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GROUND STATE SOLUTIONS FOR CHOQUARD TYPE EQUATIONS WITH A SINGULAR POTENTIAL

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Abstract. This article concerns the Choquard type equation

$$-\Delta u+V(x)u=\Big(\int_{\mathbb{R}^N}\frac{|u(y)|^p}{|x-y|^{N-\alpha}}dy\Big)|u|^{p-2}u,\quad x\in\mathbb{R}^N,$$

where $N\geq 3$, $\alpha\in((N-4)_+,N)$, $2\leq p<(N+\alpha)/(N-2)$ and $V(x)$ is a possibly singular potential and may be unbounded below. Applying a variant of the Lions' concentration-compactness principle, we prove the existence of ground state solution of the above equations.

1. Introduction

In this article, we study the Choquard type equation

$$-\Delta u+V(x)u=\Big(\int_{\mathbb{R}^N}\frac{|u(y)|^p}{|x-y|^{N-\alpha}}dy\Big)|u|^{p-2}u,\quad x\in\mathbb{R}^N,\tag{1.1}$$

where $N\geq 3$, $\alpha\in((N-4)_+,N)$, $p\in[2,\frac{N+\alpha}{N-2})$ and V is a given potential satisfying the following assumptions

- (A1) $V:\mathbb{R}^N\to\mathbb{R}$ is a measurable function;
- (A2) $V_\infty:=\lim_{|y|\to\infty}V(y)\geq V(x)$, for almost every $x\in\mathbb{R}^N$, and the inequality is strict in a non-zero measure domain;
- (A3) there exists $\bar{C}>0$ such that for any $u\in H^1(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N}(|\nabla u|^2+V(x)|u|^2)dx\geq\bar{C}\Big(\int_{\mathbb{R}^N}(|\nabla u|^2+|u|^2)dx\Big).$$

Clearly, (A3) implies $V_\infty>0$. When $N=3$, $\alpha=2$, $p=2$, (1.1) with $V\equiv 1$ just is the classical stationary Choquard equation

$$-\Delta u+u=\Big(\int_{\mathbb{R}^N}\frac{|u(y)|^2}{|x-y|}dy\Big)u\quad\text{in }\mathbb{R}^3.\tag{1.2}$$

This equation appeared at least as early as in 1954, in a work by Pekar describing the quantum mechanics of a polaron at rest [29]. In 1976, Choquard used (1.2) to describe an electron trapped in its own hole, in a certain approximation to Hartree-Fock theory of one component plasma [17]. As is known to us, the existence and

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multiplicity of radial solutions to (1.2) has been studied in [15] and [18]. Further more results for related problems can be founded in [3, 9, 11, 19, 20, 24, 28, 30, 32] and references therein, where V may be not a positive constant.

In recent years, the existence and properties of solutions for the generalized Choquard type equation (1.1) are widely considered. When the potential V is a positive constant, Ma and Zhao [23] proved the positive solutions for the generalized Choquard equation (1.1) must be radially symmetric and monotone decreasing about some point under appropriate assumptions on p, α, N. They also showed the positive solutions of (1.2) is uniquely determined, up to translations, see also [7]. Moroz and Van Schaftingen [25] obtained the existence, regularity, positivity and radial symmetry of ground state solution of (1.1), and they also derived the sharp decay asymptotic of the ground state solution. For more related problems, one can see [13, 14, 21, 26]. When the potential V is continuous and bounded below in $\mathbb{R}^N$, Alves and Yang [4] studied the multiplicity and concentration behaviour of positive solutions for quasilinear Choquard equation

$$-\epsilon^p\Delta_p u + V(x)|u|^{p-2}u = \epsilon^{\mu-N}\Big(\int_{\mathbb{R}^N}\frac{Q(y)F(u(y))}{|x-y|^\mu}dy\Big)Q(x)f(u) \text{ in } \mathbb{R}^N, \quad (1.3)$$

where Δ_p is the p-Laplacian operator, $1<p<N$, V and Q are two continuous real functions on $\mathbb{R}^N$, $F(s)$ is the primitive function of $f(s)$ and ϵ is a positive parameter. Furthermore results for related problems can be found in [5, 6, 10, 27, 34] and references therein.

To the best of our knowledge, there are only a few results on the existence of ground state solutions of (1.1) with singular potentials which may be unbounded below. In this paper, we succeed in finding a ground state solution of (1.1) under the assumptions (A1)–(A3). Here we remark the assumptions are introduced in [1] to study the singular nonlinear Schrödinger-Maxwell equations. Our aim is to extend the results in [1] to the case of Choquard type equations with some new techniques. Recall that $u \in H^1(\mathbb{R}^N)$ is said to be a ground state solution to (1.1), if u solves (1.1) and minimizes the energy functional associated with (1.1) among all possible nontrivial solutions.

Remark 1.1. According to [1, Remark 1.3], we can conclude that the potential V can be satisfied by the following type of functions which are singular and unbounded below. Let $V(x) = \gamma(x) - \lambda|x|^{-\sigma}$. Here γ satisfies (A1) and (A2), and is bounded below by a positive constant, $\sigma \in (0,2]$ and λ is a positive constant small enough. Indeed, we only need to verify (A3). By Hardy's inequality (see [12]), there exists $C>0$ such that

$$\int_{\mathbb{R}^N}\frac{u^2}{|x|^2}dx \le C\int_{\mathbb{R}^N}|\nabla u|^2dx.$$

Throughout this article, we write C for different positive constants. If $\sigma \in (0,2)$, using Hölder's inequality, we have

$$\begin{aligned}\int_{\mathbb{R}^N}\frac{u^2}{|x|^\sigma}dx &\le \Big(\int_{\mathbb{R}^N}|\frac{u^\sigma}{|x|^\sigma}|^{\frac{2}{\sigma}}dx\Big)^{\sigma/2}\Big(\int_{\mathbb{R}^N}|u^{2-\sigma}|^{\frac{2}{2-\sigma}}\Big)^{\frac{2-\sigma}{2}}\\ &\le C\Big(\int_{\mathbb{R}^N}|\nabla u|^2dx\Big)^{\sigma/2}\Big(\int_{\mathbb{R}^N}|u|^2dx\Big)^{\frac{2-\sigma}{2}} \quad (1.4)\\ &\le C\Big(\int_{\mathbb{R}^N}(|\nabla u|^2+|u|^2)dx\Big).\end{aligned}$$

By taking $\lambda > 0$ small enough, (A3) holds immediately. In particular, $\gamma(x) \equiv$ positive constant.

