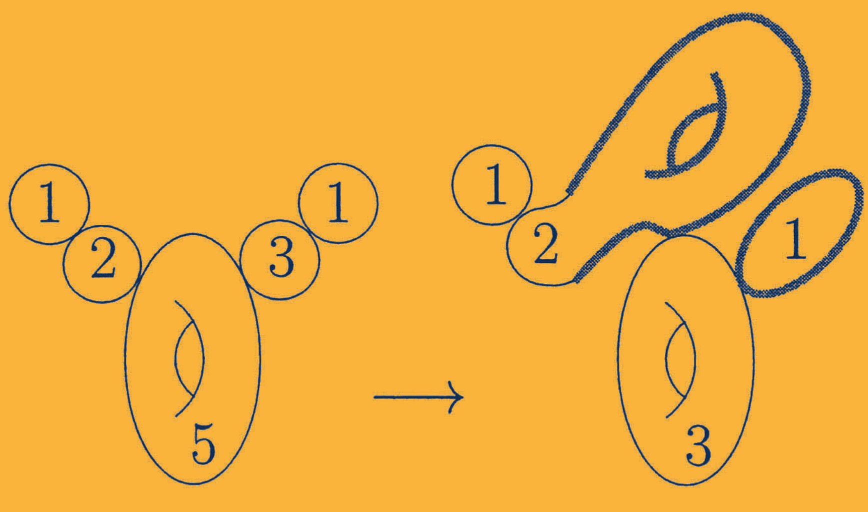
Lecture Notes in Mathematics

1886

Shigeru Takamura

Splitting Deformations of Degenerations of Complex Curves

Towards the Classification of Atoms of Degenerations, III



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Library of Congress Control Number: 2006923235

Mathematics Subject Classification (2000): 14D05, 14J15 14H15, 32S30

ISSN print edition: 0075-8434
ISSN electronic edition: 1617-9692
ISBN-10 3-540-33363-0 Springer Berlin Heidelberg New York
ISBN-13 978-3-33363-0 Springer Berlin Heidelberg New York

DOI 10.1007/b138136

Springer is a part of Springer Science+Business Media
springer.com

Printed in The Netherlands

Typesetting: by the authors and SPi using a Springer LaTeX package
Cover design: *design & production* GmbH, Heidelberg

Printed on acid-free paper SPIN: 11735212 41 3100/SPi 5 4 3 2 1 0

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Abstract

This is the third in our series of works which make a systematic study of degenerations of complex curves, and their splitting deformations. The principal aim of the present volume is to develop a new deformation theory of degenerations of complex curves. The construction of these deformations uses special subdivisors of singular fibers, which are characterized by some analytic and combinatorial properties. Intuitively speaking, given a special subdivisor, we will construct a deformation of the degeneration in such a way that the subdivisor is ‘barked’ (peeled) off from the singular fiber. The construction of these “barking deformations” are very geometric and related to deformations of surface singularities (in particular, cyclic quotient singularities) as well as the mapping class groups of Riemann surfaces (complex curves) via monodromies; moreover the positions of the singularities of a singular fiber appearing in a barking deformation is described in terms of the zeros of a certain polynomial which is expressed in terms of the Riemann theta function and its derivative. In addition to the solid foundation of the theory, we provide several applications, such as (1) a construction of interesting examples of splitting deformations which leads to the class number problem of splitting deformations and (2) the complete classification of absolute atoms of genus from 1 to 5. For genus 1 and 2 cases, this result recovers those of B. Moishezon and E. Horikawa respectively.

Introduction

Wading through,
And wading through,
Yet green mountains still.
(Santoka "Somokuto[1]")

This is the third in our series of works on degenerations of complex curves. (We here use "complex curve" instead of "Riemann surface".) The aim of the present volume is to develop a new deformation theory of degenerations of complex curves. This theory is very geometric and a particular class of subdivisors contained in singular fibers plays a prominent role in the construction of deformations. It also reveals the close relationship between the monodromy of a degeneration and existence of deformations of the degeneration. Moreover, via some diagrams, we may visually understand how a singular fiber is deformed. These deformations are called *barking deformations*, because in the process of deformation, some special subdivisor of the singular fiber looks like "barked" (peeled) off. We point out that barking deformations have a remarkable cross-disciplinary nature; they are related to algebraic geometry, low dimensional topology, and singularity theory.

We will further develop our theory: In [Ta,IV], we describe the vanishing cycles of the nodes of the singular fibers appearing in barking families; we then apply this result to give the Dehn twist decompositions of some automorphisms of Riemann surfaces. In [Ta,V], we develop the moduli theory of splitting deformations, which as a special case, includes the theory of barking deformations over several parameters (in the present volume, we mainly discuss the one-parameter deformation theory).

Background

We will give a brief survey on history and recent development of degenerations of complex curves. Our review is not exhaustive but only covers related topics to our book.

[1] Translated by Hisashi Miura and James Green.

Degenerations of complex curves

A degeneration of complex curves is a one-parameter family of smooth complex curves, which degenerates to a singular complex curve. More precisely, let $\pi : M \to \Delta$ be a proper surjective holomorphic map from a smooth complex surface M to a small disk $\Delta := \{s \in \mathbb{C} : |s| < \delta\}$ such that $\pi^{-1}(0)$ is singular and $\pi^{-1}(s)$ for $s \neq 0$ is a smooth complex curve of genus g ($g \geq 1$); so the origin $0 \in \Delta$ is the critical value of π. (In what follows, unless otherwise mentioned, complex surfaces (curves) are always supposed to be smooth.) We say that $\pi : M \to \Delta$ is a *degeneration* of complex curves of genus g with the *singular fiber* $X := \pi^{-1}(0)$. For simplicity, we sometimes say "a degeneration of genus g".

Let $f : S \to C$ be a proper surjective holomorphic map from a compact complex surface S to a compact complex curve C, and then S is called a *fibered surface* (e.g. elliptic surface). We note that a degeneration appears as a local model of a fibered surface around a singular fiber: Let X be a singular fiber of $f : S \to C$, and then the restriction of f to a sufficiently small neighborhood (germ) of X in S is a degeneration. To classify fibered surfaces, it is important to understand their local structure — degeneration — around each singular fiber. It is also important to know when the signature $\sigma(S)$ (or some other invariant) of the fibered surface concentrates on singular fibers. Namely, when does the equality $\sigma(S) = \sum_i \sigma_{\text{loc}}(M_i)$ holds?, where M_i is a germ of a singular fiber X_i in S, and $\sigma_{\text{loc}}(M_i)$ denotes the local signature of M_i, and the summation runs over all singular fibers (see a survey [AK]). These questions motivate us to study degenerations and their invariants.

Apart from the (local) signature, we have another basic invariant "monodromy" of a degeneration, which also plays an important role in studying degenerations. Given a degeneration $\pi : M \to \Delta$ of complex curves of genus g, we may associate an element h of the symplectic group $Sp(2g : \mathbb{Z})$ acting on the homology group $H_1(\Sigma_g : \mathbb{Z})$, where Σ_g is a smooth fiber of $\pi : M \to \Delta$. The element h is defined as follows. We take a circle $S^1 := \{\, |s| = r \,\}$ contained in the disk Δ, and then $R := \pi^{-1}(S^1)$ is a real 3-manifold. The map $\pi : R \to S^1$ is a fibration (all fibers are diffeomorphic); that is, R is a Σ_g-bundle over S^1, where Σ_g is a smooth fiber of $\pi : M \to \Delta$. Topologically, R is obtained from a product space $\Sigma_g \times [0, 1]$ by the identification of the boundary $\Sigma_g \times \{0\}$ and $\Sigma_g \times \{1\}$ via a homeomorphism γ of Σ_g. We say that γ is the *topological monodromy* of the degeneration $\pi : M \to \Delta$. (It measures how the complex surface M is twisted around the singular fiber X.) Then γ induces an automorphism $h := \gamma_*$ on $H_1(\Sigma_g : \mathbb{Z})$, which is called the *monodromy* of the degeneration. Note that h preserves the intersection form on $H_1(\Sigma_g : \mathbb{Z})$, and so $h \in Sp(2g : \mathbb{Z})$.

Monodromy already appeared in the early study of degenerations, notably the work of Kodaira [Ko1] on the classification of degenerations of elliptic curves (complex curves of genus 1). He showed that there are eight degenerations and determined their monodromies: The singular fibers of eight degenerations are respectively denoted by $I_n, I_n^*, II, III, IV, II^*, III^*, IV^*$. (Apart from the three types II, III, IV, each corresponds to an extended Dynkin diagram.) Kodaira also gave explicit construction of these eight degenerations.

Subsequently, Namikawa and Ueno [NU] carried out the classification of degenerations of complex curves of genus 2: there are about 120 degenerations. Namikawa and Ueno encountered with new phenomena, which did not occur in the genus 1 case: (1) The topological type of a degeneration is not necessarily determined by its singular fiber: There are topologically different degenerations of complex curves of genus 2 with the same singular fiber. (2) The monodromy does not determine the topological type of a degeneration. In fact, if $g \geq 2$, there are a lot of topologically different degenerations with the trivial topological monodromy. The reason is as follows: The mapping class group MCG_g of a complex curve of genus g has a natural homomorphism $MCG_g \to Sp(2g : \mathbb{Z})$ (homological representation), as $\gamma \in MCG_g$ induces an automorphism γ_* of $H_1(\Sigma_g : \mathbb{Z})$. The kernel of this homomorphism is the *Torelli group* T_g. (Note: If $g = 1$, then T_g is trivial (i.e. the above homomorphism is injective), whereas if $g \geq 2$, then T_g is nontrivial.) In particular, if $g \geq 2$, and the topological monodromy γ of a degeneration belongs to T_g, then $h := \gamma_*$ (monodromy) is the identity.

This fact indicates that monodromy is not powerful enough to classify degenerations. Moreover, as is suggested by Namikawa and Ueno's classification of 120 degenerations of genus 2, there seem a tremendous amount of degenerations of genus g, as g grows higher, and further classifications for genus $3, 4, ...$ got stuck. New development came from topology. Observe that in the converting process from a topological monodromy to a monodromy, some information may be lost, and hence it is natural to guess that a topological monodromy carries more information than a monodromy, and this is the starting point of the work of Matsumoto and Montesinos, which we shall explain. First of all, we note that the topological monodromy of a degeneration is a very special homeomorphism; it is either periodic or pseudo-periodic (see [Im], [ES], [ST]). Here, a homeomorphism γ of a complex curve C is *periodic* if for some positive integer m, γ^m is isotopic to the identity, and *pseudo-periodic* if for some loops (simple closed curves) $l_1, l_2, \dots, l_n$ on C, the restriction γ on $C \setminus \{l_1, l_2, \dots, l_n\}$ is periodic. A Dehn twist γ along a loop l on C is an example of a pseudo-periodic homeomorphism, as the restriction of γ to $C \setminus l$ is isotopic to the identity.

Remark 1 There is a classical study of pseudo-periodic homeomorphisms due to Nielsen [Ni1] and [Ni2]; he referred to a pseudo-periodic homeomorphism as algebraically finite type.

For a pseudo-periodic homeomorphism γ, let m be the integer as above, i.e. γ^m on $C \setminus \{l_1, l_2, \dots, l_n\}$ is isotopic to the identity. Then γ^m is generated by Dehn twists along $l_1, l_2, \dots, l_n$. According to the direction of the twist, a Dehn twist is called *right* or *left*. A pseudo-periodic homeomorphism γ is *right* or *left* provided that γ^m is generated only by right or left Dehn twists. The complex structure on a degeneration poses a strong constraint on the property of its topological monodromy. Using the theory of Teichmüller spaces, Earle–Sipe [ES] and Shiga–Tanigawa [ST] demonstrated that any topological monodromy is a right pseudo-periodic homeomorphism — in [MM2], it is called a pseudo-periodic homeomorphism of *negative type*. For example, if the singular fiber is a Lefschetz fiber (a reduced curve with one node), then the topological monodromy is a right Dehn twist along a loop l on a smooth fiber C. Note that the singular fiber is obtained from C by pinching l; in other words, l is the vanishing cycle.

Matsumoto–Montesinos theory

Matsumoto and Montesinos established the converse of the result of Earle–Sipe and Shiga–Tanigawa. Namely, given a periodic or right pseudo-periodic homeomorphism γ, they constructed a degeneration with the topological monodromy γ. Their argument is quite topological, using "open book construction". In [Ta,II], we gave algebro-geometric construction, clarifying the relationship between topological monodromies and quotient singularities.

We denote by $\mathcal{P}_g$ the set of periodic and right pseudo-periodic homeomorphisms of a complex curve of genus g, and denote by $\widehat{\mathcal{P}}_g$ the conjugacy classes of $\mathcal{P}_g$. Next, we denote by $\mathcal{D}_g$ the set of degenerations of complex curves of genus g, and denote by $\widehat{\mathcal{D}}_g$ its topologically equivalent classes. The main result of Matsumoto and Montesinos [MM2] is as follows:

Theorem 2 (Matsumoto and Montesinos [MM2]) *The elements of* $\widehat{\mathcal{P}}_g$ *are in one to one correspondence with the elements of* $\widehat{\mathcal{D}}_g$*.*

One important consequence of this theorem is that the topological classification of degenerations completely reduces to the classification of periodic and right pseudo-periodic homeomorphisms.

Matsumoto and Montesinos [MM2] also determined the shape (configuration) of the singular fiber of a degeneration in terms of the data of its topological monodromy — screw numbers and ramification data. Here, we must take care when using the word "shape", because a shape depends on the choice of model of a degeneration, and it changes under blow up or down. Algebraic geometers usually work with the relatively minimal model of a degeneration — a degeneration is *relatively minimal* if any irreducible component of its singular fiber is not an exceptional curve (a projective line with the self-intersection number -1). However, from the viewpoint of topological monodromies, the relatively minimal model is not so natural. The most natural one is the normally minimal model, because it reflects the topological monodromy very well [MM2]. We now review the definition. Express a singular fiber X as a divisor: $X = \sum_i m_i \Theta_i$ where Θ_i is an irreducible component and a positive integer m_i is its multiplicity. Then $\pi : M \to \Delta$ is called *normally minimal* if X satisfies the following conditions:

(1) the reduced curve $X_{\mathrm{red}} := \sum_i \Theta_i$ is normal crossing (i.e. any singularity of X_{red} is a node), and
(2) if Θ_i is an exceptional curve, then Θ_i intersects other irreducible components at at least three points.

We point out that a relatively minimal degeneration, after successive blow up, becomes a normally minimal one, which is uniquely determined from the relatively minimal degeneration.

In what follows, unless otherwise mentioned, we assume that a degeneration is normally minimal. According to whether the topological monodromy is periodic or pseudo-periodic, the singular fiber is *stellar* (star-shaped) or *constellar* (constellation-shaped). Here, a singular fiber X is called stellar[2] if its dual graph is stellar (star-shaped): X has a central irreducible component (*core*), and several chains of projective lines emanating from the core (see Figure 4.2.1, p61). Such a chain of projective lines is called a *branch* of X. A constellar singular fiber is obtained by bonding branches of stellar fibers, and a resulting chain of projective lines after bonding is called a *trunk*; it is a bridge joining two stellar singular fibers.

The number of the singular fibers of genus g increases rapidly, as g grows higher; this is because a constellar singular fiber is constructed from stellar singular fibers in an inductive way with respect to the genus. For instance, a constellar singular fiber of genus 2 is bonding of two stellar singular fibers of genus 1. (Precisely speaking, there is also a constellar singular fiber of genus 2 obtained from one stellar singular fiber of genus 1 by bonding its two branches.) A constellar singular fiber of genus 3 is either bonding of three stellar singular fibers of genus 1, or bonding of two stellar singular fibers of genus 1 and 2. And as g grows, the partition of the integer g increases rapidly, and accordingly the number of constellar singular fibers increases rapidly.

Based on the work of Matsumoto and Montesinos, Ashikaga and Ishizaka [AI] proposed an algorithm to carry out the topological classification of degenerations of given genus. Although the practical computation becomes difficult as genus grows higher, their algorithm settled down the topological classification problem of degenerations at least theoretically. They applied their algorithm to achieve the topological classification for the genus 3 case (see [AI]): The number of degenerations is about 1600, and among them there are about 50 degenerations with stellar singular fibers. (For any genus, the number of stellar singular fibers is much less than that of constellar singular fibers.)

Morsification

There are about 8, 120, and 1600 degenerations of genus 1, 2, and 3 respectively, and as the genus grows higher, the number of degenerations increases

[2] We have a similar notion in singularity theory, that is, a star-shaped singularity: A singularity V is *star-shaped* if the dual graph of the exceptional set in the resolution space of V is star-shaped, e.g. a singularity with $\mathbb{C}^{\times}$-action. See [OW], [Pn].

rapidly. This fact motivates us to consider another kind of classification — "classification of degenerations modulo deformations". Before we explain it, we review related materials from Morse theory, which elucidates the relationship between the shapes of smooth manifolds and smooth functions on them. One of the key ingredients of Morse theory is the Morse Lemma, asserting that we may perturb a smooth function $f : M \to \mathbb{R}$ in such a way that $f_t : M \to \mathbb{R}$ has only non-degenerate critical points. A non-degenerate critical points is stable under arbitrary perturbation, and so the Morse lemma ensures that we may split critical points of f into stable ones under perturbation. Of course, the Morse lemma is a result in the smooth category, but its spirit is carried over to the complex category, for instance, *Morsification of singularities*: When does an isolated singularity V admits a deformation $\{V_t\}$ such that V_t for $t \neq 0$ possesses only A_1-singularities? (It is known that any hypersurface isolated singularity admits a Morsification, e.g. see Dimca [Di] p82)

We next explain *Morsification of singular fibers*, which was advocated by M. Reid [Re]. First of all, we review splitting deformations.

Splitting deformations of degenerations

Let $\Delta^\dagger := \{t \in \mathbb{C} : |t| < \varepsilon\}$ be a sufficiently small disk. Suppose that $\mathcal{M}$ is a complex 3-manifold, and $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$ is a proper flat surjective holomorphic map. We set $M_t := \Psi^{-1}(\Delta \times \{t\})$ and $\pi_t := \Psi|_{M_t} : M_t \to \Delta \times \{t\}$. (Hereafter, we denote $\Delta \times \{t\}$ simply by Δ, so that $\pi_t : M_t \to \Delta$.) We say that $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$ is a *deformation family* of $\pi : M \to \Delta$ if $\pi_0 : M_0 \to \Delta$ coincides with $\pi : M \to \Delta$. In this case, $\pi_t : M_t \to \Delta$ is referred to as a *deformation* of $\pi : M \to \Delta$.

Suppose that $\pi_t : M_t \to \Delta$ for $t \neq 0$ has at least two singular fibers, say, $X_1, X_2, \ldots, X_n$ $(n \geq 2)$. Then we say that $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$ is a *splitting family* of the degeneration $\pi : M \to \Delta$, and that $\pi_t : M_t \to \Delta$ is a *splitting deformation* of $\pi : M \to \Delta$. In this case, we say that the singular fiber $X = \pi^{-1}(0)$ *splits into* $X_1, X_2, \ldots, X_n$.

To the contrary, if a singular fiber X admits no splitting deformations at all, the degeneration $\pi : M \to \Delta$ is called *atomic*. The singular fiber of the atomic degeneration is called an *atomic fiber*. (Caution: This terminology is not completely rigorous, because a singular fiber does not determine the topological type of a degeneration, so we must use it with care.) A Lefschetz fiber (i.e. a reduced curve with one node) and a multiple $m\Theta$ of a smooth curve Θ, where $m \geq 2$ is an integer, are examples of atomic fibers (see [Ta,I]).

A *Morsification* of a degeneration $\pi : M \to \Delta$ is a splitting family $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$ such that for $t \neq 0$, all singular fibers of $\pi_t : M_t \to \Delta$ are atomic fibers. Unfortunately this notion is too restrictive, as many degenerations of high genus seem to admit no Morsifications. Instead, we work with a weaker notion "a finite-stage Morsification", defined as follows. If $\pi : M \to \Delta$ is not atomic, take a splitting family $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$, say, X splits into $X_1, X_2, \ldots, X_n$ (the first-stage splitting). If all singular fibers $X_1, X_2, \ldots, X_n$

are atomic, the first-stage splitting is a Morsification. If some X_i is not atomic, then take a sufficiently small neighborhood M_i of X_i in M_t, and then consider the restriction of π_t to M_i, which is a degeneration $\pi_i : M_i \to \Delta$ (called the *fiber germ* of X_i in $\pi_t : M_t \to \Delta$). Next, take a splitting family $\Psi_i : \mathcal{M}_i \to \Delta \times \Delta^\dagger$ of $\pi_i : M_i \to \Delta$, say, X_i splits into $X_{i,1}, X_{i,2}, \ldots, X_{i,m}$ (the second-stage splitting). Repeating this process, we finally reach to a set of atomic fibers, say, $X'_1, X'_2, \ldots, X'_l$: Under the finite-stage Morsification, X splits into atomic fibers $X'_1, X'_2, \ldots, X'_l$. In this case, we obtain a smooth 4-manifold M' together with a locally holomorphic map $\pi' : M' \to \Delta$ such that (1) M' is diffeomorphic to M and (2) all singular fibers $X'_1, X'_2, \ldots, X'_l$ of π' are atomic. Here, "locally holomorphic map" means that M' has a complex structure around X'_i, and π' is holomorphic with respect to this complex structure. A finite-stage Morsification of a degeneration is useful for studying the topological types of fibered algebraic surfaces.

There is another motivation from algebraic geometry to study Morsification, inspired by the following question: How does an invariant of a degeneration (e.g. local signature, Horikawa index [AA1]) behave under splitting. Specifically, let $\mathrm{inv}(\pi)$ be some invariant of a degeneration $\pi : M \to \Delta$. Suppose that $\pi_t : M_t \to \Delta$ is a splitting deformation, which splits the singular fiber X into singular fibers $X_1, X_2, \ldots, X_n$. Then find a formula of the form

$$\mathrm{inv}(\pi) = \sum_{i=1}^{n} \mathrm{inv}(\pi_i) + c,$$

where $\pi_i : M_i \to \Delta$ is a fiber germ of X_i in M_t, and c is a "correction term". For these problems, we refer the reader to excellent surveys [AE], [AK], and also [Re].

A primary concern of the Morsification problem of degenerations is to classify all atomic degenerations. The number of atomic degenerations of genus g must be much less than that of all degenerations of genus g, and so this problem leads us to a reasonable classification — *classification of degenerations modulo deformations.*

When is a degneration atomic? Before we discuss this problem, we explain several methods to construct splitting families.

Double covering method for hyperelliptic degenerations

A hyperelliptic curve C is a complex curve which admits a double covering $C \to \mathbb{P}^1$ branched over $2g+2$ points on $\mathbb{P}^1$, where $g = \mathrm{genus}(C)$. (All complex curves of genus 1 and 2 are hyperelliptic.) A degeneration $\pi : M \to \Delta$ is called *hyperelliptic* provided that any smooth fiber $\pi^{-1}(s)$ is a hyperelliptic curve. In this case, the total space M is expressed as a double covering $M \to \mathbb{P}^1 \times \Delta$ branched over a complex curve (*branch curve*) B in $\mathbb{P}^1 \times \Delta$, and conversely from this double covering, we may recover the hyperelliptic degeneration $\pi : M \to \Delta$. (Precisely speaking, instead of M, we need to take a (singular) complex surface M' which is bimeromorphic to M.) A deformation B_t ($t \in \Delta^\dagger$) of the branch curve B induces a deformation $M_t \to \mathbb{P}^1 \times \Delta$ (a family of double coverings branched over B_t) of $M \to \mathbb{P}^1 \times \Delta$, which yields a deformation $\pi_t : M_t \to \Delta$ of the degeneration $\pi : M \to \Delta$. If we choose a suitable deformation B_t of the branch curve B, then $\pi_t : M_t \to \Delta$ is a splitting deformation. This construction is called the *double covering method*, originally due to B. Moishezon [Mo] for the genus 1 case; then applied for the genus 2 case by E. Horikawa [Ho], and finally Ashikaga and Arakawa [AA1] generalized to hyperelliptic degenerations of arbitrary genus.

Note that all degenerations of genus 1 and 2 are hyperelliptic, and so the double covering method is powerful for them. However, a complex curve of genus ≥ 3 is not necessarily hyperelliptic. Accordingly, there are non-hyperelliptic degenerations of genus ≥ 3, for which the double covering method cannot be applied.

In this book, we develop a new deformation theory, which is applicable to any degeneration, irrespective of whether it is hyperelliptic or not. Specifically, we introduce the concept of barking deformations of degenerations, and then derive their properties (here "bark" is that of a tree, not that of a dog.)

Barking deformations

The construction of barking deformations is very geometric. In the simplest case, a barking deformation is — intuitively speaking — obtained by barking (peeling) a special subdivisor of the singular fiber from the singular fiber. As applications, we (1) deduce powerful criteria for the splittability of degenerations, (2) provide interesting examples of splitting deformations which lead to the "class number problem" for degenerations, and (3) determine absolute atoms of genus 3,4, and 5. (Genus 1 and 2 case has already been known.)

Now we shall take a close look at topics of this book.

Construction of barking deformations

To simplify the explanation, for the time being, we only consider stellar singular fibers. Recall that a stellar singular fiber has a central irreducible component (a *core*), and chains of projective lines (*branches*) are attached to the core. We express $X = m_0\Theta_0 + \sum_{j=1}^{N} \mathrm{Br}^{(j)}$, where Θ_0 is the core with the multiplicity m_0 and $\mathrm{Br}^{(j)}$ is a branch: $\mathrm{Br}^{(j)}$ intersects Θ_0 transversely at one point.

The construction of a barking deformation proceeds as follows. Take a set of special subdivisors (called *crusts*) of the singular fiber X: A crust is a subdivisor contained in X satisfying certain arithmetic and analytic conditions. We then associate the set of crusts with an "initial deformation" around the core. Next, we propagate the initial deformation along all branches of X. Although the propagation is not always possible, if it is possible, we obtain a barking deformation of the degeneration $\pi : M \to \Delta$.

In general, a barking deformation is constructed from a set of crusts. When can we construct a barking deformation from a single crust? For a stellar singular fiber, we may completely answer this question by characterizing such a crust in terms of some arithmetic condition. (This is *not* the case for a constellar singular fiber, which generally has more deformations.) The answer is very simple. The subbranches of such a crust must be one of three *types* A_l*,* B_l*, and* C_l, and the converse is also valid. For the definition of types A_l, B_l, and C_l, we refer the reader to Definition 9.1.1, p154.

Moreover, we establish the following result (see p283).

Theorem 3 *Let* $\pi : M \to \Delta$ *be a linear degeneration with a stellar singular fiber* X (*see Remark below for "linear degeneration"*)*. Suppose that* X *contains a subdivisor* lY *such that* Y *is a crust and any subbranch of* Y *is either of type* A_l*,* B_l*, or* C_l*. Then* $\pi : M \to \Delta$ *admits a barking family* $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$ *which barks* lY *from* X*. Conversely if a barking family barks a subdivisor* lY *from* X*, then any subbranch of* Y *is either of type* A_l*,* B_l*, or* C_l*.*

Remark 4 Roughly speaking, a degeneration is *linear* if for any irreducible component of the singular fiber X, its tubular neighborhood is biholomorphic to its normal bundle. Essentially, we need this assumption only for irreducible components of genus ≥ 2. Indeed, for an irreducible component of genus 0 or 1 with the negative self-intersection number, its tubular neighborhood is always biholomorphic to its normal bundle (Grauert's Theorem [Gr]).

In Theorem 3, the deformation restricted to the tubular neighborhood of a branch of X is also said to be of *type* A_l*,* B_l*, or* C_l, the type corresponding to that of the subbranch of Y. These three types of deformations possess very beautiful geometric patterns. Among all, type C_l has interesting periodicity (or symmetry). See Figure 12.3.1, p221 for example.

Theorem 3 is generalized to constellar singular fibers as follows (see p332).

Theorem 5 *Let* $\pi : M \to \Delta$ *be a linear degeneration with a constellar singular fiber* X*. Suppose that* X *contains a subdivisor* lY *such that* Y *is a crust and any subbranch and subtrunk of* Y *are either of type* A_l*,* B_l *or* C_l*. Then* $\pi : M \to \Delta$ *admits a barking family* $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$ *which barks* lY *from* X*.* (The converse is not true. See §18.4, p320, and in particular Example 18.4.2.)

Based on this theorem, we introduce an important concept. Let lY be a subdivisor of X such that (1) Y is a crust and (2) any subbranch and subtrunk of Y are either of type A_l, B_l, or C_l. Then we say that Y is a *simple crust* and l is the *barking multiplicity* of Y. Using this terminology, the above theorem is simply stated as: *If a singular fiber contains a simple crust, then the degeneration admits a barking family.* We denote this barking family by $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$. For a singular fiber $X_{s,t} := \Psi^{-1}(s,t)$ in $\pi_t : M_t \to \Delta$ $(t \neq 0)$, we say that $X_{s,t}$ is the *main fiber* if $s = 0$, and a *subordinate fiber* if $s \neq 0$: The original singular fiber X splits into one main fiber and several subordinate fibers. In §16.4, p288, we describe main and subordinate fibers in details. It is noteworthy that the main fiber is generally non-reduced (some irreducible component has multiplicity at least 2); whereas each subordinate fiber is reduced, and all singularities on it are A-singularities.

Class number problem for degenerations

Assume that a degeneration $\pi : M \to \Delta$ has two splitting families $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$ and $\Psi' : \mathcal{M}' \to \Delta \times \Delta^\dagger$. We say that Ψ and Ψ' are *topologically equivalent* if there exist orientation preserving homeomorphisms $H : \mathcal{M} \to \mathcal{M}'$ and $h : \Delta\times\Delta^\dagger \to \Delta\times\Delta^\dagger$ such that $h(0,0) = (0,0)$ and the following diagrams are commutative:

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\quad H \quad} & \mathcal{M}' \\ \Psi \downarrow & & \downarrow \Psi' \\ \Delta \times \Delta^\dagger & \xrightarrow{\quad h \quad} & \Delta \times \Delta^\dagger, \end{array} \qquad \begin{array}{ccc} M_t & \xrightarrow{\quad H_t \quad} & M'_t \\ \pi_t \downarrow & & \downarrow \pi'_t \\ \Delta \times \{t\} & \xrightarrow{\quad h_t \quad} & \Delta \times \{t\}, \end{array}$$

where $H_t := H|_{M_t}$ and $h_t := h|_{\Delta\times\{t\}}$ are restrictions of H and h respectively. (Note: If Ψ and Ψ' are topologically equivalent, then for each t, $\pi_t : M_t \to \Delta$ and $\pi'_t : M'_t \to \Delta$ are topologically equivalent. But the converse is *not* true.) Barking deformations provide interesting examples of topologically different splitting deformations. For instance, we show (see §20.2, p349)

Theorem 6 *Let $\pi : M \to \Delta$ be a degeneration of elliptic curves with the singular fiber II^** (Kodaira's notation [Ko1])*. Then*

(1) *there exist splitting families Ψ and Ψ' that split II^* into III^* and I_1, but Ψ and Ψ' are topologically different, and*
(2) *there exist splitting families Ψ and Ψ' that split II^* into I_3^* and I_1, but Ψ and Ψ' are topologically different.*

Based on this result, we propose the following problem:

Problem 7 (Class number problem for degenerations) *Let $\pi : M \to \Delta$ be a degeneration. Assume that $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$ is a splitting family of $\pi : M \to \Delta$, which splits X into $X_1, X_2, \ldots, X_n$. Then how many topologically different splitting families that split X into $X_1, X_2, \ldots, X_n$ do there exist?*

(The *class number* of the splitting $X \mapsto X_1, X_2, \ldots, X_n$ is the number of topologically different splitting families that yield this splitting. It is named after the class number of an algebraic number field; roughly, it measures the deviation from unique factorizations of prime ideals.) We will explore this problem in some other paper.

Classification of atomic degenerations

We have another important application of barking deformations, namely, to the classification of atomic degenerations. Recall that a degeneration is *atomic* provided that it does not admit any splitting family at all. If a singular fiber is either a reduced curve with one node (Lefschetz fiber) or a multiple of a smooth curve, then the degeneration is atomic (see [Ta,I]). This statement is valid regardless to genus, whereas the complete classification of atomic degenerations had been known only for low genus case (genus 1 and 2); the case of genus 1 was done by B. Moishezon [Mo], and that of genus 2 by E. Horikawa [Ho] with some result of Arakawa and Ashikaga [AA1]

Remark 8 [Ho] showed that if a singular fiber of genus 2 is not a Lefschetz fiber, then it splits into singular fibers of type I_1 and type 0, where "type I_1" is a reducible Lefschetz fiber, that is, two elliptic curves intersecting at one point. On the other hand, any singular fiber of type 0 splits into irreducible Lefschetz fibers by Corollary 4.12 of [AA1].

The list of singular fibers of atomic degenerations of genus 1 and 2 is the following:

	atoms
genus 1 (Moishezon [Mo])	$m\Theta$, where $m \geq 2$ and Θ is a smooth elliptic curve, any reduced curve with one node (Lefschetz fiber)
genus 2 (Horikawa [Ho])	any reduced curve with one node (Lefschetz fiber)

What can we say about genus 3 or higher genus case? In [Re] p5, a conjecture due to Xiao Gang is stated:

"A singular fiber X is atomic precisely when X has either a single node, or is a multiple of a smooth curve, or has some other combination of singularities forced by the monodromy, or has a linear system special in the sense of moduli."

M. Reid also conjectured that an atomic fiber of genus 3 is either a Lefschetz fiber (a reduced curve with one node) or a multiple curve 2Θ where Θ is a smooth curve of genus 2.

In [Ta,I] (see §1.2, p30 of this book for the summary), we showed that a degeneration with a constellar singular fiber almost always admits a splitting family. This result is valid for any genus, and so the classification problem reduces to checking the splittability for the 'remaining case' (we explain soon). Before proceeding, we point out that for genus at least 3, there are a lot of degenerations which are topologically equivalent but analytically inequivalent: see Remark below. So, there may be two topologically equivalent degenerations such that one is atomic but another is not. This indicates that for genus at least 3, the notion of atomicness is too strong. We work instead with a weaker notion: "absolutely atomic".

Remark 9 If a singular fiber has an irreducible component, say Θ, of genus at least 2, then the tubular neighborhood of Θ in M is analytically *not* unique. To the contrary, for an irreducible component of genus 0 or 1 with the negative self-intersection number, its tubular neighborhood is always biholomorphic to its normal bundle by Grauert's Theorem [Gr].

A degeneration is called *absolutely atomic* if any degeneration with the same topological type is atomic. So, if a degeneration $\pi : M \to \Delta$ has a topologically equivalent degeneration $\pi' : M' \to \Delta$ that admits a splitting family, then $\pi : M \to \Delta$ is *not* absolutely atomic.

We proposed in [Ta,I]:

Conjucture 10 *A degeneration is absolutely atomic if and only if its singular fiber is either a reduced curve with one node, or a multiple of a smooth curve.*

Now we explain our idea to classify absolute atoms. We intend to carry it out by induction on genus. Namely, suppose that Conjecture 10 is valid for genus $\leq g-1$. According to [Ta,I], under this assumption, to classify absolutely atomic degenerations of genus g, we only have to investigate the splittability for degenerations $\pi : M \to \Delta$ such that either

(A) $X = \pi^{-1}(0)$ is stellar, or
(B) X is constellar and (B.1) X has no multiple node and (B.2) if X has an irreducible component Θ of multiplicity 1, then Θ is a projective line, and intersects other irreducible components of X only at one point (hence Θ intersects only one irreducible component).

To these cases, we apply Theorems 3 and 5 and their variants (see criteria below). Namely, we try to find a simple crust (or its generalization "a crustal set") of a singular fiber in (A) or (B): See the list of simple crusts for genus ≤ 5 in p487. As a result, we obtain the complete classification of absolute atomic degenerations of genus 3, 4, and 5 as follows.

	absolute atoms
genus 3	2Θ, where Θ is a smooth curve of genus 2, any reduced curve with one node (Lefschetz fiber)
genus 4	3Θ, where Θ is a smooth curve of genus 2, any reduced curve with one node (Lefschetz fiber)
genus 5	4Θ, where Θ is a smooth curve of genus 2, 2Θ, where Θ is a smooth curve of genus 3, any reduced curve with one node (Lefschetz fiber)

This classification also confirms the validity of Conjecture 10 for genus ≤ 5. (For the genus 6 case, we also checked the validity of this conjecture for a large class of degenerations including those with stellar singular fibers.)

We remark that T. Arakawa and T. Ashikaga [AA1], [AA2] classified absolute atoms among degenerations of "hyperelliptic" curves of genus 3; they used the double covering method.

Main criteria for splittability

Now we state our main criteria for splittability. In what follows, unless otherwise mentioned, we assume that degenerations are linear (see Remark 4). First of all, for stellar singular fibers, we shall exhibit criteria which are derived from Theorem 3. Let $\pi : M \to \Delta$ be a degeneration with a stellar singular fiber X. We denote X by

$$X = m_0\Theta_0 + \sum_{j=1}^{N} \mathrm{Br}^{(j)},$$

where $\mathrm{Br}^{(j)} = m_1^{(j)}\Theta_1^{(j)} + m_2^{(j)}\Theta_2^{(j)} + \cdots + m_{\lambda_j}^{(j)}\Theta_{\lambda_j}^{(j)}$ is a branch (a chain of projective lines) emanating from the core (the central component) Θ_0. See p284 for the following criterion.

Criterion 11 *Let $\pi : M \to \Delta$ be a degeneration with a stellar singular fiber $X = m_0\Theta_0 + \sum_{j=1}^{N} \mathrm{Br}^{(j)}$. Then the following statements hold:*

(1) *Suppose that the core Θ_0 is an exceptional curve* (i.e. *Θ_0 is a projective line such that $\Theta_0 \cdot \Theta_0 = -1$). Then $\pi : M \to \Delta$ admits a splitting family.*
(2) *Suppose that the core Θ_0 is not an exceptional curve. If X contains a simple crust Y, then $\pi : M \to \Delta$ admits a splitting family.*

(The splitting families in (1) and (2) can be explicitly described.)

See p285 for the following criterion.

Criterion 12 *Let $\pi : M \to \Delta$ be a degeneration with a stellar singular fiber $X = m_0\Theta_0 + \sum_{j=1}^{N} \mathrm{Br}^{(j)}$. Set $r = \dfrac{m_1^{(1)} + m_1^{(2)} + \cdots + m_1^{(N)}}{m_0}$. Suppose that the following conditions* (A) *and* (B) *are satisfied:*

(A) *$N_0 \cong \mathcal{O}_{\Theta_0}(-p_1^{(1)} - p_1^{(2)} - \cdots - p_1^{(r)})$ where N_0 is the normal bundle of Θ_0 in M and $p_1^{(j)} \in \Theta_0$ is the intersection point of Θ_0 and $\mathrm{Br}^{(j)}$,*
(B) *there are r branches among all branches of X, say, $\mathrm{Br}^{(1)}, \mathrm{Br}^{(2)}, \ldots, \mathrm{Br}^{(r)}$, satisfying the following conditions*:
  (B1) *for $j = 1, 2, \ldots, r$, there exists an integer e_j where $1 \le e_j \le \lambda_j$ such that $m_{e_1}^{(1)} = m_{e_2}^{(2)} = \cdots = m_{e_r}^{(r)}$, and*
  (B2) *for $j = 1, 2, \ldots, r$, each irreducible component $\Theta_i^{(j)}$ $(i = 1, 2, \ldots, e_j - 1)$ has the self-intersection number -2* (this condition is vacuous for j such that $e_j = 1$).

