_0, a_1, \dots, a_n$  in  $D^\bullet$ ,  $\text{GCD}(a_0, a_1, \dots, a_n) := \text{GCD}([a_0], [a_1], \dots, [a_n]) = [a_0 + a_1X + \dots + a_nX^n]$ .

It is classically known [23, Chap. 3, Section 5] that the integral closure of a domain with divisor theory in a finite extension of fields is again a domain with divisor theory. For integral domains with GCD–theory a stronger result holds.

**Theorem 5.3.** *Let  $D$  be an integrally closed domain with field of fractions  $K$  and let  $K \subseteq L$  be an algebraic field extension and let  $T$  be the integral closure of  $D$  in  $L$ . Then  $T$  is a  $\nu$ -domain (i.e., domain with GCD–theory) if and only if  $D$  is a  $\nu$ -domain (i.e., a domain with GCD–theory).*

The proof of the previous result is given in [92, Theorem 3.1] and it is based on the following facts:

In the situation of Theorem 5.3,

- (a) For each ideal  $I$  of  $D$ ,  $I^{\nu D} = (IT)^{\nu T} \cap K$  [90, Satz 9, p. 675];
- (b) If  $D$  is a  $\nu$ -domain, then the integral closure of  $\text{Kr}(D, \nu_D)$  in the algebraic field extension  $K(X) \subseteq L(X)$  coincides with  $\text{Kr}(T, \nu_T)$  [92, Theorem 3.3].

*Remark 5.4.* (a) The notions of GCD–theory and divisor theory, being more in the setting of monoid theory, have been given a monoid treatment [59, Exercises 18.10, 19.6 and Chap. 20].

(b) Note that a part of previous Theorem 5.3 appears also as a corollary to [61, Theorem 3.6]. More precisely, let  $\text{c}1^\nu(D) (:= \bigcup \{F^\nu : F^\nu \mid F \in f(D)\})$  be the  $\nu$ -(integral) closure of  $D$ . We have already observed (Theorem 2.4 and Remark 2.6) that an integral domain  $D$  is a  $\nu$ -domain if and only if  $D = \text{c}1^\nu(D)$ . Therefore Theorem 5.3 is an easy consequence of the fact that, in the situation of Theorem 5.3, it can be shown that  $\text{c}1^\nu(T)$  is the integral closure of  $\text{c}1^\nu(D)$  in  $L$  [61, Theorem 3.6].

(c) In [92, Section 4], the author develops a “stronger GCD–theory” in order to characterize PvMD’s. A *GCD–theory of finite type* is a GCD–theory,  $(-)$ , with the property that each divisor  $\mathfrak{a}$  in the GCD–monoid  $\mathfrak{G}$  is such that  $\mathfrak{a} = \text{GCD}((a_1), (a_2), \dots, (a_n))$  for a finite number of nonzero elements  $a_1, a_2, \dots, a_n \in D$ . For a PvMD, the group of non-zero fractional  $t$ -finite  $t$ -ideals provides a GCD–theory of finite type. (Note that the notion of a GCD–theory of finite type was introduced in [18] under the name of “quasi divisor theory”. A thorough presentation of this concept, including several characterizations of P\*MD’s, is in [59, Chap. 20].)

The analogue of Theorem 5.2 can be stated as follows: *Given an integral domain  $D$ ,  $D$  is a ring with GCD–theory of finite type if and only if  $D$  is a PvMD.* Also in this case, the GCD–theory of finite type and the GCD–monoid are constructed explicitly, via the Kronecker function ring  $\text{Kr}(D, \nu)$  (which coincides in this situation with the Nagata ring  $\text{Na}(D, \nu)$ ), for the details see [92, Theorem 4.4].

Moreover, in [92, Theorem 4.6] there is given another proof of Prüfer’s theorem [111, Section 11], analogous to Theorem 5.3: *Let  $D$  be an integrally closed domain with field of fractions  $K$  and let  $K \subseteq L$  be an algebraic field extension and let  $T$  be the integral closure of  $D$  in  $L$ . Then  $T$  is a PvMD (i.e., domain with GCD–theory of finite type) if and only if  $D$  is PvMD (i.e., domain with GCD–theory of finite type).* Recall that a similar result holds for the special case of Prüfer domains [53, Theorem 22.3].

## 6 Ideal-theoretic characterizations of $\nu$ -domains

Important progress in the knowledge of the ideal theory for  $\nu$ -domains was made in 1989, after a series of talks given by the second named author while visiting several US universities. The results of various discussions of that period are contained in the “A to Z” paper [5], which contains in particular some new characterizations of  $\nu$ -domains and of completely integrally closed domains. These characterizations were then expanded into a very long list of equivalent statements, providing further characterizations of (several classes of)  $\nu$ -domains [13].

**Proposition 6.1.** *Let  $D$  be an integral domain. Then,  $D$  is a  $\nu$ -domain if and only if  $D$  is integrally closed and  $(F_1 \cap F_2 \cap \dots \cap F_n)^\nu = F_1^\nu \cap F_2^\nu \cap \dots \cap F_n^\nu$  for all  $F_1, F_2, \dots, F_n \in f(D)$  (i.e., the  $\nu$ -operation distributes over finite intersections of finitely generated fractional ideals).*

The “if part” is contained in the “A to Z” paper (Theorem 7 of that paper, where the converse was left open). The converse of this result was proved a few years later in [96, Theorem 2].

Note that, even for a Noetherian 1-dimensional domain, the  $\nu$ -operation may not distribute over finite intersections of (finitely generated) fractional ideals. For instance, here is an example due to W. Heinzer cited in [9, Example 1.2], let  $k$  be a field,  $X$  an indeterminate over  $k$  and set  $D := k[[X^3, X^4, X^5]]$ ,  $F := (X^3, X^4)$  and  $G := (X^3, X^5)$ . Clearly,  $D$  is a non-integrally closed 1-dimensional local Noetherian domain with maximal ideal  $M := (X^3, X^4, X^5) = F + G$ . It is easy to see that  $F^\nu = G^\nu = M$ , and so  $F \cap G = (X^3) = (F \cap G)^\nu \subsetneq F^\nu \cap G^\nu = M$ .

Recently, D.D. Anderson and Clarke have investigated the star operations that distribute over finite intersections. In particular, in [8, Theorem 2.8], they proved a star operation version of the “only if part” of Proposition 6.1 and, moreover, in [8, Proposition 2.7] and [9, Lemma 3.1 and Theorem 3.2] they established several other general equivalences that, particularized in the  $\nu$ -operation case, are summarized in the following:

**Proposition 6.2.** *Let  $D$  be an integral domain.*

- (a)  $(F_1 \cap F_2 \cap \dots \cap F_n)^\nu = F_1^\nu \cap F_2^\nu \cap \dots \cap F_n^\nu$  for all  $F_1, F_2, \dots, F_n \in f(D)$  if and only if  $(F :_D G)^\nu = (F^\nu :_D G^\nu)$  for all  $F, G \in f(D)$ .
- (b) *The following are equivalent.*

- (i)  $D$  is a  $v$ -domain.
- (ii)  $D$  is integrally closed and  $(F :_D G)^v = (F^v :_D G^v)$  for all  $F, G \in f(D)$
- (iii)  $D$  is integrally closed and  $((a, b) \cap (c, d))^v = (a, b)^v \cap (c, d)^v$  for all nonzero  $a, b, c, d \in D$ .
- (iv)  $D$  is integrally closed and  $((a, b) \cap (c))^v = (a, b)^v \cap (c)$  for all nonzero  $a, b, c \in D$ .
- (v)  $D$  is integrally closed and  $((a, b) :_D (c))^v = ((a, b)^v :_D (c))$  for all nonzero  $a, b, c \in D$ .

Note that PvMD's can be characterized by "t-versions" of the statements of Proposition 6.2 (b) [9, Theorem 3.3]. Moreover, in [9], the authors also asked several questions related to distribution of the  $v$ -operation over intersections. One of these questions [8, Question 3.2(2)] can be stated as: *Is it true that, if  $D$  is a  $v$ -domain, then  $(A \cap B)^v = A^v \cap B^v$  for all  $A, B \in \mathbf{F}(D)$ ?*

In [97, Example 3.4], the author has recently answered in the negative, constructing a Prüfer domain with two ideals  $A, B \in \mathbf{F}(D)$  such that  $(A \cap B)^v \neq A^v \cap B^v$ .

In a very recent paper [6], the authors classify the integral domains that come under the umbrella of  $v$ -domains, called there *\*-Prüfer domains* for a given star operation  $*$  (i.e., integral domains such that every nonzero finitely generated fractional ideal is  $*$ -invertible). Since  $v$ -Prüfer domains coincide with  $v$ -domains, this paper provides also direct and general proofs of several relevant quotient-based characterizations of  $v$ -domains given in [13, Theorem 4.1]. We collect in the following theorem several of these ideal-theoretic characterizations in case of  $v$ -domains. For the general statements in the star setting and for the proof the reader can consult [6, Theorems 2.2 and 2.8].

**Theorem 6.3.** *Given an integral domain  $D$ , the following properties are equivalent.*

- (i)  $D$  is a  $v$ -domain.
- (ii) For all  $A \in \mathbf{F}(D)$  and  $F \in f(D)$ ,  $A \subseteq F^v$  implies  $A^v = (BF)^v$  for some  $B \in \mathbf{F}(D)$ .
- (iii)  $(A : F)^v = (A^v : F) = (AF^{-1})^v$  for all  $A \in \mathbf{F}(D)$  and  $F \in f(D)$ .
- (iv)  $(A : F^{-1})^v = (A^v : F^{-1}) = (AF)^v$  for all  $A \in \mathbf{F}(D)$  and  $F \in f(D)$ .
- (v)  $(F : A)^v = (F^v : A) = (FA^{-1})^v$  for all  $A \in \mathbf{F}(D)$  and  $F \in f(D)$ .
- (vi)  $(F^v : A^{-1}) = (FA^v)^v$  for all  $A \in \mathbf{F}(D)$  and  $F \in f(D)$ .
- (vii)  $((A + B) : F)^v = ((A : F) + (B : F))^v$  for all  $A, B \in \mathbf{F}(D)$  and  $F \in f(D)$ .
- (viii)  $(A : (F \cap G))^v = ((A : F) + (A : G))^v$  for all  $A \in \mathbf{F}(D)$  and  $F, G \in f^v(D)$   
( $:= \{H \in f(D) \mid H = H^v\}$ ).
- (ix)  $((a :_D (b)) + ((b) :_D (a)))^v = D$  for all nonzero  $a, b \in D$ .
- (x<sub>f</sub>)  $((F \cap G)(F + G))^v = (FG)^v$  for all  $F, G \in f(D)$ .
- (x<sub>F</sub>)  $((A \cap B)(A + B))^v = (AB)^v$  for all  $A, B \in \mathbf{F}(D)$ .
- (xi<sub>f</sub>)  $(F(G^v \cap H^v))^v = (FG)^v \cap (FH)^v$  for all  $F, G, H \in f(D)$ .
- (xi<sub>fF</sub>)  $(F(A^v \cap B^v))^v = (FA)^v \cap (FB)^v$  for all  $F \in f(D)$  and  $A, B \in \mathbf{F}(D)$ .
- (xii) If  $A, B \in \mathbf{F}(D)$  are  $v$ -invertible, then  $A \cap B$  and  $A + B$  are  $v$ -invertible.
- (xiii) If  $A, B \in \mathbf{F}(D)$  are  $v$ -invertible, then  $A + B$  is  $v$ -invertible.

Note that some of the previous characterizations are remarkable for various reasons. For instance, (xiii) is interesting in that while an invertible ideal (respectively,  $t$ -invertible  $t$ -ideal) is finitely generated (respectively,  $t$ -finite) a  $\nu$ -invertible  $\nu$ -ideal may not be  $\nu$ -finite. Condition  $(x_F)$  in the star setting gives  $((A \cap B)(A + B))^* = (AB)^*$  for all  $A, B \in \mathbf{F}(D)$  and for  $* = d$  (respectively,  $* = t$ ), it is a (known) characterization of Prüfer domains (respectively, PvMD's), but for  $* = \nu$  is a brand-new characterization of  $\nu$ -domains. More generally, note that, replacing in each of the statements of the previous theorem  $\nu$  with the identity star operation  $d$  (respectively, with  $t$ ), we (re)obtain several characterizations of Prüfer domains (respectively, PvMD's).

Franz Halter-Koch has recently shown a great deal of interest in the paper [6] and, at the Fez Conference in June 2008, he has presented further systematic work in the language of monoids, containing in particular the above characterizations [64].

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# Tensor product of algebras over a field

Hassan Haghighi, Massoud Tousi, and Siamak Yassemi

**Abstract** This review paper deals with tensor products of algebras over a field. Let  $k$  be a field and  $A, B$  be commutative  $k$ -algebras. We consider the following question: “Which properties of  $A$  and  $B$  are conveyed to the  $k$ -algebra  $A \otimes_k B$ ?”. This field is still developing and many contexts are yet to be explored. We will restrict the scope of the present survey, mainly, to special rings.

## 1 Introduction

In this paper, we consider the following question: “Which properties of  $A$  and  $B$  are conveyed to the  $k$ -algebra  $A \otimes_k B$ ?”. This field is still developing and many contexts are yet to be explored. We will restrict the scope of the present survey, mainly, to special rings.

Throughout this note all rings and algebras considered are commutative with identity elements, and all ring homomorphisms are unital. As well,  $k$  stands for a field and  $A$  and  $B$  are commutative  $k$ -algebras. We use  $\text{Spec}(A)$ ,  $\text{Max}(A)$ , and  $\text{Min}(A)$  to denote the sets of prime ideals, maximal ideals, and minimal prime ideals, respectively, of a ring  $A$ .

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## 1.1 Integral domain

Suppose that  $A$  and  $B$  are integral domains containing a field  $k$ . If the quotient field  $k(A)$  of  $A$  is separable over  $k$  and  $k$  is algebraically closed in  $k(A)$ , then  $A \otimes_k B$  is an integral domain, c.f. [16, p. 562, Ex. A1.2]. For example, if  $k$  is algebraically closed, and  $A$  and  $B$  are arbitrary domains containing  $k$ , then  $A \otimes_k B$  is an integral domain.

## 1.2 Unit elements

In [51], the following theorem is given: Suppose  $k$  is an algebraically closed field,  $A$  and  $B$  are commutative algebras over  $k$ , and  $k$  is algebraically closed in  $A$  and  $B$ . Then every invertible element of  $A \otimes_k B$  is of the form  $a \otimes b$ , where  $a$  is an invertible element of  $A$  and  $b$  is an invertible element of  $B$ .

## 1.3 Local rings

In [52], Sweedler showed that for commutative algebras  $A$  and  $B$  over a common field  $k$ ,  $A \otimes_k B$  is local if and only if the following hold

- (i)  $A$  and  $B$  are local,
- (ii)  $A/J(A) \otimes_k B/J(B)$  is local ( $J(-) =$  Jacobson radical),
- (iii) either  $A$  or  $B$  is algebraic over  $k$ .

In [33], it is shown that for  $A$  and  $B$  not necessarily commutative,  $A \otimes_k B$  local implies (i), (ii), and (iii), and a converse can be obtained by replacing (iii) by the condition that  $A$  or  $B$  is locally finite by  $k$ . (An algebra is called locally finite if every finite subset generates a finite dimensional subalgebra.)

In [42], it is shown that if  $A \otimes_k B$  is semilocal, then  $A$  and  $B$  are semilocal. In addition, in [32, Theorem 6] it is shown that if  $A \otimes_k B$  is semilocal, then either  $A$  or  $B$  is algebraic over  $k$ .

## 1.4 Noetherian rings

Several authors have been interested in studying when a tensor products of two  $k$ -algebras is Noetherian. In [59] Vámos showed that the tensor products of two  $k$ -algebras is not necessarily Noetherian. More precisely, let  $K$  be an extension field of  $k$  and let  $K \otimes_k K$  be the tensor product of  $K$  with itself over  $k$ . In [59, Theorem 11] Vámos proved that the following statements are equivalent:

- (i)  $K \otimes_k K$  is Noetherian,
- (ii) the ascending chain condition is satisfied by the intermediary fields between  $k$  and  $K$ ,

(iii)  $K$  is a finitely generated extension of  $k$ .

In [2], Baetica gives a different proof of the equivalence of (i) and (iii). While the proof is longer than Vámos’ proof, it is more “natural” because it considers the three cases which appear in the usual structure theory of fields:

- (1)  $K$  is a separable algebraic extension of  $k$ ,
- (2)  $K$  is purely inseparable over  $k$ ,
- (3)  $K$  is purely transcendental over  $k$ .

Moreover, in [2], some other results on when a tensor product is Noetherian are also obtained. For example, let  $k$  be a field with characteristic  $p \neq 0$ ,  $K$  a purely inseparable field extension of  $k$ , and  $A$  a  $k$ -algebra. If  $A$  is a Noetherian local ring, then  $A \otimes_k K$  is a coherent local ring with Noetherian spectrum. Furthermore,  $A \otimes_k K$  is a Noetherian ring if and only if its maximal ideal is finitely generated; moreover if the maximal ideal of  $A \otimes_k K$  is nilpotent, then  $A \otimes_k K$  is Noetherian.

## 2 Krull dimension

For a commutative ring  $R$ , let  $\dim(R)$  be the classical Krull dimension of  $R$ , i.e., the supremum of length of chains of prime ideals of  $R$ , if this supremum exists, and  $\infty$  if it does not. For an extension field  $K$  of the field  $k$ , denote by  $\text{tr.deg} K/k$  the transcendence degree of the extension or  $\infty$  if  $K$  does not have finite degree of transcendence over  $k$ . In [46], Sharp proved the following result (actually, this result appeared 10 years earlier in Grothendieck’s EGA [23, Remarque 4.2.1.4, p. 349]).

**Theorem 2.1.** *For two extension fields  $K$  and  $L$  of the field  $k$ ,  $\dim(K \otimes_k L) = \min(\text{tr deg} K/k, \text{tr deg} L/k)$ .*

Sharp and P. Vámos [47] generalized this result to the case of fields  $K_1, \dots, K_n$ , where  $n \geq 2$ . The authors showed the following.

**Theorem 2.2.** *Let  $K_1, K_2, \dots, K_n$  ( $n \geq 2$ ) be extension fields of  $k$  and for  $i = 1, \dots, n$ , let  $t_i = \text{tr.deg}(K_i/k)$ . Then*

$$\dim(K_1 \otimes \dots \otimes K_n) = t_1 + t_2 + \dots + t_n - \max\{t_i \mid 1 \leq i \leq n\}.$$

In [62], Wadsworth generalized the above results to the case where the algebras are what he calls AF-domain over  $k$  (AF stands for “altitude formula”), which is defined as follows: a commutative algebra  $D$  over  $k$  is an AF-ring if  $\text{ht}(\mathfrak{p}) + \text{tr.deg}_k(D/\mathfrak{p}) = \text{tr.deg}_k(D)$  for each prime ideal  $\mathfrak{p}$  of  $D$ . (If  $A$  is a domain  $\text{tr.deg}_k(A)$  is the transcendence degree over  $k$  of its quotient field. For a non-domain  $A$ ,  $\text{tr.deg}_k A = \sup\{\text{tr.deg}_k(A/\mathfrak{p}) \mid \mathfrak{p} \in \text{Spec}(A)\}$ .) It is worth noting that the class of AF-domains contains the most basic rings of algebraic geometry, including finitely generated  $k$ -algebras that are domains. Wadsworth showed the following result, see also [38, Theorem 4 and Remark].

**Theorem 2.3 ([62, Theorem 3.8]).** *Assume that  $\{D_i\}_{i=1}^n$  is a finite family of AF-domains that are algebras over  $k$ . Let  $t_i = \text{tr.deg}_k(D_i) < \infty$ , and let  $d_i = \dim(D_i)$ . Then*

$$\dim(D_1 \otimes \cdots \otimes D_n) = t_1 + t_2 + \cdots + t_n - \max\{t_i - d_i \mid 1 \leq i \leq n\}.$$

To show the necessity of the hypothesis that the domains  $D_i$  should be AF-domains, he gave an example of rank-one discrete valuation rings  $V_1, V_2$ , each of transcendence degree 2 over a field  $k$ , such that  $\dim(V_1 \otimes_k V_1) = \dim(V_2 \otimes_k V_2) = 3$  while  $\dim(V_1 \otimes_k V_2) = 2$ . He also stated a formula for  $\dim(A \otimes_k B)$  which holds for an AF-domain  $A$ , with no restriction on  $B$ .

Let us first recall some notation. We use  $A[n]$  to denote the polynomial ring  $A[x_1, \dots, x_n]$  and  $\mathfrak{p}[n]$  to denote the prime ideal  $\mathfrak{p}[x_1, \dots, x_n]$  of  $A[x_1, \dots, x_n]$ . Let  $d, s$  be two integers with  $0 \leq d \leq s$ . Set

$$D(s, d, A) = \max\{\text{ht}_{\mathfrak{p}}[x_1, \dots, x_s] + \text{Min}(s, d + \text{tr.deg}_k(A/\mathfrak{p})) \mid \mathfrak{p} \in \text{Spec}(A)\}.$$

**Theorem 2.4 ([62, Theorem 3.7]).** *Let  $A$  be an AF-domain. Then*

$$\dim(A \otimes_k B) = D(\text{tr.deg}_k(A), \dim(A), B).$$

In [9], Bouchiba, Girolami, and Kabbaj showed that many (but not all) of Wadsworth's results can be extended from domains to rings with zero-divisors. In particular, they provided a formula for the dimension of the tensor product  $A \otimes_k B$  where  $A$  is an AF-ring.

**Theorem 2.5 ([9, Theorem 1.4]).** *Let  $A$  be an AF-ring and  $B$  any ring. Then*

$$\dim(A \otimes_k B) = \max\{D(\text{tr.deg}_k(A_{\mathfrak{p}}), \text{ht}(\mathfrak{p}), B) \mid \mathfrak{p} \in \text{Spec}(A)\}.$$

In addition, they gave a formula for  $\dim(A_1 \otimes_k \cdots \otimes_k A_n)$ , where  $A_i$  is AF-ring for any  $i$ .

**Lemma 2.6 ([9, Lemma 1.6]).** *Assume that  $A_1, \dots, A_n$  are AF-rings that are algebras over  $k$ , with  $n \geq 2$ . For  $i = 1, \dots, n$ , we denote  $\text{tr.deg}_k((A_i)_{\mathfrak{p}_i})$  by  $t_{\mathfrak{p}_i}$  for any  $\mathfrak{p}_i \in \text{Spec}(A_i)$ . Then  $\dim(A_1 \otimes_k \cdots \otimes_k A_n)$  is equal to*

$$\max\{\text{Min}(\text{ht}(\mathfrak{p}_1) + t_{\mathfrak{p}_2} + \cdots + t_{\mathfrak{p}_n}, t_{\mathfrak{p}_1} + \text{ht}(\mathfrak{p}_2) + t_{\mathfrak{p}_3} + \cdots + t_{\mathfrak{p}_n}, \dots, \\ t_{\mathfrak{p}_1} + \cdots + t_{\mathfrak{p}_{n-1}} + \text{ht}(\mathfrak{p}_n))\},$$

where  $\mathfrak{p}_i$  runs over  $\text{Spec}(A_i)$  and  $i = 1, \dots, n$ .

Then they determined a necessary and sufficient condition under which the dimension of the tensor product of the AF-rings  $A_1, \dots, A_n$  satisfies the formula of Wadsworth's Theorem 2.3. In this way, they presented a number of applications of this result.

**Theorem 2.7 ([9, Theorem 1.8]).** Assume that  $A_1, \dots, A_n$  are AF-rings that are algebras over  $k$ , with  $t_i = \text{tr.deg}_k(A_i)$  and  $d_i = \dim(A_i)$ . Then

$$\dim(A_1 \otimes_k \dots \otimes_k A_n) = t_1 + \dots + t_n - \max\{t_i - d_i \mid 1 \leq i \leq n\}$$

if and only if for any  $i = 1, \dots, n$  there is  $\mathfrak{m}_i \in \text{Max}(A_i)$  and  $r \in \{1, \dots, n\}$  such that  $\text{ht}(\mathfrak{m}_r) = d_r$  and for any  $j \in \{1, \dots, n\} - \{r\}$ ,  $\text{tr.deg}_k((A_j)_{\mathfrak{m}_j}) = t_j$  and  $\text{tr.deg}_k(A_j/\mathfrak{m}_j) \leq \text{tr.deg}_k(A_r/\mathfrak{m}_r)$ .

**Corollary 2.8 ([9, Corollaries 1.13 and 1.14]).** Assume that  $A_1, \dots, A_n$  are AF-rings that are algebras over  $k$ , with  $t_i = \text{tr.deg}_k(A_i)$  and  $d_i = \dim(A_i)$ . Consider the following statements:

- (i)  $\text{tr.deg}(A_i/\mathfrak{p}_i) = t_i$ , for any  $i = 1, \dots, n$  and for any  $\mathfrak{p}_i \in \text{Min}(A_i)$ .
- (ii)  $A_1, \dots, A_n$  are equidimensional.

If one of (i) or (ii) holds, then

$$\dim(A_1 \otimes_k \dots \otimes_k A_n) = t_1 + \dots + t_n - \max\{t_i - d_i \mid 1 \leq i \leq n\}.$$

In [20], dimension formulas for the tensor product of two particular pullbacks are established and a conjecture on the dimension formulas for more general pullbacks is raised. In [10] such a conjecture is resolved. Consider the two pullbacks of  $k$ -algebras for  $i = 1, 2$ :

$$\begin{array}{ccc} R_i & \longrightarrow & D_i \\ \downarrow & & \downarrow \\ T_i & \longrightarrow & K_i \end{array}$$

where, for  $i = 1, 2$ ,  $T_i$  is an integral domain with maximal ideal  $M_i$ ,  $K_i = T_i/M_i$ ,  $\varphi_i$  is the canonical surjection from  $T_i$  onto  $K_i$ ,  $D_i$  is a proper subring of  $K_i$  and  $R_i = \varphi_i^{-1}(D_i)$ .

**Theorem 2.9 ([10, Theorem 1.9]).** Assume that  $T_1, T_2, D_1$  and  $D_2$  are AF-domains with  $\dim(T_1) = \text{ht}(M_1)$  and  $\dim(T_2) = \text{ht}(M_2)$ . Then

$$\dim(R_1 \otimes_k R_2) = \max\{ \text{ht}(M_1[\text{tr.deg}_k(R_2)]) + D(\text{tr.deg}_k(D_1), \dim(D_1), R_2), \text{ht}(M_2[\text{tr.deg}_k(R_1)]) + D(\text{tr.deg}_k(D_2), \dim(D_2), R_1) \}.$$

Notice that when the extension fields  $K_i$  are transcendental over the domains  $D_i$ , the pullbacks  $R_i$  are not AF-domains. In view of this, the above theorem allows us to compute Krull dimension of tensor products of two  $k$ -algebras for a large class of (not necessarily AF-domains)  $k$ -algebras.

*Question 2.10.* Let  $T_1, T_2$  be integral domains with only one of them is an AF-domain. Find  $\dim(R_1 \otimes_k R_2)$ .

However, if none of the  $T_i$ 's is an AF-domain, then the formula of Theorem 2.9 does not hold in general, see [62, Example 4.3]. In addition, [10, Example 3.4]

illustrates the fact that in Theorem 2.9 the hypothesis “ $\dim(T_i) = \text{ht}(M_i), i = 1, 2$ ” cannot be deleted.

The polynomial rings  $R_1[\text{tr.deg}_k(K_1) - \text{tr.deg}_k(D_1)]$  and  $R_2[\text{tr.deg}_k(K_2) - \text{tr.deg}_k(D_2)]$  turned out to be AF-domains [10, Proposition 2.2]. The purpose of [5] and [7] is to investigate  $\dim(A \otimes_k B)$  in the general case where  $A[n]$  is an AF-domain for some positive integer  $n$ . It is worth reminding the reader that if  $A$  is an AF-domain, then so is the polynomial ring  $A[n]$ . The converse fails in general.

Indeed, in [5], it is shown that if  $A$  is a one dimensional domain such that  $A[n]$  is an AF-domain for some positive integer  $n$  and  $B$  is any  $k$ -algebra, then one can express  $\dim(A \otimes_k B)$  entirely in terms of numerical invariants of  $A$  and  $B$ . As an application, the Krull dimension of  $A \otimes_k B$  for a new family of  $k$ -algebras is computed.

In [7], Bouchiba provides a formula for  $\dim(A \otimes_k B)$  in the case in which  $B$  is a locally Jaffard domain and  $n \leq \text{tr.deg}_k(B)$  is a positive integer such that  $A[n]$  is an AF-domain. We recall that an AF-ring is a locally Jaffard ring [19].

One of the main results proved in [6] demonstrates:

**Theorem 2.11 ([6, Theorem 1.1]).** *If  $B$  is an integral domain, then*

$$\begin{aligned} \dim(A[\text{tr.deg}_k(B)]) - (\text{tr.deg}_k(B) - \dim(B)) &\leq \dim(A \otimes_k B) \\ &\leq \dim(A[\text{tr.deg}_k(B)]). \end{aligned}$$

As a consequence, with the same assumptions, the author showed the following result.

**Corollary 2.12 ([6, Corollary 2]).** *If  $B$  is an integral domain, then*

$$\begin{aligned} \dim(A) + \dim(B) &\leq \dim(A \otimes_k B) \\ &\leq \text{tr.deg}_k(B) + (\text{tr.deg}_k(B) + 1)\dim(A). \end{aligned}$$

Note that, for  $B := k[n]$ , the previous formula gives back the classical result for polynomials proved by Seidenberg in 1954 [48, Theorem 3] and extended by Jaffard in 1960 [29, Corollary 2], that is,

$$\dim(A) + n \leq \dim(A[n]) \leq n + (n + 1)\dim(A).$$

Corollary 2.11 states the analogues of the above mentioned inequalities of polynomial rings for tensor products of  $k$ -algebras.

Theorem 2.10 yields the following consequences, which allow us to compute the Krull dimension of tensor products for a new family of  $k$ -algebras outside the scope of Wadsworth’s theorem 2.4.

**Corollary 2.13 ([6, Corollary 1.5]).** *Let  $A$  be an integral domain and  $\dim(A) = \text{tr.deg}_k(A)$ . Then*

$$\dim(A \otimes_k B) = \dim(B[\text{tr.deg}_k(A)]).$$

**Corollary 2.14 ([6, Corollary 1.6]).** *Let  $A$  be a domain and  $\mathfrak{m}$  a maximal ideal of  $A$  such that  $\text{tr.deg}_k(A/\mathfrak{m}) = \text{Min}\{\text{tr.deg}(A/M) : M \in \text{Max}(A)\}$ . Assume that  $\text{ht}(\mathfrak{m}) + \text{tr.deg}_k(A/\mathfrak{m}) = \text{tr.deg}_k(A)$ . Then*

$$\dim(A \otimes_k B) = D(\text{tr.deg}_k(A), \dim(A), B).$$

**Corollary 2.15 ([6, Corollary 1.7]).** *Let  $A$  be a domain. Let  $\mathfrak{n}$  be a maximal ideal of  $B$  such that*

$$\text{ht}(\mathfrak{n}[\text{tr.deg}_k(A)]) = \max\{\text{ht}(M[\text{tr.deg}_k(A)]) \mid M \in \text{Max}(B)\}.$$

*If  $\text{tr.deg}_k(A) \leq \text{tr.deg}_k(B/\mathfrak{n})$ , then*

$$\dim(A \otimes_k B) = \dim(B[\text{tr.deg}_k(A)]).$$

Example 2.2, Remark 2.3 and Example 2.5 of [6] showed that we are able to build  $k$ -algebras that are domains and satisfy the conditions of Corollary 2.12, 2.13, and 2.14, while they are not AF-domains. In particular, Example 2.2 of [6] illustrates the fact that the assumptions of Corollary 2.13 are actually weaker than those of Theorem 2.4.

### 3 Special rings

#### 3.1 Classical rings

Among Noetherian local rings there is a well-known chain:

Regular  $\Rightarrow$  Complete intersection  $\Rightarrow$  Gorenstein  $\Rightarrow$  Cohen–Macaulay.

These concepts are extended to non-local rings: for example a ring  $R$  is regular if for all prime ideals  $\mathfrak{p}$  of  $R$ ,  $R_{\mathfrak{p}}$  is a regular local ring.

A Noetherian local ring  $R$  is a complete intersection (ring) if its completion  $\hat{R}$  is a residue class ring of a regular local ring  $S$  with respect to an ideal generated by an  $S$ -sequence.

We investigate the cases where these properties are preserved under tensor product operations. It is well-known that the tensor product  $A \otimes_k B$  of regular rings is not regular in general. In [63], Watanabe, Ishikawa, Tachibana, and Otsuka showed that under a suitable condition, the tensor product of regular rings is a complete intersection. It is proved in [23], that the tensor product  $S \otimes_R T$  of Cohen–Macaulay rings is again Cohen–Macaulay if we assume that  $T$  and  $S$  are commutative algebras over ring  $R$  such that  $S$  is a flat  $R$ -module and  $T$  is a finitely generated  $R$ -module. In [63], it is shown that the same is true for Gorenstein rings. Also, Watanabe et al. showed that if  $A$  and  $B$  are two Gorenstein (resp., Cohen–Macaulay) rings,  $A \otimes_k B$  is Noetherian and  $A/\mathfrak{m}$  is finitely generated over  $k$  for each maximal ideal  $\mathfrak{m}$  of  $A$ , then  $A \otimes_k B$  is Gorenstein (resp., Cohen–Macaulay) ring. Recently, in [11], Bouchiba and

Kabbaj showed that if  $A \otimes_k B$  is Noetherian, then  $A \otimes_k B$  is a Cohen–Macaulay ring if and only if  $A$  and  $B$  are Cohen–Macaulay rings.

Recall that a Noetherian ring  $R$  satisfies Serre’s condition  $(S_n)$  provided  $\text{depth}R_{\mathfrak{p}} \geq \text{Min}\{n, \text{dim}(R_{\mathfrak{p}})\}$  for all prime ideals  $\mathfrak{p}$  of  $R$ ; and  $R$  satisfies condition  $(R_n)$  if  $R_{\mathfrak{p}}$  is a regular local ring for all prime ideals  $\mathfrak{p}$  with  $\text{dim}(R_{\mathfrak{p}}) \leq n$ .

In [53] we showed the following.

**Theorem 3.1.** *Let  $A$  and  $B$  be non-zero  $k$ -algebras such that  $A \otimes_k B$  is Noetherian. Then the following hold:*

- (a)  $A \otimes_k B$  is locally a complete intersection (resp., Gorenstein, Cohen–Macaulay) if and only if  $A$  and  $B$  are locally a complete intersection (resp., Gorenstein, Cohen–Macaulay).
- (b)  $A \otimes_k B$  satisfies  $(S_n)$  if and only if  $A$  and  $B$  satisfy  $(S_n)$ .
- (c) If  $A \otimes_k B$  is regular then  $A$  and  $B$  are regular.
- (d) If  $A \otimes_k B$  satisfies  $(R_n)$  then  $A$  and  $B$  satisfy  $(R_n)$ .
- (e) The converse of parts (c) and (d) hold if  $\text{char}(k) = 0$  or  $\text{char}(k) = p$  such that  $k = \{a^p | a \in k\}$ .

In view of the above result, one may ask the following:

*Question 3.2.* Let  $A$  and  $B$  be two non-zero  $k$ -algebras such that  $A \otimes_k B$  is Noetherian. Let  $A$  be a Cohen–Macaulay ring but not Gorenstein and  $B$  be a Gorenstein ring. By part (a) of Theorem 3.1, the  $k$ -algebra  $A \otimes_k B$  is Cohen–Macaulay but not Gorenstein. What can be said about the structure of  $A \otimes_k B$ ? More generally we may ask similar questions with regular, complete intersection, Gorenstein, or Cohen–Macaulay rings.

### 3.2 Locally finite-dimensional rings, (stably) strong S-property, (universal) catenarity

In order to treat Noetherian domains and Prüfer domains in a unified manner, Kaplansky [30] introduced the concepts of S(eidenberg)-domain and strong S-ring. A domain  $R$  is called an S-domain if, for each height one prime ideal  $\mathfrak{p}$  of  $R$ , the extension  $\mathfrak{p}R[x]$  to the polynomial ring in one variable also has height one. A commutative ring  $R$  is said to be a strong S-ring if  $R/\mathfrak{p}$  is an S-domain for each  $\mathfrak{p} \in \text{Spec}(R)$ . It is noteworthy that while  $R[x]$  is always an S-domain for any domain  $R$  (see [17]),  $R[x]$  need not be a strong S-ring even when  $R$  is a strong S-ring. Thus,  $R$  is said to be a stably strong S-ring (also called a universally strong S-ring) if the polynomial ring  $R[x_1, \dots, x_n]$  is a strong S-ring for each positive integer  $n$ .

Consider the following property that a ring  $R$  may satisfy.

**(Q1)** For any prime ideals  $\mathfrak{p}, \mathfrak{p}'$  of  $R$  with  $\mathfrak{p} \subset \mathfrak{p}'$ , there exists a saturated chain of prime ideals starting from  $\mathfrak{p}$  and ending at  $\mathfrak{p}'$ , and all such chains have the same finite length.



A ring  $R$  is called catenarian if it satisfies (Q1). A ring  $R$  is called universally catenarian if  $R[x_1, \dots, x_n]$  is catenarian for each positive integer  $n$ .

In [8], Bouchiba, Dobbs, and Kabbaj studied the prime ideal structure of  $A \otimes_k B$ . They sought necessary and sufficient conditions for such a tensor product to have the S-property, (stably) strong S-property and (universal) catenarity. First they investigated the minimal prime ideal structure in tensor products of  $k$ -algebras. In this direction, the following three results were given. Recall that a ring  $R$  satisfies MPC (for Minimal Primes Comaximality) if the minimal prime ideals in  $R$  are pairwise comaximal.

**Proposition 3.3 ([8, Proposition 3.1]).** *If  $A$  and  $B$  are  $k$ -algebras such that  $A \otimes_k B$  satisfies MPC, then  $A$  and  $B$  satisfy MPC.*

**Theorem 3.4 ([8, Theorem 3.3]).** *Let  $k$  be an algebraically closed field. Then  $A \otimes_k B$  satisfies MPC if and only if  $A$  and  $B$  each satisfies MPC.*

Vamós [59, Corollary 4] proved that if  $K$  and  $L$  are field extensions of  $k$ , then  $K \otimes_k L$  satisfies MPC. The third result generalizes this result in the context of integrally closed domains as follows:

**Theorem 3.5 ([8, Theorem 3.4]).** *If  $A$  and  $B$  are integrally closed domains, then  $A \otimes_k B$  satisfies MPC.*

Note that the above result can not be extended to arbitrary  $k$ -algebras. There exist a separable algebraic field extension  $K$  of finite degree over  $k$  and a  $k$ -algebra  $A$  satisfying MPC such that  $K \otimes_k A$  fails to satisfy MPC (see [8, Example 3.2]).

Bouchiba et al. extended the domain-theoretic definitions of the S-property and catenarity to the MPC context. In [8], a ring  $R$  is called an S-ring if it satisfies MPC and for each height one ideal  $\mathfrak{p}$  of  $R$ , the extension  $\mathfrak{p}R[x]$  has height one. Consider the following property that a ring  $R$  may satisfy:

(Q2)  $R$  is locally finite dimensional and  $\text{ht}(\mathfrak{p}') = \text{ht}(\mathfrak{p}) + 1$  for each containment  $\mathfrak{p} \subset \mathfrak{p}'$  of adjacent prime ideals of  $R$ .

In [8], a ring  $R$  is said to be catenarian if  $R$  satisfies MPC and (Q2). Note that  $R$  satisfies MPC and (Q1) if and only if  $R$  satisfies MPC and (Q2).

As an application, the authors established necessary and sufficient conditions for  $A \otimes_k B$  to be an S-ring:

**Theorem 3.6 ([8, Theorem 3.9]).** *Let  $A \otimes_k B$  satisfies MPC. Then  $A \otimes_k B$  is an S-ring if and only if at least one of the following statements is satisfied:*

- (1)  $A$  and  $B$  are S-rings,
- (2)  $A$  is an S-ring and  $\text{tr.deg}_k(A/\mathfrak{p}) \geq 1$  for each  $\mathfrak{p} \in \text{Min}(A)$ ,
- (3)  $B$  is an S-ring and  $\text{tr.deg}_k(B/\mathfrak{q}) \geq 1$  for each  $\mathfrak{q} \in \text{Min}(B)$ ,
- (4)  $\text{tr.deg}_k(A/\mathfrak{p})$  and  $\text{tr.deg}_k(B/\mathfrak{q}) \geq 1$  for each  $\mathfrak{p} \in \text{Min}(A)$  and  $\mathfrak{q} \in \text{Min}(B)$ .

To determine when a tensor product of  $k$ -algebras is catenarian, we first need to know when it is locally finite dimensional (LFD for short).

**Theorem 3.7 ([8, Proposition 4.1]).** *Let  $A$  and  $B$  be  $k$ -algebras. Then the following hold:*

- (a) *If  $A \otimes_k B$  is LFD, then so are  $A$  and  $B$ , and either  $\text{tr.deg}_k(A/\mathfrak{p}) < \infty$  for each prime ideal  $\mathfrak{p} \in \text{Spec}(A)$  or  $\text{tr.deg}_k(B/\mathfrak{q}) < \infty$  for each prime ideal  $\mathfrak{q}$  of  $B$ .*
- (b) *If both  $A$  and  $B$  are LFD and either  $\text{tr.deg}_k(A) < \infty$  or  $\text{tr.deg}_k(B) < \infty$ , then  $A \otimes_k B$  is LFD.*

*The converse holds provided  $A$  and  $B$  are domains.*

**Proposition 3.8 ([8, Proposition 4.4]).** *Let  $K$  be an algebraic field extension of  $k$ . If  $K \otimes_k A$  is a strong S-ring (resp., catenarian), then  $A$  is a strong S-ring (resp., catenarian).*

In [8], by giving an example, it is shown that there exists a  $k$ -algebra  $R$  which is not an S-domain (resp., catenarian domain) and a field extension  $K$  of  $k$  such that  $1 \leq \text{tr.deg}_k(K) < \infty$  and  $K \otimes_k R$  is a strong S-ring (resp., catenarian). This shows that Proposition 3.7 fails in general when the extension field  $K$  is no longer algebraic over  $k$ .

Next, the authors investigated sufficient conditions on a  $k$ -algebra  $A$  and a field extension  $K$  of  $k$ , for  $K \otimes_k A$  to inherit the (stably) strong S-property and (universal) catenarity.

In this direction, we can recall the following theorem and corollary from [8]:

**Theorem 3.9 ([8, Theorem 4.9]).** *Let  $A$  be a Noetherian domain that is a  $k$ -algebra and  $K$  a field extension of  $k$  such that  $\text{tr.deg}_k(K) < \infty$ . Then  $K \otimes_k A$  is a stably strong S-ring. If, in addition,  $K \otimes_k A$  satisfies MPC and  $A[x]$  is a catenarian, then  $K \otimes_k A$  is universally catenarian.*

**Corollary 3.10 ([8, Corollary 4.10]).** *Let  $K$  and  $L$  be field extensions of  $k$  such that  $\text{tr.deg}_k(K) < \infty$ . Then  $K \otimes_k L$  is universally catenarian.*

The main theorem of [8] asserts that:

**Theorem 3.11 ([8, Theorem 4.13]).** *Let  $A$  be an LFD  $k$ -algebra and  $K$  a field extension of  $k$  such that either  $\text{tr.deg}_k(A) < \infty$  or  $\text{tr.deg}_k(K) < \infty$ . Let  $B$  be a transcendence basis of  $K$  over  $k$ , and let  $L$  be the separable algebraic closure of  $k(B)$  in  $K$ . Assume that  $[L : k(B)] < \infty$ . If  $A$  is a stably strong S-ring (resp., universally catenarian and  $K \otimes_k A$  satisfies MPC), then  $K \otimes_k A$  is a stably strong S-ring (resp., universally catenarian).*

This result leads to new families of stably strong S-rings and universally catenarian rings. Also noteworthy is [8, Corollary 4.10], stating that the tensor product of two field extensions of  $k$ , at least one of which is of finite transcendence degree, is universally catenarian.

In [8], an example is given of a discrete rank one valuation domain  $V$  (hence universally catenarian) such that  $\text{tr.deg}_k(V) < \infty$  and  $V \otimes_k V$  not catenarian, illustrating the importance of assuming  $K$  is a field in Theorem 3.10.

*Question 3.12.* Let  $K$  be an algebraic field extension of  $k$  and  $A$  a strong S-ring (resp., catenarian such that  $K \otimes_k A$  satisfies MPC). Is  $K \otimes_k A$  a strong S-ring (resp., catenarian)?

However, for the case where  $K$  is a transcendental field extension of  $k$ , the answer is negative, as illustrated by [8, Examples 5.1 and 5.2], as follows: there exists a strong S-domain (resp., catenarian domain)  $A$  which is a  $k$ -algebra such that  $L \otimes_k A$  is a strong S-ring (resp., catenarian) for any algebraic field extension  $L$  of  $k$ , while  $K \otimes_k A$  is not a strong S-ring (resp., catenarian) for some transcendental field extension  $K$  of  $k$ .

In [54], we give positive answer to this problem in some special cases. For example, it is shown that  $K \otimes_k A$  is universally catenarian if one of the following holds:

- (a)  $A$  is universally catenarian and  $K$  a finitely generated extension field of  $k$ .
- (b)  $A$  is Noetherian universally catenarian and  $\text{tr.deg}_k(K) < \infty$ .
- (c)  $A$  is universally catenarian and  $K \otimes_k A$  is Noetherian.

Also, another result along these lines is as follows.

**Theorem 3.13 ([54, Theorem 2.7]).** *Let  $A$  be a Noetherian, catenarian and locally equidimensional ring which is a  $k$ -algebra, with  $K$  an algebraic field extension of  $k$ . If  $\mathfrak{q}_1, \mathfrak{q}_2 \in \text{Spec}(K \otimes_k A)$  are such that  $\mathfrak{q}_1 \subset \mathfrak{q}_2$ , then  $\text{ht}(\mathfrak{q}_2/\mathfrak{q}_1) = 1$  or  $\text{ht}(\mathfrak{q}_2/\mathfrak{q}_1) = \text{ht } \mathfrak{q}_2 - \text{ht } \mathfrak{q}_1$ .*

### 3.3 Approximately Cohen–Macaulay rings

Let  $(R, \mathfrak{m})$  be a Noetherian local ring with  $\dim(R) = d$ . Recall that  $R$  is a Gorenstein ring if and only if there is an element  $a$  of  $\mathfrak{m}$  such that  $R/a^n R$  is a Gorenstein ring of dimension  $d - 1$  for every integer  $n > 0$  (cf. [28]). Clearly, this is not true for arbitrary Cohen–Macaulay rings. The local ring  $R$  is called an *approximately Cohen–Macaulay* ring if either  $\dim(R) = 0$  or there exists an element  $a$  of  $\mathfrak{m}$  such that  $R/a^n R$  is a Cohen–Macaulay ring of dimension  $d - 1$  for every integer  $n > 0$  (cf. [22]). In [40], it is shown that if  $R$  is an approximately Cohen–Macaulay ring, then so is the ring  $R_{\mathfrak{p}}$  for any prime ideal  $\mathfrak{p}$ . Therefore, the concept of approximately Cohen–Macaulay can be extended to non-local rings as follows.

A non-local ring  $R$  is an approximately Cohen–Macaulay ring if for all prime ideals  $\mathfrak{p}$  of  $R$ , the ring  $R_{\mathfrak{p}}$  is an approximately Cohen–Macaulay ring.

In [40], we proved the following result.

**Theorem 3.14 ([40, Theorem 10]).** *Let  $T := A \otimes_k B$  be a non-zero Noetherian ring. Assume that  $A$  is not a Cohen–Macaulay ring. Then the following hold:*

- (i) *If  $A$  is an approximately Cohen–Macaulay ring and  $B$  is a Cohen–Macaulay ring, then  $T$  is an approximately Cohen–Macaulay ring.*

- (ii) If  $T$  is an approximately Cohen–Macaulay ring, then  $B$  is a Cohen–Macaulay ring.
- (iii) If  $A$  is a homomorphic image of a Cohen–Macaulay ring or  $k$  is algebraically closed, then the following conditions are equivalent:
  - (a)  $T$  is an approximately Cohen–Macaulay ring.
  - (b)  $A$  is an approximately Cohen–Macaulay ring and  $B$  is a Cohen–Macaulay ring.

### 3.4 Clean rings

An element in a ring  $R$  is called clean if it is the sum of a unit and an idempotent. Following Nicholson, cf. [37], we call the ring  $R$  clean if every element in  $R$  is clean. Examples of clean rings include all zero-dimensional rings (i.e., every prime ideal is maximal) and local rings. Clean rings have been studied by several authors, for example [1, 24, 37]. It is an open question whether the tensor product of two clean algebras over a field is clean, cf. [24, Question 3]. In [55] we showed that:

**Theorem 3.15.** *Let  $k$  be an algebraically closed field. Let  $A$  and  $B$  be algebras over  $k$ . If  $A$  and  $B$  have a finite number of minimal prime ideals (e.g.  $A$  and  $B$  Noetherian) then the following statements are equivalent:*

- (i)  $A \otimes_k B$  is clean.
- (ii) The following hold
  - (a)  $A$  and  $B$  are clean.
  - (b)  $A$  or  $B$  is algebraic over  $k$ .

Using this result, we gave an example of two clean algebras  $A$  and  $B$  over a field  $k$  where the tensor product  $A \otimes_k B$  is not clean.

*Example 3.16.* Assume that  $k = \mathbb{C}$  and  $A = B = \mathbb{C}[[x]]$ . Then by [1, Proposition 12]  $A$  and  $B$  are clean. We claim that  $A \otimes_k B$  is not clean. Otherwise, since  $\mathbb{C}$  is an algebraically closed field and  $A(=B)$  is Noetherian, by Theorem 3.15 we have that  $A$  or  $B$  is algebraic over  $\mathbb{C}$  and hence  $A(=B)$  is equal to  $\mathbb{C}$ . This is a contradiction.

### 3.5 Sequentially Cohen–Macaulay rings

The concept of sequentially Cohen–Macaulay module was introduced by Stanley [49, p. 87] for graded modules and studied further by Herzog and Sbarra [27]. In [13] Cuong and Nhan defined this notion for the local case as follows:

**Definition 3.17.** Let  $R$  be a Noetherian local ring. An  $R$ -module  $M$  is called a *sequentially Cohen–Macaulay* module if there exists a filtration  $0 = N_0 \subset N_1 \subset \cdots \subset N_t = M$  of submodules of  $M$  such that

- (i) Each quotient  $N_i/N_{i-1}$  is Cohen–Macaulay.
- (ii)  $\dim(N_1/N_0) < \dim(N_2/N_1) < \dots < \dim(N_t/N_{t-1})$ .

In [56], we showed the following result:

**Theorem 3.18 ([56, Corollary 12]).** *Let  $A$  and  $B$  be two  $k$ -algebras such that  $T := A \otimes_k B$  is Noetherian. Then the following holds:*

- (i) *If  $A$  is a Cohen–Macaulay ring and  $B$  is a locally sequentially Cohen–Macaulay ring then  $A \otimes_k B$  is a locally sequentially Cohen–Macaulay ring.*
- (ii) *If  $B$  is a homomorphic image of a Cohen–Macaulay ring and if  $A \otimes_k B$  is a locally sequentially Cohen–Macaulay ring then  $B$  is a locally sequentially Cohen–Macaulay ring.*
- (iii) *If  $k$  is algebraically closed and  $A$  is a Cohen–Macaulay ring and if  $A \otimes_k B$  is a locally sequentially Cohen–Macaulay ring then  $B$  is a locally sequentially Cohen–Macaulay ring.*

## 4 Combinatorial aspects

Let  $[n] := \{1, \dots, n\}$  be the vertex set and  $\Delta$  a simplicial complex on  $[n]$ . Thus  $\Delta$  is a collection of subsets of  $[n]$  such that if  $F \in \Delta$  and  $F' \subset F$ , then  $F' \in \Delta$ . Each element  $F \in \Delta$  is called a face of  $\Delta$ . The dimension of a face  $F$  is  $|F| - 1$ . Let  $d = \max\{|F| : F \in \Delta\}$  and define the dimension of  $\Delta$  to be  $\dim(\Delta) = d - 1$ . A facet is a maximal face of  $\Delta$  (with respect to inclusion). A simplicial complex is called pure if all facets have the same cardinality.

Let  $\Delta$  be an abstract simplicial complex on the vertex set  $[n]$ . The Stanley–Reisner ring  $k[\Delta]$  of  $\Delta$  over  $k$  is by definition the quotient ring  $R/I_\Delta$  where  $R = k[x_1, \dots, x_n]$  is the polynomial ring over  $k$ , and  $I_\Delta$  is a squarefree monomial ideal generated by all monomials  $x_{i_1} \cdots x_{i_r}$  such that  $\{i_1, \dots, i_r\} \notin \Delta$ . When we talk about algebraic properties of  $\Delta$  we refer to those of its Stanley–Reisner ring. Let  $\Delta'$  be a second simplicial complex whose vertex set differs from  $\Delta$ . The simplicial join  $\Delta * \Delta'$  is defined to be the simplicial complex whose simplices are of the form  $\sigma \cup \sigma'$  where  $\sigma \in \Delta$  and  $\sigma' \in \Delta'$ .

The algebraic and combinatorial properties of the simplicial join  $\Delta * \Delta'$  through the properties of  $\Delta$  and  $\Delta'$  have been studied by a number of authors (cf. [3, 18, 41], and [64]). For instance, in [18], Fröberg used the (graded)  $k$ -algebra isomorphism  $k[\Delta * \Delta'] \simeq k[\Delta] \otimes_k k[\Delta']$ , and proved that the tensor product of two graded  $k$ -algebras is Cohen–Macaulay (resp., Gorenstein) if and only if both of them are Cohen–Macaulay (resp., Gorenstein).

The approach of [43] is in the same spirit as [18], that is, via tensor product, but in a more general setting. Assume that  $A$  and  $B$  are two standard graded  $k$ -algebras, i.e., finitely generated non-negatively graded  $k$ -algebras generated over  $k$  by elements of degree 1, and  $M$  and  $N$  are two finitely generated graded modules over  $A$  and  $B$ , respectively. In [43], various sorts of Cohen–Macaulayness, cleanness, and pretty cleanness of  $A \otimes_k B$ -module  $M \otimes_k N$  through the corresponding properties of  $M$  and  $N$  are studied.

**Definition 4.1.** Let  $M$  be a finitely generated graded module over a standard graded  $k$ -algebra  $(A, \mathfrak{m})$  with  $\dim_A(M) = d$ . The  $A$ -module  $M$  is called generalized Cohen–Macaulay if the length of the local cohomology  $A$ -module  $H_{\mathfrak{m}}^i(M)$  is finite for  $i = 0, 1, \dots, d - 1$ .

**Theorem 4.2 ([43, Theorem 2.6]).** Let  $M$  and  $N$  be two finitely generated graded modules over standard graded  $k$ -algebras  $(A, \mathfrak{m})$  and  $(B, \mathfrak{n})$ , respectively. Assume that both  $\dim_A(M)$  and  $\dim_B(N)$  are positive. Then the following conditions are equivalent:

- (a)  $M \otimes_k N$  is generalized Cohen–Macaulay.
- (b)  $M \otimes_k N$  is Buchsbaum.
- (c)  $M \otimes_k N$  is Cohen–Macaulay.
- (d) Both  $M$  and  $N$  are Cohen–Macaulay.

**Corollary 4.3.** Let  $\Delta$  and  $\Delta'$  be two simplicial complexes over disjoint vertex sets. Then  $\Delta * \Delta'$  is Buchsbaum (over  $k$ ) if and only if  $\Delta$  and  $\Delta'$  are Cohen–Macaulay (over  $k$ ).

*Remark 4.4.* The notion of generalized Cohen–Macaulay module was introduced in [45]. For a simplicial complex  $\Delta$  this notion coincides with the so-called Buchsbaum property. Buchsbaum simplicial complexes have several algebraic and combinatorial characterizations (cf. [35, 50]). For instance, a simplicial complex  $\Delta$  is Buchsbaum over a field  $k$  if and only if it is pure and locally Cohen–Macaulay (i.e., the link of each vertex is Cohen–Macaulay). Recall that the link of a face  $\sigma \in \Delta$  is defined as  $\text{link}_{\Delta}(\sigma) := \{\tau \in \Delta \mid \sigma \cap \tau = \emptyset, \sigma \cup \tau \in \Delta\}$ .

For the next result we need to recall the definition of a graded sequentially Cohen–Macaulay module over a graded ring.

**Definition 4.5 (Stanley [49]).** Let  $A$  be a standard graded  $k$ -algebra. Let  $M$  be a finitely generated  $A$ -module. We say  $M$  is sequentially Cohen–Macaulay if there exists a finite filtration (called a Cohen–Macaulay filtration) of graded submodules of  $M$

$$0 = M_0 \subset M_1 \subset \dots \subset M_r = M$$

satisfying the following two conditions:

- (1) Each quotient module  $M_i/M_{i-1}$  is Cohen–Macaulay,
- (2)  $\dim(M_1/M_0) < \dim(M_2/M_1) < \dots < \dim(M_r/M_{r-1})$ , where  $\dim$  denotes the Krull dimension.

**Theorem 4.6 ([43, Theorem 2.11]).** Let  $A$  and  $B$  be two standard graded  $k$ -algebras. Let  $M$  and  $N$  be two finitely generated graded modules over  $A$  and  $B$ , respectively. Then  $M \otimes_k N$  is a sequentially Cohen–Macaulay  $A \otimes_k B$ -module if and only if  $M$  and  $N$  are sequentially Cohen–Macaulay over  $A$  and  $B$ , respectively.

**Corollary 4.7.** Let  $\Delta$  and  $\Delta'$  be two simplicial complexes over disjoint vertex sets. Then  $\Delta * \Delta'$  is sequentially Cohen–Macaulay (over  $k$ ) if and only if  $\Delta$  and  $\Delta'$  are sequentially Cohen–Macaulay (over  $k$ ).

*Remark 4.8.* (i) In [3] the authors presented a purely combinatorial argument to show that sequential Cohen–Macaulayness is preserved under simplicial join.

(ii) We refer the reader to [15, Theorem 3.3], [49, Proposition II.2.10], and [60, Proposition 1.4] for three different combinatorial characterizations of sequentially Cohen–Macaulay simplicial complexes.

Let  $\Delta$  be a simplicial complex on  $[n]$  of dimension  $d - 1$ . For each  $1 \leq i \leq d - 1$ , we define the pure  $i$ -th skeleton of  $\Delta$  to be the pure subcomplex  $\Delta(i)$  of  $\Delta$  whose facets are those faces  $F$  of  $\Delta$  with  $|F| = i + 1$ . We say that a simplicial complex  $\Delta$  is sequentially Cohen–Macaulay if  $\Delta(i)$  is Cohen–Macaulay for all  $i$ .

**Definition 4.9.** Let  $M$  be a finitely generated graded module over a standard graded  $k$ -algebra  $A$  with  $\dim_A(M) = d$ .

- The  $A$ -module  $M$  is called almost Cohen–Macaulay if  $\text{depth}M \geq d - 1$ .
- Let  $\{N_1, \dots, N_n\}$  denote a reduced primary decomposition of the  $A$ -module  $M$  where each  $N_j$  is a  $\mathfrak{p}_j$ -primary submodule of  $M$ . Let

$$\mathcal{U}_M(0) := \bigcap_{\dim(A/\mathfrak{p}_j)=d} N_j.$$

The  $A$ -module  $M$  is called approximately Cohen–Macaulay whenever it is almost Cohen–Macaulay and  $M/\mathcal{U}_M(0)$  is Cohen–Macaulay.

The original definition of the approximately Cohen–Macaulay property was given by Goto for rings (see Section 3.3). The definition here was taken from [44, Definition 4.4].

**Theorem 4.10 ([43, Theorem 2.17]).** *Let  $M$  and  $N$  be two finitely generated graded modules over standard graded  $k$ -algebras  $A$  and  $B$ , respectively. Assume that  $M$  is not Cohen–Macaulay. Then*

- (1)  $M \otimes_k N$  is almost Cohen–Macaulay if and only if  $M$  is almost Cohen–Macaulay and  $N$  is Cohen–Macaulay.
- (2)  $M \otimes_k N$  is approximately Cohen–Macaulay if and only if  $M$  is approximately Cohen–Macaulay and  $N$  is Cohen–Macaulay.

*Remark 4.11.* For simplicial complexes, the notions of almost and approximately Cohen–Macaulay have combinatorial characterizations. A simplicial complex  $\Delta$  is almost Cohen–Macaulay over a field  $k$  if and only if the codimension one skeleton of  $\Delta$  is Cohen–Macaulay over  $k$  (cf. [4, Exercise 5.1.22]). Recall that the  $r$ -skeleton of the simplicial complex  $\Delta$  is defined as  $\Delta^r := \{\sigma \in \Delta \mid \dim \sigma \leq r\}$ . By [44, Proposition 4.5], an approximately simplicial complex  $\Delta$  can be described combinatorially through the several combinatorial characterizations of sequential Cohen–Macaulayness (cf. Remark 4.8(ii)).

**Corollary 4.12.** *Let  $\Delta$  and  $\Delta'$  be two simplicial complexes over disjoint vertex sets. Then*

- (1)  $\Delta * \Delta'$  is almost Cohen–Macaulay (over  $k$ ) if and only if one of  $\Delta$  or  $\Delta'$  is Cohen–Macaulay (over  $k$ ) and the other is almost Cohen–Macaulay (over  $k$ ).

- (2)  $\Delta * \Delta'$  is approximately Cohen–Macaulay (over  $k$ ) if and only if one of  $\Delta$  or  $\Delta'$  is Cohen–Macaulay (over  $k$ ) and the other is approximately Cohen–Macaulay (over  $k$ ).

Let  $\Delta$  be a simplicial complex on  $[n]$  of dimension  $d - 1$ . We say that  $\Delta$  is shellable if  $\Delta$  is pure and its facets can be ordered as  $F_1, F_2, \dots, F_m$  such that, for all  $2 \leq m$ , the subcomplex  $\langle F_1, \dots, F_{j-1} \rangle \cap \langle F_j \rangle$  is pure of dimension  $d - 2$ . We know by a result of Dress [14] that cleanness is an algebraic counterpart of shellability. In [14, Corollary 2.9], Dress proved combinatorially that the join of two shellable simplicial complexes is shellable. The aim of [43, Section 3] is to present an algebraic way to show that shellability is preserved under tensor product.

For a nonzero finitely generated module  $M$  over a Noetherian ring  $A$ , it is well-known (cf. [34, Theorem 6.4]) that there exists a finite prime filtration

$$\mathcal{F} : 0 = M_0 \subset M_1 \subset \dots \subset M_{r-1} \subset M_r = M$$

with the cyclic quotients  $M_i/M_{i-1} \simeq A/\mathfrak{p}_i$  where  $\mathfrak{p}_i \in \text{Supp}_A(M)$ . The support of  $\mathcal{F}$  is the set of prime ideals  $\text{Supp}_A(\mathcal{F}) := \{\mathfrak{p}_1, \dots, \mathfrak{p}_r\}$ . By [34, Theorem 6.3], we have

$$\text{Min}_A(M) \subseteq \text{Ass}_A(M) \subseteq \text{Supp}_A(\mathcal{F}) \subseteq \text{Supp}_A(M).$$

Here,  $\text{Supp}_A(M)$ ,  $\text{Min}_A(M)$ , and  $\text{Ass}_A(M)$  denote the usual support of  $M$ , the set of minimal primes of  $\text{Supp}_A(M)$ , and the set of associated primes of  $M$ , respectively.

**Definition 4.13 (Dress [14]).** A prime filtration  $\mathcal{F}$  of a nonzero finitely generated module  $M$  over a Noetherian ring  $A$  is called clean if  $\text{Supp}_A(\mathcal{F}) \subseteq \text{Min}_A(M)$ . The  $A$ -module  $M$  is called clean if it admits a clean filtration.

*Remark 4.14.* This notion of “clean module” when applied to a ring is completely different with the other notion of “clean ring” which we used in 3.4.

The following result is shown in [43].

**Theorem 4.15 ([43, Theorem 3.3]).** Let  $A$  and  $B$  be two Noetherian  $k$ -algebras such that  $A \otimes_k B$  is Noetherian. Let  $M$  and  $N$  be two finitely generated modules over  $A$  and  $B$ , respectively. Assume that  $A/\mathfrak{p} \otimes_k B/\mathfrak{q}$  is an integral domain for all  $\mathfrak{p} \in \text{Ass}_A(M)$  and  $\mathfrak{q} \in \text{Ass}_B(N)$ . If  $M$  and  $N$  are clean, then  $M \otimes_k N$  is clean.

**Corollary 4.16 ([43, Corollary 3.8]).** Let  $I$  and  $J$  be two arbitrary monomial ideals in the polynomial rings  $R = k[x_1, \dots, x_n]$  and  $S = k[y_1, \dots, y_m]$ , respectively. Then  $R/I \otimes_k S/J$  is clean if and only if  $R/I$  and  $S/J$  are clean.

The aim of [43, Section 4] is to present an algebraic way to show that the notion of shellability for the simplicial multicomplexes introduced by Herzog and Popescu [26] is preserved under simplicial join of multicomplexes. Here, we recall some basic definitions and results related to multicomplexes, and we refer the reader to [26, Section 9] for more details.



For a subset  $\Gamma \subseteq \mathbb{N}_\infty^n$  where  $\mathbb{N}_\infty := \mathbb{N} \cup \{\infty\}$  with  $a < \infty$  for all  $a \in \mathbb{N}$ , the set of all maximal elements of  $\Gamma$  with respect to the componentwise partial order  $\leq$  is denoted by  $\mathcal{M}(\Gamma)$ . The subset  $\Gamma$  is called a multicomplex if it is closed under going down, and for each element  $\mathbf{a} \in \Gamma$ , there exists  $\mathbf{m} \in \mathcal{M}(\Gamma)$  with  $\mathbf{a} \leq \mathbf{m}$ .

For each multicomplex  $\Gamma \subseteq \mathbb{N}_\infty^n$ , the  $k$ -subspace in  $R = k[x_1, \dots, x_n]$  spanned by all monomials  $x_1^{a_1} \cdots x_n^{a_n}$  with  $(a_1, \dots, a_n) \notin \Gamma$  is a monomial ideal denoted by  $I_\Gamma$ . The correspondence  $\Gamma \rightsquigarrow k[\Gamma] := R/I_\Gamma$  constitutes a bijection from simplicial multicomplexes  $\Gamma$  in  $\mathbb{N}_\infty^n$  to monomial ideals inside  $R$ .

Let  $\Gamma' \subseteq \mathbb{N}_\infty^m$  be a second simplicial multicomplex. The simplicial join  $\Gamma$  with  $\Gamma'$  is the simplicial multicomplex

$$\Gamma * \Gamma' := \{\bar{\mathbf{a}} + \bar{\mathbf{b}} \mid \mathbf{a} \in \Gamma \text{ and } \mathbf{b} \in \Gamma'\} \subseteq \mathbb{N}_\infty^{n+m}$$

where  $\bar{\cdot} : \mathbb{N}_\infty^n \rightarrow \mathbb{N}_\infty^{n+m}$  and  $\bar{\cdot} : \mathbb{N}_\infty^m \rightarrow \mathbb{N}_\infty^{n+m}$  are canonical embeddings defined by  $\bar{\mathbf{a}} := (a_1, \dots, a_n, 0, \dots, 0)$  and  $\bar{\mathbf{b}} := (0, \dots, 0, b_1, \dots, b_m)$  where  $\mathbf{a} := (a_1, \dots, a_n)$  and  $\mathbf{b} := (b_1, \dots, b_m)$ . As in the case of simplicial complexes, we have the (graded)  $k$ -algebra isomorphism  $k[\Gamma * \Gamma'] \simeq k[\Gamma] \otimes_k k[\Gamma']$ .

A multicomplex  $\Gamma$  is shellable in the sense of [26, Definition 10.2] if and only if the  $k$ -algebra  $k[\Gamma]$  is pretty clean in the following sense.

**Definition 4.17 (Herzog-Popescu [26]).** A prime filtration

$$\mathcal{F} : 0 = M_0 \subset M_1 \subset \cdots \subset M_{r-1} \subset M_r = M$$

of a nonzero finitely generated module  $M$  over a Noetherian ring  $A$  with the cyclic quotients  $M_i/M_{i-1} \simeq A/\mathfrak{p}_i$  is called pretty clean if for all  $i < j$  for which  $\mathfrak{p}_i \subseteq \mathfrak{p}_j$  it follows that  $\mathfrak{p}_i = \mathfrak{p}_j$ . The module  $M$  is called pretty clean if it admits a pretty clean filtration.

**Theorem 4.18 ([43, Theorem 4.7]).** Let  $M$  and  $N$  be two finitely generated graded modules over standard graded Cohen–Macaulay  $k$ -algebras  $(A, \mathfrak{m})$  and  $(B, \mathfrak{n})$  with canonical modules  $\omega_A$  and  $\omega_B$ , respectively. Assume that  $A/\mathfrak{p} \otimes_k B/\mathfrak{q}$  is an integral domain for all  $\mathfrak{p} \in \text{Ass}_A(M)$  and  $\mathfrak{q} \in \text{Ass}_B(N)$ . If  $M$  and  $N$  are pretty clean modules with pretty clean filtrations  $\mathcal{F}_M$  and  $\mathcal{F}_N$  such that  $A/\mathfrak{p}$  and  $B/\mathfrak{q}$  are Cohen–Macaulay for all  $\mathfrak{p} \in \text{Supp}(\mathcal{F}_M)$  and  $\mathfrak{q} \in \text{Supp}(\mathcal{F}_N)$ , then  $M \otimes_k N$  is pretty clean.

As a consequence of Theorem 4.18 we have the following result.

**Corollary 4.19 ([43, Corollary 4.8]).** Let  $I$  and  $J$  be two arbitrary monomial ideals in the polynomial rings  $R = k[x_1, \dots, x_n]$  and  $S = k[y_1, \dots, y_m]$ , respectively. Then  $R/I \otimes_k S/J$  is pretty clean if and only if  $R/I$  and  $S/J$  are pretty clean.

Theorem 4.18 is not quite satisfactory because it needs a lot of hypotheses. However, in some special cases like the following, we can reduce these assumptions.

**Theorem 4.20 ([43, Theorem 4.9]).** Let  $A$  and  $B$  be two Noetherian  $k$ -algebras such that  $A \otimes_k B$  is Noetherian. Assume that  $A/\mathfrak{p} \otimes_k B/\mathfrak{q}$  is an integral domain for all  $\mathfrak{p} \in \text{Ass}(A)$  and  $\mathfrak{q} \in \text{Ass}(B)$ . If  $A$  is pretty clean and  $B$  is clean, then  $A \otimes_k B$  is pretty clean.

Following these facts about pretty clean (clean) property an interesting question is:

*Question 4.21.* Let  $A$  and  $B$  be two Noetherian  $k$ -algebras such that  $A \otimes_k B$  is Noetherian. Let  $A/\mathfrak{p} \otimes_k B/\mathfrak{q}$  be an integral domain for all  $\mathfrak{p} \in \text{Ass}(A)$  and  $\mathfrak{q} \in \text{Ass}(B)$ . If  $A \otimes_k B$  is pretty clean (resp., clean), are  $A$  and  $B$  pretty clean (resp., clean)?

## 5 Hilbert rings

In the literature, the following definitions are used interchangeably to define Hilbert (also called Jacobson) rings.

- (i) A ring  $R$  is called a Hilbert ring if each non-maximal prime ideal of  $R$  can be represented as the intersection of maximal ideals.
- (ii) A ring  $R$  is called a Hilbert ring if for each prime ideal  $P$  of  $R$ , the Jacobson radical of  $R/P$  is zero.
- (ii)' A ring  $R$  is called a Hilbert ring if for each ideal  $I$  of  $R$ , the Jacobson radical of  $R/I$  is nil.
- (iii) A ring  $R$  is called a Hilbert ring if for each maximal ideal  $\mathfrak{m}$  of the polynomial ring  $R[x]$ ,  $\mathfrak{m} \cap R$  is a maximal ideal of  $R$ .
- (iv) A ring  $R$  is called a Hilbert ring if for a prime ideal  $P$  such that the quotient field of  $R/P$  is finitely generated over  $R/P$ , then  $P$  is maximal.

These rings were independently studied by O. Goldman [21] and W. Krull [31], who named them Hilbert rings and Jacobson rings, respectively, in order to generalize the Hilbert's Nullstellensatz, which is stated for algebraically closed fields, to the general case. In [21], the algebraic version of Nullstellensatz is proved:

**Theorem 5.1.** *Let  $R$  be a Hilbert ring. Any finitely generated  $R$ -algebra is a Hilbert ring. Moreover, if  $S$  is a finitely generated  $R$ -algebra and  $\mathfrak{n} \subset S$  is a maximal ideal of  $S$  then  $\mathfrak{m} = \mathfrak{n} \cap R$  is a maximal ideal of  $R$  and  $S/\mathfrak{n}$  is a finite algebraic extension of  $R/\mathfrak{m}$ .*

The definition of Hilbert ring immediately implies that every quotient of a Hilbert ring is also a Hilbert ring. Moreover, it can be shown that if  $R$  is a Hilbert ring, then every integral extension of  $R$  is a Hilbert ring.

Rings of Krull dimension zero, ring of integers, PIDs and more generally Dedekind domains are Hilbert rings.

Even though every finitely generated algebra over a Hilbert ring is a Hilbert ring, it is not true that any subalgebra of a Hilbert ring is a Hilbert ring. In [61], it is proved if  $k$  is a countable field, then any  $k$ -algebra of transcendence degree more than two contains a  $k$ -subalgebra which is not a Hilbert ring. On the other hand, this author shows that if the  $k$ -algebra  $A$  is of transcendence degree at most two, then a  $k$ -subalgebra of  $A$  is again a Hilbert ring.

In [31], it is shown that the ring  $k[\{x_i\}_{i \in \mathbb{N}}]$  is a Hilbert ring if and only if  $k$  has uncountable cardinality. If the field  $k$  is replaced with a ring, the claim may not be true. Regarding this claim, in [25], an example of a Hilbert ring  $R$  is constructed which is uncountable but the polynomial ring  $R[\{x_i\}_{i \in \mathbb{N}}]$  is not a Hilbert ring. Furthermore, for the polynomial ring  $R[\{x_i\}_{i \in I}]$  in an infinite number of indeterminates to be a Hilbert ring, equivalent conditions are given.

Since the tensor product of  $k$ -algebras is a special kind of base change, a question similar to the one in previous sections is: under what conditions is the tensor product of two Hilbert  $k$ -algebras a Hilbert ring?

This question is settled in special cases. In [58, Theorem 5] it is proved that:

**Theorem 5.2.** *Let  $K$  and  $L$  be extension fields of  $k$  and  $T = K \otimes_k L$ . Assume that  $\text{tr.deg}_k(K) \geq \text{tr.deg}_k(L) = n < \infty$ . Then  $T$  is a Hilbert ring in which every maximal ideal has height  $n$ .*

This result is generalized in [39], and O’Carroll and Qureshi raised the following conjecture:

Let  $K_1, \dots, K_n$  be fields of finite transcendence degree  $t_1, \dots, t_n$  over a common field  $k$ , respectively, and  $n \geq 2$ . Assume  $t_1 \geq t_i$  for every  $i = 2, \dots, n$ . Then  $K_1 \otimes_k \dots \otimes_k K_n$  is a Hilbert ring such that every maximal ideal is of height  $t_2 + \dots + t_n$ .

They proved this conjecture in two particular cases: where (1)  $t_1 \geq t_2 + \dots + t_n$  and (2)  $t_i = 1$  for all  $i \geq 2$ . On the other hand, the case where  $n = 2$  is treated in Theorem 5.2.

This conjecture is proved independently in [36] and [57] with different methods. This result leads one to ask:

*Question 5.3.* Let  $A$  and  $B$  be two Hilbert  $k$ -algebras such that  $A \otimes_k B$  is Noetherian. Is  $A \otimes_k B$  a Hilbert ring? Is it equidimensional?

Since  $A \otimes_k B$  is a faithfully flat extension of  $A$  and  $B$ , the Noetherian assumption on  $A \otimes_k B$  implies both  $A$  and  $B$  are Noetherian too. If  $A$  and  $B$  are finitely generated  $k$ -algebras, or if  $A$  is a finite field extension of  $k$ , or if  $A$  is a field extension of  $k$  and  $B$  is a finitely generated  $k$  algebra, it can be shown that  $A \otimes_k B$  is a Hilbert ring.

In connection with 5.3, the paper [12] and references therein might be of interest.

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# Multiplicative ideal theory in the context of commutative monoids

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**Abstract** It is well known that large parts of multiplicative ideal theory can be derived in the language of commutative monoids. Classical parts of the theory were treated in this context in my monograph “Ideal Systems” (Marcel Dekker, 1998). The main purpose of this article is to outline some recent developments of multiplicative ideal theory (especially the concepts of spectral star operations and semistar operations together with their applications) in a purely multiplicative setting.

## 1 Introduction

General ideal theory of commutative rings has its origin in R. Dedekind’s multiplicative theory of algebraic numbers from the nineteenth century. It became an autonomous theory by the work of W. Krull and E. Noether about 1930, and it proved to be a most powerful tool in algebraic and arithmetic geometry and complex analysis. Besides this mainstream movement towards algebraic geometry, there is a modern development of multiplicative ideal theory based on the works of W. Krull and H. Prüfer.

The main objective of multiplicative ideal theory is the investigation of the multiplicative structure of integral domains by means of ideals or certain systems of ideals of that domain. In doing so, Krull’s concept of ideal systems proved to be fundamental. Its presentation in R. Gilmer’s book [23], using the notion of star operations, influenced most of the research done in this area during the last 40 years, yielding a highly developed theory of integral domains characterized by ideal-theoretic or valuation-theoretic properties.

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Fresh impetuses to the theory were given in the nineties by the concepts of spectral star operations and semistar operations. Spectral star operations were introduced by Fanggui Wang and R.L. McCasland [11, 12] and shed new light on the connection between local and global behavior of integral domains. Semistar operations were introduced by A. Okabe and R. Matsuda [39] as a generalization of the concept of star operations. This new concept proved to be more flexible and made it possible to extend the theory obtained by star operations to a larger class of integral domains.

Already in the early history of the theory, it was observed that a great deal of multiplicative ideal theory can be developed for commutative monoids disregarding the additive structure of integral domains. In an axiomatic way, this was first done by P. Lorenzen [34], and, in a more general setting, by K.E. Aubert [7]. A systematic presentation of this purely multiplicative theory was given in the volumes by P. Jaffard [32], J. Močkoř [37] and recently by the author [25].

The present article is based on the monograph [25]. Its main purpose is to outline the development of multiplicative ideal theory during the last 20 years (especially the concepts of spectral star operations and semistar operations) in the context of commutative monoids. In doing so, instead of being encyclopedic, we focus on the main results to outline the method, and we often only sketch proofs instead of giving them in full detail.

## 2 Notations and preliminaries

By a *monoid* we always mean (deviating from the usual terminology) a commutative multiplicative semigroup  $K$  containing a unit element  $1 \in K$  and a zero element  $0 \in K$  (satisfying  $0x = 0$  for all  $x \in K$ ) such that every non-zero element  $a \in K$  is cancellative (that is,  $ab = ac$  implies  $b = c$  for all  $b, c \in K$ ).

For any set  $X$ , we denote by  $X^\bullet$  the set of non-zero elements of  $X$ , by  $\mathbb{P}_f(X)$  the set of all finite subsets of  $X$ , and we set  $\mathbb{P}_f^\bullet(X) = \{E \in \mathbb{P}_f(X) \mid E^\bullet \neq \emptyset\}$ . A family  $(X_\lambda)_{\lambda \in \Lambda}$  of subsets of  $X$  is called *directed* if, for any  $\alpha, \beta \in \Lambda$  there exists some  $\lambda \in \Lambda$  such that  $X_\alpha \cup X_\beta \subset X_\lambda$ .

For a monoid  $K$ , we denote by  $K^\times$  the group of invertible elements of  $K$ . For subsets  $X, Y \subset K$ , we define  $XY = \{xy \mid x \in X, y \in Y\}$  and  $(X:Y) = (X:{}_K Y) = \{z \in K \mid zY \subset X\}$ , and for  $c \in K$  we set  $Xc = X\{c\}$  and  $(X:c) = (X:\{c\})$ .

A submonoid  $D \subset K$  is always assumed to contain 1 and 0, and a monoid homomorphism is assumed to respect 0 and 1.

*In the sequel, let  $K$  be a monoid and  $D \subset K$  a submonoid.*

A subset  $M \subset K$  is called a *D-module* if  $DM = M$ , and it is called an *ideal* of  $D$  if it is a *D-submodule* of  $D$ . A subset  $T \subset K$  is called *multiplicatively closed* if  $1 \in T$ ,  $0 \notin T$  and  $TT = T$ . For a multiplicatively closed subset  $T \subset K^\times$  and  $X \subset K$ , we define

$$T^{-1}X = \{t^{-1}x \mid t \in T, x \in X\} = \bigcup_{t \in T} t^{-1}X.$$



If  $TX = X$ , then the family  $(t^{-1}X)_{t \in T}$  is directed. If  $T \subset D$  is multiplicatively closed and  $X$  is a  $D$ -module, then  $T^{-1}D \subset K$  is a submonoid, and  $T^{-1}X = (T^{-1}D)X$  is a  $T^{-1}D$ -module. We call  $T^{-1}D$  the *quotient monoid* of  $D$  with respect to  $D$ . We call  $K$  a *quotient* of  $D$  and write  $K = q(D)$  if  $D^\bullet \subset K^\times$  and  $K = D^{\bullet -1}D$  (then  $K^\bullet = K^\times$  is a quotient group of  $D^\bullet$ ). Every monoid possesses a quotient which is unique up to canonical isomorphisms. If  $K = q(D)$ , then a subset  $X \subset K$  is called  *$D$ -fractional* if  $cX \subset D$  for some  $c \in D^\bullet$ .

An ideal  $P \subset D$  is called a *prime ideal* of  $D$  if  $D \setminus P$  is multiplicatively closed. If  $D \setminus P \subset K^\times$  and  $X \subset K$ , then we set  $X_P = (D \setminus P)^{-1}X$ .

In the following Lemma 2.1 we collect the elementary properties of quotient monoids. Proofs are easy and left to the reader.

**Lemma 2.1.** *Let  $T \subset D \cap K^\times$  a multiplicatively closed subset.*

1. *If  $J \subset D$  is an ideal of  $D$ , then  $T^{-1}J = (T^{-1}D)J \subset T^{-1}D$  is an ideal of  $T^{-1}D$ ,  $J \subset T^{-1}J \cap D$ , and  $T^{-1}J = T^{-1}D$  if and only if  $J \cap T \neq \emptyset$ .*
2. *If  $\bar{J} \subset T^{-1}D$  is an ideal of  $T^{-1}D$ , then  $\bar{J} = T^{-1}(\bar{J} \cap D)$ .*
3. *The assignment  $P \mapsto T^{-1}P$  defines a bijective map from the set of all prime ideals  $P \subset D$  with  $P \cap T = \emptyset$  onto the set of all prime ideals of  $T^{-1}D$ .*
4. *If  $P \subset D$  is a prime ideal and  $T \cap P = \emptyset$ , then  $P = T^{-1}P \cap D$ , and if  $T = D \setminus P$ , then  $T^{-1}P = PD_P = D_P \setminus D_P^\times$  is the greatest ideal of  $D_P$ .*
5. *If  $X, Y \subset K$ , then  $T^{-1}(X:Y) \subset (T^{-1}X:T^{-1}Y) = (T^{-1}X:Y)$ , and equality holds, if  $Y$  is finite.*

### 3 Definition and first properties of weak module systems

*Let  $K$  be a monoid and  $D \subset K$  a submonoid.*

**Definition 3.1.** A *weak module system* on  $K$  is a map

$$r: \begin{cases} \mathbb{P}(K) & \rightarrow \mathbb{P}(K) \\ X & \mapsto X_r \end{cases}$$

such that for all  $c \in K$  and  $X, Y \in \mathbb{P}(K)$  the following conditions are fulfilled :

- M1.**  $X \cup \{0\} \subset X_r$ .
- M2.** If  $X \subset Y_r$ , then  $X_r \subset Y_r$ .
- M3.**  $cX_r \subset (cX)_r$ .

A *module system* on  $K$  is a weak module system  $r$  on  $K$  such that, for all  $X \subset K$  and  $c \in K$ ,

- M3'.**  $cX_r = (cX)_r$ .

Let  $r$  be a weak module system on  $K$ . A subset  $A \subset K$  is called an  *$r$ -module* if  $A_r = A$ , and  $D$  is called an  *$r$ -monoid* if it is an  $r$ -module. We denote by  $\mathcal{M}_r(K)$  the set of all

$r$ -modules in  $K$ . An  $r$ -module  $A \subset K$  is called  $r$ -finite or  $r$ -finitely generated if  $A = E_r$  for some  $E \in \mathbb{P}_f(K)$ . We denote by  $\mathcal{M}_{r,f}(K)$  the set of all  $r$ -finite  $r$ -modules.

A (weak) module system  $r$  on  $K$  is called a (weak)  $D$ -module system if every  $r$ -module is a  $D$ -module, and it is called a (weak) ideal system on  $K$  if it is a (weak)  $K$ -module system. If  $r$  is a (weak) ideal system on  $K$ , then the  $r$ -modules are called  $r$ -ideals, and in this case we shall often write  $\mathcal{J}_r(K) = \mathcal{M}_r(K)$  (to be concordant with [25]).

The concept of a weak module system is a final step in a series of generalizations of the concepts of star and semistar operations on integral domains and that of Lorenzen’s  $r$ -systems and Aubert’s  $x$ -systems on commutative monoids. This concept also applies for not necessarily cancellative monoids, and in this setting it was presented in [27] (where a purely multiplicative analog of the Marot property for commutative rings was established). In this paper, however, we shall restrict to cancellative monoids.

Examples will be discussed and presented later on in 5.6. In the meantime, the interested reader is invited to consult [25, Sections 2.2 and 11.4] to see examples of (weak) ideal systems and [30] to see examples of module systems.

In the following Proposition 3.2, we gather the elementary properties of weak module systems. We shall use them freely throughout this article. Their proofs are literally identical with those for weak ideal systems as presented in [25, Propositions 2.1, 2.3 and 2.4], and thus they will be omitted.

**Proposition 3.2.** *Let  $r$  be a weak module system on  $K$  and  $X, Y \subset K$ .*

1.  $\emptyset_r = \{0\}_r$  and if  $r$  is a module system, then  $\{0\}_r = \{0\}$ .
2.  $(X_r)_r = X_r$ , and if  $X \subset Y$ , then  $X_r \subset Y_r$ . In particular,  $X_r$  is the smallest  $r$ -module containing  $X$ .
3. The intersection of any family of  $r$ -modules is again an  $r$ -module.
4. For every family  $(X_\lambda)_{\lambda \in \Lambda}$  in  $\mathbb{P}(K)$  we have

$$\bigcup_{\lambda \in \Lambda} (X_\lambda)_r \subset \left( \bigcup_{\lambda \in \Lambda} X_\lambda \right)_r = \left( \bigcup_{\lambda \in \Lambda} (X_\lambda)_r \right)_r.$$

5.  $(XY)_r = (X_r Y)_r = (XY_r)_r = (X_r Y_r)_r$ , and for every family  $(X_\lambda)_{\lambda \in \Lambda}$  in  $\mathbb{P}(K)$  we have

$$\left( \bigcup_{\lambda \in \Lambda} X_\lambda Y \right)_r = \left( \bigcup_{\lambda \in \Lambda} (X_\lambda)_r Y \right)_r = \left( \bigcup_{\lambda \in \Lambda} (X_\lambda Y)_r \right)_r.$$

Equipped with the  $r$ -multiplication, defined by  $(X, Y) \mapsto (XY)_r$ ,  $\mathcal{M}_r(K)$  is a commutative semigroup with unit element  $\{1\}_r$  and zero element  $\emptyset_r$ , and  $\mathcal{M}_{r,f}(K) \subset \mathcal{M}_r(K)$  is a subsemigroup.

6.  $(X : Y)_r \subset (X_r : Y) = (X_r : Y_r) = (X_r : Y)_r$ , and equality holds, if  $Y$  is finite. In particular, if  $X$  is an  $r$ -module, then  $(X : Y)$  is also an  $r$ -module.

**Proposition 3.3.** *Let  $r$  be a weak module system on  $K$ .*

1.  $D_r$  is an  $r$ -monoid, and if  $A \subset K$  is a  $D$ -module, then  $A_r$  is a  $D_r$ -module. In particular,  $\{1\}_r$  is the smallest  $r$ -monoid contained in  $K$ ,  $r$  is a weak  $\{1\}_r$ -module system, and if  $D \subset \{1\}_r$ , then  $\{1\}_r = D_r$ .
2. Let  $r$  be a weak  $D$ -module system. Then  $\{1\}_r = D_r$ , and if  $X \subset K$ , then  $X_r = D_r X_r = (DX)_r$ .
3.  $r$  is a weak  $D$ -module system if and only if  $cD \subset \{c\}_r$  for all  $c \in K$ , and if  $r$  is a  $D$ -module system, then  $\{c\}_r = cD_r$  for all  $c \in K$ .

*Proof.* 1. We have  $D_r D_r \subset (DD)_r = D_r \subset D_r D_r$ , and thus  $D_r = D_r D_r \subset K$  is a submonoid. If  $A \subset K$  is a  $D$ -module, then  $D_r A_r \subset (DA)_r = A_r \subset D_r A_r$ . Hence  $A_r = D_r A_r$  is a  $D_r$ -module.

2.  $\{1\}_r$  is a  $D$ -module containing 1, hence  $D \subset \{1\}_r \subset D_r$ , and thus  $\{1\}_r = D_r$ . If  $X \subset K$ , then  $X_r \subset D_r X_r \subset (DX)_r = (DX_r)_r = (X_r)_r = X_r$ , and thus equality holds.

3. If  $r$  is a weak  $D$ -module system and  $c \in K$ , then  $\{c\}_r$  is a  $D$ -module containing  $c$ , which implies  $cD \subset \{c\}_r$ . If  $r$  is a  $D$ -module system, then  $\{c\}_r = c\{1\}_r = cD_r$ . Assume now that  $cD \subset \{c\}_r$  for all  $c \in K$ , and let  $A \in \mathcal{M}_r(K)$ . Then  $A \subset DA$ , and if  $c \in A$ , then  $Dc \subset \{c\}_r \subset A_r = A$ , hence  $DA = A$ , and thus  $r$  is a weak  $D$ -module system.  $\square$

**Definition 3.4.** A weak module system  $r$  on  $K$  is called *finitary* or *of finite type* if finitary of finite type

$$X_r = \bigcup_{E \in \mathbb{P}_f(X)} E_r \quad \text{for all } X \subset K.$$

**Theorem 3.5.** *Let  $r$  be a weak module system on  $K$ . Then the following assertions are equivalent:*

- (a)  $r$  is finitary.
- (b) For all  $X \subset K$  and  $a \in X_r$  there exists a finite subset  $E \subset X$  such that  $a \in E_r$ .
- (c) For every directed family  $(X_\lambda)_{\lambda \in \Lambda}$  in  $\mathbb{P}(K)$  we have

$$\left( \bigcup_{\lambda \in \Lambda} X_\lambda \right)_r = \bigcup_{\lambda \in \Lambda} (X_\lambda)_r.$$

- (d) The union of every directed family of  $r$ -modules is again an  $r$ -module.
- (e) If  $X \subset K$ ,  $A \in \mathcal{M}_{r,f}(K)$  and  $A \subset X_r$ , then there is a finite subset  $E \subset X$  satisfying  $A \subset E_r$ .

*In particular, if  $r$  is finitary,  $X \subset K$  and  $X_r \in \mathcal{M}_{r,f}(K)$ , then there exists a finite subset  $E \subset X$  such that  $E_r = X_r$ .*

*Proof.* The equivalence of (a) and (b) is obvious, and the equivalence of (a), (c) and (d) is proved as the corresponding statements for weak ideal systems in [25, Proposition 3.1].

(b)  $\Rightarrow$  (e) Suppose that  $X \subset K$  and  $A = F_r \subset X_r$ , where  $F \in \mathbb{P}_f(K)$ . For every  $c \in F$ , there is some  $E(c) \in \mathbb{P}_f(X)$  such that  $c \in E(c)_r$ . Then

$$E = \bigcup_{c \in E} E(c) \in \mathbb{P}_f(X), \quad F \subset \bigcup_{c \in E} E(c)_r \subset E_r \quad \text{and thus} \quad A = F_r \subset E_r.$$

(e)  $\Rightarrow$  (b) If  $X \subset K$  and  $a \in X_r$ , then  $\{a\}_r \in \mathcal{M}_{r,f}(K)$  and  $\{a\}_r \subset X_r$ . Hence there exists a finite subset  $E \subset X$  such that  $a \in \{a\}_r \subset E_r$ .

The final statement follows from (e) with  $A = X_r$ .  $\square$

### Theorem 3.6.

1. Let  $r: \mathbb{P}_f(K) \rightarrow \mathbb{P}(K)$  be a map satisfying the conditions **M1**, **M2** and **M3** in Definition 3.1 for all  $X, Y \in \mathbb{P}_f(K)$  and  $c \in K$ . Then

$$\bar{r}: \mathbb{P}(K) \rightarrow \mathbb{P}(K), \quad \text{defined by} \quad X_{\bar{r}} = \bigcup_{E \in \mathbb{P}_f(X)} E_r \quad \text{for all} \quad X \subset K,$$

is the unique finitary weak module system on  $K$  satisfying  $\bar{r}|_{\mathbb{P}_f(K)} = r$ . Moreover, if  $r$  has also the property **M3'** for all  $X \in \mathbb{P}_f(K)$  and  $c \in K$ , then  $\bar{r}$  is a module system, and if  $cD \subset \{c\}_r$  for all  $c \in K$ , then  $\bar{r}$  is a weak  $D$ -module system.

2. Let  $r$  be a (weak) module system on  $K$ . Then there exists a unique finitary (weak) module system  $r_f$  on  $K$  such that  $E_r = E_{r_f}$  for all finite subsets of  $K$ . It is given by

$$X_{r_f} = \bigcup_{E \in \mathbb{P}_f(X)} E_r \quad \text{for all} \quad X \subset K,$$

it satisfies  $(r_f)_f = r_f$ ,  $X_{r_f} \subset X_r$  for all  $X \in \mathbb{P}(K)$ ,  $\mathcal{M}_{r_f,f}(K) = \mathcal{M}_{r,f}(K)$ , and if  $r$  is a (weak)  $D$ -module system, then so is  $r_f$ .

*Proof.* 1. It is easily checked that  $\bar{r}$  satisfies the conditions **M1**, **M2** and **M3** resp. **M3'** of Definition 3.1. Hence,  $\bar{r}$  is a weak module system resp. a module system, and obviously  $E_{\bar{r}} = E_r$  for all finite subsets  $E \subset K$ . Hence,  $X_{\bar{r}} = \bigcup_{E \in \mathbb{P}_f(X)} E_{\bar{r}}$  for all  $X \subset K$ , and therefore  $\bar{r}$  is finitary. If  $\tilde{r}$  is any finitary weak module system on  $K$  with  $\tilde{r}|_{\mathbb{P}_f(K)} = r$ , then

$$X_{\tilde{r}} = \bigcup_{E \in \mathbb{P}_f(X)} E_{\tilde{r}} = \bigcup_{E \in \mathbb{P}_f(X)} E_r = X_{\bar{r}}, \quad \text{which implies} \quad \tilde{r} = \bar{r}.$$

If  $cD \subset \{c\}_r = \{c\}_{\bar{r}}$  for all  $c \in K$ , then  $\bar{r}$  is a weak  $D$ -module system by Proposition 3.3.3.

2. By 1., applied for  $r|_{\mathbb{P}_f(X)}$ , there exists a unique (weak) module system  $r_f$  on  $K$  such that  $E_{r_f} = E_r$  for all  $E \in \mathbb{P}_f(X)$ . If  $X \subset K$ , then  $X_{r_f}$  is given as asserted, and if  $r$  is a (weak)  $D$ -module system, then so is  $r_f$ . By definition, we have  $\mathcal{M}_{r_f,f}(K) = \mathcal{M}_{r,f}(K)$ , and by the uniqueness of  $r_f$  it follows that  $r_f = r$  if and only if  $r$  is finitary, and, in particular,  $(r_f)_f = r_f$ .  $\square$

- Definition 3.7.** 1. Let  $r: \mathbb{P}_f(K) \rightarrow \mathbb{P}(K)$  be a map having the properties **M1**, **M2** and **M3** in Definition 3.1 for all  $X, Y \in \mathbb{P}_f(K)$  and  $c \in K$ . Then the unique weak module system on  $K$  which coincides with  $r$  on  $\mathbb{P}_f(K)$  (see Theorem 3.5.1) is called the *total system* associated with  $r$  and is again denoted by  $r$  (instead of  $\bar{r}$ ).
2. Let  $r$  be a (weak) module system on  $K$ . Then the unique finitary (weak) module system  $r_f$  on  $K$  defined in Theorem 3.5.2 is called the *finitary (weak) module system associated with  $r$* .

## 4 Comparison and mappings of weak module systems

*Let  $K$  be a monoid.*

**Definition 4.1.** Let  $r$  and  $q$  be weak module systems on  $K$ . We call  $q$  *finer* than  $r$  and  $r$  *coarser* than  $q$  and write  $r \leq q$  if  $X_r \subset X_q$  for all subsets  $X \subset K$ .

**Proposition 4.2.** *Let  $r$  and  $q$  be weak module systems on  $K$ . Then  $r_f \leq r$ , and the following assertions are equivalent:*

- (a)  $r \leq q$ .
- (b)  $X_q = (X_r)_q$  for all subsets  $X \subset K$ .
- (c)  $\mathcal{M}_q(K) \subset \mathcal{M}_r(K)$ .

*If  $r$  is finitary, then there are also equivalent:*

- (d)  $E_q \subset E_r$  for all finite subsets  $E \subset K$ .
- (e)  $\mathcal{M}_{q_f}(K) \subset \mathcal{M}_r(K)$ .
- (f)  $\mathcal{M}_{q,f}(K) \subset \mathcal{M}_r(K)$ .
- (g)  $r \leq q_f$ .

*In particular, if  $r$  and  $q$  are both finitary, then  $r = q$  if and only if  $E_r = E_q$  for all finite subsets  $E \subset K$ .*

*Proof.* Straightforward (see also [25, Proposition 5.1]). □

**Definition 4.3.** Let  $\varphi: K \rightarrow L$  a monoid homomorphism,  $r$  a weak module system on  $K$  and  $q$  a weak module system on  $L$ .

$\varphi$  is called an  $(r, q)$ -*homomorphism* if  $\varphi(X_r) \subset \varphi(X)_q$  for all subsets  $X \subset K$ . We denote by  $\text{Hom}_{(r,q)}(K, L)$  the set of all  $(r, q)$ -homomorphisms  $\varphi: K \rightarrow L$ .

**Proposition 4.4.** *Let  $\varphi: K \rightarrow L$  a monoid homomorphism,  $r$  a weak module system on  $K$  and  $q$  a weak module system on  $L$ .*

1.  $\varphi$  is an  $(r, q)$ -homomorphism if and only if  $\varphi^{-1}(A) \in \mathcal{M}_r(K)$  for all  $A \in \mathcal{M}_q(L)$ .
2. Let  $r$  be finitary and  $\varphi(E_r) \subset \varphi(E)_q$  for all  $E \in \mathbb{P}_f(K)$ . Then  $\varphi$  is an  $(r, q)$ -homomorphism.

*Proof.* 1. If  $\varphi$  is an  $(r, q)$ -homomorphism and  $A \in \mathcal{M}_q(L)$ , then it follows that  $\varphi(\varphi^{-1}(A)_r) \subset \varphi(\varphi^{-1}(A))_q = \varphi(\varphi^{-1}(A)) \subset A$ . Hence  $\varphi^{-1}(A)_r \subset \varphi^{-1}(A)$ , and thus  $\varphi^{-1}(A) = \varphi^{-1}(A)_r \in \mathcal{M}_r(K)$ .

Thus assume that  $\varphi^{-1}(A) \in \mathcal{M}_r(K)$  for all  $A \in \mathcal{M}_q(L)$ , and let  $X \subset K$ . Then  $\varphi^{-1}(\varphi(X)_q) \in \mathcal{M}_r(K)$ , and as  $X \subset \varphi^{-1}(\varphi(X)) \subset \varphi^{-1}(\varphi(X)_q)$ , it follows that  $X_r \subset \varphi^{-1}(\varphi(X)_q)$  and therefore  $\varphi(X_r) \subset \varphi(X)_q$ .

2. If  $X \subset K$  and  $a \in X_r$ , then there is some  $E \in \mathbb{P}_f(X)$  such that  $a \in E_r$ , and thus we obtain  $\varphi(a) \in \varphi(E_r) \subset \varphi(E)_q \subset \varphi(X)_q$ .  $\square$

## 5 Extension and restriction of weak module systems

*Let  $K$  be a monoid and  $D \subset K$  a submonoid.*

**Definition 5.1.** Let  $r$  be a weak module system on  $K$ . Then we define

$$r[D]: \mathbb{P}(K) \rightarrow \mathbb{P}(K) \quad \text{by} \quad X_{r[D]} = (XD)_r \quad \text{for all } X \subset K, \quad \text{and}$$

$$r_D: \mathbb{P}(D) \rightarrow \mathbb{P}(D) \quad \text{by} \quad X_{r_D} = X_{r[D]} \cap D = (XD)_r \cap D \quad \text{for all } X \subset D.$$

We call  $r[D]$  the *extension* of  $r$  by  $D$  and  $r_D$  the *weak ideal system induced by  $r$  on  $D$*  (see Proposition 5.2.4).

**Proposition 5.2.** *Let  $r$  be a (weak) module system on  $K$ .*

1.  $r[D]$  is a (weak)  $D$ -module system on  $K$ ,  $\mathcal{M}_{r[D]}(K)$  consists of all  $r$ -modules which are equally  $D$ -modules,  $r \leq r[D]$ , and  $r = r[D]$  if and only if  $r$  is a (weak)  $D$ -module system.
2.  $r_f[D]$  is finitary,  $r_f[D] \leq r[D]_f$ , and if  $r$  is finitary, then  $r[D]$  is also finitary.
3.  $r_D = r[D]_D$  is a weak ideal system on  $D$ , and if  $r$  is finitary, then  $r_D$  is also finitary.
4. Suppose that  $r$  is a weak  $D$ -module system and  $D$  is an  $r$ -monoid. Then  $r_D = r|_{\mathbb{P}(D)}$ , and if  $r$  is a module system, then  $r_D$  is an ideal system on  $D$ .
5. If  $A \in \mathcal{M}_r(K)$  is a  $D$ -module, then  $A \cap D$  is an  $r_D$ -ideal of  $D$ .
6. If  $q$  is another weak module system on  $K$  and  $r \leq q$ , then  $r[D] \leq q[D]$  and  $r_D \leq q_D$ .
7. If  $T \subset K$  is another submonoid, then  $r[D][T] = r[TD]$ .

*Proof.* 1. It is easily checked that  $r[D]$  satisfies the conditions of Definition 3.1, and thus it is a (weak) module system on  $K$ . If  $A \in \mathcal{M}_{r[D]}(K)$ , then  $A = A_{r[D]} = (AD)_r$  is a  $D_r$ -module (hence a  $D$ -module) by Proposition 3.3.1. Conversely, if  $A \in \mathcal{M}_r(K)$  is a  $D$ -module, then  $A_{r[D]} = (AD)_r = A_r = A$  and thus  $A \in \mathcal{M}_{r[D]}(K)$ . Hence  $\mathcal{M}_{r[D]}(K) \subset \mathcal{M}_r(K)$  and thus  $r \leq r[D]$ . Moreover,  $r = r[D]$  holds if and only if every  $r$ -module is a  $D$ -module, that is, if and only if  $r$  is a weak  $D$ -module system.

2. If  $X \subset K$  and  $E \in \mathbb{P}_f(XD)$ , then there exists some  $F \in \mathbb{P}_f(X)$  such that  $E \subset FD$ . Hence

$$X_{r_f[D]} = (XD)_{r_f} = \bigcup_{E \in \mathbb{P}_f(XD)} E_r \subset \bigcup_{F \in \mathbb{P}_f(X)} (FD)_r = \bigcup_{F \in \mathbb{P}_f(X)} F_{r[D]} = X_{r[D]_f},$$

and thus  $r_f[D] \leq r[D]_f$ . Applying this reasoning for  $r_f$  instead of  $r$ , we obtain  $r_f[D] = (r_f)_f[D] \leq r_f[D]_f \leq r_f[D]$ , and therefore  $r_f[D] = r_f[D]_f$  is finitary.

3. It is easily checked that  $r_D = r[D]_D$  satisfies the conditions of Definition 3.1, and thus it is a (weak) module system on  $D$ .

If  $c \in D$ , then  $cD \subset \{c\}_r D \cap D = \{c\}_{r_D}$ , and thus  $r_D$  is a weak ideal system on  $D$  by Proposition 3.3.3. If  $r$  is finitary,  $X \subset D$  and  $a \in X_{r_D} = (XD)_r \cap D$ , then there exists a finite subset  $E \subset XD$  such that  $a \in E_r \cap D$ . In particular, there exists a finite subset  $E \subset X$  such that  $a \in (ED)_r \cap D = E_{r_D}$ , and thus  $r_D$  is finitary.

4. If  $X \subset D$ , then  $X_r \subset D$ , and  $X_{r_D} = (XD)_r \cap D = X_r \cap D = X_r$  by Proposition 3.3.2. If  $r$  is a module system, then  $r_D = r|_{\mathbb{P}(D)}$  is an ideal system on  $D$ .

5. If  $A \in \mathcal{M}_r(K)$  is a  $D$ -module, then  $A = A_r = AD \in \mathcal{M}_r(K)$ , and therefore  $A \cap D \subset (A \cap D)_{r_D} = [(A \cap D)D]_r \cap D \subset (AD)_r \cap D = A \cap D$ .

6. and 7. are obvious by the definitions.  $\square$

**Proposition 5.3.** *Let  $T \subset D \cap K^\times$  be multiplicatively closed,  $r$  a finitary  $D$ -module system on  $K$  and  $X \subset K$ . Then  $T^{-1}X_r = (T^{-1}X)_r = X_{r[T^{-1}D]}$ , and if  $X \subset T^{-1}D$ , then  $X_{r_{T^{-1}D}} = T^{-1}X_{r_D}$ .*

*Proof.* Since  $TDX = DX$  and  $r$  is finitary, it follows that

$$(T^{-1}DX)_r = \left( \bigcup_{t \in T} t^{-1}DX \right)_r = \bigcup_{t \in T} (t^{-1}DX)_r = \bigcup_{t \in T} t^{-1}(DX)_r = T^{-1}(DX)_r,$$

hence  $T^{-1}X_r = T^{-1}(DX)_r = (T^{-1}DX)_r = (T^{-1}X)_r$  (by Proposition 3.3.2), and by definition we have  $(T^{-1}DX)_r = X_{r[T^{-1}D]}$ . If  $X \subset T^{-1}D$ , then  $X_{r_{T^{-1}D}} = (XT^{-1}D)_r \cap T^{-1}D = T^{-1}X_r \cap T^{-1}D = T^{-1}X_{r_D}$ .  $\square$

**Proposition 5.4.** *Assume that  $K = \mathfrak{q}(D)$ , and let  $r: \mathbb{P}(D) \rightarrow \mathbb{P}(D)$  be a module system on  $D$ .*

1. *There exists a unique module system  $r_\infty$  on  $K$  such that  $X_{r_\infty} = K$  if  $X \subset K$  is not  $D$ -fractional, and  $X_{r_\infty} = c^{-1}(cX)_r$  if  $X \subset K$  and  $c \in D^\bullet$  are such that  $cX \subset D$ . In particular,  $r_\infty|_{\mathbb{P}(D)} = r$  and  $D_{r_\infty} = D$ . Moreover,  $r_\infty$  is a  $D$ -module system if and only if  $r$  is an ideal system on  $D$ , and then  $(r_\infty)_D = r$ .*
2. *The module system  $(r_\infty)_f$  is the unique finitary module system on  $K$  satisfying  $(r_\infty)_f|_{\mathbb{P}(D)} = r_f$ . Moreover,  $(r_\infty)_f$  is a  $D$ -module system on  $K$  if and only if  $r_f$  is an ideal system on  $D$ , and then  $((r_\infty)_f)_D = r_f$ .*

*Proof.* 1. Uniqueness is obvious. To prove existence, we define  $r_\infty$  as in the assertion, making sure that for  $D$ -fractional subsets  $X \subset K$  the definition of  $X_{r_\infty}$

does not depend on the element  $c \in D^\bullet$  with  $cX \subset D$ . Then it is easily checked that  $r_\infty$  has the properties of Definition 3.1.

We obviously have  $r_\infty \mid \mathbb{P}(D) = r$ . Hence, if  $r_\infty$  is a  $D$ -module system on  $K$ , then  $r$  is an ideal system on  $D$ . Conversely, let  $r$  be an ideal system on  $D$ . If  $X \subset K$  is not  $D$ -fractional, then  $X_{r_\infty} = K$  is a  $D$ -module. If  $X \subset K$  is  $D$ -fractional and  $c \in D^\bullet$  is such that  $cX \subset D$ , then  $(cX)_r D = (cX)_r$ , and  $X_{r_\infty} D = c^{-1}(cX)_r D = c^{-1}(cX)_r = X_{r_\infty}$ . Hence,  $r_\infty$  is a  $D$ -module system, and  $(r_\infty)_D = r$  by definition.

2.  $(r_\infty)_f$  is a finitary module system on  $K$ . If  $X \subset D$ , then

$$X_{(r_\infty)_f} = \bigcup_{E \in \mathbb{P}_f(X)} E_{r_\infty} = \bigcup_{E \in \mathbb{P}_f(X)} E_r = X_{r_f}, \quad \text{hence } (r_\infty)_f \mid \mathbb{P}(D) = r_f.$$

Consequently, if  $(r_\infty)_f$  is a  $D$ -module system on  $K$ , then  $r_f$  is an ideal system on  $D$ . Conversely, let  $r_f$  be an ideal system on  $D$  and  $X \subset K$ . If  $E \in \mathbb{P}_f(X)$  and  $c \in D^\bullet$  is such that  $cE \subset D$ , then

$$E_{r_\infty} D = c^{-1}(cE)_r D = c^{-1}(cE)_{r_f} D = c^{-1}(cE)_{r_f} = c^{-1}(cE)_r = E_{r_\infty}.$$

Hence,  $X_{(r_\infty)_f} D = \bigcup_{E \in \mathbb{P}_f(X)} E_{r_\infty} D = \bigcup_{E \in \mathbb{P}_f(X)} E_{r_\infty} = X_{(r_\infty)_f}$ , thus  $(r_\infty)_f$  is a  $D$ -module system, and  $((r_\infty)_f)_D = r_f$  by definition.

It remains to prove uniqueness. Let  $x$  be any finitary module system on  $K$  satisfying  $x \mid \mathbb{P}(D) = r_f$ . If  $E \in \mathbb{P}_f(K)$  and  $c \in D^\bullet$  is such that  $cE \subset D$ , then  $E_x = [c^{-1}(cX)]_x = c^{-1}(cE)_x = c^{-1}(cE)_{r_f} = E_{r_\infty} = E_{(r_\infty)_f}$ , and thus  $x = (r_\infty)_f$  by Proposition 4.2.  $\square$

**Definition 5.5.** Assume that  $K = q(D)$ , and let  $r$  be a module system on  $D$ . Then the module system  $r_\infty$  on  $K$  constructed in Proposition 5.4 is called the *trivial extension* of  $r$  to a module system on  $K$ .

If  $r$  is a finitary module system on  $D$ , then  $(r_\infty)_f$  is called the *natural extension* of  $r$  to a module system on  $K$ . In this case, we say that  $(r_\infty)_f$  is *induced* by  $r$ , and (as there will be no risk of confusion) we write again  $r$  instead of  $(r_\infty)_f$ .

With this identification, every finitary module system  $r$  on  $D$  is a finitary module system on  $K$ , and  $r$  is even a finitary ideal system on  $D$  if and only if  $r$  is a finitary  $D$ -module system on  $K$  satisfying  $D_r = \{1\}_r = D$ .

**Examples 5.6 (Examples of ideal systems and module systems)**

1. The *semigroup system*  $s(D) : \mathbb{P}(D) \rightarrow \mathbb{P}(D)$  is defined by  $X_{s(D)} = DX$  for all  $X \subset D$ . It is a finitary ideal system on  $D$ , and  $\mathcal{M}_{s(D)}(D)$  is the set of ordinary semigroup ideals of  $D$ . For every ideal system  $r$  on  $D$ , we have  $s(D) \leq r$ .

The *identical system*  $s : \mathbb{P}(K) \rightarrow \mathbb{P}(K)$  is defined by  $X_s = X \cup \{0\}$  for all  $X \subset K$ . It is a finitary module system on  $K$ , for every subset  $X \subset K$  we have  $X_{s[D]} = DX$  (the  $D$ -module generated by  $X$ ), and  $s_D = s(D)$ .

2. Assume that  $K = q(D)$ . Then  $s(D) = s[D]$  is the finitary module system on  $K$  induced by the semigroup system  $s(D)$  (according to Definition 5.5).

The module system  $v(D)$  on  $K$  is defined by  $X_{v(D)} = (D : (D : X))$  for all subsets  $X \subset K$ . If  $X \subset K$  is not  $D$ -fractional, then  $X_{v(D)} = K$ , and thus  $v(D)$  is the trivial



extension of the classical “Vielfachensystem”  $v_D$  on  $D$  (compare [25, Section 11.4] and [23, Section 34]). Note that  $v_D$  (and thus also  $v(D)$ ) is usually not finitary. If  $X \subset K$  is  $D$ -fractional, then

$$X_{v(D)} = \bigcap_{\substack{b \in K \\ X \subset bD}} Db,$$

and for every ideal system  $r$  on  $D$  we have  $r \leq v_D$ .

The associated finitary ideal system on  $D$  (which is identified with its natural extension to a finitary module system on  $K$ ) is the classical “ $t$ -system” denoted by  $t(D) = v(D)_f$ . If  $r$  is any finitary ideal system on  $D$ , then  $r \leq t(D)$ . But note that for an overmonoid  $T \supset D$  in general  $t(D)[T] \neq t(T)$ .

3. Let  $D$  be a ring. The *Dedekind system*  $d(D) : \mathbb{P}(D) \rightarrow \mathbb{P}(D)$  is defined by  $X_{d(D)} = {}_R\langle X \rangle$  (the usual ring ideal generated by  $X$ ).

4. Let  $D$  be an integral domain and  $K = \mathfrak{q}(D)$ . The *additive system*  $d : \mathbb{P}(K) \rightarrow \mathbb{P}(K)$  is given by  $X_d = {}_{\mathbb{Z}}\langle X \rangle$  (the additive group generated by  $X$ ) for all  $X \subset K$ . It is a finitary module system on  $K$ , and  $d[D] = d(D)$  ( $X_{d[D]}$  is the  $D$ -submodule of  $K$  generated by  $X$  for every subset  $X \subset K$ ).

Recall that a *semistar operation*  $*$  on  $D$  is a map

$$\mathcal{M}_{d[D]}(K)^\bullet \rightarrow \mathcal{M}_{d[D]}(K), \quad X \mapsto X^*$$

having the following properties for all  $X, Y \subset K$  and  $c \in K$ :

$$(*1) \ (cX)^* = cX^*; \quad (*2) \ X \subset X^* = X^{**} \quad (*3) \ X \subset Y \implies X^* \subset Y^*.$$

A (semi)star operation on  $D$  is a semistar operation satisfying  $D^* = D$  (then  $*|_{\mathcal{F}(D)} \cap \mathcal{M}_{d[D]}(K)$  is a star operation in the classical sense, see [23, Section 32]).

If  $*$  is a semistar operation on  $D$ , then the map  $r^* : \mathbb{P}(K) \rightarrow \mathbb{P}(K)$ , defined by  $X_{r^*} = (X_{d[D]})^*$ , is a  $D$ -module system on  $K$  such that  $d[D] \leq r^*$  and  $r^*|_{\mathcal{M}_{d[D]}(K)^\bullet}$  coincides with  $*$ . Moreover,  $*$  is a (semi)star operation if and only if  $D$  is an  $r^*$ -monoid (then  $*|_{\mathcal{M}_{d[D]}(K)^\bullet}$  is a star operation and  $r^*|_{\mathbb{P}(D)}$  is an ideal system on  $D$ ).  $r^*$  is called the *module system induced by*  $*$ .

Conversely, let  $r$  be a module system on  $K$  such that  $d[D] \leq r$ . Then  $*_r = r|_{\mathcal{M}_{d[D]}(K)^\bullet}$  is a semistar operation on  $D$ , and  $r = r^*_{*_r}$  is the module system induced by  $*_r$ .

## 6 Prime and maximal ideals, spectral module systems

Let  $K$  be a monoid and  $D \subset K$  a submonoid.

**Proposition 6.1.** *Let  $(r_\lambda)_{\lambda \in \Lambda}$  be a family of (weak)  $D$ -module systems on  $K$ , and let  $r : \mathbb{P}(K) \rightarrow \mathbb{P}(K)$  be defined by*

$$X_r = \bigcap_{\lambda \in \Lambda} X_{r_\lambda} \quad \text{for all } X \subset K$$

(if  $\Lambda = \emptyset$ , then  $r$  is the trivial weak module system on  $K$ , defined by  $X_r = K$  for all  $X \subset K$ ).

Then  $r$  is a (weak)  $D$ -module system on  $K$ , and  $r = \inf\{r_\lambda \mid \lambda \in \Lambda\}$  is the infimum of the family  $(r_\lambda)_{\lambda \in \Lambda}$  in the partially ordered set of all weak  $D$ -module systems on  $K$  [that is, for every weak module system  $x$  on  $K$  we have  $x \leq r$  if and only if  $x \leq r_\lambda$  for all  $\lambda \in \Lambda$ ].

*Proof.* Obvious.  $\square$

**Definition 6.2.** Let  $r$  be a weak ideal system on  $D$ . We denote by  $r\text{-spec}(D)$  the set of all prime  $r$ -ideals of  $D$  and by  $r\text{-max}(D)$  the set of all maximal elements in  $\mathcal{J}_r(D) \setminus \{D\}$  (called  $r$ -maximal  $r$ -ideals). We say that  $r$  has enough primes if for every  $J \in \mathcal{J}_r(D) \setminus \{D\}$  there is some  $P \in r\text{-spec}(D)$  such that  $J \subset P$ .

**Proposition 6.3.** Let  $r$  be a finitary weak ideal system on  $D$ . Then  $r$  has enough primes. More precisely, the following assertions hold:

1. If  $J \in \mathcal{J}_r(D)$  and  $T \subset D^\bullet$  is a multiplicatively closed subset such that  $J \cap T = \emptyset$ , then the set  $\Omega = \{P \in \mathcal{J}_r(D) \mid J \subset P \text{ and } P \cap T = \emptyset\}$  has maximal elements, and every maximal element in  $\Omega$  is prime.
2. Every  $r$ -ideal  $J \in \mathcal{J}_r(D) \setminus \{D\}$  is contained in an  $r$ -maximal  $r$ -ideal of  $D$ , and  $r\text{-max}(D) \subset r\text{-spec}(D)$

*Proof.* [25, Theorems 6.3 and 6.4].  $\square$

**Proposition 6.4.** Assume that  $K = \mathfrak{q}(D)$ , let  $r$  be a finitary module system on  $K$  and  $A \in \mathcal{M}_r(K)$  a  $D$ -module. Then

$$A = \bigcap_{P \in r_D\text{-max}(D)} A_P. \text{ If } D \text{ is an } r\text{-monoid, then } D = \bigcap_{P \in r_D\text{-max}(D)} D_P.$$

*Proof.* Obviously,  $A \subset A_P$  for all  $P \in r_D\text{-max}(D)$ . Thus assume that  $z \in A_P^\bullet$  for all  $P \in r_D\text{-max}(D)$ . Then  $I = z^{-1}A \cap D$  is an  $r_D$ -ideal of  $D$ . For each  $P \in r_D\text{-max}(D)$ , there exists some  $s \in D \setminus P$  such that  $sz \in A$ , hence  $s \in I$  and  $I \not\subset P$ . Therefore we obtain  $1 \in I$  and  $z \in A$  by Proposition 6.3.  $\square$

In the sequel, we investigate two closely connected special classes of module systems, spectral and stable ones (see Definition 6.10 for a formal definition). In the case of semistar operations, they were introduced in [13] where its deep connection with localizing systems was established. For the connection with localizing systems in a purely multiplicative context we refer to [30]. In the case of integral domains, spectral module systems describe the ideal theory of generalized Nagata rings (see [19, 20]).

**Theorem 6.5.** Assume that  $K = \mathfrak{q}(D)$ , let  $q$  be a finitary  $D$ -module system on  $K$ ,  $\Delta \subset q_D\text{-spec}(D)$  and  $q_\Delta = \inf\{q[D_P] \mid P \in \Delta\}$  (see Proposition 6.1).

