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The Fields Institute for Research in Mathematical Sciences

Chris Miller
Jean-Philippe Rolin
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Editors

Lecture Notes on O-Minimal Structures and Real Analytic Geometry

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Lecture Notes on O-minimal Structures and Real Analytic Geometry

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FIELDS in the Mathematical Sciences

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Preface

The notes in this volume were produced in conjunction with the Thematic Program in O-minimal Structures and Real Analytic Geometry, held from January to June 2009 at the Fields Institute. Among the activities of our thematic program were three graduate courses, offered to participants and to graduate students from universities in the Greater Toronto Area. Each of these courses was, in turn, split into three modules, and most of these modules were taught by different instructors. Five of the six contributions to this volume arose from the modules taught by the authors: Felipe Cano on the resolution of singularities of vector fields; Chris Miller on o-minimality and Hardy fields; Jean-Philippe Rolin on the construction of o-minimal structures from quasianalytic classes; Fernando Sanzon non-oscillatory trajectories; and Patrick Speissegger on pfaffian sets. The sixth contribution, by Antongiuglio Fornasiero and Tamara Servi, is an adaptation of Wilkie's construction of o-minimal structures from total C^∞-functions to the nonstandard setting. Their adaptation was carried out concurrently with our program, and the resulting notes fit in naturally with the pfaffian portion of our lectures.

There are only a few dependencies between the contributions: Miller's is used in both Rolin's and Speissegger's, and Rolin's is used in Sanz's. In addition, familiarity with the basics is assumed for o-minimality (van den Dries [4] and Miller and van den Dries [5]) and semianalytic and subanalytic sets (Bierstone and Milman [2]). Further recommended reading are Marker [3] on model theory (basic aspects of which are used in Miller's notes) and Balser [1] on Borel-Laplace summation (used in Sanz's notes).

We thank the Fields Institute for the generous funding provided for our program, and we thank its very competent and helpful staff for making our stay there productive and very enjoyable. Participation of several US-based graduate students and junior postdoctoral researchers was partially funded by NSF Special Meetings Grant DMS-0753096.

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Contents

Blowings-Up of Vector Fields

Felipe Cano

Abstract A new proof of the reduction of singularities for planar vector fields is presented. The idea is to adapt Zariski's local uniformisation method to the vector field setting.

Mathematics Subject Classification (2010): Primary 32S65, Secondary 37F75

Introduction

These notes cover part of a course taught at the Fields Institute in January 2009, as part of the Thematic Program on O-minimal Structures and Real Analytic Geometry. I try to introduce the reader to a new proof of the reduction of singularities for vector fields in dimension two.

What is the reason for giving this new proof? Indeed, the original proof of 1968 given by Seidenberg [36] is complete and does not need much tweaking to be useful for most applications. Other proofs in dimension two were published, among them Giraud [20, 21], van den Essen [39], Dumortier [19] and one by myself [7], where I tried to recover Hironaka's way of reducing singularities.

In these notes, the idea is to recover the local uniformization method due to Zariski [42, 43], which dates back to 1940 (see also Vaquié [40] for a discussion of Zariski's method). The proof I present here can be generalized at least to dimension three, as done in joint work in progress with Roche and Spivakovsky [13, 14]. Also, as I explain later, the result in dimension three gives a global result as an application of Zariski's method.

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For a more general elementary exposition of the theory of singular holomorphic foliations the reader may look at Camacho and Lins-Neto [5], Cano and Cerveau (Introduction aux feuilletages singuliers, Unpublished lecture notes available from the authors) and Brunella [3].

Historical note. Let us give a brief historical overview of the proof of reduction of singularities for vector fields in dimensions two and three. First of all, let us indicate that there are no known results in dimension greater than or equal to four, except for the specific case of absolutely isolated singularities (see Camacho et al. [6]).

The original proof of Seidenberg is based on the behavior of the multiplicity $i(C_1, C_2; p)$ of the intersection of two plane curves C_1 and C_2 at a point p under blowing-up. More precisely, Noether's formula states that

$$i(C_1, C_2; p) = m_p(C_1)m_p(C_2) + \sum_{p' \in E} i(C_1', C_2'; p'),$$

where E is the exceptional divisor of the blowing-up with center $\{p\}$, C_1', C_2' are the strict transforms of the curves C_1, C_2 and $m_p(C)$ denotes the multiplicity of the curve C at the point p. Van den Essen's, Dumortier's and Giraud's proofs follow this same idea; Dumortier's is specific to the real case and Giraud's to the framework of Algebraic Geometry in positive characteristic.

The use of the multiplicity of the intersection as a main invariant of control is based on the fact that the singularities considered are isolated, and hence the multiplicity of the intersection of the coefficients is finite. For vector fields this invariant is called *Milnor number* and generalizes, in the Hamiltonian case, the usual Milnor number of a function. If we can assure that the Milnor number remains finite under any blowing-up, then the method generalizes to higher dimension without obstruction. This is the case for absolutely isolated singularities in any dimension, as shown in our work with Camacho and Sad.

If one wants to look at the general case in dimension three, it is necessary to develop a method not based on control of the Milnor number. In [7], I gave a proof based on the ideas of Hironaka. This method can be interpreted as follows in dimension two: first, we need an invariant acting as the *Hilbert-Samuel function*; this invariant is the logarithmic multiplicity of the vector field, together with a description of a finite list of types. Second, we need maximal contact, which acts as a kind of reduction of the dimension from two to one. Finally, we consider a more specific invariant of control for the case of maximal contact, namely, the contact exponent associated to a Hironaka's characteristic polyhedron (in this case just a line).

More precisely, the first result in ambient dimension three was given by myself in [9, 15], in the form of a positive answer to Hironaka's game. This result is of a local nature, where we allow formal centers of blowings-up. In some sense, it is a strong local uniformization result, but it has the disadvantage that formal (non-convergent) centers of blowings-up are used. The statement is as follows: we start with the germ of a vector field at $(\mathbb{C}^3, 0)$, more precisely with the germ $\mathcal{L}$ of the foliation induced by the vector field. To this vector field, we associate a *logarithmic multiplicity* at a point p, the smallest multiplicity of the coefficients of the vector field expressed in a logarithmic way with respect to a normal crossings divisor (that is, we "force" the components of the divisor to be invariant). For instance, if p is the origin, the divisor is defined by $\prod_{i=1}^{e} x_i = 0$ and the vector field is given by

$$\xi = \sum_{i=1}^{e} a_i(x) x_i \frac{\partial}{\partial x_i} + \sum_{i=e+1}^{n} a_i(x) \frac{\partial}{\partial x_i},$$

then the corresponding **logarithmic** (or **adapted**) **multiplicity** is the minimum of the multiplicities of the coefficients $a_i(x)$ at the origin, for $i = 1, 2, \ldots, n$. We say that the point p is a **log-elementary** singularity of the vector field, if the logarithmic multiplicity at p is less than or equal to 1. Now we play Hironaka's desingularization game between two players A and B (where "A" is typically interpreted as "Abhyankar" in recognition of the latter's contribution to the understanding of singularities):

1. If p is log-elementary for the vector field, player A wins; otherwise, he chooses a formal center of blowing-up.
2. Player B chooses a point p' in the preimage of p under the blowing-up.
3. The game restarts with p' in place of p.

A **winning strategy** for player A is a decision method that makes sure the game stops in a finite number of steps, independently of the choices made by player B. In [15], I presented a winning strategy for player A. In [9], I extended this strategy to so-called **elementary singularities**, that is, singularities with non-nilpotent linear part.

At this point, the problems in dimension three are the following:

(a) To obtain a result where the centers of blowings-up are analytic; that is, the geometry of the ambient space is not destroyed by a blowing-up with a formal center.
(b) To obtain a global result. Instead of blowings-up with centers adapted to the point chosen by player B, try to obtain a global morphism such that all the points on the exceptional divisor are log-elementary or, even better, elementary.

A version of Hironaka's game can be played in the case of a non-oscillatory trajectory of the germ at the origin of real vector field ξ in $\mathbb{R}^3$ (see Sanz [35]). Let γ be a non-oscillatory trajectory of ξ that approaches to the origin, that is

$$\lim_{t\to\infty} \gamma(t) = 0.$$

We assume that γ is **non-oscillatory** (that is, γ crosses any analytic hypersurface at most finitely many times) and that γ is not contained in any analytic hypersurface. Then γ acts as player B in the following way: player A chooses a blowing-up with center the origin or a nonsingular analytic curve through the origin. The lifting of γ accumulates at only one point p' of the exceptional divisor: otherwise, we could produce an algebraic hypersurface that γ crosses infinitely many times, contradicting the non-oscillatory property of γ.

In joint work with Moussu and Rolin [12], we solved Hironaka's game in the case where player B is given by a non-oscillatory trajectory of the germ at the origin of a vector field in $\mathbb{R}^3$. Since we were working over the real field, we were interested in applications that are stable under ramifications, so we allowed ourselves to do ramifications; nevertheless, all our centers of blowings-up were analytic. In this way we obtained a local, non-birational reduction of singularities method over the real field that finishes in elementary (not just log-elementary) singularities.

The techniques used in [12] have a natural interpretation in terms of Zariski's method for the local uniformization. Indeed, a non-oscillatory trajectory γ of ξ induces an identification of the field of rational functions (even of meromorphic functions) in three variables with a Hardy field, via the substitution morphism

$$\frac{F(X,Y,Z)}{G(X,Y,Z)} \mapsto \frac{F(\gamma(t))}{G(\gamma(t))}.$$

This Hardy field has a natural valuation whose centers (in the sense of Zariski) are given by the accumulation points of γ under blowing-up. Thus, player B is in this case a valuation that chooses, at each step, the center of the valuation in the corresponding model of the field of rational functions. This is precisely Zariski's point of view for the local uniformization. The difference between his point of view and Hironaka's is that, in Zariski's case, we know the nature of player B (a valuation), and we can do arguments using this particular nature of player B.

The need for ramifications was evident in [12] for passing from a special nilpotent situation to an elementary case. More precisely, an example produced by F. Sanz and F. Sancho shows that the latter is not possible in general without using formal, but nonconvergent, blowings-up. Their example is the following:

$$\xi = x\left(x\frac{\partial}{\partial x} - \alpha y\frac{\partial}{\partial y} - \beta z\frac{\partial}{\partial x}\right) + xz\frac{\partial}{\partial y} + (y - \lambda x)\frac{\partial}{\partial z}.$$

This example is discussed in detail in the introduction of Panazzolo [30]. Let me just mention that, for this ξ, using a blowing-up with center a formal ξ-invariant curve transverse to $\{x = 0\}$, one obtains elementary singularities.

We are currently working, with Roche and Spivakovsky, on a local uniformization result, in the sense of Zariski, for a general situation of algebraic geometry in characteristic zero. We obtain, via a birational transformation along a given valuation, log-elementary singularities in ambient dimension three. Moreover, these log-elementary singularities satisfy a list of axioms given by Piltant [32] that allow us to globalize the local uniformization in an ambient space of dimension three. This result represents an axiomatic version of Zariski's gluing of local uniformizations in dimension three [42, 43]. As a consequence, we obtain a global and birational way of reducing singularities in dimension three, such that the final singularities are log-elementary.

In these notes, we present the two-dimensional version of this joint work, in order to introduce the reader to the key ideas of our method.

To finish this historical note, let us point out that log-elementary singularities are far from being elementary; for instance, nilpotent singularities are always log-elementary. In fact, Panazzolo's thesis [29] deals with transforming nilpotent to elementary in a global non-birational way, via real transformations of "quasi-homogeneous" type. This important work showcases just how far log-elementary signularities are from begin elementary.

The most complete result on reduction of singularities for vector fields in dimension three is Panazzolo's [30]. This is a global result, via non-birational transformations, that obtains elementary singularities in the real case. His techniques of control and globalization in [30] are close to Hironaka's; but he also uses weighted blowings-up, with weights associated to the Newton polyhedron of the vector field. These latter ideas are, arguably, the reason for the relative simplicity of his work.

More recently, as of May 2011, some new results on these matters have appeared: first, the valuation-theoretic arguments in dimension three in [14] can be generalized to any dimension in order to get maximal contact or resonance. Both these cases represent a reduction, in a certain sense, of the ambient dimension of the problem. Second, there is a preprint of McQuillan and Panazzolo in which they apply the techniques of [29] to obtain a three-dimensional reduction of singularities for vector fields in ambient dimension three, in the framework of stack theory.

Applications. A classical application of the reduction of singularities of vector fields is the theorem of Camacho and Sad [4], which proves the existence of an invariant holomorphic curve at a singularity of a holomorphic vector field in dimension two. This result was conjectured by R. Thom, based on the intuition that the invariant hypersurfaces should "organize" the dynamics. Their proof relies on reduction of singularities and the behavior of an index, now known as the *Camacho-Sad index*. A very short proof of this result may be found in [8].

In dimension two, the reduction of the singularities for vector fields has been a central result, providing an algebraic skeleton in the study of holonomy, formal and analytic classification, deformation, integrability, etc. Introductions to these topics can be found in [16, 17, 26–28].

In dimension three, fewer applications are known, due of course to the difficulties of the result itself. There is a counterexample to the existence of an invariant analytic curve, found by Gómez-Mont and Luengo [22], based on the behavior under blowing-up of elementary singularities. Besides the geometric study of oscillation presented in [35], I would like to mention a remark of Brunella [2] that shows that any real vector field in dimension three, with an isolated singularity at the origin, has at least one trajectory arriving at or exiting from the origin.

The reader may look at the references [10, 11, 18, 23, 31, 33, 34, 38, 41] as a small seletion of papers corresponding applications of reduction of singularities and some of the technics introduced in these notes.

1 Vector Fields and Blowings-Up

Germs of vector fields. The *ambient space* M is for us of one of the following types. We can have an ambient space which is a *real analytic variety*, that is M is described by a collection of real charts such that the compatibility conditions of the charts are real analytic applications. We can also consider the case that M is a *complex analytic variety*, with the same definition as before, except for the fact that the compatibility conditions of the charts are complex analytic (holomorphic) applications. We also consider the case that $M \subset \mathbb{P}^N_{\mathbb{C}}$ is an irreducible complex projective variety, where we can eventually have singular points. Most of the properties we are going to consider have *local nature* and thus they can be explained in terms of the local ring $\mathcal{O}_{M,p}$ of the germs of functions at a point p of M, whose maximal ideal $\mathcal{M}_{M,p}$ is given by the germs of functions $f \in \mathcal{O}_{M,p}$ such that $f(p) = 0$.

Since we work either over the real numbers or over the complex numbers, we denote $k = \mathbb{R}$ or $k = \mathbb{C}$, depending on the cases we are considering.

By definition, the *germs of vector field at* $p \in M$ are the k-derivations of the local ring $\mathcal{O}_{M,p}$. That is a germ of vector field is a map

$$\xi : \mathcal{O}_{M,p} \to \mathcal{O}_{M,p}$$

which is a homomorphism of k-vector spaces and satisfies to the Leibnitz rule

$$\xi(fg) = f\xi g + g\xi f.$$

We denote $\mathrm{Der}_k\mathcal{O}_{M,p}$ the set of germs of vector fields at p. It has a natural structure of k-vector space and moreover, it is a $\mathcal{O}_{M,p}$-module, where we have $(f\xi)g = f(\xi g)$.

The set of *tangent vectors* T_pM at p is the set of "centered derivations". That is, a tangent vector at p is a map

$$v : \mathcal{O}_{M,p} \to k$$

which is a homomorphism of k-vector spaces and satisfies to the "centered" Leibnitz rule

$$v(fg) = f(p)(vg) + g(p)(vf).$$

Obviously, any germ ξ of vector field at p induces a tangent vector

$$\xi|_p \in T_pM,$$

just by putting $\xi|_p f = (\xi f)(p)$. The *tangent space* T_pM has a natural structure of k-vector space.

Assume that p is a nonsingular point of M. This is always the case when M is a real or complex analytic variety. Then the maximal ideal $\mathcal{M}_{M,p}$ of $\mathcal{O}_{M,p}$ has a set of generators $x_1, x_2, \ldots, x_n$, where n is the dimension of M. Depending on the context, this set of generators is called *regular system of parameters* or *system of* *centered local coordinates*. There are particular germs of vector field that we denote $\partial/\partial x_i$, for $i = 1, 2, \ldots, n$ defined by the properties

$$\frac{\partial}{\partial x_i}(x_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

In fact, we obtain in this way a basis of the free $\mathcal{O}_{M,p}$-module $\mathrm{Der}_k \mathcal{O}_{M,p}$. So, any germ of vector field ξ has a unique expression as

$$\xi = a_1 \frac{\partial}{\partial x_1} + a_2 \frac{\partial}{\partial x_2} + \cdots + a_n \frac{\partial}{\partial x_n},$$

where $a_1, a_2, \ldots, a_n \in \mathcal{O}_{M,p}$. Also, a k-basis of the tangent space $T_p M$ is given by $\partial/\partial x_i|_p$, for $i = 1, 2, \ldots, n$. In particular the map $\xi \mapsto \xi|_p$ is surjective.

Let us consider representatives X_i of the germs x_i for $i = 1, 2, \ldots, n$. There is an open neighborhood U of p satisfying the following property:

> For any point $q \in U$ there is a unique $\mathbf{q} = (q_1, q_2, \ldots, q_n) \in k^n$ such that the functions $X_1 - q_1, X_2 - q_2, \ldots, X_n - q_n$ define a regular system of parameters of $\mathcal{O}_{M,q}$.

In view of this property, we can consider *vector fields* defined in such neighborhoods U as expressions

$$\mathcal{V} = \sum_{i=1}^{n} A_i \frac{\partial}{\partial X_i}$$

where the $A_1, A_2, \ldots, A_n$ are functions defined in U. Obviously such a vector field $\mathcal{V}$ induces a germ of vector field $\mathcal{V}_q$ at each $q \in U$ in an evident way, as well as tangent vectors $\mathcal{V}(q) \in T_q M$.

Definition 1.1. A germ of vector field $\xi \in \mathrm{Der}_k \mathcal{O}_{M,p}$ is *non-singular* if p is a non-singular point of M and $\xi(\mathcal{M}_{M,p})$ is not contained in $\mathcal{M}_{M,p}$.

In terms of coordinates, this is equivalent to say that $\xi(x_i)(p) \neq 0$ for some of the parameters x_i. The next classical result justifies the interest of having a non-singular germ of vector field

Theorem 1.2 (Rectification). *Let $\xi \in \mathrm{Der}_k \mathcal{O}_{M,p}$ be a non-singular germ of vector field and let us assume that the ambient space M is a real or complex analytic variety. There is a choice of local coordinates $x_1, x_2, \ldots, x_n$ such that $\xi = \partial/\partial x_1$.*

Blowings-up of ambient space. Let $p \in M$ be a nonsingular point of the ambient space M. The blowing-up of M with center p is a morphism $\pi : M' \to M$ that we describe in this section.

Blowing-up of the projective space. Let us consider first the case where $M = \mathbb{P}^n_k$ is the n-dimensional projective space. Take a projective hyperplane $\Delta_\infty \subset \mathbb{P}^n_k$ such that $p \notin \Delta_\infty$. Now, we can choose homogeneous coordinates $[X_0, X_1, \ldots, X_n]$ in $\mathbb{P}^n_k$ such that $p = [1, 0, 0 \ldots, 0]$ and $\Delta_\infty = \{X_0 = 0\}$. Note that the points in Δ_∞

are of the form $[0, X_1, X_2, \dots, X_n]$ and hence $[X_1, X_2, \dots, X_n]$ can be considered as being homogeneous coordinates for Δ_∞. Let us denote by

$$\lambda : \mathbb{P}^n_k \setminus \{p\} \to \Delta_\infty$$

the linear projection defined by $\lambda(q) = (p+q) \cap \Delta_\infty$, where $p+q$ is the projective line through p and q. In terms of homogeneous coordinates, we have

$$\lambda([X_0, X_1, X_2, \dots, X_n]) = [X_1, X_2, \dots, X_n].$$

Let $G(\lambda)$ be the graph of λ and consider the topological closure

$$\overline{G(\lambda)} \subset \mathbb{P}^n_k \times \Delta_\infty.$$

The first projection $\pi : \overline{G(\lambda)} \to \mathbb{P}^n_k$ is by definition the *blowing-up* of $\mathbb{P}^n_k$ with center p. Let us note that the equations of $\overline{G(\lambda)}$ in homogeneous coordinates $[X_0, X_1, \dots, X_n]$ for $\mathbb{P}^n_k$ and $[Y_1, Y_2, \dots, Y_n]$ for Δ_∞ are

$$X_i Y_j = X_j Y_i; \text{ for } i, j = 1, 2, \dots, n.$$

We see that $\overline{G(\lambda)} \setminus \pi^{-1}(p) = G(\lambda)$ and hence π defines an isomorphism

$$\pi : \overline{G(\lambda)} \setminus \pi^{-1}(p) \to \mathbb{P}^n_k \setminus \{p\}.$$

Moreover, there is an identification between $\pi^{-1}(p)$ and Δ_∞. We say that $\pi^{-1}(p)$ is the *exceptional divisor* of π and hence each of its points corresponds to a line through p.

The transformed space $\overline{G(\lambda)}$ is a nonsingular variety. To see a chart decomposition of it, we write

$$\overline{G(\lambda)} = G(\lambda) \cup \pi^{-1}(\mathbb{P}^n_k \setminus \Delta_\infty) = \pi^{-1}(\mathbb{P}^n_k \setminus \{p\}) \cup \pi^{-1}(\mathbb{P}^n_k \setminus \Delta_\infty).$$

Now, we already know that $\pi^{-1}(\mathbb{P}^n_k \setminus \{p\})$ is identified with the open set $\mathbb{P}^n_k \setminus \{p\}$ of the projective space $\mathbb{P}^n_k$. To describe $\pi^{-1}(\mathbb{P}^n_k \setminus \Delta_\infty)$, let us first recall that there is an identification

$$\mathbb{P}^n_k \setminus \Delta_\infty \leftrightarrow \mathbb{A}^n_k = k^n,$$

given in coordinates by $[1, x_1, x_2, \dots, x_n] \leftrightarrow (x_1, x_2, \dots, x_n)$. Now, we cover $\pi^{-1}(\mathbb{A}^n_k)$ by charts $\pi^{-1}(\mathbb{A}^n_k) = \bigcup_{j=1}^n U_j$ with

$$U_j = \pi^{-1}(\mathbb{A}^n_k) \cup \{Y_j \neq 0\}.$$

Each U_j has a coordinate mapping

$$\phi_j : U_j \to \mathbb{A}^n_k; \quad (\mathbf{x}, [\mathbf{Y}]) \mapsto (x_1^{(j)}, x_2^{(j)}, \dots, x_n^{(j)}),$$

where $x_j^{(j)} = x_j$ and $x_i^{(j)} = Y_i/Y_j$ for $i \neq j$. In particular, the blowing-up π in the charts U_j has the equations

$$(x_1^{(j)}, x_2^{(j)}, \ldots, x_n^{(j)}) \mapsto (x_1, x_2, \ldots, x_n) \in \mathbb{A}_k^n,$$

where $x_j = x_j^{(j)}$ and $x_i = x_i^{(j)} x_j^{(j)}$, for $i \neq j$. Let us remark that the morphism π may be recovered starting with these equations.

Blowing-up of any variety. Let M be a variety, covered by charts $U \subset M$, that we identify as open sets $U \subset \mathbb{A}_k^n$. Take a point $p \in M$ and consider a chart U such that $p \in U$. We can shrink the other charts to assume that $p \notin U'$ for another chart U' different from U. Now, we can do the blowing-up of U with center p

$$\pi_U : \tilde{U} = \pi^{-1}(U) \to U \subset \mathbb{A}_k^n.$$

We glue the charts U' with $\tilde{U}$ by recalling the identification between $\tilde{U} \setminus \pi^{-1}(p)$ and $U \setminus \{p\}$. In this way we obtain the blow-up morphism

$$\pi : \tilde{M} \to M.$$

Blowing up along a subvariety. Let M be a variety and consider a closed subvariety $Y \subset M$. We can identify locally the pair (M, Y) with the pair $U \times V, \{\mathbf{0}\} \times V$, where U and V are open subsets $\mathbf{0} \in U \subset \mathbb{A}_k^{n-m}$ and $V \subset \mathbb{A}_k^m$. The blowing-up

$$\pi : \tilde{M} \to M,$$

of M with center π is obtained by gluing together the local blowings-up

$$\tilde{U} \times V \to U \times V,$$

where $\tilde{U} \to U$ is the blowing-up with center $\mathbf{0}$. Note that the *exceptional divisor* $\pi^{-1}(Y) \subset \tilde{M}$ is a hyper-surface covered by open sets of the form $\pi^{-1}(p) \times V$.

The universal property of the blowing-up. The above constructions seem to be highly non intrinsic. In particular one immediately sees a problem to justify the gluing procedures in the blowing-up along a subvariety. All this difficulties are solved by invoking the *universal property of the blowing-up*. In algebraic terms it can be stated as follows

> Let $\pi : \tilde{M} \to M$ be the blowing-up of M along a subvariety. Consider another proper morphism $h : M' \to M$ having the property that $h^{-1}(M \setminus Y)$ is isomorphic to $M \setminus Y$ and $h^{-1}(Y)$ is a hyper-surface (in the sense that the sheaf $\mathcal{J}_Y \mathcal{O}_{M'}$ is an inversible sheaf). Then there is a unique morphism $f : M' \to \tilde{M}$ such that $\pi \circ f = h$.

We will not insist in this property and the use of the blowing-up we will do is mainly through the equations and coordinates.

Transform of a vector field by blowings-up. Let ξ be a germ of vector field at $p \in M$. That is $\xi \in \mathrm{Der}_k \mathcal{O}_{M,p}$. Consider a blowing-up

$$\pi : \tilde{M} \to M,$$

along a subvariety $Y \subset M$ and fix a point $p' \in \pi^{-1}(p)$. We want to see if ξ defines in a natural way a germ of vector field at p'.

Remark 1.3. Let ω be a germ of differential 1-form. The standard pull-back of 1-forms by a morphism allows us to define $\pi^*\omega$ in a very natural way as a germ of differential 1-form at p'. The case of a germ of vector field is slightly more complicated.

The ring of germs of functions $\mathcal{O}'_{M,p'}$ is an extension of $\mathcal{O}_{M,p}$ through the blow-up morphism. More precisely, we can choose local coordinates $x_1, x_2, \ldots, x_n$ around $p \in M$ such that

1. The center Y of the blowing-up is locally given at p by

$$Y = \{x_1 = x_2 = \cdots = x_m = 0\},$$

where m is the codimension of Y in M.
2. There are local coordinates $x'_1, x'_2, \ldots, x'_n$ at $p' \in M'$ such that

$$x'_j = x_j / x_m, \quad j = 1, 2, \ldots, m-1.$$
$$x'_j = x_j, \quad j = m, m+1, \ldots, n.$$

(The equalities have to be interpreted locally at p' by identifying x_j with $x_j \circ \pi$).

Without doing the complete details, a necessary a sufficient condition to extend ξ to a derivation

$$\xi : \mathcal{O}_{M',p'} \to \mathcal{O}_{M',p'},$$

is that $\xi(x'_j) \in \mathcal{O}_{M',p'}$ for all $j = 1, 2, \ldots, n$. Of course, it is enough to verify that $\xi(x'_j) \in \mathcal{O}_{M',p'}$ for $1 \leq j \leq m-1$. Let us write

$$\xi = \sum_{i=1}^{n} a_i(x_1, x_2, \ldots, x_n) \frac{\partial}{\partial x_i}.$$

We have

$$\xi(x'_j) = \xi(x_j/x_m) = \frac{x_m a_j - x_j a_m}{x_m^2} = \frac{1}{x_m}\left(a_j - x'_j a_m\right).$$

That is, the condition we look for is: x'^2_m divides $x_m a_j - x_j a_m$ in the ring $\mathcal{O}_{M',p'}$, for all $1 \leq j \leq m-1$.

Proposition 1.4. *The following conditions are equivalent*

1. ξ *extends to a derivation* $\xi : \mathcal{O}_{M',p'} \to \mathcal{O}_{m',p'}$.
2. x'^2_m *divides* $x_m a_j - x_j a_m$ *in the ring* $\mathcal{O}_{M',p'}$, *for all* $1 \leq j \leq m-1$.
3. $\xi(x_i)$ *belongs to the ideal* I *of* $\mathcal{O}_{M,p}$ *generated by* $x_1, x_2, \ldots, x_m$ *(this is the ideal defining* $Y \subset M$*), for any* $i = 1, 2, \ldots, m$.

Proof. Obviously 3 implies 2. Conversely, the condition that x'^2_m divides $x_m a_j - x_j a_m$ in the ring $\mathcal{O}_{M',p'}$ is equivalent to say that $x_m a_j - x_j a_m$ is in $I^2\mathcal{O}_{M,p}$. Assume that $a_{j_0} \notin I$ for some $1 \leq j_0 \leq m-1$. Then

$$f = \frac{\partial(x_m a_{j_0} - x_{j_0} a_m)}{\partial x_m} = a_{j_0} + x_m \frac{\partial a_{j_0}}{\partial x_m} + x_{j_0} \frac{\partial a_m}{\partial x_m}$$

is not in I, contradiction, since $x_m a_j - x_j a_m$ is in $I^2\mathcal{O}_{M,p}$. If $a_m \notin I$, we do the same argument by taking the partial derivative with respect to x_j, for any $1 \leq j \leq m-1$. □

The third condition in the proposition means that Y is invariant for ξ. To be precise, we have the following definition:

Definition 1.5. Let $I \subset \mathcal{O}_{M,p}$ be a prime ideal, defining a germ of subspace $(Y, p) \subset (M, p)$. We say that (Y, p) is *invariant* for ξ if and only if $\xi(I) \subset I$.

Remark 1.6. The point $\{p\}$ is invariant for ξ if and only if ξ is singular at p (we also say that ξ has an *equilibrium point* at p). Consider the curve

$$Y = \{x_1 = x_2 = \cdots = x_{n-1} = 0\},$$

to say that Y is invariant means that the vector field is "vertical" along the curve, that is $a_i(0, 0, \ldots, 0, x_n) = 0$ for $i = 1, 2, \ldots, n-1$; in other words, the vector field is tangent to the curve at the points of Y and hence the trajectories of the integral curves of ξ starting at points in Y are contained in Y (this explains the word "invariant").

Foliations by lines. A foliation by lines $\mathcal{L}$ over M corresponds to the fact of considering locally a vector field "without velocity". The leaves will be the trajectories of the vector field, that is the images of the integral curves, where we do not consider the parametrization by the time.

To be precise, an *atlas* for a foliation $\mathcal{L}$ is a collection (U_i, ξ_i) of *foliated charts* such that the ξ_i are vector fields defined over the open sets U_i and

$$\xi|_{U_i \cap U_j} = h_{ij}\xi_j|_{U_i \cap U_j},$$

where the h_{ij} are invertible functions defined over $U_i \cap U_j$. As usual, we define the foliation by identifying it with a maximal atlas. The foliation is *reduced* if for any (nonsingular) point $p \in M$ we can write

$$\xi_i = \sum_{i=1}^{n} a_i(x) \frac{\partial}{\partial x_i}$$

where the coefficients $a_i(x) \in \mathcal{O}_{M,p}$ are without common factor. It is possible to pass from a foliation to a reduced one in a unique way just by taking the greatest common divisor of the coefficients. The singular locus Sing$\mathcal{L}$ of $\mathcal{L}$ is locally given by the singular locus of the vector fields ξ_i and it is of codimension greater or equal than two in the case of a reduced foliation.

We can also define *meromorphic foliations* as given by atlases of the form $\{(U_i, g_i^{-1}\xi_i)\}$ where $g_i \in \mathcal{O}_M(U_i)$ and the compatibility of the charts is defined as

$$g_j|_{U_i\cap U_j}\xi|_{U_i\cap U_j} = h_{ij}g_i|_{U_i\cap U_j}\xi_j|_{U_i\cap U_j}.$$

As before a meromorphic foliation gives in a unique way a reduced foliation.

Algebraic foliations. In the algebraic case we can define a meromorphic foliation in a particular way which is very convenient for the work in a bi-rational context. Let K be the field of rational functions of M, that we suppose to be an algebraic variety over a field k of characteristic zero (recall that we typically have $k = \mathbb{R}$ or $k = \mathbb{C}$). The K-vector field of derivations $\mathrm{Der}_k K$ has K-dimension $n = \dim M$. A *rational foliation by lines* is just a one dimensional K-vector subspace

$$\mathcal{L} \subset \mathrm{Der}_k K.$$

If induces a reduced foliation as follows. Let $p \in M$ be a nonsingular point. The regular local ring $\mathcal{O}_{M,p}$ has a regular system of parameters $x_1, x_2, \ldots, x_n$ (minimal set of generators of the maximal ideal) and

$$\mathrm{Der}_k\mathcal{O}_{M,p} = \sum_{i=1}^{n} \mathcal{O}_{M,p}\frac{\partial}{\partial x_i}.$$

Moreover, each germ of vector field $\xi \in \mathrm{Der}_k\mathcal{O}_{M,p}$ extends in a unique way to a derivation $\xi : K \to K$. Now $\mathcal{L} \cap \mathrm{Der}_k\mathcal{O}_{M,p}$ is a free $\mathcal{O}_{M,p}$-module of rank one generated by a germ of vector field without common factors in its coefficients. In this way we obtain a reduced foliation on M.

Blowing up foliations. We have seen that a vector field can only be blown up if the center of the blowing-up is invariant. Otherwise, we obtain a meromorphic vector field. This is not an obstruction for the blowing-up of a foliation. Hence any foliation can be transformed under a blowing-up with any center.

Dicritical vector fields. Let ξ be a germ of vector field in $p \in M$ and suppose that

$$\xi = \sum_{i=1}^{n} a_i(x_1, x_2, \ldots, x_n)\frac{\partial}{\partial x_i}$$

in local coordinates $x_1, x_2, \ldots, x_n$. Let us consider the blowing-up $\pi : M_1 \to M$ of M with center p. Assume that p is an equilibrium point of ξ and hence we have a transform $\tilde{\xi}$ of ξ by π. Let us denote by $E = \pi^{-1}(p)$ the exceptional divisor of the blowing-up π.

At each point $p_1 \in E$ we have that $\tilde{\xi} = h\xi_1'$, where $h \in \mathcal{O}_{M_1,p_1}$ and ξ_1' has no common factors in its coefficients. We have the following properties

1. The exceptional divisor E is invariant for $\tilde{\xi}$. This is a consequence of the fact that p is an equilibrium point of ξ.
2. If ξ has no common factor in its coefficients, then $h = 0$ is contained in E. More precisely, we have that either h is a unit (that is $h = 0$ is empty) or $\{h = 0\} = E$.

Let us look in a more precise way this situation. Consider the example of the radial vector field

$$R = \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i}.$$

Take a point p_1 with local coordinates $\mathbf{x}'$ such that $x_1' = x_1$ and $x_i' = x_i/x_1$ for $i \geq 2$. In this case $E = \{x_1' = 0\}$ and

$$\tilde{R} = x_1' \frac{\partial}{\partial x_1'}; \quad R_1' = \frac{\partial}{\partial x_1'}.$$

Let us note that E is not invariant for R_1'.

Definition 1.7. In the above situation we say that ξ is *dicritical* at p or that π is a *dicritical blowing-up* for ξ if and only if E is not invariant for ξ_1'.

This definition works for the case of a foliation, just by considering the reduced foliation after blowing up.

Let us give a characterization of the dicritical vector fields at p. Let r be the *order* of ξ at p, that is the minimum of the orders of the zero p of each coefficient a_i. We can decompose each coefficient $a_i(x)$ as a sum of homogeneous polynomials

$$a_i(\mathbf{x}) = A_{i,r}(x_1, x_2, \ldots, x_n) + A_{i,r+1}(x_1, x_2, \ldots, x_n) + \cdots.$$

Now, the germ of vector field ξ is dicritical at $p \in M$ if and only if the vectors $(A_{1,r}, A_{2,r}, \ldots, A_{n,r})$ and $(x_1, x_2, \ldots, x_n)$ are proportional, that is

$$x_i A_{r,j} = x_j A_{r,i}; \quad \text{for all } i, j.$$

Let us remark that being dicritical is a very particular situation. A still unsolved problem is to show that under any infinite sequence of blowings-up centered at points the resulting foliation is dicritical only finitely many times. This is true in dimension two and three, but it is not known for higher dimensions.

Invariant curves. Let $\xi = \sum_{i=1}^{n} a_i(x)\partial/\partial x_i$ be a germ of vector field at $p \in M$. A germ of analytic parameterized curve at p is just a morphism $\gamma : t \mapsto \gamma(t)$, where $\gamma(0) = 0$. The curve γ is called an *integral curve* of ξ if and only if

$$\gamma'(t) = \xi(\gamma(t))$$

for all t, where $\gamma'(t)$ means the tangent vector of γ at t. We know that there is always a unique integral curve (in the analytic context) of ξ at p. In the case that $p \in M$ is an equilibrium point, the integral curve at p is just the constant curve $t \mapsto p$.

We can also consider the definition of invariant subvariety given in a previous section. Take a germ of curve $(Y, p) \subset (M, p)$ at p, defined by the ideal $I \subset \mathcal{O}_{M,p}$. Recall that (Y, p) is invariant for ξ if and only if $\xi(I) \subset I$. It is possible to show that this is equivalent to say that (Y, p) is union of leaves, that is of images of integral curves.

In a more algebraic frame, assume that we have a Puiseux parametrization

$$x_i = \phi_i(t); \quad i = 1, 2, \ldots, n$$

of the curve (Y, p). The necessary and sufficient condition to assure that (Y, p) is invariant for ξ is that

$$a_i(\phi(t))\phi'_j(t) = a_j(\phi(t))\phi'_i(t); \quad \text{for all } i, j.$$

This condition means that $\xi(q)$ is in the tangent space of Y at each point q of Y near p.

Formal invariant curves. A formal curve $(\hat{Y}, p)$ at $p \in M$ is by definition the kernel $\hat{I} \subset \widehat{\mathcal{O}}_{M,p}$ of a morphism of complete local rings

$$\hat{\phi} : \widehat{\mathcal{O}}_{M,p} = k[[x_1, x_2, \ldots, x_n]] \to k[[t]].$$

Here we can interpret $\hat{\phi}$ as a Puiseux parametrization of $(\hat{Y}, p)$. The derivation ξ extends to a derivation $\xi : \widehat{\mathcal{O}}_{M,p} \to \widehat{\mathcal{O}}_{M,p}$. As for the convergent case we have

Proposition 1.8. *In the above situation the following properties are equivalent*

1. $\xi(\hat{I}) \subset \hat{I}$.
2. $a_i(\hat{\phi}(t))\hat{\phi}'_j(t) = a_j(\hat{\phi}(t))\hat{\phi}'_i(t)$; *for all* i, j.

If we have the equivalent properties of the above proposition, we say that $(\hat{Y}, p)$ is a *formal invariant curve* for ξ.

We shall see that there are formal invariant curves that are not convergent ones. This is one of the difficulties when doing reduction of singularities of vector fields, since the invariant objects are not necessarily convergent ones.

Definition 1.9. The formal curve $(\hat{Y}, p)$ is *non-singular* if and only if there is a Puiseux parametrization $\hat{\phi}(t)$ such that one of the $\hat{\phi}_i(t)$ has order 1.

This definition is equivalent to say that in formal coordinates, we have that $\hat{Y} = \{\hat{x}_2 = \hat{x}_3 = \cdots = \hat{x}_n = 0\}$. Moreover, if the curve is convergent, the rectification (in the analytic frame) may be done with convergent coordinates.

Behavior under blowing-up. Let $(\hat{Y}, p)$ be a formal curve at $p \in M$. Of course, a particular case is the case when $(\hat{Y}, p)$ is convergent. Consider the blowing-up

$$\pi : M_1 \to M$$

with center p. Up to do a linear change in the coordinates $x_1, x_2, \ldots, x_n$, we can assume that $\hat{Y}$ has a parametrization $\hat{\phi}(t)$ where $\hat{\phi}_1(t)$ has order d and $\hat{\phi}_i(t)$ has order $> d$ for all $i = 2, 3, \ldots, n$. Now, consider the point p_1 in the exceptional divisor E of π corresponding to the line

$$x_2 = x_3 = \cdots = x_n = 0.$$

At this point we have local coordinates $x'_1 = x_1, x'_i = x_i/x_1$, for $i = 2, 3, \ldots, n$. Now we have a Puiseux parametrization

$$x'_1 = \hat{\phi}_1(t),\ x'_i = \frac{\hat{\phi}_i(t)}{\hat{\phi}_1(t)}; \quad i = 2, 3, \ldots, n,$$

that defines a formal curve $\hat{Y}_1$ at p_1. We say that $(\hat{Y}_1, p_1)$ is the *strict transform* of $(\hat{Y}, p)$ by π and that p_1 is the *first tangent* or *first infinitesimal near point* of $(\hat{Y}, p)$.

Proposition 1.10. *Let ξ be a germ of vector field having an equilibrium point at $p \in M$ and let $(\hat{Y}, p)$ be a formal curve. Denote by $(\hat{Y}_1, p_1)$ the strict transform of (Y, p) by the blowing-up π of M with center p. We have*

1. *$(\hat{Y}_1, p_1)$ is convergent if and only if $(\hat{Y}, p)$ is convergent.*
2. *$(\hat{Y}_1, p_1)$ is invariant for ξ if and only if $(\hat{Y}, p)$ is invariant.*

Infinitely near points. Let $(\hat{Y}, p)$ be a formal curve in M. We can blow up successively $M = M_0$ to get an infinite sequence

$$\pi_{i+1} : M_{i+1} \to M_i$$

of blowings-up with centers $p_i \in M_i$, where $(\hat{Y}_{i+1}, p_{i+1})$ is the strict transform of $(\hat{Y}_i, p_i)$ and of course we put $(\hat{Y}_0, p_0) = (\hat{Y}, p)$. The points p_i are called the *iterated tangents of* $(\hat{Y}, p)$ or in another context the *infinitely near points* (although in [1] they consider only those points where the multiplicity does not drop).

Proposition 1.11 (Reduction of singularities of curves). *Given a formal curve $(\hat{Y}, p)$ in M, there is an index $N \geq 0$ such that $(\hat{Y}_i, p_i)$ is non singular for all $i \geq N$.*

Proof. Take coordinates $x_1, x_2, \ldots, x_n$ and a Puiseux expansion $\phi(t)$ such that $\phi_i(t) = t^{m_i} U_i(t)$, with $U_i(0) \neq 0$ and

$$m = m_1 < m_2 \leq m_3, \ldots, m_n$$

and moreover m does not divide m_2. Blowing up, we obtain $m'_1 = m_1$, $m'_i = m_i - m_1$, for $i \geq 2$ and the situation repeats if $m_1 < m'_2$. Note that $m_1 \neq m'_2$. After finitely many steps we get $m'_2 < m_1$ and we are done by induction on m. □

Take a (reduced) foliation by lines $\mathcal{L}$ in M locally generated at p by a vector field ξ. Let us denote by $\mathcal{L}_i$ the successive transformed foliations each one in M_i and let ξ_i be a local generator of $\mathcal{L}_i$ at p_i.

Proposition 1.12. *The following properties are equivalent:*

1. *$(\hat{Y}, p)$ is invariant for $\mathcal{L}$.*
2. *There is an index $N' \geq 0$ such that $p_i \in Sing\mathcal{L}_i$, for each $i \geq N'$.*

Proof. By reduction of singularities of the curve, we may assume that $\hat{Y}, p$ is given by $x_2 = x_3 = \cdots = x_n = 0$. Let us consider a logarithmic viewpoint relatively to $x_1 = 0$. To do this, we put $\eta_i = \xi_i$ if $x_1 = 0$ is invariant for ξ_i and $\eta_i = x_1\xi_i$ if $x_1 = 0$ is not invariant for ξ_i. We can write

$$\eta_i = b_{i1}(\mathbf{x}_i)x_{i1}\frac{\partial}{\partial x_{i1}} + \sum_{j=2}^{n} b_{ij}(\mathbf{x}_i)\frac{\partial}{\partial x_{ij}}$$

where the coefficients $b_{i1}, b_{i2}, \ldots, b_{in}$ have no common factor and the coordinates satisfy

$$x_{i1} = x_1; \; x_{ij} = x_j/x_1^i, \quad j \geq 2.$$

Let α_i be the minimum of the orders of $b_{i1}, b_{i2}, \ldots, b_{in}$ and put $\tau_i = \alpha_i$ if α_i is also the minimum of the orders of $b_{i2}, b_{i3}, \ldots, b_{in}$ and $\tau_i = \alpha_i + 1$ otherwise. We have

$$b_{i+1,1} = \frac{b_{i1}}{x_1^{\tau_i - 1}}; \quad b_{i+1,j} = \frac{b_{ij}}{x_1^{\tau_i}} - x_{i+1,j}b_{i+1,1}, \quad j \geq 2.$$

Let δ_{ij} be the order of $b_{ij}(x_1, 0, \ldots, 0) =$ and δ_i the minimum of the δ_{ij}, for $j \geq 2$. To say that (Y_i, p_i) is invariant is equivalent to say that $\delta_i = \infty$ and this implies that p_i is a singular point of ξ_i. Let us note that

$$\delta_{i+1} = \delta_i - \tau_i.$$

The only way to have a finite δ_i is that $\tau_i = 0$ for $i \geq N'$. But if $\tau_i = 0$ the point p_i is a nonsingular point of η_i and "a fortiori" of ξ_i. □

Elementary singularities. Let ξ be a germ of vector field at $p \in M$ and assume that p is an equilibrium point of ξ. That is $\xi(\mathcal{M}) \subset \mathcal{M}$, where $\mathcal{M}$ is the maximal ideal of the local ring $\mathcal{O}_{M,p}$ of M at p.

Let us recall that the quotient $\mathcal{M}/\mathcal{M}^2$ is a k-vector space (here k is the base field) of dimension n. More precisely, if $x_1, x_2, \ldots, x_n$ is a local system of coordinates at p, we have that

$$\overline{x}_j = x_j + \mathcal{M}^2; \; j = 1, 2, \ldots, n,$$

gives a k-basis of $\mathcal{M}/\mathcal{M}^2$. Now, the vector field ξ induces a k-linear map

$$L\xi : \mathcal{M}/\mathcal{M}^2 \to \mathcal{M}/\mathcal{M}^2$$

given by $f + \mathcal{M}^2 \mapsto \xi(f) + \mathcal{M}^2$. This map is called *the linear part* of ξ.

Definition 1.13. We say that p is an *elementary singularity* of ξ if and only the linear part $L\xi$ is non-nilpotent.

The study of elementary singularities is not particularly easy. Anyway, they are stable under blowing-up and, for this reason, a good objective to the reduction of singularities is to reach elementary singularities after performing well chosen blowings-up. This objective has been obtained in dimension two by Seidenberg in 1968. In higher dimensions, the situation is much more complicated. In this notes we will give some ideas in dimension three.

Lemma 1.14. *Assume that ξ has a singularity at $p \in M$ and let $\pi : M_1 \to M$ be the blowing-up with center p. Let $p_1 \in \pi^{-1}(p)$ be a singular point for the transform ξ' of ξ by the blowing-up. Then p_1 corresponds to an eigenvector of the transposed linear part $(L\xi)^t : (\mathcal{M}/\mathcal{M}^2)^* \to (\mathcal{M}/\mathcal{M}^2)^*$.*

Proof. Up to do a linear change of coordinates, we can assume that p_1 has local coordinates given by $x'_1 = x_1$ and $x'_j = x_j/x_1$ for $j \geq 2$. This means that p_1 corresponds to the projective point corresponding to

$$v_1 : \mathcal{M}/\mathcal{M}^2 \to k; \quad \overline{x}_1 \mapsto 1, \overline{x}_j \mapsto 0, \; j \geq 2.$$

If v_1 is not an eigenvector of $(L\xi)^t$, there is a coordinate x_{j_0}, $j_0 \geq 2$, such that $v_1 \circ L\xi(\overline{x}_{j_0}) \neq 0$. This is equivalent to say that

$$\xi(x_{j_0}) = \alpha x_1 + l(x_2, x_3, \ldots, x_n) + h(\mathbf{x}),$$

where $h \in \mathcal{M}^2$ and $\alpha \neq 0$. Note that

$$\xi(x'_{j_0}) = \xi(x_{j_0}/x_1) = \frac{1}{x_1}\left\{\xi(x_{j_0}) - x'_{j_0}\xi(x_1)\right\}$$

and then $\xi(x'_{j_0}) = \alpha$ modulo $\mathcal{M}'$. $\square$

Proposition 1.15 (Stability under blowing-up). *Assume that ξ has an elementary singularity at $p \in M$ and let $\pi : M_1 \to M$ be the blowing-up with center p. Let $p_1 \in \pi^{-1}(p)$ be a singular point for the transform ξ' of ξ by the blowing-up. Then p_1 is an elementary singularity for ξ'.*

Proof. Up to do a linear change of coordinates, we can assume that p_1 has local coordinates given by $x'_1 = x_1$ and $x'_j = x_j/x_1$ for $j \geq 2$. In view of the proof of the above lemma, we have that

$$\xi(x_j) = l_j(x_2, x_3, \ldots, x_n) + h(\mathbf{x}),$$

where $h \in \mathcal{M}^2$. Let $\sum_{j=1}^{n} \mu_j \overline{x}_j$ be a non-null eigenvector with a non-null eigenvalue α for $L\xi$. This means that

$$L\xi\left(\sum_{j=1}^{n} \mu_j \overline{x}_j\right) = \alpha \sum_{j=1}^{n} \mu_j \overline{x}_j$$

for $\alpha \neq 0$. Assume first that $\mu_1 \neq 0$. We have $L\xi(\overline{x}_1) = \alpha\mu_1\overline{x}_1 + \sum_{j\geq 2} \lambda_j \overline{x}_j$, since $L\xi(x_j)$ does not depend on $\overline{x}_1$. Then

$$\xi(x_1') = \xi(x_1) = x_1'(\alpha\mu_1 + h'(\mathbf{x}')); \quad h' \in \mathcal{M}'.$$

This implies that $\overline{x}_1'$ is a non-null eigenvector with a non-null eigenvalue $\alpha\mu_1$ for $L\xi'$. Assume now that $\mu_1 = 0$. Up to do a linear change in the coordinates $x_2, x_3, \ldots, x_n$, we can assume that $L\xi(\overline{x}_n) = \alpha\overline{x}_n$ and moreover, $\xi(x_1')/x_1' \in \mathcal{M}'$. We have

$$\xi(x_n') = \xi(x_n/x_1) = \frac{1}{x_1}\left\{\xi(x_n) - x_n'\xi(x_1)\right\}$$

and then $\xi(x_n') = \alpha x_n'$ modulo $\mathcal{M}'^2$. □

Simple singularities in dimension two. In the case $n = 2$ we can obtain a supplementary condition under blowings-up.

Definition 1.16. Let $p \in M$ be an elementary singularity of ξ and assume that the ambient space M has dimension two. We say that p is a *simple singularity* for ξ if and only if $\lambda/\mu \notin \mathbb{Q}_{>0}$, where λ, μ are the eigenvalues of $L\xi$, with $\mu \neq 0$.

Proposition 1.17 (Stability of simple singularities). *Let p be a simple singularity for a vector field ξ in an ambient space M of dimension two. Consider the blowing-up $\pi : M_1 \to M$ centered at p. Then:*

1. *The blowing-up π is non-dicritical.*
2. *There are exactly two singular points p_1' and p_1'' for ξ in the exceptional divisor $E = \pi^{-1}(p)$. Moreover p_1' and p_1'' are simple singularities for ξ.*

Proof. Up to a linear change of coordinates, we can make diagonal the linear part $L\xi$ of ξ and hence

$$\xi = (\lambda x + \tilde{a}(x,y))\frac{\partial}{\partial x} + (\mu y + \tilde{b}(x,y))\frac{\partial}{\partial y}; \quad \tilde{a}, \tilde{b} \in \mathcal{M}^2.$$

Let us do a blowing-up in the first chart by putting $x' = x$, $y' = y/x$. Then

$$\xi = (\lambda + x'a')x'\frac{\partial}{\partial x'} + ((\mu - \lambda)y' + x'(b' - y'a'))\frac{\partial}{\partial x}$$

where $a' = a/x'^2, b' = b/x'^2$. Note that E is invariant and the origin p_1' is a simple singularity, since the linear part is triangular with eigenvalues $\lambda, \mu - \lambda$ and hence $\lambda/(\mu - \lambda) \notin \mathbb{Q}_{>0}$. No other point of E in the first chart is a singular point. Working by symmetry, we find that the origin p_1'' of the second chart is also a simple singularity with eigenvalues $\lambda - \mu, \mu$. □

We have a more general statement as follows:

Proposition 1.18. *Let p be an elementary singularity for a vector field ξ in an ambient space M of dimension two. Consider the blowing-up $\pi : M_1 \to M$ centered at p. Denote by λ, μ, with $\mu \neq 0$ the eigenvalues of the linear $L\xi$ of ξ. If $\lambda \neq \mu$ we have:*

1. *The blowing-up π is non-dicritical.*
2. *There are exactly two singular points p_1' and p_1'' for ξ in the exceptional divisor $E = \pi^{-1}(p)$. Moreover ξ has eigenvalues $\lambda, \mu - \lambda$ at p_1' and $\lambda - \mu, \mu$ at p_1''. In particular one of them p_1' or p_1'' is a simple singularity.*

If $\lambda = \mu$, we have

1. *If the linear part $L\xi$ is diagonal, then π is a dicritical blowing-up and the transformed foliation $\mathcal{L}'$ has no singular points at the exceptional divisor.*
2. *If the linear part $L\xi$ is not diagonal (Jordan block), then π is non dicritical and there is exactly one singular point p_1' for ξ in E. Moreover p_1' is a simple singularity with eigenvalues $\mu, 0$.*

Proof. The first part is exactly as in the previous proposition. For the second part, we con choose coordinates such that

$$\xi = (\mu x + \epsilon y + \tilde{a}(x, y))\frac{\partial}{\partial x} + (\mu y + \tilde{b}(x, y))\frac{\partial}{\partial y}; \quad \tilde{a}, \tilde{b} \in \mathcal{M}^2.$$

Let us do a blowing-up in the first chart by putting $x' = x, y' = y/x$. We have

$$\xi = (\mu + x'a')x'\frac{\partial}{\partial x'} + (-\epsilon y'^2 + x'(b' - y'a'))\frac{\partial}{\partial x}.$$

If $\epsilon = 0$, then $\xi = x'\xi'$, where ξ' is non-singular in this chart and transversal to E. If $\epsilon \neq 0$, we have a simple singularity at the origin p_1' of this chart with eigenvalues $\mu, 0$. Let us put $x'' = x/y$ and $y'' = y$ and let us look at the origin of the second chart. We have

$$\xi = (-\epsilon + y'(a'' - x'b''))\frac{\partial}{\partial x} + (\mu + y'b'')y'\frac{\partial}{\partial y'}.$$

If $\epsilon = 0$ we have $\xi = y''\xi''$, where ξ'' is non singular at the origin and if $\epsilon \neq 0$ the origin is nonsingular for ξ. □

This proposition has the following corollary that allows to reduce the elementary singularities to simple singularities in dimension two.

Corollary 1.19. *Let p be an elementary singularity for a vector field ξ in an ambient space M of dimension two. There is a finite sequence of blowings-up*

$$M = M_0 \overset{\pi_1}{\leftarrow} M_1 \overset{\pi_2}{\leftarrow} \cdots \overset{\pi_N}{\leftarrow} M_N$$

centered at points $p_i \in \pi^{-1}(p_{i-1})$, where $p_0 = p$ such that all the singularities of ξ in $(\pi_1 \circ \pi_2 \cdots \circ \pi_N)^{-1}(p)$ are simple singularities.

Proof. If we have not finished, define p_{i+1} to be the only non simple point over p_i under blowing-up. Let p_i/q_i be the quotient of the eigenvalues at p_i. Assume that $q_i > p_i$. The new quotient of eigenvalues is $(q_i - p_i)/p_i$. Thus the invariant $p_i + q_i$ decreases strictly. In this way we obtain that the two eigenvalues are equal and we end by doing an additional blowing-up. ☐

Formal invariant curves at simple singularities. In this section the ambient space M has dimension two. Consider a simple singularity $p \in M$ of a vector field ξ. The blowing-up properties described in Proposition 1.17 of the previous section are enough to detect what are the formal invariant curves of ξ at p. Let us do it.

We can start by choosing local coordinates (x, y) at p such that the linear part $L\xi$ has a diagonal form in the basis $\{\overline{x}, \overline{y}\}$ of $\mathcal{M}/\mathcal{M}^2$. let us do the blowing-up

$$\pi_1 : M_1 \to M$$

of M with center p and consider p_1', p_1'', the origin of the first and second chart respectively of the blowing-up, expressed in the coordinates x, y. Assume that (Y, p) is a formal invariant curve of ξ and let (Y_1, p_1) be its strict transform. We have

The tangent p_1 of Y is either p_1' or p_1''.

To see this, note that the exceptional divisor E_1^1 is invariant and hence p_1 must be a singular point of ξ, since there are at least two invariant curves through p_1: the exceptional divisor and Y_1. Thus $p_1 = p_1'$ or $p_1 = p_1''$.

Assume that p_1' is the tangent of (Y, p). We are going to prove that (Y, p) is unique with this property. Thus we deduce

"There are at most two formal invariant curves of ξ at p."

Now, at the point p_1' we have two distinct invariant curves: the exceptional divisor E_1^1 and Y_1. Let us do the blowing-up with center p_1'

$$\pi_2 : M_2 \to M_1.$$

By Proposition 1.17, the new exceptional divisor E_2^2 is invariant, as well as the strict transform E_2^1 of E_1^1 and the strict transform Y_2 of Y_1. Let p_2'' be the point corresponding to the tangent of E_1^1, that is

$$\{p_2''\} = E_2^2 \cap E_2^1.$$

It is a simple singularity for ξ. We also have another simple singularity $p_2' \in E_2^2$. We have two possibilities for the tangent p_2 of Y_1: either $p_2 = p_2''$ or $p_2 = p_2'$.

Let us note that the point p_2'' is a *corner*, in the sense that it is in the intersection of two components of $E_2 = (\pi_1 \circ \pi_2)^{-1}(p) = E_2^2 \cup E_2^1$. The point p_2' is called a *trace point* to indicate that there is only one component of E_2 through p_2'.

Lemma 1.20. *We have $p_2' \in Y_2$, that is $p_2 = p_2'$.*

Proof. Let us show that it is not possible that p_2 is the corner p_2''. Blowing up the point p_2'' to obtain

$$\tilde{\pi}_3 : \tilde{M}_3 \to M_2.$$

The new two (simple) singularities $\tilde{p}_3'$ and $\tilde{p}_3''$ that we obtain are corners. The situation repeats. If $p_2'' \in Y_2$, we will get that all the infinitely near points of (Y_1, p_1) are corners. This is not possible. In fact, after doing finitely many blowings-up, we obtain a nonsingular curve (Z, q) passing trough a corner $E \cup F$ of a normal crossings divisor. If (Z, q) is transversal to E and F, we are done, in the next blowing-up we have a trace point. If (Z, q) is tangent to E with an order of tangency $\delta < \infty$, in the next blowing-up it has tangency order $\delta - 1$ and after finitely many transformations we obtain the transversal case. □

As a consequence of the lemma, we have a complete description o the infinitely near points $\{p_i\}_{i=0}^{\infty}$ of (Y, p). They are obtained as follows. Write $p_0 = p$, $p_1 = p_1'$. The point $p_i \in M_i$ is a trace point of the total exceptional divisor of M_i, given by

$$E_i = E_i^i \cup E_i^{i-1} \cup \cdots \cup E_i^1$$

and if we do the blowing-up

$$\pi_{i+1} : M_{i+1} \to M_i$$

with center the point p_i, then p_{i+1} is the only singularity of ξ in $\pi_{i+1}^{-1}(p_i) = E_{i+1}^{i+1}$ that is a trace point in E_{i+1}.

We have deduced that the formal invariant curve (Y, p) is necessarily given by the sequence of infinitely near points $\{p_i\}$ described above. This proves that (Y, p) is unique with the tangent p_1'. Just by the geometrical properties of this sequence of infinitely near points, we can deduce that (Y, p) is non singular. More precisely, we can find local coordinates (x_{i+1}, y_{i+1}) at p_{i+1} given by $x_{i+1} = x$ and

$$y_{i+1} = y_i/x - c_i; \quad \text{for } c_i \in k.$$

This implies that (Y, p) is necessarily the formal curve $y = \sum_{i=1}^{\infty} c_i x^i$. Indeed, this curve is invariant by Proposition 1.12.

Briot and Bouquet theorem. A natural question arises in the study of simple singularities in dimension two: may we find at least a convergent invariant curve among the two formal invariant curves of a simple singularity?

Let ξ be a germ of vector field at $p \in M$, where the ambient space M has dimension two and p is a simple singularity for ξ. Let Y_1, Y_2 be the two formal invariant curves of ξ. Choose local coordinates (x, y) at p such that

$$Y_1 = \{y - x\phi_1(x)\}; \quad Y_2 = \{x - y\phi_2(y)\}; \quad \phi_1(0) = 0 = \phi_2(0). \tag{1.1}$$

This is enough tho assure that the linear part $L\xi$ of ξ is diagonal. That is ξ is of the form

$$\xi = (\lambda x + A(x, y))\frac{\partial}{\partial x} + (\mu y + B(x, y))\frac{\partial}{\partial y},$$

where A and B have order at least two at the origin. Moreover, since p is a simple singularity, we have that $\lambda \neq 0$ or $\mu \neq 0$ and the quotient $\lambda/\mu \notin \mathbb{Q}_{>0}$.

Definition 1.21. In the above situation, we say that Y_1 is a Briot and Bouquet invariant curve for ξ, or equivalently, a *strong* invariant curve for ξ if $\lambda \neq 0$. In the same way Y_2 is a Briot and Bouquet, or a strong, invariant curve of ξ if $\mu \neq 0$.

The above definition seems to be not very intrinsical. In fact it is, we leave the verification of this to the reader. Note also that we have always that either Y_1 or Y_2 are strong.

The next theorem shows that a strong invariant curve is convergent. This is not always true if the invariant curve is not strong.

Remark 1.22 (Euler's example). The vector field

$$\xi = x^2\frac{\partial}{\partial x} + (y - x^2)\frac{\partial}{\partial y}$$

has the invariant curve $y = \sum_{n=1}^{\infty} n!x^{n+1}$, which is not convergent.

Theorem 1.23 (Briot and Bouquet). *A strong invariant curve is convergent.*

Proof. Assume that Y_1 is strong. After doing a blowing-up and multiplying ξ by a unit, we can assume that ξ is written as

$$\xi = x\frac{\partial}{\partial x} + \{\alpha y + F(x, y)\}\frac{\partial}{\partial y},$$

where $F(0, y) = 0$ and $\alpha \notin \mathbb{Q}_{>0}$. Let us write Y_1 as $y = \sum_{n\geq 1} a_n x^n$. The condition that Y_1 is invariant means that

$$\sum_{n\geq 1} n a_n x^n = \alpha \sum_{n\geq 1} a_n x^n + F\left(x, \sum_{n\geq 1} a_n x^n\right) \tag{1.2}$$

Now, write $F(x, y) = \sum_{i,j} F_{ij} x^i y^j$. Let us denote by C_n the coefficient of x^n in $F(x, \sum_{n\geq 1} a_n x^n)$. There is a polynomial P_n of nonnegative integer coefficients such that

$$C_n = P_n(\{a_1, a_2, \ldots, a_{n-1}; \{F_{ij}; i + j \leq n\}\}).$$

In this way, we have that

$$a_n = \frac{1}{n-\alpha} C_n$$

and this gives the recursive dependence of a_n from the precedent ones.

Now, we are going to apply the method of the *bounding series* . There is a positive rational number τ such that $0 < 1/(n-\alpha) < \tau$ for all n (recall that $\alpha \notin \mathbb{Q}_{>0}$). Now, consider the series with real coefficients $\Phi(t) = \sum_{\geq n} c_n t^n$ which is a solution $T = \Phi$ for the implicit problem

$$\tau T - \sum_{i,j} |F_{ij}| t^i T^j = 0.$$

This implies that Φ is convergent. Moreover, the coefficients are nonnegative and given by

$$c_n = \frac{1}{\tau} P_n(\{|c_1|, |c_2|, \ldots, |c_{n-1}|; \{|F_{ij}|; i + j \leq n\}\}).$$

This allows us to show inductively that $|a_n| \leq c_n$ and hence the series $\sum_{n\geq 1} a_n x^n$ is convergent. □

2 Two Dimensional Reduction of Singularities

Seidenberg's statement. The reduction of singularities of vector fields in dimension two has been proven by the first time in a complete way by Seidenberg in 1968. Anyway the statement is already more or less implicit in works of Poincaré, Bendixon and other authors 60 years before.

The original statement of Seidenberg is as follows. Consider an ambient space M of dimension two and let ξ be a germ of vector field at $p \in M$. Denote $\mathcal{L}$ the (germ of) reduced foliation given ξ. There is a finite sequence of blowings-up

$$M = M_0 \overset{\pi_1}{\leftarrow} M_1 \overset{\pi_2}{\leftarrow} \cdots \overset{\pi_N}{\leftarrow} M_N; \quad \pi = \pi_1 \circ \pi_2 \circ \cdots \circ \pi_N,$$

each one centered at a point $p_i \in \pi_i^{-1}(p_{i-1})$, with $p = p_0$, such that the following holds for any $q \in \pi^{-1}(p)$:

> Let $\mathcal{L}_N$ be the transform of $\mathcal{L}$ in M_N. The germ of $\mathcal{L}_N$ in q is generated by a germ of vector field $\tilde{\xi}$ at q that is either nonsingular or simple.

The original proof of Seidenberg uses Noether's formula on the multiplicity of the intersection of two plane curves after blowing up.

In this notes we give another proof that is close to the valuative structure of the infinitely near points of the foliation.

Introducing the exceptional divisor. Let us consider as in previous section an ambient space M of dimension two and a point $p \in M$. Let us do a finite sequence of blowings-up

$$\mathcal{S}^N: \quad M = M_0 \stackrel{\pi_1}{\leftarrow} M_1 \stackrel{\pi_2}{\leftarrow} \cdots \stackrel{\pi_N}{\leftarrow} M_N; \quad \sigma_i = \pi_1 \circ \pi_2 \circ \cdots \circ \pi_i,$$

each one centered at a point $p_i \in \sigma_i^{-1}(p)$, with $p_0 = p$. There is a *total exceptional divisor* $D_i = \sigma_i^{-1}(p)$ at each level M_i. Moreover D_i admits a decomposition into irreducible components

$$D_i = D_{i1} \cup D_{i2} \cup \cdots \cup D_{ii},$$

where $D_{ii} = \pi_i^{-1}(p_{i-1})$ and D_{ij} is the strict transform of $D_{i-1,j}$ by π_i. Each of the components D_{ij} is isomorphic to a projective line (the fact that the intrinsic structure of D_{ij} does not vary by a subsequent blowing-up is specific of dimension two, since a point has codimension one in D_{ij}). We also have that two $D_{ij}, D_{ij'}$ either do not intersect or they meet exactly at a point and they cross transversely at that point.

The dual graph. Sometimes we represent D_i and its irreducible components by the *dual graph* $\mathcal{D}(\mathcal{S}^i)$, weighted by the self-intersection of the components. It has a completely elementary inductive definition as follows.

The vertices, represented as black dots, correspond to the irreducible components D_{ij} of D_i. The weights ρ_{ij} that we associate to each D_{ij} are defined inductively by the following rules

1. $\rho_{ii} = -1$.
2. If $p_{i-1} \in D_{i-1,j}$, then $\rho_{ij} = \rho_{i-1,j} - 1$.
3. If $p_{i-1} \notin D_{i-1,j}$, then $\rho_{ij} = \rho_{i-1,j}$.

Finally, two vertices $D_{ij}, D_{ij'}$ are joined by an edge if and only if the components meet at one point, that is $D_{ij} \cap D_{ij'} \neq \emptyset$.

It is a good exercise to draw the dual graph in examples and to see what is the relationship between $\mathcal{G}(\mathcal{S}^{i-1})$ and $\mathcal{G}(\mathcal{S}^i)$.

The tree of infinitely near points. Instead of looking $\mathcal{S}^N$ thought the dual graph, we are going to do it by means of the *tree of infinitely near points and divisors.*

To be more coherent, let us integrate the exceptional divisor in the ambient space from the beginning. We say that a *normal crossings divisor E in M* is a subset $E \subset M$ that is a finite union

$$E = E^1 \cup E^2 \cup \cdots \cup E^k$$

of closed irreducible hyper-surfaces E^j (since M has dimension two, then each E^j is a curve) without singularities such that two E^i, E^j meet at most at one point and transversely.

Definition 2.1. A *logarithmic ambient space* is a pair (M, E), where $E \subset M$ is a normal crossings divisor. A germ $(M, E)_p$ of logarithmic ambient space at $p \in M$ is the germ of the pairs $(U, E \cap U)$, where $U \subset M$ are open sets with $p \in U$.

Consider a finite sequence of blowings-up $\mathcal{S}^N$ as in the previous section. Let (M, E) be a logarithmic ambient space. We obtain logarithmic ambient spaces (M_i, E_i) just by putting $E_0 = E$ and

$$E_i = \sigma_i^{-1}(\{p_{i-1}\} \cup E_{i-1}).$$

We also denote

$$\mathcal{S}^N[E] : \quad (M_0, E_0) \overset{\pi_1}{\leftarrow} (M_1, E_1) \overset{\pi_2}{\leftarrow} \cdots \overset{\pi_N}{\leftarrow} (M_N, E_N).$$

Let us define the *tree* $\mathcal{T}_{\mathcal{S}}^N[E]$ *of infinitely near points* associated to $\mathcal{S}^N[E]$. It is an oriented graph whose vertices are the points p_i, for $i = 0, 1, 2, \ldots, N-1$. Given two vertices p_i, p_j, with $i < j$, there is an arrow $p_j \to p_i$ if the following properties hold

1. $\pi_i \circ \pi_{i+1} \circ \cdots \circ \pi_{j-1}(p_j) = p_i$.
2. There is no k with $i < k < j$ such that $\pi_i \circ \pi_{i+1} \circ \cdots \circ \pi_{k-1}(p_k) = p_i$.

Note that if $j \to i$ we have a morphism

$$\sigma_{ji} : (M_j, E_j)_{p_j} \to (M_i, E_i)_{p_i}$$

that corresponds, up to isomorphism, to the blowing-up π_i restricted to the germ $(M_j, E_j)_{p_j}$.

The local-global argument. The discussion of the above section can also be done in the case of an infinite sequence of blowings-up

$$\mathcal{S}^\infty[E] : \quad (M_0, E_0) \overset{\pi_1}{\leftarrow} (M_1, E_1) \overset{\pi_2}{\leftarrow} \cdots ,$$

where $\mathcal{S}^N[E]$ denotes the corresponding truncation at the level N.

Definition 2.2. We say that $\mathcal{S}^\infty[E]$ is *discrete* if the tree of infinitely near points $\mathcal{T}_{\mathcal{S}}^\infty[E]$ has no points of infinite bifurcation.

Remark 2.3. The above condition is equivalent to say that the set

$$\{\pi_k \circ \pi_{k+1} \circ \cdots \circ \pi_j(p_{j+1}); j \geq k\}$$

is finite, for any $k \geq 0$.

Proposition 2.4 (Koenigs). *If $\mathcal{S}^\infty[E]$ is discrete, then the tree $\mathcal{T}_{\mathcal{S}}^\infty[E]$ has at least one infinite branch.*

By an infinite branch, or a *bamboo*, we mean a subgraph which is totally ordered. Let us give a proof of the above proposition. This argument is in fact very related with the local-global arguments in many procedures of reduction of singularities.

The proof is as follows. The tree has infinitely many vertices. Consider the root vertex v_0. Over v_0 we have an infinite tree but only finitely many vertices immediately over it. So at least one of them, say v_1 supports an infinite tree. We repeat the argument to detect v_2 and so on. This creates an infinite branch

$$v_0 \leftarrow v_1 \leftarrow v_2 \leftarrow \cdots$$

and the proof is ended.

Local global strategy. We shall use Proposition 2.4 as follows.

Let ξ be a germ of vector field at $p \in M$ and $\mathcal{L}$ the (germ of) reduced foliation given ξ. We construct a sequence of blowings-up as follows. Let

$$M = M_0 \stackrel{\pi_1}{\leftarrow} M_1$$

be the blowing-up with center p. If all the points in $\pi_1^{-1}(p)$ are simple or nonsingular for the transform $\mathcal{L}_1$ of $\mathcal{L}$, then we stop. Otherwise, choose a singular non simple point $p_1 \in \pi_1^{-1}(p)$ and let us do the blowing-up

$$M = M_1 \stackrel{\pi_2}{\leftarrow} M_2$$

with center p_1. Denote $\sigma_2 = \pi_1 \circ \pi_2$. If all the points in $\sigma_2^{-1}(p)$ are simple or nonsingular for the transform $\mathcal{L}_2$ of $\mathcal{L}$, then we stop. Otherwise, choose a singular non simple point $p_2 \in \sigma_2^{-1}(p)$. We continue in this way.

We have two possibilities, either we stop at a finite step and in this case the reduction of singularities of Seidenberg is proved, or we do not stop. Our task is to prove that this last situation does not hold. Hence, in order to find a contradiction, we assume that there is an infinite sequence of blowings-up

$$\mathcal{S}^\infty : M = M_0 \stackrel{\pi_1}{\leftarrow} M_1 \stackrel{\pi_2}{\leftarrow} M_2 \stackrel{\pi_3}{\leftarrow} \cdots$$

constructed as above.

Remark 2.5. The tree of infinitely near points $\mathcal{T}_\mathcal{S}^\infty$ is discrete. This is a consequence of the fact that the singular non simple points are isolated points (hence only finitely many in a compact).

Now, by application of Proposition 2.4, there is an infinite bamboo $\mathcal{B}_\mathcal{S}^\infty$ in the tree $\mathcal{T}_\mathcal{S}^\infty$. It corresponds to an infinite sequence of local blowings-up of points q_i.

Before continuing with the notations, we can assume that there is a normal crossings divisor $E \subset M$ given at the initial step, maybe $E = \emptyset$. Repeating all the above arguments, we obtain an infinite bamboo

$$\mathcal{B}_\mathcal{S}^\infty : (M_0, E_0)_p \stackrel{\pi_1}{\leftarrow} (M_1, E_1)_{q_1} \stackrel{\pi_2}{\leftarrow} (M_2, E_2)_{q_2} \stackrel{\pi_3}{\leftarrow} \cdots \tag{2.1}$$

(we keep the notation π_i, knowing that the indices have been altered).

Our task is to prove that $\mathcal{B}^{\infty}_{\mathcal{S}}$ cannot exist under the assumption that all the points q_i are singular non simple for $\mathcal{L}_i$.

Remark 2.6. In view of Corollary 1.19, we can assume that *the points q_i are singular non elementary.* If one of the q_i is elementary, then by Corollary 1.19 we obtain a simple singularity in a finite number of steps.

3 Types of Bamboos

We consider three types of bamboos, defined by its behavior at the infinity.

A (Combinatorial Type). There is an index N such that for any $i \geq N$ the point q_i is a corner point of the divisor E_i.
B (Formal Curve Type). There is an index N such that for any $i \geq N$ the point q_i is not a corner point of the divisor E_i.
C (Wild type). For any $N \geq 0$ there are i, j with $N \leq i, j$ such that q_i is a corner point of E_i and q_j is not a corner of E_j.

We shall see later the close relationship of these types with classical properties of valuations of the field of rational functions of the ambient space.

4 Combinatorial Situations

Assume in this section that $\mathcal{B}^{\infty}_{\mathcal{S}}[E]$ is a bamboo of combinatorial type. Up to cut it by a finite level, we may in fact assume that each q_i is a corner point of E_i.

Remark 4.1. Although it is not too important, the above reduction of the problem uses the fact that we consider a normal crossings divisor from the initial step.

Now, we need to show that $\mathcal{B}^{\infty}_{\mathcal{S}}[E]$ with the assumption that all the q_i are singular but non elementary. Our arguments are based on the *Newton Polyhedron* of a singularity and they are a strong particularization of Hironaka's weak game on characteristic polyhedra [24].

Let us select coordinates (x, y) at q_0 such that $E_0 = \{xy = 0\}$ locally at $p = q_0$. Then $\mathcal{L}_0$ is generated by a germ of vector field at q_0 that has the *logarithmic* form

$$\eta_0 = a(x,y)x\frac{\partial}{\partial x} + b(x,y)y\frac{\partial}{\partial x},$$

where a, b are without common factor. To see this it is enough to multiply a non logarithmic generator ξ_0 of $\mathcal{L}_0$ by x, respectively y or xy in the cases that $x = 0$ is not invariant, respectively $y = 0$ is not invariant or $xy = 0$ is not invariant.

Let us note that if $a(0,0) \neq 0$ or $b(0,0) \neq 0$, then p is elementary (non-nilpotent linear part). Hence we can assume that

$$a(0,0) = b(0,0) = 0.$$

The Newton polyhedron. Let us write

$$a(x,y) = \sum_{i,j} A_{ij} x^i y^j; \; b(x,y) = \sum_{i,j} B_{ij} x^i y^j.$$

We define the *Newton polyhedron* $\mathcal{N}(\eta_0; x, y) \subset \mathbb{R}^2_{\geq 0}$ to be the convex hull of $\mathrm{Supp}(\eta_0; x, y) + \mathbb{R}^2_{\geq 0}$, where

$$\mathrm{Supp}(\eta_0; x, y) = \{(i,j) \in \mathbb{Z}^2_{\geq 0};\; (A_{ij}, B_{ij}) \neq (0,0)\}.$$

Remark 4.2. Indeed, since we are working in dimension two, we could name "polygon" to the Newton Polyhedron. In fact, what we are doing in this combinatorial case has a direct generalization to any dimension. Thus in the general we deal with a polyhedron in an Euclidean space of the same dimension as the ambient space. On the other way, in the next sections we shall work with a "true" polygon, that we shall call *Newton-Puiseux polygon.*

The Newton Polyhedron has the following property:

If $\mathcal{N}(\eta; x, y)$ has only one vertex, then $\mathcal{L}$ is nonsingular or elementary at p.

This is true even if we consider $\eta' = f\eta$ instead of η. The proof is as follows. If $\mathcal{N}(\eta; x, y)$ has the only vertex (n,m), then $x^n y^m$ divides the coefficients of η' and $\mathcal{N}(x^{-n} y^{-m})\eta'; x, y$ has the only vertex $(0,0)$. This implies that one of the coefficients a or b is a unit.

The strategy is now as follows. We blow up to obtain the point q_1. At the point q_1 we obtain local coordinates (x_1, y_1) and we describe the relationship between $\mathcal{N}(\eta; x_1, y_1)$ and $\mathcal{N}(\eta; x, y)$. From this it will be evident that after finitely many steps we get a Newton Polyhedron with only one vertex.

We know that q_1 is a corner of E_1. This implies that q_1 is the origin of one of the two standard charts of the blowing-up expressed in the coordinates x, y. To be precise, we obtain local coordinates x_1, y_1 at q_1 by one of the following transformations:

$$\text{T1: } x_1 = x,\; y_1 = y/x. \tag{4.1}$$

$$\text{T2: } x_1 = x/y,\; y_1 = y. \tag{4.2}$$

Assume we have the transformation T1. Then η is given by

$$\eta = a_1(x_1, y_1) x_1 \frac{\partial}{\partial x_1} + b_1(x_1, y_1) y_1 \frac{\partial}{\partial y_1}$$

where

$$a_1(x_1, y_1) = a(x_1, x_1y_1). \tag{4.3}$$

$$b_1(x_1, y_1) = b(x_1, x_1y_1) - a(x_1, x_1y_1). \tag{4.4}$$

From this equations we deduce that for any $(i, j) \in \mathbb{Z}^2_{\geq 0}$ then

$$(i, j) \in \text{Supp}(\eta; x, y) \Leftrightarrow (i + j, j) \in \text{Supp}(\eta; x, y).$$

In other words, the new Newton polyhedron $\mathcal{N}(\eta; x_1, y_1)$ is the convex hull of $\tau_1(\mathcal{N}(\eta; x, y)) + \mathbb{R}^2_{\geq 0}$, where τ_1 is the affine transformation

$$\tau_1(i, j) = (i + 1, j).$$

In the case that we are in the second chart, that is, we have the transformation T2, we can do the same arguments, just changing the affine transformation τ_1 by τ_2 defined by

$$\tau_2(i, j) = (i, i + j).$$

Now, the problem of reduction of singularities in our combinatorial situation is reduced to give a positive answer to the next "game" (we call it "game" although there are no players in dimension two; it is the version in dimension two of Hironaka's game):

Combinatorial game of desingularization. Let $N_= \subset \mathbb{R}^2_{\geq 0}$ be a positively convex set (that is N_0 is a convex set such that $N_0 = N_0 + \mathbb{R}^2_{\geq 0}$). Assume that N_0 has only vertices with integer coordinates. Consider an infinite sequence $\epsilon_n; n = 1, 2, \ldots$ where $\epsilon_n \in \{1, 2\}$. Define inductively N_n to be the positive convex hull of $\tau_{\epsilon_n}(N_{n-1})$. Then, there is an index n_0 such that N_n has only one vertex for $n \geq n_0$.

Let us show how to give a positive answer to the game. It is obvious that N_n has no more vertices than N_{n-1}. So we can do an argument by induction on the number of vertices. Consider an arbitrary pair of vertices $v = (\alpha, \beta)$ and $v' = (\alpha', \beta')$ of N_0, where $\alpha' > \alpha$ and hence $\beta' < \beta$. Let I_0 be the sum

$$I_0 = (\alpha' - \alpha) + (\beta - \beta') \in \mathbb{Z}_{\geq 2}.$$

After one transformation, we have $I_1 < I_0$. We end by repeating the argument with the transformed vertices.

Following a formal curve. Assume in this section we assume that $\mathcal{B}^\infty_\mathcal{S}[E]$ is of a type of formal curve. Thus, up to cut the first part of the bamboo we may assume that each q_i is contained in a single irreducible component of E_i for all $i \geq 0$.

Let us interpret in terms of coordinates the above property. Let us choose local coordinates x, y at $p = q_0$ such that $E_0 = \{x = 0\}$ locally at q_0. The blowing-up π_1 is given in local coordinates x_1, y_1 at q_1 by one of the following equations

$$\text{T1-}c_1 : \quad x_1 = x; \ y_1 = y/x - c_1 \tag{4.5}$$

$$\text{T2} : \quad x_1 = x/y; \ y_1 = y. \tag{4.6}$$

If we have T1-c_1, the divisor E_1 at q_1 is $x_1 = 0$. If we have T2, then E_1 is locally given by $x_1 y_1 = 0$ at q_1. Hence we have only the case T1-c_1. We can repeat the argument at each step. In this way we obtain coefficients $c_1, c_2, \ldots$, and, by construction, all the points q_i are in the strict transform of the non-singular formal curve

$$\hat{\Gamma} = \left\{ y = \sum_{i=1}^{\infty} c_i x^i \right\}.$$

Now, we can use the arguments in the preceding section to end our proof. We can consider the new normal crossings divisor $\tilde{E} = E_0 \cup \hat{\Gamma}$ at $p = q_0$. It is maybe a formal non convergent divisor, but this is not important for our arguments. All the q_i are corners with respect to $\tilde{E}$ and we can apply the argument in the preceding section.

Wild bamboos. The most difficult case is the one corresponding to *wild bamboos*. It is also the situation where vector fields and two-variable functions or plane curves really start to be different from the viewpoint of reduction of singularities. The proof we present here is inspired in the usual method to obtain Puiseux expansions from the Newton-Puiseux polygon.

Puiseux packages. Let us denote by $e_i = e_i(E_i, q_i)$ the number of irreducible components of E_i through q_i. Up to forget the first step of the bamboo, we have that $e_i = 1$ or $e_1 = 2$; if $e_i = 2$ we have a corner point and if $e_i = 1$ we have a point that we call a *trace point*. Now we can cut the bamboo in finite sequences that we call *Puiseux's packages*. Take $i < j$. We say that

$$\mathcal{P}_{ij} : \quad (M_i, E_i)_{q_i} \overset{\pi_{i+1}}{\leftarrow} (M_{i+1}, E_{i+1})_{q_{i+1}} \overset{\pi_{i+2}}{\leftarrow} \cdots \overset{\pi_j}{\leftarrow} (M_j, E_j)_{q_j}$$

is a Puiseux package of the bamboo $\mathcal{B}_S^{\infty}[E]$ starting at the index i if and only if the next properties are satisfied:

1. $e_i = 1$ and $e_j = 1$.
2. For any k with $i < k < j$, then $e_k = 2$.

Let us note that given i with $e_i = 1$, there is a unique Puiseux package $\mathcal{P}_{ij}$.

Let us assume without loss of generality that $e_0 = 1$ (this is possible up to cut a finite first part of the bamboo). The whole bamboo may be decomposed in a unique way into Puiseux packages

$$\mathcal{P}_{0j_0}, \mathcal{P}_{i_1 j_1}, \mathcal{P}_{i_2 j_2}, \ldots$$

where $i_{s+1} = j_s$.

Definition 4.3. The Puiseux package $\mathcal{P}_{ij}$ is called to be *essential* if and only if $j \geq i + 2$. (This is equivalent to say that there is an index k with $i < k < j$ and hence $e_k = 2$.)

Remark 4.4. Since we are in the case of a wild bamboo, there are infinitely many essential Puiseux packages.

Our strategy will be the following one. We are going to attach a nonnegative integer number H_s to each Puiseux package $\mathcal{P}_{i_s j_s}$ with the property that $H_s \geq H_{s+1}$ and in the case that $\mathcal{P}_{i_s j_s}$ is essential, then $H_s > H_{s+1}$. Obviously this allow us to obtain the desired contradiction, since H_s cannot drop infinitely many times.

Newton-Puiseux Polygon. The invariant H_s will be obtained from the *Newton-Puiseux Polygon* that we introduce in this paragraph. First we need to choose local coordinates x_k, y_k at each point q_k.

We start with x, y at $p = q_0$ such that $x = 0$ is a local equation of the exceptional divisor. Now, looking at (4.5) and (4.6), we obtain x_k, y_k from x_{k-1}, y_{k-1} by one of the equations T1-λ or T2. Let us note the following remarks

1. If $e_k = 1$ then E_k is locally given by $x_k = 0$ at q_k.
2. If $e_k = 2$ then E_k is locally given by $x_k y_k = 0$ at q_k.
3. If we do T1-λ, with $\lambda \neq 0$, then $e_k = 1$.
4. If we do T2, then $e_k = 2$.
5. If we do T1-0, then $e_k = e_{k-1}$.

Now we are going to choose an non-null element of the foliation $\mathcal{L}$ of the form

$$\xi = a(x, y)x\frac{\partial}{\partial x} + b(x, y)\frac{\partial}{\partial y},$$

with the property that the coefficients a, b have no common factor except eventually powers of x. We are going to consider the (total) transform of ξ at the final-starting step of the Puiseux packages and from this we will describe and control the invariants H_s.

Let us write $\xi = \sum_{s=-1}^{\infty} y^s \eta_s$, where each η_s has the form

$$\eta_s = f_s(x)x\frac{\partial}{\partial x} + g_s(x)y\frac{\partial}{\partial y}.$$

In other words, we are saying that

$$a(x, y) = \sum_{s=0}^{\infty} f_s(x)y^s; \quad b(x, y) = \sum_{s=-1}^{\infty} g_s(x)y^{s+1}.$$

Now, let us define $\alpha_s = \operatorname{ord}(f_s(x), g_s(x))$ to be the minimum of the orders at the origin of $f_s(x)$ and $g_s(x)$ (in the case that f_s and g_s are identically zero we put $\alpha_s = +\infty$). The *Newton-Puiseux Polygon*

$$\mathcal{NP}(\xi; x, y) \subset \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq -1}$$

is by definition the convex hull of

$$\mathrm{NPSupp}(\xi; x, y) + \mathbb{R}^2_{\geq 0},$$

where $\mathrm{NPSupp}(\xi; x, y) = \{(\alpha_s, s); s = -1, 0, 1, 2, \ldots\}$.

The *main vertex* of $\mathcal{N}P(\xi; x, y)$ is the vertex with smallest abscissa. The *main height* $H(\xi; x, y)$ is the ordinate of the main vertex. That is, the main vertex has coordinates

$$(\alpha, H(\xi; x, y))$$

where any other vertex (α', h') of the Newton-Puiseux Polygon is such that $\alpha' > \alpha$ and "a fortiori" $h' < H(\xi; x, y)$.

Definition 4.5. Given a Puiseux's package $\mathcal{P}_{i_s j_s}$, we define $H_s = H(\xi; x_s, y_s)$.

Remark 4.6. Assume that $H(\xi; x, y) \leq 0$. Up to divide ξ by a power of x we can assume that $\alpha = 0$. Thus we have one of the vertices $(0, 0)$ or $(0, -1)$ in the Newton-Puiseux Polygon. If we have the vertex $(0, -1)$, the vector field ξ is non-singular. If we have the vertex $(0, 0)$, but not the vertex $(0, -1)$, the linear part of ξ is triangular with a non-null diagonal, hence it is non-nilpotent and we have an elementary singularity for ξ.

Thus, our contradiction hypothesis implies that $H_s \geq 1$ for all s.

An inessential Puiseux package. Let $\mathcal{P}_{ij}$ be a non essential Puiseux package. We know that $j = i + 1$. Denote by x, y the local coordinates at q_i and by x', y' the local coordinates at q_{i+1}. We have that

$$x' = x;\ y' = y/x - \lambda,$$

where λ may be zero or not. The control of the main height is given in this case by the following two lemmas

Lemma 4.7. *If $y^* = y + \lambda x$, then $H(\xi; x, y) = H(\xi; x, y^*)$.*

Proof. Left to the reader. □

Lemma 4.8. *If $\lambda = 0$, then $H(\xi; x', y') \leq H(\xi; x, y)$.*

Proof. Let us note that

$$x\frac{\partial}{\partial x} = x'\frac{\partial}{\partial x'} - y'\frac{\partial}{\partial y'};\quad y\frac{\partial}{\partial y} = y'\frac{\partial}{\partial y'}.$$

Let us write $\xi = \sum_{s=-1}^{\infty} y^s \eta_s$, where

$$\eta_s = f_s(x)x\frac{\partial}{\partial x} + g_s(x)y\frac{\partial}{\partial y}.$$

Then $\xi = \sum_{s=-1}^{\infty} y'^s \eta'_s$, where

$$\eta'_s = x'^s \left\{ f_s(x')x'\frac{\partial}{\partial x'} + (g_s(x') - f_s(x')y'\frac{\partial}{\partial y'} \right\}.$$

This implies that $\alpha'_s = \alpha_s + s$. If we apply this movement to the Newton Puiseux Polygon, we obtain that $H(\xi; x', y') \leq H(\xi; x, y)$. □

Now, the first lemma reduces the control of the main height to the case $\lambda = 0$ which is given by the second lemma.

Essential Puiseux packages Equations. Let us consider an essential Puiseux's package $\mathcal{P}_{i_s j_s}$. Denote by x, y the chosen local coordinates at q_{i_s} and by x', y' the local coordinates at q_{j_s}. Let us denote (x_i, y_i) the coordinates at q_i, for $i_s \leq i \leq j_s$, where $(x, y) = (x_{i_s}, y_{i_s})$ and $(x', y') = (x_{j_s}, y_{j_s})$. We know that

1. For $i = i_s + 1$ we have $x_i = x_{i-1}/y_{i-1}, y_i = y_{i-1}$. (Transformation T2.)
2. For any $i_s < i < j_s$ we have either the transformation T1-0 or the transformation T2, where

$$\text{T1-0}: x_i = x_{i-1}, y_i = y_{i-1}/x_{i-1}, \quad \text{and} \quad \text{T2}: x_i = x_{i-1}/y_{i-1}, y_i = y_{i-1}.$$

3. The last transformation is given by T1-λ, with $\lambda \neq 0$. That is, we have $x' = x_{j_s-1}, y' = y_{j_s-1}/x_{j_s-1} - \lambda$.

Lemma 4.9. *There is a unique pair of positive integer numbers p, d, with $d \geq 2$, without common factor and a scalar $\lambda \neq 0$ such that*

$$y' + \lambda = y^d/x^p.$$

Moreover, if we put $\Phi = y^d/x^p$, there are nonnegative integer numbers $\alpha, \beta, \gamma, \delta$ with $\alpha\delta - \beta\gamma = 1$ such that $x = x'^{\alpha}\Phi^{\beta};\ y = x'^{\gamma}\Phi^{\delta}$.

Proof. We proceed by inverse induction starting on j_s. We know that $y' + \lambda = y_{j_s-1}/x_{j_s-1}$. Now we respectively have

$$\frac{y_{j_s-1}}{x_{j_s-1}} = \frac{y_{j_s-2}}{x_{j_s-2}^2}, \quad \text{or} \quad \frac{y_{j_s-1}}{x_{j_s-1}} = \frac{y_{j_s-2}^2}{x_{j_s-2}}$$

if the transformation is respectively given by T1-0 or by T2. Assume that $y_{j_s-1}/x_{j_s-1} = y_i^{d_i}/x_i^{p_i}$, where d_i, p_i are without common factor. Then,

$$\frac{y_{j_s-1}}{x_{j_s-1}} = \frac{y_{i-1}^{d_{i-1}}}{x_{i-1}^{p_i-1}},$$

where we put $(d_{i-1}, p_{i-1}) = (d_i, p_i + d_i)$ in the case of a transformation T1-0 and $(d_{i-1}, p_{i-1}) = (d_i + p_i, p_i)$ in the case of a transformation T2. We see that (d_{i-1}, p_{i-1}) are without common factor and $d_{i-1} \geq d_i$, moreover $d_{i-1} > d_i$ in the case of a transformation T2. This proves the first part of the lemma.

For the second part, we proceed by induction assuming that

$$x = x_i^{\alpha_i} y_i^{\beta_i};\ x = x_i^{\gamma_i} y_i^{\delta_i}, \tag{4.7}$$

with $\alpha_i\delta_i - \beta_i\gamma_i = 1$, for $i_s \leq i < j_s - 1$. Then

$$x = x_{i+1}^{\alpha_{i+1}} y_{i+1}^{\beta_{i+1}},\ x = x_{i+1}^{\gamma_{i+1}} y_{i+1}^{\delta_{i+1}},$$

where

$$\begin{pmatrix} \alpha_{i+1} & \beta_{i+1} \\ \gamma_{i+1} & \delta_{i+1} \end{pmatrix} = \begin{pmatrix} \alpha_i + \beta_i & \beta_i \\ \gamma_i + \delta_i & \delta_i \end{pmatrix},$$

respectively

$$\begin{pmatrix} \alpha_{i+1} & \beta_{i+1} \\ \gamma_{i+1} & \delta_{i+1} \end{pmatrix} = \begin{pmatrix} \alpha_i & \alpha_i + \beta_i \\ \gamma_i & \gamma_i + \delta_i \end{pmatrix},$$

if we have respectively the transformation T1-0 or T2. The obtain that the expression (4.7) holds for $i = j_s - 1$. Now we get

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \alpha_{j_s-1} + \beta_{j_s-1} & \beta_{j_s-1} \\ \gamma_{j_s-1} + \delta_{j_s-1} & \delta_{j_s-1} \end{pmatrix}.$$

This ends the proof. □

Remark 4.10. We have that $\Phi = y^d/x^p = x'^{d\gamma - p\alpha}\Phi^{d\delta - p\beta}$. We deduce that $d\delta - p\beta = 1$ and $d\gamma - p\alpha = 0$. In particular $\gamma = p, \alpha = d$.

The critical segment. In this section we describe the effect of an essential Puiseux package on the Newton Puiseux Polygon.

Let us recall that $\xi = \sum_{s=-1}^{\infty} y^s \eta_s$, where each η_s has the form

$$\eta_s = f_s(x)x\frac{\partial}{\partial x} + g_s(x)y\frac{\partial}{\partial y}.$$

and the Newton-Puiseux Polygon $\mathcal{N}(\xi; x, y)$ is the positive convex hull of the set of (α_s, s), where α_s is the minimum of the orders of f_s, g_s, for each level $s \geq -1$. We are going to look at the contact of the Newton-Puiseux Polygon with the lines of slope $-d/p$.