Then $\pi : M \to \Delta$ admits a splitting family which is explicitly constructed from the above data. (Note: (A) is an analytic condition, while (B) is a numerical one.)

When Θ_0 is a projective line, the above criterion takes a simpler form (see p286):

Criterion 13 *Let $\pi : M \to \Delta$ be a degeneration with a stellar singular fiber $X = m_0\Theta_0 + \sum_{j=1}^{N} \mathrm{Br}^{(j)}$. Set $r = \dfrac{m_1^{(1)} + m_1^{(2)} + \cdots + m_1^{(N)}}{m_0}$. Assume that Θ_0 is a projective line. Suppose that there are r branches among all branches of X, say, $\mathrm{Br}^{(1)}, \mathrm{Br}^{(2)}, \ldots, \mathrm{Br}^{(r)}$, satisfying the following conditions*:

(B1) *for $j = 1, 2, \ldots, r$, there exists an integer e_j where $1 \le e_j \le \lambda_j$ such that $m_{e_1}^{(1)} = m_{e_2}^{(2)} = \cdots = m_{e_r}^{(r)}$,* (B2) *for* $j = 1, 2, \ldots, r$, *each irreducible component* $\Theta_i^{(j)}$ $(i = 1, 2, \ldots, e_j - 1)$ *has the self-intersection number* -2.

Then $\pi : M \to \Delta$ *admits a splitting family.*

We next exhibit splittability criteria for constellar singular fibers (see p293).

Criterion 14 (Trivial Extension Criterion) *Let* X_1 (resp. X_2) *be a stellar singular fiber of* $\pi_1 : M_1 \to \Delta$ (resp. $\pi_2 : M_2 \to \Delta$), *and let* Br_1 (resp. Br_2) *be a branch of* X_1 (resp. X_2). *Let* X *be a constellar singular fiber of* $\pi : M \to \Delta$ *obtained from* X_1 *and* X_2 *by* κ*-bonding of* Br_1 *and* Br_2, *where* κ $(\kappa \geq -1)$ *is an integer.* (Note: Br_1 and Br_2 are joined to become a "κ-trunk" Tk of X. See p293.) *Suppose that* X_1 *contains a simple crust* Y_1 *such that in the case* $\kappa = -1$,

$$\rho(\mathrm{br}_1) + 1 \leq \mathrm{length}(\mathrm{Tk}),$$

where $\rho(\mathrm{br}_1)$ *is the propagation number of the subbranch* br_1 *of* Y_1 *contained in* Br_1 (see (16.4.2), p291). *Then the barking family of* $\pi_1 : M_1 \to \Delta$ *associated with* Y_1 '*trivially*' *extends to that of* $\pi : M \to \Delta$.
(This criterion is easily generalized to the case where X is obtained by bonding an arbitrary number of stellar singular fibers.)

From Criterion 14, for a degeneration with a constellar singular fiber, we may almost always use a simple crust of some stellar singular fiber to construct its splitting family. Thus **the essential part of the classification of absolute atoms reduces to the stellar case** — precisely speaking, there are some exceptional constellar cases which are not covered by Criterion 14.

We note that stellar singular fibers are much fewer than constellar ones. For example, in genus 3 there are about 1600 singular fibers and only about 50 stellar ones among them (see [AI]). We also note that by Criterion 11 (1), if the core of a stellar singular fiber is an exceptional curve, then the singular fiber admits a splitting. Hence we only need to check the splittability of stellar singular fibers whose cores are not exceptional curves — our criteria drastically reduce the number of singular fibers whose splittability must be checked.

Finally we state a very powerful criterion (see p343).

Criterion 15 *Let* $\pi : M \to \Delta$ *be a degeneration of genus* g *with the singular fiber* X. *Then* $\pi : M \to \Delta$ *admits a splitting family if either* (1), (2), *or* (3) *below holds:*

(1) X *contains a simple crust* Y *such that either*
  (1a) Y *contains no exceptional curve, or*
  (1b) *the barking genus* $g_b(Y) \neq g$ (hence $\leq g - 1$).
(2) X *contains an exceptional curve* Θ_0 *such that*
  (2a) *at least one irreducible component of* X *intersecting* Θ_0 *is a projective line, say this component* Θ_1, *and*

(2b) *any irreducible component of X intersecting Θ_0 satisfies the tensor condition with respect to the subdivisor $Y = \Theta_0 + \Theta_1$.*

(3) *X contains an exceptional curve Θ_0 such that any irreducible component intersecting Θ_0 is a projective line.* (Note: If X is stellar, noting that Θ_0 must be the core, this condition is always satisfied.)

Organization of this book

This book is organized as follows. In Part I, after introducing basic definitions, we explain the idea of barking deformations by means of examples without mentioning much theoretical background. We also give instruction on how to draw "figures of deformations", which is extremely useful to understand geometric nature of barking deformations. We hope that Part I gives the reader a perspective of what will be going on. Part II is devoted to detail study of deformations of tubular neighborhoods of branches. Some arithmetic properties of multiplicities are deeply related to the existence of deformations. In Part III, based on the results of Part II, we introduce the notion of barking deformations for degenerations of *compact* complex curves. Theorems 3 and 5 above are proved there. Furthermore we will derive important splittability criteria of singular fibers from these theorems.

In Part IV, we describe the subordinate fibers. We show that the singularities of a subordinate fiber are A-singularities. Moreover, we give the formulas of the number of the singularities on one subordinate fiber as well as the formula of the number of all subordinate fibers in a barking family.

In Part V, we provide the list of representative crusts for a large class of singular fibers of genus from 1 to 5, which is enough for the purpose of classifying absolute atoms. As a consequence we obtain the complete classification of absolute atoms of genus from 1 to 5.

General advice: Most of chapters contain a section which computes the discriminants of deformations — the *discriminant* of a family $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$ is a plane curve in $\Delta \times \Delta^\dagger$, given by $D = \{(s,t) \in \Delta \times \Delta^\dagger \,:\, \Psi^{-1}(s,t) \text{ is singular}\}$. This section is slightly technical, and for the first reading, it may be efficient to skip it.

Without figures, it is hard to comprehend or appreciate barking deformations, and for this reason, I included representative figures. I intended to make this book accessible to researchers studying algebraic geometry, low dimensional topology, and singularity theory. I am very happy if I could share my enthusiasm on this subject with the reader.

Acknowledgment. I am extremely indebted to Professors Tadashi Ashikaga, Yukio Matsumoto and Fumio Sakai, to whom I would like to express my deep gratitude. I would like to sincerely thank Professor Oswald Riemenschneider for useful conversations on deformations of singularities. I would also like to sincerely thank Professors Masanori Ishida, Shoetsu Ogata, Masataka Tomari, Kazushi Ahara, Ikuko Awata, David De Wit, Madoka Ebihara, Toshizumi Fukui, Toru Gocho, Colin Ingalls, Masaharu Ishikawa, Mizuho Ishizaka, Toshio Ito, Yuichi Yamada, for fruitful discussions and comments. I had the opportunity to give lectures on the present work at Saitama University, Tohoku University, and Tokyo University which were of a great help in clarifying my ideas. I would like to thank the audiences there. I also would like to thank the Max-Planck-Institut für Mathematik at Bonn, and the Research Institute for Mathematical Sciences at Kyoto University for hospitality and financial support. This work was partially supported by a grant from JSPS.

Notation

1. $\Delta = \{s \in \mathbb{C} : |s| < \delta\}$ and $\Delta^\dagger = \{t \in \mathbb{C} : |t| < \varepsilon\}$
2. $\mathcal{O}_M$: the sheaf of germs of holomorphic functions on a complex manifold M
3. f_z: the derivative $\dfrac{df}{dz}$ of a function $f(z)$
4. $\mathbb{P}^1$: the projective line (Riemann sphere)
5. For a divisor $D = \sum_i m_i\Theta_i$ on a smooth complex surface,

   $D \geq 0$: D is a *nonnegative divisor*, i.e. $m_i \geq 0$ for all i

   $D > 0$: D is an *effective* (or *positive*) *divisor*, i.e. $m_i > 0$ for all i

   $D \geq D'$: $D - D'$ is a nonnegative divisor

   $D_{\text{red}} := \sum_i \Theta_i$: the underlying reduced curve of D

   $\text{Supp}(D)$: the *support* of D, i.e. D_{red} as a topological space

   We say that D is *connected* if $\text{Supp}(D)$ is connected as a topological space, and that D *intersects* D' at a point p if $\text{Supp}(D)$ intersects $\text{Supp}(D')$ at p.
6. $X = \sum_i m_i\Theta_i$: a singular fiber where m_i is the multiplicity of an irreducible component Θ_i
   $Y = \sum_i n_i\Theta_i$: a subdivisor of X, so n_i satisfies $0 \leq n_i \leq m_i$. Symbolically this condition is expressed by the notation $0 \leq Y \leq X$.
7. $X_{s,t} := \Psi^{-1}(s,t)$: a fiber of a deformation family $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$
8. $\Theta_i \cdot \Theta_i$: the *self-intersection number* of Θ_i. A projective line with the self-intersection number $-n$ is called a $(-n)$-*curve*; a (-1)-*curve* is also called an *exceptional curve* (*of the first kind*).
9. $(\Theta_i \cdot \Theta_i)_Y$: the *formal self-intersection number* of Θ_i with respect to a subdivisor Y, p65
10. $\text{Br} = m_1\Theta_1 + m_2\Theta_2 + \cdots + m_\lambda\Theta_\lambda$: an *unfringed branch*,
    $\overline{\text{Br}} = m_0\Delta_0 + m_1\Theta_1 + m_2\Theta_2 + \cdots + m_\lambda\Theta_\lambda$: a *fringed branch* ($m_0\Delta_0$ is a

fringe and Δ_0 is an open disk), p86. Both unfringed branches and fringed branches are often simply called branches.

11. $\overline{\mathrm{br}} := \overline{\mathrm{Br}} \cap Y$: a fringed subbranch p281, contained in a fringed branch $\overline{\mathrm{Br}}$ of a subdivisor Y
12. $X = m_0\Theta_0 + \sum_{j=1}^{N} \mathrm{Br}^{(j)}$: a *stellar* (star-shaped) singular fiber where Θ_0 is the central component (the *core*) of X, and $\mathrm{Br}^{(j)} = m_1^{(j)}\Theta_1^{(j)} + m_2^{(j)}\Theta_2^{(j)} + \cdots + m_{\lambda_j}^{(j)}\Theta_{\lambda_j}^{(j)}$ is a *branch*
13. For a stellar singular fiber $X = m_0\Theta_0 + \sum_{j=1}^{N} \mathrm{Br}^{(j)}$,
    - $p_1^{(j)} \in \Theta_0$: the intersection point of the core Θ_0 and a branch $\mathrm{Br}^{(j)}$, i.e. the *attachment point* of a branch to the core
    - N_0 (resp. $N_i^{(j)}$): the normal bundle of Θ_0 (resp. $\Theta_i^{(j)}$) in M
    - σ: the *standard section* of X, which is a holomorphic section of $N_0^{\otimes(-m_0)}$ such that $\mathrm{div}(\sigma) = \sum_{j=1}^{N} m_1^{(j)} p_1^{(j)}$, i.e. σ has a zero of order $m_1^{(j)}$ at each point $p_1^{(j)}$ $(j = 1, 2, \ldots, N)$
    - $Y = n_0\Theta_0 + \sum_{j=1}^{N} \mathrm{br}^{(j)}$: a *crust* of X, where $\mathrm{br}^{(j)} = n_1^{(j)}\Theta_1^{(j)} + n_2^{(j)}\Theta_2^{(j)} + \cdots + n_{e_j}^{(j)}\Theta_{e_j}^{(j)}$ is a subbranch of $\mathrm{Br}^{(j)}$
    - τ: a *core section* of a crust Y, which is a meromorphic section of $N_0^{\otimes n_0}$ with a pole of order $n_1^{(j)}$ at $p_1^{(j)}$ $(j = 1, 2, \ldots, N)$
14. $\mathbf{m} = (m_0, m_1, \ldots, m_\lambda)$ for a fringed branch $\overline{\mathrm{Br}} = m_0\Delta_0 + m_1\Theta_1 + \cdots + m_\lambda\Theta_\lambda$
15. $\mathbf{n} = (n_0, n_1, \ldots, n_e)$ for a fringed subbranch $\overline{\mathrm{br}} = n_0\Delta_0 + n_1\Theta_1 + \cdots + n_e\Theta_e$
16. DA_e : a *deformation atlas* of length e, p88
17. $DA_{e-1}(Y, d)$: a deformation atlas of length $e-1$ and weight d associated with a subbranch Y of length e, p90
18. The following continued fraction

$$r_1 - \cfrac{1}{r_2 - \cfrac{1}{r_3 - \cfrac{}{\ddots - \cfrac{1}{r_\delta}}}}$$

will be denoted by $r_1 - \dfrac{1|}{|r_2} - \dfrac{1|}{|r_3} - \cdots - \dfrac{1|}{|r_\delta}$.

19. $f_i := f(w^{p_{i-1}}\eta^{p_i})$ and $\widehat{f_i} := f(z^{p_{i+1}}\zeta^{p_i})$, $(i = 1, 2, \ldots, \lambda)$: a sequence of holomorphic functions associated with a branch $\mathrm{Br} = m_1\Theta_1 + m_2\Theta_2 + \cdots + m_\lambda\Theta_\lambda$ and a holomorphic function $f(z)$ (see p106). Here, nonnegative integers $p_0, p_1, \ldots, p_{\lambda+1}$ are inductively defined by

$$\begin{cases} p_0 = 0, \quad p_1 = 1 \quad \text{and} \\ p_{i+1} = r_i p_i - p_{i-1} \quad \text{for } i = 1,\, 2,\, \ldots,\, \lambda, \end{cases}$$

where $r_i = -\Theta_i \cdot \Theta_i$ (that is, $-r_i$ is the self-intersection number of Θ_i).

20. ${}_l\mathrm{C}_k$: the number of choices of k elements from the set of l elements, i.e. $\begin{pmatrix} l \\ k \end{pmatrix}$
21. type $B_l^\sharp$: non-proportional type B_l (this notation is used only in tables), p157
22. $\ell(A) := e_1\ell_1 + e_2\ell_2 + \cdots + e_n\ell_n$: the *length* of a waving polynomial

$$A(w, \eta, t) = w^u P_1^{e_1} P_2^{e_2} \cdots P_n^{e_n},$$

where $P_i = \prod_{j=1}^{\ell_i}(w\eta + t\beta_j^{(i)})$, p186
23. $DA_{e-1}(\mathbf{Y}, \mathbf{d})$: a deformation atlas of weight $\mathbf{d} = \{d_1, d_2, \ldots, d_l\}$ associated with a bunch $\mathbf{Y} = \{Y_1, Y_2, \ldots, Y_l\}$, p258
24. $\mathrm{div}(\tau) = \sum_i a_i p_i - \sum_j b_j q_j$: the divisor defined by a meromorphic section τ of a line bundle on a complex curve C; τ has a zero of order a_i at p_i and a pole of order b_j at q_j, p266
25. $DA_{\mathbf{e}} = \{\mathcal{W}_0\,,\, DA_{e_j}^{(j)}\}_{j=1,2,\ldots,N}$: a *deformation atlas of size* $\mathbf{e}$ for a stellar singular fiber $X = m_0\Theta_0 + \sum_{j=1}^{N} \mathrm{Br}^{(j)}$, where
   (i) $\mathcal{W}_0$ is a deformation of W_0 parameterized by $\Delta \times \Delta^\dagger$, and
   (ii) $DA_{e_j}^{(j)} = \{\mathcal{H}_i^{(j)}\,,\, \mathcal{H}_i^{(j)\prime}\; g_i^{(j)}\}_{i=1,2,\ldots,e_j}$ is a deformation atlas of length e_j for a branch $\mathrm{Br}^{(j)}$ such that under a coordinate change $(z_0, \zeta_0) = (\eta_1^{(j)}, w_1^{(j)})$ around $p_1^{(j)}$, the equation of $\mathcal{W}_0$ becomes that of $\mathcal{H}_1^{(j)}$, p270
26. $D_1 \sim D_2$: two divisors D_1 and D_2 are *linearly equivalent*, p272
27. $\rho(\mathrm{br}^{(j)})$: the *propagation number* of a subbranch $\mathrm{br}^{(j)}$ of type A_l, B_l, or C_l, defined by

$$\rho(\mathrm{br}^{(j)}) = \begin{cases} e+1 & \text{if } \mathrm{br}^{(j)} \text{ is of type } A_l \\ e & \text{if } \mathrm{br}^{(j)} \text{ is of type } B_l \\ f & \text{if } \mathrm{br}^{(j)} \text{ is of type } C_l, \end{cases}$$

where e is the length of $\mathrm{br}^{(j)}$, and for f, see the explanation following (16.4.2), p291
28. $g_b(Y)$: the *barking genus* of a simple crust Y, p295
29. $\mathrm{Tk} = m_1\Theta_1 + m_2\Theta_2 + \cdots + m_\lambda\Theta_\lambda$: an *unfringed trunk*,
   $\overline{\mathrm{Tk}} = m_0\Delta_0 + m_1\Theta_1 + m_2\Theta_2 + \cdots + m_\lambda\Theta_\lambda + m_{\lambda+1}\Delta_{\lambda+1}$: a *fringed trunk* ($m_0\Delta_0$ and $m_{\lambda+1}\Delta_{\lambda+1}$ are *fringes*, and Δ_0 and $\Delta_{\lambda+1}$ are open disks), p310. Both unfringed trunks and fringed trunks are often simply called trunks.
30. $\overline{\mathrm{tk}} := \overline{\mathrm{Tk}} \cap Y$: a fringed subtrunk p330, contained in a fringed trunk $\overline{\mathrm{Tk}}$ of a subdivisor Y
31. $X \to X_1 + X_2 + \cdots + X_n$: A singular fiber X splits into singular fibers $X_1, X_2, \ldots, X_n$, p351

Part I

Basic Notions and Ideas

1

Splitting Deformations of Degenerations

1.1 Definitions

Herein, $\Delta := \{s \in \mathbb{C} : |s| < 1\}$ stands for the unit disk. Let $\pi : M \to \Delta$ be a proper[1] surjective holomorphic map from a smooth complex surface M to Δ such that (1) $\pi^{-1}(0)$ is singular and (2) $\pi^{-1}(s)$ for nonzero s is a smooth complex curve of genus g. We say that $\pi : M \to \Delta$ is a *degeneration* of complex curves of genus g with the *singular fiber* $X := \pi^{-1}(0)$. Unless otherwise mentioned, we always assume that $g \geq 1$.

Two degenerations $\pi_1 : M_1 \to \Delta$ and $\pi_2 : M_2 \to \Delta$ are called *topologically equivalent* if there exist orientation preserving homeomorphisms $H : M_1 \to M_2$ and $h : \Delta \to \Delta$ such that $h(0) = 0$ and the following diagram is commutative:

$$\begin{array}{ccc} M_1 & \xrightarrow{H} & M_2 \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi_2} \\ \Delta & \xrightarrow{h} & \Delta. \end{array}$$

Next, we introduce basic terminology concerned with deformations of degenerations. We take another disk $\Delta^\dagger := \{t \in \mathbb{C} : |t| < \varepsilon\}$ where ε is sufficiently small. Suppose that $\mathcal{M}$ is a smooth complex 3-manifold, and $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$ is a proper flat surjective holomorphic map. (Note: Unless we pose "flatness", a fiber of Ψ is possibly 2-dimensional, e.g. blow up of $\mathcal{M}$ at one point.) We set $M_t := \Psi^{-1}(\Delta \times \{t\})$ and $\pi_t := \Psi|_{M_t} : M_t \to \Delta \times \{t\}$. Since M is smooth and $\dim \Delta^\dagger = 1$, the composite map $\mathrm{pr}_2 \circ \Psi : \mathcal{M} \to \Delta^\dagger$ is a submersion, and so M_t is smooth. We say that $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$ is a *deformation family* of $\pi : M \to \Delta$ if $\pi_0 : M_0 \to \Delta \times \{0\}$ coincides with $\pi : M \to \Delta$. By convention, we often denote $\Delta \times \{t\}$ simply by Δ, and we say that $\pi_t : M_t \to \Delta$ is a *deformation* of $\pi : M \to \Delta$.

We introduce a special class of deformation families of a degeneration. At first, we suppose that $\pi : M \to \Delta$ is *relatively minimal*, i.e. any irreducible

[1] "Proper" means that all fibers are compact.

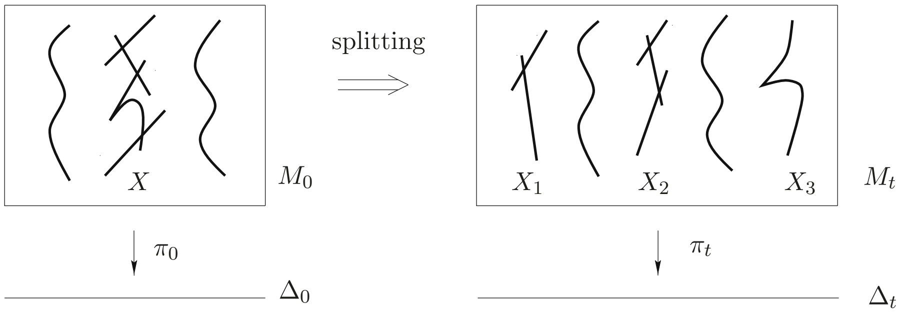

Fig. 1.1.1.

component of its singular fiber is not an *exceptional curve*[2] (a projective line with the self-intersection number -1). Then a deformation family $\Psi : M \to \Delta \times \Delta^\dagger$ is said to be a *splitting family* of $\pi : M \to \Delta$ provided that for each $t \neq 0$, $\pi_t : M_t \to \Delta$ has at least two singular fibers. In this case we say that $\pi_t : M_t \to \Delta$ is a *splitting deformation* of $\pi : M \to \Delta$, and if $X_1, X_2, \dots, X_l$ ($l \geq 2$) are singular fibers of $\pi_t : M_t \to \Delta$, then we say that X *splits into* $X_1, X_2, \dots, X_l$. See Figure 1.1.1. We remark that for sufficiently small $t \neq 0$, the number l of the singular fibers is independent of t. In fact, the discriminant $\mathsf{D} \subset \Delta \times \Delta^\dagger$ (the locus consisting of points (s,t) such that the fiber $\Psi^{-1}(s,t)$ is singular) of $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$ is a plane curve passing through $(0,0)$, and for sufficiently small $t \neq 0$, the number of the points in the intersection $\mathsf{D} \cap (\Delta \times \{t\})$ is constant, equal to the number l.

The above definition of a splitting family is too restrictive because we are actually mostly interested in the germ of degenerations, and herein we adopt a weaker definition, which allows 'shrinking' of $\pi : M \to \Delta$. Namely, we say that $\pi : M \to \Delta$ admits a splitting family if for some δ $(0 < \delta < 1)$ the restriction $\pi' : M' \to \Delta' := \{|s| < \delta\}$, where $M' := \pi^{-1}(\Delta')$ and $\pi' := \pi|_{M'}$, admits a spitting family in the above sense. For simplicity we adopt the convention to rewrite $\pi' : M' \to \Delta'$ as $\pi : M \to \Delta$. (The "shrinking procedure" for the actual case is explained in detail in [Ta,I], p133.)

Next we define the notion of a splitting family for a degeneration $\pi : M \to \Delta$ which is *not* relatively minimal. We first take a sequence of blow down maps

$$M \xrightarrow{f_1} M_1 \xrightarrow{f_2} M_2 \xrightarrow{f_3} \cdots \xrightarrow{f_r} M_r,$$

and degenerations $\pi_i : M_i \to \Delta$ $(i = 1, 2, \dots, r)$ where

(1) $f_i : M_{i-1} \to M_i$ is a blow down of an exceptional curve in M_{i-1}, and the map $\pi_i : M_i \to \Delta$ is naturally induced from $\pi_{i-1} : M_{i-1} \to \Delta$, and
(2) $\pi_r : M_r \to \Delta$ is relatively minimal.

Now given a deformation family $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$ of $\pi : M \to \Delta$, we shall construct a deformation family $\Psi_r : \mathcal{M}_r \to \Delta \times \Delta^\dagger$ of the relatively

[2] More precisely, an exceptional curve of the first kind — also called a (-1)-curve.

minimal degeneration $\pi_r : M_r \to \Delta$. First, recall Kodaira's Stability Theorem [Ko2]: Any exceptional curve in a complex surface is preserved under an arbitrary deformation of that complex surface. By assumption, $\pi : M \to \Delta$ is not relatively minimal, so that M contains an exceptional curve. Thus by Kodaira's Stability Theorem, there exists a family of exceptional curves in $\mathcal{M}$. Further by [FN], we may blow down them simultaneously to obtain a deformation family $\Psi_1 : \mathcal{M}_1 \to \Delta$ of $\pi_1 : M_1 \to \Delta$. Then, again by Kodaira's Stability Theorem, there exists a family of exceptional curves in $\mathcal{M}_1$, which we blow down simultaneously to obtain a deformation family $\Psi_2 : \mathcal{M}_2 \to \Delta$ of $\pi_2 : M_2 \to \Delta$. We repeat this process and finally we obtain a deformation family $\Psi_r : \mathcal{M}_r \to \Delta$ of $\pi_r : M_r \to \Delta$. Namely, given a deformation family $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$ of $\pi : M \to \Delta$, we constructed a deformation family $\Psi_r : \mathcal{M}_r \to \Delta \times \Delta^\dagger$ of the relatively minimal degeneration $\pi_r : M_r \to \Delta$.

After the above preparation, we give the definition of a splitting family for a degeneration $\pi : M \to \Delta$ which is not necessarily relatively minimal: A deformation family $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$ is a *splitting family* of $\pi : M \to \Delta$ provided that $\Psi_r : \mathcal{M}_r \to \Delta \times \Delta^\dagger$ is a splitting family of the relatively minimal degeneration $\pi_r : M_r \to \Delta$.

We are interested in such degenerations as are 'stable' under deformations. A degeneration is called *atomic* if it admits no splitting family at all. One of our goals is to classify all atomic degenerations.

Topological monodromies

We next define the topological monodromy of a degeneration $\pi : M \to \Delta$; see [MM2], [Ta,II] for details. We take a circle $S^1 := \{\, |s| = r \,\}$ in the unit disk Δ, where the radius $r < 1$ and we give a counterclockwise orientation on S^1. Consider a real 3-manifold $R := \pi^{-1}(S^1)$. Then the restriction $\pi : R \to S^1$ is a fibration (all fibers are diffeomorphic); that is, R is a Σ-bundle over S^1, where Σ is a smooth fiber of $\pi : M \to \Delta$. Topologically, R is obtained from a product space $\Sigma \times [0, 1]$ by the identification of the boundary $\Sigma \times \{0\}$ and $\Sigma \times \{1\}$ via a homeomorphism γ of Σ. The isotopy class of γ — it is an element of the mapping class group of Σ and it does not depend on the radius r of S^1 — is called the *topological monodromy* of $\pi : M \to \Delta$. (We usually denote this isotopy class also by γ.) The topological monodromy measures how the complex surface M is twisted around the singular fiber X.

We may also define the topological monodromy of $\pi : M \to \Delta$ geometrically. Since $\pi : R \to S^1$ is a fibration, the differential $d\pi$ has maximal rank. Thus, using a partition of unity we may construct a vector field v on R such that $d\pi(v) = r\dfrac{\partial}{\partial\theta}$ (see, for example, Theorem 4.1 of [MK]). Here (r, θ) is the polar coordinates of $S^1 = \{\, |z| = r \,\}$, that is, $z = re^{\mathrm{i}\theta}$. Integrating the vector field v, we obtain a flow on R, which defines a one-parameter family of diffeomorphisms $h_\theta : \Sigma_0 \to \Sigma_\theta$ $(0 \le \theta \le 2\pi)$: see Figure 1.1.2. We set $\gamma := h_{2\pi}$ and then the diffeomorphism $\gamma : \Sigma_0 \to \Sigma_{2\pi}\ (= \Sigma_0)$ is nothing but the topological monodromy of $\pi : M \to \Delta$.

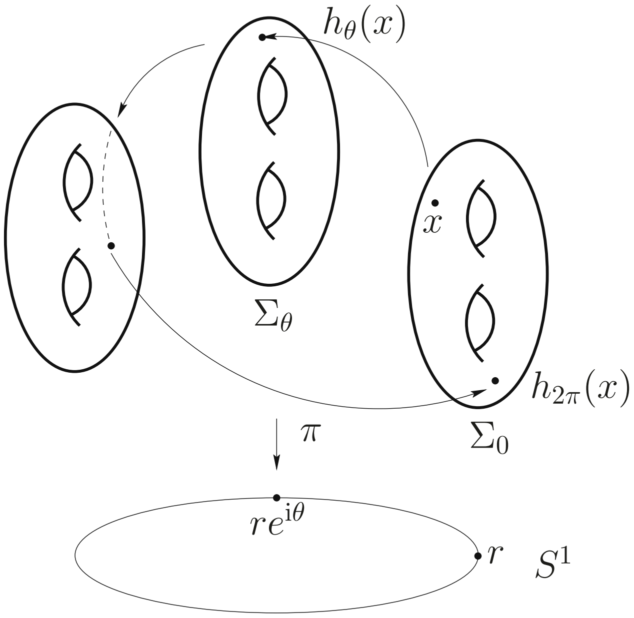

Fig. 1.1.2.

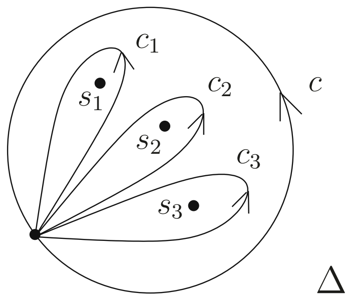

Fig. 1.1.3. $c \sim c_1c_2c_3$ (homotopic)

A splitting family induces a decomposition of the topological monodromy γ. Suppose that $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$ is a splitting family of $\pi : M \to \Delta$, say, $\pi_t : M_t \to \Delta$ $(t \neq 0)$ has singular fibers $X_1, X_2, \ldots, X_l$ where $l \geq 2$. Take loops c_i (with a counterclockwise orientation) in Δ circuiting around the points $s_i := \pi(X_i)$ $(i = 1, 2, \ldots, l)$ such that possibly after renumbering, $c \sim c_1c_2 \cdots c_l$ (homotopic) where $c = \partial\Delta$ with a counterclockwise orientation: see Figure 1.1.3.
Let γ_i be the topological monodromy around the singular fiber X_i in $\pi_t : M_t \to \Delta$ along the loop c_i. Then the topological monodromy γ of $\pi : M \to \Delta$

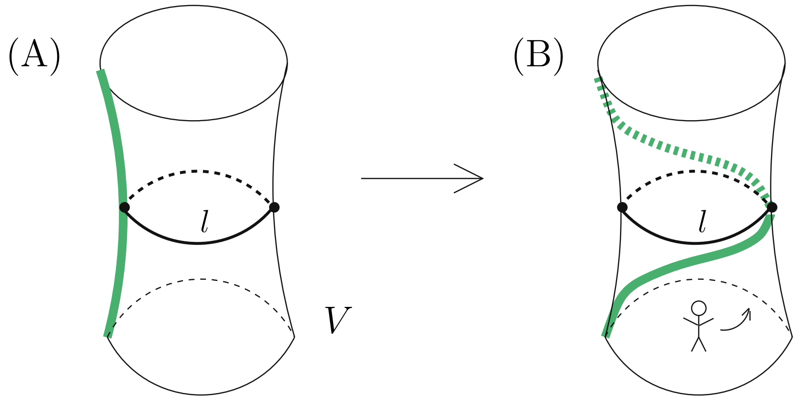

Fig. 1.1.4. Local model of right Dehn twist: As we approach the central loop l (the vanishing cycle of a node) on the annulus V, we are forced to move to the right hand side (all points on the boundary of V are fixed). For the left Dehn twist, we are forced to move to the left hand side.

admits a decomposition: $\gamma = \gamma_1\gamma_2\cdots\gamma_l$. If X_i is a reduced curve with one node (*Lefschetz fiber*), then the topological monodromy γ_i is a (right) *Dehn twist*: see Figure 1.1.4. In particular, if all $X_1, X_2, \ldots, X_l$ are Lefschetz fibers, then we obtain a decomposition of γ into Dehn twists — this is a motivation for topologists to study splitting families.

Normally minimal degenerations

We denote the singular fiber X of a degeneration $\pi : M \to \Delta$ by the divisor expression $X = \sum_i m_i\Theta_i$, where a (possibly singular) complex curve Θ_i is an irreducible component and a positive integer m_i is the multiplicity of Θ_i. We say that the degeneration $\pi : M \to \Delta$ is *normally minimal* if X satisfies the following conditions:

(1) the reduced curve $X_{\mathrm{red}} := \sum_i \Theta_i$ is normal crossing, and
(2) if Θ_i is an exceptional curve, then Θ_i intersects other irreducible components at at least three points.

In this case, we also say that the singular fiber X is normally minimal.

Herein, instead of relatively minimal degenerations, we mainly treat normally minimal degenerations, because their singular fibers reflect the topological type (and also topological monodromies) of the degenerations very well [MM2], [Ta,II]. For instance, if $\pi : M \to \Delta$ is normally minimal, then the singular fiber X is either *stellar* (star-shaped) or *constellar* (constellation-shaped). A stellar singular fiber has one irreducible component called a *core*, and several branches emanate from the core (there may be no branches): a *branch* is a chain of projective lines. A constellar singular fiber is obtained from stellar ones by 'bonding' their branches (*Matsumoto–Montesinos bonding*). When we later construct deformations of degenerations, we will explain stellar/constellar singular fibers in details.

We remark that a singular fiber X is stellar precisely when the topological monodromy γ is periodic (i.e. $\gamma^n = \mathrm{id}$ for some positive integer n); while X is constellar precisely when γ is pseudo-periodic (i.e. γ^n for some positive integer n is generated by Dehn twists). We also note that

Lemma 1.1.1 *Let $\pi : M \to \Delta$ be a normally minimal degeneration of complex curves of genus g ($g \geq 1$). Then the topological monodromy γ is trivial if and only if the singular fiber $X = \pi^{-1}(0)$ is a multiple $m\Theta$ of a smooth elliptic curve Θ, where m is an integer greater than* 1 (note: in this case, $g = 1$).

Proof. We give only the outline of the proof of the "if" part. (See [MM2] or [Ta,II] for the converse.) First, note that since X is a multiple $m\Theta$ of a smooth elliptic curve Θ, the quotient space of a smooth fiber (a torus $S^1 \times S^1$) under the γ-action is again a torus. Thus $\gamma : S^1 \times S^1 \to S^1 \times S^1$ is a 'rotation' by $\frac{2\pi}{m}$, that is, isotopic to a map of the form:

$$(z, w) \longmapsto (e^{2\pi \mathrm{i}/m} z,\ e^{2\pi \mathrm{i} n/m} w)$$

where n ($0 \leq n < m$) is an integer. This map is isotopic to the identity map via a family of maps $(z, w) \mapsto (e^{2\pi \mathrm{i} t/m} z,\ e^{2\pi \mathrm{i} tn/m} w)$, where $0 \leq t \leq 1$. Therefore, γ is isotopic to the identity map. □

The following lemma is useful.

Lemma 1.1.2 *Let $\pi : M \to \Delta$ be a normally minimal degeneration of complex curves of genus g ($g \geq 1$). Suppose that $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$ is a deformation family of $\pi : M \to \Delta$ such that $\pi_t : M_t \to \Delta$ ($t \neq 0$) has at least two normally minimal singular fibers. Then $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$ is a splitting family of $\pi : M \to \Delta$.*

Proof. We give the proof separately for two cases $g \geq 2$ and $g = 1$. We first show the assertion for $g \geq 2$. Let $\pi_r : M_r \to \Delta$ be the relatively minimal degeneration obtained from $\pi : M \to \Delta$ by blowing down, and let $\Psi_r : \mathcal{M}_r \to \Delta \times \Delta^\dagger$ be the deformation family of $\pi_r : M_r \to \Delta$ obtained from $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$ by blowing down. By assumption, there are (at least) two normally minimal singular fibers, say X_1 and X_2, of $\pi_t : M_t \to \Delta$. After blowing down, the image $\overline{X}_i$ of X_i ($i = 1, 2$) in $M_{r,t} := \Psi_r^{-1}(\Delta \times \{t\})$ has a nontrivial topological monodromy, because (i) the topological monodromy of π_t around X_i is nontrivial by the assumption $g \geq 2$ (see Lemma 1.1.1) and (ii) a topological monodromy does not change after blowing down. Therefore $\overline{X}_1$ and $\overline{X}_2$ are singular fibers of $\pi_{r,t} : M_{r,t} \to \Delta$ (if $\overline{X}_i$ is smooth, then its topological monodromy is trivial — a contradiction!). This implies that $\Psi_r : \mathcal{M}_r \to \Delta \times \Delta^\dagger$ is indeed a splitting family, and thus by definition, $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$ is a splitting family. Hence the assertion for $g \geq 2$ is proved.