1.  $q_\Delta$  is a  $D$ -module system on  $K$  satisfying  $q \leq q_\Delta$ . If  $X \subset K$ , then

$$D_P X_q = D_P X_{q_\Delta} \quad \text{for all } P \in \Delta, \quad \text{and} \quad X_{q_\Delta} = \bigcap_{P \in \Delta} D_P X_q.$$

2. For all  $A, B \in \mathcal{M}_q(K)$  we have  $(A \cap B)_{q_\Delta} = A_{q_\Delta} \cap B_{q_\Delta}$ .

3. For all  $P \in \Delta$  we have  $P_{q_\Delta} \cap D = P$  (and thus  $\Delta \subset (q_\Delta)_D\text{-spec}(D)$ ).

4. If  $J \subset D$  is an ideal such that  $1 \notin J_{q_\Delta}$ , then there exists some  $P \in \Delta$  such that  $J \subset P$ . In particular,  $(q_\Delta)_D$  has enough primes.

*Proof.* 1. By Proposition 6.1,  $q_\Delta = \inf\{q[D_P] \mid P \in \Delta\}$  is a  $D$ -module system on  $K$ . Since  $q \leq q[D_P]$  for all  $P \in \Delta$ , it follows that  $q \leq q_\Delta$ . If  $X \subset K$ , then  $X_{q[D_P]} = D_P X_q$  by Proposition 5.3, and thus

$$X_{q_\Delta} = \bigcap_{P \in \Delta} X_{q[D_P]} = \bigcap_{P \in \Delta} D_P X_q.$$

Now  $X_q \subset X_{q_\Delta} \subset D_P X_q$  implies  $D_P X_q \subset D_P X_{q_\Delta} \subset D_P D_P X_q = D_P X_q$  and thus  $D_P X_q = D_P X_{q_\Delta}$ .

2. If  $A, B \in \mathcal{M}_q(K)$ , then  $A \cap B \in \mathcal{M}_q(K)$ , and

$$(A \cap B)_{q_\Delta} = \bigcap_{P \in \Delta} D_P(A \cap B) = \bigcap_{P \in \Delta} D_P A \cap \bigcap_{P \in \Delta} D_P B = A_{q_\Delta} \cap B_{q_\Delta}.$$

3. Let  $P, Q \in \Delta$ . If  $P \not\subset Q$ , then  $D \subset D_Q = P D_Q \subset P_q D_Q$ , and if  $P \subset Q$ , then  $P_q D_Q \supset P_q D_P$ . Hence we obtain

$$P_{q_\Delta} \cap D = \bigcap_{Q \in \Delta} P_q D_Q \cap D = P_q D_P \cap D \supset P,$$

and it remains to prove that  $P_q D_P \cap D \subset P$ . If  $z \in P_q D_P \cap D$ , then there is some  $s \in D \setminus P$  such that  $sz \in P_q \cap D = P$  and therefore  $z \in P$ .

4. If  $J \subset D$  is an ideal and  $1 \notin J_{q_\Delta}$ , then  $1 \notin J_q D_P$  and thus  $1 \notin J_q D_P \cap D_P$  for some  $P \in \Delta$ . Since  $J_q D_P \cap D_P \subset D_P$  is an ideal and  $P D_P = D_P \setminus D_P^\times$ , we obtain  $J_q D_P \cap D_P \subset P D_P$  and  $J \subset J_q D_P \cap D \subset P D_P \cap D = P$ .  $\square$

**Theorem 6.6.** Let  $q$  be a finitary (weak)  $D$ -module system on  $K$ ,  $r$  a weak module system on  $K$ , and define  $r[q]: \mathbb{P}(K) \rightarrow \mathbb{P}(K)$  by

$$X_{r[q]} = \bigcup_{\substack{B \subset D \\ 1 \in B_r}} (X_q : B) \quad \text{for all } X \subset K.$$

1.  $r[q]$  is a finitary (weak)  $D$ -module system on  $K$  satisfying  $q \leq r[q]$ , and

$$X_{r[q]} = \{x \in K \mid 1 \in [(X_q : x) \cap D]_r\} \quad \text{for all } X \subset K.$$

2. For all  $X, Y \in \mathcal{M}_q(K)$  we have  $(X \cap Y)_{r[q]} = X_{r[q]} \cap Y_{r[q]}$ .

3. If  $B \subset D$  and  $1 \in B_r$ , then  $1 \in B_{r[q]}$ .
4. If  $q \leq r$ , then  $r[q] \leq r$  and  $(r[q])[q] = r[q]$ . In particular,  $q[q] = q$ .
5. If  $q \leq r$ , then  $r_D\text{-max}(D) \subset r[q]_D\text{-max}(D)$ , and equality holds if  $r_D$  has enough primes.

*Proof.* 1. Let  $X \subset K$ . Then  $(X_q : B)$  is a  $D$ -module for every  $B \subset D$ , and thus  $X_{r[q]}$  is a  $D$ -module. If  $B', B'' \subset D$  are such that  $1 \in B'_r$  and  $1 \in B''_r$ , then  $1 \in B'_r B''_r \subset (B' B'')_r$  and  $(X_q : B') \cup (X_q : B'') \subset (X_q : B' B'')$  (since  $X_q$  is a  $D$ -module). Hence  $\{(X_q : B) \mid B \subset D, 1 \in B_r\}$  is directed, and since  $q$  is finitary, it follows that

$$(X_{r[q]})_q = \bigcup_{\substack{B \subset D \\ 1 \in B_r}} (X_q : B)_q = \bigcup_{\substack{B \subset D \\ 1 \in B_r}} (X_q : B) = X_{r[q]}, \quad \text{and}$$

$$X_{r[q]} = \bigcup_{\substack{B \subset D \\ 1 \in B_r}} \left( \bigcup_{E \in \mathbb{P}_f(X)} E_q \right) : B = \bigcup_{\substack{B \subset D \\ 1 \in B_r}} \bigcup_{E \in \mathbb{P}_f(X)} (E_q : B) = \bigcup_{E \in \mathbb{P}_f(X)} E_{r[q]}.$$

We show now that  $r[q]$  satisfies the conditions of Definition 3.1. Once this is done, then by the above considerations  $r[q]$  is a finitary  $D$ -module system satisfying  $q \leq r[q]$ . Thus let,  $X, Y \subset K$  and  $c \in K$ .

**M1.** If  $B \subset D$ , then  $XB \subset XD \subset X_q$  and thus  $X \subset (X_q : B) \subset X_{r[q]}$ .

**M2.** If  $X \subset Y_{r[q]}$  and  $z \in X_{r[q]}$ , then there is some  $B \subset D$  such that  $zB \subset X_q \subset (Y_{r[q]})_q = Y_{r[q]}$  and thus  $z \in (Y_{r[q]} : B) \subset Y_{r[q]}$ , since  $Y_{r[q]}$  is a  $D$ -module.

**M3.** and **M3'.** If  $B \subset D$ , then  $((cX)_q : B) \supseteq (cX_q : B) = c(X_q : B)$ , and thus we obtain  $(cX)_{r[q]} \supseteq cX_{r[q]}$ .

It remains to prove that  $X_{r[q]} = \{x \in K \mid 1 \in [(X_q : x) \cap D]_r\}$ .

If  $x \in X_{r[q]}$ , then there is some  $B \subset D$  such that  $1 \in B_r$  and  $xB \subset X_q$ , whence  $B \subset (X_q : x) \cap D$  and  $1 \in B_r \subset [(X_q : x) \cap D]_r$ . Conversely, if  $x \in K$  and  $1 \in [(X_q : x) \cap D]_r$ , then  $B = (X_q : x) \cap D \subset D$ ,  $1 \in B_r$  and  $x \in (X_q : B) \subset X_{r[q]}$ .

2. If  $X, Y \in \mathcal{M}_q(K)$ , then obviously  $(X \cap Y)_{r[q]} \subset X_{r[q]} \cap Y_{r[q]}$ . To prove the reverse inclusion, let  $z \in X_{r[q]} \cap Y_{r[q]}$  and  $B', B'' \subset D$  such that  $1 \in B'_r$ ,  $1 \in B''_r$ ,  $zB' \subset X_q = X$  and  $zB'' \subset Y_q = Y$ . Then  $1 \in B'_r B''_r \subset (B' B'')_r$ , and since  $X$  and  $Y$  are  $D$ -modules, it follows that  $zB' B'' \subset X \cap Y$ , whence  $z \in (X \cap Y : B' B'') \subset (X \cap Y)_{r[q]}$ .

3. If  $B \subset D$  and  $1 \in B_r$ , then  $1 \in (B_q : B) \subset B_{r[q]}$ .

4. Assume that  $q \leq r$ , and let  $X \subset K$ . If  $x \in X_{r[q]}$ , then it follows that  $1 \in [(X_q : x) \cap D]_r \subset (X_r : x)_r = (X_r : x)$ , which implies  $x \in X_r$ . Hence we obtain  $X_{r[q]} \subset X_q$  and thus  $r[q] \leq r$ . Applied with  $r[q]$  instead of  $r$ , this argument shows that  $(r[q])[q] \leq r[q]$ . To prove  $r[q] \leq (r[q])[q]$ , let  $X \subset K$  and  $x \in X_{r[q]}$ . Then  $1 \in [(X_q : x) \cap D]_r \subset [(X_{r[q]} : x) \cap D]_r$ , hence  $1 \in [(X_{r[q]} : x) \cap D]_{r[q]}$  by 3. and thus  $x \in X_{(r[q])[q]}$ .

5. Assume that  $q \leq r$ , and let  $P \in r_D\text{-max}(D)$ . Then  $r[q] \leq r$  by 4., hence  $r[q]_D \leq r_D$  and thus  $P \in \mathcal{J}_{r[q]_D}(D)$ . Since  $r[q]$  (and thus also  $r[q]_D$ ) is finitary, there exists some  $P' \in r[q]_D\text{-max}(D)$  such that  $P \subset P'$ . If  $P \subsetneq P'$ , then  $1 \in P'_{r_D} \subset P'_r$ , and thus  $1 \in P'_{r[q]} \cap D = P'_{r[q]_D}$ , a contradiction. Hence it follows that  $P = P' \in r[q]_D\text{-max}(D)$ .

Assume now that  $r_D$  has enough primes, and let  $P \in r[q]_D\text{-max}(D)$ . Then  $1 \notin P = P_{r[q]} \cap D$  and thus  $1 \notin P_r \cap D = P_{r_D}$ . Therefore there exists some  $P' \in r_D\text{-spec}(D) \subset$

$\mathcal{J}_{r[q]_D}(D)$  such that  $P_{r_D} \subset P'$ . Hence  $P \subset P'$  and thus  $P = P' \in r_D\text{-spec}(D)$ . If  $P' \in r_D\text{-max}(D)$ , we are done. Otherwise, there exists some  $P'' \in \mathcal{J}_{r_D}(D) \subset \mathcal{J}_{r[q]_D}(D)$  such that  $P' \subsetneq P''$ , and then  $P \subsetneq P''$  yields a contradiction.  $\square$

**Definition 6.7.** Let  $q$  be a finitary (weak)  $D$ -module system and  $r$  a weak module system on  $K$ . The finitary (weak)  $D$ -module system  $r[q]$  defined in Theorem 6.6 is called the  $q$ -stabilizer of  $r$  on  $D$  or the spectral extension of  $q$  by  $r$  on  $D$ .

**Theorem 6.8.** Assume that  $K = \mathfrak{q}(D)$ , let  $q$  be a finitary  $D$ -module system and  $r$  a module system on  $K$ .

1.  $r[q][D_P] = q[D_P]$  for all  $P \in r_D\text{-spec}(D)$ , and if  $q \leq r$ , this holds for all  $P \in r[q]_D\text{-spec}(D)$ .
2.  $r[q] \leq \inf\{q[D_P] \mid P \in r_D\text{-spec}(D)\}$  (see Proposition 6.1). Equality holds if  $r_D$  has enough primes, and  $r[q] = \inf\{q[D_P] \mid P \in r_D\text{-max}(D)\}$  if  $r$  is finitary.
3. If  $q \leq r$ , then  $r[q] = \inf\{q[D_P] \mid P \in q_D\text{-spec}(D), 1 \notin P_r\}$ .

*Proof.* 1. Let  $P \in r_D\text{-spec}(D)$ . Then  $q \leq r[q]$  implies  $q[D_P] \leq r[q][D_P]$ . To prove the reverse inequality, we must show that  $X_{r[q][D_P]} \subset X_{q[D_P]}$  for all  $X \subset K$ . If  $X \subset K$  and  $z \in X_{r[q][D_P]} = X_{r[q]}D_P$ , let  $s \in D \setminus P$  be such that  $sz \in X_{r[q]}$ . Then there is some  $B \subset D$  such that  $1 \in B_r$  and  $szB \subset X_q$ . Since  $1 \in B_r$ , it follows that  $B \not\subset P = P_r \cap D$ , and if  $t \in B \setminus P$ , then  $stz \in X_q$  and  $z \in X_qD_P = X_{q[D_P]}$ .

Assume now that  $q \leq r$ . Then  $(r[q])[q] = r[q]$ , and we apply what we have just proved for  $r[q]$  instead of  $r$  and obtain  $r[q][D_P] = (r[q])[q][D_P] = q[D_P]$  for all  $P \in r[q]_D\text{-spec}(D)$ .

2. We must prove that  $X_{r[q]} \subset X_{q[D_P]} = X_qD_P$  for all  $P \in r_D\text{-spec}(D)$  and  $X \subset K$ . Thus let  $P \in r_D\text{-spec}(D)$ ,  $X \subset K$ ,  $x \in X_{r[q]}$  and  $B \subset D$  such that  $1 \in B_r$  and  $xB \subset X_q$ . Then it follows that  $B \not\subset P = P_r \cap D$ , and if  $s \in B \setminus P$ , then  $xs \in X_q$ , which implies  $x \in X_qD_P$ .

Assume now that  $r_D$  has enough primes and  $x \in X_{q[D_P]} = X_qD_P$  for all  $P \in r_D\text{-spec}(D)$ . For each  $P \in r_D\text{-spec}(D)$ , let  $s_P \in D \setminus P$  be such that  $s_P z \in X_q$ . Then  $B = \{s_P \mid P \in r_D\text{-spec}(D)\} \subset D$  and  $B \not\subset P$  for all  $P \in r_D\text{-spec}(D)$ . Hence  $B_{r_D} = B_r \cap D = D$ , whence  $1 \in B_r$  and  $z \in (X_q : B) \subset X_{r[q]}$ .

If  $r$  is finitary, then so is  $r_D$ . In particular,  $r_D$  has enough primes, and for every  $P \in r_D\text{-spec}(D)$  there exists some  $M \in r_D\text{-max}(D)$  such that  $P \subset M$ , hence  $D_M \subset D_P$ , and it follows that

$$\bigcap_{P \in r_D\text{-spec}(D)} X_{q[D_P]} = \bigcap_{P \in r_D\text{-max}(D)} X_{q[D_P]} \quad \text{for all } X \subset K$$

and consequently  $r[q] = \inf\{q[D_P] \mid P \in r_D\text{-max}(D)\}$ .

3. If  $q \leq r$ , then  $q_D \leq r_D$ ,  $r_D\text{-spec}(D) \subset \{P \in q_D\text{-spec}(D) \mid 1 \notin P_r\}$  and thus  $\inf\{q[D_P] \mid P \in q_D\text{-spec}(D), 1 \notin P_r\} \leq r[q]$ . To prove the reverse inequality, it suffices to show that  $r[q] \leq q[D_P]$  for all  $P \in q_D\text{-spec}(D)$  such that  $1 \notin P_r$ . Thus, let  $P \in q_D\text{-spec}(D)$ ,  $1 \notin P_r$ ,  $X \subset K$ ,  $x \in X_{r[q]}$  and  $B \subset D$  such that  $1 \in B_r$  and  $xB \subset X_q$ . Then we have  $B \not\subset P$ , and if  $x \in B \setminus P$ , then  $xs \in X_q$ , whence  $x \in X_qD_P = X_{q[D_P]}$ .  $\square$

*Remark 6.9.* Let  $D$  be an integral domain with quotient field  $K$ ,  $*$  a semistar operation on  $D$  and  $r = r^*$  the  $D$ -module system on  $K$  induced by  $*$  (see Example 5.6.4). If  $\tilde{*}$  is the spectral semistar operation associated with  $*$  (see [13]), then Theorem 6.6 implies  $r[d] = r^{\tilde{*}}$ , and in the case of star operations we also obtain  $r[d] = r^{*w}$  (where  $*_w$  is the star operation introduced in [3]) and  $t[d] = r^w = r^{\tilde{v}}$  (where  $w = \tilde{v}$  is the star operation introduced in [11]).

**Definition 6.10.** Let  $q$  be a finitary  $D$ -module system and  $r$  a module system on  $K$  such that  $q \leq r$ . Then  $r$  is called

- $q$ -stable if  $X_r \cap Y_r = (X \cap Y)_r$  for all  $X, Y \in \mathcal{M}_q(K)$ .
- $q$ -spectral if  $r = q_\Delta$  for some subset  $\Delta \subset q_D\text{-spec}(D)$  (see Theorem 6.5).

**Theorem 6.11.** Assume that  $K = \mathfrak{q}(D)$ , let  $q$  be a finitary  $D$ -module system on  $K$  such that  $D = D_q$  and  $r$  a module system on  $K$  such that  $q \leq r$ .

1. The following assertions are equivalent:

- (a)  $r = r[q]$ .
- (b)  $r$  is  $q$ -stable.
- (c)  $[(X : E) \cap D]_r = (X_r : E) \cap D_r$  for all  $E \in \mathbb{P}_f(K)$  and  $X \in \mathcal{M}_q(K)$ .

2.  $r$  is  $q$ -spectral if and only if  $r$  is  $q$ -stable and  $r_D$  has enough primes.

*Proof.* 1. (a)  $\Rightarrow$  (b) By Theorem 6.6.2.

(b)  $\Rightarrow$  (c) Let  $E \in \mathbb{P}_f(K)$  and  $X \in \mathcal{M}_q(K)$ . Then, as  $D = D_q$ ,

$$[(X : E) \cap D]_r = \left( \bigcap_{x \in E^\bullet} x^{-1} X \cap D \right)_r = \bigcap_{x \in E^\bullet} x^{-1} X_r \cap D_r = (X_r : E) \cap D_r.$$

(c)  $\Rightarrow$  (a) By Theorem 6.6 we have  $q \leq r[q] \leq r$ , and thus it suffices to prove that  $X_r \subset X_{r[q]}$  for all  $X \in \mathcal{M}_q(K)$ . Thus let  $X \in \mathcal{M}_q(K)$  and  $x \in X_r$ . Then  $1 \in (X_r : x) \cap D_r = [(X : x) \cap D]_r$  and therefore  $x \in X_{r[q]}$ .

2. If  $r$  is  $q$ -spectral, then  $r$  is  $q$ -stable and  $r_D$  has enough primes by Theorem 6.5. If  $r$  is  $q$ -stable, then  $r = r[q]$  by 1., and if  $r_D$  has enough primes, then  $r[q]$  is  $q$ -stable by Theorem 6.8.3.  $\square$

## 7 A survey on valuation monoids and GCD-monoids

Let  $K$  be a monoid and  $D \subset K$  a submonoid such that  $K = \mathfrak{q}(D)$ .

In this section, we gather several facts concerning GCD-monoids, valuation monoids and their homomorphisms. For a more concise presentation of this topic we refer to [25, Chaps. 10, 15 and 18].

**Definition 7.1.**

1. Let  $X \subset D$ . An element  $d \in D$  is called a *greatest common divisor* of  $X$  if  $dD$  is the smallest principal ideal containing  $X$  [equivalently,  $d|x$  for all  $x \in E$ , and if  $e \in D$  and  $e|x$  for all  $x \in X$ , then  $e|d$  (where the notion of divisibility in  $D$  is used in the common way)]. If  $\text{GCD}(X) = \text{GCD}_D(X)$  denotes the set of all greatest common divisors of  $X$ , then  $\text{GCD}(X) = dD^\times$  for every  $d \in X$ . If  $D$  is reduced, then  $X$  has at most one greatest common divisor, and we write  $d = \text{gcd}(X)$  instead of  $\text{GCD}(X) = \{d\}$ . If  $X = \{a_1, \dots, a_n\}$ , we set  $\text{GCD}(a_1, \dots, a_n) = \text{GCD}(X)$  resp.  $\text{gcd}(a_1, \dots, a_n) = \text{gcd}(X)$ .
2.  $D$  is called a *GCD-monoid* if  $\text{GCD}(E) \neq \emptyset$  for all  $E \in \mathbb{P}_f(D)$  [equivalently,  $\text{GCD}(a, b) \neq \emptyset$  for all  $a, b \in D^\bullet$ ].
3.  $D$  is called a *valuation monoid* if, for all  $a, b \in D$ , either  $a|b$  or  $b|a$ . If  $r$  is a module system on  $K$ , then  $D$  is called an  *$r$ -valuation monoid* (of  $K$ ) if  $D$  is a valuation monoid satisfying  $D_r = D$ .
4. A homomorphism  $\varphi: G_1 \rightarrow G_2$  of GCD-monoids is called a *GCD-homomorphism* if  $\varphi(\text{GCD}(E)) \subset \text{GCD}(\varphi(E))$  for every  $E \in \mathbb{P}_f(G_1)$ . We denote by  $\text{Hom}_{\text{GCD}}(G_1, G_2)$  the set of all GCD-homomorphisms  $\varphi: G_1 \rightarrow G_2$ .

By definition,  $D$  is a valuation monoid if and only if for every  $z \in K^\times$  either  $z \in D$  or  $z^{-1} \in D$ . If  $D$  is a valuation monoid, then every monoid  $T$  such that  $D \subset T \subset K$  is also a valuation monoid. Obviously, every valuation monoid is a GCD-monoid.

If  $D$  is a valuation monoid and  $E \in \mathbb{P}_f^\bullet(K)$ , then (after a suitable numbering)  $E = \{a_1, \dots, a_n\}$  with  $a_1D \subset a_2D \subset \dots \subset a_nD$ , hence  $ED = a_nD$ , and if  $E \subset D$ , then  $\text{GCD}(E) = a_nD^\times$ . In particular, the  $s$ -system is the only finitary ideal system on  $D$ . We identify it with its natural extension to a  $D$ -module system on  $K$ , whence  $s(D) = t(D)$  and  $D = \{1\}_{t(D)}$  (see Example 5.6.2).

**Lemma 7.2.** *Let  $D$  be a GCD-monoid.*

1. If  $E, F \in \mathbb{P}_f(D)$  and  $b \in D$ , then  $\text{GCD}(EF) = \text{GCD}(E)\text{GCD}(F)$  and  $\text{GCD}(bE) = b\text{GCD}(E)$ .
2. If  $a, b, c \in D$ ,  $\text{GCD}(a, b) = D^\times$  and  $a|bc$ , then  $a|c$ .
3. Every  $z \in K$  has a representation in the form  $z = a^{-1}b$  with  $a \in D^\bullet$  and  $b \in D$  such that  $\text{GCD}(a, b) = D^\times$ . In this representation  $aD^\times$  and  $bD^\times$  are uniquely determined by  $z$ .
4. If  $v = v(D)$ ,  $X \subset D$  and  $d \in D$ , then

$$X_v = \bigcap_{\substack{a \in D \\ X \subset aD}} aD, \quad \text{and} \quad X_v = dD \quad \text{if and only if} \quad d \in \text{GCD}(X).$$

In particular, if  $E \in \mathbb{P}_f(D)$  and  $d \in \text{GCD}(E)$ , then  $E_{t(D)} = dD$ .

5.  $\mathcal{M}_{t(D),f}(K) = \{aD \mid a \in K\}$ , and  $\mathcal{M}_{t(D),f}(K)^\bullet \cong K^\times / D^\times$  is cancellative.

*Proof.* 1., 2. and 3. are easy exercises in elementary number theory (see [25, Chap. 10]).

4. If  $a \in D$  and  $X \subset aD$ , then  $X_v \subset aD$ , which implies  $\subset$ . To prove the reverse inclusion, let  $z \in D$  be such that  $z \in aD$  for all  $a \in D$  satisfying  $X \subset aD$ . We must prove that  $z \in X_v = (D : (D : X))$ , that is,  $zx \in D$  for all  $x \in (D : X)$ . Thus, let  $x \in (D : X) \subset K$ , say  $x = c^{-1}b$ , where  $c, b \in D$  and  $\text{GCD}(b, c) = D^\times$ . Then  $c^{-1}bX \subset D$ , hence  $X \subset cb^{-1}D \cap D$ , and we assert that  $cb^{-1}D \cap D \subset cD$ . Indeed, if  $v \in D$  and  $cb^{-1}v \in D$ , then  $b|cv$ , hence  $b|v$  and thus  $cb^{-1}v \in cD$ . Now  $X \subset cD$  implies  $z \in cX$  and  $zx \in bD \subset D$ .

Hence, it follows that  $X_v = dD$  if and only if  $dD$  is the smallest principal ideal containing  $X$ , which by definition is equivalent to  $d \in \text{GCD}(X)$ .

5. If  $E \in \mathbb{P}_f(K)$ , let  $c \in D^\bullet$  be such that  $cE \subset D$  and  $d \in \text{GCD}(cE)$ . Then  $cED = dD = (cE)_{t(D)}$  and thus  $E_{t(D)} = c^{-1}dD$ . Hence the map  $\partial : K^\times \rightarrow \mathcal{M}_{t(D),f}(K)$ , defined by  $\partial(a) = aD$ , is a group epimorphism with kernel  $D^\times$  and induces an isomorphism  $\mathcal{M}_{t(D),f}(K)^\bullet \xrightarrow{\sim} K^\times / D^\times$ .  $\square$

**Lemma 7.3.** *For  $i \in \{1, 2\}$ , let  $G_i$  be a GCD-monoid,  $K_i = \mathfrak{q}(G_i)$  and  $t_i = t(G_i)$ . A monoid homomorphism  $\varphi : K_1 \rightarrow K_2$  is a  $(t_1, t_2)$ -homomorphism if and only if  $\varphi(G_1) \subset G_2$  and  $\varphi|_{G_1} : G_1 \rightarrow G_2$  is a GCD-homomorphism. In particular, there is a bijective map*

$$\text{Hom}_{(t_1, t_2)}(K_1, K_2) \rightarrow \text{Hom}_{\text{GCD}}(G_1, G_2), \quad \text{defined by } \varphi \mapsto \varphi|_{G_1}.$$

*Proof.* Let first  $\varphi$  be a  $(t_1, t_2)$ -homomorphism. Then

$$\varphi(G_1) = \varphi(\{1\}_{t_1}) \subset \{\varphi(1)\}_{t_2} = \{1\}_{t_2} = G_2.$$

Let  $E \subset G_1$  be finite,  $d_1 \in \text{GCD}(E)$  and  $d_2 \in \text{GCD}(\varphi(E))$ . Then  $E_{t_1} = d_1G_1$ , and  $\varphi(E)_{t_2} = d_2G_2$ . Since  $d_1|x$  for all  $x \in E$ , it follows that  $\varphi(d_1)|y$  for all  $y \in \varphi(E)$ , and thus  $\varphi(d_1)|d_2$ . But  $\varphi(d_1) \in \varphi(E_{t_1}) \subset \varphi(E)_{t_2} = d_2G_2$  implies  $d_2|\varphi(d_1)$  and therefore  $\varphi(d_1) \in d_2G_2^\times = \text{GCD}(\varphi(E))$ .

Assume now that  $\varphi(G_1) \subset G_2$ , and let  $\varphi|_{G_1} : G_1 \rightarrow G_2$  be a GCD-homomorphism. It is obviously sufficient to prove  $\varphi(E_{t_1}) \subset \varphi(E)_{t_2}$  for all  $E \in \mathbb{P}_f(G_1)$ . If  $E \in \mathbb{P}_f(G_1)$  and  $d \in \text{GCD}(E)$ , then  $\varphi(d) \in \text{GCD}(\varphi(E))$ , and therefore  $\varphi(E_{t_1}) = \varphi(dG_1) \subset \varphi(d)G_2 = \varphi(E)_{t_2}$ .  $\square$

**Lemma 7.4.** *Let  $r$  be a finitary module system on  $K$  and  $V \subset K$  a valuation monoid. Then  $V = V_r$  if and only if  $\text{id}_K$  is an  $(r, t(V))$ -homomorphism.*

*Proof.* If  $\text{id}_K$  is an  $(r, t(V))$ -homomorphism, the  $V \subset V_r \subset V_{t(V)} = V$ , and thus  $V = V_r$ . Conversely, assume that  $V = V_r$ . If  $E \in \mathbb{P}_f(K)$ , then Lemma 7.2.5 implies that  $E_{t(V)} = EV = aV$  for some  $a \in E$ , and therefore we obtain  $E_r \subset (aV)_r = aV = E_{t(V)}$ . Hence  $\text{id}_K$  is an  $(r, t(V))$ -homomorphism by Proposition 4.4.  $\square$

**Proposition 7.5.** *Let  $G$  be a GCD-monoid,  $K = \mathfrak{q}(G)$ ,  $V \subset K$  a submonoid and  $t = t(G)$ .*



1. Let  $V$  be a valuation monoid. Then  $V = V_t$  if and only if  $G \subset V$  and  $G \hookrightarrow V$  is a GCD-homomorphism.
2.  $V$  is a  $t$ -valuation monoid if and only if  $V = G_P$  for some  $P \in t\text{-spec}(G)$ . In particular,  $G$  is the intersection of all  $t$ -valuation monoids of  $K$ .

*Proof.* 1. By Lemma 7.4 we have  $V = V_t$  if and only if  $\text{id}_K$  is a  $(t, t(V))$ -homomorphism, and by Lemma 7.3 this holds if and only if  $G \subset V$  and  $G \hookrightarrow V$  is a GCD-homomorphism.

2. Let first  $V$  be a  $t$ -valuation monoid. By Lemma 7.4,  $j = (G \hookrightarrow V)$  is a  $(t, t(V))$ -homomorphism, and since  $t(V) = s(V)$ , it follows by Proposition 4.4 that  $P = G \setminus V^\times = j^{-1}(V \setminus V^\times) \in t\text{-spec}(G)$ . Since  $G \setminus P \subset V^\times$ , we obtain  $G_P \subset V$ . To prove the reverse inclusion, let  $z = a^{-1}b \in V$ , where  $a, b \in G$  and  $\text{GCD}_G(a, b) = G^\times$ . By 1.,  $G \hookrightarrow V$  is a GCD-homomorphism, hence  $\text{GCD}_V(a, b) = V^\times$ , and thus either  $a \in V^\times$  or  $b \in V^\times$ . If  $a \in V^\times$ , then  $a \notin P$  and thus  $z \in G_P$ . If  $b \in V^\times$ , then  $z \in V$  implies  $b \in aV$ , and hence  $a \in V$  and again  $z \in G_P$ .

Assume now that  $P \in t\text{-spec}(G)$  and  $z = a^{-1}b \in K$ , where  $a, b \in D$  and  $\text{GCD}(a, b) = D^\times$ . Then  $\{a, b\}_t = D$ , hence  $\{a, b\}_t \not\subset P = P_t$  and thus either  $a \notin P$  or  $b \notin P$ . If  $a \notin P$ , then  $z \in G_P$ , and if  $b \notin P$ , then  $z^{-1} \in G_P$ . Therefore  $G_P$  is a valuation monoid, and  $(G_P)_t = (G_t)_P = G_P$ .

By Proposition 6.4, this implies that  $G$  is the intersection of all  $t$ -valuation monoids of  $K$ .  $\square$

## 8 Integral closures and cancellation properties

*Let  $K$  be a monoid and  $D \subset K$  a submonoid.*

**Proposition 8.1.** *Let  $r$  be a weak module system on  $K$  and  $A \in \mathcal{M}_{r,f}(K)$ .*

1. *The following assertions are equivalent:*
  - (a)  *$A$  is cancellative in  $\mathcal{M}_{r,f}(K)$  (that is, for all finite subsets  $X, Y \subset K$ , if  $(AX)_r = (AY)_r$ , then  $X_r = Y_r$ ).*
  - (b) *For all finite subsets  $X, Y \subset K$ , if  $(AX)_r \subset (AY)_r$ , then  $X_r \subset Y_r$ .*
  - (c) *For all finite subsets  $X \subset K$  and  $c \in K$ , if  $cA \subset (AX)_r$ , then  $c \in X_r$ .*
  - (d) *For all finite subsets  $X \subset K$  we have  $((AX)_r : A) \subset X_r$ .*

*In each of the above assertions, the statement “for all finite subsets” can be replaced by the statement “for all  $r$ -finite  $r$ -modules”.*
2.  $\mathcal{M}_{r,f}(K)^\bullet$  *is cancellative if and only if  $((EF)_r : E) \subset F_r$  for all  $E \in \mathbb{P}_f^\bullet(K)$  and  $F \in \mathbb{P}_f(K)$ .*

*Proof.* 1. (a)  $\Rightarrow$  (b) If  $(AX)_r \subset (AY)_r$ , then

$$(AY)_r = [(AX)_r \cup (AY)_r]_r = (AX \cup AY)_r = [A(X \cup Y)]_r,$$

and therefore  $X_r \subset (X \cup Y)_r = Y_r$ .

(b)  $\Rightarrow$  (c) If  $cA \subset (AX)_r$ , then  $(A\{c\})_r = (cA)_r \subset (AX)_r$ , and thus  $c \in \{c\}_r \subset X_r$ .

(c)  $\Rightarrow$  (d) If  $z \in ((AX)_r : A)$ , then  $zA \in (AX)_r$  and therefore  $z \in X_r$ .

(d)  $\Rightarrow$  (a) If  $(AX)_r = (AY)_r$ , then  $AX_r \subset (AY)_r$  and  $AY_r \subset (AX)_r$ , hence  $X_r \subset ((AY)_r : A) \subset Y_r$  and  $Y_r \subset ((AX)_r : A) \subset X_r$ , whence  $X_r = Y_r$ .

If  $X \subset K$ , then  $(AX)_r = (AX_r)_r$ , and thus the statement “for all finite subsets” can always be replaced by the statement “for all  $r$ -finite  $r$ -modules”.

2. By 1.(d), since  $\mathcal{M}_{r,f}(K)^\bullet$  is cancellative if and only if  $E_r$  is cancellative for all  $E \in \mathbb{P}_f^\bullet(K)$ .  $\square$

**Theorem 8.2.** *Let  $r$  be a finitary weak module system on  $K$ , and let*

$$r_a : \mathbb{P}(K) \rightarrow \mathbb{P}(K) \text{ be defined by } X_{r_a} = \bigcup_{B \in \mathbb{P}_f^\bullet(K)} ((XB)_r : B).$$

1.  $r_a$  is a finitary weak module system on  $K$ ,  $r \leq r_a$ , and if  $r$  is a module system, then so is  $r_a$ .
2.  $\mathcal{M}_{r_a,f}(K)^\bullet$  is cancellative, and if  $q$  is any finitary weak module system on  $K$  such that  $r \leq q$  and  $\mathcal{M}_{q,f}(K)^\bullet$  is cancellative, then  $r_a \leq q$ . In particular,  $(r_a)_a = r_a$ , and  $\mathcal{M}_{r,f}(K)^\bullet$  is cancellative if and only if  $r = r_a$ .
3.  $r[D]_a = r_a[D]$ , and if  $r$  is a weak  $D$ -module system, then so is  $r_a$ .
4. If  $G$  is a reduced GCD-monoid and  $L = \mathfrak{q}(G)$ , then

$$\text{Hom}_{(r,l(G))}(K, L) = \text{Hom}_{(r_a,l(G))}(K, L).$$

*In particular, every  $r$ -valuation monoid of  $K$  is an  $r_a$ -valuation monoid of  $K$ .*

*Proof.* 1. If  $X \subset K$  and  $B \in \mathbb{P}_f^\bullet(K)$ , then  $X_r B \subset (XB)_r$ . Therefore we obtain  $X_r \subset ((XB)_r : B) \subset X_{r_a}$  and, since  $r$  is finitary,

$$X_{r_a} = \bigcup_{B \in \mathbb{P}_f^\bullet(K)} \left( \left( \bigcup_{E \in \mathbb{P}_f(X)} EB \right)_r : B \right) = \bigcup_{B \in \mathbb{P}_f^\bullet(K)} \bigcup_{E \in \mathbb{P}_f(X)} ((EB)_r : B) = \bigcup_{E \in \mathbb{P}_f(X)} E_{r_a}.$$

Therefore, it remains to prove that  $r_a$  is a (weak) module system, and by Theorem 3.6 we have to check the conditions of Definition 3.1 for all finite subsets  $X, Y \subset K$  and  $c \in K$ . Thus, let  $X, Y \in \mathbb{P}_f(K)$  and  $c \in K$ . The verification of **M1.**, **M3.** and **M3'** is straightforward.

**M2.** Let  $X \subset Y_{r_a}$  and  $z \in X_{r_a}$ . Then there exists some  $F \in \mathbb{P}_f^\bullet(K)$  such that  $z \in ((XF)_r : F)$ , and since  $\{((YB)_r : B) \mid B \in \mathbb{P}_f^\bullet(K)\}$  is directed, there exists some  $B \in \mathbb{P}_f^\bullet(K)$  such that  $X \subset ((YB)_r : B)$ . Then

$$zFB \subset (XF)_r B \subset (XBF)_r \subset [(YB)_r F]_r = (YFB)_r$$

and thus  $z \in ((YFB)_r : FB) \subset Y_{r_a}$ , since  $FB \in \mathbb{P}_f^\bullet(K)$ .

2. By Proposition 8.1 we must prove that  $((EF)_{r_a} : E) \subset F_{r_a}$  holds for all  $E \in \mathbb{P}_f^\bullet(K)$  and  $F \in \mathbb{P}_f(K)$ . Thus, let  $E \in \mathbb{P}_f^\bullet(K)$ ,  $F \in \mathbb{P}_f(K)$  and  $z \in ((EF)_{r_a} : E)$ . Then  $zE \subset (EF)_{r_a}$  implies  $zE \subset ((EFB)_r : B)$  for some  $B \in \mathbb{P}_f^\bullet(K)$  (since  $\{((EFB)_r : B) \mid B \in \mathbb{P}_f^\bullet(K)\}$  is directed). Hence, it follows that  $zEB \subset (EFB)_r$  and  $z \in ((EFB)_r : EB) \subset F_{r_a}$ , since  $EB \in \mathbb{P}_f^\bullet(K)$ .

Let now  $q$  be any finitary weak module system on  $K$  such that  $r \leq q$  and  $\mathcal{M}_{q,f}(K)^\bullet$  is cancellative. For any  $X \in \mathbb{P}_f(K)$  and  $B \in \mathbb{P}_f^\bullet(K)$ , Proposition 8.1 implies  $((XB)_r : B) \subset ((XB)_q : B) \subset X_q$ , and thus  $r_a \leq q$  by Proposition 4.4.2.

3. For  $X \subset K$ , it is easily checked that  $X_{r_a[D]} = X_{r[D]_a}$ .

4. Since  $r \leq r_a$ , every  $(r_a, t)$ -homomorphism is an  $(r, t)$ -homomorphism. If  $\varphi : K \rightarrow L$  is an  $(r, t)$ -homomorphism, then by Proposition 4.4.2 we must prove that  $\varphi(X_{r_a}) \subset \varphi(X)_{t(G)}$  for all  $X \in \mathbb{P}_f(K)$ . If  $X \in \mathbb{P}_f(K)$ ,  $z \in X_{r_a}$  and  $B \in \mathbb{P}_f^\bullet(K)$  are such that  $zB \subset (XB)_r$ , then

$$\varphi(z)\varphi(B) \subset \varphi((XB)_r) \subset \varphi(XB)_t = [\varphi(X)\varphi(B)]_t$$

and therefore  $\varphi(z) \in ([\varphi(X)\varphi(B)]_t : \varphi(B)) \subset \varphi(X)_t$  by Proposition 8.1 and Lemma 7.2.4.

If  $V \subset K$  is a valuation monoid, then it follows by Lemma 7.4 that  $V$  is an  $r$ - (resp.  $r_a$ -)valuation monoid if and only if  $\text{id}_K$  is an  $(r, t(V))$ - (resp.  $(r_a, t(V))$ )-homomorphism. Hence, every  $r$ -valuation monoid is an  $r_a$ -valuation monoid.  $\square$

**Definition 8.3.** Let  $r$  be a finitary weak module system on  $K$ . The finitary weak module system  $r_a$  is called the *cancellative extension* of  $r$ . An element  $a \in K$  is called  *$r$ -integral over  $D$*  if  $a \in D_{r_a}$ . A subset  $X \subset K$  is called  *$r$ -integral over  $D$*  if  $X \subset D_{r_a}$ . The monoid  $D_{r_a}$  is called the  *$r$ -closure* of  $D$ , and  $D$  is called  *$r$ -closed* if  $D = D_{r_a}$ .

*Remark 8.4.* The notion of  $r$ -integrality generalizes the concept of integral elements in commutative ring theory. If  $D$  is an integral domain and  $d = d(D)$  is the module system induced by the Dedekind system on  $K$ , then  $D_{d_a}$  is the integral closure of  $D$ . Most results of the classical theory of integral elements (transitivity and localization properties) continue to hold for  $r$ -integrality (see [25, Chap. 14] for details, [27] for a version for not necessarily cancellative monoids and [15, Example 2.1] for the history of the concept). In Krull’s ancient terminology (which is still used in the theory of semistar operations, see [23, Section 32]) ideal systems  $x$  for which  $\mathcal{M}_{x,f}(K)^\bullet$  is cancellative, are called “e.a.b.” (endlich arithmetisch brauchbar). In the case of ideal systems on monoids, the construction of  $r_a$  goes back to P. Lorenzen [34] who constructed a multiplicative substitute for the Kronecker function ring. A readable overview of the development of the concepts and results related to Kronecker function rings and semistar operations was given by M. Fontana and K.A. Loper [20].

**Definition 8.5.** Let  $r$  be a finitary module system on  $K$ . We denote by  $\Lambda_r(K) = \text{q}(\mathcal{M}_{r_a,f}(K))$  the quotient of the monoid  $\mathcal{M}_{r_a}(K)$  ( $\mathcal{M}_{r_a}(K)^\bullet$  is cancellative, see Theorem 8.2.2). The group  $\Lambda_r(K)^\times$  is a quotient group of  $\mathcal{M}_{r_a,f}(K)^\bullet$  and is called

the *Lorenzen  $r$ -group*. For  $X \in \Lambda_r(K)^\bullet$ , we denote by  $X^{[-1]}$  its inverse in the group  $\Lambda_r(K)^\times$ . Then we obtain, by the very definition,

$$\Lambda_r(K) = \{C^{[-1]}A \mid A \in \mathcal{M}_{r_a, f}(K), C \in \mathcal{M}_{r_a, f}(K)^\bullet, \}.$$

If  $A, A' \in \mathcal{M}_{r_a, f}(K)$  and  $C, C' \in \mathcal{M}_{r_a, f}(K)^\bullet$ , then  $C^{[-1]}A = C'^{[-1]}A'$  if and only if  $(AC')_{r_a} = (A'C)_{r_a}$ , and multiplication in  $\Lambda_r(K)$  is given by the formula  $(C^{[-1]}A) \cdot (C'^{[-1]}A') = (CC')_r^{[-1]}(AA')_r$ . In particular,  $D_{r_a} = \{1\}_{r_a}$  is the unit element of  $\Lambda_r(K)$ . The submonoid

$$\Lambda_r^+(K) = \{C^{[-1]}A \mid A \in \mathcal{M}_{r_a, f}(K), C \in \mathcal{M}_{r_a, f}(K)^\bullet, A \subset C\} \subset \Lambda_r(K)$$

is called the *Lorenzen  $r$ -monoid*. It is easily checked that  $\Lambda_r^+(K) \subset \Lambda_r(K)$  is really a submonoid, and  $\mathcal{M}_{r_a, f}(K) \subset \Lambda_r(K)$ . The *Lorenzen homomorphism*  $\tau_r: K \rightarrow \Lambda_r(K)$  is defined by  $\tau_r(a) = \{a\}_{r_a} = aD_{r_a} \in \mathcal{M}_{r_a, f}(K) \subset \Lambda_r(K)$  for all  $a \in K$ .

**Theorem 8.6.** *Let  $r$  be a finitary module system on  $K$ ,  $D \subset \{1\}_{r_a}$  and  $K = \mathfrak{q}(D)$ . Let  $t = t(\Lambda_r^+(K))$  be the  $t$ -system on  $\Lambda_r(K)$  induced from  $\Lambda_r^+(K)$ .*

1. *If  $A \in \mathcal{M}_{r_a, f}(K)$  and  $C \in \mathcal{M}_{r_a, f}(K)^\bullet$ , then  $C^{[-1]}A \in \Lambda_r^+(K)$  if and only if  $A \subset C$ .*
2.  *$\Lambda_r^+(K)$  is a reduced GCD-monoid, and  $\Lambda_r(K)$  is a quotient of  $\Lambda_r^+(K)$ . If  $X, Y \in \Lambda_r^+(K)$ , then there exist  $A, B \in \mathcal{M}_{r_a, f}(K)$  and  $C \in \mathcal{M}_{r_a, f}(K)^\bullet$  such that  $A \cup B \subset C$ ,  $X = C^{[-1]}A$  and  $Y = C^{[-1]}B$ . In this case, we have  $X \mid Y$  if and only if  $B \subset A$ , and  $\gcd(X, Y) = C^{[-1]}(A \cup B)_{r_a}$ .*
3. *For every  $X \in \Lambda_r^+(K)$ , there exist  $E \in \mathbb{P}_f(D)$  and  $E' \in \mathbb{P}_f^\bullet(D)$  such that  $E_{r_a} \subset E'_{r_a}$  and  $X = E'_{r_a}{}^{[-1]}E_{r_a} = \gcd(\tau_r(E'))^{[-1]} \gcd(\tau_r(E))$ .*
4. *The Lorenzen homomorphism  $\tau_r: K \rightarrow \Lambda_r(K)$  is an  $(r_a, t)$ -homomorphism and  $\tau_r|_{K^\times}: K^\times \rightarrow \Lambda_r(K)^\times$  is a group homomorphism satisfying  $\text{Ker}(\tau_r|_{K^\times}) = D_{r_a}^\times$ .*
5. *For every  $Z \subset K$  we have  $Z_{r_a} = \tau_r^{-1}[\tau_r(Z)_t] = \{c \in K \mid \{c\}_{r_a} \in \tau_r(Z)_t\}$ , and in particular  $\tau_r^{-1}(\Lambda_r^+(K)) = D_{r_a}$ .*

*Proof.* The assertions 1. to 4. follow immediately from the definitions.

5. Let now first  $Z \subset K$  be finite, say  $Z = a^{-1}A$ , where  $a \in D^\bullet$  and  $A = \{a_1, \dots, a_n\} \subset D \subset \{1\}_{r_a}$ . Then

$$\begin{aligned} A_{r_a} &= (\{a_1\}_{r_a} \cup \dots \cup \{a_n\}_{r_a})_{r_a} = \gcd(\{a_1\}_{r_a}, \dots, \{a_n\}_{r_a}) \\ &= \gcd(\tau_r(a_1), \dots, \tau_r(a_n)) = \gcd(\tau_r(A)) \end{aligned}$$

and therefore  $\tau_r(A)_t = A_{r_a} \Lambda_r^+(K)$  by Lemma 7.2.4. For  $c \in K$ , we have  $c \in \tau_r^{-1}[\tau_r(Z)_t]$  if and only if

$$\tau_r(ac) = \tau_r(a)\tau_r(c) \in \tau_r(a)\tau_r(Z)_t = \tau_r(aZ)_t = \tau_r(A)_t = A_{r_a} \Lambda_r^+(K),$$

and therefore we obtain

$$\begin{aligned} c \in \tau_r^{-1}[\tau_r(Z)_t] &\iff \tau_r(ac) \in A_{r_a} \Lambda_r^+(K) \iff A_{r_a}^{[-1]}\{ac\}_{r_a} \in \Lambda_r^+(K) \\ &\iff \{ac\}_{r_a} \subset A_{r_a} \iff ac \in A_{r_a} = aZ_{r_a} \iff c \in Z_{r_a}. \end{aligned}$$

Hence,  $Z_{r_a} = \tau_r^{-1}(\tau_r(Z)_t)$  and  $D_{r_a} = \tau_r^{-1}(\tau_r(\{1\}_t) = \tau_r^{-1}(\Lambda_r^+(K))$ . If finally  $Z \subset K$  is arbitrary, then

$$Z_{r_a} = \bigcup_{E \in \mathbb{P}_f(Z)} E_{r_a} = \bigcup_{E \in \mathbb{P}_f(Z)} \tau_r^{-1}[\tau_r(E)_t] = \tau_r^{-1}\left(\bigcup_{F \in \mathbb{P}_f(\tau_r(Z))} F_t\right) = \tau_r^{-1}(\tau_r(Z)_t).$$

In particular, it follows that  $\tau_r(Z_{r_a}) \subset \tau_r(Z)_t$ , and thus  $\tau_r$  is an  $(r_a, t)$ -homomorphism.  $\square$

*Remark 8.7.* Let  $D$  be an integral domain with quotient field  $K$ ,  $*$  a semistar operation on  $D$  and  $r = r^*$  the module system on  $K$  induced by  $*$ . Then the Lorenzen  $r$ -monoid  $\Lambda_r^+(K)$  is isomorphic to the monoid  $(\text{Kr}(D, *))$  of principal ideals of the semistar Kronecker function ring  $\text{Kr}(D, *)$  (see [19]). We recall the definition:  $\text{Kr}(D, *)$  consists of all rational functions  $f/g$  with  $f, g \in D[X]$  such that  $g \neq 0$  and there exists some  $h \in D[X]^\bullet$  satisfying  $[c(f)c(h)]^* \subset [c(g)c(h)]^*$ . An isomorphism  $(\text{Kr}(*, D)) \rightarrow \Lambda_r^+(K)$  is given by the assignment  $(f/g) \mapsto c(g)_{r_a}^{[-1]}c(f)_{r_a}$ .

**Theorem 8.8 (Universal property of the Lorenzen monoid).** *Let  $r$  be a finitary module system on  $K$ ,  $D \subset \{1\}_{r_a}$ ,  $K = \text{q}(D)$  and  $t = t(\Lambda_r^+(K))$  the  $t$ -system on  $\Lambda_r(K)$  induced from  $\Lambda_r^+(K)$ . If  $G$  is a reduced GCD-monoid and  $L = \text{q}(G)$ , then there is a bijective map*

$$\text{Hom}_{(t,t(G))}(\Lambda_r(K), L) \rightarrow \text{Hom}_{(r,t(G))}(K, L), \text{ defined by } \phi \mapsto \phi \circ \tau_r.$$

*Proof.* If  $\Phi: \Lambda_r(K) \rightarrow L$  is a  $(t, t(G))$ -homomorphism, then  $\Phi \circ \tau_r: K \rightarrow L$  is an  $(r, t(G))$ -homomorphism, since  $\tau_r$  is an  $(r_a, t)$ -homomorphism and thus also an  $(r, t)$ -homomorphism. We prove that for every  $\phi \in \text{Hom}_{(r,t(G))}(K, L)$  there is a unique  $\Phi \in \text{Hom}_{(t,t(G))}(\Lambda_r(K), L)$  such that  $\Phi \circ \tau_r = \phi$ . Thus let  $\phi \in \text{Hom}_{(r,t(G))}(K, L)$ .

By Lemma 7.3, the map  $\text{Hom}_{(t,t(G))}(\Lambda_r(K), L) \rightarrow \text{Hom}_{\text{GCD}}(\Lambda_r^+(K), G)$ , defined by  $\Phi \mapsto \Phi|_{\Lambda_r^+(K)}$ , is bijective, and for  $\Phi \in \text{Hom}_{(t,t(G))}(\Lambda_r(K), L)$  we have  $\Phi \circ \tau_r = \phi$  if and only if  $[\Phi|_{\Lambda_r^+(K)}] \circ (\tau_r|_D) = \phi|_D$  (since  $K = \text{q}(D)$ ). Hence it suffices to prove that there exists a unique  $\psi \in \text{Hom}_{\text{GCD}}(\Lambda_r^+(K), G)$  such that  $\psi \circ \tau_r(a) = \phi(a)$  for all  $a \in D^\bullet$ .

Uniqueness: If  $\psi \in \text{Hom}_{\text{GCD}}(\Lambda_r^+(K), G)$  be such that  $\psi \circ \tau_r(a) = \phi(a)$  for all  $a \in D^\bullet$  and  $X = \text{gcd}(\tau_r(E'))^{[-1]} \text{gcd}(\tau_r(E)) \in \Lambda_r^+(K)$  (where  $E \in \mathbb{P}_f(D)$ ,  $E' \in \mathbb{P}_f^\bullet(D)$  and  $E_{r_a} \subset E'_{r_a}$ ), then

$$\psi(X) = \text{gcd}[\psi(\tau_r(E'))]^{-1} \text{gcd}[\psi(\tau_r(E))] = \text{gcd}[\phi(E')]^{-1} \text{gcd}[\phi(E)],$$

and thus  $\psi$  is uniquely determined by  $\phi$ .

Existence: Define  $\psi$  provisionally by  $\psi(X) = \text{gcd}(\phi(E'))^{-1} \text{gcd}(\phi(E))$  if  $X = \text{gcd}(\tau_r(E'))^{[-1]} \text{gcd}(\tau_r(E))$  with  $E \in \mathbb{P}_f(D)$ ,  $E' \in \mathbb{P}_f^\bullet(D)$  and  $E_{r_a} \subset E'_{r_a}$ . We must prove the following assertions: **1)**  $\psi(X) \subset G$ ; **2)** the definition is independent of the choice of  $E$  and  $E'$ ; **3)**  $\psi$  is a GCD-homomorphism. The proofs are lengthy but straightforward and are left to the reader.  $\square$

**Theorem 8.9.** *Let  $r$  be a finitary module system on  $K$ ,  $D \subset \{1\}_{r_a}$  and  $K = \mathfrak{q}(D)$ . Let  $t = t(\Lambda_r^+(K))$  the  $t$ -system on  $\Lambda_r(K)$  induced from  $\Lambda_r^+(K)$ . Let  $\mathcal{V}$  be the set of all  $r$ -valuation monoids in  $K$  and  $\mathcal{W}$  the set of all  $t$ -valuation monoids in  $\Lambda_r(K)$ . Then  $\mathcal{V} = \{\tau_r^{-1}(W) \mid W \in \mathcal{W}\}$ .*

*Proof.* If  $W \in \mathcal{W}$  and  $x \in K \setminus \tau_r^{-1}(W)$ , then  $\tau_r(x)^{-1} = \tau_r(x^{-1}) \in W$  and therefore  $x^{-1} \in \tau_r^{-1}(W)$ . Hence  $\tau_r^{-1}(W)$  is a valuation monoid, and since  $\tau_r$  is an  $(r, t)$ -homomorphism, it is even an  $r$ -valuation monoid and lies in  $\mathcal{V}$ .

Let now  $V \in \mathcal{V}$  and  $\pi: K \rightarrow K/V^\times$  the canonical epimorphism. Then  $V/V^\times$  is a reduced valuation monoid,  $\mathfrak{q}(V/V^\times) = K/V^\times$ , and we denote by  $t^* = t(V/V^\times) = s(V/V^\times)$  the module system on  $K/V^\times$  which is induced by the  $t$ -system on  $V/V^\times$ . Since  $r \leq r_V = s(V)$ , it follows that  $\pi$  is an  $(r, t^*)$ -homomorphism. By Theorem 8.8, the assignment  $\Phi \mapsto \Phi \circ \tau_r$  defines a bijective map  $\text{Hom}_{(t, t^*)}(\Lambda_r(K), K/V^\times) \rightarrow \text{Hom}_{(r, t^*)}(K, K/V^\times)$ . Hence there is a unique  $(t, t^*)$ -homomorphism  $\Phi: \Lambda_r(K) \rightarrow K/V^\times$  such that  $\Phi \circ \tau_r = \pi$ , and  $\Phi$  is surjective, since  $\pi$  is surjective. Now  $W = \Phi^{-1}(V/V^\times) \subset \Lambda_r(K)$  is a valuation monoid, and  $\tau_r^{-1}(W) = (\Phi \circ \tau_r)^{-1}(V/V^\times) = \pi^{-1}(V/V^\times) = V$ . Thus, it remains to prove that  $W_t = W$ . Since  $\Phi$  is a  $(t, t^*)$ -homomorphism, it follows that  $\Phi(W_t) \subset \Phi(W)_{t^*} = (V/V^\times)_{t^*} = V/V^\times$  and  $W_t \subset \Phi^{-1}(V/V^\times) = W$ , whence  $W_t = W$ .  $\square$

**Theorem 8.10.** *Let  $r$  be a finitary module system on  $K$ ,  $D \subset \{1\}_{r_a}$  and  $K = \mathfrak{q}(D)$ . If  $\mathcal{V}_r(D)$  denotes the set of all  $r$ -valuation monoids of  $K$  containing  $D$ , then  $\mathcal{V}_r(D) = \mathcal{V}_{r_a}(D_{r_a})$  and*

$$D_{r_a} = \{1\}_{r_a} = \bigcap_{V \in \mathcal{V}_r(D)} V.$$

*Proof.* By Theorem 8.2.4, a monoid  $V \subset K$  is an  $r$ -valuation monoid if and only if it is an  $r_a$ -valuation monoid. Hence  $\mathcal{V}_r(D) = \mathcal{V}_{r_a}(D) \supset \mathcal{V}_{r_a}(D_{r_a})$ , and if  $V \in \mathcal{V}_r(D)$ , then  $\{1\}_{r_a} = D_{r_a} \subset V_{r_a} = V$  and thus  $V \in \mathcal{V}_{r_a}(D_{r_a})$ .

Let  $\tau_r: K \rightarrow \Lambda_r(K)$  be the Lorenzen homomorphism,  $t = t(\Lambda_r^+(K))$  and  $\mathcal{W}$  the set of all  $t$ -valuation monoids in  $\Lambda_r(K)$ . By Theorem 8.9 we have  $\mathcal{V}_r(D) = \{\tau_r^{-1}(W) \mid W \in \mathcal{W}\}$  and, applying Proposition 7.5.2 and Theorem 8.6.3, we obtain

$$D_{r_a} = \tau_r^{-1}(\Lambda_r^+(K)) = \tau_r^{-1}\left(\bigcap_{W \in \mathcal{W}} W\right) = \bigcap_{W \in \mathcal{W}} \tau_r^{-1}(W) = \bigcap_{V \in \mathcal{V}_r(D)} V. \quad \square$$

**Corollary 8.11.** *Let  $K = \mathfrak{q}(D)$  and  $r$  a finitary ideal system on  $D$ . Then  $D_{r_a}$  is the intersection of all  $r$ -valuation monoids in  $K$ .*

*Remark 8.12.* In the case of integral domains, Theorem 8.10 generalizes the connection between semistar Kronecker function rings and valuation overrings as developed in [18]. In particular, Corollary 8.11 contains the classical fact that the integral closure of an integral domain is the intersection of its valuation overrings (see [23, (19.8)]).

## 9 Invertible modules and Prüfer-like conditions

Let  $K$  be a monoid and  $D \subset K$  a submonoid such that  $K = q(D)$ .

This final section contains the basics of a purely multiplicative theory of semistar invertibility and semistar Prüfer domains as it was developed only recently by M. Fontana with several co-authors (see [6, 9, 15–18, 21, 22]). In particular, we refer to the examples presented in these papers which show the semistar approach covers really new classes of integral domains.

**Definition 9.1.** Let  $r$  be a module system on  $K$ . A  $D$ -module  $A \subset K$  is called ( $r$ -finitely)  $r$ -invertible (relative  $D$ ) if there exists a (finite) subset  $B \subset (D : A)$  such that  $(AB)_r = D_r$  [equivalently,  $1 \in (AB)_r$ ].

By definition,  $A$  is  $r$ -invertible if and only if  $A$  is  $r[D]$ -invertible. If  $A$  is  $r$ -invertible, then  $A$  is  $q$ -invertible for every module system  $q$  on  $K$  satisfying  $r \leq q$ , and every  $D$ -module  $A'$  with  $A \subset A' \subset A_r$  is also  $r$ -invertible.

**Lemma 9.2.** Let  $A \subset K$  be a  $D$ -module and  $B \subset K$  such that  $D = AB$ . Then  $A = aD$  for some  $a \in K$ .

*Proof.* Let  $P = D \setminus D^\times$ . Then  $PA \subset A$ , and we assert that  $PA \neq A$ . Indeed, if  $PA = A$ , then  $P = PD = PAB = AB = D$ , a contradiction. If  $a \in A \setminus AP$ , then  $aD \subset A$ , hence  $aBD \subset AB = D$ . If  $aBD \neq D$ , then  $aBD \subset P$ , since  $aBD$  is an ideal of  $D$ , and then  $a \in aD = aBD \subset AP$ , a contradiction. Hence,  $aBD = D$ , and consequently  $A = aBD = aD$ .  $\square$

**Proposition 9.3.** Let  $r$  be a module system on  $K$ ,  $c \in K^\times$ , and let  $A \subset K$  be a  $D$ -module.

1.  $A$  is  $r$ -invertible if and only if  $[A(D : A)]_r = D_r$ , and then  $(D : A)$  and  $A_{v(D)}$  are also  $r$ -invertible.
2. If  $A$  is  $r$ -invertible, then  $cA$  is also  $r$ -invertible, and  $A_r$  is cancellative in  $\mathcal{M}_r(K)$ .
3.  $A$  is  $r$ -invertible (relative  $D$ ) if and only if  $A_r$  is  $r$ -invertible (relative  $D_r$ ) and  $(D_r : A) = (D : A)_r$ .
4. If  $A_1, A_2 \subset K$  are  $D$ -modules, then  $A_1A_2$  is  $r$ -invertible if and only if  $A_1$  and  $A_2$  are both  $r$ -invertible.

*Proof.* 1. If  $[A(D : A)]_r = D_r$ , then  $A$  is  $r$ -invertible. If  $A$  is  $r$ -invertible, then there is some  $B \subset (D : A)$  such that  $(AB)_r = D_r$ , and since  $[A(D : A)]_r \subset D_r$ , it follows that  $[A(D : A)]_r = D_r$ . Hence,  $(D : A)$  is  $r$ -invertible, and (by an iteration of the argument)  $A_v = (D : (D : A))$  is also  $r$ -invertible.

2. Let  $A$  be  $r$ -invertible and  $B \subset (D : A)$  such that  $(AB)_r = D_r$ . Since  $c^{-1}B \subset (D : cA)$  and  $((cA)(c^{-1}B))_r = D_r$ , it follows that  $cA$  is also  $r$ -invertible. If  $X, Y \in \mathcal{M}_r(D)$  and  $(A_rX)_r = (A_rY)_r$ , then it follows that  $X = [(BA)_rX]_r = [B(A_rX)]_r = [B(A_rY)]_r = [(BA)_rY]_r = Y$ , and thus  $A_r$  is cancellative.

3. By Proposition 3.3.3,  $A_r$  is a  $D_r$ -module. If  $A$  is  $r$ -invertible, then  $D_r = [A(D:A)]_r \subset [A_r(D:A)]_r \subset [A_r(D_r:A)]_r = [A_r(D_r:A_r)]_r \subset D_r$ ; hence equality holds,  $A_r$  is  $r$ -invertible (relative  $D_r$ ), and since  $A_r$  is cancellative in  $\mathcal{M}_r(D)$ , it follows that  $(D:A)_r = (D_r:A)$ . To prove the converse, let  $A_r$  be  $r$ -invertible (relative  $D_r$ ) and  $(D:A)_r = (D_r:A)$ . Then it follows that  $[A(D:A)]_r = [A_r(D:A)]_r = [A_r(D_r:A)]_r = [A_r(D_r:A_r)]_r = D_r$ , and thus  $A$  is  $r$ -invertible relative  $D$ .

4. If  $A_1A_2$  is  $r$ -invertible, then there is some  $B \subset (D:A_1A_2)$  such that  $(A_1A_2B)_r = D_r$ . Since  $A_1B \subset (D:A_2)$  and  $A_2B \subset (D:A_1)$ , it follows that  $A_1$  and  $A_2$  are both  $r$ -invertible. If  $A_1$  and  $A_2$  are  $r$ -invertible, then there exist  $B_1 \subset (D:A_1)$  and  $B_2 \subset (D:A_2)$  such that  $(A_1B_1)_r = (A_2B_2)_r = D_r$ . Now  $(A_1A_2B_1B_2)_r = [(A_1B_1)_r(A_2B_2)_r]_r = D_r$  and  $B_1B_2 \subset (D:A_1A_2)$  implies that  $A_1A_2$  is  $r$ -invertible.  $\square$

**Proposition 9.4.** *Let  $r$  be a finitary module system on  $K$  and  $A \subset K$  a  $D$ -module.*

1. *The following assertions are equivalent:*
  - (a)  *$A$  is  $r$ -invertible (relative  $D$ ).*
  - (b) *There exists a finite subset  $F \subset (D:A)$  such that  $1 \in (AF)_r$ .*
  - (c) *For all  $P \in r_D\text{-max}(D)$  we have  $A(D:A) \not\subset P$ .*
2. *If  $A$  is  $r$ -invertible, then  $A_r$  is  $r[D]$ -finite and  $A$  is  $r$ -finitely  $r$ -invertible.*
3. *If  $T \subset D$  is multiplicatively closed and  $A$  is  $r$ -invertible, then  $T^{-1}A$  is  $r$ -invertible (relative  $T^{-1}D$ ).*

*Proof.* 1. (a)  $\Rightarrow$  (b) If  $B \subset (D:A)$  is such that  $1 \in (AB)_r$ , then (since  $r$  is finitary) there exists a finite subset  $F \subset B$  such that  $1 \in (AF)_r$ .

(b)  $\Rightarrow$  (c) Assume that  $A(D:A) \subset P$  for some  $P \in r_D\text{-max}(D)$ , and let  $F \subset (D:A)$  be finite such that  $1 \in (AF)_r$ . Then it follows that  $1 \in (AF)_r \cap D \subset [A(D:A)]_r \cap D \subset P_r \cap D = P$ , a contradiction.

(c)  $\Rightarrow$  (a) Since  $A(D:A) \subset [A(D:A)]_r \cap D$ , it follows that the  $r_D$ -ideal  $[A(D:A)]_r \cap D$  is contained in no  $P \in r_D\text{-max}(D)$ . Hence it follows that  $[A(D:A)]_r \cap D = D \subset [A(D:A)]_r$  and therefore  $[A(D:A)]_r = D_r$ .

2. Let  $B \subset (D:A)$  be such that  $1 \in (AB)_r$ , and let  $E \subset A$  and  $F \subset B$  be finite subsets satisfying  $1 \in (EF)_r$ . Then  $D_r \subset (DEF)_r \subset (AF)_r \subset D_r$ , which implies  $D_r = (AF)_r$ , and thus  $A$  is  $r$ -finitely  $r$ -invertible relative  $D$ . Moreover, it follows that  $A_r = D_r A_r = (DEFA)_r = (DE)_r = E_{r[D]}$ , and therefore  $A_r$  is  $r[D]$ -finite.

3. If  $B \subset (D:A)$  is such that  $(AB)_r = D_r$ , then  $B \subset (T^{-1}D : T^{-1}A)$  and  $(T^{-1}AB)_r = (T^{-1}D)_r$ . Hence  $T^{-1}A$  is  $r$ -invertible (relative  $T^{-1}D$ ).  $\square$

**Theorem 9.5.** *Let  $r$  be a finitary module system on  $K$  and  $A \subset K$  a  $D$ -module.*

1. *If  $A$  is  $r$ -invertible and  $P \in r_D\text{-spec}(D)$ , then  $A_P = aD_P$  for some  $a \in K^\times$ .*
2. *Suppose that for every  $P \in r_D\text{-max}(D)$  there is some  $a_P \in K^\times$  such that  $A_P = a_P D_P$ . If  $y$  is a finitary module system on  $K$  such that  $D_y = D$  and  $A$  is  $y$ -finite, then  $A$  is  $r$ -invertible.*



*Proof.* 1. Let  $A$  be  $r$ -invertible,  $B \subset (D:A)$  such that  $(AB)_r = D_r$  and  $P \in r_D\text{-spec}(D)$ . Then  $AB \not\subset P$ , and since  $AB \subset D$  is an ideal, we obtain  $D_P = (AB)_P = A_P B_P$ . Now the assertion follows by Lemma 9.2.

2. Suppose that  $A = E_y$  for some  $E \in \mathbb{P}_f(K)$  and that  $A$  is not  $r$ -invertible. By Proposition 9.4, there is some  $P \in r_D\text{-spec}(D)$  such that  $A(D:A) \subset P$  and thus  $a_P(D:A)_P \subset PD_P$ . Since  $D = D_y$ , it follows that

$$(D:A)_P = (D:E)_P = (D_P:E) = (D_P:A_P) = a_P^{-1}D_P$$

and thus  $PD_P \supset a_P(D:A)_P = D_P$ , a contradiction.  $\square$

**Definition 9.6.** Let  $r$  and  $y$  be finitary module systems on  $K$  such that  $y \leq r$  and  $D_y = D$ . Then  $D$  is called a  $y$ -basic  $r$ -Prüfer monoid if every  $A \in \mathcal{M}_{y,f}(K)$  is  $r$ -invertible.

*Remark 9.7.* Let  $D$  be an integral domain,  $*$  a semistar operation on  $D$  and  $r = r^*$  the  $D$ -module system on  $K$  induced by  $*$ . Then  $D$  is a  $P^*MD$  (as defined in [15]) if and only if  $D$  is a basic  $d(D)$ -Prüfer monoid.

**Theorem 9.8.** Let  $r, q$  and  $y$  be finitary module systems on  $K$  such that  $q$  is a  $D$ -module system,  $y \leq q \leq r$  and  $D_y = D$ .

1. If  $D$  is an  $y$ -basic  $r$ -Prüfer monoid, then  $D_P$  is a valuation monoid for every  $P \in r_D\text{-spec}(D)$ .
2. The following assertions are equivalent:
  - (a)  $D$  is a  $y$ -basic  $r$ -Prüfer monoid.
  - (b)  $D$  is a  $y$ -basic  $r[q]$ -Prüfer monoid.
  - (c)  $D_P$  is a valuation monoid for every  $P \in r_D\text{-max}(D)$ .

*Proof.* 1. Let  $P \in r_D\text{-spec}(D)$ . Since  $D_P = D_P^\times D$ , it suffices to prove that for all  $a, b \in D^\bullet$  we have either  $a \in bD_P$  or  $b \in aD_P$ . If  $a, b \in D^\bullet$ , then  $\{a, b\}_y$  is  $r$ -invertible by the assumption: Let  $B \subset (D:\{a, b\}_y) = (D:\{a, b\})$  be such that  $1 \in (\{a, b\}_y B)_r$ . We assert that even  $1 \in \{a, b\}BD_P$ . Indeed, if not, then  $\{a, b\}BD_P \subset PD_P$ , which implies  $\{a, b\}B \subset PD_P \cap D = P$  and  $1 \in (\{a, b\}_y B)_r \cap D = (\{a, b\}B)_r \cap D \subset P_r \cap D = P$ , a contradiction.

Now it follows that  $D_P = (\{a, b\}D_P)B$  and thus  $\{a, b\}D_P = cD_P$  for some  $c \in D_P$  by Lemma 9.2. Hence, there exist  $u, v \in D_P$  such that  $a = cu$ ,  $b = cv$ , and  $\{u, v\}D_P = D_P$ . Therefore we have either  $u \in D_P^\times$  or  $v \in D_P^\times$  and thus either  $b \in aD_P$  or  $a \in bD_P$ .

2. (a)  $\Rightarrow$  (c) By 1.

(c)  $\Rightarrow$  (a) Let  $A = E_y \in \mathcal{M}_{y,f}(K)$ , where  $E \in \mathbb{P}_f(K)$ , and assume that  $A$  is not  $r$ -invertible. By Proposition 9.4 there exists some  $P \in r_D\text{-max}(D)$  such that  $A(D:A) \subset P$ . Since  $D_P$  is a valuation monoid, we obtain  $ED_P = aD_P$  for some  $a \in E$ , and thus also  $A_P = E_y D_P = (ED_P)_y = aD_P$ . Since  $[A(D:A)]_P = A_P(D:E)_P = A_P(D_P:ED_P) = aD_P(D_P:aD_P) = D_P$ , we obtain  $PD_P \supset [A(D:A)]_P = D_P$ , a contradiction.

(a)  $\Leftrightarrow$  (b) By Theorem 6.6.5 we have  $r_D\text{-max}(D) = r[q]_D\text{-max}(D)$ . We apply the equivalence of (a) and (c) with  $r[q]$  instead of  $r$  and obtain the equivalence of (a) and (b).  $\square$

**Corollary 9.9.** *Let  $r$  and  $y$  be finitary module systems on  $K$  such that  $y \leq r$  and  $D_y = D$ . If  $D$  is an  $y$ -basic  $r$ -Prüfer monoid, then every  $y$ -monoid  $T$  satisfying  $D \subset T \subset K$  is also an  $y$ -basic  $r$ -Prüfer monoid.*

*Proof.* By Theorem 9.8 it suffices to prove that  $T_P$  is a valuation monoid if  $P \in r_T\text{-max}(T)$ . If  $P \in r_T\text{-max}(T)$ , then  $P \cap D = P_r \cap T \cap D = P_r \cap D$ . Hence,  $P \cap D \in r_D\text{-spec}(D)$ ,  $D_{P \cap D}$  is a valuation monoid, and since  $D_{P \cap D} \subset T_P$ , it follows that  $T_P$  is also a valuation monoid.

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# Projectively full ideals and compositions of consistent systems of rank one discrete valuation rings: a survey

William Heinzer, Louis J. Ratliff, Jr., and David E. Rush

**Abstract** Let  $I$  be a nonzero ideal in a Noetherian domain  $R$ . We survey recent progress on conditions under which there exists a finite integral extension domain  $A$  of  $R$  and an ideal  $J$  of  $A$  such that  $J$  is projectively full and projectively equivalent to  $IA$ . We also survey recent work on compositions of consistent systems or rank one discrete valuation rings.

## 1 Introduction

All rings in this paper are commutative with a unit  $1 \neq 0$ . For an ideal  $K$  of a ring  $R$ , we let  $K_a$  denote the *integral closure* of  $K$ ; that is  $K_a = \{x \in R \mid x \text{ satisfies an equation of the form } x^h + k_1x^{h-1} + \cdots + k_h = 0, \text{ where } k_j \in K^j \text{ for } j = 1, \dots, h\}$ . Let  $I$  be a regular proper ideal of the Noetherian ring  $R$ , that is,  $I$  contains a regular element of  $R$  and  $I \neq R$ . An ideal  $J$  of  $R$  is *projectively equivalent* to  $I$  if there exist positive integers  $m$  and  $n$  such that  $(I^m)_a = (J^n)_a$ . The concept of projective equivalence of ideals and the study of ideals projectively equivalent to  $I$  was introduced by Samuel in [S] and further developed by Nagata in [N1] and Rees in [RE]. See [CHRR4] for a recent survey. Let  $\mathbf{P}(I)$  denote the set of integrally closed ideals that are projectively equivalent to  $I$ . The ideal  $I$  is said to be *projectively full* if  $\mathbf{P}(I) = \{(I^n)_a \mid n \geq 1\}$  and  $\mathbf{P}(I)$  is said to be *projectively full* if  $\mathbf{P}(I) = \mathbf{P}(J)$  for some projectively full ideal  $J$  of  $R$ .

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The set  $\text{Rees } I$  of Rees valuation rings of  $I$  is a finite set of rank one discrete valuation rings (DVRs) that determine the integral closure  $(I^n)_a$  of  $I^n$  for every positive integer  $n$  and is the unique minimal set of DVRs having this property. Consider the minimal primes  $z$  of  $R$  such that  $IR/z$  is a proper nonzero ideal. The set  $\text{Rees } I$  is the union of the sets  $\text{Rees } IR/z$ . Thus, one is reduced to describing the set  $\text{Rees } I$  in the case where  $I$  is a nonzero proper ideal of a Noetherian integral domain  $R$ . Consider the Rees ring  $\mathbf{R} = R[t^{-1}, It]$ . The integral closure  $\mathbf{R}'$  of  $\mathbf{R}$  is a Krull domain, so  $W = \mathbf{R}'_p$  is a DVR for each minimal prime  $p$  of  $t^{-1}\mathbf{R}'$ , and  $V = W \cap F$ , where  $F$  is the field of fractions of  $R$ , is also a DVR. The set  $\text{Rees } I$  of Rees valuation rings of  $I$  is the set of DVRs  $V$  obtained in this way, cf. [SH, Section 10.1]. More information on Rees valuations is in the article by Swanson [Sw], in this volume.

If  $(V_1, N_1), \dots, (V_n, N_n)$  are the Rees valuation rings of  $I$ , then the integers  $(e_1, \dots, e_n)$ , where  $IV_i = N_i^{e_i}$ , are the *Rees integers* of  $I$ . Necessary and sufficient conditions for two regular proper ideals  $I$  and  $J$  to be projectively equivalent are that (a)  $\text{Rees } I = \text{Rees } J$  and (b) the Rees integers of  $I$  and  $J$  are proportional [CHRR, Theorem 3.4]. If  $I$  is integrally closed and each Rees integer of  $I$  is one, then  $I$  is a projectively full radical ideal.<sup>1</sup>

A main goal in the papers [CHRR, CHRR2, CHRR3, CHRR4, HRR] and [HRR2], is to answer the following question:

**Question 1.1.** Let  $I$  be a nonzero proper ideal in a Noetherian domain  $R$ . Under what conditions does there exist a finite integral extension domain  $A$  of  $R$  such that  $\mathbf{P}(IA)$  contains an ideal  $J$  whose Rees integers are all equal to one?

Progress is made on Question 1.1 in [CHRR3, HRR, HRR2]. To describe this progress, let  $I$  be a regular proper ideal of the Noetherian ring  $R$ , let  $b_1, \dots, b_g$  be regular elements in  $R$  that generate  $I$ , let  $X_1, \dots, X_g$  be indeterminates, and for each positive integer  $m > 1$  let  $A_m = R[x_1, \dots, x_g] = R[X_1, \dots, X_g]/(X_1^m - b_1, \dots, X_g^m - b_g)$  and let  $J_m = (x_1, \dots, x_g)A_m$ . Let  $(V_1, N_1), \dots, (V_n, N_n)$  be the Rees valuation rings of  $I$ . Consider the following hypothesis on  $I = (b_1, \dots, b_g)R$ :

- (a)  $b_i V_j = IV_j (= N_j^{e_j})$ , say for  $i = 1, \dots, g$  and  $j = 1, \dots, n$ .
- (b) the greatest common divisor  $c$  of  $e_1, \dots, e_n$  is a unit in  $R$ .
- (b') the least common multiple  $d$  of  $e_1, \dots, e_n$  is a unit in  $R$ .

Then the main result in [CHRR3] establishes the following:

**Theorem 1.2.** *If (a) and (b) hold, then  $A_c = R[x_1, \dots, x_g]$  is a finite free integral extension ring of  $R$  and the ideal  $J_c = (x_1, \dots, x_g)A_c$  is projectively full and projectively equivalent to  $IA_c$ . Also, if  $R$  is an integral domain and if  $z$  is a minimal prime ideal in  $A_c$ , then  $((J_c + z)/z)_a$  is a projectively full ideal in  $A_c/z$  that is projectively equivalent to  $(IA_c + z)/z$ .*

<sup>1</sup> Example 5.1 of [CHRR2] demonstrates that there exist integrally closed local domains  $(R, M)$  for which  $M$  is not projectively full. Remark 4.10 and Example 4.14 of [CHRR] show that a sufficient, but not necessary, condition for  $I$  to be projectively full is that the gcd of the Rees integers of  $I$  is equal to one.

We prove in [HRR, (3.19) and (3.20)] that if either (i)  $R$  contains an infinite field, or (ii)  $R$  is a local ring with an infinite residue field, then it is possible to choose generators  $b_1, \dots, b_g$  of  $I$  that satisfy assumption (a) of Theorem 1.2. In [HRR, (3.7)] the following is established:

**Theorem 1.3.** *If (a) and (b') hold, then for each positive multiple  $m$  of  $d$  that is a unit in  $R$  the ideal  $(J_m)_a$  is projectively full and  $(J_m)_a$  is a radical ideal that is projectively equivalent to  $IA_m$ . Also, the Rees integers of  $J_m$  are all equal to one and  $x_iU$  is the maximal ideal of  $U$  for each Rees valuation ring  $U$  of  $J_m$  and for  $i = 1, \dots, g$ . Moreover, if  $R$  is an integral domain and if  $z$  is a minimal prime ideal in  $A_m$ , then  $((J_m + z)/z)_a$  is a projectively full radical ideal that is projectively equivalent to  $(IA_m + z)/z$ .*

Examples (3.22) and (3.23) of [HRR] show that even if  $R$  is the ring  $\mathbb{Z}$  of rational integers, condition (b') of Theorem 1.3 is needed for the proof given in [HRR]. However, the following result, which is the main result in [HRR2], shows that conditions (a), (b) and (b') in Theorems 1.2 and 1.3 are not needed if  $R$  is a Noetherian domain of altitude (or in other terminology Krull dimension) one.

**Theorem 1.4.** *Let  $I$  be a nonzero proper ideal in a Noetherian integral domain  $R$ .*

1. *There exists a finite separable integral extension domain  $A$  of  $R$  and a positive integer  $m$  such that all the Rees integers of  $IA$  are equal to  $m$ .*
2. *If  $R$  has altitude one, then there exists a finite separable integral extension domain  $A$  of  $R$  such that  $\mathbf{P}(IA)$  contains an ideal  $H$  whose Rees integers are all equal to one. Therefore  $H = \text{Rad}(IA)$  is a projectively full radical ideal that is projectively equivalent to  $IA$ .*

A classical theorem of Krull, stated as Theorem 2.1 below, is an important tool in [HRR2]. We use the following terminology from [G] and [HRR2].

**Definition 1.5.** Let  $(V_1, N_1), \dots, (V_n, N_n)$  be distinct DVRs of a field  $F$  and for  $i = 1, \dots, n$  let  $K_i = V_i/N_i$  denote the residue field of  $V_i$ . Let  $m$  be a positive integer. By an  $m$ -consistent system for  $\{V_1, \dots, V_n\}$ , we mean a collection of sets  $S = \{S(V_1), \dots, S(V_n)\}$  satisfying the following conditions:

- (1)  $S(V_i) = \{(K_{i,j}, f_{i,j}, e_{i,j}) \mid j = 1, \dots, s_i\}$ , where  $K_{i,j}$  is a simple algebraic field extension of  $K_i$ ,  $f_{i,j} = [K_{i,j} : K_i]$ , and  $s_i, e_{i,j} \in \mathbb{N}_+$  (the set of positive integers).
- (2) For each  $i$ , the sum  $\sum_{j=1}^{s_i} e_{i,j} f_{i,j} = m$ .

**Definition 1.6.** The  $m$ -consistent system  $S$  for  $\{V_1, \dots, V_n\}$  as in Definition 1.5 is said to be *realizable* for  $\{V_1, \dots, V_n\}$  if there exists a separable algebraic extension field  $L$  of  $F$  such that:

- (a)  $[L : F] = m$ .
- (b) For  $1 \leq i \leq n$ ,  $V_i$  has exactly  $s_i$  extensions  $(V_{i,1}, N_{i,1}), \dots, (V_{i,s_i}, N_{i,s_i})$  to  $L$ .
- (c) The residue field  $V_{i,j}/N_{i,j}$  of  $V_{i,j}$  is  $K_i$ -isomorphic to  $K_{i,j}$ , so  $[K_{i,j} : K_i] = f_{i,j}$ , and the ramification index of  $V_{i,j}$  over  $V_i$  is  $e_{i,j}$ , so  $N_i V_{i,j} = N_{i,j} e_{i,j}$ .

If  $S$  and  $L$  are as above, we say the field  $L$  *realizes*  $S$  for  $\{V_1, \dots, V_n\}$  or that  $L$  is a *realization* of  $S$  for  $\{V_1, \dots, V_n\}$ .

In Sections 2–4, we summarize the main results in [HRR3] concerning the realizability of a consistent system  $S$  for a finite set  $\mathbf{V} = \{V_1, \dots, V_n\}$ ,  $n > 1$ , of distinct DVRs on a field  $F$ . These results are obtained by constructing and composing two realizable consistent systems that are related to  $S$ .

In Sections 5 and 6, we return to projectively full ideals. In Section 5, we summarize results in [HRR4] concerning a Rees-good basis of a regular ideal  $I$  in a Noetherian ring  $R$ . This is a basis that satisfies condition (a) above. The main result in [HRR4] shows that there always exists a finite free integral extension ring  $A$  of  $R$  such that  $IA$  has a Rees-good basis and the same Rees-integers as  $I$  (with, perhaps, greater cardinality). In Section 6 we observe that [RR] implies that the homogeneous prime spectra of the Rees rings of two filtrations related to  $I$  are isomorphic if and only if  $I$  is projectively full.

Our terminology is mainly as in Nagata [N2], so, for example, the term *altitude* refers to what is often also called dimension or Krull dimension, and a *basis* of an ideal is a set of generators of the ideal.

## 2 The realizability of $m$ -consistent systems

The following theorem of Krull is an important tool in [HRR2] and [HRR3].

**Theorem 2.1. (Krull [K]):** *Let  $(V_1, N_1), \dots, (V_n, N_n)$  be distinct DVRs with quotient field  $F$ , let  $m$  be a positive integer, and let  $S = \{S(V_1), \dots, S(V_n)\}$  be an  $m$ -consistent system for  $\{V_1, \dots, V_n\}$ , where  $S(V_i) = \{(K_{i,j}, f_{i,j}, e_{i,j}) \mid j = 1, \dots, s_i\}$  for  $i = 1, \dots, n$ . Then  $S$  is realizable for  $\{V_1, \dots, V_n\}$  if one of the following conditions is satisfied:*

- (i)  $s_i = 1$  for at least one  $i$ .
- (ii)  $F$  admits at least one DVR  $V$  distinct from  $V_1, \dots, V_n$ .
- (iii) For each monic polynomial  $X^t + a_1X^{t-1} + \dots + a_t$  with  $a_i \in \bigcap_{i=1}^n V_i = D$ , and for each  $h \in \mathbb{N}$ , there exists an irreducible separable polynomial  $X^t + b_1X^{t-1} + \dots + b_t \in D[X]$  with  $b_l - a_l \in N_i^h$  for each  $l = 1, \dots, t$  and  $i = 1, \dots, n$ .

Observe that condition (i) of Theorem 2.1 is a property of the  $m$ -consistent system  $S = \{S(V_1), \dots, S(V_n)\}$ , whereas condition (ii) is a property of the family of DVRs with quotient field  $F$ , and condition (iii) is a property of the family  $(V_1, N_1), \dots, (V_n, N_n)$ .

The result of Krull stated in Theorem 2.1 is a generalization of a classical result of Hasse [H] that shows that all  $m$ -consistent systems for a given finite set of distinct DVRs of an algebraic number field  $F$  are realizable. This has been extended further by P. Ribenboim, O. Endler and L. C. Hill, among others. For a good sampling of these results on when an  $m$ -consistent system is realizable, see [E, Sections 25–27] and [E2]. These references give several sufficient conditions on the realizability of an  $m$ -consistent system for a given finite set  $\mathbf{V} = \{V_1, \dots, V_n\}$  of distinct DVRs  $V_i$  with quotient field  $F$ .

**Remarks 2.2. (2.2.1)** There is an obvious necessary condition for an  $m$ -consistent system to be realizable. If  $n = 1$  and  $V_1$  is a Henselian DVR, then no  $m$ -consistent system  $S = \{S(V_1)\}$ , where  $S(V_1) = \{(K_1, f_1, e_1), \dots, (K_s, f_s, e_s)\}$  with  $s > 1$  is realizable for  $\{V_1\}$ , since  $V_1$  is Henselian if and only if  $V_1$  has a unique extension to each finite algebraic extension field of its quotient field  $F$ , cf. [N2, (43.12)]. It follows from Theorem 2.1(ii) that if  $V$  is a Henselian DVR, then  $V$  is the unique DVR with quotient field  $F$ . It is not true, however, that  $V$  being the unique DVR on its quotient field implies that  $V$  is Henselian. For example, using that the field  $\mathbb{Q}$  of rational numbers admits only countably many DVRs, it is possible to repeatedly use Theorem 2.1 to construct an infinite algebraic extension field  $F$  of  $\mathbb{Q}$  such that  $F$  admits a unique DVR  $V$  having quotient field  $F$  and yet  $V$  is not Henselian.

**(2.2.2)** Related to (2.2.1), it is shown in [R, Theorem 1] that, for each positive integer  $n$ , there exist fields  $F_n$  that admit exactly  $n$  DVRs  $(V_1, N_1), \dots, (V_n, N_n)$  having quotient field  $F_n$ . Moreover, the proof of [R, Theorem 1] shows that such  $F_n$  can be chosen so that there are no realizable  $m$ -consistent systems  $S$  for  $\{V_1, \dots, V_n\}$  having the property that  $m > 1$ , and, for each  $i = 1, \dots, n$ ,  $S(V_i) = \{(K_{i,j}, f_{i,j}, e_{i,j}) \mid j = 1, \dots, s_i\}$  has at least one  $j$  with  $(K_{i,j}, f_{i,j}, e_{i,j}) = (V_i/N_i, 1, 1)$ .

The following result given in [HRR3, Theorem 2.3] is a sufficient condition for realizability; by Remark 2.2.1, the hypothesis  $n > 1$  in Theorem 2.3 is essential. The proof illustrates the method of “composing” realizable systems used in [HRR2], [HRR3].

**Theorem 2.3.** *Let  $(V_1, N_1), \dots, (V_n, N_n)$ ,  $n > 1$ , be distinct DVRs with quotient field  $F$ , let  $m > 1$  be a positive integer, and let*

$$S = \{S(V_1), \dots, S(V_n)\}$$

*be an arbitrary  $m$ -consistent system for  $\{V_1, \dots, V_n\}$ , where*

$$S(V_i) = \{(K_{i,j}, f_{i,j}, e_{i,j}) \mid j = 1, \dots, s_i\},$$

*for each  $i = 1, \dots, n$ . Then  $S^* = \{S^*(V_1), \dots, S^*(V_n)\}$  is a realizable  $m^2$ -consistent system for  $\{V_1, \dots, V_n\}$ , where*

$$S^*(V_i) = \{(K_{i,j}, f_{i,j}, me_{i,j}) \mid j = 1, \dots, s_i\},$$

*for each  $i = 1, \dots, n$ .*

*Proof.* If  $s_i = 1$  for some  $i = 1, \dots, n$ , then Theorem 2.1(i) implies that  $S$  is a realizable  $m$ -consistent system and  $S^*$  is a realizable  $m^2$ -consistent system for  $\{V_1, \dots, V_n\}$ , so it may be assumed that  $s_i > 1$  for each  $i = 1, \dots, n$ .

Define  $S_1(V_i) = S(V_i)$  for  $i = 1, \dots, n - 1$  and  $S_1(V_n) = \{((V_n/N_n), 1, m)\}$ , and recall that  $n > 1$ . Theorem 2.1(i) implies that  $S_1 = \{S_1(V_1), \dots, S_1(V_n)\}$  is a realizable  $m$ -consistent system for  $\{V_1, \dots, V_n\}$ . Let  $L_1$  be a realization of  $S_1$  for  $\{V_1, \dots, V_n\}$ . Thus,  $L_1$  is a separable algebraic extension field of  $F$  of degree  $m$ . For  $i = 1, \dots, n$  let  $(W_{i,j}, N_{i,j})$  be the valuation rings of  $L_1$  that lie over  $V_i$ . It follows from the prescription of  $S_1$  that there are exactly  $s_i$  such rings for  $i = 1, \dots, n - 1$  and exactly



one such ring for  $i = n$ . Also,  $W_{i,j}/N_{i,j}$  is  $(V_i/N_i)$ -isomorphic to  $K_{i,j}$  and  $N_i W_{i,j} = N_{i,j} e_{i,j}$  for  $i = 1, \dots, n - 1$  and  $j = 1, \dots, s_i$ , while  $W_{n,1}/N_{n,1}$  is  $(V_n/N_n)$ -isomorphic to  $V_n/N_n$  and  $N_n W_{n,1} = N_{n,1}^m$ .

Let  $S_2 = \{S_2(W_{1,1}), \dots, S_2(W_{n-1,s_{n-1}}), S_2(W_{n,1})\}$ , where  $S_2(W_{i,j}) = \{(K_{i,j}, 1, m)\}$  for  $i = 1, \dots, n - 1$  and  $j = 1, \dots, s_i$ , and where  $S_2(W_{n,1}) = \{(K_{n,j}, f_{n,j}, e_{n,j}) \mid j = 1, \dots, s_n\}$ . Thus,  $S_2(W_{n,1})$  is essentially equal to  $S(V_n)$ . It is readily checked that  $S_2$  is an  $m$ -consistent system for  $\mathcal{W} := \{W_{1,1}, \dots, W_{n-1,s_{n-1}}, W_{n,1}\}$ , and by Theorem 2.1(i) it is realizable for  $\mathcal{W}$ . Let  $L$  be a realization of  $S_2$  for  $\mathcal{W}$ . Thus,  $L$  is a separable algebraic extension field of  $L_1$  of degree  $m$ , and hence a separable algebraic extension field of  $F$  of degree  $m^2$ . Moreover, for  $i = 1, \dots, n - 1$  and  $j = 1, \dots, s_i$  there exists a unique valuation ring  $(U_{i,j}, P_{i,j})$  of  $L$  that lies over  $W_{i,j}$ , and  $U_{i,j}/P_{i,j}$  is  $(W_{i,j}/N_{i,j})$ -isomorphic to  $W_{i,j}/N_{i,j}$ ; also,  $W_{i,j}/N_{i,j}$  is  $(V_i/N_i)$ -isomorphic to  $K_{i,j}$ , so  $U_{i,j}/P_{i,j}$  is  $(V_i/N_i)$ -isomorphic to  $K_{i,j}$ , and  $N_{i,j} U_{i,j} = P_{i,j}^m$ , so  $N_i U_{i,j} = N_{i,j} m e_{i,j}$ . On the other hand, for  $i = n$  there are exactly  $s_n$  valuation rings  $(U_{n,j}, P_{n,j})$  that lie over  $(W_{n,1}, N_{n,1})$ , and for  $j = 1, \dots, s_n$ ,  $U_{n,j}/P_{n,j}$  is  $(W_{n,1}/N_{n,1})$ -isomorphic to  $K_{n,j}$ , and  $W_{n,1}/N_{n,1}$  is  $(V_n/N_n)$ -isomorphic to  $V_n/N_n$ , so  $U_{n,j}/P_{n,j}$  is  $(V_n/N_n)$ -isomorphic to  $K_{n,j}$ , and  $N_{n,1} U_{n,j} = P_{n,j} e_{n,j}$ , so  $N_n U_{n,j} = P_{n,j} m e_{n,j}$ . It therefore follows that  $L$  is a realization of the  $m^2$ -consistent system  $S^* = \{S^*(V_1), \dots, S^*(V_n)\}$  for  $\{V_1, \dots, V_n\}$ , where  $S^*(V_i) = \{(K_{i,j}, f_{i,j}, m e_{i,j}) \mid j = 1, \dots, s_i\}$  for  $i = 1, \dots, n$ . Thus  $S^*$  is a realizable  $m^2$ -consistent system for  $\{V_1, \dots, V_n\}$ .  $\square$

**Corollary 2.4.** *Let  $R$  be a Noetherian domain, let  $I$  be a nonzero proper ideal in  $R$ , let  $(V_1, N_1), \dots, (V_n, N_n)$ ,  $n > 1$ , be the Rees valuation rings of  $I$ , let  $m, s_1, \dots, s_n$  be positive integers, and let  $S = \{S(V_1), \dots, S(V_n)\}$  be an arbitrary  $m$ -consistent system for  $\{V_1, \dots, V_n\}$ , say  $S(V_i) = \{(K_{i,j}, f_{i,j}, e_{i,j}) \mid j = 1, \dots, s_i\}$  for  $i = 1, \dots, n$ . Then there exists a separable algebraic extension field  $L$  of degree  $m^2$  of the quotient field of  $R$  such that, for each finite integral extension domain  $A$  of  $R$  with quotient field  $L$  and for  $i = 1, \dots, n$ ,  $IA$  has exactly  $s_i$  Rees valuation rings  $(W_{i,j}, N_{i,j})$  that extend  $(V_i, N_i)$ , and then, for  $j = 1, \dots, s_i$ , the Rees integer of  $IA$  with respect to  $W_{i,j}$  is  $m e_{i,j} e_i$  and  $[(W_{i,j}/N_{i,j}) : (V_i/N_i)] = f_{i,j}$ .*

*Proof.* By [HRR2, Remark 2.7], the extensions of the Rees valuation rings of  $I$  to the field  $L$  are the Rees valuation rings of  $IA$ , so Corollary 2.4 follows immediately from Theorem 2.3.  $\square$

Theorem 2.6, given in [HRR3, Theorem 2.7], is a sufficient condition for realizability under the hypothesis that each of the valuation rings  $(V_i, N_i)$ ,  $1 \leq i \leq n$ , has a finite residue field. For this and other results using the hypothesis that the residue fields are finite, we use the following remark.

**Remarks 2.5. (2.5.1)** Let  $F$  be a finite field. It is well known, see for example [ZS1, pp. 82–84], that the following hold: (i) Each finite extension field  $H$  of  $F$  is separable and thus a simple extension of  $F$ . (ii) If  $k$  is a positive integer and  $\overline{F}$  is a fixed algebraic closure of  $F$ , then there exists a unique extension field  $H \subseteq \overline{F}$  with  $[H : F] = k$ . (iii) If  $H, K \subseteq \overline{F}$  are finite extension fields of  $F$ , then  $H \subseteq K$  if and only if  $[H : F]$  divides  $[K : F]$ .

**(2.5.2)** There are fields other than finite fields that satisfy the three conditions given in (2.5.1). If  $E$  is an algebraically closed field of characteristic zero and  $F$  is the field

of fractions of the formal power series ring  $E[[x]]$ , then a theorem that goes back to Newton implies that  $F$  satisfies the conditions of (2.5.1) cf. [W, Theorem 3.1, p. 98].

**Theorem 2.6.** *Let  $(V_1, N_1), \dots, (V_n, N_n)$ ,  $n > 1$ , be distinct DVRs with quotient field  $F$ , where each  $V_i/N_i$  is finite. For each  $i$  let  $\overline{V_i/N_i}$  denote a fixed algebraic closure of  $V_i/N_i$ . Let  $m$  be a positive integer, and let  $S = \{S(V_1), \dots, S(V_n)\}$  be an arbitrary  $m$ -consistent system for  $\{V_1, \dots, V_n\}$ , where, for  $i = 1, \dots, n$ ,  $S(V_i) = \{(K_{i,j}, f_{i,j}, e_{i,j}) \mid K_{i,j} \subseteq \overline{V_i/N_i} \text{ and } j = 1, \dots, s_i\}$ . For  $i = 1, \dots, n$  let  $T^*(V_i) = \{(K_{i,j}^*, mf_{i,j}, e_{i,j}) \mid j = 1, \dots, s_i\}$ , where  $K_{i,j}^* \subseteq \overline{V_i/N_i}$  is the unique field extension of  $K_{i,j}$  with  $[K_{i,j}^* : K_{i,j}] = m$ . Then  $T^* = \{T^*(V_1), \dots, T^*(V_n)\}$  is a realizable  $m^2$ -consistent system for  $\{V_1, \dots, V_n\}$ .*

**Remark 2.7.** The hypothesis in Theorem 2.6 that each  $K_i = V_i/N_i$  is finite is often not essential. Specifically, if the set of extension fields of the  $K_i$  have the following properties (a)–(c), then it follows from the proof of Theorem 2.6 that the conclusion holds, even though the  $K_i$  are not finite:

- (a) For  $i = 1, \dots, n$  and  $j = 1, \dots, s_i$  there exists a field  $K_{i,j}^*$  such that  $[K_{i,j}^* : K_{i,j}] = m$ .
- (b) Each  $K_{i,j}^*$  is a simple extension of  $K_i$ .
- (c) There exists  $i \in \{1, \dots, n\}$  (say  $i = n$ ) such that there exists a simple extension field  $H_n$  of  $K_n$  of degree  $m$  such that  $H_n \subseteq K_{n,j}^*$  for  $j = 1, \dots, s_n$  (so  $[K_{n,j}^* : H_n] = f_{n,j}$  for  $j = 1, \dots, s_n$ ).

**Corollary 2.8.** *Let  $R$  be a Noetherian domain, let  $I$  be a nonzero proper ideal in  $R$ , let  $(V_1, N_1), \dots, (V_n, N_n)$ ,  $n > 1$ , be the Rees valuation rings of  $I$ , let  $m, s_1, \dots, s_n$  be positive integers, and let  $S = \{S(V_1), \dots, S(V_n)\}$  be an arbitrary  $m$ -consistent system for  $V_1, \dots, V_n$ , say  $S(V_i) = \{(K_{i,j}, f_{i,j}, e_{i,j}) \mid j = 1, \dots, s_i\}$  for  $i = 1, \dots, n$ . Assume that each  $V_i/N_i$  is finite. Then there exists a separable algebraic extension field  $L$  of  $R_{(0)}$  of degree  $m^2$  such that, for each finite integral extension domain  $A$  of  $R$  with quotient field  $L$  and for  $i = 1, \dots, n$ ,  $IA$  has exactly  $s_i$  Rees valuation rings  $(W_{i,j}, N_{i,j})$  lying over  $V_i$ , and then, for  $j = 1, \dots, s_i$ , the Rees integer of  $IA$  with respect to  $W_{i,j}$  is  $e_{i,j}e_i$  and  $[(W_{i,j}/N_{i,j}) : (V_i/N_i)] = mf_{i,j}$ .*

*Proof.* As in the proof of Corollary 2.4, this follows immediately from Theorem 2.6.  $\square$

### 3 Radical-power ideals

We use the following notation and terminology.

**Notation 3.1.** Let  $D$  be a Dedekind domain with quotient field  $F \neq D$ , let  $M_1, \dots, M_n$  be distinct maximal ideals of  $D$ , and let  $I = M_1^{e_1} \cdots M_n^{e_n}$  be an ideal in  $D$ , where  $e_1, \dots, e_n$  are positive integers. Then:

**(3.1.1)** For each finite integral extension domain  $A$  of  $D$  (including  $D$ ) let  $\mathbf{M}_I(A) = \{N \mid N \text{ is a maximal ideal in } A \text{ and } N \cap D \in \{M_1, \dots, M_n\}\}$ .

**(3.1.2)** Let  $E$  be a finite integral extension Dedekind domain of  $D$  and let  $\mathbf{V} = \{E_N \mid N \in \mathbf{M}_f(E)\}$ . If  $S$  is an  $m$ -consistent system for  $\mathbf{V}$ , then by abuse of terminology we sometimes say that  $S$  is an  $m$ -consistent system for  $\mathbf{M}_f(E)$ , and when  $N \in \mathbf{M}_f(E)$  we sometimes use  $S(N)$  in place of  $S(E_N)$ .

**Remarks 3.2.** With the notation of (3.1), let  $S = \{S(M_1), \dots, S(M_n)\}$  be a realizable  $m$ -consistent system for  $\mathbf{M}_f(D)$ , where  $S(M_i) = \{(K_{i,j}, f_{i,j}, e_{i,j}) \mid j = 1, \dots, s_i\}$  for  $i = 1, \dots, n$ . Let  $L$  be a field that realizes  $S$  for  $\mathbf{M}_f(D)$  and let  $E$  be the integral closure of  $D$  in  $L$ . Then:

**(3.2.1)**  $[L : F] = m$ , and  $L$  has distinct DVRs  $(V_{i,1}, N_{i,1}), \dots, (V_{i,s_i}, N_{i,s_i})$  such that for each  $i, j$ :  $V_{i,j} \cap F = D_{M_i}$ ;  $V_{i,j}/N_{i,j}$  is  $D/M_i$ -isomorphic to  $K_{i,j}$ ;  $[K_{i,j} : K_i] = f_{i,j}$ , where  $K_i = D/M_i$ ; and,  $M_i V_{i,j} = N_{i,j}^{e_{i,j}}$ . Also, for  $i = 1, \dots, n$ ,  $V_{i,1}, \dots, V_{i,s_i}$  are all of the extensions of  $D_{M_i}$  to  $L$ , so  $\mathbf{M}_f(E) = \{N_{i,j} \cap E \mid i = 1, \dots, n \text{ and } j = 1, \dots, s_i\}$ .

**(3.2.2)**  $E$  is a Dedekind domain that is a finite separable integral extension domain of  $D$ , and  $IE = M_1^{e_1} \dots M_n^{e_n} E = P_{1,1}^{e_1 e_{1,1}} \dots P_{n,s_n}^{e_n e_{n,s_n}}$ , where  $P_{i,j} = N_{i,j} \cap E$  for  $i = 1, \dots, n$  and  $j = 1, \dots, s_i$ .

Theorem 3.3 is proved in [HRR2, (2.11.1)], by composing  $n$  related consistent systems. In [HRR3] a different proof is given which suggests the proof of the analogous “finite-residue-field degree” result, Theorem 4.1. Notice that Theorem 3.3 shows that every ideal  $I$  as in Notation 3.1 extends to a radical-power ideal in some finite integral extension Dedekind domain.

**Theorem 3.3.** *With the notation of (3.1) and (3.2), assume that  $n > 1$ . Then the system  $S = \{S(M_1), \dots, S(M_n)\}$  is a realizable  $e_1 \dots e_n$ -consistent system for  $\mathbf{M}_f(D)$ , where, for  $i = 1, \dots, n$ ,  $S(M_i) = \{(K_{i,j}, 1, \frac{e_1 \dots e_n}{e_i}) \mid j = 1, \dots, e_i\}$ . Therefore, there exists a Dedekind domain  $E$  that is a finite separable integral extension domain of  $D$  such that  $[L : F] = e_1 \dots e_n$ , where  $L$  (resp.,  $F$ ) is the quotient field of  $E$  (resp.,  $D$ ), and, for  $i = 1, \dots, n$ , there exist exactly  $e_i$  maximal ideals  $N_{i,1}, \dots, N_{i,e_i}$  in  $E$  that lie over  $M_i$  and, for  $j = 1, \dots, e_i$ ,  $[(E/N_{i,j}) : (D/M_i)] = 1$  and  $M_i E_{N_{i,j}} = N_{i,j}^{\frac{e_1 \dots e_n}{e_i}} E_{N_{i,j}}$ , so  $IE = (\text{Rad}(IE))^{e_1 \dots e_n}$ .*

**Corollary 3.4.** *Let  $I$  be a nonzero proper ideal in a Dedekind domain  $D$ . Then there exists a finite separable integral extension Dedekind domain  $E$  of  $D$  such that  $IE = (\text{Rad}(IE))^m$  for some positive integer  $m$ .*

*Proof.* Let  $I = M_1^{e_1} \cap \dots \cap M_n^{e_n}$  be an irredundant primary decomposition of  $I$ . If  $n = 1$ , then  $I = M_1^{e_1} = (\text{Rad}(I))^{e_1}$ , so the conclusion holds with  $E = D$  and  $m = e_1$ . If  $n > 1$ , then the conclusion follows immediately from Theorem 3.3, since  $I = M_1^{e_1} \cap \dots \cap M_n^{e_n} = M_1^{e_1} \dots M_n^{e_n}$ .  $\square$

**Corollary 3.5.** *Let  $k = \pi_1^{e_1} \dots \pi_n^{e_n}$  be the factorization of the positive integer  $k > 1$  as a product of distinct prime integers  $\pi_i$ . Then there exists an extension field  $L$  of  $\mathbb{Q}$  of degree  $e_1 \dots e_n$  such that  $kE = [\prod_{i=1}^n (\prod_{j=1}^{e_i} P_{i,j})]^{e_1 \dots e_n}$ , where  $E$  is the integral closure of  $\mathbb{Z}$  in  $L$  and  $\mathbf{M}_{k\mathbb{Z}}(E) = \{p_{1,1}, \dots, p_{n,e_n}\}$ .*

Remark 3.6 shows that  $I$  sometimes extends to a radical power ideal in a simpler realizable consistent system.

**Remark 3.6.** With the notation of (3.1) and (3.2), assume<sup>2</sup> that, for  $i = 1, \dots, n$ , there exists a simple algebraic extension field  $K_i^{(1)}$  of  $D/M_i$  such that  $[K_i^{(1)} : (D/M_i)] = e_i$ . Then the system  $S^{(1)} = \{S^{(1)}(M_1), \dots, S^{(1)}(M_n)\}$ , where  $S^{(1)}(M_i) = \{(K_i^{(1)}, e_i, \frac{e_1 \cdots e_n}{e_i})\}$  for  $i = 1, \dots, n$ , is an  $e_1 \cdots e_n$ -consistent system for  $\mathbf{M}_I(D)$ . By Theorem 2.1(i), it is realizable for  $\mathbf{M}_I(D)$ . Also, if  $E$  is the integral closure of  $D$  in a realization  $L$  of  $S^{(1)}$  for  $\mathbf{M}_I(D)$ , then  $IE = J^{e_1 \cdots e_n}$ , where  $J = \text{Rad}(IE)$ . More specifically, since  $E$  is the integral closure of  $D$  in a realization  $L$  of  $S^{(1)}$  for  $\mathbf{M}_I(D)$ , for  $i = 1, \dots, n$ , there exists a unique maximal ideal  $N_i$  in  $E$  that lies over  $M_i$ , and then  $E/N_i \cong K_i^{(1)}$  and  $M_i E_{N_i} = N_i \frac{e_1 \cdots e_n}{e_i} E_{N_i}$ , so  $M_i E = N_i \frac{e_1 \cdots e_n}{e_i}$ , so  $IE = (\prod_{i=1}^n M_i^{e_i})E = \prod_{i=1}^n (N_i \frac{e_1 \cdots e_n}{e_i})^{e_i} = J^{e_1 \cdots e_n}$ , where  $J = N_1 \cdots N_n$ .

**Remark 3.7.** Let  $V_i = D_{M_i}$  and  $S = \{S(V_1), \dots, S(V_n)\}$  be an arbitrary  $m$ -consistent system for  $\mathbf{M}_I(D) = \{M_1, \dots, M_n\}$ , where, for  $i = 1, \dots, n$ ,  $S(V_i) = \{(K_{i,j}, f_{i,j}, e_{i,j}) \mid j = 1, \dots, s_i\}$ . If we consider the  $s_i$ ,  $K_{i,j}$ , and  $f_{i,j}$  as fixed in the  $m$ -consistent system for  $\mathbf{M}_I(D)$  and the  $e_{i,j}$  as variables subject to the constraint  $\sum_{j=1}^{s_i} e_{i,j} f_{i,j} = m$  for each  $i$ , then  $S$  gives a map  $\mathbb{N}_+^n \rightarrow \mathbb{N}_+^t$  (where  $t = \sum_{i=1}^n s_i$ ) defined by

$$(e_1, \dots, e_n) \mapsto (e_1 e_{1,1}, \dots, e_1 e_{1,s_1}, \dots, e_n e_{n,1}, \dots, e_n e_{n,s_n}).$$

If we are interested only in the projective equivalence class of  $IE$ , it seems appropriate to consider the induced map given by  $S : \mathbb{N}_+^n \rightarrow \mathbf{P}^t(\mathbb{N}_+) = \mathbb{N}_+^t / \sim$ , where  $(a_1, \dots, a_t) \sim (b_1, \dots, b_t)$  if  $(a_1, \dots, a_t) = (cb_1, \dots, cb_t)$  for some  $c \in \mathbb{Q}$ . In this case, Theorem 2.3 shows that the equations  $\sum_{j=1}^{s_i} e_{i,j} f_{i,j} = m$  are the only restrictions on the image of this map into  $\mathbf{P}^t(\mathbb{N}_+)$ . From this point of view, if we want an equation  $IE = (\text{Rad}(IE))^k$  for some finite separable integral extension Dedekind domain  $E$  of  $D$  and for some positive integer  $k$ , then it is not necessary to compose two realizable consistent systems, as in the proof of Theorem 3.3. Indeed, it suffices to observe that we have an  $m$ -consistent system  $S = \{S(M_1), \dots, S(M_n)\}$ , where  $m = e_1 \cdots e_n$  and  $S(M_i) = \{(K_{i,j}, 1, \frac{e_1 \cdots e_n}{e_i}) \mid j = 1, \dots, e_i\}$  for  $i = 1, \dots, n$  (realizable or not), and then apply Theorem 2.3.

Theorem 3.3 extends to ideals in Noetherian domains of altitude one by using the following result from [HRR2].

**Proposition 3.8.** [HRR2, 2.6] *Let  $R$  be a Noetherian domain of altitude one with quotient field  $F$ , let  $I$  be a nonzero proper ideal in  $R$ , let  $L$  be a finite algebraic extension field of  $F$ , let  $E$  be the integral closure of  $R$  in  $L$ , and assume there exist distinct maximal ideals  $N_1, \dots, N_n$  of  $E$  and positive integers  $k_1, \dots, k_n, h$  such*

<sup>2</sup>  $D$  may have a residue field  $D/M_i$  that has no extension field  $K_i^{(1)}$  such that  $[K_i^{(1)} : (D/M_i)] = e_i$ ; for example,  $D/M_i$  may be algebraically closed, see also Example 3 in [R].

that  $IE = (N_1^{k_1} \cdots N_n^{k_n})^h$ . Then there exists a finite integral extension domain  $A$  of  $R$  with quotient field  $L$  and distinct maximal ideals  $P_1, \dots, P_n$  of  $A$  such that, for  $i = 1, \dots, n$ :

- (i)  $P_i E = N_i$ .
- (ii)  $E/N_i \cong A/P_i$ .
- (iii)  $(IA)_a = ((P_1^{k_1} \cdots P_n^{k_n})^h)_a$ .

**Corollary 3.9.** *Let  $R$  be a Noetherian domain of altitude one, let  $I$  be a nonzero proper ideal in  $R$ , let  $R'$  be the integral closure of  $R$  in its quotient field, and let  $IR' = M_1^{e_1} \cdots M_n^{e_n}$  be a normal primary decomposition of  $IR'$ . Then there exists a finite separable integral extension domain  $A$  of  $R$  such that  $(IA)_a = ((\text{Rad}(IA))^{e_1 \cdots e_n})_a$ , and if  $A'$  denotes the integral closure of  $A$  in its quotient field, then for each  $P \in \mathbf{M}_I(A)$  we have: (i)  $PA'$  is a maximal ideal, and (ii)  $A'/PA' \cong A/P$ .*

*Proof.* If  $n = 1$ , then  $IR' = (\text{Rad}(IR'))^{e_1}$  and  $R'$  is a Dedekind domain, so the conclusion follows from Proposition 3.8.

If  $n > 1$ , then by hypothesis there are exactly  $n$  distinct maximal ideals  $M_1, \dots, M_n$  in  $R'$  that contain  $IR'$  and  $IR' = M_1^{e_1} \cdots M_n^{e_n}$ . Also,  $R'$  is a Dedekind domain, so by Theorem 3.3 there exists a finite separable integral extension Dedekind domain  $E$  of  $R'$  such that  $IE = (\text{Rad}(IE))^{e_1 \cdots e_n}$ . Then  $E$  is the integral closure of  $R$  in the quotient field of  $E$ ; the conclusions follow from this, together with Proposition 3.8.  $\square$

When the exponents  $e_1, \dots, e_n$  have no common integer prime divisors, Proposition 3.10 gives an additional way to compose realizable consistent systems to obtain a Dedekind domain  $E$  as in Theorem 3.3, but with the exponent and degree  $e_1 \cdots e_n$  of Theorem 3.3 replaced with a smaller exponent and degree  $d$ . This result is discussed in [HRR2, (2.11.2)], and [HRR3, (3.11)], and it yields corresponding different versions of Corollaries 3.5 and 3.9. (When the exponents  $e_1, \dots, e_n$  do have common integer prime divisors, see Remark 3.11.)

**Proposition 3.10.** *With the notation of (3.1) and (3.2), assume that  $n > 1$  and that no prime integer divides each  $e_i$ . Let  $d = p_1^{m_1} \cdots p_k^{m_k}$  be the least common multiple of  $e_1, \dots, e_n$ , where  $p_1, \dots, p_k$  are distinct prime integers and  $m_1, \dots, m_k$  are positive integers. Then the system  $\mathbf{S} = \{\mathbf{S}(M_1), \dots, \mathbf{S}(M_n)\}$  for  $\mathbf{M}_I(D)$ , where, for  $i = 1, \dots, n$ ,  $\mathbf{S}(M_i) = \{(K_{i,j}, 1, \frac{d}{e_i}) \mid j = 1, \dots, e_i\}$ , is a realizable  $d$ -consistent system for  $\mathbf{M}_I(D)$ . Also, if  $E$  is the integral closure of  $D$  in a realization  $L$  of  $\mathbf{S}$  for  $\mathbf{M}_I(D)$ , then  $IE = (\text{Rad}(IE))^d$ .*

**Remark 3.11.** Concerning the hypothesis in Proposition 3.10 that no prime integer divides all  $e_i$ , if, on the contrary,  $\pi$  is a prime integer that divides each  $e_i$ , then let  $c$  be the greatest common divisor of  $e_1, \dots, e_n$ . For  $i = 1, \dots, n$  define  $k_i$  by  $e_i = ck_i$ , and let  $I_0 = M_1^{k_1} \cdots M_n^{k_n}$ , so  $I_0^c = (\prod_{i=1}^n M_i^{k_i})^c = \prod_{i=1}^n M_i^{e_i} = I$  and no prime integer divides all  $k_i$ . Therefore, if the ring  $E$  of Theorem 3.3 is constructed for  $I_0$  in place of  $I$ , then  $I_0 E = (\text{Rad}(I_0 E))^d$ , where  $d$  is the least common multiple of  $k_1, \dots, k_n$ , so  $IE = (\text{Rad}(IE))^{dc}$ .

The following result, which is [HRR3, Proposition 3.13], characterizes the conditions a realizable  $m$ -consistent system  $S'$  for  $\mathbf{M}_I(D)$  must satisfy in order that  $IE = J^t$  for some radical ideal  $J$  in  $E$  and for some positive integer  $t$ .

**Proposition 3.12.** *Let  $D$  be a Dedekind domain with quotient field  $F \neq D$ , let  $M_1, \dots, M_n$  ( $n > 1$ ) be distinct maximal ideals of  $D$ , let  $I = M_1^{e_1} \dots M_n^{e_n}$  be an ideal in  $D$ , where  $e_1, \dots, e_n$  are positive integers, and let  $m$  be a positive integer. Let  $S' = \{S'(M_1), \dots, S'(M_n)\}$  be a realizable  $m$ -consistent system for  $\{D_{M_1}, \dots, D_{M_n}\}$ , where  $S'(M_i) = \{(K_{i,j}, f_{i,j}, e_{i,j}) \mid j = 1, \dots, s_i\}$  for  $i = 1, \dots, n$ , and let  $E$  be the integral closure of  $D$  in a finite separable field extension  $L$  of  $F$  which realizes  $S'$  for  $\{D_{M_1}, \dots, D_{M_n}\}$ , so  $[L : F] = m$ . Then the following hold:*

**(3.12.1)**  *$IE = J^t$  for some radical ideal  $J$  in  $E$  and for some positive integer  $t$  if and only if the products  $e_i e_{i,j}$  are equal for all  $i, j$ , and then  $J = \text{Rad}(IE)$  and  $e_i e_{i,j} = t$ .*

**(3.12.2)** *If  $IE = J^m$  (as in Theorem 3.3 and Proposition 3.10), then  $\sum_{j=1}^{s_i} f_{i,j} = e_i$  for  $i = 1, \dots, n$ .*

**(3.12.3)** *If  $IE = J^t$ , as in (3.12.1), and if no prime integer divides each  $e_i$ , then  $m$  is a positive multiple of  $t$  and  $t$  (and hence  $m$ ) is a positive multiple of each  $e_i$ .*

## 4 Finite-residue-field degree analogues

Under the assumption that each of the residue fields  $D/M_i$  is finite, “finite-residue-field degree” analogues of results in Section 3 are given in [HRR3]. For example, Theorem 4.1, which is [HRR3, Theorem 4.1], is a finite-residue-field degree analogue of Theorem 3.3.

**Theorem 4.1.** *With the notation of (3.1) and (3.2), assume that  $n > 1$  and that each  $K_i = D/M_i$  is finite. For  $i = 1, \dots, n$  let  $f_i$  be a positive integer such that  $[K_i : F_i] = f_i$  for some subfield  $F_i$  of  $K_i$ , and let  $K_i' \subseteq \overline{K}_i$  be the unique extension field of  $K_i$  of degree  $\frac{f_1 \dots f_n}{f_i}$ , where  $\overline{K}_i$  is a fixed algebraic closure of  $K_i$ . Then the system  $T = \{T(M_1), \dots, T(M_n)\}$  is a realizable  $m$ -consistent system for  $\mathbf{M}_I(D)$ , where  $m = f_1 \dots f_n$  and  $T(M_i) = \{(K_{i,j}, \frac{f_1 \dots f_n}{f_i}, 1) \mid j = 1, \dots, f_i\}$  for  $i = 1, \dots, n$  (with  $K_{i,j} = K_i'$  for  $j = 1, \dots, f_i$ ). Therefore there exists a Dedekind domain  $E$  that is a finite separable integral extension domain of  $D$  such that  $[L : F] = m$  (where  $L$  (resp.,  $F$ ) is the quotient field of  $E$  (resp.,  $D$ )) and, for  $i = 1, \dots, n$ , there exist exactly  $f_i$  maximal ideals  $N_{i,1}, \dots, N_{i,f_i}$  in  $E$  that lie over  $M_i$  and, for  $j = 1, \dots, f_i$ ,  $M_i E_{N_{i,j}} = N_{i,j} E_{N_{i,j}}$  and  $[(E/N_{i,j}) : K_i] = \frac{f_1 \dots f_n}{f_i}$ , so  $[(E/N_{i,j}) : F_i] = m$ .*

**Remark 4.2.** The hypothesis in Theorem 4.1 that each  $K_i = D_i/M_i$  is finite is often not essential. Specifically, if the set of extension fields of the  $K_i$  have the following properties (a)–(c), then it follows from the proof of Theorem 4.1 that the conclusion holds, even though the  $K_i$  are not finite:

- (a) For  $i = 1, \dots, n$ ,  $K_i$  has a subfield  $F_i$  such that  $[K_i : F_i] = f_i$ .