Remark 4.11. The slope $-d/p$ is obtained with the following valuative consideration. If we assume that the "value" of y^d/x^p is zero, then the value of y should be p/d times the value of x. Thus $-d/p$ is the anti-slope corresponding to p/d.

For each level s, let us denote $\varrho_s = \alpha_s + s(p/d)$. We denote $\varrho(\xi; x, y)$ the minimum of the ϱ_s and the *critical segment* $\mathcal{C}(\xi; x, y)$ is the set of the levels s such that $\varrho_s = \delta(\xi; x, y)$. The *critical height* $\chi(\xi; x, y)$ is the maximum of the indices s such that the level s is in the critical segment. A trivial observation is that

$$\chi(\xi; x, y) \leq H(\xi; x, y).$$

Now, let us denote by $\alpha(\xi; x, y)$ the minimum of the α_s for $s \geq -1$. It is the minimum abscissa in the Newton-Puiseux Polygon. We can now cut the vector field in two parts, the initial part corresponding to the critical segment and the rest. To be precise, put

$$(f_s, g_s) = (\rho_s, \mu_s)x^{\varrho - s(p/d)} + (\tilde{f}_s, \tilde{g}_s),$$

where $\varrho = \varrho(\xi; x, y)$ and $(\tilde{f}_s, \tilde{g}_s)$ has order strictly bigger than $\varrho - s(p/d)$. Then, we can write

$$\xi = \mathrm{In}(\xi; x, y) + \tilde{\xi},$$

where

$$\mathrm{In}(\xi; x, y) = \sum_{s=-1}^{\chi(\xi;x,y)} y^s x^{\varrho - s(p/d)} \left\{ \rho_s x \frac{\partial}{\partial x} + \mu_s y \frac{\partial}{\partial y} \right\}.$$

Note that $\varrho(\tilde{\xi}; x, y) > \varrho$, by construction.

Remark 4.12. Note also that if $(\rho_s, \mu_s) \neq (0, 0)$, then $\varrho - s(p/d)$ is an integer number. Moreover, in the case $s = \chi$, where $\chi = \chi(\xi; x, y) \in \mathbb{Z}$, we know that $(\rho_\chi, \mu_s\chi) \neq (0, 0)$. Then $\tilde{\alpha} = \varrho - \chi(p/d) \in \mathbb{Z}$ and we can write

$$\mathrm{In}(\xi; x, y) = \sum_{s=-1}^{\chi} y^s x^{\varrho - s(p/d)} \left\{ \rho_s x \frac{\partial}{\partial x} + \mu_s y \frac{\partial}{\partial y} \right\} = \tag{4.8}$$

$$= x^{\tilde{\alpha}} \sum_{s=-1}^{\chi} y^s x^{(\chi - s)(p/d)} \left\{ \rho_s x \frac{\partial}{\partial x} + \mu_s y \frac{\partial}{\partial y} \right\}. \tag{4.9}$$

Moreover, if we put $dt = (\chi - s)$, we have

$$\mathrm{In}(\xi; x, y) = x^{\tilde{\alpha}} \sum_{t=0}^{\Delta} y^{\chi - dt} x^{pt} \left\{ \tilde{\rho}_t x \frac{\partial}{\partial x} + \tilde{\mu}_t y \frac{\partial}{\partial y} \right\} = \tag{4.10}$$

$$= x^{\tilde{\alpha}} y^{\chi} (1/\Phi)^{\Delta} \sum_{t=0}^{\Delta} \Phi^{\Delta - t} \left\{ \tilde{\rho}_t x \frac{\partial}{\partial x} + \tilde{\mu}_t y \frac{\partial}{\partial y} \right\} \tag{4.11}$$

where Δ is the integer part of $(\chi + 1)/d$ and $\tilde{\rho}_t = \rho_{\chi - dt}$, $\tilde{\mu}_t = \mu_{\chi - dt}$.

Control of the main height. Let us give here a control of the evolution of the main height under an essential Puiseux package. We do it by parts, first we consider $\tilde{\xi}$ and second we do a precise computation for $\mathrm{In}(\xi; x, y)$.

Recall that our equations are $x = x'^{d}\Phi^{\beta}$, $y = x'^{p}\Phi^{\delta}$, where

$$\Phi = y^{d}/x^{p} = y' + \lambda$$

and $d\delta - p\beta = 1$. These equations imply that

$$x\frac{\partial}{\partial x} = \delta x'\frac{\partial}{\partial x'} - p(y' + \lambda)\frac{\partial}{\partial y'} \tag{4.12}$$

$$y\frac{\partial}{\partial y} = -\beta x'\frac{\partial}{\partial x'} + d(y' + \lambda)\frac{\partial}{\partial y'}. \tag{4.13}$$

Lemma 4.13. *$\alpha(\tilde{\xi}; x', y') > d\varrho$.*

Proof. If is enough to do the proof for $\tilde{\xi}$ of the form

$$\tilde{\xi} = x^{a}y^{b}\left\{\rho x\frac{\partial}{\partial x} + \mu y\frac{\partial}{\partial y}\right\}$$

where $a + b(p/d) > \varrho$, since $\tilde{\xi}$ is a combination of that kind of monomial vector fields and the corresponding α may only increase. We have

$$\tilde{\xi} = x'^{ad+bp}\Phi^{a\beta+b\delta}\left\{(\rho\delta - \mu\beta)x'\frac{\partial}{\partial x'} + (\mu d - \rho p)(y' + \lambda)\frac{\partial}{\partial y'}\right\}$$

and hence $\alpha(\tilde{\xi}; x', y') = ad + bp > d\varrho$. □

Now, let us write $\mathrm{In}(\xi; x, y)$ in the coordinates x', y'. Recalling the expression (4.11), we have

$$\mathrm{In}(\xi; x, y) = x'^{\tilde{\alpha}d+\chi p}\Phi^{\tilde{\alpha}\beta+\chi\delta+\Delta}\sum_{t=0}^{\Delta}\Phi^{\Delta-t}\left\{\rho'_t x'\frac{\partial}{\partial x'} + \mu'_t\Phi\frac{\partial}{\partial y'}\right\} \tag{4.14}$$

where $\rho'_t = \tilde{\rho}_t\delta - \tilde{\mu}_t\beta$ and $\mu'_t = \tilde{\mu}_t d - \tilde{\rho}_t p$. Recall that $\Phi = y' + \lambda$. Note also that

$$\tilde{\alpha}d + \chi p = d(\varrho - \chi(p/d)) + \chi p = d\varrho.$$

From this we already deduce that

$$\alpha(\mathrm{In}(\xi; x, y); x', y') = d\varrho.$$

From (4.14) we can write $\xi^* = \Phi^{-(\tilde{\alpha}\beta+\chi\delta+\Delta)}x'^{-d\varrho}\mathrm{In}(\xi; x, y)$ as

$$\xi^* = \sum_{t=0}^{\Delta}\Phi^{\Delta-t}\left\{\rho'_t x'\frac{\partial}{\partial x'} + \mu'_t\Phi\frac{\partial}{\partial y'}\right\} = \sum_{r=-1}^{\Delta}y'^{r}\left\{\rho^*_r x'\frac{\partial}{\partial x'} + \mu^*_r y'\frac{\partial}{\partial y'}\right\}.$$

Remark 4.14. Note that $(\rho^*_\Delta, \mu^*_\Delta) = (\rho'_0, \mu'_0) \neq (0,0)$.

Lemma 4.15. *The main height $H(\xi; x', y')$ is the minimum of the indices r such that $(\rho^*_r, \mu^*_r) \neq 0$.*

Proof. The main vertex of the Newton-Puiseux Polygon appears in $\mathrm{In}(\xi; x, y)$, since $\alpha(\mathrm{In}(\xi; x, y); x', y') = d\varrho$ and $\alpha(\tilde{\xi}; x', y') > \varrho$. Moreover, ξ^* is obtained by multiplying $\mathrm{In}(\xi; x, y)$ by $x'^{-d\varrho}$ and by a unit depending only on y'. Thus the main height $H(\xi; x', y')$ is the main height of ξ^*. This ends the proof. □

In view of Remark 4.14, we have that $H(\xi; x', y') \leq \Delta$. Recalling that Δ is the integer part of $\chi + 1$, and $d \geq 2$ and recalling also that $\chi \leq H(\xi; x, y)$, with $1 \leq H(\xi; x, y)$, we have that

1. If $H(\xi; x, y) \geq 2$, then $H(\xi; x', y') < H(\xi; x, y)$.
2. If $H(\xi; x, y) = 1$ and $\chi < 1$ then $H(\xi; x', y') < 1$.
3. If $H(\xi; x, y) = 1 = \chi$ then $H(\xi; x', y') \leq 1$.

Let us examine the last case $H(\xi; x, y) = 1 = \chi$ assuming that $H(\xi; x', y') = 1$ in order to obtain a contradiction. Note that $d = 2$. Then

$$\xi^* = \Phi \left\{ \rho'_0 x' \frac{\partial}{\partial x'} + \mu'_0 \Phi \frac{\partial}{\partial y'} \right\} + \left\{ \rho'_1 x' \frac{\partial}{\partial x'} + \mu'_1 \Phi \frac{\partial}{\partial y'} \right\}.$$

Putting $\Phi = y' + \lambda$, this implies that

$$(\rho^*_1, \mu^*_1,) = (\rho'_0, \mu'_0) \tag{4.15}$$

$$(\rho^*_0, \mu^*_0,) = (\lambda\rho'_0 + \rho'_1, 2\lambda\mu'_0 + \mu'_1) \tag{4.16}$$

$$(\rho^*_{-1}, \mu^*_{-1},) = (0, \lambda(\lambda\mu'_0 + \mu'_1)) \tag{4.17}$$

Our contradiction hypothesis implies that

$$0 = \lambda\mu'_0 + \mu'_1 = 2\lambda\mu'_0 + \mu'_1 = \lambda\rho'_0 + \rho'_1.$$

In particular, we have that $\mu'_0 = 0 = \mu'_1$. Now, let us recall that $(\rho_{-1}, \mu_{-1}) = (0, \mu_{-1})$ and on the other hand $(\tilde{\rho}_1, \tilde{\mu}_1) = (\rho_{-1}, \mu_{-1})$. Thus

$$\rho'_1 = -\mu_{-1}\beta; \ \mu'_1 = \mu_{-1}d.$$

Hence $\mu'_1 = 0$ implies that $\mu_{-1} = 0$ and $\rho'_1 = 0$. Finally we obtain that all the coefficients $\rho'_1, \rho'_0, \mu'_1, \mu'_0$ are zero. This is a contradiction.

Conclusion. The case of a wild bamboo does not occur, since otherwise, the main height decreases strictly infinitely many times.

5 Higher Dimensions

In the previous section we have presented a very particular way of reducing the singularities of foliations by lines in an ambient space of dimension two. Actually our algorithm is very "valuative" in the sense that a valuation and a bamboo are very closed concepts. In the preprint [14] we perform a generalization to ambient dimension three of these ideas.

The situation in higher dimension is much more complicated than in dimension two, but doing a drastic simplification we can recognize the analogies as follows.

We place ourselves in the algebraic case and we fix a valuation $\nu : K^* \to \Gamma$ of the field of rational functions of our ambient space M. In particular the transcendence degree of K over $\mathbb{C}$ is three. In view of the classical theory of valuations of Zariski [44], to each birational projective morphism

$$\pi : M' \to M$$

we obtain a unique center $Y' \subset M'$ of the valuation and of course $\pi(Y') = Y$, where Y is the center of ν in M. This recalls the idea of bamboo. Hironaka has done the equivalent theory for the analytic case in his "voûte étoilée" [25]. In fact, applying Hironaka's Theorem of reduction of singularities (or in a weaker way the Zariski Local Uniformization) we can do blowings-up of the ambient space in such a way that the center $P \subset M$ of ν is a nonsingular point of M and we have in addition the following situation (we take implicitly, in order to simplify the case of a zero dimensional valuation, where all the centers are points):

> There is a regular system of parameters $\mathbf{z}$ of the local ring $\mathcal{O}_{M,P}$ that we can split as follows
>
> $$\mathbf{z} = (x_1, x_2, \ldots, x_r, y_{r+1}, y_{r+2}, \ldots, y_n),$$
>
> where the values $\nu(x_i)$ are $\mathbb{Z}$-independent and generates the $\mathbb{Q}$ vector space $\Gamma \otimes_{\mathbb{Z}} \mathbb{Q}$, for $i = 1, 2, \ldots, r$.

We call $\mathcal{A} = (\mathcal{O}_{M,P}; \mathbf{z} = (\mathbf{x}, \mathbf{y}))$ a *regular parameterized local model for* K, ν.

Now, if we take a center of blowing-up given by $\{z_i, z_j\}$ where $i \leq r$, we recover a new $\mathcal{A}'$ from $\mathcal{A}$ and in this way we can try to do a *local uniformization*: that is, try to reduce the singularities of a line foliation $\mathcal{L}$ at the centers of the valuation ν in the successive models.

We have essentially three situations to consider:

Combinatorial case. This is the case when $r = n$. All the variables have independent values. We write our vector fields in a logarithmic way

$$\xi = \sum_{i=1}^{n} a_i(\mathbf{x}) x_i \frac{\partial}{\partial x_i}.$$

Now, the Newton support of the coefficients $a_i(\mathbf{x})$ allows us to draw a Newton polyhedron $\mathcal{N} \subset \mathbb{R}^n_{\geq 0}$. If we choose the center of blowing-up $\{x_i, x_j\}$ and $\nu(x_i) <$

$\nu(x_j)$ (note that we always have that $\nu(x_i) \neq \nu(x_j)$), the new Newton Polyhedron $\mathcal{N}'$ is the positive convex hull of $\sigma_{ij}(\mathcal{N})$, where σ_{ij} is the linear map given by

$$\sigma_{ij}(\mathbf{u}) = \mathbf{u}'; \quad u'_s = \begin{cases} u_s & \text{if } s \neq i \\ u_i + u_j & \text{if } s = i \end{cases}$$

In this way we define a game (that recalls Hironaka's game in [37]) consisting in choosing a center of blowing-up each time till we obtain a Newton polyhedron with a single vertex. The reader may note that it is not difficult to find a winning strategy. This combinatorial case is the parallel situation to the bamboos of corner points in dimension two.

Note that if the Newton polyhedron has a single vertex $\mathbf{v}$, then the vector field may be written

$$\xi = \mathbf{x}^{\mathbf{v}} \sum_{i=1}^{n} \tilde{a}_i(\mathbf{x}) x_i \frac{\partial}{\partial x_i}.$$

where $\tilde{a}_i(0) \neq 0$ for some indices i. In particular we have an elementary singularity of foliation.

A single dependent variable. That is $\mathbf{z} = (x_1, x_2, \ldots, x_{n-1}, y)$. In this case we can define Puiseux packages as in the case of wild bamboos and we can draw a Newton-Puiseux polygon, following the same principles as in dimension two, just taking account of the independent variables $\mathbf{x}$ on one side and the dependent variable y on the other side. To be precise, we write the vector field in a logarithmic way with respect to the independent variables

$$\xi = \sum_{i=1}^{n-1} a_i(\mathbf{x}, y) x_i \frac{\partial}{\partial x_i} + b(\mathbf{x}, y) \frac{\partial}{\partial y}$$

and we split it as follows

$$\xi = \sum_{\beta=-1}^{\infty} y^{\beta} \left(\sum_{i=1}^{n-1} a_{i,\beta}(\mathbf{x}) x_i \frac{\partial}{\partial x_i} + b_{\beta}(\mathbf{x}) \frac{\partial}{\partial y} \right).$$

We put $\alpha_\beta = \min\{\nu(a_{1,\beta}(\mathbf{x})), \ldots, \nu(a_{n-1,\beta}(\mathbf{x})), \nu(b_\beta(\mathbf{x}))\}$ and we draw a Newton-Puiseux polygon $\mathcal{P} \subset \Gamma \times \mathbb{Z}_{\geq -1}$ by taking the positive convex hull of the set of points (α_β, β). By arguments as in the two-dimensional case, we obtain that $\mathcal{P}$ has a single vertex and (after a certain work) we get elementary singularities.

Two dependent variables. We proceed by induction trying to repeat the precedent case for y_1 and after to draw a Newton-Puiseux polygon for y_2. This is possible, but we obtain the (already know) formal obstructions to get elementary singularities and finally we are able to get the so called log-elementary singularities: the order of the logarithmic coefficients is less or equal than one.

For more details and how to globalize the procedure in dimension three, the reader is referred to [14, 32].

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Basics of O-minimality and Hardy Fields

Chris Miller

Abstract This paper consists of lecture notes on some fundamental results about the asymptotic analysis of unary functions definable in o-minimal expansions of the field of real numbers.

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Preface

This paper consists of lecture notes for the first of three modules of a graduate course, "Topics in O-minimality", held at the Fields Institute (Toronto) as part of the Thematic Program on O-minimal Structures and Real Analytic Geometry (January through June of 2009). The "and" from the title indicates the intersection of the two subjects, not their union; only the most fundamental theorems and a few illustrative applications thereof are presented. Most, but not all, of the results are from my Ph.D. thesis (*Polynomially bounded o-minimal structures*, University of Illinois at Urbana-Champaign, 1994), and have also been published previously in various journals.

The reader is assumed to be familiar with basic graduate-level model theory and o-minimality. (A brief description of structures in the syntactic sense is given in the introduction for readers who might only be familiar with definability theory.) For basic model theory, see the first few chapters of Marker [7]; for model-theoretic o-minimality in particular, the survey paper by van den Dries [29] is a good start, or one can go directly to seminal papers by Pillay and Steinhorn [16]

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and Knight et al. [5]. The paper [24] by Speissegger is also useful, as it is based on lecture notes from an earlier Fields Institute program (Algebraic Model Theory, August 1996–July 1997). An extensive account of definability-theoretic o-minimality requiring little or no background in logic is found in [31]; see also van den Dries and Miller [34] for an exposition aimed at geometers.

Some global notation and conventions. The set of nonnegative integers is denoted by $\mathbb{N}$. We regard $\mathbb{R}^0$ as the one-point space $\{0\}$, and identify maps $f\colon \mathbb{R}^0 \to \mathbb{R}^n$ with the corresponding points $f(0) \in \mathbb{R}^n$.

For any additively-written abelian group G and $A \subseteq G$, we denote the nonzero elements of A by A^*. If G is ordered, then $A^{>0}$ denotes the positive elements of A.

Generally, boldface fonts used within text indicate definitions. To illustrate, **ultimately** abbreviates "for all sufficiently large positive arguments", or grammatical variants thereof as appropriate. Example of usage: Ultimately, $x^2 < e^x$.

1 Introduction

We begin with a brief description of first-order expansions in the syntactic sense of the ordered field of real numbers $\overline{\mathbb{R}} := (\mathbb{R}, <, +, -, \cdot, 0, 1)$. Readers already familiar with model theory should note that the approach here is specialized to expansions of the real field in extensions of the language of ordered rings $\{<, +, -, \cdot, 0, 1\}$ by function (including constant) symbols only.

For each $n \in \mathbb{N}$, let $\mathcal{F}_n$ be a (possibly empty) set of functions $\mathbb{R}^n \to \mathbb{R}$. We regard the real numbers as an ordered field equipped with these extra functions, that is, as the **structure** (in the syntactic sense) $\mathfrak{R} := (\overline{\mathbb{R}}, (f)_{f \in \bigcup \mathcal{F}_n})$. We also call $\mathfrak{R}$ an **expansion** (in the syntactic sense) of $\overline{\mathbb{R}}$. The **primitive functions** of $\mathfrak{R}$ are the 0-ary functions 0 and 1, the unary (that is, $\mathbb{R} \to \mathbb{R}$) function $x \mapsto -x$, the binary functions addition and multiplication, and the members of the $\mathcal{F}_n$. The **language** L of $\mathfrak{R}$ (or $L(\mathfrak{R})$ if needed for clarity) consists of the symbols $\{<, +, -, \cdot, 0, 1\}$ together with symbols representing the functions from the $\mathcal{F}_n$. All symbols are assumed to be pairwise distinct. Generally, we do not distinguish by notation the symbols in the language of $\mathfrak{R}$ and the objects that they denote.

For each $n \in \mathbb{N}$, let $\mathcal{T}_n$ be the smallest subring of functions $\mathbb{R}^n \to \mathbb{R}$ that contains the coordinate projections $x \mapsto x_i\colon \mathbb{R}^n \to \mathbb{R}$ for $i = 1, \ldots, n$ and is "closed under composition", that is, for all $m \in \mathbb{N}$, $g \in \mathcal{F}_m$ and $f_1, \ldots, f_m \in \mathcal{T}_n$, we have $g \circ (f_1, \ldots, f_m) \in \mathcal{T}_n$.

We construct collections $\mathfrak{R}_{n,k}$ of subsets of $\mathbb{R}^n$ for each $n \in \mathbb{N}$ by induction on k:

- For $k = 0$ and $n \in \mathbb{N}$ put $\mathfrak{R}_{n,0} = \{f^{-1}(0) : f \in \mathcal{T}_n\}$.
- For all $n \in \mathbb{N}$, let $\mathfrak{R}_{n,k+1}$ be the boolean algebra of subsets of $\mathbb{R}^n$ generated by

$$\mathfrak{R}_{n,k} \cup \{\mathrm{pr}(X) : X \in \mathfrak{R}_{n+1,k}\},$$

where $\mathrm{pr}\colon \mathbb{R}^{n+1} \to \mathbb{R}^n$ denotes projection on the first n coordinates.

A set $X \subseteq \mathbb{R}^n$ is **$\emptyset$-definable**[1] **in $\mathfrak{R}$** if there exists $k \in \mathbb{N}$ such that $X \in \mathfrak{R}_{n,k}$. Given $A \subseteq \mathbb{R}$, we say that X is **A-definable in $\mathfrak{R}$** if X is $\emptyset$-definable in the expanded structure $(\mathfrak{R}, (a)_{a\in A})$, where the notation indicates that we have enlarged ("expanded") $\mathcal{F}_0$ by the points in A regarded as functions $\mathbb{R}^0 \to \mathbb{R}$. For the most part, only the cases $A = \mathbb{R}$ and $A = \emptyset$ will matter to us in this paper. We usually drop the "in $\mathfrak{R}$" whenever the structure under consideration is understood, and we abbreviate "$\mathbb{R}$-definable" by just "definable".[2] If X is a singleton $\{x\}$, we usually drop the set braces and talk about the point $x \in \mathbb{R}^n$ as being A-definable (and so on). A map $f: X \to \mathbb{R}^p$ is A-definable if its graph $\{(x, f(x)) : x \in X\} \subseteq \mathbb{R}^{n+p}$ is A-definable.

The distinction between "A-definable" and "definable" is often extremely important in model-theoretic arguments. We shall be paying far more attention to this distinction than is customary in o-minimal analytic geometry.

Let $A \subseteq \mathbb{R}$. If X is A-definable, then so is its closure and interior. Given an A-definable function $f: X \to \mathbb{R}$, its domain X is A-definable, the set of points in its interior where f is differentiable is A-definable, and if X is open and f is differentiable on X, then the graph of each partial derivative of f is A-definable. Throughout this work we use many such basic facts. The proofs consist of showing that the various sets and functions are defined by formulas built up from variables, quantifiers, boolean connectives and symbols from the language of $(\mathfrak{R}, (a)_{a\in A})$. See Appendices A and B of [34] for details and more information.

A subset X of $\mathbb{R}^n$ is **described by** a collection $\mathcal{H}$ of functions $\mathbb{R}^n \to \mathbb{R}$ if X is a finite union of sets of the form $\{x \in \mathbb{R}^n : f(x) = 0, g_1(x) < 0, \ldots, g_l(x) < 0\}$ with $f, g_1, \ldots, g_l \in \mathcal{H}$.

The structure $\mathfrak{R}$:

- has (or admits) **quantifier elimination**—QE, for short—if, for all $n \in \mathbb{N}$, every $\emptyset$-definable set in $\mathbb{R}^n$ is described by $\mathcal{T}_n$;
- is **model complete** if for all $n \in \mathbb{N}$ and $\emptyset$-definable $X \subseteq \mathbb{R}^n$ there exist $p \in \mathbb{N}$ and $Y \subseteq \mathbb{R}^{n+p}$ described by $\mathcal{T}_{n+p}$ such that X is the projection of Y on the first n coordinates.
- is **polynomially bounded** if for each definable unary function f there exists $N \in \mathbb{N}$ such that ultimately $|f(x)| \leq x^N$.
- is **exponential** if it defines the real exponential function $e^x: \mathbb{R} \to \mathbb{R}$.
- is **exponentially bounded** if for each definable unary function f there exists $N \in \mathbb{N}$ such that ultimately $|f(x)| \leq \exp_N(x)$, where $\exp_N$ denotes the N-th compositional iterate of e^x.

---

[1]Often called "0-definable" in the model-theoretic literature.

[2]In many older papers in model theory, the default is that "definable" means "$\emptyset$-definable".

- is **o-minimal** if every definable subset of $\mathbb{R}$ is a finite union of points and open intervals.[3] In fact, it is enough to require that every $\emptyset$-definable subset of $\mathbb{R}$ be a finite union of points and open intervals; see 2.1 below.

Observe that QE implies model completeness (but not vice versa, as we shall soon see). Evidently, $\mathfrak{R}$ is exponentially bounded if it is polynomially bounded, and not polynomially bounded if it is exponential.

Note. Syntactic (language-dependent) notions such as quantifier elimination and model completeness are formulated in model theory for *theories* (in a given language), not *structures*. For example, rather than saying that $\mathfrak{R}$ is model complete, we should say that its complete theory $\mathrm{Th}(\mathfrak{R})$ is model complete in the language in which $\mathfrak{R}$ is presented. While this level of precision is sometimes needed in abstract model theory, we adopt the convention in this paper that whenever we apply a syntactic notion to a structure, we mean to apply it to the complete theory of the structure under consideration in some given fixed language that should be clear from context.

For $r \in \mathbb{R}$, x^r denotes the **power function**

$$x \mapsto \begin{cases} x^r, & x > 0 \\ 0, & x \leq 0. \end{cases}$$

Although the notation x^r is ambiguous for $r \in \mathbb{N}$, this will not cause any complications. The extension of the domain to $(-\infty, 0]$ is only to satisfy the technicality that primitive functions of structures must, by definition, be totally defined. Much of the time, this syntactic requirement has no impact on the underlying mathematics, where we care as usual only about the restriction to $\mathbb{R}^{>0}$. Power functions play a crucial role in this paper. The set of all $r \in \mathfrak{R}$ such that the function x^r is definable is a subfield of $\mathbb{R}$ (exercise); we call it the **field of exponents** of $\mathfrak{R}$. If $\mathfrak{R}$ is exponential, then $\mathbb{R}$ is its field of exponents (exercise).

Before proceeding any further, let us consider a few examples in order to help justify all these definitions.

The structure $\overline{\mathbb{R}}$ admits QE by the Tarski-Seidenberg theorem—see [31, Chap. 2] for an interesting proof due to Łojasiewicz—hence so does the expansion $(\overline{\mathbb{R}}, (r)_{r\in\mathbb{R}})$ (exercise); the notation indicates that $\mathcal{F}_0 = \mathbb{R}$ and all other $\mathcal{F}_n$ are empty. For $(\overline{\mathbb{R}}, (r)_{r\in\mathbb{R}})$, the sets $\mathcal{T}_n$ consist precisely of the real polynomials on $\mathbb{R}^n$, and then sets described by $\mathcal{T}_n$ are called **semialgebraic** (or sometimes $\mathbb{R}$-semialgebraic). Thus, the sets definable in $\overline{\mathbb{R}}$ are exactly the real semialgebraic sets. It follows from QE that $\overline{\mathbb{R}}$ has field of exponents $\mathbb{Q}$ and is o-minimal, as then every unary definable function is ultimately algebraic, and every set of the form

$$\{x \in \mathbb{R} : f(x) = 0, g_1(x) < 0, \ldots, g_l(x) < 0\}$$

[3] In other words, the only definable subsets of $\mathbb{R}$ are those that must be there by virtue of the usual ordering of the real line. Hence, the structure is "order-minimal", thus accounting for the abbreviation "o-minimal" and the use of a plain text font for the "o".

where $f, g_1, \ldots, g_l$ are unary polynomials is a finite union of points and open intervals.

The **real exponential field** is the structure $(\overline{\mathbb{R}}, (r)_{r\in\mathbb{R}}, e^x)$. Here, $\mathcal{F}_0 = \mathbb{R}$, $\mathcal{F}_1 = \{e^x\}$ and all other $\mathcal{F}_n$ are empty. By Wilkie [39], the structure $(\overline{\mathbb{R}}, e^x)$ is model complete, hence so is $(\overline{\mathbb{R}}, (r)_{r\in\mathbb{R}}, e^x)$ (exercise). Indeed, for every set $X \subseteq \mathbb{R}^n$ $\emptyset$-definable in $(\overline{\mathbb{R}}, e^x)$, there exist $p \in \mathbb{N}$ and a polynomial $P\colon \mathbb{R}^{2(n+p)} \to \mathbb{R}$ with integer coefficients such that X is the projection on the first n coordinates of the set $\{(x, y) \in \mathbb{R}^{n+p} : P(x, y, e^x, e^y) = 0\}$, where $e^x = (e^{x_1}, \ldots, e^{x_n})$ and $e^y = (e^{y_1}, \ldots, e^{y_p})$. By Hovanskiĭ [4], such sets have only finitely many connected components; o-minimality follows. Quantifier elimination fails for $(\overline{\mathbb{R}}, e^x)$, indeed, no proper[4] expansion of $\overline{\mathbb{R}}$ by real-analytic primitive functions has QE (van den Dries [26]; see also 4.6 below). The structure $(\overline{\mathbb{R}}, e^x)$ is exponentially bounded, but this is far from obvious; it was first established by van den Dries and Miller in [33], but a stronger and more general result was later established by Lion et al. [6].

This ends our brief description of the syntactical point of view. *From now on:* We revert to using standard terminology and conventions from mathematical logic and model theory, in particular, we allow structures to have primitive relations other than just $<$, though we tend to replace relations by characteristic functions whenever convenient. We usually employ "fraktur" font to indicate structures ($\mathfrak{A}$, $\mathfrak{B}$ and so on) with corresponding math roman caps (A, B and so on) for the underlying sets (or even underlying ordered rings, depending on context), except for $\mathfrak{R}$, which will always denote an expansion of $\overline{\mathbb{R}}$.

I next outline the main results presented in these notes. *For the rest of this section*, let $\mathfrak{R}$ be an o-minimal expansion of $\overline{\mathbb{R}}$ with field of exponents K.

There is a striking dichotomy in the asymptotic behavior of the definable unary functions:

Theorem 1.1 (Growth Dichotomy [8]). *$\mathfrak{R}$ is either polynomially bounded or exponential.*

Our focus in these notes will be on the case that $\mathfrak{R}$ is polynomially bounded. The next result is fundamental.

Theorem 1.2 (Piecewise Uniform Asymptotics [9]). *Let $\mathfrak{R}$ be polynomially bounded and $f\colon \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}$ be definable. Then there is a finite $S \subseteq K$ such that for each $a \in \mathbb{R}^m$ either the function $x \mapsto f(a, x)\colon \mathbb{R} \to \mathbb{R}$ is ultimately equal to* 0 *or there exists $r \in S$ such that* $\lim_{x\to+\infty} f(a, x)/x^r \in \mathbb{R}^*$.

An easy consequence (see 4.5 below for details):

1.3 ([10]). *If $\mathfrak{R}$ is polynomially bounded and $U \subseteq \mathbb{R}^n$ is definable, open and connnected, then the ring of all definable $\mathcal{C}^\infty$ functions $U \to \mathbb{R}$ is a quasianalytic class, that is, if $f\colon U \to \mathbb{R}$ is definable and $\mathcal{C}^\infty$, then $f = 0$ iff there exists $x_0 \in U$ such that all partials of f vanish at x_0.*

[4] A **proper** expansion of $\overline{\mathbb{R}}$ is one that defines a non-semialgebraic set.

If $K \neq \mathbb{R}$, then it is immediate from Growth Dichotomy that $\mathfrak{R}$ is polynomially bounded. We are then interested in what happens when we expand $\mathfrak{R}$ by more power functions. Let us make this precise. Given an expansion $\mathfrak{A}$ of $\overline{\mathbb{R}}$ and $S \subseteq \mathbb{R}$, put $\mathfrak{A}^S = (\mathfrak{A}, (r)_{r \in S}, (x^r)_{r \in S})$, the expansion of $\mathfrak{A}$ by constants from S and power functions with exponents from S. Since every power function is definable in $(\overline{\mathbb{R}}, e^x)$, every set definable in $\mathfrak{A}^S$ is definable in $(\mathfrak{A}, e^x)$. By o-minimality of the Pfaffian closure ([23] or [25]), $(\mathfrak{R}, e^x)$ is o-minimal, hence so is $\mathfrak{R}^S$ for any $S \subseteq \mathbb{R}$. The question: If $\mathfrak{R}$ is polynomially bounded, is the same true of $\mathfrak{R}^S$, and is the field of exponents what we would hope for, namely, the field $K(S)$? In this generality, we do not know, but we have a reasonable partial answer:

Theorem 1.4 ([9] and [12, 4.1]). *Let $\mathfrak{R}$ be polynomially bounded and $S \subseteq \mathbb{R}$ be such that the restriction $x^r \restriction [1, 2]$ is $\emptyset$-definable in $\mathfrak{R}$ for each $r \in S$.*

1. *If $\mathfrak{R}$ has QE and is $\forall$-axiomatizable*[5]*, then the same is true of $\mathfrak{R}^{K(S)}$.*
2. *If $\mathfrak{R}$ is model complete, then the same is true of $\mathfrak{R}^S$.*
3. *Every function definable in $\mathfrak{R}^{K(S)}$ is given piecewise by iterated compositions of functions definable in $\mathfrak{R}$ and powers from $K(S)$.*
4. *$\mathfrak{R}^S$ is polynomially bounded with field of exponents $K(S)$.*

The interval $[1, 2]$ is chosen only for convenience and concreteness: If $r \in \mathbb{R}$ and I is any infinite interval of positive real numbers such that $x^r \restriction I$ is $\emptyset$-definable, then $x^r \restriction [1, 2]$ is $\emptyset$-definable (by density of $\mathbb{Q}$ in $\mathbb{R}$ and the multiplicative properties of power functions).

Remark. With a bit more work, one can show that $\mathfrak{R}^S$ is o-minimal without appeal to Pfaffian closure; see [38] for the main idea (indeed, for the overarching strategy of the proof of Theorem 1.4).

Let us consider some concrete applications.

1.5 ([9]). *Put $\phi(x) = 1/(1 + x^2)$ for $x \in \mathbb{R}$. For each $S \subseteq \mathbb{R}$, all of the following are model complete and polynomially bounded with field of exponents $\mathbb{Q}(S)$:*

- $\overline{\mathbb{R}}^S$
- $(\overline{\mathbb{R}}^S, e^\phi)$
- $(\overline{\mathbb{R}}^S, \arctan)$
- $(\overline{\mathbb{R}}^S, e^\phi, \arctan)$.

Proof. By results from [27] and [39], all of the following are o-minimal, have field of exponents $\mathbb{Q}$ and are model complete:

- $(\overline{\mathbb{R}}, ((1 + \phi)^r)_{r \in S}, (r)_{r \in S})$
- $(\overline{\mathbb{R}}, ((1 + \phi)^r)_{r \in S}, (r)_{r \in S}, e^\phi)$
- $(\overline{\mathbb{R}}, ((1 + \phi)^r)_{r \in S}, (r)_{r \in S}, \arctan)$
- $(\overline{\mathbb{R}}, ((1 + \phi)^r)_{r \in S}, (r)_{r \in S}, e^\phi, \arctan)$

[5] That is, its theory is axiomatizable by universal sentences.

For each $r \in \mathbb{R}$, the function $x^r \restriction [1,2]$ is $\emptyset$-definable in $(\overline{\mathbb{R}}, (1+\phi)^r)$, and $(1+\phi)^r$ is both existentially and universally definable in $(\overline{\mathbb{R}}, x^r)$. The result is now immediate from Theorem 1.4.2. □

Remark. (i) Every compact trajectory of any real linear vector field is definable in the structure $(\overline{\mathbb{R}}, e^{\phi}, \arctan)$, which partly explains our interest in it; see [12, 28] for more information. (ii) By Bianconi [1], arctan is not definable in $(\overline{\mathbb{R}}, e^x)$, nor is e^{ϕ} definable in $(\overline{\mathbb{R}}, \arctan)$.

Result 1.5 was published in the year 1994. There is now (2011) a fairly large body of literature on model completeness and o-minimality results for expansions of $\overline{\mathbb{R}}$ by differential rings of "restricted" functions. It is fair to say that the current state of the art appears in Rolin et al. [18], see also [17], to which we refer the reader for details and history. By combining their technology with ours, we obtain:

1.6. *Let $\mathfrak{R}_0$ be the expansion of $\overline{\mathbb{R}}$ by all functions of the form*

$$x \mapsto \begin{cases} f(x), & x \in [-1,1]^n \\ 0, & x \in \mathbb{R}^n \setminus [-1,1]^n, \end{cases}$$

where $n \in \mathbb{N}$ and $f : \mathbb{R}^n \to \mathbb{R}$ is definable in $\mathfrak{R}$ and $\mathcal{C}^\infty$ on some open neighborhood of $[-1,1]^n$. Suppose that $\mathfrak{R}_0$ is polynomially bounded. Let $S \subseteq \mathbb{R}$ be such that, for each $r \in S$, the restriction $x^r \restriction [1,2]$ is $\emptyset$-definable in $\mathfrak{R}$. Then $\mathfrak{R}_0^S$ is model complete and polynomially bounded with field of exponents $\mathbb{Q}(S)$.

Proof. By 1.3 and [18], $\mathfrak{R}_0$ is model complete and has field of exponents $\mathbb{Q}$. Apply Theorem 1.4.2. □

Concrete examples of applications of Theorem 1.4.1 take a bit more work to state; for more detailed information on this material, see [9, 28]. A **convergent Weierstrass system** (over $\mathbb{R}$) is a family of rings $W := (\mathbb{R}\llcorner X_1, \dots, X_n \lrcorner)_{n \in \mathbb{N}}$ such that the following hold for all n, where $X = (X_1, \dots, X_n)$.

- $\mathbb{R}[X] \subseteq \mathbb{R}\llcorner X \lrcorner \subseteq \mathbb{R}[\![X]\!]$.
- If σ is a permutation of $\{1, \dots, n\}$ and $f(X) \in \mathbb{R}\llcorner X \lrcorner$, then

$$f(X_{\sigma(1)}, \dots, X_{\sigma(n)}) \in \mathbb{R}\llcorner X \lrcorner.$$

- If $F \in \mathbb{R}\llcorner X \lrcorner$ is a unit in $\mathbb{R}[\![X]\!]$, then it is also a unit in $\mathbb{R}\llcorner X \lrcorner$.
- (Weierstrass division) If $F \in \mathbb{R}\llcorner X, X_{n+1} \lrcorner$ and $F(0, X_{n+1}) \in \mathbb{R}[\![X_{n+1}]\!]$ is nonzero of order d, then for every $G \in \mathbb{R}\llcorner X, X_{n+1} \lrcorner$ there exist $Q \in \mathbb{R}\llcorner X, X_{n+1} \lrcorner$ and $R_0, \dots, R_{d-1} \in \mathbb{R}\llcorner X \lrcorner$ such that

$$G = QF + \sum_{i=0}^{d-1} R_i X_{n+1}^i.$$

- Every $F \in \mathbb{R}\llcorner X\lrcorner$ converges to an analytic function f on some box neighborhood U of the origin in $\mathbb{R}^n$ such that for all $a \in U$ the series

$$F(a+X) = \sum_{\mu\in\mathbb{N}^m} \frac{\partial^{|\mu|} f}{\partial X^\mu}(a)\frac{X^\mu}{\mu!} \in \mathbb{R}\llcorner X\lrcorner$$

belongs to $\mathbb{R}\llcorner X\lrcorner$.

Some examples:

- The family of rings of algebraic real power series is the smallest convergent Weierstrass system.
- The family of rings of convergent real power series is the largest convergent Weierstrass system.
- A power series $f \in \mathbb{R}[\![X_1,\dots,X_n]\!]$ is **differentially algebraic** if the integral domain $\mathbb{R}[\partial^{|\mu|} f/\partial X^\mu : \mu \in \mathbb{N}^n] \subseteq \mathbb{R}[\![X_1,\dots,X_n]\!]$ generated by the partial derivatives of f over the field $\mathbb{R}$ of constants has finite transcendence degree over $\mathbb{R}$. The family of rings of convergent differentially algebraic real power series is a convergent Weierstrass system.

Given a convergent Weierstrass system W, we obtain a structure $\overline{\mathbb{R}}_W$ by expanding $\overline{\mathbb{R}}$ by all functions of the form

$$x \mapsto \begin{cases} F(x), & x \in [-1,1]^n \\ 0, & x \in \mathbb{R}^n \setminus [-1,1]^n, \end{cases}$$

where $n \in \mathbb{N}$ and all series $F(a+X)$ belong to W for all $a \in [-1,1]^n$. (For $n = 0$, we take this to mean that we expand by all real constants.) By [27], $\overline{\mathbb{R}}_W$ is o-minimal and has field of exponents $\mathbb{Q}$. By [28], $\overline{\mathbb{R}}_W$ is model complete. By arguing as in Denef and van den Dries [2], $(\overline{\mathbb{R}}_W, x^{-1})$ has QE. Finally, by arguing as in van den Dries et al. [38, Sect. 2], $\overline{\mathbb{R}}_W^{\mathbb{Q}}$ is $\forall$-axiomatizable. Consequently, given a subfield K of $\mathbb{R}$ such that the series $\sum_{k\geq 0}\binom{r}{k}X_1^k$ belongs to $\mathbb{R}\llcorner X_1\lrcorner$ for all $r \in K$, we have by Theorem 1.4.1 that $\mathbb{R}_W^K$ admits quantifier elimination and is universally axiomatizable. This holds in particular for the case that W extends the family of rings of differentially algebraic convergent real power series. (For each $r \in \mathbb{R}$, the series $\sum_{k\geq 0}\binom{r}{k}X_1^k$ is convergent and differentially algebraic.)

We arrive at our last main result. As previously mentioned, we do not know if polynomial bounds are always preserved in passing from $\mathfrak{R}$ to $\mathfrak{R}^S$. But exponential bounds are if $\mathfrak{R}$ is polynomially bounded, indeed, the Pfaffian closure of $\mathfrak{R}$ is then exponentially bounded. We provide a part of the proof (see [25, Theorem B and Proposition 1.14] for the rest):

Theorem 1.7 ([6]). *Let $\mathfrak{R}$ be polynomially bounded, $G\colon \mathbb{R}^{N+2} \to \mathbb{R}$ be definable in $\mathfrak{R}$, and g be a solution on some ray (a,∞) to the differential equation*

$$y^{(N+1)} = G(x, y, y', \dots, y^{(N)}).$$

If $(\mathfrak{R}, g)$ *is o-minimal, then ultimately* $|g(x)| \leq C \exp_{N+1}(x^r)$ *for some* $C > 0$ *and* $r \in K$.

Consideration of the functions $g := C \exp_{N+1}(x^r)$ for $C > 0$ and $r \in K$ shows that the upper bound is optimal,[6] but the result can be sharpened in other ways; see [6].

Here is an outline of the rest of this paper. Section 2 consists of a collection of exercises for the reader, many of which are crucial for later developments. In Sect. 3, Hardy fields are introduced and their connection to o-minimal expansions of $\overline{\mathbb{R}}$ is explained, culminating in the proof of Theorem 1.1. Section 4 contains a few special results about the polynomially bounded case of Growth Dichotomy, beginning with Theorem 1.2. Sections 5 and 6 are devoted to the proofs of Theorems 1.4 and 1.7. We conclude with some suggestions for further study.

2 Some Exercises

The reader is strongly recommended to at least attempt these exercises before moving on. Many of them will be used without proof, or even further mention, in the sequel. Let $\mathfrak{R}$ be an expansion of $\overline{\mathbb{R}}$.

If $X \subseteq \mathbb{R}^n$ is definable, then there exist $m \in \mathbb{N}$, $a \in \mathbb{R}^m$ and $\emptyset$-definable $Y \subseteq \mathbb{R}^{m+n}$ such that $X = \{x \in \mathbb{R}^n : (a, x) \in Y\}$. Note that X is A-definable, where A is the union of the coordinates of a. To put this another way: Any set definable in $\mathfrak{R}$ is $\emptyset$-definable in some expansion of $\mathfrak{R}$ by finitely many constants. In practice, this observation is often used to reduce worrying about "A-definable versus definable" to the case that $A = \emptyset$ when dealing with some fixed definable set X.

Each point of a finite $\emptyset$-definable set is $\emptyset$-definable. (There is a lexicographically least element of the set if it is nonempty. And so on.)

Every polynomial map with coefficients from the real-closed subfield of $\mathbb{R}$ generated by $\mathcal{F}_0$ is $\emptyset$-definable (where $\mathcal{F}_0$ is as in the definition of "structure", given in the second paragraph of the Introduction).

Every set described by $\mathcal{T}_n$ is $\emptyset$-definable.

Let $X \subseteq \mathbb{R}^n$ be described by $\mathcal{T}_n$. Then there exist $p \in \mathbb{N}$ and $f \in \mathcal{T}_{n+p}$ such that X is the projection on the first n coordinates of $f^{-1}(0)$. (*Hint.* For functions $f, g\colon \mathbb{R}^n \to \mathbb{R}$ and $x \in \mathbb{R}^n$ we have $f(x) = 0$ and $g(x) = 0$ iff $f(x)^2 + g(x)^2 = 0$, and $f(x) = 0$ or $g(x) = 0$ iff $f(x)g(x) = 0$.)

If $\mathfrak{R}$ has QE and $A \subseteq \mathbb{R}$, then $(\mathfrak{R}, (a)_{a\in A})$ has QE, and similarly with "model complete" in place of "QE".

If an injective map $f\colon A \to B$ is definable, then the compositional inverse $f^{-1}\colon B \to A$ is definable.

The field of exponents is indeed a field.

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[6] *Exercise.* Prove it.

The field of exponents of $(\overline{\mathbb{R}}, e^x)$ is $\mathbb{R}$.

Given $A \subseteq \mathbb{R}^{n+k}$ and $x \in \mathbb{R}^n$, let A_x denote the **fiber** $\{y \in \mathbb{R}^k : (x, y) \in A\}$ of A over x. Show that if A is $\emptyset$-definable, then so are the following sets:

- A_x, if x is $\emptyset$-definable (in particular, if $x \in \mathbb{Q}^n$).
- $\{x \in \mathbb{R}^n : A_x \text{ is closed}\}$.
- $\{x \in \mathbb{R}^n : A_x \text{ is bounded}\}$.
- $\{x \in \mathbb{R}^n : A_x \text{ is discrete}\}$.
- $\{x \in \mathbb{R}^n : A_x \text{ is finite}\}$.
- $\{(x, y) \in \mathbb{R}^{n+k} : y \text{ is in the boundary of } A_x\}$.
- $\{x \in \mathbb{R}^n : \text{the boundary of } A_x \text{ is finite}\}$.
- $\{x \in \mathbb{R}^n : A_x \text{ is a finite union of points and open intervals}\}$, if $k = 1$.

2.1. If every $\emptyset$-definable subset of $\mathbb{R}$ is a finite union of points and open intervals, then $\mathfrak{R}$ is o-minimal. (By previous exercises, for every $n \in \mathbb{N}$ and $\emptyset$-definable $A \subseteq \mathbb{R}^{n+1}$, the set of all $x \in \mathbb{R}^n$ such that A_x is a finite union of points and open intervals is dense and $\emptyset$-definable. Finish by an appropriate induction on n. See [13, Proposition 1] for details. The proof works for all expansions of the real line $(\mathbb{R}, <)$ whose $\emptyset$-definable points are dense in $\mathbb{R}$. See [3, 1.14] for a rather different proof, due to van den Dries, that works for expansions of arbitrary densely ordered groups.)

2.2. Let $f: \mathbb{R} \to \mathbb{R}$. Show that:

- $Y := \{y \in \mathbb{R}^{>0} : \lim_{x\to+\infty}(f(xy) - f(x)) \in \mathbb{R}\}$ is a multiplicative group.
- $Z := \{z \in \mathbb{R} : \exists y \in \mathbb{R}^{>0},\ \lim_{x\to+\infty}(f(xy) - f(x)) = z\}$ is an additive group.
- The function $L(f)(y) = \lim_{x\to+\infty}(f(xy) - f(x)): Y \to Z$ is a surjective homomorphism. The notation is to suggest that $L(f)$ is somehow the "logarithmic part" of f, but this should not be taken too seriously, as we could easily have $Y = \{1\}$ and $Z = \{0\}$.
- The sets Y, Z and the function $L(f)$ are $\emptyset$-definable in $(\overline{\mathbb{R}}, f)$.
- If $(\overline{\mathbb{R}}, f)$ is o-minimal and $\lim_{x\to+\infty}(f(2x) - f(x)) \in \mathbb{R}^*$, then $Y = \mathbb{R}^{>0}$ and $L(f) = \log_a$ for some $a \in \mathbb{R}^{>0}$. (Recall that every subgroup of $(\mathbb{R}, +)$ is either cyclic or dense, and every endomorphism of $(\mathbb{R}, +)$ is either nowhere continuous or linear.) Conclude that log is definable, hence so is e^x.

2.3. Let $f: \mathbb{R} \to \mathbb{R}$ be ultimately nonzero. Show that:

- The sets
$$Y := \{y \in \mathbb{R}^{>0} : \lim_{x\to+\infty} f(xy)/f(x) \in \mathbb{R}\}$$
$$Z := \{z \in \mathbb{R}^{>0} : \exists y \in \mathbb{R}^{>0},\ \lim_{x\to+\infty} f(xy)/f(x) = z\}$$
are multiplicative groups.
- The function $P(f)(y) = \lim_{x\to+\infty}(f(xy)/f(x)): Y \to Z$ is a surjective homomorphism. The notation is to suggest that $P(f)$ is somehow the "power part" of f, but again, this should not be taken too seriously. We tend to write just Pf as convenient.

- The sets Y, Z and the function Pf are $\emptyset$-definable in $(\overline{\mathbb{R}}, f)$.
- If $(\overline{\mathbb{R}}, f)$ is o-minimal and $2 \in Y$, then $Y = \mathbb{R}^{>0}$ and Pf is a power function.
- If there exists $r \in \mathbb{R}$ such that $\lim_{x\to+\infty} f(x)/x^r \in \mathbb{R}^*$, then $Y = \mathbb{R}^{>0}$ and $Pf = x^r$.
- Calculate Y, Z and Pf directly for the functions $\log x$, $x^r \log x$, $(\log x)^{\log x}$.

2.4. In any expansion of $\overline{\mathbb{R}}$, the function e^x is $\emptyset$-definable if it is definable. (*Hint:* e^x is the unique solution on $\mathbb{R}$ to the initial value problem $y' = y$, $y(0) = 1$.) Similarly, the function x^r is $\emptyset$-definable if it is definable and r is $\emptyset$-definable. (*Hint:* $r = (x^r)'(1)$.) More generally: Definable solutions to initial value problems are $\emptyset$-definable if all of the data are $\emptyset$-definable.

3 Hardy Fields, O-minimality and Growth Dichotomy

All of the main results listed in the introduction are tied to the asymptotic analysis of definable unary functions, as we now begin to describe. First, define an equivalence relation on the set of all functions $\mathbb{R} \to \mathbb{R}$ by relating functions f and g if they ultimately agree. The equivalence classes are called **germs** (at $+\infty$). Working with germs instead of functions allows us to ignore all but the asymptotics (at $+\infty$), that is, to focus on the "ultimate behavior" of the functions. We regard the set of all such germs as a ring, with $\text{germ}(f) + \text{germ}(g) = \text{germ}(f + g)$, and $\text{germ}(f) \cdot \text{germ}(g) = \text{germ}(f \cdot g)$. A **Hardy field** is a field of germs that is closed under differentiation, that is, if the germ of a function f belongs to the field, then there is a differentiable function g such that the germs of g and g' belong to the field and g is ultimately equal to f. Every Hardy field is naturally ordered by setting $\text{germ}(f) < \text{germ}(g)$ iff $g - f$ is ultimately positive. Observe that the set of germs of rational constant functions is a Hardy field, indeed, it is the smallest Hardy field. Of course, germs of constant functions are not very interesting, and one usually deals at least with Hardy fields that extend the (germs of) the field $\mathbb{Q}(x)$ of rational functions.

We are now ready for a result that begins to explain the title of this paper.

3.1. *If $\mathfrak{R}$ is an expansion of $\overline{\mathbb{R}}$, then the following are equivalent*:

(1) $\mathfrak{R}$ is o-minimal.
(2) The germs of definable unary functions form a Hardy field.
(3) Every unary definable function is either ultimately zero or ultimately nonzero.

Proof. (1)$\Rightarrow$(2) is immediate from the C^1 version of the Monotonicity Theorem (a fundamental result in o-minimality). (2)$\Rightarrow$(3) is immediate from field structure and the definition of germ.

(3)$\Rightarrow$(1).[7] Let $A \subseteq \mathbb{R}$ be definable. We must show that A is a finite union of points and open intervals. For this, it suffices to show that boundary $\text{bd}(A)$ of A

[7] The argument is essentially from van den Dries et al. [38].

is finite, which in turn reduces via Bolzano-Weierstrass to showing that $\text{bd}(A)$ is bounded and discrete. (Recall that a subset of a topological space is discrete if all of its points are isolated.) Let f be the characteristic function of A. As A is definable, so is f (exercise). Then either f is ultimately equal to 1 or f is ultimately equal to 0, so there exists $b \in \mathbb{R}$ such that the ray (b, ∞) is either contained in A or disjoint from A. By the same reasoning applied to the set $\{-x : x \in A\}$, there exists $a \in \mathbb{R}$ such that $(-\infty, a)$ either is contained in A or disjoint from A. Hence, $\text{bd}(A)$ is bounded. Fix $x_0 \in \text{bd}(A)$. By arguing as above with the set $\{1/(a - x_0) : a \in A\}$, there exists $\epsilon > 0$ such that the interval $(x_0, x_0 + \epsilon)$ is either contained in A or disjoint from A, and similarly for $(x_0 - \epsilon, x_0)$. Hence, $\text{bd}(A) \cap (x_0 - \epsilon, x_0 + \epsilon) = \{x_0\}$, thus showing that $\text{bd}(A)$ is discrete. □

Exercise. The above holds with "∅-definable" in place of "definable". (Recall 2.1.)

For the rest of this section, $\mathfrak{R}$ denotes an o-minimal expansion of $\overline{\mathbb{R}}$ with field of exponents K.

Let $\mathcal{H}$ denote the Hardy field of germs at $+\infty$ of the definable unary functions. We will not distinguish between functions and their germs by notation, relying instead upon context. We regard $\mathbb{R}$ as a subfield of $\mathscr{H}$ by identifying $r \in \mathbb{R}$ with the germ of the corresponding constant function. The germ of the identity function on $\mathbb{R}$ is denoted by x. We say that $f \in \mathscr{H}$ is **infinitely increasing** if $\lim_{x \to +\infty} f(x) = +\infty$.

Next are some crucial basic facts that the reader should verify before moving on.

3.2. • If $f \in \mathscr{H}$, then $\lim_{x \to +\infty} f(x) \in \mathbb{R} \cup \{\pm\}$.
- $\{(f, g) \in \mathcal{H}^* \times \mathcal{H}^* : \lim_{x \to +\infty} f(x)/g(x) \in \mathbb{R}^*\}$ is an equivalence relation. Denote the natural quotient map by ν. The image $\nu(\mathcal{H}^*)$ is an ordered abelian group by setting $\nu(f) + \nu(g) = \nu(fg)$ and $\nu(f) > 0$ iff $\lim_{x \to +\infty} f(x) = 0$. Denote the resulting absolute value on $\nu(\mathcal{H}^*)$ by $|\ \ |$. (Be careful: This does *not* mean that $|\nu(\ \ )| = \nu(|\ \ |)$.)
- If $f, g \in \mathcal{H}^*$ and $|f| \geq |g|$, then $\nu(f) \leq \nu(g)$. (This reversal of order might strike one as perverse, but the convention is firmly established in the literature.) The converse fails.
- If $f, g \in \mathcal{H}^*$ and $f \neq -g$, then $\nu(f + g) \geq \min\{\nu(f), \nu(g)\}$, with equality if $\nu(f) \neq \nu(g)$.
- If $f \in \mathcal{H}^*$ and $r \in \mathbb{R}^*$, then $\nu(rf) = \nu(f) = \nu(|f|)$.
- If $f \in \mathcal{H}^*$ and $\nu(f) \neq 0$, then exactly one of f, $1/f$, $-f$ or $-1/f$ is infinitely increasing, and $|\nu(f)| = |\nu(-f)| = |\nu(1/f)| = |\nu(-1/f)|$
- If $f \in \mathscr{H} \setminus \mathbb{R}$, then

$$\nu\left(f'/f\right) = \nu\left((1/f)'/(1/f)\right) = \nu\left((-f)'/(-f)\right) = \nu\left((-1/f)'/(-1/f)\right).$$

Logarithmic differentiation—the map $a \mapsto a'/a \colon \mathcal{H}^* \to \mathscr{H}$—plays an important role in Hardy field theory.

3.3. *Let* $a, b \in \mathcal{H}^*$ *be such that* $0 < |\nu(a)| \leq |\nu(b)|$. *Then* $\nu(a'/a) \geq \nu(b'/b)$.

Proof. Without altering $|\nu(a)|$, $|\nu(b)|$, $\nu(a'/a)$ or $\nu(b'/b)$, we replace a by $\pm a$ or $\pm 1/a$, and b by $\pm b$ or $\pm 1/b$, to reduce to the case that a and b are infinitely increasing.

If $\nu(a) = \nu(b)$, then $\nu(a') = \nu(b')$ by l'Hôpital's Rule, so $\nu(a'/a) = \nu(a') - \nu(a) = \nu(b') - \nu(b) = \nu(b'/b)$.

Suppose that $\nu(b) < \nu(a)$. Then b/a is infinitely increasing, so all of a, a', b' and $(b/a)'$ are positive, yielding $b'/b > a'/a > 0$ by the quotient rule. Hence, $\nu(a'/a) \geq \nu(b'/b)$. □

The appeal to l'Hôpital's Rule above is worth explaining, as it might seem that we are making a classical "freshman's dream" mistake by "going the wrong way". But because we are working in a Hardy field containing both a and b, we know that the function a'/b' has a limit $l \in \mathbb{R} \cup \{\pm\infty\}$. Hence, we must have $\lim_{x\to+\infty} a(x)/b(x) = l$ by l'Hôpital. This fact—that l'Hôpital's Rule "works both ways"—is crucial in Hardy field theory and is used often.

For $f, g\colon \mathbb{R} \to \mathbb{R}$, we write $f \sim g$ if g ultimately has no zeros and $\lim_{x\to+\infty} \frac{f(x)}{g(x)} = 1$. If $f, g \in \mathcal{H}^*$, then $\nu(f) = \nu(g)$ iff $f \sim cg$ for some $c \in \mathbb{R}^*$.

3.4. *Let* $a, b \in \mathcal{H}^*$ *be such that* $\nu(a) \geq 0$ *and* $\nu(b) \neq 0$. *Then* $\nu(a') > \nu(b'/b)$.

Proof. We may assume that $\nu(a) = 0$ by replacing a with $a + 1$ if otherwise. By l'Hôpital,

$$\frac{ab}{b} \sim \frac{ab' + a'b}{b'},$$

and so

$$a = \frac{ab}{b} \sim \frac{ab' + a'b}{b'} = a + a'\frac{b}{b'}.$$

Then $1 \sim 1 + (a'/a)/(b'/b)$, yielding $\nu(a'/a) > \nu(b'/b)$. Finish by observing that $\nu(a'/a) = \nu(a') - \nu(a)$, and $\nu(a) = 0$ by assumption. □

Another important tool in Hardy field theory is **asymptotic integration**. Given $f \in \mathcal{H}^*$, there need not be $g \in \mathcal{H}^*$ such that $g' = f$. But for many purposes it is enough to have $g \in \mathcal{H}^*$ such that $g' \sim f$. Of course, there need not be such g either: Consider $\mathfrak{R} = \overline{\mathbb{R}}$ and $f = 1/x$. We do have the following result, which will be enough for the purposes of this paper.

3.5. *If* $f \in \mathcal{H}^*$ *and* $\nu(f) < \nu(1/x)$, *then there exists* $g \in \mathcal{H}^*$ *such that* $g' \sim f$.

Proof. Note that $(xf)' \neq 0$.

Suppose that $\nu(f) \geq \nu((xf)')$, that is, $\nu(f/(xf)') \geq 0$. By 3.4, $\nu((f/(xf)')') > \nu(1/x)$, that is, $\nu(x(f/(xf)')') > 0$. Put $g_1 = xf^2/(xf)'$. Basic calculus shows that $g_1'/f = 1 + x(f/(xf)')'$, so $g_1' \sim f$.

Suppose that $\nu(f) < \nu((xf)')$, equivalently, $f'/f \sim -1/x$. Check that $g_1' \neq 0$. Put $g_2 = fg_1/g_1'$ and check that $g_2' \sim f$. (*Hint:*

$$\frac{g_2'}{f} - 1 = \frac{f'/f}{g_1'/g_1} - \frac{g_1''/g_1'}{g_1'/g_1},$$

and keep 3.3 in mind.) □

Exercise. With $f = (\log x)/x$ and g_1 as above, show directly that $g_1' \not\sim f$.

3.6. So far, we have used only that $\mathscr{H}$ is a Hardy field, not that it arises from the germs of definable unary functions of an o-minimal expansion of $\overline{\mathbb{R}}$. Hence, *results established so far hold in any Hardy field.* This now begins to change.

Our Hardy field $\mathcal{H}$ is "closed under composition": If $f, g \in \mathcal{H}^*$ and f is infinitely increasing, then the composition $g \circ f$ lies in $\mathcal{H}^*$ as well. Evidently, the sign of $\nu(g \circ f)$ is the same as that of $\nu(g)$. Not all Hardy fields extending $\mathbb{R}(x)$ are closed under composition, e.g., the function field $\mathbb{R}(x, e^x)$, as a set of germs, is a Hardy field that does not contain $e^x \circ x^2$ (exercise). Also, $\mathcal{H}$ is "closed under compositional inverse": If $f \in \mathcal{H}$ and $\nu(f) < 0$, then the germ f^{-1} of the ultimately-defined compositional inverse of f also belongs to $\mathcal{H}$. (We always use fraction-bar notation for reciprocals of elements of $\mathcal{H}^*$, so there is no ambiguity in the use of $^{-1}$.) Observe that $\sqrt{x} \notin \mathbb{R}(x)$, so closure under compositional inverse is also a special property.

Exercise. The map $(r, \nu(f)) \mapsto \nu(|f|^r)\colon K \times \nu(\mathcal{H}^*) \to \nu(\mathcal{H}^*)$ is well defined. The ordered group $\nu(\mathcal{H}^*)$ together with this map is an ordered K-vector space. This is true for any Hardy field that is closed under taking powers from K of positive elements.

We are now ready to prove a stronger version of Growth Dichotomy (Theorem 1.1).

3.7. *Either $\mathfrak{R}$ is exponential or $\nu(\mathcal{H}^*) = K.\nu(x)$.*

Proof. There are two cases to consider.

Case 1. There exists $f \in \mathcal{H}^*$ such that $\nu(f'/f) \neq \nu(1/x)$ and $\nu(f) \neq 0$.

We show that $\mathfrak{R}$ is exponential. By replacing f with $-f$, we take $f > 0$. By replacing f with $1/f$, we take f to be infinitely increasing. By replacing f with f^{-1}, we suppose that $\nu(f'/f) < \nu(1/x)$. By 3.5, there exists $g \in \mathcal{H}^*$ such that $g' \sim f'/f$. Put $h = g \circ f^{-1}$; then $h' \sim 1/x$. By the mean value theorem, we have

$$h \circ (2x) - h = \frac{x}{\xi} \cdot \xi h' \circ \xi$$

for some $\xi \in \mathcal{H}$ such that $x < \xi < 2x$. (*Exercise:* Why is the ultimately-defined function ξ definable?) Note that $\nu(x/\xi) = 0 = \nu(\xi h' \circ \xi)$, the latter by substituting ξ into xh'. Thus,

$$\nu(h \circ (2x) - h) = \nu(x/\xi) + \nu(\xi h' \circ \xi) = \nu(x/\xi) + \nu(xh') = 0.$$

By 2.2, $\mathfrak{R}$ is exponential.

Case 2. For all $f \in \mathcal{H}^*$, if $\nu(f'/f) \neq \nu(1/x)$, then $\nu(f) = 0$.

We show that $\nu(\mathcal{H}^*) = K.\nu(x)$. Let $f \in \mathcal{H}^*$.

We show first that $Pf = x^r$ for some $r \in K$, where Pf is as in 2.3. This is immediate if $\nu(f) = 0$ (for then $Pf = 1 = x^0$), so assume that $\nu(f) \neq 0$. Put $g = (f \circ (2x))/f \in \mathcal{H}^*$. Observe that

$$x\frac{g'}{g} = 2x\frac{f' \circ (2x)}{f \circ (2x)} - \frac{xf'}{f}.$$

Since $\nu(f) \neq 0$, we have $\nu(xf'/f) = 0$, and so $\nu(g'/g) > \nu(1/x)$. By the case assumption, $\nu(g) = 0$, that is, $f \circ (2x) \sim cf$ for some $c \in \mathbb{R}^*$. Now apply 2.3.

To finish the proof, we now let $f \in \mathcal{H}^*$ and show that $\nu(f) = \nu(Pf)$. Since $P((Pf)/f) = 1 \restriction \mathbb{R}^{>0}$ (exercise), we are reduced to showing that if $Pf = 1 \restriction \mathbb{R}^{>0}$, then $\nu(f) = 0$. By the case assumption, it suffices to show that $\nu(xf'/f) \neq 0$. By the mean value theorem,

$$\frac{f \circ (2x)}{f} - 1 = \frac{xf' \circ \xi}{f} = \frac{x}{\xi} \cdot \frac{\xi f' \circ \xi}{f \circ \xi} \cdot \frac{f \circ \xi}{f}$$

where $\xi \in \mathcal{H}^*$ and $x < \xi < 2x$. It suffices now to show that $\nu((\xi f' \circ \xi)/f \circ \xi) \neq 0$ (for then $\nu(xf'/f) \neq 0$ as well). Since $Pf(2) = 1$, we have

$$0 = \nu\left(\frac{f \circ (2x)}{f} - 1\right) = \nu\left(\frac{x}{\xi}\right) + \nu\left(\frac{\xi f' \circ \xi}{f \circ \xi}\right) + \nu\left(\frac{f \circ \xi}{f}\right).$$

Since $\nu(\xi) = \nu(x)$, it suffices now to show that $f \circ \xi \sim f$, which follows easily from monotonicity—either $f \leq f \circ \xi \leq f \circ (2x)$ or $f \geq f \circ \xi \geq f \circ (2x)$—and that $Pf(2) = 1$ (that is, $f \circ (2x) \sim f$). □

With Growth Dichotomy now established, the next natural step is to study the two resulting cases. But for the remainder of this paper, we deal for the most part only with the polynomially bounded case.

4 Basics of Polynomial Boundedness

Throughout this section, we assume that $\mathfrak{R}$ is o-minimal and polynomially bounded with field of exponents K.

We begin with Piecewise Uniform Asymptotics (Theorem 1.2), which we restate using the notation of the preceding section.

4.1. *Let $A \subseteq \mathbb{R}^m$ and $f: A \times \mathbb{R} \to \mathbb{R}$ be definable such that, for every $a \in A$, the function $x \mapsto f(a,x)$ is ultimately nonzero. Then there is a finite $S \subseteq K$ such that $\{\nu(f(a,x)) : a \in \mathbb{R}^m\} \subseteq S.\nu(x)$.*

Proof. By 3.7, for each $a \in A$ there exists unique $\rho(a) \in K$ such that $\nu(f(a,x)) = \rho(a).\nu(x) = \nu(x^{\rho(a)})$. We must show that $\rho(A)$ is finite; for this, it suffices to show that $\rho(A)$ is definable and has no interior. As the set $\{a \in A : \nu(f(a,x)) = 0\}$ is definable, we reduce to the case that $\rho(a) \neq 0$ for all $a \in A$. For each $a \in A$, the function $x \mapsto f(a,x)$ is ultimately differentiable. By l'Hôpital,

$$\rho(a) = \lim_{x\to+\infty} \frac{x(\partial f/\partial x)(a,x)}{f(a,x)}.$$

Thus, the function $\rho: A \to \mathbb{R}$ is definable, hence so is $\rho(A)$. We next show that $\rho(A)$ has no interior. Suppose otherwise; we derive a contradiction to finish the proof. Let I be an open interval contained in $\rho(A)$; then $K = \mathbb{R}$, since $\rho(A) \subseteq K$ and K is a subfield of $\mathbb{R}$. For all $a \in A$ and $x > 0$ we have

$$\lim_{t\to+\infty} \frac{f(a,tx)}{f(a,t)} = x^{\rho(a)},$$

so $(x,a) \mapsto x^{\rho(a)}: \mathbb{R}^{>0} \times A \to \mathbb{R}$ is definable. By Definable Choice [31, p. 94], there is a definable function $g: I \to A$ such that $\rho(g(y)) = y$ for all $y \in I$. Thus, the restriction of the function x^y to $\mathbb{R}^{>0} \times I$ is definable. By dividing x^y by some power function x^r with $r \in I$, we may assume that $0 \in I$. Then for some $\epsilon > 0$ the restriction of x^y to $\mathbb{R}^{>0} \times (0,\epsilon)$ is definable. Observe that $z = x^y$ if and only if $z^{1/y} = x$ for all $x > 0$ and $y \neq 0$, so the restriction of x^y to $\mathbb{R}^{>0} \times (1/\epsilon, \infty)$ is definable. Then $y \mapsto 2^y: (1/\epsilon, \infty) \to \mathbb{R}$ is also definable, so $\lim_{y\to+\infty} 2^y/y^r = c$ for some $r \in K$ and $c \in \mathbb{R}^*$. But then

$$c = \lim_{y\to+\infty} \frac{2^y}{y^r} = \lim_{y\to+\infty} \frac{2^{y+1}}{(y+1)^r} = \lim_{y\to+\infty} \frac{22^y}{(1+(1/y))^r y^r} = 2c,$$

contradicting that $c \neq 0$. □

The following minor restatement is often useful.

4.2. *Let $A \subseteq \mathbb{R}^m$ and $f: A \times \mathbb{R}^{>0} \to \mathbb{R}$ be definable. Then there is a finite $S \subseteq K$ such that for all $a \in A$, either $f(a,t) = 0$ for all sufficiently small positive t (depending on a), or $\lim_{t\to 0^+} f(a,t)/t^r \in \mathbb{R}^*$ for some $r \in S$.*

(Apply 4.1 to the function $(a,t) \mapsto f(a, 1/t) \colon A \times \mathbb{R}^{>0} \to \mathbb{R}$.)

We need some notation before stating the next result. For $x = (x_1, \dots, x_m) \in \mathbb{R}^m$, put $|x| = \sup\{|x_1|, \dots, |x_m|\}$. Given $A \subseteq \mathbb{R}^m$, y in the closure of A, and $f \colon A \to R$, we write $f(x) = O(|x-y|^r)$ as $x \to y$ if there exist $C \in \mathbb{R}^{>0}$ and an open neighborhood $U \subseteq \mathbb{R}^m$ of y such that $|f(x)| \le C\,|x-y|^r$ for all $x \in A \cap U$. If A is unbounded, we write $f(x) = O(|x|^r)$ as $x \to \infty$ if there exist $C, M \in \mathbb{R}^{>0}$ such that $|f(x)| \le C\,|x|^r$ for all $x \in A$ with $|x| > M$. Recall the notation for fibers of sets from Sect. 2.

4.3 (Uniform Bounds on Orders of Vanishing). *Let $A \subseteq \mathbb{R}^{m+n}$ and $f \colon A \to \mathbb{R}$ be definable. Then there exists $r \in K^{>0}$ such that for all $(x,y) \in \mathbb{R}^{m+n}$, if $y \in \mathrm{cl}(A_x)$ and $f(x,z) = O(|y-z|^r)$ as $z \to y$, then $f(x,z) = 0$ for all $z \in A_x$ near y.*

Proof. For all $(x,y,t) \in \mathbb{R}^{m+n} \times (0,\infty)$, put $A(x,y,t) = \{z \in A_x : |y-z| = t\}$, and put

$$X = \left\{ (x,y,t) \in \mathbb{R}^{m+n} \times (0,\infty) : A(x,y,t) \neq \emptyset \ \& \sup_{z \in A(x,y,t)} |f(x,z)| < +\infty \right\}.$$

Define $F \colon \mathbb{R}^{m+n} \times (0,\infty) \to \mathbb{R}$ by

$$F(x,y,t) = \begin{cases} \sup\{|f(x,z)| : z \in A(x,y,t)\}, & \text{if } (x,y,t) \in X \\ 0, & \text{otherwise.} \end{cases}$$

By 4.2, there exists $r \in K$ such that for all $(x,y) \in \mathbb{R}^{m+n}$ if $F(x,y,t) = O(t^r)$ as $t \to 0^+$, then $F(x,y,t) = 0$ for all sufficiently small positive t (depending on (x,y)).

Now suppose that $(x,y) \in \mathbb{R}^{m+n}$, $y \in \mathrm{cl}(A_x)$ and $f(x,z) = O(|y-z|^r)$ as $z \to y$. Then $f(x,-)$ is bounded near y, so $F(x,y,t) = \sup_{z \in A(x,y,t)} |f(x,z)|$ for all sufficiently small $t > 0$. Then $F(x,y,t) = O(t^r)$ as $t \to 0^+$, and thus $f(x,z) = 0$ for all $z \in A_x$ sufficiently close to y. □

Remark. The "O-constant" depends on x and y.

4.4. *Let $U \subseteq \mathbb{R}^n$ be open and $f \colon U \to \mathbb{R}$ be definable. Then there exists $N \in \mathbb{N}$ such that for all $x \in U$, if f is C^N in an open neighborhood of x and all partial derivatives of f of order at most N vanish at x, then f vanishes identically on some open $V \subseteq U$ with $x \in V$.*

Proof. By 4.3 there exists $N \in \mathbb{N}$ such that for all $x \in U$, if $f(x) = O(|x-y|^N)$ as $|x-y| \to 0^+$, then f vanishes identically in a neighborhood of x. Now apply Taylor's formula. □

Quasianalyticity. For $U \subseteq \mathbb{R}^n$ a definable open set, let $C^\infty_{\mathrm{df}}(U)$ denote the ring of definable $\mathcal{C}^\infty$ functions $f \colon U \to \mathbb{R}$.

4.5. *Let $U \subseteq \mathbb{R}^n$ be definable, open and connected.*

(1) If $f \in C^\infty_{\mathrm{df}}(U)$ and all partials of f vanish at some $x_0 \in U$, then $f = 0$.
(2) $C^\infty_{\mathrm{df}}(U)$ is an integral domain.

Proof. (1). Consider the definable open set A consisting of all $x \in U$ such that $f \restriction V = 0$ for some open $V \subseteq U$ with $x \in V$. By 4.4, $x_0 \in A$. Let $x \in \mathrm{cl}(A) \cap U$. All partials of f are continuous on U and vanish identically on A. Then all partials of f vanish at x. By 4.4, $x \in A$. Thus, A is both open and closed in U, so $A = U$.

(2). Let $f, g \in C^\infty_{\mathrm{df}}(U)$ with $fg = 0$. Suppose that $g(x_0) \neq 0$ for some $x_0 \in U$. By continuity, g has no zeros in some open neighborhood of x_0. Then f vanishes identically on this neighborhood, so all partials of f vanish at x_0. Hence, $f = 0$ by (1). □

Failure of "Naive QE"

4.6. *Let $\mathfrak{R}_0$ be the reduct of $\mathfrak{R}$ generated over $\overline{\mathbb{R}}$ by $f \in \bigcup_{n\in\mathbb{N}} C^\infty_{\mathrm{df}}(\mathbb{R}^n)$. Then QE fails for $\mathfrak{R}_0$ unless every $f \in \bigcup_{n\in\mathbb{N}} C^\infty_{\mathrm{df}}(\mathbb{R}^n)$ is definable in $\overline{\mathbb{R}}$.*

Proof. Suppose that $\mathfrak{R}_0$ has QE. Let $h \colon \mathbb{R}^n \to \mathbb{R}$ be continuous and definable in $\mathfrak{R}_0$. We show that h is definable in $\overline{\mathbb{R}}$. Define $g \colon \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ by

$$g(x, y) = \begin{cases} yh(x/y), & \text{if } y \neq 0 \\ 0, & \text{otherwise.} \end{cases}$$

By QE, there is some $F \in C^\infty(\mathbb{R}^{n+2})$, definable in $\mathfrak{R}_0$ and not identically equal to 0, such that the graph of g is contained in the zero set of F. By quasianalyticity (and because g vanishes at the origin), there is a nontrivial homogeneous polynomial P of degree $d > 0$ and $G \in C^\infty(\mathbb{R}^{n+2})$ definable in $\mathfrak{R}$ such that $F = P + G$ and $\lim_{|v|\to 0^+} G(v)\,|v|^{-d} = 0$. Let $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$. We have

$$0 = F(tx, t, g(tx, t)) = F(tx, t, tg(x, 1)) = t^d P(x, 1, g(x, 1)) + G(tx, t, tg(x, 1)).$$

Since $g(x, 1) = h(x)$, we have $P(x, 1, h(x)) = -t^{-d} G(tx, t, th(x))$. Keeping x fixed and letting $t \to 0$, we have $P(x, 1, h(x)) = 0$. As x is arbitrary, the graph of h is contained in the zero set of the polynomial $Q(x, z) := P(x, 1, z)$, which is nontrivial by homogeneity of P. Thus, the zero set of Q is a finite union of nonopen cells definable in $\overline{\mathbb{R}}$. By continuity, both the set zero set of Q and the graph of h are closed. By cell decomposition in $\mathfrak{R}_0$, the graph of h is a finite union of cells definable in $\overline{\mathbb{R}}$. □

The above is a minor variant of a result due to van den Dries (generalizing the "Osgood example"):

4.7 ([26]). *The conclusion of 4.6 holds without assuming polynomial bounds or o-minimality if every* $f \in \bigcup_{n\in\mathbb{N}} C^{\infty}_{\mathrm{df}}(\mathbb{R}^n)$ *is real analytic.*

We shall close this section with a result to be used in the proofs of Theorems 1.4 and 1.7; some preliminary work is needed.

4.8. *Every power function of* $\mathfrak{R}$ *is* $\emptyset$*-definable.*

Proof. Let $r \in K$, and suppose that the function x^r is defined by $\varphi(c, x, y)$ for some $(m+2)$-ary formula φ (in the language of $\mathfrak{R}$) and $c \in \mathbb{R}^m$. The set B of all $b \in \mathbb{R}^m$ such that $\varphi(b, x, y)$ defines a solution on $\mathbb{R}^{>0}$ to the initial value problem $xy' = ry$, $y(1) = 1$ is $\{r\}$-definable. Hence, by uniqueness of solutions to ODEs, it suffices to show that r is $\emptyset$-definable. Put

$$C = \{c \in \mathbb{R}^m : \varphi(c, x, y) \text{ defines a function } f_c\colon \mathbb{R} \to \mathbb{R}\}.$$

Note that C is $\emptyset$-definable. Let $f\colon \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}$ be the function given by $f(c, x) = f_c(x)$ if $c \in C$ and $f(c, x) = 0$ otherwise. Note that f is $\emptyset$-definable. Let $S \subseteq K$ be as in the statement of Piecewise Uniform Asymptotics applied to f. The proof of Piecewise Uniform Asymptotics shows that S is $\emptyset$-definable. As S is also finite, every $s \in S$ is $\emptyset$-definable. Hence, r is $\emptyset$-definable, and we are done. □

4.9 (exercise). Let A be (the underlying set of) an ordered field.

- Define an equivalence relation on A^* by identifying $x, y \in A^*$ iff there exists $n \in \mathbb{N}$ such that $2^{-n} < |y/x| < 2^n$. Denote the natural quotient map by υ (or υ_A if needed); it is called the **archimedean valuation** on A. The image $\upsilon(A^*)$ is an ordered abelian group by setting $\upsilon(x) + \upsilon(y) = \upsilon(xy)$ and $\upsilon(x) > 0$ iff $|x| < 2^{-n}$ for all $n \in \mathbb{N}$.
- If $x, y \in A^*$ and $|x| \geq |y|$, then $\upsilon(y) \leq \upsilon(x)$. The converse fails.
- If $x, y \in A^*$ and $x \neq -y$, then $\upsilon(x+y) \geq \min\{\upsilon(x), \upsilon(y)\}$, with equality if $\upsilon(x) \neq \upsilon(y)$.
- If A is a Hardy field, then $\upsilon = \nu$ (as defined in Sect. 3).
- If A can be expanded to a model $\mathfrak{A}$ of $\mathrm{Th}(\overline{\mathbb{R}}^K)$, then the map

$$(r, \upsilon(a)) \mapsto \upsilon(|a|^r)\colon K \times \upsilon(A^*) \to \upsilon(A^*)$$

  is well defined (where x^r is interpreted in $\mathfrak{A}$), and $\upsilon(A^*)$ is an ordered K-vector space with this map as scalar multiplication. We let $r.\upsilon(a)$ denote the image of $(r, \upsilon(a))$ under this map.

4.10. *Let* $\mathfrak{A} \equiv \mathfrak{B} \equiv \mathfrak{R}$ *and* $\mathfrak{A} \subseteq \mathfrak{B}$*. If* $Y \subseteq B^*$ *and* $\upsilon(Y)$ *is* K*-linearly independent over* $\upsilon(A^*)$*, then* Y *is definably independent with respect to* $\mathrm{Th}(\mathfrak{R})$ *over* A*.*

This is a special case of a result due to van den Dries [30, Theorem C] that depends on earlier work with Lewenberg [32] as well as 4.8. A complete and self-contained (modulo 4.8) proof appears in [24].

5 Expanding by Powers: Proof of Theorem 1.4

Throughout this section, we assume that $\mathfrak{R}$ is o-minimal and polynomially bounded with field of exponents K. By 4.8, we suppose that $\mathfrak{R}$ is an expansion in the syntactic sense of $\overline{\mathbb{R}}^K$, so that the notation x^r makes sense in arbitrary models of $\mathrm{Th}(\mathfrak{R})$. Let $S \subseteq \mathbb{R}$ be such that for every $r \in S$ the restriction $x^r \restriction [1,2]$ is $\emptyset$-definable. Put $J = K(S)$. Note that $x^r \restriction [1,2]$ is $\emptyset$-definable in $\mathfrak{R}$ for every $r \in J$. Let L be the language of $\mathfrak{R}$, L^J be the extension of L by pairwise distinct new unary function symbols f_r for the $r \in J$, and T be the extension of $\mathrm{Th}(\mathfrak{R})$ by the universal closures of the following formulas:

P1. For each $r \in J$,

$$[(x = 0 \vee x < 0) \to f_r(x) = 0] \wedge [(0 < x \wedge 0 < y) \to f_r(xy) = f_r(x) \cdot f_r(y)],$$

P2. For each $r \in J^{>0}$, $\quad 1 < x \to 1 < f_r(x)$.
P3. For each $r, s \in J$, $\quad f_{rs}(x) = f_r(f_s(x)) \wedge f_{r+s}(x) = f_r(x) \cdot f_s(x)$.
P4. For each $r \in J$, a formula $\varphi_r(x, y) \to y = f_r(x)$, where φ_r is an L-formula that defines the graph of $x^r \restriction [1,2]$ in $\mathfrak{R}$.
P5. $0 < x \to f_0(x) = 1 \wedge f_1(x) = x$.

Evidently, $\mathfrak{R}^J \models T$ by interpreting f_r as x^r for $r \in J$. We are going to show that T axiomatizes $\mathfrak{R}^J$ by showing that T is complete. Our first goal is to show that T admits QE relative to $\mathrm{Th}(\mathfrak{R})$. Next is the key technical lemma.

5.1. *Suppose that $\mathfrak{R}$ admits QE and is $\forall$-axiomatizable. Let $\mathfrak{A} \subsetneq \mathfrak{B} \models T$ and $\mathfrak{B}' \succeq \mathfrak{A}$ be $|\mathfrak{B}|^+$-saturated. Then there exists $\mathfrak{D} \models T$ with $\mathfrak{A} \subsetneq \mathfrak{D} \subseteq \mathfrak{B}$ such that $\mathfrak{D}$ embeds over $\mathfrak{A}$ into $\mathfrak{B}'$.*

We have some preliminary work to do, beginning with some easy (proofs are left to the reader) consequences of ordered field properties and P1–P5.

5.2. *Let $\mathfrak{A} \models T$.*

- *For all $r \in K$, $f_r \restriction A^{>0} = x^r \restriction A^{>0}$.*
- *For all $r \in J^*$, $f_r \restriction A^{>0}$ is an automorphism of the multiplicative group $A^{>0}$, with compositional inverse $f_{1/r} \restriction A^{>0}$. If $r > 0$, then f_r is strictly increasing. If $r < 0$, then f_r is strictly decreasing.*
- *If $a \in A^{>0}$ and $s = \sum_{i=1}^{n} r_i s_i$ with $r_i \in K$ and $s_i \in J$ for $i = 1, \ldots, n$, then $f_s(a) = \prod_{i=1}^{n} f_{r_i}(f_{s_i}(a))$.*
- *If $0 < r < s$ and $a > 1$, then $f_r(a) < f_s(a)$.*
- *If $r \neq s$ and $a > 0$, then $f_r(a) = f_s(a)$ iff $a = 1$.*
- *The map $r.\upsilon(a) := (r, \upsilon(a)) \mapsto \upsilon(f_r(|a|))\colon J \times \upsilon(A^*) \to \upsilon(A^*)$ is well defined, and $\upsilon(A^*)$ is an ordered J-vector space with this map as scalar multiplication.*

We may write a^r instead of $f_r(a)$ without ambiguity whenever $r \in J$ and $a > 0$. Both notations will be used according to convenience.

Note that, as K is a subfield of J, we can also regard $\upsilon(A^*)$ as an ordered K-vector space; we shall have occasion to do so.

Put $\mathrm{Un}^+(A) = A^{>0} \cap \upsilon^{-1}(0)$. Note that if $u \in \mathrm{Un}^+(A)$ then there is some nonzero integer k such that $1 \leq u^{1/k} \leq 2$.

Given $\mathfrak{B} \models T$, $X \subseteq B$ and $S \subseteq J$, we say that X is **closed under powers from** S if $x^s \in X$ for all $x \in X$ and $s \in S$.

Given an L^J-structure $\mathfrak{A}$, let $\mathfrak{A}_L$ denote the reduct in the syntactic sense of $\mathfrak{A}$ to L. Note that if $\mathfrak{A} \models T$, then $\mathfrak{A}_L \equiv \mathfrak{R}$.

5.3. *Let $\mathfrak{B} \models T$ and $\mathfrak{A} \subseteq \mathfrak{B}_L$. If there exists $P \subseteq A^{>0}$ such that $\upsilon(P) = \upsilon(A)$ and $p^r \in A$ for all $p \in P$ and $r \in J$, then A is closed under powers from J and $(\mathfrak{A}, (x^r \restriction A)_{r\in J}) \models T$.*

Proof. Since T has a universal axiomatization, it suffices to show that A is closed under powers from J. Let $a \in A^{>0}$. (The result is trivial for $a \leq 0$ by definition of f_r.) Then there exists $u \in \mathrm{Un}^+(A)$ and $p \in P$ with $a = up$. Let $r \in J$. Since $p^r \in A$, it suffices to show that $u^r \in A$. We have some $k \in \mathbb{Z}^*$ such that $1 \leq u^{1/k} \leq 2$. Since $\mathfrak{A} \subseteq \mathfrak{B}_L \models \mathrm{Th}(\mathfrak{R})$, we have $(u^{1/k})^r \in A$ by P4. Thus, $u^r = ((u^{1/k})^r)^k \in A$. □

5.4. *Let $\mathfrak{A}, \mathfrak{B} \models T$ and $\phi\colon \mathfrak{A}_L \to \mathfrak{B}_L$ be an embedding. If there exists $P \subseteq A^{>0}$ such that $\upsilon(P) = \upsilon(A)$ and $\phi(p^r) = \phi(p)^r$ for all $r \in J$ and $p \in P$, then ϕ is an embedding $\mathfrak{A} \to \mathfrak{B}$.*

Proof. Let $a \in A^{>0}$ and $r \in J$. We must show that $\phi(a^r) = \phi(a)^r$. Since $a = up$ as in 5.3, it suffices consider the case that $\upsilon(a) = 0$. As before, we then have $1 \leq a^{1/k} \leq 2$ for some $k \in \mathbb{Z}^*$. By P4, we have $\phi(a^r) = \phi((a^{1/k})^{rk}) = \phi(a^{1/k})^{rk} = \phi(a)^{rk(1/k)} = \phi(a)^r$ as required. □

Given $\mathfrak{M} \equiv \mathfrak{R}$ and $X \subseteq M$, let $\mathfrak{M}\langle X\rangle$ denote the substructure of $\mathfrak{M}$ generated by X.

5.5. *Suppose that*:

- $\mathfrak{M}, \mathfrak{M}', \mathfrak{N}, \mathfrak{N}' \models \mathrm{Th}(\mathfrak{R})$;
- $\mathfrak{M} \preceq \mathfrak{N}$ *and* $\mathfrak{M}' \preceq \mathfrak{N}'$;
- $\phi_0\colon \mathfrak{M} \to \mathfrak{M}'$ *is an isomorphism*;
- *$X \subseteq N$ and $X' \subseteq N'$ are such that ϕ_0 extends to an order-preserving bijection $\phi_1\colon M \cup X \to M' \cup X'$.*

Then there is an isomorphism $\phi\colon \mathfrak{M}\langle X\rangle \to \mathfrak{M}'\langle X'\rangle$ extending ϕ_1.

Proof. By o-minimality, the type of an element $x \in N$ over $\mathfrak{M}$ is determined by the cut that it realizes in M, and similarly for $x' \in N'$ over $\mathfrak{M}'$. □

Proof of 5.1. In order to reduce clutter, we shall delete the superscripted $*$ from expressions like $\upsilon(A^*)$ for ordered fields A.

Suppose that $\mathfrak{R}$ admits QE and is $\forall$-axiomatizable. Note that substructures of models of $\mathrm{Th}(\mathfrak{R})$ are elementary substructures. Let $\mathfrak{A} \subsetneq \mathfrak{B} \models T$ and $\mathfrak{B}' \succeq \mathfrak{A}$ be

$|B|^+$-saturated. We must find $\mathfrak{D} \models T$ with $\mathfrak{A} \subsetneq \mathfrak{D} \subseteq \mathfrak{B}$ such that $\mathfrak{D}$ embeds over $\mathfrak{A}$ into $\mathfrak{B}'$. We proceed by a trivial case distinction: $\upsilon_B(A) = \upsilon_B(B)$ or $\upsilon_B(A) \neq \upsilon_B(B)$. The overall strategy is the same in both cases, but the tactics are more involved for the latter.

Suppose that $\upsilon_B(A) = \upsilon_B(B)$. Let $x \in B \setminus A$. By saturation, choose $y \in B' \setminus A$ realizing the same cut in A as x. Put $\mathfrak{C} = \mathfrak{A}_L\langle x\rangle$ and $\mathfrak{C}' = \mathfrak{A}_L\langle y\rangle$. Note that $\mathfrak{C} \preceq \mathfrak{B}_L$ and $\mathfrak{C}' \preceq \mathfrak{B}'_L$. By 5.5, there is an isomorphism $\phi\colon \mathfrak{C} \to \mathfrak{C}'$ fixing A pointwise and sending x to y. Trivially, $\upsilon_B(C) = \upsilon_B(A) = \upsilon_A(A) = \upsilon_{B'}(C')$. By 5.3, $\mathfrak{C}$ and $\mathfrak{C}'$ expand to submodels $\mathfrak{D}$ and $\mathfrak{D}'$ of $\mathfrak{B}$ and $\mathfrak{B}'$. By 5.4, ϕ is an L^J-embedding of $\mathfrak{D}$ into $\mathfrak{B}'$ over $\mathfrak{A}$ as desired.

Suppose that $\upsilon_B(B) \neq \upsilon_B(A)$. Take $x \in B^{>0}$ with $\upsilon_B(x) \notin \upsilon_B(A)$. Choose $y \in B'$ realizing the same cut in A as x. Fix a K-basis E for J such that $1 \in E$. Put $\mathfrak{C} = \mathfrak{A}_L\langle x^e : e \in E\rangle$ and $\mathfrak{C}' = \mathfrak{A}_L\langle y^e : e \in E\rangle$. Put $P = \{ax^r : a \in A^{>0}\ \ \&\ \ r \in J\}$ and $P' = \{ay^r : a \in A^{>0}\ \ \&\ \ r \in J\}$. As A is closed under powers from J, so are P and P'. Observe that $\upsilon_B(P) = \upsilon_B(A) + J.\upsilon_B(x)$ and $\upsilon_{B'}(P') = \upsilon_B(A) + J.\upsilon_B(y)$. Thus, it suffices now by 5.3 and 5.4 (and basic algebra) to show that $\upsilon_B(C) = \upsilon_B(A) + \sum_{e\in E} K.\upsilon_B(x^e)$, $\upsilon_{B'}(C') = \upsilon_{B'}(A) + \sum_{e\in E} K.\upsilon_{B'}(y^e)$, and there is an isomorphism $\phi\colon \mathfrak{C} \to \mathfrak{C}'$ fixing A pointwise and sending x^e to y^e for all $e \in E$. Recall that y satisfies the same cut in A as does x. Hence, the same is true of the pairs (x^e, y^e) with $e \in E$, because A is closed under powers from J. It is easy to see that $\{\upsilon_B(x^e) : e \in E\}$ is K-linearly independent over $\upsilon_B(A)$ and $\{\upsilon_{B'}(y^e) : e \in E\}$ is K-linearly independent over $\upsilon_{B'}(A) = \upsilon_B(A)$. The rest of the argument is now routine via 4.10 and 5.5. □

A useful corollary of the proof of 5.1:

5.6. *Let $\mathfrak{A} \equiv \mathfrak{B} \equiv \mathfrak{R}^J$ and $\mathfrak{A} \subseteq \mathfrak{B}$. If $Y \subseteq B^*$ and $\upsilon(Y)$ is J-linearly independent over $\upsilon(A^*)$, then Y is definably independent* (*with respect to* Th($\mathfrak{R}^J$)) *over A.*

(In other words, the conclusion of 4.10 holds with $\mathfrak{R}^J$ in place of $\mathfrak{R}$ and J in place of K.)

Proof of 1.4

(1) Suppose that $\mathfrak{R}$ has QE and is $\forall$-axiomatizable. We must show that the same is true of $\mathfrak{R}^J$. By 5.1 and general model theory (say, 2.3.9 and 4.3.28 of [7]), T has QE. Let $\mathfrak{P}$ be the prime submodel of $\mathfrak{R}$ (recall that $\mathfrak{R}$ has definable Skolem functions) and P be its underlying set. As $(P^{>0}, \cdot)$ is archimedean as an ordered group, we have $a^r \subseteq P$ for every $a \in P^{>0}$ and $r \in J$. (For every $a \in P^{>0}$ there exist $b \in [1,2)\cap P$ and $k \in \mathbb{Z}$ such that $a^r = (b^r)^k$.) Thus, $\mathfrak{P}$ expands naturally to an L^J-structure that embeds into every model of T. Hence, T is complete, so $\mathfrak{R}^J$ has QE and is axiomatized by T. Since $\mathfrak{R}$ has QE, the formulas φ_r in P4 can be taken to be quantifier free, so that all of the formulas of P1–P5 are quantifier free. Since $\mathfrak{R}$ is $\forall$-axiomatizable, so is T, hence also $\mathfrak{R}^J$.