We next show the assertion for $g = 1$. If neither X_1 nor X_2 is a multiple of a smooth elliptic curve, then the topological monodromies of X_1 and X_2 are nontrivial (Lemma 1.1.1); so we may apply the argument for $g \geq 2$ to confirm the assertion. We consider the remaining case: X_1 or X_2 is a multiple of a smooth elliptic curve. Note that if X_i is a multiple of a smooth elliptic curve, then it contains no exceptional curve (in fact, it contains no projective line at all), and so its image $\overline{X}_i$ in $M_{r,t}$ coincides with X_i itself; therefore $\overline{X}_i$ is a singular fiber of $\pi_{r,t} : M_{r,t} \to \Delta$. If X_i is not a multiple of a smooth elliptic curve, then, as we explained for the case $g \geq 2$, the image $\overline{X}_i$ in $M_{r,t}$ is also a singular fiber of $\pi_{r,t} : M_{r,t} \to \Delta$. Hence, irrespective of whether X_1 or X_2 is a multiple of a smooth elliptic curve, both images $\overline{X}_1$ and $\overline{X}_2$ in $M_{r,t}$ are singular fibers of $\pi_{r,t} : M_{r,t} \to \Delta$. So $\Psi_r : \mathcal{M}_r \to \Delta \times \Delta^\dagger$ is indeed a splitting family, and accordingly (by definition), $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$ is a splitting family.□

Fake singular fibers

We assume that $\pi : M \to \Delta$ is *not* relatively minimal, and let $\pi_r : M_r \to \Delta$ be a relatively minimal degeneration obtained from $\pi : M \to \Delta$ by blowing down. Suppose that $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$ is a deformation family of $\pi : M \to \Delta$, and let $\Psi_r : \mathcal{M}_r \to \Delta \times \Delta^\dagger$ be the deformation family of $\pi_r : M_r \to \Delta$ induced from $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$ by blowing down. We note that even if $\pi_t : M_t \to \Delta$ has at least two singular fibers — seemingly, a splitting family —, it may occur that after blowing down, $\pi_{r,t} : M_{r,t} \to \Delta$ has only one singular fiber, where we set $M_{r,t} := \Psi_r^{-1}(\Delta \times \{t\})$ and $\pi_{r,t} := \Psi_r|_{M_{r,t}}$. *After blowing down, a singular fiber X_t of $\pi_t : M_t \to \Delta$ possibly becomes a smooth fiber in $\pi_{r,t} : M_{r,t} \to \Delta$.* Such a singular fiber of $\pi_t : M_t \to \Delta$ is called *fake* (see §20.1, p349 for example). We emphasize that a deformation family $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$ is a splitting family of $\pi : M \to \Delta$ precisely when $\pi_t : M \to \Delta$ has at least two "non-fake" singular fibers. When $\pi : M \to \Delta$ is relatively minimal, any singular fiber of $\pi_t : M_t \to \Delta$ is not fake.

Topological types of splitting families

We defined topological equivalences of degenerations. We shall define a similar notion to splitting families. Suppose that a degeneration $\pi : M \to \Delta$ has two splitting families $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$ and $\Psi' : \mathcal{M}' \to \Delta \times \Delta^\dagger$. Then Ψ and Ψ' are *topologically equivalent* if there exist orientation preserving homeomorphisms $H : \mathcal{M} \to \mathcal{M}'$ and $h : \Delta \times \Delta^\dagger \to \Delta \times \Delta^\dagger$ such that $h(0,0) = (0,0)$ and the following diagrams are respectively commutative:

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{H} & \mathcal{M}' \\ \Psi \downarrow & & \downarrow \Psi' \\ \Delta \times \Delta^\dagger & \xrightarrow{h} & \Delta \times \Delta^\dagger, \end{array} \qquad \begin{array}{ccc} M_t & \xrightarrow{H_t} & M'_t \\ \pi_t \downarrow & & \downarrow \pi'_t \\ \Delta \times \{t\} & \xrightarrow{h_t} & \Delta \times \{t\}, \end{array}$$

where $H_t := H|_{M_t}$ and $h_t := h|_{\Delta\times\{t\}}$ are restrictions of H and h respectively. We also have a weaker notion; two splitting families are *weakly* topologically equivalent if for each $t \neq 0$, there exist homeomorphisms $H_t : M_t \to M'_t$ and $h_t : \Delta \to \Delta$ such that

$$\begin{array}{ccc} M_t & \xrightarrow{\quad H_t \quad} & M'_t \\ {\scriptstyle \pi_t}\downarrow & & \downarrow{\scriptstyle \pi'_t} \\ \Delta \times \{t\} & \xrightarrow{\quad h_t \quad} & \Delta \times \{t\}. \end{array}$$

Of course, if two splitting families are topologically equivalent, then they are weakly topologically equivalent. But the converse is not necessarily true.

1.2 Splitting criteria via configuration of singular fibers

For convenience, we summarize the results (splitting criteria) of [Ta,I].

Theorem 1.2.1 *Let $\pi : M \to \Delta$ be a degeneration of curves such that the singular fiber X is either* (I) *a reduced curve with one node, or* (II) *a multiple of a smooth curve of multiplicity at least* 2*. Then $\pi : M \to \Delta$ is atomic.*

Criterion 1.2.2 *Let $\pi : M \to \Delta$ be normally minimal such that the singular fiber X has a multiple node of multiplicity at least* 2*. Then there exists a splitting family of $\pi : M \to \Delta$ which splits X into X_1 and X_2, where X_1 is a reduced curve with one node and X_2 is obtained from X by replacing the multiple node by a multiple annulus.*

Criterion 1.2.3 (Multiple Criterion) *Let $\pi : M \to \Delta$ is normally minimal such that the singular fiber X contains a multiple node* (*of multiplicity* ≥ 1)*. Then $\pi : M \to \Delta$ is atomic if and only if X is a reduced curve with one node.*

Criterion 1.2.4 *Let $\pi : M \to \Delta$ be relatively minimal. Suppose that the singular fiber X has a point p such that a germ of p in X is either*

(1) *a multiple of a plane curve singularity*[3] *of multiplicity at least* 2*, or*
(2) *a plane curve singularity such that if it is a node, then $X \setminus p$ is not smooth.*

Then $\pi : M \to \Delta$ admits a splitting family.

Criterion 1.2.5 *Let $\pi : M \to \Delta$ be normally minimal. Suppose that the singular fiber X contains an irreducible component Θ_0 of multiplicity* 1 *such that $X \setminus \Theta_0$ is* (*topologically*) *disconnected. Denote by $Y_1, Y_2, \dots, Y_l$* ($l \geq 2$) *all connected components of $X \setminus \Theta_0$. Then $\pi : M \to \Delta$ admits a splitting family which splits X into $X_1, X_2, \dots, X_l$, where X_i* ($i = 1, 2, \dots, l$) *is obtained from X by 'smoothing' $Y_1, Y_2, \dots, \check{Y}_i, \dots, Y_l$. Here $\check{Y}_i$ is the omission of Y_i.*

[3] Herein, a plane curve singularity always means a reduced one.

Criterion 1.2.6 *Let $\pi : M \to \Delta$ be normally minimal such that the singular fiber X contains an irreducible component Θ_0 of multiplicity 1. Let $\pi_1 : W_1 \to \Delta$ be the restriction of π to a tubular neighborhood W_1 of $X \setminus \Theta_0$ in M. Suppose that $\pi_1 : W_1 \to \Delta$ admits a splitting family Ψ_1 which splits $Y^+ := W_1 \cap X$ into $Y_1^+, Y_2^+, \ldots, Y_l^+$. Then $\pi : M \to \Delta$ admits a splitting family Ψ which splits X into $X_1, X_2, \ldots, X_l$, where X_i is obtained from Y_i^+ by gluing $\Theta_0 \setminus (W_1 \cap \Theta_0)$ along the boundary.*

2

What is a barking?

This chapter is introductory. Using a local model of a degeneration, we illustrate an idea of "barking deformation".

2.1 Barking, I

Let m and m' be positive integers (later we will allow m or m' to be zero). We define a map $\pi : \mathbb{C}^2 \to \mathbb{C}$ by $\pi(x, y) = x^{m'} y^m$, which is considered to be a local model of a degeneration. The singular fiber X is $\pi^{-1}(0) = \{x^{m'} y^m = 0\}$: the union of the x-axis of multiplicity m and the y-axis of multiplicity m'. On the other hand, $\pi^{-1}(s)$ for nonzero s is a smooth fiber, which is

$$\begin{cases} \gcd(m, m') \text{ copies of } \mathbb{C}^\times := \mathbb{C} \setminus \{0\} & \text{if } m > 0 \text{ and } m' > 0, \\ m \text{ copies of } \mathbb{C} & \text{if } m > 0 \text{ and } m' = 0, \\ m' \text{ copies of } \mathbb{C} & \text{if } m = 0 \text{ and } m' > 0. \end{cases}$$

To construct a deformation of a 'degeneration' $\pi : \mathbb{C}^2 \to \mathbb{C}$, it is convenient to consider the graph of π:

$$\mathrm{Graph}(\pi) := \{(x, y, s) \in \mathbb{C}^2 \times \mathbb{C} \mid x^{m'} y^m - s = 0\},$$

which is regarded as a family of curves, parameterized by $s \in \mathbb{C}$. We take positive integers n and n' such that $n < m$ and $n' < m'$. Letting $t \in \mathbb{C}$ be another parameter, we define three kinds of two-parameter deformations of X as follows:

$$X_{s,t} : \quad x^{m'-n'} y^{m-n}(x^{n'} y^n + t) - s = 0, \tag{2.1.1}$$

$$X_{s,t} : \quad x^{m'} y^{m-n}(y^n + t) - s = 0 \qquad (\text{possibly } m' = 0), \tag{2.1.2}$$

$$X_{s,t} : \quad x^{m'} y^{m-n}(y^n + tx^a) - s = 0 \qquad (\text{possibly } m' = 0). \tag{2.1.3}$$

We note that in all cases, $X_{0,0} = X$ and so $X_{s,t}$ is a deformation of X. The deformation $X_{s,t}$ is referred to as a *barking deformation* of the degeneration $\pi : \mathbb{C}^2 \to \mathbb{C}$. Here "bark" is used in the context of bark of a tree (not of a dog), and the name "bark" will be clear from the subsequent description of these deformations.

Putting $s = 0$ in (2.1.1), (2.1.2), or (2.1.3), we obtain a deformation from X to $X_{0,t}$, which is respectively called a *hyperbolic barking*, *Euclidean barking*, or *parabolic barking*.

To understand these barkings geometrically, we set $Y := \{x^{n'}y^n = 0\}$. Of course, Y is the union of the x-axis of multiplicity n and the y-axis of multiplicity n'. Since $n < m$ and $n' < m'$, Y is a subdivisor of X (Notation: $Y < X$). We now give a geometric description of barkings.

Hyperbolic barking We begin with a hyperbolic barking, which is a deformation from $X : x^{m'}y^m = 0$ to $X_{0,t} : x^{m'-n'}y^{m-n}(x^{n'}y^n + t) = 0$. See Figure 2.1.1.

Note that $X_{0,t}$ for $t \neq 0$ consists of two curves, that is, $x^{m'-n'}y^{m-n} = 0$ and $x^{n'}y^n + t = 0$. We write $Z : x^{m'-n'}y^{m-n} = 0$ and $Y_t : x^{n'}y^n + t = 0$, and express $X_{0,t} = Z + Y_{0,t}$ (a sum of divisors). Of course, Z does not depend on t, while $Y_{0,t}$ does. We note that Z is a union of the x-axis of multiplicity $m - n$ and the y-axis of multiplicity $m' - n'$, and Z is a subdivisor of X.

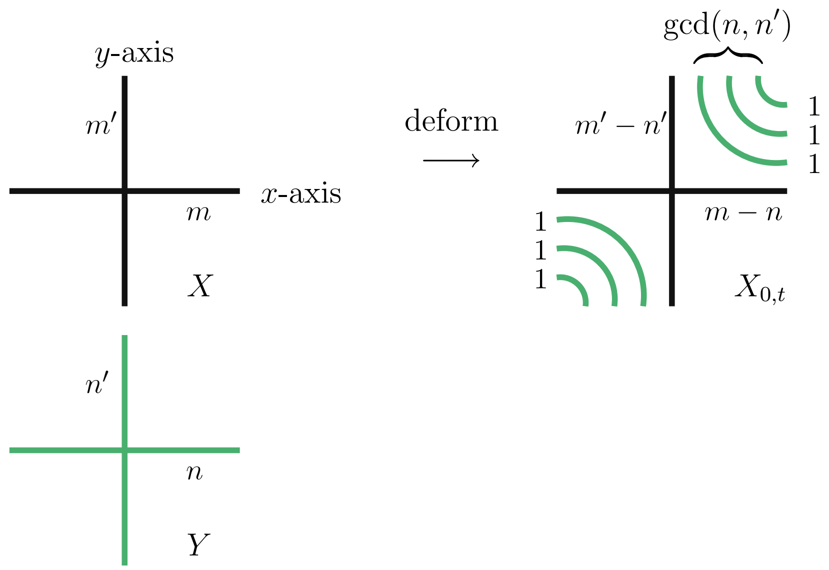

Fig. 2.1.1. Hyperbolic barking: In $X_{0,t}$, the barked part Y_t (hyperbolas) is described by gray color, while bold lines are the unbarked part Z.

We also note that Y_t admits a factorization:

$$Y_t : \quad \prod_{k=1}^{g} \left(x^{n'/g} y^{n/g} + (-t)^{1/g} e^{2\pi \mathrm{i} k/g} \right) = 0,$$

where we set $g := \gcd(n, n')$. Hence Y_t is a disjoint union of g 'hyperbolas'

$$x^{n'/g} y^{n/g} + (-t)^{1/g} e^{2\pi \mathrm{i} k/g} = 0 \qquad (k = 1, 2, \ldots, g).$$

Intuitively (look at Figure 2.1.1), in the process of the deformation from X to $X_{0,t}$, the subdivisor Y of X is "barked" (peeled) off from X to become hyperbolas Y_t, while the subdivisor Z of X is undeformed. Based on this fact, Y_t is called the *barked part* of $X_{0,t}$, whereas Z is called the *unbarked part* of $X_{0,t}$.

Euclidean barking Next we describe a Euclidean barking, which is a deformation from $X : x^{m'} y^m = 0$ to $X_{0,t} : x^{m'} y^{m-n}(y^n + t) = 0$. In this case, $X_{0,t}$ consists of two curves $Z : x^{m'} y^{m-n} = 0$ and $Y_t : y^n + t = 0$. As in the case of hyperbolic barking, Z (resp. Y_t) is called the *unbarked* (resp. *barked*) *part* of $X_{0,t}$. Clearly Z is a union of the x-axis of multiplicity $m - n$ and the y-axis of multiplicity m'. On the other hand, Y_t admits a factorization:

$$Y_t : \quad \prod_{k=1}^{n} \left(y + (-t)^{1/n} e^{2\pi \mathrm{i} k/n} \right) = 0.$$

We note that $y + (-t)^{1/n} e^{2\pi \mathrm{i} k/n} = 0$ $(k = 1, 2, \ldots, n)$ is isomorphic to the ("Euclidean") line $\mathbb{C}$, and hence Y_t is a disjoint union of n lines (The name "Euclidean" barking comes from this). In the process of the deformation, $Y : y^n = 0$ is barked off from X to become $Y_t : y^n + t = 0$. See Figure 2.1.2.

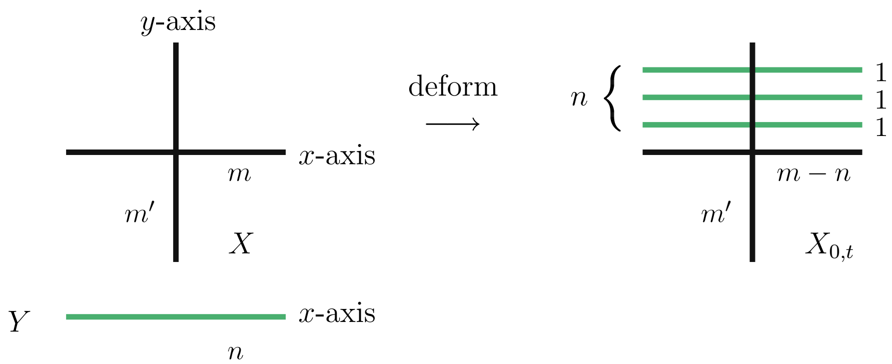

Fig. 2.1.2. Euclidean barking: In $X_{0,t}$, the barked part Y_t is described by gray colored lines, while bold lines are the unbarked part Z.

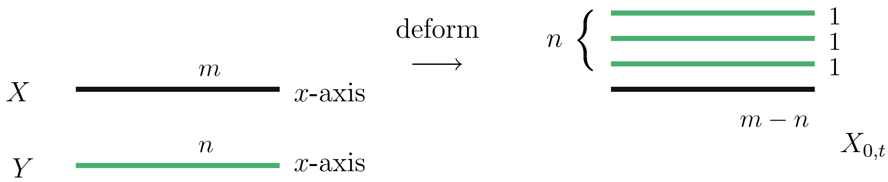

Fig. 2.1.3. Euclidean barking when $m' = 0$

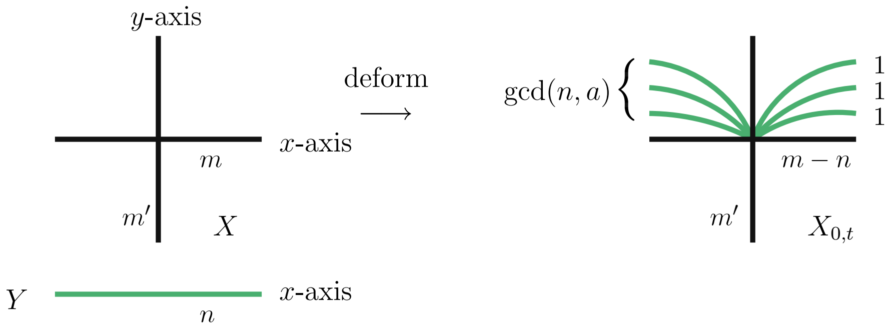

Fig. 2.1.4. Parabolic barking: In $X_{0,t}$, the barked part Y_t (plane curves passing through the origin) is described by gray color, while bold lines are the unbarked part Z.

When $m' = 0$, X is simply the x-axis of multiplicity m, and a Euclidean barking is shown in Figure 2.1.3.

Parabolic barking Finally we describe a parabolic barking, which is a deformation from $X : x^{m'}y^m = 0$ to $X_{0,t} : x^{m'}y^{m-n}(y^n + tx^a) = 0$. In this case, $X_{0,t}$ consists of two curves; $Z : x^{m'}y^{m-n} = 0$ and $Y_t : y^n + tx^a = 0$. Setting $g := \gcd(n, a)$, we have a factorization

$$Y_t : \quad \prod_{k=1}^{g} \left(y^{n/g} + (-t)^{1/g} e^{2\pi \mathrm{i} k/g} x^{a/g} \right) = 0,$$

and hence Y_t is a disjoint union of g plane curves passing through the origin:

$$y^{n/g} + (-t)^{1/g} e^{2\pi \mathrm{i} k/g} x^{a/g} = 0 \qquad (k = 1, 2, \ldots, g).$$

In the process of the deformation, $Y : y^n = 0$ is barked off from X to become $Y_t : y^n + tx^a = 0$. See Figure 2.1.4. We also draw the figure of $X_{0,t}$ for the special cases (1) $a = 1$ and (2) $a = 2$ in Figure 2.1.5 (1) and (2). Note that in (2), the barked part $y + tx^2 = 0$ is a parabola (the name "parabolic" barking comes from this).

When $m' = 0$, X is the x-axis of multiplicity m, and a parabolic barking is shown in Figure 2.1.6.

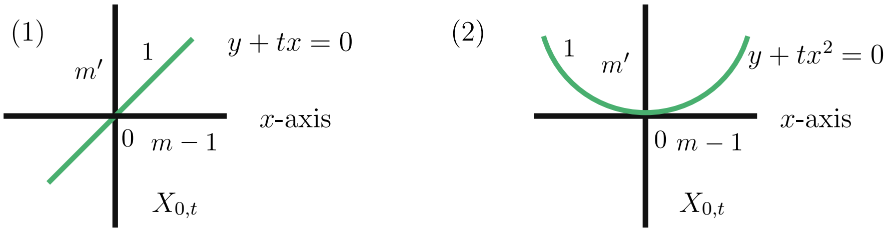

Fig. 2.1.5. Two examples of $X_{0,t}$ for parabolic barkings when (1) $n=1$ and $a=1$ and (2) $n=1$ and $a=2$ respectively.

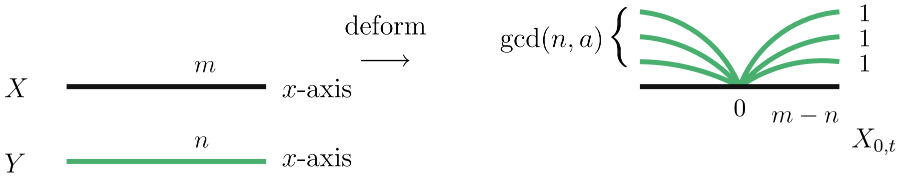

Fig. 2.1.6. Parabolic barking when $m'=0$.

2.2 Barking, II

As in the last section, we take a local model of a degeneration $x^{m'}y^m - s = 0$ (a family of curves parameterized by $s \in \mathbb{C}$). In this section we generalize the notions of hyperbolic, Euclidean, and parabolic barkings to multiple case. Namely, taking positive integers n, n' and l satisfying $m - ln > 0$ and $m' - ln' > 0$, we define two-parameter deformations as follows.

$$X_{s,t}: \quad x^{m'-ln'}y^{m-ln}(x^{n'}y^n + t)^l - s = 0, \tag{2.2.1}$$

$$X_{s,t}: \quad x^{m'}y^{m-ln}(y^n + t)^l - s = 0 \qquad \text{(possibly } m' = 0), \tag{2.2.2}$$

$$X_{s,t}: \quad x^{m'}y^{m-ln}(y^n + tx^a)^l - s = 0 \qquad \text{(possibly } m' = 0). \tag{2.2.3}$$

Note that in all cases, we have $X_{0,0} = X$; so $X_{s,t}$ is a deformation of X. The deformation $X_{s,t}$ is called a *multiple barking deformation* of the degeneration $\pi : \mathbb{C}^2 \to \mathbb{C}$.

For (2.2.1), (2.2.2), or (2.2.3), the deformation from X to $X_{0,t}$ is respectively called a *multiple hyperbolic barking*, *multiple Euclidean barking*, or *multiple parabolic barking*. The integer l is called the *barking multiplicity*. (In the previous section, we treated the case $l=1$.) The description of these deformations is the same as that for the case $l=1$, except in one aspect: in the multiple barking case, the barked part is no longer *reduced* (where “reduced” means that the multiplicity is 1), but instead has multiplicity l. To understand multiple barkings geometrically, we set $Y := \{x^{n'}y^n = 0\}$, and then $lY = \{x^{ln'}y^{ln} = 0\}$. For the hyperbolic barking, $X_{0,t}$ is a union of two curves

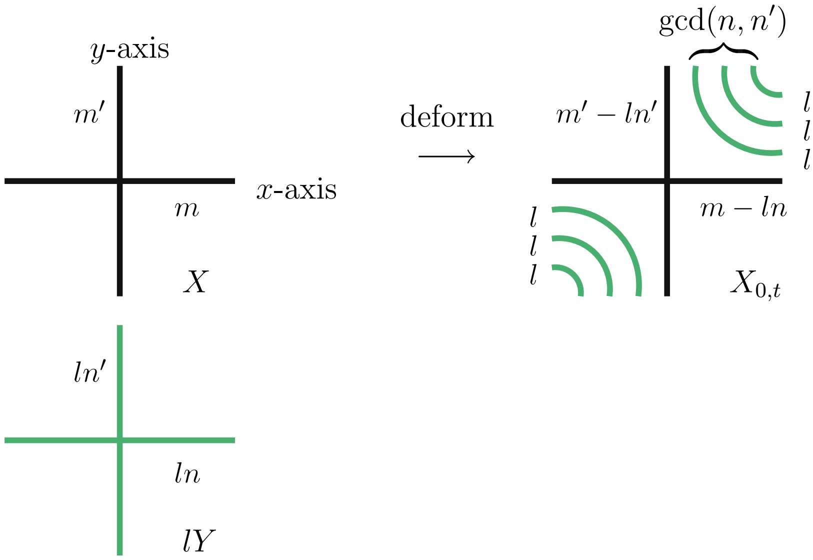

Fig. 2.2.1. Multiple hyperbolic barking: The barked part $(x^{n'}y^n + t)^l = 0$ is a multiple of hyperbolas of multiplicity l.

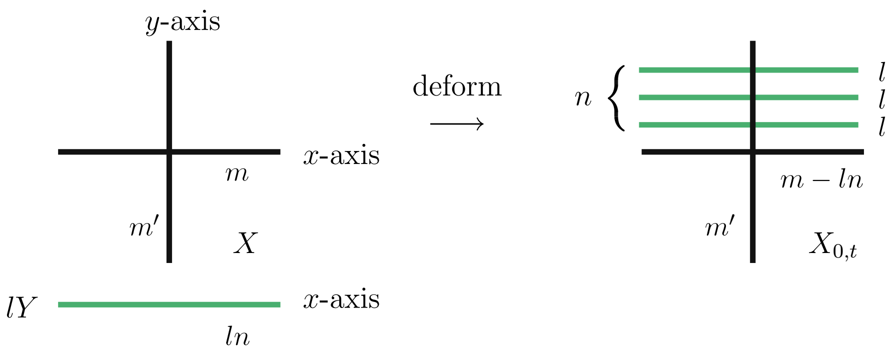

Fig. 2.2.2. Multiple Euclidean barking: The barked part $(y^n + t)^l = 0$ is a multiple of lines of multiplicity l.

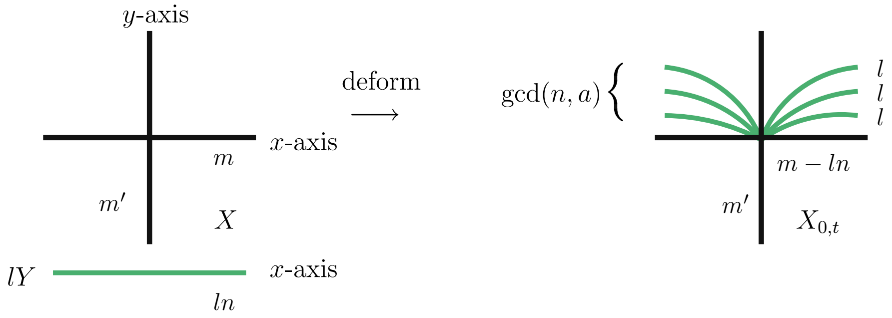

Fig. 2.2.3. Multiple parabolic barking: The barked part $(y^n+tx^a)^l=0$ is a multiple of plane curves of of multiplicity l.

$Z : x^{m'-ln'}y^{m-ln}=0$ and $lY_t : (x^{n'}y^n+t)^l=0$, where $Y_t : x^{n'}y^n+t=0$. We say that Z (resp. lY_t) is the *unbarked* (resp. *barked*) *part* of $X_{0,t}$. In the process of the deformation from X to $X_{0,t}$, the subdivisor lY of X is barked off from X to become lY_t. See Figure 2.2.1.

Similarly, the description of Euclidean and parabolic barking is shown in Figure 2.2.2 and Figure 2.2.3 respectively.

3

Semi-Local Barking Deformations: Ideas and Examples

In this chapter, for a "semi-local" model of a degeneration (to be explained below in the context), we shall construct barking deformations.

3.1 Semi-local example, I (Reduced barking)

To construct a "semi-local" model of a degeneration, we require some preparation. Let N be a line bundle on a smooth compact complex curve C. We take local trivializations $U_\alpha \times \mathbb{C}$ of N with coordinates $(z_\alpha, \zeta_\alpha) \in U_\alpha \times \mathbb{C}$, where $C = \bigcup_\alpha U_\alpha$ is an open covering. Let $\{g_{\alpha\beta}\}$ be transition functions of N, and so $\zeta_\alpha = g_{\alpha\beta}\zeta_\beta$.

Assume that k is a positive integer and $\lambda = \{\lambda_\alpha\}$ is a meromorphic section of the line bundle $N^{\otimes(-k)}$. Then $\lambda_\alpha = g_{\alpha\beta}^{-k}\lambda_\beta$. Multiplying this equation and $\zeta_\alpha^k = g_{\alpha\beta}^k \zeta_\beta^k$ together, we have $\lambda_\alpha\zeta_\alpha^k = \lambda_\beta\zeta_\beta^k$ which implies that the set $\{\lambda_\alpha\zeta_\alpha^k\}$ determines a meromorphic function on N. Thus we obtain the following.

Lemma 3.1.1 *Assume that $\lambda = \{\lambda_\alpha\}$ is a meromorphic section of the line bundle $N^{\otimes(-k)}$. Then $\{\lambda_\alpha\zeta_\alpha^k\}$ defines a meromorphic function on N. Moreover this function is holomorphic precisely when λ is holomorphic.* (Throughout, we simply denote this function by $\lambda\zeta^k$, or $\lambda(z)\zeta^k$.)

Let $\sum_{i=1}^h m_i p_i$ be an effective divisor on C, where $p_i \in C$ and m_i is a positive integer. Suppose that m is a positive integer, and N is a line bundle on C such that

$$N^{\otimes m} \cong \mathcal{O}_C(-\sum_{i=1}^h m_i p_i).$$

This condition is equivalent to the existence of a holomorphic section σ of $N^{\otimes(-m)}$ satisfying $\mathrm{div}(\sigma) = \sum_{i=1}^h m_i p_i$ (that is, σ has a zero of order m_i at each point p_i). Then by Lemma 3.1.1, we may define a holomorphic function on N by $\pi(z,\zeta) = \sigma(z)\zeta^m$. We say that $\pi : N \to \Delta$ is a *semi-local model of a degeneration*, where for consistency we denote $\mathbb{C}$ by Δ. Let C_i ($\cong \mathbb{C}$) be the fiber of the line bundle N over the point p_i, and then the singular fiber X is

$\pi^{-1}(0) = mC + \sum_{i=1}^{h} m_i C_i$. (In a terminology used later, the smooth complex curve C is the *core* of the singular fiber X.) On the other hand, $\pi^{-1}(s)$ $(s \neq 0)$ is a smooth curve. We note that all fibers of $\pi : N \to \Delta$ are non-compact.

Now we proceed to construct a deformation of the degeneration $\pi : N \to \Delta$. For this purpose, it is convenient to express N as the graph of π, which is a smooth hypersurface M in $N \times \Delta$ given by

$$M = \{(z, \zeta, s) \in N \times \Delta \,:\, \sigma(z)\zeta^m - s = 0\}.$$

Of course, N is canonically isomorphic to M by $(z, \zeta) \mapsto (z, \zeta, \sigma(z)\zeta^m)$. From now on, instead of N, we rather consider M, and then under the above isomorphism, π is given by the natural projection $(z, \zeta, s) \mapsto s$. Now we shall construct a deformation of $\pi : M \to \Delta$ under the following analytic assumption on the line bundle N:

Assumption 3.1.2 *For some integer n satisfying $0 < n < m$, the line bundle $N^{\otimes n}$ has a meromorphic section τ which has a pole of order at most m_i — possibly holomorphic — at p_i $(i = 1, 2, \ldots, h)$, and is holomorphic on $C \setminus \{p_1, p_2, \ldots, p_h\}$.*

Since σ (resp. τ) is a section of $N^{\otimes(-m)}$ (resp. $N^{\otimes n}$), the product $\sigma\tau$ is a section of $N^{\otimes(n-m)}$. We note that $\sigma\tau$ is a *holomorphic* section. Indeed, $\sigma\tau$ is holomorphic outside p_i, and since σ has a zero of order m_i at p_i, and τ has a pole of order *at most* m_i at p_i, the product $\sigma\tau$ is holomorphic also at p_i. Therefore by Lemma 3.1.1, $\sigma(z)\zeta^m$ and $\sigma(z)\tau(z)\zeta^{m-n}$ are holomorphic functions on N. We then define a holomorphic function f on $N \times \Delta \times \Delta^\dagger$ by

$$f(z, \zeta, s, t) = \sigma(z)\zeta^m - s + t\sigma(z)\tau(z)\zeta^{m-n},$$

and define a complex 3-manifold as a smooth hypersurface in $N \times \Delta \times \Delta^\dagger$, given by

$$\mathcal{M} := \{(z, \zeta, s, t) \in N \times \Delta \times \Delta^\dagger \,:\, \sigma(z)\zeta^m - s + t\sigma(z)\tau(z)\zeta^{m-n} = 0\}.$$

Letting $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$, $(z, \zeta, s, t) \mapsto (s, t)$ be the natural projection, we have $\Psi^{-1}(\Delta \times \{0\}) = M$, and $\pi = \Psi|_M$. Thus Ψ is a deformation family of π. We say that $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$ is a *barking deformation family* (or *barking family*) of $\pi : M \to \Delta$.

Remark 3.1.3 Actually, this is a very special case of barking families introduced in later chapters. In the above $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$, the defining equation of $\mathcal{M}$ contains only a single term in t, and $\mathcal{M}$ is embedded in a product space $N \times \Delta \times \Delta^\dagger$. In our later construction, the defining equation of $\mathcal{M}$ may contain higher degree terms in t, and $\mathcal{M}$ may be embedded in a more general complex 4-manifold.

We set $X_{s,t} := \Psi^{-1}(s, t)$, that is, $X_{s,t}$ is a curve defined by the equation

$$\sigma(z)\zeta^m - s + t\sigma(z)\tau(z)\zeta^{m-n} = 0.$$

We shall describe the deformation from $X = X_{0,0}$ to

$$X_{0,t}: \quad \sigma(z)\zeta^m + t\sigma(z)\tau(z)\zeta^{m-n} = 0.$$

Let $q_j \in C$ $(j = 1, 2, \ldots, k)$ be the zeros, if any, of τ, and take a small disk U_j around q_j in C, and then we consider factorizations of $\sigma\zeta^m + t\sigma\tau\zeta^{m-n}$ into holomorphic factors:

$$\sigma\zeta^m + t\sigma\tau\zeta^{m-n} = \begin{cases} \sigma\zeta^{m-n}(\zeta^n + t\tau) & z \in \bigcup_j U_j \\ \sigma\tau\zeta^{m-n}\left(\frac{1}{\tau}\zeta^n + t\right) & z \in C \setminus \{q_j\}. \end{cases}$$

(When τ has no zero, we only consider the second factorization on the whole C.) Accordingly we may write $X_{0,t}$ as a sum of two effective divisors Z_t and Y_t on $M_t \,(\cong N)$, where Z_t and Y_t are defined by

$$Z_t = \begin{cases} \sigma\zeta^{m-n} = 0, & z \in \bigcup_j U_j \\ \sigma\tau\zeta^{m-n} = 0, & z \in C \setminus \{q_j\} \end{cases}, \quad Y_t = \begin{cases} \zeta^n + t\tau = 0, & z \in \bigcup_j U_j \\ \frac{1}{\tau}\zeta^n + t = 0, & z \in C \setminus \{q_j\}. \end{cases}$$

In fact, Z_t is well-defined; the equations $\sigma\zeta^{m-n} = 0$ and $\sigma\tau\zeta^{m-n} = 0$ differ by τ, which is a non-vanishing holomorphic function on the 'overlapping' $\bigcup_j (U_j \setminus q_j)$, and hence these two equations define the same hypersurface on the overlapping. Likewise, Y_t is well-defined.

Although Z_t does not depend on the parameter t, so as to emphasize that Z_t is embedded in M_t, we write Z_t rather than Z; we note

$$Z_t = (m-n)C + \sum_{i=1}^{h}(m_i - n_i)C_i,$$

where C_i is the fiber of the line bundle $N \to C$ over p_i, and n_i $(0 \le n_i \le m_i)$ is the order of the pole of τ at p_i. In particular, $X = mC + \sum_{i=1}^{h} m_i C_i$ is written as a sum of two effective divisors

$$Z_0 = (m-n)C + \sum_{i=1}^{h}(m_i - n_i)C_i \qquad \text{and} \qquad Y_0 = nC + \sum_{i=1}^{h} n_i C_i.$$

In the process of the deformation from X to $X_{0,t}$, we note that $Z_0 \,(= Z_t)$ remains undeformed. On the other hand, Y_0 is barked off from X to become Y_t, which is a 'smoothing' of Y_0 (precisely speaking, Y_t may be singular at a zero q_j of τ). See Figure 3.1.1. We say that Y_t (resp. Z_t) is the *barked* (resp. *unbarked*) *part* of $X_{0,t}$.

The deformation from X to $X_{0,t}$ is locally a hyperbolic, Euclidean, or parabolic barking (see Figures 3.1.1 and 3.1.2). To see this, we shall describe the above deformation around p_i (a pole of τ) and q_j (a zero of τ) in more detail. If necessary, we take new coordinates to assume that $\sigma = z^{m_i}$ and $\tau = 1/z^{n_i}$ around p_i. Then $X_{0,t}$ is locally defined by

$$z^{m-n_i}\zeta^{m-n}(z^{n_i}\zeta^n + t) = 0 \quad \text{around} \quad p_i,$$

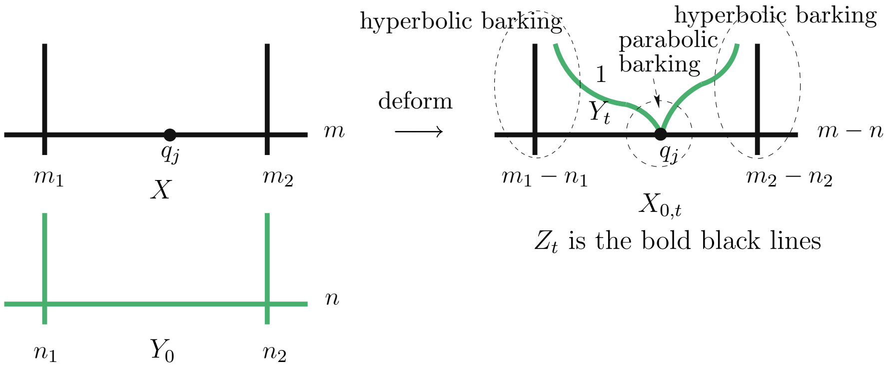

Fig. 3.1.1. Each number stands for the multiplicity.

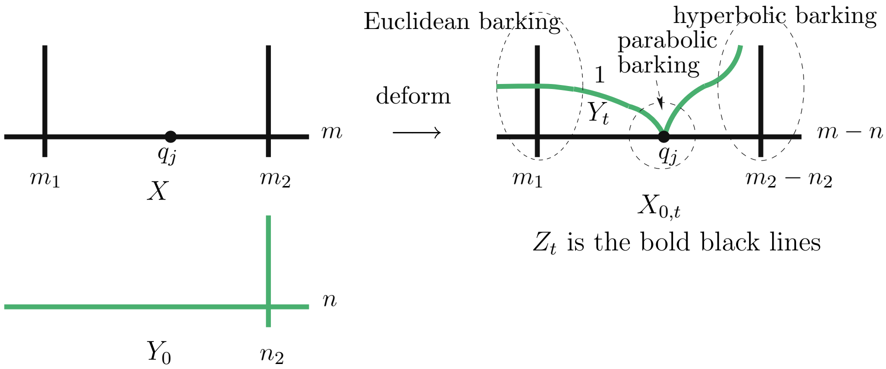

Fig. 3.1.2. Y_0 is a subdivisor of X such that Y_0 does not contain the left vertical line of X. Then Y_t intersects the left vertical line transversely at n points. (The above figure is for the case $n = 1$.)

which is a hyperbolic barking if $n_i > 0$ and a Euclidean barking if $n_i = 0$. Note that $X_{0,t}$ around p_i is a disjoint union of $z^{m-n_i}\zeta^{m-n} = 0$ (the local equation of Z_t) and $z^{n_i}\zeta^n + t = 0$ (the local equation of Y_t). The latter equation defines a (non-compact) smooth complex curve which is a smoothing of a non-reduced curve $z^{n_i}\zeta^n = 0$.