Now we are ready to state our main results.

Theorem 1.2. *Let $N \geq 3, \alpha \in ((N-4)_+, N), p \in [2, \frac{N+\alpha}{N-2})$ and suppose* (A1)–(A3) *hold. Then* (1.1) *has a ground state solution in $H^1(\mathbb{R}^N)$.*

The remainder of this article is organized as follows. In Section 2, some preliminary results are presented. In Section 3, we are devoted to the proof of our main result.

2. Preliminary results

In this article, we use the following notation.

• Let N be positive integers and B_R be an open ball of radius R centered at the origin in $\mathbb{R}^N$.

• Let $H^1(\mathbb{R}^N)$ be the usual Sobolev space with the standard norm

$$\|u\|_H = \Big(\int_{\mathbb{R}^N}(|\nabla u|^2 + |u|^2)dx\Big)^{1/2}.$$

We also use the notation

$$\|u\| = \Big(\int_{\mathbb{R}^N}(|\nabla u|^2 + V(x)|u|^2)dx\Big)^{1/2},$$

which is a norm equivalent to $\|\cdot\|_H$ in $H^1(\mathbb{R}^N)$ under (A1)–(A3) (we will prove the equivalence in Lemma 2.4).

• Let $\Omega \subset \mathbb{R}^N$ be a domain. For $1 \leq s < \infty$, $L^s(\Omega)$ denotes the Lebesgue space with the norm

$$|u|_{L^s(\Omega)} = \Big(\int_\Omega |u|^s dx\Big)^{1/s}.$$

If $\Omega = \mathbb{R}^N$, we write $|u|_{L^s} = |u|_{L^s(\Omega)}$. We can identify $u \in L^s(\Omega)$ with its extension to $\mathbb{R}^N$ obtained by setting $u = 0$ in $\mathbb{R}^N \backslash \Omega$, which ensures that we can use Hardy-Littlewood-Sobolev inequality to deal with the nonlocal problem.

• The dual space of $H^1(\mathbb{R}^N)$ is denoted by $H^{-1}(\mathbb{R}^N)$. The norm on $H^{-1}(\mathbb{R}^N)$ is denoted by $\|\cdot\|_{H^{-1}}$.

It is well known that the energy functional $I : H^1(\mathbb{R}^N) \to \mathbb{R}$ associated with (1.1) is defined by

$$I(u) = \frac{1}{2}\|u\|^2 - \frac{1}{2p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)|^p|u(y)|^p}{|x-y|^{N-\alpha}}\,dx\,dy.$$

This is a well defined $C^2(H^1(\mathbb{R}^N), \mathbb{R})$ functional whose Gateaux derivative is given by

$$I'(u)v = \int_{\mathbb{R}^N}(\nabla u \nabla v + V(x)uv)dx - \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(y)|^p|u(x)|^{p-2}u(x)v(x)}{|x-y|^{N-\alpha}}\,dx\,dy$$

for all $v \in H^1(\mathbb{R}^N)$. It is easy to see that all solutions of (1.1) correspond to critical points of the energy functional I. For simplicity of notation, we write

$$\mathbb{D}(u) = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)|^p|u(y)|^p}{|x-y|^{N-\alpha}}\,dx\,dy.$$

To study the nonlocal problems related with (1.1), we need to recall the following well-known Hardy-Littlewood-Sobolev inequality.

Lemma 2.1 (see [16]). *Let $s,t>1$ and $0<\mu<N$ with $\frac{\mu}{N}+\frac{1}{s}+\frac{1}{t}=2$, $f\in L^s(\mathbb{R}^N)$ and $h\in L^t(\mathbb{R}^N)$. There exists a sharp constant $C(N,\mu,s,t)$ independent of f,h, such that*

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{f(x)h(y)}{|x-y|^\mu}\,dx\,dy\le C(N,\mu,s,t)|f|_{L^s}|h|_{L^t}.$$

Remark 2.2. Suppose $\frac{N+\alpha}{N}\le p\le\frac{N+\alpha}{N-2}$ and $u\in L^{\frac{2Np}{N+\alpha}}(\mathbb{R}^N)$. Then according to Lemma 2.1, we have

$$\mathbb{D}(u)\le C(N,\alpha,s,t,)|u^p|_{L^s}|u^p|_{L^t}=C(N,\alpha,s,t)|u|^p_{L^{\frac{2Np}{N+\alpha}}}|u|^p_{L^{\frac{2Np}{N+\alpha}}}<\infty. \tag{2.1}$$

where C only depends on N,s,t,α and $\frac{1}{s}+\frac{1}{t}+\frac{N-\alpha}{N}=2$.

Using [25, Theorem 1 and Proposition 5], we easily obtain the following lemma.