(2) Suppose that $\mathfrak{R}$ is model complete. We must show that the same is true of $\mathfrak{R}^S$. Let $\widetilde{\mathfrak{R}}$ be the expansion of $\mathfrak{R}$ by all $\emptyset$-definable Skolem functions. Then $\widetilde{\mathfrak{R}}$ has QE and is $\forall$-axiomatizable. By Theorem 1.4.1, $\widetilde{\mathfrak{R}}^J$ has QE. Since $\mathfrak{R}$ is

model complete, its Ø-definable Skolem functions are both existentially and universally definable, so a routine syntactic argument involving "de-nesting" of terms shows that $\mathfrak{R}^J$ is model complete. For every $r \in J$, both r and x^r are existentially and universally definable in $\overline{\mathbb{R}}^S$. (There exist $m \in \mathbb{N}$, polynomials $p, q \in K[x_1, \ldots, x_m]$ and $s \in S^m$ such that $p(s) = rq(s)$.) Hence, $\mathfrak{R}^S$ is model complete.

(3) Let $\widetilde{\mathfrak{R}}$ be as in the previous paragraph. Since $\widetilde{\mathfrak{R}}$ has QE and is ∀-axiomatizable, the same is true of $(\widetilde{\mathfrak{R}}, (r)_{r\in\mathbb{R}})$. (Each real number is determined by its position in $(\mathbb{R}, <)$ relative to $\mathbb{Q}$, and each $q \in \mathbb{Q}$ is Ø-definable in $\overline{\mathbb{R}}$.) By Theorem 1.4.1, $(\widetilde{\mathfrak{R}}, (r)_{r\in\mathbb{R}})^J$ has QE and is ∀-axiomatizable. By general model theory (or see [38, 2.15]), every function Ø-definable in $(\widetilde{\mathfrak{R}}, (r)_{r\in\mathbb{R}})^J$ is given piecewise by terms.[8] Hence, every function definable in $\mathfrak{R}^J$ is given piecewise by (iterated) compositions of functions definable in $\mathfrak{R}$ and powers from J.

(4) We must show that $\mathfrak{R}^J$ is polynomially bounded with field of exponents J. Recall that $\mathfrak{R}^J$ is o-minimal. After passing to an extension by definitions, we reduce to the case that $\mathfrak{R}$ has QE and is ∀-axiomatizable.

Suppose that $\mathfrak{R}^J$ is not polynomially bounded; then it is exponential by Growth Dichotomy. Let $\mathfrak{B}$ be a proper elementary extension of $\mathfrak{R}^J$. There exists $b \in B$ such that $b > \mathbb{R}$. As b and e^b are definably dependent (where e^x is interpreted in $\mathfrak{B}$), there exists by 5.6 some $r \in K$ and $N \in \mathbb{N}$ such that $e^b \le Nb^r$, contradicting that there exists $k \in \mathbb{N}$ such that $e^x > Nx^r$ for all $x > k$.

Now we show that $\mathfrak{R}^J$ has field of exponents J. Let $s \in \mathbb{R}$ be such that x^s is definable in $\mathfrak{R}^J$. Let $\mathfrak{B}$ and b be as before. By arguing as above, we have $\upsilon(b^s) = \upsilon(b^r)$ for some $r \in K$. Then $\upsilon(b^{s-r}) = 0$, so there exists $N \in \mathbb{N}$ such that $b^{|s-r|} \le N$. Thus, $s \in J$, for if not, there exists $k \in \mathbb{N}$ such that $x^{|s-r|} > N$ for all $x > k$, contradicting that $b^{|s-r|} \le N$. □

6 Proof of Theorem 1.7

Let $\mathfrak{R}$ be o-minimal and polynomially bounded with field of exponents K. Let $N \in \mathbb{N}$, $G\colon \mathbb{R}^{N+2} \to \mathbb{R}$ be definable in $\mathfrak{R}$, and g be a solution on some ray (a, ∞) to the differential equation $y^{(N+1)} = G(x, y, y', \ldots, y^{(N)})$. Suppose that g is definable in an o-minimal expansion $\widetilde{\mathfrak{R}}$ of $\mathfrak{R}$. We must show that there exists $r \in K$ such that $|g(x)| / \exp_{N+1}(x^r)$ is ultimately bounded. (Of course, we can take $r \in \mathbb{N}$ if desired since we are working over $\mathbb{R}$.) This is obvious if $\widetilde{\mathfrak{R}}$ is polynomially bounded, so assume otherwise; then it is exponential by Growth Dichotomy. Thus, working in the Hardy field $\widetilde{\mathcal{H}}$ of $\widetilde{\mathfrak{R}}$ (recall Sect. 3), it suffices to take g infinitely increasing and show that $\nu(g) \ge \nu(\exp_{N+1}(x^r))$ for some $r \in K$.

[8] That is, if $g\colon \mathbb{R}^n \to \mathbb{R}$ is definable, then there is a finite $\mathcal{F} \subseteq \mathcal{T}_n$ such that the graph g is contained in the union of the graphs of the $f \in \mathcal{F}$.

As this statement is definability-theoretic, we make some convenient assumptions about the languages L and $\widetilde{L}$ of $\mathfrak{R}$ and $\widetilde{\mathfrak{R}}$: (i) $\{<,+,-,\cdot\} \subseteq L \subseteq \widetilde{L}$; (ii) L has constants for all real numbers; (iii) $\widetilde{L}$ has no relation symbols other than $<$; and (iv) both $\mathfrak{R}$ and $\widetilde{\mathfrak{R}}$ have QE and are $\forall$-axiomatizable (after expanding by all definable Skolem functions). Let $\mathcal{H}$ be the Hardy field of $\mathfrak{R}$. Note that $\widetilde{\mathcal{H}}$ is an extension of $\mathcal{H}$ as an ordered differential field. For an n-ary function symbol F of $\widetilde{L}$ and $f_1, \dots, f_n \in \widetilde{\mathcal{H}}$, let $F(f_1, \dots, f_n)$ denote the germ of the function $F \circ (f_1, \dots, f_n)$; then $F(f_1, \dots, f_n) \in \widetilde{\mathcal{H}}$. It follows that $\mathcal{H}$ and $\widetilde{\mathcal{H}}$ (as ordered rings) expand naturally to L-structures, with $\mathfrak{R} \subseteq \mathcal{H} \subseteq \widetilde{\mathcal{H}}$. Let $\mathfrak{H}$ be the L-substructure of $\widetilde{\mathcal{H}}$ generated by $\mathcal{H} \cup \{g^{(k)} : k \leq N\}$; then $g \in \mathfrak{H}^*$ and $\mathfrak{R} \subseteq \mathcal{H} \subseteq \mathfrak{H} \subseteq \widetilde{\mathcal{H}}$. We will show that for every $f \in \mathfrak{H}^*$ there exists $r \in K$ such that $v(f) \geq v(\exp_{N+1}(x^r))$ to finish.

We now show that the underlying ring of $\mathfrak{H}$ is a Hardy field. Let $f \in \mathfrak{H}^*$. We must show that both $1/f$ and f' belong to $\mathfrak{H}$. There exist: $n \in \mathbb{N}$; $a_1, \dots, a_n \in \mathcal{H}$; and $F\colon \mathbb{R}^{n+1+N} \to \mathbb{R}$ definable in $\mathfrak{R}$ such that $f = F(a_1, \dots, a_n, g^{(0)}, \dots, g^{(N)})$. Since $a_1, \dots, a_n$ are germs of functions definable in $\mathfrak{R}$, we may take $n = 1$ and $a_1 = x$ by replacing F with

$$(y_1, \dots, y_{N+2}) \mapsto F(a_1(y_1), \dots, a_n(y_1), y_2, \dots, y_{N+2}).$$

The function

$$(y_1, \dots, y_{N+2}) \mapsto \begin{cases} 1/F(y_1, \dots, y_{N+2}), & F(y_1, \dots, y_{N+2}) \neq 0 \\ 0, & F(y_1, \dots, y_{N+2}) = 0 \end{cases}$$

is definable in $\mathfrak{R}$, so $1/f \in \mathfrak{H}$. By cell decomposition, there is a C^1-cell $C \subseteq \mathbb{R}^{2+N}$ such that $F \restriction C$ is C^1 and the curve $(x, g^{(0)}, \dots, g^{(N)})$ ultimately lies in C. If C is open, then ultimately $f' = H(x, g^{(0)}, \dots, g^{(N)})$, where

$$H(y_1, \dots, y_{N+2}) = \nabla F(y_1, \dots, y_{N+2}) \cdot (1, y_2, \dots, y_{N+2}, G(y_1, \dots, y_{N+2})).$$

Since H is definable in $\mathfrak{R}$, we have $f' \in \mathfrak{H}$. The case that C is not open is left as an exercise. (*Hint:* Every C^1-cell is definably C^1-diffeomorphic to an open C^1-cell.)

We now show that $\mathcal{H}$, $\mathfrak{H}$ and $\widetilde{\mathcal{H}}$ are models of $\mathrm{Th}(\mathfrak{R})$. By QE and $\forall$-axiomatizability of $\mathfrak{R}$, substructures of models of $\mathrm{Th}(\mathfrak{R})$ are elementary substructures, so it suffices to show that $\widetilde{\mathcal{H}} \models \mathrm{Th}(\mathfrak{R})$; for this, it suffices by $\forall$-axiomatizability of $\mathfrak{R}$ to show that every universal $\widetilde{L}$-sentence true in $\widetilde{\mathfrak{R}}$ is true in $\widetilde{\mathcal{H}}$. Let $\varphi(v_1, \dots, v_n)$ be a quantifier-free $\widetilde{L}$-formula such that $\widetilde{\mathfrak{R}} \models \forall v_1 \dots v_n \varphi$. Let $f_1, \dots, f_n$ be unary functions definable in $\widetilde{\mathfrak{R}}$. Then $\widetilde{\mathfrak{R}} \models \forall x \varphi(f_1(x), \dots, f_n(x))$, so $\widetilde{\mathcal{H}} \models \varphi(f_1, \dots, f_n)$. Hence, $\widetilde{\mathcal{H}} \models \forall v_1 \dots v_n \varphi$.

Now, $\mathfrak{H}$ is closed under powers from K, so $v(\mathfrak{H}^*)$ is a K-linear space. As we already know that $v(\mathcal{H}^*) = K.v(x)$ by Growth Dichotomy, it follows from 4.10 and the previous paragraph that the K-linear dimension of $v(\mathfrak{H}^*)$ is at least 1 and at most $N + 2$.

Let $f_0 \in \mathfrak{H}^*$. We show there exists $r \in K$ such that $\nu(f_0) \geq \nu(\exp_{N+1}(x^r))$. (*Aside:* We shall actually need only that $\mathfrak{H}$ is a Hardy field extending $\mathbb{Q}(x)$ that is closed under powers from K and $\dim_K \nu(\mathfrak{H}^*) \leq N+2$.) It suffices to consider the case that f_0 is infinitely increasing.

If $\nu(f_0'/f_0) \geq \nu(1/x)$, then $\nu(\log f_0) \geq \nu(\log x)$ by l'Hôpital, so there exists $r \in K$ such that $\nu(f_0) \geq \nu(x^r) = \nu(\exp_0(x^r))$, and we are done. So assume that $\nu(f_0'/f_0) < \nu(1/x)$. By 3.5 and 3.6, there exists $f_1 \in \mathfrak{H}^*$ such that $f_1' \sim f_0'/f_0$, that is, $f_1 \sim \log f_0$. Observe that $\nu(f_0)$ and $\nu(f_1)$ are K-linearly independent (indeed, $\mathbb{R}$-linearly independent).

If $\nu(f_1'/f_1) \geq \nu(1/x)$, then by arguing as before there exists $r \in K$ such that $\nu(f_1) > \nu(x^r)$. By increasing r, we have $\nu(f_0) \geq \nu(e^{x^r}) = \nu(\exp_1(x^r))$, and we are done. So assume that $\nu(f_1'/f_1) < \nu(1/x)$. By arguing as before, there exists $f_2 \in \mathfrak{H}^*$ such that $f_2 \sim \log f_1 \sim \log\log f_0$; then $\{\nu(f_0), \nu(f_1), \nu(f_2)\}$ is K-linearly independent.

Continuing in this fashion, we obtain $m \leq N+1$ and $f_1, \ldots, f_m \in \mathcal{H}^*$ such that $f_m \sim \log f_{m-1} \sim \cdots \sim \log_m f_0$ and $\nu(f_m) \geq \nu(x^s)$ for some $s \in K$. Hence, there exists $r \in K$ such that $\nu(f_0) \geq \nu(\exp_m(x^r)) \geq \nu(\exp_{N+1}(x^r))$ as required. (We leave the details to the reader.) □

7 Suggestions for Further Study

We have only scratched the surface in these notes.

For a proper introduction to Hardy fields, I strongly recommend the papers [19–22] by Rosenlicht.

With just a little extra work (primarily, pinning down what should be the definition of power function), the growth dichotomy can be extended to O-minimal expansions of arbitrary ordered fields; see [11] for details. An even further extension to o-minimal expansions of arbitrary ordered groups is due to Miller and Starchenko [15], but its statement and proof are considerably more involved.

The literature on analytic-geometric properties of o-minimal expansions of $\overline{\mathbb{R}}$ is now quite extensive; [34] is a good start.

See van den Dries and Speissegger [37] for a more definability-theoretic approach to (parts of) Theorem 1.4, and [35,36] for some other interesting examples of polynomially bounded o-minimal expansions of $\overline{\mathbb{R}}$.

See [6] for a more elaborate statement of Theorem 1.7, and [14] for an exponential analogue.

One might wonder about polynomially (or exponentially) bounded expansions of $\overline{\mathbb{R}}$ that are not o-minimal, but it turns out that such structures are "almost o-minimal" in a way that can be made precise; see [3, 13] for details.

We close with just a few open questions.

If $K \neq \mathbb{Q}$ and is a subfield of $\mathbb{R}$, then QE fails for $(\overline{\mathbb{R}}, (r)_{r\in K}, ((1+x^2)^r)_{r\in K})$ by 4.7. Does QE fail for $\overline{\mathbb{R}}^K$? (Careful!)

Let $\mathfrak{R}$ be o-minimal and polynomially bounded with field of exponents $K \neq \mathbb{R}$.

1. Does $(\mathfrak{R}, x^r \restriction [1,2])$ have field of exponents K for some $r \in \mathbb{R} \setminus K$?
2. Does $(\mathfrak{R}, x^r \restriction [1,2])$ have field of exponents K for every $r \in \mathbb{R} \setminus K$?
3. Does $(\mathfrak{R}, (x^r \restriction [1,2])_{r \in \mathbb{R} \setminus K})$ have field of exponents K?
4. Does $(\mathfrak{R}, \exp \restriction [0,1])$ have field of exponents K?

Evidently, (4)⇒(3)⇒(2)⇒(1). Does (1)⇒(2)? Does (2)⇒(3)? Does (3)⇒ (4)? Is there any proper subfield K of $\mathbb{R}$—in particular, $K = \mathbb{Q}$—for which any of the above can be answered?

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Construction of O-minimal Structures from Quasianalytic Classes

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Abstract I present the method of constructing o-minimal structures based on local reduction of singularities for quasianalytic classes.

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Introduction

The notion of *o-minimal structure* was introduced by van den Dries as a convenient framework for the resolution of Hilbert's 16th problem [21]. Since then, various methods have been designed to prove that a given family of functions is definable in such a structure. The goal of these notes is to summarize one of these methods, and to recall several situations where it has been used successfully.

Among the techniques involved in proving o-minimality are establishing *quantifier elimination* or *model completeness*; famous examples of these are the *Tarski-Seidenberg theorem* for semi-algebraic sets and *Gabrielov's theorem of the complement* for globally subanalytic sets, respectively. Both techniques lead to o-minimality once finiteness of the number of connected components of quantifier-free definable sets is established. For example, finiteness properties proved by Khovanskii imply o-minimality of the structure $\mathbb{R}_{\exp}$ via Wilkie's model completeness result [25], and they lead to o-minimality of the structure $(\mathbb{R}_{\text{an}}, \exp)$ via van den Dries, Macintyre and Marker's quantifier elimination result [24].

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Van den Dries and Speissegger [23] developed a method to establish model completeness and o-minimality of the structure generated by real *Gevrey functions* (which are a subtle generalization of the functions introduced by Tougeron [20]). The outline of their method is an adaptation of a classical scheme, based on the following properties :

1. A *nonzero* Gevrey function admits a *nonzero* asymptotic expansion at the origin, which is a formal power series. This property is usually called *quasianalyticity*. Hence a nonzero Gevrey function admits a *Newton polyhedron*.
2. After a convenient blowing-up, the Newton polyhedron of a given function becomes simpler. In particular, the function may become analytic in at least one variable.
3. If a Gevrey function is analytic and regular in one variable, then Weierstrass preparation holds.

These properties allow one to prove that every existentially definable set can be described as a finite union of projections of quantifier-free definable sets with "small frontier". A generalization of the arguments involved in the proof of Gabrielov's theorem of the complement for subanalytic sets then leads to model completeness.

Having in mind to extend the former arguments to other families of functions, let us examine them more carefully. The important starting point of this strategy is the quasianalyticity property, which establishes, for every integer $n \geq 0$, a kind of dictionary between the Gevrey functions in n variables and a subalgebra of the algebra $\mathbb{R}[[X_1, \dots, X_n]]$. Hence, the algebraic operations (such as blowings-up) that simplify the formal series also simplify the corresponding function. This approach naturally leads us to wonder what the relationship between quasianalyticity and o-minimality may be.

Second, Weierstrass preparation does not hold in general in the quasianalytic framework. Fortunately, it turns out that Weierstrass preparation is only a way to accelerate the simplification process. It can be replaced by a refinement of the blowing-up process, called *(local) normalization*. (Throughout this paper, we shall omit the word "local" in this context, as no other kind of normalization is discussed.) In a first step, this process is purely formal; that is, it is applied only to formal power series. It implicitly uses the closure property of the rings of power series under some classical operations, such as composition and partial differentiation. The normalization process then transfers to the functions, thanks to quasianalyticity, under the assumption that we work with quasianalytic classes closed under the same operations.

This approach, first suggested by van den Dries, has been described in detail by Speissegger, Wilkie and the author of these notes in [17]. It is applied to the structures generated by so-called *Denjoy-Carleman* classes, in order to give a negative answer to the following questions:

1. Does every o-minimal expansion of the real field admit analytic cell decomposition?
2. Is there a "largest" o-minimal expansion of the real field?

In these notes, we give some explanations of the normalization process mentioned above, its relationship with o-minimality, and its application in various contexts. The reader should be aware that we do not have in mind a coarse statement such as "quasianalyticity implies o-minimality". In Sect. 1, after having recalled a few classical definitions, we give an explicit example of a function that generates a quasianalytic algebra but is not definable in any o-minimal expansion of the real field. The underlying ideas of the generalized version of Gabrielov's theorem of the complement are described in Sect. 2. The normalization of formal power series is explained in Sect. 3. The o-minimality of the structures generated by convenient quasianalytic classes is proved in Sect. 4. Finally, several applications of the above techniques are given in Sect. 5.

Finally, let us recall a few classical notations and definitions. Let $\mathcal{F}$ be a collection of functions $f:\mathbb{R}^m \to \mathbb{R}$, for various $m \in \mathbb{N}$. The ***structure generated by*** $\mathcal{F}$ is the smallest collection $\mathcal{S}_{\mathcal{F}}$ of subsets of the spaces $\mathbb{R}^n$, $n \in \mathbb{N}$, which contains the singletons of $\mathbb{R}$, the diagonals, the graphs of addition, multiplication and of all functions in $\mathcal{F}$, and which is closed under finite boolean operations, cartesian products and taking coordinate projections. It is an elementary exercise in predicate logic to see that a set $A \subset \mathbb{R}^n$ belongs to $\mathcal{S}_{\mathcal{F}}$ if and only if A is ***definable (with parameters)*** in the first-order expansion $\mathbb{R}_{\mathcal{F}} = \left(\mathbb{R}, <, +, \cdot, 0, 1, (f)_{f\in\mathcal{F}}\right)$ of the real field, and we shall in general use the latter terminology. Correspondingly, a map $f: A \to \mathbb{R}^n$ is ***definable in*** $\mathbb{R}_{\mathcal{F}}$ if its graph is definable in $\mathbb{R}_{\mathcal{F}}$. The structure $\mathbb{R}_{\mathcal{F}}$ is ***model complete*** if every definable set is existentially definable; in the geometric terminology above, this means that $\mathcal{S}_{\mathcal{F}}$ is also generated by $\mathcal{F}$ without using the complement operation. Finally, the structure $\mathbb{R}_{\mathcal{F}}$ is ***o-minimal*** if every subset of $\mathbb{R}^n$ that is definable in $\mathbb{R}_{\mathcal{F}}$ has finitely many connected components.

Among the now classical examples of o-minimal structures, let us mention the real field $\mathbb{R}_{\emptyset}$, whose definable sets are exactly the sermialgebraic sets (Tarski-Seidenberg); the structure $\mathbb{R}_{\text{an}}$ generated by all *restricted analytic* functions [6]; and the structure $\mathbb{R}_{\text{Pfaff}}$ generated by all *pfaffian* functions [26]. If $\mathcal{F}$ is a collection of functions, the structure generated by all restricted analytic functions and the elements of $\mathcal{F}$ is denoted here by $\mathbb{R}_{\text{an},\mathcal{F}}$.

1 Quasianalyticity Does Not Imply O-minimality

Let $\mathcal{O}_2$ denote the algebra of real analytic germs at the origin of $\mathbb{R}^2$. In this section we give an example of a solution $f : (0, \varepsilon) \to \mathbb{R}$, with $\varepsilon > 0$ and $\lim_{x\to 0} f(x) = 0$, of a polynomial differential equation such that the algebra $\mathcal{A}_f = \{F(x, f(x)), F \in \mathcal{O}_2\}$ of germs of real functions at 0^+ is quasianalytic (and hence the curve $\{(x, f(x)), x \in (0, \varepsilon)\}$ is non-oscillating), but the structure $\mathbb{R}_{\mathcal{F}}$ is not o-minimal. By ***quasianalytic***, we mean here that every nonzero element of $\mathcal{A}_f$ has a nonzero asymptotic power series expansion at the origin. By ***non-oscillating***, we mean that $\mathcal{A}_f$ is a linearly ordered set of germs.

In order to construct the function f, we first consider Euler's classical differential equation $x^2y' = y - x$ and fix one of its solution $g\colon (0,\varepsilon) \to \mathbb{R}$. It is well known that g admits the asymptotic expansion $\hat{g}(x) = \sum_{n\geq 0} n!x^{n+1}$ at the origin. Moreover, g being a pfaffian function, the structure $\mathbb{R}_g$ is o-minimal [26]. Consider now the function $f : (0,\varepsilon) \to \mathbb{R}$ defined by $f(x) = g(x) + \exp\left(-\frac{1}{x}\right)\sin\left(\frac{1}{x}\right)$. Hence f is obtained from g by adding a flat oscillating term. Despite the oscillating nature of the perturbation, the algebra $\mathcal{A}_f$ is quasianalytic: indeed, every nonzero element $F(x, f(x))$ of $\mathcal{A}_f$ admits at the origin the asymptotic expansion $F(x, \hat{f}(x)) = F(x, \hat{g}(x))$. As it follows from Puiseux's theorem that every formal power series $\hat{h}(x)$ satisfying $F\big(x, \hat{h}(x)\big) = 0$ actually converges, we have $F(x, \hat{g}(x)) \neq 0$. Hence $\mathcal{A}_f$ is quasianalytic. One immediately deduces that f is non-oscillating: for each nonzero element h of $\mathcal{A}_f$, there exist $c \in \mathbb{R}$ and $n \in \mathbb{N}$ such that $\lim_{x\to 0^+} h(x)/cx^n = 1$.

We prove that $\mathbb{R}_f$ is not o-minimal by constructing an oscillating function using definable operations. The idea is to "kill" the principal part g of f: indeed, since g is a solution of the Euler equation, we get

$$x^2 f' - f + x = -\exp\left(-\frac{1}{x}\right)\sin\frac{1}{x};$$

since f' is definable in $\mathbb{R}_f$, it follows that the latter is not o-minimal.

We remark that, while f is a solution of a polynomial differential equation (exercise), it follows from [26] and the above that f is not definable in the structure $\mathbb{R}_{\mathrm{Pfaff}}$ generated by the pfaffian functions; in particular, f is not pfaffian.

Remark 1.1. 1. Non-oscillation is implied by, but is obviously not equivalent to, quasianalyticity. For example, by Khovanskii, the curve $\{(x, \exp(-1/x)), x > 0\}$ is non-oscillating; however, the algebra $\{F(x, \exp(-1/x)), F \in \mathcal{O}_2\}$ is clearly not quasianalytic.

2. We already mentioned that the structure $\mathbb{R}_g$ is o-minimal. However, Wilkie's general result on pfaffian functions does not give any information on the possible model completeness of this structure. Moreover, while the formal series $\hat{g}$ is ***Gevrey***, the function g itself is not a ***Gevrey function*** in the terminology of [23]. Indeed, the o-minimality of $\mathbb{R}_{\mathrm{an},g}$ is not a consequence of [23]; instead, it was proved by Schaefke, Sanz and the author in [18] based on the construction discussed here. (We also refer the reader to the notes written by F. Sanz in this volume.)
3. In the above example, $\mathbb{R}_f$ is not o-minimal because some definable operations (that kill g, but not f) produce an oscillating result. This suggests two possible ways to construct a pair of o-minimal expansions of the real field that do not admit any common o-minimal expansion:

   - *First method*: find a function g with a divergent asymptotic expansion and a flat oscillating term ε such that g and $f = g + \varepsilon$ vanish under the same definable operations. Such an example, where f and g are indeed solutions of the same linear differential equation, is given in [18], and explained without details in Sect. 5.

- *Second method*: find a "sufficiently transcendental" function g, in the sense that, say, g does not satisfy any definable relation over $\mathbb{R}_{\mathrm{an}}$ (i.e., any globally subanalytic relation). An example of such a function (or more precisely of a germ of such a function) is built by Le Gal and the author in [12], and by Le Gal alone in [11]. In particular, this function is not solution of any (nontrivial) analytic differential equation. Once such a function g is constructed, we consider again $f = g + \varepsilon$, for some flat oscillating term ε.

The example given in this section illustrates the following: the quasianalyticity of the algebra $\mathcal{A}_f$ does not imply the o-minimality of the structure $\mathbb{R}_f$. On an intuitive level, this is not surprising: since o-minimality means finitely many connected components for *every definable* set, the algebraic and quasianalytic properties of $\mathcal{A}_f$ alone are not sufficient to imply this. Hence we are naturally led to only consider a family of quasianalytic algebras rich enough for this task. For instance, one might guess that these algebras need to be closed under the operations classically involved in differential geometry. Rather than "quasianalyticity implies o-minimality", we prefer "quasianalyticity of a sufficiently rich family of algebras implies o-minimality".

2 Gabrielov Property, Model Completeness and O-minimality

2.1 Gabrielov Property

The content of this section, for which the main references are [22, Sect. 2], and [2, Sect. 3], is independent of the notions and methods mentioned above, such as quasianalyticity or normalization. We describe a widely used geometric test which implies model completeness and o-minimality. The proofs given or sketched in this section are not used in the sequel.

In this section, we have to deal with coordinate projections of sets. Given $m \leq n$, we denote by $\Pi^n_m : \mathbb{R}^n \to \mathbb{R}^m$ the projection on the first m coordinates. If $\lambda \in \{1, \dots, n\}^m$ is a strictly increasing sequence, we let $\Pi^n_\lambda : \mathbb{R}^n \to \mathbb{R}^m$ be the projection defined by $\Pi^n_\lambda (x_1, \dots, x_n) = \left(x_{\lambda(1)}, \dots, x_{\lambda(m)}\right)$. Hence Π^n_λ is a linear projection onto an m-dimensional coordinate subspace of $\mathbb{R}^n$. In general, n being clear from context, we will just write Π_m for Π^n_m and Π_λ for Π^n_λ.

We mentioned, in section "Introduction", the role played by sets with "small frontier". We define the ***frontier*** of a set $A \subset \mathbb{R}^n$ by $\mathrm{fr}\,(A) = \overline{A} \setminus A$, where $\overline{A}$ is the topological closure of A. Roughly speaking, the condition of having small frontier is intended to avoid classical oscillating phenomena. For example, the frontier of the oscillating curve :

$$\left\{\left(x, \sin\left(\frac{1}{x}\right)\right) : x > 0\right\}$$

is the segment $\{(0, y) : y \in [-1, 1]\}$. In the same spirit, a trajectory of a plane vector field that admits a limit cycle as its ω-limit set is an oscillating object whose frontier (the limit cycle) is not small.

In order to give a precise meaning to this notion of small frontier, we introduce a notion of dimension. We call ***manifold*** a non empty embedded smooth (of class $\mathcal{C}^\infty$) submanifold of $\mathbb{R}^k$ (for some $k \in \mathbb{N}$) everywhere of the same dimension $\dim(M)$. The manifolds considered in [22] are analytic. However, all the results of this section hold in the smooth framework as well. We say that a set $S \subset \mathbb{R}^k$ ***has dimension*** if S is a countable union of manifolds. In that case, we put

$$\dim(S) = \max\{\dim(M) : M \subset S \text{ is a manifold}\}$$

if S is nonempty, and $\dim(\emptyset) = -\infty$.

Remark 2.1. 1. Let $n \geq m$; if $S \subset \mathbb{R}^m$ has dimension, the same is not necessarily true for $\Pi_m(S)$ (exercise). However, if S and $\Pi_m(S)$ both have dimension, then $\dim(S) \geq \dim(\Pi_m(S))$.
2. We have seen above that the frontier of a manifold A may not be small: it may even happen that $\dim(\mathrm{fr}(A))$ is greater than $\dim(A)$.

For each $n \in \mathbb{N}$, we let Λ_n be a collection of bounded subsets of $\mathbb{R}^n$, and let $\Lambda = (\Lambda_n)_{n\in\mathbb{N}}$. We call the elements of Λ the Λ***-sets***. A set $E \subset \mathbb{R}^m$ is a ***sub-***Λ***-set*** if there are $n \geq m$ and a Λ-set $A \in \Lambda_n$ such that $E = \Pi_m(A)$. A Λ***-manifold*** (*resp.* ***sub-***Λ***-manifold***) is a Λ-set (*resp.* a sub-Λ-set) that is at the same time a manifold. For example, in the classical framework of analytic geometry, we would take the Λ-sets to be the bounded semianalytic sets.

Definition 2.2. A set $A \subset \mathbb{R}^n$ has the Λ***-Gabrielov property*** if for each $m \leq n$ there are connected sub-Λ-manifolds $B_1 \subset \mathbb{R}^{n+q_1}, \dots, B_k \subset \mathbb{R}^{n+q_k}$, where $q_1, \dots, q_k \in \mathbb{N}$, such that

$$\Pi_m(A) = \Pi_m(B_1) \cup \cdots \cup \Pi_m(B_k)$$

and for each $i = 1, \dots, k$ we have:

(G1) $\mathrm{fr}(B_i)$ is contained in a closed sub-Λ-set D_i such that D_i has dimension and $\dim(D_i) < \dim(B_i)$;
(G2) $\dim(B_i) \leq m$ and there is a strictly increasing $\lambda \in \{1, \dots, m\}^d$ with $d = \dim(B_i)$ such that $\Pi_\lambda | B_i : B_i \to \mathbb{R}^d$ is an immersion.

We recognize in (G1) the condition of "small frontier". It is instructive to compare this with the analytic situation: the *Fiber-cutting Lemma* [2, Lemma 3.6], states that, if $A \subset \mathbb{R}^n$ is a bounded semianalytic set, then there are finitely many smooth semianalytic sets $B_i \subset A$ such that:

1. $\Pi_m(A) = \Pi_m(\bigcup B_i)$;
2. The restriction $\Pi_m | B_i : B_i \to \mathbb{R}^m$ is a immersion for every i;
3. The subspaces $\Pi_m(\mathrm{T}_x B_i)$, for $x \in B_i$ and every i, have a common complement in $\mathbb{R}^m$.

This fiber-cutting lemma implies that all bounded semianalytic sets have the Λ-Gabrielov property (where Λ is the collection of all bounded semianalytic sets) where, for all i in Definition 2.2, $q_i = 0$, $B_i \subset A$, and $D_i = \mathrm{fr}\,(B_i)$ is a semianalytic set. The fact that the sets D_i are semianalytic and that $\dim\,(D_i) < \dim\,(B_i)$ is not obvious, but follows from the study of the general geometric properties of these sets due to Łojaciewicz [13]. It may happen that the knowledge of the geometric properties of the Λ-sets (which play in general the role of quantifier free definable sets) is a priori pretty poor, but it can be improved via a blowing-up process. Therefore, although the general Λ-Gabrielov property may look a bit cumbersome, it offers greater flexibility in the study of various classes of functions.

We can now state the main result of this section, which describes the geometric test needed for the applications in Sect. 5.

Theorem 2.3 ([22, Corollary 2.9]). *Assume that, for every $A, B \in \Lambda_n$ and $n \in \mathbb{N}$,*

1. $\{r\} \in \Lambda_1$ *for all* $r \in I$ *and the sets*

$$\left\{(x, y, z) \in I^3 \colon x + y = z\right\} \text{ and } \left\{(x, y, z) \in I^3 \colon xy = z\right\}$$

   belong to Λ_3;
2. $\emptyset$ *and* I^n *belong to* Λ_n, *and for each pair* (i, j) *with* $1 \leq i < j \leq n$ *the diagonal* $\Delta_{ij} = \left\{x \in I^n \colon x_i = x_j\right\}$ *as well as* Δ_{ij}^c *belong to* Λ_n;
3. $A \cup B$ *and* $A \cap B$ *belong to* Λ_n;
4. $I \times A$ *and* $A \times I$ *belong to* Λ_{n+1};
5. A *has the* Λ*-Gabrielov property.*

Then the expansion $\mathbb{R}_\Lambda = (\mathbb{R}, <, 0, 1, +, -, \cdot, \Lambda)$ *of the real field by all* Λ*-sets is model complete and o-minimal.*

The main argument in the proof of this theorem is the model completeness of the expansion $I_\Lambda = (I, (A)_{A \in \Lambda_n, n \in \mathbb{N}})$ of the interval $I = [-1, 1]$; see Sect. 2.2. A few routine arguments then allow us to conclude in Sect. 2.3.

2.2 Model Completeness of I_Λ

We suppose in this subsection that the family Λ satisfies hypothesis 2.–5. of Theorem 2.3. It is worth noting that every sub-Λ-set has only finitely many connected components, which are themselves sub-Λ-sets. Indeed, according to the Λ-Gabrielov property, each sub-Λ-set $E = \Pi_m\,(A)$ is equal to a finite union $\Pi_m\,(B_1) \cup \cdots \cup \Pi_m\,(B_k)$ where the B_i's are connected sub-Λ-sets. We are therefore not far from being convinced that, in this situation, "model-completeness implies o-minimality".

The model completeness of I_Λ is an easy corollary of the next two statements (see Corollary 2.6 below).

Lemma 2.4 ([22, Lemma 2.5], [2, Lemma 3.9]). *Suppose that for a certain d the complement of each sub-Λ-set in I^d is a sub-Λ-set. Let $\lambda \in \{1,\dots m\}^d$ be strictly increasing. Let E be a sub-Λ-set in I^m and suppose there exists $M \in \mathbb{N}$ such that $\left|E \cap \Pi_\lambda^{-1}(x)\right| \le M$ for all $x \in I^d$. Then the complement E^c of E in I^m is also a sub-Λ-set.*

Proof (Sketch of proof). For simplicity, we may assume that $\lambda(i) = i$ for each i, so that $\Pi_\lambda = \Pi_d$. For $x \in I^d$, write E_x for $E \cap \Pi_\lambda^{-1}(x)$. For $k \in \mathbb{N}$, the set $C_k = \left\{x \in I^d : |E_x| \ge k\right\}$ is a sub-Λ-set in I^d. Hence the set $D_k = \left\{x \in I^d : |E_x| = k\right\} = C_k \setminus C_{k+1}$ is a sub-Λ-set. Obviously E^c is the union of the sets $\Pi_d^{-1}\left(D_k^c\right) \setminus E$, for $k = 0,\dots,M$. Finally, $\Pi_d^{-1}\left(D_k^c\right) \setminus E$ is the set of pairs $(x,y) \in I^d \times I^{m-d}$ such that y is different from the k elements of E_x. This formula defines E^c as a sub-Λ-set. (We leave the details of this proof as an exercise.) □

Theorem 2.5 ([22, Theorem 2.7]). *If $E \subset I^m$ is a sub-Λ-set, then $E^c \subset I^m$ is a sub-Λ-set.*

Proof. By induction on m (the case $m = 0$ is clear). Let $m > 0$ and assume that the theorem holds for $d < m$. Let $E \subset I^m$ be a sub-Λ-set. Hence E is the linear projection of a Λ-set B. Thanks to the Λ-Gabrielov property, we may suppose that B is a connected sub-Λ-manifold $B \subset \mathbb{R}^n$, where $m \le n$, and that B has the following properties:

1. $\mathrm{fr}(B)$ is contained in a closed sub-Λ-set $D \subset I^n$ such that D has dimension and $\dim(D) < \dim(B)$.
2. $\dim(B) = d \le m$, and there exists a strictly increasing $\lambda \in \{1,\dots m\}^d$ such that $\Pi_\lambda|B : B \to \mathbb{R}^d$ is an immersion onto $F = \Pi_\lambda(B) = \Pi_\lambda^m(E)$.

Since $\Pi_m|B$ and $\Pi_\lambda|B$ have constant rank d we have, in particular, that $\dim(B) = \dim(E) = \dim(F) = d$. The proof is divided in two cases:

1. The "small" case, where $d < m$, which leads to the hypothesis of Lemma 2.4.
2. The "large" case, where $d = m$, where E^c is proved to be the union of the complement of a sub-Λ-set contained in $\Pi_m(D)$ (which falls under the "small" case) and certain connected components of $\Pi_m(D)^c$.

Case 1. $d < m$. We claim there exists $M \in \mathbb{N}$ such that

$$\left|\left(\Pi_\lambda^m\right)^{-1}(x) \cap E\right| \le \left|\Pi_\lambda^{-1}(x) \cap B\right| \le M$$

for all $x \in I^d$. The only non-obvious inequality is the one on the right. Note that $\Pi_\lambda|_B : B \to \mathbb{R}^d$ is a local homeomorphism. Put $B_x = \Pi_\lambda^{-1}(x) \cap B$ for $x \in I^d$. We divide I^d into two sets: a "small" set $G = \Pi_\lambda(D)$, which is a closed sub-Λ-set of dimension less than d, and G^c. Since every neighborhood of every point of G contains some points of G^c, it is enough to prove the result for $x \in G^c$. For such an x, it is enough to notice that the map $\Pi_\lambda|_{B\cap\Pi_\lambda^{-1}(G^c)} : B \cap \Pi_\lambda^{-1}(G^c) \to G^c$ is *proper* (hence, being a local homeomorphism, it is a *topological covering* *map*). Indeed, if $K \subset G^c$ is compact and $(u_k)_k \in B \cap \Pi_\lambda^{-1}(K)$ converges to $u \in I^n$, then clearly $u \in \Pi_\lambda^{-1}(K)$, and $u \in B$ (otherwise, u would belong to fr$B$, so $\Pi_\lambda(u) \in G$, contradicting $\Pi_\lambda(u) \in K$). Hence $|B_x|$ takes a constant finite value on each component of G^c. By the inductive assumption, G^c is a sub-Λ-set, which has therefore only finitely many connected components. The claim is proved, and Case 1 is a consequence of Lemma 2.4.

Case 2. $d = m$. The projection $\Pi_m|_B$ is a local homeomorphism; hence $\Pi_m(B)$ is open in $\mathbb{R}^m$. Since $\Pi_m(D)$ is a (closed) sub-Λ-set of dimension less than m, $(\Pi_m(D))^c$ is a sub-Λ-set by Case 1. Now note that

$$E^c = (\Pi_m(B))^c = (\Pi_m(B \cup D))^c \cup (\Pi_m(D) \setminus (\Pi_m(B) \cap \Pi_m(D))).$$

Morover, since $(\Pi_m(B \cup D))^c = (\Pi_m(B))^c \cap (\Pi_m(D))^c$, $\Pi_m(B)$ is open and $B \cup D$ is compact, it follows that $(\Pi_m(B \cup D))^c$ is open and closed in $(\Pi_m(D))^c$ and is, therefore, a sub-Λ-set. Since

$$\dim(\Pi_m(B) \cap \Pi_m(D)) < m,$$

we now conclude by Case 1. □

Corollary 2.6. *The structure I_Λ is model complete. Its definable sets are exactly the sub-Λ-sets contained in I^n, for $n \in \mathbb{N}$.*

Proof. It is clear that the sub-Λ-sets are existentially definable in I_Λ. So it suffices to prove that every subset of I^n definable in I_Λ is a sub-Λ-set: it follows from Theorem 2.5 that the collection $S\Lambda_n$ of sub-Λ-sets of I^n, for $n \in \mathbb{N}$, is a boolean algebra. Moreover, the hypotheses of Theorem 2.3 imply that each $S\Lambda_n$ contains the diagonals Δ_{ij}, for $1 \le i < j \le n$, and that, if $A \in S\Lambda_n$, then $A \times I$ and $I \times A$ belong to $S\Lambda_{n+1}$. Finally, if $B \in S\Lambda_{n+1}$, then $\Pi_n(B) \in S\Lambda_n$. □

2.3 *From Model Completeness to O-minimality*

We can now finish the proof of Theorem 2.3. For $n \in \mathbb{N}$, the map $\tau_n : \mathbb{R}^n \to I^n$ is defined by $\tau_n(x_1, \ldots, x_n) = \left(x_1/\sqrt{1+x_1^2}, \ldots, x_n/\sqrt{1+x_n^2}\right)$. Let Σ_n be the collection of all sets $A \subset \mathbb{R}^n$ such that $\tau_n(A)$ is a sub-Λ-set. The sets of Σ_n are clearly existentially definable in $\mathbb{R}_\Lambda$. In order to prove the model completeness (and hence the o-minimality, because of the finiteness property of sub-Λ-sets) of $\mathbb{R}_\Lambda$, it remains to prove that the basic relations of $\mathbb{R}_\Lambda$ are definable in the structure $\mathbb{R}_\Sigma = (\mathbb{R}, (A)_{A \in \Sigma_n, n \in \mathbb{N}})$.

Corollary 2.6 implies that every set $A \subset \mathbb{R}^n$ that is definable in $\mathbb{R}_\Sigma$ actually belongs to Σ_n. Since classical arguments show that the graphs of addition and multiplication in $\mathbb{R}^3$ belong to Σ_3, the proof is done.

Corollary 2.7. *Under the hyotheses of Theorem 2.3, a set $A \subset \mathbb{R}^n$ is definable in $\mathbb{R}_\Lambda$ if and only if $\tau_n(A)$ is a sub-Λ-set.* □

3 Normalization of Formal Power Series

In this section, we explain the main steps of a ***normalization algorithm***, the goal of which is to simplify formal power series via certain transformations. This process is mostly based on ***blowings-up***. While inspired by Hironaka's theorem of ***resolution of singularities*** [9], our purpose (namely, to prove o-minimality and model completeness for certain quasianalytic classes) does not need the complete strength of the latter; the algorithm presented here is much simpler. For example, whereas the codimension of the ***centers*** of blowings-up may vary in the general method, it will always be equal to 2 in our process.

The material of this section mostly comes from [17, Sect. 2]. Our goal is not to repeat, or rephrase, the details of this paper. It is to make its reading easier by describing the main ideas in small dimensions ($n = 2, 3$). The method developed in [17] consists in associating, to each power series, an ***invariant*** that strictly decreases at each step and that is minimal exactly for the so-called ***normal*** power series (see Definition 3.1 below). Note that a complete proof of Hironaka's theorem controlled by such an invariant can be found in [3].

The role of our invariant is to measure "how far a series is from being normal". We do not give the formal definition of this invariant, which is a bit intricate. We prefer to explain, in a more intuitive way, how the series are simplified by blowings-up.

We recall in a first subsection the geometric definition of blowing-up. Although this definition is too general for these notes, it throws light on the terminology of this section, which would be purely technical and arid without it. We explain in the next subsection the normalization procedure in two variables. Most of the ideas underlying the process in any dimension already appear in this case. In view of the geometric applications of Sect. 4, the *closure properties* of the algebras of power series under some specific operations are emphasized. Then the general statement is given, without proof, in the next subsection. Finally, the normalization process in three variables is described in the last subsection. The explanations given in this case should make the formalism of [17] more natural.

In order to define what is actually meant by a *normal* series, we introduce the following notation. Let $X = (X_1, \dots, X_n)$ and $r = (r_1, \dots, r_n) \in \mathbb{N}^n$. We write X^r for $X_1^{r_1} \cdots X_n^{r_n}$.

Definition 3.1. A series $f \in \mathbb{R}[[X]]$ is called ***normal*** if $f(X) = X^r \cdot U(X)$ with $r \in \mathbb{N}^n$ and $U \in \mathbb{R}[[X]]$ is a unit.

A finite set $\{f_1, \dots, f_l\} \subset \mathbb{R}[[X]]$ of series is ***normal*** if $f_k(X) = X^{r_k} \cdot U(X)$, with $r_k \in \mathbb{N}^n$ and U_k is a unit for each k, and if the set of monomials $\{X^{r_1}, \dots, X^{r_l}\}$ is linearly ordered by divisibility.

Remark 3.2. Consider a $\mathcal{C}^\infty$ germ f at the origin of $\mathbb{R}^n$ such that $f(x) = x^r \cdot U(x)$, where $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, $r = (r_1, \ldots, r_n) \in \mathbb{N}^n$ and U is a $\mathcal{C}^\infty$ germ at the origin of $\mathbb{R}^n$ such that $U(0) \neq 0$. Then the zero set of f is the germ at 0 of the intersection of all sets $\{x_i = 0\}$ such that $r_i \neq 0$. This explains the terminology "normal", which has its origin in the geometric terminology "normal crossings".

3.1 Geometric Definition of Blowing-Up

We recall the definition of blowing-up as found in [2, Sect. 4] (among many other possible references; see also F. Cano's notes in this volume).

3.1.1 Blowing-Up of an Open $V \subset \mathbb{R}^m$ with Center {0}

Let $\mathbb{P}_m(\mathbb{R})$ be the real projective space of dimension m. Consider an open neighborhood V of 0 in $\mathbb{R}^m$. The ***blowing-up*** of V with ***center*** $\{0\}$ is the mapping $\pi : V' \to V$, where

$$V' = \{(x, \ell) \in V \times \mathbb{P}_{m-1}(\mathbb{R}) : x \in \ell\}$$

defined by $\pi(x, \ell) = x$. Hence $\pi^{-1}(0) = \mathbb{P}_{m-1}(\mathbb{R})$. This *proper* map restricts to a homeomorphism $V \setminus \mathbb{P}_{m-1}(\mathbb{R}) \to V \setminus \{0\}$. In order to express π in local coordinates, we introduce the affine coordinates $x = (x_1, \ldots, x_m)$ of $\mathbb{R}^m$, and the homogeneous coordinates $\xi = [\xi_1, \ldots, \xi_m]$ of $\mathbb{P}_{m-1}(\mathbb{R})$. Then

$$V' = \left\{(x, \xi) \in V \times \mathbb{P}_{m-1}(\mathbb{R}) : x_i \xi_j = x_j \xi_i \text{ for } i, j = 1, \ldots, m\right\}$$

can be covered by the coordinate charts $V_i' = \{(x, \xi) \in V' : \xi_i \neq 0\}$, $i = 1, \ldots, m$, with coordinates $(x_{i1}, \ldots, x_{im})$ defined for each i by

$$x_{ii} = x_i, \text{ and } x_{ij} = \frac{\xi_j}{\xi_i} \text{ for } j \neq i.$$

In these local coordinates, π is given by

$$\pi : (x_{i1}, \ldots, x_{im}) \longmapsto (x_1, \ldots, x_m), \text{ with } x_j = \begin{cases} x_{ii} x_{ij} & \text{if } j \neq i, \\ x_{ii} & \text{if } j = i. \end{cases}$$

3.1.2 Blowing-Up of an Open Set $V \times W \subset \mathbb{R}^m \times \mathbb{R}^{n-m}$ with Center $\{0\} \times W \subset \mathbb{R}^n$

Suppose that $n > m$ and that W is an open subset of $\mathbb{R}^{n-m}$. We still consider an open neighborhood V of $0 \in \mathbb{R}^m$. The ***blowing-up*** of $V \times W$ with ***center*** $\{0\} \times W$ is the mapping $\pi \times \mathrm{id}: V' \times W \to V \times W$, where $\pi: V' \to V$ is the blowing-up of V with center $\{0\}$.

3.1.3 Blowing-Up of a Manifold with Center a Submanifold

Let M be a real analytic manifold of dimension n and Y be a closed analytic submanifold of M of codimension m. Let $U \subset M$ be an analytic chart with coordinates given by an analytic isomorphism $\varphi: U \to V \times W$, where V, W are open neighborhoods of the origins in $\mathbb{R}^n, \mathbb{R}^{n-m}$ respectively, such that $\varphi(Y \cap U) = \{0\} \times W$. Let $\pi_0: V' \to V$ be the blowing-up of V with center $\{0\}$. The ***blowing-up*** of M with ***center*** Y is a proper analytic mapping $\pi: M' \to M$ such that:

1. π restricts to an analytic isomorphism $M' \setminus \pi^{-1}(Y) \to M \setminus Y$;
2. There is an analytic isomorphism $\varphi': \pi^{-1}(U) \to V' \times W$ such that the following diagram commutes :

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\varphi'} & V' \times W \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi_0 \times \mathrm{id}} \\ U & \xrightarrow{\varphi} & V \times W \end{array}$$

These two conditions define π uniquely, up to an isomorphism of M' commuting with π.

Example 3.3. 1. *Blowing-up of* $\mathbb{R}^3$ *with a codimension* 3 *center*. The blowing-up of $\mathbb{R}^3$ with center $\{0\}$ is the mapping

$$\pi: V' = \{(x, \ell) \in V \times \mathbb{P}_2(\mathbb{K}) : x \in \ell\} \to \mathbb{R}^3$$

defined by $\pi(x, \ell) = x$. Denote by (x, y, z) the affine coordinates of $\mathbb{R}^3$, and $[X, Y, Z]$ the homogeneous coordinates of $\mathbb{P}_2(\mathbb{R})$. Consider, for example, the chart $V_1 = \{((x, y, z), [X, Y, Z]) \in V': X \neq 0\}$, equipped with the coordinate system (u_1, v_1, w_1) defined by

$$u_1 = x,\ v_1 = \frac{Y}{X},\ w_1 = \frac{Z}{X}.$$

Then π is given in these coordinates by

$$\pi\left(u_1, v_1, w_1\right)=\left(u_1, u_1 v_1, u_1 w_1\right)=(x, y, z).$$

Obviously, if (u_2, v_2, w_2) are the coordinates in the chart

$$V_2=\left\{((x, y, z),[X, Y, Z]) \in V' : Y \neq 0\right\},$$

then $\pi\left(u_2, v_2, w_2\right)=\left(u_2 v_2, v_2, w_2 v_2\right)$, and if (u_3, v_3, w_3) are the coordinates in the chart $V_3=\{((x, y, z),[X, Y, Z]) \in V' : Z \neq 0\}$, then $\pi\left(u_3, v_3, w_3\right)=\left(u_3 w_3, v_3 w_3, w_3\right)$.

2. *Blowing-up of* $\mathbb{R}^3$ *with a codimension 2 center*. Let $Y \subset \mathbb{R}^3$ be the axis given by the equations $\{x=0, y=0\}$. The blowing-up of $\mathbb{R}^3$ with center Y is actually the blowing-up π of $\mathbb{R}^2 \times \mathbb{R}$ with center $\{0\} \times \mathbb{R}$. In the chart

$$V_1 \times \mathbb{R}=\left\{(((x, y),[X, Y]), z) : ((x, y),[X, Y]) \in V' : X \neq 0\right\},$$

equipped with the coordinate system (u_1, v_1, w_1) defined by

$$u_1=x,\ v_1=\frac{Y}{X},\ w_1=z,$$

the map π is given by

$$\pi\left(u_1, v_1, w_1\right)=\left(u_1, u_1 v_1, w_1\right)=(x, y, z).$$

In the chart

$$V_2 \times \mathbb{R}=\left\{(((x, y),[X, Y]), z) : ((x, y),[X, Y]) \in V' : Y \neq 0\right\},$$

π is given by

$$\pi(u_2, v_2, w_2)=\left(u_2 v_2, v_2, w_2\right)=(x, y, z).$$

The origin of the first chart may be translated. Instead of working in a neighborhood of the point $v_1=0$ of the projective space $\mathbb{P}_1(\mathbb{R})$, we may prefer to work in a neighborhood of the point $v_1=\lambda$, for $\lambda \in \mathbb{R}$. So we introduce the coordinate $\bar{v}_1$ defined by $v_1=\lambda+\bar{v}_1$, and the blowing-up π is expressed in these coordinates by

$$\pi\left(u_1, \bar{v}_1, w_1\right)=\left(u_1, u_1\left(\lambda+\bar{v}_1\right), w_1\right).$$

If we want to work in a neighborhood of the "point at infinity" of the first chart, it actually means that we work at the origin of the second chart.

3. *Blowing-up of* $\mathbb{R}^n$ *with a codimension 2 center.* The formulas above extend obviously to the blowing-up π of $\mathbb{R}^n$ with a center given by the equations $\left\{x_i=0, x_j=0\right\}$ for given $1 \leq i<j \leq n$. To each point of the projective space obtained by blowing-up with center the origin in the plane $\left\{x_i, x_j\right\}$ corresponds a local analytic expression of π. In the following, these expressions will be called ***blow-up substitutions***. These substitutions, which describe the action of

the blowing-up π on formal series (and then on $\mathcal{C}^{\infty}$ germs in Sect. 4), are the $\mathbb{R}$-algebra homomorphisms $\mathrm{b}_{\lambda}^{i,j} : \mathbb{R}[[X]] \to \mathbb{R}[[X]]$, $\lambda \in \mathbb{R} \cup \{\infty\}$, defined by

$$\mathrm{b}_{\lambda}^{i,j}(X_k) = \begin{cases} X_i\left(\lambda + X_j\right) & \text{if } k = j \\ X_k & \text{otherwise} \end{cases}$$

for $\lambda \in \mathbb{R}$, and

$$\mathrm{b}_{\infty}^{i,j}(X_k) = \begin{cases} X_i X_k & \text{if } k = i \\ X_k & \text{otherwise.} \end{cases}$$

4. The blowing-up of an analytic manifold with center a codimension 2 submanifold is expressed by the formulas of the previous example in a coordinate system φ such that $\varphi(Y \cap U) = \{0\} \times W$, with the notations of Sect. 3.1.3.

Remark 3.4. As was mentioned at the beginning of this section, the only blowings-up involved in our normalization process here have codimension 2 centers.

3.2 Normalization in Two Variables

Consider a nonzero series $f(X,Y) \in \mathbb{R}[[X,Y]]$. Write $f(X,Y) = \sum_{i=0}^{\infty} X^{p_i} U_i(X)Y^i$, where $U_i \in \mathbb{R}[[X]]$ are units and $p_i \in \mathbb{N}$, for $i \in \mathbb{N}$. Let $p = \min\{p_i, i \in \mathbb{N}\}$. After factoring out the monomial X^p, we may suppose that $f(X,Y)$ is ***regular of order*** d in Y, that is, $\mathrm{ord}_Y f(0,Y) = d$ with $d \in \mathbb{N}$. The property of the algebra $\mathbb{R}[[X,Y]]$ involved here is the *closure under monomial division*.

If $\mathrm{ord}_Y f(0,Y) = 0$, then f is a unit and is, therefore, normal. In order to lower $\mathrm{ord}_Y f(0,Y)$, we use the blow-up substitutions representing the charts of the blowing-up of $\mathbb{R}^2$ with center the origin (see Sect. 3.1). These substitutions are the $\mathbb{R}$-algebra homomorphisms $\mathbb{R}[[X,Y]] \to \mathbb{R}[[X,Y]]$ defined by:

$$\mathrm{b}_{\lambda}(X) = X \text{ and } \mathrm{b}_{\lambda}(Y) = X(\lambda + Y), \quad \text{for } \lambda \in \mathbb{R},$$

and

$$\mathrm{b}_{\infty}(X) = XY \text{ and } \mathrm{b}_{\infty}(Y) = Y.$$

The blowing-up of the series f is represented by the *all* $\mathrm{b}_{\lambda} f$, with $\lambda \in \mathbb{R} \cup \{\infty\}$; we therefore want to show that each $\mathrm{b}_{\lambda} f$ is, in some sense, *simpler* than f.

It is easy to see that $\mathrm{ord}_Y f(0,Y)$ does not always decrease (even up to factoring out a monomial) under blow-up substitutions. Consider for example $f(X,Y) = Y^2 + X^2Y + X^3$, which satisfies $\mathrm{ord}_Y f(X,Y) = 2$. Then

$$\begin{aligned} \mathrm{b}_0 f(X,Y) = f(XY,Y) &= X^2Y^2 + X^3Y + X^3 \\ &= X^2\left(Y^2 + XY + X\right) = X^2 \tilde{f}(X,Y) \end{aligned}$$

and $\text{ord}_Y \tilde{f}(0, Y) = 2$. The order does not decrease, but we notice that the valuation of the coefficients in X is decreasing. This example suggests a definition of an invariant that involves not only the order in Y, but also the valuation of the coefficients with respect to X.

Consider now the example

$$g(X, Y) = (Y - a_0(X))^d, \ a_0(X) = X(\lambda + a_1(X)),$$

where $a_1(X) \in \mathbb{R}[[X]]$ and $\lambda \in \mathbb{R}$. Then

$$\mathrm{b}_\lambda g(X, Y) = X^d (Y - a_1(X))^d.$$

Since $a_1(X)$ has no reason to be "simpler" than $a_0(X)$, there is no obvious invariant that is decreasing under the transformation b_λ.

We therefore need to introduce a prior transformation that prevents a series $f(X, Y)$ with $\text{ord}_Y f(0, Y) = d$ from having a root of multiplicity d, and that does not increase the "complexity" of f. This transformation is classically called the *Tschirnhausen transformation*. Since $\text{ord}_Y f(0, Y) = d$, $\partial^{d-1} f / \partial Y^{d-1}(X, Y)$ satisfies the hypothesis of the implicit function theorem: there exists $\alpha(X) \in \mathbb{R}[[X]]$ such that $\alpha(0) = 0$ and $\partial^{d-1} f / \partial Y^{d-1}(X, \alpha(X)) = 0$. Let $g(X, Y) = f(X, \alpha(X) + Y)$. According to Taylor's formula, wet get

$$g(X, Y) = \sum_{k=0}^{d-2} \frac{\partial^k f}{\partial Y^k}(X, \alpha(X)) Y^k + Y^d U(X, Y),$$

where $U(X, Y) \in \mathbb{R}[[X, Y]]$ is a unit, because $\text{ord}_Y g(0, Y) = d$. Hence we may now suppose that

$$f(X, Y) = Y^d U(X, Y) + \sum_{k \in K} X^{r_k} U_k(X) Y^k,$$

where $K \subset \{0, \ldots, d-2\}$, each $r_k \in \mathbb{N}$, and $U(X, Y) \in \mathbb{R}[[X, Y]]$ and all $U_k(X) \in \mathbb{R}[[X]]$ are units. We note that several closure properties of the algebras of power series have been used in the above transformations: closure under *translation* (namely $Y \to Y - \alpha(X)$, for $\alpha(X) \in \mathbb{R}[[X]]$ such that $\alpha(0) = 0$), *partial derivatives*, and *composition*.

If $K = \emptyset$, the series f is normal. Otherwise, $r_k > 0$ for all $k \in K$. Furthermore, after replacing X by $X^{d!}$ (a *power transformation*), we may assume that r_k is divisible by $d - k$ for each $k \in K$. In order to understand how a blowing-up with center $\{0\}$ may simplify the series, we consider the maximal integer $l \in K$ such that:

$$\frac{r_l}{d-l} \leq \frac{r_k}{d-k}, \quad \text{for all } k \in K.$$

We claim that the pair (d, r_l) is now lowered lexicographically by every blow-up substitution (after possibly factoring out some power of X). In order to prove this, we distinguish three cases represented, respectively, by (a) b_∞, (b) b_λ for $\lambda \neq 0$, and (c) b_0. The inequality $r_k \geq d - k$ for each $k \in K$ is used in the three cases. The role of Tschirnhausen's transformation is essential for case (b).

1. *Effect of the substitution* b_∞. We obtain

$$\begin{aligned}\mathrm{b}_\infty f(X,Y) &= Y^d U(XY,Y) + \sum_{k\in K} X^{r_k} U_k(XY)\, Y^{k+r_k} \\ &= Y^d \left(U(XY,Y) + \sum_{k\in K} X^{r_k} U(XY)\, Y^{r_k-(d-k)} \right).\end{aligned}$$

   Hence $\mathrm{b}_\infty f$ is normal.
2. *Effect of the substitution* b_λ, $\lambda \neq 0$. We have

$$\mathrm{b}_\lambda f(X,Y) = X^d g(X,Y),$$

   where

$$\begin{aligned}g(X,Y) &= (\lambda+Y)^d U(X, X(\lambda+Y)) + \sum_{k\in K} X^{r_k-(d-k)} U_k(X)(\lambda+Y)^k \\ &= \left(Y^d + Y^{d-1} d\lambda^{d-1}\right) U(X, X(\lambda+Y)) + \text{lower order terms.}\end{aligned}$$

   The goal of Tschirnhausen's transformation is to guarantee that the coefficient of Y^{d-1} in $f(X,Y)$ is zero. Hence the coefficient of Y^{d-1} in this expansion of $g(X,Y)$ is a unit, so that $\mathrm{ord}_Y g(0,Y) \leq d-1$.
3. *Effect of the substitution* b_0. We have

$$\begin{aligned}\mathrm{b}_0 f(X,Y) &= X^d Y^d U(X,XY) + \sum_{k\in K} X^{r_k+k} U_k(X)\, Y^k \\ &= X^d \left(Y^d U(X,XY) + \sum_{k\in K} X^{r_k-(d-k)} U_k(X)\, Y^k \right).\end{aligned}$$

   So $\mathrm{ord}_Y \mathrm{b}_0 f(0,Y) \leq d$; note moreover that

$$\frac{r_l-(d-l)}{d-l} = r_l - 1 \leq r_k - 1 \leq \frac{r_k-(d-k)}{d-k}$$

   for all $k \in K$, so $l = l(f) = l(g)$, and $r_l(g) = r_l(f) - (d-l) < r_l(f)$.

Consequently, the complete normalization of $f(X,Y)$ is achieved by the application of finitely many translations, power transformations, and blowings-up.

Remark 3.5. 1. The closure properties involved in this process are closure under: monomial division, partial derivatives, solutions of implicit equations, composition and power substitution, and of course the usual algebraic operations. The statements of Sect. 4 will be established for quasianalytic classes closed under these operations.
2. Weierstrass preparation is not involved in the above normalization process. It is replaced by Taylor's formula. This remark is useful in the quasianalytic framework in which Weierstrass preparation does not hold.

3.3 *Statement of the Normalization Theorem in Several Variables*

Notation 3.6. Let $X = (X_1, \ldots, X_n)$, and set $X' = (X_1, \ldots, X_{n-1})$ if $n > 1$. For $r = (r_1, \ldots, r_n) \in \mathbb{N}^n$, write X^r for $X_1^{r_1} \cdots X_n^{r_n}$. Given $r, s \in \mathbb{N}^n$, we write $r \leq s$ if $r_i \leq s_i$ for each i. Obviously $r \leq s$ if and only if X^s is divisible by X^r.

This general normalization process in n variables mostly follows the lines described in Sect. 3.2, extended by an induction on n:

1. Up to a linear transformation, we may suppose $\mathrm{ord}_Y f(0, X_n) = d < \infty$.
2. A Tschirnhausen transformation leads to a series $f(X)$ such that

$$\partial^{d-1} f/\partial X_n^{d-1}\left(X', 0\right) = 0.$$

3. Taylor's formula allows us to work with an expansion of f in X_n up to the order d. By induction on n, the collection of coefficients of X_n^k in this expansion, which belong to $\mathbb{R}[[X']]$, may be assumed being normal.
4. A power transformation then increases the exponents of the principal monomials of the coefficients, so that they all become divisible by $d!$. This step is an essential preparation to the next one.
5. Convenient blowings-up with codimension 2 centers simplify the coefficients, progressively leading to a series $f(X, Y)$ satisfying $\mathrm{ord}_Y f(0, Y) < d$.

With respect to normalization in two variables, the main extra difficulty is the choice of the center of each blowing-up.

We note here that, in the third step above, we need to normalize simultaneously several series instead of a single one. This little technicality is solved by the following elementary result:

Lemma 3.7 ([2, Lemma 4.7]). *Let $f_1, \ldots, f_l \in \mathbb{R}[[X]]$.*

1. *The product $f_1 \cdots f_l$ is normal if and only if each f_k is normal.*
2. *Assume that all f_k, for $k = 1, \ldots, l$, and all $f_k - f_{k'}$, for $0 \leq k < k' \leq l$, are normal. Then $\{f_1, \ldots, f_l\}$ is normal.*

List of elementary substitutions. In order to state the main result of this section, we sum up the substitutions involved in the normalization process.

(a) *Power substitution*: they are used in Step 4 above, as a necessary preparation to blow-up substitutions. For $1 \le i \le n$ and an integer $q > 0$ we let $p_{i,q}^{+}, p_{i,q}^{-} : \mathbb{R}[[X]] \to \mathbb{R}[[X]]$ be the $\mathbb{R}$-algebra homomorphisms defined by

$$p_{i,q}^{+}(X_j) = \begin{cases} X_i^q & \text{if } j = i, \\ X_j & \text{otherwise,} \end{cases} \qquad p_{i,q}^{-}(X_j) = \begin{cases} -X_i^q & \text{if } j = i, \\ X_j & \text{otherwise.} \end{cases}$$

(b) *Translation substitution*: it is used in Step 2 above for the Tschirnhausen transformation. For $1 \le i \le n$ and $\alpha \in \mathbb{R}[[X_1, \ldots, X_{i-1}]]$ such that $\alpha(0) = 0$, we let $t_\alpha : \mathbb{R}[[X]] \to \mathbb{R}[[X]]$ be the algebra homomorphism given by

$$t_\alpha(X_j) = \begin{cases} X_i + \alpha(X_1, \ldots, X_{i-1}) & \text{if } j = i, \\ X_j & \text{otherwise.} \end{cases}$$

(c) *Linear substitution*: this transformation is involved in Step 1 to turn a series regular in one variable. For $i > 0$ and $c = (c_1, \ldots, c_{i-1}) \in \mathbb{R}^{i-1}$, we let $l_{i,c} : \mathbb{R}[[X]] \to \mathbb{R}[[X]]$ be the $\mathbb{R}$-algebra homomorphism given by

$$l_{i,c}(X_j) = \begin{cases} X_j + c_j X_i & \text{if } 1 \le j < i, \\ X_j & \text{otherwise.} \end{cases}$$

(d) *Blow-up substitution*: these transformations are involved in Step 5 to lower the order of the series or of its coefficients in the last variable. For $1 \le i < j \le n$ and $\lambda \in \mathbb{R}$, we let $\mathrm{b}_\lambda^{i,j}, \mathrm{b}_\infty^{i,j} : \mathbb{R}[[X]] \to \mathbb{R}[[X]]$ be the $\mathbb{R}$-algebra homomorphisms defined by

$$\mathrm{b}_\lambda^{i,j}(X_k) = \begin{cases} X_i(\lambda + X_j) & \text{if } k = j, \\ X_k & \text{otherwise,} \end{cases}$$

and

$$\mathrm{b}_\infty^{i,j}(X_k) = \begin{cases} X_i X_j & \text{if } k = i, \\ X_k & \text{otherwise.} \end{cases}$$

Note that $\mathrm{b}_\lambda^{i,j}$ affects the j-th variable, while $\mathrm{b}_\infty^{i,j}$ affects the i-th variable.

Definition 3.8. An ***admissible substitution*** τ is any one of the following collections of $\mathbb{R}$-homomorphisms $\mathbb{R}[[X]] \to \mathbb{R}[[X]]$:

1. $\tau = \{l_{i,c}\}$ for some $1 < i \le n$ and $c \in \mathbb{R}^{i-1}$ (a ***linear substitution***);
2. $\tau = \{t_\alpha\}$ for some $1 < i \le n$ and $\alpha \in \mathbb{R}[[X']]$ with $\alpha(0) = 0$ (a ***translation substitution***);

3. $\tau = \left\{p_{i,q}^{+}, p_{i,q}^{-}\right\}$ for some $1 < i \leq n$ and integer $q > 0$ (a ***power substitution***);
4. $\tau = \mathrm{b}^{i,j} = \left\{\mathrm{b}_{\lambda}^{i,j} : \lambda \in \mathbb{R} \cup \{\infty\}\right\}$ for some $1 \leq i < j \leq n$ (a ***blow-up substitution***).

Remark 3.9. If τ is an admissible substitution and $f \in \mathbb{R}[[X]]$, then τf is a (possibly infinite) collection of series. It is easy to see that $f \in \mathbb{R}[[X]]$ is a unit if and only if every member of τf is a unit. Moreover, if $f \in \mathbb{R}[[X]]$ is normal and τ is a power substitution or a blow-up substitution, then every member of τf is normal.

One can define the ***height*** $\mathrm{h}_n(f)$ of a series $f \in \mathbb{R}[[X]]$, which measures "how far from normal" f is. The normalization is based on the analysis of $\mathrm{h}_n(f)$. Actually, $\mathrm{h}_n(f)$ belongs to $(\mathbb{N} \cup \{\infty\})^{\nu_n}$ equipped with the lexicographic ordering, where ν_n only depends on n. A proof of the following result is given in [17, Sect. 2]:

Theorem 3.10. *Let $n \geq 1$.*

1. *If $\mathrm{h}_n(f) = 0$, then f is normal.*
2. *If $\mathrm{h}_n(f) > 0$, then there is an admissible substitution τ such that $\mathrm{h}_n(g) < \mathrm{h}_n(f)$ is for each $g \in \tau f$.*

Remark 3.11. The action of an admissible substitution τ extends on a collection of series. Hence, given a series $f \in \mathbb{R}[[X]]$, there exists a finite sequence of $\{\tau_1, \ldots, \tau_N\}$ of admissible substitutions such that each element of $\tau_N \circ \cdots \circ \tau_1(f)$ is normal.

3.4 Normalization in Three Variables

In this subsection, we illustrate the normalization algorithm for a series in three variables X, Y, Z. (Note that X does not have the same meaning as in the previous subsection; now, X is a single variable.) We suppose true the statement of Theorem 3.10 for elements of $\mathbb{R}[[X, Y]]$. Let $f \in \mathbb{R}[[X, Y, Z]]$. We explain the main steps of the normalization of the series f.

Transformation into a regular series. It is well known that there exists $c \in \mathbb{R}^2$ such that $\mathrm{ord}_Z(l_{3,c} f(0, 0, Z)) < \infty$.

Tschirnhausen's transformation. We suppose from now on that

$$\mathrm{ord}_Z f(0, 0, Z) = d < \infty.$$

We have already explained the role of this transformation in Sect. 3.2; it has the exact same purpose in many variables. Since $\partial^{d-1} f/\partial Z^{d-1}(0, 0, 0) = 0$ and $\partial^{d} f/\partial Z^{d}(0, 0, 0) \neq 0$, the equation $\partial^{d-1} f/\partial Z^{d-1}(X, Y, Z) = 0$ admits a unique solution $Z = \alpha(X, Y) \in \mathbb{R}[[X, Y]]$ such that $\alpha(0, 0) = 0$. The series

$t_a f \in \mathbb{R}[[X,Y,Z]]$ satisfies $\partial^{d-1} f / \partial Z^{d-1}(X,Y,0) = 0$; so we suppose from now on that $\partial^{d-1} f / \partial Z^{d-1}(X,Y,0) = 0$. It follows from Taylor's formula that

$$f(X,Y,Z) = \sum_{k=0}^{d-2} \frac{1}{k!} \frac{\partial^k f}{\partial Z^k}(X,Y,0) Z^k + Z^d U(X,Y,Z)$$
$$= \sum_{k=0}^{d-2} f_k(X,Y) Z^k + Z^d U(X,Y,Z),$$

where $f_k(0,0) = 0$ for $k = 0,\dots,d-2$ and $U(0,0,0) \neq 0$ (U is a *unit*). This expansion could be called a "quasi-Weierstrass preparation" of f. Note that, if f is an actual polynomial in Z, the Tschirnhausen transformation is nothing other than the classical "completion of the d-th power" used in the resolution of polynomial equations.

Normalization of the coefficients. We set

$$K = \{k \in \{0,\dots d-2\} : f_k \neq 0\}.$$

The normalization of a given series (and hence of several series thanks to Lemma 3.7) in two variables has been explained in Sect. 3.2. It corresponds to the induction hypothesis made on n in the general process. Hence we may suppose that the collection $\{f_k\}_{k\in K}$ is normal. Moreover, after two power substitutions $\left\{p^+_{1,d!}, p^-_{1,d!}\right\}$, $\left\{p^+_{2,d!}, p^-_{2,d!}\right\}$ (where the indices 1 and 2 stand for the variables X and Y) we may suppose that

$$f(X,Y,Z) = Z^d U(X,Y,Z) + \sum_{k\in K} X^{r_k} Y^{s_k} U_k(X,Y) Z^k,$$

where U and U_k, for $k \in K$, are units, and the exponents r_k, s_k, for $k \in K$, are divisible by $d-j$ for all $j \in K$. Therefore

$$\left(X^{r_k} Y^{s_k} U_k(X,Y)\right)^{1/(d-k)} \in \mathbb{R}[[X,Y]]$$

for all $k \in K$, that is, the units of $\mathbb{R}[[X,Y]]$ admit a p-th root for each $p \in \mathbb{N}$. By another application of the normalization in two variables, we may suppose that the collection $\left\{\left(X^{r_k} Y^{s_k} U_k(X,Y)\right)^{1/(d-k)} : k \in K\right\}$ is normal.

Lowering the order by blowing-up. Lowering $\mathrm{ord}_Z f(0,0,Z)$ by a single blowing-up is in general hopeless. We have to lower, by successive blowings-up (with codimension 2 centers), the degree of the dominant monomial $X^{r_l} Y^{s_l}$ of some coefficient in the expansion of f. More precisely, from a practical point of view, these blowings-up will always involve the variable Z and one of the variables X and Y.

Let us recall that the collection $\{(r_k/(d-k), s_k/(d-k))\}_{k\in K}$ is linearly ordered (with respect to the partial order $(p,q) \leq (r,s)$ if and only if $p \leq r$ and $q \leq s$). Let $l = l(f) \in K$ be maximal such that

$$\left(\frac{r_l}{d-l}, \frac{s_l}{d-l}\right) \leq \left(\frac{r_k}{d-k}, \frac{s_k}{d-k}\right), \ \forall k \in K.$$

One of the exponents r_l or s_l is nonzero (otherwise we would have $\mathrm{ord}_Z f(0,0,Z) < d$). We may then suppose $s_l \neq 0$. Let us examine the effect of the admissible substitution $\mathrm{b}^{2,3}$ on the series f (where the indices 2 and 3 represent the variables Y and Z, respectively), by considering all the substitutions $\mathrm{b}_\lambda^{2,3}$, for $\lambda \in \mathbb{R} \cup \{\infty\}$.

The following will be used in all cases below: for all $k \in K$, since $s_k/(d-k) \geq s_l/(d-l) > 0$, we have $s_k \geq d-k$.

(a) *Effect of* $\mathrm{b}_\infty^{2,3}$. We claim that this substitution transforms f into a normal series. Indeed,

$$\begin{aligned}\mathrm{b}_\infty^{2,3} f(X,Y,Z) &= Z^d U(X,YZ,Z) + \sum_{k\in K} X^{r_k} Y^{s_k} U_k(X,YZ) Z^{k+s_k} \\ &= Z^d V(X,Y,Z),\end{aligned}$$

where $V(X,Y,Z) = U(X,YZ,Z) + \sum_{k\in K} X^{r_k} Y^{s_k} U_k(X,YZ) Z^{s_k-(d-k)}$ is a unit, because no monomial in the latter sum is a constant.

(b) *Effect of* $\mathrm{b}_\lambda^{2,3}$ *for* $\lambda \neq 0$. We claim that, after factoring out a monomial, we have $\mathrm{ord}_Z \mathrm{b}_\lambda f(0,0,Z) < \mathrm{ord}_Z f(0,0,Z)$. Actually, the following computation shows that, for all but finitely many $\lambda \in \mathbb{R}$, the series $\mathrm{b}_\lambda^{2,3} f(X,Y,Z)$ is normal:

$$\begin{aligned}\mathrm{b}_\lambda^{2,3} f(X,Y,Z) &= Y^d (\lambda+Z)^d U(X,Y,Y(\lambda+Z)) \\ &\quad + \sum_{k\in K} X^{r_k} Y^{s_k+k} (\lambda+Z)^k U_k(X,Y) \\ &= Y^d g(X,Y,Z),\end{aligned}$$

where

$$g(X,Y,Z) = Z^d U(X,Y,Y(\lambda+Z)) + Z^{d-1} d\lambda U(X,Y,Y(\lambda+Z)) + \cdots;$$

in particular, $\mathrm{ord}_Z g(0,0,Z) \leq d-1$. We note that the existence of a *nonvanishing* coefficient of Z^{d-1} in this expansion is a consequence of the hypothesis $\partial^{d-1} f/\partial Z^{d-1}(X,Y,0) = 0$, which was obtained by Tschirnhausen's transformation.

(c) *Effect of* $\mathrm{b}_0^{2,3}$. This substitution transforms f into the product of a monomial and a series $g \in \mathbb{R}[[X,Y,Z]]$ with $l = l(f) = l(g)$ and a lower exponent s_l. The series g is then "closer than f to being normal". More precisely, we have

$$\begin{aligned} \mathrm{b}_0^{2,3} f(X,Y,Z) &= Y^d Z^d U(X,Y,XY) + \sum_{k\in K} X^{r_k} Y^{s_k+k} U_k(X,Y) Z^k \\ &= Y^d g(X,Y,Z), \end{aligned}$$

where $g(X,Y,Z) = Z^d U(X,Y,XY) + \sum_{k\in K} X^{r_k} Y^{s_k-(d-k)} U_k(X,Y) Z^k$. Moreover, since $s_k/(d-k) \geq s_l/(d-l)$, we have $s_k/(d-k) - 1 \geq s_l/(d-l) - 1$, and hence

$$(s_k - (d-k))/(d-k) \geq (s_l - (d-l))/(d-l)$$

for all $k \in K$. This implies that $l(g) = l(f)$.

Continuation of the process. This step is straightforward. Let g be the series obtained after one of the above blow-up substitutions. Then:

1. If g is normal, we are done.
2. If $\mathrm{ord}_Z g(0,0,Z) < \mathrm{ord}_Z f(0,0,Z)$, we apply the complete process (Tschirnhausen's transformation, etc.) to g.
3. If $\mathrm{ord}_Z g(0,0,Z) = \mathrm{ord}_Z f(0,0,Z)$ and $s_l(g) < s_l(f)$, we apply repeatedly the blowing-up $\mathrm{b}^{2,3}$, until the series g obtained are either normal, or satisfy $\mathrm{ord}_Z g(0,0,Z) < \mathrm{ord}_Z f(0,0,Z)$, or satisfy $s_l(g) = 0$. In the last case, we notice that $r_k/(d-k) \geq r_l/(d-l)$ for all $k \in K$, and we proceed with the blowing-up $\mathrm{b}^{1,3}$.

Remark 3.12. In this section, the comment "up to factoring out a monomial" appears several times. The reader can check that, indeed, this operation does not affect the process.

4 Quasianalyticity, Model Completeness and O-minimality

After studying normalization on the formal side, we now have to work in the geometric framework. We consider *quasianalytic* classes of (germs of) functions. We show that, under the closure assumptions that were emphasized in Remark 3.5, these classes generate a model complete and o-minimal expansion of the real field.

In a first subsection, we introduce the quasianalytic framework, and give the list of hypotheses which allow us to transfer the formal normalization process to germs and functions. This process leads to a crucial geometric result, usually referred to as ***uniformization***. In the second subsection, we explain how this result, combined with a few classical geometric tools, leads to a version of Gabrielov's theorem of the complement. The results of Sect. 2 then allow us to conclude.

Notation 4.1. For each $n \in \mathbb{N}$, we write $x = (x_1, \ldots, x_n)$ for the affine coordinates of $\mathbb{R}^n$ and set $x' = (x_1, \ldots, x_{n-1})$ if $n > 1$. For $r = (r_1, \ldots, r_n) \in \mathbb{N}^n$ we put $x^r = x_1^{r_1} \cdots x_n^{r_n}$.

4.1 Uniformization of $\mathcal{C}$-sets

For every compact box

$$B = [a_1, b_1] \times \cdots \times [a_n, b_n],$$

with $a_i < b_i$ for $i = 1, \ldots, n$ and $n \in \mathbb{N}$, we fix an $\mathbb{R}$-algebra $\mathcal{C}_B$ of functions $f: B \to \mathbb{R}$ such that the following hold:

(C1) $\mathcal{C}_B$ contains the functions $(x_1, \ldots, x_n) \mapsto x_i$ and, for every $f \in \mathcal{C}_B$, the restriction of f to $\text{int}(B)$ is $\mathcal{C}^\infty$;
(C2) If $B' \subset \mathbb{R}^m$ is a compact box and $g_1, \ldots, g_n \in \mathcal{C}_{B'}$ are such that $g(B') \subset B$, where $g = (g_1, \ldots, g_n)$, then for every $f \in \mathcal{C}_B$ the function $y \mapsto f(g_1(y), \ldots, g_n(y))$ belongs to $\mathcal{C}_{B'}$;
(C3) For every compact box $B' \subset B$ we have $f|B' \in \mathcal{C}_{B'}$ for all $f \in \mathcal{C}_B$, and for every $f \in \mathcal{C}_B$ there is a compact box $B' \subset \mathbb{R}^n$ and $g \in \mathcal{C}_{B'}$ such that $B \subset \text{int}(B')$ and $g|B = f$.

It is important to note that (C1) and (C3) imply that every $f \in \mathcal{C}_B$ extends to a $\mathcal{C}^\infty$ function $\overline{f}: U \to \mathbb{R}$ for some open neighborhood U of B (depending on f). Therefore, for each $i = 1, \ldots, n$ we denote by $\partial f/\partial x_i$ the restriction $\partial \overline{f}/\partial x_i$. We also assume:

(C4) $\partial f/\partial x_i \in \mathcal{C}_B$ for every $f \in \mathcal{C}_B$ and each $i = 1, \ldots, n$.

Next, we complete this list of conditions satisfied by the functions $f \in \mathcal{C}_B$ by a list of conditions satisfied by the germs of the elements of $\mathcal{C}_B$.

Notation 4.2. For every polyradius $r = (r_1, \ldots, r_n) \in (0, \infty)^n$, we put

$$I_r = (-r_1, r_1) \times \cdots \times (-r_n, r_n) \quad \text{and} \quad \overline{I_r} = \text{cl}(I_r).$$

For $\varepsilon > 0$, we simply write ε for the polyradius $(\varepsilon, \ldots, \varepsilon)$. We write $\mathcal{C}_{n,r} = \mathcal{C}_{\overline{I_r}}$. We denote by $\mathcal{C}_n$ the collection of all germs at the origin of the functions in $\bigcup_{r \in (0,\infty)^n} \mathcal{C}_{n,r}$. Each $\mathcal{C}_n$ is an $\mathbb{R}$-algebra. Finally, we let $\widehat{\ }: \mathcal{C}_n \to \mathbb{R}[[X]]$ be the map that sends each $f \in \mathcal{C}_n$ to its Taylor series $\widehat{f}$ at the origin, and we denote by $\widehat{\mathcal{C}_n}$ the image of $\widehat{\ }$ in $\mathbb{R}[[X]]$.

We make the following extra assumptions:

(C5) $\widehat{\ }: \mathcal{C}_n \to \widehat{\mathcal{C}_n}$ is an $\mathbb{R}$-algebra isomorphism (***quasianalyticity***);
(C6) If $n > 1$ and $f \in \mathcal{C}_n$ is such that $f(0) = 0$ and $(\partial f/\partial x_n)(0) \neq 0$, there is an $\alpha \in \mathcal{C}_{n-1}$ with $\alpha(0) = 0$ such that $f(x', \alpha(x')) = 0$;
(C7) If $f \in \mathcal{C}_n$ and $i \leq n$ are such that $\widehat{f}(X) = X_i G(X)$ for some $G \in \mathbb{R}[[X]]$, then $f = x_i g$ for some $g \in \mathcal{C}_n$ such that $G = \widehat{g}$.

We can now adapt the language of analytic geometry to the quasianalytic framework.

Definition 4.3. 1. A set $A \subset \mathbb{R}^n$ is called a ***basic*** $\mathcal{C}$***-set*** if there are $r \in (0,\infty)^n$ and $f, g_1, \ldots, g_k \in \mathcal{C}_{n,r}$ such that

$$A = \{x \in I_r : f(x) = 0, g_1(x) > 0, \ldots, g_k(x) > 0\}.$$

2. A finite union of basic $\mathcal{C}$-sets is called a $\mathcal{C}$***-set***. We call $M \subset \mathbb{R}^n$ a $\mathcal{C}$***-manifold*** if there is an $r \in (0,\infty)^n$ such that
   - M is a basic $\mathcal{C}$-set contained in I_r;
   - There are $f_1, \ldots, f_k \in \mathcal{C}_{n,r}$ such that M is a submanifold of I_r of dimension $n-k$ on which $f_1, \ldots, f_k$ vanish identically, and the gradients $\nabla f_1(z), \ldots, \nabla f_k(z)$ are linearly independent at each $z \in M$.
3. Let $A \subset \mathbb{R}^n$. The set $A \subset \mathbb{R}^n$ is called $\mathcal{C}$***-semianalytic at a point*** $a \in \mathbb{R}^n$ if there is an $r \in (0,\infty)^n$ such that $(A-a) \cap I_r$ is a $\mathcal{C}$-set. A is $\mathcal{C}$***-semianalytic*** if A is $\mathcal{C}$-semianalytic at every point $a \in \mathbb{R}^n$. In this situation, if A is a manifold, we call A a $\mathcal{C}$***-semianalytic manifold***.

Remark 4.4. 1. Property (C7) has a simple but important consequence. Let $f \in \mathcal{C}_n$ be such that $\widehat{f} \in \mathbb{R}[[X]]$ is normal. Then there exist $r \in \mathbb{N}$ and $U(X) \in \mathbb{R}[[X]]$ a unit, such that $\widehat{f}(X) = X^r U(X)$. Hence there exists $g \in \mathcal{C}_n$ such that $g(0) \neq 0$ and $f(x) = x^r g(x)$.
2. The $\mathcal{C}$-semianalytic sets are going to play the role of the Λ-sets of Sect. 2.

Notation 4.5. 1. For $r \in (0,\infty)^n$, $f = (f_1, \ldots, f_\mu) \in (\mathcal{C}_{n,r})^\mu$, $S \subset I_r$ and a sign condition $\sigma \in \{-1,0,1\}^\mu$, we put

$$B_S(f,\sigma) = \{x \in S : \operatorname{sgn} f_1(x) = \sigma_1, \ldots, \operatorname{sgn} f_\mu(x) = \sigma_\mu\}.$$

In the following, we use the notations of Sect. 2.1.
2. Each germ $f \in \mathcal{C}_n$ admits a representative in $\mathcal{C}_{n,r}$, for some $r \in (0,\infty)^n$, which will be denoted by the same letter f. Given $g = (g_1, \ldots, g_k) \in (\mathcal{C}_n)^k$, we say that $r \in (0,\infty)^n$ is g***-small*** if g_i admits a representative in $\mathcal{C}_{n,r}$ for $i = 1, \ldots, k$.

Definition 4.6. Let $r \in (0,\infty)^n$. A set $M \subset I_r$ is $\mathcal{C}$***-trivial*** if one of the following holds:

1. $M = B_{I_r}((x_1, \ldots, x_n), \sigma)$ for some $\sigma \in \{-1,0,1\}^n$;
2. There are a permutation λ of $\{1, \ldots, n\}$, a $\mathcal{C}$-trivial $N \subset I_s$ and a $g \in \mathcal{C}_{n-1,s}$, where $s = (r_{\lambda(1)}, \ldots, r_{\lambda(n-1)})$, such that $g(I_s) \subset (-r_{\lambda(n)}, r_{\lambda(n)})$ and $\Pi_\lambda(M) = \operatorname{graph}(g|N)$.

A $\mathcal{C}$-semianalytic manifold $M \subset \mathbb{R}^n$ is called ***trivial*** if $M = N + a$ for some $\mathcal{C}$-trivial manifold $N \subset \mathbb{R}^n$ and some $a \in \mathbb{R}^n$ (thus every trivial $\mathcal{C}$-semianalytic manifold is bounded and connected).

Remark 4.7. We note that a trivial $\mathcal{C}$-manifold M is a bounded and connected $\mathcal{C}$-manifold such that $\operatorname{fr}(M) = \overline{M} \setminus M$ is a $\mathcal{C}$-set, has dimension, and $\dim(\operatorname{fr} M) <$

dim (M); this can easily be proved by induction on n. We recognize the "small frontier" condition mentioned Sect. 2.

The next statement is the first step towards obtaining the Gabrielov property.

Proposition 4.8 ([17, Proposition 3.8]). *Let* $r \in (0,\infty)^n$ *and* $f \in (\mathcal{C}_{n,r})^{\mu}$. *Then there is a neighborhood* $W \subset I_r$ *of* 0 *with the following property:*

(∗) for every sign condition $\sigma \in \{-1,0,1\}^{\mu}$, *there is an* $l \in \mathbb{N}$, *and for each* $m = 1,\dots,l$ *there are* $n_k \geq n$, $r_k \in (0,\infty)^{n_k}$ *and* $\mathcal{C}$*-trivial manifolds* $N_k \subset I_{r_k}$ *such that*

$$B_W(f,\sigma) = \Pi_n(N_1) \cup \cdots \cup \Pi_n(N_l)$$

and, for each k, *the set* $\Pi_n(N_k)$ *is a manifold and* $\Pi_n|N_k\colon N_k \to \Pi_n(N_k)$ *is a diffeomorphism.*

Remark 4.9. The complete Gabrielov property Definition 2.2 requires a statement like (∗) not only for $\mathcal{C}$-semianalytic sets, but also for their linear projections.

Proof (Proof of Proposition 4.8). We may assume that $f_j \neq 0$ for each $j = 1,\dots,\mu$. Hence $g = f_1 \cdots f_\mu \neq 0$ and, by quasianalyticity of $\mathcal{C}_n$, we have $\widehat{g} \neq 0$.

The proof is done by induction on the pair $(n, h_n(\widehat{g}))$, where h_n is the invariant defined in [17] and mentioned in Sect. 3. If $\widehat{g}$ is normal, then the statement is an easy consequence of Remark 4.4. Otherwise, there exists an admissible substitution τ such that $h_n(\widehat{\varphi}) < h_n(\widehat{g})$ for every $\widehat{\varphi} \in \tau\widehat{g}$. In these notes, we only illustrate the proof in the case where τ is a translation substitution.

Let $\alpha \in \mathcal{C}_{i-1}$, with $1 < i \leq n$ and $\alpha(0) = 0$, be such that $\tau = \{t_\alpha\}$. Let $s \in (0,\infty)^n$ be both $(t_\alpha f)$-small and t_α-small and such that $t_\alpha(I_s) \subset I_r$. By the inductive hypothesis, there is a neighborhood $V \subset I_s$ of 0 such that (∗) holds with $t_\alpha f$, s and V in place of f, r and W. Then $W = \tau_\alpha(V)$ is a neighborhood of 0, and we claim that (∗) holds with this W.

To see this, we let $\sigma \in \{-1,0,1\}^{\mu}$. Let $M_k \subset \mathbb{R}^{m_k}$, for $k = 1,\dots,p$, be the $\mathcal{C}$-trivial manifolds obtained for this σ from the inductive hypothesis applied to $t_\alpha f$. For each k, we put

$$N_k = \{(x_{<i}, t, x_{>i}, x_i) : x \in M_k \text{ and } t = x_i + \alpha(x_{<i})\},$$

where $x_{<i} = (x_1,\dots,x_{i-1})$ and $x_{>i} = \left(x_{i+1},\dots,x_{m_k}\right)$. Each N_k is a $\mathcal{C}$-trivial manifold and $B_W(f,\sigma) = \bigcup_{k=1}^{p} \Pi_n(N_k)$. Moreover, since $t_\alpha|I_s\colon I_s \to t_\alpha(I_s)$ is a diffeomorphism, it follows that $\Pi_n(N_k) = t_\alpha(\Pi_n(M_k))$ is a manifold and $\Pi_n|N_k\colon N_k \to \Pi_n(N_k)$ is a diffeomorphism, as required. □

4.2 Towards the Gabrielov Property

We want to establish the o-minimality and model completeness of structures generated by quasianalytic classes satisfying hypothesis (C1)–(C7) (see the precise statement in Sect. 4.3). This goal is achieved with the methods of Sect. 2. In particular, we have to establish the Gabrielov property in the quasianalytic framework. Roughly speaking, we have to prove the statement $(*)$ of Proposition 4.8 for linear projections of bounded $\mathcal{C}$-semianalytic sets. The following statements are proved in [17].

4.2.1 The General Statements

The following proposition is an easy consequence of Proposition 4.8.

Proposition 4.10 ([17, Corollary 4.4]). *Let $A \subset \mathbb{R}^n$ be bounded and $\mathcal{C}$-semianalytic. Then there are $n_i \geq n$ and trivial $\mathcal{C}$-semianalytic manifolds $N_i \subset \mathbb{R}^{n_i}$, for $i = 1, \ldots, k$, such that*

$$A = \Pi_n(N_1) \cup \cdots \cup \Pi_n(N_k)$$

and, for each i, the set $\Pi_n(N_i)$ is a manifold and $\Pi_n|N_i : N_i \to \Pi_n(N_i)$ is a diffeomorphism. In particular, A has dimension.

Next, let $M \subset \mathbb{R}^n$ be a $\mathcal{C}$-manifold of dimension $m \leq n$. For every strictly increasing sequence $\lambda \in \{1, \ldots, n\}^m$ (see notations in Sect. 2), we put

$$M_\lambda = \{x \in M : \Pi_\lambda|T_x M \text{ has rank } m\}.$$

It is clear that M_λ is an open subset of M and is, in fact, a $\mathcal{C}$-manifold. Moreover, we have

$$M = \bigcup \{M_\lambda : \lambda \in \{1, \ldots, n\}^m \text{ is strictly increasing}\}.$$

Let $k \leq m$ and $\lambda \in \{1, \ldots, n\}^m$ be strictly increasing. We let $m(k) \in \{0, \ldots, m\}$ be maximal such that $\lambda(m(k)) \leq k$.

The following well-known statement of analytic geometry holds as well in our quasianalytic framework.