Next, let a_j be the order of the zero of τ at q_j. After a suitable coordinate change, we may assume that $\sigma = 1$ and $\tau = z^{a_j}$ around q_j. Then $X_{0,t}$ is defined by

$$\zeta^{m-n}(\zeta^n + tz^{a_j}) = 0 \quad \text{around} \quad q_j,$$

which is a parabolic barking. Note that $X_{0,t}$ around q_j is a union of (1) $\zeta^{m-n} = 0$ (the local equation of Z_t) and (2) $\zeta^n + tz^{a_j} = 0$ (the local equation of Y_t). See Figure 3.1.1. We also note that for any t, the curve $\zeta^n + tz^{a_j} = 0$ passes through $q_j = (0,0)$; this curve is possibly singular at $(0,0)$, e.g. when $n = 3$ and $a_j = 2$.

For the description of singular fibers $X_{s,t} = \Psi^{-1}(s,t)$ such that $s, t \neq 0$, we refer to Chapter 21, p383.

Example 3.1.4 *Consider a polynomial* $f = (z^n \zeta^{n'})^a - s + t(z^n \zeta^{n'})^{a-1}$ *where* a, n, n' $(a \geq 2)$ *are positive integers. Then a curve* $X_{s,t} : f = 0$ *is singular if and only if* $s = 0$ *or* $s = b^{a-1}t^a/a$*, where we set* $b := (1-a)/a$*. In both cases,* $X_{s,t}$ *is non-reduced.*

Proof. Clearly, $X_{0,t} : (z^n\zeta^{n'})^{a-1}(z^n\zeta^{n'} + t) = 0$ is singular and non-reduced. We consider the case $s \neq 0$. Recall that $(z_0, \zeta_0) \in X_{s,t}$ is a singularity precisely when

$$\begin{cases} \dfrac{\partial f}{\partial z}(z_0,\zeta_0) = na\, z_0^{na-1}\, \zeta_0^{n'a} + t(na-n)\, z_0^{na-n-1}\, \zeta_0^{n'a-n'} = 0 \\ \dfrac{\partial f}{\partial \zeta}(z_0,\zeta_0) = n'a\, z_0^{na}\, \zeta_0^{n'a-1} + t(n'a-n')\, z_0^{na-n}\, \zeta_0^{n'a-n'-1} = 0. \end{cases}$$

These equations are equivalent to a single equation $z_0^n\zeta_0^{n'} = \dfrac{1-a}{a}t$; substituting this into the defining equation $(z^n\zeta^{n'})^a - s + t(z^n\zeta^{n'})^{a-1} = 0$ of $X_{s,t}$, we have

$$\left(\frac{1-a}{a}t\right)^a - s + t\left(\frac{1-a}{a}t\right)^{a-1} = 0,$$

and so

$$s = t^a\left(\frac{1-a}{a}\right)^{a-1}\frac{1}{a}.$$

Thus $X_{s,t}$ $(s \neq 0)$ is singular exactly when $s = b^{a-1}t^a/a$, where we set $b := (1-a)/a$.

We next prove that in this case $(s = b^{a-1}t^a/a)$, the curve $X_{s,t}$ is non-reduced; equivalently we show that when $s = b^{a-1}t^a/a$, the polynomial f has a multiple root. The equation $f = 0$ for the case $s = b^{a-1}t^a/a$ is rewritten equivalently as follows:

$$(z^n\zeta^{n'})^a - \frac{b^{a-1}}{a}t^a + t(z^n\zeta^{n'})^{a-1} = 0 \iff \left(\frac{z^n\zeta^{n'}}{t}\right)^a - \frac{b^{a-1}}{a} + \left(\frac{z^n\zeta^{n'}}{t}\right)^{a-1} = 0$$

$$\iff x^a - \frac{b^{a-1}}{a} + x^{a-1} = 0,$$

where we set $x := z^n\zeta^{n'}/t$. We claim that a polynomial $P(x) := x^a - \dfrac{b^{a-1}}{a} + x^{a-1}$ has a multiple root $x = b$. In fact, it is easy to check that $P(b) = 0$ and also the equation

$$\frac{\partial P}{\partial x}(x) = ax^{a-1} + (a-1)x^{a-2} = 0$$

has a root $x = (1-a)/a = b$. Thus $P(b) = \dfrac{\partial P}{\partial x}(b) = 0$, so that $P(x)$ has a multiple root $x = b$. This implies that f admits a factorization with a multiple factor $(z^n\zeta^{n'} - bt)^k$ for some k $(k \geq 2)$. Therefore $X_{s,t}$ is non-reduced when $s = b^{a-1}t^a/a$. □

For example, if $a = 2$, we have $s = -t^2/4$, and

$$f = z^{2n}\zeta^{2n'} + \frac{t^2}{4} + tz^n\zeta^{n'} = \left(z^n\zeta^{n'} + \frac{t}{2}\right)^2.$$

3.2 Semi-local example, II (Multiple barking)

Let $\pi : N \to \Delta$ be the semi-local model of a degeneration in the above section; so $\pi(z,\zeta) = \sigma(z)\zeta^m$ where σ is a holomorphic section of $N^{\otimes(-m)}$ with a zero of order m_i at $p_i \in C$ $(i = 1, 2, \dots, h)$, and the singular fiber $\pi^{-1}(0) = mC + \sum_{i=1}^h m_i C_i$ where $C_i\,(\cong \mathbb{C})$ is the fiber of the line bundle $N \to C$ over the point p_i; recall that C is a smooth compact complex curve. For consistency with notation of a degeneration we write M for N, and we will generalize barking families of $\pi : M \to \Delta$ to multiple ones. First, instead of Assumption 3.1.2, we pose the following assumption:

Assumption 3.2.1 *For some positive integers l and n satisfying $ln \le m$, the line bundle $N^{\otimes n}$ has a meromorphic section τ which* (1) *has a pole of order n_i at p_i $(i = 1, 2, \dots, h)$ such that $0 \le ln_i \le m_i$ and* (2) *is holomorphic on $C \setminus \{p_1, p_2, \dots, p_h\}$.*

(When $l = 1$, this assumption reduces to Assumption 3.1.2, p42.) We define a complex 3-manifold $\mathcal{M}$ which is a hypersurface in $N \times \Delta \times \Delta^\dagger$, given by

$$\mathcal{M} : \quad \sigma(z)\zeta^m - s + \sum_{k=1}^{l} {}_l\mathrm{C}_k t^k \sigma(z)\tau(z)^k \zeta^{m-kn} = 0.$$

The natural projection $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$, $(z, \zeta, s, t) \mapsto (s, t)$ is a deformation family of $\pi : M \to \Delta$, called a *barking family* of π. When $l = 1$, this construction reduces to that of the previous section.

We shall describe the deformation from X to $X_{0,t} := \Psi^{-1}(0, t)$. If τ has zeros, let $q_j \in C$ $(j = 1, 2, \dots, k)$ be the zeros of τ, and take a small disk $U_j \subset C$ around q_j. The defining equation of $X_{0,t}$ admits factorizations into holomorphic factors:

$$\begin{cases} \sigma\zeta^{m-ln}(\zeta^n + t\tau)^l = 0, & z \in \bigcup_j U_j \\ \sigma\tau^l\zeta^{m-ln}\left(\frac{1}{\tau}\zeta^n + t\right)^l = 0, & z \in C \setminus \{q_j\}. \end{cases}$$

(If τ has no zeros, it suffices to consider the latter factorization on the whole C.) Accordingly we may express $X_{0,t} = Z_t + lY_t$, where Z_t and Y_t are effective divisors in $M_t := \Psi^{-1}(\Delta \times \{t\})$ defined by

$$Z_t = \begin{cases} \sigma\zeta^{m-ln} = 0, & z \in \bigcup_j U_j \\ \sigma\tau^l\zeta^{m-ln} = 0, & z \in C \setminus \{q_j\} \end{cases}, \quad Y_t = \begin{cases} \zeta^n + t\tau = 0, & z \in \bigcup_j U_j \\ \frac{1}{\tau}\zeta^n + t = 0, & z \in C \setminus \{q_j\}. \end{cases}$$

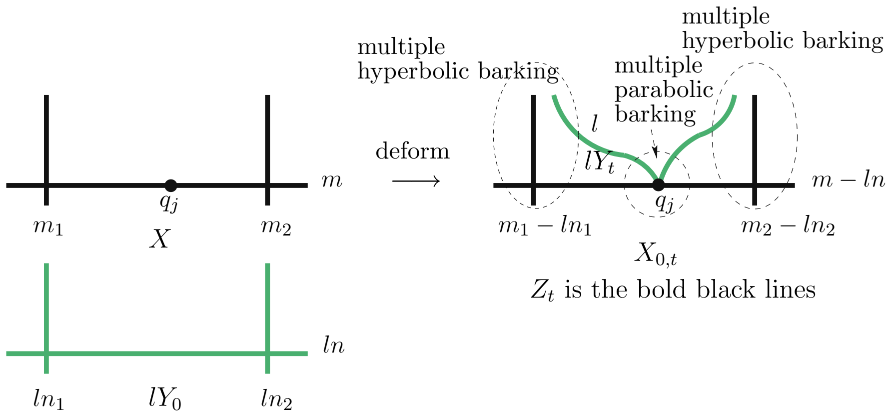

Fig. 3.2.1. lY_0 is a subdivisor of X

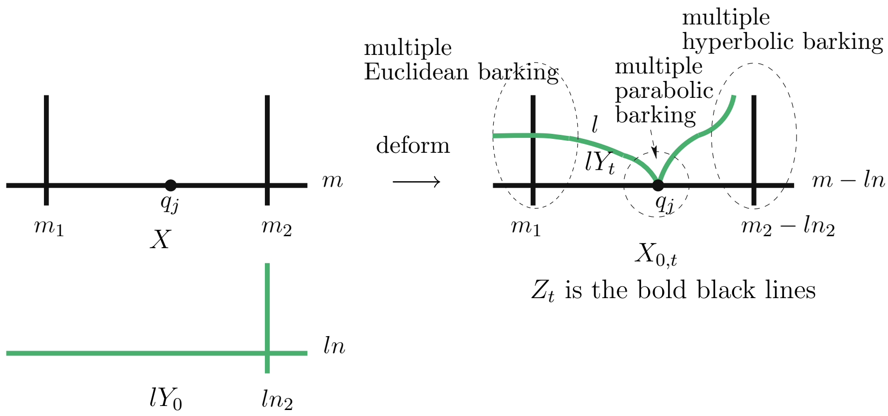

Fig. 3.2.2. lY_0 is a subdivisor of X

In the deformation from X to $X_{0,t}$, clearly $Z_0\,(=Z_t)$ remains undeformed, while lY_0 becomes lY_t (the l-multiple of a curve Y_t). See Figure 3.2.1. We also note the following expressions as divisors:

$$Z_0 = (m - ln)C + \sum_i^h (m_i - ln_i)C_i \qquad \text{and} \qquad Y_0 = nC + \sum_{i=1}^h n_i C_i.$$

The deformation from X to $X_{0,t}$ is locally a multiple hyperbolic, multiple Euclidean, or multiple parabolic barking (see Figures 3.2.1 and 3.2.2). To see this, we shall take a closer look at this deformation around p_i (a pole of τ) and q_j (a zero of τ). First, we take new coordinates so that $\sigma = z^{m_i}$ and $\tau = 1/z^{n_i}$ around p_i. Then $X_{0,t}$ is locally defined by

$$z^{m-ln_i}\zeta^{m-ln}(z^{n_i}\zeta^n + t)^l = 0 \quad \text{around} \quad p_i,$$

being a multiple hyperbolic barking if $n_i > 0$ and a multiple Euclidean barking if $n_i = 0$. Notice that Z_t and Y_t are defined locally around p_i by

$$Z_t: \quad \zeta^{m-ln} = 0 \qquad \text{and} \qquad Y_t: \quad z^{n_i}\zeta^n + t = 0.$$

If $n_i > 0$, then $X_{0,t}$ is locally a *disjoint* union of two curves around p_i, whereas if $n_i = 0$, then Y_t *intersects* Z_t transversely at the point $(z, \zeta) = (0, -t) \in M_t$ (see Figure 3.2.2).

Next we take a coordinate change so that $\sigma = 1$ and $\tau = z^{a_j}$ around q_j, where a_j is the order of the zero of τ at q_j. Then locally $X_{0,t}$ is given by

$$\zeta^{m-n}(\zeta^n + tz^{a_j}) = 0 \quad \text{around} \quad q_j,$$

which is a multiple parabolic barking, and

$$Z_t: \quad \zeta^{m-ln} = 0 \qquad \text{and} \qquad Y_t: \quad \zeta^n + tz^{a_j} = 0.$$

Note that for any t, Y_t always intersects Z_t at one point $q_j = (0, 0)$. From the above discussion, the following statement is clear.

Lemma 3.2.2 *If the meromorphic section τ of $N^{\otimes n}$ in Assumption* 3.2.1 *has no zeros, then $X_{0,t}$ is normal crossing.*

For the description of singular fibers $X_{s,t} = \Psi^{-1}(s, t)$ such that $s, t \neq 0$, we refer to Chapter 21, p383.

3.3 Semi-local example, III

We next explain an important example which is essentially different from examples we gave so far. Let N be a line bundle of degree -2 on the projective line $\mathbb{P}^1$. Take an open covering $\mathbb{P}^1 = U \cup V$ such that $w \in U$ and $z \in V$ satisfy $z = 1/w$. Then N is obtained by patching $(w, \eta) \in U \times \mathbb{C}$ with $(z, \zeta) \in V \times \mathbb{C}$ by $z = 1/w,\ \zeta = w^2\eta$. We define a map $\pi : N \to \Delta$ by

$$\begin{cases} \pi(w, \eta) = w^3\eta^2 & (w, \eta) \in U \times \mathbb{C} \\ \pi(z, \zeta) = z\zeta^2 & (z, \zeta) \in V \times \mathbb{C}, \end{cases}$$

and then the singular fiber $X := \pi^{-1}(0)$ is $2\mathbb{P}^1 + 3C_1 + C_2$ where we think of $\mathbb{P}^1$ to be embedded in N as the zero-section, and C_1 and C_2 are respectively fibers of the line bundle N over the points $w = 0$ and $z = 0$ of $\mathbb{P}^1$.

Now we shall construct a deformation of $\pi : N \to \Delta$ such that $Y = 2\mathbb{P}^1 + 2C_1$ is barked off from X. At first glance, it seems plausible to define such a deformation by

$$\begin{cases} \mathcal{H}: \quad w(w^2\eta^2 + t) - s = 0 \\ \mathcal{H}': \quad \zeta(\zeta^2 + t) - s = 0. \end{cases}$$

However this fails. Indeed, since the defining equation of $\mathcal{H}$ is rewritten as

$$w\left(\frac{1}{w^2}(w^2\eta)^2 + t\right) - s = 0,$$

the transition function $g: \ z = 1/w, \ \zeta = w^2\eta$ of N transforms $\mathcal{H}$ to

$$\frac{1}{z}(z^2\zeta^2 + t) - s = 0,$$

that is, $z\zeta^2 + \frac{t}{z} - s = 0$. This is not the defining equation of $\mathcal{H}'$; actually it cannot define a hypersurface, because this equation contains a fractional term $\frac{t}{z}$.

We make some trick to remedy this situation. Instead of the transition function $z = 1/w, \ \zeta = w^2\eta$, we take a new map (a deformation of the transition function)

$$g: \quad z = \frac{1}{w}, \quad \zeta = w^2\eta + \alpha t w,$$

where $\alpha \in \mathbb{C}$ will be determined in the course of the discussion below. Then g transforms the defining equation of $\mathcal{H}$ to

$$\frac{1}{z}\left[z^2\left(\zeta - \alpha t\frac{1}{z}\right)^2 + t\right] - s = 0,$$

that is, $z\zeta^2 - 2\alpha t\zeta + \dfrac{t + \alpha^2 t^2}{z} - s = 0$; for any value of $\alpha \in \mathbb{C}$, the left hand side necessarily contains a fractional term $\dfrac{t+\alpha^2t^2}{z}$. To remedy this situation, we modify the degree of the parameter t in $\mathcal{H}$; replacing t by t^2, $\mathcal{H}: \ w(w^2\eta^2 + t^2) - s = 0$. Then g transforms $\mathcal{H}$ to

$$z\zeta^2 - 2\alpha t\zeta + \frac{t^2 + \alpha^2 t^2}{z} - s,$$

and taking $\alpha = \sqrt{-1}$, we have $z\zeta^2 - 2\sqrt{-1}t\zeta - s$ (the fractional term vanishes). So we may define a deformation of $\pi : M \to \Delta$ by a triple

$$\begin{cases} \mathcal{H}: & w(w^2\eta^2 + t^2) - s = 0 \\ \mathcal{H}': & z\zeta^2 - 2\sqrt{-1}t\zeta - s = 0 \\ g: & z = \frac{1}{w}, \quad \zeta = w^2\eta + \sqrt{-1}tw. \end{cases} \tag{3.3.1}$$

Note that this deformation is realized *not* in the product space $N \times \Delta \times \Delta^\dagger$, but rather in a complex 4-manifold W, obtained by patching $(w, \eta, s, t) \in U \times \mathbb{C} \times \Delta \times \Delta^\dagger$ with $(z, \zeta, s, t) \in V \times \mathbb{C} \times \Delta \times \Delta^\dagger$ by $g: \ z = 1/w, \ \zeta = w^2\eta + \sqrt{-1}tw$.

Remark 3.3.1 Let N_t be the bundle obtained by patching $U \times \mathbb{C} \times \Delta$ with $V \times \mathbb{C} \times \Delta$ by $z = 1/w, \ \zeta = w^2\eta + \sqrt{-1}tw$. Then N_0 is a line bundle, whereas

N_t $(t \neq 0)$ is merely a $\mathbb{C}$-bundle (a fiber bundle with a fiber $\mathbb{C}$) and not a line bundle. The zero section $\mathbb{P}^1$ in N_0 disappears in N_t; in fact, when $t \neq 0$, two equations $\eta = 0$ and $\zeta = 0$ are *not* compatible under the gluing map $z = 1/w,\ \zeta = w^2\eta + \sqrt{-1}tw$ of N_t.

Next we shall describe the deformation from X to $X_{0,t}$. First we note factorizations

$$\begin{aligned} \mathcal{H}|_{s=0}: &\quad w(w\eta + \sqrt{-1}t)(w\eta - \sqrt{-1}t) = 0 \\ \mathcal{H}'|_{s=0}: &\quad \zeta(z\zeta - 2\sqrt{-1}t) = 0. \end{aligned}$$

Hence $\mathcal{H}|_{s=0}$ consists of three components $w = 0$, $w\eta + \sqrt{-1}t = 0$ and $w\eta - \sqrt{-1}t = 0$, while $\mathcal{H}'|_{s=0}$ consists of two components $\zeta = 0$ and $z\zeta - 2\sqrt{-1}t = 0$.

Claim 3.3.2 *The map* $g:\ z = 1/w,\ \zeta = w^2\eta + \sqrt{-1}tw$ *transforms* $\mathcal{H}|_{s=0}$ *to* $\mathcal{H}'|_{s=0}$ *in such a way that*

$$\begin{aligned} &(1) \quad w(w\eta + \sqrt{-1}t) = 0 \ \textit{to}\ \zeta = 0,\ \textit{and} \\ &(2) \quad w\eta - \sqrt{-1}t = 0 \ \textit{to}\ z\zeta - 2\sqrt{-1}t = 0. \end{aligned}$$

(See Figure 3.3.1, or 'more geometric' Figure 3.3.2).

Proof. (1): Since $w(w\eta + \sqrt{-1}t) = w^2\eta + \sqrt{-1}tw$, the map

$$g:\ z = 1/w, \quad \zeta = w^2\eta + \sqrt{-1}tw$$

transforms $w(w\eta + \sqrt{-1}t)$ to ζ. This confirms (1). Intuitively this seems to contradict that $w(w\eta + \sqrt{-1}t) = 0$ consists of two components while $\zeta = 0$ consists of one component. But this is not the case; see Figure 3.3.2 and observe that when $t \neq 0$, two curves $\eta = 0$ and $\zeta = 0$ are not compatible under g (see also Remark 3.3.1).

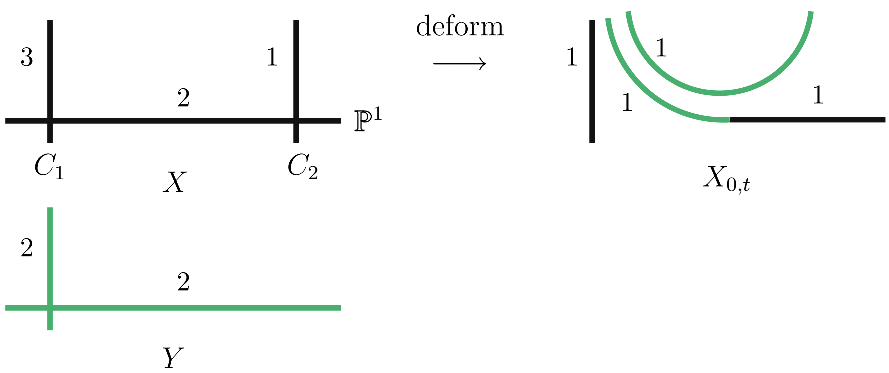

Fig. 3.3.1.

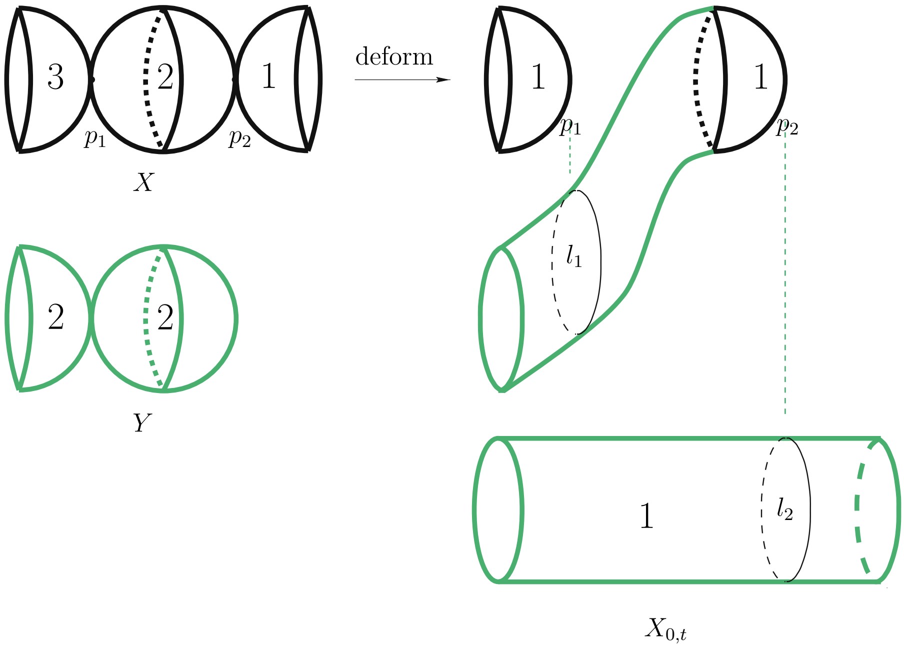

Fig. 3.3.2. We drew Figure 3.3.1 more geometrically; throughout, we will draw this kind of geometric figures to visualize barking deformations. In this figure, any intersection point is a node (i.e. two components intersect transversely at that point.) As $t \to 0$, the loops l_1 and l_2 on $X_{0,t}$ are pinched to the points p_1 and p_2 on X respectively. Namely l_1 and l_2 are *vanishing cycles*.

(2): Since $w\eta - \sqrt{-1}t = \frac{1}{w}(w^2\eta) - \sqrt{-1}t$, the map g transforms $w\eta - \sqrt{-1}t$ to

$$z\left(\zeta - \sqrt{-1}t\frac{1}{z}\right) - \sqrt{-1}t,$$

that is, $z\zeta - 2\sqrt{-1}t = 0$. This confirms (2). □

We remark that the infinitesimal deformation of the deformation (3.3.1) is obtained by taking modulo t^2. However in this process, essential information is lost; the deformation $\mathcal{H}$ contains only terms in t^2, and so $\mathcal{H}$ modulo t^2 is the trivial deformation. This indicates that the usual cohomology theory is not suitable for describing barking deformations. cf. [Ta,IV].

3.4 Supplement: Numerical condition

Suppose that m ($m \geq 2$) is an integer and N is a line bundle on a smooth compact complex curve C such that $N^{\otimes(-m)}$ has a holomorphic section σ

with a zero of order m_i at $p_i \in C$ $(i = 1, 2, \dots, h)$; equivalently $N^{\otimes m} \cong \mathcal{O}_C(-\sum_{i=1}^h m_i p_i)$. The degree of N is given by

$$\deg N = -\frac{m_1 + m_2 + \cdots + m_h}{m}. \tag{3.4.1}$$

We shall investigate the condition for the existence of a meromorphic section τ in Assumption 3.1.2, p42; recall that given integers n $(0 < n < m)$ and n_i $(0 \le n_i \le m_i)$, then τ is a meromorphic section of $N^{\otimes n}$ such that τ has a pole of order n_i at p_i and it is holomorphic outside $p_1, p_2, \dots, p_h$. The existence of τ is equivalent to the existence of a nonnegative divisor D on C satisfying $N^{\otimes n} \cong \mathcal{O}_C(-\sum_{i=1}^h n_i p_i + D)$. In fact, we express $D = \sum_{j=1}^k a_j q_j$ $(a_j \ge 0,\ q_j \in C)$, and then $N^{\otimes n}$ has a meromorphic section τ with a pole of order n_i at p_i and with a zero of order a_j at q_j. We say that $D = \sum_{j=1}^k a_j q_j$ is an *auxiliary divisor* and $q_1, q_2, \dots, q_k \in C$ are *auxiliary points*. Then

$$\deg(N) = -\frac{n_1 + n_2 + \cdots + n_h - \deg D}{n},$$

Since $\deg D = a_1 + a_2 + \cdots + a_k$, together with (3.4.1), we obtain

$$\frac{n_1 + n_2 + \cdots + n_h - (a_1 + a_2 + \cdots + a_k)}{n} = \frac{m_1 + m_2 + \cdots + m_h}{m}.$$

We thus have the following result.

Lemma 3.4.1 *Let N be a line bundle on a smooth compact complex curve C such that $N^{\otimes m} \cong \mathcal{O}_C(-\sum_{i=1}^h m_i p_i)$. If $N^{\otimes n}$ has a meromorphic section τ which has a pole of order n_i at p_i $(i = 1, 2, \dots, h)$ and a zero of order a_j at q_j $(j = 1, 2, \dots, k)$, then*

$$\frac{n_1 + n_2 + \cdots + n_h - (a_1 + a_2 + \cdots + a_k)}{n} = \frac{m_1 + m_2 + \cdots + m_h}{m}. \tag{3.4.2}$$

In particular, $\dfrac{n_1 + n_2 + \cdots + n_h}{n} \ge \dfrac{m_1 + m_2 + \cdots + m_h}{m}$, where the equality holds exactly when τ has no zeros.

Remark 3.4.2 Notice that $\dfrac{n_1 + n_2 + \cdots + n_h}{n}$ is not necessarily an integer, whereas $\dfrac{m_1 + m_2 + \cdots + m_h}{m} = \deg N$ is an integer.

When C is the projective line, the converse of Lemma 3.4.1 is valid.

Proposition 3.4.3 *Let N be a line bundle on the projective line $\mathbb{P}^1$ such that $N^{\otimes m} \cong \mathcal{O}_{\mathbb{P}^1}(-\sum_{i=1}^h m_i p_i)$. Then $N^{\otimes n}$ has a meromorphic section τ which has a pole of order n_i $(0 \le n_i \le m_i)$ at p_i and is holomorphic on $\mathbb{P}^1 \setminus \{p_1, p_2, \dots, p_h\}$ if and only if the following inequality holds:*

$$\frac{n_1 + n_2 + \cdots + n_h}{n} \ge \frac{m_1 + m_2 + \cdots + m_h}{m}. \tag{3.4.3}$$

Proof. $\Longrightarrow$: By Lemma 3.4.1.
$\Longleftarrow$: For simplicity, we set $r := \dfrac{m_1 + m_2 + \cdots + m_h}{m}$. Then r is a positive integer, and from the assumption (3.4.3),

$$n_1 + n_2 + \cdots + n_h - nr \geq 0.$$

We separate into two cases according to whether $n_1 + n_2 + \cdots + n_h - nr$ is positive (Case 1) or zero (Case 2).
Case 1: We write the positive integer $n_1 + n_2 + \cdots + n_h - nr$ as an arbitrary sum of positive integers:

$$n_1 + n_2 + \cdots + n_h - nr = a_1 + a_2 + \cdots + a_k, \qquad (a_j > 0).$$

(e.g. $a_1 = a_2 = \cdots = a_k = 1$, where we set $k := n_1 + n_2 + \cdots + n_h - nr$). Next we take k distinct points $q_1, q_2, \ldots, q_k \in \mathbb{P}^1 \setminus \{p_1, p_2, \ldots, p_h\}$, and we construct a meromorphic section τ of $N^{\otimes n}$ with a pole of order n_i at p_i $(i = 1, 2, \ldots, h)$ and with a zero of order a_j at q_j $(j = 1, 2, \ldots, k)$. First, we take a standard affine covering $C = \mathbb{P}^1 = U \cup V$ with coordinates $w \in U$ and $z \in V$ such that $z = 1/w$ on $U \cap V$. Then N is obtained by identifying $(z, \zeta) \in V \times \mathbb{C}$ with $(w, \eta) \in U \times \mathbb{C}$ by

$$z = \frac{1}{w}, \quad \zeta = w^r \eta, \qquad \text{where} \quad r := \frac{m_1 + m_2 + \cdots + m_h}{m}.$$

We define rational functions τ_U and τ_V on U and V respectively by

$$\tau_U := \frac{\prod_j (1 - q_j w)^{a_j}}{\prod_i (1 - p_i w)^{n_i}}, \qquad \tau_V := \frac{\prod_j (z - q_j)^{a_j}}{\prod_i (z - p_i)^{n_i}}. \tag{3.4.4}$$

Then the transition function $z = 1/w,\ \zeta = w^{nr}\eta$ of $N^{\otimes n}$ transforms τ_U to τ_V, and therefore τ_U and τ_V together define a meromorphic section τ of $N^{\otimes n}$.
Case 2: In Case 1, just take $\tau_U := \dfrac{1}{\prod_i (1 - p_i w)^{n_i}}$ and $\tau_V := \dfrac{1}{\prod_i (z - p_i)^{n_i}}$. □

Remark 3.4.4 As we "move" $q_1, q_2, \ldots, q_k$, or "collide" some or all of them, we may produce a new meromorphic section of $N^{\otimes n}$ (see §4.3.1, p74 for the application to the construction of deformations of degenerations).

3.5 Supplement: Example of computation of discriminant loci

We shall compute the discriminant locus of some family of curves; although this family is not a barking family, this example is heuristic so as to know how a discriminant looks like. For positive integers a, l, n and n' satisfying $a > l$, we consider a polynomial

$$f(z, \zeta, s, t) = (z^n \zeta^{n'})^a - s + \sum_{k=1}^{l} t^k\, c_k\, (z^n \zeta^{n'})^{a-k}, \tag{3.5.1}$$

where $c_1, c_2, \ldots, c_l \in \mathbb{C}$ and $c_l \neq 0$, and let $X_{s,t}$ be a curve defined by $f = 0$ (we think of s, t as parameters). As we will soon see below, in order to study the values of (s,t) such that $X_{s,t}$ is singular, the solutions of a polynomial equation of x:

$$1 + \sum_{k=1}^{l} c_k \frac{a-k}{a} x^k = 0,$$

plays an important role, which has l solutions; since $a > l$, the coefficient of the highest term x^l is nonzero, and so there are l solutions, including multiplicities.

Proposition 3.5.1 *Let $\alpha_1, \alpha_2, \ldots, \alpha_l$ be the solutions of* $1+\sum_{k=1}^{l} c_k \frac{a-k}{a} x^k = 0$, *and set* $\beta_i := \frac{1}{\alpha_i^a} + \sum_{k=1}^{l} \frac{c_k}{\alpha_i^{a-k}}$. *Then*

(A)*$X_{s,t}$ is singular if and only if* (1) $s = 0$ *or* (2) $s = \beta_i t^a$ $(i = 1, 2, \ldots, l)$.
(B)*In both cases* (1) *and* (2), *$X_{s,t}$ is non-reduced, and for the case* (2), *the equation of $X_{s,t}$ has a multiple factor* $(z^n \zeta^{n'} - t/\alpha)^d$ *for some* $d \geq 2$.

Proof. (A): Clearly if $s = 0$, then $X_{0,t}$ is non-reduced (and so singular), because

$$\begin{aligned} f(z,\zeta,0,t) &= (z^n\zeta^{n'})^a + \sum_{k=1}^{l} t^k c_k (z^n\zeta^{n'})^{a-k} \\ &= (z^n\zeta^{n'})^{a-l}\Big[(z^n\zeta^{n'})^l + \sum_{k=1}^{l} t^k c_k (z^n\zeta^{n'})^{l-k}\Big] \end{aligned}$$

has a multiple factor $(z^n\zeta^{n'})^{a-l}$ (note $a > l$ by assumption). Next we assume that $s \neq 0$. Then $(z,\zeta) \in X_{s,t}$ satisfies $z^n\zeta^{n'} \neq 0$ (otherwise, from the equation $f(z,\zeta,s,t) = 0$, we deduce $s = 0$ which gives a contradiction) and so $z, \zeta \neq 0$.

Recall that $(z_0, \zeta_0) \in X_{s,t}$ is singular if and only if

$$\begin{cases} \dfrac{\partial f}{\partial z}(z_0,\zeta_0) = an\, z_0^{an-1}\, \zeta_0^{an'} + \displaystyle\sum_{k=1}^{l} (an-kn)\, c_k\, t^k\, z_0^{an-kn-1}\, \zeta_0^{an'-kn'} = 0 \\ \dfrac{\partial f}{\partial \zeta}(z_0,\zeta_0) = an'\, z_0^{an}\, \zeta_0^{an'-1} + \displaystyle\sum_{k=1}^{l} (an'-kn')\, c_k\, t^k\, z_0^{an-kn}\, \zeta_0^{an'-kn'-1} = 0. \end{cases} \tag{3.5.2}$$

Dividing these equations respectively by $nz_0^{an-1}\zeta_0^{an'}$ and $n'z_0^{an}\zeta_0^{an'-1}$ (note that if $(z,\zeta) \in X_{s,t}$ then $z, \zeta \neq 0$ as we saw above), we see that (3.5.2) is equivalent to a single equation

$$1 + \sum_{k=1}^{l} t^k c_k \frac{a-k}{a} \cdot \frac{1}{z_0^{kn}\zeta_0^{kn'}} = 0.$$

Set $x := t/z_0^n\zeta_0^{n'}$, and write this equation as $1 + \sum_{k=1}^{l} c_k \frac{a-k}{a} x^k = 0$. For a solution α of this equation, from $\alpha = t/z_0^n\zeta_0^{n'}$, we have $z_0^n\zeta_0^{n'} = t/\alpha$. We substitute this into

$$f(z_0,\zeta_0,s,t)=(z_0^n\zeta_0^{n'})^a-s+\sum_{k=1}^{l}c_kt^k(z_0^n\zeta_0^{n'})^{a-k}=0,$$

and then we obtain

$$\left(\frac{t}{\alpha}\right)^a-s+\sum_{k=1}^{l}c_kt^k\left(\frac{t}{\alpha}\right)^{a-k}=0,$$

that is,

$$s=\left(\frac{t}{\alpha}\right)^a+\sum_{k=1}^{l}c_kt^k\left(\frac{t}{\alpha}\right)^{a-k}.$$

Therefore we have $s=\beta t^a$, where we set

$$\beta:=\left(\frac{1}{\alpha^a}+\sum_{k=1}^{l}c_k\frac{1}{\alpha^{a-k}}\right).$$

This shows the validity of (A). Next we show (B). We already proved that $X_{s,t}$ is non-reduced when $s=0$. Thus we consider the case $s=\beta t^a$, for which we shall show that the defining equation of $X_{s,t}$ admits a factorization with a multiple factor $(z^n\zeta^{n'}-t/\alpha)^d$ for some $d\geq 2$, which implies that $X_{s,t}$ is non-reduced. When $s=\beta t^a$ where $\beta=\frac{1}{\alpha^a}+\sum_{k=1}^{l}\frac{c_k}{\alpha^{a-k}}$, the defining equation of $X_{s,t}$

$$(z^n\zeta^{n'})^a-s+\sum_{k=1}^{l}t^kc_k(z^n\zeta^{n'})^{a-k}=0$$

is written as

$$(z^n\zeta^{n'})^a-t^a\left(\frac{1}{\alpha^a}+\sum_{k=1}^{l}\frac{c_k}{\alpha^{a-k}}\right)+\sum_{k=1}^{l}t^kc_k(z^n\zeta^{n'})^{a-k}=0,$$

or

$$(z^n\zeta^{n'})^a+\sum_{k=1}^{l}t^kc_k(z^n\zeta^{n'})^{a-k}-\left(\frac{t}{\alpha}\right)^a-\sum_{k=1}^{l}t^kc_k\left(\frac{t}{\alpha}\right)^{a-k}=0.$$

By Lemma 3.5.2 below, the left hand side has a factorization with a multiple factor of the form $(z^n\zeta^{n'}-t/\alpha)^d$ for some $d\geq 2$. This proves the assertion (B). □

Lemma 3.5.2 *For a complex number* α *satisfying* $1+\sum_{k=1}^{l}c_k\frac{a-k}{a}\alpha^k=0$ *(note* $\alpha\neq 0$*) where* c_k,a,l *are as in* (3.5.1)*, consider a polynomial in* X*:*

$$Q(X)=X^a+\sum_{k=1}^{l}t^kc_kX^{a-k}-\left(\frac{t}{\alpha}\right)^a-\sum_{k=1}^{l}t^kc_k\left(\frac{t}{\alpha}\right)^{a-k}.$$

Then $Q(t/\alpha)=Q'(t/\alpha)=0$*. In particular,* $Q(X)$ *has a multiple root* $X=t/\alpha$*.*

Proof. Clearly,

$$Q\left(\frac{t}{\alpha}\right) = \left(\frac{t}{\alpha}\right)^a + \sum_{k=1}^{l} t^k c_k \left(\frac{t}{\alpha}\right)^{a-k} - \left(\frac{t}{\alpha}\right)^a - \sum_{k=1}^{l} t^k c_k \left(\frac{t}{\alpha}\right)^{a-k} = 0.$$

Next we show that $Q'(t/\alpha) = 0$. Note that

$$Q'(X) = aX^{a-1} + \sum_{k=1}^{l} t^k c_k (a-k) X^{a-k-1}.$$

Dividing the right hand side by aX^{a-1}, we have

$$1 + \sum_{k=1}^{l} t^k c_k \frac{a-k}{a} \left(\frac{1}{X}\right)^k,$$

and thus

$$Q'(X) = 0 \qquad \Longleftrightarrow \qquad 1 + \sum_{k=1}^{l} c_k \frac{a-k}{a} \left(\frac{t}{X}\right)^k = 0.$$

Since α satisfies $1 + \sum_{k=1}^{l} c_k \frac{a-k}{a} \alpha^k = 0$, the equation on the right hand side has a solution $X = t/\alpha$, and so $Q'(t/\alpha) = 0$. This proves the assertion. □

4

Global Barking Deformations: Ideas and Examples

So far, we defined barking deformations only for local and semi-local models of degenerations. In this chapter, we shall explain how they may be generalized to the global case, that is, to degenerations of *compact* complex curves by exhibiting typical and illuminating examples, although the full theory will be postponed to the later chapters. We hope that the reader will grasp the essential part of the theory by these examples. We also intend the reader to get familiar with our figures describing deformations, which will be used throughout this book.