Lemma 2.3. *Let $N\ge 3$, $\alpha\in(0,N)$, $p\in(\frac{N+\alpha}{N},\frac{N+\alpha}{N-2})$. Then there exists a positive ground state solution $w\in H^1(\mathbb{R}^N)$ of* (1.1) *with $V\equiv$ positive constant.*

According to the assumptions (A1)–(A3), we have the following lemma whose proof is standard.

Lemma 2.4. *For any $u\in H^1(\mathbb{R}^N)$, there exist two positive constants C_1 and C_2 such that*

$$C_1\|u\|_H\le\|u\|\le C_2\|u\|_H. \tag{2.2}$$

In this article, we define the Nehari manifold

$$\mathcal{N}=\{u\in H^1(\mathbb{R}^N)\backslash\{0\}:I'(u)u=0\}.$$

Let

$$c:=\inf_{u\in\mathcal{N}}I(u).$$

It is easy to check that $0\notin\partial\mathcal{N}$ and $c>0$. Now we show some properties of the Nehari manifold $\mathcal{N}$.

Lemma 2.5. *Suppose* (A1)–(A3) *hold. Then the following statements hold:*
(i) For every $u\in H^1(\mathbb{R}^N)\backslash\{0\}$, there exists a unique $t_u\in(0,\infty)$ such that $t_uu\in\mathcal{N}$ and $t_u=\left(\frac{\|u\|^2}{\mathbb{D}(u)}\right)^{\frac{1}{2p-2}}$. Furthermore,

$$I(t_uu)=\sup_{t>0}I(tu)=(\frac{1}{2}-\frac{1}{2p})\Big(\frac{\|u\|^2}{\mathbb{D}^{\frac{1}{p}}(u)}\Big)^{\frac{p}{p-1}}.$$

(ii) $c=\inf_{u\in\mathcal{N}}I(u)=\inf_{u\in H^1(\mathbb{R}^N)\backslash\{0\}}\sup_{t>0}I(tu)$.

Proof. Statement (i) follows by a direct calculation. Then by (i), we have $I(t_uu)=\sup_{t>0}I(tu)\ge\inf_{u\in\mathcal{N}}I(u)$. Hence $\inf_{u\in H^1(\mathbb{R}^N)\backslash\{0\}}\sup_{t>0}I(tu)\ge\inf_{u\in\mathcal{N}}I(u)$. On the other hand, for any $u\in\mathcal{N}$,

$$I(u)=\sup_{t>0}I(tu)\ge\inf_{u\in H^1(\mathbb{R}^N)\backslash\{0\}}\sup_{t>0}I(tu).$$

This shows (ii) and completes the proof. □

Let $\lambda>0$. We define

$$I_\lambda(u)=\frac{1}{2}\int_{\mathbb{R}^N}(|\nabla u|^2+\lambda u^2)dx-\frac{1}{2p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)|^p|u(y)|^p}{|x-y|^{N-\alpha}}\,dx\,dy,$$
$$c(\lambda)=\inf_{u\in N_\lambda}I_\lambda(u), \tag{2.3}$$

where $\mathcal{N}_\lambda$ is the Nehari manifold of I_λ. Then we give some preliminary lemmas which can be proved by using the similar arguments as in [2, Lemma 2.7] with some necessary modifications.

Lemma 2.6. *Let $c(\lambda)$ be defined in (2.3). Then $c(\lambda)$ is a continuous and strictly increasing function in $(0,\infty)$.*

Proof. Let $\lambda,\delta,\lambda_n>0$. We first show $c(\lambda)$ is strictly increasing with respect to λ. To be precise, if $\lambda<\delta$, we have $c(\lambda)<c(\delta)$.

Indeed, according to Lemma 2.3, there exists $u\in H^1(\mathbb{R}^N)$ such that u is a positive critical point of I_δ and $I_\delta(u)=c(\delta)$. On the other hand, by Lemma 2.5 (i), we can find a unique $t_u>0$ such that $t_u u\in\mathcal{N}_\lambda$. Then

$$\begin{aligned} c(\delta) &= I_\delta(u) \ge I_\delta(t_u u)\\ &= I_\lambda(t_u u) + (\delta-\lambda)\int_{\mathbb{R}^N}|t_u u|^2dx\\ &\ge c(\lambda) + (\delta-\lambda)\int_{\mathbb{R}^N}|t_u u|^2dx. \end{aligned} \tag{2.4}$$

So if $\lambda<\delta$, it holds $c(\lambda)<c(\delta)$.

Now we prove $c(\lambda)$ is continuous with respect to λ. To be precise, if $\lambda_n\to\lambda$, then $c(\lambda_n)\to c(\lambda)$.

Let $\lambda_n=\lambda+h_n$, where $h_n\to 0$ as $n\to\infty$. It suffices to prove $c(\lambda+h_n)\to c(\lambda)$ as $n\to\infty$. We shall complete the proof by distinguishing two cases.

Case 1. We show that

$$c^+ := \lim_{h_n\to 0^+} c(\lambda+h_n) = c(\lambda).$$

In fact, according to the monotonicity of $c(\lambda)$, we have $c^+\ge c(\lambda)>0$. By way of contradiction, suppose

$$c^+ > c(\lambda). \tag{2.5}$$

By Lemma 2.3, there exists a positive function $u\in H^1(\mathbb{R}^N)$ such that $I'_\lambda(u)=0$ and $I_\lambda(u)=c(\lambda)$. In addition, for each n, there exists a unique $\theta_n>0$ such that $\theta_n u\in\mathcal{N}_{\lambda_n}$ due to Lemma 2.5 (i). Note that

$$\int_{\mathbb{R}^N}|\nabla u|^2+\lambda|u|^2=\mathbb{D}(u)$$

$$\int_{\mathbb{R}^N}|\nabla u|^2+\lambda_n|u|^2=\theta_n^{2p-2}\mathbb{D}(u).$$

Then a standard argument shows that $(\theta_n)_{n\ge1}$ is uniformly bounded. In addition, by Lemma 2.5 (i) and Sobolev embedding theorem, we have