- (b) With  $m = f_1 \cdots f_n$ , for  $i = 1, \dots, n$   $K_i$  has (not necessarily distinct) simple extension fields  $K_{i,1}, \dots, K_{i,f_i}$  such that  $[K_{i,j} : K_i] = \frac{m}{f_i}$ .
- (c) For  $i = 1, \dots, n - 1$ ,  $K_i$  has simple extension fields  $H_{i,j}$  such that  $[H_{i,j} : K_i] = \frac{f_1 \cdots f_{n-1}}{f_i}$  and such that  $H_{i,j} \subseteq K_{i,j}$  (so  $[K_{i,j} : H_{i,j}] = f_n$ ).

Corollary 4.3 is a special case of Theorem 4.1; it is a finite-residue-field degree analogue of Corollary 3.5.

**Corollary 4.3.** *Let  $D$  be the ring of integers of an algebraic number field  $F$  and let  $M_1, \dots, M_n$  ( $n > 1$ ) be distinct maximal ideals in  $D$ . For  $i = 1, \dots, n$  let  $\mathbb{Z}/\pi_i\mathbb{Z}$  be the prime subfield of  $D/M_i$  (possibly  $\pi_i = \pi_j$  for some  $i \neq j \in \{1, \dots, n\}$ ) and let  $f_i = [(D/M_i) : (\mathbb{Z}/\pi_i\mathbb{Z})]$ . Then there exists a Dedekind domain  $E$  that is a finite (separable) integral extension domain of  $D$  such that, for  $i = 1, \dots, n$ , there exist exactly  $f_i$  maximal ideals  $p_{i,j}$  in  $E$  that lie over  $M_i$ , and then, for  $j = 1, \dots, f_i$ ,  $M_i E_{p_{i,j}} = p_{i,j} E_{p_{i,j}}$  and  $[(E/p_{i,j}) : (\mathbb{Z}/\pi_i\mathbb{Z})] = f_1 \cdots f_n$ .*

*Proof.* This follows immediately from Theorem 4.1.

Corollary 4.4 is a finite-residue-field degree analogue of Corollary 3.9. Since hypotheses on infinite residue fields can sometimes be replaced by the hypotheses that the residue fields have cardinality greater than or equal to a given positive integer, Corollary 4.4 may be useful in this regard.

**Corollary 4.4.** *Let  $R$  be a Noetherian domain of altitude one, let  $I$  be a nonzero proper ideal in  $R$ , let  $R'$  be the integral closure of  $R$  in its quotient field, let  $IR' = M_1^{e_1} \cdots M_n^{e_n}$  ( $n > 1$ ) be a normal primary decomposition of  $IR'$ , and for  $i = 1, \dots, n$  let  $[(R'/M_i) : (R/(M_i \cap R))] = g_i$ . For  $i = 1, \dots, n$ , assume that  $R'/M_i$  is finite, let  $f_i$  be a positive integer, and assume that  $[(R/(M_i \cap R)) : F_i] = f_i$ , where  $F_i$  is a subfield of  $R/(M_i \cap R)$ . Then there exists a finite separable integral extension domain  $A$  of  $R$  such that, for all  $P \in \mathbf{M}_I(A)$ ,  $[(A/P) : F_i] = \prod_{i=1}^n f_i g_i = [A_{(0)} : R_{(0)}]$ . Also,  $A$  may be chosen so that, with  $A'$  the integral closure of  $A$  in  $A_{(0)}$ , there exist exactly  $f_i g_i$  maximal ideals  $P_{i,j}$  in  $A$  such that  $P_{i,j} A' \cap R' = M_i$  and, for all  $P \in \mathbf{M}_I(A)$  it holds that  $PA' \in \mathbf{M}_I(A')$  and  $A/P \cong A'/(PA')$ .*

*Proof.* Since  $R'$  is a Dedekind domain and  $[(R'/M_i) : F_i] = f_i g_i$  for  $i = 1, \dots, n$ , it follows from Theorem 4.1 that there exists a Dedekind domain  $E$  that is a finite separable integral extension domain of  $R'$  such that  $[A_{(0)} : R_{(0)}] = \prod_{i=1}^n f_i g_i$  and, for  $i = 1, \dots, n$ , there exist exactly  $f_i g_i$  maximal ideals  $N_{i,1}, \dots, N_{i,f_i g_i}$  in  $E$  that lie over  $M_i$  and, for  $j = 1, \dots, f_i g_i$ ,  $M_i E_{N_{i,j}} = N_{i,j} E_{N_{i,j}}$  and  $[(E/N_{i,j}) : (R'/M_i)] = \frac{f_1 g_1 \cdots f_n g_n}{f_i g_i}$ , so  $[(E/N_{i,j}) : F_i] = \prod_{i=1}^n f_i g_i$ . The conclusions follow from this, together with Proposition 3.8.  $\square$

Theorem 4.1 shows that if each residue field  $D/M_i$  is finite and  $F_i$  is a subfield of  $D/M_i$  such that  $[(D/M_i) : F_i] = f_i$ , then there exists a finite separable integral extension domain  $E$  of  $D$  such that  $[E_{(0)} : D_{(0)}] = [(E/N_{i,j}) : F_i] = f_1 \cdots f_n$  for all  $i, j$  ( $= m$ , say). Proposition 4.5 characterizes the conditions a realizable  $m$ -consistent system  $T'$  for  $\mathbf{M}_I(D)$  must satisfy in order that  $[(E/N_{i,j}) : F_i] = f_1 \cdots f_n$  for all  $i, j$ .

**Proposition 4.5.** *Let  $D$  be a Dedekind domain with quotient field  $F \neq D$ , let  $M_1, \dots, M_n$  ( $n > 1$ ) be distinct maximal ideals of  $D$ , and assume that  $K_i = D/M_i$  is finite for  $i = 1, \dots, n$ . For  $i = 1, \dots, n$  let  $f_i$  be a positive integer such that  $[K_i : F_i] = f_i$  for some subfield  $F_i$  of  $K_i$ . Let  $m$  be a positive integer and let  $T' = \{T'(M_1), \dots, T'(M_n)\}$  be a realizable  $m$ -consistent system for  $\mathbf{M}_I(D)$ , where, for  $i = 1, \dots, n$ ,  $T'(M_i) = \{(K_{i,j}, f_{i,j}, e_{i,j}) \mid j = 1, \dots, s_i\}$ , and let  $E$  be the integral closure of  $D$  in a realization  $L$  of  $T'$  for  $\mathbf{M}_I(D)$ , so  $[L : F] = m$ . Then the following hold:*

**(4.5.1)** *There exists a positive integer  $t$  such that  $[(E/N_{i,j}) : F_i] = t$  for all  $i, j$  if and only if the products  $f_i f_{i,j}$  are equal for all  $i, j$ , and then  $t = f_i f_{i,j}$ .*

**(4.5.2)** *If  $[(E/N_{i,j}) : F_i] = m$  for all  $i, j$  (as in Theorem 4.1), then  $\sum_{j=1}^{s_i} e_{i,j} = f_i$  for  $i = 1, \dots, n$ .*

**(4.5.3)** *If  $[(E/N_{i,j}) : F_i] = t$  for all  $i, j$ , as in (4.5.1), and if no prime integer divides each  $f_i$ , then  $m$  is a positive multiple of  $t$  and  $t$  (and hence  $m$ ) is a positive multiple of each  $f_i$ .*

The following theorem that combines Theorems 3.3 and 4.1 is [HRR3, Theorem 5.1].

**Theorem 4.6.** *With the notation of (3.1) and (3.2) (so  $I = M_1^{e_1} \dots M_n^{e_n}$ , where the  $e_i$  are positive integers and  $n > 1$ ), assume that each  $K_i = D/M_i$  is finite and let  $\bar{K}_i$  be a fixed algebraic closure of  $K_i$ . For  $i = 1, \dots, n$  let  $f_i$  be a positive integer such that  $K_i$  is an extension field of a subfield  $F_i$  with  $[K_i : F_i] = f_i$ , and let  $K_i^*$  be the unique extension field of  $K_i$  of degree  $\frac{f_1 \dots f_n}{f_i}$  that is contained in  $\bar{K}_i$ . Then the system  $U = \{U(M_1), \dots, U(M_n)\}$  is a realizable  $e_1 \dots e_n f_1 \dots f_n$ -consistent system for  $\mathbf{M}_I(D)$ , where, for  $i = 1, \dots, n$ ,  $U(M_i) = \{(K_{i,j}, \frac{f_1 \dots f_n}{f_i}, \frac{e_1 \dots e_n}{e_i}) \mid j = 1, \dots, e_i f_i\}$  (with  $K_{i,j} = K_i^*$  for  $j = 1, \dots, e_i f_i$ ). Therefore, there exists a separable algebraic extension field  $L$  of degree  $e_1 \dots e_n f_1 \dots f_n$  over the quotient field  $F$  of  $D$ , and a finite integral extension Dedekind domain  $E$  of  $D$  with quotient field  $L$  such that, for  $i = 1, \dots, n$ , there are exactly  $e_i f_i$  maximal ideals  $N_{i,1}, \dots, N_{i,e_i f_i}$  in  $E$  that lie over  $M_i$ , and it holds that  $[(E/N_{i,j}) : F_i] = f_1 \dots f_n$  for all  $i$  and  $j$ , and  $IE = (\text{Rad}(IE))^{e_1 \dots e_n} = (N_{1,1} \dots N_{e_n f_n})^{e_1 \dots e_n}$ .*

## 5 Rees-good bases of ideals

We introduce the following terminology in [HRR4].

**Definition 5.1.** Let  $I$  be a regular proper ideal in a Noetherian ring  $R$ . An element  $b \in I$  is said to be *Rees-good* for  $I$  in case  $bV = IV$  for all Rees valuation rings  $V$  of  $I$ . A basis  $b_1, \dots, b_g$  of  $I$  is said to be *Rees-good* in case  $b_i$  is Rees-good for  $I$  for  $i = 1, \dots, g$ .

Thus, assumption (a) of the Introduction is that the ideal  $I$  has a Rees-good basis. We summarize in this section several results concerning Rees-good bases of ideals that are proved in [HRR4].



**Notation 5.2.** Let  $I$  be a regular proper ideal in a Noetherian ring  $R$  and let  $\{(V_i, N_i)\}_{i=1}^n$  be the set of Rees valuation rings of  $I$ . For  $j \in \{1, \dots, n\}$ , let  $H_j = \{x \in I \mid xV_j \subsetneq IV_j\}$ .

**Lemma 5.3.** *With the notation of (5.2), the following hold:*

(5.3.1)  $H_i = H_i V_i \cap I$  is an ideal in  $R$  that is properly contained in  $I$  for  $i = 1, \dots, n$ .

(5.3.2) An element  $b \in I$  is Rees-good for  $I$  if and only if  $b \notin H_1 \cup \dots \cup H_n$ .

(5.3.3) If either  $I$  is principal or  $I$  has only one Rees valuation ring, then  $I$  has a Rees-good basis.

Let  $I$  be a regular proper ideal of the Noetherian ring  $R$ . H. T. Muhly and M. Sakuma prove in [MS, Lemma 3.1] that some power  $I^k$  of  $I$  contains an element  $b$  such that  $bV = I^k V$  for all Rees valuation rings  $V$  of  $I$ , or equivalently of  $I^k$ . Thus  $b$  is a Rees-good element of  $I^k$ . It follows that  $b^h$  is a Rees-good element of  $I^{kh}$  for every positive integer  $h$ . It is noted preceding Theorem 1.3 that if either (i)  $R$  contains an infinite field or (ii)  $R$  is local with an infinite residue field, then all regular proper ideals in  $R$  have a Rees-good basis.

Example 5.4 exhibits a Gorenstein local ring  $(R, M)$  of altitude one such that  $M$  contains no Rees-good elements and no power of  $M$  has a Rees-good basis.

**Example 5.4.** Let  $F$  be the field with two elements, let  $X, Y$  be independent indeterminates over  $F$ , let  $R = F[[X, Y]]/(XY(X + Y))$ , and let  $x, y$  denote the images in  $R$  of  $X, Y$ , respectively. Then  $M = (x, y)R$  has three Rees valuation rings

$$V_1 := F[[X, Y]]/(X) \quad V_2 := F[[X, Y]]/(Y) \quad V_3 := F[[X, Y]]/((X + Y)).$$

With notation as in (5.2), notice that

$$H_1 = xR + M^2 \quad H_2 = yR + M^2 \quad \text{and} \quad H_3 = (x + y)R + M^2.$$

Therefore  $M = H_1 \cup H_2 \cup H_3$ , so  $M$  does not have any Rees-good elements. Since  $xy(x + y) = 0$  and  $F$  is of characteristic two, one has  $x^2y = xy^2$ , and for  $n \geq 3$

$$x^{n-1}y = x^{n-2}y^2 = \dots = xy^{n-1}.$$

Thus  $\{x^n, x^{n-1}y, y^n\}$  is a minimal basis of  $M^n$  for every  $n \geq 2$ . It follows that the only Rees-good element for  $M^n$ , up to congruence mod  $M^{n+1}$ , is  $x^n + x^{n-1}y + y^n$ , for every  $n \geq 2$ . For  $g \in M^n$  can be written  $g = ax^n + bx^{n-1}y + cy^n + h$  with  $a, b, c \in F$  and  $h \in M^{n+1}$ , and  $g$  is a Rees-good element for  $M^n$  if and only if  $a = b = c = 1$ .

The concept of asymptotic prime divisors as in Definition 5.5 is used in [HRR4].

**Definition 5.5.** Let  $I$  be a regular proper ideal in a Noetherian ring  $R$ . The set of asymptotic prime divisors of  $I$ , denoted  $\overline{A}^*(I)$ , is the set  $\{P \in \text{Spec}(R) \mid P \in \text{Ass}(R/(I^i)_a) \text{ for some positive integer } i\}$ .

It is shown in [Mc, Proposition 3.9] that  $\overline{A}^*(I)$  is a finite set and that for each positive integer  $i$ ,  $\text{Ass}(R/(I^i)_a) \subseteq \text{Ass}(R/(I^{i+1})_a)$ .

Theorem 5.6, which is proved in [HRR4], gives another sufficient condition, besides (i) and (ii) in the Introduction, for  $I$  to have a Rees-good basis.

**Theorem 5.6.** *With the notation of (5.2) and (5.5), assume that  $n \geq 2$ . The following properties are equivalent.*

- (1)  $\text{Card}(R/p) > n$  for each  $p \in \overline{A}^*(I)$  that is a maximal ideal of  $R$ .
- (2) There exists a set  $U = \{u_1, \dots, u_n\}$  of elements in  $R$  such that the elements in  $U \cup \{u_i - u_j \mid i \neq j \text{ in } \{1, \dots, n\}\}$  are units in each Rees valuation ring of  $I$ .

If these hold, then each regular ideal  $H$  in  $R$  such that  $\bigcup\{q \mid q \in \overline{A}^*(H)\} \subseteq \bigcup\{p \mid p \in \overline{A}^*(I)\}$  and  $\text{Card}(\text{Rees } H) \leq \text{Card}(\text{Rees } I)$  has a Rees-good basis. In particular, each ideal  $H$  of  $R$  which is projectively equivalent to  $I$  has a Rees-good basis.

Theorem 5.6 yields the following corollaries.

**Corollary 5.7.** *Let  $I$  be a regular proper ideal in a Noetherian ring  $R$  and assume that no member of  $\overline{A}^*(I)$  is a maximal ideal of  $R$ . Then  $I$  has a Rees-good basis.*

**Corollary 5.8.** *Let  $R$  be a Noetherian ring and assume that  $R/M$  is infinite for all maximal ideals  $M$  in  $R$ . Then every regular proper ideal in  $R$  has a Rees-good basis.*

The concept of an unramified extension as in Definition 5.9 is used in Theorem 5.10.

**Definition 5.9.** A quasi-local ring  $(R', M')$  is *unramified* over a quasi-local ring  $(R, M)$  in case  $R$  is a subring of  $R'$ ,  $M' = MR'$ , and  $R'/M'$  is separable over  $R/M$ . A prime ideal  $p'$  of  $R'$  is *unramified* over  $p' \cap R$  in case  $R'_{p'}$  is unramified over  $R_{p' \cap R}$ .

Theorem 5.10 is the main result in [HRR4]. For a regular proper ideal  $I$  in a Noetherian ring  $R$ , Theorem 5.10 implies the existence of a finite free integral extension ring of  $R$  that satisfies the conclusions of Theorems 1.2 and 1.3 even without the assumption of hypothesis (a) of these theorems.

**Theorem 5.10.** *Let  $I$  be a regular proper ideal in a Noetherian ring  $R$ . There exists a simple free integral extension ring  $A$  of  $R$  such that:*

1. For each regular ideal  $H$  in  $R$  whose asymptotic prime divisors are contained in the union of the asymptotic prime divisors of  $I$  and for which  $\text{Card}(\text{Rees } H) \leq \text{Card}(\text{Rees } I)$ , the ring  $A_P$  is unramified over  $R_{P \cap R}$  for each asymptotic prime divisor  $P$  of  $HA$ ;
2. Each Rees valuation ring of  $HA$  is unramified over its contraction to a Rees valuation ring of  $H$ ; and
3. The ideal  $HA$  has a Rees-good basis and the same Rees integers as  $H$  (with possibly different cardinalities).

In particular, these properties hold for the ideal  $H = I$ .

## 6 Projective equivalence and homogeneous prime spectra of certain Rees rings

Rees rings of filtrations and closely related graded rings have played important auxiliary roles in many research problems in commutative algebra. There are many results about them in the literature, and a large portion of these results are concerned with the set of homogeneous ideals.

We recall the following definitions.

**Definition 6.1.** Let  $R$  be a ring.

**(6.1.1)** A *filtration*  $f = \{I_i\}_{i \geq 0}$  on  $R$  is a sequence of ideals  $I_i$  of  $R$  such that: (a)  $I_0 = R$ ; (b)  $I_i \supseteq I_{i+1}$  for all  $i \in \mathbb{N}$  (the set of nonnegative integers); and, (c)  $I_i I_j \subseteq I_{i+j}$  for all  $i, j \in \mathbb{N}$ .

**(6.1.2)** If  $f = \{I_i\}_{i \geq 0}$  is a filtration on  $R$  and if  $\mathbb{M} = \{0 = c_0, c_1, c_2, \dots\}$  is an additive submonoid of  $\mathbb{N}$  then  $f^{\mathbb{M}} = \{I_{c_i}\}_{i \geq 0}$ . (It is shown in [RR, Theorem 3.3] that  $f^{\mathbb{M}}$  is a filtration on  $R$ .)

**(6.1.3)** The *Rees ring of  $R$  with respect to a filtration  $f = \{I_i\}_{i \geq 0}$  on  $R$*  is the graded subring  $\mathbf{R}(R, f) = R[u, tf] = R[u, tI_1, t^2I_2, \dots]$  of  $R[u, t]$ , where  $t$  is an indeterminate and  $u = 1/t$ .

**(6.1.4)** The *homogeneous prime spectrum* of a graded ring  $\mathbf{A}$  is denoted by  $\text{HSpec}(\mathbf{A})$ , so  $\text{HSpec}(\mathbf{A}) = \{p \in \text{Spec}(\mathbf{A}) \mid p \text{ is homogeneous}\}$ .

It is shown in [MRS, (2.4), (2.6), (2.8), and (2.9)] that: the set  $\mathbf{P}(I)$  of all integrally closed ideals that are projectively equivalent to a regular proper ideal  $I$  in a Noetherian ring  $R$  is discrete and linearly ordered by inclusion; there exists a *unique* positive integer  $d$  such that  $\mathbf{P}(I) \subseteq \{I_{(\frac{k}{d})} \mid k \in \mathbb{N}\}$ , where  $I_{(\frac{k}{d})}$  is the integrally closed ideal  $\{x \in R \mid x^d \in (I^k)_a\}$ ; and,  $(I_{(\frac{i}{d})} I_{(\frac{j}{d})})_a = I_{(\frac{i+j}{d})}$ . And it is shown in [RR, (3.3)] that  $\mathbf{P}(I)$  (together with  $R$ ) is a filtration  $f^*$  on  $R$  which contains the filtration  $f = \{I^i\}_{i \geq 0} = \{I_{(i)}\}_{i \geq 0}$  as a subfiltration, and, in turn,  $f^*$  is a subfiltration of the filtration  $e = \{I_{(\frac{i}{k})}\}_{i \geq 0}$ . Associated with these filtrations we have graded rings  $R[u, tf] = R[u, tI_{(1)}, t^2I_{(2)}, \dots] \cong \mathbf{A} = R[u^d, t^d I_{(1)}, t^{2d} I_{(2)}, \dots] \subseteq \mathbf{B} = R[u, t^{c_1} I_{(\frac{c_1}{d})}, t^{c_2} I_{(\frac{c_2}{d})}, \dots] \subseteq \mathbf{R}(R, e) = R[u, tI_{(\frac{1}{d})}, t^2 I_{(\frac{2}{d})}, \dots]$  and these inclusion maps induce isomorphisms on the homogeneous prime spectra of these graded rings, so  $\text{HSpec}(R[u, tf]) \cong \text{HSpec}(\mathbf{A}) \cong \text{HSpec}(\mathbf{B}) \cong \text{HSpec}(R[u, te])$ . Since  $R[u, tf^*] = R[u, tI_{(\frac{c_1}{d})}, t^2 I_{(\frac{c_2}{d})}, \dots]$ , this raises the question of when  $\text{HSpec}(R[u, tf]) \cong \text{HSpec}(R[u, tf^*])$ . It is shown in [RR, (4.8)] that this holds if and only if  $c_1 = 1$  if and only if  $\mathbf{P}(I)$  is projectively full. (An earlier version of [RR] is referenced in [CHRR] under a slightly different title.)

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# Direct-sum behavior of modules over one-dimensional rings

Ryan Karr and Roger Wiegand

**Abstract** Let  $R$  be a reduced, one-dimensional Noetherian local ring whose integral closure  $\bar{R}$  is finitely generated over  $R$ . Since  $\bar{R}$  is a direct product of finitely many principal ideal domains (one for each minimal prime ideal of  $R$ ), the indecomposable finitely generated  $\bar{R}$ -modules are easily described, and every finitely generated  $\bar{R}$ -module is uniquely a direct sum of indecomposable modules. In this article we will see how little of this good behavior trickles down to  $R$ . Indeed, there are relatively few situations where one can describe all of the indecomposable  $R$ -modules, or even the torsion-free ones. Moreover, a given finitely generated module can have many different representations as a direct sum of indecomposable modules.

## 1 Finite Cohen–Macaulay type

If  $R$  is a one-dimensional reduced Noetherian local ring, the maximal Cohen–Macaulay  $R$ -modules (those with depth 1) are exactly the non-zero finitely generated torsion-free modules. One says that  $R$  has *finite Cohen–Macaulay type* provided there are, up to isomorphism, only finitely many indecomposable maximal Cohen–Macaulay modules. The following theorem classifies these rings:

**Theorem 1.1.** *Let  $(R, \mathfrak{m}, k)$  be a one-dimensional, reduced, local Noetherian ring. Then  $R$  has finite Cohen–Macaulay type if and only if*

(DR1) *The integral closure  $\bar{R}$  of  $R$  in its total quotient ring can be generated by 3 elements as an  $R$ -module; and*

(DR2)  *$\mathfrak{m}(\bar{R}/R)$  is a cyclic  $R$ -module.* □

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The two conditions above were introduced by Drozd and Roĭter in a remarkable 1967 paper [12]. They proved the theorem in the special case of a ring essentially finite over  $\mathbb{Z}$  and asserted that it is true in general. In 1978 Green and Reiner [16] gave a much more detailed proof of the theorem in this special case. In 1989 R. Wiegand [43] proved necessity of the conditions (DR), in general, and sufficiency assuming only that each residue field of  $\bar{R}$  is separable over  $k = R/\mathfrak{m}$ . Since, by (DR1), the residue field growth is of degree at most 3, this completed the proof of Theorem 1.1 except in the cases where  $\text{char}(k) = 2$  or 3. The case of characteristic 3 was handled by indirect methods in [45], leaving only the case where  $k$  is imperfect of characteristic 2. In his 1994 Ph.D. dissertation, Nuri Cimen [6] then used explicit, and very difficult, matrix reductions to prove the remaining case of the theorem.

We will sketch some of the main ingredients of the proof, though we will not touch on the matrix reductions in [16] and [6]. The pullback representation, which we describe in more generality than needed in this section, is a common theme in most of the research leading up to the proof of the theorem. For the moment, let  $R$  be any one-dimensional Noetherian ring, not necessarily local, and let  $\bar{R}$  be the integral closure of  $R$  in the total quotient ring  $K$  of  $R$ . We assume that  $\bar{R}$  is finitely generated as an  $R$ -module. (This assumption is no restriction: A reduced one-dimensional local ring is automatically Cohen–Macaulay. If, further,  $R$  has finite Cohen–Macaulay type, then  $\bar{R}$  has to be finitely generated over  $R$  (cf. [45, Lemma 1] or Proposition 1.2 below).) The conductor  $\mathfrak{f} := \{r \in R \mid r\bar{R} \subseteq R\}$  contains a non-zero-divisor of  $R$ ; therefore  $R/\mathfrak{f}$  and  $\bar{R}/\mathfrak{f}$  are Artinian rings, and we have a pullback diagram

$$\begin{array}{ccc} R & \longrightarrow & \bar{R} \\ \downarrow & & \downarrow \\ R/\mathfrak{f} & \longrightarrow & \bar{R}/\mathfrak{f} \end{array}$$

The bottom line of the pullback is an example of an *Artinian pair* [43], by which we mean a module-finite extension  $A \hookrightarrow B$  of commutative Artinian rings. Of course this pullback has the additional property that  $\bar{R}/\mathfrak{f}$  is a principal ideal ring. Given an Artinian pair  $\mathbf{A} = (A \hookrightarrow B)$ , one defines an  $\mathbf{A}$ -module to be a pair  $V \hookrightarrow W$ , where  $W$  is a finitely generated projective  $B$ -module and  $V$  is an  $A$ -submodule of  $W$  with the property that  $BV = W$ . A morphism  $(V_1 \hookrightarrow W_1) \rightarrow (V_2 \hookrightarrow W_2)$  of  $\mathbf{A}$ -modules is, by definition, a  $B$ -homomorphism from  $W_1$  to  $W_2$  that carries  $V_1$  into  $V_2$ . With submodules and direct sums defined in the obvious way, we get an additive category in which every object has finite length. We say  $\mathbf{A}$  has *finite representation type* provided there are, up to isomorphism, only finitely many indecomposable  $\mathbf{A}$ -modules. In the local case, the bottom line tells the whole story:

**Proposition 1.2 ([43, (1.9)]).** *Let  $(R, \mathfrak{m})$  be a one-dimensional, reduced, Noetherian local ring with finite integral closure  $\bar{R}$ . Then  $R$  has finite Cohen–Macaulay type if and only if the Artinian pair  $R/\mathfrak{f} \hookrightarrow \bar{R}/\mathfrak{f}$  has finite representation type.  $\square$*

The proof of this proposition is not very hard. The key ingredients are the following:

1. Krull–Remak–Schmidt: For an Artinian pair  $\mathbf{A}$ , every  $\mathbf{A}$ -module is uniquely (up to order and isomorphism of the factors) a direct sum of indecomposable  $\mathbf{A}$ -modules.
2. Dickson’s Lemma [9]:  $\mathbb{N}_0^n$  has no infinite antichains. (Here,  $\mathbb{N}_0$  is the well-ordered set of natural numbers, and  $\mathbb{N}_0^n$  has the product partial order.)
3. Given a finitely generated, torsion-free  $R$ -module  $M$ , let  $\bar{R}M$  be the  $\bar{R}$ -submodule of  $KM$  generated by  $M$ . Assume  $R \neq \bar{R}$ . Then  $M_1 \cong M_2 \iff$  the  $(R/\mathfrak{f} \hookrightarrow \bar{R}/\mathfrak{f})$ -modules  $(M_1/\mathfrak{f}M_1 \hookrightarrow \bar{R}M_1/\mathfrak{f}M_1)$  and  $(M_2/\mathfrak{f}M_2 \hookrightarrow \bar{R}M_2/\mathfrak{f}M_2)$  are isomorphic. (The fact that  $R$  is local is crucial here.)

The proof of Theorem 1.1 then reduces to the following:

**Proposition 1.3.** *Let  $\mathbf{A} = (A \hookrightarrow B)$  be an Artinian pair in which  $A$  is local, with maximal ideal  $\mathfrak{m}$  and residue field  $k$ . Assume that  $B$  is a principal ideal ring. Then  $\mathbf{A}$  has finite representation type if and only if the following conditions are satisfied:*

$$\begin{aligned}
 \text{(dr1)} \quad & \dim_k(B/\mathfrak{m}B) \leq 3 \\
 \text{(dr2)} \quad & \dim_k \frac{\mathfrak{m}B + A}{\mathfrak{m}^2B + A} \leq 1. \qquad \square
 \end{aligned}$$

Green and Reiner proved Proposition 1.3 under the additional assumption that the residue fields of  $B$  are all equal to  $k$ . There is an obvious way to eliminate residue field growth, assuming one is trying to prove the more difficult implication that (dr1) and (dr2) imply finite representation type: Adjoin roots. More precisely, we observe that by (dr1)  $B$  has at most three local components, and at most one of these has a residue field properly extending  $k$ . Moreover, the degree of the extension is at most 3. Choose a primitive element  $\theta$ , let  $f \in A[T]$  be a monic polynomial reducing to the minimal polynomial for  $\theta$ , and pass to the Artinian pair  $\mathbf{A}' := (A' \hookrightarrow B')$ , where  $A' = A[T]/(f)$  and  $B' = B \otimes_A A' = B[T]/(f)$ . The problem is that if  $\theta$  is inseparable then  $B'$  may not be a principal ideal ring, and all bets are off. If, however,  $\theta$  is separable, all is well: The Drozd–Roïter conditions ascend to  $\mathbf{A}'$ , and finite representation type descends. This is not difficult, and the details are worked out in [43]. (If  $k(\theta)/k$  is a non-Galois extension of degree 3, one has to repeat the construction one more time.) This proves sufficiency of the Drozd–Roïter conditions, except when  $k$  is imperfect of characteristic 2 or 3.

We now sketch the proof of the “if” implication in Theorem 1.1 in the case of residue field growth of degree 3. By (DR1),  $\bar{R}$  must be local, say with maximal ideal  $\mathfrak{n}$  (necessarily equal to  $\mathfrak{m}\bar{R}$ ) and residue field  $\ell$ . If  $R$  is *seminormal* (that is,  $\bar{R}/\mathfrak{f}$  is reduced), then  $\bar{R}/\mathfrak{f} = \ell$ . The ring  $B'$  described above is now a homomorphic image of  $\ell[T]$  and therefore is a principal ideal ring (even if  $\ell/k$  is not separable). The work of Green and Reiner [16] now shows that  $\mathbf{A}'$  has finite representation type, and the descent argument of [43] proves that  $R$  has finite Cohen–Macaulay type.

Suppose now that  $R$  is *not* seminormal. Then  $\mathfrak{f}$  is properly contained in  $\mathfrak{n}$ . Still assuming (DR1) and (DR2), and that  $[\ell : k] = 3$ , one can show [45, Lemma 4] that

$R$  is Gorenstein, with exactly one overring  $S$  (the seminormalization of  $R$ ) strictly between  $R$  and  $\bar{R}$ . (The argument amounts to a careful computation of lengths, and both (DR1) and (DR2) are used.) Now we use an argument that goes back to Bass’s “ubiquity” paper [3, (7.2)]: Given a maximal Cohen–Macaulay  $R$ -module  $M$ , suppose  $M$  has no free direct summand. Then  $M^* = \text{Hom}_R(M, \mathfrak{m})$ , which is a module over  $E := \text{End}_R(\mathfrak{m})$ . Clearly  $E$  contains  $R$  properly and therefore must contain  $S$ . Thus  $M^*$  is an  $S$ -module, and hence so is  $M^{**}$ , which is isomorphic to  $M$  (as  $R$  is Gorenstein and  $M$  is maximal Cohen–Macaulay). Thus every non-free indecomposable maximal Cohen–Macaulay  $R$ -module is actually an  $S$ -module. The Drozd–Roĭter conditions clearly pass to the seminormal ring  $S$ , which therefore has finite Cohen–Macaulay type. It follows that  $R$  itself has finite Cohen–Macaulay type.

The remaining case, when  $\bar{R}$  has a residue field that is purely inseparable of degree two over  $k$ , was handled via difficult matrix reductions in Cimen’s *tour de force* [6].

Next, we will prove *necessity* of the conditions (DR). This was proved in [43], but we will prove a stronger result here, giving a positive answer to the analog, in the present context, of the second Brauer–Thrall conjecture. Recall that a module  $M$  over a one-dimensional reduced Noetherian ring  $R$  has *constant rank*  $n$ , provided  $M_P \cong R_P^{(n)}$  for each minimal prime ideal  $P$ .

**Theorem 1.4.** *Let  $(R, \mathfrak{m}, k)$  be a one-dimensional, reduced, local Noetherian ring with finite integral closure. Assume that either (DR1) or (DR2) fails. Let  $n$  be an arbitrary positive integer.*

1. *There exists an indecomposable maximal Cohen–Macaulay  $R$ -module of constant rank  $n$ .*
2. *If the residue field  $k$  is infinite, there exist  $|k|$  pairwise non-isomorphic indecomposable maximal Cohen–Macaulay modules of constant rank  $n$ .*

We will prove (2) of Theorem 1.4. The additional arguments needed to prove (1) when  $k$  is finite are rather easy and are given in detail in [43]. Shifting the problem down to the bottom line of the pullback, we let  $\mathbf{A} = (A \hookrightarrow B)$ , where  $A = R/\mathfrak{f}$  and  $B = \bar{R}/\mathfrak{f}$ . We keep the notation of Proposition 1.3, so that now  $\mathfrak{m}$  is the maximal ideal of  $A$ . We assume that either (dr1) or (dr2) fails, and we want to build non-isomorphic indecomposable  $\mathbf{A}$ -modules  $V \hookrightarrow W$ , with  $W = B^{(n)}$ . Given any such  $\mathbf{A}$ -module, the module  $M$  defined by the pullback diagram

$$\begin{array}{ccc} M & \longrightarrow & \bar{R}^{(n)} \\ \downarrow & & \downarrow \\ V & \longrightarrow & W \end{array}$$

will be an indecomposable maximal Cohen–Macaulay  $R$ -module, and non-isomorphic  $\mathbf{A}$ -modules will yield non-isomorphic  $R$ -modules.

We first deal with the annoying case where (dr1) holds but (dr2) fails. (The reader might find it helpful to play along with the example  $k[[t^3, t^7]]$ .) Thus we assume, for the moment, that



$$\dim_k(B/mB) \leq 3 \tag{1}$$

$$\dim_k \frac{mB + A}{m^2B + A} \geq 2 \tag{2}$$

We claim that we actually have *equality* in (2). To see this, we note that  $m^2B \cap A$  is *properly* contained in  $m$  (lest  $mB \subseteq m^2B$ ). Computing lengths, we have

$$\ell_A \frac{m^2B + A}{m^2B} = \ell_A \frac{A}{m^2B \cap A} \geq 2. \tag{3}$$

Since  $B$  is a principal ideal ring,  $mB/m^2B$  is a cyclic  $B/mB$ -module. Therefore, (1) implies that

$$\ell_A(mB/m^2B) \leq 3. \tag{4}$$

Finally, we have  $\ell_A \frac{A+mB}{mB} = \ell_A \frac{A}{A \cap (mB)} = 1$ , and the claim now follows from (3) and (4).

Now put  $C := A + mB$ , and note that  $C/mC \cong k[X, Y]/(X^2, XY, Y^2)$ . The functor  $(V, W) \mapsto (V, BW)$  from  $(A \hookrightarrow C)$ -mod to  $\mathbf{A}$ -mod is clearly faithful; and it is full, by the requirement that  $CV = W$ . Therefore, this functor is injective on isomorphism classes, and it preserves indecomposability. Therefore we may replace  $B$  by  $C$  in this case (the only casualty being that  $B$  is now no longer a principal ideal ring).

Returning to the general case, where *either* (dr1) or (dr2) fails, we put  $D := B/mB$  when (dr1) fails, and  $D = C/mC$  otherwise. We now have either

$$D \text{ is a principal ideal ring and } \dim_k D \geq 4, \text{ or} \tag{5}$$

$$D \cong k[X, Y]/(X^2, XY, Y^2). \tag{6}$$

We now pass to the Artinian pair  $(k \hookrightarrow D)$ . The functor  $(V, W) \mapsto (\frac{V+mW}{mW}, \frac{W}{mW})$ , from  $(A \hookrightarrow B)$ -mod to  $(k \hookrightarrow D)$ -mod is surjective on isomorphism classes and reflects indecomposables. Therefore, it suffices to build our modules over the Artinian pair  $(k \hookrightarrow D)$ .

We now describe a general construction, a modification of constructions found in [7, 12, 43]. Let  $n$  be a fixed positive integer, and suppose we have chosen  $a, b \in D$  with  $\{1, a, b\}$  linearly independent over  $k$ . Let  $I$  be the  $n \times n$  identity matrix, and let  $H$  the  $n \times n$  nilpotent Jordan block with 1's on the superdiagonal and 0's elsewhere. For  $t \in k$ , we consider the  $n \times 2n$  matrix  $\Psi_t := [I \mid aI + b(tI + H)]$ . Put  $W := D^{(n)}$ , viewed as columns, and let  $V_t$  be the  $k$ -subspace of  $W$  spanned by the columns of  $\Psi_t$ .

Suppose, now, that we have a morphism  $(V_t, W) \rightarrow (V_u, W)$ , given by an  $n \times n$  matrix  $\varphi$  over  $D$ . The requirement that  $\varphi(V) \subseteq V$  says there is a  $2n \times 2n$  matrix  $\theta$  over  $k$  such that

$$\varphi \Psi_t = \Psi_u \theta. \tag{7}$$

Write  $\theta = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$ , where  $\alpha, \beta, \gamma, \delta$  are  $n \times n$  blocks. Then (7) gives the following two equations:

$$\begin{aligned} \varphi &= \alpha + a\gamma + b(uI + H)\gamma \\ a\varphi + b\varphi(tI + H) &= \beta + a\delta + b(uI + H)\delta \end{aligned} \tag{8}$$

Substituting the first equation into the second, and combining terms, we get a mess:

$$\begin{aligned} -\beta + a(\alpha - \delta) + b(t\alpha - u\delta + \alpha H - H\delta) + (a + tb)(a + ub)\gamma \\ + ab(H\gamma + \gamma H) + b^2(H\gamma H + tH\gamma + u\gamma H) = 0. \end{aligned} \tag{9}$$

In the ‘‘annoying’’ case (6), we set  $a$  and  $b$  equal to the images of  $X$  and  $Y$ , respectively. Then

$$a^2 = b^2 = ab = 0 \tag{10}$$

and from (9) and the linear independence of  $\{1, a, b\}$ , we get the equations

$$\beta = 0, \quad \alpha = \delta, \quad \alpha((t - u)I + H) = H\alpha. \tag{11}$$

If, now,  $\varphi$  is an isomorphism, we see from (8) that  $\alpha$  has to be invertible. If, in addition,  $t \neq u$ , the third equation in (11) gives a contradiction, since the left side is invertible and the right side is not. Thus,  $(V_t, W) \not\cong (V_u, W)$  if  $t \neq u$ . To see that  $(V_t, W)$  is indecomposable, we take  $u = t$  and suppose  $\varphi$ , as above, is idempotent. Squaring the first equation in (8), and comparing ‘‘1’’ and ‘‘ $a$ ’’ terms, we see that  $\alpha^2 = \alpha$  and  $\gamma = \alpha\gamma + \gamma\alpha$ . But (11) says that  $\alpha H = H\alpha$ , and it follows that  $\alpha$  is in  $k[H]$ , which is a local ring. Therefore  $\alpha = 0$  or  $1$ , and either possibility forces  $\gamma = 0$ . Thus  $\varphi = 0$  or  $1$ , as desired.

Having dealt with the annoying case, we assume from now on that that  $\dim_k D \geq 4$  and that  $D$  is a principal ideal ring. Assume, for the moment, that there exists an element  $a \in D$  such that  $\{1, a, a^2\}$  is linearly independent. Choose any element  $b \in D$  such that  $\{1, a, a^2, b\}$  is linearly independent. Then, for almost all  $t \in k$ , the set  $\{1, a, b, (a + tb)^2\}$  is linearly independent. (The set of such  $t$  is open, and it is non-empty since it contains  $0$ .) Moreover, for almost all  $t \in k$ , the set  $\{1, a, b, (a + tb)(a + ub)\}$  is linearly independent for almost all  $u \in k$ . Thus, it will suffice to show that if  $t \neq u$ , and if  $\{1, a, b, (a + tb)^2\}$  and  $\{1, a, b, (a + tb)(a + ub)\}$  are linearly independent over  $k$ , then  $(V_t \hookrightarrow W)$  is indecomposable and not isomorphic to  $(V_u \hookrightarrow W)$ .

Suppose, as before, that  $\varphi : (V_t \hookrightarrow W) \rightarrow (V_u \hookrightarrow W)$  is a homomorphism. With the same notation as in (7)–(9), we claim that  $\gamma = 0$ . To do this, we use descending induction on  $i$  and  $j$  to show that  $H^i\gamma H^j = 0$  for all  $i, j = 0, \dots, n$ . If either  $i = n$  or  $j = n$  this is clear. Assuming  $H^{i+1}\gamma H^j = 0$  and  $H^i\gamma H^{j+1} = 0$ , we multiply the mess (9) by  $H^i$  on the left and  $H^j$  on the right. In the resulting equation, the ‘‘ $ab$ ’’ and ‘‘ $b^2$ ’’ terms are  $0$  by the inductive hypothesis. Since  $\{1, a, b, (a + tb)(a + ub)\}$  is linearly independent, the ‘‘coefficient’’  $H^i\gamma H^j$  of  $(a + tb)(a + ub)$  must be  $0$ . This completes the induction and proves the claim. The rest of the proof that  $(V_t \hookrightarrow W)$  is indecomposable and not isomorphic to  $(V_u \hookrightarrow W)$  is the same as in the annoying case.

The special case, where  $\{1, a, a^2\}$  is linearly dependent for every element  $a \in A$ , is analyzed in detail in [43]. This case reduces to the following three cases:

- Case 1: There are elements  $a, b \in D$  such that  $\{1, a, b\}$  is linearly independent over  $k$  and  $a^2 = ab = b^2 = 0$ .
- Case 2: There are elements  $a, b \in D$  such that  $\{1, a, b, ab\}$  is linearly independent and  $a^2 = b^2 = 0$ .
- Case 3: The characteristic of  $k$  is 2, and there are elements  $a, b \in D$  such that  $\{1, a, b, ab\}$  is linearly independent and both  $a^2$  and  $b^2$  are in  $k$ .

We have already dealt with Case 1. In Case 2, the mess (9) again yields (11), and we proceed exactly as before. In Case 3, the mess yields the equations

$$\begin{aligned} \beta &= (a^2 + tub^2)\gamma + b^2(H\gamma H + tH\gamma + u\gamma H), & \alpha &= \delta, \\ \alpha((t - u)I + H) &= H\alpha, & (t + u)\gamma + H\gamma + \gamma H &= 0. \end{aligned} \tag{12}$$

Suppose  $t \neq u$ . Then  $t + u \neq 0$  (characteristic two), and the fourth equation shows, via the same descending induction argument as before, that  $\gamma = 0$ . Then the third equation and a now-familiar argument show that  $(V_t \hookrightarrow W) \not\cong (V_u \hookrightarrow W)$ .

Finally, we must show that  $(V_t \hookrightarrow W)$  is indecomposable in Case 3. Suppose  $t = u$  and  $\varphi^2 = \varphi$ . The third and fourth equations of (12) now show that  $\alpha$  and  $\gamma$  are in  $k[H]$ . In particular,  $\alpha\gamma = \gamma\alpha$ . Therefore, when we square the first equation of (8) and compare “ $a$ ” terms, we see that  $\gamma = 2\alpha\gamma = 0$ . Now  $\varphi = \alpha \in k[H]$ , a local ring, and it follows that  $\varphi = 0$  or 1. This completes the proof of Theorem 1.4.  $\square$

One might expect that even if  $k$  is finite one could construct a countably infinite family of pairwise non-isomorphic maximal Cohen–Macaulay modules of constant rank  $n$ . In fact, this is not the case:

**Proposition 1.5.** *With  $(R, \mathfrak{m}, k)$  as in Theorem 1.1, suppose  $k$  is a finite field. Let  $n$  be a positive integer. Then  $R$  has only finitely many isomorphism classes of maximal Cohen–Macaulay modules of constant rank  $n$ .*

*Proof.* Let  $\mathbf{A} = (R/\mathfrak{f} \hookrightarrow \overline{R}/\mathfrak{f})$  be the Artinian pair associated with  $R$ . Recall [43, (1.7)] that two maximal Cohen–Macaulay  $R$ -modules  $M_1$  and  $M_2$  are isomorphic if and only if their associated  $\mathbf{A}$ -modules  $(M_i/\mathfrak{f}M_i \hookrightarrow \overline{R}M_i/\mathfrak{f}M_i)$  are isomorphic. Therefore it is enough to show that there are only finitely many  $\mathbf{A}$ -modules  $(V \hookrightarrow W)$  with  $W = (\overline{R}/\mathfrak{f})^{(n)}$ . But this is clear because  $|W| < \infty$ .  $\square$

### 1.1 Finiteness of the integral closure

Let  $(R, \mathfrak{m})$  be a local Noetherian ring of dimension one, let  $K$  be the total quotient ring  $\{\text{non-zero-divisors}\}^{-1}R$ , and let  $\overline{R}$  be the integral closure of  $R$  in  $K$ . Suppose  $\overline{R}$  is not finitely generated over  $R$ . Then we can build an infinite ascending chain

of finitely generated  $R$ -subalgebras of  $\bar{R}$ . Each algebra in the chain is a maximal Cohen–Macaulay  $R$ -module, and it is easy to see [45, Lemma 1] that no two of the algebras are isomorphic as  $R$ -modules. It follows [45, Proposition 1] that  $\bar{R}$  is finitely generated as an  $R$ -module if  $R$  has finite Cohen–Macaulay type. If, now,  $R$  is Cohen–Macaulay and  $x$  is a non-zero nilpotent element, we claim that  $\bar{R}$  is not finitely generated over  $R$ . To see this, choose a non-zero-divisor  $t \in \mathfrak{m}$ , and note that  $R \frac{x}{t} \subset R \frac{x}{t^2} \subset R \frac{x}{t^3} \subset \dots$  is an infinite ascending chain of  $R$ -submodules of  $\bar{R}$ . We have proved:

**Proposition 1.6.** *Let  $(R, \mathfrak{m}, k)$  be a one-dimensional, Cohen–Macaulay local ring with finite Cohen–Macaulay type. Then  $R$  is reduced, and the integral closure  $\bar{R}$  is finitely generated as an  $R$ -module.  $\square$*

What if  $R$  is *not* Cohen–Macaulay? The following result, together with Theorem 1.1, gives the full classification of one-dimensional local rings of finite Cohen–Macaulay type:

**Theorem 1.7 ([45, Theorem 1]).** *Let  $(R, \mathfrak{m})$  be a one-dimensional local ring, and let  $N$  be the nilradical of  $R$ . Then  $R$  has finite Cohen–Macaulay type if and only if*

- (1)  $R/N$  has finite Cohen–Macaulay type, and
- (2)  $\mathfrak{m}^i \cap N = (0)$  for  $i \gg 0$ .  $\square$

For example,  $k[[X, Y]]/(X^2, XY)$  has finite Cohen–Macaulay type, since  $(x)$  is the nilradical and  $(x, y)^2 \cap (x) = (0)$ . However  $k[[X, Y]]/(X^3, X^2Y)$  has infinite Cohen–Macaulay type: For each  $i \geq 1$ ,  $xy^{i-1}$  is a non-zero element of  $(x, y)^i \cap (x)$ .

**Corollary 1.8 ([45, Corollary 2]).** *Let  $(R, \mathfrak{m})$  be a one-dimensional local ring. Then  $R$  has finite Cohen–Macaulay type if and only if the  $\mathfrak{m}$ -adic completion  $\widehat{R}$  has finite Cohen–Macaulay type.  $\square$*

The analogous statement can fail in higher dimensions (cf. Examples 2.1 and 2.2 of [33]).

## 1.2 Rings containing the rational numbers

For local rings containing  $\mathbb{Q}$ , the rings of finite Cohen–Macaulay type have a particularly nice classification. First, we recall the 1985 classification, by Greuel and Knörrer, of complete equicharacteristic-zero singularities of finite Cohen–Macaulay type. Recall that the *simple* (or “ADE”) plane curve singularities are the following rings corresponding to certain Dynkin diagrams:

- ( $A_n$ )  $k[[X, Y]]/(X^2 + Y^{n+1}) \quad (n \geq 1)$
- ( $D_n$ )  $k[[X, Y]]/Y(X^2 + Y^{n-2}) \quad (n \geq 4)$
- ( $E_6$ )  $k[[X, Y]]/(X^3 + Y^4)$

- (E<sub>7</sub>)  $k[[X, Y]]/X(X^2 + Y^3)$
- (E<sub>8</sub>)  $k[[X, Y]]/(X^3 + Y^5)$

**Theorem 1.9 ([17]).** *Let  $(R, \mathfrak{m}, k)$  be a one-dimensional complete local Cohen–Macaulay ring containing  $\mathbb{Q}$ , and assume that  $k$  is algebraically closed. Then  $R$  has finite Cohen–Macaulay type if and only if  $R$  birationally dominates a simple plane curve singularity.  $\square$*

To say that  $R$  *birationally dominates* a local ring  $S$  means that  $R$  sits between  $S$  and the total quotient ring of  $S$ , and that the maximal ideal of  $R$  lies over the maximal ideal of  $S$ . For example, the space curves  $k[[T^3, T^4, T^5]]$  and  $k[[T^3, T^5, T^7]]$  have finite Cohen–Macaulay type, since they birationally dominate the (E<sub>8</sub>)-singularity  $k[[T^3, T^5]]$ . To handle the case of a residue field that is not algebraically closed, we quote the following theorem (which works in all dimensions):

**Theorem 1.10 ([46, Theorem 3.3]).** *Let  $k$  be a field with separable closure  $k^s$ , and let  $f$  be a non-unit in the formal power series ring  $k[[X_0, \dots, X_d]]$ . Then  $k[[X_0, \dots, X_d]]/(f)$  has finite CM type if and only if  $k^s[[X_0, \dots, X_d]]/(f)$  has finite CM type.  $\square$*

As one might expect, inseparable extensions can cause difficulties:

*Example 1.11.* Let  $k$  be an imperfect field of characteristic 2, choose  $\alpha \in k - k^2$  and put  $K := k(\sqrt{\alpha})$ . Let  $f = X^2 + \alpha Y^2$ , and put  $R := k[[X, Y]]/(f)$ . Then  $R$  is a one-dimensional complete local domain, and the integral closure  $\bar{R}$  is generated, as an  $R$ -module, by the two elements 1 and  $\frac{x}{y}$ ; and both  $x$  and  $y$  multiply  $\frac{x}{y}$  into  $R$ . By Theorem 1.1,  $R$  has finite Cohen–Macaulay type. On the other hand, Proposition 1.6 implies that  $K[[x, y]]/(f)$  does *not* have finite Cohen–Macaulay type, since it is Cohen–Macaulay and has non-zero nilpotents.

## 2 Bounded Cohen–Macaulay type

In this section, we consider one-dimensional Cohen–Macaulay local rings  $(R, \mathfrak{m}, k)$ . We will say that  $R$  has *bounded* Cohen–Macaulay type provided there is a bound on the multiplicities of the indecomposable maximal Cohen–Macaulay  $R$ -modules. Since the notion of rank is perhaps more intuitive, we mention that if  $M$  is an  $R$  module of constant rank  $r$ , then the multiplicity  $e(M)$  of  $M$  satisfies

$$e(M) = r \cdot e(R).$$

If  $R$  is reduced, then Theorems 1.1 and 1.4 imply that finite and bounded Cohen–Macaulay types agree. In 1980 Dieterich [10] observed that the group ring  $k[[X]][G]$  has bounded Cohen–Macaulay type if  $|G| = 2$  and  $\text{char}(k) = 2$ . Of course this ring is just  $k[[X, Y]]/(Y^2)$ . In 1987 Buchweitz, Greuel and Schreyer [5] classified the indecomposable maximal Cohen–Macaulay modules over  $k[[X, Y]]/(Y^2)$  and  $k[[X, Y]]/(XY^2)$ , the  $(A_\infty)$  and  $(D_\infty)$  singularities, for every field  $k$ . A consequence

of their classification is that these singularities have bounded Cohen–Macaulay type. Of course, by Proposition 1.6, these rings do not have finite Cohen–Macaulay type. Rather surprisingly, there is, in the complete equicharacteristic case, only one additional ring with bounded but infinite Cohen–Macaulay type:

**Theorem 2.1 ([34, Theorem 2.4]).** *Let  $(R, \mathfrak{m}, k)$  be a one-dimensional local Cohen–Macaulay ring. Assume that  $R$  contains a field and that  $k$  is infinite. Then  $R$  has bounded but infinite Cohen–Macaulay type if and only if the  $\mathfrak{m}$ -adic completion  $\widehat{R}$  is  $k$ -isomorphic to one of the following:*

- (1)  $A := k[[X, Y]]/(Y^2)$
- (2)  $B := k[[X, Y]]/(XY^2)$
- (3)  $C := k[[XY, YZ, Z^2]]$ , the endomorphism ring of the maximal ideal of  $B$

*If, on the other hand,  $R$  has unbounded Cohen–Macaulay type, then  $R$  has, for each positive integer  $r$ , an indecomposable maximal Cohen–Macaulay module of constant rank  $r$ .*  $\square$

The proof of the “only if” direction of this theorem involves some rather technical ideal theory. We don’t know whether or not the theorem is correct without the assumption that  $k$  be infinite.

For the rings  $A$  and  $B$  of Theorem 2.1, we see from the explicit presentations in [5] that the indecomposable maximal Cohen–Macaulay modules are generated by at most two elements. This gives us a bound of six on the multiplicities of these modules. Since  $C = \text{End}_B(\mathfrak{m}_B)$ , where  $\mathfrak{m}_B$  is the maximal ideal of  $B$ , we see that  $C$  is a module-finite extension of  $B$ . Therefore every maximal Cohen–Macaulay  $C$ -module  $M$  is also maximal Cohen–Macaulay when viewed as a  $B$ -module. Moreover, since  $C$  is contained in the total quotient ring of  $B$  and  $M$  is torsion-free, we see that  $\text{End}_B(M) = \text{End}_C(M)$ . In particular, if  $M$  is indecomposable as a  $C$ -module, it is also indecomposable as a  $B$ -module. Thus the multiplicities of the indecomposable maximal Cohen–Macaulay  $B$ -modules are also bounded by six. The “if” direction of Theorem 2.1 now follows from the next theorem, on ascent to and descent from the completion.

**Theorem 2.2 ([34, Theorem 2.3]).** *Let  $(R, \mathfrak{m}, k)$  be a one-dimensional Cohen–Macaulay local ring with completion  $\widehat{R}$ . Assume  $R$  contains a field and that  $k$  is infinite. Then  $R$  has bounded Cohen–Macaulay type if and only if  $\widehat{R}$  has bounded Cohen–Macaulay type. Moreover, if  $R$  has unbounded Cohen–Macaulay type, then  $R$  has, for every positive integer  $r$ , an indecomposable maximal Cohen–Macaulay module of constant rank  $r$ .*  $\square$

By Lech’s Theorem [32, Theorem 1] each of the rings in Theorem 2.1 is the completion of an integral domain. Suppose, for example, that  $(R, \mathfrak{m}, k)$  is a one-dimensional local domain whose completion is  $k[[X, Y]]/(Y^2)$ . Then  $R$  has bounded but infinite Cohen–Macaulay type. Therefore the assumption, in Theorem 1.4, that  $\overline{R}$  be finitely generated over  $R$ , cannot be removed.

### 3 Modules with torsion

In this section, we consider arbitrary finitely generated modules over local rings of dimension one. Every such ring  $(R, \mathfrak{m})$  obviously has an infinite family of pairwise non-isomorphic indecomposable modules, namely, the modules  $R/\mathfrak{m}^n$ . With a little more work, one can produce indecomposable modules requiring arbitrarily many generators, as long as  $R$  is not a principal ideal domain. To see this, fix  $n \geq 1$ , let  $x$  and  $y$  be elements of  $\mathfrak{m}$  that are linearly independent modulo  $\mathfrak{m}^2$ , and let  $I$  and  $H$  be the  $n \times n$  identity and nilpotent matrices used in the proof of Theorem 1.4. Then the cokernel of the matrix  $xI + yH$  is indecomposable, and it clearly needs  $n$  generators. To prove indecomposability, one can pass to  $R/\mathfrak{m}^2$  and use an argument similar to, but much easier than, the one used in the proof of Theorem 1.4. See, for example, [21, Proposition 4.1] or [39]. Similar constructions can be found in the work of Kronecker [28] and Weierstrass [40] on classifying pairs of matrices up to simultaneous equivalence. The idea is not exactly new!

It is much more difficult to build indecomposable modules of large multiplicity. Of course it is impossible to do so if  $R$  is a principal ideal ring. More generally, recall from [29–31] that a local ring  $(R, \mathfrak{m}, k)$  is *Dedekind-like* provided  $R$  is reduced and one-dimensional, the integral closure  $\bar{R}$  is generated by at most two elements as an  $R$ -module, and  $\mathfrak{m}$  is the Jacobson radical of  $\bar{R}$ . In a long and difficult paper [30] Levy and Klingler classify the indecomposable finitely generated modules over most Dedekind-like rings. There is one exceptional case where the classification has not yet been worked out, namely, where  $\bar{R}$  is a local domain whose residue field is purely inseparable of degree two over  $k$ . We will call these Dedekind-like rings *exceptional*. The ring of Example 1.11 is such an exception. Before stating the next result, which is a consequence of the classification in [30], we note that a Dedekind-like ring has at most two minimal prime ideals and that the localization of  $R$  at a minimal prime is a field. If  $R$  has two minimal primes  $P_1$  and  $P_2$ , the *rank* of the  $R$ -module  $M$  is the pair  $(r_1, r_2)$ , where  $r_i$  is the dimension of  $M_{P_i}$  as a vector space over  $R_{P_i}$ .

**Theorem 3.1 ([30]).** *Let  $M$  be an indecomposable finitely generated module over a local Dedekind-like ring  $R$ .*

1. *If  $R$  has two minimal prime ideals  $P_1$  and  $P_2$ , then the rank of  $M$  is  $(0, 0)$ ,  $(1, 0)$ ,  $(0, 1)$  or  $(1, 1)$ .*
2. *If  $R$  is a domain and  $R$  is not exceptional, then  $M$  has rank 0, 1 or 2. □*

In a series of papers [20–22], Hassler, Klingler, and the present authors proved a strong converse to this theorem:

**Theorem 3.2 ([21, Theorem 1.2]).** *Let  $R$  be a local ring of dimension at least one, and assume  $R$  is not a homomorphic image of a Dedekind-like ring. Let  $P_1, \dots, P_s$  be an arbitrary set of pairwise incomparable non-maximal prime ideals, and let  $n_1, \dots, n_s$  be non-negative integers. Then there are  $|k|\aleph_0$  pairwise non-isomorphic indecomposable  $R$ -modules  $M_\alpha$  such that  $(M_\alpha)_{P_i} \cong R_{P_i}^{(n_i)}$  for each  $i \leq s$  and each  $\alpha$ . □*

The proof [21] of this result is rather involved. It makes heavy use of the fact [29] that the category of finite-length  $R$ -modules has wild representation type if  $R$  is not a homomorphic image of a Dedekind-like ring.

### 4 Monoids of modules

In this section, we study the different ways in which a finitely generated module can be decomposed as a direct sum of indecomposable modules. Let  $(R, \mathfrak{m}, k)$  be a local ring and  $\mathcal{C}$  a class of modules closed under isomorphism, finite direct sums, and direct summands. We always assume that  $\mathcal{C} \subseteq R\text{-mod}$ , the class of all finitely generated  $R$ -modules. There is a set  $V(\mathcal{C}) \subseteq \mathcal{C}$  of representatives; each element  $M \in \mathcal{C}$  is isomorphic to exactly one element  $[M] \in V(\mathcal{C})$ . We make  $V(\mathcal{C})$  into an additive monoid in the obvious way:  $[M] + [N] = [M \oplus N]$ . This monoid encodes information about direct-sum decompositions in  $\mathcal{C}$ . We will tacitly assume that all of our monoids are written additively, and that they are *reduced* ( $x + y = 0 \implies x = y = 0$ ).

Suppose  $R$  is complete (in the  $\mathfrak{m}$ -adic topology). Then the Krull–Remak–Schmidt theorem holds for finitely generated modules, that is, each  $M \in R\text{-mod}$  is uniquely a direct sum of indecomposable modules (up to isomorphism and ordering of the summands). In the language of monoids,  $V(R\text{-mod}) \cong \mathbb{N}_0^{(I)}$ , the free monoid with basis  $\{b_i \mid i \in I\}$ , where the  $b_i$  range over a set of representatives for the indecomposable finitely generated  $R$ -modules.

For a general local ring  $R$ , we can exploit the monoid homomorphism

$$j : V(R\text{-mod}) \rightarrow V(\widehat{R}\text{-mod})$$

taking  $[M]$  to  $[\widehat{R} \otimes_R M]$ . This homomorphism is injective [11, (2.5.8)], and it follows that the monoid  $R\text{-mod}$  is *cancellative*:  $x + z = y + z \implies x = y$ . (cf. [14, 38].) Since, in this section, we will deal only with local rings, all of our monoids are tacitly assumed to be cancellative.

The homomorphism  $j$  actually satisfies a much stronger condition. If  $x, y \in V(R\text{-mod})$  and  $j(x) \mid j(y)$ , then  $x \mid y$ . (For elements  $x$  and  $y$  in a monoid  $\Lambda$  we say  $x$  *divides*  $y$ , written “ $x \mid y$ ” provided there is an element  $\lambda \in \Lambda$  such that  $x + \lambda = y$ .) Here is a proof, given by Reiner and Roggenkamp [36] in a slightly different context: Suppose  $M'$  and  $M$  are finitely generated modules over a local ring  $R$ , and that  $\widehat{R} \otimes_R M' \mid \widehat{R} \otimes_R M$ . We identify  $\widehat{R} \otimes_R M'$  and  $\widehat{R} \otimes_R M$  with the completions  $\widehat{M}'$  and  $\widehat{M}$  of  $M'$  and  $M$ . Choose  $\widehat{R}$ -homomorphisms  $\varphi : \widehat{M}' \rightarrow \widehat{M}$  and  $\psi : \widehat{M} \rightarrow \widehat{M}'$  such that  $\psi\varphi = 1_{\widehat{M}'}$ . Since  $H := \text{Hom}_R(M', M)$  is a finitely generated  $R$ -module, it follows that  $\widehat{H} = \widehat{R} \otimes_R H = \text{Hom}_{\widehat{R}}(\widehat{M}, \widehat{N})$ . Therefore,  $\varphi$  can be approximated to any order by an element of  $H$ . In fact, order 1 suffices: Choose  $f \in \text{Hom}_R(M, N)$  such that  $\widehat{f} - \varphi \in \widehat{\mathfrak{m}}\widehat{H}$ . Similarly, we can choose  $g \in \text{Hom}_R(N, M)$  with  $\widehat{g} - \psi \in \widehat{\mathfrak{m}}\text{Hom}_{\widehat{R}}(\widehat{N}, \widehat{M})$ . Then the image of  $\widehat{g}\widehat{f} - 1_{\widehat{M}}$  is in  $\widehat{\mathfrak{m}}\widehat{M}$ , and now Nakayama’s lemma implies that  $\widehat{g}\widehat{f}$  is surjective, and therefore an isomorphism. It follows that  $\widehat{g}$  is a split surjection (with splitting map  $\widehat{f}(\widehat{g}\widehat{f})^{-1}$ ). By faithful flatness  $g$  is a split surjection.



A divisor homomorphism  $j : \Lambda_1 \rightarrow \Lambda_2$  (between reduced, cancellative monoids) is a homomorphism such that, for all  $x, y \in \Lambda_1$ ,  $j(x) \mid j(y) \implies x \mid y$ . The result we just proved is a special case of the following theorem:

**Theorem 4.1 ([24, Theorem 1.3]).** *Let  $R \rightarrow S$  be a flat local homomorphism of Noetherian local rings. Then the map  $V(R\text{-mod}) \rightarrow V(S\text{-mod})$  taking  $[M]$  to  $[S \otimes_R M]$  is a divisor homomorphism.  $\square$*

**Definition 4.2.** A Krull monoid is a monoid that admits a divisor homomorphism into a free monoid.

Every finitely generated Krull monoid admits a divisor homomorphism into  $\mathbb{N}_0^{(t)}$  for some positive integer  $t$ . Conversely, it follows easily from Dickson’s Lemma (Item 2 following Proposition 1.2) that a monoid admitting a divisor homomorphism to  $\mathbb{N}_0^{(t)}$  must be finitely generated.

Finitely generated Krull monoids are called *positive normal affine semigroups* in [4]. From [4, Exercise 6.1.10, p. 252], we obtain the following characterization of these monoids:

**Proposition 4.3.** *The following conditions on a monoid  $\Lambda$  are equivalent:*

1.  $\Lambda$  is a finitely generated Krull monoid.
2.  $\Lambda \cong G \cap \mathbb{N}^{(t)}$  for some positive integer  $t$  and some subgroup  $G$  of  $\mathbb{Z}^{(t)}$ .
3.  $\Lambda \cong W \cap \mathbb{N}^{(u)}$  for some positive integer  $u$  and some  $\mathbb{Q}$ -subspace  $W$  of  $\mathbb{Q}^{(u)}$ .
4. There exist positive integers  $m, n$  and an  $m \times n$  matrix  $\alpha$  over  $\mathbb{Z}$  such that  $\Lambda \cong \mathbb{N}^{(n)} \cap \ker(\alpha)$ .  $\square$

Item (4) says that a finitely generated Krull monoid can be regarded as the collection of non-negative integer solutions of a homogeneous system of linear equations. For this reason these monoids are sometimes called *Diophantine monoids*.

In order to study uniqueness of direct-sum decompositions, it is really enough to examine a small piece of the class  $R\text{-mod}$  of all finitely generated modules. Given a finitely generated module  $M$ , we let  $\text{add}(M)$  be the class of modules that are isomorphic to direct summands of direct sums of finitely many copies of  $M$ . We note that  $+(M) := V(\text{add}(M))$  is a finitely generated Krull monoid, since the divisor homomorphism  $j : V(R\text{-mod}) \rightarrow V(\widehat{R}\text{-mod})$  carries  $+(M)$  into the free monoid generated by the isomorphism classes of the indecomposable direct summands of  $\widehat{M}$ .

The key to understanding the monoids  $V(R\text{-mod})$  and  $+(M)$  is knowing which modules over the completion actually come from  $R$ -modules. More generally, if  $R \rightarrow S$  is a ring homomorphism, we say that the  $S$ -module  $N$  is *extended* (from  $R$ ) provided there is an  $R$ -module  $M$  such that  $S \otimes_R M \cong N$ . In dimension one, a beautiful result due to Levy and Odenthal [35] tells us exactly which  $\widehat{R}$ -modules are extended. First, we define, for any one-dimensional local ring  $(R, \mathfrak{m}, k)$  the *Artinian localization*  $\mathfrak{a}(R)$  as follows:

$$\mathfrak{a}(R) = (R - (P_1 \cup \dots \cup P_s))^{-1}R,$$

where  $P_1, \dots, P_s$  are the minimal prime ideals of  $R$  (the prime ideals distinct from  $\mathfrak{m}$ ). If  $R$  is Cohen–Macaulay, this is just the classical quotient ring. If  $R$  is *not* Cohen–Macaulay, the natural map  $R \rightarrow \mathfrak{a}(R)$  is not one-to-one.

**Theorem 4.4** ([35]). *Let  $(R, \mathfrak{m}, k)$  be a one-dimensional local ring, and let  $N$  be a finitely generated  $\widehat{R}$ -module. Then  $N$  is extended from  $R$  if and only if  $\mathfrak{a}(\widehat{R}) \otimes_{\widehat{R}} N$  is extended from  $\mathfrak{a}(R)$ .* □

We refer the reader to [24, Theorem 4.1] for the proof of a somewhat more general result.

We return now to the situation of Section 1, where  $(R, \mathfrak{m}, k)$  is a local ring whose completion  $\widehat{R}$  is reduced. The localizations at the minimal primes are then fields. If  $A := L_1 \times \dots \times L_t$  is a  $K$ -algebra, where  $K$  and the  $L_j$  are fields, a finitely generated  $A$ -module  $N$  is extended from  $K$  if and only if  $\dim_{L_i}(L_i N) = \dim_{L_j}(L_j N)$  for all  $i, j$ . Therefore, Theorem 4.4 has the following consequence:

**Corollary 4.5.** *Let  $(R, \mathfrak{m}, k)$  be a one-dimensional local ring whose completion  $\widehat{R}$  is reduced, and let  $N$  be a finitely generated  $\widehat{R}$ -module. Then  $N$  is extended from  $R$  if and only if  $\dim_{R_P}(N_P) = \dim_{R_Q}(N_Q)$  whenever  $P$  and  $Q$  are prime ideals of  $\widehat{R}$  lying over the same prime ideal of  $R$ . In particular, if  $R$  is a domain, then  $N$  is extended if and only if  $N$  has constant rank.* □

This gives us a strategy for producing strange direct-sum behavior:

- (1) Find a one-dimensional domain  $R$  whose completion has lots of minimal primes.
- (2) Build indecomposable  $\widehat{R}$ -modules with highly non-constant ranks.
- (3) Put them together in different ways to get constant-rank modules.

Suppose, for example, that  $R$  is a domain whose completion  $\widehat{R}$  has two minimal primes  $P$  and  $Q$ . Suppose we can build indecomposable  $\widehat{R}$ -modules  $U, V, W$  and  $X$ , with ranks  $(2, 0), (0, 2), (2, 1)$  and  $(1, 2)$ , respectively. Then  $U \oplus V$  is extended, say,  $U \oplus V \cong \widehat{M}$ . Similarly, there are  $R$ -modules  $N, F$  and  $G$  such that  $V \oplus W \oplus W \cong \widehat{N}, W \oplus X \cong \widehat{F}$  and  $U \oplus X \oplus X \cong \widehat{G}$ . Using the Krull–Remak–Schmidt theorem over  $\widehat{R}$ , we see easily that no non-zero proper direct summand of any of the modules  $\widehat{M}, \widehat{N}, \widehat{F}, \widehat{G}$  has constant rank. It follows from Corollary 4.5 that  $M, N, F$  and  $G$  are indecomposable, and of course no two of them are isomorphic since (again by Krull–Remak–Schmidt) their completions are pairwise non-isomorphic. Finally, we see that  $M \oplus F \oplus F \cong N \oplus G$ , since the two modules have isomorphic completions. Thus we easily obtain a mild violation of Krull–Remak–Schmidt uniqueness over  $R$ .

It’s easy to accomplish (1), getting a one-dimensional domain with a lot of splitting. In order to facilitate (2), however, we want to ensure that  $\widehat{R}/P$  has infinite Cohen–Macaulay type for each minimal prime ideal  $P$ . The following example from [47] does the job nicely:

*Example 4.6* ([47, (2.3)]). Fix a positive integer  $s$ , and let  $k$  be any field with  $|k| \geq s$ . Choose distinct elements  $t_1, \dots, t_s \in k$ . Let  $\Sigma$  be the complement of the union of the maximal ideals  $(X - t_i)k[X]$ ,  $i = 1, \dots, s$ . We define  $R = R_s$  by the pullback diagram

$$\begin{array}{ccc}
 R & \longrightarrow & \Sigma^{-1}k[X] \\
 \downarrow & & \downarrow \pi \\
 k & \longrightarrow & \frac{\Sigma^{-1}k[X]}{(X-t_1)^4 \cdots (X-t_s)^4}
 \end{array}$$

where  $\pi$  is the natural map. Then  $R$  is a one-dimensional local domain, and  $\widehat{R}$  is reduced with exactly  $s$  minimal prime ideals.

Let  $P_1, \dots, P_s$  be the minimal prime ideals of  $\widehat{R}$ . By the *rank* of a finitely generated  $\widehat{R}$ -module  $N$ , we mean the  $s$ -tuple  $(r_1, \dots, r_s)$ , where  $r_i$  is the dimension of  $N_{P_i}$  as a vector space over  $R_{P_i}$ . A jazzed-up version of the argument used to prove Theorem 1.4 yields the following:

**Theorem 4.7 ([47, (2.4)]).** *Fix a positive integer  $s$ , and let  $(r_1, \dots, r_s)$  be any non-trivial sequence of non-negative integers. Then  $\widehat{R}_s$  has an indecomposable maximal Cohen–Macaulay module  $N$  with  $\text{rank}(N) = (r_1, \dots, r_s)$ .  $\square$*

Thus even the case  $s = 2$  of Example 4.6 yields the pathology discussed after Corollary 4.5

Recalling (4) of Proposition 4.3, we say that the finitely generated Krull monoid  $\Lambda$  can be defined by  $m$  equations provided  $\Lambda = \mathbb{N}_0^{(n)} \cap \ker(\alpha)$  for some  $n$  and some  $m \times n$  integer matrix  $\alpha$ . Given such an embedding of  $\Lambda$  in  $\mathbb{N}_0^{(n)}$ , we say a column vector  $\lambda \in \Lambda$  is *strictly positive* provided each of its entries is a positive integer. By decreasing  $n$  (and removing some columns from  $\alpha$ ) if necessary, we can harmlessly assume (without changing  $m$ ) that  $\Lambda$  contains a strictly positive element (cf. [49, Remark 3.1]).

**Corollary 4.8 ([47, Theorem 2.1]).** *Fix a non-negative integer  $m$ , and let  $R$  be the ring  $R_{m+1}$  of Example 4.6. Let  $\Lambda$  be a finitely generated Krull monoid defined by  $m$  equations and containing a strictly positive element  $\lambda$ . Then there exist a maximal Cohen–Macaulay  $R$ -module  $M$  and a commutative diagram*

$$\begin{array}{ccc}
 \Lambda & \xrightarrow{\subseteq} & \mathbb{N}_0^{(n)} \\
 \varphi \downarrow \cong & & \psi \downarrow \cong \\
 \mathbb{V}(+(M)) & \xrightarrow{i} & \mathbb{V}(+(\widehat{M}))
 \end{array}
 ,$$

in which

- (1)  $i$  is the natural map taking  $[F]$  to  $[\widehat{F}]$ ,
- (2)  $\varphi$  and  $\psi$  are monoid isomorphisms, and
- (3)  $\varphi([M]) = \lambda$ .

*Proof.* We have  $\Lambda = \mathbb{N}_0^{(n)} \cap \ker(\alpha)$ , where  $\alpha = [a_{ij}]$  is an  $m \times n$  matrix over  $\mathbb{Z}$ . Choose a positive integer  $h$  such that  $a_{ij} \geq 0$  for all  $i, j$ . For  $j = 1, \dots, n$ , choose,

using Theorem 4.7, a maximal Cohen–Macaulay  $\widehat{R}$ -module  $L_j$  such that  $\text{rank}(L_j) = (a_{1j} + h, \dots, a_{mj} + h, h)$ .

Given any column vector  $\beta = [b_1 \ b_2 \ \dots \ b_n]^{\text{tr}} \in \mathbb{N}_0^{(n)}$ , put  $N_\beta = L_1^{(b_1)} \oplus \dots \oplus L_n^{(b_n)}$ . The rank of  $N_\beta$  is

$$\left( \sum_{j=1}^n (a_{1j} + h)b_j, \dots, \sum_{j=1}^n (a_{mj} + h)b_j, \left( \sum_{j=1}^n b_j \right) h \right).$$

Since  $R$  is a domain, Corollary 4.5 implies that  $N_\beta$  is in the image of  $j : \mathbb{V}(R\text{-mod}) \rightarrow \mathbb{V}(\widehat{R}\text{-mod})$  if and only if  $\sum_{j=1}^n (a_{ij} + h)b_j = (\sum_{j=1}^n b_j)h$  for each  $i$ , that is, if and only if  $\beta \in \mathbb{N}_0^{(n)} \cap \ker(\alpha) = \Lambda$ . To complete the proof, we let  $M$  be the  $R$ -module (unique up to isomorphism) such that  $\widehat{M} \cong N_\lambda$ .  $\square$

This corollary makes it very easy to demonstrate spectacular failure of Krull–Remak–Schmidt uniqueness:

*Example 4.9.* Let  $\Lambda = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{N}_0^{(3)} \mid 72x + y = 73z \right\}$ . This has three atoms (minimal non-zero elements), namely

$$\alpha := \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \beta := \begin{bmatrix} 0 \\ 73 \\ 1 \end{bmatrix}, \quad \gamma := \begin{bmatrix} 73 \\ 0 \\ 72 \end{bmatrix}.$$

Note that  $73\alpha = \beta + \gamma$ . Taking  $s = 2$  in Example 4.6, we get a local ring  $R$  and indecomposable  $R$ -modules  $M, F, G$  such that  $M^{(t)}$  has only the obvious direct-sum decompositions for  $t \leq 72$ , but  $M^{(73)} \cong F \oplus G$ .

We define the *splitting number*  $\text{spl}(R)$  of a one-dimensional local ring  $R$  by

$$\text{spl}(R) = |\text{Spec}(\widehat{R})| - |\text{Spec}(R)|.$$

The splitting number of the ring  $R_s$  in Example 4.6 is  $s - 1$ . Corollary 4.8 says that every finitely generated Krull monoid defined by  $m$  equations can be realized as  $+(M)$  for some finitely generated module over a one-dimensional local ring (in fact, a domain essentially of finite type over  $\mathbb{Q}$ ) with splitting number  $m$ . This is the best possible:

**Theorem 4.10.** *Let  $M$  be a finitely generated module over a one-dimensional local ring  $R$  with splitting number  $m$ . Then the Krull monoid  $+(M)$  is defined by  $m$  equations.*

*Proof.* Write  $\widehat{M} = V_1^{(e_1)} \oplus \dots \oplus V_n^{(e_n)}$ , where the  $V_j$  are pairwise non-isomorphic indecomposable  $\widehat{R}$ -modules and the  $e_i$  are all positive. We have an embedding  $+(M) \hookrightarrow \mathbb{N}_0^{(n)}$  taking  $[F]$  to  $[b_1 \ \dots \ b_n]^{\text{tr}}$ , where  $\widehat{F} \cong V_1^{(b_1)} \oplus \dots \oplus V_n^{(b_n)}$ , and we identify  $+(M)$  with its image  $\Lambda$  in  $\mathbb{N}_0^{(n)}$ . Given a prime  $P \in \text{Spec}(R)$  with, say,  $t$

primes  $Q_1, \dots, Q_t$  lying over it, there are  $t - 1$  homogeneous linear equations on the  $b_j$  that say that  $\widehat{N}$  has constant rank on the fiber over  $P$  (cf. Corollary 4.5). Letting  $P$  vary over  $\text{Spec}(R)$ , we obtain exactly  $m = \text{spl}(R)$  equations that must be satisfied by elements of  $\Lambda$ . Conversely, if the  $b_j$  satisfy these equations, then  $N := \bigoplus_1^{(b_1)} V_1 \oplus \dots \oplus V_n^{(b_n)}$  has constant rank on each fiber of  $\text{Spec}(\widehat{R}) \rightarrow \text{Spec}(R)$ . By Corollary 4.5,  $N$  is extended from an  $R$ -module, say  $N \cong \widehat{F}$ . Clearly  $\widehat{F} \mid \widehat{M}^{(u)}$  if  $u$  is large enough, and it follows from Theorem 4.1 that  $F \in +(M)$ , whence  $[b_1 \dots b_n]^{\text{tr}} \in \Lambda$ .  $\square$

In [27], Karl Kattchee showed that, for each  $m$ , there is a finitely generated Krull monoid  $\Lambda$  that cannot be defined by  $m$  equations. Thus, no single one-dimensional local ring can realize every finitely generated Krull monoid in the form  $+(M)$  for a finitely generated module  $M$ .

We have seen that the monoids  $+(M)$  have a very rich structure. In contrast, the monoids  $\mathbb{V}(R\text{-mod})$ , for  $R$  a one-dimensional reduced local ring, are pretty boring. For certain Dedekind-like rings we will encounter the submonoid  $\Gamma$  of the free monoid  $\mathbb{N}_0^{(\aleph_0)}$  consisting of (finitely non-zero) sequences  $[a_i]$  satisfying  $\sum_i (-1)^i a_i = 0$ . For rings that are not Dedekind-like, we fix a positive integer  $q$  and let  $v_1, v_2, v_3, \dots$  be an enumeration of the elements of  $\mathbb{Z}^{(q)}$ . Let  $F$  be the free monoid with countably infinite basis  $\{b_1, b_2, b_3, \dots\}$ , and define  $f : F \rightarrow \mathbb{Z}^{(q)}$  by  $b_i \mapsto v_i$ . Now let  $\tau$  be an infinite cardinal, and define  $g : F^{(\tau)} \rightarrow \mathbb{Z}^{(q)}$  by taking the map  $f$  on each component. We let  $\Lambda(q, \tau) = \ker(g)$ . Finally, we let  $\Lambda(0, \tau) = \mathbb{N}_0^{(\tau)}$ , the free monoid with basis of cardinality  $\tau$ .

The following theorem, from [15] and [21], is an easy consequence of Theorem 2.2 and Theorem 4.4:

**Theorem 4.11.** *Let  $(R, \mathfrak{m}, k)$  be a reduced one-dimensional local ring, with splitting number  $q = \text{spl}(R)$ . Put  $\tau = |k| \aleph_0$ .*

- (1) *If  $R$  is not Dedekind-like, then  $\mathbb{V}(R\text{-mod}) \cong \Lambda(q, \tau)$ .*
- (2) *If  $R$  is a discrete valuation ring, then  $\mathbb{V}(R\text{-mod}) \cong \Lambda(0, \aleph_0) = \mathbb{N}_0^{(\aleph_0)}$ .*
- (3) *If  $R$  is Dedekind-like,  $R$  is not a discrete valuation ring, and  $q = 0$ , then  $\mathbb{V}(R\text{-mod}) \cong \mathbb{N}_0^{(\aleph_0)}$ .*
- (4) *If  $R$  is Dedekind-like and  $q > 0$ , then  $q = 1$  and  $\mathbb{V}(R\text{-mod}) \cong \Gamma \oplus \mathbb{N}_0^{(\tau)}$ .*

*In every case, the divisor class group of  $\mathbb{V}(R\text{-mod})$  is  $\mathbb{Z}^{(q)}$ .*  $\square$

The theorem raises two questions. First, what if  $R$  has non-zero nilpotents? The problem is that we do not have, in this case, a useful criterion for an  $\widehat{R}$ -module to be extended. Theorem 4.4 reduces the problem to the case of Artinian rings, but that does not eliminate the difficulty. The interested reader is referred to [24, Section 6] for a discussion of this problem.

Secondly, is there a similar classification of the monoid  $\mathcal{C}(R)$  of isomorphism classes of maximal Cohen–Macaulay modules (say, when the completion is reduced)? If  $R$  has finite Cohen–Macaulay type, such a classification has been worked out by Nicholas Baeth and Melissa Luckas in [1] and [2]. At the other extreme,

when *each* analytic branch has infinite Cohen–Macaulay type, Andrew Crabbe and Silvia Saccon [8] have a result similar to Theorem 4.7 above, from which one can decode the structure of  $\mathcal{C}(R)$ . The intermediate case, e.g.,  $R = k[[X, Y]]/X(X^3 - Y^4)$ , where  $R$  has infinite Cohen–Macaulay type but at least one branch has finite Cohen–Macaulay type, is discussed in [8], but much less is known about the possible ranks of the indecomposables in this case.

## 5 Direct-sum cancellation

Let  $R$  be a commutative Noetherian ring. In very general terms, the direct-sum cancellation question is this: If  $M$ ,  $N$ , and  $L$  are  $R$ -modules in some fixed subcategory  $\mathcal{C} \subseteq R\text{-mod}$ , where  $R\text{-mod}$  is the category of all finitely generated  $R$ -modules, when does  $M \oplus L \cong N \oplus L$  always imply  $M \cong N$ ? When this is the case, we say that cancellation holds for  $R$  (with respect to the chosen category). Otherwise, we say cancellation fails for  $R$ .

Evans [14] and Vasconcelos [38] showed that cancellation of arbitrary finitely generated modules always holds over semilocal rings. Since the cancellation question is interesting only if the ring is not semilocal, we focus largely on *non*-semilocal rings in this section. However, the localizations of a ring  $R$  do play a role in answering some kinds of cancellation questions over  $R$  itself.

The cancellation question gained prominence in 1955 through its connection with the celebrated conjecture of Serre [37]: If  $R$  is the polynomial ring in a finite number of variables over a field, is every finitely generated projective  $R$ -module free? Serre reduced his question to a cancellation question involving projective modules: If  $P$  and  $Q$  are finitely generated projective  $R$ -modules such that  $P \oplus R \cong Q \oplus R$ , are  $P$  and  $Q$  necessarily isomorphic?

Well before the proof of Serre’s Conjecture by Quillen and Suslin in 1976, the cancellation question had taken on a life of its own. The emphasis shifted to other categories of modules and other rings. In 1962, Chase [6] studied cancellation of finitely generated torsion-free modules over two-dimensional rings. He proved, for example, that torsion-free cancellation holds for the ring  $R = k[X, Y]$  when  $k$  is an algebraically closed field of characteristic zero. He also produced non-isomorphic torsion-free modules  $A$  and  $B$  over  $R = \mathbb{R}[X, Y]$  such that  $A \oplus R \cong B \oplus R$ .

The first known failure of cancellation for finitely generated modules is perhaps due to Kaplansky, who used the non-triviality of the tangent bundle on the two-sphere to produce a module  $T$  over  $R = \mathbb{R}[X, Y]/(X^2 + Y^2 + Z^2 - 1)$  such that  $T \oplus R \cong R^3$  and yet  $T \not\cong R^2$ . For quite a while, every known failure of cancellation for finitely generated modules over commutative rings involved rings of dimension greater than one. Even as late as 1973, Eisenbud and Evans [13] raised the following question: Does cancellation hold for arbitrary finitely generated modules over one-dimensional Noetherian rings?

In the 1980s, effective techniques, such as those in [48], were developed for studying the cancellation of finitely generated torsion-free modules over one-

dimensional rings. We will sketch some of the main ideas. We assume, from now on, that all modules are finitely generated.

Borrowing from the notation we used previously for local rings, we let  $R$  be a one-dimensional domain such that the integral closure  $\bar{R}$  of  $R$  in its quotient field is a finitely generated  $R$ -module. As before, the conductor of  $\bar{R}$  in  $R$  is denoted by  $\mathfrak{f}$ . (The reader may find it helpful to refer to the pullback that precedes Proposition 1.2.) The main technique in [48] is to examine the relationship between  $M/\mathfrak{f}M$  and  $\bar{R}M/\mathfrak{f}M$  for torsion-free  $R$ -modules  $M$ .

Given a torsion-free  $R$ -module  $M$ , one defines the so-called “delta group” of  $M$ , denoted  $\Delta_M$ . This is the subgroup of  $(\bar{R}/\mathfrak{f})^\times$  consisting of determinants of automorphisms of  $\bar{R}M/\mathfrak{f}M$  that carry  $M/\mathfrak{f}M$  into itself. (See [48] for the basic properties of  $\Delta_M$ .) There are two important facts we need:

- (1)  $\Delta_{M \oplus N} = \Delta_M \cdot \Delta_N$ .
- (2) If  $M_{\mathfrak{m}} \cong N_{\mathfrak{m}}$  for each maximal ideal  $\mathfrak{m}$ , then  $\Delta_M = \Delta_N$ .

The first fact allows one to restrict attention to *indecomposable* torsion-free  $R$ -modules. The second fact says that the delta group is an invariant of the local isomorphism class of  $M$ . Now let  $\Lambda_{\mathfrak{f}}$  be the image of  $(\bar{R})^\times$  in  $(\bar{R}/\mathfrak{f})^\times$ . We call this the group of *liftable units* with respect to  $\mathfrak{f}$ . The next theorem follows directly from Lemma 1.6 and Proposition 1.9 in [48].

**Theorem 5.1.** *Let  $R$ ,  $\bar{R}$ , and  $\mathfrak{f}$  be as above. Then  $R$  has torsion-free cancellation if and only if  $(\bar{R}/\mathfrak{f})^\times \subseteq \Delta_M \cdot \Lambda_{\mathfrak{f}}$  for all torsion-free  $R$ -modules  $M$ . □*

Next, consider the cancellation question for arbitrary finitely generated modules. We shall call this the *mixed* cancellation question. Is there a result similar to the preceding theorem that pertains to the mixed cancellation question? Such a result appears in [23]. Let  $S$  denote the complement of the union of the maximal ideals of  $R$  that contain  $\mathfrak{f}$ . Then  $S^{-1}R$  is a semilocal domain of dimension one. One defines a delta group for  $S^{-1}M$ , denoted  $\Delta_{S^{-1}M}$ . From Corollary 4.4 of [23] one gets the following result, where now  $\Lambda_S$  denotes the group of units of  $S^{-1}\bar{R}$  that lift to units of  $\bar{R}$ .

**Theorem 5.2.** *Let  $R$ ,  $\bar{R}$ , and  $S$  be as above. Then  $R$  has mixed cancellation if and only if  $(S^{-1}\bar{R})^\times \subseteq \Delta_{S^{-1}M} \cdot \Lambda_S$  for every finitely generated  $R$ -module  $M$ .*

An important question one can raise at this point is whether torsion-free cancellation implies mixed cancellation. It was shown in [23] that the two kinds of cancellation are not equivalent in general. We will give an example from that paper in Section 5.2 below.

Suppose, now, that  $R$  is an order in an algebraic number field  $K$ . That is, suppose  $\mathcal{O}_K$  is the ring of algebraic integers of  $K$  and  $R$  is a subring of  $\mathcal{O}_K$  such that  $\mathbb{Q}R = K$ . (Then  $\bar{R} = \mathcal{O}_K$ .) If  $R$  is a *quadratic* order then  $R$  has finite Cohen–Macaulay type. In [41], definitive results were obtained for torsion-free cancellation over quadratic orders. In [26], one can find decisive answers to the torsion-free cancellation question

for a large family of cubic orders having finite Cohen–Macaulay type. In these two papers, each of the present authors used methods based on the calculation of delta groups. We will revisit these results in more detail below.

In [25] and [26], a connection between cancellation and finite Cohen–Macaulay type is exploited. The work is based on the idea that the failure of finite Cohen–Macaulay type often implies the failure of cancellation. In these two papers, negative answers to the torsion-free cancellation question are given for many quartic and higher-degree orders.

In the remainder of this section, we will focus on the cancellation question for one-dimensional Noetherian domains  $R$ , although many of the results given below are known to hold for other classes of rings as well, especially for reduced rings. Throughout,  $R$  will be a one-dimensional domain with quotient field  $K$ . Also,  $\bar{R}$  will always be the integral closure of  $R$  in  $K$ . We insist that  $\bar{R}$  be finitely generated as an  $R$ -module.

### 5.1 Torsion-free cancellation over one-dimensional domains

Let  $D(R)$  denote the kernel of the natural map on Picard groups  $\text{Pic}R \rightarrow \text{Pic}\bar{R}$ . If  $D(R) \neq 0$  then one can show that  $R$  has an invertible ideal  $I \not\cong R$  such that  $I \oplus \bar{R} \cong R \oplus \bar{R}$  (cf. [41, Corollary 2.4]). This is one of the easiest ways in which torsion-free cancellation can fail. For certain kinds of rings,  $D(R)$  is exactly the obstruction to torsion-free cancellation. For example, the following is from Theorem 0.1 of [44]:

**Theorem 5.3.** *Let  $R$  be as above. Assume further that  $R$  is finitely generated as a  $k$ -algebra for some infinite perfect field  $k$ . Then  $R$  has torsion-free cancellation if and only if  $D(R) = 0$ .  $\square$*

For examples of affine  $k$ -domains where  $D(R) = 0$ , we have Dedekind domains and the rings  $F + XK[X]$ , where  $k \subseteq F \subseteq K$  are field extensions of finite degree. In fact [44, (1.7)], up to analytic isomorphism, these are the only examples! In particular [41, Corollary 3.3], an affine domain over an algebraically closed field has torsion-free cancellation if and only if it is a Dedekind domain.

Another case where  $D(R)$  controls torsion-free cancellation is provided by Theorem 2.7 of [41]:

**Theorem 5.4.** *Let  $R$  be as above. Assume that every ideal of  $R$  is two-generated. Then  $R$  has torsion-free cancellation if and only if  $D(R) = 0$ .  $\square$*

In [41], the theorem above is applied to orders in quadratic number fields. We state the following classification result for imaginary quadratic orders (Theorem 4.5 of [41]):

**Theorem 5.5.** *Let  $d$  be a squarefree negative integer, and let  $R$  be an order in  $\mathbb{Q}(\sqrt{d})$ . Then  $R$  has torsion-free cancellation if and only if either  $R = \bar{R}$  or else  $R$  satisfies one of the following:*



- (1)  $R = \mathbb{Z}[\sqrt{d}]$  where  $d \equiv 1 \pmod{8}$
- (2)  $R = \mathbb{Z}[2\sqrt{-1}]$
- (3)  $R = \mathbb{Z}[\sqrt{-3}]$
- (4)  $R = \mathbb{Z}[\frac{3}{2}(1 + \sqrt{-3})]$  □

For real quadratic orders  $R$ , the situation is more complicated. The condition  $D(R) = 0$  depends on subtle arithmetical properties of the fundamental unit of  $\bar{R}$ , and it is extremely difficult to give a version of Theorem 5.5 that classifies those real quadratic orders having torsion-free cancellation. But given any *specific* real quadratic order  $R$ , a finite calculation involving the fundamental unit of  $\bar{R}$  will determine whether or not torsion-free cancellation holds.

The cancellation question can be answered decisively if one knows all the delta groups that come from indecomposable torsion-free  $R$ -modules. In cases where  $R$  has finite Cohen–Macaulay type, one has some hope of calculating these delta groups. This is indeed the case for quadratic orders. The following result is equivalent to Corollary 4.2 of [41] but is stated in terms of data intrinsic to the ring. Recall that  $f$  is the conductor of  $\bar{R}$  in  $R$ .

**Theorem 5.6.** *Suppose  $R = \mathbb{Z} + f\mathcal{O}_K$  is an order in a quadratic number field  $K$ , where  $f \in \mathbb{Z}$  is a nonzero nonunit. Then torsion-free cancellation holds for  $R$  if and only if  $(\bar{R}/f)^\times \subseteq (R/f)^\times \cdot \Lambda_f$ .* □

Let us compare this with Theorem 5.1, where the statement of the condition for cancellation to hold depends on the entire family of isomorphism classes of indecomposable torsion-free  $R$ -modules. For a quadratic order  $R$ , it is known [3] that every indecomposable torsion-free  $R$ -module has rank one. Furthermore, there are only finitely many isomorphism classes of such modules. This makes it possible to replace the condition in Theorem 5.1 with a condition that depends only on subgroups of  $(\bar{R}/f)^\times$ .

We now state some results for cubic orders. It is well known that every quadratic order  $R$  in  $K$  has the form  $R = \mathbb{Z} + f\mathcal{O}_K$  for some nonzero rational integer  $f$ . While this is not necessarily true for cubic orders, one can consider cubic orders of that same form. Now, a cubic order  $R$  having finite Cohen–Macaulay type may have indecomposable torsion-free modules of rank greater than 1. The following result is a special case of Theorem 31 in [26] and depends crucially on the existence of indecomposable torsion-free  $R$ -modules of rank two:

**Theorem 5.7.** *Suppose  $R = \mathbb{Z} + p\mathcal{O}_K$  is an order in a cubic number field  $K$ , where  $p \in \mathbb{Z}$  is nonzero. Further, suppose  $p\bar{R}$  is a prime ideal. Then torsion-free cancellation holds for  $R$  if and only if*

- (1)  $(\bar{R}/f)^\times \subseteq (R/pR)^\times \cdot \Lambda_f$ , and
- (2)  $(\bar{R}/f)^\times \subseteq ((\bar{R}/f)^\times)^2 \cdot \Lambda_f$  □

This is similar to Theorem 5.6. Once again, the torsion-free cancellation question for  $R$  is answered in terms of subgroups of  $(\overline{R}/\mathfrak{f})^\times$ . Using this result, one can find examples of cubic orders  $R$  for which  $D(R) = 0$  and yet torsion-free cancellation fails for  $R$ .

There also exist many cubic orders that do not have finite Cohen–Macaulay type. Moreover, most orders in number fields of degree four and higher do not have finite Cohen–Macaulay type. Using the Drozd–Roĭter conditions [12] (cf. Theorem 1.1 in Section 1), we easily get the following (see Proposition 19 of [26]).

**Lemma 5.8.** *Let  $K$  be a number field of degree  $d$  and suppose  $R = \mathbb{Z} + f\mathcal{O}_K$  is an order, where  $f \in \mathbb{Z}$  is a nonzero nonunit. Then  $R$  has finite Cohen–Macaulay type if and only if either (i)  $d = 2$  or (ii)  $d = 3$  and  $f$  is square-free.  $\square$*

Failure of finite Cohen–Macaulay type often leads to failure of torsion-free cancellation. Many such examples can be given using the following result, which is a specialized version of Theorem 26 in [26].

**Theorem 5.9.** *Let  $K$  be a number field of degree at least four. Suppose  $R = \mathbb{Z} + f\mathcal{O}_K$  is an order, where  $f \in \mathbb{Z}$  is a nonzero nonunit. Then torsion-free cancellation holds for  $R$  if and only if  $(\overline{R}/\mathfrak{f})^\times \subseteq \Lambda_{\mathfrak{f}}$ .  $\square$*

The condition appearing in the result above is quite satisfying, given that the category of torsion-free  $R$ -modules for these orders is generally intractable. It turns out that the condition  $(\overline{R}/\mathfrak{f})^\times \subseteq \Lambda_{\mathfrak{f}}$  is rarely satisfied. For example, the next result follows directly from Corollary 7.1 in [25].

**Corollary 5.10.** *Let  $K$  be a number field of degree four or higher. Then there are only finitely many primes  $p \in \mathbb{Z}$  for which the order  $R = \mathbb{Z} + p\mathcal{O}_K$  has torsion-free cancellation.  $\square$*

## 5.2 Mixed cancellation for one-dimensional domains

In [23], Hassler and Wiegand found a way to extend the techniques in [48] to handle arbitrary finitely generated modules. The original motivation for the work in [23] was the following question: When does torsion-free cancellation imply mixed cancellation? The following theorem gives a class of rings for which the answer is affirmative (see Theorem 6.1 of [23]):

**Theorem 5.11.** *Let  $R$  be a one-dimensional Noetherian domain. Further, suppose  $R$  is finitely generated as  $k$ -algebra, where  $k$  is an infinite field of characteristic zero. The following are equivalent:*

- (1)  $D(R) = 0$
- (2)  $R$  has torsion-free cancellation
- (3)  $R$  has mixed cancellation  $\square$

In the same paper [23], the class of Dedekind-like rings is considered. A one-dimensional, reduced, Noetherian ring  $R$  is defined to be Dedekind-like if  $R_{\mathfrak{m}}$  is Dedekind-like for all maximal ideals  $\mathfrak{m}$  of  $R$ . (See Section 3 for the definition of local Dedekind-like rings.) The following is from Corollary 6.11 in [23] and depends heavily on Levy and Klingler's classification [30] of indecomposable modules over local Dedekind-like rings:

**Theorem 5.12.** *Suppose  $R$  is a Dedekind-like order in a number field. Then torsion-free cancellation implies mixed cancellation.*  $\square$

Likewise, Hassler [18] has proved the following theorem. (We note that orders in quadratic number fields need not be Dedekind-like. For example,  $\mathbb{Z}[2\sqrt{-1}]$  is not Dedekind-like.)

**Theorem 5.13.** *Suppose  $R$  is an order in an imaginary quadratic field. Then torsion-free cancellation implies mixed cancellation.*  $\square$

Now, suppose  $R$  is an order in a real quadratic field such that  $R$  is not Dedekind-like. Does torsion-free cancellation still imply mixed cancellation over  $R$ ? The authors in [23] show that the order  $R = \mathbb{Z}\left[17\frac{1+\sqrt{17}}{2}\right]$  has torsion-free cancellation but does not have mixed cancellation!

Finally, we remark that when  $R$  is an order in a real quadratic field, Hassler has shown in [19] that the mixed cancellation question for  $R$  can often be answered by a computation that involves the fundamental unit of  $R$ . The computation is a more complicated version of the one mentioned in the paragraph that follows Theorem 5.5 above.

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# The defect

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**Abstract** We give an introduction to the valuation theoretical phenomenon of “defect”, also known as “ramification deficiency”. We describe the role it plays in deep open problems in positive characteristic: local uniformization (the local form of resolution of singularities), the model theory of valued fields, the structure theory of valued function fields. We give several examples of algebraic extensions with non-trivial defect. We indicate why Artin–Schreier defect extensions play a central role and describe a way to classify them. Further, we give an overview of various results about the defect that help to tame or avoid it, in particular “stability” theorems and theorems on “henselian rationality”, and show how they are applied. Finally, we include a list of open problems.

## 1 Valued fields

Historically, there are three main origins of valued fields:

(1) Number theory: Kurt Hensel introduced the fields  $\mathbb{Q}_p$  of  $p$ -adic numbers and proved the famous Hensel’s Lemma (see below) for them. They are defined as the completions of  $\mathbb{Q}$  with respect to the (ultra)metric induced by the  $p$ -adic valuations of  $\mathbb{Q}$ , similarly as the field of reals,  $\mathbb{R}$ , is the completion of  $\mathbb{Q}$  with respect to the usual metric induced by the ordering on  $\mathbb{Q}$ .

(2) Ordered fields:  $\mathbb{R}$  is the maximal archimedean ordered field; any ordered field properly containing  $\mathbb{R}$  will have infinite elements, that is, elements larger than all

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reals. Their inverses are infinitesimals. The classes of magnitude, called archimedean classes, give rise to a natural valuation. These valuations are important in the theory of ordered fields and in real algebraic geometry.

In connection with ordered fields and their classes of magnitude, Hans Hahn [27] introduced an important class of valued fields, the (*generalized*) *power series fields*. Take any field  $K$  and any ordered abelian group  $G$ . Let  $K((G))$  (also denoted by  $K((t^G))$ ) be the set of all maps  $\mu$  from  $G$  to  $K$  with well-ordered support  $\{g \in G \mid \mu(g) \neq 0\}$ . One can visualize the elements of  $K((G))$  as formal power series  $\sum_{g \in G} c_g t^g$  for which the support  $\{g \in G \mid c_g \neq 0\}$  is well-ordered. Using this condition one shows that  $K((G))$  is a field. Also, one uses it to introduce the valuation:

$$v_t \sum_{g \in G} c_g t^g = \min\{g \in G \mid c_g \neq 0\} \tag{1}$$

(the minimum exists because the support is well-ordered). This valuation is called the *canonical valuation* or *t-adic valuation* of  $K((G))$ , and sometimes called the *minimum support valuation*. Note that  $v_t t = 1$ . For  $G = \mathbb{Z}$ , one obtains the field of formal Laurent series  $K((t))$ .

(3) *Function fields*: if  $K$  is any field and  $X$  an indeterminate, then the rational function field  $K(X)$  has a  $p(X)$ -adic valuation for every irreducible polynomial  $p(X) \in K[X]$ , plus the  $1/X$ -adic valuation. These valuations are trivial on  $K$ . As a valuation can be extended to every extension field, these valuations together with the  $p$ -adic valuations mentioned in (1) yield that a field admits a non-trivial valuation as soon as it is not algebraic over a finite field. In particular, all algebraic function fields over  $!K$  (i.e., finitely generated field extensions of  $K$  of transcendence degree  $\geq 1$ ) admit non-trivial valuations that are trivial on  $K$ . Such valued function fields play a role in several areas of algebra and number theory, some of which we will mention in this paper. Throughout, *function field* will always mean *algebraic function field*.

If  $K$  is a field with a valuation  $v$ , then we will denote its value group by  $vK$  and its residue field by  $Kv$ . For  $a \in K$ , its value is  $va$ , and its residue is  $av$ . An extension of valued fields is written as  $(L'|L, v)$ , meaning that  $L'|L$  is a field extension,  $v$  is a valuation on  $L'$  and  $L$  is equipped with the restriction of this valuation. Then there is a natural embedding of the value group  $vL$  in the value group  $vL'$ , and a natural embedding of the residue field  $Lv$  in the residue field  $L'v$ . If both embeddings are onto (which we just express by writing  $vL = vL'$  and  $Lv = L'v$ ), then the extension  $(L'|L, v)$  is called *immediate*. For  $a \in L'$  we set  $v(a - L) := \{v(a - c) \mid c \in L\}$ . The easy proof of the following lemma is left to the reader:

**Lemma 1.1.** *The extension  $(L'|L, v)$  is immediate if and only if for all  $a \in L'$  there is  $c \in L$  such that  $v(a - c) > va$ . If the extension  $(L'|L, v)$  is immediate, then  $v(a - L)$  has no maximal element and is an initial segment of  $vL$ , that is, if  $\alpha \in v(a - L)$  and  $\alpha > \beta \in vL$ , then  $\beta \in v(a - L)$ .*

If for each  $a \in L'$  and every  $\alpha \in vL'$  there is  $c \in L$  such that  $v(a - c) > \alpha$ , then we say that  $(L, v)$  is *dense in*  $(L', v)$ . If this holds, then the extension  $(L'|L, v)$  is

immediate. The maximal extension in which  $(L, \nu)$  is dense is its *completion*  $(L, \nu)^c$ , which is unique up to isomorphism.

Every finite extension  $L'$  of a valued field  $(L, \nu)$  satisfies the *fundamental inequality* (cf. (17.5) of [18] or Theorem 19 on p. 55 of [67]):

$$n \geq \sum_{i=1}^g e_i f_i \tag{2}$$

where  $n = [L' : L]$  is the degree of the extension,  $\nu_1, \dots, \nu_g$  are the distinct extensions of  $\nu$  from  $L$  to  $L'$ ,  $e_i = (v_i L' : \nu L)$  are the respective *ramification indices* and  $f_i = [L' \nu_i : L \nu_i]$  are the respective *inertia degrees*. If  $g = 1$  for every finite extension  $L' | L$  then  $(L, \nu)$  is called *henselian*. This holds if and only if  $(L, \nu)$  satisfies *Hensel's Lemma*, that is, if  $f$  is a polynomial with coefficients in the valuation ring  $\mathcal{O}$  of  $(L, \nu)$  and there is  $b \in \mathcal{O}$  such that  $\nu f(b) > 0$  and  $\nu f'(b) = 0$ , then there is  $a \in \mathcal{O}$  such that  $f(a) = 0$  and  $\nu(b - a) > 0$ .

Every valued field  $(L, \nu)$  admits a *henselization*, that is, a minimal algebraic extension which is henselian (see Section 4 below). All henselizations are isomorphic over  $L$ , so we will frequently talk of *the* henselization of  $(L, \nu)$ , denoted by  $(L, \nu)^h$ . The henselization becomes unique in absolute terms once we fix an extension of the valuation  $\nu$  from  $L$  to its algebraic closure. All henselizations are immediate separable-algebraic extensions. If  $(L', \nu)$  is a henselian extension field of  $(L, \nu)$ , then a henselization of  $(L, \nu)$  can be found inside of  $(L', \nu)$ .

For the basic facts of valuation theory, we refer the reader to [5, Appendix], [18, 19, 59, 65, 67]. For ramification theory, we recommend [18, 19, 55]. For basic facts of model theory, see [11].

For a field  $K$ ,  $\bar{K}$  will denote its algebraic closure and  $K^{\text{sep}}$  will denote its separable-algebraic closure. If  $\text{char} K = p$ , then  $K^{1/p^\infty}$  will denote its perfect hull. If we have two subfields  $K, L$  of a field  $M$  (in our cases, we will usually have the situation that  $L \subset \bar{K}$ ) then  $K.L$  will denote the smallest subfield of  $M$  which contains both  $K$  and  $L$ ; it is called the *field compositum* of  $K$  and  $L$ .

## 2 Two problems

Let us look at two important problems that will lead us to considering the phenomenon of defect:

### 2.1 Elimination of ramification

Given a valued function field  $(F|K, \nu)$ , we want to find nice generators of  $F$  over  $K$ . For instance, if  $F|K$  is separable then it is separably generated, that is, there is a transcendence basis  $T$  such that  $F|K(T)$  is a finite separable extension, hence simple. So we can write  $F = K(T, a)$  with  $a$  separable-algebraic over  $K(T)$ .



In the presence of the valuation  $v$ , we may want to ask for more. The problem of *smooth local uniformization* is to find generators  $x_1, \dots, x_n$  of  $F|K$  in the valuation ring  $\mathcal{O}$  of  $v$  on  $F$  such that the point  $x_1v, \dots, x_nv$  is smooth, that is, the Implicit Function Theorem holds in this point. We say that  $(F|K, v)$  is *inertially generated* if there is a transcendence basis  $T$  such that  $F$  lies in the *absolute inertia field*  $K(T)^i$  (see Section 4 for its definition). A connection between both notions is given by Theorem 1.6 of [35]:

**Theorem 2.1.** *If  $(F|K, v)$  admits smooth local uniformization, then it is inertially generated.*

If  $(F|K, v)$  is inertially generated by the transcendence basis  $T$ , then  $vF = vK(T)$ , and  $Fv|K(T)v$  is separable. If this were not true, we would say that  $(F|K(T), v)$  is *ramified*. Let us consider an example.

*Example 2.2.* Suppose that  $v$  is a discrete valuation on  $F$  which is trivial on  $K$  and such that  $Fv|K$  is algebraic. So there is an element  $t \in F$  such that  $vF = \mathbb{Z}vt = vK(t)$ . Take the henselization  $F^h$  of  $F$  with respect to some fixed extension of  $v$  to the algebraic closure of  $F$ .

Assume that  $\text{trdeg} F|K = 1$ ; then  $F|K(t)$  is finite. Take  $K(t)^h$  to be the henselization of  $K(t)$  within  $F^h$ . Then  $F^h|K(t)^h$  is again finite since  $F^h = F \cdot K(t)^h$  (cf. Theorem 4.14 below). If  $\text{trdeg} F|K > 1$ , we can take  $T$  to be a transcendence basis of  $F|K$  which contains  $t$ . Then again,  $F^h|K(T)^h$  is finite, and  $vF = vK(T)$ . But does that prove that  $F|K$  is inertially generated? Well, if for instance  $K$  is algebraically closed, then it follows that  $Fv = K = K(T)v$ , so  $Fv|K(T)v$  is separable. But “inertially generated” asks for more. In order to show that  $F|K$  is inertially generated in this particular case, we would have to find  $T$  such that  $F \subseteq K(T)^h$ , that is, the extension  $F^h|K(T)^h$  is trivial (see Section 4).

Since  $K(T)^h$  is henselian, there is only one extension of the given valuation from  $K(T)^h$  to  $F^h$ . By our choice of  $T$ , we have  $e = (vF^h : vK(T)^h) = (vF : vK(T)) = 1$ , and if  $K$  is algebraically closed, also  $f = (F^hv : K(T)^hv) = (Fv : K(T))v = 1$ . Hence equality holds in the fundamental inequality (2) if and only if  $F^h|K(T)^h$  is trivial.

This example shows that it is important to know when the fundamental inequality (2) is in fact an equality, or more precisely, what the quotient  $n/ef$  is. A first and important answer is given by the *Lemma of Ostrowski*. Assume that  $(L'|L, v)$  is a finite extension and the extension of  $v$  from  $L$  to  $L'$  is unique. Then the Lemma of Ostrowski says that

$$[L' : L] = p^v \cdot (vL' : vL) \cdot [L'v : Lv] \quad \text{with } v \geq 0 \tag{3}$$

where  $p$  is the *characteristic exponent* of  $Lv$ , that is,  $p = \text{char} Lv$  if this is positive, and  $p = 1$  otherwise. The Lemma of Ostrowski can be proved using Tschirnhausen transformations (cf. [59, Theoreme 2, p. 236]). But it can also be deduced from ramification theory, as we will point out in Section 4 (see also [67, Corollary to Theorem 25, p. 78]).

The factor  $d = d(L'|L, \nu) = p^v$  is called the *defect* (or *ramification deficiency* as in [67, p. 58]) of the extension  $(L'|L, \nu)$ . If  $d = 1$ , then we call  $(L'|L, \nu)$  a *defectless extension*; otherwise, we call it a *defect extension*. Note that  $(L'|L, \nu)$  is always defectless if  $\text{char } K\nu = 0$ .

We call  $(L, \nu)$  a *defectless field*, *separably defectless field* or *inseparably defectless field* if equality holds in the fundamental inequality (2) for every finite, finite separable or finite purely inseparable, respectively, extension  $L'$  of  $L$ . One can trace this back to the case of unique extensions of the valuation; for the proof of the following theorem, see [38] (a partial proof was already given in Theorem 18.2 of [18]):

**Theorem 2.3.** *A valued field  $(L, \nu)$  is a defectless field if and only if its henselization is. The same holds for “separably defectless” and “inseparably defectless”.*

Therefore, the Lemma of Ostrowski shows:

**Corollary 2.4.** *Every valued field  $(L, \nu)$  with  $\text{char } L\nu = 0$  is a defectless field.*

The defect is multiplicative in the following sense. Let  $(L|K, \nu)$  and  $(M|L, \nu)$  be finite extensions. Assume that the extension of  $\nu$  from  $K$  to  $M$  is unique. Then the defect satisfies the following product formula

$$d(M|K, \nu) = d(M|L, \nu) \cdot d(L|K, \nu) \tag{4}$$

which is a consequence of the multiplicativity of the degree of field extensions and of ramification index and inertia degree. This formula implies:

**Lemma 2.5.**  *$(M|K, \nu)$  is defectless if and only if  $(M|L, \nu)$  and  $(L|K, \nu)$  are defectless.*

**Corollary 2.6.** *If  $(L, \nu)$  is a defectless field and  $(L', \nu)$  is a finite extension of  $(L, \nu)$ , then  $(L', \nu)$  is also a defectless field. Conversely, if there exists a finite extension  $(L', \nu)$  of  $(L, \nu)$  such that  $(L', \nu)$  is a defectless field, the extension of  $\nu$  from  $L$  to  $L'$  is unique, and the extension  $(L'|L, \nu)$  is defectless, then  $(L, \nu)$  is a defectless field. The same holds for “separably defectless” in the place of “defectless” if  $L'|L$  is separable, and for “inseparably defectless” if  $L'|L$  is purely inseparable.*

The situation of our Example 2.2 becomes more complicated when the valuations are not discrete:

*Example 2.7.* There are valued function fields  $(F|K, \nu)$  of transcendence degree 2 with  $\nu$  trivial on  $K$  such that  $\nu F$  is not finitely generated. Already on a rational function field  $K(x, y)$ , the value group of a valuation trivial on  $K$  can be any subgroup of the rationals  $\mathbb{Q}$  (see Theorem 1.1 of [42] and the references given in that paper). In such cases, if  $F$  is not a rational function field, it is not easy to find a transcendence basis  $T$  such that  $\nu F = \nu K(T)$ . But even if we find such a  $T$ , what do we know then about the extension  $(F^h|K(T)^h, \nu)$ ? For example, is it defectless?

An extension  $(L'|L, \nu)$  of henselian fields is called *unramified* if  $\nu L' = \nu L$ ,  $L'\nu|L\nu$  is separable and every finite subextension of  $(L'|L, \nu)$  is defectless. Hence if

$\text{char } Lv = 0$ , then  $(L'|L, \nu)$  is unramified already if  $\nu L' = \nu L$ . Note that our definition of “unramified” is stronger than the definition in [18, Section 22] which does not require “defectless”.

For a valued function field  $(F|K, \nu)$ , *elimination of ramification* means to find a transcendence basis  $T$  such that  $(F^h|K(T)^h, \nu)$  is unramified. According to Theorem 4.18 in Section 4 below, this is equivalent to  $F$  lying in the absolute inertia field  $K(T)^i$ . Hence,  $(F|K, \nu)$  admits elimination of ramification if and only if it is inertially generated.

If  $\text{char } K = 0$  and  $\nu$  is trivial on  $K$ , then  $(F|K, \nu)$  is always inertially generated; this follows from Zariski’s local uniformization [66] by Theorem 2.1. Since then  $\text{char } F\nu = \text{char } K\nu = \text{char } K = 0$ , Zariski did not have to deal with inseparable residue field extensions and with defect. But if  $\text{char } K > 0$ , then the existence of defect makes the problem of local uniformization much harder. This becomes visible in the approach to local uniformization that is used in the papers [34] and [35]. Local uniformization can be proved for Abhyankar places in positive characteristic because the defect does not appear [34]; we will discuss this in more detail below. For other places [35], the defect has to be “killed” by a finite extension of the function field (“alteration”).

## 2.2 Classification of valued fields up to elementary equivalence

Value group and residue field are invariants of a valued field, that is, two isomorphic valued fields have isomorphic value groups and isomorphic residue fields. But two valued fields with the same value groups and residue fields need not at all be isomorphic. For example, the valued field  $(\mathbb{F}_p(t), \nu_t)$  and  $(\mathbb{F}_p((t)), \nu_t)$  both have value group  $\mathbb{Z}$  and residue field  $\mathbb{F}_p$ , but they are not isomorphic since  $\mathbb{F}_p(t)$  is countable and  $\mathbb{F}_p((t))$  is not.

In situations where classification up to isomorphism fails, classification up to elementary equivalence may still be possible. Two algebraic structures are *elementarily equivalent* if they satisfy the same elementary (first order) sentences. For example, Abraham Robinson proved that all algebraically closed valued fields of fixed characteristic are elementarily equivalent (cf. [60, Theorem 4.3.12]). James Ax and Simon Kochen and, independently, Yuri Ershov proved that two henselian valued fields are elementarily equivalent if their value groups are elementarily equivalent and their residue fields are elementarily equivalent and of characteristic 0 (cf. [6] and [11, Theorem 5.4.12])). They also proved that all  $p$ -adically closed fields are elementarily equivalent (cf. [7, Theorem 2]). Likewise, Alfred Tarski proved that all real closed fields are elementarily equivalent (cf. [60, Theorem 4.3.3] or [11, Theorem 5.4.4]). This remains true if we consider non-archimedean real closed fields together with their natural valuations [12]. These facts (and the corresponding model completeness results) all have important applications in algebra (for instance, Nullstellensätze, Hilbert’s 17th Problem, cf. [28, Chap. A4, Section 2]). So we would like to know when classification up to elementary equivalence is possible for more general classes of valued fields.

Two elementarily equivalent valued fields have elementarily equivalent value groups and elementarily equivalent residue fields. When does the converse hold? We mentioned already that the henselization is an immediate extension. So the elementary properties of value group and residue field do not determine whether a field is henselian or not. But being henselian is an elementary property, expressed by a scheme of elementary sentences, one each for all polynomials of degree  $n$ , where  $n$  runs through all natural numbers. In our above example,  $(\mathbb{F}_p((t)), v_t)$  is henselian, but  $(\mathbb{F}_p(t), v_t)$  is not, so they are not elementarily equivalent. We see that in order to have classification up to elementary equivalence relative to value groups and residue fields, our fields need to be (at least) henselian. But if the characteristic of the residue fields is positive, then we may have proper immediate algebraic extensions of henselian valued fields, as we will see in the next section. So our fields need to be (at least) *algebraically maximal*, that is, not admitting any proper immediate algebraic extensions.

Our fields even have to be defectless. Indeed, every valued field  $(K, v)$  admits a maximal immediate extension  $(M, v)$ . Then  $(M, v)$  is maximal and therefore henselian and defectless. Since  $vK = vM$  and  $Kv = Mv$ , we want that  $(K, v) \equiv (M, v)$ . The property “henselian and defectless field” is elementary (cf. [16, 1.33], [38] or the background information in [46]), so  $(K, v)$  should be a henselian defectless field.

If  $\mathcal{L}$  is an elementary language and  $\mathcal{A} \subset \mathcal{B}$  are  $\mathcal{L}$ -structures, then we will say that  $\mathcal{A}$  is *existentially closed in*  $\mathcal{B}$  and write  $\mathcal{A} \prec_{\exists} \mathcal{B}$  if every existential sentence with parameters from  $\mathcal{A}$  that holds in  $\mathcal{B}$  also holds in  $\mathcal{A}$ . When we talk of fields, then we use the language of rings  $(\{+, -, \cdot, 0, 1\})$  or fields (adding the unary function symbol “ $\cdot^{-1}$ ”). When we talk of valued fields, we augment this language by a unary relation symbol for the valuation ring or a binary relation symbol for valuation divisibility (“ $v x \leq v y$ ”). For ordered abelian groups, we use the language of groups augmented by a binary predicate (“ $x < y$ ”) for the ordering. For the meaning of “existentially closed in” in the settings of fields, valued fields and ordered abelian groups, see [51, p. 183].