Lemma 4.11 (Fiber Cutting Lemma) [17, Lemma 4.5]. *Let $n \geq m > k \geq 0$. Let $M \subset \mathbb{R}^n$ be a $\mathcal{C}$-manifold of dimension m. Assume that $M = M_\lambda$ for some given strictly increasing $\lambda \in \{1, \ldots, n\}^m$ and that $\Pi_\lambda|M$ has constant rank $m(k)$. Then there is a $\mathcal{C}$-set $A \subset M$ such that $\dim(A) < m$ and $\Pi_k(M) = \Pi_k(A)$.*

This lemma allows us to prove:

Lemma 4.12 ([17, Lemma 4.6]). *Let $M \subset \mathbb{R}^n$ be a $\mathcal{C}$-manifold of dimension m and $k \leq m$. Then there are trivial $\mathcal{C}$-semianalytic manifolds $N_i \subset \mathbb{R}^{n_i}$, satisfying $\dim(N_i) \leq k$ and $n_i \geq n$ for $i = 1, \ldots, K$, and there are bounded $\mathcal{C}$-semianalytic sets $A_j \subset \mathbb{R}^{p_j}$, with $\dim(A_j) < m$ and $p_j \geq n$ for $j = 1, \ldots, L$, such that*

$$\Pi_k(M) = \Pi_k(N_1) \cup \cdots \cup \Pi_k(N_K) \cup \Pi_k(A_1) \cup \cdots \cup \Pi_k(A_L)$$

and, for each i, there is a strictly increasing $\lambda \in \{1,\ldots,k\}^{\dim(N_i)}$ such that $\Pi_\lambda|N_i : N_i \to \mathbb{R}^{\dim(N_i)}$ is an immersion.

We obtain the Gabrielov property from the following:

Corollary 4.13 ([17, Proposition 4.7]). *Let $A \subset \mathbb{R}^n$ be a bounded $\mathcal{C}$-semianalytic set and $k \leq n$. Then there are trivial $\mathcal{C}$-semianalytic manifolds $N_i \subset \mathbb{R}^{n_i}$, with $n_i \geq n$ for $i = 1,\ldots,J$, such that*

$$\Pi_k(A) = \Pi_k(N_1) \cup \cdots \cup \Pi_k(N_J)$$

and, for each i, $d = \dim(N_i) \leq k$ and there exists a strictly increasing $\lambda \in \{1,\ldots,k\}^d$ such that $\Pi_\lambda|N_i : N_i \to \mathbb{R}^d$ is an immersion.

4.2.2 Study in Small Dimension

As before, we do not repeat or rephrase the detailed proof given in [17, Sect. 4]. We merely illustrate the arguments of Lemma 4.12 (and Fiber Cutting Lemma 4.11) in dimension 3, under some simplifying hypotheses.

We consider a 2-dimensional $\mathcal{C}$-manifold $M \subset \mathbb{R}^3$ and its projections $\Pi_k(M)$ for $k = 1, 2$. We define

$$r_k(M) = \max\{\mathrm{rk}(\Pi_k|_{T_xM}) : x \in M\}.$$

Note that the case $r_k(M) = 0$ is obvious.

Case $k = 2$. Then $\Pi_k = \Pi_2$ denotes the projection on the (x_1, x_2)-plane. We suppose $r_2(M) = 2$. Let $M_0 = \{x \in M : \mathrm{rk}(\Pi_2|_{T_xM}) < 2\}$. Hence $M \setminus M_0$ is an open subset of M that obviously satisfies the statement of Lemma 4.12. In general, M_0 is not a $\mathcal{C}$-manifold, but is a bounded $\mathcal{C}$-set. According to Proposition 4.10, there exist finitely many integers $\nu \geq 3$ and $\mathcal{C}$-manifolds $N \subset \mathbb{R}^\nu$ such that $\Pi_3|_N : N \to \Pi_3(N) \subset M_0$ is a diffeomorphism. Hence, for all $a \in N$, we have $\mathrm{rk}\left(\Pi_2^\nu|_{T_aN}\right) \leq \mathrm{rk}\left(\Pi_2|_{T_{\Pi_3^\nu(a)}M_0}\right) < r_2(M)$. So $r_2(N) < 2$, and we leave the proof of Lemma 4.12, with N in place of M, as an exercise.

Case $k = 1$. Then $\Pi_k = \Pi_1$ denotes the projection on the x_1-axis. We suppose $r_1(M) = 1$. Hence $\Pi_1|_M : M \to \Pi_1(M)$ is not a diffeomorphism anymore, and we cannot conclude as easily as before. We denote by $\Pi_{1,2}$ the projection on the (x_1, x_2)-plane of coordinates, and by $\Pi_{1,3}$ the projection on the (x_1, x_3)-plane of coordinates. Let $M_{1,2} = \{x \in M : \mathrm{rk}(\Pi_{1,2}|_{T_xM}) = 2\}$ and $M_{1,3} = \{x \in M : \mathrm{rk}(\Pi_{1,3}|_{T_xM}) = 2\}$. These two sets are open in M. Moreover,

$$r_1(M \setminus (M_{1,2} \cup M_{1,3})) = 0,$$

so it is enough to prove Lemma 4.12 on $M_{1,2} \cup M_{1,3}$. We claim that there exists a $\mathcal{C}$-set $A \subset M_{1,2}$ such that $\Pi_1(A) = \Pi_1(M_{1,2})$ and $\dim(A) < 2$ (and similarly for $M_{2,3}$): consider $r > 0$ and $f, g_1, \ldots, g_k \in \mathcal{C}_{n,r}$ such that

$$M_{1,2} = \{x \in I_r : f(x) = 0, g_1(x) > 0, \ldots, g_k(x) > 0\}.$$

Let g be the product of all g_j, $j = 1, \ldots, k$, and all $(x_i - r_i)|I_r$ and $(r_i - x_i)|I_r$, $i = 1, 2, 3$. Note that g is strictly positive on all of $M_{1,2}$ and identically zero on $\mathrm{fr}(M_{1,2})$. For each $a \in \Pi_1(M_{1,2})$, the fiber $M_a = \Pi_1^{-1}(a) \cap M_{1,2}$ is either empty or a $\mathcal{C}$-manifold of dimension 1. Let $\pi_2 : \mathbb{R}^3 \to \mathbb{R}$ be the linear projection on the x_2-axis. Since $\Pi_2|_{M_{1,2}} : M_{1,2} \to \mathbb{R}^2$ is an immersion and $\Pi_1|_{M_a}$ is constant, the map $\pi_2|M_a$ is an immersion. If follows that, if C is a connected component of M_a, then $\pi_2(C)$ is open (and bounded) in $\mathbb{R}$, which implies that $\mathrm{fr}(C) \neq \emptyset$.

The last step in the proof is the Fiber Cutting Lemma 4.11. The function $g|M_a$ has critical points on each connected component of M_a, since g is positive on M_a and vanishes identically on $\mathrm{fr}(M_a)$. Since M_a is a $\mathcal{C}$-manifold, it follows from quasianalyticity (by the same arguments as in the analytic setting) that the set of critical points of $g|M_a$ has empty interior in M_a. Let

$$A = \{x \in M_{1,2} : x \text{ is a critical point of } g|M_a,\ a = \Pi_1(x)\}.$$

Then $\Pi_1(A) = \Pi_1(M_{1,2})$. Being a $\mathcal{C}$-set, A has dimension. Since A has empty interior in $M_{1,2}$, $\dim(A) < 2$ as required.

4.3 O-minimality and Model Completeness

We can now state the final result of this section: let $\mathcal{F} = \bigcup_{n \in \mathbb{N}} \mathcal{C}_{n,1}$.

Theorem 4.14. *The structure $\mathbb{R}_{\mathcal{F}}$ is model complete and o-minimal.*

The proof is an application of the techniques of Sect. 2: for $n \in \mathbb{N}$, we consider the collection $\Lambda_n = \{X \subset I^n : X \text{ is } \mathcal{C}\text{-semianalytic}\}$. It is easy to check that if $A \subset I^n$ is a $\mathcal{C}$-semianalytic set, then each N_i obtained from Proposition 4.13 can be taken to be a subset of I^{n_i}. Hence every Λ-set has the ***Λ-Gabrielov property***.

5 Examples and Perspectives

We describe in this last section three o-minimal structures obtained by the above methods. These examples illustrate several possible applications of Theorem 4.14. Let us summarize roughly these various points of view.

1. Certain algebras of functions are known to be quasianalytic. Such is the case for the algebras studied by Denjoy [7] and Carleman [5]; hence it remains, for our purposes, to prove that they also satisfy the closure properties (C1)–(C7) of Sect. 4.1.
2. We may also work in the opposite direction. Consider a real function H for which we suspect o-minimality (as well as model completeness and polynomial boundedness). A possible strategy is to consider the smallest collection $\mathcal{A}(H)$ of algebras of real functions containing H (we should speak actually of *germs* of functions) and closed under the operations (C1)–(C7), and to prove that these algebras are quasianalytic.
3. In order to build o-minimal structures that satisfy a given property (such as "not having smooth cell decomposition"), we look for an appropriate function H adapted to this property (such as a function that is not piecewise analytic) and such that the algebras $\mathcal{A}(H)$ are quasianalytic.

5.1 Denjoy-Carleman Classes

These families of functions have been studied in [17] from the point of view of o-minimality, and in [4] from the point of view of resolution of singularities.

Consider a sequence $M = (M_0, M_1, \ldots)$ of real numbers such that $1 \leq M_0 \leq M_1 \leq \cdots$, and assume that M is ***logarithmically convex*** (or ***log-convex*** for short), i.e., $M_i^2 \leq M_{i-1}M_{i+1}$ for all $i > 0$. To every compact box $B = [a_1, b_1] \times \cdots \times [a_n, b_n]$, with $a_i \leq b_i$ for $i = 1, \ldots n$, we associate the collection $\mathcal{C}_B^0(M)$ of all functions $f: B \to \mathbb{R}$ for which there exist an open neighborhood U of B, a $\mathcal{C}^\infty$ function $g: U \to \mathbb{R}$ such that $g|B = f$, and a constant $A > 0$ (depending of f) such that

$$\left|g^{(\alpha)}(x)\right| \leq A^{|\alpha|+1} \cdot M_{|\alpha|}, \quad \text{for all } x \in U \text{ and } \alpha \in \mathbb{N}^n,$$

where $|\alpha| = \alpha_1 + \cdots + \alpha_n$. We call $\mathcal{C}_B^0(M)$ the ***Denjoy-Carleman class on*** B ***associated to*** M. Notice that if $M_i = i!$ for all $i \geq 0$, then $\mathcal{C}_B^0(M)$ is the class of real-valued functions on B that extend analytically to an open neighborhood of B. A classical result (see [10] for example) states that, under the hypothesis

$$\sum_{i=0}^{n} \frac{M_i}{M_{i+1}} = \infty, \tag{5.1}$$

the algebra $\mathcal{C}_B^0(M)$ is ***quasianalytic***: for any $f \in \mathcal{C}_B^0(M)$ and any $x \in B$, the Taylor expansion $\widehat{f}_x$ of f at x determines f among all functions in $\mathcal{C}_B^0(M)$.

Hence a possible way to prove the o-minimality of the structure generated by the algebras $\mathcal{C}_B^0(M)$ is to prove the closure properties (C1)–(C7). Unfortunately,

$\mathcal{C}_B^0(M)$ is in general not closed under differentiation. Thus we introduce the classes $\mathcal{C}_B(M) = \bigcup_{j=0}^{\infty} \mathcal{C}_B^0\left(M^{(j)}\right)$ where $M^{(j)} = \left(M_j, M_{j+1}, \ldots\right)$; these classes are obviously closed under differentiation and still quasianalytic.

However, to obtain closure under the other operations, we need a stronger assumption on the sequence M. More precisely, we assume that M is ***strictly log-convex***, which means that the sequence $(M_i / i!)$ is log-convex. We can now state the main result about these classes, as an immediate application of Theorem 4.14: for each $n \in \mathbb{N}$ and $f \in \mathcal{C}_{[-1,1]^n}(M)$, we define $\widetilde{f}: \mathbb{R}^n \to \mathbb{R}$ by $\widetilde{f}(x) = f(x)$ if $x \in [-1,1]^n$ and $\widetilde{f}(x) = 0$ otherwise. We let $\mathbb{R}_{\mathcal{C}(M)}$ be the expansion of the real field by all $\widetilde{f}$ for $f \in \mathcal{C}_{[-1,1]^n}(M)$ and $n \in \mathbb{N}$.

Theorem 5.1. *If M is strongly log-convex and satisfies* (5.1)*, then the structure $\mathbb{R}_{\mathcal{C}(M)}$ is model complete, o-minimal, polynomially bounded and admits $\mathcal{C}^\infty$ cell decomposition.*

Some of the motivations for [17] came from van den Dries: he conjectured that

1. There is no "largest" o-minimal expansion of the real field;
2. There exist o-minimal expansions of the real field that do not admit analytic cell decomposition.

Let us explain how Theorem 5.1 leads to these statements. As a consequence of a deep result proved by Mandelbrojt [15], it is known that, given a $\mathcal{C}^\infty$ function $f: U \to \mathbb{R}$, where U is an open neighborhood of $[-1,1]^n$ and $n \in \mathbb{N}$, there exist strongly log-convex sequences M and N satisfying (5.1), and functions $f_1 \in \mathcal{C}^0_{[-1,1]^n}(M)$ and $f_2 \in \mathcal{C}^0_{[-1,1]^n}(N)$ such that $f(x) = f_1(x) + f_2(x)$ for all $x \in [-1,1]^n$ (a complete proof of this statement is given in the appendix of [17], following indications of [10, Chap. V]; it is based on a detailed analysis of the relationship between the lacunae of the Fourier spectrum of a periodic function and its quasianalyticity).

Hence, if the $\mathcal{C}^\infty$ function f is oscillating, we see that the structures $\mathbb{R}_{\mathcal{C}(M)}$ and $\mathbb{R}_{\mathcal{C}(N)}$ cannot admit any o-minimal common expansion. The first statement is proved.

In order to prove the second statement, consider a $\mathcal{C}^\infty$ function $f: [-1,1] \to \mathbb{R}$ whose Taylor series at every $x \in [-1,1]$ is divergent (actually, the set of such functions is second category in the sense of Baire, see for example [8, 4.3, p. 301]). Hence one of the two summands f_1 and f_2 of Mandelbrojt's theorem must have a divergent Taylor series at every x belonging to some open interval $I \subset [-1,1]$. The corresponding o-minimal structure does not admit analytic cell decomposition.

Remark 5.2. In a recent paper [11], Le Gal gives a new proof of these statements that completely avoids Mandelbrot's theorem. His proof is based on a generalization of the ideas explained in Sect. 5.3 below.

5.2 Differential Equations and Quasianalytic Classes

Since this topic is developed in F. Sanz's course included in this volume, we do not give many details here. We prefer to focus on a simple (though nontrivial) example as an introduction to the general result.

Consider the differential equation $x^2y' = y - x$, $y \in \mathbb{R}$, with an irregular singular point at the origin. This equation is usually called ***Euler's equation***. In a neighborhood of the origin, this equation admits solutions $H\colon (0, \varepsilon) \to \mathbb{R}$ or $H\colon (-\varepsilon, 0) \to \mathbb{R}$, for $\varepsilon > 0$. These functions are ***pfaffian***, hence they are all definable in the same common o-minimal expansion of the real field [26]. However, these results do not imply anything about the model completeness of the structures $\mathbb{R}_H$ for a fixed such H. In order to deal with this question, we establish o-minimality in a way that does not make reference to [26]. Let us begin by listing some basic properties of H:

1. For $x > 0$, the graph of such an H belongs to the so-called ***node part*** of the phase portrait. Every such H is analytic on $(0, \varepsilon)$ and admits a $\mathcal{C}^\infty$ extension on $[0, \varepsilon)$, with $H(0) = 0$. Moreover, every such H admits the same asymptotic expansion at the origin, which is the divergent ***Euler power series*** $\sum_{n\geq 0} n!x^{n+1}$.
2. For $x < 0$, only one solution, denoted by H_0, admits a $\mathcal{C}^\infty$ extension on $(-\varepsilon, 0]$. The graph of H_0 is called a ***separatrix curve*** of the phase portrait, since it separates the dynamics inside the so-called ***saddle part*** of the phase portrait. Again, H_0 is analytic on $(-\varepsilon, 0)$, and admits the Euler power series as an asymptotic expansion at the origin.
3. For $x < 0$, the other solutions can be written as

$$H(x) = H_0(x) + C \exp(-1/x),$$

for $C \in \mathbb{R} \setminus \{0\}$. Hence they diverge to $\pm\infty$ as x goes to 0.

More involved is the following observation: the series $\widehat{H}(x) = \sum_{n\geq 0} n!x^{n+1}$ is a *Gevrey series* of order 1. This means that its Borel transform

$$\mathcal{B}\widehat{H}(\zeta) = \sum_{n\geq 0} \frac{n!}{n!}\zeta^n = \frac{1}{1-\zeta}$$

is analytic at the origin. In this case, the function $\mathcal{B}\widehat{H}$ is meromorphic on $\mathbb{C}$ and bounded outside any neighborhood of $\zeta = 1$, with a simple pole at $\zeta = 1$. Thus, we can compute its Laplace transform in every direction d but the real positive axis, according to the formula

$$\widetilde{H}_d(z) = \mathcal{L}_d\mathcal{B}\widehat{H}(z) = \int_d \exp(-z\zeta)\,\frac{\mathrm{d}\zeta}{1-\zeta}.$$

The classical theory of Borel-Laplace transforms says that the function $\widetilde{H}_d$ is defined on an open sector S_d based at the origin, bisected by d, and with opening equal to π. Moreover, $\widetilde{H}_d$ is a solution of Euler's differential equation and admits the Euler series as a *Gevrey asymptotic expansion* at the origin. (We refer the reader to Balser [1] for details on Gevrey functions.)

We remark that the functions $\widetilde{H}_d$, for $d \neq \mathbb{R}_-$, have complex (non real) values when restricted to the positive axis. Therefore, *the only real solution of Euler's equation obtained by this Laplace-Borel process is* $\widetilde{H}_{\mathbb{R}_-} = H_0$, *which corresponds to the separatrix.*

It turns out that such real functions, obtained via a Borel-Laplace process, are definable in the ***Gevrey structure*** $\mathbb{R}_{\mathcal{G}}$ introduced by van den Dries and Speissegger in [23]. They prove that $\mathbb{R}_{\mathcal{G}}$ is o-minimal, model complete and polynomially bounded. This result is mostly based on the quasianalyticity of the algebras of real Gevrey functions they consider.

We can deduce—without any reference to pfaffian results—that the structure $\mathbb{R}_{H_0}$ is o-minimal and polynomially bounded. What about its possible model completeness? Based on the fact that H_0 satisfies $x^2 H_0'(x) = H_0(x) - x$ and, therefore, that H_0' is existentially definable in $\mathbb{R}_{H_0}$, there is a possible way to prove the model completeness of $\mathbb{R}_{H_0}$, along the following lines: at some point of their proof, van den Dries and Speissegger use a version of Weierstrass preparation. We can replace this tool by the normalization process of Sect. 4. Each step of this process needs the introduction of functions or sets that are existentially definable in $\mathbb{R}_{H_0}$: Taylor's formula, the implicit function theorem, monomial division, etc. Moreover the next step, which leads from normalization to the Λ-Gabrielov property, is mostly based on the ***Fiber-cutting Lemma***, which also introduces existentially definable sets. These "book-keeping" observations lead to an *explicit* version of the process described in these notes (see [17] for the details), which in turn implies the model completeness of $\mathbb{R}_{H_0}$.

Finally, what can we say for the structures $\mathbb{R}_H$ when $H \neq H_0$?

1. Consider first the solutions $H\colon (0,\varepsilon) \to \mathbb{R}$ of the node part of the saddle-node. For such H, we consider the smallest collection $\mathcal{A}(H)$ of algebras containing H and closed under the operations (C1)–(C7) of Sect. 4, and we prove that these algebras are quasianalytic (see Sanz's notes in this volume for details). Hence Theorem 4.14 implies that the corresponding structure $\mathbb{R}_H$ is o-minimal, model complete and polynomially bounded.
2. We already mentioned that the solutions $H\colon (-\varepsilon, 0) \to \mathbb{R}$ of the saddle part of the phase portrait can be written as $H(x) = H_0 + C \exp(-1/x)$. Although it is proved in [23] that the structure $\mathbb{R}_{H_0,\exp}$ is o-minimal and model complete, this does not imply that $\mathbb{R}_H$ is model complete. As of today this question is still open.

5.3 O-minimal Structures Without Smooth Cell Decomposition

Many attempts to prove the existence of such a structure were inspired by the construction of a Hardy field that contains nonsmooth elements. But it is not that easy in general to prove that the methods used in the Hardy field framework extend to the o-minimal setting. This goal was achieved in [12], using a slight variation of Theorem 4.14.

In this section, we first recall one of the possible classical constructions of a Hardy field that contains germs of not ultimately $\mathcal{C}^\infty$ functions. Then we show how some of the ideas of this construction may be adapted to the o-minimal context.

5.3.1 A Hardy Field Containing Elements That Are Not $\mathcal{C}^\infty$

Recall that a Hardy field is a differential field of germs at $+\infty$ of continuous real valued functions; for an introduction to Hardy fields, we refer the reader to [16]. Hence, if the germ of a function belongs to a Hardy field there exists, for each integer k, a neighborhood of $+\infty$ in which this function is $\mathcal{C}^k$. But this does not imply that this function is $\mathcal{C}^\infty$ in a neighborhood of $+\infty$.

One of the ideas of such a construction is the following well known fact: the germ at $+\infty$ of a function $f : \mathbb{R}_+ \to \mathbb{R}$ generates a Hardy field provided that every element of $\mathbb{R}[f, f', f'', \ldots]$ (the ring of ***differential polynomials*** in f) is ultimately never zero (and thus its germ at $+\infty$ is invertible). Here the function f will be defined as the sum of two functions $f(x) = F(x) + \mathrm{e}^{-x} g(x)$, where

1. F is a non-oscillatory analytic function at infinity. More precisely, every nonzero differential polynomial in F is equivalent at $+\infty$ to a monomial cx^{-p}, with $c \neq 0$ and $p \in \mathbb{N}$;
2. $g(x) = \left(\sin^2 x\right)^x$, for $x \in \mathbb{R}_+$.

Let us explain how F may be defined. Consider a sequence (a_i) of real numbers that are algebraically independent over $\mathbb{Q}$ and such that the radius of convergence of the series $\sum a_i z^i$ is positive. Thanks to Lindemann's theorem, such a sequence can be obtained by considering the sequence $b_i = \exp(p_i)$, where p_i is the i-th prime number, and defining $a_i = \exp(p_i)/N_i$, where the N_i are integers big enough to guarantee the convergence of the series. Finally, let $F(x) = \sum_{i=1}^{\infty} a_i x^{-i}$. In addition, another classical result of Hardy fields due to Rosenlicht [19] allows us to choose F such that every nonzero differential polynomial in F admits a nonzero principal part.

On the other hand, it can be proved (by straightforward computation) that the germ of g at $+\infty$ admits, for every positive integer k, a $\mathcal{C}^k$ representative, but that g is not ultimately $\mathcal{C}^\infty$. Moreover, for every $m \in \mathbb{N}$ and every polynomial $q \in \mathbb{R}[X_0, X_1, \ldots, X_m]$, we have

$$q\left(f, f', \ldots, f^{(m)}\right) = q\left(F, \ldots, F^{(m)}\right) + \sum_{j=1}^{N} \mathrm{e}^{-jx} Q_j\left(F, f, F', f', \ldots, F^{(m)}, f^{(m)}\right)$$

for finitely many polynomials $Q_1, \ldots, Q_N$. Hence the left-hand side is, at infinity, a flat oscillating perturbation of the differential polynomial $q\left(F, F', \ldots, F^{(m)}\right)$. Since the latter polynomial admits a nonflat principal part at $+\infty$, the same follows for $q\left(f, f', \ldots, f^{(m)}\right)$. Therefore, the germ of f at $+\infty$ generates a Hardy field. Moreover, f is not ultimately $\mathcal{C}^\infty$, since $g(x) = \mathrm{e}^x (F(x) + f(x))$ is not. This finishes the construction.

Remark 5.3. Let us point out some interesting aspects of the above example. Two goals need to be achieved simultaneously:

1. The non-oscillation of every differential polynomial in f. The dominant part F of the sum $f(x) = F(x) + \mathrm{e}^{-x} g(x)$ is designed for this purpose;
2. The fact that f is not ultimately $\mathcal{C}^\infty$. This property is satisfied thanks to the behavior of the flat perturbation $\mathrm{e}^{-x} g(x)$.

More precisely, the non-oscillatory behavior of F has to "resist" any polynomial differential operation. This *differential transcendence property* is guaranteed by the algebraic independence of the coefficients a_i. Actually, this is mostly a property of the formal power series $\sum b_i z^i$, which is finally transformed into a convergent series after the division of the coefficients b_i by convenient integers.

Having in mind the construction of an o-minimal structure, we wonder if the non-oscillation property "resists" not only the differential polynomial operations, but also any definable operation. We shall see in the next section that this is indeed the case. Finally, in order for the function f to generate an o-minimal structure, we also need to know that f has correspondingly good behavior at every point of its domain other than $+\infty$.

Thus, in order to guarantee good local behavior of the function at every point, the construction given in [12] of an o-minimal structure without smooth cell decomposition is based on a somewhat different approach. As explained in the next section, the two previous goals are achieved through a non-oscillatory function whose germ at 0 is not $\mathcal{C}^\infty$, and which has polynomial behavior everywhere else.

5.3.2 An O-minimal Structure Without Smooth Cell Decomposition

The main object of the construction proposed in [12] is a function $H : \mathbb{R} \to \mathbb{R}$ satisfying the following properties:

1. The germ of H at 0 admits, for every non-negative integer k, a $\mathcal{C}^k$ representative (we will say that H is ***weakly*** $\mathcal{C}^\infty$ for short);

2. The restriction of H to the complement of any neighborhood of 0 is piecewise given by finitely many polynomials (***piecewise polynomial*** for short);
3. The coefficients of the Taylor expansion of H at 0 are algebraically independent over $\mathbb{Q}$;
4. The germ of H at 0 is not $\mathcal{C}^\infty$.

We then prove:

Theorem 5.4. *Consider a function $H:\mathbb{R} \to \mathbb{R}$ that satisfies properties 1., 2. and 3.. Then the structure $\mathbb{R}_H$ is o-minimal and polynomially bounded.*

Remark 5.5. 1. This result can be made more precise as follows: if $\mathcal{H}$ denotes the collection of all derivatives $H^{(i)}$ of H (defined in a neighborhood I_i of 0), then the structure $\mathbb{R}_{\mathcal{H}}$ is model-complete.
2. If the function H satisfies the extra hypothesis 4., then the structure $\mathbb{R}_H$ is an o-minimal expansion of the real field that does not admit smooth cell decomposition.

Construction of the function H. Let us explain the main steps of the construction. Consider, as in Sect. 5.3.1, a sequence (a_i) of real numbers algebraically independent over $\mathbb{Q}$. The following operation is the main difference with the previous construction of a Hardy field with nonsmooth elements. We do not demand the series $\sum a_i x^i$ to have a positive radius of convergence; indeed, we are not interested at all in the possible convergence of this series.

We adapt instead a construction due to Borel, which states that, given any power series $\widehat{H}(x) \in \mathbb{R}[[x]]$, there exists a $\mathcal{C}^\infty$ function $H:\mathbb{R} \to \mathbb{R}$ whose Taylor expansion at $0 \in \mathbb{R}$ is the series $\widehat{H}(x)$ (see for example [14]). Of course, the main point in this section is precisely to avoid $\mathcal{C}^\infty$ functions. Therefore we build H in the following way. For every integer i, let P_i be the polynomial $P_i(x) = (1-x)^i(1+x)^i$. For $\varepsilon \in [0,1]$, we define

$$v_i^\varepsilon(x) = \begin{cases} x^i P_i\left(\frac{x}{\varepsilon}\right) & \text{is } x \in (-\varepsilon, \varepsilon), \\ 0 & \text{otherwise.} \end{cases}$$

The functions v_i^ε satisfy two main properties, where $I_\varepsilon := (-\varepsilon, \varepsilon)$ and $\|\cdot\|_m^\varepsilon$ denotes the mth Whitney norm on I_ε:

1. For $0 \le m \le i$, we have $v_i^\varepsilon \in \mathcal{C}^m(I)$, $\left(v_i^\varepsilon\right)^{(m)}(0) = 0$ and $\left\|v_i^\varepsilon\right\|_m^\varepsilon \to 0$ when $\varepsilon \to 0$;
2. $\left(v_i^\varepsilon\right)^{(i)}(0) = i!$.

We now define, by induction on $i \in \mathbb{N}$ (see [12] for details), numbers $\varepsilon_i > 0$ and $b_i \in \mathbb{R}$, such that $\varepsilon_i \to 0$ and the functions $h_i = \sum_{k=0}^i b_k v_k^{\varepsilon k}$ satisfy

$$\left(h_{i-1} + b_i v_i^\varepsilon\right)^{(i)}(0) = i!a_i \quad \text{and} \quad \left\|b_i v_i^{\varepsilon_i}\right\|_{i-1}^{\varepsilon_i} < \frac{1}{2^i}.$$

Therefore, for any $m \geq 0$, the series $\sum b_i v_i^{\varepsilon_i}$ is $\|\cdot\|_m^{\varepsilon_m}$-uniformly convergent in I_{ε_m}. It defines a weakly $\mathcal{C}^\infty$ function H on $(-1, 1)$ that is not $\mathcal{C}^\infty$ on the sequence ε_i.

From algebraic transcendence to quasianalyticity. The algebraic independence of the coefficients of the Taylor series of H at the origin implies actually much more than the differential transcendence of H. It implies, indeed, that the elements of the smallest collection of algebras of germs at 0, containing the germ of H and closed under the operations (C1)–(C7), are quasianalytic.

More precisely, we introduce, for every $n \in \mathbb{N}$, the algebra $\mathcal{W}_n$ of weakly $\mathcal{C}^\infty$ germs at the origin of $\mathbb{R}^n$, and we let $\mathcal{A}(H)$ be the smallest collection of algebras $\mathcal{A}_n(H) \subset \mathcal{W}_n$, $n \in \mathbb{N}$, satisfying the following conditions:

1. The germ of H belongs to $\mathcal{A}_1(H)$, and the germs of all polynomials in n variables belong to $\mathcal{A}_n(H)$;
2. If $f \in \mathcal{A}_n(H)$, and if f_i denotes the restriction of f to the hyperplane $x_i = 0$, for $i = 1, \ldots, n$, then the germ which continuously extends $(f - f_i)/x_i$ at $0 \in \mathbb{R}^m$ belongs to $\mathcal{A}_n(H)$;
3. If $g_1, \ldots, g_m \in \mathcal{A}_n(H)$ and $f \in \mathcal{A}_m(H)$, then $f(g_1 - g_1(0), \ldots, g_m - g_m(0))$ blongs to $\mathcal{A}_n(H)$;
4. If $n > 1$ and $f \in \mathcal{A}_n(H)$, let $g(x) = f(x) - f(0) - x_n \partial f/\partial x_n(0) + x_n$, so that $\partial g/\partial x_n(0) = 1$. Then the germ $\varphi \in W_{n-1}$ defined by $g(x, \varphi(x)) = 0$ belongs to $\mathcal{A}_{n-1}(H)$.

We then prove:

Theorem 5.6. *Consider a weakly $\mathcal{C}^\infty$ function $H : \mathbb{R} \to \mathbb{R}$ such that the coefficients of its Taylor expansion at* 0 *are algebraically independent over $\mathbb{Q}$. Then the algebras $\mathcal{A}_n(H)$ are quasianalytic.*

This result guarantees good behavior, in the spirit of Theorem 4.14, of the algebras of definable germs at the origin. The good behavior of the algebras of germs at any other point is a consequence of the piecewise polynomial nature of H outside the origin. The adaptation of Theorem 4.14 to this context is explained in detail in [12, Sect. 3].

Let us conclude this section by saying a few words on the proof of Theorem 5.6. The operations (C1)–(C7) are expressed in [12] in a language of operators acting on the algebras of weakly $\mathcal{C}^\infty$ germs. More precisely, an ***elementary operator*** is one of the following, where n, m denote non-negative integers:

1. The sum and the product acting on $\mathcal{W}_n \times \mathcal{W}_n$;
2. The natural embedding $\mathcal{W}_n \to \mathcal{W}_{n+1}$;
3. For any $c \in \mathbb{R}$, the constant operator $\mathcal{W}_1 \to \mathcal{W}_0$ defined by $f \mapsto c$;
4. For $1 \leq i \leq n$, the coordinate operators $\mathcal{W}_1 \to \mathcal{W}_n$ defined by $f \mapsto x_i$;
5. The monomial division operators $\mathcal{W}_n \to \mathcal{W}_n$ defined by $f \mapsto D_i(f)$, where $i \in \{1, \ldots, n\}$, $D_i(f)$ is the germ at $0 \in \mathbb{R}^n$ of the continuous extension of $(f - f_i)/x_i$ and f_i denotes the restriction of f to the hyperplane $\{x_i = 0\}$;

6. The composition operators $\mathcal{W}_m \times \mathcal{W}_n^m \to \mathcal{W}_n$ defined by
$$(f, g_1, \ldots, g_m) \longmapsto f\,(g - g_1(0), \ldots, g_m - g_m(0))\,;$$

7. The implicit function operators $\mathcal{W}_n \to \mathcal{W}_{n-1}$ defined, for $n > 1$, by $f \mapsto \varphi$, where $\varphi \in \mathcal{W}_{n-1}$ is the germ characterized by $\varphi(0) = 0$ and $g(x', \varphi(x')) = 0$, with $x' = (x_1, \ldots, x_{n-1})$ and
$$g = f - f(0) - x_n \partial f/\partial x_n(0) + x_n.$$

An ***operator*** is a finite composition of elementary operators. Note, for instance, that *partial differentiation with respect to a coordinate* is an operator. Moreover, given a germ $H \in \mathcal{W}_1$ and a positive n, for every element $g \in \mathcal{A}_n(H)$ there exists (at least) one operator $\mathcal{L}$ such that $\mathcal{L}(H) = g$.

The main tool in the proof of Theorem 5.6 is to consider, for every operator $\mathcal{L}$ acting on $\mathcal{W}_{n_1} \times \cdots \times \mathcal{W}_{n_s}$, the corresponding ***formal operator*** $\widehat{\mathcal{L}}$ acting on $\mathbb{R}[[x_1, \ldots, x_{n_1}]] \times \cdots \times \mathbb{R}[[x_1, \ldots, x_{n_s}]]$, defined in the same way as a finite composition of elementary formal operators. Indeed, the algebras $\mathcal{A}_n(H)$ are quasianalytic provided that $\widehat{\mathcal{L}(H)} = 0$ implies $\mathcal{L}(H) = 0$, for every operator $\mathcal{L}$.

Why is the algebraic independence of the coefficients of the series $\widehat{H}$ required? Roughly speaking, the idea is to "forbid" $\widehat{H}$ to belong to the kernel of any nonzero formal operator, and to deduce the quasianalyticity of the algebras $\mathcal{A}_n(H)$ from the implications
$$\widehat{\mathcal{L}}\left(\widehat{H}\right) = 0 \Longrightarrow \widehat{\mathcal{L}} = 0 \Longrightarrow \mathcal{L} = 0.$$

The second implication is proved in [12] and called *quasianalyticity for operators*. The first one holds for some operators (we have already mentioned in Sect. 5.3.1 that such a series $\widehat{H}$ is differentially transcendental), but is obviously wrong in general: for example, the Schwarz operators $\partial^2/\partial x_i \partial x_j - \partial^2/\partial x_j \partial x_i$ vanishes identically over the rings of formal power series. However, the following related result is proved in [12]:

Lemma 5.7. *Consider a formal series* $\widehat{H}(x) = \sum H_i x^i \in \mathbb{R}[[x]]$ *whose coefficients are algebraically independent over* $\mathbb{Q}$*, and let* $\widehat{\mathcal{L}}$ *be a formal operator such that* $\widehat{\mathcal{L}}\big(\widehat{H}\big) = 0$. *Then there exists an integer* $N \geq 0$ *such that*
$$\widehat{\mathcal{L}}\left(h_1 x_1 + \cdots + h_N x^N + x^{N+1}\widehat{g}(x)\right) = 0$$
for all $\widehat{g}(x) \in \mathbb{R}[[x]]$.

Thus, the formal operator
$$\widehat{\mathcal{M}}\colon \widehat{g}(x) \mapsto \widehat{\mathcal{L}}\left(h_1 x^N + \cdots + h_N x^N + x^{N+1}\widehat{g}(x)\right)$$
is identically zero. Because of the quasianalyticity property for operators mentioned above, the operator $\mathcal{M}$, which acts on $\mathcal{W}_1$, is also identically 0. In particular, if

$g \in \mathcal{W}_1$ is defined by

$$H(x) = h_1 x + \cdots + h_N x^N + x^{N+1} g(x),$$

we have $\mathcal{L}(H) = 0$. Hence the quasianalyticity of the algebras $\mathcal{A}_n(H)$ is proved.

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Course on Non-oscillatory Trajectories

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Abstract Non-oscillatory trajectories of vector fields are discussed, and some sufficient conditions are established that make the expansion of the real field by such a trajectory o-minimal.

Mathematics Subject Classification (2010): Primary 03C64, 34C08, 34M40, Secondary 34C10, 34C45, 37C10, 34E05, 34M30

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1 Introduction

These notes are framed in the general ideology of the *Thematic Program on o-minimal Structures and Real Analytic Geometry*, roughly speaking, the study of finiteness properties of "transcendental" objects arising from natural problems in mathematics. "Natural problems" is an undefined term; they can be differential equations (ordinary or partial), foliations, diffeomorphisms, normalization, integration, analytic continuation, etc., where the coefficients and parameters of the problem belong to some well-behaved category: algebraic, real analytic, holomorphic, Gevrey, quasi-analytic, etc.

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Before being able to do something reasonable, we have to specify the type of problem we have in mind. Then we have to rule out those objects arising from the problem (solutions, trajectories, leaves, accumulation sets, ...) that are apparently poorly-behaved enough to produce an infinite number of connected components when manipulated by the usual geometric constructions. We keep the remaining objects and study to what extent they really do have good geometric or topological finiteness properties. In the best case, we would like to know whether they are definable in some *o-minimal expansion* of the real field (see [31] for an introduction to o-minimal structures). In particular, we investigate whether the sets generated by those transcendental objects by standard boolean operations, cartesian products and projections have only finitely many connected components.

One example for which this program can be achieved quite satisfactorily is the case of *pfaffian sets*. They arise from pfaffian systems or codimension-one foliations with singularities, expressed locally by a differentiable integrable one-form with real analytic coefficients. The objects to study are (bounded) leaves of these foliations. We rule out those leaves that do not satisfy the so-called *Rolle condition* (see [18]). A Rolle leaf has the following non-oscillatory property: Its intersection with any relatively compact semi-analytic curve has only finitely many connected components.

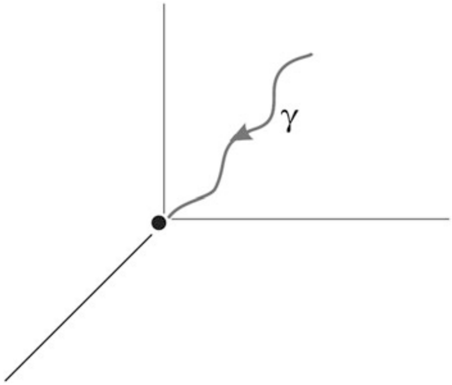
More than that, it turns out that Rolle leaves have all the good finiteness properties that we expect: all of them are definable in some o-minimal expansion of the real field [14, 28, 29].

In these notes, we are dealing with the "dual problem" in terms of the dimensions: we consider one-dimensional foliations with singularities, locally expressed by a real analytic vector field and the transcendental objects are *trajectories* of the vector field (also called *orbits* or *integral curves*). This time we have to rule out those trajectories that are oscillatory in the sense that they intersect some relatively compact analytic hypersurface infinitely many times (the non-oscillatory condition for trajectories is the dual of the corresponding non-oscillation property related to Rolle leaves).

Of course, if the ambient space has dimension two, Rolle leaves and non-oscillatory trajectories are essentially the same thing. But we will see in these notes that non-oscillatory trajectories are more complicated in higher dimensions from the point of view of definability: contrary to Rolle leaves, they may not all be definable in the same o-minimal structure (see Example 6.8 below), nor does the non-oscillation property suffice for o-minimality for individual trajectories (see Example 6.19 below).

A leaf of a one-dimensional foliation is locally an analytic curve at each of its points. We are interested in what happens at its accumulation points. In considering non-oscillatory trajectories, we skip the situation where they accumulate to a *limit cycle* or more generally a *polycycle*. This is a very important situation that is crucial for understanding the dynamics of planar polynomial or real analytic vector fields. Yet, the interesting objects arising from this situation (the natural problem) is not the spiraling one-dimensional leaves accumulating to the polycycle themselves, but the *Poincaré first return map* associated to a small (half) transversal to the polycycle.

Fig. 1 General setting



This map itself is susceptible to having tame, finiteness or o-minimal properties; see, for example [12].

Non-oscillatory trajectories cannot accumulate to several points in the ambient space and our problem is a local one. Thus, the general setting that we consider in these notes is this (Fig. 1):

> M is a real analytic smooth manifold, p is a point in M, X is a real analytic vector field in a neighborhood of p with $X(p) = 0$ and $\gamma : [0, \infty[\to M$ is an integral curve (i.e., $\frac{d\gamma}{dt}(t) = X(\gamma(t))$ for any $t \geq 0$) such that p is the unique ω-limit point
>
> $$\omega(\gamma) = \lim_{t\to\infty} \gamma(t) = p.$$

Almost all properties studied in these notes only depend on the image $|\gamma|$ of γ and not on the particular parametrization. In this way, we can (although we rarely do) replace the vector field X by the (singular) 1-dimensional foliation $\mathcal{L}_X$ generated by X. Also, we are only interested in asymptotic properties of γ at p, that is, in the germ of $|\gamma|$ near the point p.

Before going into the study of tame and o-minimal properties of non-oscillatory trajectories, we want to treat the vague question of "how bad" or "how good" can γ behave asymptotically when it approaches its limit point without making too many further assumptions on γ. More ambitiously, we want to study the problem of describing the local dynamics of real analytic vector fields in neighborhoods of singular points. The attempt to tackle this question, especially for the three dimensional case, has led quite recently to some partial but interesting results where many different techniques in the theory of dynamical systems, real geometry, reduction of singularities and asymptotic expansions are involved. A considerable part of these notes is devoted to a presentation of several aspects of these themes of research. Many details and definitions are included, so that these notes could be useful for beginners in the research of o-minimal properties of solutions of differential equations but without a deep background on geometry of dynamical systems or reduction of singularities.

The plan of these notes is the following. In Sect. 2 we motivate several concepts and definitions by reviewing the well known two-dimensional case. In Sect. 3 we introduce properly all the concepts and review some of its properties. In the next two sections we deal with the study of trajectories of a three-dimensional vector field accumulating to a singular point with the aim of finding a common qualitative behavior of a whole "package" of similar trajectories: Sect. 4 deals with oscillatory trajectories and the existence of *twister axes* and Sect. 5 deals with the existence of *linked* and *separated* packages of non-oscillatory trajectories. Finally, Sect. 6 is devoted to a general theorem about o-minimality of non-oscillatory trajectories in any dimension under certain conditions.

We have striven to make the text self-contained, but we do assume that the reader is familiar with the basic foundations of dynamical systems and differential equations; see, for example, the classical book of Hartman [8] or the more elementary book [19]. At those parts of the text where we need more specific results, such as the Theorem of the Center Manifold or summability of formal solutions of ODEs, we provide appropriate references. Also, we assume that the reader has a sufficient background on real analytic geometry (Hironaka's notes [10] are a very good source; we recommend also of Bierstone and Milman [2]). For o-minimal geometry, we recommend the book of van den Dries [31].

2 Dimension Two

Fix a trajectory $\gamma : [0,\infty[\to \mathbb{R}^2$ of a real analytic vector field X on some neighborhood of $0 \in \mathbb{R}^2$ such that $\omega(\gamma) = 0$.

Proposition 2.1 (Dichotomy in dimension 2). *Exactly one of the following holds (see Fig. 2):*

- *γ has a well defined tangent at 0; that is, there exists the limit*

$$\lim_{t\to\infty} \frac{\gamma(t)}{\|\gamma(t)\|}.$$

- *γ spirals around 0; that is, if we write $\gamma(t) = r(t)e^{i\theta(t)}$ in polar coordinates, where $r(t) = \|\gamma(t)\|$, $\theta(t) = arg(\gamma(t))$, then*

$$\lim_{t\to\infty} \theta(t) = \pm\infty.$$

There is an equivalent definition of spiraling for trajectories of analytic vector fields which will be convenient for us later. Recall that given a half-line ℓ at 0 (not containing the origin), the "local sides" of ℓ in $\mathbb{R}^2$ can be roughly defined as the two connected components of the complement of ℓ in a sufficiently small open sector in the plane with the origin as the vertex whose interior contains ℓ.

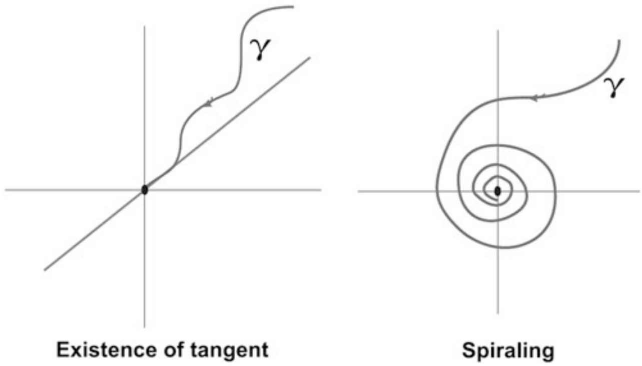
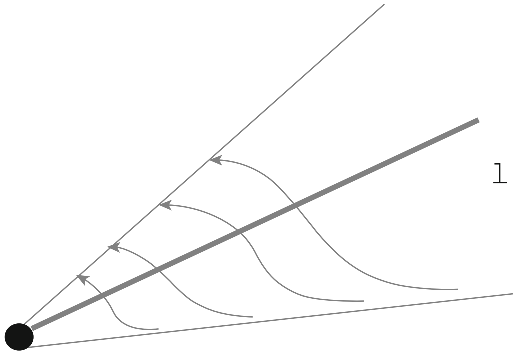


Fig. 2 Dichotomy in dimension 2

Fig. 3 Definition of spiraling



Claim. The trajectory γ spirals around 0 if and only if, for any half-line ℓ at 0, γ intersects infinitely many times ℓ and, for sufficiently large t (depending on ℓ), always transversally and in the same "direction", that is, passing from a given side to the other side of ℓ (see Fig. 3).

Proof of the claim: It is easy and standard. The "if" part is even true for parametrized analytic curves $\gamma : [0, \infty[\to \mathbb{R}^2 \setminus \{0\}$: the hypothesis implies that if we fix $\theta_0 \in \mathbb{R}$ there exists a sequence $\{t_n\}$ in $\mathbb{R}$ going to $+\infty$ with $\{t_n\} = \{t \,/\, \theta(t) \equiv \theta_0 \bmod 2\pi\mathbb{Z}\}$ and, moreover, we can suppose that $\frac{d\theta}{dt}(t_n) > 0$ for every n. Thus, since θ is continuous, $\theta(t_{n+1}) - \theta(t_n) \geq 2\pi \ \forall n$ and then $\lim_{t\to\infty} \theta(t) = +\infty$. For the "only if" part we need the hypothesis that γ is a trajectory of an analytic vector field. It goes as follows. From $\lim_{t\to\infty} \theta(t) = \pm\infty$ we infer that γ intersects infinitely many times any given half-line ℓ. In order to prove that this intersection is ultimately transversal and in the same "direction" we can use the analytic parametrization $\rho \mapsto \rho e^{i\theta_0}$, $\rho \in]0, \varepsilon[$ of ℓ and the function $\phi(\rho) =< e^{i(\theta_0 + \frac{\pi}{2})}, X(\rho e^{i\theta_0}) >$ where X is the vector field and $<, >$ denotes the usual scalar product in $\mathbb{R}^2$. It extends analytically to $\rho \in] - \varepsilon, \varepsilon[$ and it is not identically zero (otherwise ℓ would be a trajectory of X). On the other hand, ϕ vanishes at the corresponding value ρ of a point where γ does not intersect transversally ℓ. Also, ϕ has a sign at any transversal intersection point p which depends only in which local side of ℓ you came from when γ crosses ℓ through p. Thus, if γ changes the transit of sides between two intersection points, ϕ vanishes at the corresponding ρ of some intermediate point. These considerations prove that γ crosses ℓ ultimately transversally and always in the same "direction". □

Proof of Proposition 2.1.- Write the vector field X locally at 0 as $X = a(x, y)\frac{\partial}{\partial x} + b(x, y)\frac{\partial}{\partial y}$ where a, b are analytic in a neighborhood of the origin. Given a trajectory γ of X accumulating to 0, if we write in polar coordinates $r(t) =\| \gamma(t) \|$ and $\gamma(t)/r(t) = e^{i\theta(t)} \in \mathbb{S}^1$, with $\theta;]0, \infty[\to \mathbb{R}$ continuous, the functions $r(t), \theta(t)$ satisfy the system of differential equations

$$\begin{aligned} \dot{r} &= r^k(A(\theta) + O(r)) \\ \dot{\theta} &= r^{k-1}(B(\theta) + O(r)). \end{aligned} \tag{2.1}$$

Here k is the multiplicity of X at the origin (the minimum of the orders of a and b) and A, B are 2π-periodic functions. Moreover, A, B are not both identically zero (they are defined by the formulas

$$xa_k + yb_k = r^{k+1}A(\theta), \quad -ya_k + xb_k = r^{k+1}B(\theta)$$

where a_k, b_k are the homogeneous parts of a, b, respectively, of degree k). We need to show that either $\lim_{t\to\infty}\theta(t)$ exists in $\mathbb{R}$ (γ has a tangent) or $\lim_{t\to\infty}\theta(t) = \pm\infty$ (γ spirals). We have two cases to consider.

Dicritical case: $B(\theta) \equiv 0$. Both equations in the system (2.1) can be divided by r^k and then, after division, the new system is such that the points $(r, \theta) = (0, \theta_0)$ for which $A(\theta_0) \neq 0$ are non singular. On the other hand, γ is, up to reparametrization, part of a trajectory of this divided system, since the induced foliation over the half space $\{r > 0\}$ by the original system or by the divided one is exactly the same. The conclusion is that $\lim_{t\to\infty}\theta(t)$ exists and γ has a tangent.

Non-dicritical case: $B(\theta) \not\equiv 0$. The function B has finitely many zeros modulo $2\pi\mathbb{Z}$, say $0 \leq \theta_1 < \cdots < \theta_n < 2\pi$. Since $r(t)$ goes to zero while t goes to infinity, for any $\varepsilon > 0$ sufficiently small there exists $t_\varepsilon \gg 0$ such that $\theta(t)$ is monotone for every $t \geq t_\varepsilon$ for which $\theta(t)$ belongs to $\cup_j I_{j,\varepsilon} + 2\pi\mathbb{Z}$ where $I_{j,\varepsilon} = [\theta_{j-1} + \varepsilon, \theta_j - \varepsilon]$, $j = 1, \ldots, n$ (and where we set $\theta_0 = \theta_n$). Moreover, the sign of the derivative of $\theta(t)$ inside each interval $I_{j,\varepsilon} + 2\pi l$ depends only on j and not on ε or l. We have only two cases:

1. If $\theta(t)$ is increasing (or decreasing) in every $I_{j,\varepsilon}$ for $j = 1, \ldots, n$ then either the graph of θ is included in one of the strips $\mathbb{R}_{>0} \times (]\theta_j - \varepsilon, \theta_j + \varepsilon[+2\pi l)$ or θ diverges to $+\infty$ (or $-\infty$). Since ε is arbitrarily small, we conclude the result in this case.

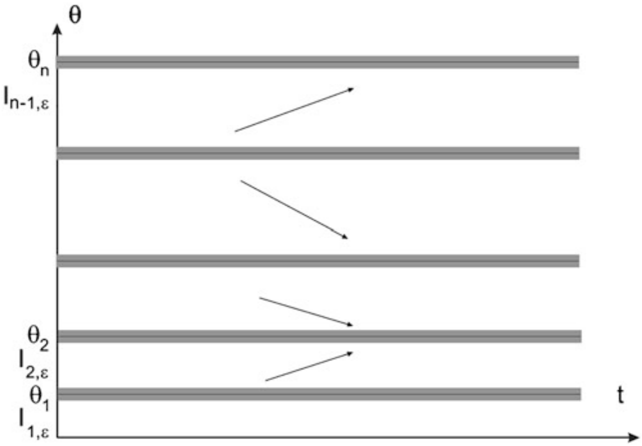


Fig. 4 Opposed monotonicity between strips implies existence of tangent

2. If $\theta(t)$ has different monotonicity when it takes values in $I_{j,\varepsilon}$ or $I_{j+1,\varepsilon}$ for some j then θ can not pass from the strip $\mathbb{R}_{>0} \times ([\theta_{j-1} + \varepsilon, \theta_j - \varepsilon] + 2\pi l)$ to $\mathbb{R}_{>0} \times ([\theta_j + \varepsilon, \theta_{j+1} - \varepsilon] + 2\pi l)$ or viceversa. This implies that $\theta(t)$ remains bounded when t goes to infinity (see Fig. 4). Since it can not have any value of $I_{j,\varepsilon} + 2\pi\mathbb{Z}$ as an accumulation value by monotonicity, if we make $\varepsilon \to 0$, we conclude that $\lim_{t\to\infty} \theta(t)$ exists and it belongs to $\{\theta_1, \ldots, \theta_n\} + 2\pi\mathbb{Z}$. □

If there exists a trajectory γ with $\omega(\gamma) = 0$ having a tangent then any other trajectory γ' with $\omega(\gamma') = 0$ has also a tangent at the origin (otherwise, γ' would be a spiraling trajectory and it would intersect infinitely many times the trajectory γ, which is not possible). Thus, the dichotomy stated in Proposition 2.1 is not just for single trajectories but for the whole set of trajectories of a given vector field that accumulate to the origin. One can go one step further in the spiraling case in the sense that a spiraling trajectory forces that any other trajectory in a neighborhood also accumulates to the origin. Let us state this result as follows:

Proposition 2.2. *If γ is a spiraling trajectory of an analytic vector field X at the origin $0 \in \mathbb{R}^2$ then there exists a neighborhood U of 0, positively invariant for X such that any trajectory of X issued of a point in $U \setminus \{0\}$ accumulates to the origin and spirals around it.*

Proof. Take a half-line ℓ through the origin and a neighborhood $\tilde{U}$ sufficiently small such that X is transverse to $\ell \cap \tilde{U}$ and 0 is the only singularity of X inside $\tilde{U}$. Let $\gamma(t_1), \gamma(t_2) \in \ell$ two consecutive points where γ intersects ℓ. The C^1-piecewise curve $\mathcal{C}$ given by $\gamma \mid_{[t_1,t_2]}$ and the segment $[\gamma(t_1), \gamma(t_2)] \subset \ell$ is a Jordan curve. Let U be the bounded connected component of $\mathbb{R}^2 \setminus \mathcal{C}$. It contains the origin. First, t_2 being greater than t_1, we must have $\gamma(t_2)$ is closer to zero than $\gamma(t_1)$ (see Fig. 5).

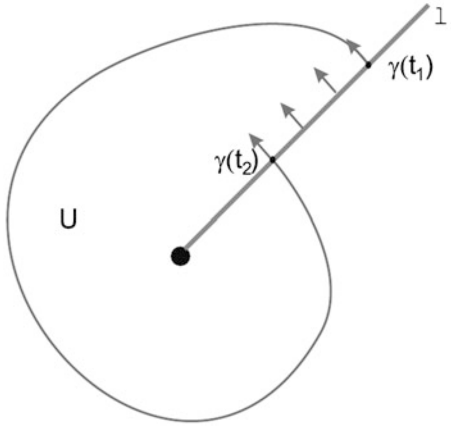


Fig. 5 Spiraling dynamics in dimension 2

On the other hand, a trajectory γ' issued of a point in U can not cross $\mathcal{C}$ forwards, so it remains in U and U is positively invariant. For such a trajectory γ', the omega-limit set $\omega(\gamma')$ is a compact connected invariant set contained in $\overline{U}$. It can not be a periodic orbit of X (otherwise, since this orbit must contain a singularity in its interior, it circumscribes 0 and then, $\mathcal{C}$ lying outside the orbit, γ must intersect it, which is impossible). Hence $\omega(\gamma') = 0$ and so γ' spirals around the origin. □

Returning to our original purpose, the only trajectories (accumulating to a single singular point) of analytic vector fields in dimension two susceptible to form moderate geometry are those having a well defined tangent in their limit point. These class of trajectories contains any relatively compact semi-analytic curve. Can we say more about trajectories with a tangent? Do they really form a moderate geometry? The answer is yes: they have all good properties that we expect from analytic or semi-analytic curves.

First of all, we have the following important consequence of having a tangent.

Proposition 2.3. *A trajectory γ with a well defined tangent at $0 \in \mathbb{R}^2$ is non-oscillatory; that is, given an analytic curve C at the origin, γ is either contained in C or it intersects C only finitely many times.*

Proof. Suppose that γ intersects infinitely many times a (local) connected component C' of $C \setminus \{0\}$ but it is not contained in C'. We can see C' as an embedded smooth one-dimensional submanifold of $\mathbb{R}^2$ which is closed in an open sector S around the origin bisected by the tangent direction of C' and of opening less than 2π. Thus $S \setminus C'$ have two connected components, the "local sides" of C'. We can moreover suppose that γ has the same tangent direction at the origin as the curve C' so that $\gamma\ |_{[t_0,\infty[}$ remains inside S for t_0 sufficiently large. Now, using the same kind of arguments that in the proof of the claim above, we find an infinite number of points in C' where C' is tangent to the vector field X: either γ intersects tangentially C' at infinitely many points or it intersects transversally changing the sides between

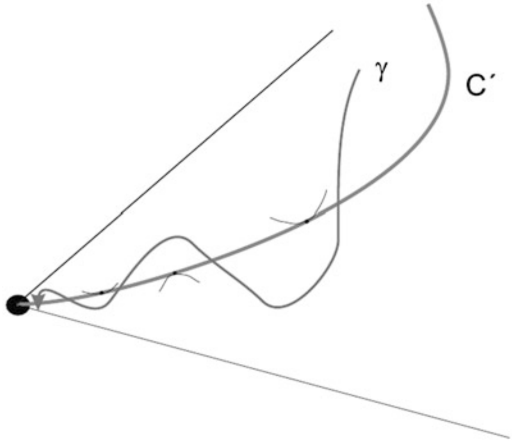


Fig. 6 Oscillation with tangent creates tangent points

two consecutive hits (see Fig. 6). But then, since the set of points of C' where X is tangent to C' is a semi-analytic subset of C', if this set contains infinitely many points it coincides with C' and C' itself is invariant by the vector field. This contradicts the fact that γ intersects C'. □

We summarize in the following statement all that we can say about finiteness properties of trajectories of vector fields in the plane.

Theorem 2.4 (Finiteness properties of trajectories in dimension 2). *A trajectory γ of an analytic vector field such that $\omega(\gamma) = 0 \in \mathbb{R}^2$ is non-oscillatory if and only if it has a well defined tangent. Moreover, there exists an o-minimal expansion of the real field in which every such trajectory is definable.*

The last sentence on this theorem is much more involved. It can be obtained using the remark that, in dimension two, a non-oscillatory trajectory of an analytic vector field is also a Rolle's leaf of the generated foliation. After that, the result follows from the o-minimality of the structure generated by Rolle's leaves, as we have mentioned in the introduction.

3 Definitions and Generalities

In this section we review briefly well known concepts and results concerning blowing-ups and vector fields that we will need throughout these notes. As well, we investigate some new concepts about trajectories in general suggested by the study of the two-dimensional case in the precedent section.

3.1 Blowing-Ups

The *orientable blowing-up at* $0 \in \mathbb{R}^n$ is the map

$$\rho : \mathbb{R} \times \mathbb{S}^{n-1} \to \mathbb{R}^n$$
$$(r, \chi) \mapsto r\chi.$$

It is a local diffeomorphism outside the *exceptional divisor* $\{0\} \times \mathbb{S}^{n-1}$ which is mapped to the origin. This map is not injective but 2-to-1 outside the divisor. In order to obtain an isomorphism outside the divisor, we have two options. If we want to maintain orientability, we consider the restriction

$$\rho^+ : \mathbb{R}_{\geq 0} \times \mathbb{S}^{n-1} \to \mathbb{R}^n$$

so that our new ambient space is not longer an analytic manifold but and analytic manifold with boundary. If we want to avoid borders, we consider the standard *(projective) blowing-up at* $0 \in \mathbb{R}^n$ defined as the map $\pi : \tilde{\mathbb{R}^n} \to \mathbb{R}^n$ obtained from ρ by identifying points in $\mathbb{R} \times \mathbb{S}^{n-1}$ according to the rule

$$(r, \chi) \sim (r', \chi') \Leftrightarrow \begin{cases} r = r' \text{ and } \chi = \chi' \\ \text{or} \\ r = -r' \text{ and } \chi = -\chi' \end{cases}$$

The quotient space $\tilde{\mathbb{R}^n}$ is a real analytic n-dimensional manifold. It is covered by affine charts $(U_k, \underline{x}^k = (x_1^k, \ldots, x_n^k))$, $k = 1, \ldots, n$ (that is, the range of $\underline{x}^k$ is the whole affine space $\mathbb{R}^n$) characterized by the fact that the blowing-up π is written in U_k as

$$\pi(\underline{x}^k) = (x_1^k x_k^k, \ldots, x_{k-1}^k x_k^k, x_k^k, x_{k+1}^k x_k^k, \ldots, x_n^k x_k^k).$$

The *exceptional divisor* of the blowing-up π is $E = \pi^{-1}(0) \simeq \mathbb{P}_{\mathbb{R}}^{n-1}$ which is given in the chart U_k by the equation $E = \{x_k^k = 0\}$.

The projective blowing-up is more suitable if we want to define it in other ground fields different from the real one or if we want to extend it to other real manifolds. More precisely, if M is an n-dimensional manifold and $p \in M$, we define the *blowing-up of M at the point p* by means of local charts as follows. Let $\phi : U \to \mathbb{R}^n$ be a local chart of M centered at p and let $V = \phi(U)$. Consider the disjoint union manifold $M \setminus \{p\} \cup \pi^{-1}(V)$ and let $\tilde{M}$ be the quotient space of this union by the identification

$$q \sim \pi^{-1}(\phi(q)), \quad q \in U \setminus \{p\}.$$

We have that $\tilde{M}$ is an analytic n-dimensional manifold and we define the blowing-up as the map $\pi_p : \tilde{M} \to M$ given by $\pi_p(q) = q$ if $q \in M \setminus \{p\} \subset \tilde{M}$ and $\pi_p(E) = p$. The definition of the blowing-up at p depends on the particular choice of the chart. However, if $\pi_p : \tilde{M} \to M$, $\pi_p' : \tilde{M}' \to M$ are two different such blowing-ups, then there exists a diffeomorphism $\phi : \tilde{M} \to \tilde{M}'$ such that $\pi_p' \circ \phi = \pi_p$; thus we can (and we will) identify them.

Let $Y = \{x_1 = \cdots = x_r = 0\} \subset \mathbb{R}^n$ with $r \geq 2$. We define the *(orientable, respectively projective) blowing-up of* $\mathbb{R}^n$ *with center* Y as the map

$$\pi_Y = \pi^r \times id_Y : \tilde{\mathbb{R}^r} \times Y \to \mathbb{R}^r \times Y = \mathbb{R}^n$$

where π^r is the (orientable, respectively projective) blowing-up of $\mathbb{R}^r$ at the origin.

Finally, let M be a real analytic manifold and $Y \subset M$ a closed embedded smooth analytic submanifold of pure codimension $r \geq 2$. Let $p \in Y$. Choose an affine coordinate chart $\phi : U \to \mathbb{R}^n$ of M at p for which $Y \cap U = \phi^{-1}(\{x_1 = \cdots = x_r = 0\})$. Then, by definition, a *(local) blowing-up of* M *with center* Y *at the point* p is the composition

$$\pi_Y^U : \tilde{\mathbb{R}^r} \times \phi(Y \cap U) \xrightarrow{\pi} \mathbb{R}^n \xrightarrow{\phi} U \xhookrightarrow{i} M$$

where i is the inclusion and π is the projective blowing-up in $\mathbb{R}^n$ with center $\phi(Y \cap U)$.[1]

3.2 Vector Fields Under Blowing-Ups

Let X be an analytic vector field in a real analytic manifold M and let $p \in M$ be a singular point of X. Let $\pi_p : \tilde{M} \to M$ be the blowing-up at p. It is not difficult to see using the charts of the blowing-up that there exists a vector field $\tilde{X}$ on $\tilde{M}$, called the *total transform of* X *by* π_p, such that

$$(\pi_p)_* \tilde{X} = X.$$

If the multiplicity of X at p is at least two then the exceptional divisor $E = \pi_p^{-1}(p)$ is contained in the singular locus of $\tilde{X}$. Usually, we do not want to have a singular locus of codimension 1. We can proceed locally as follows. If $q \in E$, there is a chart of $\tilde{M}$ at q in which E has the local equation $\{x = 0\}$ and there is a natural number $s \geq 0$ such that $\tilde{X} = x^s X'$ where X' is not identically zero at E. This vector field X' is called the *strict transform of* X *(by* π_p*) at the point* q. It is not defined in the whole manifold $\tilde{M}$.

If we want to work with global objects, it is convenient to consider the 1-dimensional foliation $\mathcal{L}'$ generated by the strict transform X' at any point which is well defined in $\tilde{M}$. The foliation $\mathcal{L}'$ is called the *strict transform (by* π_p*) of* $\mathcal{L}_X$, the 1-dimensional foliation in M generated by X. Two cases can occur:

---

[1] Of course, we can define more general concepts of blowing-ups such as the (global) blowing-up of M with center Y if Y is closed, or even only assuming that Y is a closed analytic set, possibly with singularities (see [10]). However, we will not use these generalizations in this work.

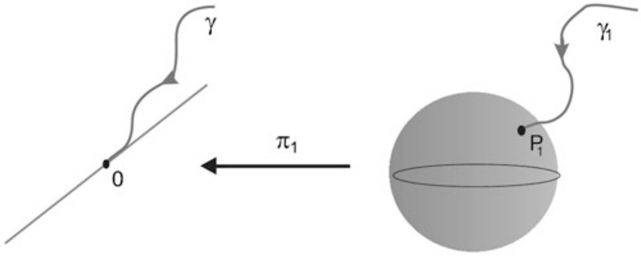


Fig. 7 Existence of tangent

1. The divisor E is invariant by $\mathcal{L}'$ (called *non-dicritical case*). In this case, leaves of $\mathcal{L}_X$, except for the singular leave $\{p\}$, correspond isomorphically to leaves of $\mathcal{L}'$ that do not intersect E.
2. The divisor E is generically transversal to $\mathcal{L}'$ (called *dicritical case*). In this case, leaves of $\mathcal{L}$, except the singular leave $\{p\}$, correspond to leaves of the restriction of $\mathcal{L}'$ to $\tilde{M} \setminus E$ and, generically, these leaves can be analytically continued through the point $\{p\}$ (and then they are semianalytic sets in M at the point p).

Similar concepts of total and strict transform of a vector field X exist for the (local) blowing-up with an analytic smooth center which is invariant for X.

3.3 *Parameterized Curves*

Let $p \in M$ and let $\gamma : [0, +\infty[\to M$ be a parameterized analytic curve such that $\gamma(t) \neq p \; \forall t$ and such that its ω-limit set

$$\omega(\gamma) := \bigcap_{t \geq 0} \overline{\gamma([t, +\infty[)}$$

is reduced to the single point p. Denote by $\mid \gamma \mid \subset M$ the image set of γ.

3.3.1 Existence of Tangent

We say that γ *has a tangent at* p if, denoting by $\pi_1 : M_1 \to M$ the blowing-up with center p, the *lifting parameterized curve* $\gamma_1 = \pi_1^{-1} \circ \gamma : [0, +\infty[\to M_1$ has a single ω-limit point $p_1 \in E_1 = \pi_1^{-1}(p)$ (see Fig. 7). We say also that p_1 is the *tangent of* γ *at* p. It is a well defined point (up to isomorphisms between different blowing-ups as we have explained in the Sect. 3.1).

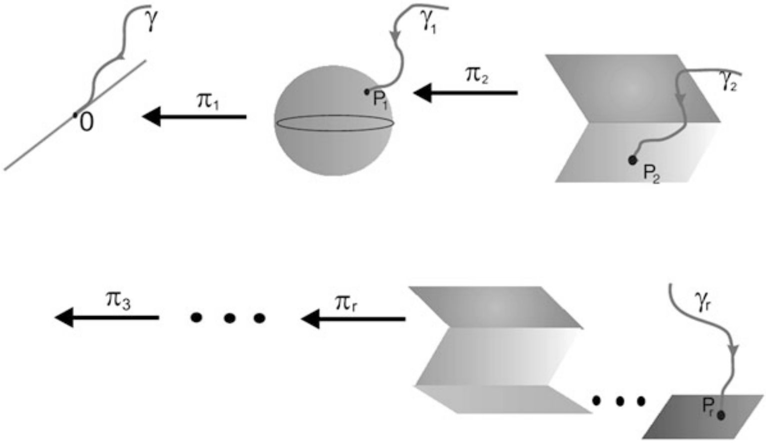


Fig. 8 Iterated tangents

3.3.2 Iterated Tangents

We say that γ has *all iterated tangents* if, recursively, γ has a tangent at p, equal to p_1, then the lifting curve γ_1 of γ has a tangent at p_1, equal to p_2, etc. (see Fig. 8). The sequence of points $IT(\gamma) = \{p_n\}_{n\geq 0}$, with $p_0 = p$, is called the *sequence of iterated tangents of* γ. As in the case of the tangent, it is well defined up to equivalencies between point blowing-ups.

3.3.3 Limit of Tangents

We say that γ has the *limit of tangents* if there is an analytic coordinate chart of M at p so that we can suppose that $M = \mathbb{R}^n$ and p is the origin and the following limit exists

$$\lim_{t\to\infty} \frac{\dot{\gamma}(t)}{|| \dot{\gamma}(t) ||} \in \mathbb{S}^{n-1},$$

where the dot denotes the derivative with respect to the parameter t. It is worth to mention that this definition does not depend on the coordinate chart.

3.3.4 Oscillation

We say that γ is *non-oscillatory at p (with respect to the analytic sets)* if for any given local analytic hypersurface H of M at p, there exists $t_0 \geq 0$ such that either $\gamma([t_0, \infty[) \subset H$ or $\gamma([t_0, \infty[) \cap H = \emptyset$ (see Fig. 9). Equivalently, we can (and we will) use the following definition: for any germ f of analytic function at p, either $f \circ \gamma \equiv 0$ or $f \circ \gamma$ is invertible in the ring of germs of continuous real functions at $t = +\infty$.

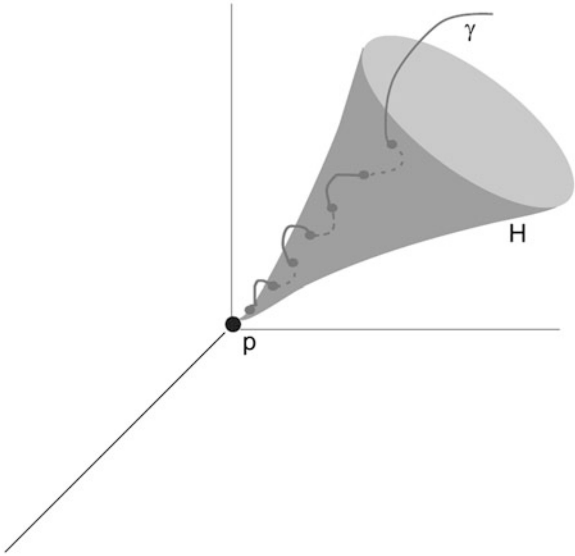


Fig. 9 Oscillating parameterized curve

3.3.5 Flat Contact

A subset $\Gamma \subset M$ is said to be an *analytic half-branch at* p if there exists a real analytic map $\alpha :]-\varepsilon, \varepsilon[\to M$ with $\alpha(0) = p$ and an open neighborhood U of p in M such that

$$\Gamma \cap U = \alpha(]0, \varepsilon[).$$

Equivalently, the germ of Γ at p is the germ of a connected component of $C \setminus \{p\}$ where C is a one-dimensional semi-analytic set in some neighborhood of p in M and $p \in C$. The local parametrization α of Γ can be supposed to be a *Puiseux's parametrization*; that is, written in some analytic coordinates at p in the form

$$\alpha(s) = (\alpha_1(s), \ldots, \alpha_{n-1}(s), s^k),$$

where $k \geq 1$ is a natural number and α_j is analytic in $]-\varepsilon, \varepsilon[$ with $\alpha_j(0) = 0$, for $j = 1, \ldots, n-1$. The minimum such k is called the *multiplicity of* Γ *at* p. We will say that Γ is *not singular at* p if, in some analytic coordinates ($\mathbf{x} = (x_1, \ldots, x_{n-1}), z$) centered at p, we have (as germs)

$$\Gamma = \{x_1 = x_2 = \cdots = x_{n-1} = 0, z > 0\}.$$

Equivalently, the multiplicity of Γ is $k = 1$.

If $\pi_1 : M_1 \to M$ is the blowing-up at the point p, $\Gamma_1 = \pi^{-1}(\Gamma)$ is called the *strict transform of* Γ *by* π_1. It is not difficult to see using a parametrization α that $\overline{\Gamma_1}$ intersects the exceptional divisor in a single point p_1, called the tangent of Γ at p, and that Γ_1 is an analytic half branch at p_1. Continuing in this way, we define the sequence $IT(\Gamma) = \{p_n\}_{n\geq 0}$ of iterated tangents and the sequence of strict transforms $\{\Gamma_n\}_{n\geq 0}$ of Γ (with $p_0 = p$ and $\Gamma_0 = \Gamma$). There is a classical result in resolution of singularities of curves (see for instance [34]) that asserts that there is n_0 such that, for any $n \geq n_0$, the point p_n in the sequence of iterated tangents of Γ is a regular point of the exceptional divisor and the strict transform Γ_n is non singular at p_n.

We say that a parameterized curve γ with a single ω-limit point $p \in M$ has a *flat contact* with an analytic half branch Γ at p if γ has all the iterated tangents and $IT(\gamma) = IT(\Gamma)$.

Remarks 3.1. (1) Having all iterated tangents does not imply flat contact: consider for instance the mentioned example of the graph of $x \mapsto x^{\sqrt{2}}$, $x > 0$, for which the sequence of iterated tangents does not correspond to the one of a real analytic curve.

(2) The name "flat contact" is motivated by the following property whose proof is left to the reader. Suppose that Γ is not singular at p, that is

$$\Gamma = \{x_1 = x_2 = \cdots = x_{n-1} = 0, z > 0\}$$

in some coordinates. Write γ in these coordinates as $\gamma(t) = (\mathbf{x}(t), z(t))$. Then γ has flat contact with Γ iff, for any $N \in \mathbb{N}$, we have

$$\lim_{t\to\infty} \frac{||\mathbf{x}(t)||}{|z(t)|^N} = 0.$$

(3) If γ has flat contact with an analytic half-branch at p then there are exactly two germs of analytic half-branches at p with respect to which γ has flat contact. To see this, we can suppose, after resolution of singularities as we have explained before, that Γ is non singular. This means that the germ of Γ is one of the two connected components of the germ of $C \setminus \{p\}$ where C is an axis of some system of coordinates. By definition, γ has flat contact with both components. Using Łojasiewicz's regular separation property [15], these are the only analytic half branches with this property.

Remark 3.2. Notice that all the concepts defined above: existence of tangent, iterated tangents, limit of tangents, non-oscillation, flat contact, depend only on the germ of the set $|\gamma|$ at the point p and not on the particular parametrization or the origin $t = 0$ of such a parametrization.

We summarize the relations between these properties in the following proposition.

Proposition 3.3. *Let $\gamma : [0,\infty[\to M$ be an analytic parameterized curve with $\omega(\gamma) = \{p\}$. Then:*

(i) $|\gamma|$ is an analytic half-branch at $p \Rightarrow \gamma$ is non-oscillatory $\Rightarrow \gamma$ has the iterated tangents $\Rightarrow \gamma$ has a tangent.

(ii) *γ has the limit of tangents $\Rightarrow \gamma$ has a tangent.*
(iii) *If γ is a trajectory of an analytic vector field on a neighborhood of p then γ is non-oscillatory $\Rightarrow \gamma$ has a limit of tangents.*
(iv) *If γ is a trajectory of an analytic vector field on a neighborhood of p and, moreover, $n = 2$, then the properties: non-oscillatory, having a tangent, having iterated tangents and having a limit of tangents are all equivalent for γ.*

Parts (ii) and (iii) are concerned with the concept of existence of limit of tangents which is not going to be used in the sequel; the proof of them, as well as examples showing that the reciprocal implication in (ii) is not true or that the hypothesis of being a trajectory in (iii) can not be dropped, can be seen in [25].

Notice that part (iv) has been already proved in Sect. 2. The hypothesis of γ being a trajectory is essential: take for instance the graph of the function $x \mapsto \exp(-1/x)\sin(1/x)$, for $x > 0$, as a parameterized curve in the plane having the property of iterated tangents but oscillating.

For part (i), the first implication is a classical result in real analytic geometry; the reciprocal of the first implication is not true in general, even for trajectories: consider the graph of the function $x \mapsto x^{\sqrt{2}}$, for $x > 0$, (the image of) a non-oscillatory trajectory of the vector field $X = -(x\partial/\partial x + \sqrt{2}y\partial/\partial y)$ which is not an analytic half branch.

The third implication in (i) is obvious and, again, the reciprocal implication is not true, even for trajectories of analytic vector fields (in dimension higher than two, by part (iv)); the reader is encouraged to look for his own examples.

In what follows, we give the details of the proof of the second implication of part (i); that is, non-oscillatory $\Rightarrow$ iterated tangents. In the next section, we study in which manner the reciprocal of this implication is not true for trajectories of analytic vector fields in dimension three.

Suppose that γ has not the property of iterated tangents. Then there exists a sequence of blowing-ups $\pi_k : M_k \to M_{k-1}$, $k = 1, \dots, r$ at points $p_{k-1} \in M_{k-1}$ (with $M_0 = M$, $p_0 = p$) such that the lifting $\gamma_k = \pi_k^{-1} \circ \gamma_{k-1}$ has a single limit point $p_k \in M_k$ for $k = 1, \dots, r-1$ but $\gamma_r = \pi_r^{-1} \circ \gamma_{r-1}$ has at least two different ω-limit points $p', p'' \in M_r$. Without loss of generality we can suppose that $M_0 = \mathbb{R}^n$ and that there exists a chart $\phi : U \to \mathbb{R}^n$ of M_r centered at p' such that the composition $\pi_1 \circ \cdots \circ \pi_r \circ \phi^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ is a polynomial map. Since $p' \in \omega(\gamma_r)$, the image of γ_r intersects any neighborhood of M_r at p'; on the other hand, since p'' is also a ω-limit point, γ_r leaves eventually any sufficiently small such neighborhood. So, there exists $\delta > 0$ such that γ_r intersects infinitely many times the hypersurface in U

$$H_r = \phi^{-1}(\{x_1^2 + x_2^2 + \cdots + x_n^2 = \delta\}).$$

By Tarski's Theorem ([30]), $H = \pi_1 \circ \cdots \circ \pi_r(H_r)$ is a semialgebraic hypersurface in $\mathbb{R}^n$ and so it is contained in an analytic hypersurface of M with respect to which γ is oscillating. □

We finish this section with the following result

Proposition 3.4. *Suppose that γ is a trajectory of an analytic vector field on M such that $\omega(\gamma) = p$ and that γ has a flat contact with an analytic half-branch Γ. Then Γ is an invariant curve of X.*

Proof. By the theorem of resolution of curves, we can suppose that Γ is non singular at p. We can also assume that there exists a transversal divisor E invariant for X. In local coordinates $(\mathbf{x} = (x_1, \ldots, x_{n-1}), z)$ we can write $\Gamma = \{\mathbf{x} = 0, z > 0\}$, $E = \{z = 0\}$ and

$$X = \sum_{i=1}^{n-1}[a_i(z) + \sum_{j=1}^{n-1} x_j a_{ij}(\mathbf{x}, z)]\frac{\partial}{\partial x_i} + za(\mathbf{x}, z)\frac{\partial}{\partial z}.$$

The branch Γ is invariant if and only if $a_i \equiv 0$ for $i = 1, \ldots, n-1$. Suppose that this is not the case. Then, there exists $N \geq 0$ such that $a_i(z) = z^N b_i(z)$ and one of the b_i satisfies $b_i(0) \neq 0$. At the tangent point p_N, we can choose the standard chart of the blowing up $(\mathbf{y} = (y_1, \ldots, y_{n-1}), z')$ such that the composition $\pi_1 \circ \cdots \pi_N$ is written as

$$\mathbf{x} = z^N \mathbf{y}, \quad z = z'.$$

A simple calculation shows that the strict transform of X at p_N is given by

$$X'_N = \sum_{i=1}^{n-1}[b_i(z') + \sum_{j=1}^{n-1} y_j b_{ij}(z^N \mathbf{y}, z')]\frac{\partial}{\partial y_i} + z'a(z^N \mathbf{y}, z')\frac{\partial}{\partial z'}$$

where the b_{ij} are analytic at the origin. We obtain by the hypothesis about the b_i that p_N is not a singular point for X'_N. But this is a contradiction with the fact that the lifting γ_N of γ is a trajectory of X'_N (up to re-parametrization) and $\omega(\gamma_N) = p_N$. □

4 Oscillation, Spiraling, Twisters

In this section we restrict ourselves to the situation of dim $M = 3$. The source for the contents of this section is the paper [3].

4.1 The Results

We have already seen that, in dimension two, a trajectory of an analytic vector field having a tangent is non-oscillatory, that these two properties are in fact equivalent and imply the existence of all iterated tangents (among other stronger properties such as o-minimality). This equivalence is not longer true in dimension three or higher: oscillation and iterated tangents can coexist for trajectories of analytic vector fields.

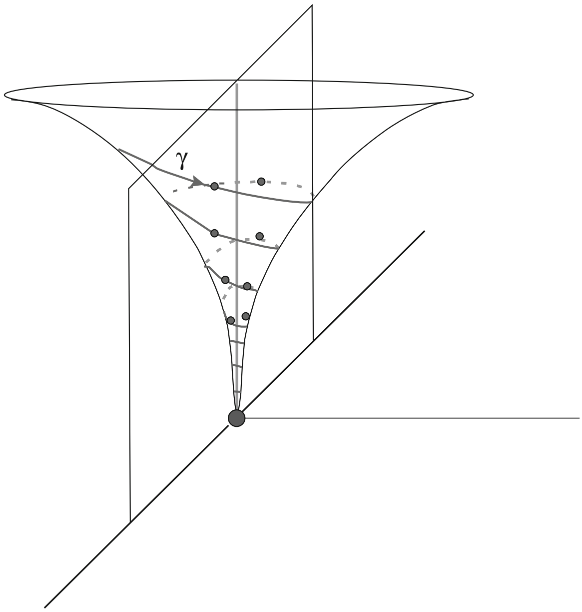


Fig. 10 Oscillating trajectories with iterated tangents

Example 4.1. Consider the following polynomial vector field in $\mathbb{R}^3$

$$X = (-x - y)\frac{\partial}{\partial x} + (-y + x)\frac{\partial}{\partial y} - z^2\frac{\partial}{\partial z}.$$

A trajectory γ of X issued of a point in the half space $\{z > 0\}$ can be explicitly parameterized by the coordinate z as

$$z \mapsto (C \exp(-1/z) \exp(i/z), z),$$

where C is a real non zero constant and where we have identified $\mathbb{R}^3$ to $\mathbb{C} \times \mathbb{R}$ in the usual sense. Then any such γ satisfies $\omega(\gamma) = 0 \in \mathbb{R}^3$, has flat contact with the analytic half axis $\{x = y = 0, z > 0\}$ (by Remark 3.1, (2)) and is oscillating with respect to any plane containing the axis (see Fig. 10).

In the example above oscillation can be somehow described in terms of "spiraling" around the vertical axis. The following results from [3] assert that, essentially, this is always the case for oscillating trajectories in dimension three if we have iterated tangents.

Theorem 4.2 (Existence of spiraling axis). *Let γ be a trajectory of a real analytic vector field in a three dimensional manifold M such that $\omega(\gamma) = p \in M$ and suppose that γ has iterated tangents at p and it is oscillating. Then there exists a (unique) analytic half branch Γ at p such that γ has flat contact with Γ and γ spirals around Γ (see Fig. 11). The half branch Γ is called the spiraling axis for γ.*

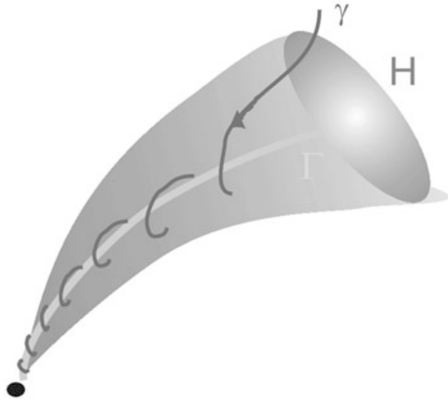


Fig. 11 Existence of the spiraling axis Γ

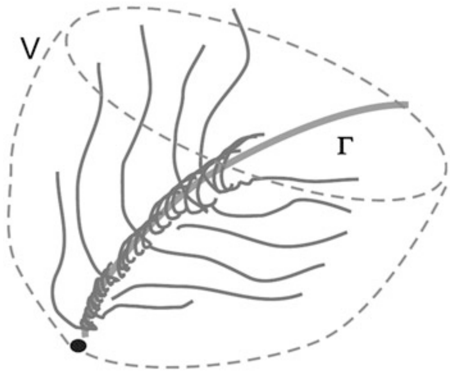


Fig. 12 Twister axis

Theorem 4.3 (Twister axis). *Suppose that an analytic half branch* Γ *is the spiraling axis for some trajectory* γ *and assume that* $\Gamma \not\subset Sing(X)$. *Then there exists an open subanalytic* [2] *set* V *of* M *containing the germ of* Γ *at* p *which is positively invariant for* X *and satisfies the property that any trajectory issued of a point in* V *accumulates to* p *and spirals around* Γ *(see Fig. 12). When this behavior occurs we say that* Γ *is a twister axis and that* V *is a twister domain of* X *at* p.

For further use, given an analytic vector field X in a neighborhood of some point $p \in M$ and an analytic half-branch Γ at p (not necessarily a spiraling axis), we will say that Γ is *non-degenerated (for* X*)* if (the germ of) Γ is not contained in the singular locus *Sing* X of X.

[2]Following carefully the details of the proof of Theorem 4.3 in [3], one can suppose, furthermore, that the twister domain V is semi-analytic.

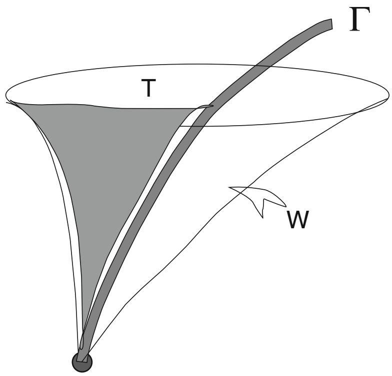


Fig. 13 Definition of semi-analytic triangle over Γ

The second result, Theorem 4.3, can be viewed as the analogous in dimension three to Proposition 2.2: the existence of a spiraling trajectory forces the rest of trajectories in certain domain to spiral. In this sense, a twister axis is for the vector field what René Thom has called a "*centre organisateur de la dynamique*".

Before starting the proof of these theorems, we need to give a precise meaning to the phrase "γ spirals around an analytic half branch Γ". The concept of spiraling trajectory in dimension three generalizes the one of spiraling trajectory in dimension two, namely the definition formulated in the claim after Proposition 2.1 in Sect. 2. We start with the definition of what we call semi-analytic triangles over Γ which play the role of half lines in dimension two.

Definition 4.4. Let $\Gamma = \alpha(]0,1[)$ be an analytic half branch at some point p of a three dimensional manifold M. A *semi-analytic triangle (over* Γ*)* is a pair (W, T) (see Fig. 13) where, for some $0 < \varepsilon < 1$,

- W is an open semi-analytic set containing $\Gamma_\varepsilon = \alpha(]0,\varepsilon[)$.
- T is a smooth connected semi-analytic embedded surface of $W \setminus \Gamma_\varepsilon$ with $\Gamma_\varepsilon \subset \partial T = \overline{T} \setminus T$.
- The triplet $(W, T, \Gamma_\varepsilon)$ is homeomorphic to

$$(\mathbb{R}^2 \times]0,\varepsilon[, \mathbb{R}^+ \times]0,\varepsilon[, \{\mathbf{0}\} \times]0,\varepsilon[).$$

We remark that, given a semi-analytic triangle (W, T), there exists an open neighborhood V of T in W which is divided by T into two connected components, V^+, V^-, called the *local sides of* T *in* W.

Definition 4.5. Given a parameterized curve γ with $\omega(\gamma) = p$ and an analytic half branch Γ at p, we say that γ *spirals around* Γ or that Γ *is the spiraling axis for* γ if for any semi-analytic triangle (W, T) over Γ there exists $t_0 \gg 0$ with the properties: $\gamma(t) \in W$ for any $t \geq t_0$ and $\gamma\mid_{[t_0,\infty[}$ intersects T infinitely many times transversally and passing from one given local side of T to the other.

4.2 Some Technical Properties of Semi-analytic Triangles

The proofs of Theorems 4.2 and 4.3 involve some known facts about the geometry of semi-analytic sets, applied to semi-analytic triangles. In what follows, we formulate these facts and will be content to indicate just the key arguments in order to prove them.

Let Γ be an analytic half branch at $p \in M$ given in some coordinates (x, y, z) by a Puiseux's parametrization

$$t \mapsto (x(t), y(t), t^n), \quad t \in]0, 1[,$$

where $n \geq 1$ and $x(t), y(t)$ are analytic functions at $t = 0$. Consider the sequence $\{W_k\}_{k \geq 0}$ of semi-analytic open sets

$$W_k = \left\{ (x, y, z) \;/\; (x - x(z))^2 + (y - y(z))^2 < z^{kn},\ 0 < z < \frac{1}{k} \right\}.$$

They are "sharper and sharper" and "smaller and smaller" while $k \to \infty$.

Proposition 4.6. *The sequence $\{W_k\}$ has the following properties.*

1. *Each W_k is an open neighborhood of the germ Γ_0 of Γ at p and for any open semi-analytic neighborhood W of Γ_0 there exists k_0 such that $W_{k_0} \subset W$.*
2. *If $A \subset M$ is a semi-analytic set and $\Gamma_0 \not\subset \overline{A}$ then there exists k_0 such that $W_k \cap A = \emptyset$ for any $k \geq k_0$.*
3. *Let $A \subset M$ be a semi-analytic set of dimension two such that $\Gamma_0 \subset \overline{A}$. There exists $k_0 \geq 0$ such that, if $R_1, \ldots, R_s$ are the connected components of $W_{k_0} \cap (A \setminus (\Gamma \cap W_{k_0}))$, then, for any $k \geq k_0$,*

   - *The pair (W_k, R_i) is a semi-analytic triangle over Γ for $i = 1, \ldots, s$.*
   - *For $i, j \in \{1, \ldots, s\}$, we have either $R_i = R_j$ or*

   $$(W_k, R_i, R_j, \Gamma \cap W_k) \cong_{top} (\mathbb{R}^2 \times \mathbb{R}^+, \mathbb{R}^+ \times \{0\} \times \mathbb{R}^+, \mathbb{R}^- \times \{0\} \times \mathbb{R}^+, \{(0,0)\} \times \mathbb{R}^+).$$

   (We will say simply that the two triangles (W_k, R_i), (W_k, R_j) are compatible).

Parts (1) and (2) are quite standard. They use the regular separation property of semi-analytic sets [15]. Part (3) is a bit more involved and uses semi-analytic stratifications adapted to the semi-analytic sets W_k, A, $\overline{A}$, Γ that satisfy good regular Whitney's conditions [9]. A complete proof of Proposition 4.6 can be seen in [25].

Corollary 4.7. *If γ spirals around Γ then γ has flat contact with Γ. Therefore, there is at most one spiraling axis of a given trajectory of an analytic vector field in a three dimensional manifold.*

Proof. By Definition 4.5, for any $k \geq 0$ there exists $t_k \geq 0$ such that $\gamma \mid_{[t_k,\infty[}$ is ultimately contained in W_k; by (1) in Proposition 4.6, using resolution of singularities, we can suppose that Γ is non singular and then Remark 3.1, (2) gives he result. □

Corollary 4.8. *The concept of axial spiralling is stable by blowing-up points: suppose that γ spirals around Γ at p and let $\pi : M' \to M$ be the blowing-up with center at p. If γ', Γ' denote the strict transform of γ, Γ by π respectively then γ' spirals around Γ' at $p' \in M'$, the tangent point of γ.*

Proof. The curve γ', considered as a trajectory of an analytic vector field up to re-parametrization, is oscillating and has iterated tangents. By Theorem 4.2, it has an spiraling axis and, by Corollary 4.7, this spiralling axis is Γ'. □

4.3 *Outline of the Proof of Theorem 4.2*

In this paragraph we sketch the proof of Theorem 4.2, complete details can be viewed in the reference [3]. We fix an oscillating trajectory γ in dimension three with iterated tangents and we have to prove that it has a spiralling axis. There are two parts in this theorem. First, the existence of an analytic half branch Γ such that γ has flat contact with Γ and, second, that Γ is actually an spiralling axis according to Definition 4.5.

4.3.1 Existence of Γ

Let H be an analytic hypersurface at $p \in M$ with respect to which γ is oscillating, that is, $\mid \gamma \mid \not\subset H$ and $\sharp(\mid \gamma \mid \cap H) = \infty$. (Notice that we are implicitly assuming that H has real dimension two as an analytic set.) Let $\{f = 0\}$ be a reduced local equation for H where f is analytic in some neighborhood of p in M. We can suppose, taking irreducible components, that the germ of f at p is irreducible.

Consider the set $Z = H \cap \{df(X) = 0\}$, called the *locus of (generalized) tangencies*. It is the set of points in H where either H or X is singular or X is tangent to H. We can suppose that Z (its germ at p) is an analytic set of dimension one: otherwise, either H is singular at any point (which is prevented by the hypothesis of irreducibility of f), or H is everywhere tangent to X (which is prevented by the fact that the trajectory γ intersects H but it is not completely contained inside H). Thus, Z consists of several analytic half branches $Z_1, Z_2, \ldots, Z_r$.

After a blowing-up $\pi : \tilde{M} \to M$ at the origin, we re-localize the problem at the tangent point p_1 of γ. Consider the strict transform X' of X at p_1 and the strict transform $H' = \overline{\pi^{-1}(H \setminus \{0\})}$ of the surface H. It is easy to observe that the new

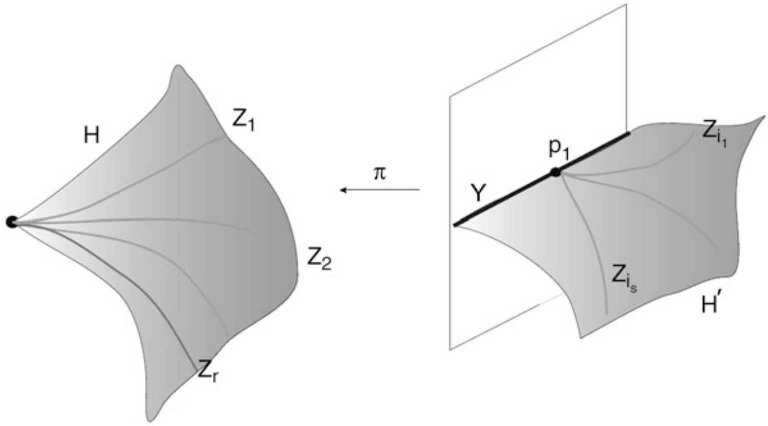


Fig. 14 Behavior of tangencies after blowing-up

locus of generalized tangencies $Z^{(1)}$ at the point p_1 is given by the strict transform of several of the branches $Z_{i_1}, \ldots, Z_{i_s}$ and, perhaps, the curve Y, intersection of H' with the exceptional divisor (see Fig. 14).