$$\begin{aligned} c^+ &\le c(\lambda+h_n) \le I_{\lambda_n}(\theta_n u)\\ &= I_\lambda(\theta_n u) + \frac{1}{2}(\lambda_n-\lambda)\int_{\mathbb{R}^N}|\theta_n u|^2\\ &\le I_\lambda(u) + \frac{1}{2}(\lambda_n-\lambda)\int_{\mathbb{R}^N}|\theta_n u|^2\\ &= c(\lambda) + \frac{1}{2}(\lambda_n-\lambda)\int_{\mathbb{R}^N}|\theta_n u|^2\\ &\le c(\lambda) + Ch_n\|\theta_n u\|_H^2. \end{aligned}$$

Letting $n \to \infty$, we conclude $c^+ \le c(\lambda)$, a contradiction to (2.5).

Case 2. We shall prove

$$c^- := \lim_{h_n \to 0^-} c(\lambda + h_n) = c(\lambda).$$

Indeed, the monotonicity of $c(\lambda)$ yields $c^- \le c(\lambda)$. By way of contradiction, we suppose

$$c^- < c(\lambda). \tag{2.6}$$

By Lemma 2.3, for each $n \ge 1$, there exists a positive function $v_n \in H^1(\mathbb{R}^N)$ such that $I'_{\lambda_n}(v_n) = 0$ and $I_{\lambda_n}(v_n) = c(\lambda_n)$. Since $c(\frac{\lambda}{2}) \le c(\lambda_n) = I_{\lambda_n}(v_n) \le c(\lambda)$ for n large enough, we can find $C_1, C_2 > 0$ such that $C_1 \le \|v_n\|_H \le C_2$ uniformly in $H^1(\mathbb{R}^N)$. By Lemma 2.5 (i), for each $n \ge 1$, there exists a unique $\theta(v_n) > 0$ such that $\theta(v_n)v_n \in \mathcal{N}_\lambda$. Note that

$$\int_{\mathbb{R}^N} |\nabla v_n|^2 + \lambda_n |v_n|^2 = \mathbb{D}(v_n)$$
$$\int_{\mathbb{R}^N} |\nabla v_n|^2 + \lambda |v_n|^2 = \theta^{2p-2}(v_n)\mathbb{D}(v_n).$$

Then a standard argument shows that $(\theta(v_n))_{n\ge1}$ is uniformly bounded. In addition, by Lemma 2.5 (i) and Sobolev embedding theorem, we have

$$\begin{aligned}
c(\lambda) &\le I_\lambda(\theta(v_n)v_n)\\
&= I_{\lambda_n}(\theta(v_n)v_n) + \frac{1}{2}(\lambda - \lambda_n)\int_{\mathbb{R}^N} |\theta(v_n)v_n|^2\\
&\le I_{\lambda_n}(v_n) + \frac{1}{2}(\lambda - \lambda_n)\int_{\mathbb{R}^N} |\theta(v_n)v_n|^2\\
&= c(\lambda_n) + \frac{1}{2}(\lambda - \lambda_n)\int_{\mathbb{R}^N} |\theta(v_n)v_n|^2\\
&\le c(\lambda_n) + Ch_n\|\theta(v_n)v_n\|_H^2.
\end{aligned}$$

Since $\lim_{n\to\infty} c(\lambda_n) = c^-$, letting $n \to \infty$, we conclude $c(\lambda) \le c^-$, a contradiction to (2.6). The proof is complete. $\square$

Lemma 2.7. *Let* (A1)–(A3) *hold. Then $c < c(V_\infty)$. Moreover, there exists $\mu > 0$ such that $c < c(V_\infty - \mu) < c(V_\infty)$.*

Proof. By Lemma 2.3, there exists a positive function $u \in H^1(\mathbb{R}^N)$ such that $I'_{V_\infty}(u) = 0$ and $I_{V_\infty}(u) = c(V_\infty)$. In addition, there exists a unique $t_u > 0$ such that $t_u u \in \mathcal{N}$. By (A2), we obtain

$$\begin{aligned}
c(V_\infty) &= I_{V_\infty}(u) \ge I_{V_\infty}(t_u u)\\
&= I(t_u u) + \int_{\mathbb{R}^N} (V_\infty - V(x))|t_u u|^2 dx\\
&\ge c + \int_{\mathbb{R}^N} (V_\infty - V(x))|t_u u|^2 dx > c.
\end{aligned} \tag{2.7}$$

By Lemma 2.6, there exists $\mu > 0$ such that

$$|c(V_\infty) - c(V_\infty - \mu)| < \frac{c(V_\infty) - c}{2},$$

which implies $c(V_\infty - \mu) > c$. In addition, we have $c(V_\infty - \mu) < c(V_\infty)$. This completes the proof. $\square$

3. Proof of Theorem 1.2

In this section, motivated by [1] and [8], we shall prove the existence of ground state solution of (1.1) by using a variant of Lions' concentration-compactness principle.

Let $(u_n)_{n\geq 1} \subset \mathcal{N}$ be a minimizing sequence such that

$$\lim_{n\to\infty} I(u_n) = c. \tag{3.1}$$

In what follows, we shall prove $(u_n)_{n\geq 1}$ is a $(PS)_c$ sequence of I.