By model theoretical tools such as Robinson’s Test, the classification problem can be transformed into the problem of finding conditions which ensure that the following *Ax–Kochen–Ershov Principle* holds:

$$(K, v) \subseteq (L, v) \wedge vK \prec_{\exists} vL \wedge Kv \prec_{\exists} Lv \implies (K, v) \prec_{\exists} (L, v). \tag{5}$$

In order to prove that  $(K, v) \prec_{\exists} (L, v)$ , we first note that existential sentences in  $L$  only talk about finitely many elements of  $L$ , and these generate a function field over  $K$ . So it suffices to show  $(K, v) \prec_{\exists} (F, v)$  for every function field  $F$  over  $K$  contained in  $L$ . One tool to show that  $(K, v) \prec_{\exists} (F, v)$  is to prove an embedding lemma: we wish to construct an embedding of  $(F, v)$  over  $K$  in some “big” (highly saturated elementary) extension  $(K^*, v^*)$  of  $(K, v)$ . Existential sentences are preserved by embeddings and will then hold in  $(K^*, v^*)$  from where they can be pulled down to  $(K, v)$ . In order to construct the embedding, we need to understand the algebraic structure of  $(F, v)$ .

*Example 2.8.* Assume that  $(K, \nu)$  is henselian (the same will then be true for  $(K^*, \nu^*)$ ) and that  $(F|K, \nu)$  is an immediate extension of transcendence degree 1. Pick an element  $x \in F$  transcendental over  $K$ . Even if we know how to embed  $(K(x), \nu)$  in  $(K^*, \nu^*)$ , how can we extend this embedding to  $(F, \nu)$ ? Practically the only tool we have for such extensions is Hensel’s Lemma. So if  $F \subset K(x)^h$ , we can use the universal property of henselizations (Theorem 4.11 below) to extend the embedding to  $K(x)^h$  and thus to  $F$ . If  $F$  is not a subfield of  $K(x)^h$ , we do not know what to do.

More generally, we have to deal with extensions which are not immediate, but for which the conditions “ $\nu K \prec_{\exists} \nu L$ ” and “ $K\nu \prec_{\exists} L\nu$ ” hold. By the saturation of  $(K^*, \nu^*)$ , they actually provide us with an embedding of  $\nu F$  over  $\nu K$  in  $\nu^* K^*$  and an embedding of  $F\nu$  over  $K\nu$  in  $K^*\nu^*$ . Using Hensel’s Lemma, they can be lifted to an embedding of  $(F, \nu)$  in  $(K^*, \nu^*)$  if  $(F, \nu)$  is inertially generated with a transcendence basis  $T$  such that  $(K(T), \nu)$  can be embedded, as we will discuss at the end of Section 5.1. If  $(F, \nu)$  is not inertially generated, we are lost again. So we see that both of our problems share the important approach of elimination of ramification.

Before we discuss the stated problems further, let us give several examples of defect extensions, in order to meet the enemy we are dealing with.

### 3 Examples for non-trivial defect

In this section, we shall give examples for extensions with defect  $> 1$ . There is one basic example which is quick at hand. It is due to F. K. Schmidt.

*Example 3.1.* We consider  $\mathbb{F}_p((t))$  with its canonical valuation  $\nu = \nu_t$ . Since  $\mathbb{F}_p((t))|\mathbb{F}_p(t)$  has infinite transcendence degree, we can choose some element  $s \in \mathbb{F}_p((t))$  which is transcendental over  $\mathbb{F}_p(t)$ . Since  $(\mathbb{F}_p((t))|\mathbb{F}_p(t), \nu)$  is an immediate extension, the same holds for  $(\mathbb{F}_p(t, s)|\mathbb{F}_p(t), \nu)$  and thus also for  $(\mathbb{F}_p(t, s)|\mathbb{F}_p(t, s^p), \nu)$ . The latter extension is purely inseparable of degree  $p$  (since  $s, t$  are algebraically independent over  $\mathbb{F}_p$ , the extension  $\mathbb{F}_p(s)|\mathbb{F}_p(s^p)$  is linearly disjoint from  $\mathbb{F}_p(t, s^p)|\mathbb{F}_p(s^p)$ ). Hence, Theorem 4.1 shows that there is only one extension of the valuation  $\nu$  from  $\mathbb{F}_p(t, s^p)$  to  $\mathbb{F}_p(t, s)$ . So we have  $e = f = g = 1$  for this extension and consequently, its defect is  $p$ .

*Remark 3.2.* This example is the easiest one used in commutative algebra to show that the integral closure of a noetherian ring of dimension 1 in a finite extension of its quotient field need not be finitely generated.

In some sense, the field  $\mathbb{F}_p(t, s^p)$  is the smallest possible admitting a defect extension. Indeed, a function field of transcendence degree 1 over its prime field  $\mathbb{F}_p$  is defectless under every valuation. More generally, a valued function field of transcendence degree 1 over a subfield on which the valuation is trivial is always a defectless field; this follows from Theorem 5.1 below.

With respect to defects, discrete valuations are not too bad. The following is easy to prove (cf. [38]):

**Theorem 3.3.** *Let  $(K, \nu)$  be a discretely valued field, that is, with value group  $\nu K \simeq \mathbb{Z}$ . Then every finite separable extension is defectless. If in addition  $\text{char } K = 0$ , then  $(K, \nu)$  is a defectless field.*

A defect can appear “out of nothing” when a finite extension is lifted through another finite extension:

*Example 3.4.* In the foregoing example, we can choose  $s$  such that  $\nu s > 1 = \nu t$ . Now we consider the extensions  $(\mathbb{F}_p(t, s^p) | \mathbb{F}_p(t^p, s^p), \nu)$  and  $(\mathbb{F}_p(t + s, s^p) | \mathbb{F}_p(t^p, s^p), \nu)$  of degree  $p$ . Both are defectless: since  $\nu \mathbb{F}_p(t^p, s^p) = p\mathbb{Z}$  and  $\nu(t + s) = \nu t = 1$ , the index of  $\nu \mathbb{F}_p(t^p, s^p)$  in  $\nu \mathbb{F}_p(t, s^p)$  and in  $\nu \mathbb{F}_p(t + s, s^p)$  must be (at least)  $p$ . But  $\mathbb{F}_p(t, s^p) \cdot \mathbb{F}_p(t + s, s^p) = \mathbb{F}_p(t, s)$ , which shows that the defectless extension  $(\mathbb{F}_p(t, s^p) | \mathbb{F}_p(t^p, s^p), \nu)$  does not remain defectless if lifted up to  $\mathbb{F}_p(t + s, s^p)$  (and vice versa).

We can derive from Example 3.1 an example of a defect extension of henselian fields.

*Example 3.5.* We consider again the immediate extension  $(\mathbb{F}_p(t, s) | \mathbb{F}_p(t, s^p), \nu)$  of Example 3.1. We take the henselization  $(\mathbb{F}_p(t, s), \nu)^h$  of  $(\mathbb{F}_p(t, s), \nu)$  in  $\mathbb{F}_p((t))$  and the henselization  $(\mathbb{F}_p(t, s^p), \nu)^h$  of  $(\mathbb{F}_p(t, s^p), \nu)$  in  $(\mathbb{F}_p(t, s), \nu)^h$ . We find that  $(\mathbb{F}_p(t, s), \nu)^h | (\mathbb{F}_p(t, s^p), \nu)^h$  is again a purely inseparable extension of degree  $p$ . Indeed, the purely inseparable extension  $\mathbb{F}_p(t, s) | \mathbb{F}_p(t, s^p)$  is linearly disjoint from the separable extension  $\mathbb{F}_p(t, s^p)^h | \mathbb{F}_p(t, s^p)$ , and by virtue of Theorem 4.14,  $\mathbb{F}_p(t, s)^h = \mathbb{F}_p(t, s) \cdot \mathbb{F}_p(t, s^p)^h$ . Also for this extension we have that  $e = f = g = 1$  and again, the defect is  $p$ . Note that by Theorem 3.3, a proper immediate extension over a field like  $(\mathbb{F}_p(t, s^p), \nu)^h$  can only be purely inseparable.

The next example is easily found by considering the purely inseparable extension  $\tilde{K} | K^{\text{sep}}$ . In comparison to the last example, the involved fields are “much bigger”, for instance, they do not have value group  $\mathbb{Z}$  anymore.

*Example 3.6.* Let  $K$  be a field which is not perfect. Then the extension  $\tilde{K} | K^{\text{sep}}$  is non-trivial. For every non-trivial valuation  $\nu$  on  $\tilde{K}$ , the value groups  $\nu \tilde{K}$  and  $\nu K^{\text{sep}}$  are both equal to the divisible hull  $\tilde{\nu K}$  of  $\nu K$ , and the residue fields  $\tilde{K} \nu$  and  $K^{\text{sep}} \nu$  are both equal to the algebraic closure of  $K \nu$  (cf. Lemma 2.16 of [42]). Consequently,  $(\tilde{K} | K^{\text{sep}}, \nu)$  is an immediate extension. Since the extension of  $\nu$  from  $K^{\text{sep}}$  to  $\tilde{K}$  is unique (cf. Theorem 4.1 below), we find that the defect of every finite subextension is equal to its degree.

Note that the separable-algebraically closed field  $K^{\text{sep}}$  is henselian for every valuation. Hence, our example shows:

**Theorem 3.7.** *There are henselian valued fields of positive characteristic which admit proper purely inseparable immediate extensions. Hence, the property “henselian” does not imply the property “algebraically maximal”.*

We can refine the previous example as follows. Let  $p > 0$  be the characteristic of the residue field  $Kv$ . An *Artin–Schreier extension* of  $K$  is an extension of degree  $p$  generated over  $K$  by a root of a polynomial  $X^p - X - c$  with  $c \in K$ . An extension of degree  $p$  of a field of characteristic  $p$  is a Galois extension if and only if it is an Artin–Schreier extension. A field  $K$  is *Artin–Schreier closed* if it does not admit Artin–Schreier extensions.

*Example 3.8.* In order that every purely inseparable extension of the valued field  $(K, v)$  be immediate, it suffices that  $vK$  be  $p$ -divisible and  $Kv$  be perfect. But these conditions are already satisfied for every non-trivially valued Artin–Schreier closed field  $K$  (see Corollary 2.17 of [42]). Hence, the perfect hull of every non-trivially valued Artin–Schreier closed field is an immediate extension.

Until now, we have only presented purely inseparable defect extensions. But our last example can give an idea of how to produce a separable defect extension by interchanging the role of purely inseparable extensions and Artin–Schreier extensions.

*Example 3.9.* Let  $(K, v)$  be a valued field of characteristic  $p > 0$  whose value group is not  $p$ -divisible. Let  $c \in K$  such that  $vc < 0$  is not divisible by  $p$ . Let  $a$  be a root of the Artin–Schreier polynomial  $X^p - X - c$ . Then  $va = vc/p$  and  $[K(a) : K] = p = (vK(a) : vK)$ . The fundamental inequality shows that  $K(a)v = Kv$  and that the extension of  $v$  from  $K$  to  $K(a)$  is unique. By Theorem 4.1 below the further extension to  $K(a)^{1/p^\infty} = K^{1/p^\infty}(a)$  is unique. It follows that the extension of  $v$  from  $K^{1/p^\infty}$  to  $K^{1/p^\infty}(a)$  is unique. On the other hand,  $[K^{1/p^\infty}(a) : K^{1/p^\infty}] = p$  since the separable extension  $K(a)|K$  is linearly disjoint from  $K^{1/p^\infty}|K$ . The value group  $vK^{1/p^\infty}(a)$  is the  $p$ -divisible hull of  $vK(a) = vK + \mathbb{Z}va$ . Since  $pva \in vK$ , this is the same as the  $p$ -divisible hull of  $vK$ , which in turn is equal to  $vK^{1/p^\infty}$ . The residue field of  $K^{1/p^\infty}(a)$  is the perfect hull of  $K(a)v = Kv$ . Hence, it is equal to the residue field of  $K^{1/p^\infty}$ . It follows that the extension  $(K^{1/p^\infty}(a)|K^{1/p^\infty}, v)$  is immediate and that its defect is  $p$ , like its degree.

Similarly, one can start with a valued field  $(K, v)$  of characteristic  $p > 0$  whose residue field is not perfect. In this case, the Artin–Schreier extension  $K(a)|K$  is constructed as in the proof of Lemma 2.13 of [42]. We leave the details to the reader.

In the previous example, we can always choose  $(K, v)$  to be henselian (since passing to the henselization does not change value group and residue field). Then all constructed extensions of  $(K, v)$  are also henselian, since they are algebraic extensions (cf. Theorem 4.14 below). Hence, our example shows:

**Theorem 3.10.** *There are henselian valued fields of positive characteristic which admit immediate Artin–Schreier defect extensions.*

If the perfect hull of a given valued field  $(K, v)$  is not an immediate extension, then  $vK$  is not  $p$ -divisible or  $Kv$  is not perfect, and we can apply the procedure of our above example. This shows:

**Theorem 3.11.** *If the perfect hull of a given valued field of positive characteristic is not an immediate extension, then it admits an immediate Artin–Schreier extension.*

An important special case of Example 3.9 is the following:

*Example 3.12.* We choose  $(K, v)$  to be  $(\mathbb{F}_p(t), v_t)$  or  $(\mathbb{F}_p((t)), v_t)$  or any intermediate field, and set  $L := K(t^{1/p^i} \mid i \in \mathbb{N})$ , the perfect hull of  $K$ . By Theorem 4.1 below,  $v = v_t$  has a unique extension to  $L$ . In all cases,  $L$  can be viewed as a subfield of the power series field  $\mathbb{F}_p((\mathbb{Q}))$ . The power series

$$\vartheta := \sum_{i=1}^{\infty} t^{-1/p^i} \in \mathbb{F}_p((\mathbb{Q})) \tag{6}$$

is a root of the Artin–Schreier polynomial

$$X^p - X - \frac{1}{t}$$

because

$$\begin{aligned} \vartheta^p - \vartheta - \frac{1}{t} &= \sum_{i=1}^{\infty} t^{-1/p^{i-1}} - \sum_{i=1}^{\infty} t^{-1/p^i} - t^{-1} \\ &= \sum_{i=0}^{\infty} t^{-1/p^i} - \sum_{i=1}^{\infty} t^{-1/p^i} - t^{-1} = 0. \end{aligned}$$

By Example 3.9, the extension  $L(\vartheta)|L$  is an immediate Artin–Schreier defect extension. The above power series expansion for  $\vartheta$  was presented by Shreeram Abhyankar in [1]. It became famous since it shows that there are elements algebraic over  $\mathbb{F}_p(t)$  with a power series expansion in which the exponents do not have a common denominator. This in turn shows that Puiseux series fields in positive characteristic are in general not algebraically closed (see also [30, 40]). With  $p = 2$ , the above was also used by Irving Kaplansky in [29, Section 5] for the construction of an example that shows that if his “hypothesis A” (see [29, Section 3]) is violated, then the maximal immediate extension of a valued field may not be unique up to isomorphism. See also [49] for more information on this subject.

Let us compute  $v(\vartheta - L)$ . For the partial sums

$$\vartheta_k := \sum_{i=1}^k t^{-1/p^i} \in L \tag{7}$$

we see that  $v(\vartheta - \vartheta_k) = -1/p^{k+1} < 0$ . Assume that there is  $c \in L$  such that  $v(\vartheta - c) > -1/p^k$  for all  $k$ . Then  $v(c - \vartheta_k) = \min\{v(\vartheta - c), v(\vartheta - \vartheta_k)\} = -1/p^{k+1}$  for all  $k$ . On the other hand, there is some  $k$  such that  $c \in K(t^{-1/p}, \dots, t^{-1/p^k}) = K(t^{-1/p^k})$ . But this contradicts the fact that  $v(c - t^{-1/p} - \dots - t^{-1/p^k}) = v(c - \vartheta_k) = -1/p^{k+1} \notin vK(t^{-1/p^k})$ . This proves that the values  $-1/p^k$  are cofinal in  $v(\vartheta - L)$ . Since  $vL$  is a subgroup of the rationals, this shows that the least upper bound of  $v(\vartheta - L)$  in  $vL$  is the element 0. As  $v(\vartheta - L)$  is an initial segment of  $vL$  by Lemma 1.1, we conclude that  $v(\vartheta - L) = (vL)^{<0}$ . It follows that  $(L(\vartheta)|L, v)$  is immediate without  $(L, v)$  being dense in  $(L(\vartheta), v)$ .



A version of this example with  $(K, v) = (\widetilde{\mathbb{F}_p}((t)), v_t)$  was given by S. K. Khanduja in [31] as a counterexample to Proposition 2' on p. 425 of [4]. That proposition states that if  $(K, v)$  is a perfect henselian valued field of rank 1 and  $a \in \widetilde{K} \setminus K$ , then there is  $c \in K$  such that

$$v(a - c) \geq \min\{v(a - a') \mid a' \neq a \text{ conjugate to } a \text{ over } K\}.$$

But for  $a = \vartheta$  in the previous example, we have that  $a - a' \in \mathbb{F}_p$  so that the right hand side is 0, whereas  $v(\vartheta - c) < 0$  for all  $c$  in the perfect hull  $L$  of  $\mathbb{F}_p((t))$ . The same holds if we take  $L$  to be the perfect hull of  $K = \widetilde{\mathbb{F}_p}((t))$ . In fact, it is Corollary 2 to Lemma 6 on p. 424 in [4] which is in error; it is stated without proof in the paper.

In a slightly different form, the previous example was already given by Alexander Ostrowski in [57], Section 57:

*Example 3.13.* Ostrowski takes  $(K, v) = (\mathbb{F}_p(t), v_t)$ , but works with the polynomial  $X^p - tX - 1$  in the place of the Artin–Schreier polynomial  $X^p - X - 1/t$ . After an extension of  $K$  of degree  $p - 1$ , it also can be transformed into an Artin–Schreier polynomial. Indeed, if we take  $b$  to be an element which satisfies  $b^{p-1} = t$ , then replacing  $X$  by  $bX$  and dividing by  $b^p$  will transform  $X^p - tX - 1$  into the polynomial  $X^p - X - 1/b^p$ . Now we replace  $X$  by  $X + 1/b$ . Since we are working in characteristic  $p$ , this transforms  $X^p - X - 1/b^p$  into  $X^p - X - 1/b$ . (This sort of transformation plays a crucial role in the proofs of Theorem 5.1 and Theorem 5.10 as well as in Abhyankar’s and Epp’s work.) Now we see that the Artin–Schreier polynomial  $X^p - X - 1/b$  plays the same role as  $X^p - X - 1/t$ . Indeed,  $vb = \frac{1}{p-1}$  and it follows that  $(v\mathbb{F}_p(b) : v\mathbb{F}_p(t)) = p - 1 = [\mathbb{F}_p(b) : \mathbb{F}_p(t)]$ , so that  $v\mathbb{F}_p(b) = \mathbb{Z}\frac{1}{p-1}$ . In this value group,  $vb$  is not divisible by  $p$ .

Interchanging the role of purely inseparable and Artin–Schreier extensions in Example 3.12, we obtain:

*Example 3.14.* We proceed as in Example 3.12, but replace  $t^{-1/p^i}$  by  $a_i$ , where we define  $a_1$  to be a root of the Artin–Schreier polynomial  $X^p - X - 1/t$  and  $a_{i+1}$  to be a root of the Artin–Schreier polynomial  $X^p - X + a_i$ . Now we choose  $\eta$  such that  $\eta^p = 1/t$ . Note that also in this case,  $a_1, \dots, a_i \in K(a_i)$  for every  $i$ , because  $a_i = a_{i+1}^p - a_{i+1}$  for every  $i$ . By induction on  $i$ , we again deduce that  $va_1 = -1/p$  and  $va_i = -1/p^i$  for every  $i$ . We set  $L := K(a_i \mid i \in \mathbb{N})$ , that is,  $L|K$  is an infinite tower of Artin–Schreier extensions. By our construction,  $vL$  is  $p$ -divisible and  $Lv = \mathbb{F}_p$  is perfect. On the other hand, for every purely inseparable extension  $L'|L$  the group  $vL'/vL$  is a  $p$ -group and the extension  $L'v|Lv$  is purely inseparable. This fact shows that  $(L(\eta)|L, v)$  is an immediate extension.

In order to compute  $v(\eta - L)$ , we set

$$\eta_k := \sum_{i=1}^k a_i \in L. \tag{8}$$

Bearing in mind that  $a_{i+1}^p = a_{i+1} - a_i$  and  $a_1^p = a_1 + 1/t$  for  $i \geq 1$ , we compute

$$\begin{aligned} (\eta - \eta_k)^p &= \eta^p - \eta_k^p = \frac{1}{t} - \sum_{i=1}^k a_i^p = \frac{1}{t} - \left( \sum_{i=1}^k a_i - \sum_{i=1}^{k-1} a_i + \frac{1}{t} \right) \\ &= a_k . \end{aligned}$$

It follows that  $v(\eta - \eta_k) = \frac{va_k}{p} = -1/p^{k+1}$ . The same argument as in Example 3.12 now shows that again,  $v(\eta - L) = (vL)^{<0}$ .

We can develop Examples 3.12 and 3.14 a bit further in order to treat complete fields.

*Example 3.15.* Take one of the immediate extensions  $(L(\vartheta)|L, v)$  of Example 3.12 and set  $\zeta = \vartheta$ , or take one of the immediate extensions  $(L(\eta)|L, v)$  of Example 3.14 and set  $\zeta = \eta$ . Consider the completion  $(L, v)^c = (L^c, v)$  of  $(L, v)$ . Since every finite extension of a complete valued field is again complete,  $(L^c(\zeta), v) = (L(\zeta).L^c, v)$  is the completion of  $(L(\zeta), v)$  for every extension of the valuation  $v$  from  $(L^c, v)$  to  $L(\zeta).L^c$ . Consequently, the extension  $(L^c(\zeta)|L(\zeta), v)$  and thus also the extension  $(L^c(\zeta)|L, v)$  is immediate. It follows that  $(L^c(\zeta)|L^c, v)$  is immediate. On the other hand, this extension is non-trivial since  $v(\zeta - L) = (vL)^{<0}$  shows that  $\zeta \notin L^c$ .

A valued field is *maximal* if it does not admit any proper immediate extension. All power series fields are maximal. A valued field is said to have *rank 1* if its value group is archimedean, i.e., a subgroup of the reals. Every complete discretely valued field of rank 1 is maximal. Every complete valued field of rank 1 is henselian (but this is not true in general in higher ranks). The previous example proves:

**Theorem 3.16.** *There are complete fields of rank 1 which admit immediate separable-algebraic and immediate purely inseparable extensions. Consequently, not every complete field of rank 1 is maximal.*

In Example 3.14, we constructed an immediate purely inseparable extension not contained in the completion of the field. Such extensions can be transformed into immediate Artin–Schreier defect extensions:

*Example 3.17.* In the situation of Example 3.14, extend  $v$  from  $L(\eta)$  to  $\tilde{L}$ . Take  $d \in L$  with  $vd \geq 1/p$ , and  $\vartheta_0$  a root of the polynomial  $X^p - dX - 1/t$ . It follows that

$$-1 = v\frac{1}{t} = v(\vartheta_0^p - d\vartheta_0) \geq \min\{v\vartheta_0^p, vd\vartheta_0\} = \min\{pv\vartheta_0, vd + v\vartheta_0\}$$

which shows that we must have  $v\vartheta_0 < 0$ . But then

$$pv\vartheta_0 < v\vartheta_0 < vd + v\vartheta_0,$$

so

$$v\vartheta_0 = -\frac{1}{p}.$$

We compute:

$$pv(\vartheta_0 - \eta) = v(\vartheta_0 - \eta)^p = v(\vartheta_0^p - \eta^p) = v(d\vartheta_0 + 1/t - 1/t) = vd + v\vartheta_0 \geq 0.$$

Hence,  $v(\vartheta_0 - \eta) \geq 0$ , and thus for all  $c \in L$ ,

$$v(\vartheta_0 - c) = \min\{v(\vartheta_0 - \eta), v(\eta - c)\} = v(\eta - c).$$

In particular,  $v(\vartheta_0 - L) = v(\eta - L) = (vL)^{<0}$ . The extension  $(L(\vartheta_0)|L, v)$  is immediate and has defect  $p$ ; however, this is not quite as easy to show as it has been before. To make things easier, we choose  $(K, v)$  to be henselian, so that also  $(L, v)$ , being an algebraic extension, is henselian. So there is only one extension of  $v$  from  $L$  to  $L(\vartheta_0)$ . Since  $v(\vartheta_0 - c) < 0$  for all  $c \in L$ , we have that  $\vartheta_0 \notin L$ . We also choose  $d = b^{p-1}$  for some  $b \in L$ . Then we will see below that  $L(\vartheta_0)|L$  is an Artin–Schreier extension. If it were not immediate, then  $e = p$  or  $f = p$ . In the first case, we can choose some  $a \in L(\vartheta_0)$  such that  $0, va, \dots, (p-1)va$  are representatives of the distinct cosets of  $vL(\vartheta_0)$  modulo  $vL$ . Then  $1, a, \dots, a^{p-1}$  are  $L$ -linearly independent and thus form an  $L$ -basis of  $L(\vartheta_0)$ . Writing  $\vartheta_0 = c_0 + c_1a + \dots + c_{p-1}a^{p-1}$ , we find that  $v(\eta - c_0) = v(\vartheta_0 - c_0) = \min\{vc_1 + va, \dots, vc_{p-1} + (p-1)va\} \notin vL$  as the values  $vc_1 + va, \dots, vc_{p-1} + (p-1)va$  lie in distinct cosets modulo  $vL$ . But this is a contradiction. In the second case,  $f = p$ , one chooses  $a \in L(\vartheta_0)$  such that  $1, av, \dots, (av)^{p-1}$  form a basis of  $L(\vartheta_0)v|Lv$ , and derives a contradiction in a similar way. (Using this method one actually proves that an extension  $(L(\zeta)|L, v)$  of degree  $p$  with unique extension of the valuation is immediate if and only if  $v(\zeta - L)$  has no maximal element.)

Now consider the polynomial  $X^p - dX - 1/t = X^p - b^{p-1}X - 1/t$  and set  $X = bY$ . Then  $X^p - dX - 1/t = b^pY^p - b^pY - 1/t$ , and dividing by  $b^p$  we obtain the polynomial  $Y^p - Y - 1/b^pt$  which admits  $\vartheta_0/b$  as a root. So we see that  $(L(\vartheta_0)|L, v)$  is in fact an immediate Artin–Schreier defect extension. But in comparison with Example 3.14, something is different:

$$\begin{aligned} v\left(\frac{\vartheta_0}{b} - L\right) &= \left\{v\left(\frac{\vartheta_0}{b} - c\right) \mid c \in L\right\} = \left\{v\left(\frac{\vartheta_0}{b} - \frac{c}{b}\right) \mid c \in L\right\} \\ &= \{v(\vartheta_0 - c) - vb \mid c \in L\} = \{\alpha \in vL \mid \alpha < vb\}, \end{aligned}$$

where  $vb > 0$ .

A similar idea can be used to turn the defect extension of Example 3.1 into a separable extension. However, in the previous example we made use of the fact that  $\eta$  was not an element of the completion of  $(L, v)$ , that is,  $v(\eta - L)$  was bounded from above. We use a “dirty trick” to first transform the extension of Example 3.1 to an extension whose generator does not lie in the completion of the base field.

*Example 3.18.* Taking the extension  $(\mathbb{F}_p(t, s)|\mathbb{F}_p(t, s^p), v)$  as in Example 3.1, we adjoin a new transcendental element  $z$  to  $\mathbb{F}_p(t, s)$  and extend the valuation  $v$  in such a way that  $vs \gg vt$ , that is,  $v\mathbb{F}_p(t, s, z)$  is the lexicographic product  $\mathbb{Z} \times \mathbb{Z}$ .

The extension  $(\mathbb{F}_p(t, s, z) | \mathbb{F}_p(t, s^p, z), \nu)$  is still purely inseparable and immediate, but now  $s$  does not lie anymore in the completion  $\mathbb{F}_p(t, s^p)((z))$  of  $\mathbb{F}_p(t, s^p, z)$ . In fact,  $\nu(s - \mathbb{F}_p(t, s^p, z)) = \{\alpha \in \nu\mathbb{F}_p(t, s^p, z) \mid \exists n \in \mathbb{N} : n\nu t \geq \alpha\}$  is bounded from above by  $\nu z$ .

Taking  $\vartheta_0$  to be a root of the polynomial  $X^p - z^{p-1}X - s^p$  we obtain that  $\nu(\vartheta_0 - c) = \nu(s - c)$  for all  $c \in \mathbb{F}_p(t, s^p, z)$  and that the Artin–Schreier extension  $(\mathbb{F}_p(t, \vartheta_0, z) | \mathbb{F}_p(t, s^p, z), \nu)$  is immediate with defect  $p$ . We leave the proof as an exercise to the reader. Note that one can pass to the henselizations of all fields involved, cf. Example 3.5.

The interplay of Artin–Schreier extensions and radical extensions that we have used in the last examples can also be transferred to the mixed characteristic case. There are infinite algebraic extensions of  $\mathbb{Q}_p$  which admit immediate Artin–Schreier defect extensions. To present an example, we need a lemma which shows that there is some *quasi-additivity* in the mixed characteristic case.

**Lemma 3.19.** *Let  $(K, \nu)$  be a valued field of characteristic 0 and residue characteristic  $p > 0$ , and with valuation ring  $\mathcal{O}$ . Further, let  $c_1, \dots, c_n$  be elements in  $K$  of value  $\geq -\frac{\nu p}{p}$ . Then*

$$(c_1 + \dots + c_n)^p \equiv c_1^p + \dots + c_n^p \pmod{\mathcal{O}}.$$

*Proof.* Every product of  $p$  many  $c_i$ 's has value  $\geq -\nu p$ . In view of the fact that every binomial coefficient  $\binom{p}{i}$  is divisible by  $p$  for  $1 \leq i \leq p - 1$ , we find that  $(c_1 + c_2)^p \equiv c_1^p + c_2^p \pmod{\mathcal{O}}$ . Now the assertion follows by induction on  $n$ .

*Example 3.20.* We choose  $(K, \nu)$  to be  $(\mathbb{Q}, \nu_p)$  or  $(\mathbb{Q}_p, \nu_p)$  or any intermediate field. Note that we write  $\nu p = 1$ . We construct an algebraic extension  $(L, \nu)$  of  $(K, \nu)$  with a  $p$ -divisible value group as follows. By induction, we choose elements  $a_i$  in the algebraic closure of  $K$  such that  $a_1^p = 1/p$  and  $a_{i+1}^p = a_i$ . Then  $\nu a_1 = -1/p$  and  $\nu a_i = -1/p^i$  for every  $i$ . Hence, the field  $L := K(a_i \mid i \in \mathbb{N})$  must have  $p$ -divisible value group under any extension of  $\nu$  from  $K$  to  $L$ . Note that  $a_1, \dots, a_i \in K(a_i)$  for every  $i$ . Since  $(\nu K(a_{i+1}) : \nu K(a_i)) = p$ , the fundamental inequality shows that  $K(a_{i+1})\nu = K(a_i)\nu$  and that the extension of  $\nu$  is unique, for every  $i$ . Hence,  $L\nu = \mathbb{Q}_p\nu = \mathbb{F}_p$  and the extension of  $\nu$  from  $K$  to  $L$  is unique.

Now we let  $\vartheta$  be a root of  $X^p - X - 1/p$ . It follows that  $\nu\vartheta = -1/p$ . We define  $b_i := \vartheta - a_1 - \dots - a_i$ . By construction,  $\nu a_i \geq -1/p$  for all  $i$ . It follows that also  $\nu b_i \geq -1/p$  for all  $i$ . With the help of the foregoing lemma, and bearing in mind that  $a_{i+1}^p = a_i$  and  $a_1^p = 1/p$ , we compute

$$\begin{aligned} 0 &= \vartheta^p - \vartheta - \frac{1}{p} = (b_i + a_1 + \dots + a_i)^p - (b_i + a_1 + \dots + a_i) - 1/p \\ &\equiv b_i^p - b_i + a_1^p + \dots + a_i^p - a_1 - \dots - a_i - 1/p = b_i^p - b_i - a_i \pmod{\mathcal{O}}. \end{aligned}$$

Since  $va_i < 0$ , we have that  $vb_i = \frac{1}{p}va_i = -1/p^{i+1}$ . Hence,  $(vK(\vartheta, a_i) : vK(a_i)) = p = [K(\vartheta, a_i) : K(a_i)]$  and  $K(\vartheta, a_i)v = K(a_i)v = \mathbb{F}_p$  for every  $i$ . If  $[L(\vartheta) : L] < p$ , then there would exist some  $i$  such that  $[K(\vartheta, a_i) : K(a_i)] < p$ . But we have just shown that this is not the case. Similarly, if  $vL(\vartheta)$  would contain an element that does not lie in the  $p$ -divisible hull of  $\mathbb{Z} = vK$ , or if  $L(\vartheta)v$  would be a proper extension of  $\mathbb{F}_p$ , then the same would already hold for  $K(\vartheta, a_i)$  for some  $i$ . But we have shown that this is not the case. Hence,  $(L(\vartheta)|L, v)$  is an Artin–Schreier defect extension.

For the partial sums  $\vartheta_k = \sum_{i=1}^k a_i$  we obtain  $v(\vartheta - \vartheta_k) = vb_k = -1/p^{k+1}$ , and the same argument as in Example 3.12 shows again that  $v(\vartheta - L) = (vL)^{<0}$ .

From this example we can derive a special case which was given by Ostrowski in [57, Section 39] (see also [8, Chap. VI, Section 8, Exercise 2]).

*Example 3.21.* In the last example, we take  $K = \mathbb{Q}_2$ . Then  $(L(\sqrt{3})|L, v)$  is an immediate extension of degree 2. Indeed, this is nothing else than the Artin–Schreier extension that we have constructed. If one substitutes  $Y = 1 - 2X$  in the minimal polynomial  $Y^2 - 3$  of  $\sqrt{3}$  and then divides by 4, one obtains the Artin–Schreier polynomial  $X^p - X - 1/2$ .

This is Ostrowski’s original example. A slightly different version was presented by Paulo Ribenboim (cf. Example 2 of Chap. G, p. 246): The extension  $(L(\sqrt{-1})|L, v)$  is immediate. Indeed, the minimal polynomial  $Y^2 + 1$  corresponds to the Artin–Schreier polynomial  $X^p - X + 1/2$  which does the same job as  $X^p - X - 1/2$ .

As in the equal characteristic case, we can interchange the role of radical extensions and Artin–Schreier extensions:

*Example 3.22.* We proceed as in Example 3.20, with the only difference that we define  $a_1$  to be a root of the Artin–Schreier polynomial  $X^p - X - 1/p$  and  $a_{i+1}$  to be a root of the Artin–Schreier polynomial  $X^p - X + a_i$ , and that we choose  $\eta$  such that  $\eta^p = 1/p$ . Note that also in this case,  $a_1, \dots, a_i \in K(a_i)$  for every  $i$ , because  $a_i = a_{i+1}^p - a_{i+1}$  for every  $i$ . By induction on  $i$ , we again deduce that  $va_1 = -1/p$  and that  $va_i = -1/p^i$  for every  $i$ . As before, we define  $b_i := \eta - a_1 - \dots - a_i$ . Using Lemma 3.19 and bearing in mind that  $a_{i+1}^p = a_{i+1} - a_i$  and  $a_1^p = a_1 + 1/p$ , we compute

$$\begin{aligned} 0 &= \eta^p - \frac{1}{p} = (b_i + a_1 + \dots + a_i)^p - 1/p \\ &\equiv b_i^p + a_1^p + \dots + a_i^p - 1/p = b_i^p + a_i \pmod{\mathfrak{O}}. \end{aligned}$$

It follows that  $v(b_i^p + a_i) \geq 0 > va_i$ . Consequently,  $vb_i^p = -1/p^{i+1}$ , that is,  $vb_i = \frac{1}{p}va_i = va_{i+1}$ . As before, we set  $L := K(a_i \mid i \in \mathbb{N})$ . Now the same arguments as in Example 3.20 show that  $(L(\eta)|L, v)$  is an immediate extension with  $v(\eta - L) = (vL)^{<0}$ .

It can happen that it takes just a finite defect extension to make a field defectless and even maximal. The following example is due to Masayoshi Nagata ([54, Appendix, Example (E3.1), pp. 206–207]):

*Example 3.23.* We take a field  $k$  of characteristic  $p$  and such that  $[k : k^p]$  is infinite, e.g.,  $k = \mathbb{F}_p(t_i | i \in \mathbb{N})$  where the  $t_i$  are algebraically independent elements over  $\mathbb{F}_p$ . Taking  $t$  to be another transcendental element over  $k$  we consider the power series fields  $k((t))$  and  $k^p((t)) = k^p((t^p))(t) = k((t))^p(t)$ . Since  $[k : k^p]$  is not finite, we have that  $k((t))|k^p((t)).k$  is a non-trivial immediate purely inseparable algebraic extension. In fact, a power series in  $k((t))$  is an element of  $k^p((t)).k$  if and only if its coefficients generate a finite extension of  $k^p$ . Since  $k^p((t)).k$  contains  $k((t))^p$ , this extension is generated by a set  $X = \{x_i | i \in I\} \subset k((t))$  such that  $x_i^p \in k^p((t)).k$  for every  $i \in I$ . Assuming this set to be minimal, or in other words, the  $x_i$  to be  $p$ -independent over  $k^p((t)).k$ , we pick some element  $x \in X$  and put  $K := k^p((t)).k(X \setminus \{x\})$ . Then  $k((t))|K$  is a purely inseparable extension of degree  $p$ . Moreover, it is an immediate extension; in fact,  $k((t))$  is the completion of  $K$ . As an algebraic extension of  $k^p((t))$ ,  $K$  is henselian.

This example proves:

**Theorem 3.24.** *There is a henselian discretely valued field  $(K, v)$  of characteristic  $p$  admitting a finite immediate purely inseparable extension  $(L|K, v)$  of degree  $p$  such that  $(L, v)$  is complete, hence maximal and thus defectless.*

For the conclusion of this section, we shall give an example which is due to Françoise Delon (cf. [16], Example 1.51). It shows that an algebraically maximal field is not necessarily a defectless field, and that a finite extension of an algebraically maximal field is not necessarily again algebraically maximal.

*Example 3.25.* We consider  $\mathbb{F}_p((t))$  with its  $t$ -adic valuation  $v = v_t$ . We choose elements  $x, y \in \mathbb{F}_p((t))$  which are algebraically independent over  $\mathbb{F}_p(t)$ . We set  $L := \mathbb{F}_p(t, x, y)$  and define

$$s := x^p + ty^p \quad \text{and} \quad K := \mathbb{F}_p(t, s).$$

Then  $s$  is transcendental over  $\mathbb{F}_p(t)$  and therefore,  $K$  has  $p$ -degree 2, that is,  $[K : K^p] = p^2$ . We take  $F$  to be the relative algebraic closure of  $K$  in  $\mathbb{F}_p((t))$ . Since the elements  $1, t^{1/p}, \dots, t^{(p-1)/p}$  are linearly independent over  $\mathbb{F}_p((t))$ , the same holds over  $F$ . Hence, the elements  $1, t, \dots, t^{p-1}$  are linearly independent over  $F^p$ . Now if  $F$  had  $p$ -degree 1, then  $s$  could be written in a unique way as an  $F^p$ -linear combination of  $1, t, \dots, t^{p-1}$ . But this is not possible since  $s = x^p + ty^p$  and  $x, y$  are transcendental over  $F$ . Hence, the  $p$ -degree of  $F$  is still 2 (as it cannot increase through algebraic extensions). On the other hand,  $vF = v\mathbb{F}_p((t)) = \mathbb{Z}$  and  $Fv = \mathbb{F}_p((t))v = \mathbb{F}_p$ , hence  $(vF : p vF) = p$  and  $[Fv : Fv^p] = 1$ . Now Theorem 6.3 shows that  $(F, v)$  is not inseparably defectless. Again from Theorem 6.3, we infer that  $F^{1/p} = F(t^{1/p}, s^{1/p})$  must be an extension of  $F$  with non-trivial defect. So  $F$  is not a defectless field.

On the other hand,  $\mathbb{F}_p((t))$  is the completion of  $F$  since it is already the completion of  $\mathbb{F}_p(t) \subseteq F$ . This shows that  $\mathbb{F}_p((t))$  is the unique maximal immediate extension of  $F$  (up to valuation preserving isomorphism over  $F$ ). If  $F$  would admit a proper immediate algebraic extension  $F'$ , then a maximal immediate extension of  $F'$  would also be a maximal immediate extension of  $F$  and would thus be isomorphic over  $F$  to  $\mathbb{F}_p((t))$ . But we have chosen  $F$  to be relatively algebraically closed in  $\mathbb{F}_p((t))$ . This proves that  $(F, v)$  must be algebraically maximal.

As  $(F, v)$  is algebraically maximal, the extension  $F^{1/p}|F$  cannot be immediate. Therefore, the defect of  $F^{1/p}|F$  implies that both  $F^{1/p}|F(s^{1/p})$  and  $F^{1/p}|F(t^{1/p})$  must be non-trivial immediate extensions. Consequently,  $F(s^{1/p})$  and  $F(t^{1/p})$  are not algebraically maximal.

Let us add to Delon's example by analyzing the situation in more detail and proving that  $F$  is the henselization of  $K$  and thus a separable extension of  $K$ . To this end, we first prove that  $K$  is relatively algebraically closed in  $L$ . Take  $b \in L$  algebraic over  $K$ . The element  $b^p$  is algebraic over  $K$  and lies in  $L^p = \mathbb{F}_p(t^p, x^p, y^p)$  and thus also in  $K(x) = \mathbb{F}_p(t, x, y^p)$ . Since  $x$  is transcendental over  $K$ ,  $K$  is relatively algebraically closed in  $K(x)$  and thus,  $b^p \in K$ . Consequently,  $b \in K^{1/p} = \mathbb{F}_p(t^{1/p}, s^{1/p})$ . Write

$$b = r_0 + r_1 s^{\frac{1}{p}} + \dots + r_{p-1} s^{\frac{p-1}{p}} \quad \text{with } r_i \in \mathbb{F}_p(t^{1/p}, s) = K(t^{1/p}).$$

By the definition of  $s$ ,

$$b = r_0 + r_1 x + \dots + r_{p-1} x^{p-1} + \dots + t^{1/p} r_1 y + \dots + t^{(p-1)/p} r_{p-1} y^{p-1}$$

(in the middle, we have omitted the summands in which both  $x$  and  $y$  appear). Since  $x, y$  are algebraically independent over  $\mathbb{F}_p$ , the  $p$ -degree of  $\mathbb{F}_p(x, y)$  is 2, and the elements  $x^i y^j$ ,  $0 \leq i < p$ ,  $0 \leq j < p$ , form a basis of  $\mathbb{F}_p(x, y)|\mathbb{F}_p(x^p, y^p)$ . Since  $t$  and  $t^{1/p}$  are transcendental over  $\mathbb{F}_p(x^p, y^p)$ , we know that  $\mathbb{F}_p(x, y)|\mathbb{F}_p(x^p, y^p)$  is linearly disjoint from  $\mathbb{F}_p(t, x^p, y^p)|\mathbb{F}_p(x^p, y^p)$  and from  $\mathbb{F}_p(t^{1/p}, x^p, y^p)|\mathbb{F}_p(x^p, y^p)$ . This shows that the elements  $x^i y^j$  also form a basis of  $L|\mathbb{F}_p(t, x^p, y^p)$  and are still  $\mathbb{F}_p(t^{1/p}, x^p, y^p)$ -linearly independent. Hence,  $b$  can also be written as a linear combination of these elements with coefficients in  $\mathbb{F}_p(t, x^p, y^p)$ , and this must coincide with the above  $\mathbb{F}_p(t^{1/p}, x^p, y^p)$ -linear combination which represents  $b$ . That is, all coefficients  $r_i$  and  $t^{i/p} r_i$ ,  $1 \leq i < p$ , are in  $\mathbb{F}_p(t, x^p, y^p)$ . Since  $t^{i/p} \notin \mathbb{F}_p(t, x^p, y^p)$ , this is impossible unless they are zero. It follows that  $b = r_0 \in K(t^{1/p})$ . Assume that  $b \notin K$ . Then  $[K(b) : K] = p$  and thus,  $K(b) = K(t^{1/p})$  since also  $[K(t^{1/p}) : K] = p$ . But then  $t^{1/p} \in K(b) \subset L$ , a contradiction. This proves that  $K$  is relatively algebraically closed in  $L$ .

On the other hand,  $t^{1/p} = y^{-1}(s^{1/p} - x) \in L(s^{1/p})$ . Hence,  $L.K^{1/p} = L(t^{1/p}, s^{1/p}) = L(s^{1/p})$  and  $[L.K^{1/p} : L] = [L(s^{1/p}) : L] \leq p < p^2 = [K^{1/p} : K]$ , that is,  $L|K$  is not linearly disjoint from  $K^{1/p}|K$  and thus not separable. Although being finitely generated,  $L|K$  is consequently not separably generated; in particular, it is not a rational function field. At the same time, we have seen that  $K(s^{1/p})$  admits a non-trivial purely inseparable algebraic extension in  $L(s^{1/p})$  (namely,  $K^{1/p}$ ). In contrast,  $K(s^{1/p})$  and  $L$  are  $K$ -linearly disjoint because  $s^{1/p} \notin L$ .

Let us prove even more: if  $K_1|K$  is any proper inseparable algebraic extension, then  $t^{1/p} \in L.K_1$ . Take such an extension  $K_1|K$ . Then there is some separable-algebraic subextension  $K_2|K$  and an element  $a \in K_1 \setminus K_2$  such that  $a^p \in K_2$ . Since  $K_2|K$  is separable and  $K$  is relatively algebraically closed in  $L$ , we see that  $K_2$  is relatively algebraically closed in  $L_2 := L.K_2$ . Hence,  $a \notin L_2$  and therefore,  $[L_2(a) : L_2] = p$ . On the other hand,  $K_2^{1/p} = K^{1/p}.K_2$  and thus,  $L_2.K_2^{1/p} = L_2.K^{1/p} = L.K^{1/p}.K_2$ . Consequently,  $[L.K^{1/p} : L] = p$  implies that  $[L_2.K_2^{1/p} : L_2] = [L.K^{1/p}.K_2 : L_2] \leq p$ . Since  $a \in K_2^{1/p} \subset L_2.K_2^{1/p}$  and  $[L_2(a) : L_2] = p$ , it follows that  $L_2.K_2^{1/p} = L_2(a)$ . We obtain:

$$t^{1/p} \in K^{1/p} \subseteq K_2^{1/p} \subseteq L_2.K_2^{1/p} = L_2(a) \subseteq L.K_1.$$

If  $F|K$  were inseparable, then  $t^{1/p} \in L.F$ , which contradicts the fact that  $L.F \subseteq \mathbb{F}_p((t))$ . This proves that  $F|K$  is separable. Since  $F$  is relatively closed in the henselian field  $\mathbb{F}_p((t))$ , it is itself henselian and thus contains the henselization  $K^h$  of  $K$ . Now  $\mathbb{F}_p((t))$  is the completion of  $K^h$  since it is already the completion of  $\mathbb{F}_p(t) \subseteq K^h$ . Since a henselian field is relatively separable-algebraically closed in its completion (cf. [65], Theorem 32.19), it follows that  $F = K^h$ .

Note that the maximal immediate extension  $\mathbb{F}_p((t))$  of  $F$  is not a separable extension since its subextension  $L.F|F$  is not linearly disjoint from  $K^{1/p}|K$ .

This example proves:

**Theorem 3.26.** *There are algebraically maximal fields which are not inseparably defectless. Hence, “algebraically maximal” does not imply “defectless”. There are algebraically maximal fields admitting a finite purely inseparable extension which is not an algebraically maximal field.*

### 4 Absolute ramification theory

Assume that  $L|K$  is an algebraic extension, not necessarily finite, and that  $v$  is a *non-trivial* valuation on  $K$ . We choose an arbitrary extension of  $v$  to the algebraic closure  $\tilde{K}$  of  $K$ . Then for every  $\sigma \in \text{Aut}(\tilde{K}|K)$ , the map

$$v\sigma = v \circ \sigma : L \ni a \mapsto v(\sigma a) \in v\tilde{K} \tag{9}$$

is a valuation of  $L$  which extends  $v$ . All extensions of  $v$  from  $K$  to  $L$  are conjugate:

**Theorem 4.1.** *The set of all extensions of  $v$  from  $K$  to  $L$  is*

$$\{v\sigma \mid \sigma \text{ an embedding of } L \text{ in } \tilde{K} \text{ over } K\}.$$

*In particular, a valuation on  $K$  has a unique extension to every purely inseparable field extension of  $K$ .*



We will now give a quick introduction to *absolute ramification theory*, that is, the ramification theory of the extension  $\tilde{K}|K$  with respect to a given valuation  $v$  on  $\tilde{K}$  with valuation ring  $\mathcal{O}_{\tilde{K}}$ . For a corresponding quick introduction to general ramification theory, see [40].

We define distinguished subgroups of the *absolute Galois group*  $G := \text{Gal } K := \text{Aut}(\tilde{K}|K) = \text{Aut}(K^{\text{sep}}|K)$  of  $K$ , with respect to a fixed extension of  $v$  to  $\tilde{K}$ , which we again call  $v$ . The subgroup

$$G^d := \{ \sigma \in G \mid v\sigma = v \text{ on } \tilde{K} \} \tag{10}$$

is called the *absolute decomposition group of  $(K, v)$*  (w.r.t.  $(\tilde{K}, v)$ ). Further, the *absolute inertia group* (w.r.t.  $(\tilde{K}, v)$ ) is defined to be

$$G^i := \{ \sigma \in G \mid \forall x \in \mathcal{O}_{\tilde{K}} : v(\sigma x - x) > 0 \}, \tag{11}$$

and the *absolute ramification group* (w.r.t.  $(\tilde{K}, v)$ ) is

$$G^r := G^r(L|K, v) := \{ \sigma \in G \mid \forall x \in \mathcal{O}_{\tilde{K}} \setminus \{0\} : v(\sigma x - x) > vx \}. \tag{12}$$

The fixed fields  $K^d$ ,  $K^i$  and  $K^r$  of  $G^d$ ,  $G^i$  and  $G^r$ , respectively, in  $K^{\text{sep}}$  are called the *absolute decomposition field*, *absolute inertia field*, and *absolute ramification field of  $(K, v)$*  (with respect to the given extension of  $v$  to  $\tilde{K}$ ).

*Remark 4.2.* In contrast to the classical definition used by other authors, we take decomposition field, inertia field and ramification field to be the fixed fields of the respective groups *in the separable-algebraic closure of  $K$* . The reason for this will become clear later.

By our definition,  $K^d$ ,  $K^i$ , and  $K^r$  are separable-algebraic extensions of  $K$ , and  $K^{\text{sep}}|K^r$ ,  $K^{\text{sep}}|K^i$ ,  $K^{\text{sep}}|K^d$  are (not necessarily finite) Galois extensions. Further,

$$1 \subset G^r \subset G^i \subset G^d \subset G \text{ and thus, } K^{\text{sep}} \supset K^r \supset K^i \supset K^d \supset K. \tag{13}$$

(For the inclusion  $G^i \subset G^d$  note that  $vx \geq 0$  and  $v(\sigma x - x) > 0$  implies that  $v\sigma x \geq 0$ .)

**Theorem 4.3.**  *$G^i$  and  $G^r$  are normal subgroups of  $G^d$ , and  $G^r$  is a normal subgroup of  $G^i$ . Therefore,  $K^i|K^d$ ,  $K^r|K^d$ , and  $K^r|K^i$  are (not necessarily finite) Galois extensions.*

First, we consider the decomposition field  $K^d$ . In some sense, it represents all extensions of  $v$  from  $K$  to  $\tilde{K}$ .

- Theorem 4.4.** (a)  $v\sigma = v\tau$  on  $\tilde{K}$  if and only if  $\sigma\tau^{-1}$  is trivial on  $K^d$ .  
 (b)  $v\sigma = v$  on  $K^d$  if and only if  $\sigma$  is trivial on  $K^d$ .  
 (c) The extension of  $v$  from  $K^d$  to  $\tilde{K}$  is unique.  
 (d) The extension  $(K^d|K, v)$  is immediate.

WARNING: It is in general not true that  $v\sigma \neq v\tau$  holds already on  $K^d$  if it holds on  $\tilde{K}$ .

Assertions (a) and (b) are easy consequences of the definition of  $G^d$ . Part (c) follows from (b) by Theorem 4.1. For (d), there is a simple proof using a trick mentioned by James Ax in [5, Appendix]; see also [5, Theorem 22, p. 70 and Theorem 23, p. 71] and [19].

Now we turn to the inertia field  $K^i$ . Let  $\mathcal{M}_{\tilde{K}}$  denote the valuation ideal of  $v$  on  $\tilde{K}$  (the unique maximal ideal of  $\mathcal{O}_{\tilde{K}}$ ). For every  $\sigma \in G^d$  we have that  $\sigma\mathcal{O}_{\tilde{K}} = \mathcal{O}_{\tilde{K}}$ , and it follows that  $\sigma\mathcal{M}_{\tilde{K}} = \mathcal{M}_{\tilde{K}}$ . Hence, every such  $\sigma$  induces an automorphism  $\bar{\sigma}$  of  $\mathcal{O}_{\tilde{K}}/\mathcal{M}_{\tilde{K}} = \tilde{K}v = \widetilde{K}v$  which satisfies  $\bar{\sigma}(av) = (\sigma a)v$ . Since  $\sigma$  fixes  $K$ , it follows that  $\bar{\sigma}$  fixes  $Kv$ .

**Lemma 4.5.** *The map*

$$G^d \ni \sigma \mapsto \bar{\sigma} \in \text{Gal } Kv \tag{14}$$

*is a group homomorphism.*

**Theorem 4.6.** (a) *The homomorphism (14) is onto and induces an isomorphism*

$$\text{Aut}(K^i|K^d) = G^d/G^i \simeq \text{Aut}(K^i v|K^d v). \tag{15}$$

(b) *For every finite subextension  $F|K^d$  of  $K^i|K^d$ ,*

$$[F : K^d] = [Fv : K^d v]. \tag{16}$$

(c) *We have that  $vK^i = vK^d = vK$ . Further,  $K^i v$  is the separable closure of  $Kv$ , and therefore*

$$\text{Aut}(K^i v|K^d v) = \text{Gal } Kv. \tag{17}$$

If  $F|K^d$  is normal, then (b) is an easy consequence of (a). From this, the general assertion of (b) follows by passing from  $F$  to the normal hull of the extension  $F|K^d$  and then using the multiplicativity of the extension degree. (c) follows from (b) by use of the fundamental inequality.

We set  $p := \text{char } Kv$  if this is positive, and  $p := 1$  if  $\text{char } Kv = 0$ . Given any abelian group  $\Delta$ , the  $p'$ -divisible hull of  $\Delta$  is defined to be the subgroup  $\{\alpha \in \tilde{\Delta} \mid \exists n \in \mathbb{N} : (p, n) = 1 \wedge n\alpha \in \Delta\}$  of all elements in the divisible hull  $\tilde{\Delta}$  of  $\Delta$  whose order modulo  $\Delta$  is prime to  $p$ .

**Theorem 4.7.** (a) *There is an isomorphism*

$$\text{Aut}(K^r|K^i) = G^i/G^r \simeq \text{Hom}(vK^r/vK^i, (K^r v)^\times), \tag{18}$$

*where the character group on the right hand side is the full character group of the abelian group  $vK^r/vK^i$ . Since this group is abelian,  $K^r|K^i$  is an abelian Galois extension.*

(b) *For every finite subextension  $F|K^i$  of  $K^r|K^i$ ,*

$$[F : K^i] = (vF : vK^i). \tag{19}$$

(c)  *$K^r v = K^i v$ , and  $vK^r$  is the  $p'$ -divisible hull of  $vK$ .*

Part (b) follows from part (a) since for a finite extension  $F|K^i$ , the group  $vF/vK^i$  is finite and thus there exists an isomorphism of  $vF/vK^i$  onto its full character group. The equality  $K^r v = K^i v$  follows from (b) by the fundamental inequality. The second assertion of part (c) follows from the next theorem and the fact that the order of all elements in  $(K^i v)^\times$  and thus also of all elements in  $\text{Hom}(vK^r/vK^i, (K^i v)^\times)$  is prime to  $p$ .

**Theorem 4.8.** *The ramification group  $G^r$  is a  $p$ -group, hence  $K^{\text{sep}}|K^r$  is a  $p$ -extension. Further,  $v\tilde{K}/vK^r$  is a  $p$ -group, and the residue field extension  $\tilde{K}v|K^r v$  is purely inseparable. If  $\text{char } Kv = 0$ , then  $K^r = K^{\text{sep}} = \tilde{K}$ .*

We note:

**Lemma 4.9.** *Every  $p$ -extension is a tower of Galois extensions of degree  $p$ . In characteristic  $p$ , all of them are Artin–Schreier extensions.*

From Theorem 4.8 it follows that there is a canonical isomorphism

$$\text{Hom}(vK^r/vK^i, (K^i v)^\times) \simeq \text{Hom}(v\tilde{K}/vK, (\tilde{K}v)^\times). \tag{20}$$

The following theorem will be very useful for our purposes:

**Theorem 4.10.** *If  $K'|K$  is algebraic, then the absolute decomposition field of  $(K', v)$  is  $K^d.K'$ , its absolute inertia field is  $K^i.K'$ , and its absolute ramification field is  $K^r.K'$ .*

From part (c) of Theorem 4.4 we infer that the extension of  $v$  from  $K^d$  to  $\tilde{K}$  is unique. On the other hand, if  $L$  is any extension field of  $K$  within  $K^d$ , then by Theorem 4.10,  $K^d = L^d$ . Thus, if  $L \neq K^d$ , then it follows from part (b) of Theorem 4.4 that there are at least two distinct extensions of  $v$  from  $L$  to  $K^d$  and thus also to  $\tilde{K} = \tilde{L}$ . This proves that the absolute decomposition field  $K^d$  is a minimal algebraic extension of  $K$  admitting a unique extension of  $v$  to its algebraic closure. So it is the minimal algebraic extension of  $K$  which is henselian. We call it the *henselization of  $(K, v)$  in  $(\tilde{K}, v)$* . Instead of  $K^d$ , we also write  $K^h$ . A valued field is henselian if and only if it is equal to its henselization. Henselizations have the following universal property:

**Theorem 4.11.** *Let  $(K, v)$  be an arbitrary valued field and  $(L, v)$  a henselian extension field of  $(K, v)$ . Then there is a unique embedding of  $(K^h, v)$  in  $(L, v)$  over  $K$ .*

From part (d) of Theorem 4.4, we obtain another very important property of the henselization:

**Theorem 4.12.** *The henselization  $(K^h, v)$  is an immediate extension of  $(K, v)$ .*

**Corollary 4.13.** *Every algebraically maximal and every maximal valued field is henselian. In particular,  $(K((t)), v_t)$  is henselian.*

We employ Theorem 4.10 again to obtain:

**Theorem 4.14.** *If  $K'|K$  is an algebraic extension, then the henselization of  $K'$  is  $K'.K^h$ . Every algebraic extension of a henselian field is again henselian.*

In conjunction with Theorems 4.6 and 4.7, Theorem 4.10 is also used to prove that there are no defects between  $K^h$  and  $K^r$ :

**Theorem 4.15.** *Take a finite extension  $K_2|K_1$  such that  $K^h \subseteq K_1 \subseteq K_2 \subseteq K^r$ . Then  $(K_2|K_1, \nu)$  is defectless.*

*Proof.* Since  $K_3 := K_2 \cap K_1^i$  is a finite subextension of  $K_1^i|K_1$ , we have by parts (b) and (c) of Theorem 4.6 that  $[K_3 : K_1] = [K_3\nu : K_1\nu]$  and  $\nu K_3 = \nu K_1$ . Since  $K_1^i|K_1$  is Galois,  $K_2$  is linearly disjoint from  $K_1^i$  over  $K_3$ . That is,  $[K_2.K_1^i : K_1^i] = [K_2 : K_3]$ . By Theorem 4.10,  $K_2 \subseteq K^r = K_1.K^r = K_1^r$ , so also  $K_2.K_1^i|K_1^i$  is a finite subextension of  $K_1^r|K_1^i$ . By part (b) of Theorem 4.7, we thus have  $[K_2.K_1^i : K_1^i] = (\nu(K_2.K_1^i) : \nu K_1^i)$ . By Theorem 4.10,  $K_1^i = K_1^i.K_3 = K_3^i$  and  $K_2.K_1^i = K_2^i$ , so by part (c) of Theorem 4.6,  $\nu K_1^i = \nu K_3^i = \nu K_3$  and  $\nu(K_2.K_1^i) = \nu K_2^i = \nu K_2$ . Therefore,

$$[K_2 : K_3] = [K_2.K_1^i : K_1^i] = (\nu(K_2.K_1^i) : \nu K_1^i) = (\nu K_2 : \nu K_3) = (\nu K_2 : \nu K_1)$$

Putting everything together, we obtain

$$\begin{aligned} [K_2 : K_1] &= [K_2 : K_3][K_3 : K_1] = (\nu K_2 : \nu K_1)[K_3\nu : K_1\nu] \\ &\leq (\nu K_2 : \nu K_1)[K_2\nu : K_1\nu] \leq [K_2 : K_1], \end{aligned}$$

so that equality must hold everywhere, which shows that  $(K_2|K_1, \nu)$  is defectless.

An algebraic extension of  $K^h$  is called *purely wild* if it is linearly disjoint from  $K^r$  over  $K^h$ . The following theorem has been proved by Matthias Pank (see Theorem 4.3 and Proposition 4.5 of [49]):

**Theorem 4.16.** *Every maximal purely wild extension  $W$  of  $K^h$  satisfies  $W.K^r = \tilde{K}$  and hence is a field complement of  $K^r$  in  $\tilde{K}$ . Moreover,  $W^r = \tilde{W}$ ,  $\nu W$  is the  $p$ -divisible hull of  $\nu K$ , and  $W\nu$  is the perfect hull of  $K\nu$ .*

**Lemma 4.17.** *If  $(L|K^h, \nu)$  is a finite extension, then its defect is equal to the defect of  $(L.K^r|K^r, \nu)$ .*

*Proof.* We put  $L_0 := L \cap K^r$ . We have  $L.K^r = L^r$  and  $L_0^r = K^r$  by Theorem 4.10. Since  $K^r|K^h$  is normal,  $L$  is linearly disjoint from  $K^r = L_0^r$  over  $L_0$ , and  $(L|L_0, \nu)$  is thus a purely wild extension.

As a finite subextension of  $(K^r|K^h, \nu)$ , the extension  $(L_0|K^h, \nu)$  is defectless. Hence by the multiplicativity of the defect (4),

$$d(L|K^h, \nu) = d(L|L_0, \nu). \tag{21}$$

It remains to show  $d(L|L_0, v) = d(L.K^r|K^r, v)$ . Since  $L|L_0$  is linearly disjoint from  $K^r|L_0$ , we have

$$[L^r : L_0^r] = [L.K^r : L_0^r] = [L : L_0]. \tag{22}$$

Since  $L|L_0$  is purely wild,  $vL/vL_0$  is a  $p$ -group and  $Lv|L_0v$  is purely inseparable. On the other hand, by Theorem 4.7,

$vL^r$  is the  $p'$ -divisible hull of  $vL$  and  $L^r v = (L v)^{\text{sep}}$ ,  
 $vL_0^r$  is the  $p'$ -divisible hull of  $vL_0$  and  $L_0^r v = (L_0 v)^{\text{sep}}$ .

It follows that

$$(vL^r : vL_0^r) = (vL : vL_0) \quad \text{and} \quad [L^r v : L_0^r v] = [L v : L_0 v]. \tag{23}$$

From (21), (22) and (23), keeping in mind that  $L.K^r = L^r$  and  $K^r = L_0^r$ , we deduce

$$\begin{aligned} d(L.K^r|K^r, v) &= d(L^r|L_0^r, v) = \frac{[L^r : L_0^r]}{(vL^r : vL_0^r)[L^r v : L_0^r v]} \\ &= \frac{[L : L_0]}{(vL : vL_0) \cdot [L v : L_0 v]} = d(L|L_0, v) = d(L|K^h, v). \end{aligned}$$

We can now describe the ramification theoretic proof for the lemma of Ostrowski (see also [67, Corollary to Theorem 25, p. 78]). Take a finite extension  $(L'|L, v)$  of henselian fields. Then  $L = L^h$ . By the foregoing theorem,  $d(L'|L, v) = d(L'.L^r|L^r, v)$ . It follows from Theorem 4.7 that  $[L'.L^r : L^r]$  is a power of  $p$ . Hence also  $d(L'.L^r|L^r, v)$ , being a divisor of it, is a power of  $p$ .

We see that non-trivial defects can only appear between  $K^r$  and  $\tilde{K}$ , or equivalently, between  $K^h$  and  $W$ . These are the areas of *wild ramification*, whereas the extension from  $K^i$  to  $K^r$  is the area of *tame ramification*. Hence, local uniformization in characteristic 0 and the classification problem for valued fields of residue characteristic 0 only have to deal with tame ramification, while the two problems we described in Section 2 also have to fight the wild ramification.

An algebraic extension  $(L'|L, v)$  of henselian fields is called *unramified* if every finite subextension is unramified. An algebraic extension  $(L'|L, v)$  of henselian fields is called *tame* if every finite subextension  $(L''|L, v)$  is defectless and such that  $(L''v|Lv)$  is separable and  $p$  does not divide  $(vL'' : vL)$ . A henselian field  $(L, v)$  is called a *tame field* if  $(\tilde{L}|L, v)$  is a tame extension, and it is called a *separably tame field* if  $(L^{\text{sep}}|L, v)$  is a tame extension. The fields  $W$  of Theorem 4.16 are examples of tame fields.

The proof of the following theorem is given in [38].

**Theorem 4.18.** *The absolute inertia field is the unique maximal unramified extension of  $K^h$  in  $(\tilde{K}, v)$ . The absolute ramification field is the unique maximal tame extension of  $K^h$  in  $(\tilde{K}, v)$ .*

Note that an extension is tame if and only if it is defectless and “tamely ramified” in the sense of [18, Section 22]. As we have already mentioned, our notion

of “unramified” is the same as “defectless” plus “unramified” in the sense of [18, Section 22] Hence for defectless valuations, the above theorem follows from [18, Corollary (22.9)].

In algebraic geometry, the absolute inertia field is often called the *strict henselization*. Theorem 2.1 can be understood as saying that the Implicit Function Theorem, or equivalently, Hensel’s Lemma, works within and only within the strict henselization. That the limit is the strict henselization and not the henselization becomes intuitively clear when one considers one of the equivalent forms of Hensel’s Lemma which states that if  $f$  has coefficients in the valuation ring of a henselian field, then every simple root of the reduced polynomial  $f_v$  (obtained by replacing the coefficients by their residues) can be lifted to a root of  $f$ . On the other hand, irreducible polynomials have only simple roots if and only if they are separable. Hence, it is clear that Hensel’s Lemma works as long as the residue field extensions are separable, which is the case between  $K^h$  and  $K^i$ .

We summarize our main results in the following table:

<i>Galois group</i>	<i>field</i>	<i>value group</i>	<i>residue field</i>
	$\tilde{K}$	$\widetilde{vK}$	$\widetilde{Kv}$
	purely inseparable		
1	$K^{\text{sep}}$	$\widetilde{vK}$	$\widetilde{Kv}$
	separable-algebraic closure	division by $p$	purely inseparable
	Galois $p$ -extension		
$G^r$	$K^r$	$\frac{1}{p'^{\infty}}vK$	$(Kv)^{\text{sep}}$
	absolute ramification field	division prime to $p$	
Char	abelian Galois $p'$ -extension, defectless		
$G^i$	$K^i$	$vK$	$(Kv)^{\text{sep}}$
	absolute inertia field		Galois
Gal $Kv$	Galois, defectless		
$G^d$	$K^h$	$vK$	$Kv$
	absolute decomposition field		
	immediate		
Gal $K$	$K$	$vK$	$Kv$

where  $\frac{1}{p'^{\infty}}vK$  denotes the  $p'$ -divisible hull of  $vK$  and Char denotes the character group (20).

## 5 Two theorems

### 5.1 The stability theorem

In this section we present two theorems about the defect which we have used for our results on local uniformization and in the model theory of valued fields in positive characteristic. The first one describes situations where no defect appears. The second one deals with certain situation where defect may well appear, but shows that the defect can be eliminated.

Let  $(L|K, v)$  be an extension of valued fields of finite transcendence degree. Then the following well known form of the “Abhyankar inequality” holds:

$$\text{trdeg}L|K \geq \text{rr}vL/vK + \text{trdeg}Lv|Kv, \tag{24}$$

where  $\text{rr}vL/vK := \dim_{\mathbb{Q}}(vL/vK) \otimes \mathbb{Q}$  is the *rational rank* of the abelian group  $vL/vK$ , i.e., the maximal number of rationally independent elements in  $vL/vK$ . This inequality is a consequence of Theorem 1 of [8, Chap. VI, Section 10.3], which states that if

$$\left\{ \begin{array}{l} x_1, \dots, x_\rho, y_1, \dots, y_\tau \in L \text{ such that} \\ vx_1, \dots, vx_\rho \text{ are rationally independent over } vK, \text{ and} \\ y_1v, \dots, y_\tau v \text{ are algebraically independent over } Kv, \end{array} \right. \tag{25}$$

then  $x_1, \dots, x_\rho, y_1, \dots, y_\tau$  are algebraically independent over  $K$ . We will say that  $(L|K, v)$  is *without transcendence defect* if equality holds in (24). In this case, every set  $\{x_1, \dots, x_\rho, y_1, \dots, y_\tau\}$  satisfying (25) with  $\rho = \text{rr}vL/vK$  and  $\tau = \text{trdeg}Lv|Kv$  is a transcendence basis of  $L|K$ .

If  $(F|K, v)$  is a valued function field without transcendence defect, then the extensions  $vF|vK$  and  $Fv|Kv$  are finitely generated (cf. [34, Corollary 2.2]).

**Theorem 5.1.** (*Generalized Stability Theorem*)

*Let  $(F|K, v)$  be a valued function field without transcendence defect. If  $(K, v)$  is a defectless field, then  $(F, v)$  is a defectless field. The same holds for “inseparably defectless” in the place of “defectless”. If  $vK$  is cofinal in  $vF$ , then it also holds for “separably defectless” in the place of “defectless”.*

If the base field  $K$  is not a defectless field, we can say at least the following:

**Corollary 5.2.** *Let  $(F|K, v)$  be a valued function field without transcendence defect, and  $E|F$  a finite extension. Fix an extension of  $v$  from  $F$  to  $\bar{K}.F$ . Then there is a finite extension  $L_0|K$  such that for every algebraic extension  $L$  of  $K$  containing  $L_0$ ,  $(L.F, v)$  is defectless in L.E. If  $(K, v)$  is henselian, then  $L_0|K$  can be chosen to be purely wild.*

Theorem 5.1 was stated and proved in [37]; the proof presented in [45] is an improved version.

The theorem has a long and interesting history. Hans Grauert and Reinhold Remmert [21, p. 119] first proved it in a very restricted case, where  $(K, v)$  is

an algebraically closed complete discretely valued field and  $(F, v)$  is discrete too. A generalization was given by Laurent Gruson [26, Théorème 3, p. 66]. A good presentation of it can be found in the book on non-archimedean analysis by Siegfried Bosch, Ulrich Güntzer and Reinhold Remmert [9, Section 5.3.2, Theorem 1]. Further generalizations are due to Michel Matignon and Jack Ohm, and also follow from results in [23] and [24]. Ohm arrived independently of [37] at a general version of the Stability Theorem for the case of  $\text{trdeg } L|K = \text{trdeg } L_v|K_v$  (see the second theorem on p. 306 of [56]). He deduces his theorem from Proposition 3 on p. 215 of [9], (more precisely, from a generalized version of this proposition which is proved but not stated in [9]).

All authors mentioned in the last paragraph use methods of non-archimedean analysis, and all results are restricted to the case of  $\text{trdeg } F|K = \text{trdeg } F_v|K_v$ . In this case we call the extension  $(F|K, v)$  *residually transcendental*, and we call the valuation  $v$  a *constant reduction* of the algebraic function field  $F|K$ . The classical origin of such valuations is the study of curves over number fields and the idea to reduce them modulo a  $p$ -adic valuation. Certainly, the reduction should again render a curve, this time over a finite field. This is guaranteed by the condition  $\text{trdeg } F|K = \text{trdeg } F_v|K_v$ , where  $F$  is the function field of the curve and  $F_v$  will be the function field of its reduction. Naturally, one seeks to relate the genus of  $F|K$  to that of  $F_v|K_v$ . Several authors proved *genus inequalities* (see, for example, [17, 23, 53] and the survey given in [22]). To illustrate the use of the defect, we will cite an inequality proved by Barry Green, Michel Matignon, and Florian Pop [23, Theorem 3.1]. Let  $F|K$  be a function field of transcendence degree 1, and  $v$  a constant reduction of  $F|K$ . We choose a henselization  $F^h$  of  $(F, v)$ ; all henselizations of subfields of  $F$  will be taken in  $F^h$ . We wish to define a defect of the extension  $(F^h|K^h, v)$  even though this extension is not algebraic. The following result helps:

**Theorem 5.3. (Independence Theorem)**

*The defect of the algebraic extension  $(F^h|K(t)^h, v)$  is independent of the choice of the element  $t \in F$ , provided that  $tv$  is transcendental over  $K_v$ .*

In [56], Ohm proves a more general version of this theorem for arbitrary transcendence degree, using his version of the Stability Theorem. The Stability Theorem tells us that in essence, the defect of a residually transcendental function field, and more generally, of a function field without transcendence defect, can only come from the base field. The following general Independence Theorem was proved in [37, Theorem 5.4 and Corollary 5.6]:

**Theorem 5.4.** *Take a valued function field  $(F|K, v)$  without transcendence defect, and set  $\rho = \text{rr } vF / vK$  and  $\tau = \text{trdeg } F_v|K_v$ . The defect of the extension*

$$(F^h|K(x_1, \dots, x_\rho, y_1, \dots, y_\tau)^h, v)$$

*is independent of the choice of the elements  $x_1, \dots, x_\rho, y_1, \dots, y_\tau$  as long as they satisfy (25). Moreover, there is a finite extension  $K'|K$  such that*

$$(F^h \cdot K'|K'(x_1, \dots, x_\rho, y_1, \dots, y_\tau)^h, v)$$



is defectless and

$$d(F^h.K'|K(x_1, \dots, x_\rho, y_1, \dots, y_\tau)^h, v) = d(K^h.K'|K^h, v).$$

A special case for simple transcendental extensions  $(K(x)|K, v)$  satisfying

$$\text{trdeg } K(x)|K = \text{rr } vK(x)/vK$$

was proved by Sudesh Khanduja in [33].

An interesting proof of Theorem 5.3 is given in [23], as it introduces another notion of defect. We take any valued field extension  $(L|K, v)$  and a finite-dimensional  $K$ -vector space  $V \subseteq L$ . We choose a system  $\mathcal{V}$  of representatives of the cosets  $va + vK, 0 \neq a \in V$ . For every  $K$ -vector space  $W \subseteq V$  and every  $\gamma \in \mathcal{V}$  we set  $W_\gamma := \{a \in W \mid va \geq 0\}$  and  $W_\gamma^\circ := \{a \in W \mid va > 0\}$ . The quotient  $W_\gamma/W_\gamma^\circ$  is in a natural way a  $Kv$ -vector space. The *vector space defect* of  $(V|K, v)$  is defined as

$$d^{vs}(V|K, v) := \sup_{W \subseteq V} \frac{\dim_K W}{\sum_{\gamma \in \mathcal{V}} \dim_{Kv} W_\gamma/W_\gamma^\circ},$$

where the supremum runs over all finite-dimensional subspaces  $W$ . For a finite extension  $(L|K, v)$ , by [23, Proposition 2.2],

$$d^{vs}(L|K, v) = \frac{[L : K]}{(vL : vK)[Lv : Kv]}$$

which is equal to the ordinary defect  $d(L|K, v)$  if the extension of  $v$  from  $K$  to  $L$  is unique.

Note that quotients of the form  $W_\gamma/W_\gamma^\circ$  also appear in the definition of the graded ring of a subring in a valued field, then often written as “ $\mathcal{P}_\gamma/\mathcal{P}_\gamma^+$ ” (see, for instance, [64, Section 2]). Graded rings are used by Bernard Teissier in his program for a “characteristic blind” local uniformization, see [62].

The following result ([23, Theorem 2.13]) implies the Independence Theorem 5.3:

**Theorem 5.5.** *For every element  $t \in F$  such that  $tv$  is transcendental over  $Kv$ ,*

$$d^{vs}(F|K, v) = (F^h|K(t)^h, v).$$

Now we are ready to cite the genus inequality for an algebraic function field  $F|K$  with distinct constant reductions  $v_1, \dots, v_s$  which have a common restriction to  $K$ . We assume in addition that  $K$  coincides with the constant field of  $F|K$  (the relative algebraic closure of  $K$  in  $F$ ). Then:

$$1 - g_F \leq 1 - s + \sum_{i=1}^s d_i e_i r_i (1 - g_i) \tag{26}$$

where  $g_F$  is the genus of  $F|K$  and  $g_i$  the genus of  $Fv_i|Kv_i$ ,  $r_i$  is the degree of the constant field of  $Fv_i|Kv_i$  over  $Kv_i$ ,  $d_i = d^{vs}(F|K, v_i)$ , and  $e_i = (v_i F : v_i K)$  (which is

always finite in the constant reduction case, see, for instance, [42, Corollary 2.7]). It follows that constant reductions  $v_1, v_2$  with common restriction to  $K$  and  $g_1 = g_2 = g_F \geq 1$  must be equal. In other words, for a fixed valuation on  $K$  there is at most one extension  $v$  to  $F$  which is a *good reduction*, that is, (i)  $g_F = g_{Fv}$ , (ii) there exists  $f \in F$  such that  $vf = 0$  and  $[F : K(f)] = [Fv : Kv(fv)]$ , (iii)  $Kv$  is the constant field of  $Fv|Kv$ . An element  $f$  as in (ii) is called a *regular function*.

More generally,  $f$  is said to have the *uniqueness property* if  $f_v$  is transcendental over  $Kv$  and the restriction of  $v$  to  $K(f)$  has a unique extension to  $F$ . In this case,  $[F : K(f)] = d \cdot e \cdot [Fv : Kv(fv)]$  where  $d$  is the defect of  $(F^h|K^h, v)$  and  $e = (vF : vK(f)) = (vF : vK)$ . If  $K$  is algebraically closed, then  $e = 1$ , and it follows from the Stability Theorem that  $d = 1$ ; hence, in this case, every element with the uniqueness property is regular.

It was proved in [24, Theorem 3.1] that  $F$  has an element with the uniqueness property already if the restriction of  $v$  to  $K$  is henselian. The proof uses Abraham Robinson’s model completeness result for algebraically closed valued fields, and ultraproducts of function fields. Elements with the uniqueness property also exist if  $vF$  is a subgroup of  $\mathbb{Q}$  and  $Kv$  is algebraic over a finite field. This follows from work in [25] where the uniqueness property is related to the *local Skolem property* which gives a criterion for the existence of algebraic  $v$ -adic integral solutions on geometrically integral varieties. This result is a special case of a theorem proved in [32] which states that elements with the uniqueness property exist if and only if the completion of  $(K, v)$  is henselian.

As an application to rigid analytic spaces, the Stability Theorem is used to prove that the quotient field of the free Tate algebra  $T_n(K)$  is a defectless field, provided that  $K$  is. This in turn is used to deduce the *Grauert–Remmert Finiteness Theorem*, in a generalized version due to Gruson; see [9, pp. 214–220] for “a simplified version of Gruson’s approach”.

In contrast to the approaches that use methods of non-archimedean analysis, we give in [37, 38] and [45] a new proof which replaces the analytic methods by valuation theoretical arguments. Such arguments seem to be more adequate for a theorem that is of (Krull) valuation theoretical nature.

Our approach has much in common with Abhyankar’s method of using ramification theory in order to reduce the question of resolution of singularities to the study of purely inseparable extensions and of Galois extensions of degree  $p$  and the search for suitable normal forms of Artin–Schreier-like minimal polynomials (cf. [2]). Given a finite separable extension  $(L'|L, v)$  of henselian fields of positive characteristic, we can study its properties by lifting it up to the absolute ramification fields. From Lemma 4.17 we know that the defect of  $(L'|L, v)$  is equal to the defect of  $(L'.L'|L', v)$ . From Lemma 4.9 we know that the extension  $L'.L'|L'$  is a tower of Artin–Schreier extensions.

Abhyankar’s ramification theoretical reduction to Artin–Schreier extensions and purely inseparable extensions is also used by Vincent Cossart and Olivier Piltant in [13] to reduce resolution of singularities of threefolds in positive characteristic to local uniformization on Artin–Schreier and purely inseparable coverings.

The Artin–Schreier extensions appearing through this reduction are not necessarily defect extensions. According to Piltant, those that are, are harder to treat than the defectless ones.

In the situation of Theorem 5.1, we have to prove for  $L = F^h$  that  $(L'|L, \nu)$  is defectless, or equivalently, that each Artin–Schreier extension in the tower is defectless. Looking at the first one in the tower, assume that it is generated by a root  $\vartheta$  of a polynomial  $X^p - X - a$  with  $a \in L'$ . Using the additivity of the Frobenius in characteristic  $p$ , we see that the element  $\vartheta - c$ , which generates the same extension, has minimal polynomial  $X^p - X - (a - c^p + c)$ . Hence, if  $a$  contains some  $p$ -th power  $c^p$ , we can replace it by  $c$  without changing the extension. Using this fact and the special structure of  $(F^h, \nu)$  given by the assumptions of Theorem 5.1 on  $(F, \nu)$ , we deduce normal forms for  $a$  which allow us to read off that the extension is defectless. This fact in turn implies that  $(F^h(\vartheta), \nu)$  is again of the same special form as  $(F^h, \nu)$ , which enables us to proceed by induction over the extensions in the tower.

Note that when algebraic geometers work with Artin–Schreier extensions they usually work with polynomials of the form  $X^p - dX - a$ . The reason is that they work over rings and not over fields. A polynomial like  $X^p - b^{p-1}X - a$  over a ring  $R$  can be transformed into the polynomial  $X^p - X - a/b^p$ , as we have seen in Example 3.17, but  $a/b^p$  does in general not lie in the ring anymore. Working with a polynomial of the form  $X^p - X - a$  is somewhat easier than with a polynomial of the form  $X^p - dX - a$ , and it suffices to derive normal forms as needed for the proof of Theorem 5.1, and of Theorem 5.10 which we will discuss below.

In the case of mixed characteristic, where the valued fields have characteristic 0 and their residue fields have positive characteristic, Artin–Schreier extensions are replaced by Kummer extensions (although re-written with corresponding Artin–Schreier polynomials), and additivity is replaced by quasi-additivity (cf. Lemma 3.19).

Related normal form results can be found in the work of Helmut Hasse, George Whaples, and in Matignon’s proof of his version of Theorem 5.1. See also Helmut Epp’s paper [20], in particular the proof of Theorem 1.3. This proof contains a gap which was filled in [43].

Let us reconsider Examples 2.2 and 2.7 in the light of Theorem 5.1. In Example 2.2 we have an extension without transcendence defect if and only the transcendence degree is 1. In this case,  $(K(t)^h, \nu)$  is a defectless field, and we have that  $F \subset K(t)^h$ . In the case of higher transcendence degree, this may not be the case, as Example 3.1 shows. At least we know that every separable extension of  $K(T)^h$  is defectless since it is discretely valued. The situation is different in Example 2.7. If the extension is without transcendence defect, then again,  $(K(t)^h, \nu)$  is a defectless field, and moreover,  $\nu F/\nu K$  and  $F\nu|K\nu$  are finitely generated ([42, Corollary 2.7]). But if  $\text{char} K > 0$ , then there are valuations  $\nu$  on  $K(x, y)$ , trivial on  $K$ , such that  $K(x, y)\nu = K$ ,  $\nu K(x, y)$  not finitely generated, and such that  $(K(x, y), \nu)$  admits an infinite tower of Artin–Schreier defect extensions ([42, Theorem 1.2]).

Applications of Theorem 5.1 are:

- *Elimination of ramification.* In [34] we use Theorem 5.1 to prove:

**Theorem 5.6.** *Take a defectless field  $(K, \nu)$  and a valued function field  $(F|K, \nu)$  without transcendence defect. Assume that  $F\nu|K\nu$  is a separable extension and  $\nu F/\nu K$  is torsion free. Then  $(F|K, \nu)$  admits elimination of ramification in the following sense: there is a transcendence basis  $T = \{x_1, \dots, x_r, y_1, \dots, y_s\}$  of  $(F|K, \nu)$  such that*

- (a)  $\nu F = \nu K \oplus \mathbb{Z}\nu x_1 \oplus \dots \oplus \mathbb{Z}\nu x_r,$
- (b)  $y_1\nu, \dots, y_s\nu$  form a separating transcendence basis of  $F\nu|K\nu.$

For each such transcendence basis  $T$  and every extension of  $\nu$  to the algebraic closure of  $F$ ,  $(F^h|K(T)^h, \nu)$  is unramified.

**Corollary 5.7.** *Let  $(F|K, \nu)$  be a valued function field without transcendence defect. Fix an extension of  $\nu$  to  $\tilde{F}$ . Then there is a finite extension  $L_0|K$  and a transcendence basis  $T$  of  $(L_0.F|L_0, \nu)$  such that for every algebraic extension  $L$  of  $K$  containing  $L_0$ , the extension  $((L.F)^h|L(T)^h, \nu)$  is unramified.*

- *Local uniformization in positive and in mixed characteristic.* We consider places  $P$  and their associated valuations  $\nu = \nu_P$  of a function field  $F|K$ , by which we mean that  $P|_K$  is the identity and hence  $\nu|_K$  is trivial. We write  $aP = a\nu$  and denote by  $\mathcal{O}$  the valuation ring of  $\nu$  on  $F$ . Rewriting our earlier definition, we say that  $P$  admits *smooth local uniformization* if there is a model for  $F$  on which  $P$  is centered at a smooth point, that is, if there are  $x_1, \dots, x_n \in \mathcal{O}$  such that  $F = K(x_1, \dots, x_n)$  and the point  $x_1P, \dots, x_nP$  is smooth. (Note that in [34] and [35] we add a further condition, which we drop here for simplicity.) The place  $P$  is called an *Abhyankar place* if equality holds in the Abhyankar inequality, which in the present case means that  $\text{trdeg } F|K = \text{rr } \nu F + \text{trdeg } FP|K.$

Theorem 5.6 is a crucial ingredient for the following result (cf. [34, Theorem 1.1], [39]):

**Theorem 5.8.** *Assume that  $P$  is an Abhyankar place of the function field  $F|K$  such that  $FP|K$  is separable. Then  $P$  admits smooth local uniformization.*

The analogous arithmetic case ([34, Theorem 1.2]) uses Theorem 5.1 in mixed characteristic. Note that the condition “ $FP|K$  is separable” is necessary since it is implied by elimination of ramification.

- *Model theory of valued fields.* In [47] we use Theorem 5.6 to prove the following Ax–Kochen–Ershov Principle:

**Theorem 5.9.** *Take a henselian defectless valued field  $(K, \nu)$  and an extension  $(L|K, \nu)$  of finite transcendence degree without transcendence defect. If  $\nu K$  is existentially closed in  $\nu L$  and  $K\nu$  is existentially closed in  $L\nu$ , then  $(K, \nu)$  is existentially closed in  $(L, \nu).$*

Let us continue our discussion from the end of Section 2.2. The conditions “ $\nu K \prec_{\exists} \nu L$ ” and “ $K\nu \prec_{\exists} L\nu$ ” imply that  $\nu F/\nu K$  is torsion free and  $F\nu|K\nu$  is a separable extension. So we can apply Theorem 5.6 to obtain the transcendence basis  $T = \{x_1, \dots, x_r, y_1, \dots, y_s\}$  of  $(F|K, \nu)$  with the properties as specified in that theorem. Because of these properties, the embeddings of  $\nu L$  in  $\nu^*K^*$  and of  $L\nu$  in  $K^*\nu^*$  lift to an embedding  $t_0$  of  $(K(T), \nu)$  in  $(K^*, \nu^*)$  over  $K$ . Using Hensel’s Lemma and the fact that  $F\nu|K(T)\nu$  is separable, one finds in  $F^h|K(T)$  a subextension  $F_1|K(T)$  with  $F_1\nu = F\nu$  and  $[F_1 : K(T)] = [F_1\nu : K(T)\nu]$  ( $F\nu|K(T)\nu$  is finite by the remark preceding Theorem 5.1). Using Hensel’s Lemma and the embedding of  $L\nu$  in  $K^*\nu^*$  again, one extends  $t_0$  to an embedding  $t_1$  of  $(F_1, \nu)$  in  $(K^*, \nu^*)$ . The extension  $(F^h|F_1^h, \nu)$  is immediate, and as it is an extension inside the unramified extension  $(F^h|K(T)^h, \nu)$ , it must be defectless and hence trivial. As  $(K^*, \nu^*)$  is henselian, being an elementary extension of the henselian field  $(K, \nu)$ , one can now use the universal property of henselizations to extend  $t_1$  to an embedding  $t_2$  of  $(F^h, \nu)$  in  $(K^*, \nu^*)$ . The restriction of  $t_2$  to  $F$  is the desired embedding which transfers every existential sentence valid in  $(F, \nu)$  into  $(K^*, \nu^*)$ .

### 5.2 Henselian rationality of immediate function fields

Let us return to Example 2.8. If  $(F, \nu)$  does not lie in the henselization  $K(x)^h$ , we are lost. This happens if and only if  $(F^h|K(x)^h, \nu)$  has non-trivial defect (the equivalence holds because  $(F|K(x), \nu)$  is finite and immediate,  $F^h = F.K(x)^h$  and henselizations are immediate extensions).

So the question arises: how can we avoid the defect in the case of immediate extensions? The answer is a theorem proved in [37] (cf. [38] and [48]):

**Theorem 5.10.** (*Henselian Rationality*)

*Let  $(K, \nu)$  be a tame field and  $(F|K, \nu)$  an immediate function field of transcendence degree 1. Then*

$$\text{there is } x \in F \text{ such that } (F^h, \nu) = (K(x)^h, \nu), \tag{27}$$

*that is,  $(F|K, \nu)$  is henselian generated. The same holds over a separably tame field  $(K, \nu)$  if in addition  $F|K$  is separable.*

Since the assertion says that  $F^h$  is equal to the henselization of a rational function field, we also call  $F$  *henselian rational* in this case. For valued fields of residue characteristic 0, the assertion is a direct consequence of the fact that every such field is defectless. Indeed, take any  $x \in F \setminus K$ . Then  $K(x)|K$  cannot be algebraic since otherwise,  $(K(x)|K, \nu)$  would be a proper finite immediate (and hence defect) extension of the tame field  $(K, \nu)$ , a contradiction to the definition of “tame”. Hence,  $F|K(x)$  is algebraic and immediate. Therefore,  $(F^h|K(x)^h, \nu)$  is algebraic and immediate too. But since it cannot have a non-trivial defect, it must be trivial. This proves that  $(F, \nu) \subset (K(x)^h, \nu)$ . In contrast to this, in the case of positive residue characteristic only a very carefully chosen  $x \in F \setminus K$  will do the job.

As for the Generalized Stability Theorem, the proof of Theorem 5.10 in positive characteristic uses ramification theory and the deduction of normal forms for Artin–Schreier extensions. This time however, all Artin–Schreier extensions are immediate and hence defect extensions. The normal forms serve a different purpose, namely, to find a suitable generator  $x$ . The proof also uses significantly a theory of immediate extensions which builds on Kaplansky’s paper [29, Sections 2 and 3].

*Open problem (HR):* Improve Theorem 5.10 by finding versions that work with weaker assumptions. For instance, can the assumption “tame” be replaced by “henselian and perfect” or just “perfect”, or can it even be dropped altogether? Then, even with a weaker assumption on  $(K, \nu)$ , can the assumption “immediate” be replaced by “ $\nu F/\nu K$  is a torsion group and  $F\nu = K\nu$ ”?

Note that in order to allow  $F\nu|K\nu$  to be any algebraic extension, a possible generalization of Theorem 5.10 would have to replace (27) by

$$\text{there is } x \in F \text{ such that } (F^i, \nu) = (K(x)^i, \nu). \tag{28}$$

Applications of Theorem 5.10 in conjunction with Theorem 5.1 are:

- *Local uniformization in positive and in mixed characteristic.* Theorem 5.10 together with Theorem 5.6 is a crucial ingredient for the proof of “local uniformization by alteration” (cf. [35, Theorem 1.2], [39]):

**Theorem 5.11.** *Assume that  $P$  is a place of the function field  $F|K$ . Then there is a finite extension  $F'|F$  and an extension  $P'$  of  $P$  from  $F$  to  $F'$  such that  $P'$  admits smooth local uniformization. The extension  $F'|F$  can be chosen to be Galois. Alternatively, it can be chosen such that  $(F', P')|(F, P)$  is purely wild, hence  $\nu_{P'}F'/\nu_{P'}F$  is a  $p$ -group and  $F'P'|FP$  is purely inseparable.*

The analogous arithmetic case ([35, Theorem 1.4]) uses Theorems 5.10 and 5.1 in mixed characteristic. While local uniformization by alteration follows from de Jong’s resolution of singularities by alteration (see [3]), the additional information on the extension  $F'|F$  does not follow. Moreover, the proofs of Theorems 5.8 and 5.11 use only valuation theory.

Recently, Michael Temkin ([61, Corollary 1.3]) proved “Inseparable Local Uniformization”:

**Theorem 5.12.** *In the setting of Theorem 5.11, the extension  $F'|F$  can also be chosen to be purely inseparable.*

It is interesting that local uniformization has now been proved up to separable alteration on the one hand, and up to purely inseparable alteration on the other. These two results are somewhat “orthogonal” to each other. Can they be put together to get rid of alteration? While this appears to be an attractive thought at first sight, one should keep in mind Example 3.17 which shows that every purely inseparable defect extension of degree  $p$  of  $(L, \nu)$  which does not lie in the completion of  $(L, \nu)$  can be transformed into an Artin–Schreier defect extension. Thus, the “same” defect may appear in a separable extension and in a purely inseparable extension (see the

next section for details), which leaves us the choice to kill it either with separable or with inseparable alteration. So this fact does not in itself indicate whether we need or do not need alteration for local uniformization.

*Open problem (LU):* Prove (or disprove) local uniformization without extension of the function field.

In fact, one reason for the extension of the function field in our approach is the fact that we apply Theorem 5.10 to fields of lower transcendence degree than the function field itself. However, subfunction fields are too small to be tame fields, so we enlarge our intermediate fields so that they become (separably) tame, and once we have found local uniformization in this larger configuration, we collect the only finitely many new elements that are needed for it and adjoin them to the original function field. So we see that if we can weaken the assumptions of Theorem 5.10, then possibly we will need smaller extensions of our function field. Temkin's work contains several developments in this direction, one of which we will discuss in more detail in the next section.

- *Model theory of valued fields.* In [47] we use Theorem 5.10 together with Theorem 5.6 to prove the following:

**Theorem 5.13.** (a) *If  $(K, v)$  is a tame field, then the Ax–Kochen–Ershov Principle (5) holds.*

(b) *The Classification Problem for valued fields has a positive solution for tame fields: If  $(K, v)$  and  $(L, v)$  are tame fields such that  $vK$  and  $vL$  are elementarily equivalent as fields and  $Kv$  and  $Lv$  are elementarily equivalent as ordered groups, then  $(K, v)$  and  $(L, v)$  are elementarily equivalent as valued fields.*

This theorem comprises several classes of valued fields for which the classification had already been known to hold, such as the already mentioned henselian fields with residue fields of characteristic 0.

*Open problem (AKE):* Prove Ax–Kochen–Ershov Principles for classes of non-perfect valued fields of positive characteristic. This problem is connected with the open problem whether the elementary theory of  $\mathbb{F}_p((t))$  is decidable (cf. [38,41,47]).

## 6 Two types of Artin–Schreier defect extensions

In this section, we assume all fields to have characteristic  $p > 0$ . In Section 3 we have given several examples of Artin–Schreier defect extensions, i.e., Artin–Schreier defect extensions with non-trivial defect. Note that every such extension is immediate. Some of our examples were derived from immediate purely inseparable extensions of degree  $p$  (Examples 3.17 and 3.18). If an Artin–Schreier defect extension is derived from a purely inseparable defect extension of degree  $p$  as in Example 3.17, then we call it a *dependent Artin–Schreier defect extension*. If it cannot be derived in this way, then we call it an *independent Artin–Schreier defect extension*.

More precisely, an Artin–Schreier defect extension  $(L(\vartheta)|L, \nu)$  with  $\vartheta^p - \vartheta \in L$  is defined to be dependent if there is a purely inseparable extension  $(L(\eta)|L, \nu)$  of degree  $p$  such that

$$\text{for all } c \in L, \nu(\vartheta - c) = \nu(\eta - c).$$

The extension  $(L(\vartheta)|L, \nu)$  constructed in Example 3.12 is an independent Artin–Schreier defect extension. This is obvious if we choose  $K = \mathbb{F}_p(t)$  or  $K = \mathbb{F}_p((t))$  because then  $L$  is the perfect hull of  $K$  and does not admit any purely inseparable defect extensions at all. But if for instance,  $K = \mathbb{F}_p(t, s)$  with  $s \in \mathbb{F}_p((t))$  transcendental over  $\mathbb{F}_p(t)$ , then  $L$  is not perfect. How do we know then that  $(L(\vartheta)|L, \nu)$  is an independent Artin–Schreier defect extension? The answer is given by the following characterization proved in [37] (see [38] and [46]):

**Theorem 6.1.** *Take an Artin–Schreier defect extension  $(L(\vartheta)|L, \nu)$  with  $\vartheta^p - \vartheta \in L$ . Then this extension is independent if and only if*

$$\nu(\vartheta - L) + \nu(\vartheta - L) = \nu(\vartheta - L). \tag{29}$$

Note that  $\nu(\vartheta - L) + \nu(\vartheta - L) := \{\alpha + \beta \mid \alpha, \beta \in \nu(\vartheta - L)\}$  and that the sum of two initial segments of a value group is again an initial segment. Equation (29) means that  $\nu(\vartheta - L)$  defines a cut in  $\nu L$  which is idempotent under addition of cuts (defined through addition of the left cut sets). If  $\nu L$  is archimedean, then there are only four possible idempotent cuts, corresponding to  $\nu(\vartheta - L) = \emptyset$  (which is impossible),  $\nu(\vartheta - L) = (\nu L)^{<0}$ ,  $\nu(\vartheta - L) = (\nu L)^{\leq 0}$ , and  $\nu(\vartheta - L) = \nu L$  (which means that  $\vartheta$  lies in the completion of  $(L, \nu)$ ).

It is important to note that  $\nu(\vartheta - K) \subseteq (\nu L)^{<0}$ . Indeed, if there were some  $c \in K$  such that  $\nu(\vartheta - c) \geq 0$ , then

$$0 \leq \nu((\vartheta - c)^p - (\vartheta - c)) \leq \nu(\vartheta^p - \vartheta - (c^p - c)).$$

But a polynomial  $X^p - X - a$  with  $va \geq 0$  splits completely in the absolute inertia field of  $(L, \nu)$  and thus cannot induce a defect extension. Therefore, if  $\nu L$  is archimedean, then (29) holds if and only if  $\nu(\vartheta - L) = (\nu L)^{<0}$ . This shows that the extension  $(L(\vartheta)|L, \nu)$  of Example 3.12 is an independent Artin–Schreier defect extension even if  $L$  is not perfect. On the other hand, the extension  $(L(\vartheta_0)|L, \nu)$  of Example 3.17, where  $\vartheta_0/b$  is a root of the polynomial  $X^p - X - 1/b^p t$ , is a dependent Artin–Schreier defect extension as it was obtained from the purely inseparable defect extension  $(L(\eta)|L, \nu)$ . And indeed,

$$\begin{aligned} \nu\left(\frac{\vartheta_0}{b} - L\right) + \nu\left(\frac{\vartheta_0}{b} - L\right) &= \{\alpha \in \nu L \mid \alpha < \nu b\} + \{\alpha \in \nu L \mid \alpha < \nu b\} \\ &\neq \{\alpha \in \nu L \mid \alpha < \nu b\} \end{aligned}$$

since  $\nu b \neq 0$ . Note that since  $\nu L$  is  $p$ -divisible, we in fact have that

$$\{\alpha \in \nu L \mid \alpha < \nu b\} + \{\alpha \in \nu L \mid \alpha < \nu b\} = \{\alpha \in \nu L \mid \alpha < 2\nu b\}.$$



This example also shows that the criterion of Theorem 6.1 only works for roots of Artin–Schreier polynomials. Indeed,  $v(\vartheta_0 - L) = v(\eta - L) = (vL)^{<0}$ , which does not contradict the theorem since the minimal polynomial of  $\vartheta_0$  is  $X^p - dX - 1/t$  with  $vd \neq 0$ .

Each of the perfect fields  $(L, v)$  from Example 3.12 provides an example of a valued field without dependent Artin–Schreier defect extensions, but admitting an independent Artin–Schreier defect extension. Valued fields without independent Artin–Schreier defect extensions but admitting dependent Artin–Schreier defect extensions are harder to find; one example is given in [46].

The classification of Artin–Schreier defect extensions and Theorem 6.1 are the main tool in the proof of the following criterion ([37]; see [38] and [46]):

**Theorem 6.2.** *A valued field  $(L, v)$  of positive characteristic is henselian and defectless if and only if it is algebraically maximal and inseparably defectless.*

This criterion is very useful when one tries to construct examples of defectless fields with certain properties, as was done in [41, Section 4]. How can one construct defectless fields? One possibility is to take any valued field and pass to its maximal immediate extension. Every maximal field is defectless. But it is in general an extension of very large transcendence degree. If we want something smaller, then we could use Theorem 5.1. But that talks only about function fields (or their henselizations). If we want to construct a field with a certain value group (as in [41]), we may have to pass to an infinite algebraic extension. If we replace that by any of its maximal immediate algebraic extensions, we obtain an algebraically maximal field  $(M, v)$ . But Example 3.25 shows that such a field may not be defectless. If, however, we can make sure that  $(M, v)$  is also inseparably defectless, then Theorem 6.2 tells us that  $(M, v)$  is defectless.

How do we know that a valued field  $(L, v)$  is inseparably defectless? In the case of finite  $p$ -degree  $[K : K^p]$  (also called *Ershov invariant of  $K$* ), Delon [16] gave a handy characterization of inseparably defectless valued fields:

**Theorem 6.3.** *Let  $L$  be a field of characteristic  $p > 0$  and finite  $p$ -degree  $[L : L^p]$ . Then for the valued field  $(L, v)$ , the property of being inseparably defectless is equivalent to each of the following properties:*

- (a)  $[L : L^p] = (vL : pvL)[Lv : Lv^p]$ , i.e.,  $(L|L^p, v)$  is a defectless extension
- (b)  $(L^{1/p}|L, v)$  is a defectless extension
- (c) Every immediate extension of  $(L, v)$  is separable
- (d) There is a separable maximal immediate extension of  $(L, v)$ .

The very useful upward direction of the following lemma was also stated by Delon ([16], Proposition 1.44):

**Lemma 6.4.** *Let  $(L'|L, v)$  be a finite extension of valued fields of characteristic  $p > 0$ . Then  $(L, v)$  is inseparably defectless and of finite  $p$ -degree if and only if  $(L', v)$  is.*

The condition of finite  $p$ -degree is necessary, as Example 3.23 shows. In that example,  $(k((t))|K, \nu)$  is a purely inseparable defect extension of degree  $p$ . Hence  $(K, \nu)$  is not inseparably defectless. But  $(k((t)), \nu)$  is, since it is a maximal and hence defectless field.

### 6.1 Work in progress

(a) An analogue of the classification of Artin–Schreier defect extensions and of Theorem 6.2 has to be developed for the mixed characteristic case (valued fields of characteristic 0 with residue fields of characteristic  $p$ ). Temkin and other authors have already done part of the work.

(b) The classification of Artin–Schreier defect extensions is also reflected in their higher ramification groups. This will be worked out in [50].

We have seen in Example 3.17 that every purely inseparable defect extension of degree  $p$  of  $(L, \nu)$  which does not lie in the completion of  $(L, \nu)$  can be transformed into a (dependent) Artin–Schreier defect extension. This can be used to prove the following result (cf. [46]):

**Proposition 6.5.** *Assume that  $(L, \nu)$  does not admit any dependent Artin–Schreier extension. Then every immediate purely inseparable extension lies in the completion of  $(L, \nu)$ .*

**Corollary 6.6.** *Every non-trivially valued Artin–Schreier closed field lies dense in its perfect hull. In particular, the algebraic closure of a non-trivially valued separable-algebraically closed field  $(L, \nu)$  lies in the completion of  $(L, \nu)$ .*

Which of the Artin–Schreier defect extensions are the more harmful, the dependent or the independent ones? There are some indications that the dependent ones are more harmful. Temkin’s work (especially [61, Theorem 3.2.3]) seems to indicate that there is a generalization of Theorem 5.10 which already works over henselian perfect instead of tame valued fields  $(K, \nu)$ . When  $K$  is perfect, then  $(K, \nu)$  does not have dependent Artin–Schreier defect extensions. The independent ones do not seem to matter here, at least when  $(K, \nu)$  has rank 1. This improvement is one of the keys to Theorem 5.12. Let us give an example which illustrates what is going on here.

*Example 6.7.* Assume that  $(F|K, \nu)$  is an immediate function field of transcendence degree 1, rank 1 and characteristic  $p > 0$ , and that we have chosen  $x \in F$  such that  $(F^h|K(x)^h, \nu)$  is of degree  $p$ . We want to improve our choice of  $x$ , that is, find  $y \in F$  such that  $F^h = K(y)^h$ . The procedure given in [37, 48] uses the fact that because  $(F|K, \nu)$  is immediate,  $x$  is a *pseudo limit* of a *pseudo Cauchy sequence*  $(a_\nu)_{\nu < \lambda}$  in  $K$  which has no pseudo limit in  $K$  ([29, Section 2]). The hypothesis that  $(K, \nu)$  be tame (or separably tame and  $F|K$  separable) guarantees that  $(a_\nu)_{\nu < \lambda}$  is of *transcendental type*, that is, if  $f$  is any polynomial in one variable over  $K$ , then the value  $\nu f(a_\nu)$  is

fixed for all large enough  $v < \lambda$  ([29, p. 306]). This is essential in our procedure. If we drop the tameness hypothesis, then  $(a_v)_{v < \lambda}$  may not be of transcendental type and may in fact have some element in some immediate algebraic extension of  $(K, v)$  as a pseudo limit. Now suppose that this element is the root  $\vartheta$  of an Artin–Schreier polynomial over  $K$ . The fact that both  $x$  and  $\vartheta$  are limits of  $(a_v)_{v < \lambda}$  implies that  $v(x - \vartheta) > v(\vartheta - K)$ . If the Artin–Schreier defect extension  $(K(\vartheta)|K, v)$  is independent, then because of our rank 1 assumption, it follows that  $v(x - \vartheta) \geq 0$ . If we assume in addition that  $Kv$  is algebraically closed, then there is some  $c \in K$  such that  $v(x - \vartheta - c) > 0$ . But then, by a special version of Krasner’s Lemma (cf. [42, Lemma 2.21]), the polynomial  $X^p - X - (\vartheta^p - \vartheta)$  splits in  $(K(x)^h, v)$ , so that  $\vartheta \in K(x)^h$ . This shows that  $K$  is not relatively algebraically closed in  $K(x)^h$ . Replacing  $K$  by its relative algebraic closure in  $K(x)^h$ , we will avoid this special case of pseudo Cauchy sequences that are not of transcendental type.

If on the other hand, the extension  $(K(\vartheta)|K, v)$  is dependent, then it does not follow that  $v(x - \vartheta) \geq 0$ . But if  $v(x - \vartheta) < 0$ , Krasner’s Lemma is of no use. However, by assuming that  $K$  is perfect we obtain that  $(K, v)$  does not admit dependent Artin–Schreier defect extensions. Assuming in addition that  $K$  is relatively algebraically closed in  $F^h$ , we obtain that  $(a_v)_{v < \lambda}$  does not admit any Artin–Schreier root  $\vartheta$  over  $K$  as a limit. This fact alone does not imply that under our additional assumptions,  $(a_v)_{v < \lambda}$  must be of transcendental type. But with some more technical effort, building on results in [46], this can be shown to be true.

Another indication may come from the paper [15] by Steven Dale Cutkosky and Olivier Piltant. They give an example of an extension  $R \subset S$  of algebraic regular local rings of dimension two over a field  $k$  of positive characteristic and a valuation on the rational function field  $\text{Quot}R$ , with  $\text{Quot}S|\text{Quot}R$  being a tower of two Artin–Schreier defect extensions, such that strong monomialization in the sense of Theorem 4.8 of their paper does not hold for  $R \subset S$  ([15, Theorem 7.38]).

Work in progress with Laura Ghezzi and Samar El-Hitti indicates that both extensions are dependent Artin–Schreier defect extensions. In fact, in Piltant’s own words, he chose the valuation in the example such that it is “very close” to [the behavior of] a valuation in a purely inseparable extension.

*Open problem (CPE):* Is there a version of the example of Cutkosky and Piltant involving independent Artin–Schreier defect extensions? Or are such extensions indeed less harmful than the dependent ones? Can strong monomialization always be proven when only independent Artin–Schreier defect extensions are involved?

## 7 Two languages

Algebraic geometers and valuation theorists often speak different languages. For example, while the defect was implicitly present already in Abhyankar’s work, it has been explicitly studied rather by the early valuation theorists like Ostrowski, and by researchers interested in the model theory of valued fields in positive characteristic or in constant reduction, most of them using a field theoretic language and “living in the Kaplansky world of pseudo Cauchy sequences” (cf. [29, Section 2], [38]).

For instance, our joint investigation with Ghezzi and ElHitti of the example given by Cutkosky and Piltant is facing the difficulty that the valuation in the example is given by means of generating sequences, whereas our criterion for dependence/independence is by nature closer to the world of pseudo Cauchy sequences, which can also be used to describe valuations.

*Open problem (CGS):* Rewrite the criterion given in Theorem 6.1 in terms of generating sequences.

As to the problem of how to describe valuations, Michel Vaquié has done much work generalizing MacLane’s approach using families of key polynomials. In this approach, he also showed how to read off defects as invariants of such families (see [63]). A closer look reveals that a set like  $v(\vartheta - L)$  can be directly determined from Vaquié’s families of key polynomials.

*Open problem (CV):* Is there an efficient algorithm to convert generating sequences into Vaquié’s families of key polynomials? More generally, find algorithms that convert between generating sequences, key polynomials, pseudo Cauchy sequences and higher ramification groups.

A lot of work has been done by several authors on the description of valuations on rational function fields, working with key polynomials or pseudo Cauchy sequences. For references, see [42].

*Open problem (RFF):* Develop a thorough theory of valuations on rational function fields, bringing the different approaches listed in (CV) together, then generalize to algebraic function fields. Understand the defect extensions that can appear over rational function fields.

Problems (CGV), (CV), and (RFF) can be understood as parts of a larger program:

*Open problem (DIC):* Develop a “dictionary” between algebraic geometry and valuation theory. This would allow us to translate results known about the defect into results in algebraic geometry, and open questions in algebraic geometry into questions in valuation theory. It would help us to investigate critical examples from several points of view and to use them both in algebraic geometry and valuation theory.

Let us conclude with the following

*Open problem (DAG):* What exactly is the meaning of the defect in algebraic geometry? How can we locate and interpret it? What is the role of dependent and independent Artin–Schreier defect extensions, e.g., in the work of Abhyankar, of Cutkosky and Piltant, or of Cossart and Piltant?

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# The use of ultrafilters to study the structure of Prüfer and Prüfer-like rings

K. Alan Loper

**Abstract** The notion of a sequence of objects converging to a limit point is very natural in the context of topology. In a collection of articles from the last couple decades ultrafilters were employed to great effect in forcing sequences of prime ideals or valuation domains to converge without reference to topology. The context of these results was the study of Prüfer domains and rings of integer-valued polynomials. More recently Fontana and Loper clarified these results by demonstrating that ultrafilters can be used to define a topology which is equivalent to the classical constructible or patch topology. The history of these results is recounted in this expository article in chronological sequence.

## 1 Introduction

Let  $S$  be an infinite set. An ultrafilter on  $S$  is a collection of subsets  $U$  of  $S$  which satisfies the following properties. (See [7] for more details concerning ultrafilters.)

1. If  $B \in U$  and  $B \subseteq C \subseteq S$  then  $C \in U$ .
2. If  $B, C \in U$  then  $B \cap C \in U$ .
3. If  $B \cup C = S$  and  $B \cap C = \Phi$  then exactly one of  $B$  and  $C$  lies in  $U$ .

It is easy to see that examples of ultrafilters exist. Choose an element  $x \in S$ . Let  $U_x$  be the collection of all subsets of  $S$  which contain  $x$ . Then  $U_x$  satisfies the axioms of an ultrafilter. We call such an ultrafilter principal. It is not as simple to show that nonprincipal ultrafilters exist. It is easy to show that collections of subsets exist which satisfy (1), (2), and the additional restriction that the empty set not be a member. (Such a collection is called a filter.) For example, the collection of all

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subsets of  $S$  which are complements of finite subsets is a filter. We can order the collection of filters by inclusion. Zorn's lemma then implies that maximal filters exist and it is easy to show that these maximal filters are ultrafilters. Moreover, a maximal filter containing a nonprincipal filter will be a nonprincipal ultrafilter.

It is not possible to explicitly list all of the subsets constituting a nonprincipal ultrafilter, but it is possible to exert some control over what subsets are in an ultrafilter. In particular, if we start with a collection  $\Lambda$  of subsets of an infinite set  $S$  such that any finite intersection of members of  $\Lambda$  is not empty then it is easy to extend  $\Lambda$  to a filter, and then to an ultrafilter. So any collection of subsets which satisfies this finite intersection property can be embedded in an ultrafilter.

Ultrafilters have been used by various authors in recent years to construct exotic examples of rings by means of what are called ultraproducts. The focus of this article will be somewhat different. Our focus is on a collection of articles (written by the present author and some coauthors) in which ultrafilters are used to analyze the structure of rings that are defined by other means. Our presentation will develop the subject in the same order in which it was developed in the literature. Also, our focus will be less on the main results of the papers cited and more on how the ultrafilters were employed. We start with a short motivating section in which we explain in heuristic terms how ultrafilters might be used to analyze ideal structure or collections of valuation overrings along with a motivating example and then give some basic theorems to apply in our investigations. In the following section we go through some results in the literature where these ideas have been used. We conclude with a section that gives some topological underpinnings to the subject and suggests some further applications that could be investigated.

## 2 Motivation

It is well known that in the classical ring of integer-valued polynomials,  $\text{Int}(Z) = \{f(X) \in \mathcal{Q}[X] \mid f(Z) \subseteq Z\}$ , the maximal ideals can be indexed naturally by the rings of  $p$ -adic integers,  $\widehat{Z}_p$ . In particular, if  $p$  is a prime number and  $\alpha$  is a  $p$ -adic integer then  $M_{p,\alpha} = \{f(X) \in \text{Int}(Z) \mid f(\alpha) \in p\widehat{Z}_p\}$  is a maximal ideal, and all maximal ideals have this form. Since  $\text{Int}(Z)$  is a Prüfer domain, the valuation overrings in which  $p$  is a nonunit can similarly be indexed by the  $p$ -adic integers. By means of this correspondence, the metric topologies on the rings of  $p$ -adic integers can be exploited to investigate the structure of  $\text{Int}(Z)$  and its overrings. The following theorem is an example.

Let  $p$  be a prime number. For any  $p$ -adic integer  $\alpha$  let  $V_{p,\alpha}$  be the valuation ring obtained by localizing  $\text{Int}(Z)$  at the maximal ideal  $M_{p,\alpha}$ . Then let  $\Lambda$  be a collection of  $p$ -adic integers. Define

$$D_\Lambda = \bigcap_{\alpha \in \Lambda} V_{p,\alpha}$$

Since  $D_\Lambda$  is an overring of  $\text{Int}(Z)$  it is a Prüfer domain. It is obvious that each valuation domain  $V_{p,\alpha}$  where  $\alpha \in \Lambda$  is an overring of  $D_\Lambda$ . It also seems plausible that there could be many other valuation overrings besides the ones from the defining collection.

**Theorem 1** ([16, Lemma 26]). *Assume the notation of the preceding paragraph. Then the valuation overrings of  $D_\Lambda$  in which  $p$  is a non-unit are exactly those of the form  $V_{p,\beta}$  corresponding to  $p$ -adic integers  $\beta$  which lie in the  $p$ -adic closure of  $\Lambda$ .*

Note in the above discussion that the maximal ideals  $M_{p,\alpha}$  are perhaps more naturally defined when  $\alpha$  is a rational integer than when it is a non-rational  $p$ -adic integer. Then we observe that the  $p$ -adic integers arise by means of convergence of sequences of rational integers. Note also that the above theorem suggests that the topological closure of sets of  $p$ -adic integers is somehow mirrored by the “closure” of sets of prime ideals or valuation domains. The question then arises of whether the notion of convergence of ideals or of overrings can be extended to a broader class of rings, where we do not necessarily have the natural connection with the  $p$ -adic integers. Perhaps, as is the case with  $\text{Int}(Z)$ , there are ideals which can be defined simply, and then other, more interesting, ideals can be obtained by means of some type of convergence. The subject is then of a topological sort, and the applicability of ultrafilters is not obvious. However, ultrafilters have a magical ability to impose convergence in situations where it is not at all apparent that it exists.

**Definition 2** Let  $D$  be a domain and let  $\Omega$  be a collection of ideals of  $D$ . Let  $U$  be an ultrafilter on  $\Omega$ . For an element  $d \in D$  we define the set  $B(d) = \{I \in \Omega \mid d \in I\}$ . We then define  $I_U = \{d \in D \mid B(d) \in U\}$ .

It is easy to see that  $I_U$  as defined above is an ideal. (Note: it may be the zero ideal.) We say that  $I_U$  is an ultrafilter limit ideal of the set  $\Omega$ . We thank the referee for pointing out that, in fact, the notion of an ultrafilter limit ideal can be expressed using the terminology of ultraproducts. The idea is to consider the ultraproduct of rings of the form  $D/I$  where  $I$  runs over all the ideals in  $\Omega$ . The kernel of the canonical map from  $D$  into this ultraproduct is then the ideal  $I_U$ .

We state a lemma next which gives tremendous power to this method in the results to follow.

**Lemma 3** *Suppose that  $D$ ,  $\Omega$  and  $U$  are as above and suppose that all of the ideals in  $\Omega$  are prime ideals. Then  $I_U$  is also a prime ideal.*

We should note that when this method was first employed in [3] and [17] the context was rings of integer-valued polynomials and rational functions, and it was not presented quite as plainly as above. In particular, the ultrafilter was placed on pairs consisting of an element and a prime ideal, and it seemed to be something whose applicability was narrowly limited to the integer-valued polynomial/rational function setting.

## 3 Applications

### 3.1 Integer-valued rational functions

In [3], the authors were interested in the ideal structure of rings of integer-valued rational functions. Let  $D$  be a domain with quotient field  $K$ . We then define  $\text{Int}^R(D) = \{\phi(X) \in K[X] \mid \phi(D) \subseteq D\}$ . Much of the paper is specialized to the case where  $D$  is a valuation domain (which we denote now by  $V$ ). In this case the ring of integer-valued polynomials  $\text{Int}(V) = \{f(X) \in K[X] \mid f(V) \in V\}$  is simply equal to  $V[X]$  except for the special case where  $V$  is a DVR with a finite residue field (in which case  $\text{Int}(V)$  is a Prüfer domain). Interesting cases arise more frequently when looking at the ring of integer-valued rational functions. For example, [3, Theorems 3.2, 3.5]  $\text{Int}^R(V)$  is a Prüfer domain for a valuation domain  $V$  if either the maximal ideal of  $V$  is principal, or if the residue field is not algebraically closed. It is interesting then to investigate ways in which Prüfer domains of integer-valued rational functions differ from Prüfer domains of integer-valued polynomials.

Let  $V$  be a DVR with finite residue field and maximal ideal  $m$ . Then, as noted above,  $\text{Int}(V)$  is a Prüfer domain. Moreover, if  $M$  is a prime ideal of  $\text{Int}(V)$  which lies over  $m$  then  $M$  has the form  $M_{m,\alpha} = \{f(X) \in \text{Int}(V) \mid f(\alpha) \in m\widehat{V}\}$  where  $\widehat{V}$  is the  $m$ -adic completion of  $V$ . It follows that each prime ideal of  $\text{Int}(V)$  which lies over  $m$  is maximal. One of the goals of [3] was to determine the extent to which this property carries over to rings of integer-valued rational functions.