We claim that γ has flat contact with one of the branches Z_j. If this is not the case, after several blowing-ups, by the previous remark about the behavior of the locus of tangencies, restricting to a ball centered at the corresponding tangent point of γ, we can assume the following situation: $M = \mathbb{R}^3$, γ accumulates to the origin and is contained in a connected component Ω of the complement in $\mathbb{R}^3$ of a union E of coordinate planes (the created divisors) and the locus of tangencies Z is contained in E. Thus, since Z contains the singularities of H, $H \cap \Omega$ is a non-singular surface closed in Ω. We conclude, since Ω is simply connected, that every connected component of $H \cap \Omega$ separates Ω in two connected components (called the *sides* of the component).

Now we can use the ubiquitous Rolle's argument. Take H_1 one of the components of $H \cap \Omega$ having infinitely many intersection points with $| \gamma |$. Since γ intersects transversally H_1, it passes from one side to the other side of H_1. Take two consecutive points $p, q \in | \gamma | \cap H_1$ and $\sigma : [0,1] \to H_1$ a C^1-path connecting them (see Fig. 15). Since X "points" in opposite directions at p and q it must be tangent at some point of $Im(\sigma)$, obtaining then the desired contradiction.

4.3.2 Γ is a Spiraling Axis for γ

We have to prove that γ spirals around Γ according to Definition 4.5. We use the properties stated in Proposition 4.6 about the family of semi-analytic neighborhoods $\{W_k\}$ of the germ Γ_0 of Γ at $p \in M$.

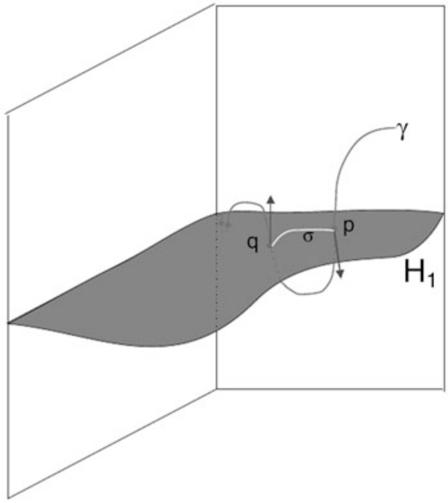


Fig. 15 Rolle's argument applied to the component H_1

Let (W, T) be a semi-analytic triangle over Γ. Property (1), says that there exists k such that $W_k \subset W$. Also, γ stays inside W_k for $t \gg 0$ since it has flat contact with Γ. On the other hand, property (3) (for $A = H \cup T$) implies that there exists a connected component R of $H \cap W_k \setminus \Gamma$ such that

1. (W_k, R) is a semi-analytic triangle.
2. (W_k, R) and $(W_k, T \cap W_k)$ are compatible.
3. γ intersects R infinitely many times.

We can, moreover, suppose that X is transversal to R and T (because for some k big enough W_k does not intersect other branches of tangencies between X and H or T but Γ). Thus γ intersects transversally R infinitely many times. We have to show that it intersects also T transversally infinitely many times and always from one given side to the other side of T in W_k. The arguments are the usual topological ones that we have already used for the spiraling in dimension two. Take two consecutive intersection points $\gamma(t_1), \gamma(t_2) \in R$ so that the arc $C =]\gamma(t_1), \gamma(t_2)[\subset W_k \setminus R$. We know also that $\gamma(t_1 + \varepsilon), \gamma(t_2 - \varepsilon)$ are in opposite sides of R inside W_k for ε sufficiently small. On the other hand, $(W \setminus R) \setminus (T \cup \Gamma)$ has two connected components Ω_1, Ω_2 and each of them contains a different side of R in W. Thus $C \cap T \neq \emptyset$ which proves that γ intersects infinitely many times T since it intersects infinitely many times R. Moreover, each side of R or T is contained in exactly one of the components Ω_1, Ω_2 and this gives us the desired conclusion that γ passes from one given side to the other of T. This finishes the proof of Theorem 4.2. □

Using similar arguments as above with semi-analytic triangles we can prove the following corollaries (see [3, 4] for details).

Corollary 4.9 (Characteristic property of spiraling). *Let γ be a trajectory of an analytic vector field in a three dimensional manifold M such that $\omega(\gamma) = p$ and let*

Γ be an analytic half-branch at p. Then γ spirals around Γ if and only if for any semi-analytic set S of dimension 2 we have

$$\Gamma_0 \subset \overline{S} \Leftrightarrow \sharp(\mid \gamma \mid \cap S) = \infty. \tag{4.1}$$

Corollary 4.10 (Spiralling around a non-singular axis). *Suppose that Γ is non singular and given in local coordinates at p by $\Gamma = \{x = y = 0, z > 0\}$. Put cylindric coordinates $x + iy = re^{i\theta}$ and write $\gamma(t) = (r(t)e^{i\theta(t)}, z(t))$ with $t \gg 0$ in these coordinates. Then γ spirals around Γ if and only if we have the properties*

$$\lim_{t\to\infty} \frac{r(t)}{z(t)^N} = 0 \ \forall N \geq 0 \text{ and } \lim_{t\to\infty} \theta(t) = \pm\infty.$$

4.4 *Outline of the Proof of Theorem 4.3*

4.4.1 Vector Fields in Pre-final Situation

First we analyze a "reduced" situation of the theorem.

Let X be an analytic vector field in a neighborhood of a point p in a three dimensional manifold and let Γ be an analytic half-branch at p which is invariant and non-degenerated for X ($\Gamma \not\subset SingX$), not necessarily a spiraling axis.

Definition 4.11. (1) We will say that X *is in pre-final situation (with respect to* Γ if there are coordinates (x, y, z) at p such that $\Gamma = \{x = y = 0, z > 0\}$ and the vector field X is written as

$$X = \sum_{i=0}^{q} z^i L_i(x, y) + z^{q+1} Y - z^{q+1} \frac{\partial}{\partial z} \tag{4.2}$$

where

- $q \geq 1$,
- $L_i(x, y) = (a_i x + b_i y)\frac{\partial}{\partial x} + (c_i x + d_i y)\frac{\partial}{\partial y}$ is a linear vector field,
- $L_0(x, y)$ is not nilpotent,
- $Y = A(x, y, z)\frac{\partial}{\partial x} + B(x, y, z)\frac{\partial}{\partial y}$ is a family of two-dimensional vector fields tangent to the fibers $\{z = cst\}$ and $A(0, 0, z) \equiv B(0, 0, z) \equiv 0$.

(2) We will say that X *is in final situation (with respect to* Γ*)* if, furthermore, we have that either every $L_k = a_k(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y})$ is radial or the first L_k, $k \in \{0, \ldots, q\}$, which is not radial has different eigenvalues (has not a two-dimensional Jordan block).

Notice that, in particular, Γ is a non-singular half-branch and then spiraling around Γ can be detected by using cylindrical coordinates $(x, y, z) = (re^{i\theta}, z)$ as in Corollary 4.10. On the other hand, since Γ is non-degenerated, Γ is the image of a trajectory that accumulates to the singular point p.

The Example 4.1 in the beginning of this section is a vector field in pre-final situation with respect to the z-axis. As we have shown, the z-axis is in this case a twister axis and the half-space $\{z > 0\}$ is a twister domain.

The rest of this paragraph is devoted to the proof of the following particular case of Theorem 4.3.

Proposition 4.12. *Suppose that X is in pre-final situation and that Γ is an spiralling axis for some trajectory γ_0, then there exists $\delta > 0$ such that the open set $D_\delta = \{x^2 + y^2 < \delta, 0 < z < \delta\}$ is a twister domain for X; that is, D_δ is positively invariant and any trajectory issued of a point in $D_\delta \setminus \Gamma$ accumulates to p and spirals around Γ.*

Proof. First, we claim that the eigenvalues of L_0 are (non-zero) complex conjugate numbers. Suppose, otherwise, that the eigenvalues of L_0 were two distinct real numbers $\lambda \neq \mu$. Choose the coordinates such that $L_0 = \lambda x \frac{\partial}{\partial x} + \mu y \frac{\partial}{\partial y}$. If we write in cylindric coordinates

$$\gamma_0(t) = (x(t), y(t), z(t)) = (r(t)e^{i\theta(t)}, z(t))$$

then we would have the equation for θ

$$\dot{\theta}(t) = (\mu - \lambda)\sin\theta(t)\cos\theta(t) + O(z(t)) + O(r(t)),$$

and thus, the function $\theta(t)$ would change its monotonicity from increasing to decreasing between the strips $[\theta_l + \varepsilon, \theta_{l+1} - \varepsilon]$ where $\theta_l = \pi l/2$, $l = 0, 1, 2, 3$ mod π and $\varepsilon > 0$. This prevents θ to diverge to $+\infty$ or $-\infty$ and thus γ_0 would not spiral.

Notice that, due to the component $-z^{q+1}\partial/\partial z$ in (4.2), any trajectory in a given domain of the form D_δ can be parameterized by z. This means that trajectories of X are in one to one correspondence with the solutions of the following system of ordinary differential equations

$$(S) \qquad z^{q+1}\begin{pmatrix} \frac{dx}{dz} \\ \frac{dy}{dz} \end{pmatrix} = -\sum_{j=0}^{q} z^j L_j \begin{pmatrix} x \\ y \end{pmatrix} - z^{q+1}\begin{pmatrix} A(x,y,z) \\ B(x,y,z) \end{pmatrix}$$

(this time we consider L_j as the matrix $\left(\begin{smallmatrix} a_j & b_j \\ c_j & d_j \end{smallmatrix}\right)$). In particular, we write $\gamma_0(z) = (x_0(z), y_0(z), z)$ where $(x_0(z), y_0(z))$ is a solution of (S). By the hypothesis that γ_0 spirals around Γ, we have that x_0, y_0 are defined for any $0 < z < \delta$ and, if we put $x_0(z) + iy_0(z) = r_0(z)e^{i\theta_0(z)}$ then $r_0(z)/z^n \to 0$ $\forall n$ and $\theta_0(z) \to \pm\infty$ when $z \to 0$.

We consider two cases.

Case Spec $L_0 = \{\lambda, \overline{\lambda}\}$ with $\lambda \neq \overline{\lambda}$ (non real eigenvalues).- First, the reader can check that, after a suitable change of coordinates

$$(x, y) \rightsquigarrow (x, y)(P_0 + zP_1 + \cdots + z^q P_q),$$

where P_i is a two-by-two matrix with P_0 invertible, we can suppose that all L_j, $j = 0, \ldots, q$, are of the form $\begin{pmatrix} a_j & -b_j \\ b_j & a_j \end{pmatrix}$. The hypotheses of non-real eigenvalues means with this notation $b_0 \neq 0$. If we put $\rho = r^2 = x^2 + y^2$, from the system of equations (S) we obtain

$$(S_\rho) \qquad \frac{z^{q+1}}{2}\frac{d\rho}{dz} = -\rho(a_0 + a_1 z + \cdots + a_q z^q) + z^{q+1} O(\rho).$$

A priori, we have the following possibilities:

(i) The first coefficient a_k different from zero is positive.
(ii) $a_0 = a_1 = \cdots = a_{q-1} = 0, a_q < 0$.
(iii) $a_0 = a_1 = \cdots = a_q = 0$.
(iv) For some $0 \leq s \leq q-1$, $a_s < 0$ and $a_j = 0$ for $0 \leq j \leq s-1$.

If we were in case (i), $\frac{d\rho}{dz}$ is negative along any trajectory if z sufficiently small; in particular, the function $\rho_0(z) = x_0(z)^2 + y_0(z)^2$ would not have limit equal to 0 when z tends to 0, contradicting the hypothesis $\omega(\gamma_0) = 0$. If we were in case (ii), $\rho_0(z) > C z^N$ for certain constants $C, N > 0$ which would contradict the fact that γ_0 has flat contact with Γ. If we were in case (iii) we could prove also that $\rho_0(z)$ does not tend to 0 when $z \to 0$.

Thus, we are necessarily in case (iv). Notice that in this case $q \geq 1$. We obtain from equation (S_ρ) that there exists some $C > 0$ such that, if $0 < \rho < \delta$ with δ sufficiently small,

$$0 < \frac{1}{\rho}\frac{d\rho}{dz} < \frac{C}{z^{s+1}}.$$

This, in turn, implies two things. On one hand, D_δ is positively invariant since X is transversal to the border of D_δ in the positive half space $\{z > 0\}$ and points into the interior of D_δ; on the other hand, for any solution $z \mapsto \rho(z)$ of equation (S_ρ) there exists $K > 0$ with $0 < \rho(z) < K \exp(-\frac{C}{sz^q})$. Thus, any trajectory γ inside D_δ, being parameterized for any $0 < z < \delta$ accumulates to the origin and has flat contact with Γ by Remark 3.1.2. Moreover, since the equation for the angle function $\theta = \arctan y/x$ is

$$(S_\theta) \quad z^{q+1}\frac{d\theta}{dz} = b_0 + O(z),$$

we have $\theta(z) \to \pm\infty$ when $z \to 0$. Using Corollary 4.10, any trajectory issued of a point in D_δ spirals around Γ.

Case Spec $L_0 = \{\lambda\}$, a single non-zero real eigenvalue.- Notice first that $\lambda < 0$, otherwise no trajectory of X in the half-space $\{z > 0\}$ can accumulate to the origin. Moreover, using similar arguments as above, for $\delta > 0$ sufficiently small, D_δ is positively invariant and any trajectory γ inside this domain has flat contact with Γ. In order to prove that γ spirals around Γ it suffices to prove that it is oscillating and then apply Theorem 4.3. Contrary to the previous case, the linear part L_0 of the vector field alone is not responsible here for the oscillation of those trajectories. We present here a proof that uses some dynamical arguments based on classical results from the theory of Center Manifolds; in the next section we will sketch another proof which only uses some algebraic manipulations of the system of ordinary differential equations (S).

Let $\pi : M' \to M$ be the polar local blowing-up with center $Y = \{x = y = 0\}$. Every trajectory γ in D_δ (except Γ itself, of course) has a lifting $\tilde{\gamma}$ in $M' \simeq \mathbb{S}^1 \times \mathbb{R} \times Y$ which is a trajectory of the total transform $\tilde{X}$ of X by π. Since $\omega(\gamma) = 0$, we have $\omega(\tilde{\gamma}) \subset E = \pi^{-1}(0) \simeq \mathbb{S}^1$. It suffices to show that $\omega(\tilde{\gamma}) = E$. We have that E is invariant by $\tilde{X}$ and either entirely composed by singularities (when L_0 is diagonalizable) or with only two singular points (when L_0 is not diagonalizable). On the other hand, from the hypothesis and taking the appropriate charts of the blowing-up, one can prove that the cylinder $C = \pi^{-1}(Y) \simeq \mathbb{S}^1 \times \mathbb{R}$ is a *global center manifold along* E; that is, an invariant manifold for X whose tangent space at any point q of E is the (generalized) eigenspace corresponding to the zero eigenvalue of the linear part of $\tilde{X}$ at q.[3] The general theory (see [6, 13] for a proof) says that for any trajectory $\tilde{\gamma}$ of $\tilde{X}$ there exists a unique trajectory $\sigma_{\tilde{\gamma}}$, called the *accompanying trajectory*, inside C to which $\tilde{\gamma}$ is exponentially close in terms of the parametrization by time, i.e., for some $A, a > 0$, we have

$$|| \tilde{\gamma}(t) - \sigma_{\tilde{\gamma}}(t) || \leq A \exp(-at). \tag{4.3}$$

Now, the accompanying trajectory $\sigma_{\tilde{\gamma}_0}$ of the lifting $\tilde{\gamma}_0$ of γ_0 must accumulates to the whole E since γ_0 is spiraling. This implies that any other trajectory inside C, in a sufficiently small neighborhood of E, also accumulates to the whole E. This forces any trajectory $\tilde{\gamma}$ to accumulate to the whole E, as we wanted to prove (see Fig. 16). □

4.5 Stability of Spiraling by Blowing-Ups

In order to prove Theorem 4.3 in the general case, the strategy is the following: given a vector field X and Γ a non-degenerated spiraling axis of X for some trajectory γ_0 at some point p, we want to *reduce* to the situation of a vector field in pre-final situation by means of analytic transformations that preserve the spiraling behavior. Then we will construct a twister domain for X at p as the image of some twister domain of the vector field in pre-final situation of the form D_δ by the map obtained by the composition of those transformations. In what follows we precise the kind

[3] Rigorously, this is only the case if $E \subset Sing(\widetilde{X})$; otherwise, we have to say, more correctly, that C is a *normally hyperbolic invariant set of* $\tilde{X}$ (see [11, 27]). The conclusions about accompanying trajectories are, however, the same.

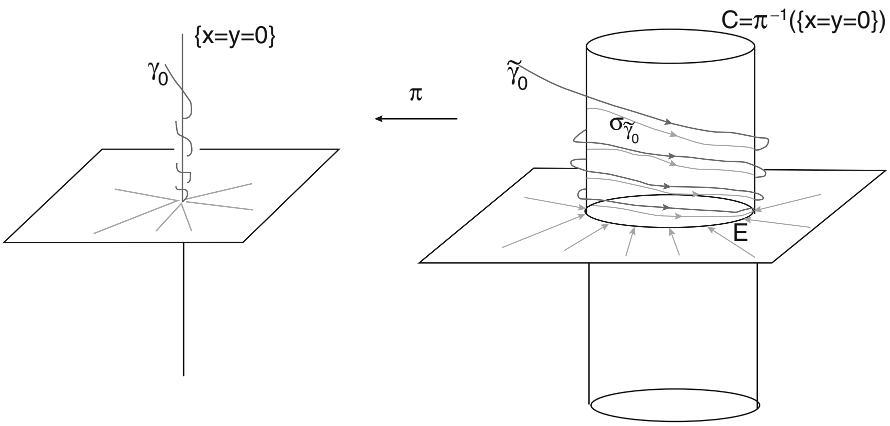


Fig. 16 Spiraling and accompanying trajectories

of transformations needed to achieve this reduction. For the most part of the time, they are blowing-ups with center either a point or a local analytic curve; eventually, we will need some real ramifications over surfaces which does not intersect the spiraling axis.

Definition 4.13. Let Γ be an analytic half-branch at some point p in a three dimensional manifold M. A map $\pi : \tilde{M} \to M$ is said to be a *basic* Γ*-admissible morphism* if it is of one of the following forms:

- π is the blowing-up at p.
- A local blowing-up along a curve: there exists a one-dimensional smooth analytic submanifold $Y \subset M$ with $Y \cap \Gamma$ and a chart $(U, (x, y, z))$ centered at p such that $Y \cap U$ is a coordinate axis and π is the composite of the blowing-up $\pi_Y : \tilde{M} \to U$ with center $Y \cap U$ and the open immersion $U \hookrightarrow M$.
- A q-ramification: there exists a chart $(U, (x, y, z))$ at p, with $\Gamma \subset \{z > 0\}$, and $q \in \mathbb{N}_{\geq 1}$ such that π is the composition of the ramification $(x, y, z) \mapsto (x, y, z^q)$ by the open immersion $U \hookrightarrow M$.

Evidently, $\tilde{\Gamma} = \pi^{-1}(\Gamma)$ is an analytic half-branch at some $\tilde{p} \in \tilde{M}$, called the *(strict) transform of* Γ *by* π. A Γ*-admissible morphism* is a composition of basic admissible morphisms for Γ and its successive transforms.

The following proposition is a consequence of the characteristic property of spiraling, Corollary 4.9, and other arguments in subanalytic geometry similar to those used in the proof of Theorem 4.2 (notice that the blowing-up with a smooth closed center is a proper map [10]).

Proposition 4.14. *Suppose that γ is a trajectory of an analytic vector field X on M such that $\omega(\gamma) = p$ and suppose that γ spirals around an analytic half-branch*

Γ. *Let* $\pi : \tilde{M} \to M$ *be a basic* Γ*-admissible morphism. Then* $\tilde{\gamma} = \pi^{-1} \circ \gamma$ *satisfies the properties:*

1. $\omega(\tilde{\gamma}) = \tilde{p} \in \tilde{M}$.
2. $\mid \tilde{\gamma} \mid$ *is a leave of a* 1*-dimensional foliation on* $\tilde{M}$*; hence, up to re-parametrization, it is a trajectory of an analytic vector field* $\tilde{X}$ *on a neighborhood of* $\tilde{p}$ *(called the strict transform of* X *by* π*).*
3. $\tilde{\gamma}$ *spirals around* $\tilde{\Gamma}$.

Moreover, Γ *is a twister axis for* X *if and only if* $\tilde{\Gamma}$ *is a twister axis for* $\tilde{X}$.

4.5.1 General Case

The reduction to the pre-final situation is given in several steps.

Step 1. By blowing-up points, using resolution of singularities of curves, we can suppose that Γ is non-singular. So we assume that p is the origin of $\mathbb{R}^3$ and that, in some local coordinates, $\Gamma = \{x = y = 0, z > 0\}$.

Step 2. We can suppose that the linear part D_0X of the vector field at the origin is not zero. In order to prove this, write

$$X = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c\frac{\partial}{\partial z}$$

and put $q + 1 = ord_z(c(0, 0, z))$. Then q is finite since Γ is non-degenerated. If $D_0X \equiv 0$, we consider the blowing-up π_1 at the origin and take standard coordinates (x_1, y_1, z_1) at the tangent point p_1 of Γ such that

$$x = x_1z_1, \quad y = y_1z_1, \quad z = z_1. \tag{4.4}$$

As we have explained in the Sect. 3.2, there exists $s_1 > 0$ such that $\pi_1^* X = X_1 = z_1^{s_1} X_1'$ where X_1' is the strict transform. We have that the number $q + 1$ drops strictly for X_1'. Thus, after a finite number of steps we can assume $D_0X \not\equiv 0$.

Step 3. With the same notations as above, we can suppose that $q \geq 1$. To see this, if $q = 0$ then Γ is tangent to an eigenvector of the linear part D_0X with non-zero eigenvalue λ. Necessarily $\lambda < 0$ since the existence of γ_0 implies that Γ is itself a trajectory accumulating to the origin.

By a point blowing-up and coordinates at the tangent point p_1 of Γ as in (4.4), the linear part transforms into

$$D_0X(x, y, z) \rightsquigarrow D_0X(x_1, y_1, z_1) + x_1\frac{\partial}{\partial x_1} + y_1\frac{\partial}{\partial y_1}.$$

After several of these blowing-ups, D_0X has two positive eigenvalues and a negative eigenvalue. This means that X has a hyperbolic singularity at the origin. By the

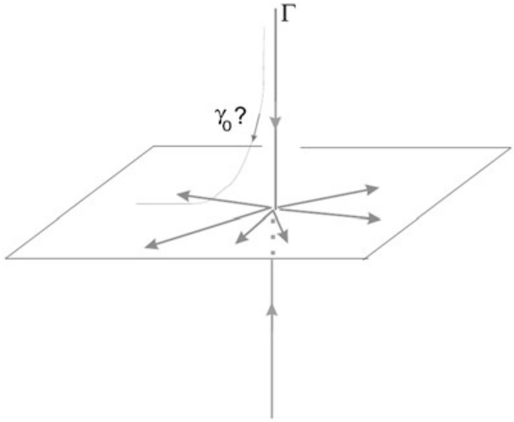


Fig. 17 Hyperbolic saddles has no spiraling axis

classical Hartman-Grobman theorem, the dynamics is topologically the same as a linear saddle in $\mathbb{R}^3$ with a two-dimensional stable variety and a one-dimensional unstable variety. This prevents the existence of γ_0 since Γ is the only trajectory of X accumulating to the origin contained in the same side of the unstable manifold as Γ (see Fig. 17).

Step 4. We can suppose that the coefficient of $\partial/\partial z$ for the vector field is just $-z^{q+1}$. This is justified by the following computations. Take coordinates as in (4.4) at the tangent point of Γ after blowing-up the origin. The total transform equals the strict transform of X since its multiplicity is equal to 1. The monomials transform according to the rules

$$z^r u^m \partial/\partial v \rightsquigarrow z_1^{r+m-1} u_1^m \partial/\partial v_1, \quad z^r u^m \partial/\partial z \rightsquigarrow z_1^{r+m} u_1^m \partial/\partial z_1,$$

where we have used the notation $u, v \in \{x, y\}$. After finitely many transformations of this type we can write

$$X = U \cdot \left[L_0(x,y) + zL_1(x,y) + \cdots + z^q L_q(x,y) - z^{q+1} \left\{ \frac{\partial}{\partial z} + Y \right\} \right], \quad (4.5)$$

where $L_j(x,y)$ is a linear vector field in the variables (x,y), $U(0,0,0) \neq 0$ and $Y = A(x,y,z)\partial/\partial x + B(x,y,z)\partial/\partial y$ with $Y(0,0,z) \equiv 0$.

Step 5. The vector field in (4.5) is already in pre-final situation if the linear part $L_0(x,y)$ is non-nilpotent. Suppose otherwise that $L_0(x,y) = y\partial/\partial x$ (by hypothesis $L_0 \not\equiv 0$). Write also, for $j \geq 1$,

$$L_j(x,y) = (a_j x + b_j y)\frac{\partial}{\partial x} + (c_j x + d_j y)\frac{\partial}{\partial y}.$$

If $c_1 = 0$, we consider the blowing-up π with center the local invariant curve $Z = \{y = z = 0\}$ and we obtain, for the standard coordinates $x = x_1, y = y_1 z_1, z = z_1$, that the strict transform of Γ is $\Gamma_1 = \{x_1 = y_1 = 0, z_1 > 0\}$ and

$$\pi^* X = z_1[L'_0(x_1, y_1) + \cdots + z_1^{q'} L_{q'}(x_1, y_1) - z_1^{q'+1}\left\{\frac{\partial}{\partial z_1} + Y'_1\right\}$$

where $L'_0 = (a_1 x_1 + y_1)\frac{\partial}{\partial x_1} + (c_2 x_1 + d_1 y_1)\frac{\partial}{\partial y_1}$ and $q' = q - 1$. The conclusion is that either L'_0 is non-nilpotent or the exponent q drops. We win after finitely many blow-ups of this kind.

If $c_1 \neq 0$, we first consider the ramification over the plane $\{z = 0\}$:

$$\rho : \mathbb{R}^3 \to \mathbb{R}^3,\ (x', y', z') \mapsto (x', y', z'^2).$$

Then $\Gamma' = \{x' = y' = 0, z' > 0\} = \rho^{-1}(\Gamma)$ is the strict transform of Γ by ρ, a non-degenerated spiraling axis for the trajectory $\rho^{-1} \circ \gamma_0$ of the transform $X' = \rho^* X$. Now we consider the blowing-up π with center the local curve $Z' = \{y' = z' = 0\}$ which is invariant for X' and does not intersect Γ'. In coordinates $x' = x_1, y' = y_1 z_1, z' = z_1$, we can write $\pi^* X' = z_1 X_1$, where

$$X_1 = y_1 \frac{\partial}{\partial x_1} + c_1 x_1 \frac{\partial}{\partial y_1} + O(z_1) - \frac{1}{2} z_1^{2q} \left\{\frac{\partial}{\partial z_1} + Y_1\right\}.$$

Despite the fact that the exponent q increases in this case, we arrive to a non-nilpotent linear part, i.e., the vector field in (4.5) in pre-final situation with respect to Γ.

4.5.2 Spiraling in Final Situation

The steps explained in the previous Sect. 4.5.1 can be considered as a special case of a reduction of singularities of the vector field X: after several transformations, the singularity of X at the point signaled by the invariant half-branch Γ has a non-nilpotent linear (more correctly we have to say that it is a *local uniformization along* Γ, see [5] and Theorem 5.9 below).

We can, nevertheless, continue this process of reduction of singularities along Γ in order to obtain a better description of the singularity of the transformed vector field, not just a non-nilpotent linear part. Namely, we can obtain a vector field in final situation as we have defined in Definition 4.11.2. The reason for considering this final situation instead of the pre-final one is that we can decide whether Γ is a twister axis of the vector field just by looking at the linear coefficients L_j of the expression (4.2) and we do not need to assume the previous existence of the trajectory γ_0 spiraling around Γ.

We summarize these commentaries in the two following results. In both statement, let X be an analytic vector field in a neighborhood of some point p in a three dimensional manifold M and let Γ be an analytic half-branch at p which is invariant and non-degenerated for X.

Proposition 4.15. *There exists a* Γ*-admissible morphism* $\pi : \tilde{M} \to M$ *such that the strict transform* $\tilde{X}$ *of* X *by* π *is in final situation with respect to the strict transform* $\tilde{\Gamma}$ *of* Γ.

Proposition 4.16. *Suppose that* X *is in final situation with respect to* Γ *and written as (5.7) with the corresponding properties of the linear terms* L_j. *Then* Γ *is a spiraling axis (and thus a twister axis) if and only if the following properties hold:*

1. $q \geq 1$,
2. *There is a first non radial term* L_k *with* $k \in \{0, \ldots, q\}$ *and for this term* $Spec(L_k) = \{\lambda, \overline{\lambda}\}$ *with* $\lambda \neq \overline{\lambda}$,
3. *There exists* $0 \leq l \leq q-1$ *with* $Spec(L_l) = \{\mu, \overline{\mu}\}$ *and* $Re(\mu) \neq 0$ *and for the first such* L_l *we have* $Re(\mu) < 0$.

Details of the proof of these results can be seen in [4]. For Proposition 4.15, one uses similar arguments as the ones described in Steps 1–4 of the Sect. 4.5.1. For Proposition 4.16, one uses some of the arguments in the theory of dynamical systems already sketched in the proof of Proposition 4.12 (compare also with Proposition 5.13 bellow).

5 Non-oscillatory Trajectories: Linked and Separated Packages

Theorem 4.3 describe non-degenerated spiraling axis as "centers" of a domain where the dynamics of the vector field is qualitatively well understood. In this section we identify other type of interesting "organizer centers" of the dynamics. Namely, they appear when the analytic axis became a purely formal non-convergent curve. The interpretation for this formal axis is that it represents a whole package of trajectories sharing the same iterated tangents and having similar asymptotic behavior.

5.1 Example of Euler's Equation

In this paragraph we analyze a classical example in dimension two and then we generalize it in dimension three. We will discover new asymptotic behavior for non-oscillatory trajectories that we will describe in general in the next paragraphs.

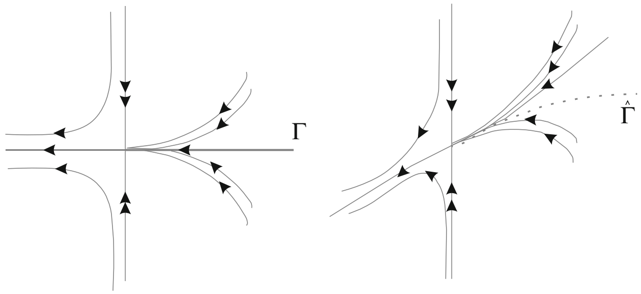


Fig. 18 Euler's Equation

5.1.1 Euler's Equation in Dimension Two

Consider the vector field

$$X_0 = -\left(x^2 \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}\right)$$

in $\mathbb{R}^2$ or, equivalently, the linear ordinary differential equation (E_0) $x^2 \frac{dy}{dx} = y$. Solutions of this equation are given by $y(x) = Ce^{-1/x}$, where $C \in \mathbb{R}$. We conclude that all trajectories of X_0 in $\{x > 0\}$ has flat contact with the analytic half-branch $\{y = 0, x > 0\}$, which can be consider as a organizer center. A slight perturbation gives *Euler's equation*:

$$X_1 = -\left(x^2 \frac{\partial}{\partial x} + (y - x) \frac{\partial}{\partial y}\right) \quad \leftrightarrow \quad (E_1) \quad x^2 \frac{dy}{dx} = y - x.$$

Solutions of (E_1) are of the form $y(x) = y_0(x) + Ce^{-1/x}$, $C \in \mathbb{R}$ where $y_0(x)$ is a particular solution. The phase portrait of X_1 is very similar to that of X_0 (see Fig. 18) and, moreover, any trajectory in $\{x > 0\}$ accumulates to the origin, is non-oscillatory and has the same sequence of iterated tangents. This sequence corresponds to the unique invariant formal curve of X_1; i.e., the graph $\widehat{\Gamma} = (x, \widehat{f}(x))$ of the (well known) unique formal solution of Euler's equation (E_1):

$$\widehat{f}(x) = x + 1!x^2 + 2!x^3 + \cdots + n!x^{n+1} + \cdots \in \mathbb{R}[[x]]. \tag{5.1}$$

More precisely, after the blowing-up at the origin, in the chart (x_1, y_1) such that $x = x_1, y = x_1 y_1$, the lifting of trajectories of $\{x > 0\}$ go to the unique singular point $p_1 = (0, 1)$ on the domain of this chart. The *strict transform* of $\widehat{\Gamma}$

$$\widehat{\Gamma}_1 = \left(x_1, \widehat{f}_1(x_1) = \frac{\widehat{f}(x_1)}{x_1}\right)$$

is a formal curve centered at p_1 where the situation repeats. In this way, the sequence of iterated tangents $IT(\widehat{\Gamma})$ can be defined and, for any trajectory γ of X_1 in $\{x > 0\}$ we have $IT(\widehat{\Gamma}) = IT(\gamma)$. On the other hand, it is easy to check that there is no invariant analytic curve of X_1 contained in $\{x > 0\}$ (the only analytic invariant curve is the y-axis).

In general, a formal curve $\widehat{\Gamma}$ of a real analytic manifold M *centered at* a point $p \in M$ is a Puiseux's parametrization of the form

$$\widehat{\Gamma} = (\widehat{x}_1(z), \ldots, \widehat{x}_{n-1}(z), z^r), \tag{5.2}$$

in some coordinates $(x_1, \ldots, x_{n-1}, z)$ at p, where $r \in \mathbb{N}$ and $\widehat{x}_j(z)$ a formal power series with $\widehat{x}_j(z) = 0$. One can define the sequence of iterated tangents $IT(\widehat{\Gamma}) = \{p_k\}$ of $\widehat{\Gamma}$ similarly as in the case of an analytic half-branch, just by blowing-up points, taking the strict transforms of the series and defining the corresponding tangent as the point defined in coordinates by the independent values of these transformed series. The formal curve Γ is called *not singular* at p if we can chose $r = 1$ in the parametrization (5.2).

Definition 5.1. Let $\gamma : [0, \infty[\to M$ be an analytic parameterized curve with $\omega(\gamma) = \{p\}$. We say that γ *has flat contact with* $\widehat{\Gamma}$ if γ has iterated tangents and $IT(\gamma) = IT(\widehat{\Gamma})$.

We have a similar result as Proposition 3.4 for formal curves.

Proposition 5.2. *If γ has flat contact with $\widehat{\Gamma}$ and γ is a trajectory of an analytic vector field X then $\widehat{\Gamma}$ is (formally) invariant for X.*

Here, formally invariant means that if we write

$$X = a_1 \frac{\partial}{\partial x_1} + \cdots + a_{n-1} \frac{\partial}{\partial x_{n-1}} + b \frac{\partial}{\partial z}$$

then the vectors of power series $(a_1(\widehat{\Gamma}(z)), \ldots, a_{n-1}(\widehat{\Gamma}(z)), b(\widehat{\Gamma}(z)))$ and $(\widehat{x}'_1(z), \ldots, \widehat{x}'_{n-1}(z), rz^{r-1})$ are colinear (in the vector space of n-uples of power series over the fractional field of $\mathbb{R}[[z]]$.

5.1.2 Euler's Equation in Dimension Three

Now we construct an example, similar to Euler's equation, in dimension three, by a perturbation of Example 4.1.

Example 5.3. Consider the following algebraic vector field in $\mathbb{R}^3$

$$X = (-x - y + z)\frac{\partial}{\partial x} + (-y + x)\frac{\partial}{\partial y} - z^2 \frac{\partial}{\partial z}.$$

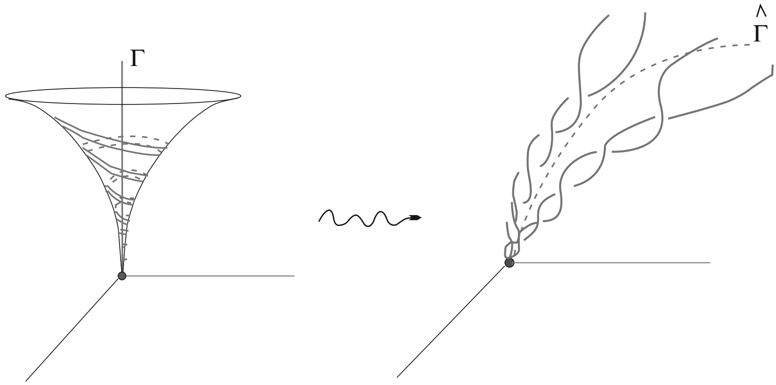


Fig. 19 Perturbation of the example of spiraling dynamics

Although we can not give explicit expressions for the trajectories of this example as in the Example 4.1, we can easily obtain the following qualitative properties for any trajectory γ in the half-space $\{z > 0\}$ (see Fig. 19):

1. γ accumulates to the origin and has iterated tangents. The calculations are similar to those that we have sketched for Euler's equation in the plane. More precisely, one can see that there is a unique formal series solution $(x, y) = (\widehat{f}_1(z)\widehat{f}_2(z)) \in \mathbb{R}[[z]]^2$ of the system of ODEs associated to the vector field:
$$\begin{cases} z^2 \dfrac{dx}{dz} = x + y - z \\ z^2 \dfrac{dy}{dz} = y - x \end{cases} \tag{5.3}$$
and that any trajectory in $\{z > 0\}$ has the same sequence of iterated tangents as the formal curve $\widehat{\Gamma} = (\widehat{f}_1(z), \widehat{f}_2(z), z)$.
2. γ is non-oscillatory: if it was oscillating, since it has iterated tangents, it would have flat contact with an analytic half-branch Γ (the spiraling axis) by Theorem 4.2; but then we would have $IT(\Gamma) = IT(\widehat{\Gamma})$ and the series $\widehat{f}_1(z), \widehat{f}_2(z)$, completely determined by the sequence of iterated tangents, would be convergent series, which is not the case (see below for a proof).
3. Given two different trajectories γ, γ' on $\{z > 0\}$, we can parameterize by the coordinate z as
$$\gamma(z) = (x(z), y(z), z),\ \ \gamma'(z) = (x'(z), y'(z), z).$$
The differences $u(z) = x(z) - x'(z), v(z) = y(z) - y'(z)$ satisfy the homogeneous system of ODEs
$$\begin{cases} z^2 \dfrac{du}{dz} = u + v \\ z^2 \dfrac{dv}{dz} = v - u \end{cases} \tag{5.4}$$

and thus the argument of the vector $(u(z), v(z)) \in \mathbb{R}^2 \setminus \{(0,0)\}$ tends to infinity when $z \to 0^+$. The two trajectories are *asymptotically linked* (see Fig. 19; we give the general definition below in Proposition 5.5).

5.1.3 Complex Euler's Equation

The relation between the examples of Euler's equation in the plane (E_1) and Example 5.3 in $\mathbb{R}^3$ is more than just an analogy. Let us see why.

Consider the complex saddle-node equation for $(t, w) \in \mathbb{C}^2$:

$$(E_\varepsilon) \quad t^2 \frac{dw}{dt} = w - \varepsilon t, \quad \varepsilon = 0, 1.$$

For $\tau = \rho e^{i\alpha} \in \{Re > 0\}$, consider the embedding $j_\tau : \mathbb{C} \times \mathbb{R} \to \mathbb{C}^2$, $j_\tau(w, z) \mapsto (w, z\tau)$. The equation induced by (E_ε) on $\mathbb{C} \times \mathbb{R}$ by j_τ is

$$(E_{\varepsilon,\tau}) \quad z^2 \frac{dw}{dt} = \frac{w}{\tau} - \varepsilon z, \quad z \in \mathbb{R}, w \in \mathbb{C}.$$

Now, put $w = x + iy$ so that $(E_{\varepsilon,\tau})$ becomes a system of two real analytic (linear) ODEs. If $\varepsilon = 0$ then the zero map $x(z) \equiv y(z) \equiv 0$ is a solution of this system and all other solutions have flat contact with this one. Their asymptotic behavior is either non-oscillatory when $\tau \in \mathbb{R}^+$ or spiralling when $\tau \notin \mathbb{R}^+$ (see Fig. 20). If $\varepsilon \neq 0$, the (unique) formal solution of $(E_{1,\tau})$, $(\widehat{f}_1^\tau(z), \widehat{f}_2^\tau(z)) \in \mathbb{R}[[z]]^2$ is such that $\widehat{f}_1^\tau(z) + i\widehat{f}_2^\tau(z) = \tilde{f}(z\tau)$, where $\tilde{f}(z)$ is the formal divergent solution (5.1) of Euler's equation (E_1). Thus $\tilde{f}_1^\tau, \tilde{f}_2^\tau$ are both divergent series. In particular, for $\tau = 1$, the formal solution of the system (5.3) is not convergent. As we have explain in the previous paragraph, solutions of $(E_{1,\tau})$ are non-oscillatory and the behavior is either "not linked" when $\tau \in \mathbb{R}^+$ or "linked" when $\tau \notin \mathbb{R}^+$ (see Fig. 21).

5.2 *Generalities About Linked Trajectories*

In this paragraph we investigate the concept of linked trajectories suggested by Example 5.3.

Definition 5.4. Let γ, γ' be two parameterized curves in a three dimensional manifold M such that $\omega(\gamma) = \omega(\gamma') = p \in M$.

(1) A system of coordinates $w = (x, y, z)$ at p is called *positive* for γ if, in a sufficiently small neighborhood of p, $|\ \gamma\ | \subset \{z > 0\}$ and γ can be parameterized with z as a parameter in the form $\gamma(z) = (x(z), y(z), z)$ with $x(z), y(z)$ analytic functions in some interval $]0, \varepsilon[$.

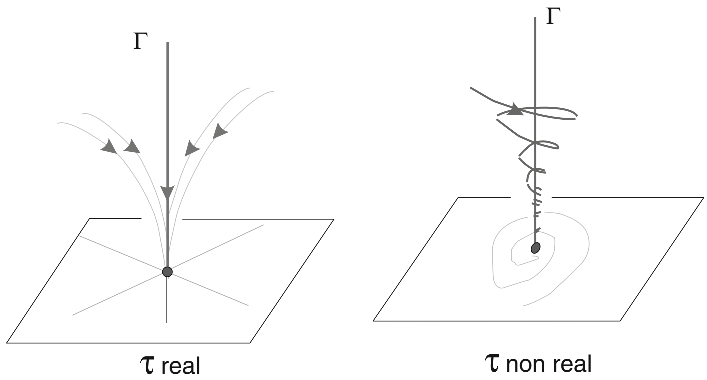


Fig. 20 Phase portrait of the equation $(E_{0,\tau})$

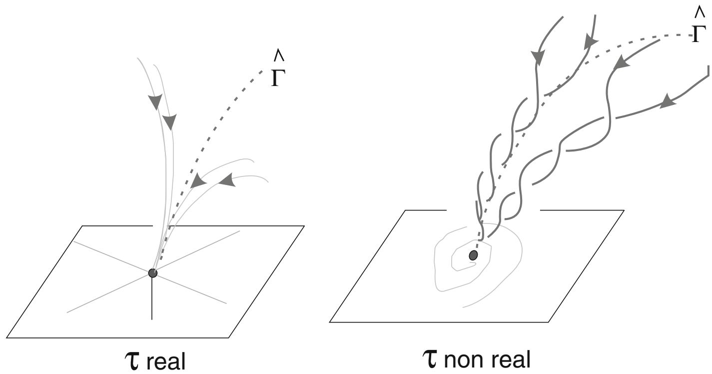


Fig. 21 Phase portrait of the equation $(E_{1,\tau})$

(2) If $w = (x, y, z)$ is positive for both γ and γ', we say that they are *w-asymptotically linked (we will write shortly w-a.l.)* if $\mid \gamma \mid \cap \mid \gamma' \mid = \emptyset$ and, writing

$$\gamma(z) = (x(z), y(z), z), \quad \gamma'(z) = (x'(z), y'(z), z), \tag{5.5}$$

we have that the plane curve $z \mapsto (x(z) - x'(z), y(z) - y'(z))$ spirals around $(0, 0) \in \mathbb{R}^2$ when $z \to 0^+$.

We are not interested in the study of the linking property for general parameterized curves but for non-oscillatory trajectories of analytic vector fields. For this kind of curves we have the following technical properties (we do not reproduce here the proof which can be seen in [4]).

Proposition 5.5. *Let γ, γ' be two non-oscillatory trajectories of an analytic vector field X in a neighborhood of a point p in a three dimensional manifold M such that $\omega(\gamma) = \omega(\gamma') = p$.*

1. *If $w = (x, y, z)$ are coordinates at p such that $\mid \gamma \mid \not\subset \{z = 0\}$ then w is positive for γ (up to change z by $-z$).*
2. *If w, w' are positive coordinates for both γ and γ', then γ, γ' are w-a.l. if and only if they are w'-a.l. We will just say in this case that γ, γ' are asymptotically linked (written a.l.).*
3. *Assume that γ, γ' are a.l. If H is an analytic surface at p then, as germs, $\mid \gamma \mid$, $\mid \gamma' \mid$ are contained in the same local connected component of $M \setminus H$.*

Notice that if γ, γ' are two different a.l. trajectories of a vector field and $p : \mathbb{R}^3 \to \mathbb{R}^2$ is a linear projection then the number of intersection points of the images $p(\mid \gamma \mid)$, $p(\mid \gamma' \mid)$ is infinite although $p(\mid \gamma \mid) \neq p(\mid \gamma' \mid)$ as germs. Despite of the fact that they are non-oscillatory and hence individually well behaved from the point of view of finiteness properties, linked trajectories exhibit bad relative behavior: the projection of one of them "oscillates" with respect to the projection of the other. In particular, no two a.l. trajectories can be definable in the same o-minimal structure. Surprisingly, as we prove in the next section, each (linked) trajectory of Example 5.3 is individually definable in an o-minimal structure.

5.3 Statement of the Main Result

In the result that follows, we want to study the possibilities for a vector field to have asymptotically linked trajectories. As we will see, this behavior will occur only when those trajectories have flat contact with a formal curve as in Example 5.3. In particular they will have the same sequence of iterated tangents. For this reason, it will be useful to introduce the following terminology.

Let γ_0 be a trajectory of a vector field X in a three dimensional manifold M such that $\omega(\gamma_0)$ is a single point $p \in M$ and such that γ_0 has all iterated tangents. We will speak of the *integral package of* γ_0 as the set $IP(\gamma_0)$ of all trajectories γ of X with $\omega(\gamma) = p$, having all iterated tangents and such that $IT(\gamma) = IT(\gamma_0)$. Although we consider trajectories as parameterized curves, we will identify two elements γ, γ' of $IP(\gamma_0)$ if the germs at p of their images $\mid \gamma \mid$, $\mid \gamma' \mid$ coincide.

Theorem 5.6. *Let $IP(\gamma_0)$ be an integral package consisting on non-oscillatory trajectories of X. Then we have one of the following possibilities:*

1. *Any two distinct elements $\gamma, \gamma' \in PI(\gamma_0)$ are asymptotically linked. (We will speak of a linked package).*
2. *For any two distinct elements γ, γ' in $IP(\gamma_0)$ there exists a bounded open neighborhood U of p in M and a subanalytic continuous map $\beta = \beta_{\gamma,\gamma'} : U \to \mathbb{R}^2$ such that, as germs at p, we have*

$$\beta(|\gamma|) \cap \beta(|\gamma'|) = \emptyset.$$

(We will speak of a separated package).

Trajectories of Example 5.3 in the half-space $\{z > 0\}$ form a linked package. The following result states that, essentially any linked package behaves like this example. It can be viewed as the formal version of Theorem 4.3.

Theorem 5.7. *Suppose that $IP(\gamma_0)$ is a linked package at $p \in M$. There exists a formal curve $\widehat{\Gamma}$ such that $IT(\gamma_0) = IT(\widehat{\Gamma})$, called the formal spiraling axis, which is (analytically) transcendental. Also, for any neighborhood W of p, there exists an open subanalytic set $V \subset W$, positively invariant for X such that*

$$\gamma \in IP(\gamma_0) \Leftrightarrow |\gamma| \cap V \neq \emptyset. \tag{5.6}$$

A formal curve with parametrization $\widehat{\Gamma}(z) = (\widehat{\mathbf{x}}(z), z^r)$ in some coordinates $(\mathbf{x}, z)$ at p is said to be *(analytically) transcendental* if given any convergent power series $f(\mathbf{x}, z) \in \mathbb{R}\{\mathbf{x}, z\}$ we have that $f(\widehat{\mathbf{x}}(z), z^r) \equiv 0$ implies $f \equiv 0$.

In the rest of this section we sketch the proof of Theorems 5.6 and 5.7 at the same time (complete details can be found in [4]). The plan is the following. First, in Sect. 5.4, we explain how to reduce the general case to the case where p is an elementary singularity of X; that is, the linear part $DX(p)$ has at least a non-zero eigenvalue. This uses the Local Uniformization Theorem of vector fields along non-oscillatory trajectories in [5]. Then, in Sect. 5.5 we study the elementary case.

5.4 Local Uniformization and Integral Packages

The Local Uniformization Theorem of vector field along a non-oscillatory trajectory γ_0 asserts that, after a finite number of "suitable" modifications of the type blowing-ups and ramifications, the lifted trajectory $\tilde{\gamma}_0$ accumulates to an elementary singularity $\tilde{p}$ of the strict transform of the vector field. Let us describe here the technical features needed in order to precise this result and to know how to use it for the proof of Theorem 5.6.

Let γ be a non-oscillatory trajectory of an analytic vector field X in a three dimensional analytic manifold M such that $\omega(\gamma) = p$. Similarly to what we have considered in Definition 4.13, a map $\pi : \tilde{M} \to M$ is said to be a *basic γ-admissible morphism* if it is, either the blowing-up at p, or a local blowing-up along a smooth curve Y through p which does not intersect γ, or a q-ramification along a plane through p. Also, similarly to Proposition 4.14, one can prove (see [4]) that, if $\pi : \tilde{M} \to M$ is a basic γ-admissible morphism then the lifting $\tilde{\gamma} = \pi^{-1} \circ \gamma$ satisfies the properties:

1. $\omega(\tilde{\gamma}) = \tilde{p} \in \tilde{M}$.
2. $\tilde{\gamma}$ is non-oscillatory.

3. $|\ \tilde{\gamma}\ |$ is a leave of a 1-dimensional foliation on $\tilde{M}$; hence, up to re-parametrization, $\tilde{\gamma}$ is a trajectory of an analytic vector field $\tilde{X}$ on a neighborhood of $\tilde{p}$ (called the *strict transform of* X *by* π).

These properties permit us to speak about a γ*-admissible morphism* as a finite composition of maps $\pi = \pi_n \circ \cdots \pi_1$ such that π_1 is basic γ-admissible and, for $i = 1, \ldots, n-1$, if γ_i denotes the lifting of γ_{i-1} by π_i with $\gamma_o = \gamma$, then π_{i+1} is basic γ_i-admissible.

The behavior of linked or subanalytically separated trajectories under admissible morphisms is summarized in the following result.

Proposition 5.8. *Let γ, γ' be two non-oscillatory trajectories such that $\omega(\gamma) = \omega(\gamma') = p \in M$ and let $\pi : \tilde{M} \to M$ be a morphism of analytic spaces.*

(i) Assume that γ, γ' are a.l. Then π is γ-admissible iff it is γ'-admissible and, in this case, the liftings $\tilde{\gamma}, \tilde{\gamma}'$ satisfy $\omega(\tilde{\gamma}) = \omega(\tilde{\gamma}') = \tilde{p}$ and they are a.l.
(ii) Reciprocally, if π is γ and γ'-admissible and $\tilde{\gamma}, \tilde{\gamma}'$ are a.l. then γ, γ' are a.l.
(iii) If π is γ and γ'-admissible and their liftings do not accumulate to the same point of $\tilde{M}$ then γ, γ' are subanalytically separated.
(iv) If π is γ and γ'-admissible then γ, γ' are subanalytically separated if and only if their liftings by π are subanalytically separated.

Proof. (See [4] for details). The proof of (iii) and (iv) are easy from the definitions and the fact that the image of a subanalytic subset by an admissible morphism is a subanalytic subset. The proof of (i) and (ii) uses, in big lines, Proposition 5.5 and the following construction for detecting the property of linking: suppose that $w = (x, y, z)$ are positive coordinates at p for both γ and γ' and parameterize $\gamma(z) = (x(z), y(z), z)$; then γ, γ' are a.l. iff γ' intersects infinitely many times the surface

$$S_w(\gamma) = \{(x(z) + t, y(z), z) \,/\, t \geq 0,\ z > 0\}$$

and the sum of the *indices of intersection* of γ' with S_w at such points diverges to $+\infty$ or $-\infty$. □

As a corollary, any pair of non-oscillatory trajectories which are asymptotically linked belong to the same integral package and they remain in the same integral package by admissible morphisms. A linked package is thus preserved by admissible morphisms. On the other hand, if some trajectory is separated from its mates in an integral package by an admissible morphism then it will be a subanalytically separated from them.

With these remarks, in order to prove Theorems 5.6 and 5.7 we can make any number of convenable admissible morphism with the purpose of finding a vector field in a simpler situation. The following important result by Cano, Moussu and Rolin on reduction of singularities of three dimensional analytic vector fields asserts that we can reduce to the case of *elementary singularity*; that is, a singularity for which the linear part of the vector field is non-nilpotent.

Theorem 5.9 (Local Uniformization along non-oscillatory trajectories, [5]). *Let γ_0 be a non-oscillatory trajectory of a vector field in a three dimensional real analytic manifold. Then there exists a γ_0-admissible morphism $\pi : \tilde{M} \to M$ such that the lifting $\tilde{\gamma_0} = \pi^{-1} \circ \gamma_0$ is, up to re-parametrization, a trajectory of a vector field $\tilde{X}$ having an elementary singularity at $\tilde{p} = \omega(\tilde{\gamma_0})$.*

5.5 *The Case of Elementary Singularity*

In this paragraph we sketch a proof of Theorems 5.6 and 5.7 when p is an elementary singularity of X; that is, the linear part $DX(p)$ is a non-nilpotent linear map from T_pM into itself.

We will make use of the classical theory of *invariant manifolds* in dynamical systems. Namely, the so called *stable W^s, center W^c, unstable W^u, center-stable W^{cs} and center-unstable W^{cu} manifold of X at p*, defined as embedded local submanifolds through p, invariant for X and whose tangent space at p is the eigenspace of $DX(p)$ corresponding to eigenvalues with negative, zero, positive, non-positive, and non-negative, respectively. The reader can consult [13] or [11] for the precise statements concerning them and used throughout this paragraph.

Fix for the rest of the paragraph a non-oscillatory trajectory γ_0 of X with $\omega(\gamma_0) = p$ and suppose that any element of the integral package $IP(\gamma_0)$ is a non-oscillatory trajectory. We start with a very general and easy result.

Lemma 5.10. *There exists $\lambda = \lambda(\gamma_0) \in Spec(DX(p))$ such that the tangent of γ_0 is an eigendirection of $DX(p)$ of eigenvalue λ.*

Proof. (The non-oscillatory condition for γ_0 is not necessary, just the existence of tangent at p.) The argument is that the tangent direction of γ_0, considered as a point in the exceptional divisor of the blowing-up at p is a singular point of the strict transform of X. On the other hand, a computation shows that these singularities are only at points corresponding to eigendirections of the linear part. □

We distinguish several cases for $\lambda(\gamma_0)$.

5.5.1 Case $\lambda(\gamma_0) \neq 0$

We show in this case that the integral package is always separated. Moreover, we show that for any $\gamma \in IP(\gamma_0)$ the image $\mid \gamma \mid$ is a *pfaffian set* so all good finiteness properties for the trajectories are true, in particular the subanalytic separation.

We have necessarily $\lambda(\gamma_0) < 0$ (otherwise $\omega(\gamma_0) \neq p$). Then any trajectory of $IP(\gamma_0)$ is tangent to the stable manifold W^s of X at p by Lemma 5.10. The classical theory of invariant manifolds says that, in fact, any such trajectory γ is actually contained in W^s. We recall also that W^s (its germ at p) is unique and analytic. We analyze the different possibilities.

If $\dim W^s = 1$ then $|\ \gamma\ | \subset W^s$ for any $\gamma \in IP(\gamma_0)$; it implies that $IP(\gamma_0)$ reduces to γ_0 and $|\ \gamma_0\ |$ is an analytic half-branch.

If $\dim W^s = 2$ then any $\gamma \in IP(\gamma_0)$ is a trajectory of the planar analytic vector field $X\ |_{W^s}$. Using the projection onto the tangent plane $T_p W^s$, we see that $IP(\gamma_0)$ is a separated integral package. Moreover, since these trajectories have a tangent, they are Rolle leaves and thus pfaffian sets.

If $\dim W^s = 3$ and all eigenvalues are real (and thus negative), we use the Dulac's analytic normal form for X (see for instance [1]): there are analytic coordinates (x, y, z) at p such that X can be written as

$$X = -x\frac{\partial}{\partial x} + f(x, y)\frac{\partial}{\partial y} + g(x, y, z)\frac{\partial}{\partial z}.$$

A trajectory of X is contained in a cylinder $|\ \sigma\ | \times \mathbb{R}$ where σ is a trajectory of the planar vector field $-x\frac{\partial}{\partial x} + f(x, y)\frac{\partial}{\partial y}$ and satisfy an ordinary differential equation over this cylinder. Since $|\ \sigma\ |$ is a Rolle leaf, we deduce that $|\ \gamma\ |$ is a pfaffian set. Thus $IP(\gamma_0)$ is a separated integral package.

Suppose, finally, that $\dim W^s = 3$ and $Spec(DX(p)) = \{\lambda(\gamma_0), a \pm ib\}$ with $a < 0$, $b \neq 0$. After a blowing-up of the point p, the strict transform of X at the singular point p' in the exceptional divisor corresponding to the tangent direction of γ_0 has eigenvalues $\lambda(\gamma_0), a - \lambda(\gamma_0) \pm ib$. By repeating the blowing-up at the iterated tangents of γ_0 we reduce to a situation where $\dim W^s = 1$.

5.5.2 Case $\lambda(\gamma_0) = 0$ and $Spec(DX(p)) = \{0, 0, \mu\}$

Let us see in this case that we obtain always that $IP(\gamma_0)$ is a separated package.

Since $DX(p)$ is non-nilpotent, μ is a non-zero real number. We analyze the two situations:

If $\mu > 0$, by classical arguments in dynamical systems, if we fix a center manifold W^c of X at p, then any γ with $\omega(\gamma) = p$ must be contained in W^c. This center manifold is the graph of a $\mathcal{C}^1$-map $(x, y) \mapsto h(x, y)$ in some (analytic) coordinates (x, y, z) at p. In this case the projection $\mathbb{R}^3 \to \mathbb{R}^2$, $(x, y, z) \mapsto (x, y)$ separates any couple of trajectories accumulating to the point p and the integral package $IP(\gamma_0)$ is separated.

If $\mu < 0$, we first use the following fact (already used in Sect. 4) from the theory of invariant manifolds: given a center manifold W^c of X at p, each trajectory γ with $\omega(\gamma) = p$ has a unique accompanying trajectory σ_γ inside W^c in the sense of (4.3). Then we use the following proposition whose proof is not reproduced here and can be seen in [4].

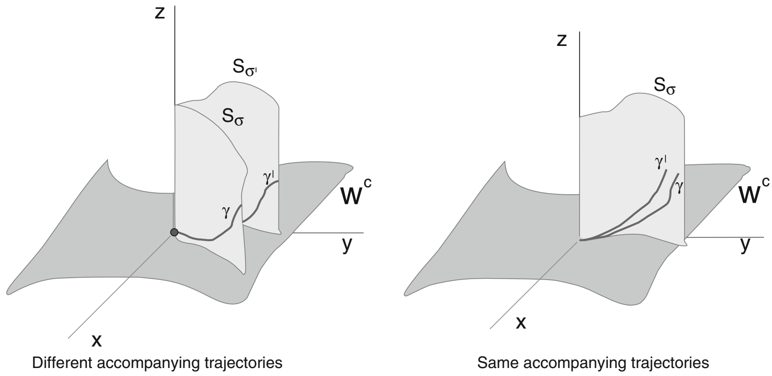


Fig. 22 Accompanying trajectories in the center manifold

Proposition 5.11. *There exists a neighborhood U of p in M such that, if σ is a trajectory inside the center manifold W^c such that $\omega(\sigma) = p$ and σ has the same tangent direction at p as γ_0, say ℓ_0, then*

$$L(\sigma) = \bigcup_{\gamma \in IP(\gamma_0),\, \sigma_\gamma = \sigma} \mid \gamma \mid \cap U$$

is an open set of a C^1-embedded surface S_σ through p. Moreover $\ell_0 \subset T_p S_\sigma$ and S_σ is transversal to W^c at p.

Now take coordinates (x, y, z) at p such that $\ell_0 = T_p(\{x = z = 0\})$, $T_p W^c = T_p(\{z = 0\})$ and $T_p S_\sigma = T_p(\{x = 0\})$ where σ is the accompanying trajectory of γ_0 in W^c. In particular, we have that W^c is a graph of a C^1-function over the (x, y)-plane and S_σ is a graph of a C^1-function over the (y, z)-plane. If γ, γ' are two different trajectories in the integral package $IP(\gamma_0)$ and let σ_γ, $\sigma_{\gamma'}$ be their respective accompanying trajectories in W^c. We have the possibilities (see Fig. 22):

(i) If $\mid \sigma_\gamma \mid \neq \mid \sigma_{\gamma'} \mid$ (as germs at p) then the linear projection $(x, y, z) \mapsto (x, y)$ separates γ, γ'.
(ii) If $\mid \sigma_\gamma \mid = \mid \sigma_{\gamma'} \mid$ then, by Proposition 5.11, $\mid \gamma \mid, \mid \gamma' \mid \subset L(\sigma_\gamma)$ and then the projection $(x, y, z) \mapsto (y, z)$ separates γ, γ'.

5.5.3 Case $\lambda(\gamma_0) = 0$ and $Spec(DX(p)) = \{0, \mu_1, \mu_2\}$ with $\mu_1 \mu_2 \neq 0$

This case is much richer than the preceding ones in the sense that both types of integral packages, either linked or separated, can appear. The following result asserts that, in any case, all such integral packages will have a formal axis (Fig. 23).

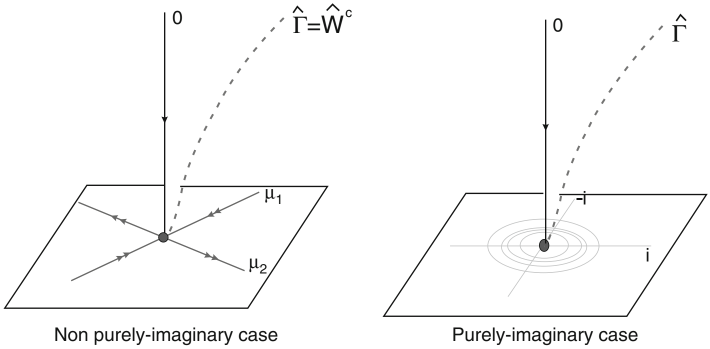


Fig. 23 Cases when $\lambda(\gamma_0) = 0$ and the other eigenvalues are non-zero

Lemma 5.12. *There exists a unique formal curve* $\widehat{\Gamma}$ *at* p *which is invariant for* X *and tangent to the eigenspace of* $DX(p)$ *corresponding to the zero eigenvalue. Moreover, we have*

(i) $\widehat{\Gamma}$ *is not singular,*
(ii) γ_0 *has flat contact with* $\widehat{\Gamma}$ *and*
(iii) $\widehat{\Gamma} \not\subset Sing\, X$.

Proof. If the real parts of μ_1, μ_2 are (both) non zero (*non-purely imaginary case*) then $\widehat{\Gamma}$ is nothing but the *formal center manifold* $\widehat{W^c}$, a formal curve at p which is uniquely determined, invariant for X and not singular (since it is a graph). Property (ii) is easily obtained in this case by repeated blowing-ups and the following observation: after the blowing-up at p, the strict transform X' of X at the singular point p_1 in the exceptional divisor, corresponding to the zero-eigenvalue satisfies $Spec(D_{p_1}X') = Spec(D_pX)$. If $Re(\mu_1) = Re(\mu_2) = 0$ (*purely imaginary case*) then the point p_1 is the only singularity on the exceptional divisor and the situation repeats. We deduce that there is a unique possible sequence of iterated tangents of a trajectory of the vector field X accumulating to p. Since formal curves at p are univocally determined by its sequence of iterated tangents, we obtain (i) and (ii).

In order to prove (iii), suppose otherwise that $\widehat{\Gamma} \subset Sing\, X$. The singular locus $SingX$ is an analytic set that we can assume to be of dimension at most one at the point p. We would have that the formal curve $\widehat{\Gamma}$ is in fact a convergent one and defines an analytic non-singular curve Y at p. In the non-purely imaginary case, Y is the center manifold (unique in this case), composed of singular points of the vector field. We use the so called Center Manifold Theorem [6, 11] which assets that, locally at p, the vector field X is topologically conjugated to a trivial product of a two-dimensional vector field (in fact a linear vector field carrying the stable and unstable dynamics) by the zero vector field in $\mathbb{R}$ (see Fig. 24). We conclude that there are no non-trivial trajectories of X accumulating to p except those contained in the stable manifold W^s. They would be then tangent to a non-zero eigenvalue and we would be in another case. In the purely-imaginary case, after the blowing-up π :

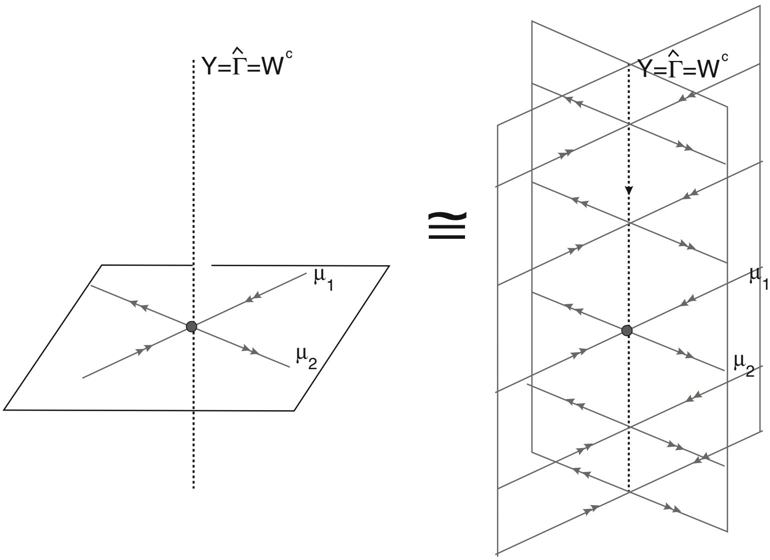


Fig. 24 Dynamics in the non-purely imaginary case if $\widehat{\Gamma} \subset Sing X$

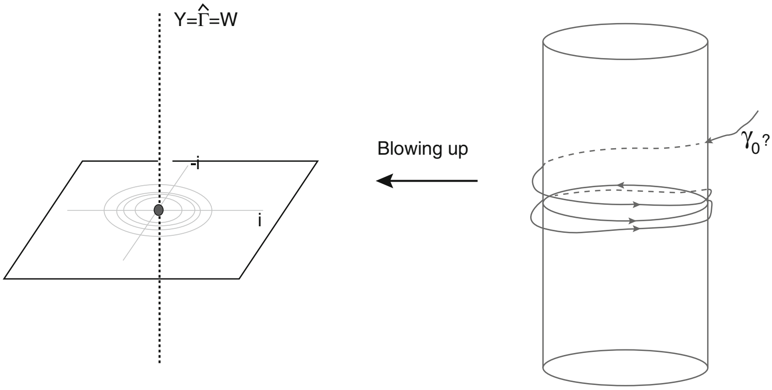


Fig. 25 Dynamics in the purely imaginary case if $\widehat{\Gamma} \subset Sing X$

$\tilde{M} \to M$ with center Y, which is admissible for γ_0, we can see that $C = \pi^{-1}(p)$ is a closed orbit of the transformed vector field. Since the lifting $\gamma_0' = \pi^{-1} \circ \gamma_0$ accumulates somewhere in C we would have that $\omega(\gamma_0) = C$ and thus γ_0 oscillating (see Fig. 25). □

Consider analytic coordinates (x, y, z) at the point p and a formal Puiseux parametrization of $\widehat{\Gamma}$ of the form

$$\widehat{\Gamma} = (\widehat{f}(z), \widehat{g}(z), z), \quad \widehat{f}(z), \widehat{g}(z) \in \mathbb{R}[[z]].$$

We write formally X in the formal coordinates $(\widehat{x} = x - \widehat{f}(z), \widehat{y} = x - \widehat{g}(z), z)$ as

$$X = \widehat{a}(\widehat{x}, \widehat{y}, z)\frac{\partial}{\partial \widehat{x}} + \widehat{b}(\widehat{x}, \widehat{y}, z)\frac{\partial}{\partial \widehat{y}} + \widehat{c}(\widehat{x}, \widehat{y}, z)\frac{\partial}{\partial z}.$$

Then we have

$$\begin{aligned} \widehat{\Gamma} \text{ invariant } &\Leftrightarrow \widehat{a}(0,0,z) \equiv \widehat{b}(0,0,z) \equiv 0, \\ \widehat{\Gamma} \not\subset SingX &\Leftrightarrow \widehat{c}(0,0,z) \not\equiv 0. \end{aligned}$$

Moreover, in our case, if we write $\widehat{c}(0,0,z) = z^{q+1}(c_0 + \cdots)$ with $c_0 \neq 0$, then $q \geq 1$ (since $\lambda(\gamma_0) = 0$) and $c_0 < 0$ (since $\omega(\gamma_0) = p$).

Following similar steps of those of the procedure presented in Sect. 4.5.1, we can reduce by means of γ_0-admissible morphisms, to the following situation (which we can call "formal final situation", see Definition 4.11): the vector field X is written in coordinates (x, y, z) as

$$X = \sum_{j=0}^{q} z^j L_j(x, y) + z^{q+1}Y - z^{q+1}\frac{\partial}{\partial z} \tag{5.7}$$

where

- $q \geq 1$ and $\widehat{\Gamma}$ is tangent to the z-axis,
- $L_j(x, y) = (a_j x + b_j y)\frac{\partial}{\partial x} + (c_j x + d_j y)\frac{\partial}{\partial y}$ is a linear vector field,
- The matrix of $L_0(x, y)$ is invertible,
- $Y = A(x, y, z)\frac{\partial}{\partial x} + B(x, y, z)\frac{\partial}{\partial y}$ is a family of two-dimensional vector fields tangent to the fibers $\{z = cst\}$ with $Y(0,0,0) = 0$ (but not necessarily $A(0,0,z) \equiv B(0,0,z) \equiv 0$, unless $\widehat{\Gamma}$ is an analytic half-branch) and
- Furthermore, either each L_j is a radial vector field $a_k(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y})$ for $0 \leq k \leq q$ or the first L_k, $k \in \{0, \ldots, q\}$, which is not radial has different eigenvalues (i.e., the discriminant $\Delta(L_k)$ is not zero).

Once we are in the final situation and γ_0 is a non-oscillatory trajectory tangent to $\widehat{\Gamma}$ at p, we can determine if the integral package $IP(\gamma_0)$ is a linked or separated package just by looking at the *q-truncated normal lineal part* of the vector field X; that is, the sum $\sum_{j=0}^{q} z^j L_j(x, y)$ in the expression (5.7). The precise statement is the following one (compare with Proposition 4.16).

Proposition 5.13. *Denote by $l = min\{min\{0 \leq j \leq q / tr(L_j) \neq 0\}, q+1\}$. Then*

1. *$IP(\gamma_0)$ is linked package iff the four following conditions hold*

$$(AL) \;\; l < q, \;\; tr(L_l) < 0, \;\; k \leq q, \;\; \Delta(L_k) < 0.$$

2. *If one of the conditions in (AL) fails then any two distinct trajectories $\gamma, \gamma' \in IP(\gamma_0)$ are separated by a linear projection onto a linear plane through the origin in the coordinates (x, y, z). In particular the integral package $IP(\gamma_0)$ is a separated package.*

Details of the proof of this proposition can be seen in [4]. It follows essentially the same lines as the proof of twister dynamics in the pre-final situation, Proposition 4.12. This time, some of the arguments of the theory of differential equations applied there to the cylindric coordinates $\rho(z), \theta(z)$ of a trajectory must be adapted to $\tilde{\rho}(z), \tilde{\theta}(z)$ with $u(z) + iv(z) = \sqrt{(\tilde{\rho}(z))}e^{i\tilde{\theta}(z)}$, where $u(z) = x(z) - x'(z), v(z) = y(z) - y'(z)$ are the differences between two trajectories in the integral package as in (5.5).

We can consider Proposition 5.13 as a refinement of Theorem 5.6 in the case where X is in formal final situation. In this situation, Theorem 5.7 is also not difficult to prove. In fact, the existence of a formal axis $\widehat{\Gamma}$ is already given by the hypothesis about the eigenvalues of the linear part (Lemma 5.12) while the existence of a system of fundamental neighborhoods of p with the property (5.6) will be a consequence of the proof of Proposition 5.13 (these neighborhoods play the role of the twister domains of the form D_δ in Proposition 4.12). Finally, the argument that proves transcendence of the formal axis of a linked package is the following. If $f(\widehat{\Gamma}(z)) \equiv 0$ for some non-zero analytic germ $f(x, y, z)$ then, since $\widehat{\Gamma}$ is invariant and not convergent, there must be an irreducible component H_1 of $H = \{f = 0\}$ of dimension 2 which is contained in the set of (generalized) tangencies, $Z = \{df(X) = 0\}$, between Z and X (otherwise Z would be an analytic curve that "contains" the formal branch $\widehat{\Gamma}$). But then, one can show that two different trajectories γ, γ' issued of points in $H_1 \cap V$, where V is an open set satisfying (5.6), can not be asymptotically linked since they are contained in an analytic surface.

6 Quasi-analytic Trajectories and O-minimal Structures

In this last section we present a result about o-minimality of certain non-oscillatory trajectories of analytic vector fields. As was announced in the introduction, the property of non-oscillation alone is not sufficient in order to obtain the stronger finiteness property of o-minimality and several extra assumptions are needed about the vector field or the trajectory. Also, we will see in this section (see Example 6.8) that each of the linked trajectories of Example 5.3 is definable in an o-minimal structure, although their "pathological" relative behavior. This example provides an uncountable family of o-minimal structures mutually incompatible; that is, there is no common o-minimal expansion of two of them (another example of such a family is given in [23]).

Complete and detailed proofs of the statements presented here can be found in the paper [24].

Consider an r-dimensional system of real analytic ODE's of the form

$$x^{q+1}\frac{d\mathbf{y}}{dx} = A(x, \mathbf{y}),\ \mathbf{y} = (y_1, \ldots, y_r) \in \mathbb{R}^r, \tag{6.1}$$

where q, called the *Poincaré rank of the system*, is greater or equal to 1 and A is a real analytic map in a neighborhood of the origin of $\mathbb{R}^{1+r}$ with values in $\mathbb{R}^r$ and such that $A(0,\mathbf{0}) = \mathbf{0}$. We want to study solutions of (6.1) near the origin; that is, analytic parameterized curves

$$H = (H_1, \dots, H_r) :]0, \varepsilon] \to \mathbb{R}^r \tag{6.2}$$

such that $\frac{dH}{dx} = A(x, H(x))$ for any $0 < x < \varepsilon$ and $\lim_{x\to 0^+} H(x) = \mathbf{0}$.

Equivalently, if $A = (A_1, \dots, A_r)$ are the components of the map A, we can consider the real analytic vector field

$$X = A_1(x, \mathbf{y})\frac{\partial}{\partial y_1} + \cdots + A_r(x, \mathbf{y})\frac{\partial}{\partial y_r} + x^{q+1}\frac{\partial}{\partial x} \tag{6.3}$$

defined in a neighborhood of the origin in $\mathbb{R}^{r+1}$. The hyperplane $\{x = 0\}$ is invariant for X and, since X is transversal to the horizontal planes $\{x = c\}_{c\neq 0}$, every trajectory of X outside $\{x = 0\}$ can be parameterized by x. We have then that solutions H of the form (6.2) correspond exactly to trajectories γ of (6.3) on $\{x > 0\}$ such that $\omega(\gamma) = \mathbf{0} \in \mathbb{R}^{r+1}$ (our general setting in these notes). This correspondence between solutions and trajectories permits to apply, with the obvious meaning, the concepts of iterated tangents, oscillation, flat contact and asymptotic linking to solutions of (6.1).

We are interested in the problem of whether the components H_i of a solution H like in (6.2) are definable in an o-minimal structure over the real field. The answer is positive provided some assumptions on the system (6.1) and/or the solution H. Of course, a first assumption is that H is non-oscillatory (at least with respect to the semialgebraic sets of $\mathbb{R}^{r+1}$). However, our assumptions will be stated for the system of differential equations (6.1) itself and in such a way that they will imply, incidentally, the non-oscillation condition for the involved solutions.

6.1 *Statement of the Main Theorem*

6.1.1 Formal Solutions and Asymptotic Solutions

Our first assumption on the system of differential equations is a generic condition concerning the spectrum $Spec(A_0) = \{\lambda_1, \dots, \lambda_r\}$ of the linear part $A_0 = \partial A/\partial \mathbf{y}(0, \mathbf{0}) \in \mathcal{M}_{r\times r}(\mathbb{R})$. We will assume

$$(DA) \quad \begin{cases} \lambda_i \neq 0, & \text{for all } i; \\ arg(\lambda_i) \neq arg(\lambda_j) \bmod 2\pi\mathbb{Z}, & \text{for } i \neq j. \end{cases}$$

There is a classical result in the theory of analytic systems of differential equations (see [35]) that asserts that, under the condition (DA), the system (6.1) has a unique formal power series solution

$$\widehat{H}(x) = (\widehat{H}_1(x), \ldots, \widehat{H}_r(x)) \in \mathbb{R}[[x]]^r, \ \widehat{H}(0) = \mathbf{0}. \tag{6.4}$$

(We can say, equivalently, that the formal curve $(x, \widehat{H}(x))$ is a formal solution of the associated vector field (6.3).)

We will study o-minimal properties of solutions H like in (6.2) having the formal solution as an *asymptotic expansion when* $x \to 0^+$. By definition this means the following: Write $\widehat{H}(x) = \sum_{n=1}^{\infty} x^n h_n$ where $h_n \in \mathbb{R}^r$; for every natural number $N \geq 0$, there is $K_N > 0$ such that,

$$\left\| H(x) - \sum_{n=1}^{N} x^n h_n \right\| \leq K_N x^{N+1}, \quad \text{for } x \in]0, \varepsilon]. \tag{6.5}$$

We will simply write $H \sim \widehat{H}$ *as* $x \to 0^+$.

6.1.2 Summability and Stokes Phenomena of the Formal Solution

In order to state our second assumption, we need first some relatively elementary facts about summability of divergent series. In our particular case where the linear part A_0 of the system of equations is non singular we only need the following result from Ramis (we use the standard notation $S(\alpha, \beta; \rho)$, $\alpha, \beta \in \mathbb{R}$, $\rho > 0$ for a sector $\{z \in \mathbb{C} \,/\, |z| < \rho, \alpha < \arg(z) < \beta\}$ *opening* $\beta - \alpha$ and of radius ρ).

Theorem 6.1. *[20] Assume that the system (6.1) of ODEs satisfy (DA) and consider the formal solution $\widehat{H}(z) \in \mathbb{C}[[z]]^r$ as a vector of complex formal power series. Then $\widehat{H}(z)$ is q-summable. More precisely, for any $\theta \in \mathbb{R}$, except (possibly) for those satisfying*

$$q\,\theta \equiv \arg(\lambda_j) \bmod 2\pi\mathbb{Z}$$

for some eigenvalue λ_j of A_0, and for any open sector $S_\theta = S(\theta - \frac{\pi}{2q} - \delta, \theta + \frac{\pi}{2q} + \delta; \rho)$ of opening slightly greater than π/q (with δ sufficiently small) there exists a unique holomorphic function $\tilde{H}^\theta = (\tilde{H}_1^\theta, \ldots, \tilde{H}_r^\theta) : S_\theta \to \mathbb{C}^r$ such that:

1. *$\tilde{H}^\theta$ satisfies the (complexified) system of equations (6.1) and*
2. *For any i, $\tilde{H}_i^\theta$ has the series $\widehat{H}_i(z)$ as the asymptotic expansion on the sector S_θ. This asymptotic expansion is, furthermore, Gevrey of order $\kappa = 1/q$.*

The function $\tilde{H}^\theta$ is called the q-sum of the series $\widehat{H}(z)$ along the direction θ.

By definition (see [21]), a bounded holomorphic function $h : S \to \mathbb{C}$ on a sector $S = S(\alpha, \beta; \rho)$ has a *Gevrey asymptotic expansion of order* κ (or a *κ-Gevrey* *asymptotic expansion*) with right hand side $\widehat{h}(z) = \sum_{k\geq 0} a_k z^k \in \mathbb{C}[[z]]$, and we write $h(z) \sim_\kappa \widehat{h}(z)$, if for any $\eta > 0$ there are constants $K, A > 0$ such that

$$\left| h(z) - \sum_{k=0}^{N-1} a_k z^k \right| \leq K\, A^N \Gamma(N\kappa + 1)|z|^N,$$

for any $N \in \mathbb{N}$ and any $z \in S(\alpha + \eta, \beta - \eta; \rho - \eta)$. We remark that $h(z) \sim_\kappa 0 + 0z + \cdots$ if and only if $h(z)$ is *exponentially small of order* $1/\kappa$, i.e., for any $\eta > 0$, there is a positive constant a such that $|h(z)\exp(a|z|^{-1/\kappa})|$ is bounded on $S(\alpha + \eta, \beta - \eta; \rho - \eta)$. This follows easily from Stirling's Formula by choosing N as the integer closest to $|z|^{-1/\kappa}$.

The excluded angles in Theorem 6.1 are called the *singular directions* (also called the *anti-Stokes directions*) of $\widehat{H}(z)$. By the hypothesis (DA), there are exactly rq different singular directions modulo $2\pi\mathbb{Z}$. Denote them by $\theta_l \in [0, 2\pi)$, $l = 0, \ldots, rq - 1$. Up to reordering of the eigenvalues λ_j, we can choose the indices such that $0 \leq \theta_0 < \cdots < \theta_{rq-1}$ and $q\,\theta_l \equiv \arg(\lambda_j) \bmod 2\pi\mathbb{Z}$ if and only if $l + 1 \equiv j \bmod r$ (we will say informally that the singular direction θ_l *corresponds* to the eigenvalue λ_j).

Two q-sums $\tilde{H}^\theta$, $\tilde{H}^{\theta'}$ of $\widehat{H}(z)$ as in Theorem 6.1 coincide in the intersection of their domains if there is no singular direction between θ and θ'. We obtain, by analytic continuation, holomorphic functions

$$\tilde{H}^l : \tilde{S}^l = S(\theta_l - \frac{\pi}{2q} + \delta, \theta_{l+1} + \frac{\pi}{2q} - \delta; \rho) \to \mathbb{C}^r, \quad l = 0, \ldots, rq - 1$$

where $\delta, \rho > 0$ are sufficiently small and where we put $\theta_{rq} = \theta_0 + 2\pi$.

In general, the functions $\tilde{H}^l$ cannot be continued analytically or change their Gevrey asymptotic behavior beyond the rays of angles $\theta_l - \frac{\pi}{2q}, \theta_{l+1} + \frac{\pi}{2q}$, $l = 0, \ldots, rq-1$, called the *Stokes directions* of $\widehat{H}(z)$. The so called *Stokes phenomenon* of the series $\widehat{H}(z)$ is the description of the behavior of the difference $\Delta^l(z) = \tilde{H}^{l+1}(z) - \tilde{H}^l(z)$ of two such consecutive functions. This difference is defined in the intersection $\tilde{S}^{l,l+1} = \tilde{S}^l \cap \tilde{S}^{l+1} = S(\theta_{l+1} - \frac{\pi}{2q} + \delta, \theta_{l+1} + \frac{\pi}{2q} - \delta; \rho)$, a sector of opening slightly smaller than π/q.

Remark 6.2. It is quite easy to show, using the Riemann removable singularity theorem, that each component of $\widehat{H}(z)$ is a convergent power series at $z = 0$ if and only if $\Delta_l \equiv 0$ for every l.

The map $\Delta^l = (\Delta^l_1, \cdots, \Delta^l_r) : \tilde{S}^{l,l+1} \to \mathbb{C}^r$ satisfies the system of linear ODEs (in the complex domain)

$$z^{q+1}\frac{d\mathbf{y}}{dz} = B_l(z)\mathbf{y}, \tag{6.6}$$

where $B_l(z)$ is the matrix of holomorphic functions on $\tilde{S}^{l,l+1}$ defined by $B_l(z) = \overline{B}(z, \tilde{H}^l(z), \tilde{H}^{l+1})$ with $\overline{B}(z, \mathbf{y}_1, \mathbf{y}_2)$ analytic and satisfying

$$A(z, \mathbf{y}_2) - A(z, \mathbf{y}_1) = \overline{B}(z, \mathbf{y}_1, \mathbf{y}_2)(\mathbf{y}_2 - \mathbf{y}_1).$$

We see that all matrices $B_l(z)$ for $l = 0, \dots, rq - 1$ have the same $\frac{1}{q}$-Gevrey asymptotic expansion on the sector $\tilde{S}^{l,l+1}$ with right hand side $\widehat{B}(z) = \overline{B}(z, \widehat{H}(z), \widehat{H}(z)) = \partial A/\partial \mathbf{y}\,(z, \widehat{H}(z))$. The initial term $\widehat{B}(0) = A_0$ has, in particular, distinct eigenvalues. We can apply a classical result in the theory of linear ordinary differential equations (see [35]) asserting that there exists a *fundamental matrix solution* of (6.6) of the form

$$Y_l(z) = G_l(z)\, \exp(Q(z))\, z^J, \tag{6.7}$$

where:

(i) $G_l(z)$ is a matrix of holomorphic functions on $\tilde{S}^{l,l+1}$.
(ii) There exists a matrix $\widehat{G}(z)$ of formal power series such that, for every l, $G_l(z)$ has $\frac{1}{q}$-Gevrey asymptotic expansion with right hand side $\widehat{G}(z)$ in $\tilde{S}^{l,l+1}$. Moreover $\det(\widehat{G}(0)) \neq 0$;
(iii) $J = \mathrm{diag}(\alpha_1, \dots, \alpha_r)$ is a constant diagonal matrix and
(iv) $Q(z) = \mathrm{diag}(Q_1(z), \dots, Q_r(z))$ is a diagonal matrix where

$$Q_j(z) = -\frac{\lambda_j}{q} z^{-q} + \cdots \in \mathbb{C}[z^{-1}]$$

are polynomials in the variable z^{-1} of degree q without constant term.

The singular directions (respectively the Stokes directions) are precisely the rays where the initial term of some of the polynomials $Q_j(z)$ is real negative (respectively purely imaginary). Denote the columns of the matrix $G_l(z)$ by $G_{lj}(z)$, $j = 1, \dots, r$. Then, the particular solution Δ^l of (6.6) can be written as

$$\Delta_l(z) = \sum_j c_{lj} G_{lj}(z) e^{Q_j(z)} z^{\alpha_j}, \tag{6.8}$$

where $c_l = (c_{l1}, \dots, c_{lr}) \in \mathbb{C}^r$ is some constant vector.

Lemma 6.3. *Given* $l \in \{0, \dots, rq - 1\}$ *and* $\mu = \mu(l) \in \{1, \dots, r\}$ *defined by* $l + 1 \equiv \mu \bmod r$, *we have that* $c_{lj} = 0$ *for every* $j \neq \mu$.

Proof. This is a classical result. It is due to the fact that for every $j \neq \mu$, the function $\exp(Q_j(z))$ is exponentially large on some ray in $\tilde{S}^{l,l+1}$ while Δ_l remains bounded on that sector. □

Thus, in the expression (6.8), only the j-th coefficient c_{lj}, where λ_j is the eigenvalue corresponding to θ_l, can have a non-trivial contribution in order to compute Δ^l. This coefficient merits a definition.

Definition 6.4. The coefficient $\gamma_l = c_{l\mu(l)}$ is called the *Stokes multiplier* of the solution $\widehat{H}(z)$ associated to the singular direction θ_{l+1}.

6.1.3 Transcendence, Quasi-analyticity and Statement of the Main Theorem

It seems reasonable that, as long as $\widehat{H}$ is transcendental (far from being a convergent series), the solutions H which are asymptotic to $\widehat{H}$ "separates" from any given analytic object and will have then finiteness properties with respect to analytic functions. As a confirmation of this impression, we propose the following proposition, to be used later. It is stated for formal power series in general, irrespectively of the fact that they are or not solutions of a system of ODEs. (Recall the definition of (analytically) transcendental power series after the statement of Theorem 5.7.)

Proposition 6.5. *Let $\widehat{H}(x)$ be an r-uple of real formal power series as in (6.4) and let $H :]0, \varepsilon] \to \mathbb{R}^r$ be an analytic parameterized curve such that $H \sim \widehat{H}$ as $x \to 0^+$. If $\widehat{H}$ is a transcendental power series then H is non-oscillatory.*

Proof. Given $f(x, \mathbf{y}) \in \mathbb{R}\{x, \mathbf{y}\}$ non zero with $f(0, \mathbf{0}) = 0$ then, by hypothesis of transcendence, $f(x, \widehat{H}(x)) = x^N \widehat{u}(x)$ with a series $\widehat{u}(x)$ such that $\widehat{u}(0) = u_0 \neq 0$. From the definition of asymptotic expansion we obtain that $f(x, H(x)) = x^N u(x)$ where $u :]0, \varepsilon] \to \mathbb{R}$ is bounded and $\lim_{x \to 0^+} u(x) = u_0$. This implies that the function $x \mapsto f(x, H(x))$ for $x > 0$ has no zeroes accumulating to $x = 0$. This gives, by definition, the property of non-oscillation for H. □

Proposition 6.5 means that the germs of functions of the form $x \mapsto f(x, H(x))$, $x > 0$, where f is analytic at the origin, are completely determined by its formal Taylor expansion at $x = 0$; in other words, it is a result of *quasi-analyticity* (QA for short) for that class of germs. In the statement of our main result, we impose a condition on the series $\widehat{H}(x)$ that makes it to be transcendent enough (see Definition 6.15 bellow) so that we can infer a QA property for a wider class of germs constructed from analytic functions and the components of a solution H of (6.1) satisfying $H \sim \widehat{H}(x)$ as $x \to 0^+$. This property on quasi-analyticity imply, by itself, o-minimality of the components of H.

Transcendence of a series implies in particular divergence. Thus, according to Remark 6.2, we need some Stokes multiplier to be non zero. The imposed condition bellow is intended to achieve transcendence as much as possible by requiring a non-zero Stokes multiplier for any eigenvalue. More precisely, we will denote by (SD) (for *singularity at any direction* $\arg(\lambda_j)$) the following condition:

(SD) For any $\mu \in \{1, \ldots, r\}$ there exists $l \in \{0, \ldots, qr-1\}$ with $l + 1 \equiv \mu \bmod r$ such that $\gamma_l \neq 0$.

We can now state the main result of this section.

Theorem 6.6. *Consider a system of ordinary differential equations (6.1) with real analytic coefficients and assume that the linear part A_0 satisfies the condition (DA).*

Assume, moreover, that the formal power series solution $\widehat{H}(z)$ *satisfies the condition (SD) about its Stokes multipliers. Then, given a solution* $H\ :]0,\varepsilon] \to \mathbb{R}^r$ *such that* $H \sim \widehat{H}(z)$ *as* $x \to 0^+$*, the components* H_j*,* $j = 1,\ldots,r$ *of* H *are simultaneously definable in an o-minimal expansion of the real field* $(\mathbb{R},+,\cdot,0,1)$*. More precisely, the structure* $\mathbb{R}_{an,H}$*, expansion of the real field by the restricted analytic functions and the components of* H *is o-minimal, model-complete and polynomially bounded.*

A restricted analytic function is a real function in $\mathbb{R}^n$ for some n which takes value 0 outside the compact cube $I^n = [-1,1]^n$ and coincides in I^n with a convergent real power series with radius of convergence greater than 1 in each variable. The structure $\mathbb{R}_{\text{an}}$ generated over the real field by the restricted analytic functions is o-minimal and model-complete and the family of its definable sets in each $\mathbb{R}^n$ coincides with the family of sets which are subanalytic in $(\mathbb{P}^1_{\mathbb{R}})^n$. All these properties can be seen in [33].

We analyze here several applications of Theorem 6.6.

Example 6.7 (Plane Pfaffian curves). A first consequence is an o-minimality and model completeness result for certain planar trajectories of vector fields or, equivalently, planar pfaffian curves. Consider a real analytic vector field

$$X = a(x,y)\frac{\partial}{\partial x} + b(x,y)\frac{\partial}{\partial y}$$

in some neighborhood of $0 \in \mathbb{R}^2$ with $a(0,0) = b(0,0) = 0$ and let γ be a trajectory of X accumulating to the origin and having a well defined tangent there. By virtue of Proposition 2.3, and up to changing the variables x, y, the image $\mid \gamma \mid$ is the graph of a non-oscillatory solution $H\ :]0,\epsilon] \to \mathbb{R}$ tending to 0 as $x \to 0$ of the differential equation $b(x,y)\frac{dy}{dx} = -a(x,y)$. The o-minimality of the structure $\mathbb{R}_{\text{an,H}}$ is a particular case of the results proved in [14, 36] since this graph is a Rolle's leave of an analytic foliation. Model completeness of the structure $\mathbb{R}_{\text{an,H}}$ is, however, unknown in general. Using Theorem 6.6 we can prove this property in the case that γ has not flat contact with an analytic half-branch at the origin. The proof of this claim is as follows. Analytic changes of coordinates or blow-ups are obviously inessential for the question of model completeness. Therefore, by a classical theorem on the reduction of singularities [26], we can reduce to one of the following situations:

1. H is analytic also at $x = 0$. In this case, $\mathbb{R}_{an,H} = \mathbb{R}_{\text{an}}$ and the result is well known ([7, 33]).
2. $H(x) = x^\lambda$ for some irrational $\lambda > 0$. In this case, the model completeness is proved in [17].
3. H is a solution of a *saddle-node equation*

$$x^{q+1}\frac{dy}{dx} = y + A_1(x,y), \quad q \geq 1,$$

where A_1 is analytic at the origin and $A_1(0,0) = \frac{\partial A_1}{\partial y}(0,0) = 0$. In this case, H is asymptotic to the unique formal power series $\widehat{H}(x)$ solution. Since we have supposed that γ has not flat contact with and analytic half-branch, $\widehat{H}(x)$ is divergent. We have that the hypothesis in Theorem 6.6 are satisfied (there is a single eigenvalue and some Stokes multiplier of the formal solution $\widehat{H}(x)$ must be non-zero by Remark 6.2). The structure $\mathbb{R}_{\mathrm{an,H}}$ is thus model complete (and polynomially bounded).

Example 6.8 (Two dimensional systems with non-real eigenvalues). Consider a system (6.1) with $r = 2$ such that the linear part $A_0 = \partial A/\partial \mathbf{y}(0)$ has two non real conjugate eigenvalues $\lambda_1, \lambda_2 = \overline{\lambda_1}$ (in particular, it satisfies condition (DA)). Let $\widehat{H}(x) \in \mathbb{R}[[x]]^2$ be its formal power solution. The singular directions $\theta_{l,j}$ of $\widehat{H}(x)$ satisfy $\theta_{l,1} = -\theta_{k,2}$ if $l \equiv -k \mod q$ and, since $\widehat{H}(x)$ has real coefficients, the corresponding Stokes coefficients $c_{l,1}$, $c_{k,2}$ are complex conjugate. Thus, condition (SD) is equivalent to the existence of a non zero Stokes coefficient for some singular direction. By Remark 6.2, this is equivalent to the divergence of the series $\widehat{H}(x)$ (of at least one of its components). In this situation, Theorem 6.6 applies and, for any solution $H :]0,\varepsilon] \to \mathbb{R}^2$ with $H(x) \sim \widehat{H}(x)$ as $\mathbb{R}^+ \ni x \to 0$, the structure $\mathbb{R}_{\mathrm{an,H}}$ is o-minimal and model complete. A concrete example is the system of ODEs (5.3) with asymptotically linked solutions: the linear part has eigenvalues $1 \pm i$ and, as we have seen, its formal power series solution is not convergent. Thus, any solution is definable in an o-minimal structure over the real field. However, since all these solutions form a linked package, no two different solutions can be definable in the same o-minimal structure.