Definition 3.1. We say that $(u_n)_{n\geq 1} \subset H^1(\mathbb{R}^N)$ is $(PS)_c$ sequence of I, if $(u_n)_{n\geq 1}$ satisfies

$$I(u_n) \to c, I'(u_n) \to 0, \quad \text{as } n \to \infty. \tag{3.2}$$

Lemma 3.2. *If $(u_n)_{n\geq 1} \subset \mathcal{N}$ is a minimizing sequence such that* (3.1) *holds, then $(u_n)_{n\geq 1}$ is a $(PS)_c$ sequence of I.*

Proof. The outline of the proof is as follows.

Step 1. We shall show $G'(u_n) \neq 0$ for any $n \geq 1$. In fact, it is easy to check $(u_n)_{n\geq 1}$ is uniformly bounded in $H^1(\mathbb{R}^N)$. Let $G : H^1(\mathbb{R}^N) \to \mathbb{R}$ defined by

$$G(u) = \|u\|^2 - \mathbb{D}(u).$$

By standard arguments, we deduce G is of class $\mathcal{C}^2(H^1(\mathbb{R}^N), \mathbb{R})$ and its Gateaux derivative is given by

$$G'(u)v = 2\int_{\mathbb{R}^N} (\nabla u \nabla v + V(x)uv)\,dx\,dy - 2p\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(y)|^p |u(x)|^{p-2}u(x)v(x)}{|x-y|^{N-\alpha}}\,dx\,dy$$

for all u, v in $H^1(\mathbb{R}^N)$. Since $(u_n)_{n\geq 1}$ is uniformly bounded, by (2.1), we deduce $G'(u_n)$ is uniformly bounded in $H^{-1}(\mathbb{R}^N)$. Note that for any $u \in \mathcal{N}$,

$$I(u) = (\frac{1}{2} - \frac{1}{2p})\|u\|^2 \geq c.$$

Then, for all $u \in \mathcal{N}$,

$$G'(u)u = 2\|u\|^2 - 2p\mathbb{D}(u) = (2-2p)\|u\|^2 \leq -4pc < 0.$$

Hence $G'(u_n) \neq 0$.

Step 2. We shall prove $I'(u_n) \to 0$ as $n \to \infty$. Let $J_\lambda(u_n) := \|I'(u_n) - \lambda G'(u_n)\|_{H^{-1}}$. By [33, Theorem 8.5], we assume

$$\min_{\lambda\in\mathbb{R}} J_\lambda(u_n) \to 0, \quad \text{as } n \to \infty.$$

Then up to a subsequence, we have

$$\min_{\lambda\in\mathbb{R}} J_\lambda(u_n) < \frac{1}{2n}.$$

On the other hand, for each n, we can find $\lambda_n \in \mathbb{R}$ such that

$$|J_{\lambda_n}(u_n) - \min_{\lambda\in\mathbb{R}} J_\lambda(u_n)| < \frac{1}{2n}\,.$$

Therefore,

$$J_{\lambda_n}(u_n) = \|I'(u_n) - \lambda_n G'(u_n)\|_{H^{-1}} \to 0, \quad \text{as } n \to \infty. \tag{3.3}$$

Note that

$$|I'(u_n)u_n - \lambda_n G'(u_n)u_n| \leq \|I'(u_n) - \lambda_n G'(u_n)\|_{H^{-1}}\|u_n\|. \tag{3.4}$$

Since $I'(u_n)u_n = 0$ and $G'(u_n)u_n \neq 0$, we conclude from (3.3) that $\lambda_n \to 0$ as $n \to \infty$. Note that $G'(u_n) \neq 0$ by Step 1. Then we have

$$\|I'(u_n)\|_{H^{-1}} \leq \|I'(u_n) - \lambda_n G'(u_n)\|_{H^{-1}} + \|\lambda_n G'(u_n)\|_{H^{-1}} \to 0$$

as $n \to \infty$. This completes the proof. $\square$

Next we need some compactness on the minimizing sequence $(u_n)_{n\geq 1}$ defined in (3.1) in order to prove our existence results. Define the functional $J : H^1(\mathbb{R}^N) \to \mathbb{R}$ by

$$J(u) = (\frac{1}{2} - \frac{1}{2p}) \int_{\mathbb{R}^N} |\nabla u|^2 + V(x)|u|^2.$$

It is easy to check that for any $u \in \mathcal{N}$, we have $I(u) = J(u)$. Using a variant of Lion's concentration-compactness principle presented in [8, Lemma 6.1] (see also [22, Proposition 3.1]), we have the following lemma. Throughout this section, $O(\mu)$ denotes a constant depending on μ such that $|\frac{O(\mu)}{\mu}| \leq C$.

Lemma 3.3. *For any $\epsilon > 0$, there exists $\bar{R} = \bar{R}(\epsilon) > 0$ such that for any $n \geq \bar{R}$,*

$$\int_{|x|>\bar{R}} (|\nabla u_n|^2 + |u_n|^2) < \epsilon.$$

Proof. By way of contradiction, we suppose that there exist $\epsilon_0 > 0$ and a subsequence $(u_k)_{k\geq 1}$ such that for any $k \geq 1$,

$$\int_{|x|>k} (|\nabla u_k|^2 + |u_k|^2) \geq \epsilon_0. \tag{3.5}$$

Let

$$\rho_k(\Omega) = \int_{\Omega} (|\nabla u_k|^2 + |u_k|^2).$$

Fix $l > 1$ and define

$$A_r := \{x \in \mathbb{R}^N | r \leq |x| \leq r + l\}, \quad \text{for any } \ r > 0.$$

We shall finish the proof by distinguishing four steps.

Step 1. We shall show that for any $\mu, R > 0$, there exists $r = r(\mu, R) > R$ such that $\rho_k(A_r) < \mu$ for infinitely many k.