The basic idea is as follows. Given a degeneration $\pi : M \to \Delta$, we identify M with the graph of π, i.e. $\{(x,t) \in M \times \Delta : \pi(x) - s = 0\}$. We then have a natural embedding $M \subset W := M \times \Delta$. Next, we construct a complex 4-manifold $\mathcal{W}$ which is a deformation of W, and realize a certain complex 3-manifold $\mathcal{M}$ as a hypersurface in $\mathcal{W}$. (The construction of $\mathcal{W}$ is a kind of generalization of deformations of Hirzebruch (ruled) surfaces.) Then $\mathcal{M}$ admits a natural projection $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$, which is a barking family of $\pi : M \to \Delta$.

4.1 Preparation: Simplification lemmas

This section is devoted to some technical preparation. Given a hypersurface, it is convenient if we could transform its defining equation into a simpler form by some coordinate change. We will give several conditions, under which this transformation is possible.

Assume that $f = f(z,\zeta)$ and $g = g(z,\zeta)$ are non-vanishing holomorphic functions around $(0,0)$. Let m, m', n, n' and l be nonnegative integers satisfying

$$l \geq 1, \qquad m - ln \geq 0, \qquad m' - ln' \geq 0.$$

When we construct deformations, we often encounter hypersurfaces of the form $F(z,\zeta,s,t) = 0$, where F is a holomorphic function around $(0,0,0,0)$, given by

$$F = z^m \zeta^{m'} g - s + \sum_{k=1}^{l} c_k t^{d_k} z^{m-kn} \zeta^{m'-kn'} g f^k,$$

where $c_k \in \mathbb{C}$ and d_k is a positive integer. Under a coordinate change $\zeta' = \zeta g^{1/m'}$ where we choose some branch $g^{1/m'}$, the function F is written as

$$F = z^m (\zeta')^{m'} - s + \sum_{k=1}^{l} c_k t^{d_k} z^{m-kn} (\zeta')^{m'-kn'} (g^{n'/m'} f)^k.$$

Rewriting $g^{n'/m'} f$ by f, we may assume

$$F = z^m \zeta^{m'} - s + \sum_{k=1}^{l} c_k t^{d_k} z^{m-kn} \zeta^{m'-kn'} f^k.$$

We shall further simplify F under some assumption. The following lemma (and also its variant) is referred to as *Simplification Lemma.*

Lemma 4.1.1 *If $mn' - m'n \neq 0$, then by a coordinate change*

$$z' = z f^{m'/(mn'-m'n)}(z, \zeta), \qquad \zeta' = \zeta f^{-m/(mn'-m'n)}(z, \zeta),$$

a function $F = z^m \zeta^{m'} - s + \sum_{k=1}^{l} c_k t^{d_k} z^{m-kn} \zeta^{m'-kn'} f^k$ is expressed as

$$(z')^m (\zeta')^{m'} - s + \sum_{k=1}^{l} c_k t^{d_k} (z')^{m-kn} (\zeta')^{m'-kn'}. \tag{4.1.1}$$

Proof. Firstly, we shall show that there exist non-vanishing holomorphic functions $u = u(z, \zeta)$ and $v = v(z, \zeta)$ around $(0, 0)$ such that

$$u^m v^{m'} = 1, \tag{4.1.2}$$

$$u^{m-kn} v^{m'-kn'} = f^k, \qquad (k = 1, 2, \dots, l). \tag{4.1.3}$$

Note that u and v satisfying the following equations gives a solution of (4.1.2) and (4.1.3):

$$u^m v^{m'} = 1, \qquad u^n v^{n'} = \frac{1}{f}. \tag{4.1.4}$$

In fact from (4.1.4) we may deduce an equation $\dfrac{u^m v^{m'}}{(u^n v^{n'})^k} = f^k$, which is nothing but (4.1.3); while (4.1.2) is just the equation on the left hand side of (4.1.4).

Since $mn' - m'n \neq 0$ by assumption, the equations (4.1.4) have a solution

$$u(z, \zeta) = f^{m'/(mn'-m'n)}, \qquad v(z, \zeta) = f^{-m/(mn'-m'n)},$$

where we choose some branches. Since u and v, as we saw above, fulfill (4.1.2) and (4.1.3), it follows that in new coordinates $z' = zu(z, \zeta)$, $\zeta' = \zeta v(z, \zeta)$, the function F is expressed as

$$(z')^m(\zeta')^{m'} - s + \sum_{k=1}^{l} c_k t^{d_k}(z')^{m-kn}(\zeta')^{m'-kn'}.$$

This proves the assertion. □

Remark 4.1.2 *Under the same assumption $mn'-m'n \neq 0$, it is also possible to convert $F = z^m\zeta^{m'}g - s + \sum_{k=1}^{l} c_k t^{d_k} z^{m-kn}\zeta^{m'-kn'} g f^k$ directly into* (4.1.1) *by a coordinate change $z' = z\varphi(z,\zeta)$, $\zeta' = \zeta\psi(z,\zeta)$, where*

$$(*) \quad \begin{cases} \varphi(z,\zeta) = f^{m'/(mn'-m'n)} g^{n'/(mn'-m'n)}, \\ \psi(z,\zeta) = f^{-m/(mn'-m'n)} g^{-n/(mn'-m'n)}. \end{cases}$$

Note that φ and ψ satisfy $\varphi^m\psi^{m'} = g$ and $\varphi^n\psi^{n'} = 1/f$, from which we have $\varphi^m\psi^{m'} = g$ and $\varphi^{m-kn}\psi^{m'-kn'} = gf^k$ $(k = 1, 2, \ldots, l)$. This insures the expression (4.1.1) *under the coordinate change* $(*)$.

In Lemma 4.1.1, we consider a special case $c_k = {}_l\mathrm{C}_k$ and $d_k = k$, in which case F admits a 'factorization'

$$F = z^{m-ln}\zeta^{m'-ln'}(z^n\zeta^{n'} + tf)^l - s.$$

As a consequence of Lemma 4.1.1, we also obtain the following.

Corollary 4.1.3 *If $mn' - m'n \neq 0$, then by a coordinate change*

$$z' = zf^{m'/(mn'-m'n)}(z,\zeta), \qquad \zeta' = \zeta f^{-m/(mn'-m'n)}(z,\zeta),$$

the function $F = z^{m-ln}\zeta^{m'-ln'}(z^n\zeta^{n'} + tf)^l - s$ is expressed as

$$(z')^{m-ln}(\zeta')^{m'-ln'}\Big((z')^n(\zeta')^{n'} + t\Big)^l - s.$$

Similarly, we can show

Lemma 4.1.4 *Consider a function $F = z^a\zeta^b - s + t^k z^c\zeta^d f$, where $f = f(z,\zeta)$ is a non-vanishing holomorphic function around $(0,0)$ and a, b, c, d are non-negative integers satisfying $ad - bc \neq 0$. Then by a coordinate change*

$$z' = zf(z,\zeta)^{-b/ad-bc}, \qquad \zeta' = \zeta f(z,\zeta)^{a/ad-bc},$$

the function F is expressed as $(z')^a(\zeta')^b - s + t^k(z')^c(\zeta')^d$.

The next lemma is often useful for the case $ad - bc = 0$.

Lemma 4.1.5 *Consider a function $F = z^a\zeta^b f^m - s + t^k z^c\zeta^d f^n$, where $f = f(z,\zeta)$ is a non-vanishing holomorphic function around $(0,0)$, and a, b, c, d and m, n are nonnegative integers such that* (A) *$an - cm = 0$ and $a \neq 0$, or* (B) *$bn - dm = 0$ and $b \neq 0$. Then after some coordinate change, the function F is expressed as $(z')^a(\zeta')^b - s + t^k(z')^c(\zeta')^d$.*

Proof. For Case (A), take a new coordinate $z' := zf^{m/a}$, and then F is expressed as $(z')^a\zeta^b - s + t^k(z')^c\zeta^d$. Likewise, for Case (B), take a new coordinate $\zeta' := \zeta f^{m/b}$, and then F is expressed as $z^a(\zeta')^b - s + t^k z^c(\zeta')^d$. □

We give an example which are slightly different from the situation in the above results, but we often encounter similar examples in later construction of deformations.

Example 4.1.6 *Consider a hypersurface*

$$z^{m_0}(1-z)^{bm}\zeta^{cm} - s + \sum_k a_k\, t^{d_k}\, z^{m_0-n_{0,k}}\, (1-z)^{bm-bn_k}\, \zeta^{cm-cn_k} = 0,$$

where $m_0, m, n_{0,k}, n_k, b, c$ *are positive integers such that* $m_0 \geq n_{0,k}$ *and* $m \geq n_k$*. Then under a coordinate change* $\zeta' = (1-z)^{b/c}\zeta$*, this equation is expressed as*

$$z^{m_0}\,(\zeta')^{cm} - s + \sum_k a_k\, t^{d_k}\, z^{m_0-n_{0,k}}\, (\zeta')^{cm-cn_k} = 0.$$

We give some comment on the coordinate change of Lemma 4.1.1:

$$z' = z f^{m'/(mn'-m'n)}(z,\zeta), \qquad \zeta' = \zeta f^{-m/(mn'-m'n)}(z,\zeta), \tag{4.1.5}$$

which we used for simplifying a function

$$F = z^m \zeta^{m'} - s + \sum_{k=1}^{l} c_k t^{d_k} z^{m-kn} \zeta^{m'-kn'} f^k,$$

where for brevity, we assume $f(0,0) = 1$; if $f(0,0) = a\,(\neq 0)$, then rewriting $c_k a^k$ by c_k, and f/a by f, we have $f(0,0) = 1$.

We consider such a situation as a positive integer d ($d \geq 2$) divides both m and m'. Then taking an arbitrary d-th root of unity μ, we may choose a branch $f^{-m/(mn'-m'n)}$ so that $f^{-m/(mn'-m'n)}(0,0) = \mu$. In fact, since $f^{-m}(0,0) = 1$ and d divides $mn' - m'n$, we can always choose such a branch. Similarly, we take a branch $f^{m'/(mn'-m'n)}$ so that $f^{m'/(mn'-m'n)}(0,0) = 1$. Then the coordinate change (4.1.5) has the following form:

$$\begin{cases} z' = z\Big(1 + (\text{higher order terms in } z \text{ and } \zeta)\Big), \\ \zeta' = \mu\zeta\Big(1 + (\text{higher order terms in } z \text{ and } \zeta)\Big). \end{cases} \tag{4.1.6}$$

In other words, we can 'twist' the coordinate change ζ' of ζ by an arbitrary d-th root of unity. This fact will be used for 'twisting' a deformation to produce a new deformation.

4.2 Typical examples of barking deformations

In this section, unless otherwise mentioned, $\pi : M \to \Delta$ is a degeneration of elliptic curves such that its singular fiber X is shown in Figure 4.2.1; in Kodaira's notation [Ko1], X is of type IV^*. We express $X = 3\Theta_0 + \mathrm{Br}^{(1)} +$

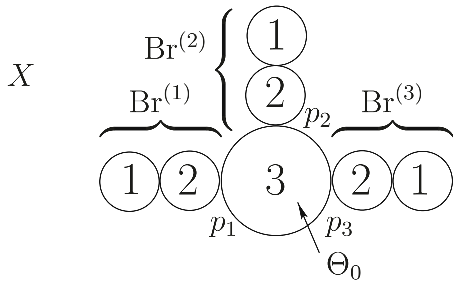

Fig. 4.2.1. Each circle is a projective line, the number stands for its multiplicity, and each intersection point is a *node* (i.e. locally two lines intersect transversely at that point).

$\mathrm{Br}^{(2)} + \mathrm{Br}^{(3)}$ where a projective line Θ_0 is the *core* of X, and $\mathrm{Br}^{(j)}$ is a *branch* of X attached to Θ_0 at a point p_j. A branch is a chain of projective lines, and we write $\mathrm{Br}^{(j)} = 2\Theta_1^{(j)} + \Theta_2^{(j)}$, where $\Theta_1^{(j)}$ and $\Theta_2^{(j)}$ are projective lines. We assume that $p_1 = 1$, $p_2 = 0$ and $p_3 = \infty$.

Now we explicitly construct a *linear degeneration* with the singular fiber X (see §15.1 and [Ta,II] for more detail). First, take an open covering $\Theta_0 = U_0 \cup U_0'$ by two complex lines with coordinates $w_0 \in U_0$ and $z_0 \in U_0'$ such that $z_0 = 1/w_0$ on $U_0 \cap U_0'$. Similarly, we take an open covering $\Theta_i^{(j)} = U_i^{(j)} \cup U_i^{(j)\prime}$ by two complex lines with coordinates $w_i^{(j)} \in U_i^{(j)}$ and $z_i^{(j)} \in U_i^{(j)\prime}$ such that $z_i^{(j)} = 1/w_i^{(j)}$ on $U_i^{(j)} \cap U_i^{(j)\prime}$. For brevity, if it is clear from the context, we often omit sub-/superscripts, such as $z = z_i^{(j)}$.

Patching $U_0 \times \mathbb{C}$ with $U_0' \times \mathbb{C}$ by $z = 1/w$, $\zeta = w^2\eta$ where $(w, \eta) \in U_0 \times \mathbb{C}$ and $(z, \zeta) \in U_0' \times \mathbb{C}$, we then obtain a line bundle N_0 of degree -2 on Θ_0. Similarly, we patch $U_i^{(j)} \times \mathbb{C}$ with $U_i^{(j)\prime} \times \mathbb{C}$ by $z = 1/w$, $\zeta = w^2\eta$, where $(w, \eta) \in U_i^{(j)} \times \mathbb{C}$ and $(z, \zeta) \in U_i^{(j)\prime} \times \mathbb{C}$. Then we obtain a line bundle $N_i^{(j)}$ of degree -2 on $\Theta_i^{(j)}$.

Next, we define complex surfaces H_0 and H_0' as hypersurfaces in $U_0 \times \Delta$ and $U_0' \times \Delta$ respectively, given by

$$\begin{cases} H_0: & w^2(w-1)^2\eta^3 - s = 0 \\ H_0': & z^2(1-z)^2\zeta^3 - s = 0. \end{cases}$$

Lemma 4.2.1 (1) *H_0 and H_0' are smooth.* (2) *The transition function $z = 1/w$, $\zeta = w^2\eta$ of the line bundle N_0 transforms H_0 to H_0'.*

Proof. (1) follows from $\dfrac{\partial H_0}{\partial s} = -1 \neq 0$ and $\dfrac{\partial H_0'}{\partial s} = -1 \neq 0$. We next show (2). Since H_0 is rewritten as

$$\frac{1}{w^4}(w-1)^2(w^2\eta)^3 - s = 0,$$

the transition function $z = 1/w$, $\zeta = w^2\eta$ transforms H_0 to

$$z^4\left(\frac{1}{z}-1\right)^2\zeta^3 - s = 0,$$

that is, to $H_0' : z^2(1-z)^2\zeta^3 - s = 0$. This proves (2). □

Therefore the hypersurfaces H_0 and H_0' together define a smooth complex surface M_0 in $N_0 \times \Delta$. Similarly we define a smooth complex surface $M_1^{(j)}$, which is a hypersurface in $N_1^{(j)} \times \Delta$ determined by two hypersurfaces

$$\begin{cases} H_1^{(j)} : \quad w^3\eta^2 - s = 0 & \text{in } U_1^{(j)} \times \Delta \\ H_1^{(j)\prime} : \quad z\zeta^2 - s = 0 & \text{in } U_1^{(j)\prime} \times \Delta. \end{cases}$$

Also, we define a hypersurface $M_2^{(j)}$ in $N_2^{(j)} \times \Delta$ by two hypersurfaces

$$\begin{cases} H_2^{(j)} : \quad w^2\eta - s = 0 & \text{in } U_2^{(j)} \times \Delta \\ H_2^{(j)\prime} : \quad \zeta - s = 0 & \text{in } U_2^{(j)\prime} \times \Delta. \end{cases}$$

Now we shall patch complex surfaces M_0, $M_1^{(j)}$ and $M_2^{(j)}$ $(j = 1, 2, 3)$; we first note that after some coordinate change, H_0' is written as $z^2\zeta^3 - s = 0$ around p_j. (e.g. take new coordinates $z' = z(1-z)$, $\zeta' = \zeta$, and then $H_0' : z^2(1-z)^2\zeta^3 - s = 0$ becomes $(z')^2(\zeta')^3 - s = 0$.) We then glue M_0 with $M_1^{(j)}$ by $(z, \zeta, s) = (\eta_1^{(j)}, w_1^{(j)}, s)$. Similarly, we glue $M_1^{(j)}$ with $M_2^{(j)}$ by $(z_1^{(j)}, \zeta_1^{(j)}, s) = (\eta_2^{(j)}, w_2^{(j)}, s)$. Then we obtain a smooth complex surface M, and the natural projection $\pi : M \to \Delta$ is a degeneration with the singular fiber X in Figure 4.2.1.

Now we shall provide five representative examples of "barking deformations" of $\pi : M \to \Delta$. Roughly speaking, the construction is carried out in two steps: (1) Construct a deformation around the core Θ_0, and (2) propagate the deformation in (1) along branches $\mathrm{Br}^{(1)}$, $\mathrm{Br}^{(2)}$, $\mathrm{Br}^{(3)}$.

Example 4.2.2 (Reduced barking 1)

We begin with an example such that a subdivisor Y of X is barked off from X as shown in Figure 4.2.2 (actually the figure of $X_{0,t}$ is too simplified; the correct one is shown in Figure 4.2.3).

First, we construct a deformation around the core Θ_0. Consider two smooth hypersurfaces, respectively given by

$$\begin{cases} \mathcal{H}_0 : \quad w(w-1)\eta^2\Big[w(w-1)\eta + t\Big] - s = 0 & \text{in } U_0 \times \Delta \times \Delta^\dagger, \\ \mathcal{H}_0' : \quad z^2(1-z)\zeta^2\Big[(1-z)\zeta + t\Big] - s = 0 & \text{in } U_0' \times \Delta \times \Delta^\dagger. \end{cases} \qquad (4.2.1)$$

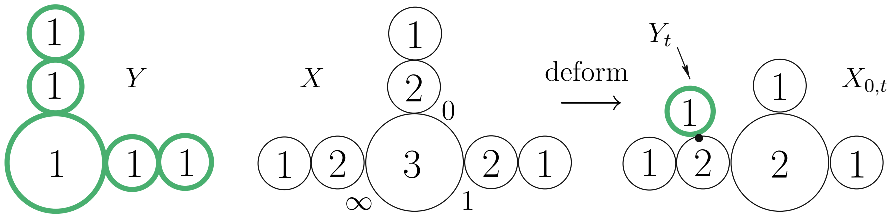

Fig. 4.2.2. Y is a subdivisor of X. The singular fiber $X_{0,t} := \Psi^{-1}(0,t)$ is called of *type* I_1^*.

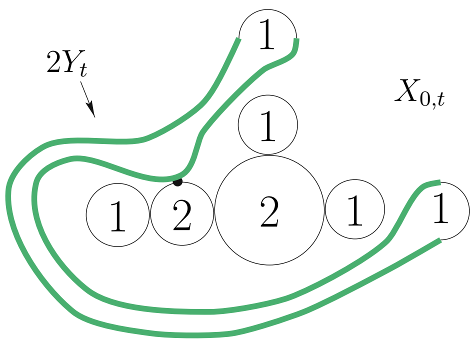

Fig. 4.2.3. A more geometrically precise figure of $X_{0,t}$ in Figure 4.2.2; see also Remark 4.2.3.

(Note that $\mathcal{H}_0|_{t=0} = H_0$ and $\mathcal{H}'_0|_{t=0} = H'_0$.) Since the equation of $\mathcal{H}_0$ is written as

$$\frac{1}{w^2}\left(1-\frac{1}{w}\right)(w^2\eta)^2\left[\left(1-\frac{1}{w}\right)(w^2\eta)+t\right]-s=0,$$

the transition function $g_0: \ z=1/w,\ \zeta=w^2\eta$ of N_0 transforms $\mathcal{H}_0$ to $\mathcal{H}'_0$, and so the hypersurfaces $\mathcal{H}_0$ and $\mathcal{H}'_0$ together determine a complex 3-manifold $\mathcal{M}_0$ in $N_0\times\Delta\times\Delta^\dagger$. We think of $\mathcal{M}_0$ as a deformation of M_0, parameterized by $\Delta\times\Delta^\dagger$.

Next, we shall 'propagate' the deformation $\mathcal{M}_0$ to a deformation around each branch of X; this process is referred to as *propagation along branches*.

Step 1: Propagation along $\mathrm{Br}^{(1)}$ By a coordinate change $z'=z(1-z)^{-1/2}$ and $\zeta'=\zeta(1-z)$, the hypersurface $\mathcal{H}'_0$ is written as $(z')^2(\zeta')^2(\zeta'+t)-s=0$ around $p_1=1\in\Theta_0$. (Throughout this section, we often change coordinates so as to make equations into simpler forms. See §4.1, p57 for the systematic account of choices of new coordinates.) We define two smooth hypersurfaces, respectively given by

$$\begin{cases} \mathcal{H}_1^{(1)}: \quad w^2\eta^2(w+t)-s=0 & \text{in } U_1^{(1)}\times\Delta\times\Delta^\dagger \\ \mathcal{H}_1^{(1)\prime}: \quad z\zeta^2(1+tz)-s=0 & \text{in } U_1^{(1)\prime}\times\Delta\times\Delta^\dagger. \end{cases} \tag{4.2.2}$$

(Note: $\mathcal{H}_1^{(1)'}$ is the *trivial family* of $H_1^{(1)'}$. Indeed, in new coordinates $z' = z(1+tz)$, $\mathcal{H}_1^{(1)'}$ is $z'\zeta^2 - s = 0$, independent of t.) Here $\mathcal{H}_1^{(1)}$ is equal to $w^3\eta^2 + tw^2\eta^2 - s = 0$, and since

$$w^3\eta^2 + tw^2\eta^2 - s = \frac{1}{w}(w^2\eta)^2 + t\frac{1}{w^2}(w^2\eta)^2 - s,$$

the transition function $g_1^{(1)} : z = 1/w,\ \zeta = w^2$ of $N_1^{(1)}$ transforms $\mathcal{H}_1^{(1)}$ to $\mathcal{H}_1^{(1)'}$. Thus $\mathcal{H}_1^{(1)}$ and $\mathcal{H}_1^{(1)'}$ together determine a complex 3-manifold $\mathcal{M}_1^{(1)}$ in $N_1^{(1)} \times \Delta \times \Delta^\dagger$. Similarly, we define a complex 3-manifold $\mathcal{M}_2^{(1)}$ in $N_2^{(1)} \times \Delta \times \Delta^\dagger$ by

$$\begin{cases} \mathcal{H}_2^{(1)} : & w^2\eta(1+t\eta) - s = 0, \\ \mathcal{H}_2^{(1)'} : & \zeta(1+tz^2\zeta) - s = 0. \end{cases}$$

(Note: $\mathcal{H}_2^{(1)}$ and $\mathcal{H}_2^{(1)'}$ are respectively the trivial families of $H_2^{(1)}$ and $H_2^{(1)'}$.) We glue $\mathcal{M}_0$ with $\mathcal{M}_1^{(1)}$ by plumbing $(z_0, \zeta_0, s, t) = (\eta_1^{(1)}, w_1^{(1)}, s, t)$. Likewise, we glue $\mathcal{M}_1^{(1)}$ with $\mathcal{M}_2^{(1)}$ by plumbing $(z_1^{(1)}, \zeta_1^{(1)}, s, t) = (\eta_2^{(1)}, w_2^{(1)}, s, t)$. We then obtain a complex 3-manifold resulting from a propagation of the deformation $\mathcal{M}_0$ along the branch $\mathrm{Br}^{(1)}$.

Step 2: Propagation along $\mathrm{Br}^{(2)}$ By Simplification Lemma (Lemma 4.1.1), after some coordinate change, $\mathcal{H}_0$ in (4.2.1) is written as $w\eta^2(w\eta + t) - s = 0$ around $p_2 = 0 \in \Theta_0$. Here we slightly change the notation. For consistency with the discussion in Step 1, we write $\mathcal{H}_0$ in (z, ζ)-coordinates, i.e. $\mathcal{H}_0 :\ z\zeta^2(z\zeta + t) - s = 0$ around p_2. We then define a complex 3-manifold $\mathcal{M}_1^{(2)}$ in $N_1^{(2)} \times \Delta \times \Delta^\dagger$ by

$$\begin{cases} \mathcal{H}_1^{(2)} : & w^2\eta(w\eta + t) - s = 0, \\ \mathcal{H}_1^{(2)'} : & \zeta(z\zeta + t) - s = 0. \end{cases}$$

To propagate this to a deformation around $\Theta_2^{(2)}$, we first define a deformation $\mathcal{N}_2^{(2)}$ of the line bundle $N_2^{(2)}$; it is a complex 3-manifold obtained by patching $U_2^{(2)} \times \Delta^\dagger$ with $U_2^{(2)'} \times \Delta^\dagger$ by $z = 1/w$, $\zeta = w^2\eta + tw$. Then the following two smooth hypersurfaces together determine a smooth hypersurface in $\mathcal{N}_2^{(2)} \times \Delta^\dagger$:

$$\begin{cases} \mathcal{H}_2^{(2)} : & w(w\eta + t) - s = 0 & \text{in } U_2^{(2)} \times \Delta^\dagger, \\ \mathcal{H}_2^{(2)'} : & \zeta - s = 0 & \text{in } U_2^{(2)'} \times \Delta^\dagger. \end{cases}$$

In fact, $z = 1/w$, $\zeta = w^2\eta + tw$ transforms $\mathcal{H}_2^{(2)}$ to $\mathcal{H}_2^{(2)'}$, and thus $\mathcal{H}_2^{(2)}$ and $\mathcal{H}_2^{(2)'}$ together determine a complex 3-manifold $\mathcal{M}_2^{(2)}$ in $\mathcal{N}_2^{(2)} \times \Delta$.

We glue $\mathcal{M}_0$ with $\mathcal{M}_1^{(2)}$ by plumbing $(z_0, \zeta_0, s, t) = (\eta_1^{(2)}, w_1^{(2)}, s, t)$. Likewise, we glue $\mathcal{M}_1^{(2)}$ with $\mathcal{M}_2^{(2)}$ by plumbing $(z_1^{(2)}, \zeta_1^{(2)}, s, t) = (\eta_2^{(2)}, w_2^{(2)}, s, t)$, yielding a complex 3-manifold which results from a propagation of the deformation $\mathcal{M}_0$ along the branch $\mathrm{Br}^{(2)}$.

Step 3: Propagation along $\mathrm{Br}^{(3)}$ A propagation of $\mathcal{M}_0$ along the branch $\mathrm{Br}^{(3)}$ is carried out in the same way as the above propagation along $\mathrm{Br}^{(2)}$; we leave this as an exercise for the reader.

After the above three steps, we obtain a complex 3-manifold $\mathcal{M}$. The natural projection $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$ is a deformation family of $\pi : M \to \Delta$ such that the deformation from X to $X_{0,t} := \Psi^{-1}(0, t)$ is shown in Figure 4.2.2 (see also Remark 4.2.3 below). We say that $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$ is a *barking family* of $\pi : M \to \Delta$ which *barks* Y from X. The subdivisor Y is called a *crust* of X.

We give several comments on the above construction.

(1) *The barking family* $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$ *is a splitting family which splits* X *into* $X_{0,t}$ *and a projective line with one node.* In fact, suppose that the singular fiber X (type IV^*) splits into $X_1, X_2, \ldots, X_n$ where $X_1 = X_{0,t}$ (type I_1^*). From $\chi(M) = \chi(M_t)$ where $\chi(M)$ stands for the (topological) Euler characteristic of M, we may deduce $\chi(X) = \chi(X_1) + \chi(X_2) + \cdots + \chi(X_n)$ (Lemma 20.2.1, p350). Since $\chi(X) = 8$ and $\chi(X_1) = 7$, we conclude that $n = 2$ and $\chi(X_2) = 1$; here $\chi(X_2) = 1$ implies that X_2 is a projective line with one node.

(2) Let N_t be a fiber bundle on a projective line Θ, obtained by patching $(w, \eta) \in U \times \Delta$ with $(z, \zeta) \in V \times \Delta$ by $g_t :\ z = 1/w,\ \zeta = w^r \eta + tw$. When $t = 0$, N_0 is a line bundle of degree $-r$ on Θ, and the projective line Θ is embedded in N_0 as the zero-section. On the other hand, N_t for $t \neq 0$ is no longer a line bundle, and "zero-section" does not make sense because the equations $\zeta = 0$ and $\eta = 0$ are not compatible with the gluing map g_t. Geometrically speaking, under the deformation from N_0 to N_t, the zero-section Θ in N_0 *disappears* (Remark 3.3.1).

(3) In Step 2 of the above construction (propagation along $\mathrm{Br}^{(2)}$), the deformation $\mathcal{M}_2^{(2)}$ is realized not in the trivial product space $N_2^{(2)} \times \Delta \times \Delta^\dagger$ but in $\mathcal{N}_2^{(2)} \times \Delta^\dagger$, where $\mathcal{N}_2^{(2)}$ is a deformation of the line bundle $N_2^{(2)}$. Compare this with the propagation along $\mathrm{Br}^{(1)}$ in which case the deformation is realized in the trivial product. This difference is explained in terms of a notion of "formal self-intersection number". Given an irreducible component Θ_i of Y, we set $(\Theta_i \cdot \Theta_i)_Y := -\dfrac{\sum_j n_j}{n_i}$, where the summation runs over all j such that Θ_j intersects Θ_i. We say that $(\Theta_i \cdot \Theta_i)_Y$ is the *formal self-intersection number* of Θ_i in Y; the 'usual' self-intersection number is given by $\Theta_i \cdot \Theta_i = -\dfrac{\sum_j m_j}{m_i}$. As long as $(\Theta_i \cdot \Theta_i)_Y \leq \Theta_i \cdot \Theta_i$, we do not need to deform the line bundle N_i when we propagate a deformation. However if $(\Theta_i \cdot \Theta_i)_Y > \Theta_i \cdot \Theta_i$, we must deform N_i; in Step 2 of the above construction, $(\Theta_2^{(2)} \cdot \Theta_2^{(2)})_Y = -1 > \Theta_2^{(2)} \cdot \Theta_2^{(2)} = -2$, and in order to propagate a deformation, we require to deform the line bundle $N_2^{(2)}$.

Remark 4.2.3 We briefly explain how to draw Figure 4.2.3; the fiber $X_{0,t} = \Psi^{-1}(0,t)$ is described in the following procedure: (1) Put $s=0$ in the equations of $\mathcal{H}_0, \mathcal{H}'_0$, and $\mathcal{H}_i^{(j)}$, and then (2) draw a figure according to the 'factorizations' of the equations in (1). For instance, since

$$\begin{cases} \mathcal{H}_0|_{s=0}: \quad w(w-1)\eta^2\Big[w(w-1)\eta + t\Big] = 0, \\ \mathcal{H}'_0|_{s=0}: \quad z^2(1-z)\zeta^2\Big[(1-z)\zeta + t\Big] = 0, \end{cases}$$

under this deformation, a singular curve $w(w-1)\eta = 0$ (resp. $(1-z)\zeta = 0$) is deformed (barked off) to be a smooth curve $w(w-1)\eta + t = 0$ (resp. $(1-z)\zeta + t = 0$). Further, for $\mathcal{H}_1^{(1)}|_{s=0}: \ w^2\eta^2(w+t) = 0$ in (4.2.2), a curve $w^2\eta^2 = 0$ remains undeformed, whereas a curve $w = 0$ is barked off to become a curve $w + t = 0$ which intersects the curve $w^2\eta^2 = 0$ transversely at one point $(w,\eta) = (-t, 0)$.

Example 4.2.4 (Reduced barking 2)

In the above example, $(\Theta_0 \cdot \Theta_0)_Y = \Theta_0 \cdot \Theta_0 = -2$ where $\Theta_0 \cdot \Theta_0$ is the self-intersection number of Θ_0, while $(\Theta_0 \cdot \Theta_0)_Y$ is the formal one of Θ_0 in Y. Next we provide an example such that $(\Theta_0 \cdot \Theta_0)_Y < \Theta_0 \cdot \Theta_0$. (cf. the case $(\Theta_0 \cdot \Theta_0)_Y \geq \Theta_0 \cdot \Theta_0$; then we need to deform the line bundle N_0 so as to construct a barking deformation.) Taking Y as in Figure 4.2.4, then we have $(\Theta_0 \cdot \Theta_0)_Y = -3$ and $\Theta_0 \cdot \Theta_0 = -2$, and so $(\Theta_0 \cdot \Theta_0)_Y < \Theta_0 \cdot \Theta_0$. Therefore we do *not* need to deform N_0, and the construction of a barking deformation is almost the same as the above example. However there is one important difference; as $\Theta_0 \cdot \Theta_0 \neq (\Theta_0 \cdot \Theta_0)_Y$, we are not able to take an 'obvious' deformation around Θ_0. Here by an 'obvious' deformation, we mean a deformation defined by

$$\begin{cases} \mathcal{H}_0: \quad w(w-1)\eta^2\Big[w(w-1)\eta + t\Big] - s = 0, \\ \mathcal{H}'_0: \quad z(1-z)\zeta^2\Big[z(1-z)\zeta + t\Big] - s = 0. \end{cases}$$

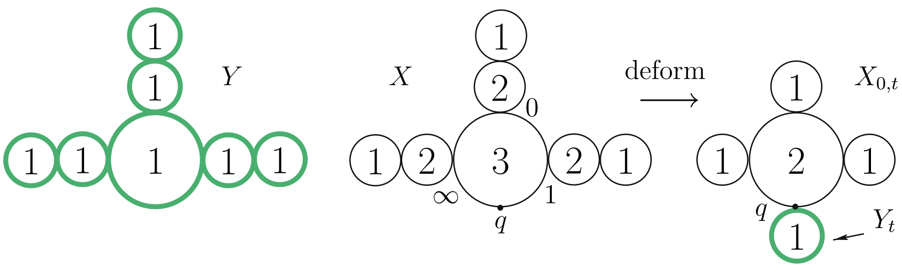

Fig. 4.2.4. Y is a subdivisor of X. See Figure 4.2.5 for a more geometrically precise figure of $X_{0,t} := \Psi^{-1}(0,t)$.

Unfortunately, such $\mathcal{H}_0$ and $\mathcal{H}'_0$ do *not* determine a hypersurface in $N_0 \times \Delta \times \Delta^\dagger$, because $\mathcal{H}_0$ is not compatible with $\mathcal{H}'_0$ under the transition function of N_0. Indeed, writing the equation of $\mathcal{H}_0$ as

$$\frac{1}{w^2}\left(1-\frac{1}{w}\right)(w^2\eta)^2\left[\left(1-\frac{1}{w}\right)(w^2\eta)+t\right]-s=0,$$

we see that the transition function $g_0: \ z = 1/w, \ \zeta = w^2\eta$ transforms $\mathcal{H}_0$ to

$$z^2(1-z)\zeta^2\Big[(1-z)\zeta+t\Big]-s=0,$$

which is *not* $\mathcal{H}'_0$. This phenomenon is due to $(\Theta_0\cdot\Theta_0)_Y < \Theta_0\cdot\Theta_0$. To remedy this situation, we add some additional terms for $\mathcal{H}_0$ and $\mathcal{H}'_0$; taking $q \in \mathbb{C}$ (an *auxiliary point*), we set

$$\begin{cases} \mathcal{H}_0: \quad w(w-1)\eta^2\Big[w(w-1)\eta+t(w-q)\Big]-s=0, \\ \mathcal{H}'_0: \quad z(1-z)\zeta^2\Big[z(1-z)\zeta+t(1-qz)\Big]-s=0. \end{cases}$$

It is easy to check that the transition function g_0 of N_0 transforms $\mathcal{H}_0$ to $\mathcal{H}'_0$, and so $\mathcal{H}_0$ and $\mathcal{H}'_0$ together determine a complex 3-manifold $\mathcal{M}_0$ in $N_0 \times \Delta \times \Delta^\dagger$. The propagation of $\mathcal{M}_0$ along each branch of the singular fiber X is carried out in the same way as Step 2 in Example 4.2.2. We then obtain a barking family $\Psi: \mathcal{M} \to \Delta \times \Delta^\dagger$ such that in the process of the deformation from X to $X_{0,t}$, the subdivisor Y is barked from X to become a smooth projective line Y_t as shown in Figure 4.2.4. We remark that in Example 4.2.2, Y_t does *not* intersect Θ_0, whereas in the present example, Y_t intersects Θ_0; regardless of the value of t, $\mathcal{H}_0|_{s=0}$ passes through a point $(w,\eta) = (q,0)$ on Θ_0. (hence $\mathcal{H}'_0|_{s=0}$ passes through a point $(z,\zeta) = (\frac{1}{q},0)$ on Θ_0.) *This difference arises from* $(\Theta_0\cdot\Theta_0)_Y = \Theta_0\cdot\Theta_0$ *in Example* 4.2.2*, whereas* $(\Theta_0\cdot\Theta_0)_Y < \Theta_0\cdot\Theta_0$ *in the present example.*

We point out that the barking family of this example is interesting in that it does *not* arise from any deformation of the E_6-singularity. To elucidate this, we first note that from the singular fiber X of type IV^* (Figure 4.2.2), we delete one irreducible component at the edge of any branch to obtain an exceptional set of the E_6-singularity. Namely, a divisor $X' := X \setminus \Theta_3^{(j)}$, where j is arbitrary ($j = 1, 2$ or 3), is the exceptional set of the E_6-singularity. By Riemenschneider's Theorem [Ri3], any deformation of the resolution space R of the E_6-singularity can be simultaneously blown down to a deformation of the E_6-singularity, and conversely by Brieskorn's Theorem [Br1], any deformation of the E_6-singularity admits a simultaneous resolution. Hence there is a correspondence between deformations of the E_6-singularity and deformations of its resolution space. Taking this fact into account, it seems plausible that from a deformation of the E_6-singularity, we can recover the barking deformation of the above example. However this is false; in the above barking deformation, all of three

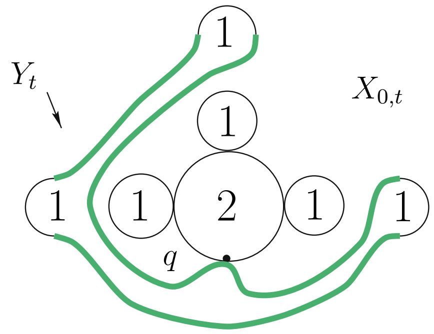

Fig. 4.2.5. A more geometrically precise figure of $X_{0,t}$ in Figure 4.2.4

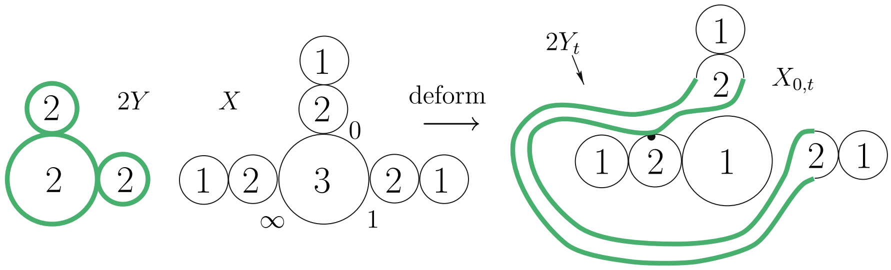

Fig. 4.2.6.

irreducible components $\Theta_3^{(1)}, \Theta_3^{(2)}, \Theta_3^{(3)}$ disappear (Figure 4.2.5 and Remark 3.3.1), and so this deformation can *not* arise from a deformation of the E_6-singularity.