Let  $V$  be a valuation domain with quotient field  $K$  such that the maximal ideal  $m$  is not principal and the residue field is not algebraically closed. We know that  $\text{Int}^R(V)$  is a Prüfer domain. As we observed above in the classical case of  $\text{Int}(Z)$ , it is easy to define maximal ideals of  $\text{Int}^R(V)$  by choosing an element  $d \in V$  and then defining  $M_{m,d} = \{\phi(X) \in \text{Int}^R(V) \mid \phi(d) \in m\}$ . In the case of  $\text{Int}(Z)$  it is easy to talk about convergence of the maximal ideals containing a prime  $p$  because of the correspondence with the compact metric space of  $p$ -adic numbers. In the current setting things are not as clear, because in the valuation domain  $V$ , the fact that  $m$  is not principal means that there are sequences of elements of  $V$  for which it does not seem to make any sense to talk about convergence. In particular, let  $\{d_1, d_2, d_3, \dots\}$  be a sequence of elements in  $m$  such that the sequence  $\{v(d_1), v(d_2), \dots\}$  converges to zero where  $v$  is a valuation corresponding to  $V$ . The sequence (and every subsequence) of elements does not converge (at least not in the standard  $m$ -adic topology). However, we can assign a maximal ideal of  $\text{Int}^R(V)$  to each element  $d_i$  by defining  $M_i = \{\phi(X) \in \text{Int}^R(V) \mid \phi(d_i) \in m\}$  as above. We can then use an ultrafilter to force the sequence  $\{M_1, M_2, M_3, \dots\}$  to converge.

We list the steps in our analysis of this process:

1. Let  $U$  be an ultrafilter on the collection  $\{M_1, M_2, M_3, \dots\}$  of maximal ideals of  $\text{Int}^R(V)$ . As we noted in the introduction, we cannot specify all of the sets in an ultrafilter, but we can specify some of them. In particular, we choose here

to insure that every set of the form  $\{M_i, M_{i+1}, M_{i+2}, \dots\}$  is in the ultrafilter. This is possible because this collection of subsets satisfies the finite intersection condition.

2. We define the ideal  $M_U$  as in Definition 2. We know that  $M_U$  is a prime ideal of  $\text{Int}^R(V)$  by Lemma 3. It is also easy to see that each  $M_i$  lies over  $m$  and then it follows easily from the definition that  $M_U$  also lies over  $m$ .
3. Let  $\phi(X) \in \text{Int}^R(V)$ . We can write  $\phi(X) = \frac{f(X)}{g(X)}$  with  $f, g \in V[X]$ . Further, recalling that  $V$  is a valuation domain, we can write any polynomial  $h(X) \in V[X]$  as a product  $h(X) = (d)(k(X))$  where  $d \in V$ ,  $k(X) \in V[X]$  and one of the coefficients of  $k$  is a unit in  $V$  – simply factor out the coefficient with smallest value. Finally, we can write any rational function  $\phi(X) \in \text{Int}^R(V)$  in the form  $\phi(X) = \frac{df(X)}{g(X)}$  where  $f, g \in V[X]$  and each has a unit coefficient and  $d \in K$ . Such a representation is not unique, but it is clear that the value of  $d$  is determined by choice of  $\phi$ . Let  $M^*$  be the collection of all  $\phi(X) \in \text{Int}^R(V)$  for which the choices of the constant  $d$  associated with  $\phi$  lie in  $m$ .
4. It is easy to show that  $M^*$  is a prime ideal. It is also easy to show that  $M^* \subseteq M_U$ . To see this choose a rational function  $\phi(X) = \frac{df(X)}{g(X)} \in M^*$  as above. Then consider elements of the form  $\phi(d_i)$ . Since both  $f$  and  $g$  have a coefficient which is a unit, we can make the values of  $f(d_i)$  and  $g(d_i)$  as small as we wish by choosing a sufficiently large value of  $i$ . However, the value of  $d$  is constant and positive and so the value of  $\phi(d_i)$  must be positive for sufficiently large  $i$ . Hence,  $\phi(X) \in M_i$  for all sufficiently large  $i$ , which is sufficient for  $\phi(X)$  to lie in  $M_U$ .
5. We have proven that  $M^* \subseteq M_U$ . It is also obvious from the definition that  $M^*$  lies over  $m$ . We next investigate whether these two prime ideals are the same. Recall that  $M_U$  was defined to be essentially a limit of the maximal ideals  $M_i$ . Thus, we can view  $M_U$  as being the collection of rational functions which map a sufficiently large set of the elements  $d_i$  into  $m$ . Note that the polynomial  $\phi(X) = X$  maps every  $d_i$  into  $m$  and so  $X \in M_U$ . But  $X \notin M^*$ . It follows that prime ideals of  $\text{Int}^R(V)$  which lie over  $m$  are not necessarily maximal.

**Proposition 4** ([3, Theorem 6.6]). *Let  $V$  be a valuation domain with non-principal maximal ideal  $m$  and with residue field not algebraically closed. Then the ring  $\text{Int}^R(V)$  of integer-valued rational functions over  $V$  contains prime ideals which lie over  $m$  but are not maximal.*

### 3.2 Sequence domains

Let  $D$  be a domain with quotient field  $K$ . The ring of integer-valued polynomials over  $D$  is defined by  $\text{Int}(D) = \{f(X) \in K[X] \mid f(D) \subseteq D\}$ . One of the key questions concerning integer-valued polynomials for a long time was to determine necessary and sufficient conditions on  $D$  in order that  $\text{Int}(D)$  should be a Prüfer domain. Chabert proved in [5] that a necessary condition was that  $D$  should be an almost

Dedekind domain (i.e. the localization  $D_M$  is a DVR for each maximal ideal  $M$  of  $D$ ) with all residue fields finite. If  $V$  is a DVR with a finite residue field and maximal ideal  $m$  then  $\text{Int}(V)$  is a Prüfer domain and the maximal ideals which lie over  $m$  are naturally indexed by the elements of the  $m$ -adic completion of  $V$  – generalizing the situation with the  $p$ -adic numbers and  $\text{Int}(Z)$  described above. Suppose however that  $D$  is an almost Dedekind domain (not a valuation domain) with all residue fields finite. We know that  $\text{Int}(D_M)$  is a Prüfer domain for each maximal ideal  $M$ . If we also knew that  $\text{Int}(D)_M = \text{Int}(D_M)$  (where  $\text{Int}(D)_M$  is  $\text{Int}(D)$  localized at the multiplicatively closed set  $D - M$ ) for each maximal ideal  $M$  then we could prove that  $\text{Int}(D)$  was a Prüfer domain. Chabert called this equality  $\text{Int}(D)_M = \text{Int}(D_M)$  (for all maximal ideals  $M$ ) good behavior under localization and questioned whether this was a necessary and sufficient condition for  $\text{Int}(D)$  to be a Prüfer domain.

One of the goals of [17] was to demonstrate that good behavior under localization is not necessary for  $\text{Int}(D)$  to be a Prüfer domain. Again, the core idea was to utilize the notion of convergence of maximal ideals both in  $D$  and in  $\text{Int}(D)$ . Since this notion of convergence was still far from clearly understood it seemed wise to construct the domain  $D$  in such a way that the convergence there was plain. So we considered domains which satisfied the following properties. (The paper actually considered a class of domains somewhat larger than that defined below – which we referred to as sequence domains.)

1.  $D$  is an almost Dedekind domain with quotient field  $K$  and with all residue fields finite.
2.  $D$  contains a countable number of maximal ideals which we designate as  $\{P_1, P_2, P_3, \dots\}$  and  $P^*$ .
3. We designate  $q_i$  to be the order of the residue field of  $P_i$  for each  $i$  and  $q^*$  to be the order of the residue field of  $P^*$ .
4. Each residue field has the same characteristic  $p$ .
5. The set  $\{q_i \mid i \in \mathbb{Z}^+\}$  is bounded.
6. Each  $P_i$  is finitely generated while  $P^*$  is not finitely generated.
7. We designate  $v_i$  to be a valuation on  $K$  corresponding to  $D_{P_i}$  and  $v^*$  to be a valuation on  $K$  corresponding to  $D_{P^*}$ . The value groups of the  $v_i$ 's are not necessarily assumed to be the additive group of the integers.  $v^*$  is assumed to have the integers as value group.
8. The sequence  $\{v_i(d) \mid i \in \mathbb{Z}^+\}$  is eventually constant for each nonzero element  $d \in D$ .
9. For each  $i$  the valuation  $v_i^N$  is the normed valuation (with  $Z$  as value group) corresponding to  $D_{P_i}$ .
10. The set  $\{v_i^N(d) \mid i \in \mathbb{Z}^+\}$  is bounded for each nonzero element  $d \in D$ .
11. For any nonzero element  $d \in K$  we have  $v^*(d) = \lim_{i \rightarrow \infty} (v_i(d)) \in \mathbb{Z}^+ \cup \{0\}$ .
12. There exists an element  $\pi \in D$  such that  $v_i(\pi) = 1$  for each  $i$ .

The convergence of the sequence  $\{P_1, P_2, P_3, \dots\}$  to  $P^*$  is sufficiently clear from the strong restrictions placed on the structure of  $D$  that we were able to analyze many properties of  $\text{Int}(D)$  without leaning on ultrafilters. In particular, we proved that if  $D$  satisfies all of the conditions above then  $\text{Int}(D)$  is a Prüfer domain. We did

use the ultrafilters for the question of behaving well under localization however. The key theorem proven [17, Theorem 5.13] was

**Theorem 5** *Let  $D$  be a sequence domain defined as above. Then  $\text{Int}(D)$  behaves well under localization if and only if both of the following conditions hold.*

1.  $q_i = |D/P_i| = |D/P^*| = q^*$  for all but finitely many  $i \in \mathbb{Z}^+$ .
2.  $v_i = v_i^{(N)}$  for all but finitely many  $i \in \mathbb{Z}^+$ .

We sketch the basic idea of the argument. Suppose that  $V$  is a DVR with maximal ideal  $m$  with residue field of order  $q$ . Suppose also that  $m = dV$  for some element  $d \in V$ . And suppose that  $M$  is a maximal ideal of  $\text{Int}(V)$  which lies over  $m$ . Then  $|\text{Int}(V)/M| = q$  and  $M\text{Int}(V)_M$  is generated by  $d$ . The point of behaving well under localization is that these structural similarities between  $V$  and  $\text{Int}(V)$  should globalize to similarities between the structure of  $D$  and  $\text{Int}(D)$  when  $\text{Int}(D)$  is a Prüfer domain. So we suppose that we do not have the conditions of Theorem 5 and investigate what goes wrong. The basic idea is to choose a sequence of maximal ideals of  $\text{Int}(D)$  lying over the ideals  $P_i$  which converges to a maximal ideal lying over  $P^*$ . If such a sequence is chosen carefully then the structure of the limit prime will mirror the properties of the maximal ideals which converge to it and thereby cannot match the maximal ideals which correspond to those of  $\text{Int}(D_{P^*})$ .

First suppose that  $q_i = |D/P_i| > |D/P^*|$  for an infinite number of values of  $i$ . The finite generation of the  $P_i$ 's is sufficient to show that the maximal ideals of  $\text{Int}(D)$  which lie over them are well behaved. In particular, they can all be defined by means of the  $P_i$ -adic completions of  $D$  as described above. So we choose a collection of positive integers  $n_1 < n_2 < n_3 < \dots$  such that  $q_{n_i} = |D/P_{n_i}| > |D/P^*|$  for each  $n_i$ . The residue field of  $P_{n_i}$  has order  $q_{n_i}$  which means that the multiplicative group is a cyclic group of order  $q_{n_i} - 1$ . Choose an element  $d_i$  in  $D$  which corresponds to a generator of this cyclic group. Then for each  $i$  we define a maximal ideal  $M_i$  of  $\text{Int}(D)$  which lies over  $P_{n_i}$  corresponding to  $d_i$ . We then place a nonprincipal ultrafilter  $U$  on the collection of ideals  $\{M_1, M_2, M_3, \dots\}$  and construct the ultrafilter limit prime  $M_U$ . The structure of  $D$  insures that  $M_U$  lies over  $P^*$ . However,  $M_U$  is defined as a collection of polynomials which sends (ultrafilter) large collections of elements  $d_i$  into the corresponding maximal ideals  $P_{n_i}$ . Then the choice of the elements  $d_i$  guarantees that the residue field of  $M_U$  will have order larger than  $q^*$  – the order of the residue field of  $P^*$ . It follows that  $M_U$  cannot be defined by means of an element in the  $P^*$ -adic completion of  $D$ . It follows that  $D$  is not well-behaved under localization although  $\text{Int}(D)$  is a Prüfer domain.

The problems that arise if condition (2) above are violated are similar. Essentially, the idea is that all of the maximal ideals of  $\text{Int}(D_{P^*})$  which lie over  $P^*$  are locally generated by the element  $\pi \in D$ . But, if the condition  $v_i = v_i^{(N)}$  for all but finitely many  $i \in \mathbb{Z}^+$  fails then we can choose a sequence of maximal ideals  $M_1, M_2, \dots$ , with  $M_i$  lying over  $P_i$ , such that  $\pi$  is not a local generator for any of them. Hence, an ultrafilter limit prime  $M_U$  will lie over  $P^*$  but will not have  $\pi$  as a local generator. And again, we will have proven that  $D$  is not well behaved under localization with  $\text{Int}(D)$  a Prüfer domain.

### 3.3 Characterizing when $\text{Int}(D)$ is a Prüfer domain

In [18], a leap forward was taken in understanding the nature of the convergence of prime ideals that had been employed as described above. The question was to characterize the domains  $D$  such that  $\text{Int}(D)$  is a Prüfer domain. Chabert proved that a necessary condition is that  $D$  be an almost Dedekind domain with all residue fields finite. On the other hand, Gilmer [14] and Chabert [6] each constructed examples of almost Dedekind domains with finite residue fields such that  $\text{Int}(D)$  was not Prüfer. In each case, the proof that  $\text{Int}(D)$  is not a Prüfer domain is accomplished by demonstrating that  $\text{Int}(D) \subseteq D_P[X]$  for some maximal ideal  $P$  of  $D$ . The almost Dedekind domains involved in these proofs were obtained by means of countably infinite integral extensions of Dedekind domains (in particular, rings of algebraic integers). Properties of the field extensions made clear which prime ideal  $P$  was the offending prime which would satisfy  $\text{Int}(D) \subseteq D_P[X]$ . Chabert hypothesized a two-part boundedness condition as being necessary and sufficient for  $\text{Int}(D)$  to be a Prüfer domain. We give the condition here.

- Let  $D$  be an almost Dedekind domain with quotient field  $K$  and with all residue fields finite. Suppose that  $K$  has characteristic 0. (The characteristic  $p$  case is similar – we will not go into details here.)
- For each maximal ideal  $P$  of  $D$  let  $v_p^{(N)}$  represent the normalized valuation (i.e. the value group is the additive group of the integers.) on  $K$  associated with  $P$ .
- Note that the finiteness of the residue fields implies that each maximal ideal contains a prime number  $p$ .
- For each prime number  $p$  which is not a unit in  $D$  we define two sets
  - $F_p = \{D/P \mid p \in P\}$
  - $E_p = \{v_p^{(N)}(p) \mid p \in P\}$

**Definition 6** Let  $D$  be an almost Dedekind domain as described above. We say that  $D$  is doubly-bounded provided  $E_p$  and  $F_p$  are bounded sets for each prime  $p$  which is a nonunit in  $D$ .

This double-boundedness condition is essentially the same as the one suggested by Chabert as a necessary and sufficient condition for  $\text{Int}(D)$  to be a Prüfer domain. The sufficiency is essentially proven in [2, Proposition VI.4.4]. The statement there is more modest, but the proof is easily extended to the general case. The necessity was more problematic. As noted above, the method used previously to show that  $\text{Int}(D)$  was not a Prüfer domain was to find a maximal ideal  $P$  of  $D$  such that  $\text{Int}(D) \subseteq D_P[X]$ . The difficulty in the general setting is to identify which maximal ideal  $P$  is the offending prime which will make the proof work. The use of ultrafilters along with the double-boundedness condition make it easy to identify the bad prime. We outline the proof.

1. Suppose that  $D$  is an almost Dedekind domain as described above except we assume that  $D$  is not doubly-bounded.

2. First, suppose that  $F_p$  is not bounded for some prime  $p$ . Then we can find a collection of maximal ideals  $\{P_1, P_2, P_3, \dots\}$  of  $D$  such that each residue field has characteristic  $p$  and such that  $|D/P_{i+1}| > |D/P_i|$  for each  $i$ . Then we can place a nonprincipal ultrafilter on the collection  $\{P_1, P_2, P_3, \dots\}$  and define the ultrafilter limit prime  $P_U$ . It is then possible to prove that  $\text{Int}(D) \subseteq D_{P_U}[X]$ .
3. Next, we suppose that  $E_p$  is not bounded for some prime  $p$ . We then find a collection of maximal ideals  $\{P_1, P_2, P_3, \dots\}$  of  $D$  such that each residue field has characteristic  $p$  and such that  $v_{P_{i+1}}^{(N)}(p) > v_{P_i}^{(N)}(p)$  for each  $i$ . Then as above we place a nonprincipal ultrafilter on the collection  $\{P_1, P_2, P_3, \dots\}$  and define the ultrafilter limit prime  $P_U$ . Again we can prove that  $\text{Int}(D) \subseteq D_{P_U}[X]$ .

The proof that  $\text{Int}(D) \subseteq D_{P_U}[X]$  in both of the above cases is accomplished by means of some technical lemmas regarding coefficients of integer-valued polynomials. The key use of the ultrafilters was in identifying the appropriate maximal ideal. The use of the ultrafilters in this paper represented a departure from the way they had been used in the previous papers described. Previously, ultrafilters considered pairs consisting of prime ideals and elements of the coefficient ring  $D$  which made the application of the method peculiar to the province of integer-valued polynomials. In this paper, however, the ultrafilters are applied to a sequence of maximal ideals in an almost Dedekind domain in a manner that could be generalized to prime ideals in any ring.

### 3.4 Characterizing when $\text{Int}(D)$ is a PvMD

In [4], the main goal was to characterize the domains  $D$  such that  $\text{Int}(D)$  is a PvMD. Recall that a domain  $D$  is a PvMD provided  $D_P$  is a valuation domain for each maximal  $t$ -ideal  $P$  of  $D$ . Recall also that any domain is equal to the intersection of the localizations at the maximal  $t$ -ideals. A necessary condition for  $\text{Int}(D)$  to be a PvMD is that  $D$  itself be a PvMD. To accomplish the characterization the prime ideals of  $D$  are partitioned into two groups, the *int* primes and the *polynomial* primes. A prime  $P$  of  $D$  is an *int* prime provided  $\text{Int}(D) \not\subseteq D_P[X]$  and is a *polynomial* prime otherwise. Given these relevant collections of ideals there were two fundamental results concerning ultrafilters which have potentially broad use beyond the context of [4].

**Theorem 7 ([4, Proposition 2.5]).** *Let  $D$  be a domain and let  $\Lambda = \{I_\alpha\}$  be an infinite collection of  $t$ -ideals of  $D$ . Let  $U$  be an ultrafilter on the set  $\Lambda$  and define the ultrafilter limit ideal  $I_U$ . If  $I_U$  is not the zero ideal then it is a  $t$ -ideal.*

(Note: In [10], the above theorem was generalized to the case where the  $t$ -operation is replaced by any star operation of finite type. The proof is the same as in [4].)



**Theorem 8** ([4, Proposition 2.8]). *Let  $D$  be the intersection  $D = \bigcap_{\lambda \in \Lambda} V_\lambda$  of a family of valuation domains. For each  $\lambda \in \Lambda$ , denote by  $P_\lambda$ , the center in  $D$  of the maximal ideal of  $V_\lambda$ .*

1. *If  $I$  is a  $t$ -ideal of  $D$ , then  $I$  is contained in the limit prime  $P_U$  of the family  $\{P_\lambda\}$  with respect to some ultrafilter  $U$ .*
2. *If moreover,  $I$  is maximal, or if  $I$  is  $t$ -maximal and every  $V_\lambda$  is essential (that is,  $V_\lambda = D_{P_\lambda}$ ), then  $I = P_U$ .*

The characterization given in [4] using int and polynomial primes is technical. It is easy to explain, however, the major application of ultrafilters given.

1. Let  $D$  be a domain.
2. Let  $\{P_i \mid i \in \Omega\}$  be a collection of prime ideals of  $D$ .
3. For each element  $d \in D$  define  $B(d) = \{i \in \Omega \mid d \in P_i\}$ .
4.  $P$  is a nonzero prime ideal of  $D$ .

**Proposition 9** ([4, Lemma 2.2]). *Assume the notation in (1)–(4) above. Then the following are equivalent.*

1.  *$P$  is contained in the limit prime  $P_U$  of the family  $\{P_i \mid i \in \Omega\}$  with respect to some ultrafilter  $U$ .*
2. *The finite intersections of the sets of the form  $B(d)$  for  $d \in P$  are not empty.*
3. *For each finitely generated ideal  $J$  contained in  $P$  there is some ideal  $P_i$  containing  $J$ .*

The proof of this result is straightforward. The key element is to note that the sets  $B(d)$  for  $d \in P$  satisfy the finite intersection property and hence can be extended to an ultrafilter on  $\Omega$ .

This result is then key in proving that if  $\text{Int}(D)$  is a PvMD then each polynomial prime contains a finitely generated ideal which is contained in no int prime. If it were otherwise then we could use Proposition 9 above to find an ultrafilter limit of int primes which contains a polynomial prime – which is impossible. This property of polynomial primes is a key element in the characterization given in the paper. For our purposes however the key points to glean from this paper are the ways ultrafilters are used to obtain results. In particular, we used the fact that nonzero ultrafilter limit ideals of  $t$ -ideals are still  $t$ -ideals (note that the ideals here need not be prime). We learned that when we have a representation of a domain as an intersection of valuation domains then we can obtain certain prime ideals as ultrafilter limits of the centers of the defining family. It is not stated above, but it is also proven that when  $\text{Int}(D)$  is a PvMD it follows that an ultrafilter limit of int primes is again an int prime. What the results of this paper demonstrated was that given a collection of ideals of a domain there are many important properties of ultrafilter limit ideals which can be deduced from knowledge of the ideals in the collection.

Another step forward that occurred in [4] was the realization that the notion of ultrafilter limit could be applied to a collection of rings that were all subrings of one large fixed ring. In particular, this is interesting with regard to valuation overrings of domains.

**Definition 10** Let  $K$  be a field and let  $\{D_i \mid i \in \Omega\}$  be a collection of quasi-local domains with quotient field  $K$ . Let  $U$  be an ultrafilter on  $\Omega$ . For an element  $d \in K$  we define the set  $B(d) = \{i \in \Omega \mid d \in D_i\}$ . We then define  $D_U = \{d \in K \mid B(d) \in U\}$ .

It is easy to see that  $D_U$  as defined above is again a quasi-local domain with quotient field  $K$ . (Note: it may actually be  $K$ .) We say that  $D_U$  is an ultrafilter limit ring of the set  $\{D_i\}$

**Lemma 11** *Suppose that  $K$ ,  $\Omega$  and  $U$  are as above and suppose that all of the domains in  $\Omega$  are valuation rings. Then  $D_U$  is also a valuation ring.*

This notion of convergence of valuation domains rather than ideals was put to frequent use in [19] where the goal was to classify integrally closed rings of polynomials which lie between  $Z_p[X]$  and  $Q[X]$  where  $p$  is a prime number. The notion was not key to proving a powerful theorem as in earlier papers. Rather it provides a convenient way of describing/identifying certain valuation domains which play important roles in the results. For example, a partial order was placed on all valuation overrings of  $Z[X]$  which contain a given prime  $p$ . It is easy to see then that ultrafilters could be of use in proving the existence of minimal valuation domains within certain subcollections – the ultrafilter limit process respects the ordering and the limit of valuation domains is again a valuation domain.

## 4 Topology and questions

The original motivation for using ultrafilters on collections of prime ideals came from the behavior of maximal ideals in  $\text{Int}(Z)$  which they inherited from the compact metric spaces of  $p$ -adic integers. It is certainly unreasonable to expect that every space of prime ideals of a ring (or valuation overrings) is similarly governed by a metric space topology. It is true, however, that the space of prime ideals of a ring or valuation overrings of a domain can be viewed naturally as a compact Hausdorff space. We next give a quick review of some results on topological structures which have been naturally defined on spaces of prime ideals or valuation domains.

**Definition 12** ([9, Chap. I]).

1. Let  $R$  be a ring and let  $\text{Spec}(R)$  denote the collection of prime ideals of  $R$ . For any ideal  $I$  of  $R$  we let  $V(I)$  be the set of all prime ideals  $P$  of  $R$  such that  $I \subseteq P$ . Then the sets of the form  $V(I)$  comprise the closed sets of a topology which we call the Zariski topology on  $\text{Spec}(R)$ .

2. Let  $D$  be an integral domain with quotient field  $K$  and let  $X(D)$  denote the set of all valuation overrings of  $D$ . For any finite collection  $\{d_1, d_2, \dots, d_n\}$  of elements of  $K$  we let  $E(d_1, \dots, d_n)$  denote the collection of valuation overrings  $V$  of  $D$  such that  $\{d_1, d_2, \dots, d_n\} \subseteq V$ . Then the sets  $E(d_1, \dots, d_n)$  comprise a basis for the open sets of a topology on  $X(D)$  which we call the Zariski

In both of the cases above, the Zariski topology is well-known to be quasi-compact but is generally far from being Hausdorff. There is a refinement of the Zariski topology however, which always yields a compact Hausdorff space.

**Definition 13** [15] Let  $R$  be a ring and let  $\text{Spec}(R)$  be the collection of prime ideals as above. Start with the Zariski topology as above. Then for every element  $d \in R$  consider the set  $V((d))$ . The coarsest topology which refines the Zariski topology and includes every set  $V((d))$  as above as both an open set and a closed set is known as the patch topology on  $\text{Spec}(R)$ . In a similar fashion we can start with the Zariski topology on the space  $X(R)$  of valuation overrings of a domain  $D$  and determine that the sets of the form  $E(d)$  where  $d \in K$  should be closed as well as open. The topological space generated from this process is also known as the patch topology.

There is another well known description of the patch topology [13, pp. 337–339] or [1, Chap. 3, Exercises 27, 28, and 30]. Let  $R$  be a ring and let  $f : R \rightarrow T$  be a homomorphism from  $R$  to another ring  $T$ . Then the set  $\{f^{-1}(P) \mid P \in \text{Spec}(T)\}$  is a subset of  $\text{Spec}(R)$ . If we consider all such sets for all possible homomorphisms into all possible rings  $T$  the resulting collection of subsets of  $\text{Spec}(R)$  comprises the closed sets of a topology known as the constructible topology.

**Proposition 14** *For a ring  $R$ , the patch topology and the constructible topology on the set  $\text{Spec}(R)$  of prime ideals of  $R$  are identical.*

It should be noted that the definition of the constructible topology actually predated the definition of the patch topology. An early discussion of the constructible topology can be found in [12].

In [8], it is demonstrated that for an integral domain  $D$  the Zariski topology on the space  $X(D)$  of valuation overrings of  $D$  can be naturally identified with the ordinary Zariski topology on the prime spectrum of the Kronecker function ring  $Kr(D)$ . Hence, we can naturally define the constructible topology (which will again be identical to the patch topology) on  $X(D)$  as well.

Given the above topologies on  $\text{Spec}(R)$  or  $X(D)$  it is natural to ask whether the convergence properties we have observed involving ultrafilters are related to these topologies. In [11] we use ultrafilters to define a topology on  $\text{Spec}(R)$  and prove that the resulting topology is the same as the patch/constructible topology.

**Definition 15** Let  $R$  be a ring and let  $C$  be an infinite subset of  $\text{Spec}(R)$ . Let  $U$  be an ultrafilter on  $C$  and let  $P_U$  be the ultrafilter limit prime as defined in the previous section. If  $U$  is a principal ultrafilter based on a prime  $P \in \text{Spec}(R)$  then  $P = P_U$ .

Let  $\overline{C}$  be  $C$  together with its ultrafilter limit primes (considering all possible ultrafilters). We then say that  $C$  is ultrafilter closed if  $C = \overline{C}$ . The collection of all ultrafilter closed subsets of  $\text{Spec}(R)$  (together with the empty set) are the closed sets of a topology which we call the ultrafilter topology.

The main goal of [11] is to connect the ultrafilter topology with the patch/constructible topology on a ring.

**Theorem 16** *Let  $R$  be a commutative ring. The ultrafilter topology and the patch/constructible topologies are identical.*

The primary tool used in proving the above result is an absolutely flat ring  $T(R)$  which can be associated with a commutative ring  $R$  such that the constructible/patch topologies on  $R$  and  $T(R)$  are homeomorphic and such that on  $T(R)$  the Zariski and constructible/patch topologies are identical. Detail concerning this ring can be found in [20].

As above, recall that we can identify the space of valuation overrings of an integral domain  $D$  with the prime ideals of the Kronecker function ring of  $D$ . We then take a collection of valuation overrings of  $D$  and define ultrafilter limit primes of the collection and then define closed sets of valuation domains as those that contain all of their ultrafilter limit points. As with the topologies on  $\text{Spec}(R)$  this ultrafilter topology is the same as the classical patch topology.

So we have proven that the convergence properties we observed are actually consequences of a well-known Hausdorff topology. It seems, however, that the ultrafilter viewpoint has some advantages over the classical definitions. In particular, it is possible to have a great deal of control over convergent sequences with the ultrafilter context. In fact, the ultrafilters give the topology a localized character since closure is defined one ultrafilter at a time. We now make note of some topics left to be investigated that might be of future interest.

1. Much of the utility of the ultrafilter techniques developed so far have to do with properties of ultrafilter limit primes which are inherited from the collection of primes on which the ultrafilter is placed. It would be nice to have a better description of what properties are transferred from a collection of primes to a limit prime. For example, say that a prime ideal  $P$  of a domain  $D$  is essential provided  $D_P$  is a valuation domain. Let  $D$  be a domain and let  $\{P_i \mid i \in \Omega\}$  be a collection of essential primes of  $D$ . Place a nonprincipal ultrafilter on  $\Omega$ . The ultrafilter can be viewed as being on either the collection of primes or on the collection of valuation overrings arising by means of localization. If we look at this from the valuation overring point of view, the ultrafilter limit will be a limit of a collection of valuation domains – which is again a valuation domain. It is not at all clear, however, that the corresponding ultrafilter limit prime within  $D$  will again be an essential prime. It would be interesting to know if or when such a prime is essential.
2. In our original motivating example, portions of the maximal spectrum of a Prüfer domain were identified with a well studied compact metric space. Let  $D$  be a quasi-local domain with maximal ideal  $M$ . Consider the space of all

valuation overrings of  $D$  whose maximal ideals lie over  $M$ . It is too much to ask that such a space always be metrizable. But it seems clear that it will sometimes be metrizable. For example, if  $D$  is a countable ring then this space should be metrizable. Perhaps, this is true if  $D$  is a Noetherian local ring – countable or not. It would be nice to know when this space is metrizable and if it is whether there is a natural description of the metric.

3. In defining the ultrafilter topology we allowed the use of any ultrafilter defined on a given collection of prime ideals. If only the principal ultrafilters are used then we could use the same construction and end up with something very close to the discrete topology. Perhaps other interesting topologies which lie between the patch/constructible and the discrete topologies could be constructed by using collections of ultrafilters which contain all the principal ultrafilters but not all ultrafilters.
4. As well as giving an alternate description of the patch topology, the ultrafilters generalize it. The classical patch topology is generally defined on sets of prime ideals or valuation domains. If  $R$  is a ring and  $M$  is an  $R$ -module then the ultrafilter topology can be defined naturally on the collection of all  $R$ -submodules of  $M$ . So, for example, we can define the topology on the collection of all ideals of a ring  $R$ , or if  $D$  is a domain with quotient field  $K$  then we can define the topology on the set of all  $D$ -submodules of  $K$ . In fact, we can define the ultrafilter topology on the set of subgroups of an infinite group. (Note that the ultrafilter limit of a collection of normal subgroups is again a normal subgroup.) There are surely applications beyond what is outlined in this paper in these more general settings.

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# Intersections of valuation overrings of two-dimensional Noetherian domains

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**Abstract** We survey and extend recent work on integrally closed overrings of two-dimensional Noetherian domains, where such overrings are viewed as intersections of valuation overrings. Of particular interest are the cases where the domain can be represented uniquely by an irredundant intersection of valuation rings, and when the valuation rings can be chosen from a Noetherian subspace of the Zariski-Riemann space of valuation rings.

## 1 Introduction

This article is motivated by the problem of trying to understand the integrally closed domains that can occur between an arbitrary Noetherian domain  $D$  of Krull dimension 2 and its quotient field. In general such a ring need not be Noetherian, and from the idiosyncratic standpoint of this article, the Noetherian rings are less interesting than the non-Noetherian ones. This is because we approach integrally closed rings from the “outside,” and seek to describe them as an intersection of valuation overrings, and in particular how they are cut out of the quotient field by these intersections. It is known, thanks to a 1969 theorem of Heinzer which we state below, how the integrally closed Noetherian overrings of  $D$  are cut out in this way: they are precisely the overrings having a finite character representation consisting of DVRs. Recall that a collection of valuation overrings of a domain  $H$  has *finite character* if each nonzero element of  $H$  is a unit in all but at most finitely many valuation rings in the collection. A *DVR* is a discrete rank one valuation ring, or, equivalently, a local PID. An *overring* of  $H$  is a ring between  $H$  and its quotient field. Heinzer’s theorem then states:

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**Theorem 1.1 (Heinzer [7]).** *Let  $H$  be an integrally closed overring of a two-dimensional Noetherian domain. Then  $H$  is a Noetherian domain if and only if  $H$  is a finite character intersection of DVR overrings of  $H$ .<sup>1</sup>*

From well-known properties of Krull domains, we deduce then that an integrally closed overring  $H$  of a two-dimensional Noetherian domain is Noetherian if and only if  $H$  can be written *uniquely* as an *irredundant* finite character intersection of DVRs. Thus, each integrally closed Noetherian overring is determined in a precise way by a unique set of DVR overrings. This gives additional evidence that from our peculiar point of view, the Noetherian case is particularly transparent.

In this article, we consider other classes of rings that can be represented in a similar way. We are particularly interested in existence and uniqueness of nice representations of overrings of two-dimensional Noetherian domains. Of course, the richness and complexity of the class of two-dimensional integrally closed Noetherian domains, despite the transparent way in which they are assembled from DVRs, suggest that intersections of even “more” valuation overrings which need not be DVRs, should produce even more complicated rings. So our goal is not so much to shed light on the ideal theory or the internal structure of the integrally closed overrings of  $D$ , as it is to understand better the different sorts of representations of these rings one can encounter in the vast expanse between a two-dimensional Noetherian domain and its quotient field.

In keeping with the theme of Heinzer’s theorem, we intersect valuation rings coming from topologically “small” (in actuality, Noetherian) subspaces of the Zariski–Riemann space of all valuation overrings of  $D$ , and we show that a fairly complete and satisfactory account can be given of these intersections. In so doing, since finite character collections of valuation rings are Noetherian spaces, we also generalize Heinzer’s theorem, and classify the integrally closed overrings of  $D$  that are finite character intersections of arbitrary valuation overrings of  $D$ , not just DVRs.

Our approach to these topics is from the following more general point of view. With  $D$  a two-dimensional Noetherian domain, let  $R$  be an integrally closed overring of  $D$ . If  $H$  is an integrally closed overring of  $D$  such that  $D \subseteq H \subseteq R$ , then there exists a collection  $\Sigma$  of valuation overrings of  $D$ , none of which contain  $R$ , such that  $H = (\bigcap_{V \in \Sigma} V) \cap R$ . When not many such valuation rings are needed, i.e., when  $\Sigma$  can be chosen a Noetherian subspace of the space of all valuation overrings of  $D$ , then we can say quite a lot about  $H$  in terms of  $R$ , but of course, what we can say is limited by our knowledge of  $R$ . However, choosing  $R$  to be the quotient field of  $D$ , we then obtain the setting described in the preceding paragraph, and our results are more definitive.

The most success with this sort of approach has been achieved by K. A. Loper and F. Tartarone in their very interesting recent study [13] of the integrally closed rings between  $\mathbb{Z}[X]$  and  $\mathbb{Q}[X]$ . They use MacLane’s notion of key polynomials to

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<sup>1</sup> Heinzer gave a direct proof of this result in [7]. Later, in 1976, Nishimura showed that Heinzer’s theorem was a quick corollary of a more general result: *If  $H$  is a Krull domain such that  $H/P$  is a Noetherian domain for each height 1 prime ideal  $P$  of  $H$ , then  $H$  is a Noetherian domain* [16]. So to obtain Heinzer’s theorem, apply Nishimura’s criterion, along with Proposition 1.2 below.



place a tree structure on the valuation overrings of  $\mathbb{Z}[X]$ , and from this structure deduce information about integrally closed rings between  $\mathbb{Z}[X]$  and  $\mathbb{Q}[X]$ , such as when such rings are Prüfer, Noetherian or Mori. The framework they introduce to consider such questions is beyond the scope of the present paper, but we do recall one of their main results below, in Theorem 2.11.

By way of introduction, we review next some general properties of integrally closed overrings of two-dimensional Noetherian domains, and discuss the valuation theory of these domains. The following proposition, which can be found in [19] and is an easy consequence of well-known results, perhaps misleads one to believe that the class of integrally closed overrings of  $D$  might have more tractability than it does, in that the factor rings and localizations at nonzero nonmaximal prime ideals are Noetherian domains. Yet even given these strong constraints, there exist numerous examples of complicated non-Noetherian integrally closed overrings of two-dimensional Noetherian domains, as is suggested by the rings we consider later in Section 4.

**Proposition 1.2 ([19, Proposition 2.3]).** *Let  $H$  be an integrally closed overring of the two-dimensional Noetherian domain  $D$ , and suppose that  $P$  is a nonzero prime ideal of  $H$ . Then  $H$  has Krull dimension  $\leq 2$ ,  $H/P$  is a Noetherian domain, and if  $P$  is not a maximal ideal of  $H$ , then  $H_P$  is a DVR.*

Applying the proposition to the special case of valuation overrings of  $D$ , we see that every such valuation overring  $V$  has Krull dimension at most 2, and if  $V$  has Krull dimension 2, then for  $\mathfrak{P}$  the height 1 prime ideal of  $V$ , we have that both  $V/\mathfrak{P}$  and  $V_{\mathfrak{P}}$  are DVRs; equivalently, the value group of  $V$  is isomorphic to  $\mathbb{Z} \oplus \mathbb{Z}$ , ordered lexicographically. This classifies the valuation overrings of  $D$  of Krull dimension 2. In fact the basic valuation theory of two-dimensional Noetherian domains can be described rather briefly. For ease of reference we restate first our classification of the two-dimensional case.

**(1.3) Valuation rings of Krull dimension 2.** A valuation overring  $V$  of a two-dimensional Noetherian domain has Krull dimension 2 if and only if there is a prime ideal  $\mathfrak{P}$  of  $V$  such that both  $V/\mathfrak{P}$  and  $V_{\mathfrak{P}}$  are DVRs.

**(1.4) Rational and irrational valuation rings.** A valuation domain is *rational* if its value group is isomorphic as a totally ordered abelian group to a nonzero subgroup of the rational numbers. A valuation domain is *irrational* if it is not rational and its value group is isomorphic as a totally ordered abelian group to a nonzero subgroup of the real numbers. A valuation ring has Krull dimension 1 if and only if its value group is isomorphic as a totally ordered group to a subgroup of the real numbers [4]; hence a valuation ring has Krull dimension 1 if and only if it is rational or irrational.

**(1.5) Prime divisors.** Among the DVR overrings of the two-dimensional Noetherian domain  $D$ , the *essential prime divisors* of  $D$  are those of the form  $\overline{D}_P$ , where  $\overline{D}$  is the integral closure of  $D$  in its quotient field and  $P$  is a height one prime ideal of  $\overline{D}$ . The *hidden prime divisors* are those valuation overrings  $V$  that are DVRs having maximal ideals contracting to a maximal ideal of  $D$  and such that the residue field

of  $V$  has transcendence degree 1 over the residue field of its center in  $D$ . A hidden prime divisor has the property that its residue field is a finitely generated extension of the residue field of its center  $\mathfrak{M}_V \cap D$  in  $D$  [1, Theorem 1(4)]. Moreover, a hidden prime divisor of  $D$  cannot be an essential prime divisor of  $D$ . The classes of essential and hidden prime divisors do not generally account for all the DVR overrings of  $D$ .<sup>2</sup> There remains the class of DVRs having maximal ideals contracting to a height 2 maximal ideal of  $D$  and such that the residue field of  $V$  is algebraic over the residue field of its center in  $D$ . See [23, p. 102] for explicit examples of such DVRs.

*Notation.* Most of our notation is standard, with a few possible exceptions. When  $H$  is a domain,  $R$  is an overring of  $H$  and  $P$  is a prime ideal of  $H$ , we write  $R_P$  for the ring  $RH_P = (H \setminus P)^{-1}R$ . We also always denote the maximal ideal of a valuation ring  $V$  by  $\mathfrak{M}_V$ , and when  $V$  has a height 1 prime ideal, we denote this by  $\mathfrak{P}_V$ . Of course it can happen that  $\mathfrak{P}_V = \mathfrak{M}_V$ .

## 2 Strongly irredundant representatives

An integral domain is integrally closed if and only if it is an intersection of valuation overrings, but in general no one valuation overring is special enough to be necessary in this representation. However, the phenomenon of being necessary in the representation does occur, as we will see often throughout the rest of this article. To formalize this idea, we introduce first some terminology. Let  $R$  be an overring of a domain  $H$ . A collection  $\Sigma$  of valuation overrings of  $H$  is an  $R$ -representation of  $H$  if  $H = (\bigcap_{V \in \Sigma} V) \cap R$ . When  $R$  is the quotient field of  $H$ , then  $H = \bigcap_{V \in \Sigma} V$ , and we say simply that  $\Sigma$  is a representation of  $H$ . As discussed in the introduction, the added generality here in allowing  $R$  to range over various integrally closed overrings is a useful framework for considering the case where  $R$  is an integrally closed overring of the two-dimensional Noetherian domain  $D$ , and  $H$  is an integrally closed domain with  $D \subseteq H \subseteq R$ , since then  $H$  has an  $R$ -representation consisting of valuation rings not containing  $R$ . For this reason, most of what follows is phrased in terms of  $R$ -representations, but by choosing  $R$  to be the quotient field of  $D$ , one effortlessly obtains stronger consequences.

The existence of an  $R$ -representation of the domain  $H$  is a triviality, since one may always choose the set  $\text{Zar}(H)$  of all valuation overrings of  $H$ , or even the set  $\{V \in \text{Zar}(H) : R \not\subseteq V\}$  for the representation. Thus, a more interesting issue is the existence of “nice”  $R$ -representations. In particular, in this section we are interested in when such a representation is unique; uniqueness, as usual, presupposes irredundance: An  $R$ -representation  $\Sigma$  of  $H$  is *irredundant* if no proper subset of  $\Sigma$  is an  $R$ -representation of  $H$ . The main question we address then is: *When can an integrally*

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<sup>2</sup> However, if  $D$  is a two-dimensional complete local Noetherian domain, then every DVR overring of  $D$  is either a hidden prime divisor or an essential prime divisor [11, Corollary 2.4]

closed overring of  $D$  have two different irredundant  $R$ -representations? The short answer is: “frequently,” but we will see that this is because the question is not quite posed correctly.

The following easy example makes this clearer. Let  $P$  be a height 1 prime ideal of an integrally closed local Noetherian domain  $H$  of Krull dimension 2. Let  $X$  be the collection of height 1 prime ideals of  $H$  distinct from  $P$ , and let  $R = \bigcap_{Q \in X} H_Q$ . If  $W$  is any valuation overring of  $H$  of Krull dimension 2 such that  $H \subseteq W \subseteq H_P$ , then  $H = W \cap R = H_P \cap R$  and  $W$  and  $H_P$  are irredundant in these representations, yet  $W \neq H_P$ .

Thus, uniqueness fails in the example because one of the valuation rings,  $W$ , can be replaced in the representation by one of its valuation overrings, namely  $H_P$ . So we refine our notion of irredundance to exclude the phenomenon in the example, and we say that an  $R$ -representation  $\Sigma$  of  $H$  is *strongly irredundant* if no member  $V$  of  $\Sigma$  can be replaced with a proper overring  $V_1$  of  $V$ . More precisely,  $\Sigma$  is a strongly irredundant  $R$ -representation of  $H$  if for every  $V \in \Sigma$  and proper overring  $V_1$  of  $V$ ,  $\{V_1\} \cup (\Sigma \setminus \{V\})$  is not an  $R$ -representation of  $H$ .

The more interesting question then is: *When can an integrally closed overring of  $D$  have two different strongly irredundant  $R$ -representations?* We will see below that the answer for a large class of strongly irredundant  $R$ -representations is “never,” and hence that for this class, strongly irredundant representations are always unique. We postpone till the next section the very relevant question of when such representations occur. But even putting aside this issue of existence, the finite case is still interesting. For a consequence of the uniqueness theorem, Theorem 2.8, is that if  $R$  is an integrally closed overring of  $D$ , and  $V_1, \dots, V_n, W_1, \dots, W_m$  are valuation overrings of  $D$  not containing  $R$  such that:

$$V_1 \cap \dots \cap V_n \cap R = W_1 \cap \dots \cap W_m \cap R,$$

then by throwing out any  $V_i$  or  $W_j$  that is not needed, and by replacing wherever possible  $V_i$  or  $W_j$  with an overring (finiteness here allows all this), we may assume that these intersections are strongly irredundant. Hence, from the uniqueness theorem, we have cancellation: namely,  $\{V_1, \dots, V_n\} = \{W_1, \dots, W_m\}$ .

Uniqueness depends heavily on our hypotheses here. For example, it fails in dimension higher than 2: Let  $K$  be a field, and let  $X, Y, Z$  be indeterminates for  $K$ . Define  $D = K[X, Y]$ ,  $U = D[Z]_{(Z)}$  and  $H = K + ZU$ . Then  $H$  is an integrally closed overring of a three-dimensional Noetherian domain that has uncountably many distinct strongly irredundant representations [18, Example 6.2]. Uniqueness can also fail if  $R$  is not integrally closed. Briefly, following Example 6.1 in [18], where more details are included, let  $D = \mathbb{Q}[X, Y]$ , and let  $P = (Y^2 - X^3 - X + 1)\mathbb{Q}[X, Y]$ . Then  $P$  is a prime ideal. Let  $A = (D + PD_P)/PD_P$ . Then the quotient field of  $A$  is  $D_P/PD_P$ . Write  $A = \mathbb{Q}[x, y]$ , where  $x, y \in A$  and  $y^2 = x^3 + x - 1$ . Consider the subring  $\mathbb{Q}[x]$  of  $D_P/PD_P$ , and let  $U = \mathbb{Q}[x]_{(x-1)}$ . Then there exist two distinct valuation rings  $U_1$  and  $U_2$  with quotient field  $D_P/PD_P$  such that  $U = U_1 \cap \mathbb{Q}(x) = U_2 \cap \mathbb{Q}(x)$ . Moreover, there exists a subring  $R$  of  $D_P$  such that  $PD_P \subseteq R \subseteq D_P$  and  $R/PD_P = \mathbb{Q}(x)$ , and there exist valuation overrings  $V_1$  and  $V_2$  of  $D$  such that  $V_1/PD_P = U_1$  and

$V_2/PD_P = U_2$ . Now  $(V_1 \cap R)/PD_P = U_1 \cap \mathbb{Q}(x) = U = U_2 \cap \mathbb{Q}(x) = (V_2 \cap R)/PD_P$ . Hence,  $V_1 \cap R = V_2 \cap R$ , but  $V_1 \neq V_2$  since  $U_1 \neq U_2$ . In light of the uniqueness theorem, the problem here— the reason that uniqueness fails— is that  $R$  is not integrally closed.

The above two examples involve valuation rings of Krull dimension  $>1$ , and it is in dealing with these valuation rings where it is important that we work over two-dimensional Noetherian domains. But if one restricts to valuation rings of Krull dimension 1, then this hypotheses is not needed, as long as the collections are assumed to have finite character. This is part of a theorem due to Heinzer and Ohm, which we state shortly. But first we introduce some terminology: Let  $H$  be a domain, and let  $R$  be an overring of  $H$ . If  $V$  is a valuation overring of  $H$ , then we say that  $V$  is a (strongly) *irredundant  $R$ -representative* of  $H$  if there exists an  $R$ -representation  $\Sigma$  of  $H$  such that  $V \in \Sigma$  and  $V$  is (strongly) irredundant in this representation. Thus  $V$  is a strongly irredundant  $R$ -representative of  $H$  if and only if there exists an integrally closed overring  $R_1$  of  $H$  such that  $H = V \cap R_1 \cap R$  and  $V$  is strongly irredundant in this intersection. In the case where  $R$  is the quotient field of  $H$ , we simply say that  $V$  is a *strongly irredundant representative* of  $H$ . Thus,  $V$  is a strongly irredundant representative of  $H$  if and only if there exists an integrally closed overring  $R_1$  of  $H$  such that  $H = V \cap R_1$  and  $V$  is strongly irredundant in this intersection. In particular, a strongly irredundant  $R$ -representative is a strongly irredundant representative.

A valuation ring that has Krull dimension 1 is, trivially, an irredundant representative if and only if it is a strongly irredundant representative. As noted above, it is in treating valuation overrings of Krull dimension  $>1$  where we need to work over a two-dimensional Noetherian domain in order to obtain the strongest results. But restricting for the moment to one-dimensional valuation rings, there exist some very general results, the first of which concerns rational valuation rings:

**Proposition 2.1 (Heinzer–Ohm [9, Lemma 1.3]).** *Let  $H$  be a domain. If there exists a rational valuation overring  $V$  of  $H$  such that  $H = V \cap R$  for some overring  $R \neq H$ , then  $V$  is a localization of  $H$ . In particular, every rational valuation overring of  $H$  that is a strongly irredundant representative of  $H$  is a localization of  $H$ .*

A slight technical generalization of the proposition is proved in Lemma 3.1 of the article [18]: *If  $H = V \cap R$ , where  $V$  is a valuation overring of  $H$  not necessarily of Krull dimension 1, and there exists a nonmaximal prime ideal  $\mathfrak{P}$  of  $V$  such that  $V/\mathfrak{P}$  is a rational valuation ring and  $V \subseteq H_{\mathfrak{P} \cap H}$ , then  $V$  is a localization of  $H$  or  $H = V_{\mathfrak{P}} \cap R$ . The relevance of this more general version is that if  $H$  is an overring of the two-dimensional Noetherian domain  $D$ , and  $V$  has Krull dimension 2, then  $V/\mathfrak{P}$  is a DVR, hence a rational valuation ring, where  $\mathfrak{P}$  is the nonzero nonmaximal prime ideal of  $V$ . This is the main step in obtaining the following useful characterization.*

**Proposition 2.2 ([18, Propositions 3.2 and 3.4]).** *Let  $H$  be an overring of the two-dimensional Noetherian domain  $D$ , and suppose there exists a valuation overring  $V$  of Krull dimension 2 such that  $H = V \cap R$  for some overring  $R \neq H$ . Then  $V$  is strongly irredundant in  $H = V \cap R$  if and only if either  $V$  is a localization of  $H$  or the nonzero prime ideals of  $V$  contract to a maximal ideal of  $H$ .*

A domain  $H$  is *completely integrally closed* if for every  $x$  in the quotient field,  $x$  belongs to  $H$  whenever the powers  $x^n$  ( $n \geq 0$ ) are contained in a finitely generated  $H$ -submodule of the quotient field of  $H$ . For example, an intersection of valuation rings of Krull dimension 1 is completely integrally closed. We observe in the next proposition, which will be useful in Section 4, that the only strongly irredundant representatives a completely integrally closed domain possibly can have are those of Krull dimension 1.

**Proposition 2.3.** *Let  $H$  be a domain, and let  $R$  be an overring of  $H$ . Suppose there exists a valuation overring  $V$  of Krull dimension  $> 1$  such that  $H = V \cap R$ , and  $V$  is strongly irredundant in this intersection. Then  $H$  is not completely integrally closed.*

*Proof.* Let  $\mathfrak{P}$  be a nonzero nonmaximal prime ideal of  $V$ , and let  $U = V_{\mathfrak{P}}$ . Then  $H \subsetneq U \cap R$ , since  $V$  is strongly irredundant in  $H = V \cap R$ . Choose  $0 \neq h \in \mathfrak{P} \cap H$ . Then  $h(U \cap R) \subseteq hU \cap R \subseteq \mathfrak{P} \cap R \subseteq V \cap R = H$ . Choose  $x \in (U \cap R) \setminus H$ . Then  $hH[x] \subseteq h(U \cap R) \subseteq H$ , so that  $H[x] \subseteq h^{-1}H$ . Since  $x \notin H$  and  $h^{-1}H$  is a finitely generated  $H$ -module, we see that  $H$  cannot be completely integrally closed.  $\square$

In particular, an integrally closed Noetherian domain  $D$ , since it is completely integrally closed, cannot have a strongly irredundant representative of Krull dimension  $> 1$ . In fact, if  $D$  has Krull dimension 2, the only strongly irredundant representatives of  $D$  are its essential prime divisors [19, Corollary 3.7].

Returning to the general setting where we do not assume  $H$  is an overring of a two-dimensional Noetherian domain, and continuing with the theme of identifying when irredundant representatives are localizations, we see in the special case where  $R$  has nonzero Jacobson radical, then not only are irredundant representatives localizations, but so is the ring  $R$ :

**Proposition 2.4 (Heinzer [8, Corollary 1.16]).** *Let  $H$  be an integrally closed domain, and let  $R$  be an overring of  $H$  having nonzero Jacobson radical. If  $H = V_1 \cap V_2 \cap \dots \cap V_n \cap R$ , where  $V_1, \dots, V_n$  are valuation overrings of Krull dimension 1 that are irredundant in this intersection, then  $R$  and the valuation rings  $V_i$  are all localizations of  $H$ .*

The next proposition also concerns valuation overrings of Krull dimension 1 that are irredundant representatives. The  $V$ -value referred to in the proposition is the value of a given element under the valuation corresponding to the valuation ring  $V$ . In light of Proposition 2.1, the real content of the proposition is that if  $V$  is an irrational irredundant representative, then its maximal ideal is generated as an ideal of  $V$  by its center in  $H$ .

**Proposition 2.5 (Heinzer–Ohm [10, Theorem 1.1]).** *Let  $H$  be a domain, and suppose  $V$  is a valuation overring of Krull dimension 1 such that  $H = V \cap R$  for some overring  $R$  with  $H \neq R$ . Then either  $V$  is a DVR, or  $H$  contains elements of arbitrarily small  $V$ -value.*

An example due to Ohm in [17, Example 5.3] illustrates the proposition. Let  $p$  be a prime integer, and let  $v_p$  be the  $p$ -adic valuation on  $\mathbb{Q}$ . Extend  $v_p$  to a mapping  $v$

on  $\mathbb{Q}[X]$  by defining for each  $\sum_{i=0}^n a_i X^i \in \mathbb{Q}[X]$ ,  $v(\sum_{i=0}^n a_i X^i) = \inf\{v_p(a_i) + i\pi\}$ . Then, by defining  $v(f/g) = v(f) - v(g)$  for all  $f, g \in \mathbb{Q}[X]$ ,  $g \neq 0$ , we have that  $v$  is a valuation on  $\mathbb{Q}(X)$  that extends  $v_p$  [4, Theorem 2.2.1]. Let  $H = V \cap \mathbb{Q}[X]$ , where  $V$  is the valuation ring corresponding to  $v$ . Then  $V$  is an irrational valuation ring irredundant in this representation. Moreover, if  $M = \mathfrak{M}_V \cap H$ , then by the proposition,  $MV$  is the maximal ideal of  $V$ . Ohm’s original motivation for introducing this example was to show that a domain that is an (irredundant) intersection of valuation overrings of Krull dimension 1 need not have the property that every nonunit is contained in a height one prime ideal. Indeed, in the example, the maximal ideal  $M$  is the radical of  $pH$  in  $H$  [17, Corollary 5.6].

We turn now to the framing of our main uniqueness result, Theorem 2.8. The dimension 1 case, which is due to Heinzer and Ohm, works without assumptions on being an overring of a two-dimensional Noetherian domain. Note that the theorem asserts both existence and uniqueness:

**Theorem 2.6 (Heinzer–Ohm [10, Corollary 1.4]).** *Let  $H$  be a domain, let  $R$  be an overring of  $H$  and suppose that  $H$  has a finite character  $R$ -representation  $\Sigma$  consisting of valuation overrings of Krull dimension 1. Then every irredundant  $R$ -representative of  $H$  of Krull dimension 1 is a member of  $\Sigma$ , and the set of all such representatives of  $H$  is a finite character  $R$ -representation of  $H$ . In particular,  $H$  has a unique irredundant  $R$ -representation of  $H$  consisting of valuation rings of Krull dimension 1.*

As alluded to earlier, the valuation overrings of Krull dimension  $> 1$  cause more difficulties. But in the context of overrings of two-dimensional Noetherian domains, as discussed in (1.3), a valuation overring of Krull dimension  $> 1$  must be a discrete valuation ring of Krull dimension 2. Let  $V$  be such a valuation ring, and let  $\mathfrak{P}$  denote the nonzero nonmaximal prime ideal of  $V$ . Then  $V/\mathfrak{P}$  is a DVR with quotient field  $V_{\mathfrak{P}}/\mathfrak{P}$ . If also  $\mathfrak{P}$  contracts to a height 2 maximal ideal of  $D$ , then, as in (1.5),  $V_{\mathfrak{P}}/\mathfrak{P}$  is a finitely generated field extension of  $D/(\mathfrak{P} \cap D)$  of transcendence degree 1, a fact that allows the strong approximation theorem for projective curves to be exploited in the treatment of  $V$ . This is done in [18] to obtain one of the main results of that article:

**Theorem 2.7 ([18, Theorem 5.3]).** *Let  $H \subseteq R$  be integrally closed overrings of the two-dimensional Noetherian domain  $D$ , and let  $V$  be a valuation overring of  $H$  of Krull dimension 2. Then  $V$  is a strongly irredundant  $R$ -representative of  $H$  if and only if  $V$  is a member of every  $R$ -representation of  $H$ .*

The next theorem, our main uniqueness theorem, appears in slightly weaker form in Corollary 5.6 of [18]. It is a consequence of Theorems 2.6 and 2.7. We need the added strength of the present version in order to prove Corollary 2.12 below.

**Theorem 2.8.** *Let  $H \subseteq R$  be integrally closed overrings of the two-dimensional Noetherian domain  $D$ . Suppose that  $\Sigma$  and  $\Gamma$  are collections of valuation overrings of  $H$  such that*

$$H = \left( \bigcap_{V \in \Sigma} V \right) \cap R = \left( \bigcap_{V \in \Gamma} V \right) \cap R,$$

and let  $\Delta$  be the subcollection of irrational valuation rings in  $\Gamma$ . If each maximal ideal of  $H$  has at most finitely many members of  $\Delta$  centered on it (which is the case if  $\Delta$  has finite character), and the members of  $\Sigma$  and  $\Gamma$  are strongly irredundant in these intersections, then  $\Sigma = \Gamma$ .

*Proof.* We claim that  $H = (\bigcap_{V \in \Sigma \cap \Gamma} V) \cap R$ . For if this is the case, then since  $\Sigma \cap \Gamma \subseteq \Sigma$ , and  $\Sigma$  is an irredundant  $R$ -representation of  $H$ , necessarily  $\Sigma \cap \Gamma = \Sigma$ , and a similar argument shows that  $\Sigma \cap \Gamma = \Gamma$ ; hence  $\Sigma = \Gamma$ . By way of contradiction, suppose that  $H \neq (\bigcap_{V \in \Sigma \cap \Gamma} V) \cap R$ . By Theorem 2.7, a valuation ring of Krull dimension 2 is in  $\Sigma$  if and only if it is in  $\Gamma$ . Thus, by replacing  $R$  with  $(\bigcap_{V \in \Sigma \cap \Gamma} V) \cap R$ ,  $\Sigma$  with  $\Sigma \setminus (\Sigma \cap \Gamma)$  and  $\Gamma$  with  $\Gamma \setminus (\Sigma \cap \Gamma)$ , we may assume without loss of generality that  $\Sigma \cap \Gamma$  is empty and every valuation ring in  $\Sigma$  and  $\Gamma$  has Krull dimension 1. We will derive a contradiction to the assumption that  $\Sigma \cap \Gamma$  is empty, and by so doing conclude that  $\Sigma = \Gamma$ .

Let  $V \in \Sigma$ . Since  $V$  is a strongly irredundant  $R$ -representative of  $H$ , there exists an integrally closed ring  $R_1 \subseteq R$  such that  $H = V \cap R_1$  and  $V$  is strongly irredundant in this intersection. Let  $P = \mathfrak{M}_V \cap H$ . We show that  $P$  is a nonmaximal prime ideal of  $H$ . Suppose otherwise. By Lemma 5.2 of [18], since  $H = V \cap R_1$ , the set  $\Gamma(P) := \{V \in \Gamma : P \subseteq \mathfrak{M}_V\}$  is an  $R_1$ -representation of  $H$ , so if  $U \in \Gamma(P)$ , then necessarily, since  $P$  is a maximal ideal of  $H$ ,  $P = \mathfrak{M}_U \cap H$ . Moreover, if some member  $U \in \Gamma(P)$  is a rational valuation ring, then by Proposition 2.1,  $U = H_P$ , and hence the irredundancy of  $\Gamma$  implies that  $U$  is the only member of  $\Gamma$  centered on  $P$ ; i.e.,  $\Gamma(P) = \{U\}$ . Otherwise, every member of  $\Gamma(P)$  is an irrational valuation ring, so by assumption there are at most finitely many members of  $\Gamma$  centered on the maximal ideal  $P$ . Hence, in every case,  $\Gamma(P)$  is a finite set. Thus since  $V$  is an irredundant  $R_1$ -representative of  $H$ , and  $\Gamma(P)$  is a finite  $R_1$ -representation of  $H$ , we have by Theorem 2.6 that  $V \in \Gamma(P) \subseteq \Gamma$ . But we have assumed that  $\Sigma \cap \Gamma$  is empty, so this contradiction shows that every valuation ring in  $\Sigma$  is centered on a nonmaximal prime ideal of  $H$ .

Now let  $W \in \Gamma$ . Then there exists an integrally closed ring  $R_2 \subseteq R$  such that  $H = W \cap R_2$  and  $W$  is strongly irredundant in this intersection. Let  $Q = \mathfrak{M}_W \cap H$ . By Lemma 5.2 of [18],  $\Sigma(Q)$  is an  $R_2$ -representation of  $H$ ; in particular  $\Sigma(Q)$  is nonempty, and we may choose  $U \in \Sigma(Q)$ . As we have shown above, every member of  $\Sigma$  is centered on a nonmaximal prime ideal of  $H$ , so necessarily, since  $Q \subseteq \mathfrak{M}_U \cap H$  and  $H$  has Krull dimension at most 2 (Proposition 1.2), it must be that  $Q = \mathfrak{M}_U \cap H$ , and  $Q$  is a nonmaximal prime ideal of  $H$ . But  $H_Q \subseteq W \cap U$ , and since by Proposition 1.2,  $H_Q$  is a DVR, we have  $W = U \in \Sigma$ . This contradicts the assumption that  $\Sigma \cap \Gamma$  is nonempty, so we conclude that  $\Sigma = \Gamma$ .  $\square$

Theorem 2.8 does not appear to generalize in any obvious way to higher dimensions, a fact which is not too surprising given our reliance, as discussed above, on a reduction to the case of projective curves. Example 6.2 of [18],

which was recalled earlier in this section, demonstrates in fact that uniqueness of strongly irredundant representations fails rather dramatically for overrings of three-dimensional Noetherian domains. Example 6.1 of [18], which was also recalled earlier, shows that Theorem 2.8 can fail if  $R$  is not integrally closed. However, I do not know of any examples that show the restriction on  $\Delta$  is needed in the theorem. This raises:

*Question 2.9.* Can the restriction on  $\Delta$  in Theorem 2.8 be omitted? That is, suppose that  $R$  is an integrally closed overring of the two-dimensional Noetherian domain  $D$ , and  $\Sigma$  and  $\Gamma$  are collections of valuation overrings of  $D$  such that  $(\bigcap_{V \in \Sigma} V) \cap R = (\bigcap_{V \in \Gamma} V) \cap R$ . If the members of  $\Sigma$  and  $\Gamma$  are strongly irredundant in these intersections, is  $\Sigma = \Gamma$ ?

To answer the question in the affirmative, the proof of Theorem 2.8 shows that it is enough to prove the question has a positive answer in the case where  $\Sigma$  and  $\Gamma$  consist of irrational valuation rings. This raises a more general question, one in which we do not restrict to overrings of two-dimensional Noetherian domains.

*Question 2.10.* Suppose that  $H$  is a domain,  $R$  is an overring of  $H$  and  $\Sigma$  and  $\Gamma$  are collections of valuation overrings of  $H$  of Krull dimension 1 such that  $(\bigcap_{V \in \Sigma} V) \cap R = (\bigcap_{V \in \Gamma} V) \cap R$ . If the members of  $\Sigma$  and  $\Gamma$  are irredundant in these intersections, is  $\Sigma = \Gamma$ ? Theorem 2.6 shows that the answer to the question is yes if  $\Sigma$  has finite character.

See also Question 4.3 for another approach to answering Question 2.9. In lieu of not knowing the answer to Question 2.9, we give next a specific instance in which the restriction on  $\Delta$  in Theorem 2.8 is automatically satisfied. We consider integrally closed rings between  $V[X]$  and  $F[X]$ , where  $V$  is a DVR with finite residue field and  $F$  is the quotient field of  $V$ . In [13], Loper and Tartarone prove the remarkable fact that locally, each such domain has an  $F[X]$ -representation requiring only one valuation ring:

**Theorem 2.11 (Loper–Tartarone [13, Corollary 3.3]).** *Let  $V$  be a DVR having finite residue field and quotient field  $F$ , and let  $X$  be an indeterminate for  $F$ . Let  $H$  be an integrally closed domain such that  $V[X] \subseteq H \subseteq F[X]$ . Then for each prime ideal  $P$  of  $H$ , there exists a valuation overring  $V$  of  $H$  such that  $H_P = V \cap F[X]_P$ .*

Using the theorem, we can prove for  $R = F[X]$  a uniqueness result that on the surface appears stronger than Theorem 2.8.

**Corollary 2.12.** *Let  $V$  be a DVR having finite residue field and quotient field  $F$ , and let  $X$  be an indeterminate for  $F$ . Suppose that  $\Sigma$  and  $\Gamma$  are collections of valuation overrings of  $V[X]$  such that*

$$\left( \bigcap_{V \in \Sigma} V \right) \cap F[X] = \left( \bigcap_{V \in \Gamma} V \right) \cap F[X],$$

*and the members of  $\Sigma$  and  $\Gamma$  are strongly irredundant in these intersections. Then  $\Sigma = \Gamma$ .*



*Proof.* Let  $H = (\bigcap_{V \in \Sigma} V) \cap F[X]$ . To apply Theorem 2.8 we need only show that there are at most finitely many members of  $\Gamma$  centered on any given maximal ideal of  $H$ . Let  $M$  be a maximal ideal of  $H$  such that some member, say  $W$ , of  $\Gamma$ , is centered on  $M$ . By Theorem 2.11,  $H_M = V \cap F[X]_M$  for some valuation overring  $V$  of  $H$ . On the other hand, since  $W$  is a strongly irredundant  $F[X]$ -representative of  $H$ , there exists an integrally closed ring  $R \subseteq F[X]$  such that  $H = W \cap R$  and  $W$  is strongly irredundant in this intersection. Moreover, since  $W$  is centered on  $M$ ,  $W$  is strongly irredundant in  $H_M = W \cap R_M$  [18, Proposition 3.2]; in particular,  $H_M \neq R_M$ . Since  $R \subseteq F[X]$ , we have  $H_M = V \cap R_M$ , and by possibly replacing  $V$  with an overring of  $V$ , since  $H_M \neq R_M$  we may assume without loss of generality that  $V$  is strongly irredundant in this representation. Thus, we have  $V \cap R_M = H_M = W \cap R_M$ , with both  $V$  and  $W$  strongly irredundant in their respective representations. Consequently, by Theorem 2.8,  $V = W$ . This then shows that for each valuation ring  $W$  in  $\Gamma$  centered on  $M$ ,  $W$  is equal to an overring of  $V$ , and since  $V$  has at most three overrings, we conclude that only finitely many members of  $\Gamma$  are centered on  $M$ .  $\square$

### 3 Noetherian representations

We consider in this section the special case in which the overring  $H$  of the two-dimensional Noetherian domain  $D$  has a *Noetherian  $R$ -representation*; i.e., an  $R$ -representation that is a Noetherian subspace of the space of all valuation overrings of  $D$ . The *Zariski–Riemann space* of a domain  $H$  is the set  $\text{Zar}(H)$  of all valuation overrings of  $H$  endowed with the topology whose basic open sets are of the form

$$U(x_1, \dots, x_n) := \{V \in \text{Zar}(H) : x_1, \dots, x_n \in V\},$$

where  $x_1, \dots, x_n$  are in the quotient field of  $H$ ; cf. [23, Chap. VI, Section 17]. We are particularly interested in Noetherian subspaces of  $\text{Zar}(H)$ , where a topological space  $X$  is *Noetherian* if  $X$  satisfies the ascending chain condition for open sets. One of our main motivations for considering the class of Noetherian subspaces of  $\text{Zar}(H)$  is that it includes finite character collections:

**Proposition 3.1** ([21, Proposition 3.2 and Theorem 3.4]). *If  $\Sigma$  is a finite character collection of valuations overrings of the domain  $H$ , then  $\Sigma$  is a Noetherian subspace of  $\text{Zar}(H)$ . Conversely, if  $\Sigma$  is a Noetherian space, then the set of all valuation overrings  $V$  of  $H$  of Krull dimension 1 with  $U \subseteq V$  for some  $U \in \Sigma$  has finite character.*

Thus for  $\Sigma$  a collection of valuation overrings of Krull dimension 1,  $\Sigma$  is a Noetherian space if and only if  $\Sigma$  has finite character. But if  $\Sigma$  is a Noetherian space that contains valuation rings of Krull dimension 2, then  $\Sigma$  may not have finite character, even in the special case where  $\Sigma$  consists of valuation overrings of a two-dimensional Noetherian domain. This is a consequence of the following proposition, which also provides a useful source of examples of Noetherian subspaces of  $\text{Zar}(D)$ , where  $D$  is a two-dimensional Noetherian domain. For  $\Sigma \subseteq \text{Zar}(D)$ , we define

$$\Sigma_1 = \{U \in \text{Zar}(D) : U \text{ has Krull dimension 1 and } V \subseteq U \text{ for some } V \in \Sigma\}.$$

By Proposition 3.1 and the fact that every valuation overring of  $D$  has Krull dimension  $\leq 2$ , if  $\Sigma$  is a Noetherian space, then  $\Sigma_1$  has finite character. The converse is true in the following special case.

**Proposition 3.2 ([18, Lemma 4.3]).** *Suppose that  $\Sigma$  is a collection of valuation overrings of the two-dimensional Noetherian domain  $D$ , and suppose that among the valuation rings in  $\Sigma_1$ , there are only finitely many essential prime divisors of  $D$ . Then  $\Sigma$  is a Noetherian subspace of  $\text{Zar}(D)$  if and only if  $\Sigma_1$  has finite character.*

It follows from the proposition that a Noetherian space of valuation rings need not have finite character. For example, suppose that  $D$  is a two-dimensional Noetherian domain, and  $V$  is a hidden prime divisor of  $D$ . Then by the proposition, the collection of all valuation rings  $U$  with  $D \subseteq U \subseteq V$  is a Noetherian subspace of  $\text{Zar}(D)$ , and it is easy to see that since by (1.5), the field  $V/\mathfrak{M}_V$  has transcendence degree 1 over the residue field  $D/(\mathfrak{M}_V \cap D)$ , there are infinitely many such valuation rings  $U$ , and since each contains  $\mathfrak{M}_V$ , the collection of all such rings  $U$  does not have finite character.

The proposition suggests the question of whether  $\Sigma_1$  having finite character always implies that  $\Sigma$  is a Noetherian space. This is not the case, as follows from (4.5) and Proposition 4.6 in the next section, where there is exhibited an integrally closed overring  $H$  of a two-dimensional Noetherian domain such that  $\Sigma_1$  has finite character but  $H$  does not have a Noetherian representation. In fact, the domain  $H$  in (4.5) does not have any irredundant representatives. By contrast, if  $H$  can be written as an intersection of valuation rings from a Noetherian subspace of  $\text{Zar}(H)$ , this phenomenon of having no irredundant representatives can never happen:

**Theorem 3.3 ([18, Corollary 5.7]).** *Let  $H \subsetneq R$  be integrally closed overrings of the two-dimensional Noetherian domain  $D$ . If  $H$  has a Noetherian  $R$ -representation, then the collection of all strongly irredundant  $R$ -representatives of  $H$  is a Noetherian  $R$ -representation of  $H$  and it is the unique strongly irredundant  $R$ -representation of  $H$ .*

Unpacking the theorem, we see that a Noetherian  $R$ -representation of an overring of a two-dimensional Noetherian domain (a) can always be replaced by a strongly irredundant Noetherian  $R$ -representation, (b) this  $R$ -representation is unique, and (c) it consists of all the strongly irredundant  $R$ -representatives of  $H$ . Statement (a) holds much more generally (see below), while statement (b) follows from Theorem 2.8. Statement (c) is a bit more subtle, in that strongly irredundant  $R$ -representatives of  $H$  are defined “locally”:  $V$  is a strongly irredundant  $R$ -representative if  $H = V \cap R_1$  for some integrally closed overring  $R_1 \subseteq R$ . Thus (c) asserts that the collection of all such representatives is large enough to form a representation of  $H$  yet small enough to be strongly irredundant, a fact which is not obvious.

The existence of a strongly irredundant  $R$ -representation in Theorem 3.3 is a consequence of a general result: *If a domain  $H$  with overring  $R$  has a Noetherian (resp., finite character)  $R$ -representation  $\Sigma$  of valuation overrings, then  $H$  has a strongly irredundant Noetherian (resp., finite character)  $R$ -representation  $\Gamma$  of valuation overrings. Moreover, every member of  $\Gamma$  can be chosen a valuation overring*

of a member of  $\Sigma$  [21, Theorem 4.3]. Following Brewer and Mott in [2], where this result was proven for finite character collections and in the case where  $R$  is the quotient field, the main idea behind the proof of this result is to pass to a Kronecker function ring of  $H$ , and to sort things out there, which is easier to do since this ring is a *Prüfer domain*, meaning that each valuation overring is a localization. In fact, the argument given in [21] works for collections of integrally closed domains, not just valuation rings.

One of the consequences of Theorem 3.3 is that to exhibit a (the!) strongly irredundant  $R$ -representation of  $H$ , one must find all the strongly irredundant  $R$ -representatives of  $H$ ; these then form the unique strongly irredundant  $R$ -representation of  $H$ . Identifying which valuation overrings are strongly irredundant  $R$ -representatives is in general not easy. The next proposition considers a special sort of Noetherian  $R$ -representation, one which contains only finitely many essential prime divisors, and for which  $R$  is “close” to  $H$ . In this case,  $H$  has a very transparent strongly irredundant  $R$ -representation. In the theorem, we use the notation  $\mathfrak{P}_V$  to denote the height 1 prime ideal of a valuation ring  $V$  of Krull dimension  $\geq 1$ .

**Proposition 3.4 ([18, Lemma 4.3 and Theorem 4.5]).** *Let  $H \subseteq R$  be integrally closed overrings of the two-dimensional Noetherian domain  $D$ . If there is a Noetherian  $R$ -representation  $\Sigma$  of  $H$  such that there are at most finitely many essential prime divisors of  $D$  contained in  $\Sigma_1$ , and  $R \subseteq U$  for each  $U \in \Sigma_1$ , then  $\{V \in \text{Zar}(H) : R \not\subseteq V, R \subseteq V_{\mathfrak{P}_V}\}$  is a strongly irredundant Noetherian  $R$ -representation of  $H$ .*

In particular, suppose  $A$  is an integrally closed overring of the two-dimensional Noetherian domain  $D$ , suppose that  $A$  is not completely integrally closed, and let  $R$  be the intersection of all the valuation overrings of  $A$  of Krull dimension 1. Then  $A \neq R$ , since  $A$  is not completely integrally closed. Choose any collection  $\Sigma$  of valuation overrings of  $A$  such that  $\Sigma_1$  is finite, and define  $H = (\bigcap_{V \in \Sigma} V) \cap R$ . Then, assuming some  $V$  in  $\Sigma$  does not contain  $R$ , the theorem implies that  $\{V \in \text{Zar}(H) : R \not\subseteq V\}$  is the unique strongly irredundant  $R$ -representation of  $H$ ; see Corollary 4.8 of [18] for more details and a stronger version of this fact.

Another quick consequence of the theorem is that if  $V$  and  $W$  are distinct valuation overrings of  $D$  such that  $W$  has Krull dimension 2 with  $V \cap R \subseteq W$  and  $R \subseteq V_{\mathfrak{P}_V} \cap W_{\mathfrak{P}_W}$ , then necessarily  $R \subseteq W$ . More generally:

**Corollary 3.5 ([18, Corollary 4.6]).** *Let  $R$  be an integrally closed overring of the two-dimensional Noetherian domain  $D$ , and let  $V$  and  $W$  be valuation overrings of the two-dimensional Noetherian  $D$ . If  $V \cap R \subseteq W$ , then  $V \subseteq W$  or  $V_{\mathfrak{P}_V} \cap W_{\mathfrak{P}_W} \cap R \subseteq W$ .*

Along these same lines, the theorem also implies the following corollary, which is interesting when  $H$  is not completely integrally closed, and hence  $H \neq H'$ .

**Corollary 3.6 ([18, Corollary 4.7]).** *Let  $H$  be an overring of the two-dimensional Noetherian domain  $D$ , let  $H'$  be the intersection of all the valuation overrings of  $H$  of Krull dimension 1, and let  $V$  be a valuation overring of  $H$ . Then  $\text{Zar}(H' \cap V) = \{V\} \cup \text{Zar}(H')$ .*

We turn now to the classification of quasilocal overrings  $H$  of the two-dimensional Noetherian domain  $D$  that have a Noetherian  $R$ -representation. This restriction to the quasilocal case is a reasonable reduction to make because of the next proposition. It follows from general principles, so we do not restrict to the case where  $H$  is an overring of a two-dimensional Noetherian domain.

**Proposition 3.7** ([21, Theorems 3.5 and 3.7]). *Let  $\Sigma$  be a Noetherian collection of valuation overrings of the domain  $H$ . If  $S$  is a multiplicatively closed subset of  $H$ , then  $H_S = \bigcap_{V \in \Sigma} V_S$  and  $\{V_S : V \in \Sigma\}$  is a Noetherian subspace of  $\text{Zar}(H)$ .*

Thus, if  $H$  has a Noetherian (resp., finite character)  $R$ -representation, and  $M$  is a maximal ideal of  $H$ , then by Proposition 3.7,  $H_M$  has a Noetherian (resp., finite character)  $R_M$ -representation, and for this reason we are interested in the quasilocal case. One of the main results of the article [19] is the following characterization.

**Theorem 3.8** ([19, Corollary 6.7]). *Let  $H \subseteq R$  be integrally closed overrings of the two-dimensional Noetherian domain  $D$ , with  $H$  quasilocal but not a valuation domain, and let  $E = \{r \in R : rM \subseteq M\}$ , where  $M$  is the maximal ideal of  $H$ . The following statements are equivalent.*

- (1)  $H$  has a Noetherian (resp., finite character)  $R$ -representation.
- (2)  $E/M$  is a Noetherian ring (resp., finitely generated  $H/M$ -algebra) and  $E = A \cap B \cap R$ , where  $B$  is an integrally closed Noetherian overring of  $H$  and  $A$  is either the quotient field of  $H$  or a finite intersection of irrational valuation overrings of  $H$ .

In the special case where  $R$  is the quotient field of  $D$ , Theorem 3.8 can be formulated more succinctly using Corollary 8.3 of [19], which states that the ring  $H$  has a Noetherian representation (that is, a Noetherian  $R$ -representation where  $R$  is the quotient field of  $H$ ) if and only if  $\text{End}(M)$  has a Noetherian representation for some maximal ideal  $M$  of  $H$ . (For an ideal  $I$  of a domain  $H$  with quotient field  $F$ , we let  $\text{End}(I) = \{q \in F : qI \subseteq I\}$ ).

**Corollary 3.9.** *Suppose that  $H$  is a quasilocal integrally closed overring of the two-dimensional Noetherian domain  $D$ . Let  $M$  denote the maximal ideal of  $H$ . Then  $H$  has a Noetherian representation if and only if  $H$  is a valuation domain, a Noetherian domain or  $\text{End}(M) = A \cap B$ , where  $B$  is an integrally closed Noetherian domain and  $A$  is either the quotient field of  $H$  or a finite intersection of irrational valuation overrings of  $H$ .*

The reader familiar with pullbacks will recognize Theorem 3.8 as asserting that the ring  $H$  is a pullback of the Noetherian ring  $E/M$  and the ring  $E = A \cap B \cap R$ . This insight, as indicated by the next corollary, is useful in locating examples of rings with Noetherian  $R$ -representations.

**Corollary 3.10** ([19, Corollaries 4.9 and 5.4]). *Let  $H \subseteq R$  be integrally closed overrings of the two-dimensional Noetherian domain  $D$ , and suppose that  $J$  is an ideal of  $R$  such that  $D \cap J$  is a maximal ideal of  $D$  and  $R/J$  is a reduced indecomposable ring*

having finitely many minimal prime ideals. Then every integrally closed overring  $C$  of  $D$  such that  $J \subseteq C \subseteq R$  has a Noetherian  $R$ -representation. If also  $R/J$  is a finitely generated  $D/(D \cap J)$ -algebra, then every such ring  $C$  has a finite  $R$ -representation.

For example, suppose that  $R$  is an integrally closed overring of the two-dimensional Noetherian domain  $D$ , and  $R$  is a finitely generated  $D$ -algebra. Choose a maximal ideal  $\mathfrak{m}$  of  $D$ , and let  $J$  be an intersection of prime ideals of  $R$  minimal over  $\mathfrak{m}R$  such that  $R/J$  is an indecomposable ring (e.g., choose  $J$  to be a prime ideal of  $R$  minimal over  $\mathfrak{m}R$ ). Then by the corollary, the integral closure of  $D + J$  in its quotient field has a finite  $R$ -representation.

The next theorem is more or less implicit in Theorems 6.5 and 6.6 of [19]. It gives an extrinsic classification of the quasilocal overrings of a two-dimensional Noetherian domain having a Noetherian  $R$ -representation, with emphasis on how these rings are assembled from valuation rings.

**Theorem 3.11.** *Let  $H$  be a quasilocal integrally closed overring of the two-dimensional Noetherian domain  $D$  that is not a valuation domain, and let  $R$  be an integrally closed overring of  $H$ . Then  $H$  has a Noetherian  $R$ -representation if and only if there exists a Noetherian integrally closed overring  $B$  of  $H$  such that one of the following statements holds.*

- (a)  $H = B \cap R$
- (b) There exist unique irrational valuation rings  $V_1, \dots, V_n$  such that:

$$H = V_1 \cap \dots \cap V_n \cap B \cap R,$$

and each  $V_i$  is irredundant in this intersection

- (c) There exist unique irrational valuation rings  $V_1, \dots, V_n$  and a unique collection  $\Gamma$  of valuation overrings of  $H$  of Krull dimension 2 such that:  $\Gamma_1$  is finite,

$$H = V_1 \cap \dots \cap V_n \cap \left( \bigcap_{V \in \Gamma} V \right) \cap B \cap R,$$

and each member of  $\{V_1, \dots, V_n\} \cup \Gamma$  is strongly irredundant in this intersection; or

- (d) There exists a unique collection  $\Gamma$  of valuation overrings of  $H$  of Krull dimension 2 such that:  $\Gamma_1$  is finite,

$$H = \left( \bigcap_{V \in \Gamma} V \right) \cap B \cap R,$$

and each member of  $\Gamma$  is strongly irredundant in this intersection.

*Proof.* Let  $E = \{r \in R : rM \subseteq M\}$ . By Theorem 3.8,  $H$  has a Noetherian  $R$ -representation if and only if  $E/M$  is a Noetherian ring and  $E = A \cap B \cap R$ , where  $B$  is an integrally closed Noetherian overring of  $E$  and  $A$  is either the quotient field of  $H$  or a finite intersection of valuation overrings of  $H$ . Suppose that  $H$  has a Noetherian  $R$ -representation. If  $H = E$ , then  $H = A \cap B \cap R$ , and if  $A$  can be omitted

from this representation, then  $H = B \cap R$ , and hence (a) is satisfied. If  $A$  cannot be omitted, then there exist finitely many irrational valuation overrings  $V_1, \dots, V_n$  of  $H$  such that  $H = V_1 \cap \dots \cap V_n \cap B \cap R$ . We may assume in fact that no  $V_i$  is redundant in this intersection, and hence by Theorem 2.8 each  $V_i$  is unique. Thus, in the case where  $H = E = A \cap B \cap R$ , and  $A$  cannot be omitted, we see that (b) is satisfied.

Otherwise, suppose that  $H \neq E$ . Then since  $H$  is not a valuation domain, Proposition 4.6 and Corollary 4.7 of [19] imply there exists a unique collection  $\Gamma$  of two-dimensional valuation overrings of  $H$  such that  $\Gamma_1$  is finite and  $\Gamma$  is a strongly irredundant  $E$ -representation of  $H$ . In particular,

$$H = \left(\bigcap_{V \in \Gamma} V\right) \cap E = \left(\bigcap_{V \in \Gamma} V\right) \cap A \cap B \cap R.$$

If  $A$  can be omitted from this intersection, then (d) is satisfied. If, on the other hand,  $A$  cannot be omitted, then since  $A$  is a finite intersection of irrational valuation overrings, we may throw away those not needed in the representation of  $H$  and obtain (c), where uniqueness follows from Theorem 2.8. Conversely, in (c) and (d), since  $\Gamma_1$  finite, we have by Proposition 3.2 that  $\Gamma$  is a Noetherian space. Thus it is clear since a finite union of Noetherian spaces is Noetherian that each of (a)–(d) implies  $H$  has a Noetherian  $R$ -representation.  $\square$

Examples given in the next section illustrate each case in Theorem 3.11. From the theorem we draw the following corollary, which is implicit in Theorems 6.5 and 6.6 of [19], and classifies quasilocal overrings of two-dimensional Noetherian domains having a finite character  $R$ -representation.

**Corollary 3.12.** *Let  $H$  be a quasilocal integrally closed overring of the two-dimensional Noetherian domain  $D$ , and let  $R$  be an integrally closed overring of  $H$ . Then  $H$  has a finite character  $R$ -representation if and only if (a), (b), (c) or (d) of Theorem 3.11 hold, with the additional assumption that  $\Gamma$  in (c) and (d) must be a finite set.*

*Proof.* Suppose that  $H$  has a finite character  $R$ -representation. Then since by Proposition 3.1 a finite character  $R$ -representation is a Noetherian  $R$ -representation, we have by Theorem 3.11 that one of (a)–(d) holds. Suppose that (d) holds. Then necessarily, since  $\Gamma_1$  is finite, we have by Proposition 3.2 that  $\Gamma$  is a strongly irredundant Noetherian  $(B \cap R)$ -representation of  $H$ . In particular, every member of  $\Gamma$  is a strongly irredundant  $R$ -representative of  $H$ . However, since  $H$  has a finite character  $R$ -representation, say  $\Sigma$ , then every strongly irredundant  $R$ -representative of  $H$  is a member of  $\Sigma \cup \Sigma_1$ , with  $\Sigma \cup \Sigma_1$  a finite character collection [18, Corollary 5.8]. Thus, since  $\Gamma \subseteq \Sigma \cup \Sigma_1$ , the set  $\Gamma$  has finite character, and since each member of  $\Gamma$  is centered on the maximal ideal of  $H$  (indeed, by Proposition 1.2, a valuation ring centered on a nonmaximal prime ideal is a DVR),  $\Gamma$  must be finite. Similarly, if (c) holds, then  $\Gamma$  is a strongly irredundant Noetherian  $(A \cap B \cap R)$ -representation of  $H$ , and as above,  $\Gamma$  must be finite. Conversely, since a finite union of finite character collections of valuation overrings has finite character, it follows that (a), (b), (c) and (d), with  $\Gamma$  finite in (c) and (d), each implies that  $R$  has a finite character  $R$ -representation.  $\square$

The next theorem considers Noetherian representations (i.e., Noetherian  $R$ -representations, with  $R$  the quotient field of  $H$ ) of one-dimensional overrings of two-dimensional Noetherian domains.

**Theorem 3.13** ([19, Theorem 8.6]). *Suppose that  $H$  is an integrally closed overring of the two-dimensional Noetherian domain  $D$ , and that  $H$  is not a valuation domain. Then the following statements are equivalent.*

- (1)  $H$  is a quasilocal domain of Krull dimension 1 that has a Noetherian representation.
- (2) There exists a hidden prime divisor  $U$  of  $D$  of such that  $H = \bigcap_{V \subseteq U} V$ , where  $V$  ranges over the valuation overrings of  $D$  contained in  $U$ .
- (3) There exists a hidden prime divisor  $U$  of  $D$  of such that  $H$  is the integral closure of the ring  $D + \mathfrak{M}_U$  in its quotient field.

To close this section, we mention that it is somewhat complicated to describe  $\text{Spec}(H)$  when  $H$  has a Noetherian  $R$ -representation: in general some prime ideals are contracted from  $R$ , others are contracted from prime ideals of the valuation rings in the representation, and some arise from neither of these sources. We omit the details of how to account for these prime ideals, and refer instead to Section 7 of [19]. However, the analysis of the particular case where  $R$  is the quotient field of  $D$  does lead to some nice consequences.

**Theorem 3.14** ([19, Theorem 8.1, Corollary 8.2 and Theorem 8.9]). *Let  $H$  be a quasilocal overring of the two-dimensional Noetherian domain  $D$  such that  $H$  has a Noetherian representation and  $H$  is not a field. Then  $\text{Spec}(H)$  is a Noetherian space, and for each nonzero radical ideal  $J$  of  $H$ ,  $H/J$  is a Noetherian ring. Moreover,  $H$  is a Noetherian domain if and only if  $H$  has a height 1 finitely generated prime ideal.*

## 4 (Counter)examples

In this section, we give examples of some ill-behaved integrally closed overrings of two-dimensional Noetherian domains, “ill-behaved” because they lie outside the reach of our preceding results or prevent us from tightening our results further, but they are not atypical. Indeed, loosely speaking, one should expect that “most” integrally closed overrings of two-dimensional Noetherian domains belong in this section, rather than to the orderly classification in the previous ones.

First we prove a theorem to illustrate that all the cases in Theorem 3.11 can occur, and that it is possible to have more than one strongly irredundant representative centered on a maximal ideal of an overring of a two-dimensional Noetherian domain. This is in dramatic contrast to the quasilocal rings  $H_P$  in Theorem 2.11, which have a Noetherian representation (indeed the representation consists of a single valuation ring) yet have only one strongly irredundant representative centered on a maximal ideal. We discuss the consequences of the theorem in more detail after the proof.

**Theorem 4.1.** *Let  $K$  be a field of characteristic 0, let  $X$  and  $Y$  be indeterminates for  $K$ , and let  $n > 1$ . Then there exists a finitely generated  $K$ -subalgebra  $D$  of  $K[X, Y]$  having quotient field  $K(X, Y)$  and valuation overrings  $V_1, \dots, V_n$  of  $D$  such that each  $V_i$  is strongly irredundant in  $H := V_1 \cap \dots \cap V_n \cap K[X, Y]$ , and each  $V_i$  is centered on the same maximal ideal of  $H$ . Moreover, each  $V_i$  may be chosen to be either an irrational valuation ring or a valuation ring of Krull dimension 2.*

*Proof.* Without loss of generality we assume that  $K$  contains  $\mathbb{Q}$ , the field of rational numbers. For each  $i = 1, 2, \dots, n$ , let  $G_i$  be a totally ordered free abelian group of rational rank 2. Then either (a)  $G_i$  can be viewed as a free subgroup of the real numbers, at least one of whose generators is an irrational number, or (b)  $G_i$  is isomorphic as a totally ordered abelian group to  $\mathbb{Z} \oplus \mathbb{Z}$  ordered lexicographically. We will construct a valuation  $v_i$  having value group  $\Gamma_i \subseteq G_i$ , such that either, in case (a), the valuation ring of  $v_i$  is an irrational valuation, or, in case (b), the valuation ring of  $v_i$  has Krull dimension 2.

Choose rationally independent generators  $\sigma_i$  and  $\tau_i$  of  $G_i$  with  $0 < \tau_i < \sigma_i$  such that  $\tau_i$  is contained in the smallest nontrivial convex subgroup of  $G_i$  (possibly this subgroup is all of  $G_i$ ). For each  $i = 1, 2, \dots, n$ , since  $K[X, Y] = K[X + iY, Y]$ , we may write each  $f(X, Y)$  in  $K[X, Y]$  in the form

$$f(X, Y) = \sum_{k, \ell} \alpha_{k, \ell} (X + iY)^k Y^\ell,$$

where each  $\alpha_{k, \ell} \in K$ . With this in mind we define a mapping  $v_i : K[X, Y] \rightarrow G_i \cup \{\infty\}$  by  $v_i(0) = \infty$  and

$$v_i(f(X, Y)) = \min\{kn\sigma_i - \ell\tau_i : \alpha_{k, \ell} \neq 0\},$$

for each  $0 \neq f(X, Y) = \sum_{k, \ell} \alpha_{k, \ell} (X + iY)^k Y^\ell \in K[X, Y]$ . For each  $g, h \in K[X, Y]$ , with  $h \neq 0$ , let  $v_i(g/h) = v_i(g) - v_i(h)$ . Then  $v_i$  defines a valuation on  $K(X, Y)$  with value group  $\Gamma_i \subseteq G_i$  generated by  $n\sigma_i$  and  $\tau_i$  [4, Theorem 2.2.1]. In particular, if  $G_i$  is chosen a subgroup of the reals, then  $v_i$  has an irrational value group  $\Gamma_i$ , while if  $G_i$  is chosen to be  $\mathbb{Z} \oplus \mathbb{Z}$  ordered lexicographically, then  $\Gamma_i$  also is isomorphic to  $\mathbb{Z} \oplus \mathbb{Z}$  ordered lexicographically, and the valuation ring corresponding to  $v_i$  has Krull dimension 2.

For each  $i$ , let  $V_i$  be the valuation ring associated to  $v_i$ . First observe that for  $i \neq j$  in  $\{1, 2, \dots, n\}$ , we have  $v_j(X + iY) = -\tau_j$ . Indeed, suppose by way of contradiction that  $v_j(X + iY) \neq -\tau_j$ . Then, since  $v_j(Y) = -\tau_j$ , we have  $v_j(X + iY) \neq v_j(Y)$ . Since the value of the sum of two elements, each having distinct values, is the minimum of the values of the elements, it follows that:

$$\begin{aligned} n\sigma_j &= v_j(X + jY) = v_j(X + iY + (j - i)Y) \\ &= \min\{v_j(X + iY), v_j((j - i)Y)\} \\ &= \min\{v_j(X + iY), -\tau_j\} \leq -\tau_j. \end{aligned}$$

But  $-\tau_j < 0 < n\sigma_j$ , a contradiction which shows that for each  $j \neq i$ ,  $v_j(X + iY) = -\tau_j$ .



Now we show that  $H := V_1 \cap \dots \cap V_n \cap K[X, Y]$  is an irredundant intersection. Define  $g(X, Y) = (X + Y)(X + 2Y) \cdots (X + nY)$ , and for each  $i = 1, \dots, n$ , define  $g_i(X, Y) = (X + iY)^{-1}g(X, Y)$ . We claim that for each  $i$ ,  $g_i(X, Y) \in \bigcap_{j \neq i} V_j$  but  $g_i(X, Y) \notin V_i$ . To see that this is the case, observe that:

$$\begin{aligned} v_j(g_i(X, Y)) &= \left( \sum_{k=1}^n v_j(X + kY) \right) - v_j(X + iY) \\ &= \sum_{k \neq i} v_j(X + kY) \end{aligned}$$

Thus, if  $j = i$ , then

$$v_i(g_i(X, Y)) = \sum_{k \neq i} v_i(X + kY) = \sum_{k \neq i} -\tau_i = -(n - 1)\tau_i.$$

Otherwise, if  $j \neq i$ , then

$$\begin{aligned} v_j(g_i(X, Y)) &= \sum_{k \neq i} v_j(X + kY) \\ &= v_j(X + jY) + \sum_{k \neq i, j} v_j(X + kY) \\ &= n\sigma_j - (n - 2)\tau_j. \end{aligned}$$

In the former case,  $v_i(g_i(X, Y)) = -(n - 1)\tau_i < 0$ ; in the latter case:

$$v_j(g_i(X, Y)) = n\sigma_j - (n - 2)\tau_j > n\sigma_j - n\tau_j > 0,$$

since  $\tau_j$  was chosen with  $0 < \tau_j < \sigma_j$ . Therefore,  $g_i(X, Y) \in \bigcap_{j \neq i} V_j$ , while  $g_i(X, Y) \notin V_i$ . This proves that  $H = V_1 \cap \dots \cap V_n \cap K[X, Y]$  is an irredundant intersection.

We in fact claim that each  $V_i$  is strongly irredundant in this intersection. If each  $V_i$  has Krull dimension 1, then the claim is clear. So suppose that some  $V_i$ , say  $V_1$ , has Krull dimension 2. To prove that  $V_1$  is strongly irredundant, we must show that  $(V_1)_{\mathfrak{P}} \cap (\bigcap_{j=2}^n V_j) \cap K[X, Y] \not\subseteq V_1$ , where  $\mathfrak{P}$  is the height 1 prime ideal of  $V_1$ . The value group  $\Gamma_1$  of  $v_1$ , since  $V_1$  has Krull dimension 2, has rank 2, and so there exists a unique proper nontrivial convex subgroup  $\Delta$  of  $\Gamma_1$ . Moreover, the mapping  $w : K(X, Y) \rightarrow \Gamma_1 / \Delta \cup \{\infty\}$  defined by  $w(0) = \infty$  and  $w(q(X, Y)) = v_1(q(X, Y)) + \Delta$  for all  $0 \neq q(X, Y) \in K(X, Y)$  is a valuation having valuation ring  $(V_1)_{\mathfrak{P}}$  [4, p. 44]. By assumption,  $0 < \tau_1 < \sigma_1$ , with  $\tau_1 \in \Delta$ . From our above calculations then, we see that since  $v_1(g_1(X, Y)) = -(n - 1)\tau_1$ , we have

$$w(g_1(X, Y)) = v_1(g_1(X, Y)) + \Delta = -(n - 1)\tau_1 + \Delta = 0 + \Delta,$$

and hence  $g_1(X, Y) \in (V_1)_{\mathfrak{P}}$ . We verified earlier that  $g_1(X, Y) \in \bigcap_{j=2}^n V_j$  and  $g_1(X, Y) \notin V_1$ , so we conclude that  $g_1(X, Y) \in (V_1)_{\mathfrak{P}} \cap (\bigcap_{j=2}^n V_j) \cap K[X, Y]$ , but  $g_1(X, Y) \notin V_1$ . This proves that each  $V_i$  is strongly irredundant in the representation  $H = V_1 \cap \dots \cap V_n \cap K[X, Y]$ .

Next we show that each  $V_i$  is centered on the same maximal ideal of  $H$ . Each  $V_i$  is necessarily centered on a maximal ideal of  $H$ , since otherwise by Proposition 1.2,  $V_i$  is a DVR. Let  $M = \mathfrak{M}_{V_1} \cap H$ . Then we claim that  $M \subseteq \mathfrak{M}_{V_i} \cap H$  for all  $i = 1, 2, \dots, n$ , and to prove this it suffices to show that for each  $f \in M$ ,  $v_i(f) > 0$ . This in turn is equivalent to proving that for each  $f \in H$ , if  $v_i(f) = 0$  for some  $i$ , then  $v_1(f) = 0$  also. We in fact prove the slightly stronger claim: If  $f(X, Y) \in H$ , and  $v_i(f(X, Y)) = 0$  for some  $i = 1, 2, \dots, n$ , then  $v_i(f(X, Y)) = 0$  for all  $i = 1, 2, \dots, n$ .

Let  $f(X, Y) \in H$ , and suppose that  $v_i(f(X, Y)) = 0$  for some  $i \in \{1, 2, \dots, n\}$ . Since  $f(X, Y) \in K[X, Y] = K[X + iY, Y]$ , we may write

$$f(X, Y) = \sum_{k, \ell} \alpha_{k, \ell} (X + iY)^k Y^\ell,$$

where each  $\alpha_{k, \ell} \in K$ . Therefore,

$$\begin{aligned} 0 = v_i(f(X, Y)) &= \min\{v_i(\alpha_{k, \ell} (X + iY)^k Y^\ell) : \alpha_{k, \ell} \neq 0\} \\ &= \min\{kn\sigma_i - \ell\tau_i : \alpha_{k, \ell} \neq 0\}, \end{aligned}$$

and hence there exist  $k$  and  $\ell$  such that  $\alpha_{k, \ell} \neq 0$  and  $0 = kn\sigma_i - \ell\tau_i$ . Since  $\sigma_i$  and  $\tau_i$  are rationally independent and  $n > 0$ , it follows that  $k = \ell = 0$ , so that  $\alpha_{0, 0} \neq 0$ . Therefore, since

$$f(0, 0) = \sum_{k, \ell} \alpha_{k, \ell} (0 + i \cdot 0)^k 0^\ell = \alpha_{0, 0},$$

we conclude  $f(0, 0) \neq 0$ .

Now, using the fact that  $f(0, 0) \neq 0$ , we claim that  $v_j(f(X, Y)) = 0$  for all  $j = 1, \dots, n$ . Let  $j$  be such an integer, and write  $f(X, Y) = \sum_{k, \ell} \beta_{k, \ell} (X + jY)^k Y^\ell$ , where  $\beta_{k, \ell} \in K$ . As above, we have:

$$\begin{aligned} v_j(f(X, Y)) &= \min\{v_j(\beta_{k, \ell} (X + jY)^k Y^\ell) : \beta_{k, \ell} \neq 0\} \\ &= \min\{nk\sigma_j - \ell\tau_j : \beta_{k, \ell} \neq 0\}. \end{aligned}$$

By our above calculation,  $0 \neq f(0, 0)$ , so

$$0 \neq f(0, 0) = \sum_{k, \ell} \beta_{k, \ell} (0 + j \cdot 0)^k 0^\ell = \beta_{0, 0}.$$

Therefore, since  $\beta_{0, 0} \neq 0$ ,

$$v_j(f(X, Y)) = \min\{nk\sigma_j - \ell\tau_j : \beta_{k, \ell} \neq 0\} \leq 0.$$

But by assumption,  $f(X, Y) \in H \subseteq V_j$ , so  $v_j(f(X, Y)) = 0$ , which proves the claim that each  $V_i$  is centered on the same maximal ideal of  $H$ .

Finally, we claim that  $H$  is an overring of a finitely generated  $K$ -subalgebra  $D$  of  $K[X, Y]$  having quotient field  $K(X, Y)$ . Let

$$D = K[g(X, Y), (X + Y)g(X, Y), (X + 2Y)g(X, Y)].$$

For each  $i, j \in \{1, 2, \dots, n\}$ , we have:

$$\begin{aligned} v_i((X + jY)g(X, Y)) &= v_i(X + jY) + v_i(g(X, Y)) \\ &= v_i(X + jY) + \sum_{k=1}^n v_i(X + kY) \\ &= 2v_i(X + jY) + \sum_{k \neq j} v_i(X + kY). \end{aligned}$$

Thus, if  $i = j$ , we have

$$\begin{aligned} v_i((X + jY)g(X, Y)) &= 2v_i(X + jY) + \sum_{k \neq j} v_i(X + kY) \\ &= 2n\sigma_i - (n - 1)\tau_i > n\sigma_i - n\tau_i > 0, \end{aligned}$$

while if  $i \neq j$ , we have

$$\begin{aligned} v_i((X + jY)g(X, Y)) &= 2v_i(X + jY) + \sum_{k \neq j} v_i(X + kY) \\ &= -2\tau_i + v_i(X + iY) + \sum_{k \neq i, j} v_i(X + kY) \\ &= -2\tau_i + n\sigma_i - (n - 2)\tau_i = n\sigma_i - n\tau_i > 0. \end{aligned}$$

Thus, in the case  $j = 1$ , we have for all  $i$ ,  $v_i((X + Y)g(X, Y)) > 0$ , so that  $(X + Y)g(X, Y) \in V_i$ . Similarly, for  $j = 2$ , we have for all  $i$ ,  $(X + 2Y)g(X, Y) \in V_i$ . Also, as above, for each  $i$ ,  $v_i(g(X, Y)) = n\sigma_i - (n - 1)\tau_i > n\sigma_i - n\tau_i > 0$ , so we conclude that  $D \subseteq H$ . Moreover,  $X + Y$  and  $X + 2Y$  are in the quotient field of  $D$ , so  $Y = X + 2Y - (X + Y)$  and  $X = (X + Y) - Y$  are in the quotient field of  $D$ . Therefore, since  $D \subseteq K(X, Y)$ ,  $D$  has quotient field  $K(X, Y)$ .  $\square$

We apply the theorem to illustrate the various cases that occur in Theorem 3.11:

**(4.2)** Let  $K$  be a field of characteristic 0, and choose  $m, n > 0$ . Then by the theorem, there exist a two-dimensional Noetherian domain  $D \subseteq K[X, Y]$  with quotient field  $K(X, Y)$ , irrational valuation overrings  $V_1, \dots, V_n$  and valuation rings  $W_1, \dots, W_m$  of Krull dimension 2 such that each of these valuation rings is strongly irredundant in the intersection:

$$H := V_1 \cap \dots \cap V_n \cap W_1 \cap \dots \cap W_m \cap K[X, Y],$$

and each  $V_i$  and  $W_j$  is centered on the same maximal ideal  $M$  of  $H$ . Localizing at  $M$  we have

$$H_M = V_1 \cap \dots \cap V_n \cap W_1 \cap \dots \cap W_m \cap K[X, Y]_M.$$

The valuation rings  $V_1, \dots, V_n, W_1, \dots, W_m$  remain strongly irredundant in this localization [18, Proposition 3.2], so  $H_M$  illustrates case (c) of Theorem 3.11. If in our application of the theorem we had omitted the  $W_j$ 's, we would have obtained an example of case (b) of the theorem. Similarly, by omitting each  $V_i$  we would

exhibit case (d). This shows that all the cases of Theorem 3.11 can occur, and also, interestingly, there is no bound on the number of irrational valuation overrings needed to represent a *quasilocal* overring of a two-dimensional Noetherian domain. (I am not aware of other examples of this last phenomenon, or examples of Theorem 3.11(c), in the literature.) Case (d) also is illustrated by the one-dimensional domains occurring in Theorem 3.13.

In summary, let  $H$  be a quasilocal integrally closed overring of a two-dimensional Noetherian domain, and let  $M$  denote the maximal ideal of  $H$ . If  $H$  has a strongly irredundant representative  $V$  that is a rational valuation ring centered on  $M$ , then  $H = V$  (Proposition 2.1). Thus, if  $H$  is not a valuation ring, then a strongly irredundant representative of  $H$  centered on  $M$  (assuming one exists) is either an irrational valuation ring or a valuation ring of Krull dimension 2. Theorem 3.13 shows that  $H$  may have infinitely many strongly irredundant representatives of Krull dimension 2 centered on  $M$ , while the preceding discussion shows that for each  $m, n \geq 0$ ,  $H$  can have precisely  $n$  strongly irredundant irrational representatives centered on  $M$  and precisely  $m$  strongly irredundant representatives of Krull dimension 2 centered on  $M$ . Thus one question that remains is:

*Question 4.3.* Does there exist an integrally closed overring  $H$  of a two-dimensional Noetherian domain such that  $H$  has infinitely many strongly irredundant representatives that are irrational valuation rings and all lie over the same maximal ideal of  $H$ ? (If the answer is negative, then it follows from Theorem 2.8 that the answer to Question 2.9 is affirmative.)

As hinted at in the previous discussion, it is possible that an overring of a two-dimensional Noetherian domain has no irredundant representatives. Such an example is given by a construction of Nagata. Krull conjectured in 1936 that a quasilocal completely integrally closed domain of Krull dimension 1 had to be a valuation ring. Later, in 1952, Nagata in [14] (but see also [15]) intersected a large number of valuation rings together to form a counterexample to this conjecture. We apply his construction in the next proposition, which gives another example of how large classes of valuation overrings of two-dimensional Noetherian domains can intersect in complicated ways.

**Proposition 4.4** (cf. Nagata [14, 15]). *Let  $k$  be a field, and let  $U$  and  $X$  be indeterminates over  $k$ . Then there exists a quasilocal completely integrally closed domain  $H$  of Krull dimension 1 that is an overring of  $k[U, X]$  and is not a valuation domain. Moreover,  $H$  has no irredundant representatives.*

*Proof.* In Theorem 1 of [14], Nagata shows that if  $K$  is an algebraically closed field having a nontrivial valuation  $v$  whose value group  $G$  is a proper subgroup of the real numbers, then there exists a collection  $\Sigma$  of valuation rings extending  $v$  and having quotient field  $K(X)$  whose value groups are subgroups of the reals, and such that  $A = \bigcap_{V \in \Sigma} V$  is a quasilocal completely integrally closed domain of Krull dimension 1 that is not a valuation domain and has quotient field  $K(X)$ . The set  $\Sigma$  is constructed in the following way. Choose a positive real number  $\alpha$  not in the value group  $G$  of  $v$ .

For every element  $e \in K$  such that  $\alpha < v(e) < 2\alpha$  (since  $K$  is algebraically closed,  $2\alpha \notin G$ ), define a valuation  $v_e$  of  $K(X)$  such that

$$v_e \left( \sum_{i=0}^n a_i (X + e)^i \right) = \min\{v(a_i) + 2\alpha i\} \quad (a_i \in K).$$

Also, for every real number  $\lambda$  with  $\alpha \leq \lambda \leq 2\alpha$ , define a valuation  $v_\lambda$  such that

$$v_\lambda \left( \sum_{i=0}^n a_i X^i \right) = \min\{v(a_i) + \lambda i\} \quad (a_i \in K).$$

Then  $\Sigma$  consists of the valuation rings corresponding to all the  $v_e$ 's and  $v_\lambda$ 's.

We apply this result to show that  $k[U, X]$  has an overring that is a quasilocal completely integrally closed domain of Krull dimension 1 but is not a valuation ring. To do so, we imitate aspects of the proof Theorem 2 in [14]. Let  $K$  be the algebraic closure of  $k(U)$ , and let  $v_0$  be the valuation corresponding to  $k[U]_{(U)}$ . Then  $v_0$  extends to a valuation  $v$  of  $K$  whose value group is the group of rational numbers. Thus, by Nagata's theorem there exists a collection  $\Sigma$  of valuation rings extending  $v$  and having quotient field  $K(X)$  whose value groups are subgroups of the reals, and such that  $A = \bigcap_{V \in \Sigma} V$  is a quasilocal completely integrally closed domain of Krull dimension 1 that is not a valuation domain and the quotient field of  $A$  is  $K(X)$ . Let  $\bar{A}$  be the integral closure of  $A$  in  $\overline{K(X)}$ , the algebraic closure of  $K(X)$ , and let  $N$  be a maximal ideal of  $\bar{A}$ . Let  $H = \bar{A}_N \cap k(U, X)$ . Then since  $A$  is completely integrally closed, so is  $\bar{A}_N$  [14, Lemma 3]. Therefore, since  $H = \bar{A}_N \cap k(U, X)$ ,  $H$  is completely integrally closed also. Similarly, since  $\bar{A}_N$  is quasilocal of Krull dimension 1, so is  $H$ . But also  $\bar{A}_N \cap K(X) = A$  [14, Lemma 1], and since  $A$  is not a valuation domain,  $\bar{A}_N$  is also not a valuation domain. If  $H$  is a valuation ring, then the integral closure  $\bar{H}$  of  $H$  in  $\overline{K(X)}$  is a Prüfer domain [5], and since  $\bar{H} \subseteq \bar{A}_N$ , it follows that  $\bar{A}_N$  is a valuation domain, a contradiction. Therefore,  $H$  is a quasilocal completely integrally closed domain of Krull dimension 1 that is not a valuation domain. Examination of the valuations used above to build  $\Sigma$  shows that  $U, X \in A$ . Indeed, it is clear that  $v_e(U), v_\lambda(U) > 0$  for all  $e$  and  $\lambda$ . Also, for  $e \in K$  such that  $0 < \alpha < v(e) < 2\alpha$ , since  $v_e(X + e) = 2\alpha$ , we have

$$v_e(X) = v_e(X + e - e) = \min\{v(X + e), v(e)\} = v(e) > 0.$$

Thus, it follows that  $k[U, X] \subseteq H \subseteq k(U, X)$ , and hence  $H$  is an overring of  $k[U, X]$ .

If  $H$  has an irredundant representative  $V$ , then by possibly replacing  $V$  with a proper overring of  $V$ , we may assume without loss of generality that  $H$  has a strongly irredundant representative. To see that  $H$  has no strongly irredundant representatives, suppose by way of contradiction that there exists a valuation overring  $V$  of  $H$  and an integrally closed overring  $R$  such that  $H = V \cap R$ , with  $V$  strongly irredundant in this intersection. Since  $H$  is quasilocal of Krull dimension 1,  $R$  has nonzero Jacobson radical, and  $V$  is necessarily centered on the maximal ideal of  $H$ . Thus by Proposition 2.4, since  $H$  is not a valuation ring, it must be that  $V$  has

Krull dimension 2. But since  $H$  is completely integrally closed,  $H$  has no strongly irredundant representatives of Krull dimension 2 (Proposition 2.3). This contradiction shows that  $H$  has no irredundant representatives.  $\square$

We recall next from [18] a different sort of example of having no irredundant representatives. It has the additional property of having a representation  $\Sigma$  such that  $\Sigma_1$  has finite character, and hence  $H$  is in some sense close to having a Noetherian representation. The ring is a *Prüfer domain*, meaning that each valuation overring is a localization.

**(4.5)** Let  $K$  be a field of characteristic 0 that is not algebraically closed, and let  $D$  be a two-dimensional integrally closed local Noetherian domain with maximal ideal  $\mathfrak{m}$  such that  $D$  has residue field  $K$  and  $D$  is the localization of a finitely generated  $K$ -algebra. Let  $\Sigma$  be the set of all valuation overrings  $V$  of  $D$  of Krull dimension 2 such that the residue field of  $V$  is  $K$  and  $V \subseteq D_{\mathfrak{p}}$  for some height one prime ideal  $\mathfrak{p}$  of  $D$ . Let  $H = \bigcap_{V \in \Sigma} V$ , and observe that  $\Sigma_1$  has finite character. It is shown in Example 6.4 of [18] using resolution of singularities that  $H$  is a two-dimensional Prüfer domain having no irredundant representatives.

We discuss another naturally occurring example exhibiting some of the same traits as (4.5). Let  $R$  be a domain with quotient field  $F$ . Then the *ring of  $R$ -valued polynomials* is defined to be  $\text{Int}(R) = \{f(X) \in F[X] : f(R) \subseteq R\}$ . The ring  $\text{Int}(\mathbb{Z})$  is a two-dimensional Prüfer (hence non-Noetherian) overring of the two dimensional Noetherian domain  $\mathbb{Z}[X]$ , and has a rich and well-studied structure; see [3]. In the next proposition, we wish to consider  $\text{Int}(\widehat{\mathbb{Z}}_p)$ , where  $\widehat{\mathbb{Z}}_p$  denote the ring of  $p$ -adic integers, and  $\widehat{\mathbb{Q}}_p$  denotes its quotient field. This ring is a two-dimensional completely integrally closed Prüfer domain that cannot be written as an intersection of valuation overrings of Krull dimension 1 [3, Propositions VI.2.1, VI.2.2 and Remark VI.1.8].

**Proposition 4.6.** *Let  $H = \text{Int}(\widehat{\mathbb{Z}}_p)$ , and let  $\Sigma = \text{Zar}(H)$ . Then  $\Sigma_1 = \text{Zar}(\widehat{\mathbb{Q}}_p[X])$ , and  $\Sigma_1$  has finite character but  $H$  does not have a Noetherian representation, and  $H$  has no irredundant  $\widehat{\mathbb{Q}}_p[X]$ -representatives.*

*Proof.* The proof of Proposition VI.2.2 in [3] shows that the intersection of all the valuation overrings of  $H$  of Krull dimension 1 is  $\widehat{\mathbb{Q}}_p[X]$ . Thus,  $H$  has a  $\widehat{\mathbb{Q}}_p[X]$ -representation consisting of valuation rings of Krull dimension 2. Moreover, the prime spectrum of  $H$  is not a Noetherian space [3, Proposition VI.2.8], and hence  $H$  does not have a Noetherian representation (Theorem 3.14), which in turn implies since  $\widehat{\mathbb{Q}}_p[X]$  is a Dedekind domain that  $H$  does not have a Noetherian  $\widehat{\mathbb{Q}}_p[X]$ -representation. Since  $\widehat{\mathbb{Q}}_p[X]$  is the intersection of all the valuation overrings of  $H$  of Krull dimension 1, we have  $\Sigma_1 = \text{Zar}(\widehat{\mathbb{Q}}_p[X])$ , and hence  $\Sigma_1$  has finite character. Finally, since  $H$  is completely integrally closed,  $H$  has no strongly irredundant representatives of Krull dimension 2 (Proposition 2.3), so since every valuation overring of  $H$  of Krull dimension 1 contains  $\widehat{\mathbb{Q}}_p[X]$ , we conclude that  $H$  has no strongly irredundant  $\widehat{\mathbb{Q}}_p[X]$ -representatives. Since an irredundant  $\widehat{\mathbb{Q}}_p[X]$ -representative that is not a strongly irredundant  $\widehat{\mathbb{Q}}_p[X]$ -representative can be replaced with a proper

overring, the existence of irredundant  $\widehat{\mathbb{Q}}_p[X]$ -representatives implies the existence of strongly irredundant  $\widehat{\mathbb{Q}}_p[X]$ -representatives. Having ruled out the latter possibility, the claim is proved.  $\square$

We collect below a few more examples from other sources. As with the last two examples, these are also interesting instances of Prüfer overrings of two-dimensional Noetherian domains.

**(4.7)** Let  $K$  be a field of characteristic 0 that is not algebraically closed, let  $D = K[X, Y]$  and let  $H$  be the intersection of all the valuation overrings of  $D$  having residue field  $K$ . Then  $H$  is a Prüfer domain of Krull dimension 2 having no irredundant representatives (apply [6, Lemma 1.6 and Theorem 1.7] and [20, Theorem 4.7(i)]). This ring is also interesting in that it is a Hilbert ring such that for each nonzero proper finitely generated ideal  $I$  of  $H$ , there exist for each  $h = 1, 2$  and  $d = 0, 1$ , infinitely many prime ideals minimal over  $I$  of height  $h$  and dimension  $d$  [20, Proposition 3.11 and Theorem 4.7]. All these claims remain true if  $D$  is assumed to be a two-dimensional affine  $K$ -domain such that  $K$  is existentially closed in the quotient field of  $D$ ; moreover, similar results hold in higher dimensions; see [20]. See also [12] for a way to create similar examples with no restriction on whether  $K$  is algebraically closed, and using all the valuation overrings of  $D$ , not just those with residue field  $K$ : the caveat is that these valuation overrings must be extended to  $F(T)$ , where  $F$  is the quotient field of  $D$  and  $T$  is an indeterminate for  $F$ . But the ring so created remains an overring of a two-dimensional Noetherian domain, namely,  $D(T)$ , where  $D(T)$  is the Nagata function ring of  $D$ .

**(4.8)** This example, which is taken from [22], gives a somewhat natural construction of a Prüfer overring of a two-dimensional Noetherian domain that has a strongly irredundant representation but does not have a Noetherian representation. Let  $K$  be a field that is not algebraically closed, and let  $D = K[X, Y]$ . We recall the notion of an order valuation: Let  $\mathfrak{m}$  be a maximal ideal of  $D$ , and define a mapping  $\text{ord}_{\mathfrak{m}} : D_{\mathfrak{m}} \rightarrow \mathbb{Z} \cup \{\infty\}$  by  $\text{ord}_{\mathfrak{m}}(0) = \infty$  and  $\text{ord}_{\mathfrak{m}}(f) = \sup\{k : f \in \mathfrak{m}^k\}$  for all  $f \in D_{\mathfrak{m}}$ . Since  $D_{\mathfrak{m}}$  is a regular local ring, the mapping  $\text{ord}_{\mathfrak{m}}$  extends to a rank one discrete valuation (the *order valuation* with respect to  $\mathfrak{m}$ ) on the quotient field of  $D$ . Let  $E$  be a subset of  $K^2$ . Then the *order holomorphy ring with respect to  $E$*  is the ring  $H = \bigcap_{p \in E} V_p$ , where for each  $p = (a, b) \in E$ ,  $V_p$  is the order valuation ring of  $D_{(X-a, Y-b)}$ . The representation  $\{V_p : p \in E\}$  of  $H$  is strongly irredundant [22, Theorem 2.3], yet if  $E$  is chosen so that it intersects with some algebraic set in  $K^2$  in infinitely many points, then  $H$  does not have a Noetherian representation. The reason is that  $H$  is necessarily an *almost Dedekind domain* (that is,  $H_M$  is a DVR for each maximal ideal  $M$ ) [22, Theorem 2.6], yet if  $H$  has a Noetherian representation, then by Theorem 3.14,  $\text{Spec}(H)$  is a Noetherian space, so that necessarily  $H$  is a Dedekind domain. But then every element of  $D$  is contained in at most finitely many maximal ideals in  $\{(X - a, Y - b)D : (a, b) \in E\}$ , which is impossible if  $E$  meets some algebraic set in infinitely many points. In fact, there is a one-to-one correspondence between the irreducible algebraic sets that  $E$  meets in infinitely many points and the valuation overrings of  $H$  that are not strongly irredundant representatives [22, Theorem 2.6].