We argue by contradiction. Suppose there exist $\mu_0 > 0$ and $\tilde{R} \in \mathbb{N}$ such that, for any $m \geq \tilde{R}$, there exists a strictly increasing sequence $\{p(m)\}_{m\geq\tilde{R}} \subset (0, \infty)$ such that

$$\rho_k(A_m) \geq \mu_0, \quad \text{for any } k \geq p(m).$$

By applying this fact, we have

$$\|u_k\|_H^2 \geq \rho_k(B_m \backslash B_{\tilde{R}}) \geq (\frac{m - \tilde{R}}{l})\mu_0, \quad \text{for any } m \geq \tilde{R},\ k \geq p(m).$$

Take $m > \tilde{R} + \frac{l}{\mu_0} \sup_{n\geq 1} \|u_n\|_H^2$. Then

$$\|u_{p(m)}\|_H^2 > \sup_{n\geq 1} \|u_n\|_H^2 \geq \|u_{p(m)}\|_H^2,$$

which is a contradiction.

Step 2. We shall show there exists $\mu_0 \in (0, 1)$ such that for any $\mu \in (0, \mu_0)$, we have the following results:

(i) It holds

$$c < c(V_\infty - \mu) < c(V_\infty). \tag{3.6}$$

(ii) There exists $R_\mu > 0$ such that for almost every $|x| > R_\mu$,

$$V(x) \ge V_\infty - \mu > 0. \tag{3.7}$$

(iii) There exists $r > R_\mu$ such that, going if necessary to a subsequence,

$$\rho_k(A_r) < \mu, \quad \text{for all } k \ge 1. \tag{3.8}$$

(iv) It holds

$$\int_{A_r} |\nabla u_k|^2 + V(x)|u_k|^2 = O(\mu), \quad \text{for all } k \ge 1, \tag{3.9}$$

$$\int_{A_r}\int_{\mathbb{R}^N} \frac{|u_k(x)|^p|u_k(y)|^p}{|x-y|^{N-\alpha}}\,dx\,dy = O(\mu), \quad \text{for all } k \ge 1. \tag{3.10}$$

Indeed, (i) follows from Lemma 2.7. By $(V2)$, we can find $R_\mu > 0$ such that for almost every $|x| > R_\mu$, (3.7) holds. Consider μ and R_μ satisfying (3.6) and (3.7). Then by Step 1, we can take $r > R_\mu$ such that, going if necessary to a subsequence, (3.8) is valid. According to (A2), (3.7)and (3.8), we can easily obtain (3.9). Since $\mu \in (0,1)$, combining (2.1), we have (3.10).

Step 3. We shall first give some estimates. Let $\eta \in C^\infty(\mathbb{R}^N)$ such that $\eta = 1$ in B_r and $\eta = 0$ in B^c_{r+l}, $0 \le \eta \le 1$ and $|\nabla\eta| \le 2$, where r is defined in Step 2 (iii). Define $v_k = \eta u_k$ and $w_k = (1-\eta)u_k$.

It follows from (3.7) and (3.9) that

$$\begin{gathered}\int_{A_r} |\nabla v_k|^2 + V(x)|v_k|^2 = O(\mu),\\ \int_{A_r} |\nabla w_k|^2 + V(x)|w_k|^2 = O(\mu).\end{gathered} \tag{3.11}$$

This, combined with (3.9), implies that

$$\begin{aligned}
&\int_{\mathbb{R}^N} |\nabla u_k|^2 + V(x)|u_k|^2\\
&= \int_{A_r} |\nabla u_k|^2 + V(x)|u_k|^2 + \int_{B_r} |\nabla v_k|^2 + V(x)|v_k|^2 + \int_{B^c_{r+l}} |\nabla w_k|^2 + V(x)|w_k|^2\\
&= \int_{\mathbb{R}^N} |\nabla v_k|^2 + V(x)|v_k|^2 + \int_{\mathbb{R}^N} |\nabla w_k|^2 + V(x)|w_k|^2 + \int_{A_r} |\nabla u_k|^2 + V(x)|u_k|^2\\
&\quad - \int_{A_r} |\nabla v_k|^2 + V(x)|v_k|^2 - \int_{A_r} |\nabla w_k|^2 + V(x)|w_k|^2\\
&= \int_{\mathbb{R}^N} |\nabla v_k|^2 + V(x)|v_k|^2 + \int_{\mathbb{R}^N} |\nabla w_k|^2 + V(x)|w_k|^2 + O(\mu).
\end{aligned} \tag{3.12}$$

According to Step 1 above, we can take $l > 0$ appropriately large such that

$$\int_{B_r}\int_{B^c_{r+l}} \frac{|u_k(x)|^p|u_k(y)|^p}{|x-y|^{N-\alpha}}\,dx\,dy = O(\mu).$$

Then we conclude from (3.9), (3.10) and (3.11) that

$$\begin{aligned}
&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_k(x)|^p|u_k(y)|^p}{|x-y|^{N-\alpha}}\,dx\,dy\\
&=\int_{B_r}\int_{B_r}\frac{|u_k(x)|^p|u_k(y)|^p}{|x-y|^{N-\alpha}}\,dx\,dy+2\int_{B_r}\int_{B_{r+l}^c}\frac{|u_k(x)|^p|u_k(y)|^p}{|x-y|^{N-\alpha}}\,dx\,dy\\
&\quad+\int_{B_r}\int_{A_r}\frac{|u_k(x)|^p|u_k(y)|^p}{|x-y|^{N-\alpha}}\,dx\,dy+\int_{B_{r+l}^c}\int_{B_{r+l}^c}\frac{|u_k(x)|^p|u_k(y)|^p}{|x-y|^{N-\alpha}}\,dx\,dy\\
&\quad+\int_{B_{r+l}^c}\int_{A_r}\frac{|u_k(x)|^p|u_k(y)|^p}{|x-y|^{N-\alpha}}\,dx\,dy+\int_{A_r}\int_{\mathbb{R}^N}\frac{|u_k(x)|^p|u_k(y)|^p}{|x-y|^{N-\alpha}}\,dx\,dy\\
&=\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|v_k(x)|^p|v_k(y)|^p}{|x-y|^{N-\alpha}}\,dx\,dy+\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|w_k(x)|^p|w_k(y)|^p}{|x-y|^{N-\alpha}}\,dx\,dy+O(\mu).
\end{aligned}\tag{3.13}$$