Example 4.2.5 (Multiple barking)

Thus far, we only gave examples of barking families such that a subdivisor Y of X becomes a *reduced* irreducible component (i.e. of multiplicity 1) of $X_{0,t}$. Next we provide a barking family such that a subdivisor of X becomes a *non-reduced* irreducible component (i.e. of multiplicity at least 2) of $X_{0,t}$ (Figure 4.2.6).

First, we construct a deformation around the core Θ_0. Define two smooth hypersurfaces by

$$\begin{cases} \mathcal{H}_0 : \quad \eta\Big[w(w-1)\eta + t\Big]^2 - s = 0 \qquad \text{in } U_0 \times \mathbb{C} \times \Delta \times \Delta^\dagger \\ \mathcal{H}_0' : \quad z^2\zeta\Big[(1-z)\zeta + t\Big]^2 - s = 0 \qquad \text{in } U_0' \times \mathbb{C} \times \Delta \times \Delta^\dagger. \end{cases}$$

Since $\mathcal{H}_0$ is written as

$$\frac{1}{w^2}(w^2\eta)\left[\left(1-\frac{1}{w}\right)(w^2\eta)+t\right]^2-s=0,$$

the transition function $g_0: \ z=1/w, \ \zeta=w^2\eta$ of N_0 transforms $\mathcal{H}_0$ to $\mathcal{H}_0'$, and thus they together define a complex 3-manifold $\mathcal{M}_0$ in $N_0\times\Delta\times\Delta^\dagger$.

We then propagate $\mathcal{M}_0$ along each branch of the singular fiber X.

Step 1: Propagation along $\mathrm{Br}^{(1)}$ By a coordinate change $z'=z(1-z)^{-1/2}$ and $\zeta'=\zeta(1-z)$, the hypersurface $\mathcal{H}_0'$ is written as $z^2\zeta(\zeta+t)^2-s=0$ around $p_1=1\in\Theta_0$. We then define a complex 3-manifold $\mathcal{M}_1^{(1)}$ in $N_1^{(1)}\times\Delta\times\Delta^\dagger$ by

$$\begin{cases} \mathcal{H}_1^{(1)}: & w\eta^2(w+t)^2-s=0, \\ \mathcal{H}_1^{(1)\prime}: & z\zeta^2(1+tz)^2-s=0. \end{cases}$$

Note that $\mathcal{H}_1^{(1)}|_{s=0}$ consists of $w\eta^2=0$ and $(w+t)^2=0$, where the curve $(w+t)^2=0$ intersects the curve $w\eta^2=0$ at one point $(w,\eta)=(-t,0)$ transversely; the bold point in the right figure of Figure 4.2.6 is $(-t,0)$. Also note that $\mathcal{H}_1^{(1)\prime}$ is the trivial family of $H_1^{(1)}$. In fact, take a new coordinate $z'=z(1+tz)^2$, and then $\mathcal{H}_1^{(1)\prime}$ is rewritten as $z'\zeta^2-s=0$, which does not depend on t.

Now we define a complex 3-manifold $\mathcal{M}_2^{(1)}$ in $N_2^{(1)}\times\Delta\times\Delta^\dagger$ by

$$\begin{cases} \mathcal{H}_2^{(1)}: & w^2\eta(1+t\eta)^2-s=0, \\ \mathcal{H}_2^{(1)\prime}: & \zeta(1+tz^2\zeta)^2-s=0. \end{cases}$$

Then $\mathcal{M}_1^{(1)}$ and $\mathcal{M}_2^{(1)}$ together give a propagation of $\mathcal{M}_0$ along the branch $\mathrm{Br}^{(1)}$.

Step 2: Propagation along $\mathrm{Br}^{(2)}$ By Simplification Lemma (Lemma 4.1.1), after some coordinate change, $\mathcal{H}_0$ is written as $\eta(w\eta+t)^2-s=0$ around $p_2=0\in\Theta_0$. For consistency with the discussion in Step 1, we write $\mathcal{H}_0$ in (z,ζ)-coordinates: $\zeta(z\zeta+t)^2-s=0$ around p_2, and then set

$$\mathcal{H}_1^{(2)}: \quad w(w\eta+t)^2-s=0.$$

Here we note that the transition function $z=1/w, \ \zeta=w^2\eta$, unfortunately, does *not* transform $\mathcal{H}_1^{(2)}$ to a hypersurface. Indeed, since $\mathcal{H}_1^{(2)}$ is written as

$$w\left(\frac{1}{w}(w^2\eta)+t\right)^2-s=0,$$

the transition function $z=1/w, \ \zeta=w^2\eta$ transforms $\mathcal{H}_1^{(2)}$ to $\frac{1}{z}(z\zeta+t)^2-s=0$. This does *not* define a hypersurface, because the right hand side contains a fractional term. How can we remedy this situation? The answer is to deform the line bundle $N_1^{(2)}$ as well. Set $g_1^{(2)}:\ z = 1/w,\ \zeta = w^2\eta + tw$, and define

$$\begin{cases} \mathcal{H}_1^{(2)}: & w(w\eta + t)^2 - s = 0, \\ \mathcal{H}_1^{(2)\prime}: & z\zeta^2 - s = 0. \end{cases}$$

(Note: $\mathcal{H}_2^{(1)}$ and $\mathcal{H}_2^{(1)\prime}$ are respectively the trivial families of $H_2^{(1)}$ and $H_2^{(1)\prime}$.) We claim that $g_1^{(2)}$ transforms $\mathcal{H}_1^{(2)}$ to $\mathcal{H}_1^{(2)\prime}$. To see this, write $\mathcal{H}_1^{(2)}$ as

$$w\left(\frac{1}{w}(w^2\eta) + t\right)^2 - s = 0.$$

On the other hand, from the expression of $g_1^{(2)}$, we have $w^2\eta = \zeta - tw = \zeta - t\frac{1}{z}$, and so $g_1^{(2)}$ transforms $\mathcal{H}_1^{(2)}$ to

$$\frac{1}{z}\left[z\left(\zeta - t\frac{1}{z}\right) + t\right]^2 - s = 0,$$

that is, to $\frac{1}{z}[z\zeta]^2 - s = 0$ which equals $\mathcal{H}_1^{(2)\prime}: z\zeta^2 - s = 0$, and hence $g_1^{(2)}$ transforms $\mathcal{H}_1^{(2)}$ to $\mathcal{H}_1^{(2)\prime}$. Now we 'trivially' propagate this deformation to that around $\Theta_2^{(2)}$:

$$\begin{cases} \mathcal{H}_2^{(2)}: & w^2\eta - s = 0, \\ \mathcal{H}_2^{(2)\prime}: & \zeta - s = 0. \end{cases}$$

Clearly, the transition function $z = 1/w$, $\zeta = w^2\eta$ of $N_2^{(2)}$ transforms $\mathcal{H}_2^{(2)}$ to $\mathcal{H}_2^{(2)\prime}$, and so we obtain a propagation of $\mathcal{M}_0$ along the branch $\mathrm{Br}^{(2)}$.

Step 3: Propagation along $\mathrm{Br}^{(3)}$ A propagation of $\mathcal{M}_0$ along the branch $\mathrm{Br}^{(3)}$ is carried out in the same way as the propagation along $\mathrm{Br}^{(2)}$.

The three steps in the above yield a complex 3-manifold $\mathcal{M}$. The natural projection $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$ is a barking family of $\pi : M \to \Delta$ such that the deformation from X to $X_{0,t}$ is shown in Figure 4.2.6.

Example 4.2.6 (Application of multiple barking)

The barking family of Example 4.2.5 has an interesting application; since its restriction around irreducible components $\Theta_2^{(1)}$, $\Theta_2^{(2)}$ and $\Theta_2^{(3)}$ is trivial, it is trivially extensible to a deformation of some degeneration with a constellar (constellation-shaped) singular fiber, which is obtained by 'bonding' the singular fiber X with another singular fiber. See Figure 4.2.7 for example ("Tk" in the figure is a *trunk*). The detailed account of this construction will be given later in §16.5, p292.

Example 4.2.7 (Compound barking)

So far, in each construction of barking families, we used only *one* subdivisor of the singular fiber X. In the next construction, we use *two* subdivisors Y_1

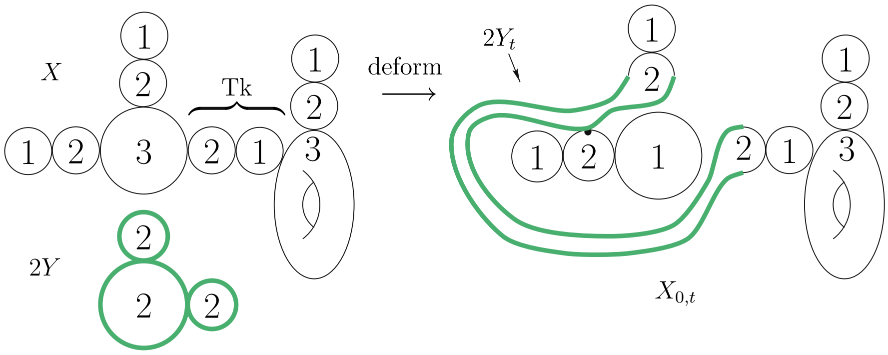

Fig. 4.2.7.

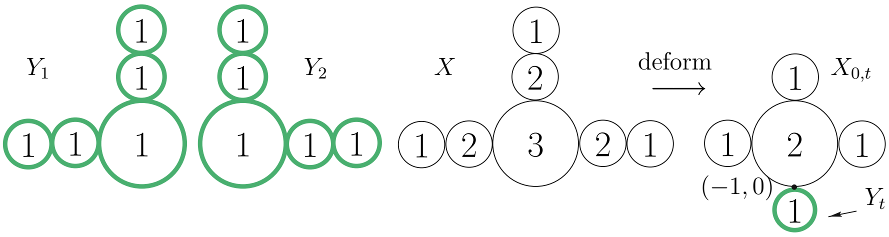

Fig. 4.2.8. Y_1 and Y_2 are subdivisors of X

and Y_2 of X simultaneously, where Y_1 and Y_2 are shown in Figure 4.2.8. First of all, we define a complex 3-manifold $\mathcal{M}_0$ in $N_0 \times \Delta \times \Delta^\dagger$ by patching the following smooth hypersurfaces by $g_0 : \ z = 1/w, \ \zeta = w^2\eta$:

$$\begin{cases} \mathcal{H}_0 : & w^2(w-1)^2\eta^3 - s + atw(w-1)^2\eta^2 + btw(w-1)\eta^2 = 0 \\ \mathcal{H}_0' : & z^2(1-z)^2\zeta^3 - s + atz(1-z)^2\zeta^2 + btz^2(1-z)\zeta^2 = 0, \end{cases} \tag{4.2.3}$$

where $a, b \in \mathbb{C} \setminus \{0\}$. The term with a (resp. b) 'corresponds' to Y_1 (resp. Y_2). In fact, if we set $a = 0$, then (4.2.3) (after propagation along branches) reduces to a deformation such that Y_2 is barked off from X (Example 4.2.2), while if we set $b = 0$, then (4.2.3) (after propagation along branches) reduces to a deformation such that Y_1 is barked off from X.

Now we shall propagate $\mathcal{M}_0$ along the branches of X. We carry this out separately for two cases $a \neq b$ and $a = b$; we will see that the resulting deformations for the two cases are totally different.

Case I $a \neq b$**:** First, we note the following factorizations of $\mathcal{H}_0$ and $\mathcal{H}_0'$:

$$\begin{cases} \mathcal{H}_0 : & w(w-1)\eta^2\Big[w(w-1)\eta + t(aw + b - a)\Big] - s = 0 \\ \mathcal{H}_0' : & z(1-z)\zeta^2\Big[z(1-z)\zeta + t\Big(a + (b-a)z\Big)\Big] - s = 0. \end{cases} \tag{4.2.4}$$

For simplicity, we assume $a = 1$ and $b = 2$:

$$\begin{cases} \mathcal{H}_0: & w(w-1)\eta^2\Big[w(w-1)\eta + t(w+1)\Big] - s = 0 \\ \mathcal{H}'_0: & z(1-z)\zeta^2\Big[z(1-z)\zeta + t(1+z)\Big] - s = 0. \end{cases} \tag{4.2.5}$$

We then construct a propagation of $\mathcal{H}'_0$ along $\mathrm{Br}^{(1)}$. In suitable coordinates, $\mathcal{H}'_0$ is written as $z\zeta^2(z\zeta + t) - s = 0$ around $p_1 = 1 \in \Theta_0$ by Simplification Lemma (Lemma 4.1.1). Then a propagation along $\mathrm{Br}^{(1)}$ is given by the following data — in what follows, we simply write a triple (two hypersurfaces and their gluing map) because the triple completely determines a deformation:

$$\begin{cases} \mathcal{H}_1^{(1)}: & w^2\eta(w\eta + t) - s = 0, \\ \mathcal{H}_1^{(1)\prime}: & \zeta(z\zeta + t) - s = 0, \\ g_1^{(1)}: & z = \dfrac{1}{w}, \quad \zeta = w^2\eta, \end{cases}$$

$$\begin{cases} \mathcal{H}_2^{(1)}: & w(w\eta + t) - s = 0, \\ \mathcal{H}_2^{(1)\prime}: & \zeta - s = 0, \\ g_2^{(1)}: & z = \dfrac{1}{w}, \quad \zeta = w^2\eta + tw. \end{cases}$$

We may also perform a propagation along $\mathrm{Br}^{(3)}$ in the same way as above. Along the branch $\mathrm{Br}^{(2)}$, we carry out a propagation of $\mathcal{H}_0$ as follows. In suitable coordinates, $\mathcal{H}_0$ is written as $w\eta^2(w\eta+t) - s = 0$ around $p_2 = 0 \in \Theta_0$ by Simplification Lemma (Lemma 4.1.1). For consistency, we write this in (z, ζ)-coordinates: $\mathcal{H}_0 : z\zeta^2(z\zeta + t) - s = 0$. Then a propagation along $\mathrm{Br}^{(2)}$ is given by

$$\begin{cases} \mathcal{H}_1^{(2)}: & w^2\eta(w\eta + t) - s = 0, \\ \mathcal{H}_1^{(2)\prime}: & \zeta(z\zeta + t) - s = 0, \\ g_1^{(2)}: & z = \dfrac{1}{w}, \quad \zeta = w^2\eta, \end{cases}$$

$$\begin{cases} \mathcal{H}_2^{(2)}: & w(w\eta + t) - s = 0, \\ \mathcal{H}_2^{(2)\prime}: & \zeta - s = 0, \\ g_2^{(2)}: & z = \dfrac{1}{w}, \quad \zeta = w^2\eta + tw. \end{cases}$$

In this way, we achieve propagations along all branches — *surprisingly, the propagations along* $\mathrm{Br}^{(1)}$*,* $\mathrm{Br}^{(2)}$ *and* $\mathrm{Br}^{(3)}$ *are all the same* — and we obtain a barking family. Note that for each t, $\mathcal{H}'_0|_{s=0}$ in (4.2.5) always passes through a point $(z, \zeta) = (-1, 0) \in \Theta_0$. See Figure 4.2.8; a more geometrically precise figure of $X_{0,t}$ is Figure 4.2.5.

Case II $a = b$**:** For simplicity, we assume $a = b = 1$. Then (4.2.4) is

$$\begin{cases} \mathcal{H}_0 : & w^2(w-1)\eta^2\Big[(w-1)\eta + t\Big] - s = 0 \\ \mathcal{H}_0' : & z(1-z)\zeta^2\Big[z(1-z)\zeta + t\Big] - s = 0. \end{cases}$$

In suitable coordinates, $\mathcal{H}_0'$ is written as $z\zeta^2(z\zeta+t)-s=0$ around $p_1 = 1 \in \Theta_0$ (Lemma 4.1.1). Then a propagation of $\mathcal{H}_0'$ along $\mathrm{Br}^{(1)}$ is given by

$$\begin{cases} \mathcal{H}_1^{(1)} : & w^2\eta(w\eta + t) - s = 0, \\ \mathcal{H}_1^{(1)\prime} : & \zeta(z\zeta + t) - s = 0, \\ g_1^{(1)} : & z = \frac{1}{w}, \quad \zeta = w^2\eta, \end{cases} \tag{4.2.6}$$

$$\begin{cases} \mathcal{H}_2^{(1)} : & w(w\eta + t) - s = 0, \\ \mathcal{H}_2^{(1)\prime} : & \zeta - s = 0, \\ g_2^{(1)} : & z = \frac{1}{w}, \quad \zeta = w^2\eta + tw. \end{cases} \tag{4.2.7}$$

(Note that this propagation is the same as the propagation along $\mathrm{Br}^{(1)}$ in Case I.) We may carry out a propagation along $\mathrm{Br}^{(3)}$ in the same way as the above (4.2.6), (4.2.7). Finally we construct a propagation of $\mathcal{H}_0$ along $\mathrm{Br}^{(2)}$. In suitable coordinates, the hypersurface $\mathcal{H}_0$ is written as $w^2\eta^2(\eta+t)-s=0$ around $p_2 = 0 \in \Theta_0$ (Lemma 4.1.1). For consistency, write this in (z,ζ)-coordinates: $\mathcal{H}_0 : z^2\zeta^2(\zeta + t) - s = 0$ around p_2. Then a propagation along $\mathrm{Br}^{(2)}$ is given by

$$\begin{cases} \mathcal{H}_1^{(2)} : & w^2\eta^2(w + t) - s = 0, \\ \mathcal{H}_1^{(2)\prime} : & z\zeta^2(1 + tz) - s = 0, \\ g_1^{(2)} : & z = \frac{1}{w}, \quad \zeta = w^2\eta, \end{cases}$$

$$\begin{cases} \mathcal{H}_2^{(2)} : & w^2\eta(1 + t\eta) - s = 0, \\ \mathcal{H}_2^{(2)\prime} : & \zeta(1 + tz^2\zeta) - s = 0, \\ g_2^{(2)} : & z = \frac{1}{w}, \quad \zeta = w^2\eta. \end{cases}$$

Hence we obtain propagations along all branches; accordingly we establish a barking family. Note that $\mathcal{H}_1^{(2)}|_{s=0}$ intersects $\Theta_1^{(2)}$ at one point $w = -t$ (the bold point in the right figure of Figure 4.2.9). We also note that in contrast with Case I (Figure 4.2.8), Y_t in this case does *not* intersect Θ_0.

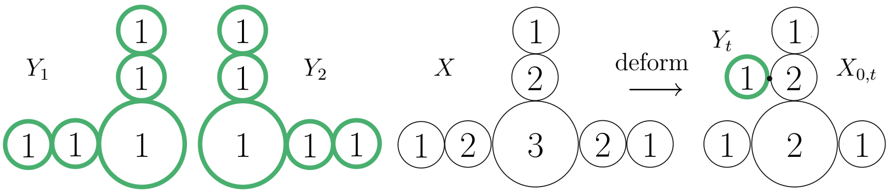

Fig. 4.2.9. See also Figure 4.2.10.

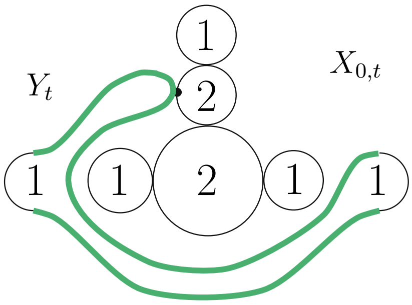

Fig. 4.2.10. A more geometrically precise figure of $X_{0,t}$ in Figure 4.2.9.

4.3 Supplement: Collision and Symmetry

4.3.1 Collision, I

In the previous section, to *one* subdivisor Y of a singular fiber, we associated *one* barking family, except for the final example where we took two subdivisors Y_1 and Y_2. However to one subdivisor, it is sometimes possible to associate different barking families by *collision of points*. We explain this by examples. Consider a degeneration of elliptic curves with the singular fiber X of type I_0^* (Figure 4.3.1).

Given a subdivisor Y in Figure 4.3.1, we shall construct two different barking families such that Y is barked off from X in different ways. The first construction is as follows. Taking "auxiliary points" $q_1, q_2 \in \Theta_0$ distinct from the attachment points p_1, p_2, p_3, p_4 of the branches, we define a deformation around the core Θ_0 by

$$\begin{cases} \mathcal{H}_0: \ \eta\Big[(w-p_1)(w-p_2)(w-p_3)(w-p_4)\eta + t(w-q_1)(w-q_2)\Big] - s = 0 \\ \mathcal{H}_0': \ \zeta\Big[(1-p_1z)(1-p_2z)(1-p_3z)(1-p_4z)\zeta + t(1-q_1z)(1-q_2z)\Big] - s = 0 \\ g_0: \ \ z = \dfrac{1}{w}, \quad \zeta = w^2\eta. \end{cases} \tag{4.3.1}$$

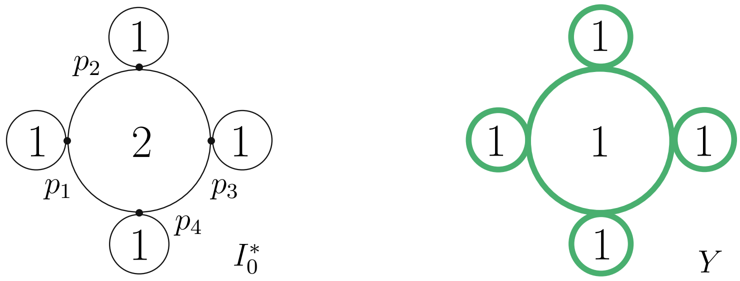

Fig. 4.3.1.

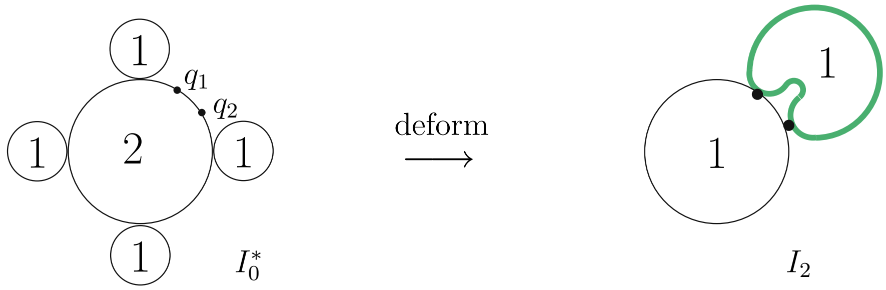

Fig. 4.3.2. See also Figure 4.3.3.

We propagate this along the branch $\mathrm{Br}^{(1)}$; for simplicity, we assume $p_1 = 0$. By Simplification Lemma (Lemma 4.1.1), after some coordinate change, $\mathcal{H}_0$ is of the form $\eta(w\eta+t)-s=0$ around p_1. For consistency with other examples, we write $\mathcal{H}_0$ in (z,ζ)-coordinates: $\zeta(z\zeta+t)-s=0$ which, by a coordinate change $(w,\eta)=(\zeta,z)$, becomes $w(w\eta+t)-s=0$. We then consider

$$\begin{cases} \mathcal{H}_1^{(1)}: & w(w\eta+t)-s=0, \\ \mathcal{H}_1^{(1)\prime}: & \zeta - s = 0, \\ g_1^{(1)}: & z=\dfrac{1}{w}, \quad \zeta = w^2\eta + tw. \end{cases}$$

Evidently, $g_1^{(1)}$ transforms $\mathcal{H}_1^{(1)}$ to $\mathcal{H}_1^{(1)\prime}$, and so the above data gives a propagation of $\mathcal{H}_0$ along the branch $\mathrm{Br}^{(1)}$. Similarly, we can construct propagations along other branches of X, yielding a barking family. In this family, $X = I_0^*$ is deformed to $X_{0,t} = I_2$. In fact, after some coordinate change, $\mathcal{H}_0|_{s=0}$ is locally $\eta(\eta+tw)=0$ around q_i $(i=1,2)$ where we choose coordinates such that $q_i=0$. Hence locally $X_{0,t}$ consists of two lines intersecting transversely at q_i $(i=1,2)$. See Figure 4.3.2.

Collision The second construction is obtained from the above one by making two points q_1 and q_2 collide to one point q (compare Figure 4.3.2 with

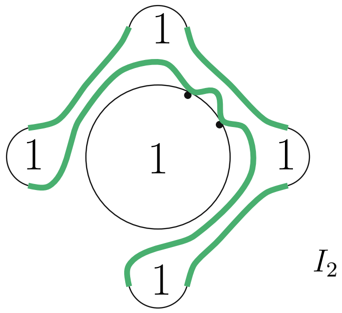

Fig. 4.3.3. A more geometrically precise figure of I_2 in Figure 4.3.2. Observe that all branches disappear under the deformation.

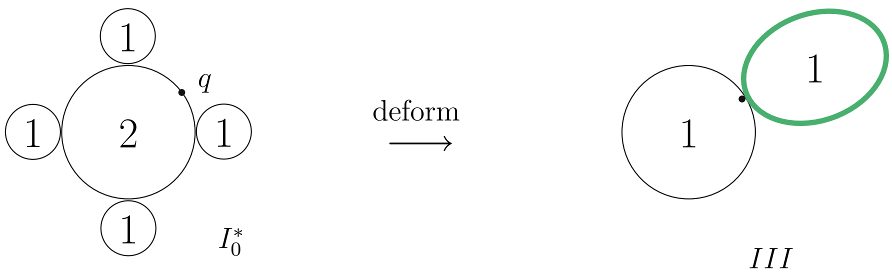

Fig. 4.3.4. See Figure 4.3.5 for a more geometrically precise figure.

Figure 4.3.4), meaning that we define a deformation around the core by

$$\begin{cases} \mathcal{H}_0: & \eta\Big[(w-p_1)(w-p_2)(w-p_3)(w-p_4)\eta + t(w-q)^2\Big] - s = 0 \\ \mathcal{H}_0': & \zeta\Big[(1-p_1z)(1-p_2z)(1-p_3z)(1-p_4z)\zeta + t(1-qz)^2\Big] - s = 0 \\ g_0: & z = \dfrac{1}{w}, \quad \zeta = w^2\eta. \end{cases}$$

We leave the reader to propagate this along the branches of the singular fiber X. In the resulting family, $X = I_0^*$ is deformed to $X_{0,t} = III$. In fact, after some coordinate change, $\mathcal{H}_0|_{s=0}$ is locally given by $\eta(\eta + tw^2) = 0$ around $q = 0$, and thus locally $X_{0,t}$ consists of two lines $\eta = 0$ and $\eta + tw^2 = 0$ intersecting at the point q with the second order contact, and so $X_{0,t}$ is of type III.

4.3.2 Collision, II

The above construction is based on the collision of two points q_1 and q_2 to one point q, where q is *not* an attachment point of a branch to the core. Next, we shall provide a slightly different construction using the collision of q_1 and q_2 to

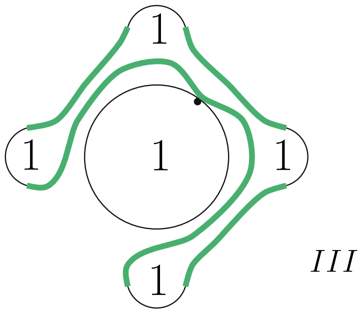

Fig. 4.3.5. A more geometrically precise figure of III in Figure 4.3.4. Observe that all branches disappear under the deformation.

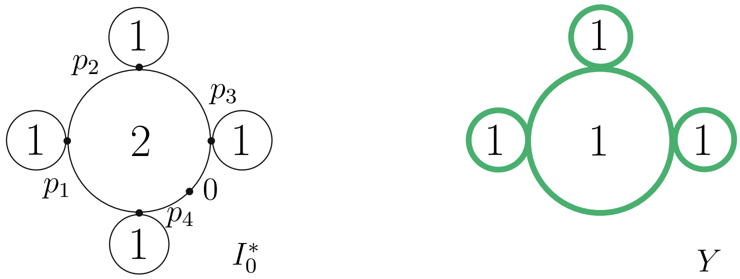

Fig. 4.3.6.

the attachment point of some branch. We take a subdivisor Y in Figure 4.3.6; we assume that $p_i \neq 0$ $(i = 1, 2, 3, 4)$ where p_i are the attachment points of branches to the core.

Initially, we construct a barking family which barks Y so that the singular fiber $X = I_0^*$ is deformed to $X_{0,t} = I_3$, and then by means of collision, we construct another barking family. First define a deformation around the core Θ_0 by

$$\begin{cases} \mathcal{H}_0 : & (w - p_4)\eta\Big[(w - p_1)(w - p_2)(w - p_3)\eta + tw\Big] - s = 0 \\ \mathcal{H}_0' : & (1 - p_4 z)\zeta\Big[(1 - p_1 z)(1 - p_2 z)(1 - p_3 z)\zeta + t\Big] - s = 0 \\ g_0 : & z = \dfrac{1}{w}, \quad \zeta = w^2\eta. \end{cases} \tag{4.3.2}$$

(Specializing $q_1 = p_4$ and $q_2 = 0$ in (**??**), we obtain (4.3.2).) We leave the reader to propagate this along the branches of the singular fiber X. The resulting barking family is described as follows. In a new coordinate $\eta' = (w - p_1)(w - p_2)(w - p_3)\eta$, the hypersurface $\mathcal{H}_0|_{s=0}$ becomes

$$\frac{(w - p_4)}{(w - p_1)(w - p_2)(w - p_3)}\,\eta'\,(\eta' + tw) = 0,$$

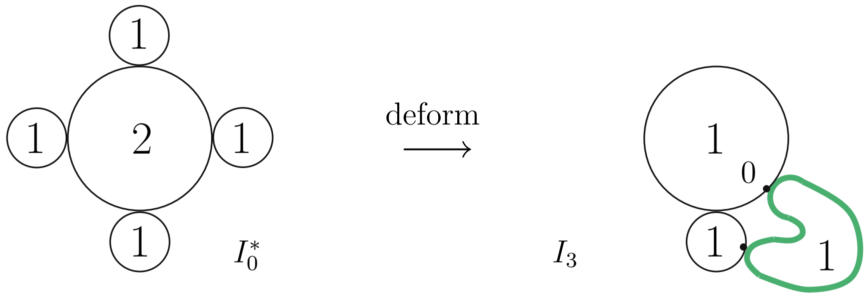

Fig. 4.3.7. See Figure 4.3.8 for a more geometrically precise figure.

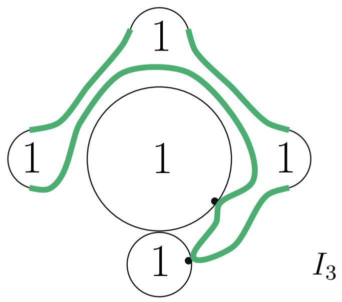

Fig. 4.3.8. A more geometrically precise figure of I_3 in Figure 4.3.7. Observe that except for $\mathrm{Br}^{(4)}$, other branches disappear under the deformation.

that is, $\eta'(\eta' + tw) = 0$ around $(w, \eta') = (0, 0)$. Thus $\mathcal{H}_0|_{s=0}$ is locally given by $\eta'(\eta' + tw) = 0$ around $(w, \eta') = (0, 0)$, where the barked part $\eta' + tw = 0$ intersects Θ_0 at one point $(w, \eta') = (0, 0)$ — a bold point marked by 0 in Figure 4.3.7 — transversely.

Collision Next we modify the above construction; we collide two points, say p_4 and 0 on the left figure in Figure 4.3.6, and then we obtain a new deformation:

$$\begin{cases} \mathcal{H}_0 : & w\eta\Big[(w-p_1)(w-p_2)(w-p_3)\eta + tw\Big] - s = 0 \\ \mathcal{H}_0' : & \zeta\Big[(1-p_1z)(1-p_2z)(1-p_3z)\zeta + t\Big] - s = 0 \\ g_0 : & z = \dfrac{1}{w}, \quad \zeta = w^2\eta. \end{cases}$$

Note that $\mathcal{H}_0|_{s=0} : w\eta\Big[(w-p_1)(w-p_2)(w-p_3)\eta + tw\Big] = 0$, which locally defines three lines passing through $p_4 = 0 \in \Theta_0$, and hence p_4 is the ordinary triple point (the bold point in the right figure of Figure 4.3.9).

4.3.3 Construction based on symmetry

In §4.3.2, we constructed a barking family which deforms $X = I_0^*$ to $X_{0,t} = III$. Using the 'symmetry' of I_0^* (see Y_1 and Y_2 in Figure 4.3.11), we shall

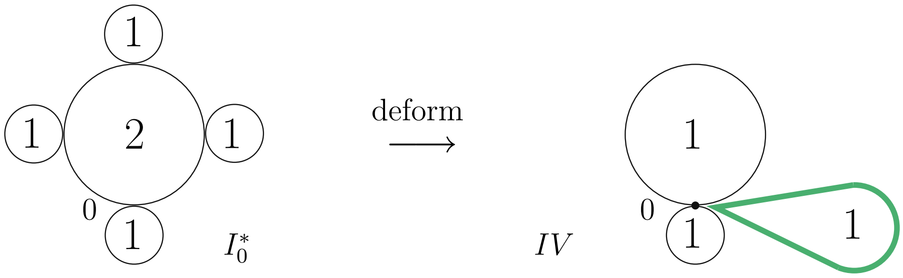

Fig. 4.3.9. Very hard to write a figure! Actually, the projective line with the gray color is *smooth*, and three smooth projective lines intersect at one point p_4 which is the ordinary triple point. See also Figure 4.3.10.

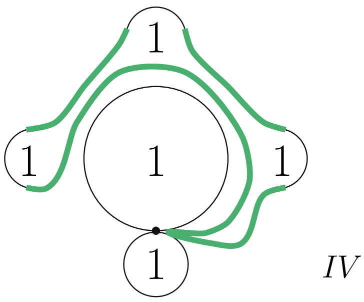

Fig. 4.3.10. A slightly more precise figure of IV in Figure 4.3.9. (The projective line with the gray color is actually smooth.) Observe that except for $\mathrm{Br}^{(4)}$, other branches disappear under the deformation.

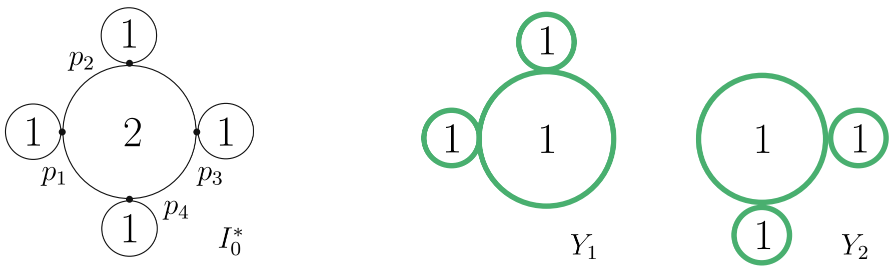

Fig. 4.3.11.

give another barking family which also deforms $X = I_0^*$ to $X_{0,t} = III$. First define a deformation around the core by

$$\begin{cases} \mathcal{H}_0: & \Big[(w-p_1)(w-p_2)\eta + t\Big]\Big[(w-p_3)(w-p_4)\eta + t\Big] - s = 0 \\ \mathcal{H}_0': & \Big[(1-p_1z)(1-p_2z)\zeta + t\Big]\Big[(1-p_3z)(1-p_4z)\zeta + t\Big] - s = 0 \\ g_0: & z = \dfrac{1}{w}, \quad \zeta = w^2\eta. \end{cases} \tag{4.3.3}$$

We leave the reader to propagate this along the branches of the singular fiber X. The resulting barking family is described as follows. Put $s = 0$ in (4.3.3), and then $\mathcal{H}_0|_{s=0}$ and $\mathcal{H}_0'|_{s=0}$ together give a smoothing of the subdivior Y_1 (resp. Y_2) to a projective line $Y_{1,t}$ (resp. $Y_{2,t}$), as illustrated in Figure 4.3.12 (cf. Figure 4.3.4), where $Y_{1,t}$ and $Y_{2,t}$ around the core Θ_0 are respectively given by

$$Y_{1,t}: \begin{cases} (w-p_1)(w-p_2)\eta + t = 0 \\ (1-p_1z)(1-p_2z)\zeta + t = 0, \end{cases} \tag{4.3.4}$$

$$Y_{2,t}: \begin{cases} (w-p_3)(w-p_4)\eta + t = 0 \\ (1-p_3z)(1-p_4z)\zeta + t = 0. \end{cases} \tag{4.3.5}$$

The following lemma confirms that $X_{0,t}$ is of type III.