(4.9) Here is another example, taken from Example 6.3 of [18], that like (4.8) shows a strongly irredundant representation is not enough in general to guarantee the existence of a Noetherian representation. Let  $K$  be an infinite field that is not algebraically closed, and let  $D$  be a two-dimensional regular local ring with maximal ideal  $\mathfrak{m}$  such that  $K \subseteq D$  and  $K = D/\mathfrak{m}$ . Let  $x, y \in \mathfrak{m}$  be such that  $\mathfrak{m} = (x, y)D$ , and define  $B = D[x/y]$ . Then  $B/yB$  is isomorphic to the polynomial ring  $K[Z]$ , where  $Z$  is an indeterminate for  $K$ . Hence, since  $K$  is an infinite field, there exists an infinite collection  $\mathcal{P}$  of maximal ideals of  $B$  such that for each  $\mathfrak{n} \in \mathcal{P}$ ,  $\mathfrak{m} \subseteq \mathfrak{n}$  and  $B/\mathfrak{n} = K$ . For each  $\mathfrak{n} \in \mathcal{P}$  choose a valuation overring  $V$  of  $B$  such that  $\mathfrak{P}_V \cap B = \mathfrak{n}$ , where  $\mathfrak{P}_V$  is the height one prime ideal of  $V$ , and the residue field of  $V$  is  $K$ . As discussed in [18], one may in fact choose  $V$  to be of Krull dimension 1 or 2. Let  $\Sigma$  be the collection of these valuation rings, one for each member of  $\mathcal{P}$ , and define  $H = \bigcap_{V \in \Sigma} V$ . Then, as is shown in Example 6.3 of [18],  $H$  is a Prüfer domain, and  $\Sigma$  is a strongly irredundant representation of  $H$ , but there does not exist a Noetherian representation of  $H$ .

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# Almost perfect domains and their modules

Luigi Salce

**Abstract** The discussion of almost perfect domains and their modules presented in this survey paper is divided in two parts. In the first part the ring-theoretical properties of almost perfect domains are investigated, and different possibilities concerning the behaviour of their archimedean valuation overrings are shown. Connections of almost perfect domains with other well known classes of integral domains are established. The second part illustrates the influence of the property of a domain  $R$  of being almost perfect on the category of  $R$ -modules, focusing on the subcategories of torsion modules, flat modules, and divisible modules. A final section summarizes the module-theoretical results in the frame of the cotorsion pairs. The two parts are separated by a section containing concrete examples of different kinds of almost perfect domains. A final section on open questions concludes the paper.

## 1 Introduction

Almost perfect domains emerged in the investigation of the existence of strongly flat covers of modules over commutative integral domains, that Silvana Bazzoni and the author undertook in 2001, trying to answer the third of the five questions posed by Jan Trlifaj in the notes of the Cortona 2000 workshop (see [37, p. 36]). The main result obtained in [5] was the proof of the equivalence of the following conditions for a commutative integral domain  $R$ :

- (i) Every  $R$ -module has a strongly flat cover.
- (ii) Every flat module is strongly flat.
- (iii) Every proper homomorphic image of  $R$  is a perfect ring.

We called the domains satisfying condition (iii) *almost perfect*, in analogy with other classes of domains defined in terms of a property satisfied by all their proper factors,

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e.g., almost maximal, almost henselian, almost Bézout or almost Dedekind domains (see [15]). Actually, in the local case almost perfect domains have been investigated 40 years ago by J. R. Smith [35], under the name of domains with topologically  $T$ -nilpotent radical (for short,  $TTN$ -domains). As in the case of almost maximal domains (see [10] or [15]), the passage to the global case involves the property of  $h$ -locality.

Some further investigations of almost perfect domains have been performed since 2001, both of ring-theoretical and module-theoretical type, mainly by Silvana Bazzoni, Paolo Zanardo and the author, in different combinations (see the references). Recently, Fuchs and Lee [14], the author [30], and Bazzoni[3] found more characterizations of almost perfect domains in terms of weak-injective and divisible modules, which gave more consistency to the theory.

This survey paper is the first attempt to present in an organized form a large portion of the results obtained up to now on almost perfect rings. The paper is well balanced between ring theory and module theory. Sections 1–4 are devoted to describe the ring-theoretical properties of almost perfect rings and their connections with other well known classes of integral domains. Section 5 acts as a watershed between rings and modules, providing numerous examples of almost perfect domains with different features. Sections 6–8 consider three classes of modules whose structure is heavily influenced by the property of the domain of being almost perfect, that is, torsion modules, flat and divisible modules. Actually, specific properties of each one of these three classes provide a characterization of almost perfect domains. Section 9 includes the results of the previous two sections in the frame of cotorsion pairs. Finally, Section 10 collects some open questions, both of ring-theoretical and module-theoretical nature.

We state here the theorem collecting the main characterizations of almost perfect domains that will be proved in different sections, where the definitions of the involved notions will be given.

**Main Theorem.** *For an integral domain  $R$  with field of quotients  $Q \neq R$  the following conditions are equivalent:*

- (1)  $R$  is almost perfect.
- (2)  $R$  is  $h$ -local and every localization of  $R$  at a maximal ideal is almost perfect.
- (3)  $Q/R$  is semiartinian and isomorphic to  $\bigoplus_P(Q/R_P)$ , where  $P$  ranges over the maximal spectrum of  $R$ .
- (4) Every torsion  $R$ -module  $T$  is semiartinian and isomorphic to  $\bigoplus_P T_P$ , where  $P$  ranges over the maximal spectrum of  $R$ .
- (5) Every flat  $R$ -module is strongly flat.
- (6) Every weakly cotorsion  $R$ -module is cotorsion.
- (7) Every  $R$ -module of weak dimension  $\leq 1$  has projective dimension  $\leq 1$ .
- (8) Every divisible  $R$ -module is weak injective.  $\square$

Conditions (5) and (6) can be expressed by saying that the cotorsion pair  $(\mathcal{SF}, \mathcal{WC})$ , consisting of strongly flat and weakly cotorsion modules, coincides with the cotorsion pair  $(\mathcal{F}, \mathcal{C})$ , consisting of flat and cotorsion modules. Similarly, conditions (7) and (8) can be expressed by saying that the cotorsion pair  $(\mathcal{F}_1, \mathcal{WJ})$ ,

consisting of modules of flat dimension  $\leq 1$  and weak injective modules, coincides with the cotorsion pair  $(\mathcal{P}_1, \mathcal{D})$ , consisting of modules of projective dimension  $\leq 1$  and divisible modules.

In this paper, we cannot provide long and technical proofs. However, we will present some short and simple proofs that help the understanding of the subject, and some details for certain examples. Some topics are not covered, as universal test modules for strong flatness, whose existence in ZFC characterizes almost perfect domains among Matlis domains (see [16]). Some other topic is just touched on, for instance, the three topologies investigated in [28] and [29], especially the Prüfer topology and its connections with the commutativity of the endomorphism ring of the minimal injective cogenerator in the local case, and with the classical rings (see Vámos [38]), are briefly discussed at the end of Section 6. Our basic reference text for modules over commutative integral domains is the monograph [15].

## 2 Commutative perfect rings and almost perfect rings

The origin of our topic dates back to the 1960s, when perfect rings have been introduced by H. Bass [2] in order to characterize the rings all whose modules admit a projective cover. Actually, often and also here, the property that every left module has a projective cover is taken as definition of left-perfect rings.

Among the many characterizations of left-perfect rings, it is worthwhile to recall the following one:  $R/J$  is semisimple (where  $J$  denotes the Jacobson radical of  $R$ ) and  $J$  is a left  $T$ -nilpotent ideal. Recall that a left ideal  $I$  of a ring  $R$  is left  $T$ -nilpotent if, given any sequence  $\{x_n\}_{n \in \mathbb{N}}$  of elements in  $I$ , there exists an index  $n \in \omega$  such that the product  $x_1 \cdot \dots \cdot x_n$  vanishes. A  $T$ -nilpotent ideal is obviously contained in the nilradical, hence in the Jacobson radical, of  $R$ .

In this paper, we will consider commutative rings only, so the term “ring” will always mean “commutative ring”.

Since the property of the ideals of being  $T$ -nilpotent plays a crucial role in the whole theory of perfect and almost perfect rings, it is convenient to furnish the following useful characterization of  $T$ -nilpotency, whose proof can be found for instance in [21, Theorem 23.16]. It shows that the multiplication by a  $T$ -nilpotent ideal produces the same effect on an arbitrary  $R$ -module as the multiplication by an arbitrary ideal contained in the Jacobson radical produces on a finitely generated  $R$ -module, according to the Nakayama’s Lemma.

**Proposition 2.1.** *For an ideal  $I$  of a commutative ring  $R$  the following conditions are equivalent:*

- (1)  $I$  is  $T$ -nilpotent.
- (2) if  $M$  is an  $R$ -module, then  $IM = M$  implies that  $M = 0$ .
- (3) if  $N$  is a submodule of an  $R$ -module  $M$  such that  $IM + N = M$ , then  $N = M$ .
- (4) if  $M$  is an  $R$ -module, then  $M[I] = 0$  implies that  $M = 0$ .  $\square$

In Proposition 2.1(4), as usual, we set  $M[I] = \{x \in M \mid Ix = 0\}$ .

Recall that by a local (respectively, semilocal) ring we mean a commutative ring, not necessarily Noetherian, with only a unique maximal ideal (respectively, with finitely many maximal ideals), and that a module is semiartinian if every non-zero quotient of it has non-zero socle, that is, there are non-zero elements whose annihilator ideal is a maximal one.

The following theorem collects the most useful characterizations of perfect commutative rings; for its proof see [2, Theorem P], [21, Theorems 23.20 and 23.24] and [36, Proposition 5.1].

**Theorem 2.2.** *For a commutative ring  $R$  the following conditions are equivalent:*

- (1)  $R$  is a perfect ring.
- (2)  $R$  satisfies the DCC on principal ideals.
- (3)  $R$  is a finite direct product of local rings with  $T$ -nilpotent maximal ideals.
- (4)  $R$  is semilocal and every localization of  $R$  at a maximal ideal is a perfect ring.
- (5)  $R$  is semilocal and semiartinian.
- (6) All flat  $R$ -modules are projective.
- (7) The class of projective  $R$ -modules is closed under direct limits.  $\square$

As a direct consequence of point (3) we deduce the following

**Corollary 2.3.** *A commutative integral domain is perfect if and only if it is a field.*  
 $\square$

Corollary 2.3 shows that only rings with non-zero zero-divisors furnish non-trivial examples of perfect rings, being the fields considered “trivial” examples. A homomorphic image of a perfect ring is still perfect, by Theorem 2.2(5), as both the properties of being semilocal and semiartinian are inherited by homomorphic images. It is an interesting fact that a ring with non-zero zero-divisors is perfect provided all its proper homomorphic images are perfect.

**Proposition 2.4.** *Let  $R$  be a ring such that all its proper homomorphic images are perfect. If  $R$  is not a domain, then  $R$  is perfect.*

*Proof.* We just sketch the proof; a detailed demonstration can be found in [6]. First note that  $R$  is 0-dimensional since, if  $L$  is a non-zero prime ideal, then  $R/L$  is a perfect domain, hence a field. Then one proves that  $R$  is semilocal, so a reduction to the local case is possible by Theorem 2.2(4). For  $R$  a 0-dimensional local ring with maximal ideal  $P$ , one can show that, if  $R/aR$  is perfect for some element  $a \in P$ , then  $R/a^2R$  is perfect. Then the conclusion easily follows, since  $P$  is the nilradical of  $R$ .  $\square$

Proposition 2.4 motivates the following

**Definition.** A ring  $R$  is said to be *almost perfect* if all its proper homomorphic images are perfect.

So Proposition 2.4 says that almost perfect rings which are not domains are indeed perfect rings. This result is reminiscent of similar results proved for almost-maximal valuation rings (see [17] or [15, Proposition 6.4, Chap. II]), or almost-Henselian local rings (see [39] or [15, p. 85]).

Proposition 2.4 also enables us to restrict the investigation of almost perfect rings and their modules to integral domains, as perfect commutative rings and their modules have been investigated since a long while (for a comprehensive exposition of this subject we refer to [21, Sections 23 and 24]). Thus, from now on, the objects of our investigation will be the almost perfect domains, that we will denote for short by APD.

### 3 Intrinsic properties of APDs

Let  $R$  be a domain with field of quotients  $Q \neq R$ . Recall that  $R$  is a Matlis domain if  $\text{p.d.}_R(Q) = 1$ , and that this property is equivalent to the fact that every divisible module is  $h$ -divisible, i.e., a quotient of an injective module (see [15, Theorem 2.8, Chap. VII]).

A domain  $R$  is said to be  $h$ -local if it satisfies the following two conditions:

- (i) Every proper factor ring of  $R$  is semilocal.
- (ii) Every proper factor domain of  $R$  is local.

Obviously, for 1-dimensional (i.e., of Krull dimension 1) domains  $h$ -locality can be checked just looking at condition (i), since condition (ii) is automatically satisfied. A ring  $R$  satisfying condition (i) is said to have *finite character*.

**Lemma 3.1.** *If  $R$  is a local 1-dimensional domain with maximal ideal  $P$ , then, for every  $0 \neq a \in P$ ,  $Q/R = \bigcup_n a^{-n}R/R$ .*

*Proof.* An ideal  $L$  of  $R$  maximal with respect to  $L \cap \{a^n\}_{n \in \mathbb{N}} = \emptyset$  is a prime ideal, hence  $L = 0$ . Therefore, for every  $0 \neq r \in R$  there exists an  $n \geq 1$  such that  $a^n \in rR$ , equivalently,  $r^{-1} \in a^{-n}R$ . The conclusion easily follows.  $\square$

Note the difference, focused by the proof of Lemma 3.1, between local 1-dimensional domains and local almost perfect domains: the former domains satisfy the property that, fixed  $0 \neq r \in R$  and called  $P$  the maximal ideal, given a sequence  $\{a_n\}_{n \in \mathbb{N}}$ , with  $a_n$  constantly equal to a fixed element  $a \in P$ , there exists a  $k$  such that the product  $a_1 \cdot \dots \cdot a_k = a^k \in rR$  (we say that  $P$  is almost nil); the latter domains satisfy the same property but for an arbitrary sequence  $\{a_n\}_{n \in \mathbb{N}}$  contained in  $P$ , that is,  $P$  is almost  $T$ -nilpotent. This difference is clear if one considers valuation domains.

*Example 3.2.* Let  $R$  be a 1-dimensional valuation domain. Then  $R$  is an APD if and only if  $R$  is discrete. The sufficiency being obvious, assume that  $R$  is non-discrete. The value group  $\Gamma$  of  $R$  is a dense subgroup of the reals, so it contains a sequence of positive elements  $\gamma_n$  ( $n \geq 1$ ) such that  $\gamma_n < 2^{-n}$ . If the  $a_n$  ( $n \geq 1$ ) are elements of  $R$  of value  $\gamma_n$ , then every product  $a_1 \cdot \dots \cdot a_k$  has value  $< 1$ , so  $R$  cannot be almost perfect.  $\square$

The property of the maximal ideal  $P$  of a local domain of being almost nilpotent (that is,  $P^n \leq I$  for every non-zero ideal  $I \leq R$ , for some  $n$  depending on  $I$ ) is stronger than almost  $T$ -nilpotency, hence domains satisfying this property are necessarily APDs. We anticipate in the next Lemma 3.3 a characterization of these domains, that will be used especially in Section 6. We need the notion of Loewy series of a module  $M$ , which is the transfinite ascending sequence of submodules

$$0 = L_0(M) \leq L_1(M) \leq \dots \leq L_\lambda(M) \leq \dots$$

defined inductively as follows:  $L_{\alpha+1}(M)$  is the inverse image in  $M$  of the socle of  $M/L_\alpha(M)$ , and, if  $\beta$  is a limit ordinal,  $L_\beta(M)$  is the union of the  $L_\alpha(M)$  for  $\alpha < \beta$ . By cardinality reasons, there exists a first ordinal  $\lambda$  such that  $L_\lambda(M) = L_{\lambda+1}(M)$ , and this ordinal is the *Loewy length* of  $M$ , denoted by  $l(M)$ . The module  $M$  is semiartinian if and only if  $M = L_\lambda(M)$  for  $\lambda = l(M)$ .

**Lemma 3.3.** *For a local APD  $R$  with maximal ideal  $P$  the following conditions are equivalent:*

- (1)  $P$  is almost nilpotent.
- (2) There exists an element  $0 \neq a \in P$  such that  $P^n \leq aR$  for some  $n$ .
- (3) The Loewy length  $l(Q/R)$  of  $Q/R$  is  $\omega$ .  $\square$

The proof of the lemma is straightforward.

Thus, for a maximal ideal  $P$  of a local domain  $R$ , we have the following implications, none of which can be reversed:

almost nilpotent  $\Rightarrow$  almost  $T$ -nilpotent ( $R$  APD)  $\Rightarrow$   
 almost nil ( $R$  1-dimensional).

Lemma 3.1 shows that, for a local 1-dimensional domain  $R$ , the field of quotients  $Q$  is countably generated, hence, by a well known result,  $R$  is a Matlis domain. This fact extends to  $h$ -local domains.

**Lemma 3.4.** *Let  $R$  be a 1-dimensional  $h$ -local domain. Then  $R$  is a Matlis domain.*

*Proof.* We must show that  $\text{p.d.}_R Q = 1$ , equivalently, that  $\text{p.d.}_R(Q/R) = 1$ . The  $h$ -locality of  $R$  is equivalent (by [24], see also [15, Theorem 3.7, Chap. IV]) to the direct decomposition  $Q/R \cong \bigoplus_P(Q/R_P)$ , where  $P$  ranges over  $\text{Max}(R)$ . Since  $R_P$  is local and 1-dimensional, by Lemma 3.1 we have that, for every  $0 \neq x \in PR_P$ ,  $Q/R_P = \bigcup_n x^{-n}R_P/R_P$ . In particular,  $Q/R_P$  is countably generated as  $R_P$ -module, so that  $\text{p.d.}_{R_P}(Q/R_P) = 1$ . The  $h$ -locality of  $R$  implies that  $R$ -submodules of torsion  $R_P$ -modules are  $R_P$ -modules too (see [15, Lemma 3.8, Chap. IV]), hence the elements  $x^{-n}$  ( $n \geq 0$ ) generate  $Q/R_P$  also as  $R$ -module. Consequently  $\text{p.d.}_R(Q/R_P) = 1$  for every maximal ideal  $P$ , therefore  $\text{p.d.}_R(Q/R) = \text{p.d.}_R(\bigoplus_P(Q/R_P)) = 1$ .  $\square$

The assumption that  $R$  is  $h$ -local in Lemma 2.4 is essential. In fact, Olberding proved in [26], as a byproduct of the development of the notion of ‘‘Prüfer sections’’ of Noetherian domains, that there exists an almost Dedekind domain  $R$  that is not a Matlis domain. Note that such a domain  $R$  satisfies the property that

every localization  $R_P$  at a maximal ideal  $P$  is a DVR, hence an APD, but  $R$  itself is not an APD, in view of the next result.

**Proposition 3.5.** *Let  $R$  be an APD. Then  $R$  is a 1-dimensional,  $h$ -local and Matlis domain.*

*Proof.* Let  $L$  be a non-zero prime ideal of  $R$ . Then  $R/L$  is a perfect domain, hence a field, by Corollary 2.3. Thus,  $L$  is maximal and consequently  $R$  is 1-dimensional.  $R$  is  $h$ -local, since every proper factor ring is semilocal, by Theorem 2.2. The property of being Matlis follows from Lemma 3.4.  $\square$

We are now in the position to prove the first characterization of APDs in the Main Theorem. As usual for other properties, a domain  $R$  is said to be *locally almost perfect* if every localization  $R_P$  at a maximal ideal  $P$  is an APD.

**Theorem 3.6.** *A domain  $R$  is an APD if and only if it is  $h$ -local and locally almost perfect.*

*Proof.* Assume  $R$  an APD. It is  $h$ -local by Proposition 3.5. If  $P$  is a maximal ideal of  $R$ , let  $J$  be a non-zero proper ideal of  $R_P$ . Then  $J = IR_P$  for some non-zero ideal  $I \leq P$  of  $R$ . But  $R_P/IR_P \cong R/I \otimes_R R_P \cong (R/I)_{P/I}$ , and the last ring, as a localization at the maximal ideal  $P/I$  of the perfect ring  $R/I$ , is also perfect, by Theorem 2.2(4).

Conversely, assume  $R$   $h$ -local and locally almost perfect. Then, fixed an element  $0 \neq r \in R$ ,  $R/rR \cong \bigoplus_P (R_P/rR_P)$ , where  $P$  ranges over the finite set of the maximal ideals containing  $r$ . Hence,  $R/rR$  is a finite product of local perfect rings, hence the conclusion follows from Theorem 2.2(3).  $\square$

We will show now that there exist locally almost perfect domains which fail to be  $h$ -local. The construction of the Bézout domain  $R$  in the next Example 3.7 is due to Heinzer–Ohm [20], the remark that every cyclic torsion  $R$ -module is semiartinian is due to Bazzoni–Salce [6].

*Example 3.7.* Consider the lattice-ordered group  $\mathbb{Z}^{\mathbb{N}}$  with the pointwise ordering. Let  $\Gamma$  be its subgroup consisting of the eventually constant sequences.  $\Gamma$  properly contains the direct sum of the  $\mathbb{Z}$ 's and becomes a lattice-ordered group under the induced ordering. Let  $R$  be a Bézout domain with divisibility group  $\Gamma$  (such a domain does exist by the Kaplansky–Jaffard–Ohm theorem, see [15, Theorem 5.3, Chap. III]). The maximal ideals of  $R$  are the ideals  $P_n$  ( $n \in \mathbb{N}$ ), corresponding to the filters  $F_n$  consisting of the vectors in  $\Gamma^+$  whose  $n$ -th coordinate is  $> 0$ , and one more, denoted by  $P_\infty$ , corresponding to the filter  $F_\infty$  consisting of the vectors in  $\Gamma^+$  that are eventually strictly positive.  $R$  has no other non-zero prime ideals, hence it is 1-dimensional. All the localizations at maximal ideals of  $R$  are rank one discrete valuation domains, so  $R$  is an almost Dedekind domain, hence, in particular,  $R$  is locally almost perfect, by Example 3.2.  $R$  is not Noetherian, since the ideal  $P_\infty$  is not finitely generated, and it is not  $h$ -local, because the principal ideal of  $R$  corresponding to the element  $(1, 1, 1, \dots) \in \Gamma$  is contained in all the maximal ideals of  $R$ . Finally, it is not difficult to show that  $R$  satisfies also the condition that, for every non-zero ideal  $I$ , the cyclic  $R$ -module  $R/I$  has non-zero socle.  $\square$



It is worthwhile to observe that the domain  $R$  in Example 3.7 satisfies the condition that the  $R$ -module  $Q/R$  is semiartinian, as one can derive from the next Proposition 7.1. This condition implies that  $R$  is locally almost perfect. The converse is not true, as shown in [13], where the locally almost perfect domains  $R$  with  $Q/R$  semiartinian are characterized in Theorem 5.2. Furthermore, examples of almost Dedekind domains (hence, in particular, locally almost perfect domains) are provided which fail to have  $Q/R$  semiartinian (see Example 5.5 in [13]).

Theorem 3.6 reduces the investigation of APDs to the local case. Actually, as recalled in the Introduction, local APDs have been introduced by J. R. Smith [35] under the name of domains with TTN (that is, with topologically  $T$ -nilpotent radical).

## 4 Valuation overrings of APDs

Let  $R$  be a local domain with field of quotients  $Q$  and maximal ideal  $P$ . A local ring  $V$  with maximal ideal  $M$  is said to *dominate*  $R$  if  $R \subseteq V \subseteq Q$  and  $M \cap R = P$ . A classical result [9, p. 92] states that every local domain  $R$  is dominated by a valuation domain  $V$ . Recall that a 1-dimensional valuation domain is called archimedean. J. R. Smith improved the above result [35, p. 235] by proving the following:

**Theorem 4.1.** *Let  $R$  be a 1-dimensional local domain with field of quotients  $Q$  and maximal ideal  $P$ . Then there exists an archimedean valuation domain  $V$  dominating  $R$ . Furthermore, if  $R$  is an APD,  $V$  necessarily satisfies the condition that the elements in  $P$  have value (in the valuation of  $V$ ) larger or equal to a positive element  $\gamma$  of the value group of  $V$ .  $\square$*

The next result is a useful consequence of the last claim in Theorem 4.1; it states that the  $P$ -adic topology on a local APD with maximal ideal  $P$  is Hausdorff.

**Corollary 4.2.** *Given a local APD with maximal ideal  $P$ ,  $\bigcap_n P^n = 0$ .*

*Proof.* Using the notation of Theorem 4.1, an element in  $P^n$  has value  $\geq n \cdot \gamma$ ; so if  $r \in \bigcap_n P^n$ , then its value is  $\geq n \cdot \gamma$  for all  $n \geq 1$ , hence it is  $\infty$ , consequently  $r = 0$ .  $\square$

One could ask when there exists a unique archimedean valuation domain dominating a local APD. The answer is given in the following result, whose proof, valid for arbitrary 1-dimensional local domains, can be found in [40, Theorem 3.2 and Proposition 3.1].

**Theorem 4.3.** *Let  $R$  be a local APD. The following conditions are equivalent:*

- (1) *There exists a unique archimedean valuation domain  $V$  dominating  $R$*
- (2) *The integral closure  $\bar{R}$  of  $R$  contains an ideal of  $V$ .*

*If these equivalent conditions hold, then  $\bar{R}$  is local and shares the maximal ideal with  $V$ , hence  $\bar{R}$  is a pseudo-valuation domain.  $\square$*

Instead of starting, as in Theorem 4.1, with a local almost perfect domain and look for archimedean valuation overrings dominating it, we can consider the converse process, namely, start with an archimedean valuation domain  $V$  and look for local subrings  $R$  dominated by  $V$  which are almost perfect. Zanardo obtained [40, Theorem 3.1] the following satisfactory result.

**Theorem 4.4.** *Let  $V$  be an archimedean valuation domain. Then there exists an APD dominated by  $V$ .*

*Proof.* We can assume that  $V$  is not discrete, otherwise we can set  $R = V$ , by Example 3.2. Let  $Q$  be the field of quotients of  $V$ ,  $v$  the valuation on  $Q$  determined by  $V$ , and  $P$  the maximal ideal of  $V$ . The construction of an APD  $R$  inside  $V$  is of two different types, depending on whether  $V$  contains a field  $K$  or not. Note that the first case always happens in case  $V$  has positive characteristic. We skip the details of the proof, for which we refer to [40, Theorem 3.1], and just indicate how the ring  $R$  is defined.

Assume first that  $V$  contains the field  $K$ . Choose an element  $z \in P$  and set  $R = K + zV$ . Then one can prove that  $R$  is an APD with unique maximal ideal  $zV$ , and that  $V$  dominates  $R$ . Then assume that  $V$  does not contain a field, so  $V$  has characteristic 0 and contains, up to isomorphism, the ring  $\mathbb{Z}$  of the integers. There exists a unique prime integer  $p$  belonging to  $P$ . Consider the multiplicative part of  $V$ :  $S = \{m + px \mid m \in \mathbb{Z} \setminus p\mathbb{Z}, x \in V\}$ . The subring of  $V$ :

$$R = \{(n + py)/s \mid n \in \mathbb{Z}, y \in V, s \in S\}$$

is an APD with  $p\mathbb{Z}$  as unique maximal ideal, and is still dominated by  $V$ .  $\square$

Note that in both the constructions in the proof of Theorem 4.4 the maximal ideal  $M$  of  $R$  not only satisfies the condition stated by Smith in Theorem 4.1 (namely,  $v(M) \geq \gamma$ , where  $\gamma$  is a positive element of the value group of  $V$ ), but also the additional condition that it contains (actually it equals) an ideal of  $V$  properly contained in  $P$ . This last condition has been shown by J. R. Smith to suffice to force a local subring  $R$  of  $V$  (with maximal ideal  $M$  such that  $v(M) \geq \gamma > 0$ ) to be an APD. However, the condition is not necessary, as the next example shows (see [40, Example 3.1]), which is a slight generalization of [6, Example 3.8]). For another example see [35, pp. 236–238].

*Example 4.5.* Let  $M_1 < M_2 < M_3 < \dots$  be a strictly increasing sequence of positive integers. Let us define a sequence of additive subsemigroups of the rational numbers as follows:

$$\Sigma_0 = \{0\}; \quad \Sigma_n = \{m/2^n \mid m \in \mathbb{N}, m/2^n \geq M_n\}, \quad n \geq 1.$$

Each  $\Sigma_n$  is a submonoid of  $\Sigma_{n+1}$  and  $\Sigma = \bigcup_n \Sigma_n$  is a semigroup generating the group  $\Gamma = \{m/2^n \mid m \in \mathbb{Z}, n \in \mathbb{N}\}$ . Let  $F$  be a field of positive characteristic and  $K$  a purely transcendental extension of  $F$ . Let  $K[\Sigma]$  and  $K[\Gamma]$  denote the rings of formal polynomials over  $K$  with exponents respectively in the semigroups  $\Sigma$  and  $\Gamma^+$ . For every  $f \in K[\Gamma]$ , denote by  $f(0)$  its constant term.

Let us consider now the two subrings of  $K[\Gamma]$ :

$$V = \{f/g \mid f, g \in K[\Gamma], g(0) = 1\}$$

$$R = \{f/g \mid f, g \in K[\Sigma], f(0) \in F, g(0) = 1\}$$

and denote by  $Q$  their common field of fractions. Then one can show that

- $R$  is an almost perfect domain (the proof is not easy; one needs to use the special properties of  $\Sigma$ ; see ([6, Lemma 3.6]).
- $R$  is dominated by the valuation domain  $V$ .
- $R$  does not contain an ideal of  $V$ ; this depends on the fact that there are arbitrarily large elements of  $\Gamma^+$  which do not belong to  $\Sigma$ ; in fact, fixed  $0 < h \in \mathbb{N}$ , choose  $n \geq 2$  such that  $M_n > h$ . Then there exists an odd natural number  $m$  satisfying  $2^n < m < 2^n M_n$ . It is immediate to check that  $m/2^n \notin \Sigma$ .

From Theorem 4.3 we deduce that the ring  $V$  is the unique archimedean valuation domain dominating  $R$ . In fact, the integral closure of  $R$  is the ring  $\bar{R} = \{f/g \mid f, g \in K[\Gamma], f(0) \in F, g(0) = 1\}$ , which contains the maximal ideal  $P$  of  $V$ .  $\square$

Up to now we have not seen examples of APDs dominated by more than one archimedean valuation domain. If we look for finitely many of them, the following lemma is very useful.

**Lemma 4.6.** *Let  $V_1, \dots, V_n$  be archimedean valuation domains with the same field of quotients  $Q$ , and with respective maximal ideals  $P_1, \dots, P_n$ . Then there exists a non-zero element  $z \in P_1 \cap \dots \cap P_n$ .  $\square$*

Lemma 4.6 can be readily proved by an application of the Approximation Theorem (see [25]), or directly using induction (see [40, Lemma 3.3]); it is the starting point in the following construction. Take the archimedean valuation domains  $V_1, \dots, V_n$  and let  $0 \neq z \in P_1 \cap \dots \cap P_n$  be the element, whose existence is ensured by the lemma. Assume now, as in the first case of the proof of Theorem 4.3, that all the domains  $V_i$  contain the same field  $K$ ; generalizing the construction of that proof, consider the ring  $R = K + (zV_1 \cap \dots \cap zV_n)$ . In this notation, one can prove the following result (see [40, Theorem 4.1]).

**Theorem 4.7.** *The domain  $R = K + (zV_1 \cap \dots \cap zV_n)$  is an APD with field of quotients  $Q$ , and the rings  $V_1, \dots, V_n$  are exactly the archimedean valuation domains in  $Q$  dominating  $R$ .  $\square$*

To complete the panorama, we need an example of an APD with infinitely many archimedean valuation domains dominating it. In the next example, we sketch the construction of such a domain; for details we refer to [40, Example 4.3].

*Example 4.8.* Let  $Q = K(Y, X_n \mid n \in \mathbb{N})$ , where  $K$  is an arbitrary field and  $Y$  and the  $X_n$ 's are indeterminates. For all integers  $i$  set  $D_i = K(X_n \mid n \neq i)[Y, X_i]$ , and let  $v_i$  be the discrete valuation on  $Q$  defined in the following way: if  $f(Y, X_i)$  is a non-zero polynomial in  $Y$  and  $X_i$  with coefficients in  $K(X_n \mid n \neq i)$ , set  $v_i(f)$  to be the minimal

degree of the homogeneous components of  $f$ . Then  $v_i$  can be extended to a rank one discrete valuation of  $Q$ . Let  $V_i$  be the corresponding valuation domain, with maximal ideal  $P_i$ . Set  $M = \bigcap_{n \in \mathbb{N}} P_n$  and consider the ring  $R = K + M$ . Then one can prove that  $R$  is an APD and that every valuation overring  $V_n$  dominates  $R$ . It turns out that  $R$  fails to be integrally closed.  $\square$

## 5 Connections with other classes of domains

For people working in module theory, three of the most important classes of rings are the class of coherent rings and its subclasses of Prüfer and Noetherian rings. So our first goal is to compare APDs with coherent and Noetherian domains.

**Proposition 5.1.** (1) *A 1-dimensional Noetherian domain is an APD.*  
 (2) *A coherent APD is Noetherian.*

*Proof.* (1) If  $R$  is a 1-dimensional Noetherian domain, then, for every non-zero element  $r \in R$ ,  $R/rR$  is artinian, hence perfect. Therefore,  $R$  is an APD.

(2) Let  $R$  be coherent and almost perfect. Since an  $h$ -local domain is Noetherian if and only if it is locally Noetherian, we can assume  $R$  to be local. Pick a non-zero element  $r \in R$ ; then  $R/rR$  is perfect, so it has a minimal ideal isomorphic to  $R/P$ ; by coherency,  $P/rR$  is finitely generated, so the same holds for  $P$ . The conclusion follows by Cohen’s theorem.  $\square$

An immediate consequence is the following

**Corollary 5.2.** *Let  $R$  be a Prüfer domain. Then  $R$  is almost perfect if and only if it is a Dedekind domain.*  $\square$

Thus, the domains which are both Prüfer and almost perfect are necessarily Noetherian. Note however that there exist APDs which are Noetherian but not integrally closed, so not Dedekind (see next Example 6.1).

At this point a natural question is: when is an APD  $R$  Noetherian? As we have seen in the proof of Proposition 5.1, one can consider the local case only, by the  $h$ -locality of  $R$ . In this case, being  $P$  the unique non-zero prime ideal, a trivial answer is, by Cohen’s theorem: exactly when the maximal ideal  $P$  is finitely generated. More information and a slight improvement are in the following proposition, whose proof can be found in [28] and [34, Theorem 5.2].

**Proposition 5.3.** *Let  $R$  be a local APD with maximal ideal  $P$ . Then:*

- (1)  *$R$  is a DVR if and only if  $P$  is principal.*
- (2)  *$R$  is Noetherian if and only if  $P/P^2$  is a finitely generated  $R$ -module.*  $\square$

There exists another characterization of local Noetherian APDs in terms of the overring  $R_1 = P : P$ , where  $P$  is the maximal ideal of  $R$ . Note that, if  $P$  is a principal ideal, then  $R_1 = R$ ; otherwise,  $R_1$  is a fractional ideal of  $R$  and  $R_1/R$  coincides with the socle of  $Q/R$ .

**Proposition 5.4.** *A local APD  $R$  with maximal ideal  $P$  is Noetherian if and only if  $Q/R$  has finitely generated socle and  $R_1 = p_1^{-1}P \cap \cdots \cap p_n^{-1}P$  for suitable elements  $p_i \in P$ .  $\square$*

For the proof see [28, Theorem 2.3.3]. It is an open question (see Question 3) whether the hypothesis that  $Q/R$  has finitely generated socle in Proposition 5.4 implies that  $R_1 = p_1^{-1}P \cap \cdots \cap p_n^{-1}P$  for suitable elements  $p_i \in P$ . The opposite implication is certainly false, as we will see in the Example 6.1.

There are local APDs  $R$ , with almost nilpotent maximal ideal  $P$ , which have a very simple ideal structure, called  *$P$ -chained domains*; they are defined by the additional condition that, for every proper non-zero ideal  $I$ , there exists an  $n$  such that  $P^{n+1} \leq I \leq P^n$ . This is equivalent to say that  $P$  is almost nilpotent and  $pP = P^2$  for every  $p \in P \setminus P^2$ , or that the overring  $R_1 = P : P$  is a DVR with maximal ideal  $P$  (see [29, Proposition 4.1]). It is easy to deduce that, given two incomparable ideals  $I$  and  $J$ , the inclusion  $PJ \leq I$  holds; hence, from a result by Hedstrom–Houston [19, Theorem 1.4], we obtain the following proposition (for the proof of the converse see [29, Proposition 4.3]).

**Proposition 5.5.** *A  $P$ -chained domain  $R$  is a pseudo-valuation domain. If  $R$  is Noetherian, the converse holds.  $\square$*

If we require that in a local domain  $R$  with maximal ideal  $P$ ,  $pP = P^2$  does not hold for every  $p \in P \setminus P^2$ , as in the case of  $P$ -chained domains, but just for a selected  $p \in P \setminus P^2$ , then we obtain a larger class of APDs, called  *$P$ -stable*. This notion agrees with the notion of stability of Eakin–Sathaye [11], and is stronger than the stability notion of Sally–Vasconcelos [33], who define  $P$  to be stable if it is projective in its endomorphism ring.

It is an easy exercise to prove (see [29, Lemma 2.3.5]) that  $R$  is  $P$ -stable exactly if  $P$  is a principal ideal of the overring  $R_1 = P : P$ , and in this case  $R_1/P \cong P^n/P^{n+1}$  for all  $n$ .

**Proposition 5.6.** *A 1-dimensional  $P$ -stable local domain has  $P$  almost nilpotent, hence it is an APD. It is Noetherian if and only if  $Q/R$  has finitely generated socle.  $\square$*

For the simple proof see [29, 2.3.6 and 2.3.7]). An example of  $P$ -stable APD is given below.

*Example 5.7.* Let  $F$  be a field and  $K$  a proper extension of  $F$ . Fix  $n \geq 1$  and set  $S_n = F + FX + FX^2 + \cdots + FX^{n-1} + X^nK[[X]]$ .  $S_n$  is local 1-dimensional with maximal ideal  $P_n = FX + FX^2 + \cdots + FX^{n-1} + X^nK[[X]]$ . Then  $P_n = XS_n$  and  $P_n : P_n = S_{n-1}$  (obviously  $S_0 = K[[X]]$ ). Each  $S_n$  is a  $P_n$ -stable APD, not  $P$ -chained if  $n > 1$ .  $S_0$  is the unique archimedean valuation domain dominating  $S_n$  for each  $n$ .  $\square$

## 6 Examples of APDs

Three main sources of examples of APDs are the  $D + M$  construction, the simple integral extensions of APDs, and the semigroup rings over submonoids of the non-negative real numbers. In this section, we will present examples of each type.

*Example 6.1.* Let  $V = K[[X]]$  be the DVR of the power series over an arbitrary field  $K$ , and let  $F$  be a proper subfield of  $K$ . The ring  $R = F + XK[[X]]$  of the power series with constant term in  $F$  is a local domain with the same maximal ideal  $P = XK[[X]]$  of  $V$ , which fails to be a valuation domain. The ideal  $P$  of  $R$  is almost nilpotent, so  $R$  is an APD, and, by Lemma 3.3, the Loewy length of  $Q/R$  is  $\omega$ .  $R$  is Noetherian if and only if the degree  $[K : F]$  is finite, and it is integrally closed provided  $F$  is algebraically closed in  $K$ . So there exist integrally closed APDs which are not valuation domains. Note that  $(Q/R)[P]$  is infinitely generated if  $[K : F]$  is infinite and  $P : P = X^{-1}P$ . Clearly  $R$  is a  $P$ -chained domain.  $\square$

*Example 6.2.* In this example, we already start with a local APD  $R$  with maximal ideal  $P$ , and construct a larger APD containing  $R$ . If  $Q$  denotes the field of quotients of  $R$ , let  $F$  be a field extension of  $Q$ . Let us assume that there exists an element  $x \in F \setminus Q$  which is integral over  $R$ . Let  $f(X) = X^{n+1} + r_nX^n + \dots + r_1X + r_0$  be a monic polynomial in  $R[X]$  of minimum degree  $n + 1 > 1$  such that  $f(x) = 0$ . Then  $R[x]$  is an APD if and only if all the coefficients  $r_i$  belong to  $P$ . If this happens,  $R[x]$  is local with maximal ideal  $M = P + xR + x^2R + \dots + x^nR$ ;  $R[x]$  is Noetherian if and only if  $R$  is Noetherian. If  $P$  is almost nilpotent, also  $M$  is almost nilpotent. The proof of these facts are in Smith [35, Section 5] and [6, Proposition 3.5]. The property of being  $P$ -chained is not inherited in general passing from  $R$  to  $R[x]$  (see [29, Example 4.6]).  $\square$

A concrete ring obtained as in Example 6.2 is  $R = \mathbb{Z}_p[p\sqrt{p}]$ , where  $\mathbb{Z}_p$  is the localization of the ring of the integers at the prime  $p$ . Its field of quotients  $F = \mathbb{Q}[\sqrt{p}]$  is algebraic over  $\mathbb{Q}$  and  $x = p\sqrt{p}$  is integral over  $\mathbb{Z}_p$ , with minimal polynomial  $X^2 - p^3$ , and  $p^3 \in P = p\mathbb{Z}_p$ . Note that  $\mathbb{Z}_p[p\sqrt{p}]$  is Noetherian but not integrally closed, since  $\sqrt{p} \notin R$  is integral over  $R$ , and the maximal ideal of  $R$  is  $M = p\mathbb{Z}_p + xR$ .

The general technique developed in the next example have been already used in a particular case in Example 4.5.

*Example 6.3.* Let  $\Sigma$  be a submonoid of the additive monoid of the non-negative real numbers, and denote by  $\Sigma^+$  its subset of the positive elements. Given a field  $K$  of positive characteristic  $p$ , consider the semigroup ring  $K[\Sigma]$ , which is a domain, since  $\Sigma$  is cancellative torsionfree. Let  $R$  be the localization of  $K[\Sigma]$  at the maximal ideal  $M$  generated by the elements  $X^\sigma$ , for  $\sigma \in \Sigma^+$ . Denote by  $P$  the maximal ideal of  $R$ , that is,  $P = MK[\Sigma]_M$ . The same construction as above, but using the group  $\Gamma$  generated by  $\Sigma$ , instead of  $\Sigma$ , produces a valuation domain  $V$  dominating  $R$ . If  $\Sigma$  is assumed to satisfy the following property:

- Given  $\sigma > \tau$  in  $\Sigma$ ,  $m(\sigma - \tau) \in \Sigma$  for all integers  $m$  large enough

then the ring  $R$  is 1-dimensional. If, furthermore,  $\Sigma$  is assumed to satisfy the additional property:

- If  $\sigma_1, \dots, \sigma_n \in \Sigma^+$ , there exists  $\sigma_0 \in \Sigma^+$  such that  $\sigma_1 + \dots + \sigma_n - \sigma_0 \in \Sigma$

then the maximal ideal  $P$  of  $R$  is almost nilpotent, hence  $R$  is almost perfect, and is dominated by  $V$ . The positivity of the characteristic of  $K$  is needed to prove that any two principal proper ideals of  $R$  have the same radical, so that  $R$  is 1-dimensional, an argument borrowed by [1]. It is easy to see that, under the above assumptions on  $\Sigma$ ,  $R$  is not  $P$ -chained, but  $P$  is almost nilpotent.  $\square$

All the examples of local APDs  $R$  exhibited in this and in the previous sections satisfy the condition that the maximal ideal  $P$  is almost nilpotent, or, equivalently, by Lemma 3.3,  $Q/R$  has Loewy length  $\omega$ . As we will see in Section 7, the Loewy length of  $Q/R$  is the supremum of the Loewy lengths of all torsion  $R$ -modules, hence  $l(Q/R) = \omega$  implies that the structure of all torsion  $R$ -modules, from the point of view of the Loewy series, is relatively simple. Thus, a crucial question is: are there local APDs such that  $l(Q/R) > \omega$ ?

A simple argument shows that the Loewy length of a divisible module is a limit ordinal. The first example of a local APD  $R$  with  $l(Q/R) = \omega \cdot 2$  was given in [6, pp. 12–16], using the technique illustrated in Example 6.4, that is, constructing a subsemigroup  $\Sigma$  of the non-negative real numbers satisfying particular properties, and then considering the semigroup ring  $K[\Sigma]$  over a field  $K$  of positive characteristic. A completely different example of local APD  $R$  with  $l(Q/R) = \omega \cdot 2$  was constructed in [40, Example 4.4], modifying the definition of the valuations  $v_i$  in Example 4.8.

A more complicated construction was presented in [32], a paper entirely devoted to prove that, for every positive integer  $n$ , there exists a local APD  $R$  such that  $l(Q/R) = \omega \cdot n$ . The construction elaborates on that in [40, Example 4.4], and it is too technical to be illustrated here. We just state the result, for further references.

**Theorem 6.4.** *For every positive integer  $n$  there exists a local APD  $R$  such that  $Q/R$  has Loewy length  $\omega \cdot n$ .  $\square$*

Up to now, we don't have any example of non-Noetherian APD which is not local. The next example fills this gap, providing an APD with uncountably many maximal ideals, all whose localizations are DVR's except one, which is non-Noetherian. Since the example appears here for the first time, we furnish all the details.

*Example 6.5.* Let  $K$  be an algebraically closed field,  $F$  a proper subfield such that  $[K : F] = \infty$ ,  $X$  an indeterminate. Let  $R = F + XK[X]$ . We will prove that  $R$  is a non-Noetherian almost perfect domain with infinitely many maximal ideals, which are  $P = XK[X]$ , and those of the form  $P_a = (1 - aX)R$ , for all  $0 \neq a \in K$ . First note that a polynomial  $f \in K[X]$  lies in  $R$  if and only if  $f(0) \in F$ . We have a canonical factorization of the elements of  $R$ . In fact, since  $K$  is algebraically closed, we have two possibilities:

- (i) If  $f(0) \neq 0$  then we may uniquely write  $f = c \prod_{i=1}^m (1 - a_i X)$ , for suitable  $c \in F$  and  $a_i \in K$
- (ii) If  $f(0) = 0$  then we may uniquely write  $f = bX^k \prod_{i=1}^m (1 - a_i X)$ , for suitable  $k > 0, b \in K$  and  $a_i \in K$  (here  $bX \in R$ ). Note that, presently,  $f \in P$ .

It is clear that  $P$  is a maximal ideal of  $R$ . The fact that  $P_a = (1 - aX)R$  is also a maximal ideal, for every  $0 \neq a \in K$ , follows from the equality  $P_a = (1 - aX)K[X] \cap R$  and from the isomorphisms:

$$R/P_a \cong (R + (1 - aX)K[X]) / (1 - aX)K[X] = K[X] / (1 - aX)K[X] \cong K.$$

Let us show that there are no other maximal ideals. Assume, by way of contradiction, that  $J$  is a maximal ideal different from  $P$  and from all the  $P_a$ . Pick  $0 \neq f \in J$ . We consider the factorization of  $f$ . If  $f = c \prod_{i=1}^m (1 - a_i X)$ , as in (i), then  $J$  prime implies that  $1 - a_h X \in J$ , for some  $h \leq m$ . It follows that  $J$  contains  $P_{a_h}$ , impossible, since  $P_{a_h}$  is maximal. Thus, necessarily,  $f = bX^k \prod_{i=1}^m (1 - a_i X)$ , as in (ii). It follows that  $f \in P$ ; since  $f \in J$  was arbitrary, we conclude that  $J \subseteq P$ , another contradiction.

Since the factorizations in (i) and (ii) are unique, we see that any element of  $R$  is contained in only finitely many maximal ideals. Hence, to conclude that  $R$  is  $h$ -local, it suffices to show that  $R$  is 1-dimensional. Let  $H$  be a nonzero prime ideal of  $R$ , and pick  $0 \neq f \in H$ . Arguing as above, if  $f$  is as in (i), we get that  $H \geq P_{a_h}$ , for some  $h \leq m$ ; otherwise,  $f$  is as in (ii), and we derive that  $H \leq P$ . So it remains to prove that  $P$  does not properly contain a nonzero prime ideal. Equivalently, the local domain  $R_P$  is 1-dimensional. Using (ii), we see that, in the present situation, every element  $z$  of  $PR_P$  has the form  $bX^k u$ , where  $b \in K$  and  $u$  is a unit of  $R_P$ . Then an easy exercise shows that, for any given nonzero  $z_1, z_2 \in R_P$ , there exists  $n > 0$  such that  $z_1^n \in z_2 R_P$ , whence  $R_P$  1-dimensional follows. Finally,  $R$  is not Noetherian, since  $P$  is not finitely generated. In fact, take a basis  $\{a_\lambda\}_{\lambda \in \Lambda}$  of  $K$  over  $F$ ; by assumption, this basis is infinite. A direct check shows that the elements  $a_\lambda X$  of  $P$  are linearly independent modulo  $P^2$ , and the desired conclusion follows.

We remark that  $R_{P_a}$  is a DVR for every  $0 \neq a \in K$ , while  $R_P$  is a non-Noetherian local APD. Moreover, if  $i \notin F$ ,  $X$  is an irreducible element which is not prime; the equality  $X \cdot X = (iX)(-iX)$  provides a non unique factorization of  $X^2$ .  $\square$

## 7 APDs and torsion modules

APDs can be characterized in terms of two properties of their torsion modules  $T$ : the existence of the primary decomposition  $T \cong \bigoplus_{P \in \text{Max}(R)} T_P$ , and the semiartinianity. The first property is equivalent, by a Matlis' classical theorem (see [24] and [15, Theorem 3.7, Chap. IV]), to the  $h$ -locality of  $R$ ; the latter is equivalent to some other properties, as shown in the next result.

**Proposition 7.1.** *For a domain  $R$  the following conditions are equivalent:*

- (1) *Every non-zero torsion  $R$ -module contains a simple  $R$ -module.*
- (2) *Every torsion  $R$ -module is semiartinian.*



- (3)  $Q/R$  is semiartinian.
- (4) For every non-zero  $R$ -submodule  $A$  of  $Q$ ,  $Q/A$  contains a simple module.
- (5) For every non-zero ideal  $I$  of  $R$ ,  $R/I$  contains a simple  $R$ -module.

If  $R$  is almost perfect, all the previous conditions hold and each one implies that  $R$  is locally almost perfect.

*Proof.* The proof of the equivalence of the five conditions can be found in [12, Theorem 4.4.1]. If  $R$  is almost perfect, then condition (5) holds by Theorem 2.2. If  $Q/R$  is a semiartinian  $R$ -module, then  $Q/R_P$ , as a quotient of it, is also a semiartinian  $R$ -module. The only simple  $R$ -modules contained in an  $R_P$ -module are isomorphic to  $R/P \cong R_P/PR_P$ , hence  $Q/R_P$  is also a semiartinian  $R_P$ -module. Therefore  $R_P$  is an APD.  $\square$

From Theorems 2.2, 3.6 and Proposition 7.1 one easily deduces the equivalence of conditions (1), (3) and (4) in the Main Theorem.

**Theorem 7.2.** *A domain  $R$  is an APD if and only if it is  $h$ -local and satisfies one of the equivalent conditions of Proposition 7.1. In particular, if  $R$  is local, these conditions are all equivalent to the fact that  $R$  is an APD.  $\square$*

In a torsion module  $M$  over a local APD  $R$ , beside the ascending Loewy series  $0 = L_0(M) \leq L_1(M) \leq \dots \leq L_\lambda(M) \leq \dots$  we have also the descending chain of submodules, called the  $P$ -series of  $M$ :

$$M \geq PM \geq P^2M \geq \dots \geq P^\lambda M \geq \dots$$

where  $P^\lambda M = P(P^{\lambda-1}M)$  if  $\lambda$  is a successor ordinal, otherwise  $P^\lambda M = \bigcap_{\sigma < \lambda} P^\sigma M$ . A simple but crucial result concerning this chain is the next Proposition 9.1(2), which states that  $M$  is divisible if and only if  $PM = M$ . Hence the divisible submodule  $d(M)$  of the module  $M$  is  $P^\lambda M$ , where  $\lambda$  is the first ordinal such that  $P^\lambda M = P^{\lambda+1}M$ . In general,  $d(M)$  is not a direct summand of  $M$ , unless  $R$  is a DVR, by the well known fact that all torsion divisible modules are injective if and only if the domain is Dedekind.

Recall that a relevant phenomenon concerning torsion modules over a local APD  $R$ , arising when the maximal ideal  $P$  is not almost nilpotent, is that  $Q/R$  can have Loewy length  $\omega \cdot n$ , for any positive integer  $n$  (see Theorem 6.4). Another simpler phenomenon arising when the maximal ideal  $P$  is not almost nilpotent is that a bounded module  $M$  (i.e.,  $rM = 0$  for some  $0 \neq r \in R$ ) can very well have an infinite  $P$ -series.

*Example 7.3.* Let  $R$  be a local APD with maximal ideal  $P$  not almost nilpotent. By Lemma 3.3, for every  $0 \neq a \in R$ ,  $P^n$  is not contained in  $aR$  for all  $n$ , hence  $(P^n + aR)/aR$  is not zero for all  $n$ . On the other hand,  $(P^n + aR)/aR$  is not divisible, since it is bounded, hence it strictly contains  $P((P^n + aR)/aR) = (P^{n+1} + aR)/aR$ .  $\square$

One main question for a torsion module  $M$  over local APDs is: what is its Loewy length  $l(M)$ ? We can give easily an upper bound for these lengths.

**Proposition 7.4.** *Let  $R$  be a local APD and  $M$  a torsion  $R$ -module. Then  $l(M) \leq l(Q/R)$ .*

*Proof.* The Loewy length does not increase passing to submodules and quotients. So  $l(M) \leq l(E(M))$ , where  $E(M)$  is the injective envelope of  $M$ . Being  $R$  a Matlis domain, divisible  $R$ -modules are  $h$ -divisible, so  $Q/R$  is a generator for the full subcategory of  $\text{Mod}(R)$  consisting of the torsion divisible  $R$ -modules (see [15, Theorem 2.8, Chap. VII]), hence  $E(M)$  is a quotient of a direct sum of copies of  $Q/R$ . Therefore,  $l(M) \leq l(E(M)) \leq l(Q/R)$ .  $\square$

Among torsion modules over a local APD  $R$ , beside  $Q/R$  there is another module which deserves a particular attention, namely  $E(R/P)$ , the minimal injective cogenerator. We summarize here the known results on  $E(R/P)$ , referring to [28] for a more precise discussion on this subject.

The fact that  $l(E(R/P)) = \omega$  is equivalent to say that the  $P$ -adic topology of  $R$  is finer than the Prüfer topology (which has as basis of neighborhoods of 0 the family of non-zero ideals  $I$  such that  $R/I$  has finitely generated socle; see [29, Proposition 2.2]); this fact is certainly true when  $P$  is almost nilpotent, that is, when  $l(Q/R) = \omega$ , by Lemma 3.3. We don't know whether  $l(E(R/P)) = l(Q/R)$  is always true (see Question 5).

The behavior of  $E(R/P)$  and of its endomorphism ring  $\text{End}_R(E(R/P))$  is completely different when  $R$  is a Noetherian APD with respect to the non-Noetherian case. In the former case, Matlis' theory of injective modules developed in [23] applies;  $Q/R$  is injective exactly when  $R$  is divisorial or, equivalently,  $R : P$  is 2-generated (see [15, Proposition 5.8, Chap. IV]); the  $P$ -adic topology coincides with the  $R$ -topology (with neighborhoods of zero all the non-zero ideals) and also with the Prüfer topology; and, finally, the two modules  $E(R/P)$  and  $Q/R$  both have endomorphism ring isomorphic to the completion of  $R$  in these topologies.

When  $R$  is not Noetherian, the three above topologies can be different,  $Q/R$  cannot be injective (see [6, Proposition 2.6]), and  $\text{End}_R(E(R/P))$  is not commutative in general; its center is isomorphic to the completion of  $R$  in the Prüfer topology, by a result of Vámos [38, Proposition 1.6], and it contains the completion of  $R$  in the  $R$ -topology (which is the endomorphism ring of  $Q/R$  for any domain  $R$ , see [15, Chap. VIII]).

A concrete example of local APD  $R$  with  $\text{End}_R(E(R/P))$  not commutative is obtained by taking  $R$  to be a simple integral extension of a  $P$ -chained non-Noetherian APD, as in Example 6.3 (see [28, Theorem 4.8]).

## 8 APDs and flat modules

The discovery of APDs was made in [5], where a completely satisfactory solution was given to the following problem: characterize the integral domains such that every  $R$ -module has a strongly flat cover. The problem was settled in [37] after the Flat Cover Conjecture was positively solved by Bican-El Bashir-Enochs.

In fact, in [8] it was proved in two different ways that all modules over any ring have a flat cover. In order to state a similar result, replacing flat modules by strongly flat modules in the case of integral domains, we must restrict to the class of APDs; in which case, flat modules turn out to be strongly flat, so they have a particularly nice structure (see next Proposition 8.3).

We recall some basic definitions. Recall that a module  $F$  over a domain  $R$  with field of quotients  $Q$  is *strongly flat* if  $\text{Ext}_R^1(F, M) = 0$  for all modules  $M$  such that  $\text{Ext}_R^1(Q, M) = 0$ ; this is equivalent to saying that  $F$  is a direct summand of a module which is an extension of a free module by a divisible torsionfree module. Clearly, such a module is flat.

Let  $\mathcal{C}$  be a class of  $R$ -modules closed under isomorphisms and direct summands. Given a module  $M$ , a map  $f: C \rightarrow M$ , where  $C \in \mathcal{C}$ , is said to be a  $\mathcal{C}$ -precover of  $M$  if, for every  $C' \in \mathcal{C}$ , the induced map  $\text{Hom}_R(C', C) \rightarrow \text{Hom}_R(C', M)$  is surjective. The  $\mathcal{C}$ -precover  $C$  is a  $\mathcal{C}$ -cover provided  $f \cdot g = f$  for an endomorphism  $g$  of  $C$  implies that  $g$  is an automorphism.

The main result in [5] proves *inter alia* the equivalence of the conditions (1) and (5) in the Main Theorem.

**Theorem 8.1.** *For a domain  $R$  the following are equivalent:*

- (1)  $R$  is almost perfect.
- (2) Every flat  $R$ -module is strongly flat.
- (3) Every  $R$ -module has a strongly flat cover.
- (4) The class of strongly flat modules is closed under direct limits.  $\square$

The proof of the core part in Theorem 8.1, namely, (1)  $\Leftrightarrow$  (2), is complicated and makes use of several intermediate results (see Proposition 4.4 in [5]).

Another interesting result concerning the existence of strongly flat covers proved in [5, Theorem 4.9] is that a domain  $R$  is  $h$ -local if and only if every finitely generated torsion  $R$ -module has a strongly flat cover. The proof of this fact relies on the analogous result proved by Bass [2, Theorem 2.1], stating that a ring is semiperfect if and only if all finitely generated  $R$ -modules have a projective cover; recall that the ring  $R$  is semiperfect if  $R/J(R)$  is semisimple and the idempotents lift modulo  $J(R)$ ; when  $R$  is a commutative domain, this is equivalent to the fact that  $R$  is  $h$ -local (see [5, Theorem 4.9]).

From the equivalence of (1) and (7) in the Main Theorem (discussed in the next two Sections) we derive the following fact concerning subprojective modules over APDs.

**Corollary 8.2.** *Let  $R$  be an APD. Then a flat submodule  $A$  of a projective module  $B$  is projective.*

*Proof.* The weak dimension of  $B/A$  is  $\leq 1$ . By (7) in the Main Theorem, we have that the projective dimension of  $(B/A)$  is  $\leq 1$ , therefore  $A$  is projective.  $\square$

If  $R$  is only a Matlis domain, one can prove that a strongly flat submodule  $A$  of a projective  $R$ -module  $B$  is projective, but with the additional assumption that  $B/A$  is torsion (see [7, Proposition 2.5]).

We close this section recalling that a deep investigation of strongly flat modules over a domain  $R$ , hence of flat modules in case  $R$  is an APD, is made in [7]. In particular, from [7, Theorem 2.1 and Corollary 2.7] one deduces the following:

**Proposition 8.3.** *Let  $R$  be an APD and  $M$  an  $R$ -module. The following conditions are equivalent:*

- (1)  $M$  is flat.
- (2) The completion of  $M$  in the  $R$ -topology is a summand of the completion of a free module.
- (3)  $M \otimes_R (Q/R)$  is isomorphic to a summand of a direct sum of copies of  $Q/R$ .

If  $R$  is local, then “a summand of” can be cancelled in (2) and (3).  $\square$

### 9 APDs and divisible modules

A first relevant property of divisible modules over APDs is that they are all  $h$ -divisible, that is, homomorphic images of injective modules. This property depends only on (actually, it is equivalent to) the fact that an APD is a Matlis domain. In this section we will see that more specific properties of divisible modules over APDs are available.

We start with a characterization of injective and divisible modules over APDs; the latter have been obtained in the local case by J. R. Smith in [35].

**Proposition 9.1.** *Let  $R$  be an APD. Then*

- (1) An  $R$ -module  $E$  is injective if and only if  $\text{Ext}_R^1(R/P, E) = 0$  for every maximal ideal  $P$ .
- (2) An  $R$ -module  $M$  is divisible if and only if  $PM = M$  for every maximal ideal  $P$ .

*Proof.* (1) The Baer’s criterion says that  $E$  is injective if and only if  $\text{Ext}_R^1(R/I, E) = 0$  for every non-zero ideal  $I$ . But the factors of the Loewy series of  $R/I$  are semisimple modules, so a ubiquitous lemma on  $\text{Ext}$  (see [15, Lemma 2.5, Chap. VI]) applies.

(2) If  $M$  is divisible, then obviously  $PM = M$  for every maximal ideal  $P$ . To prove the viceversa, let  $0 \neq r \in R$ ; we want to show that  $M = rM$ , that is,  $M/rM = 0$ . Being  $R$   $h$ -local, we have that  $M/rM \cong \bigoplus_{1 \leq i \leq n} M_{P_i}/rM_{P_i}$ , where  $P_1, \dots, P_n$  are the maximal ideals containing  $r$ . So we are led to prove the claim for  $R$  local. Let  $P$  be the maximal ideal; then  $\bar{P} = P/rP$  is a  $T$ -nilpotent ideal of the perfect ring  $R/rR$  and  $\bar{M} = M/rM = \bar{P}\bar{M}$ . From Proposition 2.1 we get that  $\bar{M} = 0$ , as desired.  $\square$

We recall some basic definitions. A module  $M$  over a domain  $R$  with field of quotients  $Q$  is said to be *weak injective* if  $\text{Ext}_R^1(F, M) = 0$  for every  $R$ -module  $F$  of weak (i.e., flat) dimension  $\leq 1$ . Such a module is divisible, in view of the characterization of divisible modules  $D$  by the property that  $\text{Ext}_R^1(R/I, D) = 0$  for all invertible (i.e., projective) ideals  $I$  (see [15, Lemma 7.2, Chap. I]).

Let  $\mathcal{C}$  be a class of  $R$ -modules closed under isomorphisms and direct summands. Given a module  $M$ , a map  $f: M \rightarrow C$ , where  $C \in \mathcal{C}$ , is said to be a  $\mathcal{C}$ -preenvelope of  $M$  if, for every  $C' \in \mathcal{C}$ , the induced map  $\text{Hom}_R(C, C') \rightarrow \text{Hom}_R(M, C')$  is surjective. The  $\mathcal{C}$ -preenvelope  $C$  is a  $\mathcal{C}$ -envelope provided  $g \cdot f = f$  for an endomorphism  $g$  of  $C$  implies that  $g$  is an automorphism.

Recently, Fuchs and Lee proved [14, Theorem 6.4] that a domain that is not a field has global weak injective dimension 1 if and only if it is an APD. The first condition amounts to say that, given any module  $M$ ,  $E(M)/M$  is weak injective ( $E(M)$  denotes the injective envelope of  $M$ ); Lee proved [22, Lemma 3.6] that this is equivalent to the fact that all divisible modules are weak injective. So we can formulate the result in the form appearing in our Main Theorem, with an additional characterization; a proof can be found in [30].

**Theorem 9.2.** *For a domain  $R$  the following are equivalent:*

- (1)  $R$  is almost perfect.
- (2) Every divisible  $R$ -module is weak injective.
- (3) For every flat  $R$ -module  $F$ , the embedding of  $F$  into its injective envelope  $E(F)$  is a divisible envelope.  $\square$

Trlifaj (see [37] and [18, Theorem 4.1.3]) and Lee [22, Lemma 4.1] proved that the class of weak injective modules over any ring is an enveloping class, that is, every  $R$ -module has a weak injective envelope. Since for APDs the two classes of divisible and weak injective modules coincide, we derive the following:

**Corollary 9.3.** *All modules over an APD have a divisible envelope.*  $\square$

We conjectured in [30] that for no other domains the class of divisible modules is an enveloping class, on the ground that, if this happens for an almost maximal Prüfer domain  $R$ , then  $R$  must be a Dedekind domain, that is,  $R$  must be an APD. Very recently, Bazzoni proved in [3] that this conjecture is true, by showing that a domain  $R$  whose divisible modules form an enveloping class must be  $h$ -local and locally almost perfect.

We consider now two more classes of divisible modules. A module is *finitely injective* if every finitely generated submodule is contained in an injective summand; a module  $M$  is *FP-injective* (or absolutely pure) if  $\text{Ext}_R^1(F, M) = 0$  for all finitely presented modules  $F$ . It is easy to see that finitely injective modules are *FP-injective* and that *FP-injective* modules are divisible. Over Noetherian rings the two classes coincide. Over general APDs, we have the following result, whose proof can be found in [31, Theorem 3.4].

**Theorem 9.4.** *Let  $R$  be a countable APD. Then every *FP-injective* module is finitely injective if and only if  $R$  is Noetherian.*  $\square$

## 10 APDs and cotorsion pairs

The frame of the cotorsion pairs is very useful to prove results for classes of modules that are right or left Ext-orthogonal of other classes (the original name of cotorsion pairs in [27] was “cotorsion theories”, in analogy with the “torsion theories” which use the functor Hom instead of Ext).

A cotorsion pair is a pair  $(\mathcal{A}, \mathcal{B})$  of classes of modules (over any ring  $R$ ) such that

$$\begin{aligned} \mathcal{A} &= \{M \in \text{Mod}(R) \mid \text{Ext}_R^1(M, B) = 0 \text{ for all } B \in \mathcal{B}\} \\ \mathcal{B} &= \{M \in \text{Mod}(R) \mid \text{Ext}_R^1(A, M) = 0 \text{ for all } A \in \mathcal{A}\}. \end{aligned}$$

Given two cotorsion pairs  $(\mathcal{A}, \mathcal{B})$  and  $(\mathcal{A}', \mathcal{B}')$ , we set  $(\mathcal{A}, \mathcal{B}) \leq (\mathcal{A}', \mathcal{B}')$  if  $\mathcal{A} \subseteq \mathcal{A}'$ , equivalently, if  $\mathcal{B}' \subseteq \mathcal{B}$ . So  $(\text{Mod}(R), \mathcal{I})$  is the maximal cotorsion pair, where  $\mathcal{I}$  denotes the class of the injective  $R$ -modules, and  $(\mathcal{P}, \text{Mod}(R))$  is the minimal one, where  $\mathcal{P}$  denotes the class of the projective  $R$ -modules. For a systematic treatment of the subject of the cotorsion pairs we refer to the monograph [18].

Of course, given two cotorsion pairs  $(\mathcal{A}, \mathcal{B})$  and  $(\mathcal{A}', \mathcal{B}')$ , the two equalities  $\mathcal{A} = \mathcal{A}'$  and  $\mathcal{B} = \mathcal{B}'$  are each other equivalent. This fact is used to derive in the Main Theorem that condition (5) is equivalent to condition (6), and condition (8) is equivalent to condition (7).

Therefore, in order to prove that (5) is equivalent to (6), it is enough to know that, over a domain  $R$ :

- (i)  $(\mathcal{F}, \mathcal{C})$  is a cotorsion pair, where  $\mathcal{F}$  denotes the class of flat modules, and  $\mathcal{C}$  that of cotorsion modules  $C$ , defined by  $\text{Ext}_R^1(F, C) = 0$  for all flat modules  $F$  (this is true over any ring).
- (ii)  $(\mathcal{SF}, \mathcal{WC})$  is a cotorsion pair, where  $\mathcal{SF}$  denotes the class of strongly flat modules, and  $\mathcal{WC}$  that of weakly cotorsion modules  $M$ , defined by the property that  $\text{Ext}_R^1(Q, M) = 0$ , where  $Q$  is the field of quotients of  $R$  (see [24]).

Note that  $(\mathcal{SF}, \mathcal{WC}) \leq (\mathcal{F}, \mathcal{C})$ . Similarly, in order to prove that (7) is equivalent to (8), it is enough to know that, over a domain  $R$ :

- (iii)  $(\mathcal{P}_1, \mathcal{D})$  is a cotorsion pair, where  $\mathcal{P}_1$  denotes the class of modules of projective dimension  $\leq 1$ , and  $\mathcal{D}$  that of divisible modules.
- (iv)  $(\mathcal{F}_1, \mathcal{WJ})$  is a cotorsion pair, where  $\mathcal{F}_1$  denotes the class of modules of weak dimension  $\leq 1$ , and  $\mathcal{WJ}$  that of weak injective modules.

Note that  $(\mathcal{P}_1, \mathcal{D}) \leq (\mathcal{F}_1, \mathcal{WJ})$ . The proofs of (i), (ii) and (iv) can be found in [18]. The fact that  $(\mathcal{P}_1, \mathcal{D})$  is a cotorsion pair was proved only very recently in a remarkable paper by Bazzoni–Herbera [4], and it does not hold over general rings. The cotorsion pair naturally associated to the class  $\mathcal{D}$  of divisible modules is  $(\mathcal{CS}, \mathcal{D})$ , where  $\mathcal{CS}$  denotes the class of the direct summands of modules admitting a filtration of cyclically presented modules (see [18, p. 136]). One of the main consequences obtained in [4] is that, if  $R$  is a commutative integral domain, then the class  $\mathcal{CS}$  consists exactly of the modules of projective dimension  $\leq 1$ .

## 11 Open questions

Throughout this section  $R$  denotes an integral domain and  $Q$  its field of quotients. If  $R$  is assumed to be local, then  $P$  will denote its maximal ideal.  $l(M)$  denotes the Loewy length of a module  $M$ ,

**Question 1.** Is every perfect commutative ring  $R$  the quotient of an APD?

**Question 2.** Is a local APD  $R$ , such that  $Q/R$  has finitely generated socle, necessarily Noetherian?

**Question 3.** Does a local APD  $R$  exist such that  $l(Q/R)$  is  $\geq \omega^2$ ?

**Question 4.** Does a local APD  $R$  exist such that  $l(Q/R)$  is strictly bigger than  $l(E)$ , where  $E = E(R/P)$  is the minimal injective cogenerator?

**Question 5.** Is a local APD  $R$  such that  $\text{End}_R(E(R/P))$  is commutative necessarily Noetherian?

**Question 6.** Is there a functorial relationship between modules over a local APD  $R$  and modules over an archimedean valuation domain  $V$  dominating it?

**Question 7.** Is it true that for every APD  $R$  (without the countability assumption of Theorem 8.3), all  $FP$ -injective modules are finitely injective exactly if  $R$  is Noetherian?

**Question 8.** Is it possible to extend the theory of almost perfect rings to the non-commutative setting?

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# Characteristic $p$ methods in characteristic zero via ultraproducts

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**Abstract** In recent decades, by exploiting the algebraic properties of the Frobenius in positive characteristic, many so-called homological conjectures and intersection conjectures have been established, culminating into the powerful theory of tight closure and big Cohen–Macaulay algebras. In the present article, I give a survey of how these methods also can be applied directly in characteristic zero by taking ultraproducts, rather than through the cumbersome lifting/reduction techniques. This has led to some new results regarding rational and log-terminal singularities, as well as some new vanishing theorems. Even in mixed characteristic, we can get positive results, albeit only asymptotically.

## 1 Introduction

In the last three decades, all the so-called Homological Conjectures have been settled completely for Noetherian local rings containing a field by work of Peskine–Szpiro, Hochster–Roberts, Hochster, Evans-Griffith, et. al. (some of the main papers are [21, 28, 30, 43, 58]; for an overview, see [11, Section 9] or [85]). More recently, Hochster and Huneke have given more simplified proofs of most of these results by means of their elegant tight closure theory, including a more canonical construction of big Cohen–Macaulay algebras (see [19, 36, 41]; for a survey, see [45, 77, 83]). However, tight closure theory also turned out to have applications to other fields, including the study of rational singularities; see for instance [34, 35, 37, 38, 47, 80–82].

Most of these results have in common that they are based on characteristic  $p$  methods, where results in characteristic zero are then obtained by reduction to characteristic  $p$ . To control the behavior under this reduction in its greatest generality, strong forms of Artin Approximation [59, 84, 86] are required, rendering the theory highly non-elementary. Moreover, there are plenty technical difficulties, which

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offset the elegance of the characteristic  $p$  method. It is the aim of this survey paper to show that when using ultraproducts as a means of transfer from positive to zero characteristic, the resulting theory is, in comparison, (a) easier and more elementary (at worst, we need Rotthaus's version of Artin Approximation [63]); (b) more elegant; and (c) more powerful. In Section 4, I will substantiate the former two claims, and in Section 5, the latter. In a final section, I discuss briefly the status in mixed characteristic.

## 2 Characteristic $p$ methods

Let  $A$  be a ring of prime characteristic  $p$ . One feature that distinguishes it immediately from any ring in characteristic zero is the presence of the *Frobenius* morphism  $x \mapsto x^p$ . We will denote this ring homomorphism by  $\text{Frob}_A$ , or, when there is little room for confusion, by  $\text{Frob}$ .<sup>1</sup> In case  $A$  is a domain, with field of fractions  $K$ , we fix an algebraic closure  $\bar{K}$  of  $K$ , and let  $A^+$  be the integral closure of  $A$  in  $\bar{K}$ . We call  $A^+$  the *absolute integral closure* of  $A$ ; it is uniquely defined up to isomorphism. Although no longer Noetherian, it has many good properties. We start with a result, the proof of which we will discuss below (the reader be warned that we are presenting the results in a reversed logical, as well as historical, order).

**Theorem 2.1.** *If  $A$  is an excellent regular local ring of characteristic  $p$ , then  $A^+$  is a flat  $A$ -algebra.*

From this fact, many deep theorems can be deduced. To discuss these, we need a definition.

### 2.1 Big Cohen–Macaulay algebras

Given a Noetherian local ring  $R$ , we call an  $R$ -module  $M$  a *big Cohen–Macaulay module*,<sup>2</sup> if there exists a system of parameters<sup>3</sup> which is an  $M$ -regular sequence. If every system of parameters is  $M$ -regular, then we say that  $M$  is a *balanced big Cohen–Macaulay module*. There are big Cohen–Macaulay modules which are not balanced, but this can be overcome by taking their completion ([11, Corollary 8.5.3]). We will make use of the following criterion (whose proof is rather straightforward).

<sup>1</sup> The key fact about this map, of course, is its additivity:  $(a+b)^p = a^p + b^p$ , since in the expansion of the former, all non-trivial binomials  $\binom{p}{i}$  are divisible by  $p$ , whence zero in  $A$ .

<sup>2</sup> The nomenclature is meant to emphasize that the module need not be finitely generated.

<sup>3</sup> A tuple of the same length as the dimension of  $R$  is called a *system of parameters* if it generates an  $\mathfrak{m}$ -primary ideal; such an ideal is then called a *parameter ideal*.

**Lemma 2.2 ([3, Lemma 4.8]).** *If  $M$  is a big Cohen–Macaulay module in which every permutation of an  $M$ -regular sequence is again  $M$ -regular, then  $M$  is a balanced big Cohen–Macaulay module.  $\square$*

If  $M$  has moreover the structure of an  $R$ -algebra, we call it a (*balanced*) big Cohen–Macaulay algebra. Hochster [28] proved the existence of big Cohen–Macaulay modules in equal characteristic, and showed how they imply several homological conjectures (we will give an example below). These ideas went back to the characteristic  $p$  methods introduced by Peskine and Szpiro [58], which together with Kunz’s theorem [50] and the Hochster–Roberts theorem [43] form the precursors of tight closure theory (see [45, Chap. 0]).

**Theorem 2.3.** *Every Noetherian local ring of characteristic  $p$  admits a balanced big Cohen–Macaulay algebra.*

*Proof.* Since completion preserves systems of parameters, it suffices to prove this for  $R$  a complete Noetherian local ring. Killing a minimal prime of maximal dimension, we may moreover assume that  $R$  is a domain. Let  $(x_1, \dots, x_d)$  be a system of parameters. By Cohen’s structure theorem, there exists a regular subring  $S \subseteq R$  with maximal ideal  $(x_1, \dots, x_d)S$  such that  $R$  is finite as an  $S$ -module. In particular,  $R^+ = S^+$ . By Theorem 2.1, the map  $S \rightarrow S^+$  is flat, and hence  $(x_1, \dots, x_d)$  is a regular sequence in  $S^+ = R^+$ .  $\square$

*Remark 2.4.* We cheated by deriving the existence of big Cohen–Macaulay algebras from Theorem 2.1, since currently the only known proof of the latter theorem is via big Cohen–Macaulay algebras. Here is the correct logical order: Hochster and Huneke show in [36], by different, and rather technical means, that  $R^+$  is a balanced big Cohen–Macaulay  $R$ -algebra whenever  $R$  is an excellent local domain (see [45, Chap. 7] or [46]). This result in turn implies the flatness of  $R^+$  if  $R$  is regular, by the following flatness criterion ([45, Theorem 9.1] or [71, Theorem IV.1]).

**Proposition 2.5.** *A module over a regular local ring is a balanced big Cohen–Macaulay module if and only if it is flat.*

*Proof.* One direction is immediate since flat maps preserve regular sequences. So let  $M$  be a balanced big Cohen–Macaulay module over the  $d$ -dimensional regular local ring  $R$ . Since all modules have finite projective dimension, the functors  $\text{Tor}_i^R(M, \cdot)$  vanish for  $i \gg 0$ . Let  $e$  be maximal such that  $\text{Tor}_e^R(M, N) \neq 0$  for some finitely generated  $R$ -module  $N$ . We need to show that  $e = 0$ , so, by way of contradiction, assume  $e \geq 1$ . Using that  $N$  admits a filtration  $0 = N_0 \subseteq N_1 \subseteq \dots \subseteq N_s = N$  in which each subsequent quotient has the form  $R/\mathfrak{p}_i$  with  $\mathfrak{p}_i$  a prime ideal ([17, Proposition 3.7]), we may assume that  $N = R/\mathfrak{p}$  for some prime  $\mathfrak{p}$ . Let  $h$  be the height of  $\mathfrak{p}$ , and choose a system of parameters  $(x_1, \dots, x_d)$  in  $R$  such that  $\mathfrak{p}$  is a minimal prime of  $I := (x_1, \dots, x_h)R$ . Since  $\mathfrak{p}$  is then an associated prime of  $R/I$ , we can find a short exact sequence

$$0 \rightarrow R/\mathfrak{p} \rightarrow R/I \rightarrow C \rightarrow 0$$

for some finitely generated  $R$ -module  $C$ . Tensoring the above exact sequence with  $M$ , yields part of a long exact sequence

$$\mathrm{Tor}_{e+1}^R(M, C) \rightarrow \mathrm{Tor}_e^R(M, R/\mathfrak{p}) \rightarrow \mathrm{Tor}_e^R(M, R/I). \tag{1}$$

The first module in (1) is zero by the maximality of  $e$ . The last module is isomorphic to  $\mathrm{Tor}_e^{R/I}(M/IM, R/I) = 0$  since  $(x_1, \dots, x_h)$  is both  $R$ -regular and  $M$ -regular. Hence the middle module is zero too, contradiction.  $\square$

Since the Frobenius preserves regular sequences, we immediately get one half of Kunz’s theorem [50]:

**Corollary 2.6 (Kunz).** *The Frobenius is flat on a regular ring.*  $\square$

To illustrate the power of the existence of big Cohen–Macaulay modules, let me derive from it one of the so-called Homological Conjectures:

**Theorem 2.7 (Monomial Conjecture).** *In a Noetherian local ring  $R$  of characteristic  $p$ , every system of parameters  $(x_1, \dots, x_d)$  is monomial, in the sense that  $(x_1x_2 \cdots x_d)^t \notin (x_1^{t+1}, \dots, x_d^{t+1})R$ , for all  $t$ .*

*Proof.* Assume that the statement is false for some system of parameters  $(x_1, \dots, x_d)$  and some  $t$ . Let  $B$  be a balanced big Cohen–Macaulay  $R$ -algebra. Hence,  $(x_1x_2 \cdots x_d)^t$  belongs to  $(x_1^{t+1}, \dots, x_d^{t+1})B$ . However,  $(x_1, \dots, x_d)$  is  $B$ -regular, and it is not hard to prove that a regular sequence is always monomial, leading to the desired contradiction.  $\square$

## 2.2 Tight closure

Let  $I$  be an ideal in a Noetherian ring  $A$  of characteristic  $p$ . We denote the ideal generated by the image of  $I$  under  $\mathrm{Frob}_A$  by  $\mathrm{Frob}_A(I)A$  (one also writes  $I^{[p]}$ ). If  $I = (f_1, \dots, f_s)A$ , then  $\mathrm{Frob}(I)A = (f_1^p, \dots, f_s^p)A$ . In particular,  $\mathrm{Frob}(I)A \subseteq I^p$ , but most of the time, this is a strict inclusion. Hochster and Huneke defined the tight closure of an ideal as follows. Let  $A^\circ$  be the multiplicative set in  $A$  of all elements not contained in any minimal prime ideal of  $A$ . An element  $z$  belongs to the *tight closure*  $\mathrm{cl}_A(I)$  of  $I$  (in the literature, the tight closure is more commonly denoted  $I^*$ ), if there exists some  $c \in A^\circ$  such that

$$c\mathrm{Frob}_A^n(z) \in \mathrm{Frob}_A^n(I)A \tag{2}$$

for all  $n \gg 0$ . Using that  $A^\circ$  is multiplicative, one easily verifies that  $\mathrm{cl}(I)$  is an ideal, containing  $I$ , which itself is *tightly closed*, meaning that it is equal to its own tight closure. Equally easy to see is that, if  $I \subseteq J$ , then  $\mathrm{cl}(I) \subseteq \mathrm{cl}(J)$ .

*Remark 2.8.* The following observations all follow very easily from the definitions (see ([45, Section 1] for details).

1. If  $A$  is a domain, then the only restriction on  $c$  is that it be non-zero. We may always reduce to the domain case since  $z$  belongs to the tight closure of  $I$  if and only if the image of  $z$  belongs to the tight closure of  $I(A/\mathfrak{p})$ , where  $\mathfrak{p}$  runs over all minimal prime ideals of  $A$ .
2. If  $A$  is reduced – a situation we may always reduce to by Remark 2.8(1) – or if  $I$  has positive height, then we may require (2) to hold for all  $n$ .
3. It is crucial to note that  $c$  is independent from  $n$ . If  $A$  is a domain, then we may take  $p^n$ -th roots in (2), to get  $c^{1/p^n} z \in IA^{1/p^\infty}$ , for all  $n$ , where  $A^{1/p^\infty}$  is the subring of  $A^+$  consisting of all  $p^n$ -th roots of elements of  $A$ . If we think of  $c^{1/p^n}$  approaching 1 as  $n$  goes to infinity, then the condition says, loosely speaking, that in the limit  $z$  belongs to  $IA^{1/p^\infty}$ . The strictly weaker condition that  $z \in IA^{1/p^\infty}$  is equivalent with requiring  $c$  to be 1 in (2), and leads to the notion of *Frobenius closure*. This latter closure does not have properties as good as tight closure.
4. A priori,  $c$  does depend on  $z$  as well as  $I$ . However, in many instances there is a single  $c$  which works for all  $z$ , all  $I$ , and all  $n$ ; such an element is called a *test element*. Unlike most properties of tight closure, the existence of test elements is a more delicate issue (see, for instance, [37] or [45, Section 2]). Fortunately, for most of applications, it is not needed.

*Example 2.9.* It is instructive to look at an example. Let  $K$  be a field of characteristic  $p > 3$ , and let  $A := K[x, y, z]/(x^3 - y^3 - z^3)K[x, y, z]$  be the projective coordinate ring of the cubic Fermat curve. Let us show that  $x^2$  is in the tight closure of  $I := (y, z)A$ . For a fixed  $e$ , write  $2pe = 3h + r$  for some  $h \in \mathbb{N}$  and some remainder  $r \in \{1, 2\}$ , and let  $c := x^3$ . Hence

$$cx^{2pe} = x^{3(h+1)+r} = x^r(y^3 + z^3)^{h+1}.$$

A quick calculation shows that any monomial in the expansion of  $(y^3 + z^3)^{h+1}$  is a multiple of either  $y^{p^e}$  or  $z^{p^e}$ , showing that (2) holds for all  $e$ , and hence that  $(x^2, y, z)A \subseteq \text{cl}(I)$ .

It is often much harder to show that an element does not belong to the tight closure of an ideal. By Theorem 2.10 below, any element outside the integral closure is also outside the tight closure. Since  $(x^2, y, z)A$  is integrally closed, we conclude that it is equal to  $\text{cl}(I)$ .

The following five properties all have fairly simple proofs, yet are powerful enough to deduce many deeper theorems.

**Theorem 2.10.** *Let  $A$  and  $B$  be Noetherian rings of prime characteristic  $p$ , and let  $I$  be an ideal in  $A$ .*

*(weak persistence)* In an extension of domains  $A \subseteq B$ , tight closure is preserved in the sense that  $\text{cl}_A(I)B \subseteq \text{cl}_B(IB)$ .

*(regular closure)* If  $A$  is a regular local ring, then  $I$  is tightly closed.

*(plus closure)* If  $A$  is a domain, then  $IA^+ \cap A \subseteq \text{cl}_A(I)$ .

*(colon capturing)* If  $A$  is a homomorphic image of a local Cohen–Macaulay ring then  $((x_1, \dots, x_i)A : x_{i+1}) \subseteq \text{cl}((x_1, \dots, x_i)A)$ , for each  $i$  and each system of parameters  $(x_1, \dots, x_d)$ .

(integral closure) *Tight closure*,  $\text{cl}(I)$ , is contained in integral closure,  $\bar{I}$ ; if  $I$  is principal, then  $\text{cl}(I) = \bar{I}$ .

*Proof.* Weak persistence is immediate from the fact that (2) also holds, by functoriality of the Frobenius, in  $B$ , and  $c$  remains non-zero in  $B$ . In fact, the much stronger property, *persistence*, where the homomorphism does not need to be injective, holds in many cases. However, to prove this, one needs test elements (see Remark 2.8(4)).

To prove the regularity property, suppose  $A$  is regular but  $I$  is not tightly closed. Hence, there exists  $z \in \text{cl}(I)$  not in  $I$ . In particular,  $(I : z)$  is contained in the maximal ideal  $\mathfrak{m}$  of  $A$ . By definition, there is some non-zero  $c$  such that  $c\text{Frob}^n(z) \in \text{Frob}^n(I)A$  for all  $n \gg 0$ . Since the Frobenius is flat on a regular ring by Corollary 2.6, and since flat maps commute with colons (see for instance [77]), we get

$$c \in (\text{Frob}^n(I)A : \text{Frob}^n(z)) = \text{Frob}^n(I : z)A$$

for all  $n$ . Since  $(I : z) \subseteq \mathfrak{m}$ , we get  $c \in \text{Frob}^n(\mathfrak{m})A \subseteq \mathfrak{m}^n$ , for all  $n$ , yielding the contradiction that  $c = 0$  by Krull’s intersection theorem.

To prove the plus closure property, let  $z \in IA^+ \cap A$ . Hence, there exists a finite extension  $A \subseteq B \subseteq A^+$  such that already  $z \in IB$ . Choose an  $A$ -linear (module) morphism  $g : B \rightarrow A$  sending 1 to a non-zero element  $c \in A$ .<sup>4</sup> Applying the Frobenius to  $z \in IB$ , yields  $\text{Frob}_A^n(z) \in \text{Frob}_B^n(IB)B$  for all  $n$ . Applying  $g$  to the latter shows  $c\text{Frob}_A^n(z) \in \text{Frob}_A^n(I)A$ , for all  $n$ , that is to say,  $z \in \text{cl}_A(I)$ .

Colon capturing knows many variants. Let me only discuss the special, but important case that  $A$  is moreover complete. By Cohen’s structure theorem, we can find a regular local subring  $(S, \mathfrak{n})$  of  $A$  such that  $A$  is finite as an  $S$ -module and  $\mathfrak{n}A = (x_1, \dots, x_d)A$ . Suppose  $zx_{i+1} \in (x_1, \dots, x_i)A$ . Applying powers of Frobenius, we get

$$\text{Frob}^n(zx_{i+1}) \in (\text{Frob}^n(x_1), \dots, \text{Frob}^n(x_i))A \tag{3}$$

for all  $n$ . Let  $R$  be the  $S$ -subalgebra of  $A$  generated by  $z$ , and as above, choose an  $R$ -linear morphism  $g : A \rightarrow R$  with  $c := g(1) \neq 0$ . Applying  $g$  to (3) yields a relation

$$c\text{Frob}^n(z)\text{Frob}^n(x_{i+1}) \in (\text{Frob}^n(x_1), \dots, \text{Frob}^n(x_i))R \tag{4}$$

for all  $n$ . Since  $R$  is a hypersurface ring, it is Cohen–Macaulay. In particular,  $(x_1, \dots, x_d)$ , being a system of parameters in  $R$ , is  $R$ -regular, and so is therefore the sequence  $(\text{Frob}^n(x_1), \dots, \text{Frob}^n(x_d))$ . This allows us to cancel  $\text{Frob}^n(x_{i+1})$  in (4), getting the tight closure relations  $c\text{Frob}^n(z) \in (\text{Frob}^n(x_1), \dots, \text{Frob}^n(x_i))R$ . Weak persistence then shows that  $z$  also belongs to the tight closure of  $(x_1, \dots, x_i)A$ , as we needed to show.

Finally, the containment  $\text{cl}(I) \subseteq \bar{I}$  is immediate from the integrality criterion that  $z \in \bar{I}$  if and only if  $cz^n \in I^n$  for some  $c \in A^\circ$  and infinitely many  $n$  (note that  $\text{Frob}^n(I)A \subseteq I^n$ ). If  $I$  is principal, then  $\text{Frob}^n(I)A = I^n$ .  $\square$

<sup>4</sup> Let  $K$  be the field of fractions of  $A$ . Embed  $B$  in a finite dimensional vector space  $K^n$  and choose a projection  $K^n \rightarrow K$  so that the image of 1 under the composition is non-zero. The required map  $B \rightarrow A$  is obtained from this composition by clearing denominators.

*Remark 2.11.* It follows from the last property that the tight closure of an ideal is contained in its radical. In particular, radical ideals are tightly closed. It had been conjectured that  $IA^+ \cap A$  is equal to the tight closure of  $I$ , but this has now been disproved by the counterexample in [9]. Nonetheless, for parameter ideals they are the same by [79]; see also Remark 2.14(3).

To convince the reader of the strength of these properties, I provide a short tight closure proof of the following celebrated theorem of Hochster and Roberts (see also Theorem 4.5 below):

**Theorem 2.12 (Hochster–Roberts [43]).** *If  $R \rightarrow S$  is a cyclically pure extension of Noetherian local rings (that is to say, if  $IS \cap R = I$  for all ideals  $I \subseteq R$ ), and if  $S$  is regular, then  $R$  is Cohen–Macaulay.*

*Proof.* We leave it to the reader to verify that all properties pass to the completion, and so we may assume that  $R$  and  $S$  are moreover complete. Let  $(x_1, \dots, x_d)$  be a system of parameters in  $R$ . We have to show that it is  $R$ -regular, that is to say, that  $(J_i : x_{i+1})$  is equal to  $J_i := (x_1, \dots, x_i)R$ , for all  $i$ . By colon capturing, the former ideal is contained in the tight closure of  $J_i$ , whence by weak persistence (note that  $R$  is a domain since  $R \rightarrow S$  is in particular injective) in the tight closure of  $J_i S$ . Since  $S$  is regular, the latter ideal is tightly closed, showing that  $(J_i : x_{i+1})$  is contained in  $J_i S \cap R = J_i$ , where the last equality follows from cyclic purity.  $\square$

### 2.3 Tight closure and singularities

The regularity property suggests the following paradigm: the larger the collection of tightly closed ideals in a Noetherian ring  $A$  of prime characteristic  $p$ , the closer it is to being regular.

**Definition 2.13.** If every ideal is tightly closed,  $A$  is called *weakly F-regular*; if every localization is weakly F-regular, then  $A$  is called *F-regular*. If  $A$  is local and some parameter ideal is tightly closed, the ring is called *F-rational*. If  $A$  is reduced and the Frobenius is pure on  $A$ , that is to say, each base change of the Frobenius is injective, then  $A$  is called *F-pure*.

Table 1: Correspondence (partly conjectural)

F-singularity	Classical singularity
F-rational	Rational singularities
F-pure	Log-canonical
F-regular	Log-terminal



*Remark 2.14.* In Section 5, I will discuss the connection with the singularities in the above table. For now, let me just make some remarks on the tight closure versions.

1. Regular rings are F-regular by Theorem 2.10, but these are not the only ones. In fact, the proof of Theorem 2.12 shows that any cyclically pure subring of a regular local ring is F-regular. It is a major open question whether in general F-regular and weakly F-regular are the same. This is tied in with the problem of the behavior of tight closure under localization (only recently, [9], tight closure has been shown to not always commute with localization).
2. Notwithstanding, the property of being F-rational is preserved under localization. Therefore, any weakly F-regular ring is F-rational. Moreover, in an F-rational local domain, every parameter ideal, and more generally, every ideal generated by part of a system of parameters is tightly closed ([45, Theorem 4.2]). In particular, every principal ideal is tightly closed, since it is generated by a parameter. Hence every principal ideal is integrally closed by the last property in Theorem 2.10, from which it follows that an F-rational ring is normal. Moreover, by colon capturing, any system of parameters is regular (same argument as in the proof of Theorem 2.12), proving that an F-rational ring is Cohen–Macaulay.
3. Smith has shown in [79] that for a local domain  $A$ , the tight closure of a parameter ideal  $I$  is equal to  $IA^+ \cap A$ , and hence such an  $A$  is F-rational if and only if some parameter ideal is contracted from  $A^+$ .
4. A weakly F-regular ring is F-pure: given an ideal  $I \subseteq A$ , we have

$$IA^{1/p} \cap A \subseteq IA^+ \cap A \subseteq \text{cl}(I),$$

by Remark 2.11. Hence, by weak F-regularity, the latter ideal is just  $I$ . This shows that the Frobenius is cyclically pure. Since  $A$  is normal by remark (2) above, this in turn implies the purity of the Frobenius by [29, Theorem 2.6].

### 3 Difference closure and the ultra-Frobenius

From the proofs of the five basic properties of tight closure listed in Theorem 2.10, we extract the following three key properties of the Frobenius: its functoriality, its contractive nature (sending a power of an ideal into a higher power of the ideal),<sup>5</sup> and its preservation of regular sequences. Moreover, it is not necessary that the Frobenius acts on the ring itself; it suffices that it does this on some faithfully flat overring. So I propose the following formalization of tight closure.

**Definition 3.1 (Difference hull).** Let  $\mathcal{C}$  be a category of Noetherian rings (at this point we do not need to make any characteristic assumption). A *difference*

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<sup>5</sup> The Frobenius is a contractive homeomorphism on the metric space given by the maximal adic topology; this is not to be confused with tight closure coming from contraction in finite extensions.

*hull* on  $\mathfrak{C}$  is a functor  $D(\cdot)$  from  $\mathfrak{C}$  to the category of difference rings,<sup>6</sup> and a natural transformation  $\eta$  from the identity functor to  $D(\cdot)$ , with the following three additional properties:

1. each  $\eta_A : A \rightarrow D(A)$  is faithfully flat;
2. the endomorphism  $\sigma_A$  of  $D(A)$  preserves  $D(A)$ -regular sequences;
3. for any ideal  $I \subseteq A$ , we have  $\sigma_A(I) \subseteq I^2 D(A)$ .

Functoriality here means that, for each  $A$  in  $\mathfrak{C}$ , we have a ring  $D(A)$  together with an endomorphism  $\sigma_A$ , and a ring homomorphism  $\eta_A : A \rightarrow D(A)$ , such that for each morphism  $A \rightarrow B$  in  $\mathfrak{C}$ , we get an induced morphism of difference rings  $D(A) \rightarrow D(B)$  for which the diagrams commute.

$$\begin{array}{ccccc}
 A & \xrightarrow{\eta_A} & D(A) & \xrightarrow{\sigma_A} & D(A) \\
 \downarrow & & \downarrow & & \downarrow \\
 B & \xrightarrow{\eta_B} & D(B) & \xrightarrow{\sigma_B} & D(B).
 \end{array}$$

Since  $\eta_A$  is in particular injective, we will henceforth view  $A$  as a subring of  $D(A)$  and omit  $\eta_A$  from our notation. Given a difference hull  $D(\cdot)$  on some category  $\mathfrak{C}$ , we define the *difference closure*  $\text{cl}^D(I)$  of an ideal  $I \subseteq A$  of a member  $A$  of  $\mathfrak{C}$  as follows: an element  $z \in A$  belongs to  $\text{cl}^D(I)$  if there exists  $c \in A^\circ$  such that

$$c\sigma^n(z) \in \sigma^n(I)D(A) \tag{5}$$

for all  $n \gg 0$ . Here,  $\sigma^n(I)D(A)$  denotes the ideal in  $D(A)$  generated by all  $\sigma^n(y)$  with  $y \in I$ , where  $\sigma$  is the endomorphism of the difference ring  $D(A)$ . It is crucial here that  $c$  belongs to the original ring  $A$ , although the membership relations in (5) are inside the bigger ring  $D(A)$ . An ideal that is equal to its difference closure will be called *difference closed*. One easily checks that  $\text{cl}^D(I)$  is a difference closed ideal containing  $I$ .

*Example 3.2 (Frobenius hull).* It is clear that our definition is inspired by the membership test (2) for tight closure, and indeed, this is just a special case. Namely, for a fixed prime number  $p$ , let  $\mathfrak{C}_p$  be the category of all Noetherian rings of characteristic  $p$  and let  $D(\cdot)$  be the functor assigning to a ring  $A$  the difference ring  $(A, \text{Frob}_A)$ . It is easy to see that this makes  $D(\cdot)$  a difference hull in the above sense, and the difference closure with respect to this hull is just the tight closure of the ideal.

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<sup>6</sup> A *difference ring* is a ring with an endomorphism; a *morphism of difference rings* is a ring homomorphism between difference rings that commutes with the respective endomorphisms.

*Remark 3.3.* Let  $\mathcal{C}$  be a category with a difference hull  $D(\cdot)$ , consisting of local rings. We can extend this difference hull to any Noetherian ring  $A$  all of whose localizations belong to  $\mathcal{C}$ . Indeed, let  $D(A)$  be the Cartesian product of all  $D(A_{\mathfrak{m}})$  where  $\mathfrak{m}$  runs over all maximal ideals of  $A$ . The product of all  $\sigma_{A_{\mathfrak{m}}}$  is then an endomorphism satisfying the conditions of difference hull in Definition 3.1. In particular, this yields a difference closure on  $A$  as well.

Before we discuss how we will view tight closure in characteristic zero as a difference closure, we discuss the difference analogue of three of the five key properties of tight closure (for the remaining two, one needs some additional assumptions, which I will not discuss here).

**Theorem 3.4.** *Let  $\mathcal{C}$  be a category endowed with a difference hull  $D(\cdot)$ , and let  $A \rightarrow B$  be a morphism in  $\mathcal{C}$  with  $B$  a domain.*

*(weak persistence)* If  $A \subseteq B$  is injective, then  $\text{cl}^D(I)B \subseteq \text{cl}^D(IB)$ , for each ideal  $I \subseteq A$ .

*(regular closure)* If  $A$  is a regular local ring, then every ideal is difference closed.

*(color capturing)* If  $A$  is a homomorphic image of a local Cohen–Macaulay ring, then  $((x_1, \dots, x_i)A : x_{i+1}) \subseteq \text{cl}^D((x_1, \dots, x_i)A)$ , for each  $i$  and each system of parameters  $(x_1, \dots, x_d)$ .

*Proof.* The arguments in the proof of Theorem 2.10 carry over easily, once we have shown that  $\sigma : A \rightarrow D(A)$  is flat whenever  $A$  is regular. This follows from the fact that  $\sigma$  preserves regular sequences, so that  $D(A)$  is a balanced big Cohen–Macaulay  $A$ -algebra, whence flat by Proposition 2.5.  $\square$

*Remark 3.5.* Note that exactly these three properties were required to deduce Theorem 2.12.

### 3.1 Lefschetz rings

By a *Lefschetz ring*,<sup>7</sup> we mean a ring of characteristic zero which is realized as the ultraproduct of rings of prime characteristic. More precisely, let  $W$  be an infinite index set and  $A_w$  a ring, for each  $w \in W$ . Let  $A_\infty$  be the Cartesian product of the  $A_w$ . We may view  $A_\infty$  as a  $\mathbb{Z}_\infty$ -algebra, where  $\mathbb{Z}_\infty$  is the corresponding Cartesian power of  $\mathbb{Z}$ . We call a prime ideal  $\mathfrak{p}$  in  $\mathbb{Z}_\infty$  *non-standard*, if it is a minimal prime ideal of the direct sum ideal  $\bigoplus_w \mathbb{Z} \subseteq \mathbb{Z}_\infty$ . An *ultraproduct* of the rings  $A_w$  is any residue ring of the form  $A_{\mathfrak{p}} := A_\infty / \mathfrak{p}A_\infty$ , where  $\mathfrak{p} \subseteq \mathbb{Z}_\infty$  is some non-standard prime ideal. Given elements  $a_w \in A_w$ , we call the image of the sequence  $(a_w)_w$  in  $A_{\mathfrak{p}}$  the *ultraproduct* of the  $a_w$ . If each  $A_w$  is a  $Z$ -algebra, for some ring  $Z$ , then so is  $A_{\mathfrak{p}}$ . The structure map  $Z \rightarrow A_{\mathfrak{p}}$  is given as follows: if we write  $z_w$  for  $z$  viewed as an element of  $A_w$ , then we send  $z$  to the ultraproduct  $z_{\mathfrak{p}}$  of the  $z_w$ .

<sup>7</sup> The designation alludes to an old heuristic principle in algebraic geometry regarding transfer between positive and zero characteristic, which Weil [88] attributes to Lefschetz.

Although a simple and elegant algebraic definition, the aforesaid is not the usual definition of an ultraproduct, and to formulate the main properties and prove them, we need to turn to its classical definition from logic. Namely, let  $\mathcal{U}$  be an ultrafilter on  $W$  – that is to say, a collection of infinite subsets of  $W$  closed under finite intersections and supersets, and such that any subset of  $W$  or its complement belongs to  $\mathcal{U}$ . Let  $\mathfrak{a}_{\mathcal{U}}$  be the ideal in  $A_{\infty}$  of all sequences almost all of whose entries are zero (a property is said to hold *for almost all*  $w$  if the subset of all indices  $w$  for which it holds belongs to the ultrafilter). We call the residue ring  $A_{\mathfrak{a}_{\mathcal{U}}} := A_{\infty}/\mathfrak{a}_{\mathcal{U}}$  the *ultraproduct*<sup>8</sup> of the  $A_w$ . To connect this to our previous definition, one then shows that  $\mathfrak{a}_{\mathcal{U}}$  is of the form  $\mathfrak{p}A_{\infty}$  for some non-standard prime  $\mathfrak{p}$ . More precisely,  $\mathfrak{p}$  is generated by all characteristic functions  $1_W$  (viewed as elements in  $\mathbb{Z}_{\infty}$ ) with  $W \notin \mathcal{U}$ . The main property of an ultraproduct is the following version of what logicians call Łos’ Theorem (its proof is a straightforward verification of the definitions [77, Theorem 2.3.1]).

**Proposition 3.6 (Equational Łos’ Theorem).** *Let  $A_{\mathfrak{a}_{\mathcal{U}}}$  be the ultraproduct of rings  $A_w$ , let  $\mathbf{a}_w$  be a tuple of length  $n$  in  $A_w$  and let  $\mathbf{a}_{\mathfrak{a}_{\mathcal{U}}}$  be their ultraproduct in  $A_{\mathfrak{a}_{\mathcal{U}}}$ . Given a finite set of polynomials  $f_1, \dots, f_s \in \mathbb{Z}[x]$  in  $n$  indeterminates  $x$ , we have that  $f_1(\mathbf{a}_w) = \dots = f_s(\mathbf{a}_w) = 0$  in  $A_w$  for almost all  $w$  if and only if  $f_1(\mathbf{a}_{\mathfrak{a}_{\mathcal{U}}}) = \dots = f_s(\mathbf{a}_{\mathfrak{a}_{\mathcal{U}}}) = 0$  in  $A_{\mathfrak{a}_{\mathcal{U}}}$ .*

*Remark 3.7.* Instead of just equations, we may also include inequations. If all  $A_w$  are  $Z$ -algebras, over some ring  $Z$ , then so is  $A_{\mathfrak{a}_{\mathcal{U}}}$ , and we may take the polynomials  $f_i$  with coefficients over  $Z$ . The full, model-theoretic, version, Łos’ Theorem, allows for arbitrary first-order sentences, which are obtained from equational formulae by taking finite Boolean combinations and quantification (for in-depth discussions of ultraproducts, see [12, 18, 44]; for a brief review see [69, Section 2] or [77, Section 2]). However, the above version is often sufficient to prove transfer results between an ultraproduct and its components. For instance, one easily deduces from it that  $A_{\mathfrak{a}_{\mathcal{U}}}$  is reduced (respectively a domain, or a field), if and only if almost all components  $A_w$  are. Indeed, reducedness follows from the equation  $x^2 = 0$  only having the zero solution. Unfortunately, one of the most fundamental properties used in commutative algebra, Noetherianity, is rarely preserved under ultraproducts (the case of fields mentioned above is a providential exception). One of our main tasks, therefore, will be to circumvent this major obstacle.

*Remark 3.8.* One can also define ultraproducts sheaf-theoretically as follows (see [77, Section 2.6]). Let  $A_w$  be a collection of rings indexed by an infinite set  $W$ . Viewing  $W$  in the discrete topology, we can encode this as a sheaf of rings  $\mathcal{A}$  on  $W$ , whose stalk  $\mathcal{A}_w$  at  $w \in W$  is equal to  $A_w$ . There exists a unique Hausdorff compactification  $i: W \rightarrow W^{\vee}$ , called the *Stone-Čech compactification* of  $W$ , where  $W^{\vee}$  is the set of all maximal filters on  $W$ . Let  $\mathcal{A}^{\vee} := i_*\mathcal{A}$  be the direct image sheaf of  $\mathcal{A}$  under

<sup>8</sup> More generally, if  $A_w$  are certain algebraic, or more precisely, first-order structures, then their ultraproduct is defined in a similar way, by taking the quotient of the Cartesian product  $A_{\infty}$  modulo the equivalence relation that two sequences are equivalent if and only if almost all their entries are the same.

*i.* Then there is a one-one correspondence between all different ultraproducts  $A_{\mathfrak{h}}$  of  $A_w$  and all stalks of  $\mathcal{A}^\vee$  at boundary points, that is to say, at points in  $W^\vee - W$ .

Let  $A_{\mathfrak{h}}$  be an ultraproduct of rings  $A_w$  of positive characteristic  $p_w$ . Using the above theorem, one can show that  $A_{\mathfrak{h}}$  has equal characteristic zero if and only if the  $p_w$  are *unbounded*, meaning that for every  $N$ , almost all  $p_w > N$ . A ring with this property will be called a *Lefschetz ring*. Our key example is:

**Proposition 3.9.** *The field of complex numbers is a Lefschetz ring.*

*Proof.* For each prime number  $p$ , let  $\overline{\mathbb{F}}_p$  be the algebraic closure of the  $p$ -element field, and let  $F_{\mathfrak{h}}$  be their ultraproduct (with respect to some ultrafilter on the set of prime numbers). By our above discussions  $F_{\mathfrak{h}}$  is again a field of characteristic zero. Since one can express in terms of equations that a field is algebraically closed, Proposition 3.6 proves that  $F_{\mathfrak{h}}$  is algebraically closed. One checks that its cardinality is that of the continuum. So we may invoke Steinitz's theorem to conclude that it must be the unique algebraically closed field of characteristic zero of that cardinality, to wit,  $\mathbb{C}$ .  $\square$

*Remark 3.10.* It is clear from the above proof that the isomorphism  $F_{\mathfrak{h}} \cong \mathbb{C}$  is far from explicit. This is the curse when working with ultraproducts: they are highly non-constructive; after all, the very existence of ultrafilters hinges on the Axiom of Choice. Steinitz's theorem holds of course also in higher cardinalities, and we may therefore extend the above result to: every algebraically closed field of characteristic zero of cardinality  $2^\kappa$  for some infinite cardinal  $\kappa$  is Lefschetz.<sup>9</sup> In particular, any field of characteristic zero is contained in some Lefschetz field. No countable field can be Lefschetz because of cardinality reasons. In particular, the algebraic closure of  $\mathbb{Q}$  is an example of an algebraically closed field of characteristic zero which is not Lefschetz.

Our main interest in Lefschetz rings comes from the following observation. Let  $\text{Frob}_\infty := \prod_w \text{Frob}_{A_w}$  be the product of the Frobenii on the components. It is not hard to show that any non-standard prime ideal  $\mathfrak{p}$  is generated by idempotents, and hence  $\text{Frob}_\infty(\mathfrak{p}A_\infty) = \mathfrak{p}A_\infty$ . In particular, we get an induced homomorphism on  $A_{\mathfrak{h}}$ , which we call the *ultra-Frobenius* of  $A_{\mathfrak{h}}$  and which we continue to denote by  $\text{Frob}_\infty$ . In other words, Lefschetz rings are difference rings in a natural way, and this is our point of departure to define tight closure in characteristic zero.

## 4 Tight closure in characteristic zero

As mentioned above, we will use Lefschetz rings as difference hulls to define tight closure in characteristic zero. However, before we describe the theory, let us see how Hochster and Huneke arrive at a tight closure operation in characteristic zero, which

<sup>9</sup> If one assumes the (generalized) Continuum Hypothesis, then this just means any uncountable algebraically closed field.

for emphasis, we will denote  $\text{cl}^{\text{HH}}(\cdot)$ .<sup>10</sup> Their method goes back, once more, to the seminal work [58] of Peskine and Szpiro: use generic flatness and Artin Approximation to lift results in characteristic  $p$  to characteristic zero. The method was elaborated upon further by Hochster [28], and can be summarized briefly as follows (see also [11, Chap. 8] or [85]). Given a Noetherian ring  $A$  of equal characteristic zero we first construct a suitable finitely generated subalgebra  $A_0 \subseteq A$  and then reduce modulo  $p$ , to obtain the rings  $A_0/pA_0$  of characteristic  $p$ . Of course, we must do this in such way that properties of  $A$  are reflected by properties of  $A_0$  and these in turn should be reflected by properties of the closed fibers  $A_0/pA_0$ . The former requires Artin Approximation (see below) and the latter generic flatness. Moreover, due to these two techniques, only properties expressible by systems of equations stand a chance of being transferred. To carry this out, quite some machinery is needed, which unfortunately drowns tight closure’s elegance in technical prerequisites; see [45, Appendix] or [41].

So, let us describe the ultraproduct method, in which transfer will be achieved mainly through Łos’ Theorem. Given a Noetherian ring of equal characteristic zero, we must construct a Lefschetz ring  $L(A)$  containing  $A$  in such a way that the functor  $L(\cdot)$  constitutes a difference hull; we will call  $L(\cdot)$  a *Lefschetz hull*. Condition 3.1.(3) is clear, whereas a simple application of Łos’ Theorem can be used to prove that any ultraproduct of regular sequences is again a regular sequence, showing that also condition 3.1.(2) is automatically satisfied. So, apart from functoriality, remains to construct  $L(A)$  so that the embedding  $A \subseteq L(A)$  is faithfully flat. Part of functoriality is easily obtained: if we have a Lefschetz hull  $L(A)$  for  $A$ , and  $I \subseteq A$  is an ideal, then we can take  $L(A/I) := L(A)/IL(A)$  as a Lefschetz hull for  $A/I$ . Indeed,  $L(A)/IL(A)$  is again Lefschetz by Lemma 4.1 below; all three properties in Definition 3.1 now follow by base change.

**Lemma 4.1.** *Any residue ring of a Lefschetz ring modulo a finitely generated ideal is again Lefschetz.*

*Proof.* Let  $B_{\mathfrak{q}}$  be an arbitrary Lefschetz ring and  $J \subseteq B_{\mathfrak{q}}$  a finitely generated ideal. By assumption,  $B_{\mathfrak{q}}$  is the ultraproduct of rings  $B_w$  of positive characteristic. Let  $f_1, \dots, f_s \in B_{\mathfrak{q}}$  generate  $J$ , and choose  $f_{i_w} \in B_w$  so that for each  $i$ , the ultraproduct of the elements  $f_{i_w}$  is equal to  $f_i$ . Let  $J_w := (f_{1_w}, \dots, f_{s_w})B_w$ . It is an easy but instructive exercise on Łos’ Theorem to show that the image of  $\prod J_w \subseteq B_{\infty}$  in  $B_{\mathfrak{q}}$  is equal to  $J$ , and that the ultraproduct of the  $B_w/J_w$  is equal to  $B_{\mathfrak{q}}/J$ , proving in particular that the latter is again Lefschetz.  $\square$

With notation as in the proof, we call  $J$  the *ultraproduct* of the  $J_w$ . A note of caution: infinitely generated ideals in  $B_{\mathfrak{q}}$  need not be realizable as an ultraproduct of ideals, and so their residue rings need not be Lefschetz. An example is the ideal of infinitesimals, introduced in Proposition 4.10 below.

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<sup>10</sup> In fact, there are several candidates for tight closure in characteristic zero, depending on the choice of a base field, which are only conjecturally equivalent;  $\text{cl}^{\text{HH}}(\cdot)$  is the smallest of these variants.

We will prove the existence of Lefschetz hulls for two classes of rings: affine algebras over a field  $K$  of characteristic zero,<sup>11</sup> and equicharacteristic zero Noetherian local rings. Since the only issue is the flatness of the hull, we may always pass to a faithfully flat extension of the ring  $A$ . Hence, in either case, we can find, by Remark 3.10, a sufficient large algebraically closed Lefschetz field  $K$ , such that in the affine case,  $A$  is finitely generated over  $K$ , and in the local case,  $A$  is complete, with residue field  $K$ . By Noether Normalization and Cohen’s structure theorem,  $A$  is a residue ring of respectively the polynomial ring  $K[x]$  or the power series ring  $K[[x]]$ , where  $x$  is a finite tuple of indeterminates. By our above discussion on residue rings, this then reduces the problem to finding a Lefschetz hull of the polynomial ring and the power series ring respectively. We start with the easiest case, the polynomial case.

### 4.1 Tight closure for affine algebras

Let  $K$  be an (algebraically closed) Lefschetz field, realized as the ultraproduct of (algebraically closed) fields  $K_w$  of positive characteristic, and let  $x$  be a finite tuple of indeterminates. Put  $A_w := K_w[x]$  and let  $A_{\mathfrak{I}}$  be their ultraproduct. Clearly,  $K$  is a subring of  $A_{\mathfrak{I}}$ . For each  $i$ , let us write also  $x_i$  for the ultraproduct of the constant sequence  $x_i$ . By Łoś’ Theorem, each  $x_i$  is transcendental over  $K$ , and hence the polynomial ring  $A = K[x]$  embeds in  $A_{\mathfrak{I}}$ . So remains to show that this embedding is faithfully flat. This fact was first observed by van den Dries in [15], and used by him and Schmidt in [64] to deduce several uniform bounds in polynomial rings; further extensions based on this method can be found in [65,66,75]; for an overview, see [77].

**Proposition 4.2.** *The embedding  $A \subseteq A_{\mathfrak{I}}$  is faithfully flat.*

*Proof.* Since  $K$  is algebraically closed, every maximal ideal  $\mathfrak{m} \subseteq A$  is of the form  $(x_1 - a_1, \dots, x_n - a_n)A$  by Hilbert’s Nullstellensatz. After a change of coordinates, we may assume that  $\mathfrak{m} = (x_1, \dots, x_d)A$ . Since  $\mathfrak{m}A_{\mathfrak{I}}$  is the ultraproduct of the maximal ideals  $\mathfrak{m}_w := (x_1, \dots, x_d)A_w$ , it too is maximal, and  $(A_{\mathfrak{I}})_{\mathfrak{m}A_{\mathfrak{I}}}$  is the ultraproduct of the localizations  $(A_w)_{\mathfrak{m}_w}$ . It suffices, therefore, to show that the homomorphism  $A_{\mathfrak{m}} \rightarrow (A_{\mathfrak{I}})_{\mathfrak{m}A_{\mathfrak{I}}}$  is flat. We already remarked that ultraproducts preserve regular sequences, showing that  $(x_1, \dots, x_d)$  is  $(A_{\mathfrak{I}})_{\mathfrak{m}A_{\mathfrak{I}}}$ -regular. Hence,  $(A_{\mathfrak{I}})_{\mathfrak{m}A_{\mathfrak{I}}}$  is a big Cohen–Macaulay  $A_{\mathfrak{m}}$ -algebra. Moreover, every permutation of an  $(A_{\mathfrak{I}})_{\mathfrak{m}A_{\mathfrak{I}}}$ -regular sequence is again regular, since this is true in each  $(A_w)_{\mathfrak{m}_w}$ . So  $(A_{\mathfrak{I}})_{\mathfrak{m}A_{\mathfrak{I}}}$  is a balanced big Cohen–Macaulay algebra by Lemma 2.2. Since  $A_{\mathfrak{m}}$  is regular,  $A_{\mathfrak{m}} \rightarrow (A_{\mathfrak{I}})_{\mathfrak{m}A_{\mathfrak{I}}}$  is flat by Proposition 2.5.  $\square$

*Remark 4.3.* The original proof in [15] employed an induction on the number of indeterminates based on classical arguments of Hermann from constructive

<sup>11</sup> We call an algebra  $A$  *affine* if it is finitely generated over a field.

commutative algebra. The above approach via big Cohen–Macaulay algebras has the advantage that one can extend this method to other situations, like Proposition 4.7 below. Yet another approach, through a coherency result due to Vasconcelos, can be found in [2].

By our previous discussion, we have now constructed a Lefschetz hull  $L(C)$  for any affine algebra  $C$  over a field  $k$  of characteristic zero. Namely, with notation as above, if  $C \otimes_k K \cong A/I$ , then  $L(C) := A_{\mathfrak{q}}/IA_{\mathfrak{q}}$ . The positive characteristic affine algebras  $C_w$  whose ultraproduct equals  $L(C)$  will be called *approximations* of  $C$ . It is justified to call  $L(C)$  a hull, since any  $K$ -algebra homomorphism  $C \rightarrow B_{\mathfrak{q}}$  into an ultraproduct  $B_{\mathfrak{q}}$  of finitely generated  $K_w$ -algebras  $B_w$ , induces a unique homomorphism of Lefschetz rings  $L(C) \rightarrow B_{\mathfrak{q}}$  (that is to say, an ultraproduct of  $K_w$ -algebras  $C_w \rightarrow B_w$ ). It follows that  $L(C)$  only depends on the choice of algebraically closed Lefschetz field  $K$ , and on the way we represent the latter as an ultraproduct of algebraically closed fields of positive characteristic, that is to say, on the choice of ultrafilter (see Remark 3.8 for how the different choices are related). An example due to Brenner and Katzman [8] indicates that different choices of ultrafilter may lead to different tight closure notions: this is true for ultra-closure as defined below in Section 4.11, and most likely also for tight closure to be defined shortly; see [8, Remark 4.10]. Nonetheless, this dependence on the ultrafilter is in all what we will do with Lefschetz hulls of no consequence, and we will henceforth pretend that, once  $K$  has been fixed, the Lefschetz hull is unique. Nonetheless, even when fixing the ultrafilter, the approximations  $C_w$  are not uniquely defined by  $C$ : given a second approximation  $C'_w$ , we can at best conclude that  $C_w \cong C'_w$  for almost all  $w$ . Again, this seems not to matter. The approximations do carry a lot of the structure of the affine algebra (to a much larger extent than the characteristic  $p$  reductions of  $C$  used in the Hochster–Huneke tight closure  $\text{cl}^{\text{HH}}(\cdot)$  in characteristic zero; see [73, Section 2.17]), and we summarize this in the following theorem, stated without proof.

**Theorem 4.4 ([69, Theorem 4.18]).** *Let  $K$  be an algebraically closed Lefschetz field, let  $C$  be a  $K$ -affine algebra, and let  $C_w$  be approximations of  $C$ . Then  $C$  has the same dimension and depth as almost all  $C_w$ . Moreover,  $C$  is a domain, normal, regular, Cohen–Macaulay, or Gorenstein, if and only if almost all  $C_w$  are.  $\square$*

In particular, we can extend the Lefschetz hull to any localization of a  $K$ -affine algebra: if  $\mathfrak{p}$  is a prime ideal in  $C$ , then by construction,  $L(C)/\mathfrak{p}L(C)$  is a Lefschetz hull of  $C/\mathfrak{p}$ . By Theorem 4.4, the approximations of  $C/\mathfrak{p}$  are domains, and hence, by Łos’ Theorem, so is their ultraproduct, proving that  $\mathfrak{p}L(C)$  is a prime ideal. Hence, we can take  $L(C)_{\mathfrak{p}L(C)}$  as a Lefschetz hull of  $C_{\mathfrak{p}}$ . Since we will treat the local case below, I skip the details.