Next, observe that for $k\ge 1$ large enough, there exists $\epsilon'>0$ such that

$$\int_{\mathbb{R}^N}|\nabla w_k|^2+V(x)|w_k|^2\ge\epsilon'.\tag{3.14}$$

Indeed, we can conclude from (3.5) and (3.7) that for $k>r+l$, it holds that

$$\begin{aligned}
&\int_{\mathbb{R}^N}|\nabla w_k|^2+V(x)|w_k|^2\\
&\ge\int_{B_{r+l}^c}|\nabla w_k|^2+(V_\infty-\mu)|w_k|^2\\
&=\int_{|x|>k}|\nabla w_k|^2+(V_\infty-\mu)|w_k|^2+\int_{B_k\setminus B_{r+l}}|\nabla w_k|^2+(V_\infty-\mu)|w_k|^2\\
&\ge\int_{|x|>k}|\nabla u_k|^2+(V_\infty-\mu)|u_k|^2\\
&\ge\min\{1,V_\infty-\mu\}\epsilon_0.
\end{aligned}$$

Hence (3.14) holds.

Therefore, by (3.12) and (3.14), the following equality and inequality hold.

$$J(u_k)=J(v_k)+J(w_k)+O(\mu),\tag{3.15}$$

$$J(u_k)\ge J(w_k)+O(\mu),\tag{3.16}$$

$$J(u_k)-C\epsilon'\ge J(v_k)+O(\mu).\tag{3.17}$$

Step 4. Recall $G(u)$ defined in Lemma 3.2. By (3.12) and (3.13), we deduce

$$0=G(u_k)=G(v_k)+G(w_k)+O(\mu).\tag{3.18}$$

We shall complete the proof by distinguishing three cases.

Case 1. Up to a subsequence, $G(v_k)\le 0$. By Lemma 2.5 (i), for any $k\ge 1$, there exists a unique $t_k>0$ such that $t_kv_k\in\mathcal{N}$. Then

$$\int_{\mathbb{R}^N}|\nabla v_k|^2+V(x)|v_k|^2=t_k^{2p-2}\mathbb{D}(v_k).\tag{3.19}$$

Note that

$$\int_{\mathbb{R}^N}|\nabla v_k|^2+V(x)|v_k|^2\le\mathbb{D}(v_k).$$

This, combined with (3.19), implies that $t_k \le 1$ uniformly. By (3.17), we obtain

$$\begin{aligned} c &\le I(t_k v_k) = J(t_k v_k) \le J(v_k) \\ &\le J(u_k) - C\epsilon' + O(\mu) = c - C\epsilon' + O(\mu) + o_k(1). \end{aligned} \tag{3.20}$$

Here and in the following part, we point out $o_k(1) \to 0$ as $k \to \infty$. By letting $\mu \to 0$ and $k \to \infty$, (3.20) yields a contradiction.

Case 2. Up to a subsequence, $G(w_k) \le 0$. For any $k \ge 1$, there exists $s_k > 0$ such that $s_k w_k \in \mathcal{N}$. Arguing as in Case 1, we have $s_k \le 1$ uniformly. Define $\bar{w}_k = s_k w_k$. Then there exists $\theta_k > 0$ such that $\theta_k \bar{w}_k \in \mathcal{N}_{V_\infty - \mu}$. By (3.7), we have

$$\int_{\mathbb{R}^N} |\nabla \bar{w}_k|^2 + (V_\infty - \mu)|\bar{w}_k|^2 \le \int_{\mathbb{R}^N} |\nabla \bar{w}_k|^2 + V(x)|\bar{w}_k|^2 = \mathbb{D}(\bar{w}_k),$$

which implies that $\theta_k \le 1$ uniformly. Hence, by (3.16), we deduce

$$\begin{aligned} c(V_\infty - \mu) &\le I_{V_\infty - \mu}(\theta_k \bar{w}_k) \\ &\le (\frac{1}{2} - \frac{1}{2p}) \int_{\mathbb{R}^N} |\nabla \bar{w}_k|^2 + (V_\infty - \mu)|\bar{w}_k|^2 \\ &\le (\frac{1}{2} - \frac{1}{2p}) \int_{\mathbb{R}^N} |\nabla \bar{w}_k|^2 + V(x)|\bar{w}_k|^2 \\ &\le J(w_k) \\ &\le J(u_k) + O(\mu) \\ &= c + o_k(1) + O(\mu). \end{aligned} \tag{3.21}$$

Letting $\mu \to 0$ and $k \to \infty$, we obtain a contradiction with (3.6).

Case 3. Up to a subsequence, $G(v_k) > 0$ and $G(w_k) > 0$. According to (3.18), we have

$$G(w_k) = O(\mu) > 0, \; G(v_k) = O(\mu) > 0.$$

For any $k \ge 1$, there exists $s_k > 0$ such that $s_k w_k \in \mathcal{N}$ and then $G(s_k w_k) = 0$. So that

$$\begin{gathered} \int_{\mathbb{R}^N} |\nabla w_k|^2 + V(x)|w_k|^2 = s_k^{2p-2} \mathbb{D}(w_k), \\ \int_{\mathbb{R}^N} |\nabla w_k|^2 + V(x)|w_k|^2 - \mathbb{D}(w_k) = O(\mu) > 0, \end{gathered}$$

which implies $s_k \ge 1$ uniformly. Since $(w_k)_{k\ge1}$ is bounded, by (3.14), we have

$$s_k^{2p-2} = \frac{\int_{\mathbb{R}^N} |\nabla w_k|^2 + V(x)|w_k|^2}{\mathbb{D}(w_k)} \le \frac{C}{\epsilon' - O(\mu)}.$$

Hence $(s_k)_{k\ge1}$ is uniformly bounded when μ is small enough.