Lemma 4.3.1 *Two projective lines $Y_{1,t}$ and $Y_{2,t}$ $(t \neq 0)$ intersect only at one point with the second order contact.*

Proof. Letting (w, η) be an intersection point of $Y_{1,t}$ and $Y_{2,t}$, then from the first equation of (4.3.4) and that of (4.3.5), we have

$$(w-p_1)(w-p_2)\eta + t = (w-p_3)(w-p_4)\eta + t, \tag{4.3.6}$$

and hence

$$\Big[-(p_1+p_2)w + p_1p_2\Big]\eta = \Big[-(p_3+p_4)w + p_3p_4\Big]\eta. \tag{4.3.7}$$

Here notice that $\eta \neq 0$; indeed if $\eta = 0$, then from $(w-p_1)(w-p_2)\eta + t = 0$, we obtain $t = 0$, yielding a contradiction to $t \neq 0$. So we may divide (4.3.7)

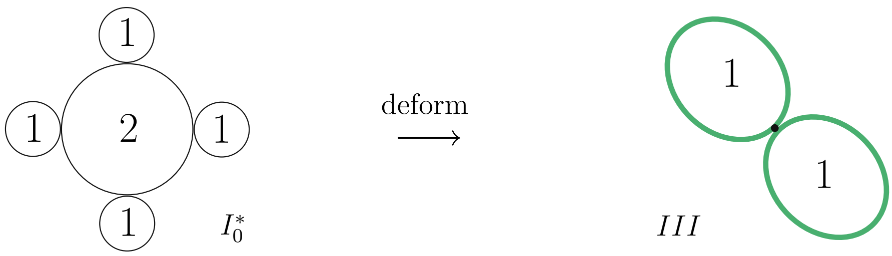

Fig. 4.3.12.

by η to deduce

$$(p_1 + p_2 - p_3 - p_4)w = p_1p_2 - p_3p_4. \tag{4.3.8}$$

We claim that $p_1+p_2-p_3-p_4 \neq 0$. In fact, if $p_1+p_2-p_3-p_4 = 0$ (equivalently $p_1 + p_2 = p_3 + p_4$), then from (4.3.8), we have $p_1p_2 - p_3p_4 = 0$. However the two equations $p_1+p_2 = p_3+p_4$ and $p_1p_2 = p_3p_4$ imply that $\{p_1, p_2\} = \{p_3, p_4\}$ (equal as a set), because both of $\{p_1, p_2\}$ and $\{p_3, p_4\}$ are the solutions of the *same* equation $x^2 - (p_1 + p_2)x + p_1p_2 = 0$. This contradicts that p_1, p_2, p_3, p_4 are distinct (they are the attachment points of the branches of the singular fiber X to the core), and thus $p_1 + p_2 - p_3 - p_4 \neq 0$.

Now by (4.3.8), we have

$$w = \frac{p_1p_2 - p_3p_4}{p_1 + p_2 - p_3 - p_4}. \tag{4.3.9}$$

Substitute this into the first equation (4.3.4) of $Y_{1,t}$, which gives

$$\left[\frac{p_1p_2 - p_3p_4}{p_1 + p_2 - p_3 - p_4} - p_1\right]\left[\frac{p_1p_2 - p_3p_4}{p_1 + p_2 - p_3 - p_4} - p_2\right]\eta + t = 0. \tag{4.3.10}$$

Here note the following equations:

$$\frac{p_1p_2 - p_3p_4}{p_1 + p_2 - p_3 - p_4} - p_1 = -\frac{(p_1 - p_3)(p_1 - p_4)}{p_1 + p_2 - p_3 - p_4},$$

$$\frac{p_1p_2 - p_3p_4}{p_1 + p_2 - p_3 - p_4} - p_2 = -\frac{(p_2 - p_3)(p_2 - p_4)}{p_1 + p_2 - p_3 - p_4}.$$

Thus from (4.3.10), we have

$$\eta = -\frac{(p_1 + p_2 - p_3 - p_4)^2 t}{(p_1 - p_3)(p_1 - p_4)(p_2 - p_3)(p_2 - p_4)}. \tag{4.3.11}$$

By (4.3.9) and (4.3.11), we conclude

$$(w, \eta) = \left(\frac{p_1p_2 - p_3p_4}{p_1 + p_2 - p_3 - p_4}\ ,\ -\frac{(p_1 + p_2 - p_3 - p_4)^2 t}{(p_1 - p_3)(p_1 - p_4)(p_2 - p_3)(p_2 - p_4)}\right).$$

This point is a unique solution of the equation (4.3.6), and thus the intersection $Y_{1,t} \cap Y_{2,t}$ is just one point (w, η). Furthermore since (4.3.6) is quadratic in w, two projective lines $Y_{1,t}$ and $Y_{2,t}$ have the second order contact at this point. This completes the proof of the assertion.

Remark 4.3.2 For the deformation (4.3.3), we may include additional parameters $a,\ b \in \mathbb{C}$ as follows:

$$\begin{cases} \mathcal{H}_0: & \Big[(w - p_1)(w - p_2)\eta + at\Big]\Big[(w - p_3)(w - p_4)\eta + bt\Big] - s = 0 \\ \mathcal{H}_0': & \Big[(1 - p_1z)(1 - p_2z)\zeta + at\Big]\Big[(1 - p_3z)(1 - p_4z)\zeta + bt\Big] - s = 0 \\ g_0: & z = \dfrac{1}{w}, \quad \zeta = w^2\eta. \end{cases}$$

When $b = 0$ (resp. $a = 0$), this deformation barks Y_1 (resp. Y_2) in Figure 4.3.11 from X.

Part II

Deformations of Tubular Neighborhoods of Branches

5

Deformations of Tubular Neighborhoods of Branches (Preparation)

The construction of barking deformations for degenerations of *compact* complex curves is outlined as follows: First we construct deformations separately for ambient spaces (tubular neighborhoods) of branches, trunks, and cores. Then we glue these deformations together. In this chapter we introduce several important notions (multiplicity sequences, tame or wild subbranches), which will play a prominent role when we construct barking deformations of tubular neighborhoods of branches.

5.1 Branches

We begin by constructing a degeneration whose singular fiber is a branch. Suppose that $\mathbf{m} = (m_0, m_1, \dots, m_\lambda)$ is a sequence of positive integers with $\lambda \geq 1$ such that

(i) $m_0 > m_1 > \cdots > m_\lambda > 0$,
(ii) $r_i := \dfrac{m_{i-1} + m_{i+1}}{m_i}$ $\quad (i = 1, 2, \dots, \lambda - 1)$ $\quad$ and $\quad r_\lambda := \dfrac{m_{\lambda-1}}{m_\lambda}$ are integers greater than 1.

Note: $(m_0, m_1, \dots, m_\lambda)$ is an arithmetic progression precisely when $r_1 = r_2 = \cdots = r_{\lambda-1} = 2$ (r_λ may not be 2.) In fact, from (ii), $m_{i+1} = r_i m_i - m_{i-1}$, and putting $r_i = 2$, we deduce $m_{i+1} - m_i = m_i - m_{i-1}$.

We take λ copies $\Theta_1, \Theta_2, \dots, \Theta_\lambda$ of the projective line $\mathbb{P}^1$. Let $\Theta_i = U_i \cup U_i'$ be an open covering by two complex lines with coordinates $z_i \in U_i'$ and $w_i \in U_i$ satisfying $z_i = 1/w_i$ on $U_i \cap U_i'$. Next, we consider a line bundle N_i on Θ_i obtained by gluing $(z_i, \zeta_i) \in U_i' \times \mathbb{C}$ with $(w_i, \eta_i) \in U_i \times \mathbb{C}$ by

$$z_i = \frac{1}{w_i}, \quad \zeta_i = w_i^{r_i} \eta_i.$$

Note that $N_i \cong \mathcal{O}_{\Theta_i}(-r_i)$. We patch the line bundles $N_1, N_2, \dots, N_\lambda$ by *plumbing*, that is, we patch N_i with N_{i+1} ($i = 1, 2, \dots, \lambda - 1$) by $(z_i, \zeta_i) = (\eta_{i+1}, w_{i+1})$. We then obtain a smooth complex surface M.

Next we shall define a holomorphic function on M. For this purpose, we first define a holomorphic function π_i on the line bundle N_i $(i = 1, 2, \ldots, \lambda)$ by

$$\begin{cases} \pi_i(w,\eta) = w^{m_{i-1}}\eta^{m_i} & \text{on } U_i \times \mathbb{C} \\ \pi_i(z,\zeta) = z^{m_{i+1}}\zeta^{m_i} & \text{on } U_i' \times \mathbb{C}, \end{cases}$$

where $m_{\lambda+1} = 0$ by convention, and we omit the subscripts i of $w_i, \eta_i, z_i, \zeta_i$ for simplicity. Let us check that π_i is well-defined, i.e. the above two functions on $U_i \times \mathbb{C}$ and $U_i' \times \mathbb{C}$ are compatible with the transition function $z = 1/w$, $\zeta = w^{r_i}\eta$ of N_i. This is confirmed as follows:

$$\begin{aligned} w^{m_{i-1}}\eta^{m_i} &= w^{m_{i-1}-r_i m_i}(w^{r_i}\eta)^{m_i} = z^{r_i m_i - m_{i-1}}\zeta^{m_i} \\ &= z^{m_{i+1}}\zeta^{m_i}, \end{aligned} \tag{5.1.1}$$

where in the last equality, we used $r_i = \dfrac{m_{i-1} + m_{i+1}}{m_i}$.

Since the holomorphic functions π_i $(i = 1, 2, \ldots, \lambda)$ are compatible with the patchings of M, they together determine a holomorphic function π on M, and $\pi : M \to \Delta$ is a degeneration with a singular fiber (Figure 5.1.1)

$$\pi^{-1}(0) = m_0\Delta_0 + m_1\Theta_1 + m_2\Theta_2 + \cdots + m_\lambda\Theta_\lambda,$$

where $\Delta_0 \cong \mathbb{C}$ while $\Theta_1, \Theta_2, \ldots, \Theta_\lambda$ are projective lines such that Θ_i and Θ_{i+1} (resp. Δ_0 and Θ_1) intersect transversely at one point (we will sometimes shrink M so that Δ_0 becomes a small open disk). Note that a smooth fiber is not necessarily connected; it is a disjoint union of m_λ copies of $\mathbb{C}$ where we note $m_\lambda = \gcd(m_0, m_1, \ldots, m_\lambda)$.
The singular fiber $X := \pi^{-1}(0)$ is called a *branch* (of *length* λ).

Remark 5.1.1 Precisely speaking, X is a *fringed* branch ($m_0\Delta_0$ is its fringe), while $m_1\Theta_1 + \cdots + m_\lambda\Theta_\lambda$ is an *unfringed* branch. However for the sake of brevity, we often refer to both unfringed and fringed branches simply as branches. In later chapters, occasionally we need to distinguish unfringed and fringed branches, in which case we use notations Br and $\overline{\mathrm{Br}}$ respectively for them.

For the subsequent discussion, it is convenient to regard M as the graph of π:

$$\mathrm{Graph}(\pi) = \{(x,s) \in M \times \Delta \,:\, \pi(x) - s = 0\}.$$

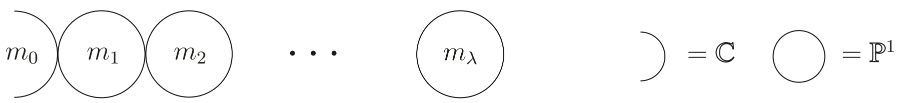

Fig. 5.1.1. m_i stands for a multiplicity.

We identify Graph(π) with M via an isomorphism given by $(x, s) \in \mathrm{Graph}(\pi) \mapsto x \in M$. Under this canonical isomorphism, π is given by the projection $(x, s) \in \mathrm{Graph}(\pi) \to s \in \Delta$. We remark that it is also possible to construct Graph(π) by patching complex surfaces as follows. Consider complex surfaces H_i and H_i' $(i = 1, 2 \ldots, \lambda)$:

$$H_i = \{(w_i, \eta_i, s) \in U_i \times \mathbb{C} \times \Delta \, : \, w_i^{m_{i-1}} \eta_i^{m_i} - s = 0\},$$
$$H_i' = \{(z_i, \zeta_i, s) \in U_i' \times \mathbb{C} \times \Delta \, : \, z_i^{m_{i+1}} \zeta_i^{m_i} - s = 0\}.$$

Here $m_{\lambda+1} = 0$ by convention. For each i, H_i and H_i' together determine a complex surface M_i in $N_i \times \Delta$. Gluing $N_i \times \Delta$ with $N_{i+1} \times \Delta$ $(i = 1, 2, \ldots, \lambda)$ by plumbing, that is, by $(z_i, \zeta_i, s) = (\eta_{i+1}, w_{i+1}, s)$, we obtain a complex 3-manifold $M \times \Delta$, and then $M_1, M_2, \ldots, M_\lambda$ together determine the complex surface Graph(π) in $M \times \Delta$. Henceforth, identifying M with Graph(π) via the canonical isomorphism, we write M for Graph(π).

Remark 5.1.2 The complex surface M is the minimal resolution space of a cyclic quotient singularity V, and $\Theta_1 + \Theta_2 + \cdots + \Theta_\lambda$ is the exceptional set of M. More explicitly, let m_1^* $(0 < m_1^* < m_0)$ be the integer satisfying $m_1 m_1^* \equiv 1 \mod m_0$, and then $V = \mathbb{C}^2/G$, where G is a cyclic group action generated by

$$(x, y) \longmapsto (e^{2\pi \mathrm{i}/m_0} x, e^{2\pi \mathrm{i} m_1^*/m_0} y).$$

See [Ta,II] for details. Since any quotient singularity is *taut* (see [La2]), the complex structure on M is unique, and so without loss of generality, we may assume that M is obtained by plumbing as above. (However, there are many choices of maps $\pi : M \to \Delta$ with their singular fiber $\pi^{-1}(0) = m_0\Delta_0 + m_1\Theta_1 + \cdots + m_\lambda\Theta_\lambda$.)

5.2 Deformation atlas

Next, we shall introduce the notion of a deformation atlas which plays an important role in the construction of deformations of degenerations. First, we consider a complex 3-manifold $\mathcal{H}_i$, given by a hypersurface in $U_i \times \mathbb{C} \times \Delta \times \Delta^\dagger$:

$$\mathcal{H}_i : \quad w^{m_{i-1}} \eta^{m_i} - s + \varphi_i(w, \eta, t) = 0,$$

where φ_i is a holomorphic function on $U_i \times \mathbb{C} \times \Delta^\dagger$ satisfying $\varphi_i(w, \eta, 0) = 0$. We say that $\mathcal{H}_i$ is a *deformation* of H_i. Note that $\mathcal{H}_i$ is smooth because

$$\frac{\partial}{\partial s}\Big(w_i^{m_{i-1}} \eta_i^{m_i} - s + \varphi_i(w, \eta, t)\Big) = -1 \neq 0.$$

In the case $\varphi_i \equiv 0$, $\mathcal{H}_i$ is called the *trivial* deformation. Similarly, a complex 3-manifold $\mathcal{H}_i'$ is called a *deformation* of H_i' provided that it is a smooth hypersurface in $U_i' \times \mathbb{C} \times \Delta \times \Delta^\dagger$ of the form:

$$\mathcal{H}_i' : \quad z^{m_{i+1}} \zeta^{m_i} - s + \psi_i(z, \zeta, t) = 0$$

such that ψ_i is holomorphic with $\psi_i(z, \zeta, 0) = 0$.

Next, we consider a gluing map g_i of $U_i \times \mathbb{C} \times \Delta \times \Delta^\dagger$ with $U_i' \times \mathbb{C} \times \Delta \times \Delta^\dagger$:

$$g_i : \quad z = \frac{1}{w}, \quad \zeta = w^{r_i}\eta + h_i(w, \eta, t),$$

where h_i is a holomorphic function on $(U_i \cap U_i') \times \mathbb{C} \times \Delta^\dagger$ such that $h_i(w, \eta, 0) = 0$. We say that g_i is a *deformation* of the transition function $z = 1/w,\ \zeta = w^{r_i}\eta$ of N_i. For the case $h_i \equiv 0$, the map g_i is called the *trivial* deformation of the transition function.

Remark 5.2.1 After a coordinate change, we may assume that g_i is of the "standard form"

$$z = \frac{1}{w}, \quad \zeta = w^{r_i}\eta + \alpha_1(t)w + \alpha_2(t)w^2 + \cdots + \alpha_{r_i - 1}(t)w^{r_i - 1},$$

where $\alpha_k(t)$ $(k = 1, 2, \ldots, r_i - 1)$ is a holomorphic function in t with $\alpha_k(0) = 0$. See [Ri3] and §5.5.1, p98.

Letting e be an integer satisfying $1 \le e \le \lambda$, we consider the following data: For $i = 1, 2, \ldots, e$,

$$\begin{cases} \mathcal{H}_i : & \text{a deformation of } H_i \\ \mathcal{H}_i' : & \text{a deformation of } H_i' \\ g_i : & \text{a deformation of the transition function of } N_i. \end{cases}$$

A set $DA_e := \{\mathcal{H}_i, \mathcal{H}_i', g_i\}_{i=1,2,\ldots,e}$ is referred to as a *deformation atlas* of *length* e if the following two conditions are satisfied:

g_i $(i = 1, 2, \ldots, e)$ transforms $\mathcal{H}_i$ to $\mathcal{H}_i'$, and (5.2.1)

by a coordinate change $(z_i, \zeta_i, s, t) = (\eta_{i+1}, w_{i+1}, s, t)$, $\mathcal{H}_i'$ becomes $\mathcal{H}_{i+1}$ $(i = 1, 2, \ldots, e-1)$. (5.2.2)

For any integer e' $(0 < e' < e)$, $DA_{e'} := \{\mathcal{H}_i, \mathcal{H}_i', g_i\}_{i=1,2,\ldots,e'}$ is a deformation atlas of length e'. We say that DA_e is an *e-th propagation* of $DA_{e'}$. For the case $e = \lambda$, we say that DA_λ is a *complete propagation* of $DA_{e'}$, and DA_λ is a *complete* deformation atlas. We often simply write $DA_e = \{\mathcal{H}_i, \mathcal{H}_i', g_i\}$ instead of $DA_e = \{\mathcal{H}_i, \mathcal{H}_i', g_i\}_{i=1,2,\ldots,e}$.

We next give a useful lemma, quoted as the *Propagation Lemma.*

Lemma 5.2.2 *If $m_\lambda = 1$, then any deformation atlas of length $\lambda - 1$ admits a complete propagation.*

Proof. In fact, the following data gives a complete propagation of $DA_{\lambda-1}$:

$$\begin{cases} \mathcal{H}_\lambda : & w^{m_{\lambda-1}}\eta - s + h_{\lambda-1}(w,\eta,t) = 0 \\ \mathcal{H}'_\lambda : & \zeta - s = 0 \\ g_\lambda : & z = \dfrac{1}{w}, \quad \zeta = w^{m_{\lambda-1}}\eta + h_{\lambda-1}(w,\eta,t). \end{cases}$$

(Note: since $m_\lambda = 1$, we have $r_\lambda := m_{\lambda-1}/m_\lambda = m_{\lambda-1}$.) □

Given a complete deformation atlas $DA_\lambda = \{\mathcal{H}_i, \mathcal{H}'_i, g_i\}_{i=1,2,\dots,\lambda}$, we may construct a deformation of $\boldsymbol{\pi} : M \to \Delta$ as follows. First, let $\mathcal{N}_i$ be a complex 4-manifold obtained by patching $U_i \times \mathbb{C} \times \Delta \times \Delta^\dagger$ with $U'_i \times \mathbb{C} \times \Delta \times \Delta^\dagger$ by g_i $(i = 1,2,\dots,\lambda)$. Then $\mathcal{H}_i$ and $\mathcal{H}'_i$ together determine a complex 3-manifold $\mathcal{M}_i$ in $\mathcal{N}_i$. Next, we patch $\mathcal{N}_i$ with $\mathcal{N}_{i+1}$ $(i = 1,2,\dots,\lambda-1)$ by $(z_i, \zeta_i, s, t) = (\eta_{i+1}, w_{i+1}, s, t)$, which yields a complex 4-manifold $\mathcal{W}$. By the conditions (5.2.1) and (5.2.2), the complex 3-manifolds $\mathcal{M}_i$ $(i = 1,2,\dots,\lambda)$ together determine a complex 3-manifold $\mathcal{M}$ in $\mathcal{W}$. Then the natural projection $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$ is a deformation family of $\boldsymbol{\pi} : M \to \Delta$, because $\Psi^{-1}(\Delta \times \{0\}) = M$ and $\Psi|_M = \boldsymbol{\pi}$. We say that $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$ is a *barking family* of $\boldsymbol{\pi} : M \to \Delta$ obtained from DA_λ.

Note: The natural projection $p_i : \mathcal{N}_i \to \Delta \times \Delta^\dagger$ is considered as a deformation of the line bundle $N_i = p_i^{-1}(0,0)$, parameterized by $\Delta \times \Delta^\dagger$. Note that $p_i^{-1}(0,t)$ $(t \neq 0)$ is not necessarily a line bundle.

5.3 Subbranches

For a branch $X = m_0\Delta_0 + m_1\Theta_1 + \cdots + m_\lambda\Theta_\lambda$, remember that the sequence of multiplicities $\mathbf{m} = (m_0, m_1, \dots, m_\lambda)$ satisfies $m_0 > m_1 > \cdots > m_\lambda > 0$, and $r_i := \dfrac{m_{i-1} + m_{i+1}}{m_i}$ $(i = 1,2,\dots,\lambda-1)$ and $r_\lambda := \dfrac{m_{\lambda-1}}{m_\lambda}$ are integers with $r_i \geq 2$ $(i = 1,2,\dots,\lambda)$. Then the self-intersection number $\Theta_i \cdot \Theta_i = -r_i$, and the condition $r_i \geq 2$ implies that Θ_i is *not* an exceptional curve, i.e. $\Theta_i \cdot \Theta_i \neq -1$. Now let $Y = n_0\Delta_0 + n_1\Theta_1 + \cdots + n_e\Theta_e$ be a connected subdivisor of X, so $e \leq \lambda$ and $0 < n_i \leq m_i$. Symbolically we express $0 < Y \leq X$.

Definition 5.3.1 A connected subdivisor $Y = n_0\Delta_0 + n_1\Theta_1 + \cdots + n_e\Theta_e$ is called a *subbranch* of X if (1) $e = 0$ or 1, or (2) $e \geq 2$ and the following equations hold:

$$\frac{n_{i-1} + n_{i+1}}{n_i} = r_i, \qquad i = 1,2,\dots,e-1. \tag{5.3.1}$$

We say that e is the *length* of the subbranch Y; unless otherwise mentioned we assume $e \geq 1$, omitting the trivial case $Y = n_0\Delta_0$ in the subsequent discussion. Note that for a subbranch Y and a positive integer l, if $lY \leq X$, then lY is a subbranch as well.

A sequence of positive integers $\mathbf{n} = (n_0, n_1, \ldots, n_e)$ satisfying (5.3.1) is called a *multiplicity sequence*. The condition (5.3.1) is intended for the use of the following lemma:

Lemma 5.3.2 *Let a, b, c and r be positive integers satisfying $\dfrac{a+c}{b} = r$. Then a map $g : z = 1/w,\ \zeta = w^r\eta$ transforms a monomial $w^a\eta^b$ to a monomial $z^c\zeta^b$.*

Proof. Indeed, $w^a\eta^b = w^{a-br}(w^r\eta)^b = z^{br-a}\zeta^b = z^c\zeta^b$, where the last equation follows from $br = a + c$. □

To a subbranch $Y = n_0\Delta_0 + n_1\Theta_1 + \cdots + n_e\Theta_e$, we shall associate a deformation atlas of length $e-1$.

Lemma 5.3.3 *Let Y be a subbranch of X and d be an arbitrary positive integer. Then the following data gives a deformation atlas $DA_{e-1}(Y, d)$ of length $e-1$: For $i = 1, 2, \ldots, e-1$,*

$$\begin{cases} \mathcal{H}_i: & w_i^{m_{i-1}-n_{i-1}}\eta_i^{m_i-n_i}(w_i^{n_{i-1}}\eta_i^{n_i} + t^d) - s = 0 \\ \mathcal{H}'_i: & z_i^{m_{i+1}-n_{i+1}}\zeta_i^{m_i-n_i}(z_i^{n_{i+1}}\zeta_i^{n_i} + t^d) - s = 0 \\ g_i: & \textit{the transition function } z = 1/w,\ \zeta = w^{r_i}\eta \textit{ of } N_i. \end{cases}$$

Proof. From the definition of branches and subbranches,

$$m_{i-1} + m_{i+1} = r_i m_i, \qquad n_{i-1} + n_{i+1} = r_i n_i. \tag{5.3.2}$$

Setting

$$a := m_{i-1} - n_{i-1}, \qquad b := m_i - n_i, \qquad c := m_{i+1} - n_{i+1},$$

then we have

$$\begin{aligned} \frac{a+c}{b} &= \frac{(m_{i-1} - n_{i-1}) + (m_{i+1} - n_{i+1})}{m_i - n_i} = \frac{(m_{i-1} + m_{i+1}) - (n_{i-1} + n_{i+1})}{m_i - n_i} \\ &= \frac{(r_i m_i) - (r_i n_i)}{m_i - n_i} \qquad \text{by (5.3.2)} \\ &= r_i. \end{aligned}$$

Thus by Lemma 5.3.2, g_i transforms $w^{m_{i-1}-n_{i-1}}\eta^{m_i-n_i}$ to $z^{m_{i+1}-n_{i+1}}\zeta^{m_i-n_i}$. Likewise, g_i also transforms $w^{n_{i-1}}\eta^{n_i}$ to $z^{n_{i+1}}\zeta^{n_i}$. Hence g_i transforms $\mathcal{H}_i$ to $\mathcal{H}'_i$. On the other hand, $\mathcal{H}'_i$ becomes $\mathcal{H}_{i+1}$ by a coordinate change $(w_{i+1}, \eta_{i+1}, s, t) = (\zeta_i, z_i, s, t)$. Thus $DA_{e-1}(Y, d) = \{\mathcal{H}_i, \mathcal{H}'_i, g_i\}_{i=1,2,\ldots,e-1}$ is a deformation atlas of length $e-1$. □

When d plays no role, we frequently omit d to write $DA_{e-1}(Y)$ for $DA_{e-1}(Y, d)$.

5.4 Dominant subbranches

To a subbranch Y of length e, we have associated a deformation atlas $DA_{e-1}(Y)$ of length $e-1$. Since we would ideally like to construct a complete deformation atlas, we are interested in a subbranch with 'maximal' length, and a deformation atlas associated with it. In this section, we will introduce a class of subbranches called *dominant* which, in some sense, have maximal lengths.

We begin with arithmetic preparation. Given two positive integers n_0 and n_1 satisfying $m_0 \geq n_0$ and $m_1 \geq n_1$, let us construct a multiplicity sequence with the maximal length among the multiplicity sequences whose first two terms are n_0 and n_1. For this purpose, we first define a sequence $n_2, n_3, \ldots, n_\lambda$ of integers inductively by

$$n_i = r_{i-1}n_{i-1} - n_{i-2}, \qquad i = 2, 3, \ldots, \lambda. \tag{5.4.1}$$

Let e $(1 \leq e \leq \lambda)$ be the maximal integer such that $0 < n_i \leq m_i$, $(i = 1, 2, \ldots, e)$. From (5.4.1), we have

$$\frac{n_{i-1} + n_{i+1}}{n_i} = r_i, \quad i = 1, 2, \ldots, e-1,$$

and so $\mathbf{n} = (n_0, n_1, \ldots, n_e)$ is a multiplicity sequence. We say that $\mathbf{n}$ is a *dominant sequence* with the first two terms n_0 and n_1; by construction, $\mathbf{n}$ is of maximal length among the multiplicity sequences with first two terms n_0 and n_1. For the case $e = 0$, the 'sequence' $\mathbf{n} = (n_0)$ is conventionally regarded as a dominant sequence. Henceforth unless otherwise stated, we suppose $e \geq 1$. We summarize several properties of dominant sequences.

Lemma 5.4.1 *Let* $\mathbf{n} = (n_0, n_1, \ldots, n_e)$ *be a dominant sequence. If* $n_0 \geq n_1$ *and* $r_1 = r_2 = \cdots = r_{e-1} = 2$*, then* $n_0 \geq n_1 \geq n_2 \geq \cdots \geq n_e$. (Note: in this case, $(m_1, m_2, \ldots, m_e)$ is an arithmetic progression.)

Proof. We show $n_i \geq n_{i+1}$ $(i = 0, 1, \ldots, e-1)$ by induction. First, $n_0 \geq n_1$ by assumption. Next, assuming $n_{i-1} \geq n_i$, we show $n_i \geq n_{i+1}$. Since $r_i = 2$, we have $n_{i+1} = 2n_i - n_{i-1}$, and hence

$$n_i - n_{i+1} = n_i - (2n_i - n_{i-1}) = n_{i-1} - n_i \geq 0,$$

where the last inequality follows from the inductive hypothesis $n_{i-1} \geq n_i$. Therefore $n_i \geq n_{i+1}$, and this completes the inductive step. □

Remark 5.4.2 By the same argument, if $r_i = r_{i+1} = \cdots = r_j = 2$, then $n_{i-1} \geq n_i \geq \cdots \geq n_j$.

We point out that unless $r_1 = r_2 = \cdots = r_\lambda = 2$, the inequalities

$$n_0 \geq n_1 \geq n_2 \geq \cdots \geq n_e > 0$$

is *not* necessarily correct. For instance,

(a) $\mathbf{m} = (15, 7, 6, 5, 4, 3, 2, 1)$ and $\mathbf{n} = (5, 3, 4, 5)$
(b) $\mathbf{m} = (90, 20, 10)$ and $\mathbf{n} = (2, 2, 8)$.

We next provide a result 'opposite' to Lemma 5.4.1.

Lemma 5.4.3 *Let* $\mathbf{n} = (n_0, n_1, \ldots, n_e)$ *be a multiplicity sequence such that there exists* k $(1 \le k \le e)$ *satisfying* $n_{k-1} < n_k$ (resp. $n_{k-1} \le n_k$). *Then*

$$n_{k-1} < n_k < n_{k+1} < \cdots < n_e \qquad (\text{resp. } n_{k-1} \le n_k \le n_{k+1} \le \cdots \le n_e).$$

(In particular, retaking $k := \min\{i : n_{i-1} < n_i\}$, we have the following inequalities:

$$n_0 > n_1 > \cdots > n_{k-1} < n_k < n_{k+1} < \cdots < n_e.$$

The same is valid for replacing "$<$" by "$\le$".)

Proof. We show this by induction. From the definition of a multiplicity sequence,

$$n_{i+1} = r_i n_i - n_{i-1}, \qquad (i = 1, 2, \ldots, e-1). \tag{5.4.2}$$

Suppose that $n_i > n_{i-1}$, and then

$$\begin{aligned} n_{i+1} &= r_i n_i - n_{i-1} \\ &\ge 2n_i - n_{i-1} \qquad \text{by } r_i \ge 2 \\ &= n_i + (n_i - n_{i-1}) \\ &> n_i, \end{aligned}$$

where the last inequality follows from the inductive hypothesis $n_i > n_{i-1}$. Hence we have $n_{i+1} > n_i$, completing the inductive step. □

We now return to subbranches.

Definition 5.4.4 A subbranch $Y = n_0\Delta_0 + n_1\Theta_1 + \cdots + n_e\Theta_e$ is called *dominant* if the multiplicity sequence $\mathbf{n} = (n_0, n_1, \ldots, n_e)$ is dominant.

Let $Y = n_0\Delta_0 + n_1\Theta_1 + \cdots + n_e\Theta_e$ be a subbranch. Then for any integer e' $(0 \le e' < e)$, we say that a subbranch $Y' = n_0\Delta_0 + n_1\Theta_1 + \cdots + n_{e'}\Theta_{e'}$ *is contained in* Y.

Proposition 5.4.5 *Any subbranch* Y *is contained in a unique dominant subbranch.*

Proof. We write $Y = n_0\Delta_0 + n_1\Theta_1 + \cdots + n_f\Theta_f$, and then from the definition of a subbranch,

$$n_i = r_{i-1}n_{i-1} - n_{i-2}, \qquad (i = 1, 2, \ldots, f).$$

We next define a sequence $n_{f+1}, n_{f+2}, \ldots, n_e$ inductively by $n_i = r_{i-1}n_{i-1} - n_{i-2}$, where e $(e \le \lambda)$ is the maximal integer such that $0 < n_i \le m_i$ holds for each $i = f+1, f+2, \ldots, e$. Then $Z = n_0\Delta_0 + n_1\Theta_1 + \cdots + n_e\Theta_e$ is a dominant subbranch containing Y. The uniqueness of Z is immediate from the uniqueness of a dominant sequence with the first two terms n_0 and n_1. □

5.5 Tame and wild subbranches

Let $DA_{e-1}(Y,d)$ be the deformation atlas associated with a dominant subbranch Y of length e. We would ideally like to construct its complete propagation. Unfortunately, this is *not* always possible and depends on some arithmetic properties ("tame" and "wild" to be defined below) of the multiplicities of Y.

Lemma 5.5.1 *Let $Y = n_0\Delta_0 + n_1\Theta_1 + \cdots + n_e\Theta_e$ be a dominant subbranch. If $e \leq \lambda - 1$, then precisely one of the following inequalities holds:*

$$\frac{n_{e-1}}{n_e} \geq r_e \tag{5.5.1}$$

$$r_e > \frac{n_{e-1} + m_{e+1}}{n_e}. \tag{5.5.2}$$

(If we formally define $m_{\lambda+1}$ as 0, then this lemma is also valid for $e = \lambda$.)

Proof. From the definition of dominance, e is the maximal integer such that $0 < n_i \leq m_i$ holds for $i = 1, 2, \ldots, e$. Therefore by the recursive relation $n_i = r_{i-1}n_{i-1} - n_{i-2}$, the integer $n_{e+1}\,(= r_e n_e - n_{e-1})$ satisfies either (i) $0 \geq n_{e+1}$ or (ii) $n_{e+1} > m_{e+1}$. Namely

$$\text{(i)}\quad 0 \geq r_e n_e - n_{e-1} \quad \text{or} \quad \text{(ii)}\quad r_e n_e - n_{e-1} > m_{e+1}.$$

The assertion follows immediately from these inequalities. □

In the above lemma, the reader may wonder that the converse to the inequality $\frac{n_{e-1}}{n_e} \geq r_e$ is given by $\frac{n_{e-1}}{n_e} < r_e$. But when Y is dominant, $\frac{n_{e-1}}{n_e} < r_e$ implies a stronger inequality $\dfrac{n_{e-1} + m_{e+1}}{n_e} < r_e$.

Remark 5.5.2 In (5.5.1), n_{e-1}/n_e is not necessarily an integer, e.g. $\mathbf{m} = (6,2)$ and $\mathbf{n} = (3,2)$. Also note that for the case $e = \lambda$, it may happen that $r_\lambda > n_{\lambda-1}/n_\lambda$. For example,

(i) $\mathbf{m} = (7,5,3,1)$ and $\mathbf{n} = (1,1,1,1)$
(ii) $\mathbf{m} = (6,2)$ and $\mathbf{n} = (2,1)$ or $(2,2)$.

Now we introduce important notions related to the propagatability of deformation atlases.

Definition 5.5.3 Let $Y = n_0\Delta_0 + n_1\Theta_1 + \cdots + n_e\Theta_e$ be a dominant subbranch. Then Y is called (1) *tame* if $\frac{n_{e-1}}{n_e} \geq r_e$, and (2) *wild* if one of the following conditions is satisfied:

(W.1) $e \leq \lambda - 1$ and $\dfrac{n_{e-1} + m_{e+1}}{n_e} < r_e$

(W.2) $e = \lambda$ and $\frac{n_{\lambda-1}}{n_\lambda} < r_\lambda$.

By Lemma 5.5.1*, any dominant subbranch is either tame or wild.*

(When Y is not dominant, “tame” and “wild” will be defined in Definition 5.5.5 below.) When a dominant subbranch Y is wild, the irreducible component Θ_e is called *wild*. According to whether a dominant subbranch Y is tame or wild, the multiplicity sequence $\mathbf{n} = (n_0, n_1, \dots, n_e)$ is called *tame* or *wild*.

Tame example: $\mathbf{m} = (4, 3, 2, 1)$ and $\mathbf{n} = (3, 2, 1)$.

Then $\frac{n_1}{n_2} = r_2 = 2$, and so $\mathbf{n}$ satisfies (1) of Definition 5.5.3.

Wild example 1: $\mathbf{m} = (4, 3, 2, 1)$ and $\mathbf{n} = (2, 2, 2)$.

Then $r_2 = 2 > \frac{n_1 + m_2}{n_2} = \frac{3}{2}$, and so $\mathbf{n}$ satisfies (W.1) of Definition 5.5.3.

Wild example 2: $\mathbf{m} = (10, 6, 2)$ and $\mathbf{n} = (3, 2, 1)$.

Then $r_2 = 3 > \frac{n_1}{n_2} = 2$, and so $\mathbf{n}$ satisfies (W.2) of Definition 5.5.3.

Remark 5.5.4 If a dominant sequence $\mathbf{n} = (n_0, n_1, \dots, n_e)$ satisfies $n_{e-1} < n_e$, then $\mathbf{n}$ is wild. In fact, $\frac{n_{e-1}}{n_e} < 1 < r_e$, because $r_e \geq 2$. (Recall that $r_i \geq 2$ for any $i = 1, 2, \dots, \lambda$.)

As we will show in Proposition 5.5.11, “wildness” is, in some sense, an obstruction for the propagatability of a deformation atlas.

Next, we define “tame” and “wild” for (not necessarily dominant) subbranches. Recall that any subbranch is contained in a unique dominant subbranch (Proposition 5.4.5).

Definition 5.5.5 A subbranch Y is called *tame* (resp. *wild*) if the dominant subbranch containing Y is tame (resp. wild).

As long as we consider a deformation atlas associated with *one* subbranch, it is enough to consider a dominant subbranch. But we will later construct a deformation atlas associated with a *set* of subbranches, in which case we need to consider subbranches which are not necessarily dominant. Until then, we mainly work with dominant subbranches.

We now introduce a quantity which will play an important role for describing deformations.

Definition 5.5.6 Suppose that $Y = n_0\Delta_0 + n_1\Theta_1 + \cdots + n_e\Theta_e$ is a dominant subbranch. Then an integer $q(Y) := n_{e-1} - r_e n_e$ is called the *slant* of Y.

Besides the description of deformations, the slant $q(Y)$ is also used for characterizing the type (tame or wild) of the dominant subbranch as follows.

Lemma 5.5.7 *Let $Y = n_0\Delta_0 + n_1\Theta_1 + \cdots + n_e\Theta_e$ be a dominant subbranch of X. Set $q = q(Y) = n_{e-1} - r_e n_e$, and then the following equivalences hold:*
(1) *if $e \leq \lambda - 1$, then*

(1.a) $\qquad Y$ is tame $\iff q \geq 0,$

(1.b) $\qquad Y$ is wild $\iff 0 > m_{e+1} + q.$

(2) *if* $e = \lambda$, *then*

$$\text{(2.a)} \qquad Y \text{ is tame} \iff q \geq 0,$$

$$\text{(2.b)} \qquad Y \text{ is wild} \iff 0 > q.$$

(If we formally define $m_{\lambda+1}$ as 0, then (2) is a special case of (1).)