In any case, we may apply the difference closure theory from Section 3, to define the *tight closure*<sup>12</sup>  $\text{cl}_C(I)$  of an ideal  $I \subseteq C$  as the collection of all  $z \in C$  for which there exists  $c \in C^\circ$  such that  $c\text{Frob}_\infty^n(z) \in \text{Frob}_\infty^n(I)L(C)$  for all  $n \gg 0$ , where  $\text{Frob}_\infty$

<sup>12</sup> I previously referred to it as *non-standard* tight closure.



is the ultra-Frobenius on the Lefschetz hull  $L(C)$ . We remind the reader that the analogues of all five properties in Theorem 2.10 hold in characteristic zero (we did not give details for two of these; they can be found in [69]).<sup>13</sup> In particular, any ideal in a polynomial ring, and more generally, in a regular  $K$ -algebra, is tightly closed.

As in positive characteristic, we immediately get the characteristic zero version of the Hochster–Roberts theorem for affine algebras (see Remark 3.5). To state the original version, let us call an affine scheme  $X$  a *quotient singularity*, if there exists a smooth scheme  $Y = \text{Spec}(A)$  over  $\mathbb{C}$ , and a linearly reductive algebraic group  $G$  (meaning, that  $G$  is the complexification of a compact real Lie group), acting  $\mathbb{C}$ -rationally on  $Y$  by  $\mathbb{C}$ -algebra automorphisms, so that  $X$  is the quotient  $Y/G$  given as the affine scheme  $\text{Spec}(A^G)$ , where  $A^G$  is the subring of  $A$  of  $G$ -invariant elements.

**Theorem 4.5 (Hochster–Roberts, [43]).** *A quotient singularity is Cohen–Macaulay.*

*Proof.* With notation as above, from Lie theory or a general argument about linearly reductive groups, we get the so-called *Reynolds operator*  $\rho_G: A \rightarrow A^G$ , that is to say, a homomorphism of  $A^G$ -modules. In particular, the inclusion  $A^G \subseteq A$  is split whence cyclically pure, and the result now follows, after localization, from the analogue of Theorem 2.12.

For another application in characteristic zero, first proven using deep methods from birational geometry [16], but subsequently reproved and generalized by a simple tight closure argument in characteristic  $p$  in [42], see [67]. Yet another problem requiring sophisticated methods for its solution, but now admitting a very simple tight closure proof, is the Briançon–Skoda Theorem.<sup>14</sup>

**Theorem 4.6.** *The following Briançon–Skoda type properties hold:*

(Briançon–Skoda, [10]) *If  $f$  is a power series without constant term in  $s$  variables  $x$  over  $\mathbb{C}$ , then  $f^s$  lies in the Jacobian ideal of  $f$ , that is to say,  $f^s \in (\partial f/\partial x_1, \dots, \partial f/\partial x_s)\mathbb{C}[[x]]$ ;*

(Lipman–Sathaye, [52]) *If  $R$  is a regular local ring and  $\mathfrak{a} \subseteq R$  an ideal generated by  $s$  elements, then the integral closure  $\overline{\mathfrak{a}^s}$  of  $\mathfrak{a}^s$  is contained in  $\mathfrak{a}$ ;*

(Hochster–Huneke, [35]) *If  $A$  is a Noetherian ring containing a field and if  $\mathfrak{a} \subseteq A$  is an ideal generated by  $s$  elements, then  $\overline{\mathfrak{a}^s} \subseteq \text{cl}(\mathfrak{a})$ .*

*Proof.* We start with the last assertion. Assume first that  $A$  has characteristic  $p$ . Let  $\mathfrak{a} = (a_1, \dots, a_s)A$  and assume  $z$  lies in the integral closure of  $\mathfrak{a}^s$ . Hence,

$$cz^N \in \mathfrak{a}^{sN}, \tag{6}$$

for all  $N$  and some  $c \in A^\circ$ . One easily verifies that  $\mathfrak{a}^{sN}$  is contained in  $(a_1^N, \dots, a_s^N)A$ . In particular, (6) with  $N = p^n$  yields the tight closure relation (2), showing that  $z \in$

<sup>13</sup> In fact, the plus closure property is almost trivial in characteristic zero, since for  $A$  normal, we always have  $IA^+ \cap A = I$  (see [11, Remark 9.2.4]).

<sup>14</sup> According to Wall [87], the question was originally posed by Mather.

$\text{cl}(I)$ . Suppose next that  $A$  is an affine algebra over a field of characteristic zero with approximations  $A_w$  and Lefschetz hull  $L(A)$ . Choose  $s$ -generated ideals  $\mathfrak{a}_w \subseteq A_w$  whose ultraproduct equals  $\mathfrak{a}L(A)$  (see the discussion following Lemma 4.1), and choose  $z_w \in A_w$  with ultraproduct equal to  $z$  viewed as an element in  $L(A)$ . By the previous argument, we have a tight closure relation

$$c_w \text{Frob}_{A_w}^n(z_w) \in \text{Frob}_{A_w}^n(\mathfrak{a}_w)A_w$$

in each  $A_w$ , for some  $c_w \in A_w^\circ$ . Taking ultraproducts, we get a relation

$$c_{\mathfrak{I}} \text{Frob}_{\infty}^n(z) \in \text{Frob}_{\infty}^n(I)L(A) \tag{7}$$

in  $L(A)$ , where  $c_{\mathfrak{I}}$  is the ultraproduct of the  $c_w$ . A priori,  $c_{\mathfrak{I}}$  does not belong to the subring  $A$  of  $L(A)$ , so that (7) is not a true tight closure relation. Nevertheless, in [69, Proposition 8.4], I show that there always exist test elements  $c_w \in A_w$  such that their ultraproduct lies in  $A^\circ$ . By a more careful bookkeeping, we can circumvent this complication altogether, at least when  $\mathfrak{a}$  has positive height, the only case of interest. Namely, an easy calculation (see [3, Theorem 6.13] or [77, Theorem 5.4.1]) yields the following variant of (6): for all  $N$ , we have an inclusion  $\mathfrak{a}^{sd}z^N \subseteq \mathfrak{a}^{sN}$ , where  $d$  is the degree of an integral equation exhibiting  $z \in \overline{\mathfrak{a}^s}$ . By Łos’ Theorem, we may choose the  $z_w$  to satisfy an integral equation of the same degree, and hence in the ultraproduct we get a relation  $\mathfrak{a}^{sd}\text{Frob}_{\infty}^n(z) \subseteq \text{Frob}_{\infty}^n(I)L(A)$ . Taking, therefore, any  $c$  in  $\mathfrak{a}^{sd} \cap A^\circ$  yields a true tight closure relation, proving that  $z \in \text{cl}(\mathfrak{a})$ . We will shortly define tight closure for Noetherian local rings containing  $\mathbb{Q}$ , and by Remark 3.3, we may then extend this to any Noetherian ring containing the rationals (see also [3, Section 6.17]); the previous argument is still applicable, thus completing the proof of the last assertion.

From this the validity of the second property in equal characteristic follows immediately,<sup>15</sup> since any ideal is tightly closed in a regular ring. To obtain the first property, a nice little exercise on the chain rule – which, incidentally, requires us to be in characteristic zero – and using that an element lies in the integral closure of an ideal if and only if it lies in the extension of the ideal under any  $\mathbb{C}$ -algebra homomorphism  $\mathbb{C}[[x]] \rightarrow \mathbb{C}[[t]]$  for  $t$  a single variable (see [70, Fact 5.1]), shows that  $f$  lies in the integral closure of its Jacobian ideal. The original Briançon–Skoda theorem then follows from the second property applied to the regular local ring  $\mathbb{C}[[x]]$  with  $\mathfrak{a}$  equal to the Jacobian ideal of  $f$  (see [70] for a different argument deducing the characteristic zero case from the characteristic  $p$  case using ultraproducts).  $\square$

### 4.2 Local case

To extend tight closure to an arbitrary Noetherian local ring containing  $\mathbb{Q}$ , we need to construct a Lefschetz hull for  $A := K[[x]]$ , where  $K$  is an algebraically closed

<sup>15</sup> So far, no tight closure proof in mixed characteristic exists.

Lefschetz field, given as the ultraproduct of algebraically closed fields  $K_w$  of prime characteristic. In analogy with the affine case, we expect the Lefschetz hull to be the ultraproduct  $A_{\mathfrak{I}}$  of the power series rings  $A_w := K_w[[x]]$ . However, there is no immediate  $K[x]$ -algebra embedding of  $A$  into  $A_{\mathfrak{I}}$ . The very existence of such an embedding, in fact, has non-trivial ramifications, as we shall see.

**Proposition 4.7.** *There exists an ultraproduct  $L(A)$  of the power series rings  $A_w$  and a faithfully flat  $K[x]$ -algebra embedding of  $A$  into  $L(A)$ .*

*Proof.* Once we have defined a  $K[x]$ -algebra homomorphism  $A \rightarrow L(A)$ , its flatness follows by the same argument as in the proof of Proposition 4.2. To prove the existence of  $A \rightarrow L(A)$ , we use Proposition 4.8 below, with  $Z := K[x]$  and  $B = A_{\mathfrak{I}}$ . To apply this result, let  $f_1(y) = \dots = f_s(y) = 0$  be a system of equations in the unknowns  $y$  with coefficients in  $Z$ . Given a solution  $\mathbf{y}$  in  $A = K[[x]]$ , we need to construct a solution  $\mathbf{z}$  in  $A_{\mathfrak{I}}$ . By Artin Approximation ([1, Theorem 1.10]), there exists already a solution  $\tilde{\mathbf{y}}$  in the Henselization  $Z^{\sim}$  of  $Z$ . Recall that the Henselization of  $Z$  (at the maximal ideal  $(x_1, \dots, x_n)Z$ ) is the smallest Henselian subring  $Z^{\sim}$  of  $A$  containing  $Z$  (see for instance [57]), and is equal to the ring of algebraic power series over  $K$ . Since the  $A_w$  are complete, they are in particular Henselian, and hence, by Łos' Theorem, so is their ultraproduct  $A_{\mathfrak{I}}$ . By the universal property of Henselization, we have a (unique)  $Z$ -algebra homomorphism  $Z^{\sim} \rightarrow A_{\mathfrak{I}}$ . The image of  $\tilde{\mathbf{y}}$  under this homomorphism is then the desired solution in  $A_{\mathfrak{I}}$ . By Proposition 4.8, there is therefore a  $Z$ -algebra homomorphism from  $A$  to some ultrapower  $L(A)$  of  $A_{\mathfrak{I}}$ . Since an ultraproduct of ultraproducts is itself an ultraproduct,  $L(A)$  is a Lefschetz ring.  $\square$

In the above proof, we used the following result, which originates with Henkin [25], and has proven to be useful in other situations related to Artin Approximation; for instance, see [6, Lemma 1.4] and [85, Lemma 12.1.3].

**Proposition 4.8 ([3, Corollary 2.5] or [77, Theorem 7.1.1]).** *For a Noetherian ring  $Z$ , and  $Z$ -algebras  $A$  and  $B$ , the following are equivalent:*

1. *Every system of polynomial equations with coefficients from  $Z$  which is solvable in  $A$ , is solvable in  $B$ ;*
2. *There exists a  $Z$ -algebra homomorphism  $A \rightarrow B_{\mathfrak{I}}$ , where  $B_{\mathfrak{I}}$  is some ultrapower of  $B$ .  $\square$*

Proposition 4.7, which was proven using Artin Approximation, in turn implies the following stronger form of Artin Approximation.

**Theorem 4.9 (Uniform strong Artin Approximation, [5, Theorem 4.3]).** *There exists a function  $N: \mathbb{N}^2 \rightarrow \mathbb{N}$  with the following property. Let  $K$  be a field, put  $Z := K[x]$  with  $x$  an  $n$ -tuple of indeterminates, and let  $\mathfrak{m}$  be the ideal generated by these indeterminates. Let  $f_1(y) = \dots = f_s(y) = 0$  be a polynomial system of equations in the  $n$  unknowns  $y$  with coefficients from  $Z$ , such that each  $f_i$  has total degree at most  $d$  (in  $x$  and  $y$ ). If there exists some  $\mathbf{y}$  in  $Z$  such that  $f_i(\mathbf{y}) \equiv 0 \pmod{\mathfrak{m}^{N(n,d)}Z}$  for all  $i$ , then there exists  $\mathbf{z}$  in  $K[[x]]$  such that  $f_i(\mathbf{z}) = 0$  for all  $i$ .*

*Proof.* Towards a contradiction, assume such a bound does not exist for the pair  $(d, n)$ , so that for each  $w \in \mathbb{N}$  we can find a counterexample consisting of a field  $K_w$ , and polynomial equations  $f_{1w}(y) = \cdots = f_{sw}(y) = 0$  in the unknowns  $y$  over  $Z_w := K_w[x]$  of total degree at most  $d$ , admitting an approximate solution  $\mathbf{x}_w$  in  $Z_w$  modulo  $\mathfrak{m}^w Z_w$  but no actual solution in  $A_w := K_w[[x]]$ . Note that the size,  $s$ , of these systems can be bounded in terms of  $d$  and  $n$  only (see for instance [77, Lemma 4.4.2]), and hence, in particular, can be taken independent from  $w$ . Let  $K$  and  $A_{\mathfrak{q}}$  be the ultraproduct of the  $K_w$  and  $A_w$  respectively, and let  $f_i$  and  $\mathbf{x}$  be the ultraproduct of the  $f_{iw}$  and  $\mathbf{x}_w$  respectively. Since ultraproducts commute with finite sums, each  $f_i$  is again a polynomial over  $K$  of total degree at most  $d$ . Moreover, by Łos' Theorem,  $f_i(\mathbf{x}) \equiv 0 \pmod{\mathfrak{m}^N R_{\mathfrak{q}}}$  for all  $N$ . By Proposition 4.10 below, we have an epimorphism  $A_{\mathfrak{q}} \rightarrow A := K[[x]]$  having kernel equal to the intersection of all  $\mathfrak{m}^N A_{\mathfrak{q}}$ . In particular, the image of  $\mathbf{x}$  under this surjection is a solution in  $A$  of the system  $f_1 = \cdots = f_s = 0$ .

Since there exists a  $Z$ -algebra homomorphism  $A \rightarrow L(A)$  by Proposition 4.7, where  $L(A)$  is some ultrapower of  $A_{\mathfrak{q}}$  (note that nowhere in the proof we used that the fields were algebraically closed nor that they had a certain characteristic), the image of  $\mathbf{x}$  in  $L(A)$  remains a solution of this system, and hence by Łos' Theorem, we can find, contrary to our assumptions, for almost each  $w$  (with respect to the larger ultrafilter defining  $L(A)$ ), a solution of  $f_{1w}(y) = \cdots = f_{sw}(y) = 0$  in  $A_w$ .  $\square$

**Proposition 4.10.** *There is a canonical epimorphism  $A_{\mathfrak{q}} \rightarrow A$  whose kernel is the ideal of infinitesimals  $\mathfrak{I}_{A_{\mathfrak{q}}} := \bigcap_N \mathfrak{m}^N A_{\mathfrak{q}}$ .*

*Proof.* Given  $f \in A_{\mathfrak{q}}$ , choose  $f_w \in A_w$  whose ultraproduct is equal to  $f$ , and expand as a power series

$$f_w = \sum_{v \in \mathbb{N}^n} a_{v,w} x^v$$

for some  $a_{v,w} \in K_w$ . For each  $v$ , let  $a_v \in K$  be the ultraproduct of the  $a_{v,w}$  and define

$$\tilde{f} := \sum_{v \in \mathbb{N}^n} a_v x^v \in A.$$

One checks that the map  $f \mapsto \tilde{f}$  is well-defined (that is to say, independent of the choice of the  $f_w$ ), and is a ring homomorphism, which is surjective, with kernel equal to the ideal of infinitesimals (see [77, Proposition 7.1.7]).  $\square$

In Section 6.2 below, we will rephrase this as  $A$  is the cataproduct  $A_{\mathfrak{q}}$  of the  $A_w$ . For some other uniform versions of Artin Approximation proven using ultraproducts, see [13, 14]. Returning to the issue of defining a Lefschetz hull, whence a tight closure operation, on the category of Noetherian local rings containing  $\mathbb{Q}$ , there is, however, a catch. Let  $\tilde{x}$  be a subtuple of  $x$ . Put  $\tilde{A} := K[[\tilde{x}]]$ , and let  $\tilde{A}_{\mathfrak{q}}$  be the ultraproduct of the  $\tilde{A}_w := K_w[[\tilde{x}]]$ . In the polynomial case, the inclusion  $K[\tilde{x}] \subseteq K[x]$  extends to a homomorphism of Lefschetz rings  $\tilde{B}_{\mathfrak{q}} \rightarrow B_{\mathfrak{q}}$ , where  $\tilde{B}_{\mathfrak{q}}$  and  $B_{\mathfrak{q}}$  are the respective ultraproducts of the  $K_w[\tilde{x}]$  and  $K_w[x]$ , making the whole construction functorial. However, it is no longer true that the inclusion  $\tilde{A} \subseteq A$  leads to a similar homomorphism  $\tilde{A}_{\mathfrak{q}} \rightarrow A_{\mathfrak{q}}$ . In fact, in [3, Section 4.33] we give a counterexample based on an observation of Roberts in [61] – which itself

was intended as a counterexample to an attempt of Hochster [31] to generalize tight closure via the notion of solid closure; see footnote 21. Nonetheless, such a homomorphism does exist if  $\tilde{x}$  is an initial tuple of  $x$ . To prove this, and thus salvage the functoriality of the construction, we have to prove a filtered version of Proposition 4.8. To apply this filtered version, however, a deeper Artin Approximation result, due to Rotthaus [63], is needed. In turn, we derive a filtered version of Theorem 4.9. All this needs some work and is explained in full detail in [3]; for a weaker form of functoriality, still sufficient for applications, see [77, Section 7.3]. Although functoriality is essential for applications, we will not say more about it here in order to keep the exposition transparent, but see the discussion preceding Theorem 5.3 on the relative Lefschetz hull, which recovers some of this functoriality required for defining ultra-cohomology.

In sum, we have now a tight closure operation on any equicharacteristic Noetherian local ring, and by Remark 3.3, even on any Noetherian ring containing a field. It has the five properties listed in Theorem 2.10. So, in equal characteristic zero, there are many, potentially different notions: the tight closure  $\text{cl}^{HH}(\cdot)$  (and its variants) introduced by Hochster and Huneke, our notion  $\text{cl}(\cdot)$  (which a priori depends on the choice of ultrafilter), and some variants that I will now discuss briefly. Let  $A$  be either an affine algebra or a local ring, with Lefschetz hull  $L(A)$ , realized as the ultraproduct of positive characteristic rings  $A_w$ , called *approximations* of  $A$ . Given an ideal  $I \subseteq A$ , choose  $I_w \subseteq A_w$  with ultraproduct equal to  $IL(A)$ .

**Definition 4.11 (Ultra-tight closure).** We define the *ultra-tight closure*<sup>16</sup> of  $I$  as the ideal  $\text{ultra-cl}(I) := J_{\mathfrak{q}} \cap A$ , where  $J_{\mathfrak{q}} \subseteq L(A)$  is the ultraproduct of the  $J_w := \text{cl}_{A_w}(I_w)$ .

In other words,  $z \in \text{ultra-cl}(I)$ , if almost each  $z_w$  belongs to the tight closure of  $I_w$ , for some choice of  $z_w \in A_w$  with ultraproduct equal to  $z$ . We have the following comparison between these three notions

$$\text{cl}^{HH}(I) \subseteq \text{ultra-cl}(I) \subseteq \text{cl}(I) \tag{8}$$

where the latter inclusion holds under some mild conditions; see [69, Theorems 8.5 and 10.4] for the affine, and [3, Corollaries 6.23 and 6.26] for the local case.

Another variant is derived from the observation that a single power of the ultra-Frobenius has all the contractive power, in the sense of Definition 3.1(3), needed to prove that regular rings are F-regular. So we may define the *simple tight closure* of an ideal  $I$  as the collection of all elements  $z \in A$  such that there exists  $c \in A^\circ$  for which  $c\text{Frob}_\infty(z) \in \text{Frob}_\infty(I)L(A)$ . Clearly, simple tight closure contains tight closure. All these variants satisfy the five main properties listed in Theorem 2.10, except that I do not know whether simple tight closure admits strong persistence. To define a last variant, we turn again to big Cohen–Macaulay algebras.

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<sup>16</sup> Elsewhere, I called this *generic* tight closure.

### 4.3 Big Cohen–Macaulay algebras

Hochster and Huneke [39] also construct (balanced) big Cohen–Macaulay algebras in equal characteristic zero, but their construction is highly non-canonical.<sup>17</sup> Using ultraproducts, we can restore canonicity modulo the choice of ultrafilter (for the affine case, see [72]).

**Theorem 4.12.** *Any equal characteristic Noetherian local ring  $A$  admits a balanced big Cohen–Macaulay algebra  $\mathcal{B}(A)$ .*

*Proof.* By the same argument as in Theorem 2.3, we may reduce to the case that  $A$  is a complete  $d$ -dimensional Noetherian local domain with algebraically closed residue field of characteristic zero. The local analogue of Theorem 4.4 gives the same transfer results between the ring  $A$  and its approximations  $A_w$  (see [3, Section 5]). In particular, by the same argument as in the discussion after Theorem 4.4, almost each approximation  $A_w$  is a domain. Hence, almost each absolute integral closure  $A_w^+$  is a balanced big Cohen–Macaulay  $A_w$ -algebra (see Remark 2.4). Let  $\mathcal{B}(A)$  be the ultraproduct of the  $A_w^+$ , so that we have a canonical homomorphism  $L(A) \rightarrow \mathcal{B}(A)$ . Hence,  $\mathcal{B}(A)$  is an  $A$ -algebra via the composition  $A \rightarrow L(A) \rightarrow \mathcal{B}(A)$ . Let  $\mathbf{x}$  be a system of parameters in  $A$ , and choose tuples  $\mathbf{x}_w$  over  $A_w$  with ultraproduct equal to  $\mathbf{x}$  (as a tuple in  $L(A)$ ). Since  $L(A)/\mathbf{x}L(A)$  is the Lefschetz hull of  $A/\mathbf{x}A$ , almost all  $A_w/\mathbf{x}_wA_w$  have, like  $A/\mathbf{x}A$ , dimension zero. Since almost each  $A_w$  has dimension  $d$ , almost each  $\mathbf{x}_w$  is therefore a system of parameters in  $A_w$ , whence  $A_w^+$ -regular by the proof of Theorem 2.3 (see Remark 2.4). By Łos’ Theorem, their ultraproduct  $\mathbf{x}$  is then  $\mathcal{B}(A)$ -regular.  $\square$

This proves the characteristic zero version of Theorem 2.7, as well as all the other Homological Conjectures that follow from the existence of big Cohen–Macaulay modules (see [28] for an exhaustive list). The big Cohen–Macaulay algebra construction is even weakly functorial (for details see [3, Section 7]). Unlike the Hochster–Huneke construction, we can preserve some of the good properties of the absolute integral closure from positive characteristic:  $\mathcal{B}(A)$  is absolutely integrally closed (though not integral over  $A$ !). In particular, the sum of prime ideals is either the unit ideal or again prime (see [72, Proposition 3.2 and Corollary 3.3]).

One can also define a closure operation using the big Cohen–Macaulay algebra  $\mathcal{B}(A)$  by taking for closure of the ideal  $I \subseteq A$ , the ideal  $\text{cl}^{\mathcal{B}}(I) := I\mathcal{B}(A) \cap A$ . It satisfies the five main properties of tight closure ([72, Theorem 4.2] and [3, Theorem 7.14]), and it even commutes with localization in certain cases ([72, Theorems 4.3 and 5.2]). If  $A$  is a complete local domain, then  $\text{cl}^{\mathcal{B}}(I) \subseteq \text{ultra-cl}(I)$ , with equality, by [79], if  $I$  is a parameter ideal.

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<sup>17</sup> The naive guess that absolute integral closure also yields big Cohen–Macaulay algebras in characteristic zero is manifestly false; see footnote 13.

## 5 Rational singularities

So far in our discussion, the tight closure theory in characteristic zero via ultraproducts has not brought anything new to the table: it merely gives an alternative, more streamlined, theory than the Hochster–Huneke constructions, but anything proven in our theory can also be proven with theirs. However, this is no longer true when it comes to  $F$ -singularities (the definitions in 2.13 extend verbatim to characteristic zero). Since the Hochster–Huneke tight closure is contained in ours, to be  $F$ -regular or  $F$ -rational in their theory is a priori weaker than in ours. For instance, it is not known whether  $F$ -rational in their sense implies rational singularities, but it does for our notion. They also introduced the notions of  $F$ -regular type and  $F$ -rational type, which do characterize the corresponding singularity notions given in Table (1), but these notions do not (a priori) behave well enough with respect to quotients.<sup>18</sup>

**Definition 5.1 (Rational Singularities).** An equicharacteristic zero excellent local domain  $R$  is said to have *rational singularities* if it is normal, analytically unramified, and Cohen–Macaulay, and the canonical embedding

$$H^0(W, \omega_W) \rightarrow H^0(X, \omega_X) \tag{9}$$

is surjective (it is always injective), where  $W \rightarrow X := \text{Spec}R$  is a resolution of singularities, and where in general,  $\omega_Y$  denotes the canonical sheaf on a scheme  $Y$ . To make the definition in the absence of a resolution of singularities, one calls  $(R, \mathfrak{m})$  (in either characteristic) *pseudo-rational* if the canonical map

$$\delta_W : H_{\mathfrak{m}}^d(R) \rightarrow H_E^d(\mathcal{O}_W) \tag{10}$$

is injective (it always is surjective), for all proper birational maps  $\pi : W \rightarrow X$  with  $W$  normal, where  $d$  is the dimension of  $R$  and  $E = \pi^{-1}(\mathfrak{m})$  the closed fiber of  $\pi$ .

Note that in (9) we take sheaf cohomology, whereas in (10) we take cohomology with support, which in the local case amounts to local cohomology. By [53, Section 2, Remark (a) and Example (b)], if  $\delta_W$  in (10) is injective for some non-singular  $W$ , then  $R$  is pseudo-rational, and, in fact, has rational singularities. By Matlis duality, therefore, if  $R$  is essentially of finite type over a field of characteristic zero then  $R$  has rational singularities if and only if it is pseudo-rational.

The key to study rational singularities using tight closure theory is the following result due to Smith:

**Theorem 5.2 ([80]).** *A  $d$ -dimensional excellent local ring  $(R, \mathfrak{m})$  of characteristic  $p$  is  $F$ -rational if and only if  $H_{\mathfrak{m}}^d(R)$  admits no non-trivial submodule closed under the action of Frobenius.  $\square$*

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<sup>18</sup> In [73], I prove that in the affine case, they are actually equivalent with the notions in this paper, and hence admit the desired properties; this, however, relies on a deeper result due to Hara [22].

Note that the top local cohomology  $H_m^d(R)$  is the cokernel of the final map in the Čech complex  $R_{y_1} \oplus \cdots \oplus R_{y_d} \rightarrow R_y$  where  $y = x_1x_2 \cdots x_d$  and  $y_i = y/x_i$ , for  $(x_1, \dots, x_d)$  a system of parameters in  $R$ . In particular, the Frobenius acts on these localizations, whence on the top local cohomology. To formulate the analogue in characteristic zero, we define the *ultra-local cohomology*  $\text{UH}_m^\bullet(R)$  of  $R$  as the ultraproduct of the local cohomology of its approximations. To describe this as the cohomology of a complex over  $R$ , we need the notion of a *relative Lefschetz hull*  $L_R(S)$ , for  $S$  a finitely generated  $R$ -algebra (we do not need this in case  $R$  itself is essentially of finite type over a field). The construction is a relative version of the affine Lefschetz hull. More precisely, it suffices to make the construction for a polynomial ring  $R[x]$ , as follows: let  $R_w$  be approximations of  $R$ , and define  $L_R(R[x])$  as the ultraproduct of the  $R_w[x]$ . For  $S$  arbitrary, say, of the form  $R[x]/I$ , we put  $L_R(S) := L_R(R[x])/IL_R(R[x])$ . The natural maps  $R[x] \rightarrow L(R)[x] \rightarrow L_R(R[x])$ , induce by base change a homomorphism  $S \rightarrow L_R(S)$ . By the same argument as in the affine case,  $L_R(\cdot)$  is a difference hull on the category of finitely generated  $R$ -algebras ([77, Proposition 7.4.3]). We can now also realize  $\text{UH}_m^d(R)$  as the cokernel of

$$L_R(R_{y_1}) \oplus \cdots \oplus L_R(R_{y_d}) \rightarrow L_R(R_y).$$

In particular, there is a natural morphism  $H_m^d(R) \rightarrow \text{UH}_m^d(R)$ .

Since the ultra-Frobenius acts on relative hulls, it also acts on  $\text{UH}_m^d(R)$ . The analogue of Theorem 5.2 in characteristic zero is then that  $R$  is *ultra-F-rational*, meaning that some parameter ideal is equal to its ultra-closure, if and only if  $\text{UH}_m^d(R)$  has no non-trivial submodule closed under the action of the ultra-Frobenius. From this characterization, we get:

**Theorem 5.3.** *If an equicharacteristic excellent local ring is F-rational, then it is pseudo-rational.*

*Proof.* Let  $R$  be an equicharacteristic excellent F-rational local ring. By Remark 2.14(2), and its characteristic zero analogue,  $R$  is Cohen–Macaulay and normal. Since  $R$  is excellent, it is therefore also analytically unramified. Moreover,  $R$  is ultra-F-rational by (8). Let  $\pi: W \rightarrow \text{Spec}R$  be a proper birational map with  $W$  normal. By functoriality, the submodule of  $\text{UH}_m^d(R)$  generated by the kernel of the surjection  $\delta_W$  in (10) is invariant under the action of Frobenius, whence has to be trivial by Theorem 5.2 and our previous discussion.  $\square$

Smith<sup>19</sup> [80] proved Theorem 5.3 in characteristic  $p$  and Hara [22] has proven its converse; in the affine case, I proved that having rational singularities is equivalent with being ultra-F-rational ([72, Theorem 5.11]); I do not know whether this is also true in general, nor do I know whether ultra-F-rational and F-rational are equivalent. From the discussion at the end of the previous section it follows that  $R$  is ultra-F-rational if and only if  $I = \text{cl}^B(I)$ , for some parameter ideal  $I \subseteq R$ .

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<sup>19</sup> Much of the work discussed in this section is based on Karen Smith’s ideas, and grew out from some stimulating conversations I had with her.



Soon after Hochster and Roberts proved Theorem 4.5, Boutot, using some deep vanishing theorems, improved this by showing that quotient singularities have rational singularities. This is just a special case of the following more general result:

**Theorem 5.4 ([76, Main Theorem A]).** *Let  $A \rightarrow B$  be a cyclically pure homomorphism of Noetherian rings containing  $\mathbb{Q}$ . If  $B$  is regular, then  $A$  is pseudo-rational.*

*Proof.* As in the proof of Theorem 4.5, we may reduce to the local case. Remark 2.14(1) yields that  $A$  is F-rational, whence pseudo-rational by Theorem 5.3.  $\square$

At present, no proof of this using Hochster–Huneke tight closure is known. Boutot actually proves a stronger version in the affine case, in which he only assumes that  $B$  itself has rational singularities. I gave a tight closure proof of this more general result under the additional assumption that  $B$  is Gorenstein ([72, Section 5.14]), but I do not know whether this also holds in the general case.

We already observed that a cyclically pure subring of a regular ring is in fact F-regular, which is stronger than being F-rational. According to Table (1), we expect quotient singularities therefore to be actually log-terminal. This was proven for a quotient modulo a finite group by Kawamata [48]. I will now discuss an extension of this result.

**Definition 5.5 ( $\mathbb{Q}$ -Gorenstein Singularities).** Let  $R$  be an equicharacteristic zero Noetherian local domain and put  $X := \text{Spec}R$ . We say that  $R$  is  $\mathbb{Q}$ -Gorenstein if it is normal and some positive multiple of the canonical divisor  $K_X$  is Cartier; the least such positive multiple is called the *index* of  $R$ . If  $R$  is the homomorphic image of an excellent regular local ring (which is for instance the case if  $R$  is complete), then  $X$  admits an *embedded resolution of singularities*  $f: Y \rightarrow X$  by [26]. If  $E_i$  are the irreducible components of the exceptional locus of  $f$ , then the canonical divisor  $K_Y$  is numerically equivalent to  $f^*(K_X) + \sum a_i E_i$  (as  $\mathbb{Q}$ -divisors), for some  $a_i \in \mathbb{Q}$ . The rational number  $a_i$  is called the *discrepancy* of  $X$  along  $E_i$ ; see [49, Definition 2.22]. If all  $a_i > -1$ , we call  $R$  *log-terminal* (in case we only have a weak inequality, we call  $R$  *log-canonical*).

If  $r$  is the index of the  $\mathbb{Q}$ -Gorenstein ring  $R$ , then  $\mathcal{O}_X(rK_X) \cong \mathcal{O}_X$ , where  $X := \text{Spec}R$  and  $K_X$  the canonical divisor of  $X$ . This isomorphism induces an  $R$ -algebra structure on

$$R^* := H^0(X, \mathcal{O}_X \oplus \mathcal{O}_X(K_X) \oplus \cdots \oplus \mathcal{O}_X((r-1)K_X)),$$

which is called the *canonical cover* of  $R$ ; see [48]. To relate log-terminal singularities to F-singularities, we use the following characterization:

**Proposition 5.6 ([48, Proposition 1.7]).** *Let  $R$  be a homomorphic image of an excellent regular local ring (e.g.,  $R$  is complete). If  $R$  has equal characteristic zero and is  $\mathbb{Q}$ -Gorenstein, then it has log-terminal singularities if and only if its canonical cover is rational.*  $\square$

Unfortunately, we cannot use this characterization directly to conclude that a cyclically pure local subring  $R$  of a regular ring  $S$  has log-terminal singularities. For starters, we do not know whether the assumptions imply that  $R$  is  $\mathbb{Q}$ -Gorenstein. We resolve this by simply adding this as an additional assumption on  $R$ . Now, by our previous discussion,  $R$  is F-regular – in fact, it is easy to show that it is also *ultra-F-regular* in the sense that any ideal is equal to its ultra-closure in any localization of  $R$ . However, in order to show that  $R$  has log-terminal singularities, we would like to invoke Proposition 5.6, and so it would suffice to show that its canonical cover  $R^*$  is (ultra-)F-regular too, whence has rational singularities by Theorem 5.3. We know that F-regularity is preserved under étale extensions,<sup>20</sup> but the canonical cover  $R \rightarrow R^*$  is only étale in codimension one (see for instance [81, 4.12]). It was Smith’s brilliant observation that a strengthening of the F-regularity condition, however, is preserved under this type of maps.

### 5.1 Strong F-regularity

Let  $R$  be an equicharacteristic excellent normal local ring. If  $R$  has characteristic  $p$ , then we call it *strongly F-regular*, if for any non-zero  $c$ , there exists an  $n := n(c)$  with the property that for any element  $z \in R$  and any ideal  $I \subseteq R$ , if  $c\text{Frob}_R^n(z) \in \text{Frob}_R^n(I)R$ , then  $z \in I$ . In other words, for each given  $c$ , a single tight closure equation (2) of a sufficiently high power implies already ideal membership. In particular, a strongly F-regular ring is F-regular. The converse is conjectured to hold, but is currently only known in the graded case [54]. If this condition holds just for  $c = 1$ , then we call  $R$  *strongly F-pure*.

To make the definition in characteristic zero, we must allow non-standard powers of the ultra-Frobenius: let  $\alpha$  be a positive element in  $\mathbb{Z}_{\mathfrak{p}}$ , the ultrapower of  $\mathbb{Z}$  (in other words,  $\alpha$  is an ultraproduct of positive integers  $\alpha_w$ ). We define  $\text{Frob}_{\infty}^{\alpha}$  as the homomorphism  $R \rightarrow L(R)$  sending  $x \in R$  to the ultraproduct of the  $\text{Frob}_{R_w}^{\alpha_w}(x_w)$ , where  $R_w$  are approximations of  $R$  and  $x_w \in R_w$  with ultraproduct equal to  $x$  (viewed as an element in  $L(R)$ ). One checks that this yields a well-defined homomorphism. We can now define similarly what it means for  $R$  to be *strongly F-regular* (respectively, *strongly F-pure*) in characteristic zero: for every non-zero  $c$  (for  $c = 1$ ), there exists some positive  $\alpha := \alpha(c) \in \mathbb{Z}_{\mathfrak{p}}$  with the property that for any element  $z \in R$  and any ideal  $I \subseteq R$ , if  $c\text{Frob}_{\infty}^{\alpha}(z) \in \text{Frob}_{\infty}^{\alpha}(I)L(R)$ , then  $z \in I$ . We have:

**Proposition 5.7 ([81, Theorem 4.15] and [76, Proposition 7.8]).** *Let  $R \subseteq S$  be a finite extension of equicharacteristic excellent normal local rings which is étale in codimension one. If  $R$  is strongly F-regular, then so is  $S$ .  $\square$*

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<sup>20</sup> A finite extension  $(R, \mathfrak{m}) \subseteq (S, \mathfrak{n})$  of equal characteristic zero Noetherian local rings is *étale* if it is flat, and *unramified*, meaning that  $\mathfrak{m}S = \mathfrak{n}$ .

**Theorem 5.8 ([76, Main Theorem B]).** *Let  $R \rightarrow S$  be a cyclically pure homomorphism of equicharacteristic zero excellent local rings with  $S$  regular and  $R$  a homomorphic image of a regular local ring. If  $R$  is  $\mathbb{Q}$ -Gorenstein, then it is log-terminal.*

*Proof.* The regular local ring  $S$  is strongly F-regular since its ultra-Frobenius is flat (use the argument in the proof of Theorem 2.10). Moreover, it is easy to check that  $R$ , being a cyclically pure subring, is then also strongly F-regular. Therefore, its canonical cover  $R^*$  is strongly F-regular by Proposition 5.7. In particular,  $R^*$  is F-rational whence has rational singularities by Theorem 5.3. Proposition 5.6 implies then that  $R$  is log-terminal.  $\square$

### 5.2 Kawamata–Viehweg vanishing

As a final application of our methods, I discuss some vanishing theorems and a conjecture of Smith on quotients of Fano varieties. Let  $X$  be a connected projective scheme of finite type over  $\mathbb{C}$ ; a *projective variety*, for short. Choose an ample line bundle  $\mathcal{P}$  on  $X$ , and let  $S$  be *section ring* of the pair  $(X, \mathcal{P})$ , defined as the graded ring

$$S := \bigoplus_{n \in \mathbb{Z}} H^0(X, \mathcal{P}^n). \tag{11}$$

The section ring is a finitely generated, positively graded  $\mathbb{C}$ -algebra, which encodes both the projective variety, to wit,  $X = \text{Proj}(S)$ , as well as the ample line bundle, to wit,  $\mathcal{P} \cong \widetilde{S}(1)$ . The *vertex* of  $S$  is the localization of  $S$  at its irrelevant maximal ideal  $S_{>0}$ . By a *vertex* of  $X$ , we then mean the vertex of some section ring associated to some ample line bundle. It is common wisdom that (global) properties of the projective variety are often already captured by the (local) properties of one of its vertices. Following Smith, we define:

**Definition 5.9.** Let  $X$  be a projective variety. We call  $X$  *globally F-regular* if it has some strongly F-regular vertex. We call  $X$  *globally F-pure* if it has some strongly F-pure vertex.

If  $R$  is globally F-regular (respectively, globally F-pure), then any vertex is strongly F-regular (respectively strongly F-pure); see [82, Theorem 3.10] or [73, Remark 6.3]. In particular, the vertex is Cohen–Macaulay, whence so is  $X$ . The main technical result, inspired by Smith’s work, is:

**Theorem 5.10 ([73, Theorem 6.5 and Remark 6.6]).** *Let  $X$  be a projective variety over  $\mathbb{C}$ , let  $i > 0$ , and let  $\mathcal{L}$  be an invertible  $\mathcal{O}_X$ -module. Each of the following two conditions implies the vanishing of  $H^i(X, \mathcal{L})$ :*

1.  $X$  is globally F-pure and  $H^i(X, \mathcal{L}^n) = 0$  for all  $n \gg 0$ ;
2.  $X$  is globally F-regular and for some effective Cartier divisor  $D$ , all  $H^i(X, \mathcal{L}^n(D)) = 0$  for  $n \gg 0$ .  $\square$

Using this we can now derive the following vanishing theorems:

**Theorem 5.11.** *Let  $X$  be a projective variety over  $\mathbb{C}$  and let  $\mathcal{L}$  be an invertible  $\mathcal{O}_X$ -module. Then  $H^i(X, \mathcal{L}^{-1})$  vanishes for all  $i < \dim X$ , in the following two cases:*

- (Kodaira Vanishing)  $X$  is globally F-pure and Cohen–Macaulay, and  $\mathcal{L}$  is ample;*
- (Kawamata-Viehweg Vanishing)  $X$  is globally F-regular, and  $\mathcal{L}$  is big and numerically effective.*

*Proof.* To prove the first vanishing theorem, observe that by Serre duality ([23, III. Corollary 7.7]), the dual of  $H^i(X, \mathcal{L}^{-n})$  is  $H^{d-i}(X, \omega_X \otimes \mathcal{L}^n)$  where  $d$  is the dimension of  $X$  and  $\omega_X$  the canonical sheaf on  $X$ . Because  $\mathcal{L}$  is ample, the latter cohomology group vanishes for large  $n$ , and hence so does the first. The conclusion then follows from Theorem 5.10(1).

To prove the second vanishing theorem, fix some  $i < d$ . Because  $\mathcal{L}$  is big and numerically effective, we can find an effective Cartier divisor  $D$  such that  $\mathcal{L}^m(-D)$  is ample for all  $m \gg 0$ , by [49, Proposition 2.61]. Hence

$$H^i(X, (\mathcal{L}^{-m}(D))^n) = H^i(X, (\mathcal{L}^m(-D))^{-n}) = 0$$

for all sufficiently large  $m$  and  $n$ , where the vanishing follows from Serre duality and the fact that  $\mathcal{L}^m(-D)$  is ample. Hence, for fixed  $m$ , Theorem 5.10(1) yields the vanishing of  $H^i(X, \mathcal{L}^{-m}(D))$ . Since this holds for all large  $m$ , Theorem 5.10(2) then gives  $H^i(X, \mathcal{L}^{-1}) = 0$ .  $\square$

Recall that a normal projective variety  $X$  is called *Fano*, if its anti-canonical sheaf  $\omega_X^{-1}$  is ample. The following was conjectured by Smith:

**Theorem 5.12.** *Any quotient of a smooth Fano variety by a reductive group (in the sense of Geometric Invariant Theory) admits Kawamata-Viehweg Vanishing.*

*Proof.* The key fact is that a Fano variety  $X$  is globally F-regular ([73, Theorem 7.1]). This rests on some deep result due to Hara [22], which itself was proven using Kodaira vanishing. Assuming this fact, let  $S$  be a section ring with strongly F-regular vertex. If  $G$  is a reductive group acting algebraically on  $X$ , then  $S^G$  is a section ring of the GIT quotient  $X//G$ . Since  $S^G \subseteq S$  is split (see the proof of Theorem 4.5), whence cyclically pure, the vertex of  $S^G$  is strongly F-regular too, showing that  $X//G$  is globally F-regular. The result now follows from Theorem 5.11.  $\square$

*Remark 5.13.* Because of the analogy with the notion of *Frobenius split* (see [82, Proposition 3.1]) and the fact that a Schubert variety has this property [56, Theorem 2], it is reasonable to expect that a Schubert variety is globally F-pure. This is known in characteristic  $p$  by [51]. In particular, if this result on Schubert varieties also holds in characteristic zero, then we get Kodaira Vanishing for any GIT quotient of a Schubert variety.

## 6 Mixed characteristic

Tight closure and big Cohen–Macaulay modules are two powerful tools, as we showed above, but unfortunately, neither one is currently available in mixed characteristic. Hochster [31, 32] has made some attempts to define a closure operation akin to tight closure in mixed characteristic, called *solid closure*,<sup>21</sup> and some amendments have been made by Brenner [7], but as of yet, the theory is not powerful enough to derive any new result in mixed characteristic. Some of the homological theorems are now known in mixed characteristic due to work of Roberts [60, 62], but the preferred method, through big Cohen–Macaulay modules is only available in dimension at most three by work of Heitmann [24] and Hochster [33].<sup>22</sup>

Neither has the ultraproduct method being able to prove any of the outstanding problems in mixed characteristic. At best, we obtain *asymptotic* results, meaning that a certain property holds if the residual characteristic of the ring is large with respect to other invariants associated to the problem. There are essentially two approaches, which I will now sketch briefly. From this, the inherent asymptotic nature of the results should then also become clear.

### 6.1 Protoproducts

In a first approach, we use a mixed characteristic analogue of Proposition 3.9, the celebrated Ax–Kochen–Ershov Principle [4, 19, 20]: for each  $w$ , let  $\mathfrak{D}_w^{\text{mix}}$  be a complete discrete valuation ring of mixed characteristic with residue field  $K_w$  of characteristic  $p_w$ . To each  $\mathfrak{D}_w^{\text{mix}}$ , we associate a corresponding equicharacteristic complete discrete valuation ring with the same residue field, by letting  $\mathfrak{D}_w^{\text{eq}} := K_w[[t]]$ , where  $t$  is a single indeterminate.

**Theorem 6.1 (Ax–Kochen–Ershov).** *If the residual characteristics  $p_w$  are unbounded, then the ultraproduct of the  $\mathfrak{D}_w^{\text{eq}}$  is isomorphic (as a local ring) with the ultraproduct of the  $\mathfrak{D}_w^{\text{mix}}$ .*  $\square$

Let  $\mathfrak{D}_{\mathfrak{q}}$  be this common ultraproduct. It is an equal characteristic zero (non-discrete) valuation ring with principal maximal ideal, such that  $\mathfrak{D}_{\mathfrak{q}}/\mathfrak{I}_{\mathfrak{D}_{\mathfrak{q}}}$  is a complete discrete valuation ring. Fix a tuple of indeterminates  $x$ . Let  $A_w^{\text{eq}} := \mathfrak{D}_w^{\text{eq}}[x]$ , and let  $A_{\mathfrak{q}}^{\text{eq}}$  be their ultraproduct. Since  $\mathfrak{D}_{\mathfrak{q}} \subseteq A_{\mathfrak{q}}^{\text{eq}}$  and the  $x$  are algebraically independent in  $A_{\mathfrak{q}}^{\text{eq}}$ , we have an inclusion  $\mathfrak{D}_{\mathfrak{q}}[x] \subseteq A_{\mathfrak{q}}^{\text{eq}}$ . We call  $\mathfrak{D}_{\mathfrak{q}}[x]$  the *protoproduct* of the  $A_w^{\text{eq}}$ ; it is the subring of all ultraproducts  $f_{\mathfrak{q}}$  of elements  $f_w \in A_w^{\text{eq}}$

<sup>21</sup> Solid closure also intended to provide an alternative approach in characteristic zero, avoiding any reference to reductions modulo  $p$ . A counterexample due to Roberts [61], however, has seriously undermined this approach.

<sup>22</sup> Some earlier attempts that alas led nowhere were made by Hochster in [27].

having bounded degree.<sup>23</sup> The inclusion  $\mathfrak{D}_{\mathfrak{h}}[x] \subseteq A_{\mathfrak{h}}^{\text{eq}}$  is almost a difference hull as in Definition 3.1, except that it is flat, but not faithfully flat ([74, Theorem 4.2]). Using instead the mixed characteristic discrete valuation rings, we get  $A_w^{\text{mix}} := \mathfrak{D}_w^{\text{mix}}[x]$ , whose ultraproduct  $A_{\mathfrak{h}}^{\text{mix}}$  contains  $\mathfrak{D}_{\mathfrak{h}}[x]$  as a flat subring. A note of caution: the Ax–Kochen–Ershov principle is false in higher dimensions:  $A_{\mathfrak{h}}^{\text{eq}}$  and  $A_{\mathfrak{h}}^{\text{mix}}$  are no longer isomorphic. Therefore, the transfer between the  $A_w^{\text{eq}}$  and the  $A_w^{\text{mix}}$  is achieved via their common subring  $\mathfrak{D}_{\mathfrak{h}}[x]$ , and we may thus think of them as respectively the mixed and equal characteristic approximations of  $\mathfrak{D}_{\mathfrak{h}}[x]$ . That the latter is not Noetherian causes quite some headaches; for details, see [68, 74]. The main technical result is that any local ring  $R$  which is essentially of finite type over  $\mathfrak{D}_{\mathfrak{h}}$  admits the analogue of a big Cohen–Macaulay algebra. This enables us to prove some non-Noetherian analogues of the homological conjectures over  $R$ , which then descend to its mixed characteristic approximations. Since the transfer requires the degree to be bounded, we can only get an asymptotic version: the residual characteristic has to be sufficiently large with respect to the degrees of the data involved. For instance, we get:

**Theorem 6.2 (Asymptotic Monomial Conjecture, [74, Corollary 9.5]).** *For each  $N$ , we can find a bound  $\mu(N)$  with the following property. Let  $\mathfrak{D}$  be a mixed characteristic discrete valuation ring and let  $R$  be a finite extension of the localization  $S := \mathfrak{D}[x]_{(x_1, \dots, x_d)\mathfrak{D}[x]}$ . If  $R$  is defined by at most  $N$  polynomials of degree at most  $N$  over  $S$ , then the tuple  $(x_1, \dots, x_d)$ , viewed as a system of parameters in  $R$ , is monomial, provided the residual characteristic of  $\mathfrak{D}$  is at least  $\mu(N)$ .  $\square$*

## 6.2 Cataproducts

In the second approach, rather than subrings, we look for nice residue rings. Let  $(R_w, \mathfrak{m}_w)$  be Noetherian local rings and let  $R_{\mathfrak{h}}$  be their ultraproduct. We already observed that  $R_{\mathfrak{h}}$  is hardly ever Noetherian, and hence the usual methods from commutative algebra do not apply. Nonetheless, Proposition 4.10 and the property of the Ax–Kochen–Ershov ring  $\mathfrak{D}_{\mathfrak{h}}$  are not isolated events; there is often a Noetherian residue ring lurking in the background:

**Proposition 6.3.** *If the  $R_w$  have bounded embedding dimension,<sup>24</sup> then their cataproduct  $R_{\#} := R_{\mathfrak{h}}/\mathfrak{J}_{R_{\mathfrak{h}}}$  is a complete Noetherian local ring.*

*Proof (Sketch; see [77, Theorem 8.1.4] or [78, Lemma 5.6]).* By Łos’ Theorem,  $R_{\mathfrak{h}}$  has a finitely generated maximal ideal. By saturatedness of ultraproducts, every Cauchy sequence in  $R_{\mathfrak{h}}$  has a limit. Hence the Hausdorffication of  $R_{\mathfrak{h}}$ , that is to say,  $R_{\#}$ , is complete. Now, a complete local ring with finitely generated maximal ideal is Noetherian by [55, Theorem 29.4].  $\square$

<sup>23</sup> We may similarly view an affine algebra over a field of characteristic zero as the protoproduct of its approximations; this is the point of view in [77, Chap. 9].

<sup>24</sup> The *embedding dimension* of a local ring is the minimal number of generators of its maximal ideal.

Moreover, the  $R_w$  share many properties with their cataproduct, and a transfer result, albeit weaker than Theorem 4.4, holds (see, for instance, [78, Corollaries 8.3 and 8.7, and Theorem 8.10]). Suppose the  $R_w$  have mixed characteristic. If their residual characteristics are unbounded, then by Łos' Theorem, their cataproduct has equal characteristic zero (since the ultraproduct of the residue fields of the  $R_w$  is the residue field of  $R_{\sharp}$ ). In the remaining case, almost all  $R_w$  have residual characteristic  $p$ , for some  $p$ , and in that case  $R_{\sharp}$  has characteristic  $p$  if the *ramification indices* of the  $R_w$  are unbounded, that is to say, if for all  $N$ , we have  $p \in \mathfrak{m}_w^N$  for almost all  $w$ , whence  $p \in \mathfrak{I}_{R_{\sharp}}$ . So in either case, we get an equicharacteristic cataproduct with a tight closure operation and a balanced big Cohen–Macaulay algebra. However, neither construction descends to the components  $R_w$  (intuitively, an ultraproduct can only transfer finitely many information). I conclude with an application of the method, and a discussion how this could potentially lead to the full conjecture.

Given a Noetherian local ring  $R$ , let  $F_{\bullet}$  be a complex of length  $s$  consisting of finite free  $R$ -modules. We say that the *rank* of  $F_{\bullet}$  is at most  $r$ , if each free  $R$ -module  $F_i$  in  $F_{\bullet}$  has rank at most  $r$ ; we say that  $F_{\bullet}$  has *homological complexity* at most  $l$ , if each homology group  $H_i(F_{\bullet})$  for  $i > 0$  has length at most  $l$ , and  $H_0(F_{\bullet})$  has a minimal generator generating a submodule of length at most  $l$ .

**Theorem 6.4 (Asymptotic Improved New Intersection Theorem, [78, Theorem 13.6]).** *For each triple of non-negative integers  $(m, r, l)$ , there exists a bound  $v(m, r, l)$  with the following property. Let  $R$  be a mixed characteristic Noetherian local ring of embedding dimension at most  $m$ . If  $F_{\bullet}$  is a finite free complex of rank at most  $r$  and homological complexity at most  $l$ , then its length is at least the dimension of  $R$ , provided the residual characteristic or the ramification index of  $R$  is at least  $v(m, r, l)$ .  $\square$*

If we can show that the above bound grows slowly enough, then we can even deduce the full version from this. The idea is to reach a contradiction from a minimal counterexample by increasing its ramification, but controlling the growth of the other data. Without proof, I quote:

**Theorem 6.5 ([78, Theorem 13.8]).** *If for each fixed  $(m, r)$  the bound  $v(m, r, l)$  from the previous theorem grows sub-linearly in  $l$ , in the sense that there exists some  $0 \leq \alpha := \alpha_{m,r} < 1$  and  $c > 0$  such that  $v(m, r, l) \leq c \cdot l^{\alpha}$  for all  $l$ , then the Improved New Intersection Theorem holds.  $\square$*

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# Rees valuations

Irena Swanson

**Abstract** This expository paper contains history, definitions, constructions, and the basic properties of Rees valuations of ideals. A section is devoted to one-fibered ideals, that is, ideals with only one Rees valuation. Cutkosky [17] proved that there exists a two-dimensional complete Noetherian local integrally closed domain in which no zero-dimensional ideal is one-fibered. However, no concrete ring of this form has been found. An emphasis in this paper is on bounding the number of Rees valuations of ideals. A section is on the projective equivalence of ideals, with the discussion of “rational powers” of ideals. The last section is about the Izumi–Rees Theorem, which establishes comparability of Rees valuations with the same center. Several examples are computed explicitly. More on Rees valuations can be done via the projective equivalence of ideals, and there have been many articles along that line. See the latest article by Heinzer, Ratliff, and Rush, in this volume.

All rings in this paper are commutative with identity, and most are Noetherian domains. The following notation will be used throughout:

- $Q(R)$  denotes the field of fractions of a domain  $R$ .
- For any prime ideal  $P$  in a ring  $R$ ,  $\kappa(P)$  denotes the field of fractions of  $R/P$ .
- If  $V$  is a valuation ring,  $\mathfrak{m}_V$  denotes its unique maximal ideal, and  $v$  denotes an element of the equivalence class of valuations naturally determined by  $V$ .
- We say that a Noetherian valuation is **normalized** if its value group is a subset of  $\mathbb{Z}$  whose greatest common divisor is 1.
- If  $R$  is a ring and  $V$  is a valuation overring, then the **center** of  $V$  on  $R$  is  $\mathfrak{m}_V \cap R$ .
- A valuation ring  $V$  (or a corresponding valuation  $v$ ) is said to be **divisorial** with respect to a subdomain  $R$  if  $Q(R) = Q(V)$  and if  $\text{tr.deg}_{\kappa(P)} \kappa(\mathfrak{m}_V) = \text{ht } p - 1$ , where  $p = \mathfrak{m}_V \cap R$ . It is a fact that every divisorial valuation with respect to a Noetherian ring  $R$  is essentially of finite type over  $R$ .

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- If  $I$  is an ideal in a ring  $R$ , the  $I$ -adic **order** is the function  $\text{ord}_I : R \rightarrow \mathbb{Z}_{\geq 0} \cup \{\infty\}$  given by

$$\text{ord}_I(x) = \sup\{n \in \mathbb{Z} : x \in I^n\}.$$

- If  $R$  is a domain and  $v$  is a valuation on  $Q(R)$ , we allow  $v$  to be defined on all of  $R$  by setting  $v(0) = \infty$ .
- For any Noetherian valuation  $v$  that is non-negative on a domain  $R$ , and for any ideal  $I$  in  $R$ ,  $v(I)$  is  $\min\{v(r) : r \in I\}$ .

Many of the omitted proofs can be found in Chap. 10 in [12].

## 1 Introduction

David Rees was the first to systematically study the valuations associated to an ideal that were later called Rees valuations. Rees proved their existence, uniqueness, he proved that they determine the integral closures of the powers of the given ideal, and over the years he and others proved many applications.

Recall that if  $I$  is an ideal in a ring  $R$ , we denote by  $\bar{I}$  the integral closure of  $I$ , namely,

$$\bar{I} = \{r \in R : r^n + a_1 r^{n-1} + \cdots + a_n = 0 \text{ for some positive } n \text{ and some } a_j \in I^j\}.$$

It is well-known that this equals

$$\bar{I} = \bigcap_V IV \cap R,$$

as  $V$  varies over the valuation rings that are  $R$ -algebras, or alternatively, as  $V$  varies over the valuation rings between  $R/P$  and  $Q(R/P)$ , and  $P$  varies over the set  $\text{Min}(R)$  of minimal prime ideals in  $R$ . In case  $R$  is Noetherian, the valuation rings may all be restricted to be Noetherian valuation rings. If  $R/I$  is Artinian, by the descending chain property, there exists a finite set  $S$  of valuation rings such that  $\bar{I} = \bigcap_{V \in S} IV \cap R$ . Finiteness of the needed set of valuations is a desirable property in general, as it simplifies existence proofs and algorithmic computations. Rees valuations aim for more: there are finitely many valuations that suffice in the sense above not just for  $I$  but also for  $I, I^2, I^3, I^4, \dots$  simultaneously. Here is a formal definition:

**Definition 1.1.** Let  $R$  be a ring and  $I$  an ideal in  $R$ . A set of **Rees valuation rings** of  $I$  is a set  $\{V_1, \dots, V_s\}$  consisting of valuation rings, subject to the following conditions:

1. Each  $V_i$  is Noetherian and is not a field.
2. For each  $i = 1, \dots, s$ , there exists a minimal prime ideal  $P_i$  of  $R$  such that  $V_i$  is a ring between  $R/P_i$  and  $Q(R/P_i)$ .
3. For all  $n \in \mathbb{N}$ ,  $\bar{I}^n = \bigcap_{i=1}^s (I^n V_i) \cap R$ .
4. The set  $\{V_1, \dots, V_s\}$  satisfying the previous conditions is minimal possible.

By a slight abuse of notation, the notation  $\mathcal{RV}(I)$  stands for a set of Rees valuation rings of  $I$  – even though the set is in general not uniquely determined. (Uniqueness is discussed in Section 2.)

For each valuation ring there is a natural corresponding equivalence class of valuations. A set of representatives of valuations for a set of Rees valuation rings is called a set of **Rees valuations**. Typically, we take the normalized representatives.

By the standard abuse of notation, each valuation  $v$  is defined on the whole ring  $R$  and takes on in addition the value  $\infty$ , with  $\{r \in R : v(r) = \infty\}$  being a prime ideal. If a valuation  $v_i$  corresponds to a Rees valuation ring  $V_i$ , then  $\{r \in R : v_i(r) = \infty\}$  is precisely the minimal prime ideal  $P_i$  of  $R$  as in condition 2. above. With this notation, condition 3. translates to:

$$\bar{I}^n = \{r \in R : v_1(r) \geq nv_1(I), \dots, v_s(r) \geq nv_s(I)\} \text{ for all } n \geq 0.$$

All Rees valuations are constructed as localizations of the integral closures of finitely many finitely generated  $R$ -algebras contained in  $Q(R)$ , as we explain in Section 2. One idea of where Rees valuations might be found is contained in the following observation: If  $\{V_1, \dots, V_s\}$  is a set of Rees valuation rings of  $I$  (unique or not), then for all  $n$ ,

$$\bar{I}^n = \bigcap_{j=1}^s (I^n V_j \cap R)$$

is a (possibly redundant) primary decomposition of  $\bar{I}^n$ , and thus

$$\bigcup_{n \geq 1} \text{Ass}(R/\bar{I}^n) \subseteq \{\{r \in R : rV_j \neq V_j\} : j = 1, \dots, s\} \tag{1}$$

$$= \{\mathfrak{m}_{V_j} \cap R : j = 1, \dots, s\} \tag{2}$$

is a finite set.

It is straightforward to verify that for all ideals  $I$ ,  $\mathcal{RV}(I) = \mathcal{RV}(\bar{I})$ .

A basic property of Rees valuations of an ideal is that they localize, in the sense that for any multiplicatively closed set  $W$  in  $R$ ,  $\mathcal{RV}(W^{-1}I) = \{V \in \mathcal{RV}(I) : \mathfrak{m}_V \cap W = \emptyset\}$ . This follows in a straightforward way from the definitions.

How do Rees valuations behave under extending the ideal to an overring? Let  $R \rightarrow S$  be a ring homomorphism of rings such that  $S$  is either faithfully flat over  $R$  or  $S$  is integral over  $R$ . Then for any ideal  $I$  in  $R$ ,  $\bar{I} = \bar{I}S \cap R$  (proofs can be found in Propositions 1.6.1 and 1.6.2 of [12]). If Rees valuations exist for  $IS$ , this implies that

$$\mathcal{RV}(I) \subseteq \{V \cap Q(R) : V \in \mathcal{RV}(IS)\}.$$

If  $S$  is the integral closure of the Noetherian domain  $R$  in its field of fractions, even equality holds, i.e.,  $\mathcal{RV}(I) = \{V \cap Q(R) : V \in \mathcal{RV}(IS)\} = \mathcal{RV}(IS)$ . Furthermore, if  $(R, \mathfrak{m})$  is a Noetherian local ring and  $I$  is an  $\mathfrak{m}$ -primary ideal of  $R$ , then  $\widehat{IR}$  is the integral closure of  $I\widehat{R}$ , whence also in this case,  $\mathcal{RV}(I) = \{V \cap Q(R) : V \in \mathcal{RV}(I\widehat{R})\}$ . If in addition  $\widehat{R}$  is a domain, no two Noetherian valuations on  $Q(\widehat{R})$  centered on

$m\widehat{R}$  contract to the same valuation on  $Q(R)$ , so in that case the numbers of Rees valuations of  $I$  and of  $I\widehat{R}$  are the same if  $I$  is  $\mathfrak{m}$ -primary. More generally, Katz and Validashti [16, Theorem 5.3] proved the following:

**Theorem 1.2. (Katz and Validashti [16, Theorem 5.3]).** *Let  $I$  be an ideal in a Noetherian local ring  $(R, \mathfrak{m})$  that is not contained in any minimal prime ideal. Let  $w$  be a Rees valuation of  $I\widehat{R}$  with center  $m\widehat{R}$ , and let  $Q$  be the corresponding minimal prime ideal in  $\widehat{R}$  such that  $w$  is a valuation on  $\kappa(Q)$ . Then  $w$  restricted to  $\kappa(Q \cap R)$  is a Rees valuation of  $I$  with center  $\mathfrak{m}$ . The function*

$$w \mapsto w|_{\kappa(\{r \in R : w(r) = \infty\})}$$

*from Rees valuations of  $I\widehat{R}$  with center on  $m\widehat{R}$  to Rees valuations of  $I$  with center on  $\mathfrak{m}$  is a one-to-one and onto function.*

The results above were in the direction of where to search for Rees valuations of a given ideal; the following result searches for an ideal for which a given valuation is a Rees valuation:

**Proposition 1.3.** *Let  $R$  be a Noetherian domain. Let  $V$  be a divisorial valuation ring with respect to  $R$ . Then there exists an ideal  $I$  in  $R$ , primary to  $P = \mathfrak{m}_V \cap R$ , such that  $V$  is one of its Rees valuation rings.*

*Conversely, let  $J$  be an ideal and  $W$  a Rees valuation ring of  $J$ . Set  $P = \mathfrak{m}_W \cap R$  and assume that  $R_P$  is formally equidimensional. Then  $W$  is a divisorial valuation ring with respect to  $R$  and  $R_P$ .*

(A proof can be found for example in [12, Propositions 10.4.3 and 10.4.4].)

For this reason, on Noetherian locally formally equidimensional domains, the Rees valuations of non-zero ideals are the same as the divisorial valuations with respect to  $R$ .

**Examples**

1. A maximal ideal  $\mathfrak{m}$  in a regular ring has only one Rees valuation, namely the  $\mathfrak{m}$ -adic valuation. The  $\mathfrak{m}$ -adic valuation ring equals  $R[\frac{\mathfrak{m}}{x}]_{(x)R[\frac{\mathfrak{m}}{x}]}$  for any  $x \in \mathfrak{m} \setminus \mathfrak{m}^2$ .
2. Let  $R = k[X_1, \dots, X_d]$  be a polynomial ring over a field  $k$ . For any monomial ideal  $I$  in  $R$ , the convex hull of the set  $\{(a_1, \dots, a_d) \in \mathbb{N}^d : X_1^{a_1} \cdots X_d^{a_d} \in I\}$  in  $\mathbb{R}^d$  is called the **Newton polyhedron** of  $I$ , and is denoted  $\text{NP}(I)$ . The Newton polyhedron of  $I$  contains the information on the integral closure of  $I$ :

$$\bar{I} = (X_1^{a_1} \cdots X_d^{a_d} \mid (a_1, \dots, a_d) \in \text{NP}(I) \cap \mathbb{N}^d).$$

By Carathéodory’s Theorem the convex hull is bounded by the coordinate hyperplanes and by finitely many other faces/hyperplanes, each of the form  $c_1X_1 + \cdots + c_dX_d = 1$  for some  $c_i \in \mathbb{Q}$ . The corresponding hyperplane bounding the Newton polyhedron of  $I^n$  has the form  $c_1X_1 + \cdots + c_dX_d = n$ . Thus

$$\bar{I}^n = (X_1^{a_1} \cdots X_d^{a_d} : c_1 a_1 + \cdots + c_d a_d \geq n,$$

as  $(c_1, \dots, c_d)$  varies over the hyperplane coefficients as above).

For each such irredundant bounding hyperplane  $c_1 X_1 + \cdots + c_d X_d = 1$ , define a **monomial valuation**  $v : R \rightarrow \mathbb{Q}$  by  $v(X_1^{a_1} \cdots X_d^{a_d}) = c_1 a_1 + \cdots + c_d a_d$  and by  $v(\sum_{i=1}^n m_i) = \min \{v(m_i) : i = 1, \dots, n\}$ , where the  $m_i$  are products of non-zero elements of  $k$  with distinct monomials. By above, these valuations are the Rees valuations of  $I$ . (Hübl and the author generalized this in [11] to all ideals generated by monomials in a regular system of parameters.)

3. In particular, in a polynomial ring  $k[X_1, \dots, X_n]$ , every ideal of the pure power form  $(X_1^{a_1}, \dots, X_m^{a_m})$  has only one Rees valuation.
4. The above can easily be worked on the monomial ideal  $(X, Y)$  in the polynomial ring  $R = k[X, Y, Z]$  over a field  $k$ , to get that the only Rees valuation of  $(X, Y)$  is the monomial valuation  $v_1$ , which takes value 1 on  $X$  and  $Y$  and value 0 on  $Z$ . Similarly, the only Rees valuation of  $(X, Z)k[X, Y, Z]$  is the monomial valuation  $v_2$  taking value 1 on  $X$  and  $Z$  and value 0 on  $Y$ .

The valuations  $v_1$  and  $v_2$  are not the only Rees valuations of the product of the two ideals  $(X, Y)$  and  $(X, Z)$ , by the following reasoning:  $v_1((X, Y)(X, Z)) = 1$  since  $v_1(XZ) = 1$ , and similarly  $v_2((X, Y)(X, Z)) = 1$ . However,  $v_1(X) = v_2(X) = 1$  as well, but  $X \notin \overline{(X, Y)(X, Z)}$ .

Related to the last example is the following:

**Proposition 1.4.** (Cf. *Muhly–Sakuma [23]*) *For any non-zero ideals  $I$  and  $J$  in a Noetherian domain  $R$ ,  $\mathcal{RV}(I) \cup \mathcal{RV}(J) \subseteq \mathcal{RV}(IJ)$ . If  $I$  is locally principal or if  $R$  is Noetherian locally formally equidimensional of dimension at most 2, then  $\mathcal{RV}(I) \cup \mathcal{RV}(J) = \mathcal{RV}(IJ)$ .*

The example above the proposition shows that the inclusion  $\mathcal{RV}(I) \cup \mathcal{RV}(J) \subseteq \mathcal{RV}(IJ)$  may be proper.

On page 423 it was mentioned that  $\mathcal{RV}(I) = \mathcal{RV}(J)$  if the integral closures of  $I$  and  $J$  coincide. It is similarly clear that for ideals  $I \subseteq J$ ,  $\bar{I} = \bar{J}$  if and only if for every Rees valuation ring  $V$  of  $I$ ,  $IV = JV$ . It is not much harder to prove that for every positive integer  $n$ ,  $\mathcal{RV}(I) = \mathcal{RV}(I^n)$ . Furthermore, if  $\bar{I}^m = \bar{J}^n$  for some positive integers  $m, n$ , then  $\mathcal{RV}(I) = \mathcal{RV}(J)$ . However, the converse may fail, namely  $\mathcal{RV}(I) = \mathcal{RV}(J)$  does not imply that  $\bar{I}^m = \bar{J}^n$  for some positive integers  $m, n$ . For example, let  $I = (X^2, Y)^5 \cap (X, Y^2)^4$  and  $J = (X^2, Y)^4 \cap (X, Y^2)^5$ , and use the monomial ideal method above for finding the Rees valuations. The two ideals  $I$  and  $J$  have the same two monomial Rees valuations, yet the integral closures of powers of  $I$  do not coincide with the integral closures of powers of  $J$ .

We recall some more vocabulary: an ideal  $J$  is a **reduction** of an ideal  $I$  if  $J \subseteq I$  and  $\bar{J} = \bar{I}$ . The first crucial paper on reductions is [24], by Northcott and Rees.



## 2 Existence and uniqueness

There is the question of existence and uniqueness of Rees valuation rings. For the zero ideal in a domain, any one Noetherian valuation ring  $V$  between  $R$  and  $Q(R)$  is the Rees valuation ring. Thus we have existence but not uniqueness in this case.

The first case of the existence and uniqueness of Rees valuations was proved by David Rees in [25], for zero-dimensional ideals in equicharacteristic Noetherian local rings. The second case was proved by Rees in [27] for arbitrary ideals  $I$  in Noetherian domains for which the following Artin–Rees-like assumption holds: there exists an integer  $t$  such that for all sufficiently large  $n$ ,  $I^{n+1} \cap I^n \subseteq I^{n+1}$ . Neither of the two cases of existence in [25] and [27] is covered by the other. The general existence theorem, for all ideals in [26] Noetherian rings, was proved by Rees in [28]. Uniqueness was proved in [25]. A good reference is also [26]. Here is a summary general result (for a proof, see for example [12, Theorems 10.1.6 and 10.2.2]):

**Theorem 2.1. (Existence and uniqueness of Rees valuations)** *Let  $R$  be a Noetherian ring. Then for any ideal  $I$  of  $R$ , there exists a set of Rees valuation rings, and if  $I$  is not contained in any minimal prime ideal of  $R$ , then the set of Rees valuation rings is uniquely determined.*

The main case of the proof of the existence is actually when  $R$  is a Noetherian domain and  $I$  is a non-zero principal ideal. In that case, by the Mori–Nagata Theorem, the integral closure  $\bar{R}$  of  $R$  is a Krull domain, so that the associated primes of  $I^n \bar{R}$ , as  $n$  varies, are all minimal over  $I$ , there are only finitely many of them, and the localizations of  $\bar{R}$  at these primes are Noetherian valuation domains. These finitely many valuation rings are then the Rees valuation rings of  $I\bar{R}$ , and hence of  $I$ .

The reduction of the existence proof in general to the non-zero principal ideal case is via the extended Rees algebra  $R[It, t^{-1}]$  (cf. [12, Exercise 10.6]):

$$\mathcal{RV}(I) = \{V \cap Q(R) : V \in \mathcal{RV}(t^{-1}R[It, t^{-1}])\}.$$

The reduction to the domain case relies on the fact that the integral closure of ideals is determined by the integral closures when passing modulo the minimal primes.

It turns out that in a Noetherian ring, all minimal prime ideals play a role in the Rees valuations of all ideals of positive height, in the sense that for every ideal  $I$  of positive height and every  $P \in \text{Min}(R)$  there exists a Rees valuation  $v$  of  $I$  such that  $\{r \in R : v(r) = \infty\} = P$ . But even more is true (and does not seem to be in the literature):

**Proposition 2.2.** *Let  $R$  be a Noetherian ring and  $I$  an ideal in  $R$  not contained in any minimal prime ideal. For each  $P \in \text{Min}(R)$ , let  $T_P$  be the set of Rees valuations of  $I(R/P)$ . By abuse of notation, these valuations are also valuations on  $R$ , with  $\{r \in R : v(r) = \infty\} = P$ . Then  $\cup_P T_P$  is the set of Rees valuations of  $I$ .*

*Proof.* The standard proofs of the existence of Rees valuations show that  $\mathcal{RV}(I) \subseteq \cup_P T_P$ . We need to prove that no valuation in  $\cup_P T_P$  is redundant.

Let  $Q \in \text{Min}(R)$  and  $v \in T_Q$ . By the minimality of Rees valuations of  $I(R/Q)$ , there exist  $n \in \mathbb{N}$  and  $r \in R$  such that for all  $w \in T_Q \setminus \{v\}$ ,  $w(r) \geq nw(I)$ , yet  $r \notin \bar{I}^n(R/Q)$  (i.e.,  $v(r) < nv(I)$ ). Let  $r'$  be an element of  $R$  that lies in precisely those minimal prime ideals that do not contain  $r$ . Then  $r + r'$  is not contained in any minimal prime ideal, for all  $w \in T_Q \setminus \{v\}$ ,  $w(r + r') \geq nw(I)$ , and  $v(r + r') < nv(I)$ . Let  $J'$  be the intersection of all the minimal primes other than  $Q$ , let  $J''$  be the intersection of all the centers of  $w \in T_Q$ , and let  $s \in J' \cap J'' \setminus Q$ . By assumption on  $r$ , there exists a positive integer  $k$  such that for all  $w \in T_Q \setminus \{v\}$ ,

$$\frac{v(s)}{v(I)} - \frac{w(s)}{w(I)} + 1 < k \left( \frac{w(r + r')}{w(I)} - \frac{v(r + r')}{v(I)} \right).$$

Note that for all  $w \in \cup_{P \neq Q} S_P$ ,  $w(s) = \infty$ . Thus for all  $w \in \cup_P T_P \setminus \{v\}$ ,  $\frac{v(s)}{v(I)} - \frac{w(s)}{w(I)} + 1 < k \left( \frac{w(r + r')}{w(I)} - \frac{v(r + r')}{v(I)} \right)$ . Then with  $m = \lfloor \frac{v(s(r + r')^k)}{v(I)} \rfloor$ ,  $s(r + r')^k \notin \bar{I}^{m+1}$ , yet for all  $w \in \cup_P T_P \setminus \{v\}$ ,  $w(s(r + r')^k) \geq (m + 1)w(I)$ . This proves that  $v$  is not redundant.  $\square$

How does one construct the Rees valuation rings in practice? The steps indicated above of computing the integral closure of  $R[It, t^{-1}]$  require an additional variable over  $R$ , and afterwards one needs to take the intersections of the obtained valuation rings with  $Q(R)$ . There is an **alternative construction** that eliminates these two steps of extending and intersecting, namely a construction using blowups: if  $I = (a_1, \dots, a_r)$ , then

$$\mathcal{RV}(I) = \bigcup_{j=1}^r \mathcal{RV} \left( a_j R \left[ \frac{I}{a_j} \right] \right).$$

Just as the first construction, this one also reduces to the case of principal ideals. As announced, this construction avoids introducing a new indeterminate and then intersecting the valuation rings with a field, but it instead requires the computation of  $r$  integral closures of rings. This can also be computationally daunting. If there is a way of making  $r$  smaller, the task gets a bit easier. Since  $\mathcal{RV}(J) = \mathcal{RV}(I)$  whenever  $\bar{I} = \bar{J}$ , one can replace  $I$  in this alternative construction with  $J$ , if  $J$  has fewer generators than  $I$ . A standard choice for  $J$  is a minimal reduction of  $I$ , or even a minimal reduction of a power of  $I$ . It is known that in a Noetherian local ring  $R$ , every ideal has a power that has a reduction generated by at most  $\dim R$  elements. In case the residue field is infinite, the ideal itself has a reduction generated by at most  $\dim R$  elements (see [24] or [12, Propositions 8.3.7, 8.3.8]).

Even better, if  $R$  contains an infinite field, or if  $R$  is local with an infinite residue field, there exists a “sufficiently general” element  $a \in I$  such that  $\mathcal{RV}(I) = \mathcal{RV}(aR[\frac{I}{a}])$ . In fact,  $\mathcal{RV}(I) = \mathcal{RV}(aR[\frac{I}{a}])$  whenever  $aV = IV$  for all  $V \in \mathcal{RV}(I)$ . Unfortunately, the “sufficient generality” is not so easily determined, and might be doable only after the Rees valuations have already been found.

Sally proved a determinate case when only one affine piece of the blowup suffices for finding the Rees valuations:

**Theorem 2.3.** (*Sally [33, page 438]*) *Let  $(R, \mathfrak{m})$  be a Noetherian formally equidimensional local domain of dimension  $d > 0$ , and let  $I$  be an  $\mathfrak{m}$ -primary ideal minimally generated by  $d$  elements. Then for any  $a \in I$  that is part of a minimal generating set, any Rees valuation ring  $V$  of  $I$  is the localization of the integral closure of  $R[\frac{I}{a}]$  at a height one prime ideal minimal over  $a$ .*

Similarly, if  $x_1, \dots, x_r$  is a regular sequence in a Noetherian domain  $R$ , then for every Rees valuation ring  $V$  of  $I = (x_1, \dots, x_r)$ , and for every  $i = 1, \dots, r$ ,  $x_i V = IV$ , and  $V$  is the localization of the integral closure of  $S = R[\frac{I}{x_i}]$  at a height one prime ideal containing  $x_i$ .

Thus, there are occasions when the alternative construction of Rees valuations using blowups only requires the computation of the integral closure of one ring, and sometimes this one ring is known a priori.

There is yet **another construction** of Rees valuations, this one via the (ordinary) Rees algebra  $R[It]$ : the set of Rees valuation rings of  $I$  equals the set of all rings  $\overline{R[It]}_P \cap Q(R)$ , as  $P$  varies over the prime ideals in  $\overline{R[It]}$  that are minimal over  $IR[It]$ .

A consequence of this formulation is a criterion for recognizing when the associated graded ring and the associated “integral” graded ring are reduced:

**Theorem 2.4.** (*Hübl–Swanson [10]*) *If  $R$  is integrally closed, then  $\text{gr}_I(R)$  is reduced if and only if all the powers of  $I$  are integrally closed and if for each (normalized integer-valued) Rees valuation  $v$  of  $I$ ,  $v(I) = 1$ . Also,  $R/\bar{I} \oplus \bar{I}/\bar{I}^2 \oplus \bar{I}^2/\bar{I}^3 \oplus \dots$  is a reduced ring if and only if for each (integer-valued) Rees valuation  $v$  of  $I$ ,  $v(I) = 1$ .*

Here is an example illustrating this result. Let  $X, Y, Z$  be variables over  $\mathbb{C}$ , and  $R = \mathbb{C}[X, Y, Z]/(X^2 + Y^3 + Z^5)$ . Then  $R$  is an integrally closed domain. By Flenner [5, 3.10],  $R$  is a rational singularity ring, so that by Lipman [17], all the powers of the maximal ideal  $\mathfrak{m} = (X, Y, Z)R$  are integrally closed, and the blowup rings in the construction of Rees valuations of  $\mathfrak{m}$  are integrally closed. As  $X \in (\overline{Y, Z})R$ , it follows that  $\mathcal{R}\mathcal{V}(\mathfrak{m}) = \mathcal{R}\mathcal{V}((Y, Z))$ , and by Sally’s Theorem 2.3 above,  $\mathcal{R}\mathcal{V}(\mathfrak{m}) = \mathcal{R}\mathcal{V}\left(YR\left[\frac{(Y, Z)}{Y}\right]\right)$ . Certainly,  $\frac{X}{Y}$  is integral over  $R\left[\frac{(Y, Z)}{Y}\right]$ , so that  $\mathcal{R}\mathcal{V}(\mathfrak{m}) = \mathcal{R}\mathcal{V}\left(YR\left[\frac{(X, Y, Z)}{Y}\right]\right)$ . By the cited Lipman’s result, with  $X' = \frac{X}{Y}$  and  $Z' = \frac{Z}{Y}$ ,

$$R\left[\frac{(X, Y, Z)}{Y}\right] \cong R\left[\frac{\mathfrak{m}}{Y}\right] \cong \frac{\mathbb{C}[X', Y, Z']}{((X')^2 + Y + Y^3(Z')^5)}$$

is integrally closed, and there is only one minimal prime over  $(Y)$ , namely  $(X', Y)$ , so that  $\mathfrak{m}$  has only one Rees valuation. Locally at  $(X', Y)$ , the maximal ideal is generated by  $X'$ , so that if  $v$  is the natural corresponding valuation,  $v(X') = 1$ ,  $v(Z') = 0$ , from  $Y(1 + Y^2(Z')^5) = -(X')^2$  we get that  $v(Y) = 2$ , hence  $v(X) = v(X') + v(Y) = 3$ ,  $v(Z) = v(Z') + v(Y) = 2$ , whence  $v(\mathfrak{m}) \geq 2$ . However, notice that  $\text{gr}_{\mathfrak{m}}(R) = R/\overline{\mathfrak{m}} \oplus \overline{\mathfrak{m}}/\overline{\mathfrak{m}}^2 \oplus \dots \cong \mathbb{C}[x, y, z]/(x^2)$  is not reduced.

The listed constructions of Rees valuations above give several methods for finding the unique set. All the methods require passing to finitely generated ring extensions, then taking the integral closure of the ring, followed by finding minimal primes over some height one ideals. The whole procedure may be fairly challenging: the integral closures of rings and primary decompositions of ideals are in practice hard to compute.

The following theorem gives a method for finding the centers of the Rees valuations without going through the full construction of the valuations. Recall (see [24]) that the **analytic spread** of an ideal  $I$  in a Noetherian local ring  $(R, \mathfrak{m})$  is the Krull dimension of  $(R/\mathfrak{m}) \oplus (I/\mathfrak{m}I) \oplus (I^2/\mathfrak{m}I^2) \oplus \dots$ . If  $R/\mathfrak{m}$  is infinite, this number is the same as the number of generators of any ideal minimal reduction of  $I$ , and in general, there always exists a power of  $I$  with a reduction generated minimally by this number of elements.

**Theorem 2.5. (Burch [1], McAdam [20])** *Let  $R$  be a Noetherian ring,  $I$  an ideal in  $R$  and  $P$  a prime ideal in  $R$ . If the analytic spread of  $IR_P$  equals  $\dim(R_P)$ , then  $P$  is the center of a Rees valuation of  $I$ . If  $R$  is locally formally equidimensional and  $P$  is the center of a Rees valuation of  $I$ , then the analytic spread of  $IR_P$  equals  $\dim(R_P)$ .*

I end this section with another example of where Rees valuations appear. In [30], Rees defined the **degree function** of an  $\mathfrak{m}$ -primary ideal  $I$  in  $(R, \mathfrak{m})$  as follows: for any  $x \in \mathfrak{m}$  such that  $\dim(R/(x)) = \dim(R) - 1$ , set  $d_I(x) = e_{R/(x)}(I(R/(x)))$ , i.e.,  $d_I(x)$  is the multiplicity of the ideal  $I$  in the ring  $R/(x)$ . Rees proved that for all allowed  $x$ ,

$$d_I(x) = \sum_{v \in \mathcal{RV}'(I)} d(I, v)v(x)$$

for some positive integers  $d(I, v)$  depending only on  $I$  and  $v$ , where  $\mathcal{RV}'(I)$  is the set of those Rees valuations of  $I$  that are divisorial with respect to  $R$ . (Rees avoided the name “Rees valuation”).

### 3 One-fibered ideals

**Definition 3.1.** An ideal  $I$  is called **one-fibered** if  $\mathcal{RV}(I)$  has exactly one element.

By the constructions of Rees valuations, this means that  $I$  is one-fibered if and only if the radical of  $t^{-1}R[\overline{It}, t^{-1}]$  is a prime ideal, which holds if and only if the radical of  $IR[\overline{It}]$  is a prime ideal.

Zariski [35] proved that if  $(R, \mathfrak{m})$  is a two-dimensional regular local ring, then for every divisorial valuation  $v$  on  $R$  that is centered on  $\mathfrak{m}$ , there exists an ideal  $I$  such that  $v$  is the only Rees valuation of  $I$ . Namely, by Proposition 1.3,  $v$  is a Rees valuation of some ideal  $I$  of height two. This ideal may be assumed to be integrally closed, since  $\mathcal{RV}(I) = \mathcal{RV}(\overline{I})$ . Zariski proved that in a two-dimensional regular local ring the product of any two integrally closed ideals is integrally closed, and each integrally closed ideal factors uniquely (up to order) into a product of

simple integrally closed ideals. Then by Proposition 1.4,  $v$  is a Rees valuation of some simple integrally closed ideal  $I$  of height two. It is a fact that a simple integrally closed  $\mathfrak{m}$ -primary ideal in a two-dimensional regular local ring has only one Rees valuation, so  $v$  is the only Rees valuation of  $I$ .

Lipman [17] generalized Zariski’s result to all two-dimensional local rational singularity rings, and Göhner [G] proved it for two-dimensional complete integrally closed rings with torsion class group. Muhly [22] showed that there are two-dimensional analytically irreducible local domains for which Zariski’s conclusion fails. Recall that a Noetherian local ring  $(R, \mathfrak{m})$  is **analytically irreducible** if the  $\mathfrak{m}$ -adic completion of  $R$  is a domain.

More strongly, Cutkosky [2] proved that there exists a two-dimensional complete integrally closed local domain  $(R, \mathfrak{m})$  in which every  $\mathfrak{m}$ -primary ideal has at least two Rees valuations. However, no concrete example of such a ring with no one-fibered zero-dimensional ideals has been found. One of the quests, alas unfulfilled, of the following section, is to find such a ring.

We discuss in this section what, if any, restrictions on the ring does the existence of a one-fibered ideal impose, and we also discuss various criteria for one-fiberedness.

**Theorem 3.2. (Sally [33])** *Let  $(R, \mathfrak{m})$  be a Noetherian local ring whose  $\mathfrak{m}$ -adic completion is reduced (i.e.,  $R$  is **analytically unramified**), and that has a one-fibered  $\mathfrak{m}$ -primary ideal  $I$ . Then  $R$  is analytically irreducible, i.e., the  $\mathfrak{m}$ -adic completion of  $R$  is a domain.*

Katz more generally proved in [15] that if  $(R, \mathfrak{m})$  is a formally equidimensional Noetherian local ring, then for any  $\mathfrak{m}$ -primary ideal  $I$ , the number of Rees valuations is bounded below by the number of minimal prime ideals in the  $\mathfrak{m}$ -adic completion of  $R$ . Thus, if  $R$  has a one-fibered  $\mathfrak{m}$ -primary ideal, then  $\hat{R}$  has only one minimal prime ideal. The converse fails by Cutkosky’s example [2] mentioned earlier in this section.

By the more recent work of Katz and Validashti, see Theorem 1.2, if  $(R, \mathfrak{m})$  is a Noetherian local ring of positive dimension, the number of Rees valuations of an  $\mathfrak{m}$ -primary ideal  $I$  is the same as the number of Rees valuations of  $I\hat{R}$ . By Proposition 2.2, the number of Rees valuations of  $I\hat{R}$  is at least the number of the minimal primes in  $\hat{R}$ , so that if  $R$  has a one-fibered ideal, the completion must have only one minimal prime ideal. If in addition the completion is assumed to be reduced, this forces the completion to be a domain, thus proving Theorem 3.2.

Another proof of Theorem 3.2, without assuming Theorem 1.2 and Proposition 2.2, goes as follows: Rees proved in [29] that since  $R$  is analytically unramified, there exists an integer  $k$  such that for all  $n$ ,  $\overline{I^{n+k}} \subseteq I^n$ . Let  $V$  be the Rees valuation ring of  $I$ , and let  $r$  be an integer such that  $IV = \mathfrak{m}_V^r$ . As  $I$  is  $\mathfrak{m}$ -primary,  $\mathfrak{m}$  is the center of  $V$  on  $R$ . Then for all  $n$ ,  $\mathfrak{m}^{r(n+k)} \subseteq \mathfrak{m}_V^{r(n+k)}V \cap R = I^{n+k}V \cap R = \overline{I^{n+k}} \subseteq I^n \subseteq \mathfrak{m}^n$ , so that the  $\mathfrak{m}$ -adic completion of  $R$  is contained in the  $\mathfrak{m}_V$ -adic completion of  $V$ . But the latter is a domain since  $V$  is regular.

An arbitrary Noetherian local ring may have a zero-dimensional one-fibered ideal, yet not be analytically irreducible or even analytically unramified (of course

the example is due to Nagata, see [12, Exercise 4.11]): Let  $k_0$  be a perfect field of characteristic 2, let  $X, Y, X_1, Y_1, X_2, Y_2, \dots$  be variables over  $k_0$ , let  $k$  be the field  $k_0(X_1, Y_1, X_2, Y_2, \dots)$ ,  $f = \sum_{i=1}^{\infty} (X_i X^i + Y_i Y^i)$ , and  $R = k^2[[X, Y]][k][f]$ . Then  $R$  is an integrally closed Noetherian local domain whose completion  $\hat{R}$  is isomorphic to  $k[[X, Y, Z]]/(Z^2)$ . The integral closures of powers of  $(X, Y)R$  are contracted from the integral closures of powers of  $(X, Y)\hat{R}$ , hence clearly  $(X, Y)R$  has only one Rees valuation.

As proved in Fedder–Huneke–Hübl [4, Lemma 1.3], the following are equivalent for an analytically unramified one-dimensional local domain  $R$ :

1. The integral closure  $\bar{R}$  of  $R$  is local.
2.  $R$  has a non-zero one-fibered ideal.
3. Every non-zero ideal in  $R$  is one-fibered.

How does one find one-fibered ideals in arbitrary Noetherian local domains? In case  $\hat{R}$  is a domain,  $R$  has a one-fibered  $\mathfrak{m}$ -primary ideal if and only if the integral closure of  $\hat{R}$  has a one-fibered zero-dimensional ideal (see Sally [33, page 440]).

If  $(R, \mathfrak{m})$  is a Noetherian local analytically irreducible domain, and  $I$  an  $\mathfrak{m}$ -primary ideal, then  $I$  is one-fibered if and only if there exists an integer  $b$  such that for all positive integers  $n$  and all  $x, y \in R$ ,  $xy \in I^{2n+b}$  implies that either  $x$  or  $y$  lies in  $I^n$  (see Hübl–Swanson [10]). A question that appeared in the same paper and has not yet been answered may be worth repeating:

**Question.** Let  $I$  be an  $\mathfrak{m}$ -primary ideal in an analytically irreducible Noetherian local domain  $(R, \mathfrak{m})$ . Suppose that for all  $n \in \mathbb{N}$  and all  $x, y$  such that  $xy \in I^{2n}$ , either  $x$  or  $y$  lies in  $I^n$ . Or even suppose that for all  $x \in R$  such that  $x^2 \in I^{2n}$ , necessarily  $x \in I^n$ . Are all the powers of  $I$  then integrally closed?

Another criterion of one-fiberedness was observed first by Sally [4, page 323] in dimension one, and the more general case below appeared in [9, page 3510]:

**Theorem 3.3.** *Let  $(R, \mathfrak{m})$  be a Noetherian  $d$ -dimensional analytically unramified local ring, and let  $l$  be a positive integer satisfying the following:*

*If  $f \in \mathfrak{m} \setminus I^l$ , then there exist  $g_2, \dots, g_d \in I$  such that  $I^{n+l} \subseteq (f, g_2, \dots, g_d)$ , and for all Rees valuations  $v$  of  $I$ ,  $v(I^{n+l}) \geq v(f)$ .*

*Then  $I$  is one-fibered.*

*Remark 3.4.* Lipman [17] proved that the quadratic transformations of two-dimensional rational singularity rings are integrally closed. However, this does not mean that the maximal ideal has only one Rees valuation. For example, let  $c \geq 3$  and take  $R$  to be the localization of  $\frac{\mathbb{C}[X, Y, Z]}{(X^2 + Y^2 + Z^c)}$  at  $(X, Y, Z)$ . By Flenner [5], Korollar (3.10),  $R$  is a rational singularity ring. The quadratic transformation  $S = R\left[\frac{x}{z}, \frac{y}{z}\right]$  is isomorphic to a localization of  $\frac{\mathbb{C}[Z, A, B]}{(A^2 + B^2 + Z^{c-2})}$  and is integrally closed, so that the primes in  $S$  minimal over  $ZS$  are  $(Z, A + iB)$ ,  $(Z, A - iB)$ . By the blowup construction of Rees valuations, this says that  $(X, Y, Z)$  has at least two Rees valuations. In fact, since  $(Y, Z)$  is a minimal reduction of  $(X, Y, Z)$ , Sally’s result [33, page 438] shows that  $(X, Y, Z)$  has exactly two Rees valuations, the two arising from the two obtained prime ideals.

Muhly and Sakuma [23, Lemma 4.1] proved the following result on one-fibered ideals  $I_1, \dots, I_r$  in a two-dimensional universally catenary Noetherian integral domain  $R$ : If for  $j = 1, \dots, r$ ,  $\mathcal{RV}(I_j) = \{V_j\}$ , and the corresponding valuations  $v_1, \dots, v_r$  are pairwise not equivalent, then  $\det(v_i(I_j)_{i,j}) \neq 0$ . Here is a sketch of the proof. Let  $A$  be the  $r \times r$  matrix  $(v_i(I_j)_{i,j})$ . Suppose that  $\det A = 0$ . Then the columns of  $A$  are linearly dependent over  $\mathbb{Q}$ , and we can find integers  $a_1, \dots, a_r$ , not all zero, such that for all  $i$ ,  $\sum_j a_j v_i(I_j) = 0$ . By changing indices, we assume that  $a_1, \dots, a_t, -a_{t+1}, \dots, -a_r$  are non-negative integers. Let  $I = I_1^{a_1} \cdots I_t^{a_t}$ ,  $J = I_{t+1}^{-a_{t+1}} \cdots I_r^{-a_r}$  for some positive integer  $t < r$ . By Proposition 1.4,  $\mathcal{RV}(I) = \{V_1, \dots, V_t\}$ ,  $\mathcal{RV}(J) = \{V_{t+1}, \dots, V_r\}$ . Since for all  $i = 1, \dots, r$ ,  $v_i(I) = v_i(J)$ , we have  $\bar{I}^n = \bar{J}^n$  for all  $n$ , so  $I$  and  $J$  have the same Rees valuations, which is a contradiction.

### 4 Upper bounds on the number of Rees valuations

The main goal of this section is to bound above the number of Rees valuations of ideals in an arbitrary Noetherian local ring  $(R, \mathfrak{m})$ , and to find  $\mathfrak{m}$ -primary ideals with only one Rees valuation, if possible.

If  $R$  is a one-dimensional Noetherian semi-local integral domain, then it follows easily from the constructions of Rees valuations that  $\{\bar{R}_P : P \in \text{Max } \bar{R}\} = \cup_I \mathcal{RV}(I)$  is finite, as  $I$  varies over all the ideals of  $R$ . Thus the total number of Rees valuations of all possible ideals in a one-dimensional Noetherian semi-local ring is finite, and this number is a desired upper bound on the number of Rees valuations of any one ideal.

In higher dimensions, there is no upper bound on the number of all possible Rees valuations of ideals. For one thing, there are infinitely many prime ideals of height one, and each one of these has at least one Rees valuation centered on the prime itself. But more importantly, even if we restrict the ideals to  $\mathfrak{m}$ -primary ideals, there is no upper bound on the number of Rees valuations:

**Proposition 4.1.** *Let  $(R, \mathfrak{m})$  be a Noetherian local domain of dimension  $d > 1$ . Let  $(x_1, \dots, x_d)$  be an  $\mathfrak{m}$ -primary ideal. Then  $\cup_n \mathcal{RV}(x_1^n, x_2^n, \dots, x_d^n)$  is not a finite set. Furthermore, there is no upper bound on  $|\mathcal{RV}(I)|$  as  $I$  varies over  $\mathfrak{m}$ -primary ideals.*

*Proof.* The last statement follows from the first one, by Proposition 1.4.

Suppose that the set  $S$  of all Rees valuations of  $(x_1^n, x_2^n, \dots, x_d^n)$ , as  $n$  varies, is finite. Let  $N$  be a positive integer such that for all  $v \in S$ ,  $Nv(x_1) \geq v(x_2), \dots, v(x_d)$ . Then for all  $n \geq N$ , the integral closure of  $(x_1^n, x_2^n, \dots, x_d^n)$  is independent of  $n$ , whence  $x_1^N \in \overline{(x_1^n, x_2^n, \dots, x_d^n)}$ . Let  $'$  denote the images modulo  $(x_2, \dots, x_d)$ . Then in the one-dimensional Noetherian ring  $R'$ ,  $x_1'$  is a parameter, and  $x_1'^N \in \overline{(x_1')^n}$  for all  $n \geq N$ . We may even pass to the completion of  $R'$  and then go modulo a minimal prime ideal to get a one-dimensional complete Noetherian local domain  $A$  and a parameter  $x$  such that for all  $n \geq N$ ,  $x^N \in \overline{(x^n)}$ . Since  $A$  is analytically unramified,

by Rees [29] there exists an integer  $l$  such that for all  $n \geq l$ ,  $\overline{(x^n)} \subseteq (x^{n-l})$ . Thus  $x^N \subseteq \bigcap_{n \geq N, l} (x^{n-l}) = (0)$ , which is a contradiction.  $\square$

As already mentioned, a maximal ideal in a regular ring has only one Rees valuation. For zero-dimensional monomial ideals (in polynomial or power series rings, or even zero-dimensional monomial ideals in a regular system of parameters in a regular local ring), the number of Rees valuations is exactly the number of bounding non-coordinate hyperplane faces of the Newton polyhedron. By Carathéodory’s Theorem, each of these hyperplanes is determined by  $d = \dim R$  of the exponent vectors of the generators of  $I$ . With this geometric consideration one obtains a very crude upper bound on the number of Rees valuations of a zero-dimensional monomial ideal in terms of its generators:

$$|\mathcal{RV}(I)| \leq \binom{\text{number of generators of } I}{\dim(R)}.$$

In practice, this upper bound is much too generous. I thank Ezra Miller for providing the following much better upper bound: the number of Rees valuations of  $I$  is at most

$$\begin{cases} 2 \left( 1 + m + \binom{m+1}{2} + \binom{m+2}{3} + \dots + \binom{m-1+\frac{d-1}{2}}{\frac{d-1}{2}} \right); & \text{if } d \text{ is odd;} \\ \binom{m-1+\frac{d}{2}}{\frac{d}{2}} + 2 \left( 1 + m + \binom{m+1}{2} + \binom{m+2}{3} + \dots + \binom{m-1+\frac{d-2}{2}}{\frac{d-2}{2}} \right); & \text{if } d \text{ is even,} \end{cases}$$

where  $n$  is the number of generators of  $I$  and  $m = n - d - 1$ . In particular, for fixed  $d$ , this upper bound on the number of Rees valuations of monomial ideals in  $d$  variables is a polynomial in the number of generators of degree  $\lfloor d/2 \rfloor$ . This follows among others from the Upper Bound Theorem for simplicial complexes. The relevant ingredients using Hilbert functions can be found in Lemma 16.19, Exercise 14.34, and Definition 16.32 in [13].

In general, however, the number of Rees valuations is not a function of the number of generators: at least when  $R$  is a polynomial ring over an infinite field (see [19]), or if  $(R, \mathfrak{m})$  is a Noetherian local ring with infinite residue field (see [24]), every ideal  $I$  has a minimal reduction  $J$  generated by  $\dim(R)$  elements, we already know that  $\mathcal{RV}(I) = \mathcal{RV}(\bar{I}) = \mathcal{RV}(\bar{J}) = \mathcal{RV}(\bar{J})$ , yet the number of Rees valuations is unbounded when the ring dimension is strictly bigger than one.

We now concentrate on finding upper bounds on the number of Rees valuations of ideals in Noetherian local rings. Let  $(R, \mathfrak{m})$  be a Noetherian local ring. For every ideal  $I$  in  $R$ ,  $\bar{I} = \overline{I\hat{R}} \cap R$ , so that the number of Rees valuations of  $I\hat{R}$  is an upper bound on the number of Rees valuations of  $I$ . By Proposition 2.2, it then suffices to find an upper bound on the number of Rees valuations of  $I(\hat{R}/Q)$  for each  $Q \in \text{Min}(\hat{R})$  (and adding them), so finding bounds on the number of Rees valuations of  $I$  reduces to the ring being a complete local domain. These reductions preserve the property of ideals being primary to the maximal ideal. As established on page 427, we may replace  $I$  by its power, and in particular, by a power that has a  $d$ -generated reduction, where  $d$  is the dimension of the ring. (Or alternatively, we could first pass in the standard way to  $R[X]_{\mathfrak{m}_R[X]}$ , which is a faithfully flat extension of  $(R, \mathfrak{m})$



with an infinite residue field, to have the existence of  $d$ -generated reductions for all  $m$ -primary ideals.) We have thus reduced to finding upper bounds on the number of Rees valuations of a  $d$ -generated  $m$ -primary ideal in a complete Noetherian local domain  $(R, \mathfrak{m})$  of dimension  $d$ . To simplify matters, we now **restrict** our attention to the case where  $R$  contains a field. In that case, by the Cohen Structure Theorem, there exists a regular local subring  $A = k[[X_1, \dots, X_d]]$  of  $R$ , with  $k \cong R/\mathfrak{m}$  a field and  $X_1, \dots, X_d$  variables over  $k$ , such that  $R$  is module-finite over  $A$  and such that  $JR = I$ , where  $J = (X_1, \dots, X_d)A$ . If the field of fractions of  $R$  is separable over that of  $A$ , there exists an element  $z \in R$  such that  $A[z] \subseteq R$  is a module-finite extension of domains with identical fields of fractions. Necessarily,  $A[z]$  is a hypersurface ring, and the Rees valuations of  $JA[z]$  are the Rees valuations of  $JR = I$  (see page 423). Under the separable assumption we have thus reduced to finding upper bounds on the number of Rees valuations of the parameter ideal  $(X_1, \dots, X_d)$  in the complete local hypersurface domain

$$R = \frac{A[Z]}{(Z^n + a_1 Z^{n-1} + \dots + a_n)} = \frac{k[[X_1, \dots, X_d, Z]]}{(Z^n + a_1 Z^{n-1} + \dots + a_n)},$$

where  $a_i \in A$ . Since without loss of generality  $z$  may be replaced by any  $A$ -multiple of  $z$ , we may assume that  $z$  is in the integral closure of  $I$ , so that we may assume that  $a_i \in J^i A$  for all  $i$ . We can even control the degree  $n$ : since  $1 = e_A((X_1, \dots, X_d)A) = e_R(I) [R/\mathfrak{m} : k]/[Q(R) : Q(A)] = e_R(I)/n$ , we get  $n = e_R(I)$ . Now we handle the general case, not assuming that  $R$  is separable over  $A$ . By the standard field theory, there exists a purely inseparable field extension  $k'$  of  $k$  and a positive integer  $m$  such that the field of fractions of  $B[R] = R[k'][X_1^{1/p^m}, \dots, X_d^{1/p^m}]$  is finite and separable over the field of fractions of  $B = k'[X_1^{1/p^m}, \dots, X_d^{1/p^m}]$ . Note that  $B[R]$  is a module finite (hence integral) extension of  $R$ , and by page 423, an upper bound on the set of Rees valuations of  $IB[R]$  is an upper bound on the set of Rees valuations of  $I$ . Thus, it suffices to replace  $R$  by  $B[R]$ . The field of fractions of this ring is separably generated over that of  $B$ , and the extension  $B \subseteq B[R]$  has the same form as the extension  $A \subseteq R$ , so we are in the situation as above. In this case, the degree of the integral extension from  $B$  to  $B[R]$  is  $[Q(B[R]) : Q(B)] = e_{B[R]}(IB[R])[k' : k] = e_R(I) [Q(B[R]) : Q(R)]$ .

In summary, in all cases of Noetherian local rings containing a field, we reduce the computation of the bounds on the number of Rees valuations of  $I$  to the computation of upper bounds on the number of Rees valuations of the ideal  $(X_1, \dots, X_d)R$  in the domain

$$R = \frac{k[[X_1, \dots, X_d, Z]]}{(Z^n + a_1 Z^{n-1} + \dots + a_n)},$$

where  $a_i \in (X_1, \dots, X_d)^i k[[X_1, \dots, X_d]]$ . We can even control the degree  $n$  as  $e_R(I)$  if  $R$  is separably generated over  $k$ , say in characteristic 0.

**Proposition 4.2.** *With notation as above,*

$$|\mathcal{RV}((X_1, \dots, X_d)R)| \leq n.$$

*Proof.* Let  $S = R[\frac{X_2}{X_1}, \dots, \frac{X_d}{X_1}]$ . Then

$$S \cong \frac{k[[X_1, \dots, X_d, Z]][T_2, \dots, T_d]}{(Z^n + b_1 Z^{n-1} + \dots + b_n, X_1 T_2 - X_2, \dots, X_1 T_d - X_d)},$$

for some  $b_i \in X_1^i k[[X_1, \dots, X_d, Z]][T_2, \dots, T_d]$ . By Theorem 2.3, all the Rees valuations of  $(X_1, \dots, X_d)R$  are of the form  $(\bar{S})_P$ , where  $\bar{S}$  is the integral closure of  $S$ , and  $P$  is a height one prime ideal in  $\bar{S}$  containing  $X_1$ .

Let  $P$  be such a prime ideal, and let  $p = P \cap S$ . Since  $R$  is formally equidimensional, by the Dimension Formula,  $\text{ht}(p) = \text{ht}(P) = 1$ . Necessarily  $p$  is a prime ideal in  $S$  minimal over  $X_1 S$ , hence  $p = (X_1, \dots, X_d, Z)$ . Thus, it suffices to prove that the number of prime ideals in  $(\bar{S})_{S \setminus p}$  that contract to  $p$  in  $S$  is at most  $n$ . By [12, Proposition 4.8.2], it suffices to prove that the number of minimal primes in the completion  $\widehat{S}_p$  of  $S_p$  is at most  $n$ . Let  $T = k[[X_1, \dots, X_d]][\frac{X_2}{X_1}, \dots, \frac{X_d}{X_1}]$ , and let  $q = (X_1, \dots, X_d)T$ . Then  $T_q$  is a regular local ring of dimension 1, and the maximal ideal is generated by  $X_1$ . The  $q$ -adic completion of  $T_q$  is  $T_q[[Y]]/(X_1 - Y)$ , which is a regular local ring of dimension 1 with maximal ideal generated by  $Y$ . But  $\widehat{S}_p$  is  $T_q[[Z]][[Y]]/(Z^n + b_1 Z^{n-1} + \dots + b_n, X_1 - Y)$ , which has at most  $n$  minimal primes.  $\square$

Here is a table for the number of Rees valuations of the maximal ideal in  $R = \frac{k[[X, Y, Z]]}{(X^a + Y^b + Z^c)}$  (with  $2 \leq a \leq b \leq c$ ) that illustrates the proposition above, showing that the number of Rees valuations of  $(X, Y, Z)$  is at most  $a$ . By page 425, it suffices to bound the number of Rees valuations of the ideal  $(X, Y, Z)$  in the ring  $R = \frac{\mathbb{C}[[X, Y, Z]]}{(X^a + Y^b + Z^c)}$ . Some of the calculations below were done with Anna Guerrieri.

$a, b, c$	$\#\mathcal{RV}(X, Y, Z)$
2, 2, 2	1
2, 2, $c \geq 3$	1 if $k = \mathbb{R}$ , 2 if $k = \mathbb{C}$
2, 3, $c \geq 3$	1
2, 4, 4	1
2, 4, $c \geq 5$	1 if $k = \mathbb{R}$ , 2 if $k = \mathbb{C}$
2, 5, $c \geq 5$	1
2, 6, 6	1
2, 6, $c \geq 7$	1 if $k = \mathbb{R}$ , 2 if $k = \mathbb{C}$
3, 3, 3	1
3, 3, $c \geq 4$	2 if $k = \mathbb{R}$ , 3 if $k = \mathbb{C}$

On the list above, do all of the rings have an  $(X, Y, Z)$ -primary ideal with only one Rees valuation? Can one find examples that are generated by monomials in  $X, Y, Z$ ? This is indeed the case:

**Proposition 4.3.** *Let  $R = \frac{\mathbb{C}[[X, Y, Z]]}{(X^a + Y^b + Z^c)}$  or  $R = \frac{\mathbb{C}[[X, Y, Z]]}{(X^a + Y^b + Z^c)}$ , with  $2 \leq a \leq b \leq c$  integers. Then the ideal  $(X^a, Y^b, Z^c)$  has exactly one Rees valuation.*

*Proof.* By page 423, it suffices to prove the proposition for the ideal  $I = (X^a, Y^b, Z^c)$  in the ring  $R = \frac{\mathbb{C}[X,Y,Z]}{(X^a+Y^b+Z^c)}$ . Let  $S$  be obtained from  $R$  by adjoining a  $(bc)$ th root  $x$  of  $X$ , an  $(ac)$ th root  $y$  of  $Y$ , and an  $(ab)$ th root  $z$  of  $Z$ . Then  $S = \frac{\mathbb{C}[x,y,z]}{(x^{abc}+y^{abc}+z^{abc})}$  and  $IS = (x^{abc}, y^{abc}, z^{abc})S$ . By reductions on page 423, it suffices to prove that  $IS$  has only one Rees valuation. But  $\mathcal{RV}(IS) = \mathcal{RV}(\overline{IS}) = \mathcal{RV}((x,y,z)^{abc}S) = \mathcal{RV}((x,y,z)S)$ , so it suffices to prove that the ideal  $(x,y,z)$  in the ring  $S = \mathbb{C}[x,y,z]/(x^d + y^d + z^d)$  has only one Rees valuation. Since  $(x,y)$  is a reduction of  $(x,y,z)$ , by Theorem 2.3, all the Rees valuations of  $(x,y,z)$  are of the form  $\overline{T}'_P$ , as  $P$  varies over the height one prime ideals in  $\overline{T}'$  that are minimal over  $x\overline{T}'$ , where  $T' = S[y/x]$ . Since clearly  $z/x$  is integral over  $T'$ , all the Rees valuations of  $(x,y,z)$  are of the form  $\overline{T}_P$ , as  $P$  varies over the minimal prime ideals over  $Y\overline{T}$ , and  $T = S[y/x, z/x]$ . Note that

$$T \cong \frac{\mathbb{C}[x,y,z,U,V]}{(xU - y, xV - z, 1 + U^d + V^d)}.$$

By the Jacobian criterion,  $T$  is an integrally closed domain, so that each Rees valuation corresponds to a prime ideal in  $T$  minimal over  $xT$ , but  $xT$  is a prime ideal.  $\square$

### 5 The Izumi–Rees theorem

The Izumi–Rees theorem is a very powerful and possibly surprising theorem, saying that all divisorial valuations over a good ring  $R$  with the same center are comparable, in the sense that if  $v$  and  $w$  are such valuations, there exists a constant  $C$  such that for all  $x \in R$ ,  $v(x) \leq Cw(x)$ . Since over good rings divisorial valuations are the same as Rees valuations (of possibly different ideals), this theorem enables us to compare Rees valuations with the same center. The surprising part of the Izumi–Rees Theorem is the contrast with the fact that if  $v$  and  $w$  are any two non-equivalent integer-valued valuations on a field  $K$  (such as on  $Q(R)$ ), then for any integers  $n, m \in \mathbb{Z}$  there exists  $x \in K$  such that  $v(x) = n$  and  $w(x) = m$ . The difference between this result and the Izumi–Rees Theorem is that the former takes elements from the field of fractions, but the Izumi–Rees Theorem only from the (good) subring.

Izumi [14] characterized analytically irreducible local domains, in the context of analytic algebras, without passing to the completion of the domains. Rees [31] generalized Izumi’s result to the following two versions:

**Theorem 5.1.** (Rees [31, (C)]) *A Noetherian local ring  $(R, \mathfrak{m})$  is analytically irreducible if for a least one  $\mathfrak{m}$ -primary ideal  $I$ , and only if, for all  $\mathfrak{m}$ -primary ideals  $I$ , there exist constants  $C$  and  $C'$ , depending only on  $I$ , such that*

$$\text{ord}_I(xy) - \text{ord}_I(y) \leq C \text{ord}_I(x) + C', \text{ for all non-zero } x, y \in R.$$

**Theorem 5.2. (Rees [31, (E)])** *Let  $(R, \mathfrak{m})$  be a complete Noetherian local domain and let  $v, w$  be divisorial valuations centered on  $\mathfrak{m}$ . Then there exists a constant  $C$  such that for all non-zero  $x \in R$ ,  $v(x) \leq Cw(x)$ .*

Rees’s proof first reduces to the proof in Krull dimension two, and then uses the existence of desingularizations and intersection numbers of  $\mathfrak{m}$ -adic valuations: in case the intersection number  $[v, w]$  of  $v$  and  $w$  is non-zero, the constant  $C$  in Rees’s theorem may be taken to be  $C = -[w, w]/[v, w]$ .

A stronger version of Theorem 5.2 was stated in Hübl–Swanson [10]: whenever  $(R, \mathfrak{m})$  is an analytically irreducible excellent local domain and whenever  $v$  is a divisorial valuation centered on  $\mathfrak{m}$ , there exists a constant  $C$  such that for all divisorial valuations  $w$  centered on  $\mathfrak{m}$  and all non-zero  $x \in R$ ,  $v(x) \leq Cw(x)$ . In recent conversations with Shuzo Izumi, we removed the excellent assumption above, analytically irreducible assumption suffices.

A version of the Izumi–Rees Theorem for affine rings, with explicit bounds for comparisons of valuations in terms of MacLane key polynomials, was given by Moghaddam in [21].

One of the consequences of the Izumi–Rees Theorem is a form of control of zero divisors modulo powers of ideals:

**Theorem 5.3. (Criterion of analytic irreducibility [10, Theorem 2.6])** *Let  $(R, \mathfrak{m})$  be a Noetherian local ring. The following are equivalent:*

1.  *$R$  is analytically irreducible.*
2. *There exist integers  $a$  and  $b$  such that for all  $n \in \mathbb{N}$ , whenever  $x, y \in R$  and  $xy \in \mathfrak{m}^{an+b}$ , then either  $x \in \mathfrak{m}^n$  or  $y \in \mathfrak{m}^n$ .*
3. *For every  $\mathfrak{m}$ -primary ideal  $I$  there exist integers  $a$  and  $b$  such that for all  $n \in \mathbb{N}$ , whenever  $x, y \in R$  and  $xy \in I^{an+b}$ , then either  $x \in I^n$  or  $y \in I^n$ .*

In [34], the author used the Izumi–Rees Theorem to prove the following: Let  $R$  be a Noetherian ring and  $I, J$  ideals in  $R$  such that the topology determined by  $\{I^n : J^\infty\}_n$  is equivalent to the  $I$ -adic topology. Then the two topologies are equivalent linearly, i.e., there exists an integer  $k$  such that for all  $n$ ,  $I^{kn} : J^\infty \subseteq I^n$ . In particular, if  $I$  is a prime ideal for which the topology determined by the symbolic powers is equivalent to the  $I$ -adic topology, then there exists an integer  $k$  such that  $I^{(kn)} \subseteq I^n$ . However, one cannot read  $k$  from the proof.

Subsequently, Ein, Lazarsfeld, and Smith in [3], and Hochster and Huneke in [7] proved that in a regular ring containing a field, the constant  $k$  for the prime ideal  $I$  may be taken to be the height of  $I$ . (The two papers [3] and [7] prove much more general results).

In short, the Izumi–Rees Theorem has proved to be a powerful tool for handling powers of ideals.

Rond used the Izumi–Rees Theorem in a very different context: he proved in [32] that the Izumi–Rees Theorem is equivalent to a bounding of the Artin functions by a special upper bound of a certain family of polynomials. Also, Rond used the Izumi–Rees Theorem to bound other Artin functions.

## 6 Adjoints of ideals

In this final section, I present yet another construction that is related to Rees valuations, and I end with an open question.

As already mentioned, Ein, Lazarsfeld and Smith [3] proved that for any prime ideal  $P$  in a regular ring containing a field of characteristic 0,  $P^{(nh)} \subseteq P^n$  for all integers  $n$ , where  $h$  is the height of  $P$ . Hochster and Huneke [7] extended this to regular rings containing a field of positive prime characteristic, but no corresponding result is known in mixed characteristic. Hochster and Huneke used tight closure, and Ein, Lazarsfeld and Smith used the multiplier ideals. A possible approach to proving such a result in mixed characteristic is to use the adjoint ideals. The adjoint and multiplier ideals agree whenever they are both defined. However, the theory of multiplier ideals has access to powerful vanishing theorems, whereas adjoint ideals do not.

**Definition 6.1.** (Lipman [18]) Let  $R$  be a regular domain with field of fractions  $K$ . The **adjoint** of an ideal  $I$  in  $R$  is the ideal

$$\text{adj} I = \bigcap_V \{r \in K : rJ_{V/R} \subseteq IV\},$$

where  $V$  varies over all the divisorial valuations with respect to  $R$ , and  $J_{V/R}$  denotes the Jacobian ideal of the essentially finite-type extension  $R \subseteq V$ .

The adjoint  $\text{adj} I$  is an integrally closed ideal in  $R$  containing the integral closure of  $I$ , and hence containing  $I$ . Also,  $\text{adj}(I) = \text{adj}(\bar{I})$ , and if  $x \in R$ , then  $\text{adj}(xI) = x \cdot \text{adj}(I)$ . In particular, the adjoint of every principal ideal is the ideal itself.

In general, adjoints are not easily computable. One problem is the apparent need to use infinitely many valuations in the definition. The emphasis in the rest of this section is on limiting the number of necessary valuations, and the connection with Rees valuations.

Howald [8] proved that if  $I$  is a monomial ideal in  $k[X_1, \dots, X_d]$ , then  $\text{adj} I = (\underline{X}^{\underline{e}} : \underline{e} \in \mathbb{N}^d, \underline{e} + (1, \dots, 1) \in \text{NP}^\circ(I))$ , where  $\text{NP}^\circ(I)$  is the interior of the Newton polyhedron of  $I$ . Hübl and Swanson [11] extended this to all ideals generated by monomials in an arbitrary permutable regular sequence  $X_1, \dots, X_d$  in a regular ring  $R$  such that for every  $i_1, \dots, i_s \in \{1, \dots, d\}$ , the ring  $R/(X_{i_1}, \dots, X_{i_s})$  is a regular domain. Furthermore, [11] proved that for such  $I$ ,  $\text{adj} I = \bigcap_V \{r \in K : rJ_{V/R} \subseteq IV\}$ , where  $V$  varies only over the finite set of Rees valuations of  $I$ .

In addition, [11] proved that for all ideals  $I$  in a two-dimensional regular local ring,  $\text{adj} I = \bigcap_V \{r \in K : rJ_{V/R} \subseteq IV\}$ , where  $V$  varies only over the finite set of Rees valuations.

However, in general, Rees valuations do not suffice for computing the adjoints of ideals, see [11].

**Question.** Given an  $\mathfrak{m}$ -primary ideal  $I$  in a regular local ring  $(R, \mathfrak{m})$ , does there exist a finite set  $S$  of valuations such that the adjoint of all the (integer) powers of  $I$  can be computed by using only the valuations from  $S$ ?

If there is such a set  $S$ , by [11] it always contains the set of Rees valuations of  $I$ . In general,  $S$  contains other valuations as well. There is as yet no good criterion on what the other needed valuations might be.

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# Weak normality and seminormality

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**Abstract** In this survey article we outline the history of the twin theories of weak normality and seminormality for commutative rings and algebraic varieties with an emphasis on the recent developments in these theories over the past 15 years. We develop the theories for general commutative rings, but specialize to reduced Noetherian rings when necessary. We hope to acquaint the reader with many of the consequences of the theories.

All rings in this paper are commutative with identity, all modules are unitary, and ring homomorphisms preserve the identity.