As we have commented in the introduction, the example of the system (5.3) contributes to the belief that non-oscillatory trajectories are more complicated than Rolle's leafs, from the point of view of finiteness properties: while any bounded Rolle's leave is definable in a common o-minimal structure, the *pfaffian closure* $\mathbb{R}_{\mathrm{an,Pfaff}}$ of $\mathbb{R}_{\mathrm{an}}$, we cannot say the same thing for the non-oscillatory trajectories. (We recall the result in [28] on the construction of the o-minimal pfaffian closure of any o-minimal structure over the real field). More than that, it is clear that at most one of the structures $\mathbb{R}_{\mathrm{an,H}}$ for H a solution of (5.3) could be a reduct of $\mathbb{R}_{\mathrm{an,Pfaff}}$. We claim that it is actually the case for none of these structures.

Proposition 6.9. *Let $H : x \to H(x) = (H_1(x), H_2(x))$ be any given solution of the system (5.3) defined for x in some interval $(0,\varepsilon]$. Then H is not definable in $\mathbb{R}_{an,Pfaff}$.*

Proof. Assume, on the contrary, that for some solution H the structure $\mathbb{R}_{an,H}$ is a reduct of $\mathbb{R}_{\mathrm{an,Pfaff}}$. Then, for any other solution G, the pfaffian closure $\mathbb{R}_{\mathrm{an,G,Pfaff}}$ (which is of course an extension of $\mathbb{R}_{\mathrm{an,Pfaff}}$) would be a common o-minimal extension of $\mathbb{R}_{\mathrm{an,H}}$ and $\mathbb{R}_{\mathrm{an,G}}$, which is impossible. □

The proof of Theorem 6.6 is organized in several steps, each of them having its own interest by itself. In the next paragraphs we sketch the main ideas involved. Complete details can be found in [24].

6.2 Step 1: Quasi-analyticity and O-minimality

Consider a system (6.1) of ODEs with real analytic coefficients and assume that it has a formal power series solution

$$\widehat{H}(x) = (\widehat{H}_1(x), \ldots, \widehat{H}_r(x)) \in \mathbb{R}[[x]].$$

Fix a solution $H :]0, \varepsilon] \to \mathbb{R}^r$ such that $H \sim \widehat{H}(x)$ as $x \to 0^+$.

In general, the property (6.5) of having a real asymptotic expansion is not preserved for the respective derivatives of a given function H and of a power series $\widehat{H}(x)$. However, in our situation, since H is a solution of a system of ODEs, we can easily obtain that $d^n H/dx^n \sim \widehat{H}^{(n)}(x)$ as $x \to 0^+$ for any n. Using this result and up to the ramification $x \mapsto H(x^2)$, we can suppose that H is defined and of class $\mathcal{C}^\infty$ in the closed interval $[-\varepsilon, \varepsilon]$ and that its formal Taylor expansion at $x = 0$ is precisely $\widehat{H}(x)$.

Let $\mathcal{A}_H = \{\mathcal{A}_m\}_{m \geq 0}$ be the smallest collection of germs of $\mathcal{C}^\infty$ functions at the origin of the euclidean spaces $\mathbb{R}^m$, for any m, satisfying:

- A1 $\mathbb{R}\{x_1, \ldots, x_m\} \subset \mathcal{A}_m$ for any m and the germ of the components H_i of H at $x = 0$ belong to $\mathcal{A}_1$.
- A2 If $f \in \mathcal{A}_m$ and $g_1, \ldots, g_m \in \mathcal{A}_l$ with $g_j(\mathbf{0}) = 0$ for any j, then $f(g_1, \ldots, g_m) \in \mathcal{A}_l$.
- A3 If $f \in \mathcal{A}_m$ and $f(0, x_2, \ldots, x_m) \equiv 0$ then there exists $g \in \mathcal{A}_m$ such that $x_1 g = f$.
- A4 If $f \in \mathcal{A}_m$ with $f(\mathbf{0}) = 0$ and $\partial f/\partial x_m(\mathbf{0}) \neq 0$ then the solution $h(x_1, \ldots, x_{m-1})$ of the implicit equation $\{f(x_1, \ldots, x_{m-1}, h) = 0, h(\mathbf{0}) = 0\}$ belongs to $\mathcal{A}_{m-1}$.

If $f \in \mathcal{A}_m$, we denote by

$$\widehat{f} = \sum_{\alpha \in \mathbb{N}^m} \frac{1}{\alpha!} \frac{\partial^\alpha f}{\partial x^\alpha}(\mathbf{0}) X^\alpha \in \mathbb{R}[[X]], \; X = (X_1, \ldots, X_m)$$

its formal Taylor expansion at the origin.

An essential key for the proof of Theorem 6.6 is the following result of Rolin-Speissegger-Wilkie that allows to obtain new o-minimal structures from quasi-analytic classes.[4]

Theorem 6.10 ([23]). *If $\mathcal{A}_H$ is quasi-analytic; that is, if the following property holds*

$$(QA) \quad \forall f \in \mathcal{A}_m, \; \widehat{f} \equiv 0 \Rightarrow f \equiv 0,$$

then the structure $\mathbb{R}_{an,H}$ is o-minimal, model-complete and polynomially bounded.

[4] This result is discussed in [22].

Thus our goal now is to prove (QA) for our class of functions $\mathcal{A}_H$. The following proposition permits to reduce considerably this property.

Proposition 6.11. *The class $\mathcal{A}_H$ is quasi-analytic if property (QA) holds for the family $\mathcal{A}_1$ of one-dimensional germs.*

Proof. Let $f \in \mathcal{A}_m$ such that $\widehat{f} \in \mathbb{R}[[X]]$ vanishes identically. We first claim that there exists a representative F of f defined in some neighborhood U of the origin $\mathbf{0} \in \mathbb{R}^m$ satisfying the property that for any $a \in U$, the germ F_a of F at a is equal to $g(x-a)$ for some $g \in \mathcal{A}_m$. In order to prove this claim, consider the class $\tilde{\mathcal{A}}_H$ of germs at the origin satisfying this condition; we can check that this class satisfy conditions A1–A4 and thus $\mathcal{A}_H \subset \tilde{\mathcal{A}}_H$. Now, assume that $U = B(0,\varepsilon)$ for some $\varepsilon > 0$. If we restrict F to a line $\ell_z : t \mapsto tz$ with $z \in \mathbb{S}^{m-1}$, the germ $(F \circ \ell_z)_t$ of $F \circ \ell_z$ at the point tz belong to $\mathcal{A}_1$ (after translation). It has zero Taylor expansion at $t = 0$ since $\widehat{f} \equiv 0$ and thus $(F \circ \ell_z)_0 \equiv 0$ by hypothesis. By usual topological arguments we conclude that the germ $(F \circ \ell_z)_t$ is identically zero for every t. So $F \circ \ell_z \equiv 0$ and, since $z \in \mathbb{S}^{m-1}$ is arbitrary, we conclude that $f \equiv 0$. □

6.3 Step 2: Reduction to Simple Functions

The class $\mathcal{A}_1$, although consisting on one-dimensional germs, can be a priori very complicated to handle: it has composites of transcendental functions, for instance of the form $H_j(H_k(x))$ where H_j, H_k are components of the vector solution H. We describe a simpler subclass which will play an important role in the sequel.

We will use the notation

$$T_k\phi(x) = (\phi(x) - J_k\phi(x))/x^k$$

for ϕ a $\mathcal{C}^\infty$ function or a formal power series in a single variable x, where $J_k\phi(x) \in \mathbb{R}[x]$ denotes its k-jet at $x = 0$.

Definition 6.12. Let $H = (H_1, \dots, H_r) : [-\varepsilon, \varepsilon] \to \mathbb{R}^r$ be a $\mathcal{C}^\infty$ function. A germ $\varphi(x)$ of a $\mathcal{C}^\infty$ function at $0 \in \mathbb{R}$ will be called a *simple function* (relatively to H) if there exists $n \geq 0$, an analytic function $f \in \mathbb{R}\{x, z_{11}, \dots, z_{rn}\}$, polynomials $P_j(x)$ with $P_j(0) = 0$ for $j = 1, \dots, n$ and an integer $k \geq 0$ such that

$$\varphi(x) = f(x, T_kH(P_1(x)), \dots, T_kH(P_n(x))). \tag{6.9}$$

The family of simple functions is an algebra denoted by $\mathcal{S}_H$.

Our problem of quasi-analyticity reduces considerably using the following result.

Theorem 6.13. *Let H $:]0,\varepsilon] \to \mathbb{R}^r$ be a solution of a system of ODEs (6.1) with real analytic coefficients and assume that H has an asymptotic expansion $H \sim \widehat{H}(x)$ as $x \to 0^+$. If the class of simple functions $\mathcal{S}_H$ has the property (QA) then $\mathcal{S}_H = \mathcal{A}_1$.*

In this statement, we do not assume any condition about the linear part A_0 of the system or about the Stokes phenomena of the formal solution $\widehat{H}(x)$. The only existence of the formal solution $\widehat{H}(x)$ and the asymptotic expansion of the actual solution H permits to construct the class of simple functions $\mathcal{S}_H$. So, Theorem 6.13 could be applied to other situations where we do not have a priori the hypothesis of Theorem 6.6. For instance, if the formal series solution $\widehat{H}(x)$ is *multisummable* in the positive real direction (see [16]) then the restriction $H\ :]0,\varepsilon] \to \mathbb{R}^r$ of its multisum will have the required properties so that the associated class of simple functions $\mathcal{S}_H$ is quasi-analytic. (We remark that the o-minimality of the structure $\mathbb{R}_{an,H}$ in this case is already proved in [32]; however, model-completeness can not easily deduced from the proofs in that paper).

The proof of Theorem 6.13 is quite technical (see [24] for details). One main ingredient is the following quite general result in the theory of ordinary differential equations.

Proposition 6.14. *Let $H\ :]0,\varepsilon] \to \mathbb{R}^r$ be a solution of the system (6.1) such that* $\lim_{x\to 0^+} H(x) = \mathbf{0}$. *Assume that $A(0,\mathbf{0}) = \mathbf{0}$. For any $L > 0$ there exists a neighborhood V of $(0,\mathbf{0}) \in \mathbb{R}^{1+r}$, $\delta_L > 0$ and an analytic map $B : [-L, L] \times V \to \mathbb{R}^r$ such that*

$$H(x + x^{q+1}z) = B(z, x, H(x)), \quad |\, z \,|\leq L,\ 0 < x < \delta_L. \tag{6.10}$$

Proof. Let V_1 be a neighborhood of the origin of $\mathbb{R}^{1+r}$ for which the map

$$\tilde{A} : (-L-1, L+1) \times V_1 \to \mathbb{R}^r,\ (z, (x, \mathbf{w})) \mapsto A(x + x^{q+1}z, \mathbf{w})$$

is well defined and take $\delta_1 > 0$ such that $(x, z) \mapsto 1 + x^q z$ does not vanish for $0 <\mid x \mid< \delta_1, \mid z \mid\leq L + 1$. Consider the (non-singular) system of ODEs on the variable z parameterized by x

$$(E_x) \quad \frac{d\mathbf{w}}{dz} = (1 + x^q z)^{-1} \tilde{A}(x, z, \mathbf{w}).$$

The zero map is a solution of (E_0). Using the theorem of analytic dependence of solutions of differential equations on parameters and initial values, we can see that there exist a smaller neighborhood $V \subset V_1$ such that, given $(x_0, \mathbf{w}_0) \in V$, the solution $\phi_{(x_0,\mathbf{w}_0)}$ of (E_{x_0}) with initial condition $\phi(0) = \mathbf{w}_0$ exists and is analytic on the interval $[-L, L]$. Moreover, the map

$$B : [-L, L] \times V \to \mathbb{R}^r,\ (z, (x_0, \mathbf{w}_0)) \to \phi_{(x_0,\mathbf{w}_0)}(z)$$

is analytic. Choose finally $\delta_L > 0$ such that $(x, H(x)) \in V$ for $0 < x \leq \delta_L$. If we fix such an x, the map $z \mapsto B(z, x, H(x))$ is the solution of (E_x) with initial condition $H(x)$. This solution, for z small, can also be expressed as $z \mapsto H(x + x^{q+1}z)$. We obtain the (6.10) by unicity of solutions. □

6.4 Step 3: Strong Analytic Transcendence

Consider a system (6.1) of ODEs with real analytic coefficients and assume that it has a formal solution $\widehat{H}(x) = (\widehat{H}_1(x), \ldots, \widehat{H}_r(x)) \in \mathbb{R}[[x]]^r$.

Definition 6.15. We say that $\widehat{H}(x)$ is *strongly analytically transcendental* if it satisfies the following condition:

(SAT) If $k \geq 0, n \geq 0$, an analytic function $f \in \mathbb{R}\{x, z_{11}, \ldots, z_{rn}\}$ with $f(0) = 0$ and distinct polynomials $P_1(x), \ldots, P_n(x)$ with

$$\deg P_l < (q+1)\text{ord}\, P_l \quad \text{and} \quad P_l^{(\text{ord}\, P_l)}(0) > 0 \tag{6.11}$$

are given, then one has $f(x, \{T_k \widehat{H}_j\,(P_l(x))\}_{j,l}) \equiv 0 \implies f \equiv 0$.

(Recall the notation $T_k\phi\,(x) = (\phi(x) - J_k\phi\,(x))/x^k$ where $J_k\phi\,(x) \in \mathbb{R}[x]$ denotes the k-jet at $x = 0$).

With the aim of simplifying the exposition, a real polynomial $P_l(x) \in \mathbb{R}[x]$ satisfying the properties in (6.11) will be called a *q-short positive polynomial*. The first condition about the bound on the degree is justified by Proposition 6.14. The second condition guarantees that P_l takes positive values for small positive x and will play an important role below.

The (SAT) condition is considerably stronger than the condition of transcendence proposed in Proposition 6.5. We can see easily that it implies quasi-analyticity of the class $\mathcal{S}_H$ of simple functions associated to a solution $H\ :]0,\varepsilon] \to \mathbb{R}^r$ with $H \sim \widehat{H}(x)$ as $x \to 0^+$.

Summarizing, by virtue of Theorem 6.10, Proposition 6.11 and Theorem 6.13, we can state the following

Theorem 6.16. *Let $\widehat{H}(x) \in \mathbb{R}[[x]]^r$ be a formal power solution of a system (6.1) and suppose that it holds the (SAT) property. Then, for any solution $H\ :]0,\varepsilon] \to \mathbb{R}^r$ with $H \sim \widehat{H}(x)$ as $x \to 0^+$, the expansion $\mathbb{R}_{an,H}$ is o-minimal, model-complete and polynomially bounded.*

Finally, to complete the proof of Theorem 6.6 we need to prove the following result.

Theorem 6.17. *Assume that conditions (DA) and (SD) hold for the system (6.1). Then the (unique) formal power solution $\widehat{H}(x)$ has the (SAT) property.*

Before entering in the ideas of th proof of Theorem 6.17, let us discuss here the pertinence of hypothesis (DA) and (SD) with some examples.

Example 6.18. Let $\widehat{E}(x) \in \mathbb{R}[[x]]$ be the Euler series as in (5.1), solution of the Euler's equation $x^2 dy/dx = y - x$. Let $E : x \mapsto E(x)$, for $x > 0$, be a particular solution of this equation and put

$$H(x) = \left(E(x) + \exp\left(-\frac{1}{x}\right), E(2x)\right).$$

It is a solution of the system

$$\begin{cases} x^2 \dfrac{dy_1}{dx} = y_1 - x \\ x^2 \dfrac{dy_1}{dx} = \dfrac{y_2}{2} - x \end{cases}$$

which does not satisfy (DA). The asymptotic expansion of H at $x = 0$ is $\widehat{H}(x) = (\widehat{E}(x), \widehat{E}(2x))$. The Euler series has the (SAT) property (since it has the (SD) condition, see Example 6.7). However, it is evident from construction that the (vector) series $\widehat{H}(x)$ has not the (SAT) property. Thus condition (DA) in Theorem 6.17 cannot be completely removed. Notice also that $\widehat{H}(x)$ is divergent (both components are divergent series) and, moreover, it is analytically transcendental in the sense of what we have defined in Proposition 6.5; thus (SAT) property is strictly stronger than just the transcendental property.

Example 6.19. Consider the example of the system (5.3), better written in our current notation as

$$\begin{cases} x^2 \dfrac{dy_1}{dx} = y_1 + y_2 - x \\ x^2 \dfrac{dy_2}{dx} = y_2 - y_1 \end{cases}$$

We have already seen (see Example 6.8) that the formal power series solution $\widehat{H}(x) \in \mathbb{R}[[x]]^2$ satisfies the condition (SD) on the Stokes phenomena and thus, by Theorem 6.17, it has the (SAT) property. Let $H, G :]0, \varepsilon] \to \mathbb{R}^2$ be two different solutions with $\lim_{x\to 0^+} H(x) = \lim_{x\to 0^+} G(x) = \mathbf{0}$. The map

$$H^* :]0, \varepsilon] \to \mathbb{R}^4, \quad H^*(x) = (H(x), G(2x))$$

is a solution of a system of four ODEs whose linear part has eigenvalues $1 \pm i$, $\frac{1}{2} \pm \frac{i}{2}$. It has an asymptotic expansion as $x \to 0^+$ equal to

$$\widehat{H^*}(x) = (\widehat{H}(x), \widehat{H}(2x)).$$

By the (SAT) property for $\widehat{H}$, we deduce that $\widehat{H^*}(x)$ is transcendental and thus, by Proposition 6.5, that H^* is a non-oscillatory solution. However, since H and G are asymptotically linked, the components of H^* can not all be simultaneously definable in an o-minimal expansion over the real field.

In terms of vector fields, this example shows that, in dimension greater or equal to 5, there are non-oscillatory trajectories of analytic vector fields which do not generate an o-minimal structure over the real field. Such an example cannot exist in dimension two since non-oscillatory trajectories in the plane are pfaffian sets. It is an open question to know whether these kind of example may exist for vector fields in dimension three or four.

6.5 Step 4: Proof of the (SAT) Property

Let us sketch in this paragraph the proof of Theorem 6.17. In order to simplify the exposition we will consider a particular case where all the main ideas involved already appear.

Consider a system of real analytic ODEs as in (6.1)

$$x^{q+1}\frac{d\mathbf{y}}{dx} = A(x,\mathbf{y}),$$

where $\mathbf{y} = (y_1, y_2)$ (that is $r = 2$) and

$$A_0 = \frac{\partial A}{\partial \mathbf{y}}(0,\mathbf{0}) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

It satisfies the condition (DA) on the eigenvalues of the linear part. Let $\widehat{H}(x) = (\widehat{H}_1(x), \widehat{H}_2(x)) \in \mathbb{R}[[x]]^2$ be the formal power series solution.

The general discussion in Sect. 6.1.2 gives for this particular situation the following description of the Stokes phenomena of $\widehat{H}(z)$, considered as a complex power series. The set of singular directions are denoted as

$$\{\theta_l^1, \theta_l^2\}_{0\leq l\leq q-1}, \quad \theta_l^1 = \frac{2l\pi}{q}, \theta_l^2 = \frac{(2l+1)\pi}{q}.$$

For each θ_l^e, $e = 1,2$, $l = 0,\ldots,q-1$, the difference $\Delta_l^e = (\Delta_{l1}^e, \Delta_{l2}^e)$ between two q-sums of $\widehat{H}(z)$ corresponding to directions slightly above and slightly below of θ_l^e is holomorphic in some open sector bisected by the direction θ_l^e and of opening slightly smaller than $\pi/2q$, as shown in Fig. 26. There exist complex polynomials of degree q

$$Q_1(z) = -\frac{1}{q}z^q + \cdots, \quad Q_2(z) = \frac{1}{q}z^q + \cdots$$

without constant terms, complex numbers $\alpha_1, \alpha_2 \in \mathbb{C}$ and a 2×2 matrix $\widehat{G}(z)$ whose entries are complex power series and for which $\widehat{G}(0)$ is non-singular such that, for each $(e,l) \in \{1,2\} \times \{0,\ldots,q-1\}$,

$$\Delta_l^e(z) = c_l^e \exp(Q_e(1/z)) z^{\alpha_e} G_l^e(z), \tag{6.12}$$

where $c_l^e \in \mathbb{C}$ and $G_l^e(z)$ is a two-dimensional vector of holomorphic functions in the corresponding sector having there the e-th column of $\widehat{G}(z)$ as Gevrey asymptotic expansion of order $1/q$.

We assume that the formal power series $\widehat{H}(z)$ has the (SD) condition. In our particular case, it means that there exist $l(1), l(2) \in \{0,\ldots,q-1\}$ such that $c_{l(1)}^1$, $c_{l(2)}^2$ are both non-zero. Let us see that it has the (SAT) property.

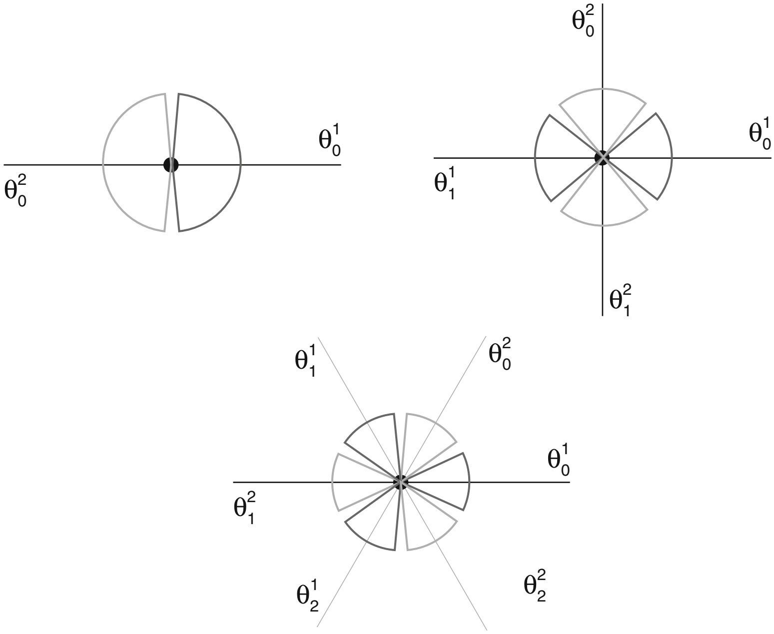


Fig. 26 Singular directions for different values of the Poincaré rank q

Suppose, by contradiction, that there exists $k \geq 0$, a non-zero analytic germ f and distinct q-short positive polynomials $P_1, \dots, P_n$ as in Definition 6.15 such that the power series

$$f(z, T_k \widehat{H}\,(P_l(x)), \dots, T_k \widehat{H}\,(P_n(x))) \equiv 0. \tag{6.13}$$

First, taking into account that $T_k \widehat{H}(z)$ is the unique formal solution of a system of equations analogous to (6.1), we can suppose without loss of generality that $k = 0$.

Denote by $(z, \{z_{ij}\}_{1 \leq i \leq 2, 1 \leq j \leq n})$ the variables involved in f (the index i for the two components of $\widehat{H}(z)$ and the index j for the different polynomials $P_j(x)$) and let $\Lambda \subset \{1, 2\} \times \{1, \dots, n\}$ be the subset of indices (i, j) for which f depends effectively on the variable z_{ij}. Assume that Λ has minimal cardinality among all the non-zero germs f satisfying (6.13).

Lemma 6.20. *Fix* $(k, l) \in \Delta$ *and denote* $\widehat{K}(z) = (z, \widehat{H}\,(P_l(z)), \dots, \widehat{H}\,(P_n(z)))$. *We can suppose that* $\partial f / \partial z_{kl}(\widehat{K}(z)) \not\equiv 0$.

Proof. First, there exists $s \geq 1$ such that $\partial^s f / \partial z_{kl}^s(\widehat{K}(z)) \not\equiv 0$: otherwise the power series

$$f(z, z_{kl}, \{\widehat{H}_i(P_j(z))\}_{(i,j) \neq (k,l)}) = \sum_{m \geq 0} \frac{1}{m!} \frac{\partial^m f}{\partial z_{kl}^m}(\widehat{K}(z))(z_{kl} - \widehat{H}_k(P_l(z)))^m$$

in $\mathbb{R}[[z, z_{kl}]]$ would be identically zero. Thus any of its coefficients as a series in $\mathbb{R}[[z]][[z_{kl}]]$ is zero. On the other hand, these coefficients arise from plugging $(z, \{\widehat{H}_i(P_j(z))\}_{(i,j)\neq(k,l)})$ in the coefficients f_m of f as a convergent power series in the variable z_{kl}; that is,

$$f(z, \{z_{ij}\}_{(i,j)\in\Lambda}) = \sum_{m\geq 0} f_m(z, \{z_{ij}\}_{(i,j)\neq(k,l)}) z_{kl}^m.$$

Since each f_m is a convergent power series, by minimality of the cardinal of Λ, we would have $f_m \equiv 0$ for every m and thus $f \equiv 0$. Now, consider $g = \partial^{s-1} f / \partial z_{kl}^{s-1}$ where s is the minimum satisfying the condition above. The germ g is non-zero and satisfies the required properties of the lemma. □

Denote $\nu_j = \operatorname{ord} P_j \geq 1$ for $j = 1, \ldots, n$ and $\nu = \min\{\nu_1, \ldots, \nu_n\}$. We have (see [20]) that $\widehat{H}(P_j(z)) \in \mathbb{C}[[z]]^2$ is $\nu_j q$-summable whose (possibly) singular directions are the ν_j-th roots of the singular directions of $\widehat{H}(z)$. Let

$$\Omega = \{\varphi \in [0, 2\pi[\,/\, \nu_j\varphi \equiv \theta_l^i \bmod 2\pi\mathbb{Z} \text{ for some } i, j, l\}$$

be the set of representatives of the singular directions of all $\widehat{H}(P_j(z))$ modulo 2π and order its elements as

$$0 \leq \varphi_1 < \varphi_1 < \cdots < \varphi_N < 2\pi.$$

For any $\varphi \notin \Omega$, we put $F^{\varphi}(z) = f(z, \{\tilde{H}_i^{\nu_j\varphi}(P_j(z))\}_{(i,j)\in\Lambda})$, a holomorphic function defined in some open sector V_φ bisected by φ and of opening $\pi/\overline{\nu}q$ where $\overline{\nu} = max\{\nu_1, \ldots, \nu_n\}$. (Recall that $\tilde{H}^\theta = (\tilde{H}_1^\theta, \tilde{H}_2)$ denotes the q-sum of the formal power series $\widehat{H}(z)$ along the non singular direction θ). For any $(i, j, k) \in \{1, 2\} \times \{1, 2, \ldots, n\} \times \{1, 2, \ldots, N\}$, denote

$$h_{ijk}(z) = \tilde{H}_i^{\nu_j\varphi_k^+}(P_j(z)) - \tilde{H}_i^{\nu_j\varphi_k^-}(P_j(z))$$

where $\varphi_k^-, \varphi_k^+ \notin \Omega$ are close to φ_k and $\varphi_k^- < \varphi_k < \varphi_k^+$. Using the Taylor expansion of the function f, we can write, for each $k = 1, \ldots, N$,

$$\Sigma_k(z) = F^{\varphi_k^+}(z) - F^{\varphi_k^-}(z) = \sum_{ij} D_{ijk}(z) h_{ijk}(z), \tag{6.14}$$

where D_{ijk} is a holomorphic function in some sector V_k bisected by φ_k and of opening slightly smaller than $\pi/\overline{\nu}q$ where it satisfies the Gevrey asymptotics

$$D_{ijk} \sim_{\frac{1}{\overline{\nu}q}} \frac{\partial f}{\partial z_{ij}}(\widehat{K}(z)). \tag{6.15}$$

The contradiction (and thus the end of the proof) will be found after proving the following two incompatible results:

(I) If $f(\widehat{K}(z)) \equiv 0$ then every F^{φ} is exponentially small of order strictly greater than $q\nu$ in V_{φ}.
(II) If $f \not\equiv 0$ then there exists some k_0 such that Σ_{k_0} is exponentially small of order exactly $q\nu$ along at least some ray in the sector V_{k_0}.

The result (I) is an avatar of the Relative Watson's Lemma, a quite general result in the theory of *multisummable series*. We do not enter here in the proof (see [24] and the references there).

Let us focus finally in the proof of (II).

For any $(j,k) \in \{1,\dots,n\} \times \{1,\dots,N\}$, there exists $e = e(j,k) \in \{1,2\}$ and $l = l(j,k)$ with $\nu_j \varphi_k = \theta^{e(j,k)}_{l(j,k)}$ such that

$$h_{ijk}(x) = \Delta^e_{l,i}(P_j(z)) = c^e_l \exp(Q_e(1/P_j(z)))z^{\alpha_e} G^e_{l,i}(z). \tag{6.16}$$

Up to reordering the polynomials P_j, we can suppose that

$$\nu = \nu_1 = \cdots = \nu_{n_1} < \nu_{n_1+1} \leq \cdots \leq \nu_n.$$

Using the positiveness of the first coefficient of the polynomials P_j, we can write, using (6.16), the function Σ_k in (6.14) as

$$\Sigma_k(z) = \sum_{i=1,2;\, 1\leq j\leq n_1} D_{ijk}(z)\Delta^e_{l,i}(P_j(z)) + O\left(\exp\left(-\frac{K}{\mid z \mid^{q\nu+\varepsilon}}\right)\right),$$

where $K, \varepsilon > 0$ and ε is sufficiently small. Denote by $T_k(z)$ the first summand in the above expression. It remains to prove that, for some k_0, T_{k_0} is exponentially small of order exactly $q\nu$ along some ray.

Notice that, by definition,

$$l(1,k) = \cdots = l(n_1,k) = l(k), \quad e(1,k) = \cdots = e(n_1,k) = e(k).$$

Write, using (6.12),

$$T_k(z) = c^{e(k)}_{l(k)} \sum_{j=1}^{n_1} E_{jk}(z)(P_j(z))^{\alpha_{e(k)}} \exp(Q_{e(k)}(1/P_j(z)), \tag{6.17}$$

where

$$E_{jk}(z) = D_{1jk}(z)G^{e(k)}_{l(k),1}(P_j(z)) + D_{2jk}(z)G^{e(k)}_{l(k),2}(P_j(z)). \tag{6.18}$$

Claim. We can suppose that there exists $e_0 \in \{1,2\}$ such that, for any k with $e(k) = e_0$, $E_{1k}(z) \sim \widehat{E_{1k}}(z) \not\equiv 0$ as a formal asymptotic expansion.

Proof of the claim.- First notice that

$$\widehat{E_{jk}}(z) = (\widehat{D_{1jk}}(z) \;\; \widehat{D_{2jk}}(z))\, \widehat{G}^{e(k)}(z),$$

where $\widehat{G}^e(z)$ is the e-th column of the matrix $\widehat{G}(z)$ defined at the beginning of this paragraph. On the other hand, by Lemma 6.20 and (6.15), we can take f such that

$$\widehat{D_{1jk}}(z) = \frac{\partial f}{\partial z_{11}}(\widehat{K}(z)) \not\equiv 0.$$

Now use the fact that $\widehat{G}(0)$ is non-singular in order to assure that, for some e_0, the series $(\widehat{D_{1jk}}(z) \;\; \widehat{D_{2jk}}(z))\, \widehat{G}^{e_0}(z)$ is not identically zero.

Once we have chosen $e_0 \in \{1, 2\}$, we use the hypothesis (SD) so that we can chose $l_0 \in \{0, \ldots, q-1\}$ such that $c_{l_0}^{e_0} \neq 0$. Take, finally $k_0 \in \{1, \ldots, N\}$ such that $e_0 = e(k_0)$ and $l_0 = l(k_0)$ (that is, k_0 is defined by $\nu\varphi_{k_0} \equiv \theta_{l_0}^{e_0} \bmod 2\pi\mathbb{Z}$).

We use the expression (6.17) for $k = k_0$. We know that at least E_{1k_0} has a non zero asymptotic expansion. We can suppose that this is so for E_{jk_0} if $j \in \{1, \ldots, n_1\}$ (otherwise, if $E_{j_0k_0} \sim \widehat{0}$ for such a j_0, since this expansion is Gevrey of order $1/q\nu$, this would imply that $E_{j_0k_0}$ is exponentially small of order at least $q\nu$ and, together with the exponential term $\exp(Q_{j_0}(1/P_{j_0}(z)))$, will produce a summand in (6.17) negligible in front of those for which $E_{jk_0} \not\sim \widehat{0}$).

The fact that the P_j are different q-short polynomials permits to assure that, if $j_1 \neq j_2$, then

$$Q_{e_0}(1/P_{j_1}(z)) - Q_{e_0}(1/P_{j_2}(z))$$

is a meromorphic function with non zero principal part. (The proof is an easy exercise in the algebra of polynomials).

Now, up to have chosen conveniently the first index for the polynomials $P_1, \ldots, P_{n_1}$ (and thus have adapted the claim above to that index), we can suppose that, for $j > 1$,

$$\beta_j(z) = \frac{\exp\left(Q_{e_0}(1/P_j(z))\right)}{\exp\left(Q_{e_0}(1/P_1(z))\right)} = \exp\left(Q_{e_0}(1/P_j(z)) - Q_{e_0}(1/P_1(z))\right)$$

is exponentially small along the ray of angle φ_{k_0} as $|\, z \,| \to 0^+$. We have finally, along the ray $\arg z = \varphi_{k_0}$,

$$\begin{aligned} \frac{T_{k_0}(z)}{\exp\left(Q_{e_0}(1/P_1(z))\right)} &= c_{l_0}^{e_0} E_{1,k_0}(z) P_1(z)^{\alpha_{e_0}} + \sum_{j=2}^{n_1} E_{j,k_0}(z) P_j(z)^{\alpha_{e_0}} \beta_j(z) \\ &= z^{M_0}(a_0 + o(1)) \end{aligned}$$

where M_0 some positive constant and $a_0 \in \mathbb{C} \setminus \{0\}$. We deduce that T_{k_0}, along that ray, is exponentially small of order exactly $q\nu$, the exponential order of the function $\exp(Q_{e_0}(1/P_1(z)))$.

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Pfaffian Sets and O-minimality

Patrick Speissegger

Abstract Recent developments in the theory of pfaffian sets are presented from a model-theoretic point of view. In particular, the current state of affairs for Van den Dries's model-completeness conjecture is discussed in some detail. I prove the o-minimality of the pfaffian closure of an o-minimal structure, and I extend a weak model completeness result, originally proved as Theorem 5.1 in (J.-M. Lion and P. Speissegger, Duke Math J 103:215–231, 2000), to certain reducts of the pfaffian closure, such as the reduct generated by a single pfaffian chain.

Keywords Pfaffian functions • O-minimal structures • Model completeness

Mathematics Subject Classification (2010): Primary 14P15, 58A17, Secondary 03C64

Introduction

These notes are the result of a month-long course taught during the Thematic Program on o-minimal structures and Real Analytic Geometry, held at the Fields Institute from January to June of 2009. They present an introduction to pfaffian sets and functions with a model-theoretic perspective.

Introduced by Khovanskii [9] in the late 1970s, pfaffian sets are of interest to many areas of mathematics; see for instance the Conclusion of [9], Moussu and Roche [21] and Karpinski and Macintyre [7]. Van den Dries conjectured in the early

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1980s that pfaffian functions generate a model complete and o-minimal structure over the real field, and he proposed proving this via a model-completeness argument based on Khovanskii's description of pfaffian sets. The first breakthrough in this direction came with Wilkie's model-completeness result for the real exponential function [25]. While the significance of this work goes beyond that of pfaffian functions in general, it also contains a proof of Van den Dries's conjecture for *restricted pfaffian functions*, that is, pfaffian functions restricted to compact boxes. A geometric proof of the latter was given later by Gabrielov [3].

The second breakthrough came again from Wilkie in his proof of o-minimality for unrestricted pfaffian functions [26], followed shortly by a similar result of Lion and Rolin [14] for pfaffian functions over $\mathbb{R}_{an}$. The ideas in these two papers were then adapted to working over any o-minimal expansion of the real field in [22]. The first goal of these notes is to prove the main theorem of the latter, stated as Theorem A below.

The proofs of Wilkie's second theorem, of Lion and Rolin's theorem and of Theorem A do not quite follow the strategy proposed by Van den Dries. They are based instead on ideas of Charbonnel; see [26] for a detailed account. These ideas amount to allowing a certain limit operation (represented here by the *pfaffian limits* in Sect. 5) in the description of the definable sets, in addition to the usual first-order operations, and establishing a version of model completeness using this additional operation on definable sets. However, this limit operation does not set any limits on the quantifier complexity for definable sets in terms of the usual first-order operations, so the model-completeness aspect of Van den Dries's conjecture remains open.

Gabrielov reiterated the model completeness conjecture in [3], and he established [4] a variant of Wilkie's second theorem that produces a bound on the quantifier complexity of sets definable by pfaffian functions. Around the same time, Lion and I tinkered with Nash blowings-up (what we called "blowing up in jet space") to try to rewrite sets obtained by the above limit operation in terms of only the first-order operations. In [15], we were only partially successful and obtained a weak model-completeness result, stated as Theorem B below. The proof of this result is the second goal of these notes. This weak model completeness turns out to be useful for two geometric applications found in [15, 19], implied by Theorem B and Proposition 1.14 below.

More recently, Lion and I were able to use Nash blowings-up to prove model completeness for *nested pfaffian sets*, see [18]. Nested pfaffian sets were already introduced in [9] and represent a natural language for pfaffian sets. While the proof of this model-completeness result goes beyond the scope of these notes, the approach taken there allowed me to streamline the proofs of Theorems A and B given here. (Note, for instance, the short proof of Proposition 6.5 below, which represented the key step in all versions of Theorem A published before the year 2000. Its short proof given here is due to a more careful treatment of Hausdorff limits in Sects. 4 and 5, which allows me to "hide" the use of Baire category theory in a convenient notion of dimension.) As a result, these notes also provide an introduction to some of the ideas used in [18] in a somewhat less involved setting.

Moreover, this approach is what allows me to establish Theorem B for certain reducts of $\mathcal{P}(\mathcal{R})$, such as the expansion of $\mathcal{R}$ by a single pfaffian chain over $\mathcal{R}$.

These notes are organized as follows: in Sect. 1, I introduce the necessary terminology, state the main results and discuss some related issues, such as the existence of pfaffian functions and various versions of Van den Dries's model-completeness conjecture. Proposition 1.14, being independent of the other material in these notes, is proved in Sect. 2. In Sect. 3, I return to the pfaffian setting to develop Khovanskii theory over an o-minimal structure, and Sects. 4 and 5 discuss pfaffian limits in the same setting. Theorem A is proved in Sect. 6, together with a characterization (Corollary 6.12) of the sets definable in reducts of $\mathcal{R}_1$. In preparation for Theorem B, the effect of Nash blowings-up on pfaffian limits is studied in Sect. 7 under specific hypotheses. The key point in the proof of Theorem B is that we can definably reduce to these specific hypotheses, modulo a certain subset of the given pfaffian limit. This subset is shown in Sect. 8 to be obtained from pfaffian limits of smaller dimension than the given pfaffian limit; this is achieved via a fiber-cutting lemma for pfaffian limits (Proposition 8.2). Theorem B is proved in Sect. 9.

I thank Gareth O. Jones for many helpful suggestions and discussions during the writing of these notes.

Conventions. Throughout these notes, all functions, maps, manifolds, etc. are assumed to be of class C^1 unless indicated otherwise.

Let $\mathcal{R}$ denote a fixed, but arbitrary, expansion of the real field $\overline{\mathbb{R}} := (\mathbb{R}, <, 0, 1, +, \cdot)$. Unless indicated otherwise, we use "definable" to mean "definable in $\mathcal{R}$ with parameters from $\mathbb{R}$". Following model-theoretic convention, if $\mathcal{A} = \{a_1, \ldots, a_k\}$ is a finite collection of real constants, real-valued functions on Euclidean space and subsets of Euclidean space, we denote by $(\mathcal{R}, a_1, \ldots, a_k)$ the expansion of $\mathcal{R}$ by all elements of $\mathcal{A}$ in the sense of model theory.

For $x \in \mathbb{R}^n$, we put $|x| := \sup\{|x_1|, \ldots, |x_n|\}$ and $\|x\| := \sqrt{x_1^2 + \cdots + x_n^2}$, and for $r > 0$ we set $B(x, r) := \{ y \in \mathbb{R}^n : |y - x| < r \}$.

1 Pfaffian Functions and Rolle Leaves

I first introduce the notions of pfaffian functions and Rolle leaves over $\mathcal{R}$, then state the theorems proved in these notes and some related open questions. I finish this section with a brief discussion concerning the existence of Rolle leaves over $\mathcal{R}$.

Definition 1.1. A tuple of functions $f = (f_1, \ldots, f_k) : \mathbb{R}^n \longrightarrow \mathbb{R}^k$ is a **pfaffian chain of length k over $\mathcal{R}$** if there are definable functions $P_{l,i} : \mathbb{R}^{n+l} \longrightarrow \mathbb{R}$ such that

$$\frac{\partial f_l}{\partial x_i}(x) = P_{l,i}(x, f_1(x), \ldots, f_l(x)) \quad \text{for } l = 1, \ldots, k,\ i = 1, \ldots, n \text{ and } x \in \mathbb{R}^n. \tag{1.1}$$

A function $g : \mathbb{R}^n \longrightarrow \mathbb{R}$ is **pfaffian over** $\mathcal{R}$ if (g) is a pfaffian chain of length 1 over $\mathcal{R}$.

Examples 1.2. (1) The function log is not pfaffian over $\overline{\mathbb{R}}$ (because log is not total, i.e., defined on all of $\mathbb{R}$), but the function $x \mapsto \log\left(1 + x^2\right)$ is pfaffian over $\overline{\mathbb{R}}$. Similarly, the function arctan is pfaffian over $\overline{\mathbb{R}}$.

(2) Every antiderivative of a definable function from $\mathbb{R}$ to $\mathbb{R}$ is pfaffian over $\mathcal{R}$, but not necessarily definable: $\log\left(1 + x^2\right)$ is not definable in $\overline{\mathbb{R}}$ by quantifier elimination and analytic continuation.

(3) Let $f = (f_1, \dots, f_k) : \mathbb{R}^n \longrightarrow \mathbb{R}^k$ be a pfaffian chain over $\mathcal{R}$. Then the function f_1 is pfaffian over $\mathcal{R}$, and for $i = 2, \dots, k$, the function f_i is pfaffian over $(\mathcal{R}, f_1, \dots, f_{i-1})$.

A more general way to define pfaffian functions comes from differential geometry. For $k, l \in \mathbb{N}$, identify the real vector space $M_{k,l}(\mathbb{R})$ of all real-valued $(k \times l)$-matrices with $\mathbb{R}^{kl}$ via the map $A = (a_{ij}) \mapsto z_A = (z_1, \dots, z_{kl})$ defined by $a_{ij} = z_{k(i-1)+j}$. As usual, I write $M_n(\mathbb{R})$ in place of $M_{n,n}(\mathbb{R})$.

Let $l \leq n$. I denote by G_n^l the **Grassmannian** of all l-dimensional vector subspaces of $\mathbb{R}^n$. This G_n^l is a real algebraic variety with a natural embedding into the vector space $M_n(\mathbb{R})$: each l-dimensional vector space E is identified with the unique matrix A_E (with respect to the standard basis of $\mathbb{R}^n$) corresponding to the orthogonal projection on the orthogonal complement of E (see Sect. 3.4.2 of [1]); in particular, $E = \ker(A_E)$. I identify G_n^l with its image in $\mathbb{R}^{n^2}$ under the above map. Note that the sets $G_n^0, \dots, G_n^n$ are the connected components of $G_n := \bigcup_{p=0}^n G_n^p$ and, under the above identification, G_n is definable in $\mathcal{R}$.

Definition 1.3. Let M be a C^2-submanifold of $\mathbb{R}^n$. A map $d : M \longrightarrow G_n$ is a **distribution on** M if $d(x) \subseteq T_x M$ for all $x \in M$. A distribution d on M is an l**-distribution** if $d(M) \subseteq G_n^l$; in this situation, I say that d **has dimension** and set $\dim d := l$.

For example, the distribution g_M on M defined by $g_M(x) := T_x M$, called the **Gauss map** of M, has dimension $\dim M$.

Definitions 1.4. Let M be a C^2-submanifold of $\mathbb{R}^n$ and d be an l-distribution on M.

(1) An manifold V immersed in M of dimension l is an **integral manifold** of d if $T_x V = d(x)$ for $x \in V$. An integral manifold V of d is **closed** if V is a closed subset of M.

For example, for $x \in \mathbb{R}^2$ we let $d_{\exp}(x)$ be the kernel of the 1-form $dx_2 - x_2 dx_1$, that is, $d_{\exp}(x)$ is the orthogonal complement of the vector $(-x_2, 1)$ in $\mathbb{R}^2$. Then the graph of exp is a closed integral manifold of $d_{\exp}$. On the other hand, for $x \in \mathbb{R}^2 \setminus \{0\}$, we let $d_{\text{spiral}}(x)$ be the kernel of $(x_1 - x_2)dx_1 + x_2 dx_2$. Then the image of every trajectory of the vector field $\chi_{\text{spiral}} := -x_2 \partial/\partial x_1 + (x_1 - x_2)\partial/\partial x_2$ in $\mathbb{R}^2 \setminus \{0\}$ is an integral manifold of d_{spiral}.

(2) A **leaf** of d is a maximal connected integral manifold of d.

For example, the graph of exp is a leaf of $d_{\exp}$, and the image of every maximal trajectory of χ_{spiral} in $\mathbb{R}^2 \setminus \{0\}$ is a leaf of d_{spiral}.

(3) An integral manifold V of d **has the Rolle property** (see Moussu and Roche [21]) if for every curve $\gamma : [0,1] \longrightarrow M$ such that $\gamma(0), \gamma(1) \in V$, there exists a $t \in [0,1]$ such that $\gamma'(t) \in d(\gamma(t))$. A leaf V of d is a **Rolle leaf of** d if V is a closed and embedded leaf of d that has the Rolle property.

For example, M is a Rolle leaf of g_M. The image of every maximal trajectory of χ_{spiral} in $\mathbb{R}^2 \setminus \{0\}$ is a leaf of d_{spiral} that does not have the Rolle property, because $(0,\infty) \times \{0\}$ is transverse to d_{spiral}. On the other hand, define $d_{\text{horizontal}}(x) := \mathbb{R} \times \{0\}$ for all $x \in \mathbb{R}^2$. Rolle's Theorem means exactly that every horizontal line is a Rolle leaf of $d_{\text{horizontal}}$.

Definition 1.5. If d is a definable $(n-1)$-distribution on $\mathbb{R}^n$ and V is a Rolle leaf of d, then V is called a **Rolle leaf over** $\mathcal{R}$.

The connection between pfaffian functions and Rolle leaves is established by Lemmas 1.6 and 1.8 below.

Lemma 1.6 (Khovanskii [8]). *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be pfaffian over $\mathcal{R}$. Then the graph* gr f *of f is a Rolle leaf over $\mathcal{R}$.*

Proof. Let $P_1, \dots, P_n : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}$ be definable such that $\partial f/\partial x_i(x) = P_i(x, f(x))$ for all $x \in \mathbb{R}^n$. For $(x,y) \in \mathbb{R}^{n+1}$, we let $d(x,y)$ be the kernel of the 1-form $\omega := dy - P_1 dx_1 - \cdots - P_n dx_n$. Note that d is definable. Since f is C^1 and total, each of the sets $C_1 := \{(x,y) \in \mathbb{R}^{n+1} : y < f(x)\}$ and $C_2 := \{(x,y) \in \mathbb{R}^{n+1} : y > f(x)\}$ is connected and gr f is a closed leaf of d. Let $\gamma : [0,1] \longrightarrow \mathbb{R}^{n+1}$ be a curve with $\gamma(0), \gamma(1) \in$ gr f. Without loss of generality, we may assume that $\omega(\gamma(0))\big(\gamma'(0)\big)$ and $\omega(\gamma(1))\big(\gamma'(1)\big)$ are both nonzero and $\gamma\big((0,1)\big)$ is contained in either C_1 or C_2.

We now claim that $\omega(\gamma(0))\big(\gamma'(0)\big)$ and $\omega(\gamma(1))\big(\gamma'(1)\big)$ must have opposite signs. For if $\omega(\gamma(0))\big(\gamma'(0)\big) > 0$, say, there is an $\epsilon > 0$ such that $\gamma\big((0,\epsilon)\big) \subseteq C_1$, and so by the above $\gamma\big((0,1)\big) \subseteq C_1$; but if also $\omega(\gamma(1))\big(\gamma'(1)\big) > 0$, there is a $\delta > 0$ such that $\gamma\big((\delta,1)\big) \subseteq C_2$, so that $\gamma\big((0,1)\big) \subseteq C_2$, a contradiction. We obtain a similar contradiction if both $\omega(\gamma(0))\big(\gamma'(0)\big)$ and $\omega(\gamma(1))\big(\gamma'(1)\big)$ are negative, so the claim is proved.

It follows from the claim and Rolle's Theorem that there exists a $t \in (0,1)$ such that $\omega(\gamma(t))\big(\gamma'(t)\big) = 0$. This is equivalent to saying that $\gamma'(t) \in d(\gamma(t))$, so the lemma is proved. □

Corollary 1.7. *Let $f = (f_1, \dots, f_k)$ be a pfaffian chain over $\mathcal{R}$. Then for each $i = 1, \dots, k$, the graph of f_i is a Rolle leaf over $(\mathcal{R}, f_1, \dots, f_{i-1})$.*

Proof. Combine Lemma 1.6 with Example 1.2(3). □

The converse to Lemma 1.6 is true locally around each point of a Rolle leaf over $\mathcal{R}$ after a definable rescaling:

Lemma 1.8. *Let d be a definable n-distribution on $\mathbb{R}^{n+1}$, and assume that $\Pi_n \restriction d(x, y)$ is an immersion for every $(x, y) \in \mathbb{R}^{n+1}$. Let L be a Rolle leaf of d such that $\Pi_n(L) = \mathbb{R}^n$. Then L is the graph of a pfaffian function over $\mathcal{R}$.*

Proof. Since for every $x \in \mathbb{R}^n$, the line $\{x\} \times \mathbb{R}$ is nowhere tangent to d by hypothesis, it follows that L is the graph of a function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$. Also by hypothesis, for each $(x, y) \in \mathbb{R}^{n+1}$ the space $d(x, y)$ is the graph of a linear function determined by an $n \times n$-matrix $l(x, y)$, with respect to the standard basis of $\mathbb{R}^n$, such that the map $(x, y) \mapsto l(x, y)$ is definable. Now the fact that L is an integral manifold of d means that for any unit vector $v \in \mathbb{R}^n$, the derivative $\partial_v f$ of f in the direction v is given by $l(x, f(x)) \cdot v$; in particular, f is pfaffian over $\mathcal{R}$. □

Definition 1.9. Let $\mathcal{L}(\mathcal{R})$ be the set of all Rolle leaves over $\mathcal{R}$. Define $\mathcal{R}_i$ by induction on $i \in \mathbb{N}$: $\mathcal{R}_0 := \mathcal{R}$, and for $i > 0$ we let $\mathcal{R}_i$ be the expansion of $\mathcal{R}_{i-1}$ by all Rolle leaves over $\mathcal{R}_{i-1}$. The **pfaffian closure of** $\mathcal{R}$ is the expansion $\mathcal{P}(\mathcal{R})$ of $\mathcal{R}$ by all Rolle leaves in $\bigcup_{i \in \mathbb{N}} \mathcal{L}(\mathcal{R}_i)$. We call $\mathcal{R}$ **pfaffian closed** if every Rolle leaf over $\mathcal{R}$ is definable in $\mathcal{R}$.

Remark. Every Rolle leaf over $\mathcal{P}(\mathcal{R})$ is quantifier-free definable in $\mathcal{P}(\mathcal{R})$; in particular, $\mathcal{P}(\mathcal{R})$ is pfaffian closed.

The first main goal of these notes is to prove:

Theorem A ([22]). *If $\mathcal{R}$ is o-minimal, then so is $\mathcal{P}(\mathcal{R})$.*

As mentioned in the introduction, the proof of Theorem A does not provide meaningful insight about quantifier simplification:

Question 1. Is $\mathcal{R}_1$ model complete?

Note that every definable C^2-cell is definably diffeomorphic to a Rolle leaf over $\mathcal{R}$. It follows that every definable set is existentially definable in $\mathcal{R}_1$. Therefore, "model completeness of $\mathcal{R}_1$" is the same as "model completeness of $\mathcal{R}_1$ relative to $\mathcal{R}$", and $\mathcal{P}(\mathcal{R})$ is model complete by definition. On the other hand, it seems unlikely that every pfaffian chain over $\mathcal{R}$ is definable in $\mathcal{R}_1$—although I am not aware of any specific counterexamples—and hence that $\mathcal{R}_1$ and $\mathcal{P}(\mathcal{R})$ are interdefinable.

Question 2. Is there a pfaffian chain over $\mathcal{R}$ that is not definable in $\mathcal{R}_1$?

The following reduct of $\mathcal{P}(\mathcal{R})$ is more naturally defined than $\mathcal{P}(\mathcal{R})$ itself: let $\mathcal{R}_{\text{pfaff}}$ be the expansion of $\mathcal{R}$ by all graphs of component functions of pfaffian chains over $\mathcal{R}$. By Corollary 1.7, $\mathcal{R}_{\text{pfaff}}$ is a reduct of $\mathcal{P}(\mathcal{R})$ and so is o-minimal.

Question 3. Is $\mathcal{R}_{\text{pfaff}}$ model complete? Is $\mathcal{R}_{\text{pfaff}}$ a proper reduct of $\mathcal{P}(\mathcal{R})$?

I do not know whether any of these open questions implies any other. The first attempt of Lion and myself to find an answer to some of these questions led to the following:

Definition 1.10. A set $V \subseteq \mathbb{R}^n$ is an **integral manifold over** $\mathcal{R}$ if there are a definable C^2-manifold $M \subseteq \mathbb{R}^n$, $l \le \dim M$ and a definable l-distribution d on M such that V is an integral manifold of d.

Fact 1.11 ([15]). *Let $X \subseteq \mathbb{R}^n$ be definable in $\mathcal{P}(\mathcal{R})$. Then there is a $q \in \mathbb{N}$, and for $p = 1, \dots, q$, there are $n_p \ge n$ and an integral manifold $U_p \subseteq \mathbb{R}^{n_p}$ over $\mathcal{R}$ that is a cell definable in $\mathcal{P}(\mathcal{R})$ and of dimension at most* $\dim X$ *such that $X = \Pi_n(U_1) \cup \dots \cup \Pi_n(U_q)$.*

Fact 1.11 is a model-completeness result, albeit in a language that has no intrinsic description over $\mathcal{R}$. Nevertheless, this language has an advantage over the one used to define $\mathcal{P}(\mathcal{R})$, because its predicates are integral manifolds of *definable* distributions. This turns out to be useful in certain geometric situations and leads to the following:

Definition 1.12. An expansion $\mathcal{R}'$ of $\mathcal{R}$ is $\mathcal{R}$**-differentially model complete** if for every $X \subseteq \mathbb{R}^n$ definable in $\mathcal{R}'$, there is a $q \in \mathbb{N}$, and for $p = 1, \dots, q$, there are $n_p \ge n$ and an integral manifold $U_p \subseteq \mathbb{R}^{n_p}$ over $\mathcal{R}$ definable in $\mathcal{R}'$ and of dimension at most $\dim X$ such that $X = \Pi_n(U_1) \cup \dots \cup \Pi_n(U_q)$.

Exercise 1.13. Let $\mathcal{R}'$ be an *o-minimal* expansion of $\mathcal{R}$. Show that $\mathcal{R}'$ is $\mathcal{R}$-differentially model complete if and only if for every $X \subseteq \mathbb{R}^n$ definable in $\mathcal{R}'$, there is a $q \in \mathbb{N}$, and for $p = 1, \dots, q$, there are $n_p \ge n$ and an integral manifold $U_p \subseteq \mathbb{R}^{n_p}$ over $\mathcal{R}$ *that is a cell* definable in $\mathcal{R}'$ and of dimension at most $\dim X$ such that $X = \Pi_n(U_1) \cup \dots \cup \Pi_n(U_q)$.

To illustrate the use of $\mathcal{R}$-differential model completeness, I show the following: recall that $\mathcal{R}$ is **polynomially bounded** if for every definable function $f : \mathbb{R} \longrightarrow \mathbb{R}$, there exist $N \in \mathbb{N}$ and $a > 0$ such that $|f(x)| \le x^N$ for all $x > a$. Similarly, $\mathcal{R}$ is **exponentially bounded** if for every definable function $f : \mathbb{R} \longrightarrow \mathbb{R}$, there exist $N \in \mathbb{N}$ and $a > 0$ such that $|f(x)| \le e_N(x)$ for all $x > a$.

Proposition 1.14. *Let $\mathcal{R}'$ be an $\mathcal{R}$-differentially model complete expansion of $\mathcal{R}$, and assume that $\mathcal{R}'$ is o-minimal.*

(1) If $\mathcal{R}$ admits analytic cell decomposition, then so does $\mathcal{R}'$.
(2) If $\mathcal{R}$ is polynomially bounded, then $\mathcal{R}'$ is exponentially bounded.

Other uses of $\mathcal{R}$-differential model completeness can be found for instance in Jones [6]. Note that, by Exercise 1.13, Fact 1.11 is equivalent to stating that $\mathcal{P}(\mathcal{R})$ is $\mathcal{R}$-differentially model complete.

In these notes, I obtain something better (see Theorem B below): let $\mathcal{R}'$ be a reduct of $\mathcal{P}(\mathcal{R})$ that expands $\mathcal{R}$. Let $\mathcal{L}' \subseteq \bigcup \mathcal{L}(\mathcal{R}_j)$ be such that $\mathcal{R}'$ is the expansion of $\mathcal{R}$ by all leaves in $\mathcal{L}'$. For $j \in \mathbb{N}$, put $\mathcal{L}'_j := \mathcal{L}' \cap \bigcup_{i=0}^{j-1} \mathcal{L}(\mathcal{R}_i)$ and let $\mathcal{R}'_j$ be the expansion of $\mathcal{R}$ by all leaves in $\mathcal{L}'_j$. Then $\mathcal{R}'_0 = \mathcal{R}$ and for each $j > 0$, $\mathcal{R}'_j$ is a reduct of $\mathcal{R}_j$ that expands $\mathcal{R}'_{j-1}$.

Definition 1.15. $\mathcal{R}'$ is **chain-closed** if, for $j > 0$, we have $\mathcal{L}'_j \subseteq \mathcal{L}(\mathcal{R}'_{j-1})$, that is, every leaf in $\mathcal{L}'_j$ is a Rolle leaf over $\mathcal{R}'_{j-1}$.

Examples 1.16. (1) If $\mathcal{R}'$ is a reduct of $\mathcal{R}_1$, then $\mathcal{R}'$ is chain-closed.

(2) $\mathcal{P}(\mathcal{R})$ is chain-closed.

(3) Let $(f_1, \dots, f_k) : \mathbb{R}^n \longrightarrow \mathbb{R}^k$ be a pfaffian chain over $\mathcal{R}$. Then by Corollary 1.7, the expansion $(\mathcal{R}, \text{gr } f_1, \dots, \text{gr } f_k)$ is a chain-closed reduct of $\mathcal{P}(\mathcal{R})$. In particular, $\mathcal{R}_{\text{pfaff}}$ is chain-closed.

The second main goal of these notes is to prove:

Theorem B. *Assume that $\mathcal{R}'$ is chain-closed. Then $\mathcal{R}'$ is $\mathcal{R}$-differentially model complete.*

As an immediate corollary to Theorem B and Proposition 1.14, one obtains Theorem 3 in Lion et al. [19] and a generalization of the main theorem of [15]:

Corollary 1.17. *Assume that $\mathcal{R}'$ is chain-closed.*

(1) If $\mathcal{R}$ admits analytic cell decomposition, then so does $\mathcal{R}'$.

(2) If $\mathcal{R}$ is polynomially bounded, then $\mathcal{R}'$ is exponentially bounded. □

More recently, Lion and I have obtained a model completeness result for $\mathcal{P}(\mathcal{R})$ in a natural language, that of *nested Rolle leaves over* $\mathcal{R}$ (see [18] for details), under the additional hypothesis that $\mathcal{R}$ admits analytic cell decomposition. Indeed, we let $\mathcal{N}(\mathcal{R})$ be the expansion of $\mathcal{R}$ by all nested Rolle leaves over $\mathcal{R}$, and we prove that if $\mathcal{R}$ admits analytic cell decomposition, then $\mathcal{N}(\mathcal{R})$ is model complete. It follows that $\mathcal{N}(\mathcal{R})$ and $\mathcal{P}(\mathcal{R})$ are interdefinable.

To finish this section, I point out (without proofs) that there are many Rolle leaves and hence pfaffian functions. To explain why, I first need to introduce the notion of integrability, also used throughout these notes. Let d be an l-distribution on a C^2-submanifold M of $\mathbb{R}^n$, and assume that d is of class C^p with $p \geq 1$. Let $\mathcal{V}(M, d)$ be the collection of all vector fields on M tangent to d, and put

$$I(d) := \{ x \in M : [v, w](x) \in d(x) \text{ for all } v, w \in \mathcal{V}(M, d) \},$$

where $[v, w]$ denotes the Lie bracket of the vector fields v and w. The distribution d is **integrable** if $I(d) = M$, and d is **nowhere integrable** if $I(d) = \emptyset$.

Exercise 1.18. Let M be a C^2-submanifold of $\mathbb{R}^n$.

(1) Show that every 1-distribution on M is integrable.

(2) Prove that if d is a definable l-distribution on M, then the set $I(d)$ is definable.

(3) Let d be an l-distribution on M. Prove that every integral manifold of d is a subset of $I(d)$.

(4) Let d be an integrable l-distribution on M. Show that if V is an *embedded* leaf of d, then V is closed.

(5) Let d be an integrable $(m-1)$-distribution on M, where $m := \dim M$, and let V be a leaf of d with the Rolle property. Prove that V is a Rolle leaf.

Fact 1.19 (Froebenius, see Spivak [23]). *Let d be an l-distribution on an open set $U \subseteq \mathbb{R}^n$. Then d is integrable if and only if for every $x \in U$, there is an integral manifold V of d such that $x \in V$ and V is of class C^2.*

It follows in particular that, in the situation of the previous fact, every point of U belongs to a leaf of d. Since two leaves of d are either equal or disjoint, this means that U is partitioned by the leaves of d. (This partition is called the *foliation* associated to d; see Camacho and Lins-Neto [2] for details.) This observation, together with the next fact, shows that there are plenty of analytic Rolle leaves.

Fact 1.20 (Haefliger [5], see also [21]). *Let d be an analytic $(m-1)$-distribution on an analytic submanifold M of $\mathbb{R}^n$ of dimension m, and assume that M is simply connected and d is integrable. Then every leaf of d is a Rolle leaf of d.*

Haefliger's Theorem is false without the assumption of analyticity, even in the o-minimal context:

Reeb foliation (Lion [13], see also [17]). *There is an integrable 2-distribution d on $\mathbb{R}^3$ that is of class C^∞ and definable in $(\mathbb{R}_{an}, \exp)$ such that d has a leaf L that is not a Rolle leaf of d.*

However, analyticity is not necessary to produce Rolle leaves. The following weaker version of Haefliger's Theorem is true in the o-minimal context:

Fact 1.21 ([17]). *Assume $\mathcal{R}$ is o-minimal, and let d be a definable and integrable $(m-1)$-distribution on a C^2-submanifold M of $\mathbb{R}^n$ of dimension m. Then M can be covered by finitely many definable open subsets M_i such that every leaf of the restriction d_i of d to M_i is a Rolle leaf of d_i.*

2 Proof of Proposition 1.14

Let $\mathcal{R}'$ be an o-minimal and $\mathcal{R}$-differentially model complete expansion of $\mathcal{R}$.

Lemma 2.1. *Assume that $\mathcal{R}$ admits analytic cell decomposition, and let $S \subseteq \mathbb{R}^n$ be definable in $\mathcal{R}'$. Then S is a finite union of analytic manifolds that are definable in $\mathcal{R}'$.*

Proof. By induction on $p := \dim S$; the case $p = 0$ is trivial, so we assume $p > 0$ and the lemma holds for lower values of p. Since $\mathcal{R}'$ is $\mathcal{R}$-differentially model complete, we may assume $S = \Pi^N_n(V)$, where $N \geq n$, $M \subseteq \mathbb{R}^N$ is a definable manifold, d is a definable distribution on M and V an integral manifold of d that is a cell definable in $\mathcal{R}'$ of dimension at most p. By the inductive hypothesis, we may assume that $\dim V = p = \dim S$; in particular, V is the graph of a function $f : V \longrightarrow \mathbb{R}^{N-n}$. Let $\mathcal{C}$ be an analytic cell decomposition compatible with d such that for $C \in \mathcal{C}$, d^C is analytic and either d^C is integrable or d^C is nowhere integrable. Replacing M, d and V by C, d^C and $V \cap C$ for each $C \in \mathcal{C}$ such that $C \cap V \neq \emptyset$ we may assume, by Exercise 1.18(3), that M and d are analytic and d is integrable. The Frobenius integrability theorem in the analytic setting then implies that V is an analytic manifold. Since V is the graph of the function f, it follows that S an analytic manifold. □

Proof (Proof of Proposition 1.14(1)). Assume that $\mathcal{R}$ admits analytic cell decomposition. We show by induction on n that if $\mathcal{S}$ is a finite collection of subsets of $\mathbb{R}^n$ definable in $\mathcal{R}'$, then there is a decomposition of $\mathbb{R}^n$ into analytic cells definable in $\mathcal{R}'$ that is compatible with each member of $\mathcal{S}$. The cases $n = 0, 1$ are trivial, so we assume that $n > 1$. Let $f : S \longrightarrow \mathbb{R}$ be a function definable in $\mathcal{R}'$ with $S \subseteq \mathbb{R}^{n-1}$ a cell; by cell decomposition, it now suffices to show that S can be partitioned into analytic cells $S_1, \dots, S_k$ definable in $\mathcal{R}'$ such that $f \restriction S_j$ is analytic for each j.

To see this, we apply Lemma 2.1 to $\mathrm{gr}(f)$. The resulting analytic manifolds $S_1', \dots, S_L' \subseteq \mathbb{R}^n$ are the graphs of analytic functions $g_j : \Pi_{n-1}(S_j') \longrightarrow \mathbb{R}$. Now use the inductive hypothesis to obtain a partition of $\Pi_{n-1}(S)$ into analytic cells $S_1, \dots, S_k$ definable in $\mathcal{R}'$ compatible with each $\Pi_{n-1}(S_j')$. □

The proof of Proposition 1.14(2) is based on a version of a conjecture of E. Borel established in [19] (see also Miller [20] for an exposition of this conjecture) and uses an argument found in the proof of [12, Proposition 4].

Proof (Proof of Proposition 1.14(2)). Assume that $\mathcal{R}$ is polynomially bounded, and let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be definable in $\mathcal{R}'$. By $\mathcal{R}$-differential model completeness and o-minimality, there are a definable manifold $M \subseteq \mathbb{R}^{1+n}$ with $n \geq 1$, a definable 1-distribution d on M, an integral manifold V of d that is a cell definable in $\mathcal{R}'$ and an $a > 0$ such that $\mathrm{gr}(f \restriction (a, \infty)) = \Pi_2(V)$. Thus $V = \mathrm{gr}(F)$ for some $F = (f_1, \dots, f_n) : (a, \infty) \longrightarrow \mathbb{R}^n$ definable in $\mathcal{R}'$ such that $f_1 = f \restriction (a, \infty)$. Moreover, since V is an integral manifold of d, there is a definable vector field ξ on M such that V is a trajectory of ξ: for $x \in M$, $\xi(x)$ is the unique vector contained in $d(x)$ satisfying $\Pi_1(\xi(x)) = 1$.

We now let (t, x) range over $\mathbb{R}^{1+n}$, with $t \in \mathbb{R}$ and $x \in \mathbb{R}^n$. Put $m := \dim M$ and define $u_0, \dots, u_{m-1} : M \longrightarrow \mathbb{R}$ as follows: $u_0(t, x) := x_1$ and for $j > 0$, let $u_j(t, x) := L_\xi(u_{j-1})(t, x) = (du_{j-1} \cdot \xi)(t, x)$ be the Lie derivative of u_{j-1} with respect to ξ (see [23]). Consider the definable set

$$E := \{ (t, u_0(t, x), \dots, u_{m-1}(t, x), x) : \ (t, x) \in M \}$$

and put $D := \Pi_{1+m}(E)$. Then D is of dimension at most m and by construction, after increasing a if necessary, $f \restriction (a, \infty)$ is of class C^{m-1} and the graph of the function $t \mapsto \big(f(t), \dots, f^{(m-1)}(t)\big) : (a, \infty) \longrightarrow \mathbb{R}^m$ is contained in D. By cell decomposition and after increasing a if necessary, we may assume that D is a cell. Then D is not an open cell, so there exists $k < m$ such that $\Pi_k(D) = \mathrm{gr}\, g$ for some definable function $g : \Pi_{k-1}(D) \longrightarrow \mathbb{R}$. Note that $\Pi_1(D) = (a, \infty)$, so that $k > 1$. Hence $f^{(k)}(t) = g\big(t, f(t), \dots, f^{(k-1)}(t)\big)$ for $t > a$; it follows from Corollary 1 in [19] that f is bounded at $+\infty$ by some finite iterate of the exponential function. □

3 Khovanskii Theory

Khovanskii theory relative to $\mathcal{R}$, the fundamental result of which is Proposition 3.5 below, is the basis for Theorems A and B. Let M be a C^2-submanifold of $\mathbb{R}^n$ of dimension m.

I sometimes need to work with maps from M to G_n that are not necessarily distributions on M. I shall use the following terminology: Given two maps $d, e : M \longrightarrow G_n$, I write $d \cap e : M \longrightarrow G_n$ for the map defined by $(d \cap e)(x) := d(x) \cap e(x)$, and I write $d \subseteq e$ if $d(x) \subseteq e(x)$ for all $x \in M$. A map $d : M \longrightarrow G_n$ **has dimension** if $d(M) \subseteq G_n^m$ for some $m \leq n$; in this situation, I write $\dim d := m$. Note that, by linear algebra, if M and $d, e : M \longrightarrow G_n$ are of class C^p, with $p \in \mathbb{N} \cup \{\infty, \omega\}$, and if $d \cap e$ has dimension, then $d \cap e$ is of class C^p.

Definition 3.1. Let $N \subseteq \mathbb{R}^l$ be a C^2-manifold and $f : N \longrightarrow M$ a C^2-map, and let d be a distribution on M. The **pull-back** of d on N by f is the distribution f^*d on N defined by

$$f^*d(y) := (df_y)^{-1}\big(d(f(y))\big),$$

where $df_y : T_yN \longrightarrow T_{f(y)}M$ denotes the linear map defined by the jacobian matrix of f at y and $(df_y)^{-1}(S)$ denotes the inverse image of S under this map for any $S \subseteq T_{f(y)}M$.

Remark. In the situation of the previous definition, the pull-back f^*d is a distribution on N of class C^1. It is for this reason that I usually work on C^2-manifolds and with C^2-cell decompositions in these notes.

If N is a C^2-submanifold of M and $f : N \longrightarrow M$ is the inclusion map, the pull-back f^*d is simply the distribution $g_N \cap d\restriction_N$ on N, which I shall denote by d^N.

Definition 3.2 ([21]). Let $\mathcal{D}$ be a set of distributions on M. A C^2-submanifold N of M is **compatible with** $\mathcal{D}$ if the pull-back $\big(\bigcap_{e\in\mathcal{E}} e\big)^N$ has dimension for every $\mathcal{E} \subseteq \mathcal{D}$. A collection $\mathcal{C}$ of submanifolds of M is **compatible with** $\mathcal{D}$ if every $C \in \mathcal{C}$ is compatible with $\mathcal{D}$.

For the remainder of this section, I assume that M is definable. I also fix a finite set $\mathcal{D}$ of definable distributions on M and set $d_{\mathcal{D}} := \bigcap_{d\in\mathcal{D}} d$.

Proposition 3.3. *Let $A_1, \dots, A_k \subseteq \mathbb{R}^n$ be definable and $p \geq 2$. Then there is a finite partition (stratification, Whitney stratification) $\mathcal{P}$ of M into definable C^p-cells such that $\mathcal{P}$ is compatible with each A_j as well as $\mathcal{D}$ and d^N is of class C^p for each $d \in \mathcal{D}$ and $N \in \mathcal{P}$.*

Proof. We proceed by induction on $m = \dim M$; the case $m = 0$ is trivial. So assume $m > 0$ and the lemma holds for lower values of m. By cell decomposition, we may assume that $\{A_1, \dots, A_k\}$ is a partition of M into definable C^p-cells such that $d\restriction A_j$ is of class C^p for each j and each $d \in \mathcal{D}$. In particular, for each $x \in M$, there is a unique $l(x) \in \{1, \dots, k\}$ such that $x \in A_{l(x)}$. For $x \in M$ and $\mathcal{E} \subseteq \mathcal{D}$ we write $T_x\mathcal{E} := T_xA_{l(x)} \cap d_{\mathcal{E}}(x)$, and for each $\mathcal{E} \subseteq \mathcal{D}$, $j \in \{1, \dots, k\}$ and $i \in \{0, \dots, m\}$, we define the set

$$M_{\mathcal{E},j,i} := \{ x \in A_j : \dim T_x\mathcal{E} = i \}.$$

For each $\mathcal{E}$, the sets $M_{\mathcal{E},j,i}$ form a partition of M, and since each $d \in \mathcal{D}$ is definable, each set $M_{\mathcal{E},j,i}$ is definable. Let $\mathcal{C}$ be a partition (stratification, Whitney stratification) of M into definable C^p-cells compatible with each $M_{\mathcal{E},j,i}$. Then for each $C \in \mathcal{C}$, there is a unique $j(C) \in \{1,\dots,k\}$ such that $C \subseteq A_{j(C)}$. Fix a $C \in \mathcal{C}$. If $\dim C = m$, then for each $\mathcal{E} \subseteq \mathcal{D}$ there is a unique $i(C,\mathcal{E}) \in \{0,\dots,m\}$ such that $C \subseteq M_{\mathcal{E},j(C),i(C,\mathcal{E})}$. Since C is open in M, it follows that $(d_{\mathcal{E}})^C$ has dimension $i(C,\mathcal{E})$ for every $\mathcal{E} \subseteq \mathcal{D}$. On the other hand, if $\dim C < m$, then the inductive hypothesis applied to C and $\mathcal{D}^C := \{ d^C : d \in \mathcal{D} \}$ in place of M and $\mathcal{D}$ produces a partition (stratification, Whitney stratification) $\mathcal{P}_C$ of C compatible with each A_j as well as $\mathcal{D}^C$. Now it is straightforward to see that the collection

$$\mathcal{P} := \{ C \in \mathcal{C} : \dim C = m \} \cup \bigcup_{C \in \mathcal{C},\ \dim C < m} \mathcal{P}_C$$

is a partition with all the required properties. □

Recall that if $C \subseteq \mathbb{R}^n$ is a manifold, a function $\phi : C \longrightarrow (0,\infty)$ is a **carpeting function on** C if ϕ is proper and satisfies $\lim_{x \to y} \phi(x) = 0$ whenever $y \in \operatorname{fr} C$, where the frontier is taken in $\mathbb{R}^n \cup \{\infty\}$. For instance, given positive real numbers $u_1,\dots,u_n$, the function

$$x \mapsto \phi_u(x) := \frac{1}{1 + u_1 x_1^2 + \cdots + u_n x_n^2}$$

is a carpeting function on $\mathbb{R}^n$.

Lemma 3.4. *Let $N \subseteq M$ be a definable C^2-cell compatible with $\mathcal{D}$, and suppose that* $\dim(d_{\mathcal{D}})^N > 0$*. Then there is a definable C^2-carpeting function ϕ on N such that the definable set*

$$B := \{ x \in N : (d_{\mathcal{D}})^N(x) \subseteq \ker d\phi(x) \}$$

has dimension less than $\dim N$.

Proof. By van den Dries and Miller [24] there is a definable diffeomorphism $\sigma : \mathbb{R}^{\dim N} \longrightarrow N$ of class C^2. Replacing n by $\dim N$, N by $\mathbb{R}^{\dim N}$ and each d^N by its pull-back $\sigma^* d^N$, we reduce to the case where $N = M = \mathbb{R}^n$ and $d_{\mathcal{D}}$ has dimension with $\dim d_{\mathcal{D}} > 0$. Then for $u = (u_1,\dots,u_n) \in \mathbb{R}^n$ with all $u_i > 0$, put

$$B_u := \{ x \in \mathbb{R}^n : d_{\mathcal{D}}(x) \subseteq \ker d\phi_u(x) \},$$

where ϕ_u is the carpeting function defined on $\mathbb{R}^n$ before the lemma. If $\dim B_u < n$ for some u as above, the proof is finished. So assume for a contradiction that $\dim B_u = n$ for all u as above. Then $\dim B = 2n$, where

$$B := \{ (u,x) \in \mathbb{R}^n \times \mathbb{R}^n : u_1 > 0, \ldots, u_n > 0, x \in B_u \},$$

so there are nonempty open $V \subseteq (0,\infty)^n$ and $W \subseteq \mathbb{R}^n$ such that $V \times W \subseteq B$. Fix some $x \in W$ with all $x_i \neq 0$ and let u range over V. Note that

$$d\phi_u(x) = -\frac{2u_1x_1dx_1(x) + \cdots + 2u_nx_ndx_n(x)}{(1+u_1x_1^2+\cdots+u_nx_n^2)^2}.$$

Therefore the vector space generated by all $d\phi_u(x)$ as u ranges over V has dimension n, that is, the intersection of all $\ker d\phi_u(x)$ as u ranges over V is trivial, which contradicts $\dim d_{\mathcal{D}} > 0$. □

I assume for the remainder of this section that every $d \in \mathcal{D}$ has dimension $m-1$.

Proposition 3.5. *Let $A \subseteq \mathbb{R}^n$ be a definable set. Then there exists a $K \in \mathbb{N}$ such that, whenever L_d is a Rolle leaf of d for each $d \in \mathcal{D}$, the set $A \cap \bigcap_{d\in\mathcal{D}} L_d$ is a union of at most K connected manifolds.*

Proof. We proceed by induction on $\dim A$ and the cardinality $|\mathcal{D}|$ of $\mathcal{D}$. The cases $\dim A = 0$ or $|\mathcal{D}| = 0$ being trivial, we assume that $\dim A > 0$ and $|\mathcal{D}| > 0$ and that the result holds for lower values of $\dim A$ or $|\mathcal{D}|$. By Proposition 3.3, it suffices to consider the case where $A = N$ is a C^2-cell contained in M and compatible with $\mathcal{D}$. For each $d \in \mathcal{D}$, let L_d be a Rolle leaf of d, and put $L := \bigcap_{d\in\mathcal{D}} L_d$; then $N \cap L$ is an integral manifold of $(d_{\mathcal{D}})^N$.

Case. $\dim(d_{\mathcal{D}})^N = 0$. Choose any $e \in \mathcal{D}$ and put $\mathcal{D}' := \mathcal{D} \setminus \{e\}$ and $L' := \bigcap_{d\in\mathcal{D}'} L_d$. Then $N \cap L'$ is an integral manifold of $(d_{\mathcal{D}'})^N$ of dimension at most 1. By the inductive hypothesis, there is a $K \in \mathbb{N}$ (depending only on N and Δ', but not on the particular Rolle leaves) such that the manifold $N \cap L'$ has at most K components. Thus, if $\dim(N \cap L') = 0$, the proposition follows from the inductive hypothesis, so assume that $\dim(N \cap L') = 1$. Since N is compatible with $\mathcal{D}'$, it follows that $\dim(d_{\mathcal{D}'})^N = 1$ as well.

Let C be a component of $N \cap L'$. If $C \cap L_e$ contains more than one point, then by the Rolle property of L_e and the fact that C is a connected submanifold of M of dimension 1, C is tangent at some point $x \in C$ to e, which contradicts the assumption that $\dim(d_{\mathcal{D}})^N = 0$. So $C \cap L_e$ contains at most one point for each component C of $N \cap L'$. Hence $N \cap L$ consists of at most K points.

Case. $\dim(d_{\mathcal{D}})^N > 0$. Let ϕ and B be obtained from Lemma 3.4. Then $\dim B < \dim A$, so by the inductive hypothesis, there is a $K \in \mathbb{N}$, independent of the particular Rolle leaves chosen, such that $B \cap L$ has at most K components. Since $N \cap L$ is a closed, embedded submanifold of N, ϕ attains a maximum on every component of $N \cap L$, and any point in $N \cap L$ where ϕ attains a local maximum belongs to B. Hence $N \cap L$ has at most K components. □

Corollary 3.6. *(1) Let $\mathcal{C}$ be a partition of M into definable C^2-cells compatible with $\mathcal{D}$. Then there is a $K \in \mathbb{N}$ such that whenever $C \in \mathcal{C}$ and L_d is a Rolle* *leaf of d for each $d \in \mathcal{D}$, the set $C \cap \bigcap_{d\in\mathcal{D}} L_d$ is a union of at most K integral manifolds $L_1, \dots, L_K$ of $(d_{\mathcal{D}})^C$ of the form $L_i = \bigcap_{d\in\mathcal{D}} L_{i,d}$, where each $L_{i,d}$ is a Rolle leaf of d^C.*

(2) *Let $\mathcal{A}$ be a definable family of sets. Then there is a $K \in \mathbb{N}$ such that whenever $A \in \mathcal{A}$ and L_d is a Rolle leaf of d for each $d \in \mathcal{D}$, the set $A \cap \bigcap_{d\in\mathcal{D}}$ is a union of at most K connected manifolds.*

Proof. Part (1) follows from Propositions 3.3 and 3.5. For (2), let $A \subseteq \mathbb{R}^{m+n}$ be definable such that $\mathcal{A} = \{ A_z : z \in \mathbb{R}^m \}$, where $A_z := \{ x \in \mathbb{R}^n : (z, x) \in A \}$ is the **fiber** of A over z. Replace M by $M' := \mathbb{R}^m \times M$ and each $d \in \mathcal{D}$ by the distribution e_d on M' defined by $e_d(z, x) := \mathbb{R}^m \times d(x)$, and put $\mathcal{E} := \{ e_d : d \in \mathcal{D} \} \cup \{\ker dz_1 \restriction M', \dots, dz_m \restriction M'\}$. By Proposition 3.5, there is a K' such that whenever L_e is a Rolle leaf of e for each $e \in \mathcal{E}$, the set $A \cap \bigcap_{e\in\mathcal{E}} L_e$ has at most K' components. But for every Rolle leaf L_d of d with $d \in \mathcal{D}$, the set $\mathbb{R}^m \times L_d$ is a Rolle leaf of e_d; and for every $z \in \mathbb{R}^m$, $i \in \{1, \dots, m\}$ and each component C of M, the set $\mathbb{R}^{i-1} \times \{z_i\} \times \mathbb{R}^{m-i} \times C$ is a Rolle leaf of $\ker dz_i \restriction M'$. Thus, we take $K := K' \cdot l$, where l is the number of components of M. □

Definition 3.7. A set $X \subseteq \mathbb{R}^n$ is **basic pfaffian over** $\mathcal{R}$ if there are a definable C^2-submanifold N of $\mathbb{R}^n$ of dimension l, a finite set $\mathcal{E}$ of definable $(l-1)$-distributions on M, a Rolle leaf L_e of e for each $e \in \mathcal{E}$ and a definable set $A \subseteq \mathbb{R}^n$ such that $X = A \cap \bigcap_{e\in\mathcal{E}} L_e$. A **pfaffian set over** $\mathcal{R}$ is a finite union of basic pfaffian sets over $\mathcal{R}$.

Proposition 3.8. *Let $X_1 \subseteq \mathbb{R}^{n_1}$ and $X_2 \subseteq \mathbb{R}^{n_2}$ be pfaffian over $\mathcal{R}$.*

(1) *If $n_1 = n_2$, then $X_1 \cap X_2$ is pfaffian over $\mathcal{R}$.*
(2) *The product $X_1 \times X_2$ is pfaffian over $\mathcal{R}$.*

Proof. (1) It suffices to consider X_1 and X_2 basic pfaffian over $\mathcal{R}$. Let $M_1, M_2 \subseteq \mathbb{R}^n$ be definable manifolds with $n = n_1 = n_2$, and for $p = 1, 2$, let $\mathcal{D}^p$ be finite sets of definable distributions on M_p, L_d^p be a Rolle leaf of d for each $d \in \mathcal{D}^p$ and $A_p \subseteq \mathbb{R}^n$ be definable such that $X_p = A_p \cap \bigcap_{d\in\mathcal{D}^p} L_d^p$. Let $\mathcal{C}$ be a C^2-cell decomposition of $\mathbb{R}^n$ compatible with $M_1, M_2, M_1 \cap M_2, A_1$ and A_2. Refining $\mathcal{C}$ if necessary, we may assume that if $C \in \mathcal{C}$ is such that $C \subseteq M_1 \cap M_2$, then C is compatible with $\mathcal{D} := \mathcal{D}^1 \cup \mathcal{D}^2$. Then by Corollary 3.6(1), we may even assume that $M_1 = M_2 = C$ for each such $C \in \mathcal{C}$. In this case, $X_1 \cap X_2$ is a finite union of basic pfaffian sets over $\mathcal{R}$ by Corollary 3.6(1).

(2) Arguing as in the proof of Corollary 3.6(2), but without adding the extra distributions $\ker dz_i \restriction M'$ there, it follows that $\mathbb{R}^{n_1} \times X_2$ and $X_1 \times \mathbb{R}^{n_2}$ are pfaffian over $\mathcal{R}$. Hence $X_1 \times X_2 = (X_1 \times \mathbb{R}^{n_2}) \cap (\mathbb{R}^{n_1} \times X_2)$ is pfaffian over $\mathcal{R}$ by (1). □

Question 4. Are the components of pfaffian sets over $\mathcal{R}$ also pfaffian over $\mathcal{R}$? The corresponding question for *nested* pfaffian sets has an affirmative answer [18].

Finally, similar arguments yield a fiber cutting lemma for pfaffian sets over $\mathcal{R}$, see Proposition 3.10 below.

Lemma 3.9. *Let $A \subseteq \mathbb{R}^n$ be definable and $\mathcal{E} \subseteq \mathcal{D}$. Then there is a finite collection $\mathcal{P}$ of pairwise disjoint definable C^2-cells contained in A and compatible with $\mathcal{D}$ such that*

(i) $\dim(d_{\mathcal{E}})^N = 0$ *for every $N \in \mathcal{P}$;*
(ii) Whenever L is a closed and embedded integral manifold of $d_{\mathcal{E}}$, each component of $A \cap L$ intersects some cell in $\mathcal{P}$.

Proof. By induction on $\dim A$; if $\dim A = 0$, there is nothing to do, so assume $\dim A > 0$ and the corollary is true for lower values of $\dim A$. By Proposition 3.3 and the inductive hypothesis, we may assume that $A = N$ is a definable C^2-cell contained in M and compatible with $\mathcal{D}$. Thus, if $\dim(d_{\mathcal{E}})^N = 0$, the lemma is proved; otherwise, let ϕ and B be as in Lemma 3.4 with $\mathcal{E}$ in place of $\mathcal{D}$.

Let now L be a closed, embedded integral manifold of $d_{\mathcal{E}}$; it suffices to show that every component of $N \cap L$ intersects B. However, since $(d_{\mathcal{E}})^N$ has dimension, $N \cap L$ is a closed, embedded submanifold of N. Thus, ϕ attains a maximum on every component of $N \cap L$, and any point in $N \cap L$ where ϕ attains a local maximum belongs to B. □

For each $m \leq n$, the map $\Pi^n_m : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ denotes the projection on the first m coordinates; and for every strictly increasing $\lambda : \{1, \dots, m\} \longrightarrow \{1, \dots, n\}$, the map $\Pi^n_\lambda : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ denotes the projection $\Pi^n_\lambda(x_1, \dots, x_n) := (x_{\lambda(1)}, \dots, x_{\lambda(m)})$. When n is clear from context, I usually write Π_m and Π_λ in place of Π^n_m and Π^n_λ, respectively.

Proposition 3.10. *Let $A \subseteq \mathbb{R}^n$ be definable and $m \leq n$. Then there is a finite collection $\mathcal{P}$ of pairwise disjoint definable C^2-cells contained in A and compatible with $\mathcal{D}$ such that whenever L_d is a Rolle leaf of d for each $d \in \mathcal{D}$ and $L := \bigcap_{d \in \mathcal{D}} L_d$, we have $\Pi_m(A \cap L) = \bigcup_{N \in \mathcal{P}} \Pi_m(N \cap L)$ and for every $N \in \mathcal{P}$, the set $N \cap L$ is a submanifold of M, $\Pi_m \restriction N \cap L$ is an immersion and for every $n' \leq n$ and every strictly increasing $\lambda : \{1, \dots, n'\} \longrightarrow \{1, \dots, n\}$, the projection $\Pi_\lambda \restriction N \cap L$ has constant rank.*

Proof. Apply Lemma 3.9 with $\mathcal{D} \cup \{\ker dx_1, \dots, \ker dx_n\}$ in place of $\mathcal{D}$ and $\mathcal{E} = \mathcal{D} \cup \{\ker dx_1, \dots, \ker dx_m\}$. □

4 Hausdorff Limits of Lipschitz Manifolds

A key ingredient in the proof of Theorem A is the representation of the frontier of a pfaffian set over $\mathcal{R}$ in terms of certain Hausdorff limits. In this section, I introduce the notion of Hausdorff limit and establish some basic facts about limits of certain sequences of manifolds.

For $A \subseteq \mathbb{R}^n$ and $x \in \mathbb{R}^n$, put $d(x, A) := \inf_{y \in B} |x - y|$. For sets $A, B \subseteq \mathbb{R}^n$, the **Hausdorff distance** is defined as

$$d(A,B) := \sup\left\{\sup_{x\in A} d(x,B), \sup_{y\in B} d(y,A),\right\},$$

where the sup is taken in the set $[0,\infty]$. Let $\mathcal{K}_n$ be the set of all compact subsets of $\mathbb{R}^n$ equipped with the Hausdorff distance; note that $d(A,\emptyset) = \infty$ for all nonempty $A \in \mathcal{K}_n$ and $d(\emptyset,\emptyset) = 0$. I refer the reader to Kuratowski [10, 11] for the classical results about $\mathcal{K}_n$; in particular, I shall often use without reference the following facts:

Exercise 4.1. (1) $\mathcal{K}_n$ is a metric space.
(2) Every bounded sequence in $\mathcal{K}_n$ contains a convergent subsequence.
(3) Assume $A_\iota \to A$ in $\mathcal{K}_n$ as $\iota \to \infty$ in $\mathbb{N}$. Then A is the set of all limits of convergent sequences (x_ι) with $x_\iota \in A_\iota$ for each ι.
(4) Assume that $A_{\iota,\kappa} \to A_\iota$ in $\mathcal{K}_n$ as $\kappa \to \infty$ in $\mathbb{N}$, for each ι, and that $A_\iota \to A$ in $\mathcal{K}_n$ as $\iota \to \infty$. Then there is a subsequence $(\kappa(\iota))_\iota$ such that $A = \lim_\iota A_{\iota,\kappa(\iota)}$.

Given a sequence $(A_\iota)_{\iota\in\mathbb{N}}$ of bounded subsets of $\mathbb{R}^n$, I say that (A_ι) **converges to** $C \in \mathcal{K}_n$ if the sequence $(\mathrm{cl} A_\iota)$ converges in $\mathcal{K}_n$ to C, and in this situation I write $C = \lim_\iota A_\iota$ and call C the **Hausdorff limit** of the sequence (A_ι).

I am interested in Hausdorff limits of sequences of manifolds of the following kind: for $m \le n$, let I_n^m be the set of all $E \in G_n^m$ such that $\Pi_m \restriction E$ is an immersion. For every $E \in I_n^m$, there is a matrix $L_E \in M_{m,n-m}(\mathbb{R})$ such that $E = \{(u, L_E u) : u \in \mathbb{R}^m\}$, and I define $\mathbf{n}(E) := \|L_E\|$. If $e_1,\dots,e_n$ denote the standard basis vectors of $\mathbb{R}^n$ and $A_E \in M_n(\mathbb{R})$ denotes the matrix represented by $E \in I_n^m$, it follows that

$$\mathbf{n}(E) = \sqrt{\|A_E e_1\|^2 + \cdots + \|A_E e_m\|^2}\,;$$

in particular, $\mathbf{n} : I_n^m \longrightarrow [0,\infty)$ is a definable, real analytic map. For convenience, I extend $\mathbf{n}$ to all of G_n^m by putting $\mathbf{n}(E) := \infty$ if $E \notin I_n^m$. Note that $|L_E| \le \mathbf{n}(E) \le n|L_E|$ for $E \in I_n^m$.

Definition 4.2. Let $M \subseteq \mathbb{R}^n$ be a manifold of dimension m and $\eta > 0$. Then M is η**-bounded** if $\mathbf{n}(T_x M) \le \eta$ for every $x \in M$.

I now fix a submanifold M of $\mathbb{R}^n$ of dimension m. For $x \in \mathbb{R}^n$ and $p \le n$, set $x_{\le p} := (x_1,\dots,x_p)$ and $x_{>p} := (x_{p+1},\dots,x_n)$.

Lemma 4.3. *Let $\eta > 0$, and let V be an η-bounded submanifold of M of dimension $p \le m$. Let $x \in V$, and let $\epsilon > 0$ be such that $\big(B(x_{\le p},\epsilon)\times B(x_{>p}, p\eta\epsilon)\big)\cap \mathrm{fr}\, V = \emptyset$. Then the component of $V \cap \big(B(x_{\le p},\epsilon)\times B(x_{>p}, p\eta\epsilon)\big)$ that contains x is the graph of a function $g : B(x_{\le p},\epsilon) \longrightarrow B(x_{>p}, p\eta\epsilon)$ that is $p\eta$-Lipschitz with respect to $|\cdot|$.*

Proof. Set $W := B(x_{\le p},\epsilon)$ and $W' := B(x_{>p}, p\eta\epsilon)$, and denote by C the component of $V \cap (W \times W')$ that contains x. Since C is η-bounded, the map $\Pi_p \restriction C : C \longrightarrow W$ is a local homeomorphism onto its image. By general topology, it is therefore enough to show that $\Pi_p(C) = W$; we do this by showing that there is a function $g : W \longrightarrow W'$ such that $\operatorname{gr} g \subseteq C$.