Now we need to distinguish two cases.

Case 3-(i). Up to a subsequence, if $\lim_{k\to\infty} s_k = 1$, for k large enough, $1 \le s_k \le 1 + O(\mu)$. Using similar arguments as in Case 2, we have

$$\begin{aligned}
c(V_\infty - \mu) &\le I_{V_\infty-\mu}(\theta_k \bar{w}_k)\\
&\le (\frac{1}{2} - \frac{1}{2p}) \int_{\mathbb{R}^N} |\nabla \bar{w}_k|^2 + (V_\infty - \mu)|\bar{w}_k|^2\\
&\le (\frac{1}{2} - \frac{1}{2p}) \int_{\mathbb{R}^N} |\nabla \bar{w}_k|^2 + V(x)|\bar{w}_k|^2\\
&\le (1 + O(\mu))^2 J(w_k)\\
&\le (1 + O(\mu))^2 (J(u_k) + O(\mu))\\
&= (1 + O(\mu))^2 (c + o_k(1) + O(\mu)).
\end{aligned} \tag{3.22}$$

Letting $\mu \to 0$ and $k \to \infty$, we obtain a contradiction with (3.6).

Case 3-(ii). Up to a subsequence, if $\lim_{k\to\infty} s_k = s_0 > 1$, for k large enough, $s_k > 1$. On the other hand, we have

$$\begin{aligned}
O(\mu) = G(w_k) &= \int_{\mathbb{R}^N} |\nabla w_k|^2 + V(x)|w_k|^2 - \mathbb{D}(w_k)\\
&= (1 - s_k^{\frac{1}{2p-2}}) \int_{\mathbb{R}^N} |\nabla w_k|^2 + V(x)|w_k|^2.
\end{aligned}$$

Hence

$$\int_{\mathbb{R}^N} |\nabla w_k|^2 + V(x)|w_k|^2 = O(\mu),$$

which contradicts (3.14). The proof is complete. □

Proof of Theorem 1.2. Since $(u_n)_{n\ge1}$ is uniformly bounded, going if necessary to a subsequence, there exists $u_0 \in H^1(\mathbb{R}^N)$ such that $u_n \rightharpoonup u_0$ in $H^1(\mathbb{R}^N)$ and $u_n \to u_0$ a.e. in $\mathbb{R}^N$. By Lemma 3.2, we have $I'(u_n) \to 0$ as $n \to \infty$, and then $I'(u_0) = 0$ because of [31, Lemma 2.6].

Now we show $u_0 \ne 0$. According to Lemma 3.3, for any $\epsilon > 0$, there exists $r > 0$ such that, up to a subsequence,

$$\|u_n\|_{H^1(B_r^c)} < \epsilon, \quad \text{for any } n \ge 1.$$

Let $s \in [2, \frac{2N}{N-2})$. For $n \ge 1$ large enough, we have

$$\begin{aligned}
|u_n - u_0|_{L^s(\mathbb{R}^N)} &= |u_n - u_0|_{L^s(B_r)} + |u_n - u_0|_{L^s(B_r^c)}\\
&\le \epsilon + C_0(\|u_n\|_{H^1(B_r^c)} + \|u_0\|_{H^1(B_r^c)})\\
&\le (1 + 2C_0)\epsilon.
\end{aligned}$$

Then we deduce $u_n \to u_0$ in $L^s(\mathbb{R}^N)$ for any $s \in [2, \frac{2N}{N-2})$. This, combined with Brézis-Lieb Lemma (see [33, Theorem 1.32]), implies

$$|u_n|^p \to |u_0|^p, \quad \text{in } L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N).$$

Hence using (2.1), we obtain

$$
\begin{aligned}
&|\mathbb{D}(u_n) - \mathbb{D}(u_0)| \\
&\le \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\big||u_n(x)|^p - |u_0(x)|^p\big||u_n(y)|^p}{|x-y|^{N-\alpha}}\,dx\,dy \\
&\quad + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\big||u_n(y)|^p - |u_0(y)|^p\big||u_0(x)|^p}{|x-y|^{N-\alpha}}\,dx\,dy. \\
&\le C\big||u_n|^p - |u_0|^p\big|_{L^{\frac{2N}{N+\alpha}}} |u_n|^p_{L^{\frac{2Np}{N+\alpha}}} + C\big||u_n|^p - |u_0|^p\big|_{L^{\frac{2N}{N+\alpha}}} |u_0|^p_{L^{\frac{2Np}{N+\alpha}}} \\
&\to 0.
\end{aligned} \tag{3.23}
$$

Note that $I'(u_n)u_n = 0$ and $I'(u_0)u_0 = 0$. Then

$$
I(u_n) = (\frac{1}{2} - \frac{1}{2p})\mathbb{D}(u_n),
$$
$$
I(u_0) = (\frac{1}{2} - \frac{1}{2p})\mathbb{D}(u_0).
$$

Since $I(u_n) \to c$ as $n \to \infty$, we conclude from (3.23) that $I(u_n) \to I(u_0) = c$. Therefore, u_0 is a ground state solution of (1.1). This completes the proof. $\square$

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