## 1 Introduction

The operation of *weak normalization* was formally introduced in 1967 by A. Andreotti and F. Norguet [3] in order to solve a problem that arose while constructing a certain parameter space associated with a complex analytic variety. Their construction was dependent on the embedding of the space in complex projective space. In this setting the normalization of the parameter space is independent of the embedding, but no longer parametrizes what it was intended to since one point may split into several in the normalization. To compensate one “glues” together points on the normalization that lie over a single point in the original space. This leads to the weak normalization of the parameter space, a new space whose underlying point set is in one-to-one correspondence with the point set of the parameter space. A few years later weak normalization was introduced in the context of schemes and their morphisms by A. Andreotti and E. Bombieri. For an integral extension  $B$  of a local ring  $A$ , they first introduced the notion of *gluing* the prime ideals of  $B$  lying over the unique maximal ideal of  $A$ , mirroring the complex analytic construction. This notion of gluing, which we will refer to

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as *weak gluing*, appears in Lemma 3.1 and is formally defined in Definition 3.2. The weak gluing of  $A$  in  $B$  results in a local ring, integral over  $A$ , and whose residue field is purely inseparable over the residue field of  $A$  [2]. For a general integral extension  $A \subset B$ , an element  $b \in B$  is in the weak normalization of  $A$  in  $B$  if and only if for every point  $x \in \text{Spec}(A)$ , the image of  $b$  is in the weak gluing of  $A_x$  in  $B_x$  over  $x$ . Andreotti and Bombieri then turned their attention to schemes and their structure sheaves. They assumed they were working with preschemes, however, what was called a prescheme in those days (e.g. see Mumford's Red Book [32]) is today called a scheme so we dispense with the prescheme label. They defined the *weak subintegral closure* of the structure sheaf pointwise using the notion of gluing they already defined for local rings. They next defined the *weak normalization*  $\sigma: X^* \rightarrow X$  of a reduced algebraic scheme  $X$  over an arbitrary field so that the scheme  $X^*$  represents the weak subintegral closure of the structure sheaf of  $X$ . Andreotti and Bombieri established a universal mapping property of this pair  $(X^*, \sigma)$ , which we discuss in Section 3.4.

At about the same time C. Traverso [50] introduced the closely related notion of the *seminormalization* of a commutative ring  $A$  in an integral extension  $B$ . Like in the Andreotti–Bombieri construction, given a local ring  $A$  one glues the prime ideals of  $B$  lying over the unique maximal ideal of  $A$  (i.e. the maximal ideals of  $B$ ) but this time in a way that results in a local ring with residue field *isomorphic* to that of  $A$  (see Definition 2.2). For an arbitrary integral extension of rings  $A \subset B$ , an element  $b \in B$  is in the seminormalization  ${}^{\dagger}_B A$  of  $A$  in  $B$  if and only if for every point  $x \in \text{Spec}(A)$ , the image of  $b$  is in the gluing of  $A_x$  in  $B_x$  over  $x$ . Traverso showed [50, Theorem 2.1] that for a finite integral extension of Noetherian rings  $A \subset B$ , the seminormalization  ${}^{\dagger}_B A$  of  $A$  in  $B$  is obtained by a finite sequence of gluings. Traverso defined a *seminormal ring* to be a ring that is equal to its seminormalization in  $\bar{A}$ , where  $\bar{A}$  denotes the integral closure of  $A$  in its total ring of quotients. Traverso showed that for a reduced Noetherian ring  $A$  with finite normalization and a finite number of indeterminates  $T$ , the ring  $A$  is seminormal if and only if the canonical homomorphism  $\text{Pic}(A) \rightarrow \text{Pic}(A[T])$  is surjective. Here,  $\text{Pic}(A)$  denotes the *Picard group* of  $A$ , namely the group of isomorphism classes of rank one projective modules over  $A$  with  $\otimes_A$  as the group law.

A few years after Traverso's paper appeared E. Hamann showed that a seminormal ring  $A$  contains each element  $a$  of its total quotient ring such that  $a^n, a^{n+1} \in A$  for some positive integer  $n$  (see [23, Prop. 2.10]). A ring that satisfies this property is said to be  $(n, n+1)$ -closed. Hamann showed that this property is a characterization of seminormality for a pseudogeometric ring [23, Prop. 2.11]; today pseudogeometric rings are more commonly known as Nagata rings. More generally, Hamann's criterion characterizes seminormal reduced Noetherian rings. It also can be used to characterize arbitrary seminormal extensions, as we shall see. The integers  $n, n+1$  that appear in Hamann's characterization can be replaced by any pair  $e, f$  of relatively prime positive integers [28, Prop. 1.4]. A long-known construction by Schanuel shows that if for a reduced ring  $A$ , the canonical map  $\text{Pic}(A) \rightarrow \text{Pic}(A[T])$  is an isomorphism, then  $A$  is  $(2, 3)$ -closed (see [11, Appendix A]).

A slight but significant refinement of Hamann's criterion was made in 1980 by R. Swan [47]. Swan realized that the condition given by Hamann could be revised so that it became an internal condition on a general ring  $A$  and that with the revamped definition an arbitrary reduced ring is seminormal if and only if the canonical homomorphism  $\text{Pic}(A) \rightarrow \text{Pic}(A[T])$  is surjective. Swan defined a *seminormal ring* as a reduced ring  $A$  such that whenever  $b, c \in A$  satisfy  $b^3 = c^2$  there exists  $a \in A$  such that  $b = a^2, c = a^3$ . For reduced Noetherian rings (or, more generally, any reduced ring whose total quotient ring is a product of fields), Swan's definition of a seminormal ring is equivalent to Traverso's. It was pointed out by D.L. Costa [12] that a ring that satisfies Swan's criterion is necessarily reduced so the  $b^3 = c^2$  criterion alone characterizes seminormality. Swan showed in [47, Theorem 1] that  $\text{Pic}(A) \cong \text{Pic}(A[T])$  for some finite set of indeterminates  $T$  is equivalent to the seminormality of  $A_{\text{red}}$ . In [19] Gilmer and Heitmann presented an example of a non-Noetherian reduced ring that is equal to its own total ring of quotients but such that  $\text{Pic}(A)$  is not canonically isomorphic to  $\text{Pic}(A[T])$ . Thus, according to Traverso's original definition of a seminormal ring, the seminormality of  $A$  isn't equivalent  $\text{Pic}A \cong \text{Pic}A[T]$ . In [47] Swan constructed the seminormalization of a general commutative ring in a way that is reminiscent of the construction of the algebraic closure of a field. He went on to show that any reduced commutative ring has an essentially unique seminormalization (i.e., a subintegral extension  ${}^+A$  of  $A$  such that  ${}^+A$  is a seminormal ring).

Early in the 1980s, both Greco-Traverso [20] and Leahy-Vitulli [28] published pivotal papers that linked the earlier work of Andreotti-Norguet-Bombieri to the work of Traverso and looked at the singularities of schemes and varieties. Both pairs of authors showed that a reduced, complex analytic space is weakly normal at a point  $x \in X$  in the sense of Andreotti-Norguet if and only if the local ring of germs of holomorphic functions  $\mathcal{O}_{X,x}$  is seminormal in the sense of Traverso (cf. [28, Prop. 2.24] and [20, Cor. 5.3]). Leahy-Vitulli defined a weakly normal singularity called a *multicross* (see [28] and [29]). Briefly a point  $x$  on a variety  $X$  is a multicross if  $x \in X$  is analytically isomorphic to  $z \in Z$  where  $Z$  is the union of linearly disjoint linear subspaces. Most singularities of a weakly normal variety are of this type in the sense that the complement of the set of multicrosses forms a closed subvariety of codimension at least 2 [29, Theorem 3.8]. Leahy and Vitulli worked with algebraic varieties over an algebraically closed field of characteristic 0.

A few years later, Yanagihara gave an intrinsic definition of a *weakly normal ring* analogous to the Swan definition of a seminormal ring (see Definition 3.12 or [57]). Yanagihara said that a reduced ring  $A$  is weakly normal provided that the ring is seminormal in the sense of Swan and another condition involving rational primes holds.

Before we can talk about the next set of results we need to recall a pair of definitions. An integral extension of rings  $A \subset B$  is called a (*weakly*) *subintegral extension* if for each prime ideal  $P$  of  $A$  there is a unique prime ideal  $Q$  of  $B$  lying over  $P$  and the induced map of residue fields  $A_P/PA_P \rightarrow B_Q/QB_Q$  is an isomorphism (a purely inseparable extension) (see Definitions 2.5 and 3.4). An element  $b \in B$  is said to be (*weakly*) subintegral over  $A$  provided that the extension  $A \subset A[b]$  is (*weakly*) subintegral.

In a series of papers by L. Reid, L. Roberts, and B. Singh that appeared in the mid to late 1990s, a criterion for an element to be subintegral over a ring was introduced and developed. At first Reid, Roberts, and Singh worked with  $\mathbb{Q}$ -algebras and introduced what they called a *system of subintegrality* (SOSI). They proved that an element  $b$  in an integral extension  $B \supset A$  admits a SOSI if and only if  $b$  is subintegral over  $A$ . A system of subintegrality is global in nature and makes no reference to local gluings. We view the SOSI as the first true “element-wise” criterion for subintegrality over a  $\mathbb{Q}$ -algebra. Over the years Reid, Roberts, and Singh showed each of many conditions is equivalent to the existence of a system of subintegrality. Among the equivalent conditions is the existence of a highly-structured sequence of monic polynomials of various degrees such that the element  $b$  is a common root of each of these polynomials.

Reid, Roberts, and Singh were eventually able to drop the assumption that the rings are  $\mathbb{Q}$ -algebras, but only if they talked about weak subintegrality rather than subintegrality. However they still talked about SOSIs, which perhaps should have been renamed systems of *weak* subintegrality.

In 1999, Roberts observed that the lower degree polynomials are rational multiples of the polynomial of highest degree. Gaffney and Vitulli [18, Prop. 2.2 and Prop. 2.3] have recently expanded this observation to make some geometric sense out of the sequence of equations that appears in the work of Reid, Roberts, and Singh. Proposition 3.26 in this paper is an original result that gives a new element-wise characterization of weak subintegrality. Our result also provides a simple, purely algebraic explanation for why an element satisfying the sequence of equations introduced by Roberts, Reid, and Singh is necessarily weakly subintegral over the base ring.

The element-wise criteria of Reid, Roberts, and Singh enabled the current author and Leahy to talk about when an element  $b$  of a ring  $B$  is *weakly subintegral over an ideal*  $I$  of a subring  $A$  and define the weak subintegral closure of an ideal either in the containing ring  $A$  or an integral extension ring  $B$  [54]. Vitulli and Leahy showed that for an extension  $A \subset B$  of rings, an ideal  $I$  of  $A$ , and  $b \in B$ , the element  $b$  is weakly subintegral over  $I^m$  if and only if  $bI^m$  is weakly subintegral over the Rees ring  $A[It]$  [54, Lemma 3.2]. Thus, the weak subintegral closure  ${}^*I$  of  $I$  in  $B$  is an ideal of the weak subintegral closure  ${}^*A$  of  $A$  in  $B$  [54, Prop. 2.11]. In particular, the weak subintegral closure of  $I$  in  $A$  is again an ideal of  $A$ , which we denote simply by  ${}^*I$ . Vitulli and Leahy showed that given an ideal  $I$  in a reduced ring  $A$  with finitely many minimal primes and total quotient ring  $Q$ , it holds that  ${}^*(A[It]) = \bigoplus_{n \geq 0} {}^*(I^n)t^n$  [54, Cor. 3.5]. Gaffney and Vitulli [18] further developed the theory of weakly subintegrally closed ideals in both the algebraic and complex analytic settings. They defined a subideal  $I_{>}$  of the weak subintegral closure  ${}^*I$  of an  $I$  in a Noetherian ring that can be described solely in terms of the Rees valuations of the ideal and related this subideal to the minimal reductions of  $I$ . Gaffney and Vitulli also proved a valuative criterion for when an element is in the weak subintegral closure of an ideal. H. Brenner [8] has proposed another valuative criterion in terms of maps into the appropriate test rings.

We now describe the contents of this paper. Section 2 deals with seminormality and seminormalization. In Section 2.1, we recall Traverso's notion of gluing the maximal ideals of an integral extension of a local ring and prove the fundamental properties that the ring so obtained enjoys. We give Traverso's definition of the seminormalization of a ring  $A$  in an integral extension  $B$  of  $A$  and provide proofs of the fundamental properties of seminormalization. In Section 2.2, we take a look at Swan's revamped notion of a seminormal ring and his construction of the seminormalization of a ring. We discuss seminormality and Chinese Remainder Theorems in Section 2.3. In Section 2.4, we explore the connections between seminormality and the surjectivity of the canonical map  $\text{Pic}(A) \rightarrow \text{Pic}(A[T])$ . In Section 2.5, we recall some characterizations of seminormal one-dimensional local rings. In Section 3 we develop the theories of weakly subintegral extensions and weak normality. We discuss weakly subintegral and weakly normal extensions and the operation of weak normalization in Section 3.1. We look at systems of subintegrality in Section 3.2. We offer a new algebraic criterion for the weak subintegrality of an element over a subring in Section 3.3. We give the reader a very brief glimpse of some geometric aspects of weakly normal varieties in Section 3.4. In Section 3.5, we recall a couple of Chinese Remainder Theorem results for weakly normal varieties. Our final section, Section 3.6, is devoted to the notions of weak subintegrality over an ideal and the weak subintegral closure of an ideal. We recall the Reid-Vitulli geometric and algebraic characterizations of the weak subintegral closure of a monomial ideal. We introduce the ideal  $I_{>}$  that was defined by Gaffney and Vitulli and cite some of their results connected with this ideal.

## 2 Seminormality and seminormalization

In Section 2.1, we will outline the development of gluings of prime ideals, subintegral extensions, and the seminormalization of a ring in an integral extension ring, a relative notion dependent on the extension ring. In Section 2.2, we will deal with the absolute notions of a seminormal ring and seminormalization; these notions do not depend on particular integral extensions. In Section 2.3, we discuss seminormality in relation to various versions of the Chinese Remainder Theorem. In Section 2.4, we discuss some Picard group results connected with seminormality. Most of the original proofs of these results were  $K$ -theoretic in nature, but Coquand's recent treatment, which we sketch, is more elementary. In Section 2.5, we discuss some results on one-dimensional seminormal rings.

We work with arbitrary commutative rings in this section. We do not assume the rings we are discussing are reduced, Noetherian, or have any other special properties.

### 2.1 Subintegral and seminormal extensions and seminormalization relative to an extension

We start by giving the local construction that leads to the seminormalization of a ring in an integral extension. For a ring  $B$  we let  $R(B)$  denote the Jacobson radical of  $B$ . We notationally distinguish between a prime ideal  $P = P_x$  of  $B$  and the corresponding point  $x \in \text{Spec}(B)$ , when this is convenient. For an element  $b \in B$  and point  $x \in \text{Spec}(B)$  we let  $b(x)$  denote the image of  $b$  in the residue field  $\kappa(x) = B_x/P_xB_x$ . If  $A \subset B$  are rings and  $x \in \text{Spec}(A)$  we let  $B_x$  denote the ring obtained by localizing the  $A$ -module  $B$  at the prime ideal  $x$ .

**Lemma 2.1.** *Let  $(A, \mathfrak{m})$  be a local ring and  $A \subset B$  be an integral extension. Set  $A' = A + R(B) \subset B$  and  $\mathfrak{m}' = R(B)$ . For each maximal ideal  $\mathfrak{n}_i$  of  $B$  let  $\omega_i: A/\mathfrak{m} \rightarrow B/\mathfrak{n}_i$  be the canonical homomorphism. The following assertions hold.*

1.  $(A', \mathfrak{m}')$  is a local ring and the canonical homomorphism  $A/\mathfrak{m} \rightarrow A'/\mathfrak{m}'$  is an isomorphism;
2.  $A'$  is the largest intermediate local ring  $(C, \mathfrak{n})$  such that  $A/\mathfrak{m} \cong C/\mathfrak{n}$ ; and
3. An element  $b \in B$  is in  $A'$  if and only if
  - a.  $b(x_i) \in \omega_i(\kappa(x))$  for all closed points  $x_i$  of  $\text{Spec}(B)$ , and
  - b.  $\omega_i^{-1}(b(x_i)) = \omega_j^{-1}(b(x_j))$  for all  $i, j$ .

*Proof.* 1. We first show that  $A' = A + R(B)$  is local with unique maximal ideal  $\mathfrak{m}' = R(B)$ . It is clear that the canonical map  $A/\mathfrak{m} \rightarrow A'/\mathfrak{m}'$  is an isomorphism and hence  $\mathfrak{m}'$  is a maximal ideal of  $A'$ . Suppose that  $\mathfrak{n}$  is any maximal ideal of  $A'$ . By Lying Over there exists a necessarily maximal ideal  $\mathfrak{n}_i$  of  $B$  lying over  $\mathfrak{n}$ . Since we have  $\mathfrak{n} = \mathfrak{n}_i \cap A' \supset R(B) \cap A' = R(B) = \mathfrak{m}'$  we may conclude that  $\mathfrak{m}'$  is the unique maximal ideal of  $A'$ .

2. Suppose that  $A \subset C \subset B$  are rings and that  $(C, \mathfrak{n})$  is local with residue field isomorphic to that of  $A$ . As in the proof of 1. we must have  $\mathfrak{n} \subset R(B)$  and since  $A/\mathfrak{m} \rightarrow C/\mathfrak{n}$  is an isomorphism we may conclude that  $C \subset A + R(B)$ .

3. Suppose that  $b \in B$  satisfies (a) and (b) above. Choose  $a \in A$  such that  $\omega_i(a(x)) = b(x_i)$  for some  $i$ , and hence for every  $i$ . Then  $b - a \in \mathfrak{n}_i$  for every maximal ideal  $\mathfrak{n}_i$  of  $B$ , and hence,  $b \in A + R(B) = A'$ . It is clear that if  $b \in A'$  then  $b$  satisfies conditions a. and b. above.  $\square$

**Definition 2.2.** Let  $(A, \mathfrak{m})$  be a local ring and  $A \subset B$  be an integral extension. We say that  $A + R(B)$  is the ring obtained from  $A$  by gluing the maximal ideals in  $B$  over  $\mathfrak{m}$  or that  $A + R(B)$  is the ring obtained from  $A$  by gluing in  $B$  over  $\mathfrak{m}$ . Letting  $x \in \text{Spec}(A)$  denote the point corresponding to the maximal ideal  $\mathfrak{m}$ , this ring is sometimes denoted by  $\overset{\dagger}{x}A$ .

We can also glue the prime ideals of an integral extension ring lying over an arbitrary prime ideal. We state the result and leave the proof up to the reader.

**Lemma 2.3.** *Let  $A \subset B$  be an integral extension of rings and  $x \in \text{Spec}(A)$ . For each point  $x_i \in \text{Spec}(B)$  lying over  $x$ , let  $\omega_i: \kappa(x) \rightarrow \kappa(x_i)$  denote the canonical*

homomorphism of residue fields. Set  $A' = \{b \in B \mid b_x \in A_x + R(B_x)\}$ . The following assertions hold.

1. There is exactly one point  $x' \in \text{Spec}(A')$  lying over  $x$  and the canonical homomorphism  $\kappa(x) \rightarrow \kappa(x')$  is an isomorphism.
2.  $A'$  is the largest intermediate ring with one point lying over  $x$  and with isomorphic residue field at the corresponding prime; and
3. An element  $b \in B$  is in  $A'$  if and only if
  - a.  $b(x_i) \in \omega_i(\kappa(x))$  for all points  $x_i$  of  $\text{Spec}(B)$  lying over  $x$ , and
  - b.  $\omega_i^{-1}(b(x_i)) = \omega_j^{-1}(b(x_j))$  for all  $i, j$ .

We will now recall Traverso’s global definition and its first properties.

**Definition 2.4.** Let  $A \subset B$  be an integral extension of rings. We define the *seminormalization*  ${}^{\dagger}_B A$  of  $A$  in  $B$  to be

$${}^{\dagger}_B A = \{b \in B \mid b_x \in A_x + R(B_x) \text{ for all } x \in \text{Spec}(A)\}. \tag{1}$$

Before stating the first properties of  ${}^{\dagger}_B A$ , it will be useful at this point to recall the notion of a subintegral extension. Such extensions were first studied by Greco and Traverso in [20], who called them quasi-isomorphisms. In his 1980 paper, R. Swan [47] called them subintegral extensions and this is what they are called today.

**Definition 2.5.** A *subintegral extension of rings* is an integral extension  $A \subset B$  such that the associated map  $\text{Spec}(B) \rightarrow \text{Spec}(A)$  is a bijection and induces isomorphisms on the residue fields. An element  $b \in B$  is said to be *subintegral* over  $A$  provided that  $A \subset A[b]$  is a subintegral extension.

Notice that there are no proper subintegral extensions of fields. We now look at some first examples of subintegral extensions.

*Example 2.6.* Let  $A \subset B$  be an extension of rings,  $b \in B$  and  $b^2, b^3 \in A$ . Then,  $A \subset A[b]$  is easily seen to be a subintegral extension. If  $A$  is any ring and  $b, c \in A$  satisfy  $b^3 = c^2$  then the extension  $A \subset A[x] := A[X]/(X^2 - b, X^3 - c)$ , where  $X$  is an indeterminate, is thus a subintegral extension.

**Definition 2.7.** An *elementary subintegral extension* is a simple integral extension  $A \subset A[b]$ , such that  $b^2, b^3 \in A$ .

The seminormalization  ${}^{\dagger}_B A$  of  $A$  in an integral extension  $B$  is the filtered union of all subrings of  $B$  that can be obtained from  $A$  by a finite sequence of elementary subintegral extensions. Here by filtered union we mean that given any rings

$C_1, C_2$  with the property there is a third ring  $C$  with the property and satisfying  $A \subset C_i \subset C \subset B$  ( $i = 1, 2$ ). In particular, the union of all subring of  $B$  with this property is again a ring.

The notion of a seminormal extension, which we now define, is *complementary* to that of a subintegral extension.

**Definition 2.8.** If  $A \subset B$  is an integral extension of rings, we say  $A$  is *seminormal* in  $B$  if there is no subextension  $A \subset C \subset B$  with  $C \neq A$  and  $A \subset C$  subintegral.

We point out that  $A$  is seminormal in  $B$  if and only if  $A = \overset{+}{B}A$ . We now recall Traverso’s results regarding the seminormalization of  $A$  in an integral extension  $B$ .

**Theorem 2.9. [50, (1.1)]** *Let  $A \subset B$  be an integral extension of rings. The following assertions hold.*

1. *The extension  $A \subset \overset{+}{B}A$  is subintegral;*
2. *If  $A \subset C \subset B$  and  $A \subset C$  is subintegral, then  $C \subset \overset{+}{B}A$ ;*
3. *The extension  $\overset{+}{B}A \subset B$  is seminormal; and*
4.  *$\overset{+}{B}A$  has no proper subrings containing  $A$  and seminormal in  $B$ .*

*Proof.* 1. This follows immediately from Lemma 2.3.

2. Now suppose that  $A \subset C \subset B$  and that  $A \subset C$  is subintegral. Let  $P \in \text{Spec}(A)$  and let  $Q \in \text{Spec}(C)$  be the unique prime lying over  $P$ . Since  $A_P/PA_P \rightarrow C_P/QC_P$  is an isomorphism and  $QC_P \subset R(B_P)$  we must have  $C_P \subset A_P + R(B_P)$ . Since  $P$  was arbitrary,  $C \subset A'$ .

3. This follows immediately from parts 1. and 2.

4. Suppose that  $A \subset C \subset \overset{+}{B}A$  and that  $C \subset B$  is seminormal. Observe that  $C \subset \overset{+}{B}A$  is necessarily subintegral and hence  $C = \overset{+}{B}A$ .  $\square$

Paraphrasing this result,  $\overset{+}{B}A$  is the unique largest subintegral extension of  $A$  in  $B$  and is minimal among the intermediate rings  $C$  such that  $C \subset B$  is seminormal.

We mention some fundamental properties of seminormal extensions.

**Proposition 2.10.** *Let  $A \subset B \subset C$  be integral extensions of rings.*

1. *If  $A \subset B$  is a seminormal extension, then the colon ideal  $A : B$  is a radical ideal of  $B$  (contained in  $A$ ).*
2.  *$A \subset B$  is seminormal if and only if  $A : A[b]$  is a radical ideal of  $A[b]$  for every  $b \in B$ .*
3. *If  $A \subset B$  and  $B \subset C$  are seminormal extensions, then so is  $A \subset C$ .*

*Proof.* For the proof of the first statement see [50, Lemma 1.3] and for the second and third see [28, Prop. 1.4 and Cor. 1.5].  $\square$

Before recalling a criterion due to Hamann and its generalizations we recall some definitions.

**Definition 2.11.** Let  $A \subset B$  be an integral extension of rings and  $m, n$  be positive integers. We say that  $A$  is  *$n$ -closed* in  $B$  if  $A$  contains each element  $b \in B$  such that  $b^n \in A$ . We say that  $A$  is  *$(m, n)$ -closed* in  $B$  if  $A$  contains each element  $b \in B$  such that  $b^m, b^n \in A$ .

Hamann [23, Prop. 2.10] showed that a ring  $A$  that is seminormal in  $\bar{A}$ , the integral closure of  $A$  in its total quotient ring, is  $(n, n + 1)$ -closed in  $\bar{A}$ . She also showed that this property is a characterization of seminormality for pseudogeometric rings (also known as Nagata rings) ([23, Prop. 2.11]. We will say that a ring  $A$  (respectively, an integral extension  $A \subset B$ ) satisfies Hamann’s criterion provided that  $A$  is  $(n, n + 1)$ -closed in  $\bar{A}$  (respectively,  $B$ ), for some positive integer  $n$ . Seminormality arose in Hamann’s study of  $R$ -invariant and steadfast rings. An  $R$ -algebra  $A$  is  $R$ -invariant provided that whenever  $A[x_1, \dots, x_n] \cong_R B[y_1, \dots, y_n]$  for indeterminates  $x_i, y_j$ , we must have  $A \cong_R B$ . Finally,  $R$  is called *steadfast* if the polynomial ring in one variable  $R[x]$  is  $R$ -invariant. Hamann used Traverso’s work on seminormality and the Picard group to show that a pseudogeometric local reduced ring  $A$  that is seminormal in  $\bar{A}$  is steadfast ([23, Theorem 2.4]) and a domain  $A$  that is seminormal in  $\bar{A}$  is steadfast [23, Theorem 2.5]. Hamann’s criterion characterizes reduced Noetherian rings that are seminormal in  $\bar{A}$ . It also can be used to characterize arbitrary seminormal extensions as we shall see. The integers  $n, n + 1$  that appear in Hamann’s characterization can be replaced by any pair  $e, f$  of relatively prime positive integers ([28, Prop. 1.4].

The following was proven by Leahy and Vitulli.

**Proposition 2.12.** [28, Prop. 1.4] *Let  $A \subset B$  be an integral extension of rings. The following are equivalent.*

1.  $A$  is seminormal in  $B$ ;
2.  $A$  is  $(n, n + 1)$ -closed in  $B$  for some positive integer  $n$ ; and
3.  $A$  is  $(m, n)$ -closed in  $B$  for some relatively prime positive integers  $m, n$ .

The reader should note if  $A$  is seminormal in  $B$ , then  $A$  is  $(m, n)$ -closed in  $B$  for every pair of relatively prime positive integers  $m$  and  $n$ . Recall from the introduction Hamann’s results that assert that a ring  $A$  that is seminormal in  $\bar{A}$  is  $(n, n + 1)$ -closed for every positive integer  $n$  and that a pseudogeometric ring  $A$  that is  $(n, n + 1)$ -closed is necessarily seminormal in  $\bar{A}$ .

The most commonly cited version of 2. above is with  $n = 2$ , that is, the integral extension  $A \subset B$  is seminormal if and only if it is  $(2, 3)$ -closed (see [23, Props. 2.10 and 2.11].

We point out that a normal integral domain  $A$  is  $(2, 3)$ -closed, and hence seminormal in any reduced extension ring  $B$ . For if  $0 \neq b \in B$  and  $b^2, b^3 \in A$ , then  $b = b^3/b^2$  is in the quotient field of  $A$  and is integral over  $A$ , hence  $b \in A$ . In particular, if  $K$  is a field, then  $K[x^2] \subset K[x]$  is a seminormal extension.

With the previous characterization of a seminormal extension, it is easy to show that seminormal extensions are preserved under localization.

**Proposition 2.13.** [28, Cor. 1.6] *If  $A \subset B$  is a seminormal extension and  $S$  is any multiplicative subset, then  $S^{-1}A \subset S^{-1}B$  is again seminormal. Moreover, the operations of seminormalization and localization commute.*

Swan gave another useful characterization of a subintegral extension.



**Lemma 2.14.** [47, Lemma 2.1] *An extension  $A \subset B$  is subintegral if and only if  $B$  is integral over  $A$  and for all homomorphisms  $\varphi: A \rightarrow F$  into a field  $F$ , there exists a unique extension  $\psi: B \rightarrow F$ .*

*Proof.* This follows from noting that  $\varphi$  is determined by specifying a prime  $Q$  (which will serve as  $\ker \psi$ ) lying over  $P = \ker \varphi$  and an extension of  $\kappa(P) \rightarrow F$  to a map  $\kappa(Q) \rightarrow F$ .  $\square$

## 2.2 Seminormal rings, seminormalization, and Swan's refinements

Traverso's original definition of a seminormal extension of rings and his construction of the seminormalization of a ring  $A$  in an integral extension  $B$  of  $A$  are still accepted by specialists in the field today. Since this conceptualization of seminormalization depends on the extension ring we'll refer to it as the *relative* notion. However, the current *absolute* notion of a seminormal ring and the construction of the seminormalization of a ring are the result of Swan's insights. As the reader shall see, Swan defined a seminormal ring without any mention of an extension ring. To say it differently, the relative theory is still due to Traverso but the absolute theory commonly used today is due to Swan. We recall from the introduction that Traverso defined a seminormal ring as a ring  $A$  that is equal to its seminormalization in the integral closure of  $A$  in its total quotient ring. We will not formally give Traverso's definition, but rather will follow Swan's approach. One compelling reason for modifying the definition is that the modification enabled Swan to prove that  $A_{\text{red}}$  is seminormal if and only if every rank one projective module over the polynomial ring  $A[T]$  is extended from  $A$  [47, Theorem 1]. The reader is referred to Swan's original paper [47] for more details. Lemma 2.16 below illustrates that under mild assumptions, Swan's absolute notion of a seminormal ring is equivalent to the earlier notion of Traverso. We also present the Swan example that shows that, in general, the definitions disagree. We mention the Gilmer–Heitmann example that show these definitions disagree even for reduced rings.

For a ring  $A$  we let  $\bar{A}$  denote the integral closure of  $A$  in its total ring of quotients.

Notice that if  $A$  is a ring and  $b$  is an element of the total ring of quotients of  $A$  such that  $b^2, b^3 \in A$ , then  $b$  is necessarily integral over  $A$ . Thus a ring  $A = \frac{\dagger}{A}A$  if and only if it is (2,3)-closed in its total quotient ring.

We now present Swan's definition of a seminormal ring.

**Definition 2.15.** A *seminormal ring* is a ring  $A$  such that whenever  $b, c \in A$  satisfy  $b^3 = c^2$  there exists  $a \in A$  such that  $b = a^2, c = a^3$ .

D. L. Costa [12] observed that if  $A$  is seminormal in this sense, then  $A$  is necessarily reduced. Just suppose  $A$  is not reduced. Then, there is some nonzero element  $b \in A$  such that  $b^2 = 0$ . Then,  $b^3 = b^2$  and hence there exists  $a \in A$  such that  $b = a^2, b = a^3$ . Hence,  $b = a^3 = aa^2 = ab = aa^3 = b^2 = 0$ , a contradiction. Swan's original

definition of a seminormal ring stipulated that the ring is reduced, but in light of the previous remark this is redundant and we have omitted it from our definition.

Suppose that  $A$  is seminormal and let  $K$  denote its total ring of quotients. Assume that  $b \in K$  and  $b^2, b^3 \in A$ . Then, there exists an element  $a \in A$  such that  $a^2 = b^2$ ,  $a^3 = b^3$ . Then,  $(a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3 = a^3 - 3b^3 + 3a^3 - b^3 = 0$  and hence  $b = a \in A$ , since  $A$  is reduced. Thus  $A = \overset{+}{A}A$ . This same argument shows that for a seminormal ring  $A$  and elements  $b, c \in A$  satisfying  $b^3 = c^2$ , there exists a *unique* element  $a \in A$  such that  $b = a^2, c = a^3$ . If  $K$  is a product of fields and  $A$  is seminormal in  $\bar{A}$ , then  $A$  is a seminormal ring, as we will now show.

**Lemma 2.16.** *Suppose that  $A$  is a ring whose total ring of quotients  $K$  is a product of fields and let  $B = \bar{A}$ . Then,  $A = \overset{+}{B}A$  if and only if  $A$  is seminormal.*

*Proof.* Let  $K = \prod K_i$ , where  $K_i$  are fields, and let  $\rho_i: A \rightarrow K_i$  be the inclusion followed by the projection. Notice that  $A$  is reduced. Assume that  $A = \overset{+}{B}A$ . Suppose that  $b, c \in A$  and  $b^3 = c^2$ . If  $b = 0$  then  $b = 0^2, c = 0^3$ . Now assume  $b \neq 0$ . Consider the element  $\alpha = (\alpha_i) \in K$  where

$$\alpha_i = \begin{cases} 0 & \text{if } \rho_i(b) = 0 \\ \rho_i(c)/\rho_i(b) & \text{if } \rho_i(b) \neq 0 \end{cases}$$

One can check that  $\alpha^2 = b, \alpha^3 = c \in A$ . Since  $A = \overset{+}{B}A$ ,  $\alpha = (\rho_i(a))$  for some  $a \in A$ . Then,  $a^2 = b, a^3 = c$ , as desired.

The converse was observed to be true in the preceding paragraph.  $\square$

We wish to define the seminormalization  $\overset{+}{A}$  of a reduced ring  $A$  to be a seminormal ring such that  $A \subset \overset{+}{A}$  is a subintegral extension. It turns out that any such extension has a universal mapping property that makes it essentially unique; we will make this precise after stating Swan’s theorem. In light of Lemma 2.16 and Theorem 2.9, if the ring  $A$  is reduced and its total ring of quotients  $K$  is a product of fields, then we may define  $\overset{+}{A} = \overset{+}{\bar{A}}A$ , where  $\bar{A}$  is the normalization of  $A$ . If the total quotient ring isn’t a product of fields, this doesn’t always produce a seminormal ring as the following example that appeared in Swan [47] illustrates.

*Example 2.17.* A ring that is seminormal in its integral closure in its total quotient ring need not be seminormal. Let  $(B, \mathfrak{n}, k)$  be a local ring and set  $A = B[X]/(\mathfrak{n}X, X^2) = B[x]$ . Notice that  $A$  is local with unique maximal ideal  $\mathfrak{m} = (\mathfrak{n}, x)$ . Since  $\mathfrak{m}$  consists of zero divisors for  $A$  the ring  $A$  is equal to its own total ring of quotients  $K$  and hence  $A = \overset{+}{K}A$ . However,  $A$  is not a seminormal ring, since it isn’t reduced.

Gilmer and Heitmann [19, Example 2.1] constructed an example of a *reduced* local ring  $A$  that is equal to its own quotient ring, but the canonical map  $\text{Pic}(A) \rightarrow \text{Pic}(A[T])$  is not surjective. As we shall see in the next section, this implies  $A$  is not a seminormal ring. However, since  $A$  is equal to its own quotient ring,  $A$  is seminormal in its integral closure in its quotient ring. Gilmer and Heitmann’s construction and proof that the canonical map  $\text{Pic}(A) \rightarrow \text{Pic}(A[T])$  is not surjective is more complicated; we refer the reader to their original paper for the details.

Swan constructed the seminormalization of a reduced ring using elementary subintegral extensions and mimicking the construction of the algebraic closure of a field. He also proved the following universal mapping property.

**Theorem 2.18.** [47, Theorem 4.1] *Let  $A$  be any reduced ring. Then there is a subintegral extension  $A \subset B$  with  $B$  seminormal. Any such extension is universal for maps of  $A$  to seminormal rings: If  $C$  is seminormal and  $\varphi : A \rightarrow C$ , then  $\varphi$  has a unique extension  $\psi : B \rightarrow C$ . Furthermore  $\psi$  is injective, if  $\varphi$  is.*

Thus if there exist subintegral extensions  $A \subset B$  and  $A \subset C$  of a reduced ring  $A$  with  $B$  and  $C$  seminormal, then there is a unique ring homomorphism  $\psi : B \rightarrow C$  which is the identity map on  $A$ , and  $\psi$  is an isomorphism. In this strong sense,  $A$  admits an essentially unique seminormalization.

Mindful of Hamann’s examples in [23], Swan calls an extension of rings  $A \subset B$  *p*-seminormal if  $b \in B, b^2, b^3, pb \in A$  imply  $b \in A$ ; here  $p$  denotes a positive integer. Swan proves in [47, Theorem 9.1] that a reduced commutative ring is steadfast if and only if it is *p*-seminormal for all rational primes  $p$ . This assertion had been proven for domains by Asanuma [4]. Greither [21, Theorem 2.3] proved that a projective algebra in one variable over a seminormal ring is the symmetric algebra of an  $R$ -projective module. Before we conclude this section we’d like to mention some results of Greco and Traverso on faithfully flat homomorphisms, pull-backs, and completions.

**Definition 2.19.** A *Mori ring* is a reduced ring  $A$  whose integral closure  $\bar{A}$  in its total ring of quotients is a finite  $A$ -module.

We point out that an affine ring is Mori. More generally, a reduced excellent ring is Mori.

**Theorem 2.20.** *Consider a faithfully flat ring homomorphism  $f : A \rightarrow A'$ . Let  $B$  be a finite overring of  $A$  and put  $B' = A' \otimes_A B$ .*

1. *If  $A'$  is seminormal in  $B'$ , then  $A$  is seminormal in  $B$ .*
2. *If  $A'$  is a seminormal Mori ring, then  $A$  is a seminormal Mori ring.*

*Proof.* See [20, Theorem 1.6 and Cor. 1.7].  $\square$

Greco and Traverso also proved this result on pull-backs in order to prove a result for reduced base change.

**Lemma 2.21.** [20, Lemma 4.2] *Let  $R$  be a ring and let*

$$\begin{array}{ccc}
 A & \xrightarrow{f} & B \\
 g \downarrow & & \downarrow u \\
 C & \xrightarrow{v} & D
 \end{array}$$

*be a pull-back diagram of  $R$ -algebras, that is,*

$$A = B \times_D C := \{ (b, c) \in B \times C \mid u(b) = v(c) \}.$$

Assume that the horizontal arrows are injective and finite, and that the vertical ones are surjective. If  $C$  is seminormal in  $D$ , then  $A$  is seminormal in  $B$ .

This lemma is used to prove the following base change result. Recall that a homomorphism  $f: A \rightarrow A'$  is *reduced* if it is flat and all of its fibers  $A' \otimes_A \kappa(P)$  are geometrically reduced. The latter is equivalent to saying that for any prime  $P$  of  $A$  and any finite extension  $L$  of  $\kappa(P)$ , the ring  $B \otimes_A L$  is reduced.

**Theorem 2.22.** [20, Theorem 4.1] *Let  $A \subset B$  be a finite integral extension and let  $A \rightarrow A'$  be a reduced homomorphism. If  $A$  is seminormal in  $B$ , then  $A'$  is seminormal in  $B' = A' \otimes_A B$ .*

We conclude this section by recalling a result on seminormality and completions. A proof of this result for algebro-geometric rings can be found in Leahy–Vitulli [28, Theorem 1.21].

**Theorem 2.23.** [20, Cor. 5.3] *An excellent local ring is seminormal if and only if its completion is seminormal.*

### 2.3 Seminormality and Chinese remainder theorems

Let  $A$  be a commutative ring and  $I_1, \dots, I_n$  be ideals of  $A$ . We say that the Chinese Remainder Theorem (CRT) holds for  $\{I_1, \dots, I_n\}$  provided that given  $a_1, \dots, a_n \in A$  such that  $a_i \equiv a_j \pmod{I_i + I_j}$  ( $i \neq j$ ), there exists an element  $a \in A$  such that  $a \equiv a_i \pmod{I_i}$  for all  $i$ .

Notice that if the ideals  $I_j$  are co-maximal we get the statement of the classical Chinese Remainder Theorem.

Note too that in the generalized sense the CRT holds for any pair of ideals since the sequence

$$0 \rightarrow A/(I_1 \cap I_2) \xrightarrow{\alpha} A/I_1 \times A/I_2 \xrightarrow{\beta} A/(I_1 + I_2) \rightarrow 0$$

where  $\alpha(a + I_1 \cap I_2) = (a + I_1, a + I_2)$  and  $\beta(a_1 + I_1, a_2 + I_2) = a_1 - a_2 + I_1 + I_2$  is always exact.

The generalized Chinese Remainder Theorem need not hold for 3 ideals. Consider the ideals  $I_1 = (X), I_2 = (Y), I_3 = (X - Y) \subset \mathbb{C}[X, Y]$ . There isn't an element  $f \in \mathbb{C}[X, Y]$  such that  $f \equiv Y \pmod{(X)}, f \equiv X \pmod{(Y)}$  and  $f \equiv 1 \pmod{(X - Y)}$ .

We now mention a few of the results on seminormality, weak normality, and results in the spirit of the Chinese Remainder Theorem that are due to Dayton, Dayton–Roberts, and Leahy–Vitulli. These results appeared at about the same time. The results in Leahy–Vitulli were stated for algebraic varieties over an algebraically closed field of characteristic 0. The first result appeared both in [17, Theorem 2] and in [16, Theorem A], where a direct proof of the result is given.

**Theorem 2.24.** *Let  $A$  be a commutative ring and  $I_1, \dots, I_n$  be ideals of  $A$ . Suppose that  $I_i + I_j$  is radical for  $i \neq j$  and let  $B = \prod A/I_i$ . Then, the CRT holds for  $\{I_1, \dots, I_n\}$  if and only if  $A/\cap I_i$  is seminormal in  $B$ .*

Another form of the CRT appears in [17, Theorem 2]. To put this result in some context we remind the reader that if  $A \subset B$  is a seminormal extension of rings, then the colon ideal  $A : B$  is a radical ideal of  $A$ . A general ring extension  $A \subset B$  is seminormal if and only if  $A : A[b]$  is a radical ideal of  $A[b]$  for every  $b \in B$ . Both statements about colon ideals (once called conductors) appeared in Proposition 2.10. The theorem below is preceded by a lemma on colon ideals.

**Lemma 2.25.** *Let  $A$  be a commutative ring and  $I_1, \dots, I_n$  ideals of  $A$  with  $\cap I_i = 0$ . Let  $B = \prod A/I_i$  and set  $\mathfrak{c} = A : B$ . Then,  $\mathfrak{c} = \sum_i (\cap_{j \neq i} I_j) = \cap_i (I_i + \cap_{j \neq i} I_j)$ .*

**Theorem 2.26.** *Let  $A$  be a commutative ring and  $I_1, \dots, I_n$  ideals of  $A$  with  $\cap I_i = 0$ . Let  $B = \prod A/I_i$ ,  $\mathfrak{c} = A : B$  and  $\mathfrak{c}_i$  the projection of  $\mathfrak{c}$  in  $A/I_i$ . Assume also that  $A/I_i$  is seminormal for each  $i$  and that  $I_j + I_k$  is radical all  $j, k$ . Then conditions 1. and 2. below are equivalent.*

1.  $A$  is seminormal.
2. The CRT holds for  $\{I_1, \dots, I_n\}$ .

*These imply the next 5 conditions, which are equivalent.*

3. For each  $i$ ,  $\cap_{j \neq i} (I_i + I_j) = I_i + \cap_{j \neq i} I_j$ .
4. For each  $i$ ,  $I_i + \cap_{j \neq i} I_j$  is a radical ideal in  $A$ .
5. For each  $i$ ,  $\mathfrak{c}_i = \cap_{j \neq i} ((I_i + I_j)/I_i)$ .
6. For each  $i$ ,  $\mathfrak{c}_i$  is a radical ideal in  $A/I_i$ .
7.  $\mathfrak{c}$  is a radical ideal in  $B$ .

*The above imply the next 2 conditions, which are equivalent.*

6.  $\mathfrak{c}$  is a radical ideal in  $A$ .
7.  $\mathfrak{c} = \cup_{j \neq k} (I_j + I_k)$ .

*If, in addition,  $A/\cap_{j \neq i} I_j$  is seminormal for any  $i$  then 7.  $\Rightarrow$  1.*

There are various results in the spirit of the Chinese Remainder Theorem given in [28, Section 2], where they are stated for varieties over an algebraically closed field of characteristic 0. These results appeared at the same time as the algebraic results cited above. Their proofs use an algebro-geometric characterization of weakly normal varieties that will be introduced in Section 3.4.

## 2.4 Seminormality and the Traverso-Swan Picard group result with Coquand's simplification

Recall that  $\text{Pic}(A)$  denotes the Picard group of  $A$ , i.e., the group of isomorphism classes of rank one projective modules over  $A$ . Given a homomorphism of rings

$\phi : A \rightarrow B$  we get a homomorphism of groups

$$\text{Pic}(\phi) : \text{Pic}(A) \rightarrow \text{Pic}(B), \tag{2}$$

defined by extending scalars to  $B$ . When  $B = A[X]$  and  $\phi$  is the natural inclusion, evaluation at zero defines a retract  $\rho : A[X] \rightarrow A$ . We thus have group homomorphisms

$$\text{Pic}(A) \xrightarrow{\text{Pic}(\phi)} \text{Pic}(A[X]) \xrightarrow{\text{Pic}(\rho)} \text{Pic}(A), \tag{3}$$

whose composition is the identity map. We point out that the first homomorphism is always injective and the second always surjective. The maps  $\text{Pic}(\phi)$  and  $\text{Pic}(\rho)$  are isomorphisms if and only if the first is surjective if and only if the second is injective. Therefore to show that  $\text{Pic}(\phi) : \text{Pic}(A) \rightarrow \text{Pic}(A[X])$  is an isomorphism is equivalent to showing: if  $P$  is a rank one projective over  $A[X]$  and  $P(0) := \text{Pic}(\rho)(P)$  is a free  $A$ -module, then  $P$  is a free  $A[X]$ -module.

Various people have proven in different cases that  $\text{Pic}(\phi)$  is an isomorphism if and only if  $A$  is seminormal. This was proven by Traverso in case  $A$  is a reduced Noetherian ring with finite normalization [50, Theorem 3.6], by Gilmer and Heitmann for an arbitrary integral domain [19], by Rush for a reduced ring with finitely many minimal primes [43, Theorems 1 and 2], and by Swan for an arbitrary reduced ring [47, Theorem 1]. Both Traverso and Swan used standard  $K$ -theoretic results to prove the ‘if’ direction and a construction of Schanuel to prove the ‘only if’ direction. A recent paper by Coquand [11] simplified the connection between seminormality and the Picard group result by giving a self-contained proof of Swan’s result. For a finite integral extension of reduced rings  $A \subset B$ , Coquand’s work suggests an algorithmic approach to finding a sequence of elements  $a_1, a_2, \dots, a_n \in B$  such that  $a_{i+1}^2, a_{i+1}^3 \in A[a_1, \dots, a_i]$  for  $i = 1, \dots, n - 1$  and  ${}^+_B A = A[a_1, \dots, a_n]$ . Another algorithm was recently given by Barhoumi and Lombardi [5].

Since  $\text{Pic}(A) \cong \text{Pic}(A_{\text{red}})$  and  $A$  seminormal implies  $A$  is reduced we will assume that our base ring  $A$  is reduced in the remainder of this section.

As pointed out by Swan [47] one can replace finitely-generated projective modules by “projection matrices,” as we now explain. If

$$0 \rightarrow K \rightarrow A^n \xrightarrow{\pi} P \rightarrow 0$$

is a presentation of a finitely-generated projective  $A$ -module and  $\rho : P \rightarrow A^n$  is a section of  $\pi$  then  $\rho \circ \pi : A^n \rightarrow A^n$  is given by an  $n \times n$  idempotent matrix  $M$  such that  $\text{im}(M) = P$ ; this matrix is referred to as a *projection matrix* and is said to present  $P$ .

Suppose that  $M, M'$  are two idempotent matrices over the ring  $A$ , not necessarily of the same size. We write  $M \cong_A M'$  to express that  $M$  and  $M'$  present isomorphic  $A$ -modules. We write  $M \cong_A 1$  to express that  $M$  presents a free  $A$ -module. Let  $P_n$  denote the  $n \times n$  matrix  $[p_{ij}]$  such that  $p_{11} = 1$  and all other entries are 0. Let  $I_n$  denote the  $n \times n$  identity matrix.

Coquand shows that given an  $n \times n$  idempotent matrix  $M$  with entries in  $A[X]$ , one has  $M(0) \cong_A 1$  only if  $M \cong_{A[X]} 1$ . He does this by establishing a sequence of straightforward lemmas that culminate in the main result. He first shows that for  $A$  seminormal,  $M \in A[X]^{n \times n}$  a rank one projection matrix such that  $M(0) = P_n$ , it must be the case that  $M \cong_{A[X]} 1$ . Finally he shows that for  $A$  seminormal,  $M \in A^{n \times n}$  a rank one projection matrix such that  $M(0) \cong_A 1$ , it must be the case that  $M \cong_{A[X]} 1$ . The reader is referred to Coquand’s paper for the proofs of these results. To give a flavor of the statements of these results and to present Schanuel’s example, we mention 2 of Coquand’s results.

**Lemma 2.27.** *[11, Lemma 1.1] Let  $M$  be a projection matrix of rank one over a ring  $A$ . We have  $M \cong_A 1$  if and only if there exist  $x_i, y_j \in A$  such that  $m_{ij} = x_i y_j$ . If we write  $x$  for the column vector  $(x_i)$  and  $y$  for the row vector  $(y_j)$  this can be written as  $M = xy$ . Furthermore the column vector  $x$  and the row vector  $y$  are uniquely defined up to a unit by these conditions: if we have another column vector  $x'$  and row vector  $y'$  such that  $M = x'y'$ , then there exists a unit  $u \in A$  such that  $x = ux'$  and  $y' = uy$ .*

Coquand then proves a result which is well known to many, that asserts that if  $f, g \in A[X]$ ,  $A$  reduced, and  $fg = 1$  then  $f = f(0), g = g(0)$  in  $A[X]$ . Next he proves the following.

**Lemma 2.28.** *[11, Corollary 1.3] Let  $A'$  be an extension of the reduced ring  $A$ . Let  $M$  be an  $n \times n$  projection matrix over  $A[X]$  such that  $M(0) = P_n$ . Assume that  $f_i, g_j \in A'[X]$  are such that  $m_{ij} = f_i g_j$  and  $f_1(0) = 1$ . If  $M \cong_{A[X]} 1$  then  $f_i, g_j \in A[X]$ .*

We state the main theorem of Coquand’s paper, which was previously established by Swan. Notice that this result gives a direct proof for rank one projective modules of the theorem of Quillen-Suslin settling Serre’s Conjecture.

**Theorem 2.29.** *[47, Theorem 1][11, Theorem 2.5] If  $A$  is seminormal then the canonical map  $\text{Pic}(A) \rightarrow \text{Pic}(A[X_1, \dots, X_n])$  is an isomorphism.*

It is straightforward to see, using a construction of Schanuel that originally appeared in a paper by Bass [6], more recently appeared in [11, Appendix A.], and in different garb in both [50] and [47], that if  $\text{Pic}(\phi): \text{Pic}(A) \rightarrow \text{Pic}(A[X])$  is an isomorphism, then  $A$  is seminormal. We give a quick proof.

**Lemma 2.30.** *If  $A$  is a reduced ring and  $\text{Pic}(\phi): \text{Pic}(A) \rightarrow \text{Pic}(A[X])$  is an isomorphism, then  $A$  is seminormal.*

*Proof.* Suppose that  $b, c \in A$  satisfy  $b^3 = c^2$  and let  $B$  be a reduced extension of  $A$  containing an element  $a$  such that  $b = a^2, c = a^3$ . Consider the following polynomials with coefficients in  $B$ .

$$f_1 = 1 + aX, f_2 = bX^2, g_1 = (1 - aX)(1 + bX^2), g_2 = bX^2 \tag{4}$$

The matrix  $M = (f_i g_j)$  is a projection matrix of rank one over  $A[X]$  such that  $M(0) = P_2$ . Assuming that  $\text{Pic}(\phi)$  is an isomorphism we may conclude that this

matrix presents a free module over  $A[X]$ . By Lemma 2.28 this implies  $f_i, g_j \in A[X]$  and hence  $a \in A$ .  $\square$

To conclude this section we would like to mention related work of J. Gubeladze [22] that settled a generalized version of Anderson's Conjecture [1], which postulated that finitely-generated projectives are free over normal monomial subalgebras of  $k[X_1, \dots, X_n]$ . In what follows, by a *seminormal monoid* we mean a commutative, cancellative, torsion-free monoid  $M$  with total quotient group  $G$  such that whenever  $x \in G$  and  $2x, 3x \in M$  we must have  $x \in M$ .

**Theorem 2.31.** [22] *Let  $R$  be a PID and  $M$  be a commutative, cancellative, torsion-free, seminormal monoid. Then all finitely-generated projective  $R[M]$ -modules are free.*

The original proof of Gubeladze was geometric in nature. Swan gave an algebraic proof of Gubeladze's result in [48]. To read more about Serre's Conjecture and related topics the reader is referred to the recent book by Lam [27].

## 2.5 Seminormal local rings in dimension one

A basic fact about a reduced, Noetherian 1-dimensional local ring  $A$  is that  $A$  is normal if and only if it is a discrete rank one valuation ring. This leads to a well-known characterization of the normalization of a Noetherian integral domain  $A$  with quotient field  $K$ . Namely, an element  $b \in K$  is in the normalization of  $A$  if and only if  $b$  is in every valuation subring  $V$  of  $K$  containing  $A$ .

Reduced, Noetherian 1-dimensional seminormal local rings with finite normalization are also fairly well-behaved, particularly in the algebro-geometric setting. Let  $e(R)$  and  $\text{emdim}(R)$  denote the multiplicity and embedding dimension of a local ring  $(R, \mathfrak{m})$ , respectively. The following theorem was proven by E. D. Davis.

**Theorem 2.32.** [15, Theorem 1] *Let  $(R, \mathfrak{m})$  be a reduced, Noetherian, 1-dimensional local ring with finite normalization  $S$ . The following are equivalent.*

1.  $R$  is seminormal.
2.  $\text{gr}_{\mathfrak{m}}(R)$  is reduced and  $e(R) = \text{emdim}(R)$ .
3.  $\text{Proj}(\text{gr}_{\mathfrak{m}}(R))$  is reduced and  $e(R) = \text{emdim}(R)$ .

If the local ring comes from looking at an algebraic curve at a closed point one can say more. The following result of Davis generalizes earlier results of Salmon [44] for plane curves over an algebraically closed field and Bombieri [7] for arbitrary curves over an algebraically closed field.

**Corollary 2.33.** [15, Corollary 1] *Let  $x$  be a closed point of an algebraic (or algebroid) curve at which the Zariski tangent space has dimension  $n$ . Then:*

1.  $x$  is seminormal if, and only if, it is an  $n$ -fold point at which the projectivized tangent cone is reduced.



- For an algebraically closed ground field,  $x$  is seminormal if, and only if, it is an ordinary  $n$ -fold point (i.e., a point of multiplicity  $n$  with  $n$  distinct tangents).

Thus analytically, a 1-dimensional algebro-geometric seminormal local ring over an algebraically closed field  $K$  looks like  $K[X_1, \dots, X_n]/(X_i X_j \mid i \neq j)$ . Here are some specific examples.

*Example 2.34.* Let  $K$  be an algebraically closed field. Consider the curve  $C = \{(x, y) \in K^2 \mid xy - x^6 - y^6 = 0\}$ . Let  $f = xy - x^6 - y^6$  and  $P = (0, 0)$ . Since  $P$  is a point of multiplicity 2 with 2 distinct tangent lines (the  $x$ - and  $y$ - axes), it is an ordinary double point (also called a node) and the local ring of  $A = K[x, y]/(f)$  at  $P$  is seminormal by Corollary 2.33. Since  $P$  is the only singular point of  $A$  we may conclude that  $A$  is seminormal. This curve is drawn below in Fig. 1a.

*Example 2.35.* Let  $K$  be an algebraically closed field. Consider the curve  $C = \{(x, y) \in K^2 \mid x^2 = x^4 + y^4\}$ . Let  $f = x^2 - x^4 - y^4$  and  $P = (0, 0)$ . Since  $P$  is a point of multiplicity 2 with 1 tangent line (the  $y$ -axis) occurring with multiplicity 2, the local ring of  $A = K[x, y]/(f)$  at  $P$  fails to be seminormal by Corollary 2.33. This curve is called a tacnode and is sketched below in Fig. 1b.

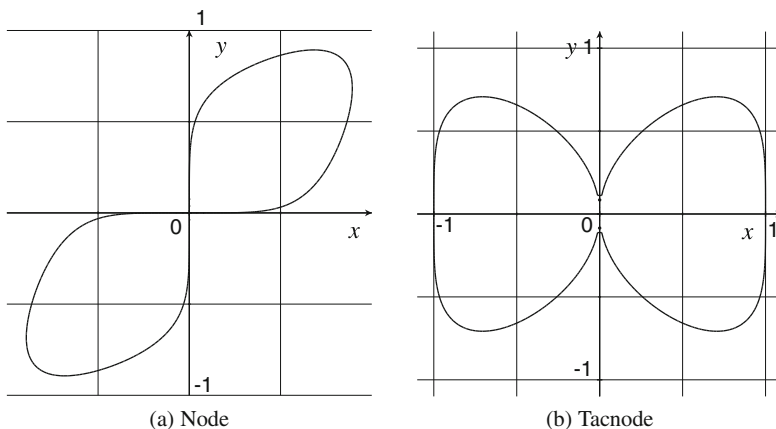


Fig. 1: A Seminormal and a nonseminormal curve

### 3 Weak normality and weak normalization

Let's change gears and speak about weak subintegrality and weak normalization. The reader should keep in mind that weak normalization and seminormalization are *not* the same concept in positive characteristic.

### 3.1 Weakly subintegral and weakly normal extensions and weak normalization relative to an extension

We now recall the Andreotti–Bombieri definition of weak normalization and its first properties. Recall that the *characteristic exponent*  $e$  of a field  $k$  is 1 if  $\text{char}(k) = 0$  and is  $p$  if  $\text{char}(k) = p > 0$ . As earlier we only distinguish between a prime ideal  $P = P_x$  of a ring  $B$  and the corresponding point  $x \in \text{Spec}(B)$  when it is convenient to do so.

**Lemma 3.1.** [2, Prop. 2] *Let  $(A, \mathfrak{m}, k)$  be a local ring,  $e$  be the characteristic exponent of  $k$ , and  $A \subset B$  an integral extension. Set*

$$A' = \{b \in B \mid b^{e^m} \in A + R(B) \text{ for some } m \geq 0\}$$

and let  $\mathfrak{m}' = R(A')$ . For each maximal ideal  $\mathfrak{n}_i$  of  $B$  let  $\omega_i: A/\mathfrak{m} \rightarrow B/\mathfrak{n}_i$  be the canonical homomorphism. The following assertions hold.

1.  $(A', \mathfrak{m}', k')$  is a local ring and the induced extension of residue fields  $k = \kappa(x) \subset k' = \kappa(x')$  is purely inseparable;
2.  $A'$  is the largest intermediate ring that is local and has residue field a purely inseparable extension of  $k$ .
3. An element  $b \in B$  is in  $A'$  if and only if there exists some integer  $m \geq 0$  such that
  - a.  $b(x_i)^{e^m} \in \omega_i(\kappa(x))$  for all closed points  $x_i$  of  $\text{Spec}(B)$ , and
  - b.  $\omega_i^{-1}(b(x_i)^{e^m}) = \omega_j^{-1}(b(x_j)^{e^m})$  for all  $i, j$ .

*Proof.* Either see the proof of [2, Prop. 2] or modify the proof of Lemma 2.1 above.  $\square$

**Definition 3.2.** We will refer to the ring  $A'$  described above as the *weak gluing* of  $A$  in  $B$  over  $\mathfrak{m}$ . Letting  $x \in \text{Spec}(A)$  denote the point corresponding to the maximal ideal  $\mathfrak{m}$ , this ring is sometimes denoted by  ${}^*_x A$ .

The weak normalization of a ring  $A$  in an integral extension  $B$  is defined analogously to the seminormalization (see (1)).

**Definition 3.3.** Let  $A \subset B$  be an integral extension of rings. We define the *weak normalization*  ${}^*_B A$  of  $A$  in  $B$  to be

$${}^*_B A = \{b \in B \mid \forall x \in \text{Spec}(A), \exists m \geq 0 \text{ such that } (b_x)^{e^m} \in A_x + R(B_x)\}. \quad (5)$$

In this description,  $e$  denotes the characteristic exponent of the residue field of  $A_x$ .

Notice that from the definition we have  $A \subset {}^+_B A \subset {}^*_B A$  for an arbitrary integral extension  $A \subset B$ .

Before stating some fundamental properties of weak normalization we introduce some additional terminology.

**Definition 3.4.** An integral extension of rings  $A \subset B$  is said to be *weakly subintegral* if the associated map  $\text{Spec}(B) \rightarrow \text{Spec}(A)$  is a bijection and induces purely inseparable extensions of the residue fields. An element  $b \in B$  is said to be *weakly subintegral over B* provided that  $A \subset A[b]$  is a weakly subintegral extension.

Notice that a weakly subintegral extension of fields is a purely inseparable extension. A subintegral extension must be weakly subintegral (by definition), but the converse need not be true since a purely inseparable extension of fields is weakly subintegral but not subintegral. The following lemma is the basis for what Yanagihara calls an elementary weakly subintegral extension. We include Yanagihara’s proof for the convenience of the reader.

**Lemma 3.5.** [57, Lemma 3] *Let  $A \subset A[b]$  be a simple extension of rings, where  $p$  is a rational prime and  $pb, b^p \in A$ . Then,  $A \subset B$  is weakly subintegral.*

*Proof.* Let  $P \subset A$  be a prime ideal. First suppose that  $p \notin P$ . Then,  $b = pb/p \in A_p$ , which implies  $A_p = B_p$  and the residue fields are isomorphic. Now suppose that  $p \in P$ , so that  $A/P$  is an integral domain of characteristic  $p$ . By the Lying Over Theorem,  $PB \cap A = P$ . Suppose  $Q_1, Q_2$  are prime ideals of  $B$  lying over  $P$ . Consider the extension  $A' := A/P \subset B' := B/PB$  and let  $Q'_i = Q_i/PB$  ( $i = 1, 2$ ). Then,  $Q'_1$  and  $Q'_2$  both lie over the zero ideal in  $A'$ . Notice that  $f^p \in A'$  for any element  $f \in B'$ . Consider  $f \in Q'_1$ . Then,  $f^p \in Q'_1 \cap A' = \{0\} \subset Q'_2 \Rightarrow f \in Q'_2$ . Hence  $Q'_1 \subset Q'_2$ , which implies  $Q_1 \subset Q_2$ . Similarly,  $Q_2 \subset Q_1$  and, hence,  $Q_1 = Q_2$ . Thus there is a unique prime ideal  $Q$  of  $B$  lying over  $P$  and  $\kappa(P) \subset \kappa(Q)$  is purely inseparable.  $\square$

**Definition 3.6.** An *elementary weakly subintegral extension* is a simple extension  $A \subset A[b]$  of rings such that  $pb, b^p \in A$ , for some rational prime  $p$ .

The notion of a weakly normal extension, which we now define, is *complementary* to that of a weakly subintegral extension.

**Definition 3.7.** If  $A \subset B$  is an integral extension of rings, we say  $A$  is *weakly normal* in  $B$  if there is no subextension  $A \subset C \subset B$  with  $C \neq A$  and  $A \subset C$  weakly subintegral.

The following result is well known and can be proven by modifying the proof of Theorem 2.9.

**Theorem 3.8.** *Let  $A \subset B$  be an integral extension of rings. The following assertions hold.*

1. *The extension  $A \subset {}^*_B A$  is weakly subintegral;*
2. *If  $A \subset C \subset B$  and  $A \subset C$  is weakly subintegral, then  $C \subset {}^*_B A$ ;*
3. *The extension  ${}^*_B A \subset B$  is weakly normal; and*
4.  *${}^*_B A$  has no proper subrings containing  $A$  and weakly normal in  $B$ .*

The reader will note that  ${}^*_B A$  is the unique largest weakly subintegral extension of  $A$  in  $B$  and is minimal among the intermediate rings  $C$  such that  $C \subset B$  is weakly normal. With this result in hand, we see that an integral extension  $A \subset B$

is weakly normal if and only if  $A = {}^*_B A$ . In light of the inclusions  $A \subset {}^+_B A \subset {}^*_B A$ , a weakly normal extension is necessarily seminormal, but the converse is not true as Example 3.11 will illustrate.

H. Yanagihara proved a weakly subintegral analogue of Swan’s characterization of subintegral extensions in terms of extensions of maps into fields, which we now state.

**Lemma 3.9.** [57, Lemma 1] *Let  $A \subset B$  be an integral extension of rings. It is a weakly subintegral extension if and only if for all fields  $F$  and homomorphisms  $\varphi: A \rightarrow F$  there exists at most one homomorphism  $\psi: B \rightarrow F$  extending  $\varphi$ .*

The positive integer  $p$  that appears in the work of Hamann and Swan also appears in Yanagihara’s characterization of a weakly normal extension. Yanagihara showed [56] that if a ring  $A$  contains a field of positive characteristic, an integral extension  $A \subset B$  is weakly normal provided that  $A \subset B$  is seminormal and  $A$  contains every element  $b \in B$  such that  $b^p$  and  $pb \in A$ , for some rational prime  $p$ . This result has been widely cited but didn’t appear in print until much later. Subsequently, Itoh [25, Proposition 1] proved the above-mentioned characterization holds for any integral extension  $A \subset B$  and Yanagihara gave an alternate proof [56, Theorem 1]. We now recall the generalized result.

**Proposition 3.10.** *Let  $A \subset B$  be an integral extension of rings. Then,  $A$  is weakly normal in  $B$  if and only if the following hold:*

1.  $A$  is seminormal in  $B$ ; and
2.  $A$  contains each element  $b \in B$  such that  $b^p, pb \in A$ , for some rational prime  $p$ .

Thus,  $A$  is weakly normal in  $B$  if and only if  $A$  doesn’t admit any proper (elementary) subintegral or weakly subintegral extensions in  $B$ . In fact, this characterizes a weakly normal extension by [57, Corollary to Lemma 4].

For an integral extension  $A \subset B$  the weak normalization  ${}^*_B A$  is the filtered union of all subrings of  $B$  that can be obtained from  $A$  by a finite sequence of elementary weakly subintegral extensions. We now offer an example of a seminormal extension of rings that isn’t weakly normal. Indeed, this example shows that a normal domain need not be weakly normal in an extension domain.

*Example 3.11.* Let  $K$  be a field of characteristic 2,  $X$  be an indeterminate, and consider the integral extension  $A := K[X^2] \subset B := K[X]$ . Since  $X^2$  and  $2X = 0$  are both in  $A$  but  $X$  is not in  $A$ , this extension is not weakly normal. However, it is a seminormal extension, since  $K[X^2]$  is a normal domain and  $B$  is reduced.

Inspired by Swan’s intrinsic definition of a seminormal ring, Yanagihara made the following definition in [57].

**Definition 3.12.** A ring  $A$  is said to be *weakly normal* if  $A$  is reduced and the following conditions hold:

1. For any elements  $b, c \in A$  with  $b^3 = c^2$ , there is an element  $a \in A$  with  $a^2 = b$  and  $a^3 = c$ ; and

2. For any element  $b, c, e \in A$  and any non-zero divisor  $d \in A$  with  $c^p = bd^p$  and  $pc = de$  for some rational prime  $p$ , there is an element  $a \in A$  with  $b = a^p$  and  $e = pa$ .

As the reader may recall, condition 1. implies that the ring  $A$  is reduced so we omitted that requirement in the definition of a weakly normal ring. Since  $A$  is reduced, the element  $a$  that occurs in condition 1. is necessarily unique. Since this definition is quite complicated we will look at some special instances of weakly normal rings.

**Lemma 3.13.** *A product of fields is weakly normal.*

*Proof.* We first show that a field  $K$  is weakly normal. We start by observing that  $K$  is seminormal since it is (2,3)-closed in itself. Now suppose that  $b, c, e \in K$  and  $0 \neq d \in K$  with  $c^p = bd^p$  and  $pc = de$  for some rational prime  $p$ . Setting  $a = c/d$  we have  $b = a^p, e = pa$ . Now let  $K = \prod K_i$  be a product of fields. Again,  $K$  is seminormal. Suppose  $b = (b_i), c = (c_i) \in K$  and  $b^3 = c^2$ . Setting  $a_i = c_i/b_i$  whenever  $b_i \neq 0$  and  $a_i = 0$  whenever  $b_i = 0$ , we have  $b = a^2, c = a^3$ . Now suppose that  $b, c, e \in K$  and  $d \in K$  a non-zero divisor with  $c^p = bd^p$  and  $pc = de$  for some rational prime  $p$ . Since  $d = (d_i)$ , where each  $d_i \neq 0$  we can let  $a_i = c_i/d_i$  and set  $a = (a_i)$ . Then,  $b = a^p, e = pa$ .  $\square$

**Lemma 3.14.** *Let  $A$  be a reduced ring whose total quotient ring is a product of fields. If  $A$  is normal, then  $A$  is weakly normal.*

*Proof.* Let  $A$  be a normal ring whose total quotient ring is a product of fields. Since a ring  $A$  with such a total quotient ring is seminormal if and only if it is equal to  ${}^+_B A$ , where  $B$  is the normalization of  $A$ , we may conclude that  $A$  is seminormal. Now suppose that  $b, c, e \in A$  and  $d \in A$  is not a zero divisor such that  $c^p = bd^p$  and  $pc = de$  for some rational prime  $p$ . Since  $(c/d)^p = b \in A$  and  $A$  is normal, we must have  $c/d \in A$  so we may let  $a = c/d$  as in the above example.  $\square$

We now recall a pair of results by Yanagihara that we will find helpful.

**Proposition 3.15.** *[57, Propositions 3 and 4] Let  $A$  and  $B$  be reduced rings.*

1. *If  $A$  is weakly normal and  $B$  is a subring of the total quotient ring of  $A$  containing  $A$ , then  $A$  is weakly normal in  $B$ .*
2. *If  $A$  is a subring of a weakly normal ring  $B$  such that any non-zero divisor in  $A$  is also not a zero divisor in  $B$  and  $A$  is weakly normal in  $B$ , then  $A$  is weakly normal.*

The next result is a weakly normal analog of the fact that a ring whose total quotient ring is a product of fields is a seminormal ring if and only if it is equal to its seminormalization in  $\bar{A}$ .

**Corollary 3.16.** *Let  $A$  be a reduced ring whose total quotient ring is a product of fields and let  $B$  denote the normalization of  $A$ . The following are equivalent.*

1.  *$A$  is weakly normal;*
2.  *$A = {}^*_B A$ ; and*
3.  *$A$  is weakly normal in  $B$ .*

*Proof.* The last two items are equivalent as remarked after the proof of Theorem 3.8. The equivalence of the first and last statements follows from Lemma 3.14 and the second part of 3.15.  $\square$

When the base ring is reduced and contains a field of positive characteristic, weak normality has a simple characterization. Notice that to say a reduced ring  $A$  is  $n$ -closed in its normalization for some positive integer  $n$  is the same thing as saying that  $A$  contains each element  $b$  of its total quotient ring such that  $b^n \in A$ . If the latter condition holds we will say  $A$  is  $n$ -closed in its total quotient ring  $K$  even though  $A \subset K$  need not be an integral extension.

**Corollary 3.17.** *Suppose that  $A$  is a reduced ring whose total quotient ring is a product of fields and that  $A$  contains a field of characteristic  $p > 0$ . Then,  $A$  is weakly normal if and only if  $A$  is  $p$ -closed in its total quotient ring.*

*Proof.* Let  $A$  be a reduced ring whose total quotient ring  $K$  is a product of fields and suppose that  $A$  contains a field  $k$  of characteristic  $p > 0$ . By Lemma 3.16 and Hamman’s criterion it suffices to show that  $A$  is (2,3)-closed in  $K$  and contains each element  $b \in K$  such that  $qb, b^q \in A$  for some rational prime  $q$ . Suppose that  $b \in K$  and  $qb, b^q \in A$  for some rational prime  $q$ . If  $q \neq p$ , then  $b = qb/q \in A$ . Suppose  $q = p$ . Since  $A$  is  $p$ -closed in  $K$  by assumption,  $b \in A$ . Now suppose that  $b$  is an element of  $K$  and  $b^2, b^3 \in A$ . Then,  $b^n \in A$  for all  $n \geq 2$ . In particular,  $b^p \in A$  and hence  $b \in A$ .  $\square$

Looking at the complementary notion of weak subintegrality in positive characteristic we have another simple characterization, which we now recall.

**Proposition 3.18.** [38, Theorems 4.3 and 6.8] *Let  $A \subset B$  be an extension of commutative rings and suppose that  $A$  contains a field of characteristic  $p > 0$ . Then  $b \in B$  is weakly subintegral over  $A$  if and only if  $b^{p^n} \in A$  for some  $n \geq 0$ .*

We mention a few more results of Yanagihara regarding localization, faithfully flat descent, and pull-backs.

**Proposition 3.19.** [57, Prop. 7] *A ring  $A$  is weakly normal if all localizations of  $A$  at maximal ideals are weakly normal.*

**Proposition 3.20.** [57, Prop. 5] *Let  $B$  be a weakly normal ring and  $A$  be a subring of  $B$  such that  $A$  has only a finite number of minimal prime ideals. If  $B$  is faithfully flat over  $A$ , then  $A$  is weakly normal.*

**Proposition 3.21.** [55, Prop. 1] *Let*

$$\begin{array}{ccc}
 D & \xrightarrow{\alpha} & A \\
 \beta \downarrow & & \downarrow f \\
 B & \xrightarrow{g} & C
 \end{array}$$

*be a pull-back diagram of commutative rings. Assume that  $A$  is weakly normal domain, that  $B$  is a perfect field of positive characteristic  $p$ , and that  $C$  is reduced. Then,  $D$  is weakly normal.*

Yanagihara does not construct the weak normalization of a general commutative ring.

Itoh went on to prove a result like Greither’s. He showed [25, Theorem] that a finite extension  $A \subset B$  of reduced Noetherian rings is a weakly normal extension if and only if for every  $A$ -algebra  $C$ ,  $B \otimes_A C \cong \text{Sym}_B(Q)$ ,  $Q \in \text{Pic}(B)$ , implies  $C \cong \text{Sym}_A(P)$ , for some  $P \in \text{Pic}(A)$ .

### 3.2 Systems of (weak) subintegrality

The reader may have noticed that the only way to determine if an element  $b$  in an integral extension ring  $B$  of a general ring  $A$  is weakly subintegral over  $A$  is to check if the ring extension  $A \subset A[b]$  is weakly subintegral. Beginning in 1993, in a series of papers [34–38,41,42] by Reid, Roberts, and Singh, a genuine element-wise criterion for weak subintegrality was introduced and developed. At first they worked with  $\mathbb{Q}$ -algebras, where weak subintegrality and subintegrality coincide, and discussed a criterion for subintegrality that involved what they called a *system of subintegrality* or SOSI. Over time, they were able to handle general rings, but only if they discussed weak subintegrality rather than subintegrality. However, since the term system of subintegrality had already been used extensively in their earlier papers they kept it instead of redefining the system as a system of *weak* subintegrality. Reid, Roberts and Singh constructed in [38, Section 2] a “universal weakly subintegral extension” based on systems of subintegrality. Later, Roberts [40] wrote a paper that helped to elucidate the earlier work. Much more recently Gaffney and the current author [18] gave a more intuitive geometric description of the element-wise criterion for weak subintegrality for rings that arise in the study of classical algebraic varieties or complex analytic varieties. We offer a new algebraic criterion for weak subintegrality in this section.

We begin with a recap of the work of Reid, Roberts, and Singh.

**Theorem 3.22.** [38, Theorems 2.1 ,5.5, and 6.10] *Let  $A \subset B$  be an extension of rings,  $b \in B$ , and  $q$  a nonnegative integer. The following statements are equivalent.*

1. *There exists a positive integer  $N$  and elements  $c_1, \dots, c_q \in B$  such that*

$$b^n + \sum_{i=1}^q \binom{n}{i} c_i b^{n-i} \in A \text{ for all } n \geq N.$$

2. *There exists a positive integer  $N$  and elements  $c_1, \dots, c_q \in B$  such that*

$$b^n + \sum_{i=1}^q \binom{n}{i} c_i b^{n-i} \in A \text{ for } N \leq n \leq 2N + 2q - 1.$$

3. *There exist elements  $c_1, \dots, c_q \in B$  such that*

$$b^n + \sum_{i=1}^q \binom{n}{i} c_i b^{n-i} \in A \text{ for all } n \geq 1.$$

4. There exist elements  $a_1, a_2, a_3, \dots \in A$  such that

$$b^n + \sum_{i=1}^n (-1)^i \binom{n}{i} a_i b^{n-i} = 0 \text{ for all } n > q.$$

5. There exist elements  $a_1, \dots, a_{2q+1}$  in  $A$  such that

$$b^n + \sum_{i=1}^n (-1)^i \binom{n}{i} a_i b^{n-i} = 0 \text{ for } q + 1 \leq n \leq 2q + 1. \tag{6}$$

6. The extension  $A \subset A[b]$  is weakly subintegral.

Notice condition 2. in the above theorem is a finite version of condition 1. and that condition 3. is a special case of condition 1. with  $N = 1$ . Similarly, condition 5. is a finite version of condition 4.

We now recall the definition that was the basis of the work of Reid, Roberts, and Singh.

**Definition 3.23.** Let  $A \subset B$  be an extension of rings and  $b \in B$ . A system of subintegrality (SOSI) for  $b$  over  $A$  consists of a nonnegative integer  $q$ , a positive integer  $N$ , and elements  $c_1, \dots, c_q \in B$  such that

$$b^n + \sum_{i=1}^q \binom{n}{i} c_i b^{n-i} \in A \text{ for all } n \geq N. \tag{7}$$

Originally, an element  $b$  admitting a SOSI was called *quasisubintegral* over  $A$  but Reid, Roberts, and Singh later dropped this terminology for reasons which will soon be clear.

In light of the equivalence of conditions 1. and 6. of Theorem 3.22 we say an element  $b \in B$  is *weakly subintegral* over  $A$  provided that  $b$  admits a SOSI over  $A$ ; we will not mention a quasisubintegral element again. By this theorem, an element  $b$  is weakly subintegral over  $A$  if and only if  $b$  satisfies a highly structured sequence (6) of equations of integral dependence.

Let  $F(T) = T^n + \sum_{i=1}^q (-1)^i \binom{n}{i} c_i T^{n-i}$ . Notice that  $F'(T) = nF_{n-1}(T)$ . So the equations that appear in part 5. of Theorem 3.22 are rational multiples of the derivatives of the equation of highest degree. In the section to follow we will show directly that this system of equations implies that  $b$  is weakly subintegral over  $A$ . Hopefully our proof will shed more light on why this system of equations of integral dependence implies weak subintegrality.

### 3.3 A new criterion for weak subintegrality

We will now develop a new criterion for an element to be weakly subintegral over a subring. First, we look at the case of an extension of fields and state our result in such a way that it will generalize to an extension of arbitrary rings.



**Lemma 3.24.** *Suppose that  $K \subset L$  is an extension of fields and that  $x \in L$ . Then,  $x$  is weakly subintegral over  $K$  if and only if  $x$  is a root of some monic polynomial  $F(T) \in K[T]$  of degree  $n > 0$  and its first  $\lfloor \frac{n}{2} \rfloor$  derivatives.*

*Proof.* First assume that  $x$  is a root of some monic polynomial  $F(T) \in K[T]$  of degree  $n > 0$  and its first  $\lfloor \frac{n}{2} \rfloor$  derivatives. Let  $E \supset L$  be an extension field over which  $F(T)$  splits. Our condition says that, counting multiplicities, more than half the roots of  $F(T)$  in  $E$  are  $x$ . Let  $f(T)$  be the minimal polynomial of  $x$  over  $K$  and write  $F(T) = f(T)^e g(T)$ , where  $g(T) \in K[T]$  and  $g(x) \neq 0$ . Then, counting multiplicities, more than half the roots of  $f(T)$  in  $E$  are  $x$ . Recall that every root of  $f(T)$  occurs with the same multiplicity. So  $f(T)$  can have only one distinct root. Therefore, either  $x \in K$  or  $x$  is purely inseparable over  $K$ .

Now assume that  $x$  is weakly subintegral over  $K$ . If  $K$  has characteristic 0, then  $x \in K$  and  $F(T) = T - x$  is the required polynomial. If  $K$  has characteristic  $p > 0$  then  $x^{p^m} \in K$  for some  $m > 0$  and  $F(T) = T^{p^m} - x^{p^m}$  is the required polynomial.  $\square$

An immediate consequence of the above proof is the following.

**Corollary 3.25.** *Let  $K \subset L$  be an extension of fields and  $x \in L$  be algebraic over  $K$ . Then,  $x$  is weakly subintegral over  $K$  if and only if the minimal polynomial of  $x$  over  $K$  is  $f(T) = T^{e^m} - x^{e^m}$  for some positive integer  $m$ , where  $e$  is the characteristic exponent of  $K$ .*

Next, we present our new element-wise criterion for weak subintegrality and give an elementary self-contained proof that an element satisfying a system of (6) is weakly subintegral over the base ring of an arbitrary integral extension.

**Proposition 3.26.** *Let  $A \subset B$  be an integral extension of rings and  $b \in B$ . Then,  $b$  is weakly subintegral over  $A$  if and only if there is a monic polynomial  $F(T)$  in  $A[T]$  of degree  $n > 0$  such that  $b$  is a root of  $F(T)$  and its first  $\lfloor \frac{n}{2} \rfloor$  derivatives.*

*Proof.* First, assume that  $F(T)$  in  $A[T]$  is a monic polynomial of degree  $n > 0$  such that  $b$  is a root of  $F(T)$  and its first  $\lfloor \frac{n}{2} \rfloor$  derivatives. Replacing  $B$  by  $A[b]$  we may and shall assume that  $A \subset B$  is a finite integral extension. Let  $P \in \text{Spec}(A)$  and  $e$  be the characteristic exponent of  $\kappa(P)$ . We will show that  $b_p^{e^m} \in A_P + R(B_P)$  for some  $m \geq 1$ . Replacing  $A, B$ , and  $b$  by  $A_P, B_P$ , and  $b_P$ , respectively, we may and shall assume that  $(A, P, K)$  is local and  $K$  has characteristic exponent  $e$ . Let  $Q_i$  denote the maximal ideals of  $B$ , and  $\omega_i: K = \kappa(P) \rightarrow \kappa(Q_i)$  the canonical injections of residue fields. Let  $\bar{F}$  denote the polynomial obtained from  $F$  by reducing coefficients modulo  $P$ . Fix a maximal ideal  $Q_i$  of  $B$  and let  $b(Q_i)$  denote the image of  $b$  in  $\kappa(Q_i) = B/Q_i$ . By Lemma 3.24 we may conclude  $b(Q_i)^{e^m} \in \omega_i(\kappa(P))$  for some  $m$ . Since there are finitely many maximal ideals in  $B$  there is some positive integer  $m$  such  $b(Q_i)^{e^m} \in \omega_i(\kappa(P))$  for all  $i$ . Since  $\bar{F}$  has at most one root in  $\kappa(P) = K$  of multiplicity at least  $\lfloor \frac{n}{2} \rfloor$  we may conclude that  $\omega_i^{-1}(b(Q_i)^{e^m}) = \omega_j^{-1}(b(Q_j)^{e^m})$  for all  $i, j$ . Hence,  $b^{e^m} \in A + R(B)$ , as desired.

Now assume that  $b$  is weakly subintegral over  $A$ . Then, there exist  $q \geq 0$  in  $\mathbb{Z}$  and elements  $a_1, \dots, a_{2q+1}$  in  $A$  such that  $b^m + \sum_{i=1}^m (-1)^i \binom{n}{i} a_i b^{m-i} = 0$  for all integers  $q$  with  $q + 1 < m < 2q + 1$  by part 5. of Theorem 3.22. Let  $n = 2q + 1$  and

$F(T) = T^n + \sum_{i=1}^n (-1)^i \binom{n}{i} a_i T^{n-i}$ . By our earlier remarks,  $b$  is a root of  $F(T)$  and its first  $\lfloor \frac{n}{2} \rfloor$  derivatives.  $\square$

We feel that there should be a direct proof of the ‘only if’ part of the above result, i.e., a proof that doesn’t depend on Theorem 3.22 or [38, Theorem 6.8].

### 3.4 First geometric properties of weakly normal varieties

In the introduction, we traced the history of weak normality and weak normalization, beginning with its roots in complex analytic space theory. We’d like to further discuss the history of weakly normal complex analytic spaces and then take a brief glimpse at the theory of weakly normal algebraic varieties from a geometric point of view.

Weak normalization first was defined for complex analytic spaces by Andreotti and Norguet [3]. Some say that their quotient space construction was already implicit in the work of Cartan [10]. The sheaf  $\mathcal{O}_X^c$  of  $c$ -regular functions on a complex analytic space  $X$  is defined by setting its sections on an open subset  $U \subset X$  to be those continuous complex-valued functions on  $U$  which are holomorphic at the regular points of  $U$ ; this coincides with setting  $\Gamma(U, \mathcal{O}_X^c)$  equal to the set of continuous complex-valued functions on  $U$  that become regular when lifted to the normalization of  $U$ . The weak normalization  $X^w$  of  $X$  represents the sheaf  $\mathcal{O}_X^c$  in the category of complex analytic spaces, i.e.,  $\mathcal{O}_X^c$  becomes the sheaf of germs of holomorphic functions on  $X^w$ . A complex analytic space is weakly normal if and only if the sheaf of  $c$ -regular functions on  $X$  coincides with the sheaf of regular functions on  $X$ . The weak normalization of a what is today called a scheme was introduced by A. Andreotti and E. Bombieri [2]. After introducing the notion of “gluing” the prime ideals of  $B$  lying over the unique maximal ideal of a local ring  $A$ , where  $A \subset B$  is an integral extension, Andreotti and Bombieri constructed the weak normalization of the structure sheaf of a scheme pointwise using the gluing they previously defined for local rings. They then turned their attention to defining and constructing the weak normalization  $\sigma: X^* \rightarrow X$  of a reduced algebraic scheme  $X$  over an arbitrary field  $K$ . They showed that  $(X^*, \sigma)$  is maximal among all pairs  $(Z, g)$  consisting of an algebraic scheme  $Z$  over  $K$  and a  $K$ -morphism  $g: Z \rightarrow X$  that is birational and a universal homeomorphism, where the latter means that all maps  $Z' \rightarrow X'$  obtained by base change are homeomorphisms. If we study algebraic varieties rather than schemes we can either proceed as in the case of schemes or we can capture some of the ideas that permeated the original complex analytic theory of weakly normal spaces. We will build on the complex analytic viewpoint.

In this section, let  $K$  be a fixed algebraically closed field of characteristic 0. When we speak of an algebraic variety over  $K$  we assume that the underlying topological space is the set of closed points of a reduced, separated scheme of finite type over  $K$ . By an affine ring (over  $K$ ) we mean the coordinate ring of an affine variety (over  $K$ ). By assuming  $\text{char } K = 0$  we avoid all inseparability problems and hence the operations of seminormalization and weak normalization coincide. We will use the latter terminology.

Consider an algebraic variety  $X$  defined over  $K$ . One might expect that if you define the sheaf  $\mathcal{O}_X^c$  of  $c$ -regular functions on  $X$  so that its sections on an open subset  $U \subset X$  are the continuous  $K$ -valued functions that are regular at the nonsingular points of  $U$  then  $\mathcal{O}_X^c$  becomes the sheaf of regular functions on the weak normalization of  $X$  (see [28, Definition 2.4] and the corrected definition by the current author [52, Definition 3.4]). This isn't the case due to the very special nature of the Zariski topology in dimension one [52, Example 3.3]. We recall the corrected definition and an alternate characterization now.

**Definition 3.27.** [52, Definition 3.4] Let  $X$  be a variety defined over  $K$  and let  $\pi : \tilde{X} \rightarrow X$  be the normalization of  $X$ . The sheaf of  $c$ -regular functions on  $X$ , denoted by  $\mathcal{O}_X^c$ , is defined as follows. For an open subset  $U \subset X$ , we let

$$\Gamma(U, \mathcal{O}_X^c) = \{ \phi : U \rightarrow K \mid \phi \circ \pi \in \Gamma(\pi^{-1}(U), \mathcal{O}_{\tilde{X}}) \}.$$

Notice that a  $c$ -regular function on an affine variety  $X$  may be identified with a regular function on the normalization  $\tilde{X}$  of  $X$  that is constant on the fibers of  $\pi : \tilde{X} \rightarrow X$ . This observation and Theorem 3.30 below show that the  $c$ -regular functions on an affine variety may be identified with the regular functions on the weak normalization of  $X$ . We now describe a  $c$ -regular function without reference to the normalization. In this result the underlying topology is the Zariski topology.

**Theorem 3.28.** [52, Theorem 3.9] Let  $X$  be an affine variety defined over  $K$  without any 1-dimensional components and consider a function  $\phi : X \rightarrow K$ . Then,  $\phi$  is  $c$ -regular if and only if every polynomial in  $\phi$  with coefficients in  $\Gamma(X, \mathcal{O}_X)$  is continuous and the graph of  $\phi$  is closed in  $X \times K$ .

We wish to present some characterizations of the weak normalization of an affine ring. First we recall a variant of a well-known result.

**Lemma 3.29.** [46, Theorem 7, p. 116] Let  $\pi : Y \rightarrow X$  be a dominating finite morphism of irreducible varieties and let  $n = [K(Y) : K(X)]$  denote the degree of the extension of fields of rational functions. Then there is a nonempty open subset  $U$  of  $X$  such that for each  $x \in U$  the fiber  $\pi^{-1}(x)$  consists of  $n$  distinct points.

With this result in hand one can prove the following result, which has an immediate corollary.

**Theorem 3.30.** [28, Theorem 2.2] Let  $A \subset B$  be a finite integral extension of affine rings and define  $A'$  by  $A' = \{ b \in B \mid b_x \in A_x + R(B_x) \ \forall x \in X = \text{Var}(A) \}$ . Then  $A' = \frac{1}{B}A$ . Thus if  $\pi : Y = \text{Var}(B) \rightarrow X$  is the induced morphism, then  ${}^*_B A$  consists of all regular functions  $f$  on  $Y$  such that  $f(y_1) = f(y_2)$  whenever  $\pi(y_1) = \pi(y_2)$ .

If you work with an integral extension  $A \subset B$  of affine rings over an algebraically closed field of characteristic 0, the condition that the residue fields are isomorphic in the definition of a seminormal extension is redundant, as we shall now see. Furthermore, you only need to verify that for each maximal ideal of  $A$  there exists a unique prime ideal of  $B$  lying over  $A$ .

**Corollary 3.31.** *Let  $A \subset B$  be an integral extension of affine rings over  $K$ . The extension is weakly subintegral if and only if the induced map of affine varieties  $\text{Var}(B) \rightarrow \text{Var}(A)$  is a bijection.*

*Proof.* The only if direction follows from the definition. Now suppose that the induced map of varieties is a bijection. Let  $Q \in \text{Spec}(B)$  and apply Lemma 3.29 to the induced map  $\text{Var}(B/Q) \rightarrow \text{Var}(A/(Q \cap A))$  to deduce that the induced map of fields of rational functions is an isomorphism. Since  $Q$  was arbitrary, the extension is weakly subintegral.  $\square$

We now present an example of a weakly normal surface.

**Example 3.32. [The Whitney Umbrella]** Let  $A = K[u, uv, v^2] \subset B = K[u, v]$ , where  $K$  is an algebraically closed field of characteristic 0. Notice that  $B$  is the normalization of  $A$ . We claim that  $A$  is weakly normal. Let  $\pi : \mathbb{A}^2 \rightarrow \mathbb{A}^3$  be given by  $\pi(u, v) = (u, uv, v^2)$  and let  $X$  denote the image of  $\pi$ . Then,  $\pi : \mathbb{A}^2 \rightarrow X$  is the normalization of  $X$ . Suppose  $f \in B$  agrees on the fibers of  $\pi$ . Write  $f = \sum_{i=0}^m g_i(u)v^i, g_i \in K[u]$ . Then  $f(0, c) = f(0, -c)$  for all  $c \in K$  and hence  $\sum_{i=0}^m g_i(0)v^i = \sum_{i=0}^m (-1)^i g_i(0)v^i$ . Thus,  $g_i(0) = 0$  whenever  $i$  is odd. Then  $f = \sum [g_i(u) - g_i(0)]v^i + \sum_{i \text{ even}} g_i(0)v^i$  is in  $A$  since  $g_i(u) - g_i(0) \in uB \subset A$ . This example generalizes to higher dimensions (see [28, Prop. 3.5].) The zero set of  $y^2 = x^2z$  is  $X$  together with the negative  $z$ -axis and is called the Whitney umbrella. Here is a sketch of the umbrella minus its handle, which is the negative  $z$ -axis.

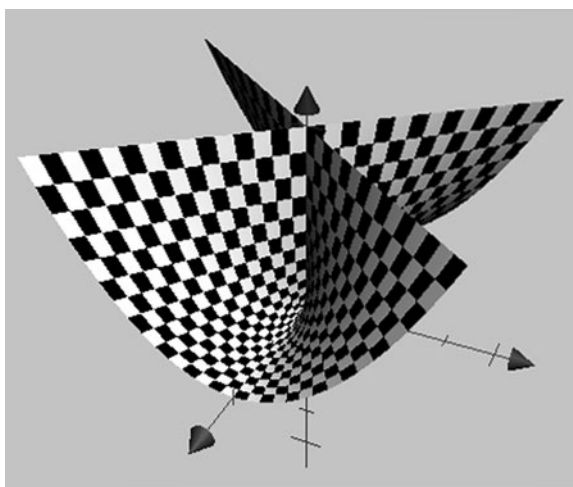


Fig. 2: Whitney Umbrella

Let  $X$  be an algebraic variety over  $K$ . In the introduction we said that a point  $x \in X$  is a multicross if  $x \in X$  is analytically isomorphic to  $z \in Z$  where  $Z$  is the union of linearly disjoint linear subspaces and  $z$  is the origin. We now recall the precise definition and cite a theorem, which asserts that a variety is weakly normal

at a multicross singularity. Then we cite a result about generic hyperplane sections of weakly normal varieties whose proof relies on various characterizations of the multicrosses due to Leahy and the current author.

**Definition 3.33.** [see [28, Definition 3.3] and [29, Definition 3.1]] Let  $\mathcal{C} = \{T_1, \dots, T_r\}$  be a nonempty collection of disjoint subsets of  $\{1, \dots, p\}$  and let  $z_1, \dots, z_p$  be indeterminates. We let  $R_{\mathcal{C}}$  denote the complete local ring defined by

$$R_{\mathcal{C}} = K[[z_1, \dots, z_p]] / (z_{\alpha}z_{\beta} \mid \alpha \in T_i, \beta \in T_j, i \neq j). \tag{8}$$

We say a point  $x$  on a variety  $X$  is a multicross if  $\widehat{\mathcal{O}}_{X,x}$  is isomorphic (as a  $K$ -algebra) to  $R_{\mathcal{C}}$  for some  $\mathcal{C}$  as above.

We cite two fundamental results about the multicrosses.

**Proposition 3.34.** [28, Prop. 3.4] *Let  $X$  be an algebraic variety over an algebraically closed field of characteristic 0. If  $x \in X$  is a multicross then  $\mathcal{O}_{X,x}$  is weakly normal.*

**Theorem 3.35.** [29, Theorem 3.8] *Suppose  $X$  is a weakly normal variety over an algebraically closed field of characteristic 0 and let  $Z$  denote the complement of the set of multicrosses. Then,  $Z$  is a closed subvariety of codimension at least two.*

The following result of the current author parallels well-known results of Seidenberg [45, see Theorems 7, 7', and 14] on generic hyperplane sections of irreducible normal varieties. The theorem was independently proven by C. Cumino, S. Greco, and M. Manaresi in [13]. The conclusion of this theorem fails in positive characteristic as illustrated by a class of examples that Cumino, Greco, and Manaresi introduced in [14]. In the statement of the next theorem, for a point  $a \in \mathbb{A}^{m+1}$  by  $H_a$  we mean the hyperplane  $a_0 + a_1X_1 + \dots + a_mX_m = 0$  in  $\mathbb{A}^{m+1}$ .

**Theorem 3.36.** [51, Theorem 3.4] *Let  $X \subset \mathbb{A}^m$  be an equidimensional weakly normal affine variety over an algebraically closed field of characteristic 0. Then there exists a dense open subset  $U$  of  $\mathbb{A}^{m+1}$  such that  $X \cap H_a$  is weakly normal whenever  $a \in U$ .*

Instead of asking about whether a property that a variety has is also enjoyed by a general hyperplane section, one can ask that if a general hyperplane section has a property, does that property “lift” to the original variety. Cumino and Manaresi defined a WN1 variety as a weakly normal variety such that the normalization  $\bar{X} \rightarrow X$  is unramified in codimension one. In [14] Cumino, Greco, and Manaresi showed that if the general hyperplane sections are WN1 then so is the original variety. They used this fact to show that in positive characteristic, the general hyperplane section of a weakly normal variety is not weakly normal.

More recently R. Heitmann proved a strong lifting result for *seminormality*, which we now state. Since we are dealing with general rings now we can not blur the distinction between seminormal and weakly normal rings.

**Theorem 3.37.** [24, Main Theorem] *If  $(R, \mathfrak{m})$  is a Noetherian local ring,  $y$  is a regular element in  $\mathfrak{m}$ , and  $R/yR$  is seminormal, then  $R$  is seminormal.*

### 3.5 Weak normality and Chinese remainder theorems

In Section 2.3, we reported on seminormality and the Chinese Remainder Theorem. We now mention a couple of results by Leahy and Vitulli that connect weak normality and the Chinese Remainder Theorem. All varieties are taken over an algebraic closed field of characteristic 0. The reader should see what these results say about affine rings and contrast the results to Theorems 2.24 and 2.26.

For a closed subvariety  $Y$  of an algebraic variety  $X$  we let  $\mathcal{J}_Y$  denote the sheaf of ideals defining  $Y$ . For a point  $X \in X$  we let  $T_{X,x}$  denote the tangent space of  $X$  at  $x$ .

**Theorem 3.38.** [28, Prop. 2.19] *Let  $X = X_1 \cup \dots \cup X_n$  where each  $X_i$  is a closed subvariety and suppose that  $X_i$  is weakly normal for each  $i$ . Further assume that  $X_i \cap X_j = Y$  whenever  $i \neq j$ . Then  $X$  is weakly normal if and only if  $\mathcal{J}_Y = \mathcal{J}_{X_i} + \mathcal{J}_{X_1 \cup \dots \cup X_{i-1}}$  for  $i = 2, \dots, n$ .*

If, in addition, we assume that the common intersection  $X_i \cap X_j = Y$  is weakly normal we get the following result. This result can be used to show that a variety is weakly normal at a multicross singularity.

**Theorem 3.39.** [28, Prop. 2.23] *Let  $X = X_1 \cup \dots \cup X_n$  be a union of closed subvarieties and assume that  $X_i \cap X_j = Y$  whenever  $i \neq j$ , where  $Y$  is weakly normal. The following assertions are true.*

1.  $X$  is weakly normal if and only if each  $X_i$  is weakly normal and  $\mathcal{J}_Y = \mathcal{J}_{X_i} + \mathcal{J}_{X_1 \cup \dots \cup X_{i-1}}$  for  $i = 2, \dots, n$ .
2. Suppose in addition that  $Y$  is nonsingular. Then  $X$  is weakly normal if and only if each  $X_i$  is weakly normal and  $T_{Y,x} = T_{X_i,x} \cap T_{X_1 \cup \dots \cup X_{i-1},x}$  for all  $x \in Y$  and  $i = 2, \dots, n$ .

### 3.6 The weak subintegral closure of an ideal

In this section, we discuss the weak subintegral closure of an ideal. We use the definition proposed by the current author and Leahy [54], which in turn is based on the criterion of Reid, Roberts and Singh [38]. Our definition stands in the same relation to the definition of Reid, Roberts, and Singh, as the definition of the integral closure of an ideal does to the normalization of a ring. In our definition we absorb the factor of  $(-1)^n$  that appears in (6).

**Definition 3.40.** Consider an  $A$ -ideal  $I$  and a ring extension  $A \subset B$ . We say an element  $b \in B$  is *weakly subintegral over  $I$*  provided that there exist  $q \in \mathbb{N}$  and elements  $a_i \in I^i$  ( $1 \leq i \leq 2q + 1$ ), such that

$$b^n + \sum_{i=1}^n \binom{n}{i} a_i b^{n-i} = 0 \quad (q + 1 \leq n \leq 2q + 1). \tag{9}$$

We let

$$*_B I = \{b \in B \mid b \text{ is weakly subintegral over } I\}.$$

We call  $*_B I$  the *weak subintegral closure of  $I$  in  $B$* . We write  $*I$  instead of  $*_A I$  and refer to  $*I$  as the *weak subintegral closure of  $I$* .

The paper [54] contains an important link between weak normalization of a graded ring and weak subintegral closure of an ideal, which we now recall. Suppose that  $A \subseteq B$  are rings,  $I$  is an ideal in  $A$ , and  $b \in B$ . Then,  $b$  is weakly subintegral over  $I^m$  if and only if the element  $bt^m \in B[t]$  is weakly subintegral over the Rees ring  $A[It]$  by [54, Lemma 3.2]. Thus  $*_B I$  is an ideal of  $*_B A$  [54, Prop. 2.11]. In particular,  $*I$  is an ideal of  $A$ . Vitulli and Leahy also showed that for an ideal  $I$  in a reduced ring  $A$  with finitely many minimal primes and total quotient ring  $Q$ , we have  $*(A[It]) = \bigoplus_{n \geq 0} *_Q(I^n)t^n$  by [54, Corollary 3.5].

Let’s make a quick observation about a sufficient condition for an element to be weakly normal over an ideal.

**Lemma 3.41.** *Suppose that  $I$  is an ideal of a ring  $A$ ,  $b \in A$  and  $b^n \in I^n$  for all sufficiently high powers of  $n$ . Then,  $b \in *I$ .*

*Proof.* Suppose  $q$  is such that  $b^n \in I^n$  for all  $n > q$ . Set  $a_i = 0$  for  $i = 1, \dots, q$ . Define  $a_{q+1} = -b^{q+1}$ . Suppose  $a_{q+1}, \dots, a_{n-1}$  have been defined for  $q + 1 \leq n - 1 < 2q + 1$ . Set  $a_n = -[b^n + \sum_{i=q+1}^{n-1} \binom{n}{i} a_i b^{n-i}]$ . Since  $a_i$  is an integer multiple of  $b^i$  for  $i = 1, \dots, n - 1$ , we have  $a_n \in I^n$  and  $b^n + \sum_{i=1}^n \binom{n}{i} a_i b^{n-i} = 0$ . By induction, we can define coefficients  $a_i$  so that (9) are satisfied.  $\square$

We can compare the weak subintegral closure of an ideal to what is usually called the Ratliff–Rush closure and was introduced by Ratliff and Rush in [33].

**Corollary 3.42.** *Let  $I$  be an ideal in a Noetherian ring  $A$  containing a regular element of  $A$  and let  $\tilde{I}$  denote the Ratliff-Rush ideal associated with  $I$ , that is,  $\tilde{I} = \cup_{n \geq 0} (I^{n+1} : I^n)$ . Then,  $\tilde{I} \subset *I$ .*

*Proof.* This follows immediately from the preceding corollary and the fact that  $\tilde{I}^n = I^n$  for all sufficiently high powers of  $n$ , which was proven in [33].  $\square$

For an ideal  $I$  of a ring  $A$  we have inclusions  $I \subset *I \subset \bar{I} \subset \sqrt{I}$  and if  $I$  is a regular ideal (i.e,  $I$  contains a regular element) of a Noetherian ring we also have  $I \subset \tilde{I} \subset *I \subset \bar{I} \subset \sqrt{I}$ .

Vitulli and Reid [39] algebraically characterized weakly normal monomial ideals in a polynomial ring over a field. Recall that the integral closure of a monomial ideal  $I$  in a polynomial ring  $K[x_1, \dots, x_n]$  is generated by all monomials  $x^\gamma$  such that  $x^{m\gamma} \in I^m$  for some positive integer  $m$ . This is independent of the characteristic of  $K$ . The condition to be in the weak subintegral closure of a monomial ideal is slightly stronger and is characteristic dependent as we now explain.

**Proposition 3.43.** [39, Proposition 3.3] *Let  $I$  be a monomial ideal in a polynomial ring  $K[x_1, \dots, x_n]$  in  $n$  indeterminates over a field  $K$ .*

1. *If  $\text{char}(K) = 0$ , then  $*I$  is the monomial ideal generated by all monomials  $x^\gamma$  such that  $x^{m\gamma} \in I^m$  for all sufficiently large positive integers  $m$ .*

2. If  $\text{char}(K) = p > 0$ , then  ${}^*I$  is the monomial ideal generated by all monomials  $x^\gamma$  such that  $x^{p^m \gamma} \in I^{p^m}$  for some nonnegative integer  $m$ .

Reid and Vitulli [39, Theorem 4.10] also presented a geometric characterization of the weak subintegral closure of a monomial ideal over a field of characteristic 0 in terms of the Newton polyhedron  $\text{conv}(\Gamma)$  of the exponent set  $\Gamma = \Gamma(I)$  of the monomial ideal  $I$ . We recall that  $\Gamma(I)$  consists of all exponents of monomials in  $I$  and  $\text{conv}(\Gamma)$  is the convex hull of  $\Gamma = \Gamma(I)$ . We write  $\overline{\Gamma}$  for the set of integral points in  $\text{conv}(\Gamma)$ ; thus,  $\overline{\Gamma}$  is the exponent set of the integral closure  $\overline{I}$  of  $I$ . We present a streamlined version of their characterization due to the current author. First we recall the pertinent definitions.

**Definition 3.44.** For a polyhedron  $P$  we define the *relative interior* of  $P$  by

$$\text{relint}(P) := P - \cup E,$$

where the union is taken over all facets  $E$  of  $P$ .

**Definition 3.45.** For a face  $F$  of the Newton polyhedron  $\Sigma = \text{conv}(\Gamma)$  of a monomial ideal, define

$${}^*F = \left\{ \mathbf{x} \in \text{relint}(F) \mid \mathbf{x} = \sum n_i \gamma_i, \gamma_i \in F \cap \Gamma, n_i \in \mathbb{Z} \right\}.$$

That is,  ${}^*F$  is the intersection of the group generated by  $F \cap \Gamma$  with the relative interior of  $F$ .

*Remark 3.46.* In [39] the additional requirement that  $\sum n_i = 1$  was part of the definition of  ${}^*F$ . Lemma 3.5 of [53] shows that this condition may and shall be deleted.

Before going further, we look at some examples.

*Example 3.47.* Consider  $I = (x^n, x^2y^{n-2}, y^n) \subset K[x, y]$ , where  $n = 2m + 1$ , let  $\Gamma = \Gamma(I)$ , and  $\Sigma = \text{conv}(\Gamma)$ . Notice that

$$(n, 0) = (0, n) + n(1, -1) \tag{10}$$

$$(2, n - 2) = (0, n) + 2(1, -1). \tag{11}$$

Subtracting  $m$  times (11) from (10) we get

$$(n, 0) + (m - 1)(0, n) - m(2, n - 2) = (1, -1). \tag{12}$$

With this one can check that every lattice point on the line segment  $E$  from  $(0, n)$  to  $(n, 0)$ , which is a face of  $\Sigma$ , is in the group generated by  $E \cap \Gamma$ . We may conclude that  ${}^*E = \{(1, n - 1), (2, n - 2), \dots, (n - 1, 1)\}$ .



*Example 3.48.* Consider  $I = (x^n, x^2y^{n-2}, y^n) \subset K[x, y]$ , where  $n$  is even, let  $\Gamma = \Gamma(I)$ , and  $\Sigma = \text{conv}(\Gamma)$ .

Using the preceding example (after dividing all coordinates by 2) we see that every lattice point with *even* coordinates on the line segment  $E$  from  $(0, n)$  to  $(n, 0)$ , which is a face of  $\Sigma$ , is in the group generated by  $E \cap \Gamma$  so that  ${}^*E = \{(2, n-2), \dots, (n-2, 2)\}$ .

Here, is the geometric characterization of the weak subintegral closure of a monomial ideal in characteristic 0.

**Theorem 3.49.** *Let  $\Gamma = \Gamma(I)$  be the exponent set of a monomial ideal  $I$  in a polynomial ring  $K[x_1, \dots, x_n]$  over a field  $K$  of characteristic 0. Then, an integral point  $\gamma \in \text{conv}(\Gamma)$  is in the exponent set of  ${}^*I$  if and only if  $\gamma \in \bigcup {}^*F$ , where the union is taken over all faces  $F$  of  $\text{conv}(\Gamma)$ .*

We include an example that illustrates the distinction between the integral closure and weak subintegral closure of a bivariate monomial ideal over a field  $K$  of characteristic 0. By  $\text{pos}(X)$  we mean the positive cone of the subset  $X$  of  $\mathbb{R}^n$ .

*Example 3.50.* Let  $I = (x^6, x^2y^4, y^6) \subset K[x, y]$ ,  $\Gamma = \Gamma(I)$ ,  ${}^*\Gamma = \Gamma({}^*I)$  and  $\Sigma = \text{conv}(\Gamma)$ . The exponent set  $\Gamma$  consists of all lattice points on or above the thick-lined staircase figure and  $\Sigma$  is the 1st quadrant with the lower left-hand corner clipped, i.e.,  $\Sigma$  consists of all points on or above the oblique line joining  $V_1$  and  $V_2$ . Observe that  $\Sigma$  and is the sole 2-dimensional face of  $\text{conv}(\Gamma)$ ,  $E_1 = \{0\} \times [6, \infty)$ ,  $E_2 = [(0, 6), (6, 0)]$ , and  $E_3 = [6, \infty) \times \{0\}$  are the edges and  $V_1 = \{(0, 6)\}$  and  $V_2 = \{(6, 0)\}$  are the vertices of  $\Sigma$ . The sets  $\Gamma$ ,  $\Sigma$ ,  $E_1, E_2, E_3, V_1, V_2$  are depicted below. The lattice points depicted by open circles are in  $\Sigma \setminus {}^*\Gamma$ . The lattice points depicted by filled circles are in  ${}^*\Gamma$ .

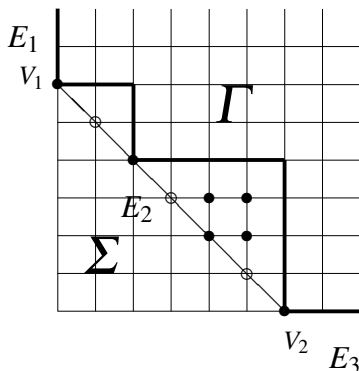


Fig. 3: Weak Subintegral Closure of  $(x^6, x^2y^4, y^6)$

Observe that

$$\begin{aligned}
 * \Sigma &= (\Gamma \setminus (E_1 \cup E_2 \cup E_3) \cap \Gamma) \cup \{(4, 3), (5, 2), (5, 3)\}; \\
 * E_1 &= \{(0, 7), (0, 8), (0, 9), \dots\}; \\
 * E_2 &= \{(2, 4), (4, 2)\}; \\
 * E_3 &= \{(7, 0), (8, 0), (9, 0), \dots\}; \\
 * V_i &= V_i \quad (i = 1, 2); \\
 * \Gamma &= \Gamma \cup \{(4, 2), (4, 3), (5, 2), (5, 3)\}; \text{ and} \\
 \bar{\Gamma} &= \Gamma \cup \{(1, 5), (3, 3), (4, 2), (4, 3), (5, 1), (5, 2), (5, 3)\}.
 \end{aligned}$$

Thus we have  $\Gamma \subset * \Gamma \subset \bar{\Gamma}$ , where both containments are proper.

Notice that all of the points in the relative interior of the Newton polyhedron are in the weak subintegral closure of  $I$ . An analog of this observation is true for arbitrary ideals by Prop. 3.57 below, which we will present after we introduce some necessary notation. For more details the reader should consult [18].

Let  $I$  be a monomial ideal over a field of arbitrary characteristic. The proof of Theorem 3.49 shows that if  $\alpha$  is in  $*F$  for some face  $F$  of  $\text{conv}(\Gamma(I))$ , then  $x^{\alpha} \in I^n$  for all sufficiently large  $n$ . Thus we may conclude that  $x^{\alpha} \in *I$  by Lemma 3.41. We now take a look at the ideal with the same monomial generators as in Example 3.50 but over a field of characteristic 2.

*Example 3.51.* Assume that  $\text{char}(K) = 2$ , let  $I = (x^6, x^2y^4, y^6) \subset K[x, y]$ ,  $\Gamma = \Gamma(I)$ ,  $*\Gamma = \Gamma(*I)$  and  $\Sigma = \text{conv}(\Gamma)$ . Notice that

$$\begin{aligned}
 (xy^5)^2 &= (x^2y^4)y^6 \in I^2 \\
 (x^3y^3)^2 &= x^6y^6 \in I^2 \\
 (x^4y^2)^2 &= x^6(x^2y^4) \in I^2 \\
 (x^5y)^4 &= (x^6)^3(x^2y^4) \in I^4
 \end{aligned}$$

Combining these calculations with those in Example 3.48 we see that  $*I = \bar{I}$ . This explicit example illustrates that unlike integral closure, the weak subintegral closure of a monomial ideal depends on the characteristic of the base field.

We now turn our attention to more general ideals in Noetherian rings.

**Notation 3.52.** For an ideal  $I$  of a Noetherian ring  $A$  and an element  $a \in A$  we write  $\text{ord}_I(a) = n$  if  $a \in I^n \setminus I^{n+1}$  and  $\text{ord}_I(a) = \infty$  if  $a \in \bigcap_{n \geq 1} I^n$ . Next we define

$$\bar{v}_I(a) = \lim_{n \rightarrow \infty} \frac{\text{ord}_I(a^n)}{n}.$$

The indicated limit always exists (possibly being  $\infty$ ; ([31, Prop. 11.1] or [49, Theorem 10.1.6]) and  $\bar{v}_I$  is called the *asymptotic Samuel function* of  $I$ .

Let  $I$  be a regular ideal in a Noetherian ring  $A$  and  $a \in A$ . The asymptotic Samuel function  $\bar{v}_I$  is determined by the Rees valuations  $v_j$  of  $I$ . Namely,

$$\bar{v}_I(a) = \min_j \left\{ \frac{v_j(a)}{v_j(I)} \right\},$$

where  $v_j(I) = \min \{v_j(b) \mid b \in I\}$  (see [49, Lemma 10.1.5]). We refer the reader to Chap. 10 of [49] for the fundamentals on Rees valuations of ideals. Recall that  $\bar{v}_I = \bar{v}_J$  whenever  $\bar{J} = \bar{I}$  (see [31, Cor. 11.9]). This immediately implies that  $J_{>} = I_{>}$  whenever  $\bar{J} = \bar{I}$ .

**Notation 3.53.** For an ideal  $I$  in a Noetherian ring  $A$  we let

$$I_{>} = \{a \in A \mid \bar{v}_I(a) > 1\}.$$

By elementary properties of the asymptotic Samuel function,  $I_{>}$  is an ideal of  $A$  and a subideal of  $\bar{I}$ . It does not contain the original ideal  $I$  by definition of  $\bar{v}_I$ .

*Example 3.54.* Let  $K$  be a field and  $I = (x^6, x^2y^4, y^6) \subset K[x, y]$ . Then,  $I$  has one Rees valuation, namely the monomial valuation defined by  $v(x^a y^b) = a + b$  (see Chap. 10 of [49] for the fundamentals on monomial valuations). We have  $v(I) = 6 = v(y^6)$  and  $I_{>} = \{f \in K[x, y] \mid v(f) > 6\} = (x, y)^7 \subsetneq I$ .

The ideal  $I_{>}$  plays an important role in conditions from stratification theory such as Whitney’s condition  $A$  and Thom’s condition  $A_f$ ; the reader can learn more about these conditions in [26]. To give the reader a little more feeling for the ideal  $I_{>}$  we cite some lemmas that were proven in [18].

**Lemma 3.55.** [18, Lemma 4.2] *Let  $I$  be a regular ideal in a Noetherian ring  $A$ . Then,*

$$I_{>} = \bigcap_i \mathfrak{m}_i V_i \cap A,$$

where the intersection is taken over all Rees valuation rings  $(V_i, \mathfrak{m}_i)$  of  $I$ . In particular,  $I_{>}$  is an integrally closed ideal.

**Lemma 3.56.** [18, Lemma 4.3] *Let  $I$  be a nonzero monomial ideal in a polynomial ring over a field. Then,  $I_{>}$  is again a monomial ideal.*

We now present a generalization of what was known as Lantz’s conjecture (after a talk D. Lantz gave in 1999 at a Route 81 Conference in upstate New York) and illustrate the result with an example. Lantz conjectured that if  $I$  is an  $\mathfrak{m}$ -primary ideal in a 2-dimensional regular local ring  $(A, \mathfrak{m})$ , then  $*I$  contains all elements  $a \in A$  such that  $\bar{v}_I(a) > 1$ .

**Proposition 3.57.** [18, Prop. 4.4] *Let  $I$  be an ideal of a Noetherian ring  $A$ . Then,  $I_{>} \subseteq *I$ .*

*Example 3.58.* Let  $K$  be a field of characteristic 0, let  $n = 2m + 1$ , and consider  $I = (x^n, x^2y^{n-2}, y^n)$ . We claim that  $*I = \bar{I}$ . By Example 3.47 we may conclude that for the facet  $E$  of  $\text{conv}(\Gamma)$ ,  $*E = \{(1, n - 1), (2, n - 2), \dots, (n - 2, 2), (n - 1, 1)\}$ . This, together with the fact that  $I_{>} \subset *I$ , implies that  $*I = \bar{I}$ .

*Example 3.59.* Let  $K$  be a field of characteristic 0, let  $n = 2m$ , and consider  $I = (x^n, x^2y^{n-2}, y^n)$ . Using Example 3.48 and Theorem 3.49 we see that the weak subintegral closure is  $*I = (x^n, x^{n-2}y^2, \dots, x^2y^{n-2}, y^n)$ .

One consequence of the preceding proposition is the following connection between the ideal  $I_{>}$  and the minimal reductions  $\mathcal{MR}(I)$  of the ideal  $I$ .

**Corollary 3.60.** [18, Cor. 4.5] *Let  $I$  be an ideal of a Noetherian ring  $A$ . Then,*

$$I_{>} \subseteq \bigcap_{J \in \mathcal{MR}(I)} *J.$$

We mention another interesting occurrence of the ideal  $I_{>}$ . For more results in the same spirit the reader should consult [18].

**Theorem 3.61.** [18, Theorem 4.6] *Let  $(A, \mathfrak{m}, k)$  be a Noetherian local ring such that  $k$  is algebraically closed of characteristic 0. Suppose that  $I$  is an  $\mathfrak{m}$ -primary ideal. If  $J$  is any minimal reduction of  $I$  then  $J + I_{>} = *J$ .*

It is well known that the integral closure of an ideal  $I$  in an integral domain  $A$  can be characterized in terms of valuation rings. Namely,  $\bar{I} = \bigcap_V IV \cap A$ , where the intersection is taken over all valuation rings of the quotient field of  $A$  that contain  $A$  (see [49, Prop. 6.8.2]). This can be rephrased in terms of maps into valuation rings and for Noetherian rings in terms of maps into discrete rank one valuation rings. More precisely, an element  $f$  in a Noetherian ring  $A$  is in  $\bar{I}$  if and only if  $\rho(f) \in \rho(I)V$  for every homomorphism  $\rho : A \rightarrow V$ , where  $V$  is a discrete rank one valuation ring. For algebro-geometric local rings that we can limit which discrete rank one valuation rings we look at. The following is an analog of the complex analytic criterion involving germs of morphisms from the germ of the pointed complex unit disk  $(\mathbb{D}, 0)$  to the germ  $(X, x)$  (e.g., see [30, 2.1 Théorème]).

**Proposition 3.62.** [18, Prop. 5.4] *Let  $(A, \mathfrak{m}, k)$  be the local ring of an algebraic variety over an algebraically closed field  $k$ ,  $I$  an ideal of  $A$ , and  $h \in A$ . Then,  $h \in \bar{I} \Leftrightarrow$  for every local homomorphism of  $k$ -algebras  $\rho : A \rightarrow k[[X]]$  we have  $\rho(h) \in \rho(I)k[[X]]$ .*

In the criteria, involving maps into discrete valuation rings, one thinks of the targets of those maps as test rings for determining integral dependence. Recently H. Brenner [8] suggested a valuative criterion for an element to be in the weak subintegral closure of an ideal in an algebro-geometric local ring. Brenner used certain 1-dimensional seminormal local rings as test rings and the results of [18]. Gaffney-Vitulli took another approach to developing a valuative criterion in both the algebraic and complex analytic settings (see [18, Props. 5.7 and 5.8]).

We end our account here. We hope we have left you, the reader, with a better idea of the many ramifications of the closely related notions of weak and seminormality.

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