Since V is η-bounded, there are $\delta > 0$ and a $p\eta$-Lipschitz function

$$g_x : B(x_{\le p}, \delta) \longrightarrow W'$$

such that $\operatorname{gr} g_x \subseteq C$. Extend g to all of W as follows: for each $v \in \operatorname{bd} W$, let v' be the point in the closed line segment $[x_{\le p}, v]$ closest to v such that g_x extends to a $p\eta$-Lipschitz function g_v on the half-open line segment $[x_{\le p}, v')$ satisfying $\operatorname{gr} g_v \subseteq V \cap (W \times W')$. Then the proportion of the sidelengths of W and W', the η-boundedness of V and the fact that $(W \times W') \cap \operatorname{fr} V = \emptyset$ imply that $v' = v$ for each $v \in \operatorname{bd} W$. Let $g : W \longrightarrow W'$ be defined by $g(y) := g_v(y)$ if $y \in [x_{\le p}, v]$. Since $\operatorname{gr} g$ is connected and contains x, it follows that $\operatorname{gr} g \subseteq C$, as required. □

Proposition 4.4. *Assume that M is bounded, and let (V_ι) be a sequence of submanifolds of M of dimension $p \le m$. Let $\eta > 0$, and assume that each V_ι is η-bounded. Moreover, assume that both $K = \lim_\iota V_\iota$ and $K' = \lim_\iota \operatorname{fr} V_\iota$ exist and there is a $\nu \in \mathbb{N}$ such that for every ι and every open box $U \subseteq \mathbb{R}^n$, the set $V_\iota \cap U$ has at most ν components. Then for every $x \in K \setminus K'$, there are a box $U \subseteq \mathbb{R}^n$ containing x and $p\eta$-Lipschitz functions $f_1, \dots, f_\nu : \Pi_p(U) \longrightarrow \mathbb{R}^{n-p}$ such that $\operatorname{gr} f_i \subseteq K \setminus K'$ for each i and*

$$K \cap U = (\operatorname{gr} f_1 \cap U) \cup \cdots \cup (\operatorname{gr} f_\nu \cap U).$$

Proof. We write "lim" in place of "$\lim_\iota$" throughout this proof. Let $x \in K \setminus K'$, and choose $\epsilon > 0$ such that $\big(B(x_{\le p}, 3\epsilon) \times B(x_{>p}, 3p\eta\epsilon)\big) \cap \operatorname{fr} V_\iota = \emptyset$ for all ι (after passing to a subsequence if necessary). Let $U := B(x_{\le p}, \epsilon) \times B(x_{>p}, p\eta\epsilon)$, $W := B(x_{\le p}, \epsilon)$ and $W' := B(x_{>p}, 3p\eta\epsilon)$. Then for each ι, the assumptions and Lemma 4.3 imply, with 2ϵ in place of ϵ and each $z \in U \cap V_\iota$ in place of x, that there are definable $p\eta$-Lipschitz functions $f_{1,\iota}, \dots, f_{\nu,\iota} : W \longrightarrow \mathbb{R}^{n-p}$ such that every connected component of $V_\iota \cap (W \times W')$ intersecting U is the graph of some $f_{\lambda,\iota}$. Moreover, either $f_{\lambda,\iota} = f_{\lambda',\iota}$ or $\operatorname{gr} f_{\lambda,\iota} \cap \operatorname{gr} f_{\lambda',\iota} = \emptyset$, for all $\lambda, \lambda' \in \{1, \dots, \nu\}$, and

$$V_\iota \cap U = (\operatorname{gr} f_{1,\iota} \cap U) \cup \cdots \cup (\operatorname{gr} f_{\nu,\iota} \cap U).$$

Passing to a subsequence if necessary, we may assume that each sequence $(f_{\lambda,\iota})_\iota$ converges to a $p\eta$-Lipschitz function $f_\lambda : W \longrightarrow \mathbb{R}^{n-p}$; then $\operatorname{gr} f_\lambda \subseteq K \setminus K'$. On the other hand, if $x' \in K \cap U$, then $x' \in \lim(V_\iota \cap U)$, so by the above $x' \in \lim(\operatorname{gr} f_{\lambda,\iota} \cap U)$ for some λ, that is, $x' \in \operatorname{gr} f_\lambda$. □

Definitions 4.5. Call $N \subseteq \mathbb{R}^n$ a C^0**-manifold of dimension** p if $N \ne \emptyset$ and each point of N has an open neighbourhood in N homeomorphic to $\mathbb{R}^p$; in this case p is uniquely determined (by a theorem of Brouwer), and we write $p = \dim(N)$. Correspondingly, a set $S \subseteq \mathbb{R}^n$ **has dimension** if S is a countable union of C^0-manifolds, and in this case put

$$\dim(S) := \begin{cases} \max\{\dim(N) : N \subseteq S \text{ is a } C^0\text{-manifold}\} & \text{if } S \neq \emptyset \\ -\infty & \text{otherwise.} \end{cases}$$

It follows (by a Baire category argument) that, if $S = \bigcup_{i \in \mathbb{N}} S_i$ and each S_i has dimension, then S has dimension and $\dim(S) = \max\{\dim(S_i) : i \in \mathbb{N}\}$. Thus, if N is a C^1-manifold of dimension p, then N has dimension in the sense of this definition and the two dimensions of N agree.

Corollary 4.6. *In the situation of Proposition 4.4, the set $K \setminus K'$ is either empty or has dimension p.* □

The following situation is central to my use of Hausdorff limits:

Lemma 4.7. *Assume that M is bounded and has a carpeting function ϕ. Let V be a closed subset of M, and assume that $V \cap U$ has finitely many components for every open box $U \subseteq \mathbb{R}^n$. Then for every sequence $(r_\kappa)_{\kappa \in \mathbb{N}}$ of positive real numbers satisfying $r_\kappa \to 0$ as $\kappa \to \infty$, we have*

$$\operatorname{fr} V = \lim_\kappa \left(V \cap \phi^{-1}(r_\kappa)\right).$$

Proof. Let $r_\kappa \to 0$ as $\kappa \to \infty$. It suffices to show that

$$\operatorname{fr} V = \lim_{\kappa(j)} \left(\phi^{-1}(r_{\kappa(j)}) \cap V\right)$$

for every subsequence $(\kappa(j))_{j \in \mathbb{N}}$ of (κ) such that the limit on the right-hand side exists in $\mathcal{K}_n$, that is, we may assume that the sequence $\left(\phi^{-1}(r_\kappa) \cap V\right)$ converges in $\mathcal{K}_n$. The properties of ϕ then imply that $\operatorname{fr} V \supseteq \lim_\kappa \left(\phi^{-1}(r_\kappa) \cap V\right)$. Conversely, let $x \in \operatorname{fr} V$. Since $V \cap B(x,1)$ has finitely many components, there is a component C of $V \cap B(x,1)$ such that $x \in \operatorname{fr} C$. Then $C \cup \{x\}$ is connected, so there is a continuous curve $\gamma : [0,1] \longrightarrow C \cup \{x\}$ such that $\gamma([0,1)) \subseteq C$ and $\gamma(1) = x$. Hence $\phi \circ \gamma : [0,1) \longrightarrow (0,\infty)$ is continuous and satisfies $\lim_{t \to 1} \phi(\gamma(t)) = 0$, so the intermediate value theorem implies that the image $\gamma([0,1))$ intersects $\phi^{-1}(r_\iota)$ for all sufficiently large ι, so that $x \in \lim_\iota \left(\phi^{-1}(r_\iota) \cap V\right)$. Hence $\operatorname{fr} V = \lim_\iota \left(\phi^{-1}(r_\iota) \cap V\right)$. □

Definition 4.8. I abbreviate the conclusion of Lemma 4.7 by the statement

$$\operatorname{fr} V = \lim_{r \to 0} \left(V \cap \phi^{-1}(r)\right).$$

In these notes, sequences of η-bounded manifolds arise as sequences of integral manifolds of a distribution on M: Let d be a p-distribution on M and $\eta > 0$.

Definition 4.9. The distribution d is called **η-bounded at** $x \in M$ if $\mathbf{n}(d(x)) \leq \eta$. The distribution d is **η-bounded** if d is η-bounded at every $x \in M$.

Remark. If d is η-bounded, then every integral manifold of d is η-bounded.

Let Σ_n be the collection of all permutations of $\{1,\dots,n\}$. For $\sigma \in \Sigma_n$, I write $\sigma : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ for the map defined by $\sigma(x_1,\dots,x_n) := \big(x_{\sigma(1)},\dots,x_{\sigma(n)}\big)$.

I assume for the remainder of this section that M is of class C^2. Given a permutation $\sigma \in \Sigma_n$, the set $\sigma^{-1}(M)$ is a manifold and the pull-back $\sigma^* d$ is a distribution on $\sigma^{-1}(M)$; define

$$M_{\sigma,\eta} := \{\, x \in M : \mathbf{n}\left(\sigma^* d(\sigma^{-1}(x))\right) < \eta \,\}.$$

Lemma 4.10. *(1) If $\eta > n$, then $M = \bigcup_{\sigma \in \Sigma_n} M_{\sigma,\eta}$.*

(2) If ϕ is a carpeting function on M, then the function $\phi_{\sigma,\eta} : M_{\sigma,\eta} \longrightarrow (0,\infty)$ defined by $\phi_{\sigma,\eta}(x) := \left(\eta - \mathbf{n}\left(\sigma^* d(\sigma^{-1}(x))\right)\right)\phi(x)$ is a carpeting function on $M_{\sigma,\eta}$.

(3) If d is definable, then so is each $M_{\sigma,\eta}$.

Proof. Part (1) follows from the following elementary observation: let $E \subseteq \mathbb{R}^n$ be a linear subspace of dimension d. Then there exists a $\sigma \in \Sigma_n$ such that $\mathbf{n}(\sigma^{-1}(E)) \le n$.

To see this, given a basis $\{v_1,\dots,v_d\}$ of E and $\sigma \in \Sigma_n$, denote by $(v_1,\dots,v_d)_\sigma$ the signed volume of the parallelepiped in $\mathbb{R}^d$ spanned by the vectors $\Pi_d(\sigma(v_1))$, $\dots$, $\Pi_d(\sigma(v_d))$, and choose a $\sigma_0 \in \Sigma$ such that $|(v_1,\dots,v_d)_{\sigma_0}|$ is maximal. Since the map $(v_1,\dots,v_d) \mapsto (v_1,\dots,v_d)_\sigma$ is d-linear for each σ, σ_0 is independent of the particular basis considered; we claim that the lemma works with $\sigma = \sigma_0$.

To prove the claim, assume for simplicity of notation that σ_0 is the identity map on $\mathbb{R}^n$. Then $\Pi_d(E) = \mathbb{R}^d$, so there is a matrix $L = (l_{i,j}) \in \mathrm{M}_{n-d,d}(\mathbb{R})$ (with respect to the standard bases for $\mathbb{R}^d$ and $\mathbb{R}^{n-d}$) such that $E = \{\,(u, Lu) : u \in \mathbb{R}^p\,\}$. Let $\{e_1,\dots,e_d\}$ be the standard basis of $\mathbb{R}^d$, and consider the vectors $v_k = (e_k, Le_k) \in E$ for $k = 1,\dots,d$; clearly $\{v_1,\dots,v_d\}$ is a basis of E. For $i \in \{1,\dots,n-d\}$ and $j \in \{1,\dots,d\}$, denote by $\sigma_{i,j} \in \Sigma$ the permutation that exchanges the j-th and the $(p+i)$-th coordinates. Then $l_{i,j} = (v_1,\dots,v_d)_{\sigma_{i,j}}$ for all i, j, and the maximality of $|(v_1,\dots,v_d)_{\sigma_0}|$ gives $|l_{i,j}| \le |(v_1,\dots,v_d)_{\sigma_0}| = 1$ for all i and j, and hence $|L| \le 1$, as required.

For part (2), assume for simplicity of notation that σ is the identity map and note that $\eta - \mathbf{n}(d(x)) < \eta$ for all $x \in M_{\sigma,\eta}$. It is straightforward to see that $\lim_{y\to x} \phi_{\sigma,\eta}(x) = 0$ for $x \in \operatorname{fr} M_{\sigma,\eta}$; in particular, the function $\psi_{\sigma,\eta} : M \longrightarrow [0,\infty)$, defined by $\psi_{\sigma,\eta}(x) := \phi_{\sigma,\eta}(x)$ if $x \in M_{\sigma,\eta}$ and $\psi_{\sigma,\eta}(x) := 0$ otherwise, is continuous. Thus for all $a > 0$, the set $\phi_{\sigma,\eta}^{-1}([a,0)) = \psi_{\sigma,\eta}^{-1}([a,\infty))$ is a closed subset of M contained in the compact set $\phi^{-1}([a/\eta,\infty))$, hence is itself compact.

For part (3), note that the set of all $E \in G_n^m$ satisfying $\mathbf{n}(E) < \eta$ is semialgebraic. □

5 Pfaffian Limits

In this section, I introduce the pfaffian limits over $\mathcal{R}$, which are used to describe the frontiers of pfaffian sets. I then establish several regularity properties for pfaffian limits over $\mathcal{R}$, which allow me to give a quick proof of Theorem A in Sect. 6 and to prepare the terrain for the proof of Theorem B.

Let $M \subseteq \mathbb{R}^n$ be a *bounded*, definable C^2-submanifold of dimension m. Let $\mathcal{D}_0$ be a finite set of definable $(m-1)$-distributions on M, and let $1 \le l \le m$ and d_0 be a definable l-distribution on M. Put $\mathcal{D} := \mathcal{D}_0 \cup \{d_0\}$, and if N is a C^2-submanifold of M that is compatible with $\mathcal{D}$, I set $\mathcal{D}^N := \{\, d^N : d \in \mathcal{D} \}$.

I assume in this section that $\mathcal{D}$ is compatible with M.

Definition 5.1. Let V be an integral manifold of $d_{\mathcal{D}}$. Then V is **admissible** if there are Rolle leaves $W_d = W_d(V)$ of d for each $d \in \mathcal{D}_0$ and a definable, closed integral manifold $B = B(V)$ of d_0 such that $V = W \cap B$, where $W = W(V) := M \cap \bigcap_{d \in \mathcal{D}_0} W_d$ (in particular, $W = M$ if $\mathcal{D}_0 = \emptyset$).

Note that in this situation, W is uniquely determined by V and $\mathcal{D}$, while B is not; I call W the **core** of V and B a **definable part** of V. Note also that every admissible integral manifold of $d_{\mathcal{D}}$ is closed, so that its frontier is a subset of fr M.

Remark 5.2. Let N be a definable C^2-submanifold of M compatible with $\mathcal{D}$, let W_d be a Rolle leaf of d for each $d \in \mathcal{D}_0$, and put $W := M \cap \bigcap_{d \in \mathcal{D}_0} W_d$. By Corollary 3.6(1), $W \cap N$ is a finite union of closed integral manifolds $W_1^N, \dots, W_q^N$ of $\left(d_{\mathcal{D}_0}\right)^N$ of the form $W_p^N = N \cap \bigcap_{d \in \mathcal{D}_0} W_{p,d}^N$, where each $W_{p,d}^N$ is a Rolle leaf of d^N.

Let now V be an admissible integral manifold of $d_{\mathcal{D}}$ with core W and definable part B. Writing $V_p^N := W_p^N \cap B$ for $p = 1, \dots, q$, it follows that $V \cap N = V_1^N \cup \dots \cup V_q^N$ and each V_p^N is an admissible integral manifold of $(d_{\mathcal{D}})^N$ with core W_p^N and definable part $B \cap N$.

Definition 5.3. Let $(V_\iota)_{\iota \in \mathbb{N}}$ be a sequence of integral manifolds of $d_{\mathcal{D}}$.

(1) The sequence (V_ι) is **admissible** if each V_ι is admissible with same core W and there is a definable family $\mathcal{B}$ of closed integral manifolds of d_0 such that $B(V_\iota) \in \mathcal{B}$ for all ι. In this situation, W is the **core** of (V_ι) and $\mathcal{B}$ is a **definable part** of (V_ι).
(2) Assume (V_ι) is admissible with core W and definable part $\mathcal{B}$. If (V_ι) converges to $K \in \mathcal{K}_n$, I call K a **pfaffian limit over** $\mathcal{R}$ and say that K **is obtained from** $\mathcal{D}$ with **core** W.

I think of the core W in Definition 5.3 as representing the non-definable content of the admissible sequence (V_ι). It is crucial to the arguments in this section that only the definable parts of the V_ι are allowed to vary with ι.

Exercise 5.4. Let (V_ι) be an admissible sequence of integral manifolds of $d_{\mathcal{D}}$ with core W and definable part $\mathcal{B}$.

(1) Let $\mathcal{C}$ be a partition of M into definable C^2-cells compatible with $\mathcal{D}$, and assume that $K = \lim_\iota V_\iota$ exists and $K^N = \lim_\iota(V_\iota \cap N)$ exists for each $N \in \mathcal{C}$. Then $K = \bigcup_{N \in \mathcal{C}} K^N$.
(2) Let N be a C^2-submanifold of M compatible with $\mathcal{D}$, and adopt the notations of Remark 5.2. Assume that $K^N = \lim_\iota(V_\iota \cap N)$ and each $K_p^N = \lim_\iota(V_\iota)_p^N$ exist. Then each sequence $(V_\iota)_p^N$ is admissible with core W_p^N and finite part $\mathcal{B}^N := \{B \cap N : B \in \mathcal{B}\}$ and $K^N = K_1^N \cup \cdots \cup K_q^N$; in particular, K^N is a finite union of pfaffian limits obtained from $\mathcal{D}^N$ with cores among $W_1^N, \ldots, W_q^N$.

Remark. The sets W_p^N of the previous exercise are definable in $\mathcal{R}(W)$. More precisely, let $\mathcal{L}_\mathcal{R}$ be the language containing a relation symbol for every definable set. Then each W_p^N is quantifier-free definable in the language $\mathcal{L}_\mathcal{R}(W)$.

To understand the frontier of an admissible integral manifold in terms of pfaffian limits over $\mathcal{R}$, I assume there is a definable C^2-carpeting function ϕ on M and define $g_\phi : M \longrightarrow G_n$ by $g_\phi(x) := \ker d\phi(x) \subseteq T_x M$. Let $\mathcal{C}$ be a Whitney stratification of M by definable C^2-cells compatible with $\mathcal{D} \cup \{g_\phi\}$, as obtained from Proposition 3.3, and put

$$\mathcal{C}^\phi := \{ C \in \mathcal{C} : (d_\mathcal{D})^C \not\subseteq g_\phi \}.$$

Then $\dim \left(d_\mathcal{D} \cap g_\phi\right)^C < \dim d_\mathcal{D}$ for every $C \in \mathcal{C}^\phi$. Set $d_0^\phi := d_0 \cap g_\phi$ and $\mathcal{D}^\phi := \mathcal{D}_0 \cup \{d_0^\phi\}$.

Lemma 5.5. *The union of all cells in $\mathcal{C}^\phi$ is an open subset M^ϕ of M that is compatible with $\mathcal{D}^\phi$ and* $\dim(d_{\mathcal{D}^\phi})^{M^\phi} < \dim d_\mathcal{D}$.

Proof. Note first that if $C, D \in \mathcal{C}$ are such that $D \subseteq \operatorname{fr} C$, then by the Whitney property of the pair (C, D), as defined on p. 502 of [24], for every sequence $(x_i)_{i \in \mathbb{N}}$ of points in C that converges to a point $y \in D$ and for which $T := \lim_i T_{x_i} C$ exists in $G_n^{\dim C}$, the inclusion $T_y D \subseteq T$ holds. Since $d_\mathcal{D}$ and g_ϕ are continuous, it follows that the union of all cells in $\mathcal{C} \setminus \mathcal{C}^\phi$ is a closed subset of M; hence M^ϕ is an open subset of M. Finally, the definition of M^ϕ implies that $d_\mathcal{D}(x) \not\subseteq g_\phi(x)$ for all $x \in M^\phi$, and the lemma is proved. □

Next, let W_d be a Rolle leaf of d for each $d \in \mathcal{D}_0$ and put $W := M \cap \bigcap_{d \in \mathcal{D}_0} W_d$. I adopt here the notations of Remark 5.2 corresponding to each $N \in \mathcal{C} \cup \{M^\phi\}$; to simplify notation, I assume that the corresponding q is the same for each of these N by not requiring that the sets $W_1^N, \ldots, W_q^N$ be pairwise distinct. Then for $N \in \mathcal{C}^\phi \cup \{M^\phi\}$, $p = 1, \ldots, q$, $r > 0$ and every admissible integral manifold V of $d_\mathcal{D}$ with core W and definable part B, the set $\left(V_p^N \cap \phi^{-1}(r_\kappa)\right)_\kappa$ is an admissible integral manifold of $(d_{\mathcal{D}^\phi})^N$ with core W_p^N and definable part $B \cap N \cap \phi^{-1}(r)$.

Lemma 5.6. *Let V be an admissible integral manifold of $d_\mathcal{D}$ with core W, and let $(r_\kappa)_{\kappa \in \mathbb{N}}$ be a sequence of positive real numbers such that $r_\kappa \to 0$ and $K_p^{M^\phi}(V) :=$*

$\lim_\kappa \left(V_p^{M^\phi} \cap \phi^{-1}(r_\kappa)\right)$ *exists for each* p. *Then each* $K_p^{M^\phi}(V)$ *is a pfaffian limit over* $\mathcal{R}$ *obtained from* $(\mathcal{D}^\phi)^{M^\phi}$ *with core* $W_p^{M^\phi}$ *and* $\operatorname{fr} V = K_1^{M^\phi}(V) \cup \cdots \cup K_q^{M^\phi}(V)$.

Proof. Passing to a subsequence if necessary, we may assume

$$K_p^N := \lim_\kappa \left(V_p^N \cap \phi^{-1}(r_\kappa)\right)$$

exists for each p and $N \in \mathcal{C}^\phi$ as well; note that

$$\bigcup_{p=1}^{q} K_p^{M^\phi}(V) = \bigcup_{N \in \mathcal{C}^\phi} \bigcup_{p=1}^{q} K_p^N(V).$$

Let $x \in \lim_\kappa \left(V \cap \phi^{-1}(r_\kappa)\right)$; by Lemma 4.7 and the above, it suffices to show that $x \in K_p^N(V)$ for some $N \in \mathcal{C}^\phi$ and $p \in \{1, \dots, q\}$. Let $x_\kappa \in V \cap \phi^{-1}(r_\kappa)$ be such that $\lim x_\kappa = x$. Let $N \in \mathcal{C}$ be such that infinitely many x_κ belong to N; passing to a subsequence, we may assume that $x_\kappa \in N$ for all κ. Then $N \in \mathcal{C}^\phi$: otherwise $g_N \cap d_\mathcal{D} \subseteq d_\phi$, which implies that $N \cap V \cap \phi^{-1}(r) = \emptyset$ for all but finitely many $r > 0$. Thus, after again passing to a subsequence if necessary, we may assume that there is a p such that $x_\kappa \in V_p^N \cap \phi^{-1}(r_\kappa)$ for all κ. Hence $x \in K_p^N(V)$, as required. □

Proposition 5.7. *Let* (V_ι) *be an admissible sequence of integral manifolds of* $d_\mathcal{D}$ *with core* W, *and assume that* $K' := \lim_\iota \operatorname{fr} V_\iota$ *exists. Then* K' *is a finite union of pfaffian limits obtained from* $(\mathcal{D}^\phi)^{M^\phi}$ *with cores among* $W_1^{M^\phi}, \dots, W_q^{M^\phi}$.

Proof. Let $\mathcal{B}$ be a definable part of (V_ι). Let $(r_\kappa)_{\kappa \in \mathbb{N}}$ be a sequence of positive real numbers such that $r_\kappa \to 0$ and $K_p^{M^\phi}(V_\iota) := \lim_\kappa \left((V_\iota)_p^{M^\phi} \cap \phi^{-1}(r_\kappa)\right)$ exists for each p and each ι. Passing to a subsequence if necessary, we may assume that $\lim_\iota K_p^{M^\phi}(V_\iota)$ exists for each p. Then by the previous lemma,

$$K' = \lim_\iota \left(\lim_\kappa \left(V_\iota \cap \phi^{-1}(r_\kappa)\right)\right) = \bigcup_{p=1}^{q} \lim_\iota K_p^{M^\phi}(V_\iota).$$

So by Exercise 4.1(4), $K' = \bigcup_{p=1}^q \lim_\iota \left((V_\iota)_p^{M^\phi} \cap \phi^{-1}(r_{\kappa(\iota)})\right)$ for some subsequence $(\kappa(\iota))_\iota$. Since for each p, the sequence $\left((V_\iota)_p^{M^\phi} \cap \phi^{-1}(r_{\kappa(\iota)})\right)$ is an admissible sequence of integral manifolds of $(d_{\mathcal{D}^\phi})^{M^\phi}$ with core $W_p^{M^\phi}$ and definable part $\mathcal{B}_\phi^{M^\phi}$, the proposition follows. □

Proposition 5.8. *Let* K *be a pfaffian limit obtained from* $\mathcal{D}$. *Then* K *has dimension and* $\dim K \leq \dim d_\mathcal{D}$.

Proof. Let (V_ι) be an admissible sequence of integral manifolds of $d_\mathcal{D}$ such that $K = \lim_\iota V_\iota$. We proceed by induction on $\dim d_\mathcal{D}$. If $\dim d_\mathcal{D} = 0$, Corollary 3.6(2) gives a uniform bound on the cardinality of V_ι, so K is finite. So assume $\dim d_\mathcal{D} > 0$

and the corollary holds for all pfaffian limits obtained from finite sets $\mathcal{D}'$ of definable distributions on manifolds M' that are compatible with M' and satisfy $\dim d_{\mathcal{D}'} < \dim d_{\mathcal{D}}$.

By Proposition 3.3 and Exercise 5.4(2), we may assume that M is a definable C^2-cell; in particular, there is a definable C^2-carpeting function ϕ on M. For each $\sigma \in \Sigma_n$, let $M_{\sigma,2n}$ be as before Lemma 4.10 with $d_{\mathcal{D}}$ in place of d. Then by that lemma, $M = \bigcup_{\sigma\in\Sigma} M_{\sigma,2n}$ and each $M_{\sigma,2n}$ is an open subset of M. Hence $\mathcal{D}$ is compatible with each $M_{\sigma,2n}$, and after passing to a subsequence if necessary, we may assume that $K_\sigma = \lim_\iota(V_\iota \cap M_{\sigma,2n})$ exists for each σ. As in Exercise 5.4(2), it follows that $K = \bigcup_{\sigma\in\Sigma_n} K_\sigma$, so by Lemma 4.10(2), after replacing M with each $\sigma^{-1}(M_{\sigma,2n})$, we may assume that $d_{\mathcal{D}}$ is $2n$-bounded. Passing to a subsequence again, we may assume that $K' := \lim_\iota \operatorname{fr} V_\iota$ exists as well. Then by Corollary 4.6, the set $K \setminus K'$ has dimension at most $\dim d_{\mathcal{D}}$, while by Proposition 5.7 the set K' is a finite union of pfaffian limits obtained from a finite set $\mathcal{D}'$ of definable distributions on a definable manifold M' that is compatible with M' and satisfies $\dim d_{\mathcal{D}'} < \dim d_{\mathcal{D}}$. So K' has dimension with $\dim K' < \dim d_{\mathcal{D}}$ by the inductive hypothesis, and the proposition is proved. □

Definition 5.9. A pfaffian limit $K \subseteq \mathbb{R}^n$ obtained from $\mathcal{D}$ is **proper** if $\dim K = \dim d_{\mathcal{D}}$.

Exercise 5.10. Let $K \subseteq \mathbb{R}^n$ be a pfaffian limit over $\mathcal{R}$ with core W. Prove that $K = K_1 \cup \cdots \cup K_l$, where each K_j is a proper pfaffian limit over $\mathcal{R}$ whose core is definable in $\mathcal{R}(W)$. [Hint: proceed as in the proof of Proposition 5.8.]

Finally, pfaffian limits over $\mathcal{R}$ are well behaved with respect to intersecting with closed definable sets. To see this, define $\mathbf{M} := M \times (0,1)$ and write (x,ϵ) for the typical element of $\mathbf{M}$ with $x \in M$ and $\epsilon \in (0,1)$. I also consider $\mathcal{D}$ as a set of distributions on $\mathbf{M}$ in the obvious way, and I set $\mathbf{d}_0 := d_0 \cap (\ker d\epsilon)_{\restriction \mathbf{M}}$ and $\mathbf{D} := \mathcal{D}_0 \cup \{\mathbf{d}_0\}$. For $d \in \mathcal{D}$, the set $\mathbf{W}_d := W_d \times (0,1)$ is a Rolle leaf of d, and I put $\mathbf{W} := \bigcap_{d\in\mathcal{D}} \mathbf{W}_d$; then $\mathbf{W}$ is definable in $\mathcal{R}(W)$.

Note that $\mathbf{M}$ is compatible with $\mathbf{D}$, and whenever (V_ι) is an admissible sequence of integral manifolds of $d_{\mathcal{D}}$ with core W and $\epsilon_\iota \in (0,1)$ for $\iota \in \mathbb{N}$, the sequence $\big(V_\iota \times \{\epsilon_\iota\}\big)$ is an admissible sequence of integral manifolds of $d_{\mathbf{D}}$ with core $\mathbf{W}$.

Proposition 5.11. *Let K be a pfaffian limit obtained from $\mathcal{D}$ with core W, and let $C \subseteq \mathbb{R}^n$ be a definable closed set. Then there is a definable open subset $\mathbf{N}$ of $\mathbf{M}$ and there are $q \in \mathbb{N}$ and pfaffian limits $K_1, \dots, K_q \subseteq \mathbb{R}^{n+1}$ obtained from $\mathbf{D}^{\mathbf{N}}$ with cores definable in $\mathcal{R}(W)$ such that $K \cap C = \Pi_n(K_1) \cup \cdots \cup \Pi_n(K_q)$.*

Proof (Sketch of proof). For $\epsilon > 0$ put $T(C,\epsilon) := \{\, x \in \mathbb{R}^n : d(x,C) < \epsilon \,\}$. Note first that $K \cap C = \bigcap_{\epsilon>0} \big(K \cap T(C,\epsilon)\big)$, and the latter is equal to $\lim_{\epsilon\to 0} \big(K \cap T(C,\epsilon)\big)$ in the sense of Definition 4.8. Next, let (V_ι) be an admissible sequence of integral manifolds of $d_{\mathcal{D}}$ such that $K = \lim_\iota V_\iota$. Then for every $\epsilon > 0$, there is a subsequence $(\iota(\kappa))$ of (ι) such that the sequence $(V_{\iota(\kappa)} \cap T(C,\epsilon))$ converges to some compact set K_ϵ. Note that $K_\epsilon \cap T(C,\epsilon) = K \cap T(C,\epsilon)$, since $T(C,\epsilon)$ is an open set.

Fix a sequence (ϵ_κ) of positive real numbers approaching 0, and for each κ, choose $\iota(\kappa)$ such that $d(V_{\iota(\kappa)} \cap T(C,\epsilon_\kappa), K_{\epsilon_\kappa}) < \epsilon_\kappa$. Passing to a subsequence if necessary, we may assume that $\lim_\kappa K_{\epsilon_\kappa}$ and $\lim_\kappa \left(V_{\iota(\kappa)} \cap T(C,\epsilon_\kappa)\right)$ exist; note that these limits are then equal. Hence by the above, $K \cap C = \lim_\kappa \left(K \cap T(C,\epsilon_\kappa)\right) = \lim_\kappa \left(K_{\epsilon_\kappa} \cap T(C,\epsilon_\kappa)\right) \subseteq \lim_\kappa K_{\epsilon_\kappa} = \lim_\kappa \left(V_{\iota(\kappa)} \cap T(C,\epsilon_\kappa)\right)$. The reverse inclusion is obvious, so $K \cap C = \lim_\kappa \left(V_{\iota(\kappa)} \cap T(C,\epsilon_\kappa)\right)$. Therefore, put

$$\mathbf{N} := \{\, (x,\epsilon) \in \mathbf{M} : \ d(x,C) < \epsilon \,\}.$$

Then $\mathbf{N}$ is an open, definable subset of $\mathbf{M}$ and by the above $K \cap C = \lim_\kappa (V_{\iota(\kappa)} \cap \mathbf{N}^{\epsilon_\kappa})$, where $\mathbf{N}^\epsilon := \{\, x \in M : \ (x,\epsilon) \in \mathbf{N} \,\}$. Hence $K \cap C = \lim_\kappa \Pi_n \left((V_{\iota(\kappa)} \times \{\epsilon_\kappa\}) \cap \mathbf{N}\right)$. Since $\lim_\kappa \epsilon_\kappa = 0$, it follows that $K \cap C = \Pi_n \left(\lim_\kappa \left((V_{\iota(\kappa)} \times \{\epsilon_\kappa\}) \cap \mathbf{N}\right)\right)$. Since the sequence $\left(V_{\iota(\kappa)} \times \{\epsilon_\kappa\}\right)$ is an admissible sequence of integral manifolds of $\mathbf{d}$, the proposition now follows from Remark 5.2. ☐

Exercise 5.12. Let $\mathcal{B}$ and $\mathcal{C}$ be two definable families of closed subsets of $\mathbb{R}^n$. Prove that the pfaffian limits in the previous proposition depend uniformly on $C \in \mathcal{C}$, for all pfaffian limits obtained from $\mathcal{D}$ with definable part $\mathcal{B}$. That is, there are $\mu, q \in \mathbb{N}$, a bounded, definable manifold $\mathbf{M} \subseteq \mathbb{R}^{n+\mu+1}$, a finite set $\mathbf{D}$ of distributions on $\mathbf{M}$ and a definable family $\mathbf{B}$ of subsets of $\mathbb{R}^{n+\nu+1}$ such that whenever K is a pfaffian limit obtained from $\mathcal{D}$ with definable part $\mathcal{B}$ and $C \in \mathcal{C}$, there are pfaffian limits $K_1, \dots, K_q \subseteq \mathbb{R}^{n+\nu+1}$ obtained from $\mathbf{D}$ with definable part $\mathbf{B}$ such that $K \cap C = \Pi_n(K_1) \cup \dots \cup \Pi_n(K_q)$.

6 O-minimality

I now fix an arbitrary reduct $\mathcal{R}'$ of $\mathcal{R}_1$ that expands $\mathcal{R}$.

Definition 6.1. A set $X \subseteq \mathbb{R}^k$ is a **basic** $\Lambda(\mathcal{R}')$**-set** if there exist $n \geq k$, a definable, bounded C^2-manifold $M \subseteq \mathbb{R}^n$ of dimension m, a finite set $\mathcal{D}_0$ of definable $(m-1)$-distributions on M and a definable l-distribution d_0 on M, and for each $\kappa \in \mathbb{N}$ an admissible sequence $(V_{\kappa,\iota})_\iota$ of integral manifolds of $d_\mathcal{D}$ with core W and definable part $\mathcal{B}$ independent of κ, where $\mathcal{D} := \mathcal{D}_0 \cup d_0$, such that:

(i) The core W is definable in $\mathcal{R}'$;
(ii) For each κ, the limit $K_\kappa := \lim_\iota V_{\kappa,\iota}$ exists in $\mathcal{K}_n$;
(iii) The sequence $(\Pi_k(K_\kappa))_\kappa$ is increasing and has union X.

In this situation, I say that X is **obtained from** $\mathcal{D}$ with **core** W and **definable part** $\mathcal{B}$. A $\Lambda(\mathcal{R}')$**-set** is a finite union of basic $\Lambda(\mathcal{R}')$-sets. I denote by $\Lambda(\mathcal{R}')_k$ the collection of all $\Lambda(\mathcal{R}')$-sets in $\mathbb{R}^k$ and put $\Lambda(\mathcal{R}') := (\Lambda(\mathcal{R}')_k)_{k\in\mathbb{N}}$.

Whenever $\mathcal{R}'$ is clear from context, I shall simply write "Λ" instead of "$\Lambda(\mathcal{R}')$".

Proposition 6.2. *In the situation of Definition 6.1, there is an $N \in \mathbb{N}$, depending only on $\mathcal{D}$ and $\mathcal{B}$ but not on W, such that every basic Λ-set obtained from $\mathcal{D}$ with* *core W and definable part $\mathcal{B}$ has at most N components. In particular, if $X \subseteq \mathbb{R}^k$ is a Λ-set and $l \in k$, there is an $N \in \mathbb{N}$ such that for every $a \in \mathbb{R}^l$ the fiber X_a has at most N components.*

Proof. Let N be a bound on the number of components of the sets $W \cap B$ as $W = \bigcap_{d \in \mathcal{D}_0} W_d$ ranges over all intersections of Rolle leaves W_d of $d \in \mathcal{D}_0$ and B ranges over $\mathcal{B}$. Let X be a basic Λ-set as in Definition 6.1. Then each $V_{\kappa,\iota}$ has at most N components, so each K_κ has at most N components, and hence X has at most N components. Combining this observation with Exercise 5.12 yields, for every Λ-set $X \subseteq \mathbb{R}^k$, a uniform bound on the number of connected components of the fibers of X. □

Proposition 6.3. *(1) Any coordinate projection of a pfaffian limit over $\mathcal{R}$ whose core is definable in $\mathcal{R}'$ is a Λ-set.*
(2) Every bounded definable set is a Λ-set.
(3) Let d be a definable $(n-1)$-distribution on $M := (-1,1)^n$ and L be a Rolle leaf of d definable in $\mathcal{R}'$. Then L is a Λ-set.

Proof. (1) is obvious. For (2), let $C \subseteq \mathbb{R}^n$ be a bounded, definable cell. By cell decomposition, it suffices to show that C is a Λ-set. Let ϕ be a definable carpeting function on C. Then $C = \bigcup_{i=1}^{\infty} \mathrm{cl}\left(\phi^{-1}((1/i,\infty))\right)$, so let $\mathbf{C} := \{(x,r) \in C \times (0,1) : \phi(x) > r\}$ and put $\mathbf{d}_0 := \ker dr \restriction \mathbf{C}$ and $\mathbf{D} := \{\mathbf{d}_0\}$. Then for $r > 0$, the set $\mathbf{C}^r = \phi^{-1}((r,\infty)) \times \{r\}$ is an admissible integral manifold of $d_{\mathbf{D}}$ with core $\mathbf{C}$ and definable part $\mathbf{C}^r$, so $\mathrm{cl}(\mathbf{C}^r)$ is a pfaffian limit obtained from $\mathcal{D}$ with definable core.

(3) Let ϕ be a carpeting function on M. Then $L = \bigcup_{i=1}^{\infty} \mathrm{cl}\left(L \cap \phi^{-1}((1/i,\infty))\right)$, so let $\mathbf{M} := \{(x,r) \in M \times (0,1) : \phi(x) > r\}$ and put $\mathbf{d}_0 := \ker dr \restriction \mathbf{M}$, $\mathbf{d} := d \restriction \mathbf{M}$ and $\mathbf{D} := \{\mathbf{d}, \mathbf{d}_0\}$. Let $L_1, \dots, L_q$ be the components of $(L \times (0,1)) \cap \mathbf{M}$; note that each L_p is a Rolle leaf of $\mathbf{d}$. Thus for $r > 0$ and each p, the set $L_p \cap \phi^{-1}((r,\infty))$ is an admissible integral manifold of $d_{\mathbf{D}}$ with core L_p and definable part $\mathbf{M}^r = \phi^{-1}((r,\infty)) \times \{r\}$. □

Proposition 6.4. *The collection of all Λ-sets is closed under taking finite unions, finite intersections, coordinate projections, cartesian products, permutations of coordinates and topological closure.*

Proof. Closure under taking finite unions, coordinate projections and permutations of coordinates is obvious from the definition and the properties of pfaffian sets over $\mathcal{R}$.

For topological closure, let $X \subseteq \mathbb{R}^k$ be a basic Λ-set with associated data as in Definition 6.1. Then $\mathrm{cl}(X) = \lim_\kappa \Pi_k(K_\kappa) = \Pi_k(\lim_\kappa \lim_\iota V_{\kappa,\iota}) = \Pi_k(\lim_\kappa V_{\kappa,\iota(\kappa)})$ for some subsequence $(\iota(\kappa))_\kappa$; in particular, $\mathrm{cl}(X)$ is a projection of a pfaffian limit over $\mathcal{R}$ with same core as X.

For cartesian products, let $X_1 \subseteq \mathbb{R}^{k_1}$ and $X_2 \subseteq \mathbb{R}^{k_2}$ be basic Λ-sets, and let $M^i \subseteq \mathbb{R}^{n_i}$, $\mathcal{D}^i$ and $\left(V^i_{\iota,\kappa}\right)$ be the data associated to X_i as in Definition 6.1, for $i = 1, 2$. Denote their cores by $W_1 \subseteq \mathbb{R}^{n_1}$ and $W_2 \subseteq \mathbb{R}^{n_2}$, respectively. We assume that both M^1 and M^2 are connected; the general case is easily reduced to this situation. Define

$$\mathbf{M} := \{ (x, y, u, v) : (x, u) \in M^1 \text{ and } (y, v) \in M^2 \},$$

where x ranges over $\mathbb{R}^{k_1}$, y over $\mathbb{R}^{k_2}$, u over $\mathbb{R}^{n_1-k_1}$ and v over $\mathbb{R}^{n_2-k_2}$. We interpret $\mathcal{D}^1$ and $\mathcal{D}^2$ as sets of distributions on $\mathbf{M}$ correspondingly and put $\mathbf{D} := \mathcal{D}^1 \cup \mathcal{D}^2$. Since M^1 and M^2 are connected, each set

$$V_{\kappa,\iota} := \{ (x, y, u, v) : (x, u) \in V^1_{\kappa,\iota} \text{ and } (y, v) \in V^2_{\kappa,\iota} \}$$

is an admissible integral manifold of $d_{\mathbf{D}}$ with core

$$\mathbf{W} := \{ (x, y, u, v) : (x, u) \in W_1 \text{ and } (y, v) \in W_2 \}.$$

It is now easy to see that for each κ, the limit $K_\kappa := \lim_\iota V_{\kappa,\iota}$ exists in $\mathcal{K}_{n_1+n_2}$, and that the sequence $\left(\Pi_{k_1+k_2}(K_\kappa)\right)$ is increasing and has union $X_1 \times X_2$.

For intersections, let $X_1, X_2 \subseteq \mathbb{R}^k$ be basic Λ-sets. Then $X_1 \cap X_2 = \Pi_k((X_1 \times X_2) \cap \Delta)$, where $\Delta := \{ (x, y) \in \mathbb{R}^k \times \mathbb{R}^k : x_i = y_i \text{ for } i = 1, \dots, k \}$. Therefore, we let $X \subseteq \mathbb{R}^k$ be a basic Λ-set and $C \subseteq \mathbb{R}^k$ be closed and definable, and we show that $X \cap C$ is a Λ-set. Let the data associated to X be as in Definition 6.1, and let $\mathbf{M}$, $\mathbf{D}$ and $\mathbf{W}$ be associated to that data as before Proposition 5.11. Let also $\mathbf{N}$ be the open subset of $\mathbf{M}$ given by that proposition with $C' := C \times \mathbb{R}^{n-k}$ in place of C. Then by that proposition, there is a $q \in \mathbb{N}$ such that for every κ the set $K_\kappa \cap C'$ is the union of the projections of pfaffian limits $K^1_\kappa, \dots, K^q_\kappa$ obtained from $\mathbf{D}^{\mathbf{N}}$ with cores definable in $\mathcal{R}'$. Note that each K^j_κ is the limit of an admissible sequence of integral manifolds of $\mathbf{D}^{\mathbf{N}}$ whose core depends only on j but not on κ. Replacing each sequence $\left(K^j_\kappa\right)$ by a (possibly finite) subsequence if necessary, we may assume that each sequence $\left(\Pi_k(K^j_\kappa)\right)$ is increasing. Then each $X_j := \bigcup_\kappa K^j_\kappa$ is a basic Λ-set and $X \cap C = X_1 \cup \dots \cup X_q$. □

Proposition 6.5. *Let $X \subseteq \mathbb{R}^k$ be a Λ-set. Then $\mathrm{bd}(X)$ is contained in a closed Λ-set with empty interior.*

Proof. Let the data associated to X be given as in Definition 6.1. Note that

$$\mathrm{bd}(X) \subseteq \lim_\kappa \mathrm{bd}(\Pi_k(K_\kappa)).$$

Fix an arbitrary κ; since $\Pi_k(K_\kappa) = \lim_\iota \Pi_k(V_{\kappa,\iota})$ we may assume, by Proposition 3.10, Exercise 5.4 and after replacing M if necessary, that $\Pi_k \restriction d_{\mathcal{D}}$ is an immersion and has constant rank $r \le k$; in particular, $\dim(V_{\kappa,\iota}) \le k$. If $r < k$, then each $\Pi_k(K_\kappa)$ has empty interior by Proposition 4.6, so

$$\lim_\kappa \mathrm{bd}(\Pi_k(K_\kappa)) = \lim_\kappa \Pi_k(K_\kappa) = \Pi_k(\lim_\kappa K_\kappa) = \Pi_k(\lim_\kappa V_{\kappa,\iota(\kappa)})$$

for some subsequence $(\iota(\kappa))$, and we conclude by Propositions 5.8 and 6.3(1) in this case. So assume that $r = k$; in particular, $\Pi_k(V_{\kappa,\iota})$ is open for every κ and ι. In this case, since M is bounded, we have $\mathrm{bd}(\Pi_k(K_\kappa)) \subseteq \Pi_k(\lim_\iota \mathrm{fr}\, V_{\kappa,\iota})$ for each κ. Hence

$$\lim_\kappa \mathrm{bd}(\Pi_k(K_\kappa)) \subseteq \Pi_k(\lim_\kappa \lim_\iota \mathrm{fr}\, V_{\kappa,\iota}) = \Pi_k(\lim_\kappa \mathrm{fr}\, V_{\kappa,\iota(\kappa)})$$

for some subsequence $(\iota(\kappa))$, and we are done by Propositions 5.7 and 6.3(1). □

Recall that for $S \subseteq \mathbb{R}^k$, $l \in \{1, \dots, k\}$ and $a \in \mathbb{R}^k$, we put

$$S_a := \{ y \in \mathbb{R}^{k-l} : (a, y) \in S \}.$$

Corollary 6.6. *Let $X \subseteq \mathbb{R}^k$ be a Λ-set, and let $1 \le l \le k$. Then the set*

$$B := \{ a \in \mathbb{R}^l : \mathrm{cl}(X_a) \ne \mathrm{cl}(X)_a \}$$

has empty interior.

Proof. It suffices to show that the corollary holds with $X \cap \left((-R, R)^l \times \mathbb{R}^{k-l}\right)$ in place of X, for each $R > 0$, so we assume that $\Pi_l(X)$ is bounded. For each $a \in B$ there is a box $U \subseteq \mathbb{R}^{k-l}$ such that $\mathrm{cl}(X_a) \cap U = \emptyset$, but $\mathrm{cl}(X)_a \cap U \ne \emptyset$. Hence $B = \bigcup_U B_U$, where U ranges over all rational boxes in $\mathbb{R}^{k-l}$ and

$$B_U := \{ a \in \mathbb{R}^l : \mathrm{cl}(X_a) \cap U = \emptyset,\ \mathrm{cl}(X)_a \cap U \ne \emptyset \}.$$

Each B_U is contained in the frontier of the bounded Λ-set $\Pi_l\left(X \cap \left(\mathbb{R}^l \times U\right)\right)$. So by the previous proposition $B_U \subseteq Y_U$ for some closed Λ-set Y_U with empty interior. Since each Y_U is compact, we conclude that B has empty interior. □

Proposition 6.7. *Let $X \subseteq [-1, 1]^k$ be a Λ-set. Then $[-1, 1]^k \setminus X$ is also a Λ-set.*

Proof. Set $I := [-1, 1]$. Let $X \subseteq I^k$ be a Λ-set. We establish the following two statements by induction on k:

- $(\mathrm{I})_k$ If $\mathrm{int}(X) = \emptyset$, then X can be partitioned into finitely many Λ-sets $G_1, \dots, G_K$ in such a way that, for each $i \in \{1, \dots, K\}$, there is a permutation π_i of $\{1, \dots, k\}$ such that $\pi_i(G_i)$ is the graph of a continuous function $f_i : \Pi_{k-1}(\pi_i(G_i)) \longrightarrow \mathbb{R}$.
- $(\mathrm{II})_k$ The complement $I^k \setminus X$ is a Λ-set, and the components of both X and $I^k \setminus X$ are Λ-sets.

The case $k = 1$ follows from Proposition 6.2; so assume $k > 1$ and that the two statements hold for lower values of k. First we establish the following

Claim. Assume there is a Λ-set $Z \subseteq I^k$ with empty interior such that $X \subseteq Z$ and $(\mathrm{I})_k$ and $(\mathrm{II})_k$ hold with Z in place of X. Then $(\mathrm{I})_k$ and $(\mathrm{II})_k$ hold.

To prove the claim, let $G_1, \dots, G_K$ be as in $(\mathrm{I})_k$ with Z in place of X. Clearly $(\mathrm{I})_k$ then also holds for X, since each $\pi_i(G_i \cap X)$ is the graph of the continuous function $f_i \restriction \Pi_{k-1}(\pi_i(G_i \cap X))$. Since the G_i partition Z and $(\mathrm{II})_k$ holds with Z in place of X, it suffices to prove for each i that the complement $G_i \setminus X$ as well as the components of both $G_i \cap X$ and $G_i \setminus X$ are Λ-sets. Fix an i; by Proposition 6.4, we reduce to the case that π_i is the identity map. Then by the inductive hypothesis, the set $\Pi_{k-1}(G_i \setminus X)$ as well as the components of both $\Pi_{k-1}(G_i \cap X)$ and $\Pi_{k-1}(G_i \setminus X)$ are Λ-sets, so the claim follows.

We now return to the proof of the theorem; there are two cases to consider.

Case 1. X has empty interior. Consider the Λ-sets

$$C_i := \{ a \in I^{k-1} : |X_a| \geq i \} \quad \text{for } i \in \mathbb{N}.$$

By $(\mathrm{II})_{k-1}$ the sets

$$D_i := C_i \setminus C_{i+1} = \{ a \in I^{k-1} : |X_a| = i \}$$

are also Λ-sets, and by Proposition 6.2 there is an $N \in \mathbb{N}$ such that $C_i = C_{N+1}$ for all $i > N$. Let $X_1 := X \cap ((C_0 \setminus C_{N+1}) \times I)$ and $X_2 := X \cap (C_{N+1} \times I)$; by the inductive hypothesis both X_1 and X_2 are Λ-sets. The next two paragraphs then finish the proof of Case 1.

First, note that $|(X_1)_a| \leq N$ for every $a \in \Pi_{k-1}(X_1)$. For $1 \leq j \leq i \leq N$, define the Λ-sets

$$X_{i,j} := \{ (a, y) \in D_i \times I : y \text{ is the } j^{\text{th}} \text{ element of } (X_1)_a \},$$
$$S_{i,j} := \{ a \in D_i : \left|(\mathrm{cl}(X_{i,j}))_a\right| \geq 2 \},$$

and put $S := \bigcup_{1 \leq j \leq i \leq N} S_{i,j}$. (Here we use the fact that the collection of Λ-sets is closed under taking topological closure.) Note that each $X_{i,j}$ is by construction the graph of a function that is continuous away from S. Thus, $(\mathrm{I})_k$ holds with $X_1 \setminus (S \times I)$ in place of X by construction, and the corresponding $(\mathrm{II})_k$ then follows easily from the inductive hypothesis (and since the order $<$ is semialgebraic). Note that

$$S_{i,j} \subseteq \{ a \in \mathbb{R}^{k-1} : \mathrm{cl}((X_{i,j})_a) \neq (\mathrm{cl}(X_{i,j}))_a \};$$

it follows from Corollary 6.6 that each $S_{i,j}$ has empty interior, so S has empty interior. Therefore $(\mathrm{I})_{k-1}$ and $(\mathrm{II})_{k-1}$ hold with S in place of X by the inductive hypothesis, and so $(\mathrm{I})_k$ and $(\mathrm{II})_k$ hold with $S \times I$ in place of X. The claim implies now that $(\mathrm{I})_k$ and $(\mathrm{II})_k$ also hold with $X_1 \cap (S \times I)$ in place of X, and with $(S \times I) \setminus X_1$ in place of X. Therefore, $(\mathrm{I})_k$ and $(\mathrm{II})_k$ hold with X_1 in place of X, and with $((C_0 \setminus C_{N+1}) \times I) \setminus X_1$ in place of X.

Second, note that $C_{N+1} = \Pi_{k-1}(X_2)$. Thus every fiber $(X_2)_a \subseteq I$ with $a \in C_{N+1}$ is infinite and hence (by Proposition 6.2 again) contains an interval. Since X_2 has empty interior, it follows that C_{N+1} has empty interior. $(\mathrm{I})_k$ and $(\mathrm{II})_k$, with X_2 as well as with $(C_{N+1} \times I) \setminus X_2$ in place of X, now follow from the claim by a similar argument as in the previous subcase.

Case 2. X has nonempty interior. By Proposition 6.5 there is a closed Λ-set $Y \subseteq I^k$ such that $\mathrm{bd}(X) \subseteq Y$ and Y has empty interior. By Case 1 applied to Y, both $(\mathrm{I})_k$ and $(\mathrm{II})_k$ hold with Y in place of X. Note that if C is a component of $I^k \setminus Y$ and $C \cap X \neq \emptyset$, then $C \subseteq X$. It follows that each component of $I^k \setminus Y$ is either contained in $X \setminus Y$ or is disjoint from $X \cup Y$. On the other hand, by the claim the statements $(\mathrm{I})_k$ and $(\mathrm{II})_k$ hold with $X \cap Y$ in place of X. Thus $(\mathrm{II})_k$ follows easily. □

For $m \in \mathbb{N}$, let $\mathcal{T}_m = \mathcal{T}(\mathcal{R}')_m$ be the collection of all Λ-sets $X \subseteq I^m$.

Corollary 6.8. *The collection $\mathcal{T} = \mathcal{T}(\mathcal{R}') := (\mathcal{T}_m)_m$ forms an o-minimal structure on I.* □

Proof (Proof of Theorem A). For each m, let $\tau_m : \mathbb{R}^m \longrightarrow (-1,1)^m$ be the (definable) homeomorphism given by

$$\tau_m(x_1,\dots,x_m) := \left(\frac{x_1}{1+x_1^2},\dots,\frac{x_m}{1+x_m^2}\right),$$

and let $\mathcal{S}_m = \mathcal{S}(\mathcal{R}')_m$ be the collection of sets $\tau_m^{-1}(X)$ with $X \in \mathcal{T}_m$. By Corollary 6.8, the collection $\mathcal{S} = \mathcal{S}(\mathcal{R}') := (\mathcal{S}_m)_m$ gives rise to an o-minimal expansion $\mathcal{R}_\mathcal{S}$ of $\mathcal{R}$. By Proposition 6.3(2), every definable set is definable in $\mathcal{R}_\mathcal{S}$. But every $L \in \mathcal{L}(\mathcal{R})$ that is definable in $\mathcal{R}'$ is definable in $\mathcal{R}_\mathcal{S}$ as well: if L is a Rolle leaf of a definable $(n-1)$-distribution d on $\mathbb{R}^n$, then $\tau_n(L)$ is a Rolle leaf of the pullback $(\tau_n^{-1})^* d$. It follows from Proposition 6.3(3) that $\tau_n(L) \in \mathcal{T}_n$, so L is definable in $\mathcal{R}_\mathcal{S}$. Since $L \in \mathcal{L}(\mathcal{R})$ was arbitrary, it follows that $\mathcal{R}'$ is a reduct of $\mathcal{R}_\mathcal{S}$ in the sense of definability; in particular, $\mathcal{R}'$ is o-minimal. The theorem now follows by taking $\mathcal{R}' = \mathcal{R}_1$ and from the definition of $\mathcal{P}(\mathcal{R})$. □

Corollary 6.9. *Let $K \subseteq \mathbb{R}^n$ be a pfaffian limit over $\mathcal{R}$ whose core is definable in $\mathcal{R}'$. Then K is definable in $\mathcal{R}'$.*

Proof. Let W be the core of K and $\mathcal{B}$ be the definable part of K. Then the family of all intersections $W \cap B$ with $B \in \mathcal{B}$ is a family definable in $\mathcal{R}'$. The corollary follows from the Marker-Steinhorn Theorem, see [16]. □

Exercise 6.10. Let K be a pfaffian limit over $\mathcal{R}$. Prove that the o-minimal dimension of K is equal to $\dim K$.

Exercise 6.11. Let $X \subseteq \mathbb{R}^n$ be definable in $\mathcal{R}_\mathcal{S}$.

(1) Show that, if X is bounded, then X is a Λ-set.
(2) Show that, if X is compact, then there are pfaffian limits $K_p \subseteq \mathbb{R}^{n_p}$ over $\mathcal{R}$ with $n_p \geq n$, for $p = 1,\dots,q$, such that each K_p has core definable in $\mathcal{R}'$ and $X = \Pi_n(K_1) \cup \cdots \cup \Pi_n(K_q)$; in particular, X is definable in $\mathcal{R}'$.

Corollary 6.12. *The structures $\mathcal{R}'$ and $\mathcal{R}_\mathcal{S}$ are interdefinable; in particular, every bounded set definable in $\mathcal{R}'$ is a Λ-set.*

Proof. Let X be a bounded cell definable in $\mathcal{R}_{\mathcal{S}}$; since $\mathcal{R}_{\mathcal{S}}$ is o-minimal, it suffices to show that X is definable in $\mathcal{R}'$. But both $\mathrm{cl}X$ and $\mathrm{fr}\,X$ are compact and hence definable in $\mathcal{R}'$ by the previous exercise. □

7 Blowing-Up Along a Distribution

In this section, I establish a criterion for generic portions (in the sense of dimension) of pfaffian limits over $\mathcal{R}$ to be integral manifolds of definable distributions. I fix a bounded, definable manifold $M \subseteq \mathbb{R}^n$ of dimension m, a finite set $\mathcal{D}_0$ of definable $(m-1)$-distributions on M and a definable l-distribution d_0 on M, and I assume that all are of class C^2. I put $\mathcal{D} := \mathcal{D}_0 \cup \{d_0\}$ and $k := \dim d_{\mathcal{D}}$, and I assume that $\mathcal{D}$ is compatible with M.

Definition 7.1. Put $n_1 := n + n^2$ and let $\Pi : \mathbb{R}^{n_1} \longrightarrow \mathbb{R}^n$ denote the projection on the first n coordinates. I define

$$M^1 := \mathrm{gr}\, d_{\mathcal{D}} \subseteq M \times G_n^k \subseteq \mathbb{R}^{n_1}, \text{ the graph of the distribution } d_{\mathcal{D}},$$

$$d^1 := (\Pi \restriction M^1)^* d, \text{ the pull-back to } M^1 \text{ of } d \text{ via } \Pi, \text{ for } d \in \mathcal{D} \cup \{d_{\mathcal{D}}\}.$$

I call $\mathcal{D}^1 := \{d^1 : d \in \mathcal{D}\}$ the **blowing-up of** $\mathcal{D}$ **(along** $d_{\mathcal{D}}$**)**; note that M^1 is of class C^2, while d^1 is of class C^1. Finally, for $d \in \mathcal{D} \cup \{d_{\mathcal{D}}\}$ and an integral manifold V of d, I define

$$V^1 := (\Pi \restriction M^1)^{-1}(V),$$

the **lifting of** V **(along** $d_{\mathcal{D}}$**)**. Note that, in this situation, V^1 is an integral manifold of d^1, and if $d = d_{\mathcal{D}}$, then V^1 is the graph of the Gauss map g_V.

Next, I write $M = \bigcup M_\sigma$, where σ ranges over Σ_n and the $M_\sigma := M_{\sigma,2n}$ are as in Lemma 4.10 with d and η there equal to $d_{\mathcal{D}}$ and $2n$ here.

Definition 7.2. For an integral manifold $V \subseteq M$ of $d_{\mathcal{D}}$ and $\sigma \in \Sigma_n$, I put $V_\sigma := V \cap M_\sigma$. Then V_σ is an integral manifold of $d_{\mathcal{D}}$, and I define

$$F^1 V := \bigcup_{\sigma \in \Sigma_n} \mathrm{fr}\ V_\sigma^1.$$

For the criterion, I let $D \subseteq \mathrm{cl}M^1$ be a definable C^2-cell such that $C := \Pi(D)$ has the same dimension as D and C is compatible with M_σ and $\mathrm{fr}\,M_\sigma$ for every $\sigma \in \Sigma_n$. Then $D = \mathrm{gr}\,g$, where $g : C \longrightarrow G_n^k$ is a definable map, and I assume that the following hold:

(i) The map $g \cap g_C$ has dimension and hence is a distribution on C;
(ii) If $g = g \cap g_C$, then either g is integrable or g is nowhere integrable.

I also assume that there is a definable set $S \subseteq \mathrm{cl} M^1$ such that $S \cap D = \emptyset$ and both S and $S \cup D$ are open in $\mathrm{cl} M^1$. In this situation, for any sequence (V_ι) of integral manifolds of $d_\mathcal{D}$ such that $K := \lim_\iota V_\iota^1$ and $K' := \lim_\iota F^1 V_\iota$ exist, I put

$$L_{(V_\iota)} := (D \cap K) \setminus \big(K' \cup \mathrm{fr}\,(S \cap K)\big).$$

Remark. Assume that (V_ι) is an admissible sequence of integral manifolds of $d_\mathcal{D}$ with core W such that $K := \lim_\iota V_\iota^1$ and $K' := \lim_\iota F^1 V_\iota$ exist, and assume that K is proper. Then $L_{(V_\iota)}$ is a generic subset of K in the following sense: $\left(V_\iota^1\right)$ is an admissible sequence of integral manifolds of $d_\mathcal{D}^1$ with core W^1. Thus by Exercise 5.4 and Propositions 5.8 and 5.7, K' is a finite union of pfaffian limits over $\mathcal{R}$ with cores definable in $\mathcal{R}(W)$ and of dimension less than $\dim K$. Moreover, by cell decomposition in $\mathcal{R}$ and Corollary 8.4 below, there is a finite union $F \subseteq \mathbb{R}^{n_1+2}$ of pfaffian limits over $\mathcal{R}$ with cores definable in $\mathcal{R}(W)$ and of dimension less than $\dim K$ such that $\mathrm{fr}(S \cap K) \subseteq \Pi_{n_1}(F)$.

Finally, let $g^1 : D \longrightarrow G_{n_1}^k$ be the pull-back of $g \cap g_C$ to M^1 via Π.

Proposition 7.3. *In this situation, exactly one of the following holds:*

(1) $L_{(V_\iota)} = \emptyset$ *for every admissible sequence* (V_ι) *of integral manifolds of* $d_\mathcal{D}$ *such that* $\lim_\iota V_\iota^1$ *and* $\lim_\iota F^1 V_\iota$ *exist;*
(2) g *is an integrable distribution on* C*, and for every admissible sequence* (V_ι) *of integral manifolds of* $d_\mathcal{D}$ *such that* $\lim_\iota V_\iota^1$ *and* $\lim_\iota F^1 V_\iota$ *exist, the set* $L_{(V_\iota)}$ *is an embedded integral manifold of* g^1 *and an open subset of* $\lim_\iota V_\iota^1$.

In particular, if D *is an open subset of* M^1 *and* (V_ι) *is an admissible sequence of integral manifolds of* $d_\mathcal{D}$ *such that* $\lim_\iota V_\iota^1$ *exists, then* $D \cap \lim_\iota V_\iota^1$ *is a finite union of leaves of* $\left(d_\mathcal{D}^1\right)^D$.

I need the following observation for the proof of Proposition 7.3.

Remark. Let $\sigma \in \Sigma_n$. Then σ induces a diffeomorphism $\sigma : G_n \longrightarrow G_n$ defined, in the notation of Sect. 1, by $\sigma(y) := A_{\sigma(\ker y)}$; define $\sigma^1 : \mathbb{R}^n \times G_n \longrightarrow \mathbb{R}^n \times G_n$ by $\sigma^1(x, y) := (\sigma(x), \sigma(y))$. Note that σ^1 is also just a permutation of coordinates. The map $g^\sigma : \sigma(C) \longrightarrow G_n^k$ defined by $g^\sigma(\sigma(x)) := \sigma(g(x))$ satisfies $(g^\sigma)^1 = \sigma^1 \circ g^1 \circ (\sigma^1)^{-1}$. Moreover, if (V_ι) is a sequence of integral manifolds of $d_\mathcal{D}$ such that $\lim_\iota V_\iota^1$ exists, then $\lim_\iota \sigma\left(V_\iota^1\right)$ also exists and $\sigma^1(D) \cap \lim_\iota \sigma^1\left(V_\iota^1\right) = \sigma^1\left(D \cap \lim_\iota V_\iota^1\right)$.

Proof (Proof of Proposition 7.3). By the previous remark and Remark 5.2, after replacing M by $\sigma(M_\sigma)$ and W by $\sigma^1(W \cap \mathrm{cl} M_\sigma^1)$ for each $\sigma \in \Sigma_n$ satisfying $C \subseteq \mathrm{cl} M_\sigma$, we may assume for the rest of this proof that $d_\mathcal{D}$ is $2n$-bounded and prove the proposition with $\mathrm{fr}\, V_\iota^1$ in place of $F^1(V_\iota)$. Thus, let (V_ι) be an admissible sequence of integral manifolds of $d_\mathcal{D}$ such that $K := \lim_\iota V_\iota^1$ and $K' := \lim_\iota \mathrm{fr}\, V_\iota^1$ exist, and put

$$L := (D \cap K) \setminus \big(K' \cup \mathrm{fr}\,(S \cap K)\big).$$

For the remainder of this proof, we write "lim" in place of "$\lim_\iota$". By Corollary 3.6(2), there is a $\nu \in \mathbb{N}$ such that for every open box $U \subseteq \mathbb{R}^n$ and every $\iota \in \mathbb{N}$, the set $U \cap V_\iota$ has at most ν connected components. We assume that $L \neq \emptyset$; we then need to show that g is an integrable distribution on C and that L is an embedded integral manifold of g^1 and an open subset of K.

To do so, choose an arbitrary $(x, y) \in L$ with $x \in \mathbb{R}^n$ and $y \in G_n$. Since $S \cup D$ is open in $\text{cl}M^1$, there is a bounded open box $B \subseteq \mathbb{R}^{n_1}$ such that $(x, y) \in B$ and

$$\text{cl}B \cap K \subseteq D \setminus (K' \cup \text{fr}(S \cap K));$$

in particular, $(x, y) \in D$. Write $B = B_0 \times B_1$ with $B_0 \subseteq \mathbb{R}^n$ and $B_1 \subseteq \mathbb{R}^{n^2}$. Since D is the graph of the continuous map g and C is locally closed, we may also assume, after shrinking B_0 if necessary, that $D \cap \left(\text{cl}B_0 \times \text{fr}\, B_1\right) = \emptyset$.

On the other hand, after passing to a subsequence if necessary, we may assume that $\lim\left(B \cap V_\iota^1\right)$, $\lim V_{\iota,B}^1$ and $\lim \text{fr}\, V_{\iota,B}$ exist, where $V_{\iota,B} := \{x \in V_\iota : (x, T_x V_\iota) \in B\}$. Then

$$B \cap K = B \cap \lim\left(B \cap V_\iota^1\right) = B \cap \lim V_{\iota,B}^1.$$

We now claim that $x \notin \lim \text{fr}\, V_{\iota,B}$: in fact, since $\text{fr}\, V_\iota^1 \cap \text{cl}B = \emptyset$ for all sufficiently large ι, it follows that $\text{fr}\, V_{\iota,B}^1 \subseteq \text{fr}\, B$ for all sufficiently large ι. Also, $\lim V_{\iota,B}^1 \subseteq \text{cl}B \cap \lim V_\iota^1$ is disjoint from $\text{cl}B_0 \times \text{fr}\, B_1$ by the previous paragraph, so $\text{cl}V_{\iota,B}^1$ is disjoint from $\text{cl}B_0 \times \text{fr}\, B_1$ for all sufficiently large ι. Hence $\text{fr}\, V_{\iota,B}^1 \subseteq \text{fr}\, B_0 \times B_1$ for all sufficiently large ι. Since B is bounded, $\text{fr}\, V_{\iota,B} \subseteq \Pi_n\left(\text{fr}\, V_{\iota,B}^1\right)$ holds, and it follows that $\text{fr}\, V_{\iota,B} \subseteq \text{fr}\, B_0$ for all sufficiently large ι, which proves the claim.

Since each V_ι is an embedded, closed submanifold of M, apply Lemma 4.4 with $V_{\iota,B}$ in place of V_ι and $\eta = 2n$, to obtain a corresponding open neighbourhood $U \subseteq B_0$ of x and $f_1, \ldots, f_\nu : \Pi_k(U) \longrightarrow \mathbb{R}^{n-k}$. Let $\lambda \in \{1, \ldots, \nu\}$ be such that $x \in \text{gr}\, f_\lambda$. We claim that for every $x' \in \text{gr}\, f_\lambda \cap U$, the map f_λ is differentiable at $z' := \Pi_k(x')$ with $T_{x'}\, \text{gr}\, f_\lambda = g(x')$; since x' is arbitrary, the claim implies that $\text{gr}\, f_\lambda$ is an embedded, connected integral manifold of g. Assumption (ii) and Exercise 1.18(3) then imply that g is an integrable distribution on C. Since $(x, y) \in L$ was arbitrary, it follows that L is an embedded integral manifold of g^1, as desired.

To prove the claim, let $f_{\lambda,\iota} : \Pi_k(U) \longrightarrow \mathbb{R}^{n-k}$ be the functions corresponding to f_λ as in the proof of Lemma 4.4. After a linear change of coordinates if necessary, we may assume that $g(x') = \mathbb{R}^k \times \{0\}$ (the subspace spanned by the first k coordinates). It now suffices to show that f_λ is η-Lipschitz at x' for every $\eta > 0$, since then $T_{x'}\, \text{gr}\, f_\lambda = \mathbb{R}^k \times \{0\}$. So let $\eta > 0$; since $\lim V_{\iota,B}^1 \subseteq D = \text{gr}\, g$ and $x' \in C$, and because C is locally closed and g is continuous, there is a neighborhood $U' \subseteq U$ of x' such that $\text{gr}\, f_{\lambda,\iota} \cap U'$ is (η/k)-bounded for all sufficiently large ι. Thus by Lemma 4.4 again, f_λ is η-Lipschitz at x', as required.

Finally, if D is open in M^1, then $g = d_{\mathcal{D}} \restriction C$ and we can take $S := \emptyset$. Since C is open in M and compatible with M_σ and $\text{fr}\, M_\sigma$ for $\sigma \in \Sigma_n$, the equality $C \cap \text{fr}\, M_\sigma = \emptyset$ holds for $\sigma \in \Sigma_n$. Hence $F^1(V_\iota) \cap D = \emptyset$, and it follows that $L_{(V_\iota)} = D \cap \lim V_\iota$ in this case. □

8 Fiber Cutting for Pfaffian Limits

What about the set $\mathrm{fr}(S \cap K)$ that appears in the previous section? I show first—similar to Proposition 5.11, but using the o-minimality of $\mathcal{R}_1$—that this set is a finite union of projections of pfaffian limits over $\mathcal{R}$.

Lemma 8.1. *Let $K \subseteq \mathbb{R}^n$ be a pfaffian limit over $\mathcal{R}$ with core W, and let $C \subseteq \mathbb{R}^n$ be a definable cell. Then there are pfaffian limits $K_1, \dots, K_p \subseteq \mathbb{R}^{n+2}$ over $\mathcal{R}$ whose cores are definable in $\mathcal{R}(W)$ such that* $\mathrm{fr}(K \cap C) = \Pi_n(K_1) \cup \cdots \cup \Pi_n(K_q)$.

Proof. Let $M \subseteq \mathbb{R}^n$ be a definable C^2-manifold of dimension m, $\mathcal{D}_0$ a finite collection of definable $(m-1)$-distributions on M and d_0 a definable l-distribution on M, and put $\mathcal{D} := \mathcal{D}_0 \cup \{d_0\}$. Assume that M is compatible with $\mathcal{D}$ and K is obtained from $\mathcal{D}$, and put $k := \dim d_{\mathcal{D}}$.

Define $\mathbf{M} := M \times (0,1)^2$ and write (x, r, ϵ) for the typical element of $\mathbf{M}$ with $x \in M$ and $r, \epsilon \in (0,1)$. Set $\mathbf{d}_0 := d_0 \cap \ker dr \cap \ker d\epsilon$, $\mathbf{D} := D_0 \cup \{\mathbf{d}_0\}$ and $\mathbf{W} := W \times (0,1)$. Note that whenever (V_ι) is an admissible sequence of integral manifolds of $d_{\mathcal{D}}$ with core W and $(r_\iota, \epsilon_\iota) \in (0,1)^2$ for $\iota \in \mathbb{N}$, the sequence $\big(V_\iota \times \{(r_\iota, \epsilon_\iota)\}\big)$ is an admissible sequence of integral manifolds of $d_{\mathbf{D}}$ with core $\mathbf{W}$. Let ϕ be a definable carpeting function on C and put

$$\mathbf{N} := \{\, (x, r, \epsilon) \in \mathbf{M} :\ d(x, \phi^{-1}(r)) < \epsilon \,\}.$$

Then $\mathbf{N}$ is an open, definable subset of $\mathbf{M}$, and since K is compact and definable in the o-minimal structure $\mathcal{R}_1$, we obtain from Lemma 4.7 that $\mathrm{fr}(K \cap C) = \lim_{r \to 0}(K \cap \phi^{-1}(r))$. Moreover, let (V_ι) be an admissible sequence of integral manifolds of $d_{\mathcal{D}}$ such that $K = \lim_\iota V_\iota$. Then for $r > 0$, the family of sets $\big\{\lim_\iota(V_\iota \cap N^{r,\epsilon}) :\ \epsilon > 0\big\}$ is decreasing in ϵ, where $\mathbf{N}^{r,\epsilon} := \{\, x \in M :\ (x, r, \epsilon) \in \mathbf{N} \,\}$, so

$$\phi^{-1}(r) \cap K = \lim_{\epsilon \to 0} \lim_{\iota} (V_\iota \cap \mathbf{N}^{r,\epsilon}).$$

Hence, after passing to a subsequence of (V_ι) if necessary, there are $r_\iota \to 0$ and $\epsilon_\iota \to 0$ such that

$$\mathrm{fr}(K \cap C) = \lim_{\iota} (V_\iota \cap \mathbf{N}^{r_\iota, \epsilon_\iota}) = \lim_{\iota} \Pi_n\big((V_\iota \times \{r_\iota, \epsilon_\iota)\}) \cap \mathbf{N}\big).$$

Since $\lim_\iota (r_\iota, \epsilon_\iota) = (0,0)$, the right-hand side above is equal to $\Pi_n\big(\lim_\iota \big((V_\iota \times \{(r_\iota, \epsilon_\iota)\}) \cap \mathbf{N}\big)\big)$. Since the sequence $\big(V_\iota \times \{(r_\iota, \epsilon_\iota)\}\big)$ is an admissible sequence of integral manifolds of $d_{\mathbf{D}}$ with core $\mathbf{W}$, Remark 5.2 now implies the lemma. □

The problem with the previous lemma, and with Exercise 6.11 as well, for their use in the proof of Theorem B is that $\dim d_{\mathbf{D}} = \dim d_{\mathcal{D}}$, so it is possible that $\dim K_p > \dim \mathrm{fr}(K \cap C)$ for some p. To remedy this, we need a fiber cutting lemma for pfaffian limits over $\mathcal{R}$:

Proposition 8.2. *Let $K \subseteq \mathbb{R}^n$ be a pfaffian limit over $\mathcal{R}$ with core W and $\nu \leq n$. Then there are $q \in \mathbb{N}$ and proper pfaffian limits $K_1, \dots, K_q \subseteq \mathbb{R}^n$ over $\mathcal{R}$ with cores definable in $\mathcal{R}(W)$ such that*

$$\Pi_\nu(K) = \Pi_\nu(K_1) \cup \cdots \cup \Pi_\nu(K_q)$$

and $\dim K_p = \dim \Pi_\nu(K_p) \leq \dim K$ *for each* p.

The following is needed in the proof of this proposition:

Exercise 8.3. Let $S \subseteq \mathbb{R}^k$ be definable in an o-minimal expansion $\mathcal{S}$ of the real field and put $s := \dim S$. Then there is a set $Y \subseteq S$, definable in $\mathcal{S}$, such that $S \subseteq \mathrm{cl} Y$, and for every $x \in Y$ there is a strictly increasing $\lambda : \{1, \dots, s\} \longrightarrow \{1, \dots, k\}$ such that x is isolated in $S \cap \Pi_\lambda^{-1}(\Pi_\lambda(x))$.

Proof (Proof of Proposition 8.2). Let $M \subseteq \mathbb{R}^n$ be a definable C^2-manifold of dimension m, $\mathcal{D}_0$ a finite collection of definable $(m-1)$-distributions on M and d_0 a definable l-distribution on M, and put $\mathcal{D} := \mathcal{D}_0 \cup \{d_0\}$. We assume that M is compatible with $\mathcal{D}$ and K is obtained from $\mathcal{D}$, and we proceed by induction on m. The case $m = 0$ is trivial, so assume $m > 0$ and the proposition holds for lower values of m. Let (V_ι) be an admissible sequence of integral manifolds of d with core W such that $K = \lim_\iota V_\iota$. Choosing a suitable C^2-cell decomposition of M compatible with $\mathcal{D}$, and using Remark 5.2 and the inductive hypothesis, we reduce to the case where M is a definable C^2-cell such that for every $s \leq \nu$ and every strictly increasing map $\lambda : \{1, \dots, s\} \longrightarrow \{1, \dots, \nu\}$, the rank of $\Pi_\lambda^n \restriction d_{\mathcal{D}}(x)$ is constant for $x \in M$; we denote this rank by r_λ. Putting $\mathcal{D}(\lambda) := \mathcal{D} \cup \{ (\ker dx_{\lambda(1)})^M, \dots, (\ker dx_{\lambda(s)})^M \}$ and $k := \dim d_{\mathcal{D}}$, this means that $d_{\mathcal{D}(\lambda)}$ has dimension $k - r_\lambda$. It follows from the rank theorem and the fact that admissible integral manifolds of $d_{\mathcal{D}}$ are closed in M that $V_\iota \cap (\Pi_\lambda^n)^{-1}(y)$ is a closed integral manifold of $d_{\mathcal{D}(\lambda)}$ with core W, for $\iota \in \mathbb{N}$ and $y \in \Pi_\lambda^n(V_\iota)$.

Let $s := \dim \Pi_\nu(K)$; then $s \leq k$ by Proposition 5.8. If $s = k$, we are done, so we assume from now on that $s < k$. Let $\lambda : \{1, \dots, s\} \longrightarrow \{1, \dots, \nu\}$ be strictly increasing; since $s < k$,

$$\dim d_{\mathcal{D}(\lambda)} \geq k - s > 0;$$

in particular, $r_\lambda < k$. Hence by Lemma 3.4 and because each fiber $V_\iota \cap (\Pi_\lambda^n)^{-1}(y)$ is a closed submanifold of M, there is a closed, definable set $B_\lambda \subseteq M$ such that $\dim B_\lambda < m$ and

- For $y \in \mathbb{R}^s$ and $\iota \in \mathbb{N}$, each component of the fiber $V_\iota \cap (\Pi_\lambda^n)^{-1}(y)$ intersects the fiber $B_\lambda \cap (\Pi_\lambda^n)^{-1}(y)$.

In particular, $\Pi_\lambda^n(V_\iota \cap B_\lambda) = \Pi_\lambda^n(V_\iota)$ for all ι, and for all $y \in \mathbb{R}^s$, every component of $\Pi_\nu(V_\iota) \cap (\Pi_\lambda^\nu)^{-1}(y)$ intersects the fiber $\Pi_\nu(V_\iota \cap B_\lambda) \cap (\Pi_\lambda^\nu)^{-1}(y)$.

We now denote by Λ the set of all strictly increasing $\lambda : \{1, \dots, s\} \longrightarrow \{1, \dots, \nu\}$. Passing to a subsequence if necessary, we may assume for $\lambda \in \Lambda$ that the sequence $(V_\iota \cap B_\lambda)_\iota$ converges to a compact set K^λ. Choosing a suitable C^2-cell decomposition of B_λ and using again Remark 5.2, it follows from the inductive hypothesis that the proposition holds with each K^λ in place of K. It therefore remains to show that $\Pi_\nu(K) = \bigcup_{\lambda\in\Lambda} \Pi_\nu\left(K^\lambda\right)$. To see this, fix a $\lambda \in \Lambda$; since each $\Pi_\nu\left(K^\lambda\right)$ is closed, it suffices by Exercise 8.3 to establish the following

Claim. Let $y \in \Pi^n_\lambda(K)$, and let $x \in \Pi_\nu(K) \cap (\Pi^\nu_\lambda)^{-1}(y)$ be isolated. Then $x \in \Pi_\nu\left(K^\lambda\right)$.

To prove the claim, note that $\Pi_\nu(K) = \lim_\iota \Pi_\nu(V_\iota)$ since M is bounded. Let $x_\iota \in \Pi_\nu(V_\iota)$ be such that $\lim_\iota x_\iota = x$, and put $y_\iota := \Pi^\nu_\lambda(x_\iota)$. Let $C_\iota \subseteq \mathbb{R}^\nu$ be the component of $\Pi_\nu(V_\iota)\cap\left(\Pi^\nu_\lambda\right)^{-1}(y_\iota)$ containing x_ι, and let x'_ι belong to $C_\iota\cap\Pi_\nu(V_\iota\cap B_\lambda)$. Since also $\Pi_\nu\left(K^\lambda\right) = \lim_\iota \Pi_\nu(V_\iota \cap B_\lambda)$ we may assume, after passing to a subsequence if necessary, that $x' := \lim_\iota x'_\iota \in \Pi_\nu\left(K^\lambda\right)$. We show that $x' = x$, which then proves the claim. Assume for a contradiction that $x' \neq x$, and let $\delta > 0$ be such that $\delta \leq |x - x'|$ and

$$B(x,\delta) \cap \Pi_\nu(K) \cap (\Pi^\nu_\lambda)^{-1}(y) = \{x\}. \tag{8.1}$$

Then for all sufficiently large ι, there is an $x''_\iota \in C_\iota$ such that $\delta/3 \leq |x''_\iota - x_\iota| \leq 2\delta/3$, because $x_\iota, x'_\iota \in C_\iota$ and C_ι is connected. Passing to a subsequence if necessary, we may assume that $x'' := \lim_\iota x''_\iota \in \Pi_\nu(K)$. Then $x'' \in B(x,\delta)$ with $x'' \neq x$, and since $x''_\iota \in C_\iota$ implies that $\Pi^\nu_\lambda(x''_\iota) = y_\iota$, it follows that $\Pi^\nu_\lambda(x'') = y$, contradicting (8.1). ☐

Combining Lemma 8.1 and Exercise 6.11 with Proposition 8.2 gives:

Corollary 8.4. *(1) Let $K \subseteq \mathbb{R}^n$ be a pfaffian limit over $\mathcal{R}$ with core W, and let $C \subseteq \mathbb{R}^n$ be a definable cell. Then there are pfaffian limits $K_1, \dots, K_p \subseteq \mathbb{R}^{n+2}$ over $\mathcal{R}$ with cores definable in $\mathcal{R}(W)$ such that* $\mathrm{fr}(K \cap C) = \Pi_n(K_1) \cup \dots \cup \Pi_n(K_q)$ *and* $\dim K_p < \dim K$ *for each p.*

(2) Let $\mathcal{R}'$ be a reduct of $\mathcal{R}_1$ that expands $\mathcal{R}$, and let $X \subseteq \mathbb{R}^n$ be definable in $\mathcal{R}'$ and compact. Then there are pfaffian limits $K_p \subseteq \mathbb{R}^{n_p}$ over $\mathcal{R}$ with $n_p \geq n$ and $\dim K_p \leq \dim X$*, for $p = 1, \dots, q$, such that each K_p has core definable in $\mathcal{R}'$ and $X = \Pi_n(K_1) \cup \dots \cup \Pi_n(K_q)$.* ☐

9 Proof of Theorem B

The main ingredient is Theorem 9.2 below, which in turn is based on the following:

Proposition 9.1. *Let $K \subseteq \mathbb{R}^n$ be a pfaffian limit over $\mathcal{R}$ with core W. Then there is a $q \in \mathbb{N}$, and for $p = 1, \dots, q$, there are $n_p \geq n$ and embedded integral manifolds $U_p \subseteq \mathbb{R}^{n_p}$ over $\mathcal{R}$ definable in $\mathcal{R}(W)$ and of dimension at most* $\dim K$ *such that $K \subseteq \Pi_n(U_1) \cup \dots \cup \Pi_n(U_q)$.*

Proof. By induction on $k := \dim K$, simultaneously for all n. If $k = 0$, then K is finite and the proposition is trivial. So we assume $k > 0$ and the proposition holds for all pfaffian limits K' over $\mathcal{R}$ satisfying $\dim K' < k$. By Exercise 5.10, we may assume that K is a proper pfaffian limit over $\mathcal{R}$. Let $M \subseteq \mathbb{R}^n$ be a definable C^2-manifold of dimension m, $\mathcal{D}_0$ be a finite set of definable $(m-1)$-distributions on M, d_0 be a definable l-distribution on M and put $\mathcal{D} := \mathcal{D}_0 \cup \{d_0\}$. Let (V_ι) be an admissible sequence of integral manifolds of $d_{\mathcal{D}}$ with core W such that $K = \lim_\iota V_\iota$ and $\dim d_{\mathcal{D}} = k$. By C^{m+2}-cell decomposition and Remark 5.2, we may assume that M is a C^{m+2}-cell and each $d \in \mathcal{D}$ is of class C^{m+2}.

We now blow up $m+1$ times along $d_{\mathcal{D}}$, that is, put $n_0 := n$, $M^0 := M$ and $d^0 := d$ for each $d \in \mathcal{D} \cup \{d_{\mathcal{D}}\}$ and $\mathcal{D}^0 := \mathcal{D}$, and put $V^0 := V$ and $F^0 V := \bigcup_{\sigma \in \Sigma_n} \operatorname{fr} V_\sigma$ for every integral manifold V of $d_{\mathcal{E}}$ with $\mathcal{E} \subseteq \mathcal{D} \cup \{d_{\mathcal{D}}\}$. By induction on $j = 1, \dots, m+1$, define $n_j := (n_{j-1})_1 = n_{j-1} + n_{j-1}^2$, $M^j := \left(M^{j-1}\right)^1 = \operatorname{gr} d_{\mathcal{D}}^{j-1}$, $d^j := \left(d^{j-1}\right)^1$ for each $d \in \mathcal{D} \cup \{d_{\mathcal{D}}\}$ and $\mathcal{D}^j := \left(\mathcal{D}^{j-1}\right)^1$, and define the corresponding liftings $V^j := \left(V^{j-1}\right)^1$ and $F^j V := F^1 V^{j-1}$ for every integral manifold V of $d_{\mathcal{E}}$ with $\mathcal{E} \subseteq \mathcal{D} \cup \{d_{\mathcal{D}}\}$. For each $0 \le i \le j \le m+1$, we also let $\pi_i^j : \mathbb{R}^{n_j} \longrightarrow \mathbb{R}^{n_i}$ be the projection on the first n_i coordinates.

Passing to a subsequence if necessary, we may assume that $K^j := \lim_\iota V_\iota^j$ and $\lim_\iota F^j V_\iota$ exist for $j = 0, \dots, m$ (so $K^0 = K$). Then $\pi_0^j\left(K^j\right) = K$ for each j, and since K is proper with core W, each K^j is proper with core W^j, and the latter is definable in $\mathcal{R}(W)$. It follows from Exercise 5.4(2), Proposition 5.7 and the inductive hypothesis that

(I) The proposition holds with each $\lim_\iota F^j V_\iota$ in place of K.

For $j = 0, \dots, m+1$, we write $M_\sigma^j := (M^j)_{\sigma,2n}$ as in Lemma 4.10 with M, d and η there equal to M^j, $d_{\mathcal{D}}^j$ and $2n$ here. Let $\mathcal{C}^j$ be a C^2-cell decomposition of $\operatorname{cl} M^j$ compatible with the sets M^j, $\operatorname{fr} M^j$, M_σ^j and $\operatorname{fr} M_\sigma^j$, for $\sigma \in \Sigma_{n_j}$. Refining each $\mathcal{C}^j$ in order of decreasing $j \in \{0, \dots, m\}$ if necessary, we may assume for each such j that

(i) $\mathcal{C}^j$ is a stratification compatible with $\{\pi_j^{j+1}(C) : C \in \mathcal{C}^{j+1}\}$; and for every $D \in \mathcal{C}^{j+1}$ that is the graph of a map $g : C \longrightarrow G_{n_j}^k$, where $C := \pi_j^{j+1}(D)$, that

(ii) The map $g \cap g_C$ has dimension and hence is a distribution on C;

(iii) If $g = g \cap g_C$, then either g is integrable or g is nowhere integrable.

By Corollary 8.4(1) and the inductive hypothesis,

(II) For $j = 0, \dots, m$ and $E \in \mathcal{C}^j$, the proposition holds with $\operatorname{fr}\left(K^j \cap E\right)$ in place of K.

We now fix $j \in \{0, \dots, m\}$ and a cell $C \in \mathcal{C}^j$ such that $\dim C \ge j$.

Claim. There is a $q \in \mathbb{N}$, and for $p = 1, \dots, q$, there are $n_p \ge n$ and an integral manifold $U_p \subseteq \mathbb{R}^{n_p}$ over $\mathcal{R}$ definable in $\mathcal{R}(W)$ and of dimension at most $\dim K$ such that $K^j \cap C \subseteq \Pi_n(U_1) \cup \dots \cup \Pi_n(U_q)$.

The proposition follows by applying this claim to each $C \in C^0$. To prove the claim, we proceed by reverse induction on $\dim C \le m$. Let

$$\mathcal{D}_C := \{ D' \cap \left(\pi_j^{j+1}\right)^{-1}(C) :\ D' \in \mathcal{C}^{j+1},\ C \subseteq \pi_j^{j+1}(D') \},$$

and fix an arbitrary $D \in \mathcal{D}_C$; it suffices to prove the claim with K^{j+1} and D in place of K^j and C. Let $D' \in \mathcal{C}^{j+1}$ be such that $D \subseteq D'$; if $\dim D' > \dim C$, then the claim with K^{j+1} and D in place of K^j and C follows from the inductive hypothesis, so we may assume that $\dim D' = \dim C$. Then D is open in D', and since $M^{j+1} \subseteq \mathbb{R}^{n_j} \times G^k_{n_j}$ and $G^k_{n_j}$ is compact, there is a definable map $g : C \longrightarrow G^k_{n_j}$ such that $D = \operatorname{gr} g$. Let

$$S := \bigcup \{ E \in \mathcal{C}^{j+1} :\ \dim E > \dim C \};$$

since $\mathcal{C}^{j+1}$ is a stratification, both S and $S \cup D'$ are open in $\mathrm{cl} M^{j+1}$, and since D is open in D', the set $S \cup D$ is also open in $\mathrm{cl} M^{j+1}$. Hence by Proposition 7.3, the set $\left(K^{j+1} \cap D\right) \setminus \left(\lim_\iota F^{j+1} V_\iota \cup \mathrm{fr}\left(S \cap K^{j+1}\right)\right)$ is an embedded integral manifold of g^1. But $D \cap \mathrm{fr}\left(S \cap K^{j+1}\right) \subseteq F$, where

$$F := \bigcup \{ \mathrm{fr}\left(K^{j+1} \cap E\right) :\ E \in \mathcal{C}^{j+1} \text{ and } \dim E > \dim C \}$$

is compact, so the set

$$L := \left(K^{j+1} \cap D\right) \setminus \left(\lim_\iota F^{j+1}(V_\iota) \cup F\right)$$

is an embedded integral manifold of g^1 definable in $\mathcal{R}(W)$. The claim with K^{j+1} and D in place of K^j and C now follows (I) and (II), which finishes the proof of the proposition. □

Theorem 9.2. *Let $\mathcal{R}'$ be a reduct of $\mathcal{R}_1$ that expands $\mathcal{R}$. Then $\mathcal{R}'$ is $\mathcal{R}$-differentially model complete.*

Proof. By induction on $\dim X$; the case $\dim X = 0$ is trivial, so assume $\dim X > 0$ and the corollary holds for lower values of $\dim X$. Using semi-algebraic diffeomorphisms, we may also assume that X is bounded. Then $\mathrm{cl} X$ is compact and hence, by Corollary 8.4(2), a finite union of projections of pfaffian limits over $\mathcal{R}$ whose cores are definable in $\mathcal{R}'$, each of dimension at most $\dim X$. So by Proposition 9.1, there is a $q \in \mathbb{N}$, and for $p = 1, \dots, q$, there are $n_p \ge n$ and an integral manifold $U_p \subseteq \mathbb{R}^{n_p}$ over $\mathcal{R}$ definable in $\mathcal{R}'$ and of dimension at most $\dim X$ such that $\mathrm{cl}(X) \subseteq \Pi_n(U_1) \cup \cdots \cup \Pi_n(U_q)$. For each $p \in \{1, \dots, q\}$, let $\mathcal{C}_p$ be a partition of U_p by cells definable in $\mathcal{R}'$ such that the collection $\{\Pi_n(C) :\ C \in \mathcal{C}_p\}$ is compatible with X. Since U_p is a submanifold of $\mathbb{R}^{n_p}$, each cell in $\mathcal{C}_p$ of dimension $\dim U_p$ is an open subset of U_p and hence an integral manifold of e_p. The corollary therefore follows by applying the inductive hypothesis to the union of all $\Pi_n(C)$ such that $C \in \mathcal{C}_p$ with $p \in \{1, \dots, q\}$, $\Pi_n(C) \subseteq X$ and $\dim C < \dim U_p$. □

Proof (Proof of Theorem B). Let $j \in \mathbb{N}$ be such that X is definable in $\mathcal{R}'_j$; we claim a slightly stronger statement: there is a $q \in \mathbb{N}$, and for $p = 1, \dots, q$, there are $n_p \geq n$ and an integral manifold $U_p \subseteq \mathbb{R}^{n_p}$ over $\mathcal{R}$ that is a cell definable in $\mathcal{R}'_j$ and of dimension at most $\dim X$ such that $X = \Pi_n(U_1) \cup \cdots \cup \Pi_n(U_q)$. We prove this claim by induction on j and $k := \dim(X)$. If $k = 0$, the claim is trivial and if $j = 1$, the claim follows from Theorem 9.2; so assume that $j > 1$ and $k > 0$ and the claim holds for lower values of j or k. By the inductive hypothesis and Theorem 9.2 with $\mathcal{R}'_{j-1}$ in place of $\mathcal{R}$, and after increasing n if necessary, we may assume there are a C^2-manifold $M \subseteq \mathbb{R}^n$ of dimension $m \geq k$ definable in $\mathcal{R}'_{j-1}$ and a k-distribution d on M definable in $\mathcal{R}'_{j-1}$ such that X is a cell that is also an integral manifold of d.

By the inductive hypothesis, there are a $\nu \geq n+n^2$, a definable C^2-manifold $N \subseteq \mathbb{R}^\nu$, a definable m-distribution e on N and an integral manifold W of e that is a cell definable in $\mathcal{R}'_{j-1}$ such that $\mathrm{gr}(d) = \Pi_{n+n^2}(W)$. Note, in particular, that W is the graph of a function $g_W : M \longrightarrow \mathbb{R}^{\nu-n}$, because W is a cell, $\dim W = \dim M$ and $\Pi_n(W) = M$. Let $\sigma : \mathbb{R}^{n+n^2} \longrightarrow \mathbb{R}^{n^2}$ be the projection on the last n^2 coordinates, put $N' := \left\{x \in N : \sigma \circ \Pi_{n+n^2}(x) \in G_n^k\right\}$ and set $H(x) := \sigma \circ \Pi_{n+n^2}(x)$ for $x \in N'$. Note that N' is definable and $W \subseteq N'$, because $H(x) = d(\Pi_{n+n^2}(x))$ for $x \in W$. Let $\mathcal{C}$ be a C^2-cell decomposition of N' compatible with e and definable in $\mathcal{R}$ such that, for $C \in \mathcal{C}$,

$(*)_C$ the dimensions of the spaces $\Pi_n\big(e^C(x)\big)$ and $\Pi_n\big(e^C(x)\big) \cap H(x)$ are constant as x ranges over C; denote them by r_C and s_C, respectively.

Let $C \in \mathcal{C}$ be such that $\mathcal{C} \cap W \neq \emptyset$; it now suffices to prove the claim with $X^C := \Pi_n(C \cap W) \cap X$ in place of X. Note that the set $C \cap W$ is an integral manifold of e^C and the graph of the restriction of g_W to $\Pi_n(C \cap W)$; in particular, by $(*)_C$, $\Pi_n \restriction e^C(x)$ is an immersion for $x \in C$. Also by $(*)_C$, the map $f^C : C \longrightarrow G_n$ defined by $f^C(x) := e^C(x) \cap \Pi_n^{-1}(H(x))$ is a definable distribution on C, and the set $V := \Pi_n^{-1}(X) \cap W \cap C$ is an embedded integral manifold of f^C definable in $\mathcal{R}'_j$ such that $\Pi_n(V) = X^C$. Assumption $(*)_C$ also implies that $\dim X^C = s_C \leq k$, so by the inductive hypothesis, we may assume $s_C = k$, that is, $H(x) \subseteq \Pi_n(e^C(x))$ for $x \in C$. Since $\Pi_n \restriction e^C(x)$ is an immersion for $x \in C$, it follows that $\dim f^C = k$. Applying cell decomposition in $\mathcal{R}'_j$ to V and using the inductive hypothesis one more time now finishes the proof of the claim. □

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Theorems of the Complement

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Abstract This is an expository paper on a Theorem of the Complement, due to Wilkie, and its generalisations. Wilkie (Sel Math (NS) 5:397–421, 1999) gave necessary and sufficient conditions for an expansion of the real field by C-infinity functions to be o-minimal. Karpinski and Macintyre (Sel Math (NS) 5:507–516, 1999) weakened the original smoothness hypotheses of Wilkie's theorem. Here we exhibit the proof of a generalised Wilkie's result, where we further weaken the smoothness assumptions and show that the proof can be carried out not only over the real numbers but more generally in a non-Archimedean context, i.e. for definably complete Baire structures, which we introduced in 2008 and which form an axiomatizable class. Furthermore we give necessary and sufficient conditions for a definably complete Baire expansion of an o-minimal structure by C-infinity functions to be o-minimal.

Keywords Pfaffian functions • Definably complete structures • Baire spaces • O-minimality

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1 Introduction

In [13], Wilkie proved a Theorem of the Complement (see Theorem 3.2 for the precise statement), which allowed him to derive the following result: given an expansion $\mathcal{R}$ of the real field with a family of $\mathcal{C}^\infty$ functions, if there are bounds (uniform in the parameters) on the number of connected components of quantifier free definable sets, then $\mathcal{R}$ is o-minimal. The first application of this theorem is the following: recall that a sequence $f_1, \dots, f_s : \mathbb{R}^n \to \mathbb{R}$ of $\mathcal{C}^\infty$ functions forms a *Pfaffian chain* if

$$\frac{\partial f_i}{\partial x_j} = p_{ij}(\overline{x}, f_1, \dots, f_i) \quad i = 1, \dots, s; \ j = 1, \dots, n,$$

where $p_{ij} \in \mathbb{R}[\overline{x}, y_1, \dots, y_i]$; a function $f : \mathbb{R}^n \to \mathbb{R}$ is Pfaffian if it appears in some Pfaffian chain. Thanks to a well known finiteness result in [7], Wilkie's theorem implies that the structure generated by all real Pfaffian functions is o-minimal.

Let $\mathcal{R}_0$ be an o-minimal expansion of the real field. We say that a sequence $f_1, \dots, f_s : \mathbb{R}^n \to \mathbb{R}$ of $\mathcal{C}^\infty$ functions forms a *generalized Pfaffian chain* over $\mathcal{R}_0$ if

$$\frac{\partial f_i}{\partial x_j} = g_{ij}(\overline{x}, f_1, \dots, f_i) \quad i = 1, \dots, s; \ j = 1, \dots, n,$$

where g_{ij} are $\mathcal{R}_0$-definable $\mathcal{C}^\infty$ functions; a function $f : \mathbb{R}^n \to \mathbb{R}$ is a generalized Pfaffian function over $\mathcal{R}_0$ if it appears in some generalized Pfaffian chain.

In [6], the authors generalized Wilkie's Theorem of the Complement (by weakening the smoothness assumption), and apply it to derive the following result: the expansion of $\mathcal{R}_0$ by all generalized Pfaffian functions over $\mathcal{R}_0$ is o-minimal (see Example 6.4).

By a different method (relying very indirectly on Wilkie's work), Speissegger proved in [9] a stronger result (the proof of which is discussed in [10]), namely the o-minimality of the so called *Pfaffian closure* of $\mathcal{R}_0$ (where, roughly speaking, we allow the functions f_i and g_{ij} in the above definition to be $\mathcal{C}^1$ and not necessarily total).[1]

Finally, in [1] the authors proved an effective version of Wilkie's Theorem of the Complement, which allowed them to deduce the following result: given an expansion $\mathcal{R}$ of the real field with a family of $\mathcal{C}^\infty$ functions, if there are *recursive* uniform bounds on the number of connected components of quantifier free definable sets, then there are recursive uniform bounds on the number of connected components of all definable sets, i.e. the theory of $\mathcal{R}$ is *recursively o-minimal*.

[1] In [6] an alternative proof of Speissegger's result is claimed. However, the proof contains a gap (see [4]), which forces us to conclude that only the weaker statement given above is fully proved in [6].

Here we present a further generalized version of Wilkie's and Karpinski and Macinyre's Theorems of the Complement. We generalize the original statements in two ways: firstly, we further weaken the smoothness assumptions made in [6]; secondly, we adapt the arguments in the proof to the non-Archimedean situation, i.e. we consider not only expansions of the real field, but more generally *definably complete Baire* expansions of ordered fields (see Definition 2.1). The main motivation for proving such a general statement is that it allows us to obtain the following first order version (proved in [4]) of Speissegger's theorem mentioned above: the Pfaffian closure of an o-minimal structure *inside* a definably complete Baire structure is again o-minimal.

For the reader who is only interested in expansions of the real field, we remark that this latter result can be used to give an alternative proof of Speissegger's theorem (in the spirit of the original [6]): for an exposition of the proof of the real case, see also [11]. Moreover, we recover the main result in [1], but with a more uniform axiomatization: let $\mathcal{R}$ be an expansion of the real field by a (classical) Pfaffian chain f of functions; then the (recursive) subtheory of the complete theory of $\mathcal{R}$, axiomatized by the axioms of definably complete Baire structure and the differential equations satisfied by f, is o-minimal.

The main result of this paper is Theorem 3.8. The main ideas and the structure of the proof are due to [13]. The basic properties of the Charbonnel closure (the first part of Sect. 5) are developed following [8], whereas the proof of the Theorem of the boundary 5.5 is inspired to [6] and the treatment given in [1].

In the final section we give an application of the main theorem. In Theorem 6.2 we give a necessary and sufficient condition for a definably complete Baire expansion of an o-minimal structure by $\mathcal{C}^\infty$ function to be o-minimal.

2 Preliminaries About Definably Complete Baire Structures

Throughout this paper, $\mathbb{K}$ is a (first-order) structure expanding an ordered field. We use the word "definable" as a shorthand for "definable in $\mathbb{K}$ with parameters from $\mathbb{K}$".

We denote by $x, y, z, \ldots$ the points in $\mathbb{K}^n$. When we want to stress the fact that they are tuples, we write $\overline{x}, \overline{y}, \overline{z}, \ldots$, where $\overline{x} = (x_1, \ldots, x_n)$, etc.

For convenience, on $\mathbb{K}^m$ instead of the usual Euclidean distance we will use the equivalent distance

$$d : (x, y) \mapsto \max_{i=1,\ldots,m} |x_i - y_i| .$$

For every $\delta > 0$ and $x \in \mathbb{K}^m$, we define by $B^m(x;\delta) := \{y \in \mathbb{K}^m : d(x, y) < \delta\}$ the open "ball" of center x and "radius" δ and its closure by $\overline{B}^m(x;\delta)$; we will drop the superscript m if it is clear from the context.

We define $\Pi^{m+n}_m : \mathbb{K}^{m+n} \to \mathbb{K}^m$ as the projection onto the first m coordinates.

We write $\mathrm{bd}(X) := \overline{X} \setminus \mathring{X}$ for the boundary of X.

We introduced definably complete Baire structures in [3].

Definition 2.1. An expansion $\mathbb{K}$ of an ordered field is a *definably complete Baire structure* if the two following (first-order) conditions hold:

1. Every definable subset of $\mathbb{K}$ has a supremum in $\mathbb{K} \cup \{\pm\infty\}$.
2. $\mathbb{K}$, as a set, is not *definably meager*, i.e. $\mathbb{K}$ is not the union of a definable increasing family of nowhere dense sets.

We refer the reader to [3, Sects. 1.2 and 2.1] for the precise definitions and preliminary results about definably complete Baire structures.[2] Every expansion of the real field and every o-minimal structure are definably complete and Baire; for the (easy) proof of the latter fact, together with more examples of definably complete Baire structure, see [3, Sect. 2.3]. The reader who is mostly interested in expansion of $\mathbb{R}$ can recognize in the following results some "definable" analogues of well-known topological properties of $\mathbb{R}$.

Definition 2.2. Let $X \subseteq \mathbb{K}^n$. X is in $\mathcal{F}_\sigma$ (or, "X is an $\mathcal{F}_\sigma$-set") if X is the union of a definable increasing family of closed subsets of $\mathbb{K}^n$, indexed by $\mathbb{K}$.

The following three results, corresponding to [3, Lemmas 3.5, 5.4 and Corollary 3.8], will be used in the following sections.

Proposition 2.3 (Baire's category theorem). *Assume that $D \subset \mathbb{K}^n$ is in $\mathcal{F}_\sigma$. Then, D is definably meager iff $\mathring{D} = \emptyset$.*

Proposition 2.4 (Kuratowski-Ulam's theorem). *Let D be an $\mathcal{F}_\sigma$ subset of $\mathbb{K}^{m+n}$; for every $x \in \mathbb{K}^m$, let $D_x = \{y \in \mathbb{K}^n : (x, y) \in D\}$ be the fiber of D over x and $T(D) := \{x \in \mathbb{K}^m : D_x$ is definably meager$\}$. Then, D is definably meager iff $\mathbb{K}^m \setminus T(D)$ is definably meager.*

Proposition 2.5. *Let $C \subseteq \mathbb{K}^m$ be in $\mathcal{F}_\sigma$, and $f : C \to \mathbb{K}^d$ be definable and continuous. Assume that for every $x \in C$ there exists $V_x \subseteq C$ neighbourhood of x, such that $f\left(C \cap V_x\right)$ is definably meager. Then, $f(C)$ is definably meager.*

Finally, recall the following definitions.

Definition 2.6. $X \subseteq \mathbb{K}^m$ is *definably compact* (d-compact for short) if it is definable, closed in $\mathbb{K}^m$, and bounded.

Definition 2.7. A definable set $X \subset \mathbb{K}^n$ is *definably connected* if it can not be expressed as a union of two *definable* non-empty disjoint open sets. A subset $C \subseteq X$ is a *definably connected component* of X if it is a maximal definably connected subset of X.

Every definable and connected set is definably connected. If $\mathbb{K}$ does not expand $\mathbb{R}$, then it is totally disconnected; hence, every infinite subset (definable or not)

[2] After this article was submitted, P. Hieronymi proved in [5] that, for expansions of ordered fields, Condition (1) implies Condition (2): that is, every definably complete structure expanding a field is also definably Baire.

of $\mathbb{K}^n$ is not connected. An example of an expansion $\mathbb{K}$ of the real field, such that $\mathbb{K}$ defines a set which is definably connected but not connected, is given in [2]. On the other hand, if $\mathbb{K}$ is an o-minimal expansion of $\mathbb{R}$ and X is definable in $\mathbb{K}$, then X is connected iff it is definably connected.

Definition 2.8. A n-dimensional definable embedded $\mathcal{C}^N$ $\mathbb{K}$-manifold $V \subseteq \mathbb{K}^d$ (which we will simply call n-dimensional $\mathbb{K}$-manifold) is a definable subset V of $\mathbb{K}^d$, such that for every $x \in V$ there exists a definable neighbourhood U_x of x (in $\mathbb{K}^d$), and a definable $\mathcal{C}^N$ diffeomorphism $f_x : U_x \simeq \mathbb{K}^d$, such that $U_x \cap V = f_x^{-1}\big(\mathbb{K}^n \times \{0\}\big)$.

3 The Generalized Theorem of the Complement

Definition 3.1 (Weak structure). Let $\mathbb{K}$ be definably complete and Baire. A collection $\mathcal{S}$ of $\mathbb{K}$-definable subsets of $\bigcup_{n\in\mathbb{N}} \mathbb{K}^n$ is a **weak structure** (over $\mathbb{K}$) if $\mathcal{S}$ contains all zero-sets of polynomials and is closed under finite intersection, Cartesian product and permutation of the variables. $\mathcal{S}$ is **semi-closed** if every set in $\mathcal{S}$ is a projection of some closed set in $\mathcal{S}$. $\mathcal{S}$ is **o-minimal** if for every $A \in \mathcal{S}$ there exists a natural number N such that, for every $\mathbb{K}$-affine set L, the number of definably connected componentsof $A \cap L$ is at most N. $\mathcal{S}$ is **determined by its smooth functions** (DSF) if every set in $\mathcal{S}$ is a projection of the zero-set of some $\mathcal{C}^\infty$ function whose graph lies in $\mathcal{S}$.

In [13] the following Theorem of the Complement is proved:

Theorem 3.2. *If $\mathbb{K}$ expands the real field and $\mathcal{S}$ is an o-minimal weak structure determined by its smooth functions, then $\mathcal{S}$ generates an o-minimal (first-order) structure.*

In [6] it is proved that this result still holds if one weakens the DSF assumption and allows every set in $\mathcal{S}$ to be, for every $N \in \mathbb{N}$, a projection of the zero-set of some $\mathcal{C}^N$ function (with a further uniformity condition). Here we weaken further the assumptions and allow $\mathcal{S}$ to be determined, not only by its smooth or its $\mathcal{C}^N$ functions, but by its $\mathcal{C}^N$ **admissible correspondences** (roughly, partial multi-valued functions with finitely many values at each point). Moreover, we do not restrict ourselves to working with the real numbers, but we allow $\mathcal{S}$ to be a collection of sets definable in a definably complete Baire structure.

To be able to state exactly the result we want to prove, we need to give some definitions.

Definition 3.3 (Charbonnel closure). Let $\mathcal{S}$ be an o-minimal weak structure (over $\mathbb{K}$). The **Charbonnel closure** $\widetilde{\mathcal{S}} = \langle \widetilde{\mathcal{S}}_n : n \in \mathbb{N}^+ \rangle$ is obtained from $\mathcal{S}$ by closing under the following *Charbonnel operations*: finite union, intersection with $\mathbb{K}$-affine sets, projection and topological closure.

Definition 3.4 (Admissible correspondence). A correspondence $f : \mathbb{K}^n \rightsquigarrow \mathbb{K}^m$ is a $\mathbb{K}$-definable partial function from $\mathbb{K}^n$ to the set of finite subsets of $\mathbb{K}^m$. We denote by $F \subset \mathbb{K}^n \times \mathbb{K}^m$ the graph of f, i.e. the set $F = \{(\overline{x}, \overline{y}) \in \mathbb{K}^n \times \mathbb{K}^m : f(\overline{x}) \neq \emptyset \wedge \overline{y} \in f(\overline{x})\}$. Given $1 \leq N \in \mathbb{N}$, a $\mathcal{C}^N$ **admissible correspondence** is a correspondence $f : \mathbb{K}^n \rightsquigarrow \mathbb{K}^m$, satisfying the following conditions.

1. F is $\mathbb{K}$-definable and has a finite number of definably connected components;
2. F is a $\mathcal{C}^N$ *closed* embedded submanifold of $\mathbb{K}^{n+m}$, of dimension n;
3. For every $\overline{x} \in F$, the normal space $N_{\overline{x}}F$ to F at $\overline{x}$ is transversal to the coordinate space $\mathbb{K}^n$; equivalently, the restriction to F of the projection map Π^{n+m}_n is a local diffeomorphism between F and $\mathbb{K}^n$.

Definition 3.5. Let $f : \mathbb{K}^n \rightsquigarrow \mathbb{K}^m$ be a $\mathcal{C}^N$ admissible correspondence. For every $C \subseteq \mathbb{K}^m$, denote by $f^{-1}(C)$ the preimage of C under f, that is $f^{-1}(C) := \{\overline{x} \in \mathbb{K}^n : \exists \overline{y} \in C\ (\overline{x}, \overline{y}) \in F\}$. Define $V(f) := f^{-1}(\{0\})$. Define the domain of f to be $\mathrm{dom}(f) := f^{-1}(\mathbb{K}^m)$. For every $A \subseteq \mathbb{K}^n$, denote by $f(A) := \{\overline{y} \in \mathbb{K}^m : \exists \overline{x} \in A\ (\overline{x}, \overline{y}) \in F\}$, the image of A under f. For every $\overline{x} \in \mathbb{K}^n$, we define $f(\overline{x}) := f\big(\{\overline{x}\}\big)$.

Example 3.6. 1. Every $\mathcal{C}^N$ function is an admissible correspondence.
2. The correspondence $\sqrt{x}$ is not admissible.
3. Define $g : \mathbb{R} \rightsquigarrow \mathbb{R}$ to be the correspondence with graph $G := \{(x, y) \in \mathbb{R} : y = x^2 \vee y = x^2 - 1\}$. g is $\mathcal{C}^\infty$ admissible, it is definable in the real field, but it is not a partial function.
4. Define $g : \mathbb{R} \rightsquigarrow \mathbb{R}$, $g(x) := 1/x$, defined for $x \neq 0$. g is an admissible $\mathcal{C}^\infty$ partial function. The domain of g is not closed, and therefore it is not true that the preimage of a closed set is closed.

Definition 3.7. Let $\mathcal{S}$ be a semi-closed o-minimal weak structure. $\mathcal{S}$ is **determined by its $\mathcal{C}^N$ admissible correspondences** (DAC^N) for all $N \in \mathbb{N}$ if for each $A \in \mathcal{S}_n$, there exist $m \geq n$ and $r \geq 1$, such that, for each N, there exists a set $S_N \subseteq \mathbb{K}^m$, which is a *finite union* of sets, each of which is an intersection of at most r sets of the form $V(f_{N,i})$, where each $f_{N,i} : \mathbb{K}^m \rightsquigarrow \mathbb{K}$ is an *admissible $\mathcal{C}^N$* correspondence in $\widetilde{\mathcal{S}}$, and $A = \Pi^m_n(S_N)$.

Theorem 3.8 (Generalized Theorem of the Complement). *Suppose that $\mathcal{S}$ is a semi-closed o-minimal weak structure over $\mathbb{K}$, which is DAC^N for all N. Then the Charbonnel closure $\widetilde{\mathcal{S}}$ of $\mathcal{S}$ is an o-minimal weak structure over $\mathbb{K}$, which is closed under complementation. Hence, $\widetilde{\mathcal{S}}$ is an o-minimal first-order structure.*

Remark 3.9. In Definition 3.7, note that:

1. Each set S_N is of the form $S_N = \bigcup_{0 \leq j < k_N} S_{N,j}$ (for some natural number k_N), where each set $S_{N,j}$ is of the form

$$S_{N,j} = \bigcap_{0 \leq i < r} V(f_{N,j,i}).$$

2. m and r do *not* depend on N; however, the number of sets forming the union (and therefore the total number of correspondences $f_{N,i,j}$) might depend on N. This property will be crucial in the proof of Proposition 5.9, which is an essential step in the proof of Theorem 5.5.
3. We only ask the correspondences $f_{N,i,j}$ to be in $\widetilde{\mathcal{S}}$, not in $\mathcal{S}$, and only that they are admissible correspondences, instead of total functions. Thus, the condition above is weaker than the one formulated in [6], even for $\mathbb{K}$ an expansion of the real field. Moreover, $\mathcal{S}$ satisfying DAC^N for all N does not imply that $\mathcal{S}$ is semi-closed.
4. By Example 3.6(1), DSF implies DAC^N for all N.
5. Notice that if f, g are functions, then $V(f) \cap V(g) = V(f^2 + g^2)$ and $V(f) \cup V(g) = V(fg)$; hence if each $f_{N,i,j}$ is a (total single-valued) function, we can replace the functions $f_{N,i,j}$ by a single function f_N, obtained from the $f_{N,i,j}$ using products and sums of squares; this is the reason why in [6] only one function f_N is used (and in [13] one $\mathcal{C}^\infty$ function f). However, admissible correspondences do not form a ring (in particular, the square of an admissible correspondence is not admissible in general), so in our case we can not reduce to a single correspondence.
6. Let $\mathcal{S}$ be a semi-closed o-minimal weak structure satisfying DAC^N for all N. Then it is harmless to assume every set in $\mathcal{S}$ to be closed: let $\mathcal{S}'$ is the collection of all closed sets in $\mathcal{S}$; then, $\mathcal{S}'$ is clearly an o-minimal weak structure; moreover the Charbonnel closures of $\mathcal{S}$ and $\mathcal{S}'$ coincide (since $\mathcal{S}$ is contained in the closure of $\mathcal{S}'$ under projection); it follows that $\mathcal{S}'$ satisfies DAC^N for all N.

4 Admissible Correspondences

Before proving the Theorem 3.8 we need to give some preliminary results about admissible correspondences.

Proviso. For the rest of this section, $f : \mathbb{K}^n \rightsquigarrow \mathbb{K}^m$ is a $\mathcal{C}^N$ admissible correspondence, with graph F.

Lemma 4.1. *1. For every $C \subseteq \mathbb{K}^m$ d-compact, $f^{-1}(C)$ is closed (in $\mathbb{K}^n$). In particular, $V(f)$ is closed.*
2. For every $U \subseteq \mathbb{K}^m$ open and $\mathbb{K}$-definable, $f^{-1}(U)$ is open. In particular, dom(f) *is open.*

Proof. Let $x \in \overline{f^{-1}(C)}$. We have to prove that $x \in f^{-1}(C)$. Let $D := \left(\overline{F \cap (\mathbb{K}^n \times C)}\right)_x$. Notice that $D \subseteq \overline{C} = C$, and therefore $D = D \cap C$. Since $x \in \overline{f^{-1}(C)}$, we have that for every U neighbourhood of x there exists $y \in U$, such that $f(y) \cap C \neq \emptyset$, i.e. the section $\left(F \cap (\mathbb{K}^n \times C)\right)_y$ is non-empty. Since C is d-compact, D is non-empty. Since F and C are closed, we have $\overline{F \cap (\mathbb{K}^n \times C)} = F \cap (\mathbb{K}^n \times C)$, and therefore

$$F_x \cap C = \left(F \cap (\mathbb{K}^n \times C)\right)_x = D.$$

Since $D \neq \emptyset$, we have that $x \in f^{-1}(C)$. □

Remark 4.2. If F is the graph of an admissible $\mathcal{C}^N$ correspondence, then every definably connected component of F is the graph of an admissible $\mathcal{C}^N$ correspondence. Conversely, if F_1 and F_2 are the graphs of two admissible $\mathcal{C}^N$ correspondences and F_1 and F_2 are disjoint, then $F_1 \cup F_2$ is the graph of an admissible $\mathcal{C}^N$ correspondence.

Lemma 4.3. *Let $g : \mathbb{K}^n \to \mathbb{K}^m$ be a $\mathbb{K}$-definable partial function, with definably connected domain. Then, g is admissible $\mathcal{C}^N$ iff:*

1. *The domain of g is an open set U;*
2. *$g : U \to \mathbb{K}^m$ is a $\mathcal{C}^N$ function;*
3. *For every $\overline{x} \in \mathrm{bd}(U)$,*

$$\lim_{\substack{\overline{y}\to\overline{x},\\ \overline{y}\in U}} |g(\overline{y})| = +\infty.$$

We conjecture that, if F is definably connected and $\mathrm{dom}(f) = \mathbb{K}^n$, then, f is a (total and single-valued) function.

The reader can check that the following properties of admissible correspondences hold.

Lemma 4.4.

- *Let $\phi : \mathbb{K}^m \to \mathbb{K}^m$ be a $\mathcal{C}^N$ $\mathbb{K}$-definable diffeomorphism. Then, $\phi \circ f : \mathbb{K}^n \rightsquigarrow \mathbb{K}^m$ is $\mathcal{C}^N$ and admissible.*
- *Let $\theta : \mathbb{K}^n \to \mathbb{K}^n$ be a $\mathcal{C}^N$ $\mathbb{K}$-definable diffeomorphism. Then, $f \circ \theta : \mathbb{K}^n \rightsquigarrow \mathbb{K}^m$ is $\mathcal{C}^N$ and admissible.*
- *Let $\theta : \mathbb{K}^n \to \mathbb{K}^n$ be a $\mathcal{C}^N$ $\mathbb{K}$-definable function. If $f \circ \theta : \mathbb{K}^n \rightsquigarrow \mathbb{K}^m$ has a finite number of definably connected components, then it is a $\mathcal{C}^N$ admissible correspondence.*

Notice that in the above lemma we can not drop the hypothesis that ϕ is a diffeomorphism, and replace it with the hypothesis that it is a $\mathcal{C}^N$ function, and similarly we cannot drop the additional conditions on θ.

- In fact, if $m = 1, n > 1$, and $\phi(x) = x^2$, it might happen that the graph of $\phi \circ f$ is not a submanifold (because it "self-intersects"). For example, let g be defined as in Example 3.6(3). Then the graph of g^2 is not a submanifold.
- For instance, let $\mathbb{K}$ be an expansion of $\mathbb{R}$ where the sine function is defined; let $f(x) := 1/x$, and $\theta(t) := \sin t$. Then, $f \circ \theta = 1/\sin(t)$ is not admissible.

Lemma 4.5 (Difference). *Let $m = 1$, and define $g : \mathbb{K}^{n+1} \rightsquigarrow \mathbb{K}$ as $g(\overline{x}, y) := y - f(\overline{x})$. That is, the graph of g is $G := \{(\overline{x}, y, z) \in \mathbb{K}^{n+2} : (\overline{x}, z - y) \in F\}$. Then, g is $\mathcal{C}^N$ and admissible.*

Lemma 4.6 (Extension). *Given $g : \mathbb{K}^n \to \mathbb{K}$ a (total and single-valued) $\mathcal{C}^N$ and $\mathbb{K}$-definable function, define the correspondence $h := \langle f, g\rangle : \mathbb{K}^n \rightsquigarrow \mathbb{K}^{m+1}$; that is,* *the graph of h is* $H := \{(\overline{x}, \overline{y}, z) \in \mathbb{K}^{n+m+1} : (\overline{x}, \overline{y}) \in F \;\&\; z = g(\overline{x})\}$*. Then, h is* $\mathcal{C}^N$ *and admissible.*

Definition 4.7 (Differential). For every $(\overline{x}, \overline{y}) \in F$, it makes sense to define $Df(\overline{x}; \overline{y})$, the differential of f at the point $(\overline{x}, \overline{y})$ (the notational difference with the usual case when f is a function is that here we have to specify at which $\overline{y} \in f(\overline{x})$ we compute Df). As usual, we say that $(\overline{x}, \overline{y})$ is a regular point for f if $Df(\overline{x}; \overline{y})$ has maximal rank, otherwise $(\overline{x}, \overline{y})$ is singular. Similarly, $\overline{y} \in \mathbb{K}^n$ is a regular value if, for every $\overline{x} \in f^{-1}(y)$, $(\overline{x}, \overline{y})$ is a regular point; otherwise, $\overline{y}$ is a singular value.

Moreover, we have a correspondence on $\mathbb{K}^n$, which assign to every point $\overline{x}$ the values of $Df(\overline{x}; \overline{y})$, as $\overline{y}$ varies in $f(\overline{x})$. This correspondence in general is not admissible, even if $N \geq 2$, because its graph might not be a manifold. The following lemma addresses this point.

Lemma 4.8. *Assume that* $N \geq 2$.

- *Let* $\tilde{D} f$ *be the correspondence* $\langle f, Df \rangle$ *on* $\mathbb{K}^n$*. That is, the graph of* $\tilde{D} f$ *is* $H := \{(\overline{x}, \overline{y}, \overline{z}) : (\overline{x}, \overline{y}) \in F \;\&\; \overline{z} = Df(\overline{x}; \overline{y})\}$*. Then,* $\tilde{D} f$ *is* $\mathcal{C}^{N-1}$ *and admissible.*
- *Assume that* $n = m + k$*, with* $k \geq 1$*. Fix* $1 \leq i_1 < \cdots < i_k \leq n$*. Then, the correspondence*

$$\left\langle f, \det\left(\frac{\partial\,(f_1, \ldots, f_k)}{\partial\left(x_{i_1}, \ldots, x_{i_k}\right)}\right)^2 \right\rangle$$

  is admissible.

The two previous lemmas are particular cases of the following:

Lemma 4.9 (Composition). *Let* $g : F \to \mathbb{K}^k$ *be a* $\mathcal{C}^N$ $\mathbb{K}$*-definable function. Let* $h := \langle f, g \rangle$*; that is, the graph of h is* $H := \{(\overline{x}, \overline{y}, \overline{z}) : (\overline{x}, \overline{y}) \in F \;\&\; z = g(\overline{x}, \overline{y})\}$*. Then, h is* $\mathcal{C}^N$ *and admissible.*

Proof. Since g is continuous, H is closed in $F \times \mathbb{K}^k$. Since F is closed in $\mathbb{K}^{n+m}$, H is closed in $\mathbb{K}^{n+m+k}$. □

Lemma 4.10 (Product). *For* $i = 1, 2$*, let* $f_i : \mathbb{K}^{n_i} \rightsquigarrow \mathbb{K}^{m_i}$ *be an admissible* $\mathcal{C}^N$ *correspondence, with graph* F_i*. The, the correspondence* $f_1 \times f_2 : \mathbb{K}^{n_1+n_2} \rightsquigarrow \mathbb{K}^{m_1+m_2}$*, with graph* $F_1 \times F_2$*, is an admissible* $\mathcal{C}^N$ *correspondence.*

Definition 4.11. Given a correspondence $g : \mathbb{K}^n \rightsquigarrow \mathbb{K}^m$, we denote by $|g|$ the correspondence $|g| : \mathbb{K}^n \rightsquigarrow \mathbb{K}$, with graph $|G| := \{(x, t) : \exists \overline{y} \in \mathbb{K}^m \; (\overline{x}, \overline{y}) \in F \;\&\; |\overline{y}| = t\}$.

Definition 4.12. Given $C \subseteq \mathbb{K}^n$ and $g : \mathbb{K}^n \rightsquigarrow \mathbb{K}$ correspondence with graph G, and $\overline{x} \in C$, we say that g reaches the minimum on C at $\overline{x}$, if there exists $y \in g(\overline{x})$ such that, for every $(\overline{x}', y') \in G$, if $\overline{x}' \in C$, then $y \leq y'$; moreover, y is the minimum of g on C.

We also define $\inf_{\overline{x} \in C} g(\overline{x}) := \inf g(C) \in \mathbb{K} \cup \{\pm\infty\}$.

Notice that $\inf_{\overline{x} \in C} g(\overline{x}) = +\infty$ iff $g(C) = \emptyset$.

Lemma 4.13. *Let $f : \mathbb{K}^n \rightsquigarrow \mathbb{K}$ be admissible, and $C \subseteq \mathbb{K}^n$ be $\mathbb{K}$-definable, d-compact and such that $f(C)$ is non-empty. Then,*

1. *$|f|$ achieves its minimum (but not necessarily its maximum) on C;*
2. *If $\inf_{x \in C} f(\overline{x}) \neq -\infty$, then f achieves its minimum on C (and similarly for the maximum).*

Proof. The graph $|F|$ of $|f|$ is closed in $C \times [0, +\infty)$, and C is d-compact; hence, $\pi(|F|)$ is closed in $[0, +\infty)$, where $\pi : C \times [0, +\infty) \to [0, +\infty)$ is the projection onto the second coordinate. □

5 The Theorem of the Boundary

In this section we prove Theorem 3.8. The proof is shaped on Wilkie's original proof, which proceeds as follows: let $\mathcal{S}$ be a semi-closed o-minimal weak structure which is DSF. Then $\widetilde{\mathcal{S}}$ is an o-minimal weak structure and it is closed under complementation (hence, it coincides with the structure generated by $\mathcal{S}$). The proof of the closure under complementation uses measure theoretic arguments (which here we replace with definable Baire category arguments): mainly, Fubini's Theorem and a strong version of Sard's Lemma which holds for $\mathcal{C}^1$ functions with graph in $\mathcal{S}$. With these tools the author shows how to approximate the boundary of a set in $\widetilde{\mathcal{S}}$ with smooth manifolds also in $\widetilde{\mathcal{S}}$ (this is where the DSF condition is used) and concludes by a cell-decomposition argument.

The first step is to establish the following.

Theorem 5.1. *If $\mathcal{S}$ is a semi-closed o-minimal weak structure, then its Charbonnel closure $\widetilde{\mathcal{S}}$ is a semi-closed o-minimal weak structure.*

Notice that if $\mathcal{S}$ is as in the above theorem and $X \in \mathcal{S}$, then X has a finite number of definably connected components, but we do not know whether such components are in $\widetilde{\mathcal{S}}$.

The proof of this statement can be found in [8, Sect. 1].

It does not use specific properties of $\mathbb{R}$, and can be reformulated in any definably complete structure (the Baire property is not needed here).

Proviso 5.2. We fix for the rest of the section a semi-closed o-minimal weak structure $\mathcal{S}$ over a definably complete Baire structure $\mathbb{K}$ which is DAC^N for all N.

The next two results do not need the DAC^N assumption.

Lemma 5.3. *Let $A \in \widetilde{\mathcal{S}}$. Then A is definably meager $\Leftrightarrow$ A has empty interior $\Leftrightarrow$ A is nowhere dense.*

Proof. We first observe that, $\widetilde{\mathcal{S}}$ being semi-closed, every set in $\widetilde{\mathcal{S}}$ is an $\mathcal{F}_\sigma$-set. In particular, by Proposition 2.3, the first equivalence is proved. The other equivalence can be proved as in [8, Lemma 2.7], where we conclude by using Proposition 2.4 and the previous observation, instead of Fubini's Theorem. □

Theorem 5.4 (Strong Sard's Lemma). *Suppose that $m \geq 1$ and $f : \mathbb{K}^n \rightsquigarrow \mathbb{K}^m$ is a $\mathcal{C}^1$ admissible correspondence whose graph F is in $\widetilde{\mathcal{S}}$. Then the set of singular values of f is in $\widetilde{\mathcal{S}}$ and has empty interior (hence, it is definably meager in $\mathbb{K}^m$).*

Proof. If f is a function, then we can apply, *mutatis mutandis*, the proof of Wilkie [13, Lemma 2.7]. Otherwise, by the Implicit Function Theorem and the definition of F, every point $(\overline{x}, \overline{y}) \in F$ has a $\mathbb{K}$-definable neighbourhood $U = U_1 \times U_2 \subset \mathbb{K}^n \times \mathbb{K}^m$ such that $U \cap F$ is the graph of a $\mathcal{C}^1$ function $f_U : U_1 \to U_2$. By reducing U, if necessary, we can ensure that U_1 and U_2 are in $\widetilde{\mathcal{S}}$ (in fact, we can assume they are boxes), so that $f_U \in \widetilde{\mathcal{S}}$. We can apply the analogue statement for functions to f_U and obtain that the set of its singular values is definably meager. Now, since the set Σ_f of the singular values of f is given by $\bigcup_U \Sigma_{f_U}$, we can apply Proposition 2.5 to the projection $\pi : \mathbb{K}^n \times \mathbb{K}^m \to \mathbb{K}^m$ onto the second factor and obtain that Σ_f is definably meager. It is clear that $\Sigma_f \in \widetilde{\mathcal{S}}$. □

The next, and most difficult, step is to prove the following Theorem of the Boundary, corresponding to [13, Theorem 3.1]. Here it will be crucial that $\mathcal{S}$ satisfies DAC^N for all N.

Theorem 5.5 (Boundary). *Let $A \in \widetilde{\mathcal{S}}_n$ be closed. Then there exists a closed set $B \in \widetilde{\mathcal{S}}_n$ such that B has empty interior and* $\mathrm{bd}(A) \subseteq B$.

Notice that, even without the DAC^N hypothesis, the following is true: if A is a closed set in $\widetilde{\mathcal{S}}$, then $\mathrm{bd}(A)$ has empty interior. The missing information is whether $\mathrm{bd}(A)$ is in $\widetilde{\mathcal{S}}$ or not.

We will follow the outline of Wilkie [13, Sect. 3], but we will use [1] for some definitions and proofs. The two approaches are equivalent, but we find the latter easier to read.

Definition 5.6.

- $\mathbb{K}_+ := \{x \in \mathbb{K} : x > 0\}$.
- Given $\overline{x} \in \mathbb{K}^n$, let $|\overline{x}| := \max\{|x_1|, \ldots, |x_n|\}$, and $\|\overline{x}\| := \sqrt{x_1^2 + \cdots + x_n^2}$. Notice that $\overline{x} \mapsto \|\overline{x}\|^2$ is a $\mathcal{C}^\infty$ function, and so is the function $\overline{x} \mapsto \frac{1}{1+\|\overline{x}\|^2}$.
- Given $A \subseteq \mathbb{K}^n$ and $\varepsilon \in \mathbb{K}_+$, define the *$\varepsilon$-neighborhood* A^ε of A as the set $\{x \in \mathbb{K}^n \mid \exists y \in A\ d(x, y) < \varepsilon\}$.
- (The quantifier "for all sufficiently small") Given a formula ϕ, we write $\forall^s \varepsilon \phi$ as a shorthand for $(\exists \mu)(\forall \varepsilon < \mu)\phi$, where μ, ε are always assumed to range in $\mathbb{K}_+$. If $\overline{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_n)$, then $\forall^s \overline{\varepsilon}$ is an abbreviation for $\forall^s \varepsilon_1 \ldots \forall^s \varepsilon_n$.
- (Sections) Given $S \subseteq \mathbb{K}^n \times \mathbb{K}_+^k$ and given $\overline{\varepsilon} \in \mathbb{K}_+^k$, we define $S_{\overline{\varepsilon}}$ as the set $\{x \in \mathbb{K}^n \mid (x, \overline{\varepsilon}) \in S\}$.
- Let $A \subseteq \mathbb{K}^n$, $S \subseteq \mathbb{K}^n \times \mathbb{K}_+^k$. S approximates A from below ($S \leq A$) if

$$\forall^s \varepsilon_0 \forall^s \varepsilon_1 \ldots \forall^s \varepsilon_k (S_{\varepsilon_1, \ldots, \varepsilon_k} \subseteq A^{\varepsilon_0}).$$

- Let $A \subseteq \mathbb{K}^n, S \subseteq \mathbb{K}^n \times \mathbb{K}^k_+$. S approximates A from above on bounded sets ($S \geq A$) if

$$\forall^s \varepsilon_0 \forall^s \varepsilon_1 \dots \forall^s \varepsilon_k (A \cap B(0, 1/\varepsilon_0) \subseteq (S_{\varepsilon_1,\dots,\varepsilon_k})^{\varepsilon_0}).$$

Definition 5.7. Let k and $N \geq 1$ be natural numbers. An $\widetilde{\mathcal{S}}(N)$-*constituent of complexity* k is a subset $S \subseteq \mathbb{K}^n \times \mathbb{K}^k_+$ of the form

$$\{(\overline{x}, \overline{\varepsilon}) \in \mathbb{K}^n \times \mathbb{K}^k_+ : \exists \overline{y} \in \mathbb{K}^{k-1}\ f(\overline{x}, \overline{y}) \ni \overline{\varepsilon}\}$$

where $f : \mathbb{K}^n \times \mathbb{K}^{k-1} \rightsquigarrow \mathbb{K}^k$ is *admissible*, $\mathcal{C}^N$ and in $\widetilde{\mathcal{S}}$. An $\widetilde{\mathcal{S}}(N)$-*set* of complexity k is a finite union of $\widetilde{\mathcal{S}}(N)$-constituents of complexity k.

In the definition of $\widetilde{\mathcal{S}}(N)$-constituents of complexity k, we can relax somehow the condition on the dimension of the domain of the correspondence.

Remark 5.8. Let $0 \leq d < k$, and $f : \mathbb{K}^n \times \mathbb{K}^d \rightsquigarrow \mathbb{K}^k$ be an admissible $\mathcal{C}^N$ correspondence in $\widetilde{\mathcal{S}}$. Let $S(f) := \{(\overline{x}, \overline{\varepsilon}) \in \mathbb{K}^n \times \mathbb{K}^k_+ : \exists \overline{y} \in \mathbb{K}^d\ f(\overline{x}, \overline{y}) \ni \overline{\varepsilon}\}$. Then, $S(f)$ is an $\widetilde{\mathcal{S}}(N)$-constituent of complexity k. In fact, $S(f)$ is of the form $\{(\overline{x}, \overline{\varepsilon}) \in \mathbb{K}^n \times \mathbb{K}^k_+ : \exists \overline{z} \in \mathbb{K}^{k-1} \tilde{f}(\overline{x}, \overline{z}) \ni \overline{\varepsilon}\}$, where $\tilde{f} : \mathbb{K}^n \times \mathbb{K}^{k-1} \rightsquigarrow \mathbb{K}^k$ is an admissible $\mathcal{C}^N$ correspondence in $\widetilde{\mathcal{S}}$; in particular, $S(f)$ is an $\widetilde{\mathcal{S}}(N)$-constituent. The graph of $\tilde{f}$ is

$$\tilde{F} := \{(\overline{x}, \overline{z}, \overline{w}) \in \mathbb{K}^n \times \mathbb{K}^{k-1} \times \mathbb{K}^k : (\overline{x}, z_1, \dots, z_d, \overline{w}) \in F\}.$$

We will show that, to obtain Theorem 5.5, it is enough to prove the following:

Proposition 5.9. *For each* $n \in \mathbb{N}$, $A \in \widetilde{\mathcal{S}}_n$, *and each* $N \geq 1$, *the following holds:* (Φ_N): *There exist* $k \geq 1$ (*the* $\widetilde{\mathcal{S}}(N)$-complexity *of* A) *and a* $\widetilde{\mathcal{S}}(N)$-*set* S *of complexity* k, *such that* S *both approximates* $\text{bd}(\overline{A})$ *from above on bounded sets and approximates* $\overline{A}$ *from below.*

A set S as in the above proposition is called an $\widetilde{\mathcal{S}}(N)$-*approximant* of A.

The following statement is a remark at the end of Wilkie [13, Sect. 4].

Lemma 5.10. *Given* $N \geq 1$, *every* $\widetilde{\mathcal{S}}(N)$-*set has empty interior.*

Proof. It suffices to show that each $\widetilde{\mathcal{S}}(N)$-constituent S has empty interior. S is of the form $\text{Im}(g) \cap (\mathbb{K}^n \times \mathbb{K}^k_+)$, where

$$\begin{aligned} g : \mathbb{K}^{n+k-1} &\rightsquigarrow \mathbb{K}^{n+k} \\ (\overline{x}, \overline{y}) &\mapsto (\overline{x}, f(\overline{x}, \overline{y})), \end{aligned}$$

for some $f : \mathbb{K}^{n+k-1} \rightsquigarrow \mathbb{K}^k$ admissible, $\mathcal{C}^N$ and in $\widetilde{\mathcal{S}}$. Since g is in $\widetilde{\mathcal{S}}$, Theorem 5.4 implies that the image of g has empty interior. □

The proof of the following statement can be obtained from the proof of Wilkie [13, Lemma 3.3] by using Proposition 2.3 and Lemma 5.3 instead of Fubini's Theorem and [13, Theorem 2.1].

Lemma 5.11. *Let $S \subseteq \mathbb{K}^n \times \mathbb{K}_+^k$ be in $\widetilde{\mathcal{S}}_{n+k}$. Denote by T the section $T := \overline{S}_{\overline{0}} = \{\overline{x} \in \mathbb{K}^n |\ (\overline{x}, \overline{0}) \in \overline{S}\} \in \widetilde{\mathcal{S}}_n$. If S has empty interior, then T also has empty interior. $A \subseteq \mathbb{K}^n$ (not necessarily definable) such that $S \geq A$. Then, $A \subseteq T$. Let $A \in \widetilde{\mathcal{S}}_n$, $S \in \widetilde{\mathcal{S}}_{n+k}$. Suppose that S has empty interior and is an $\widetilde{\mathcal{S}}(N)$-approximant for A. Then so does the section $\overline{S}_{\overline{0}} = \{\overline{x} \in \mathbb{K}^n |\ (\overline{x}, \overline{0}) \in \overline{S}\} \in \widetilde{\mathcal{S}}_n$.*

Theorem 5.5 follows immediately from the proposition and the two previous lemmas. In fact, let $A \in \widetilde{\mathcal{S}}_n$ be closed. By Proposition 5.9, there exists $S \in \widetilde{\mathcal{S}}_{n+k}$ such that $S \geq \mathrm{bd}(A)$ and S is an $\widetilde{\mathcal{S}}(1)$-set. By Lemma 5.10, S has empty interior, and therefore, by Lemma 5.11, $B := \overline{S}_0$ has also empty interior, is in $\widetilde{\mathcal{S}}_n$, and $\mathrm{bd}(A) \subseteq B$. Notice that in proving Theorem 5.5 we did not use the full power of the Proposition 5.9, but only the case $N = 1$ and the fact that S approximates A from above on bounded sets; however, the proof of Proposition 5.9 will be by induction, and we need the stronger form as inductive hypothesis.

The remainder of this section is devoted to the proof of Proposition 5.9. The proof of Proposition 5.9 follows the pattern of Wilkie [13, Statement 3.6]; however, we need to prove some more intermediate steps, due to the fact that we are dealing with several, not just one, correspondences in Definition 3.7.

Lemma 5.12 (Union). *Let $N, r, n \geq 1$, $A_1, \dots, A_r$ be subsets of $\mathbb{K}^n$, and, for $i = 1, \dots, r$, let $S_i \subseteq \mathbb{K}^n \times \mathbb{K}_+^{k_i}$ be an $\widetilde{\mathcal{S}}(N)$-approximant for A_i. Then, $A := \bigcup_i A_i$ has an $\widetilde{\mathcal{S}}(N)$-approximant.*

Proof. We may suppose that all the S_i have the same complexity k; then, $\bigcup_i S_i$ is an $\widetilde{\mathcal{S}}(N)$-approximant of A. □

Lemma 5.13. *Let $f : \mathbb{K}^n \rightsquigarrow \mathbb{K}$ be an admissible $\mathcal{C}^N$ correspondence, and define $S := \{(\overline{x}, t) \in \mathbb{K}^n \times \mathbb{K}_+ : |f(\overline{x})| \ni t\}$. Then S approximates $\mathrm{bd}[V(f)]$ from above on bounded sets.*

Proof. Fix $\varepsilon > 0$, and let $V := V(f)$. Let $X := \mathrm{bd}(V) \cap \overline{B(0; 1/\varepsilon)}$, and $Y_t := X \setminus \left(|f|^{-1}(t)^{\varepsilon}\right)$. Note that X and Y_t are d-compact. Let

$$P := \{t \in \mathbb{K} : t > 0 \;\&\; Y_t \neq \emptyset\}.$$

Assume for contradiction that the conclusion is false. This implies that P has arbitrarily small elements, if we chose ε small enough. Let $\overline{x} \in \mathrm{acc}_{t \to 0^+} Y_t$ ($\overline{x}$ exists, because each Y_t is contained in the d-compact set X), and $U := B(\overline{x}; \varepsilon/2)$. Note that V is closed (because f is admissible), and that $\overline{x} \in \mathrm{bd}(V)$.

By shrinking ε if necessary, we may assume that there exists $\delta > 0$, such that $F \cap \big(U \times (-\delta, \delta)\big)$ is the graph of a $\mathcal{C}^N$ function $g : U \to (-\delta, \delta)$, such that $g(\overline{x}) = 0$. Since $\overline{x} \in \mathrm{bd}(V)$, $|g|$ assumes a positive value γ on U. Since U is definably connected and g is continuous, $|g|$ assumes all values in the interval $[0, \gamma]$ in U.

Choose $t_0 \in P$ such that $Y_{t_0} \cap U \neq \emptyset$, and $t_0 < \gamma$. Since $t_0 < \gamma$, $U \cap |g|^{-1}(t_0) \neq \emptyset$; therefore, $U \subseteq |g|^{-1}(t_0)^{\varepsilon}$, and thus $Y_{t_0} \cap U = \emptyset$, a contradiction. □

Lemma 5.14 (Zero-set of correspondences). *If $f : \mathbb{K}^n \rightsquigarrow \mathbb{K}$ is admissible, $\mathcal{C}^N$ and in $\widetilde{\mathcal{S}}$, then its zero set $V(f)$ has an $\widetilde{\mathcal{S}}(N)$-approximant $S \in \widetilde{\mathcal{S}}_{n+2}$.*

Proof. Define the following two sets S_+ and S_-:

$$S_{\pm} := \{(\overline{x}, \varepsilon_1, \varepsilon_2) \in \mathbb{K}^n \times \mathbb{K}_+^2 : 1 + \|\overline{x}\|^2 \le 1/\varepsilon_1 \ \&\ f(\overline{x}) \ni \pm\varepsilon_2\},$$

and $S := S_+ \cup S_-$. By Lemma 4.6, $\langle \pm f, \phi \rangle$ are $\mathcal{C}^N$ and admissible, where $\phi : \mathbb{K}^{n+1} \to \mathbb{K}$, $(\overline{x}, y) \mapsto (1 + \|\overline{x}\|^2 + y^2)^{-1}$ (and in $\widetilde{\mathcal{S}}$). Thus, S is an $\widetilde{\mathcal{S}}(N)$-set.

We prove that S approximates $V(f)$ from below, namely $\forall^s \varepsilon_0 \forall^s \varepsilon_1 \forall^s \varepsilon_2\ S_{\varepsilon_1,\varepsilon_2} \subseteq V(f)^{\varepsilon_0}$. Let $K := \{\overline{x} \in \mathbb{K}^n : 1 + \|\overline{x}\|^2 \le 1/\varepsilon_1\}$, and $H := K \setminus V(f)^{\varepsilon_0}$. Note that K and H are d-compact, and $S_{\varepsilon_1,\varepsilon_2} \subseteq K$.

Claim. $|f|$ has a positive minimum on H, if $f(H)$ is non-empty.

If not, then, by Lemma 4.13, there exists $\overline{x} \in H$ such that $|f(\overline{x})| \ni 0$; however, this means that $\overline{x} \in V(f) \cap H$, contradicting the definition of H.

Thus, if we choose ε_2 smaller than the minimum of $|f|$ on H (or arbitrarily if H is empty), then $S_{\varepsilon_1,\varepsilon_2} \cap H = \emptyset$, and therefore $S_{\varepsilon_1,\varepsilon_2} \subseteq K \cap V(f)^{\varepsilon_0} \subseteq V(f)^{\varepsilon_0}$.

We prove that S approximates $\mathrm{bd}\big(V(f)\big)$ from above on bounded sets. Fix ε_0 and choose ε_1 so that the set K considered above contains $B(0, 1/\varepsilon_0)$. By Lemma 5.13, for all sufficiently small ε_2, setting $g = |f|$, we have $g^{-1}(\varepsilon_2)^{\varepsilon_0} \supseteq \mathrm{bd}(V(f)) \cap B(0, 1/\varepsilon_0)$. Thus $\mathrm{bd}(V(f)) \cap B(0, 1/\varepsilon_0) \subseteq S^{\varepsilon_0}_{\varepsilon_1,\varepsilon_2}$. □

The most difficult step of the proof of Proposition 5.9 concerns projections. We will need some preliminary definitions and lemmas.

Definition 5.15. If $f : \mathbb{K}^{n+k} \rightsquigarrow \mathbb{K}^k$ is an admissible correspondence, $\overline{a} \in \mathbb{K}^k$, and $\varepsilon_0 > 0$, then we denote by $V[\varepsilon_0]$ the ε_0-critical part of $V = V(f - \overline{a})$, i.e. the set of points $\overline{x}$ in $f^{-1}(\overline{a})$, such that one of the following conditions is satisfied for some $1 \le i_1 \le \cdots \le i_k \le n + k$:

- Either $1 + \|(x_{n+1}, \dots x_{n+k})\|^2 = 1/\varepsilon_0$,
- Or $\det\big(\frac{\partial f}{\partial(x_{i_1},\dots,x_{i_k})}(\overline{x}; \overline{a})\big)^2 = \varepsilon_0$.

Proposition 5.16. *Let $n, k \ge 1$, $f = \langle f_1, \dots, f_k \rangle : \mathbb{K}^{n+k} \to \mathbb{K}^k$ be an admissible $\mathcal{C}^1$ correspondence in $\widetilde{\mathcal{S}}$, and $V := V(f)$. Suppose further that 0 is a regular value of f, and that U is an open ball in $\mathbb{K}^n$ with the property that the set $\mathrm{bd}(\pi V) \cap U$ is non-empty and bounded, where $\pi := \Pi^{n+k}_n$. Then for every sufficiently small $\varepsilon > 0$, U intersects $\pi(V[\varepsilon])$.*

Proof. The proof proceeds as in the original [13, Corollary 2.9]. □

Lemma 5.17. *Let $X \subseteq \mathbb{K}^n$ be d-compact. Fix $0 < \varepsilon \in \mathbb{K}$, and $N \subseteq \mathbb{K}$ $\mathbb{K}$-definable and cofinal. Let $(A(t))_{t \in N}$ be a definable family of subsets of $\mathbb{K}^n$. The following are equivalent*

1. $\forall x \in X\ \forall t \in N$ *large enough* $X \cap B(x;\varepsilon) \subseteq A(t)$;
2. $\forall t \in N$ *large enough* $X \subseteq A(t)$.

Proof. That (2) implies (1) is clear.

Conversely, assume that (1) is true. Suppose, for contradiction, that (2) is false. Let $D(t) := X \setminus A(t)$. Let $N' := \{t \in N : D(t) \neq \emptyset\}$. Since (2) is false, N' is cofinal in N. Let C be the set of accumulation points of $\big(D(t)\big)_{t\to+\infty}$; that is, $x \in C$ iff $(\forall r \in \mathbb{K}^m)$ $(\forall \varepsilon \in \mathbb{K}_+)$ $(\exists y > r)$ $y \in N$ and $d(D(y), x) < \varepsilon$.

It is easy to see that $C \neq \emptyset$; let $x \in C$. By (1), if t is large enough, then $X \cap B(x;\varepsilon) \subseteq A(t)$. Choose $t \in N'$ such that $X \cap B(x;\varepsilon) \subseteq A(t)$ and $d(x, D(t)) < \varepsilon$. Let $y \in D(t)$ such that $d(x, y) < \varepsilon$. Since $y \in D(t)$, we have $y \notin A(t)$. Since $y \in X \cap B(x;\varepsilon)$, we have $y \in A(t)$, a contradiction. ☐

Lemma 5.18 (Projection). *Let* $N \geq 1$. *If* $A \subseteq \mathbb{K}^{n+1}$ *has an* $\widetilde{\mathcal{S}}(N+1)$*-approximant* $S \subseteq \mathbb{K}^{n+1} \times \mathbb{K}^k_+$, *then there is an* $\widetilde{\mathcal{S}}(N)$*-approximant* $S' \subseteq \mathbb{K}^n \times \mathbb{K}^{k+1}_+$ *for* $\Pi^{n+1}_n A \subseteq \mathbb{K}^n$.

The drop in regularity in the above lemma from $N+1$ to N is due to the fact that the definition of S' involves the derivatives of the functions defining S.

Proof. We will give some of the details of the case when S has only one $\widetilde{\mathcal{S}}(N)$-constituent:

$$S = \{(\overline{x}, \varepsilon) \in \mathbb{K}^{n+1} \times \mathbb{K}^{k-1}_+ : \exists \overline{y} \in \mathbb{K}^k\ f(\overline{x}, \overline{y}) \ni \overline{\varepsilon}\},$$

were $f : \mathbb{K}^{n+1} \times \mathbb{K}^{k-1} \rightsquigarrow \mathbb{K}^k$ is some admissible $\mathcal{C}^{N+1}$ correspondence. Define $\overline{\varepsilon} := (\varepsilon_1, \dots, \varepsilon_k)$. We let $S' \subseteq \mathbb{K}^n \times \mathbb{K}^{k+1}$ to be the set whose sections $S'_{\overline{\varepsilon},\varepsilon_{k+1}} \subseteq \mathbb{K}^n$ are given by:

$$S'_{\overline{\varepsilon},\varepsilon_{k+1}} := \Pi^{n+1}_n\big(S_{\overline{\varepsilon}}[\varepsilon_{k+1}]\big) = \Pi^{n+k}_n\big(\{(\overline{x}, \overline{y}) \in \mathbb{K}^{n+1} \times \mathbb{K}^{k-1} : f(\overline{x}, \overline{y}) \ni \overline{\varepsilon}\}[\varepsilon_{k+1}]\big).$$

By Lemmas 4.8 and 4.6, S' is an $\widetilde{\mathcal{S}}(N)$-set. Let us see that S' approximates $\overline{\Pi^{n+1}_n A}$ from below. From the definition of S' it follows that $S'_{\overline{\varepsilon},\varepsilon_{k+1}} \subseteq \Pi^{n+1}_n S_{\overline{\varepsilon}}$. On the other hand since S approximates $\overline{A}$ from below, given $\varepsilon_0 > 0$, we have $\forall^s \overline{\varepsilon}$ $S_{\overline{\varepsilon}} \subseteq (\overline{A})^{\varepsilon_0}$. It follows that $\forall \varepsilon_0 > 0 \forall^s \overline{\varepsilon}$ we have $S'_{\overline{\varepsilon},\varepsilon_{k+1}} \subseteq \Pi^{n+1}_n S_{\overline{\varepsilon}} \subseteq (\overline{\Pi^{n+1}_n A})^{\varepsilon_0}$.

It remains to prove that S' approximates $\mathrm{bd}(\overline{\Pi^{n+1}_n A})$ from above on bounded sets.

We can use [13, Lemma 3.4] to prove that $\forall^s \overline{\varepsilon}$, $\overline{\varepsilon}$ is a regular value of f.

Fix $\varepsilon_0 > 0$. Let $X := \mathrm{bd}(\overline{\Pi^{n+1}_n A}) \cap \overline{B(0; 1/\varepsilon_0)}$; note that X is d-compact. Let $\overline{x} \in X$, and U be the open ball of center $\overline{x}$ and radius ε_0.

Then U intersects $\Pi^{n+1}_n \mathrm{bd}(\overline{A})$, and since S approximates $\mathrm{bd}(\overline{A})$ from above on bounded sets, it easily follows that $\forall^s \overline{\varepsilon}$ U intersects $\Pi^{n+1}_n S_{\overline{\varepsilon}}$. On the other hand since S approximates $\overline{A}$ from below and $U \not\subseteq \overline{\Pi^{n+1}_n A}$, it is easy to see that U is not included in $\Pi^{n+1}_n S_{\overline{\varepsilon}}$, and therefore must intersect its frontier. Thus by Proposition 5.16, $\forall^s \overline{\varepsilon} \forall^s \varepsilon_{k+1}$ U intersects $\Pi^{n+1}_n S_{\overline{\varepsilon}}[\varepsilon_{k+1}] = S'_{\varepsilon_1,\dots,\varepsilon_{k+1}}$ and hence we see that $\forall^s \varepsilon_1 \dots \forall^s \varepsilon_{k+1}\ U \subseteq (S'_{\varepsilon_1,\dots,\varepsilon_{k+1}})^{\varepsilon_0}$. Using Lemma 5.17, we deduce that $\forall^s \varepsilon_1 \dots \forall^s \varepsilon_{k+1}\ X \subseteq (S'_{\varepsilon_1,\dots,\varepsilon_{k+1}})^{\varepsilon_0}$, which is the conclusion. □

Lemma 5.19 (Product). *Let $n_1, n_2, k_1, k_2, N \geq 1$. For $i = 1, 2$, let $A_i \in \widetilde{\mathcal{S}}_{n_i}$, such that A_i has empty interior (in $\mathbb{K}^{n_i}$). Assume that each A_i has an $\widetilde{\mathcal{S}}(N)$-approximant $S^i \subset \mathbb{K}^{n_i} \times \mathbb{K}^{k_i}_+$. Then, $A_1 \times A_2$ has an $\widetilde{\mathcal{S}}(N)$-approximant $S \subset \mathbb{K}^{n_1+n_2} \times \mathbb{K}^{k_1+k_2}_+$. Moreover, up to permutation of variables, $S = S_1 \times S_2$.*

Proof. W.l.o.g., each S^i has only one $\widetilde{\mathcal{S}}(N)$-constituent, that is, it is of the form $S^i := \{(\overline{x}, \overline{\varepsilon}) \in \mathbb{K}^{n_i} \times \mathbb{K}^{k_i} : \exists \overline{y} \in \mathbb{K}^{k_i-1}\ f_i(\overline{x}, \overline{y}) \ni \overline{\varepsilon}\}$, for some $\mathcal{C}^N$ admissible correspondence $f_i : \mathbb{K}^{n_i} \times \mathbb{K}^{k_i-1} \rightsquigarrow \mathbb{K}^{k_i}$, $i = 1, 2$. Define

$$S := \{(\overline{x}_1, \overline{x}_2, \overline{\varepsilon}_1, \overline{\varepsilon}_2) \in \mathbb{K}^{n_1} \times \mathbb{K}^{n_2} \times \mathbb{K}^{k_1}_+ \times \mathbb{K}^{k_2}_+ : \\ \exists \overline{y}_1 \in \mathbb{K}^{k_1-1}\ \exists \overline{y}_2 \in \mathbb{K}^{k_2-1}\ f_1(\overline{x}_1, \overline{y}_1) \ni \overline{\varepsilon}_1 \ \&\ f_2(\overline{x}_2, \overline{y}_2) \ni \overline{\varepsilon}_2\};$$

By Lemma 4.10 and Remark 5.8, S is an $\widetilde{\mathcal{S}}(N)$-set in $\widetilde{\mathcal{S}}_{n_1+n_2+k_1+k_2}$. Since each A_i has empty interior, also the $\overline{A_i}$ have empty interiors; therefore, $\operatorname{bd}(\overline{A_i}) = \overline{A_i}$, and we have $A_i \leq S^i$, and $S^i \leq A_i$. The reader can check that we can conclude that S approximates $A_1 \times A_2$. □

Lemma 5.20 (Linear intersection). *Given $N, n, k \geq 1$, let $A \in \widetilde{\mathcal{S}}_n$ have an $\widetilde{\mathcal{S}}(N)$-approximant $S \subset \mathbb{K}^n \times \mathbb{K}^k_+$, and suppose Y is an $(n-1)$-dimensional $\mathbb{K}$-affine subset of $\mathbb{K}^n$; suppose further that $\mathring{\overline{A}} \cap Y = \emptyset$. Then, there is an $\widetilde{\mathcal{S}}(N)$-approximant $S' \subseteq \mathbb{K}^n \times \mathbb{K}^{k+2}_+$ for $\overline{A} \cap Y$.*

Proof. We will use the following easy observation: let $A, B \subseteq \mathbb{K}^n$ be closed sets, and let $K \subseteq \mathbb{K}^n$ be d-compact. Then $\forall^s \varepsilon_1 \forall^s \varepsilon_2\ A^{\varepsilon_2} \cap B^{\varepsilon_2} \cap K \subseteq (A \cap B)^{\varepsilon_1}$.

By assumption, $\overline{A} \cap Y \subseteq \operatorname{bd}(\overline{A})$, hence we only need to worry about a subset of $\operatorname{bd}(\overline{A})$. Suppose Y is the zero-set of a linear polynomial l with coefficients in $\mathbb{K}$. The sections $S_{\varepsilon_1,\dots,\varepsilon_k} \subseteq \mathbb{K}^n$ of S have the form:

$$S_{\varepsilon_1,\dots,\varepsilon_k} = \Pi^{n+k-1}_n\{f_1^{-1}(\varepsilon_1, \dots, \varepsilon_k)\} \cup \dots \cup \Pi^{n+k-1}_n\{f_s^{-1}(\varepsilon_1, \dots, \varepsilon_k)\},$$

where $f_i : \mathbb{K}^{n+k} \to \mathbb{K}^k$ is a C^∞ function in $\widetilde{\mathcal{S}}$.

Define $S' \subseteq \mathbb{K}^{n+k+2}$ as the set whose sections $S'_{\varepsilon_1,\dots,\varepsilon_{k+2}} \subseteq \mathbb{K}^n$ have the form:

$$S'_{\varepsilon_1,\dots,\varepsilon_{k+2}} = S_{\varepsilon_3,\dots,\varepsilon_{k+2}} \cap Y_{\varepsilon_2} \cap K_{\varepsilon_1},$$

where $K_{\varepsilon_1} = \{\overline{x} \in \mathbb{K}^n \mid \|1, x_1, \dots, x_n\|^2 \leq 1/\varepsilon_1\} = \{\overline{x} \mid \exists x_{n+k}\ (1 + \sum_{i=1}^n x_i^2 + x_{n+k}^2)^{-1} = \varepsilon_1\}$ and $Y_{\varepsilon_2} = \{\overline{x} \mid \exists x_{n+k+1}\ l(x_1, \dots, x_n)^2 + x_{n+k+1}^2 = \varepsilon_2\}$. By Lemma 4.6 the set S' is indeed an $\widetilde{\mathcal{S}}(N)$-set.

Let us prove that S' approximates $\overline{A} \cap Y$ from below. By the above observation,

$$\forall^s \varepsilon_1 \forall^s \varepsilon_2\ \overline{A}^{\varepsilon_2} \cap B(0; \varepsilon_1^{-1}) \cap Y_{\varepsilon_2} \subseteq (\overline{A} \cap Y)^{\varepsilon_1}.$$

Since S approximates $\overline{A}$ from below we have:

$$\forall^s \varepsilon_0 \forall^s \varepsilon_1 \dots \forall^s \varepsilon_{k+2} \;\; S'_{\varepsilon_1,\dots,\varepsilon_{k+2}} \subseteq S_{\varepsilon_3,\dots,\varepsilon_{k+2}} \subseteq \overline{A}^{\varepsilon_2}.$$

From the definition it follows that $S'_{\varepsilon_1,\dots,\varepsilon_{k+2}} \subseteq K_{\varepsilon_1} \cap Y_{\varepsilon_2}$, hence combining all these equations we get

$$\forall^s \varepsilon_0 > 0 \forall^s \varepsilon_1 \dots \forall^s \varepsilon_{k+2} \;\; S'_{\varepsilon_1,\dots,\varepsilon_{k+2}} \subseteq (\overline{A} \cap Y)^{\varepsilon_0}.$$

It remains to prove that S' approximates $\overline{A} \cap Y$ from above on bounded sets. Since S approximates $\overline{A}$ from above on bounded sets we have:

$$\forall^s \varepsilon_2 \dots \forall^s \varepsilon_{k+2} \;\; \mathrm{bd}(\overline{A}) \cap B(0; \varepsilon_2^{-1}) \subseteq S^{\varepsilon_2}_{\varepsilon_3 \dots \varepsilon_{k+2}}.$$

Since $\forall^s \varepsilon_0 \forall^s \varepsilon_1 \; B(0, \varepsilon_2^{-1}) \subseteq K_{\varepsilon_1}$ and by our hypothesis $\mathrm{bd}(A) \cap Y = \overline{A} \cap Y$, we obtain, using again the above observation:

$$\forall^s \varepsilon_0 \forall^s \varepsilon_1 \dots \forall^s \varepsilon_{k+2} \;\; \overline{A} \cap Y \subseteq (S_{\varepsilon_3,\dots,\varepsilon_{k+2}} \cap Y_{\varepsilon_2} \cap K_{\varepsilon_1})^{\varepsilon_0}.$$

This concludes the proof of the lemma. ☐

Lemma 5.21 (Small intersection). *Let $n, k_1, k_2, N \geq 1$; define $M := N + n$. For $i = 1, 2$, let A_i be closed sets in $\widetilde{\mathcal{S}}_{n_i}$. Assume that each A_i has an $\widetilde{\mathcal{S}}(M)$-approximant $S^i \subset \mathbb{K}^n \times \mathbb{K}_+^{k_i}$. Assume moreover that each A_i has empty interior. Then, $A := A_1 \cap A_2$ has an $\widetilde{\mathcal{S}}(N)$-approximant $S \subset \mathbb{K}^n \times \mathbb{K}_+^{3n+k_1+k_2}$.*

Proof. $A = \Pi_n^{2n}\big((A_1 \times A_2) \cap \Delta\big)$, where Δ is the diagonal of $\mathbb{K}^n \times \mathbb{K}^n$. By Lemma 5.19, $A_1 \times A_2$ has an $\widetilde{\mathcal{S}}(M)$-approximant in $\mathcal{S}_{2n+k_1+k_2}$. By hypothesis, $A_1 \times A_2$ has empty interior, hence we can apply Lemma 5.20 n times, and therefore $(A_1 \times A_2) \cap \Delta$ has an $\widetilde{\mathcal{S}}(M)$-approximant in $\mathcal{S}_{2n+k_1+k_2+2n}$. Finally, by Lemma 5.18, A has an $\widetilde{\mathcal{S}}(M-n)$-approximant in $\mathcal{S}_{4n+k_1+k_2}$. ☐

Lemma 5.22. *Let $f : \mathbb{K}^n \rightsquigarrow \mathbb{K}$ be an admissible $\mathcal{C}^N$ correspondence in $\widetilde{\mathcal{S}}$. Let $A := V(f) \times \{0\} \subset \mathbb{K}^{n+1}$. Then, A has a $\widetilde{\mathcal{S}}(N)$-approximant in $\mathcal{S}_{n+4}$.*

Proof. Define $S := \{(\overline{x}, z, \varepsilon_1, \varepsilon_2, \varepsilon_3) \in \mathbb{K}^{n+1} \times \mathbb{K}_+^3 : 1 + \|\overline{x}\|^2 \leq 1/\varepsilon_1 \;\&\; |z|^2 \leq \varepsilon_2 \;\&\; f(\overline{x}) + z \ni \varepsilon_3\}$. Notice that S is an $\widetilde{\mathcal{S}}(N)$ set (with only one constituent): in fact, $S = \{(\overline{x}, z, \varepsilon_1, \varepsilon_2, \varepsilon_3) : \exists y_1 y_2 \; 1/(1 + \|\overline{x}\|^2) + y_1^2 = \varepsilon_1 \;\&\; z^2 + y_2^2 = \varepsilon_2 \;\&\; f(\overline{x}) + z \ni \varepsilon_3\}$. Notice also that A has empty interior. We claim that $A \leq S$ and $S \leq A$, proving the conclusion. For fixed $t > 0$, let $K(t) := \{\overline{x} \in \mathbb{K}^n : 1 + \|\overline{x}\|^2 \leq 1/t\}$.

Claim 1. $S \leq A$, i.e. $\forall^s \varepsilon_0 \forall^s \overline{\varepsilon} \; S_{\overline{\varepsilon}} \subseteq A^{\varepsilon_0}$, where $\overline{\varepsilon} := (\varepsilon_1, \varepsilon_2, \varepsilon_3)$.

Let $(\overline{x}, z) \in S_{\overline{\varepsilon}}$. Define $I := [-\sqrt{\varepsilon_2}, \sqrt{\varepsilon_2}]$ and $H := K(\varepsilon_1) \setminus V(f)^{\varepsilon_0/4}$. I, H and $K(\varepsilon_1)$ are d-compact, and $S_{\overline{\varepsilon}} \subseteq K(\varepsilon_1) \times I$.

We claim that $|f|$ has a positive minimum on H, if $f(H)$ is non-empty. Otherwise, by Lemma 4.13, there exists $\overline{x} \in H$ such that $f(\overline{x}) \ni 0$, contradicting the definition of H. Let $\delta > 0$ be such minimum (or $\delta = 1$ if $f(H)$ is empty). If we choose ε_3 smaller than δ, then $\overline{x} \in K(\varepsilon_1) \setminus H$, and therefore $\overline{x} \in V(f)^{\varepsilon_0/4}$. Now choose ε_2 smaller than ${\varepsilon_0}^2/4$, and obtain $(\overline{x}, z) \in V(f)^{\varepsilon_0/4} \times [-\varepsilon_0/4, \varepsilon_0/4] \subseteq A^{\varepsilon_0}$.

Claim 2. $A \leq S$, i.e. $\forall^s \varepsilon_0 \forall^s \overline{\varepsilon}\ A \cap B(0; 1/\varepsilon_0) \subseteq (S_{\overline{\varepsilon}})^{\varepsilon_0}$.

Fix $\varepsilon_0 > 0$, and choose $1 > \delta_1 > 0$ such that $B(0; 1/\varepsilon_0) \subseteq K(\delta_1)$. Let $\delta_2 := \varepsilon_0/2$. For any ε_2 such that $0 < \varepsilon_2 < \delta_2$, let $\delta_3 := \varepsilon_2/2$. Finally, choose any ε_3 such that $0 < \varepsilon_3 < \delta_3$. Let $\overline{y} := (\overline{x}, z) \in A \cap B(0; 1/\varepsilon_0)$. We prove that, for ε_0 and $\overline{\varepsilon}$ chosen as above, $\overline{y} \in (S_{\overline{\varepsilon}})^{\varepsilon_0}$. First, notice $z = 0$ and $\overline{x} \in V(f)$. Let $\overline{w} := (\overline{x}, \varepsilon_3)$. Notice that $d(\overline{y}, \overline{w}) = \varepsilon_3 < \varepsilon_0$, and that $\overline{w} \in S_{\overline{\varepsilon}}$, and therefore $\overline{y} \in (S_{\overline{\varepsilon}})^{\varepsilon_0}$. Hence,

$$\forall \varepsilon_0 \exists \delta_1 \forall \varepsilon_1 < \delta_1 \exists \delta_2 \forall \varepsilon_2 < \delta_2 \exists \delta_3 \forall \varepsilon_3 < \delta_3 \left(A \cap B(0; 1/\varepsilon_0) \subseteq (S_{\overline{\varepsilon}})^{\varepsilon_0}\right). \qquad \square$$

Proof (Proof of Proposition 5.9). Since every set in $\widetilde{\mathcal{S}}$ can be obtained from a finite number of sets in $\mathcal{S}$ by performing a finite sequence of Charbonnel operations, we can prove the statement inductively.

First, we prove the basic case, i.e. when $A \in \mathcal{S}_n$. Fix $N \geq 1$. Let M be large enough (how large will be clear from the rest of the proof).

By hypothesis, there exist $m \geq n$ and $r \geq 1$, such that $A = \Pi^m_n(S_M)$, for some $S_M \subseteq \mathbb{K}^m$ of the form $S_M = \bigcup_{0 \leq j < k_M} S_{M,j}$ where each set $S_{M,j}$ is of the form

$$S_{M,j} = \bigcap_{0 \leq i < r} V(f_{M,i,j}),$$

and each $f_{M,i,j} : \mathbb{K}^m \rightsquigarrow \mathbb{K}$ is a $\mathcal{C}^M$ admissible correspondence in $\widetilde{\mathcal{S}}$.

Let $A_j := \Pi^m_n(S_{M,j})$. If we prove that each A_j satisfies (Φ_N), then, by Lemma 5.12, A also satisfies (Φ_N). Therefore, w.l.o.g., $k_M = 1$, i.e. $S_M = \bigcap_{0 \leq i < r} V_{M,i}$, where $V_{M,i} := V(f_{M,i})$ (where each $f_{M,i} : \mathbb{K}^m \rightsquigarrow \mathbb{K}$ is a $\mathcal{C}^M$ admissible correspondence in $\widetilde{\mathcal{S}}$). By Lemma 5.14, each $V_{M,i}$ satisfies (Φ_M). We need to prove that S_M satisfies $(\Phi_{M'})$ (for a suitable M'). If all the $V_{M,i}$ were with empty interior, we could apply Lemma 5.21. Otherwise, for every i, define $W_{M,i} := V_{M,i} \times \{0\} \subset \mathbb{K}^{m+1}$. By Lemma 5.22, each $W_{M,i}$ has an $\widetilde{\mathcal{S}}(M)$-approximant in $\mathcal{S}_{m+4}$; moreover, each $W_{M,i}$ has empty interior, and therefore, by Lemma 5.21, $W_M := \bigcap_i W_{M,i}$ has an $\widetilde{\mathcal{S}}(M - (r-1)(m+1))$-approximant in $\mathcal{S}_{(3 \cdot 2^r - 2)m + 4 \cdot 2^r - 5}$. Since $S_M = \Pi^{m+1}_m(W_M)$, S_M has an $\widetilde{\mathcal{S}}(M - rm + m - r)$-approximant in $\mathcal{S}_{(3 \cdot 2^r - 2)m + 4 \cdot 2^r - 5}$. Finally, by Lemma 5.18, A has an $\widetilde{\mathcal{S}}(M - rm + n - r)$-approximant in $\mathcal{S}_{(3 \cdot 2^r - 2)m + 4 \cdot 2^r - 5}$.

The basic case when $A \in \mathcal{S}$ is the only place where we use the DAC^N hypothesis. Notice that we have to take

$$M \geq N + rm - n + r. \tag{5.1}$$

Now we can explain the reason why we had to take m and r independent from N in Definition 3.7: in fact, if instead $m = m(N)$ and $r = r(N)$ depended on N, then in (5.1) we would have to take $r = r(M)$ and $m = m(M)$; but we would not be able to do it, since we don't know the value of M yet.

To prove the inductive step, suppose $A \in \widetilde{\mathcal{S}}_n$.

If A is described as $A_1 \cup A_2$, then an $\widetilde{\mathcal{S}}(N)$-approximant for A is given by the union of the $\widetilde{\mathcal{S}}(N)$-approximants for A_1 and A_2, respectively. The reason why this arguments works is that topological closure commutes with union. The same is not true with intersection instead of union, and this is the reason why we will need a more complicated argument for the intersection.

If A is described as $\Pi_n^{n+h}[A_1]$, then an iterated use of Lemma 5.20 tells us what to do.

If A is described as $\overline{B}$, then it is trivial since by definition an $\widetilde{\mathcal{S}}(N)$-approximant for B is an $\widetilde{\mathcal{S}}(N)$-approximant for A.

So, the only case which requires more care is the case when A is described as $A_1 \cap L$, where L is $\mathbb{K}$-affine. We need to analyze all subcases.

If A_1 is described as a set in $\mathcal{S}$, then A too can be described as a set in $\mathcal{S}$ and we already know how to deal with these sets. If A_1 is obtained as a union, then by the distributivity laws for $\cup, \cap$, by inductive hypothesis and by an application of the argument above on how to approximate unions, we know how to approximate A. If $A_1 = \Pi_n^m[U]$, then we use the equation

$$\Pi_n^m[U] \cap L = \Pi_n^m[(U \times L) \cap (\Delta \times \mathbb{R}^{m-n})],$$

where $\Delta \subset \mathbb{K}^{2n}$ is the diagonal, and we conclude again by an application of Lemma 5.18 and by inductive hypothesis. If A_1 is obtained by by intersection with a $\mathbb{K}$-affine set, then we conclude by the inductive hypothesis (as the intersection of two $\mathbb{K}$-affine sets is $\mathbb{K}$-affine). The only difficult case is when A_1 is described as $\overline{U}$. Let $L = Y_1 \cap \ldots \cap Y_m$, where Y_i is a $\mathbb{K}$-affine set of codimension 1. Notice that

$$\overline{U} \cap Y_1 = \overline{U \cap Y_1} \cup (\overline{U \cap Y_1^+} \cap Y_1) \cup (\overline{U \cap Y_1^-} \cap Y_1),$$

where, Y_1 is the zero set of a linear polynomial l over $\mathbb{K}$, $Y_1^+ = \{\overline{x} \in \mathbb{K}^n \mid l(\overline{x}) > 0\}$, and Y_1^- is defined similarly by $l < 0$.

Now, Y_1 does not meet the interior of $\overline{U \cap Y_1^\pm}$ (since it does not meet the interior of $\overline{Y_1^\pm}$), hence to approximate the sets $\overline{U \cap Y_1^\pm} \cap Y_1$ we can use Lemma 5.20; while by inductive hypothesis we can get an approximant for the set $U \cap Y_1$. Now notice that $\overline{U} \cap Y_1$ has empty interior, so that we can make use of Lemma 5.20 for $(\overline{U} \cap Y_1) \cap Y_2$, and continue this way until we complete the proof of the theorem. □

5.1 Cell Decomposition

We conclude the proof of Theorem 3.8 by a cell decomposition argument.

Theorem 5.23. *Let $\mathcal{S}$ be a semi-closed o-minimal weak structure on a definably complete Baire structure $\mathbb{K}$, satisfying the following condition:*

(*) *For every closed set in $\widetilde{\mathcal{S}}$, there exists a closed set $B \in \widetilde{\mathcal{S}}$, such that B has empty interior and* $\mathrm{bd}(A) \subseteq B$.

Then, $\widetilde{\mathcal{S}}$ is closed under complementation (and hence is an o-minimal structure).

Notice that, by Theorem 5.5, if $\mathcal{S}$ satisfies DAC^N for all N, then $\mathcal{S}$ satisfies the assumption (*), and therefore $\widetilde{\mathcal{S}}$ is an o-minimal structure; hence, the above theorem implies Theorem 3.8.

If one is interested only in the case when $\mathbb{K}$ is an expansion of the real field, then [13, Sect. 4] already proves the result. We will now sketch how to modify Wilkie's original proof to adapt it to the case when $\mathbb{K}$ is not an expansion of $\mathbb{R}$.

We assume that $\mathcal{S}$ satisfies the Condition (*). Our aim is now to prove the analogue of the $\widetilde{\mathcal{S}}$-cell Decomposition Theorem 4.5 in [13]; the reader can refer to [13, Definitions 4.1 and 4.3], where we replace $\mathbb{R}$ by $\mathbb{K}$, for the definition of $\widetilde{\mathcal{S}}$-cell and of $\widetilde{\mathcal{S}}$-cell decomposition (the only unusual aspect in these definitions is that an $\widetilde{\mathcal{S}}$-cell is assumed to be bounded).

Proposition 5.24. *Let $n \geq 1$ and suppose that D is an $\widetilde{\mathcal{S}}$-cell in $\mathbb{K}^n$ and that A is a set in $\widetilde{\mathcal{S}}_n$. Suppose further that A is a subset of D which is also closed in D. Then, there exists an $\widetilde{\mathcal{S}}$-cell decomposition $\mathcal{D}$ of D which is compatible with A.*

Once established this result, we see that Theorem 5.23 follows easily, as explained in the remarks preceding the proof of Wilkie [13, Theorem 4.5].

Instead of repeating Wilkie's proof of Proposition 5.24, we will give the details of the proof of the following weaker form, which shows the main ideas involved and the points we need to modify in the non-Archimedean settings, but requires much simpler formal scaffolds.

Proposition 5.25. *Let $n \geq 1$ and suppose that D is an $\widetilde{\mathcal{S}}$-cell in $\mathbb{K}^n$ and that A is a set in $\widetilde{\mathcal{S}}_n$. Suppose further that A is a subset of D which is also closed in D. Then, there exists a finite partition $\mathcal{D}$ of D into $\widetilde{\mathcal{S}}$-cells which is compatible with A.*

The proof of Proposition 5.25 proceeds by induction on n. The case $n = 0$ is trivial. Therefore, assume that we have proved the conclusion for n, and let us prove it for $n + 1$. If D is not an open cell, then, by the usual tricks, we can lower n, and conclude by using the inductive hypothesis. Therefore, we can assume that D is an open cell. We want to further reduce to the case when A has empty interior. If A has non-empty interior, let $B \in \widetilde{\mathcal{S}}_{n+1}$ such that B has empty interior and $\mathrm{bd}(A) \subseteq B$. Assume that we are able to find a finite partition $\mathcal{D}$ of D into $\widetilde{\mathcal{S}}$-cells which is compatible with B. Then, since A is closed in D and every cell is definably connected, $\mathcal{D}$ is also compatible with A, and we are done. Hence, w.l.o.g. A has empty interior (this is the only point where we use the condition (*)). Let $C := \pi(D)$, where $\pi := \Pi_n^{n+1}$, and $D = (f, g)_C$.

Claim 1. For each $i \geq 1$, consider the set

$$A_i := \left\{ \overline{x} \in C : \exists y_1, \ldots, y_i \ (y_1 < \cdots < y_i \wedge \bigwedge_{j=1}^{i} (\overline{x}, y_j) \in A) \right\}.$$

Then each set A_i lies in $\widetilde{\mathcal{S}}_n$, and A_N has empty interior in $\mathbb{K}^n$ for some $N \geq 1$.

Proof. The definition of A_i implies immediately that $A_i \in \widetilde{\mathcal{S}}_n$.

Let $N := \gamma(A) + 1$, and fix $\overline{x} \in C$. Note that if the fibre A_x has cardinality greater or equal to N, then it has non-empty interior.

Since A has empty interior, it is definably meager. Therefore, by Proposition 2.4, the set of those points $\overline{x} \in C$ such that $A_{\overline{x}}$ has non-empty interior is definably meager. Thus, A_N is definably meager, and hence it has empty interior. □

Fix N as in the above claim, and define $\tilde{H}$ etc. as in [13, p. 418]. More precisely, define

$$H := \{(\overline{x}, \varepsilon) \in C \times \mathbb{K}_+ : \exists y \in \mathbb{K}\ (\overline{x}, y) \in D \ \&\ (x, y + \varepsilon) \in D\}$$
$$H_f := \{(\overline{x}, \varepsilon) \in C \times \mathbb{K}_+ : \exists y \in \mathbb{K}\ (\overline{x}, f(\overline{x}) + \varepsilon) \in D\}$$
$$H_g := \{(\overline{x}, \varepsilon) \in C \times \mathbb{K}_+ : \exists y \in \mathbb{K}\ (\overline{x}, g(\overline{x}) - \varepsilon) \in D\},$$

and the following subsets of C

$$\tilde{H} := C \cap (\overline{H})_0,$$
$$\tilde{H}_f := C \cap (\overline{H}_f)_0,$$
$$\tilde{H}_g := C \cap (\overline{H}_g)_0.$$

Notice that the inductive hypothesis implies the following statement:
Let $A_1, \ldots, A_l$ be subsets of C which are closed in C and in $\widetilde{\mathcal{S}}$. Then, there exists a finite partition $\mathcal{C}$ of C into $\widetilde{\mathcal{S}}$-cells which is compatible with each A_i.

In particular, there exists a finite partition $\mathcal{C}$ of C into $\widetilde{\mathcal{S}}$-cells which is compatible with $\tilde{H}$, $\tilde{H}_f$, $\tilde{H}_g$, and with each $\overline{A_i} \cap C$, for $i = 1, \ldots, N$.

Let $C' := C_j$ be a cell in $\mathcal{C}$. Now, we want to define a partition $\mathcal{D}' := \mathcal{D}'_j$ of $(C' \times \mathbb{K}) \cap D$ into $\widetilde{\mathcal{S}}$-cells which is compatible with $(C' \times \mathbb{K}) \cap A$. Once we have these partitions, we can define $\mathcal{D} := \bigcup_j \mathcal{D}_j$ and prove the conclusion.

If C' is not open, we use the inductive hypothesis to find the desired partition.

Hence, we only need to consider the case when C' is an open cell in $\mathcal{C}$. Notice that $C' \cap \overline{A_N}$ is empty, because $\overline{A_N}$ has empty interior. If $(C' \times \mathbb{K}) \cap A = \emptyset$, we choose $\mathcal{D}' := \{(f, g)_{C'}\}$, and we are done. Otherwise, $C' \cap \overline{A_1}$ is non-empty; choose $k < N$ maximal such that $C' \cap \overline{A_k} \neq \emptyset$ (and therefore $C' \subseteq \overline{A_k}$).

Claim 2. $C' \subseteq A_k$.

Proof. As in Wilkie's proof, we conclude that $\tilde{H}$, $\tilde{H}_f$ and $\tilde{H}_g$ have empty interior, and therefore are disjoint from C'.

Consider a point $\overline{x} \in C'$. Let M be the cardinality of the fibre $A_{\overline{x}}$; note that, by definition of k, $M \leq k$. Let $y_0 := f(\overline{x})$, $y_{M+1} := g(\overline{x})$, and, for $1 \leq i \leq M$, y_i be the i-th point of $\mathbb{K}$ such that $(\overline{x}, y_i) \in A$.

If $1 \leq i \leq M$, since $\overline{x} \notin \tilde{H}$, we may find open neighbourhoods V_i of $\overline{x}$ in $\mathbb{K}^n$ and J_i of y in $\mathbb{K}$, such that for each $\overline{x}' \in V_i$ there is at most one $y' \in J_i$ such that $(\overline{x}', y') \in A$.

Similarly, if $i = 0$ or $i = M + 1$ then, since $\overline{x} \notin \tilde{H}_f \cup \tilde{H}_g$, we may choose V_i and J_i such that $(V_i \times J_i) \cap A = \emptyset$. Let $T := \{y \in \mathbb{K} : (\overline{x}, y) \in A \;\&\; y \notin \bigcup_i J_i\}$, and $T' := \{\overline{x}\} \times T$. Note that T' is a d-compact subset of D, and that A is a closed subset of D disjoint from T'. Hence, the distance between T' and A is some positive number $d > 0$. Let $U := \bigcap_{i=0}^{M+1} V_i \cap \{\overline{x}' \in C' : d(\overline{x}', \overline{x}) < d\}$.

Therefore, for every $\overline{x}' \in U$,

$$\left|(\{\overline{x}'\} \times \mathbb{K}) \cap A\right| \leq \left|(\{\overline{x}\} \times \mathbb{K}) \cap A\right| = M. \tag{5.2}$$

We conclude as in [13]: as $\overline{x} \in \overline{A_k}$, we may choose $\overline{x}' \in U \cap A_k$ here, from which it follows (using the maximality of k) that $M = k$. Hence $\overline{x} \in A_k$ and the claim is justified. □

Thus, for each $i = 1, \ldots, k$, we may define the function $f_i : C' \to K$ is $\widetilde{\mathcal{S}}$ by $f_i(\overline{x}) = y$ iff y is the i-th point of $\mathbb{K}$ such that $(\overline{x}, y_i) \in A$.

Claim 3. Each function f_i is continuous.

Proof. Let $\overline{x} \in C'$. Let U, V_i and J_i be defined as in the proof of the previous claim, for $i = 1, \ldots, k$. Let $\overline{x}' \in U$. Note that, since we have equality in (5.2), then, for every $i = 1, \ldots, k$, there is exactly one $y_i' \in J_i$ such that $(\overline{x}', y_i') \in A$. Note also that $y_i' = f_i(\overline{x}')$. Fix i such that $1 \leq i \leq k$, and fix J neighbourhood of $y_i = f_i(\overline{x})$. In the construction of V_i and J_i, we could have chosen J_i such that $J_i \subseteq J$, and then found a corresponding V_i. Proceeding in the construction, we see that, for every J neighbourhood of $f_i(\overline{x}')$, we can find U neighbourhood of $\overline{x}$ such that $f_i(U) \subseteq J$, which is equivalent to the definition of f_i being continuous at $\overline{x}$. Since $\overline{x} \in U$ is arbitrary, the claim is proved. □

Now, using the functions f, g, $f_1, \ldots, f_k$, we can define a cell decomposition of $(C' \times \mathbb{K}) \cap D$ which is compatible with $(C' \times \mathbb{K}) \cap A$, and we are done.

6 Expansions of O-minimal Structures by Total $\mathcal{C}^\infty$ Functions

In this section we give an application of Theorem 3.8. Let $\mathbb{K}$ be a definably complete Baire structure, $\mathbb{K}_0$ be an o-minimal reduct of $\mathbb{K}$, expanding the field structure, and $\mathcal{F}$ be a family of total $\mathcal{C}^\infty$ functions definable in $\mathbb{K}$. We assume that $\mathcal{F}$ is closed under permutation of variables, contains the coordinate functions $(x_1, \dots, x_n) \mapsto x_i$, and that if $f \in \mathcal{F}$, then $(\overline{x}, y) \mapsto f(\overline{x})$ is also in $\mathcal{F}$. Let $\mathbb{K}_0(\mathcal{F})$ be the reduct of $\mathbb{K}$ generated by $\mathbb{K}_0$ and $\mathcal{F}$. We give necessary and sufficient conditions for $\mathbb{K}_0(\mathcal{F})$ to be an o-minimal structure.

Definition 6.1. Let $\mathcal{G}_0$ be the set of all total continuous functions definable in $\mathbb{K}_0$, and $\mathcal{G}$ be the set of functions of the form $h \circ f$, for some $f : \mathbb{K}^n \to \mathbb{K}^m$ in $\mathcal{F}^m$ and some $h : \mathbb{K}^m \to \mathbb{K}$ in $\mathcal{G}_0$.

For every $n \in \mathbb{N}$, let $\mathcal{S}_n$ be the family of subsets of $\mathbb{K}^n$ of the form $V(g)$, for some $g : \mathbb{K}^n \to \mathbb{K}$ in $\mathcal{G}$, and let $\mathcal{S} := (\mathcal{S}_n)_{n \in \mathbb{N}}$.

Notice that $\mathcal{F} \subseteq \mathcal{G}$, because, for every $n \in \mathbb{N}$, $\mathcal{F}^n$ contains the identity function on $\mathbb{K}^n$.

Theorem 6.2. $\mathbb{K}_0(\mathcal{F})$ *is o-minimal iff, for every X in $\mathcal{S}$ there exists a natural number N, such that, for every $\mathbb{K}$-affine set A, $X \cap A$ has less than N definably connected components.*

We will need the following result about o-minimal structures.

Let $\mathbb{F}$ be an o-minimal structure expanding a (real closed) field.

Proposition 6.3. *For every $N \geq 1$ and every $Y \subseteq \mathbb{F}^n$ closed and $\mathbb{F}$-definable there exists $h : \mathbb{F}^n \to [0, 1]$ $\mathbb{F}$-definable and $\mathcal{C}^N$, such that $Y = V(h)$. In particular, since every $\mathbb{F}$-definable set is a finite union of $\mathbb{F}$-cells, and every cell is the intersection of an open and a closed set, $\mathbb{F}$ is generated by its $\mathcal{C}^N$ definable functions.*

Moreover, if Z is a closed $\mathbb{F}$-definable subset of $\mathbb{F}^n$ disjoint from Y, then we can also require that $Z = V(1 - h)$.

Proof. We can use [12, Corollary C.12], since the proof works also for o-minimal structures expanding any real closed field, not just $\mathbb{R}$. □

Proof (Proof of Theorem 6.2). Notice that $\mathcal{S}$ is a closed weak structure. It is obvious that every set in $\mathcal{S}$ is definable in $\mathbb{K}_0(\mathcal{F})$. Conversely, since $\mathbb{K}_0$ is o-minimal, Proposition 6.3 and the fact that $\mathcal{G}_0 \subseteq \mathcal{G}$ imply that the structure generated by $\mathcal{S}$ expands $\mathbb{K}_0$; since moreover $\mathcal{F} \subseteq \mathcal{G}$, $\mathcal{S}$ generates $\mathbb{K}_0(\mathcal{F})$.

Hence, by Theorem 3.8, it suffices to show that $\mathcal{S}$ satisfies DAC^N for all N. That is, let $n \in \mathbb{N}$ and fix $A \in \mathcal{S}_n$. It is enough to prove the following:
(*) For every $N \in \mathbb{N}$, A is of the form $V(g_N)$ for some $g_N : \mathbb{K}^m \to \mathbb{K}$ in $\mathcal{G}$ and $\mathcal{C}^N$.

Let $g \in \mathcal{G}$ such that $A = V(g)$. Hence, $g = h \circ f$, for some $f : \mathbb{K}^n \to \mathbb{K}^m$ in $\mathcal{F}^m$ and some $h : \mathbb{K}^m \to \mathbb{K}$ in $\mathcal{G}_0$. Let $h_N : \mathbb{K}^m \to \mathbb{K}$ be $\mathcal{C}^N$ and definable in $\mathbb{K}_0$, such that $V(h) = V(h_N)$ (the existence of h_N is given by Proposition 6.3), and define $g_N := h_N \circ f : \mathbb{K}^n \to \mathbb{K}$. Note that g_N is $\mathcal{C}^N$ and in $\mathcal{G}$. Note moreover that

$$A = V(g) = f^{-1}\big(V(h)\big) = f^{-1}\big(V(h_N)\big) = V(g_N),$$

and we are done. □

Example 6.4. Let $\mathcal{R}_0$ be o-minimal expansion of the real field, and let $\mathcal{F}$ be the family of generalized Pfaffian functions over $\mathcal{R}_0$.

Then, $\mathcal{R}_0(\mathcal{F})$ satisfies the assumption of Theorem 6.2 (see [10]), and hence it is o-minimal.

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