Proof. (1.a) and (1.b) respectively restate (1) and (W.1) of Definition 5.5.3, while (2.a) and (2.b) respectively restate (1) and (W.2) of Definition 5.5.3. □

We turn to discuss deformation atlases associated with subbranches, and study their propagatability. Let $Y = n_0\Delta_0 + n_1\Theta_1 + \cdots + n_e\Theta_e$ be a subbranch of a branch $X = m_0\Delta_0 + m_1\Theta_1 + \cdots + m_\lambda\Theta_\lambda$. For the deformation atlas $DA_{e-1}(Y)$ associated with Y, we would like to construct an e-th propagation of $DA_{e-1}(Y)$. We already know that if $m_\lambda = 1$, then $DA_{\lambda-1}$ is always propagatable to a complete one by Propagation Lemma (Lemma 5.2.2). Thus we assume that $e \leq \lambda - 1$, or $e = \lambda$ and $m_\lambda \geq 2$.

Lemma 5.5.8 *Consider a hypersurface*

$$\mathcal{H}_e : \ w^{m_{e-1}-n_{e-1}}\eta^{m_e-n_e}(w^{n_{e-1}}\eta^{n_e} + t) - s = 0.$$

Then the transition function $z = 1/w$, $\zeta = w^{r_e}\eta$ *of* N_e *transforms the equation of* $\mathcal{H}_e$ *to*

$$\begin{cases} z^{m_{e+1}}\zeta^{m_e-n_e}(\zeta^{n_e} + tz^q) - s & \text{if } e \leq \lambda - 1 \\ \zeta^{m_\lambda-n_\lambda}(\zeta^{n_\lambda} + tz^q) - s & \text{if } e = \lambda. \end{cases} \tag{5.5.3}$$

Proof. Note that (5.5.3) is written as

$$\begin{cases} z^{m_{e+1}}\zeta^{m_e} - s + tz^{m_{e+1}+q}\zeta^{m_e-n_e} & \text{if } e \leq \lambda - 1 \\ \zeta^{m_\lambda} - s + tz^q\zeta^{m_\lambda-n_\lambda} & \text{if } e = \lambda, \end{cases}$$

where $q = n_{e-1} - r_e n_e$. First we consider the case $e \leq \lambda - 1$, in which case the map $z = 1/w$, $\zeta = w^{r_e}\eta$ transforms $w^{m_{e-1}-n_{e-1}}\eta^{m_e-n_e}$ to $z^{m_{e+1}+q}\zeta^{m_e-n_e}$. In fact,

$$\begin{aligned} w^{m_{e-1}-n_{e-1}}\eta^{m_e-n_e} &= w^{m_{e-1}-n_{e-1}-r_e(m_e-n_e)}\,(w^{r_e}\eta)^{m_e-n_e} \\ &= z^{r_e(m_e-n_e)-(m_{e-1}-n_{e-1})}\zeta^{m_e-n_e} \\ &= z^{m_{e+1}+q}\zeta^{m_e-n_e}, \end{aligned}$$

where the last equality is derived from

$$\begin{aligned} r_e(m_e - n_e) - (m_{e-1} - n_{e-1}) &= (r_e m_e - m_{e-1}) + (n_{e-1} - r_e n_e) \\ &= m_{e+1} + q. \end{aligned}$$

This confirms the assertion for $e \leq \lambda - 1$. For the case $e = \lambda$, just set $m_{\lambda+1} = 0$ in the above computation. □

From Lemma 5.5.8 with (1.a) and (2.a) of Lemma 5.5.7, if Y is tame, then (5.5.3) are polynomials (i.e. with no fractional terms), and an e-th propagation of $DA_{e-1}(Y)$ is given by

$$\mathcal{H}'_e : \begin{cases} z^{m_{e+1}}\zeta^{m_e-n_e}(\zeta^{n_e}+tz^q)-s=0 & \text{if } e\le\lambda-1 \\ \zeta^{m_\lambda-n_\lambda}(\zeta^{n_\lambda}+tz^q)-s=0 & \text{if } e=\lambda \end{cases} \tag{5.5.4}$$

with g_e being the transition function of N_e. Furthermore we will show in Theorem 6.1.1, p99 that $DA_{e-1}(Y)$ for tame Y always admits a complete propagation. To the contrary, we have the following result for the wild case.

Corollary 5.5.9 *If Y is wild, then the transition function g_e of N_e does not transform $\mathcal{H}_e$ to a hypersurface.*

Remark 5.5.10 Instead of a single subbranch Y, we will later use a set of subbranches for the construction of a deformation atlas. Then the situation is quite different. See Remark 20.3.1, p368, and also Remark 12.5.1, p230.

Proof. By Lemma 5.5.8, g_e transforms $\mathcal{H}_e$ to

$$\mathcal{H}'_e : \begin{cases} z^{m_{e+1}}\zeta^{n_e}+tz^{m_{e+1}+q}\zeta^{m_e-n_e}-s=0 & \text{if } e\le\lambda-1 \\ \zeta^{m_\lambda}+tz^q\zeta^{m_\lambda-n_\lambda}-s=0 & \text{if } e=\lambda, \end{cases}$$

where $q=n_{e-1}-r_en_e$. However, this is *not* a hypersurface; since Y is wild, from (1.b) and (2.b) of Lemma 5.5.7, we have

$$\begin{cases} m_{e+1}+q<0 & \text{if } e\le\lambda-1 \\ q<0 & \text{if } e=\lambda, \end{cases}$$

and therefore the exponents of $z^{m_{e+1}+q}$ for the case $e\le\lambda-1$ and z^q for the case $e=\lambda$ are negative. This means that $\mathcal{H}'_e$ is not well-defined as a hypersurface. □

We summarize the above results as follows.

Proposition 5.5.11 *The following data*

$$\begin{cases} \mathcal{H}_e: & w^{m_{e-1}-n_{e-1}}\eta^{m_e-n_e}(w^{n_{e-1}}\eta^{n_e}+t)-s=0 \\ \mathcal{H}'_e: & z^{m_{e+1}}\zeta^{m_e-n_e}(\zeta^{n_e}+tz^q)-s=0 \\ & \text{(by convention, } m_{\lambda+1}=0 \text{ if } e=\lambda) \\ g_e: & \text{the transition function } z=1/w,\ \zeta=w^{r_e}\eta \text{ of } N_e. \end{cases}$$

defines an e-th propagation of $DA_{e-1}(Y)$ if and only if Y is tame.

This statement implies that for the wild case, in order to construct an e-th propagation of $DA_{e-1}(Y)$, we must deform the transition function of N_e in such a way that it maps $\mathcal{H}_e$ to some hypersurface. Unfortunately, this is *not* always possible, as is seen from the following example.

Example 5.5.12 Consider a branch

$$X=15\Delta_0+7\Theta_1+6\Theta_2+5\Theta_3+4\Theta_4+3\Theta_5+2\Theta_6+\Theta_7,$$

and take a dominant subbranch $Y = 5\Delta_0 + 3\Theta_1 + 4\Theta_2 + 5\Theta_3$. Notice that Y is wild, because

$$\frac{n_2 + m_4}{n_3} = \frac{4+4}{5} < r_3 = 2.$$

Let $DA_2(Y, d)$ be the deformation atlas associated with Y, and then

$$\mathcal{H}_3 : \ w^6\eta^5 - s + ct^d w^2 = 0, \qquad \text{where } c \in \mathbb{C} \text{ is nonzero.}$$

We take the "standard form" $g_3 : \ z = 1/w, \ \zeta = w^2\eta + \alpha(t)w$ of a deformation of the transition function of N_3 (see §5.5.1 below). Then g_3 transforms $\mathcal{H}_3$ to

$$\begin{aligned}
&w^6\eta^5 - s + ct^d w^2 \\
&= \frac{1}{w^4}(w^2\eta)^5 - s + ct^d w^2 \\
&= z^4\left(\zeta - \alpha(t)\frac{1}{z}\right)^5 - s + ct^d\frac{1}{z^2} \\
&= \left(z^4\zeta^5 - 5\alpha(t)z^3\zeta^4 + 10\alpha(t)^2z^2\zeta^3 - 10\alpha(t)^3z\zeta^2 + 5\alpha(t)^4\zeta - \alpha(t)^5\frac{1}{z}\right) \\
&\qquad - s + ct^d\frac{1}{z^2}.
\end{aligned}$$

The last expression contains fractional terms $-\alpha(t)^5\frac{1}{z} + ct^d\frac{1}{z^2}$ (recall $c \neq 0$ while possibly $\alpha(t) \equiv 0$), and hence it cannot define a hypersurface; hence $DA_2(Y, d)$ does not admit a further propagation.

It is heuristic to check the above statement also for g_3 of a non-standard form, that is, with higher or lower order terms, e.g. g_3 :

$$z = \frac{1}{w} + \left(tw^2 + t^3\frac{1}{w} + t(t+1)\eta\right), \quad \zeta = w^2\eta + \alpha(t)w + \left(tw\eta + t\frac{1}{w^4} + t^3\eta^2\right),$$

where the terms inside the parentheses are "non-standard" terms; they may have a pole at $w = 0$ but is necessarily holomorphic at $\eta = 0$, because we glue $(w, \eta) \in (U \setminus \{0\}) \times \mathbb{C}$ and $(z, \zeta) \in (U' \setminus \{0\}) \times \mathbb{C}$ (or shrinkage of these spaces) via g_3. We claim that g_3 does not transform $\mathcal{H}_3$ to any hypersurface; there appear fractional terms after transformation. In fact, note that g_3^{-1} has a form:

$$g_3^{-1} : \quad w = \frac{1}{z} + \sum_{i,j}\gamma_{i,j}(t)z^i\zeta^j, \qquad \eta = z^2\zeta - \alpha(t)z + \sum_{i,j}\beta_{i,j}(t)z^i\zeta^j,$$

where $\alpha(0) = \beta_{i,j}(0) = \gamma_{i,j}(0) = 0$, and so the map g_3 transforms $\mathcal{H}_3$ to

$$\begin{aligned}
&w^6\eta^5 - s + ct^d w^2 \\
&= \left(\frac{1}{z} + \sum_{i,j}\gamma_{i,j}(t)z^i\zeta^j\right)^6 \left(z^2\zeta - \alpha(t)z + \sum_{i,j}\beta_{i,j}(t)z^i\zeta^j\right)^5 \\
&\qquad - s + ct^d\left(\frac{1}{z} + \sum_{i,j}\gamma_{i,j}(t)z^i\zeta^j\right)^2,
\end{aligned}$$

which, after expansion, contains fractional terms.

5.5.1 Supplement: Riemenschneider's work

Possibly after a coordinate change, we may assume that a deformation of a transition function $z = 1/w$, $\zeta = w^r \eta$ is of the form

$$z = \frac{1}{w}, \quad \zeta = w^{r_i}\eta + \alpha_1(t)w + \alpha_2(t)w^2 + \cdots + \alpha_{r_i-1}(t)w^{r_i-1},$$

where $\alpha_k(t)$ $(k = 1, 2, \ldots, r_i - 1)$ is a holomorphic function in t with $\alpha_k(0) = 0$. This map is related to a *versal deformation* of the resolution space of a cyclic quotient singularity. To explain this, we review some terminologies in deformation theory. Let V be an analytic object, such as a germ of an isolated singularity, the resolution space of a surface singularity, or a compact complex manifold. By a deformation of V, we mean a flat holomorphic map $f : \mathcal{V} \to B$ where $\mathcal{V}$ and B are complex analytic spaces with the base point $0 \in B$ such that $f^{-1}(0) = V$. Then $\mathcal{V} \to B$ is called *versal* (or *semi-universal*) if for any deformation $\mathcal{V}' \to B'$ of V, there exists a holomorphic map $h : B' \to B$ such that $\mathcal{V}' \to B'$ is the pull-back of $\mathcal{V} \to B$ via h; notice that h with this property may *not* be unique. For any deformation $\mathcal{V}' \to B'$, if we could always take a *unique* holomorphic map h with the above property, then we say that $\mathcal{V} \to B$ is *universal*. When V is a local object (e.g. a germ of an isolated singularity) or non-compact (e.g. the resolution space of a surface singularity), it often occurs that V admits many automorphisms, so that h is not unique and accordingly $\mathcal{V} \to B$ is not universal.

We turn to discuss a tubular neighborhood M of a branch. Let N_i be a line bundle obtained by patching $(w, \eta) \in U_i \times \mathbb{C}$ with $(z, \zeta) \in V_i \times \mathbb{C}$ via $z = 1/w$, $\zeta = w^{r_i}\eta$, and then M is a complex surface obtained by plumbing line bundles $N_1, N_2, \ldots, N_\lambda$. Note that M is the resolution space of a cyclic quotient singularity (Remark 5.1.2, p87), and so by the work of Riemenschneider, we can construct the versal deformation of M as follows (see [Ri3] for details). First consider $B := \prod_{i=1}^{\lambda} \mathbb{C}^{r_i-1}$ with coordinates

$$\prod_{i=1}^{\lambda} (t_{i,1},\, t_{i,2},\, \ldots\, , t_{i,r_i-1}) \in B.$$

Then construct a complex manifold $\mathcal{N}_i$ by patching $U_i \times \mathbb{C} \times B$ with $V_i \times \mathbb{C} \times B$ by

$$g_i : \quad z = \frac{1}{w}, \quad \zeta = w^{r_i}\eta + t_{i,1}w + t_{i,2}w^2 + \cdots + t_{i,r_i-1}w^{r_i-1}.$$

Next patch $\mathcal{N}_i$ and $\mathcal{N}_{i+1}$ $(i = 1, 2, \ldots, \lambda - 1)$ by plumbings, i.e. $(z, \zeta) = (\eta, w)$, which yields a complex manifold $\mathcal{M}_{\mathrm{ver}}$ and the natural projection $\mathcal{M}_{\mathrm{ver}} \to B$ is the versal deformation of M.

6

Construction of Deformations by Tame Subbranches

We have classified subbranches into two types: tame and wild ones. Also, we showed that any subbranch is contained in a unique dominant subbranch. In this chapter, from dominant *tame* subbranches we will construct complete deformation atlases.

6.1 Construction of deformations by tame subbranches

Suppose that $X = m_0\Delta_0 + m_1\Theta_1 + \cdots + m_\lambda\Theta_\lambda$ is a branch, and $Y = n_0\Delta_0 + n_1\Theta_1 + \cdots + n_e\Theta_e$ is a dominant subbranch of X; as before we adopt the convention $m_{\lambda+1} = 0$. Taking an arbitrary positive integer d, let $DA_{e-1}(Y, d)$ be the deformation atlas associated with Y of weight d, i.e. for $i = 1, 2, \ldots, e-1$,

$$\begin{cases} \mathcal{H}_i : & w^{m_{i-1}-n_{i-1}}\eta^{m_i-n_i}(w^{n_{i-1}}\eta^{n_i} + t^d) - s = 0 \\ \mathcal{H}'_i : & z^{m_{i+1}-n_{i+1}}\zeta^{m_i-n_i}(z^{n_{i+1}}\zeta^{n_i} + t^d) - s = 0 \\ g_i : & \text{the transition function } z = 1/w,\ \zeta = w^{r_i}\eta \text{ of } N_i. \end{cases}$$

Theorem 6.1.1 *If Y is tame, then $DA_{e-1}(Y, d)$ admits a complete propagation.*
(We let $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$ be the barking family associated with this complete deformation atlas. In the course of the proof, we will also describe the deformation from X to $X_{0,t} := \Psi^{-1}(0, t)$.)

Proof. For simplicity, we write $DA_{e-1} = DA_{e-1}(Y, d)$. We only show the statement for the case $d = 1$ (the proof below also works for arbitrary d). First of all, setting $q := n_{e-1} - r_e n_e$ (the slant of Y), then by Lemma 5.5.7, we have

$$q \geq 0. \tag{6.1.1}$$

We give the proof of the assertion separately for[1]: Case 1. $e = \lambda$ and Case 2. $e < \lambda$.

[1] For example, Case 1 $\mathbf{m} = (8, 6, 4, 2)$, $\mathbf{n} = (4, 3, 2, 1)$ Case 2 $\mathbf{m} = (9, 7, 5, 3, 1)$, $\mathbf{n} = (8, 5, 2)$.

Case 1 $e = \lambda$**:** We set

$$\begin{cases} \mathcal{H}_\lambda : & w^{m_{\lambda-1}-n_{\lambda-1}}\eta^{m_\lambda - n_\lambda}(w^{n_{\lambda-1}}\eta^{n_\lambda} + t) - s = 0 \\ \mathcal{H}'_\lambda : & \zeta^{m_\lambda - n_\lambda}(\zeta^{n_\lambda} + tz^q) - s = 0 \\ g_\lambda : & \text{the transition function } z = 1/w, \ \zeta = w^{r_\lambda}\eta \text{ of } N_\lambda, \end{cases} \tag{6.1.2}$$

where $q = n_{\lambda-1} - r_\lambda n_\lambda$. By (6.1.1), the exponent q of z^q is nonnegative. Moreover since $m_{\lambda-1} - n_{\lambda-1} \geq 0$, $m_\lambda - n_\lambda \geq 0$, $n_{\lambda-1} > 0$ and $n_\lambda > 0$, all exponents of the terms in the defining equations of $\mathcal{H}_\lambda$ and $\mathcal{H}'_\lambda$ are nonnegative. Hence $\mathcal{H}_\lambda$ and $\mathcal{H}'_\lambda$ are well-defined as hypersurfaces. Next, by Lemma 5.5.8, g_λ transforms $\mathcal{H}_\lambda$ to $\mathcal{H}'_\lambda$, and so (6.1.2) gives a complete propagation of $DA_{\lambda-1}$. This proves the assertion for the case $e = \lambda$.

Let $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$ be the barking family associated with this complete deformation atlas. Then according to whether $q = 0$ or $q \geq 1$, the singular fiber X is deformed to $X_{0,t} := \Psi^{-1}(0, t)$ as illustrated in Figure 6.1.1 or Figure 6.1.2 respectively. See Remark 4.2.3, p65 for how to draw figures.

Case 2 $e < \lambda$**:** By Proposition 5.5.11, we may define an e-th propagation of DA_{e-1} as follows:

$$\begin{cases} \mathcal{H}_e : & w^{m_{e-1}-n_{e-1}}\eta^{m_e - n_e}(w^{n_{e-1}}\eta^{n_e} + t) - s = 0 \\ \mathcal{H}'_e : & z^{m_{e+1}}\zeta^{m_e - n_e}(\zeta^{n_e} + tz^q) - s = 0 \\ g_e : & \text{the transition function } z = 1/w, \ \zeta = w^{r_e}\eta \text{ of } N_e, \end{cases} \tag{6.1.3}$$

where we recall $q = n_{e-1} - r_e n_e$. We claim that the following data gives an $(e+1)$-st propagation of DA_{e-1}:

Case 1
$(e = \lambda,\ m_\lambda \geq 2)$
(1) $q = 0$

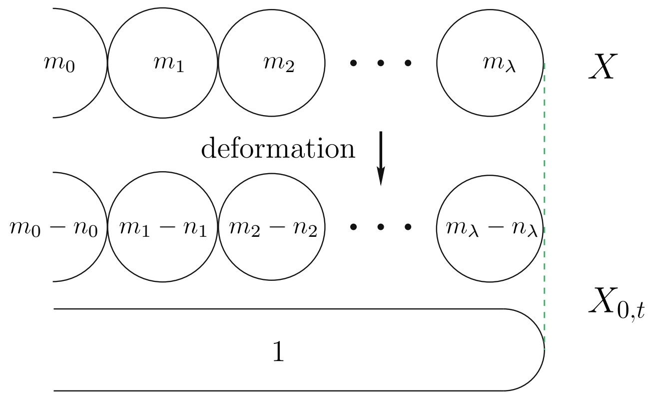

Fig. 6.1.1. The above figure is for the case $n_\lambda = 1$. When $n_\lambda \geq 2$, there are disjoint n_λ connected components of multiplicity 1. Note that when $q = 0$, $\mathcal{H}'_\lambda|_{s=0}$: $\zeta^{m_\lambda - n_\lambda}(\zeta^{n_\lambda} + t) = 0$ is a disjoint union of two curves $C_1 : \zeta^{m_\lambda - n_\lambda} = 0$ and $C_2 : \zeta^{n_\lambda} + t = 0$, where C_2 for $t \neq 0$ consists of disjoint n_λ copies of a line $\mathbb{C}$.

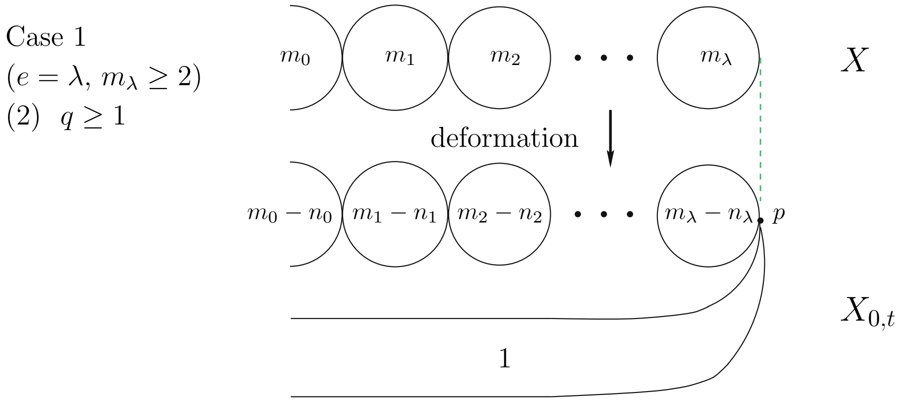

Fig. 6.1.2. This figure is for the case $n_\lambda = 1$. When $n_\lambda \geq 2$, there are n_λ connected components of multiplicity 1, and they intersect at a unique point $(z_\lambda, \zeta_\lambda) = (0,0)$, denoted by p in the figure. For the case $q \geq 1$, $\mathcal{H}'_\lambda|_{s=0}: \ \zeta^{m_\lambda - n_\lambda}(\zeta^{n_\lambda} + tz^q) = 0$. In particular, if $q = 1$, then $\mathcal{H}'_\lambda|_{s=0}$ consists of two curves $C_1 : \zeta^{m_\lambda - n_\lambda} = 0$ and $C_2 : \zeta^{n_\lambda} + tz = 0$, where C_2 for $t \neq 0$ is a line intersecting C_1 at one point p transversely, and if furthermore $m_\lambda = 2$ and $n_\lambda = 1$, then $\mathcal{H}'_\lambda|_{s=0}: \ \zeta(\zeta + tz) = 0$ defines an ordinary double point (in this situation, we may apply a splitting criterion in [Ta,I] to obtain a splitting deformation of $X_{0,t}$).

$$\begin{cases} \mathcal{H}_{e+1}: & w^{m_e - n_e}\eta^{m_{e+1}}(w^{n_e} + t\,\eta^q) - s = 0 \\ \mathcal{H}'_{e+1}: & z^{m_{e+2}}\zeta^{m_{e+1}}(1 + t\,z^{r_{e+1}\,q + n_e}\,\zeta^q) - s = 0 \\ g_{e+1}: & \text{the transition function } z = 1/w, \ \zeta = w^{r_{e+1}}\eta \text{ of } N_{e+1}. \end{cases} \tag{6.1.4}$$

In fact, since $q \geq 0$ (6.1.1), the defining equations of $\mathcal{H}_{e+1}$ and $\mathcal{H}'_{e+1}$ do not contain fractional terms, and so $\mathcal{H}_{e+1}$ and $\mathcal{H}'_{e+1}$ are well-defined as hypersurfaces. We next show that g_{e+1} transforms $\mathcal{H}_{e+1}$ to $\mathcal{H}'_{e+1}$. To see this, we shall rewrite $\mathcal{H}_{e+1}$ as

$$\mathcal{H}_{e+1}: \quad w^{m_e - n_e}\,\frac{1}{w^{\,r_{e+1}\,m_{e+1}}}\left(w^{\,r_{e+1}}\,\eta\right)^{m_{e+1}}\left(w^{n_e} + t\,\frac{1}{w^{\,r_{e+1}\,q}}\,(w^{\,r_{e+1}}\eta)^q\right) - s = 0.$$

Since $\dfrac{m_e + m_{e+2}}{m_{e+1}} = r_{e+1}$, we have $r_{e+1}m_{e+1} - m_e = m_{e+2}$, and so

$$w^{m_e - n_e}\,\frac{1}{w^{\,r_{e+1}\,m_{e+1}}} = \frac{1}{w^{\,r_{e+1}m_{e+1}\,-m_e\,+n_e}} = \frac{1}{w^{\,m_{e+2}\,+\,n_e}}.$$

Therefore

$$\mathcal{H}_{e+1}: \quad \frac{1}{w^{\,m_{e+2}\,+\,n_e}}\left(w^{\,r_{e+1}}\,\eta\right)^{m_{e+1}}\left(w^{n_e} + t\,\frac{1}{w^{\,r_{e+1}\,q}}\,(\,w^{\,r_{e+1}}\eta\,)^q\right) - s = 0,$$

and hence by $g_{e+1} : z = 1/w,\ \zeta = w^{r_{e+1}}\eta$, the hypersurface $\mathcal{H}_{e+1}$ is transformed to

$$z^{m_{e+2}+n_e}\,\zeta^{m_{e+1}}\left(\frac{1}{z^{n_e}} + t\,z^{r_{e+1}\,q}\,\zeta^q\right) - s = 0,$$

that is, $z^{m_{e+2}}\,\zeta^{m_{e+1}}\,(1 + t\,z^{r_{e+1}\,q+n_e}\,\zeta^q) - s = 0$. This is nothing but $\mathcal{H}'_{e+1}$, and thus g_{e+1} transforms $\mathcal{H}_{e+1}$ to $\mathcal{H}'_{e+1}$. Therefore (6.1.4) gives an $(e+1)$-st propagation of DA_{e-1}.

To construct further propagations of DA_{e-1}, we define integers a_i ($i = e+1, e+2, \dots, \lambda+1$) inductively by

$$\begin{cases} a_{e+1} := q, \quad a_{e+2} := r_{e+1}q + n_e \qquad \text{and} \\ a_{i+1} := r_i a_i - a_{i-1} \qquad \text{for} \quad i = e+2,\ e+3,\ \dots,\ \lambda. \end{cases} \tag{6.1.5}$$

Note 1. $a_i \geq 0$, in fact, $a_{\lambda+1} > a_\lambda > \cdots > a_{e+1} \geq 0$. This is shown by induction. Assuming that $a_i > a_{i-1}$, then we have

$$\begin{aligned} a_{i+1} &= r_i a_i - a_{i-1} \\ &\geq 2a_i - a_{i-1} \qquad\qquad \text{by} \quad r_i \geq 2 \\ &= a_i + (a_i - a_{i-1}) \\ &> a_i, \end{aligned}$$

namely $a_{i+1} > a_i$. This completes the inductive step.

Note 2. The transition function $g_i : z = 1/w,\ \zeta = w^{r_i}\eta$ of N_i transforms $w^{a_{i-1}}\eta^{a_i}$ to $z^{a_{i+1}}\zeta^{a_i}$, which is derived from Lemma 5.3.2, because $\dfrac{a_{i-1} + a_{i+1}}{a_i} = r_i$. Now for $i = e+2, e+3, \dots, \lambda$, we set

$$\begin{cases} \mathcal{H}_i : \quad w^{m_{i-1}}\,\eta^{m_i}\,(1 + t\,w^{a_{i-1}}\,\eta^{a_i}) - s = 0 \\ \mathcal{H}'_i : \quad z^{m_{i+1}}\,\zeta^{m_i}\,(1 + t\,z^{a_{i+1}}\,\zeta^{a_i}) - s = 0 \\ g_i : \quad \text{the transition function } z = 1/w,\ \zeta = w^{r_i}\eta \text{ of } N_i, \end{cases} \tag{6.1.6}$$

where $m_{\lambda+1} = 0$ by convention. We assert that (6.1.6) gives a complete propagation of DA_{e-1}. To show this, it suffices to confirm the following two claims:

Claim A $\mathcal{H}_i$ and $\mathcal{H}'_i$ are well-defined as hypersurfaces, in other words, the equations of $\mathcal{H}_i$ and $\mathcal{H}'_i$ do not contain fractional terms,

Claim B g_i transforms $\mathcal{H}_i$ to $\mathcal{H}'_i$.

Claim A follows from Note 1: $a_i \geq 0$, while Claim B follows from Note 2: the transition function g_i of N_i transforms $w^{a_{i-1}}\eta^{a_i}$ to $z^{a_{i+1}}\zeta^{a_i}$. Therefore, (6.1.6) defines a complete propagation of DA_{e-1}.

Finally, let $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$ be the barking family obtained from the above complete deformation atlas. We set $X_{s,t} := \Psi^{-1}(s,t)$. According to whether $q = 0$ or $q \geq 1$, X is deformed to $X_{0,t}$ as illustrated in Figure 6.1.3 or Figure 6.1.4. This completes the proof of Theorem 6.1.1. □

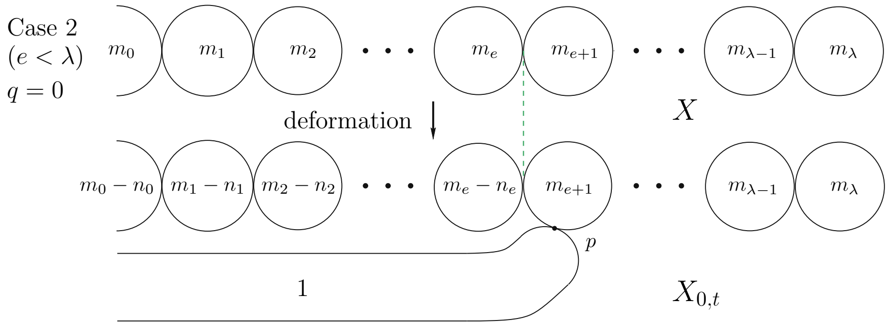

Fig. 6.1.3. The above figure is for $n_e = 1$. When $n_e \geq 2$, there are disjoint n_e connected components of multiplicity 1, which intersect Θ_{e+1} at n_e points (when $n_e = 1$, at one point p in the figure); each connected component intersects Θ_{e+1} at one point. In fact, $\mathcal{H}'_e: \ z^{m_{e+1}}\zeta^{m_e - n_e}(\zeta^{n_e} + t) = 0$ (6.1.3) is a union of two curves $C_1 : z^{m_{e+1}}\zeta^{m_e-n_e} = 0$ and $C_2 : \zeta^{n_e} + t = 0$, where C_2 for $t \neq 0$ consists of n_e copies of a line $\mathbb{C}$, and C_1 and C_2 intersect ζ-axis at n_e points.

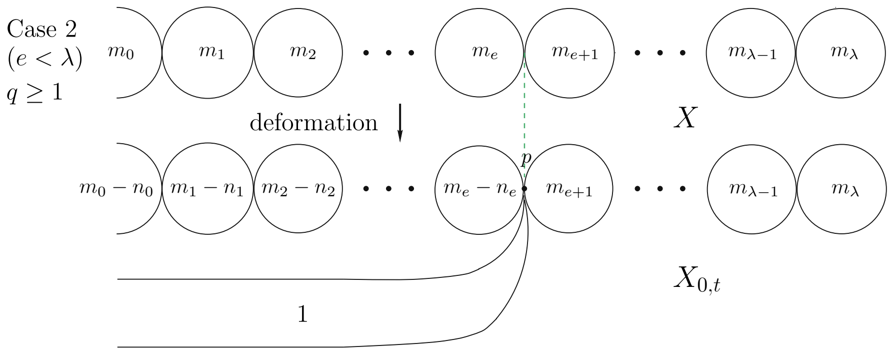

Fig. 6.1.4. The above figure is for $n_e = 1$. When $n_\lambda \geq 2$, there are n_λ connected components of multiplicity 1, which intersect at a unique point $(z_e, \zeta_e) = (0, 0)$, denoted by p in the figure. Note that $\mathcal{H}'_e|_{s=0}: \ z^{m_{e+1}}\zeta^{m_e-n_e}(\zeta^{n_e} + tz^q) = 0$ (6.1.3), and in particular, if $n_e = 1$, then $\mathcal{H}'_e|_{s=0}$ consists of $z^{m_{e+1}}\zeta^{m_e-n_e} = 0$ and a line $\zeta + tz^q = 0$. If furthermore, $m_e = 2$ and $m_{e+1} = 1$ (so $\lambda = e + 1$), then $\mathcal{H}'_e$ has an ordinary triple point (in this situation, we may apply a splitting criterion in [Ta,I] to obtain a splitting deformation of $X_{0,t}$).

Remark 6.1.2 When $|t|$ and $|w^{a_{i-1}}\eta^{a_i}|$ are sufficiently small, $1 + tw^{a_{i-1}}\eta^{a_i} \neq 0$, and so $\mathcal{H}_i$ ($i = e{+}2, e{+}3, \ldots, \lambda$) in (6.1.6) is equivalent to $w^{m_{i-1}}\eta^{m_i} + s = 0$ (e.g. by a coordinate change $w' = w$, $\eta' = \eta(1 + tw^{a_{i-1}}\eta^{a_i})^{1/m_i}$). Similarly, when $|t|$ and $|z^{a_{i+1}}\zeta^{a_i}|$ are sufficiently small, $1 + tz^{a_{i+1}}\zeta^{a_i} \neq 0$, and so $\mathcal{H}'_i$ is equivalent to $z^{m_{i+1}}\zeta^{m_i} + s = 0$. Therefore the barking family obtained from the complete deformation atlas in Case 2 ($e < \lambda$) is trivial around Θ_i ($i = e + 2, e + 3, \ldots, \lambda$).

6.2 Supplement for the proof of Theorem 6.1.1

6.2.1 Alternative construction

We give an alternative construction of a complete propagation of DA_{e-1} for Case 2 in the proof of Theorem 6.1.1. Recall that when $e \le \lambda - 1$, we defined an $(e+1)$-st propagation of DA_{e-1} as follows (see (6.1.4)):

$$\begin{cases} \mathcal{H}_{e+1}: & w^{m_e - n_e}\,\eta^{m_{e+1}}\,(w^{n_e} + t\eta^q) - s = 0 \\ \mathcal{H}'_{e+1}: & z^{m_{e+2}}\,\zeta^{m_{e+1}}\,(1 + t\,z^{r_{e+1}\,q + n_e}\,\zeta^q) - s = 0, \\ g_{e+1}: & \text{the transition function } z = 1/w,\ \zeta = w^{r_{e+1}}\eta \text{ of } N_{e+1}. \end{cases} \tag{6.2.1}$$

Notice that $\mathcal{H}'_{e+1}$ is equivalent to the trivial family of H'_{e+1}. Indeed, in new coordinates

$$(z'_e, \zeta'_e) = \Big(z_e\,,\ \zeta_e\,(1 + t\,z_{e+1}^{r_e\,q + n_e}\,\zeta_{e+1}^q)^{1/m_{e+1}}\Big),$$

the hypersurface $\mathcal{H}'_{e+1}$ is written as $(z')^{m_{e+2}}(\zeta')^{m_{e+1}} - s = 0$, and so we assume that $\mathcal{H}'_{e+1}$ is of this form. We then define further propagations 'trivially': for $i = e+2, r+3, \ldots, \lambda$,

$$\begin{cases} \mathcal{H}_i & w^{m_{i-1}}\eta^{m_i} - s = 0, \\ \mathcal{H}'_i: & z^{m_{i+1}}\zeta^{m_i} - s = 0, \\ g_i: & \text{the transition function } z = 1/w,\ \zeta = w^{r_i}\eta \text{ of } N_i. \end{cases}$$

This data clearly defines a complete propagation of DA_{e-1}.

Remark 6.2.1 This construction is much shorter than that we previously gave in the proof of Theorem 6.1.1. However, for later application the previous construction is more useful (see the proof of Theorem 14.2.7, p260).

6.2.2 Generalization

So far, we have treated only a deformation atlas of the form: for $i = 1, 2, \ldots, e-1$,

$$\begin{cases} \mathcal{H}_i: & w^{m_{i-1} - n_{i-1}}\eta^{m_i - n_i}(w^{n_{i-1}}\eta^{n_i} + t^d) - s = 0 \\ \mathcal{H}'_i: & z^{m_{i+1} - n_{i+1}}\zeta^{m_i - n_i}(z^{n_{i+1}}\zeta^{n_i} + t^d) - s = 0 \\ g_i: & \text{the transition function } z = 1/w,\ \zeta = w^{r_i}\eta \text{ of } N_i. \end{cases}$$

However, for our later application, we have to consider more general deformation atlases; let $f(z)$ be a non-vanishing holomorphic function on a domain $\{z \in \mathbb{C} : |z| < \varepsilon\}$. (Here ε is possibly ∞, e.g. when f is constant.) Then we would like to have a deformation atlas such that $\mathcal{H}_1$ is given by

$$w_1^{m_{1-1} - n_{1-1}}\eta_1^{m_1 - n_1}\Big(w_1^{n_{1-1}}\eta_1^{n_1} + t^d f(\eta_1)\Big) - s = 0. \tag{6.2.2}$$

(Precisely speaking, according to the value of ε, we have to 'shrink' $\pi : M \to \Delta$.) We introduce a sequence of integers $p_0, p_1, \ldots, p_{\lambda+1}$, inductively given by

$$\begin{cases} p_0 = 0, \quad p_1 = 1 \quad \text{and} \\ p_{i+1} = r_i p_i - p_{i-1} \quad \text{for } i = 1, 2, \ldots, \lambda. \end{cases} \tag{6.2.3}$$

Then the following inequalities hold.

$$p_{\lambda+1} > p_\lambda > \cdots > p_1 > p_0 = 0. \tag{6.2.4}$$

This is seen by induction. Suppose that $p_i > p_{i-1}$, and then

$$\begin{aligned} p_{i+1} &= r_i p_i - p_{i-1} \\ &\geq 2p_i - p_{i-1} \qquad \text{by } r_i \geq 2 \\ &= p_i + (p_i - p_{i-1}) \\ &> p_i, \end{aligned}$$

where the last inequality follows from the inductive hypothesis $p_i > p_{i-1}$. Therefore $p_{i+1} > p_i$, and so we complete the inductive step.

Lemma 6.2.2 *The transition function $g_i : z = 1/w,\ \zeta = w^{r_i}\eta$ of N_i transforms $w^{p_{i-1}}\eta^{p_i}$ to $z^{p_{i+1}}\zeta^{p_i}$.*

Proof. Recall that if a, b, c are positive integers satisfying

$$\frac{a+c}{b} = r_i,$$

then $g_i : z = 1/w,\ \zeta = w^{r_i}\eta$ transforms $w^a\eta^b$ to $z^c\zeta^b$ (Lemma 5.3.2, p90). From $\dfrac{p_{i+1} + p_{i-1}}{p_i} = r_i$ (6.2.3), the assertion follows. □

Next, we define domains $\Omega_i = \Omega_i(\varepsilon)$ and $\Omega_i' = \Omega_i'(\varepsilon)$ $(i = 1, 2, \ldots, \lambda)$ by

$$\begin{aligned} \Omega_i &= \{(w_i, \eta_i) \in U_i \times \mathbb{C} \quad : \quad |w_i^{p_{i-1}}\eta_i^{p_i}| < \varepsilon\}, \\ \Omega_i' &= \{(z_i, \zeta_i) \in U_i' \times \mathbb{C} \quad : \quad |z_i^{p_{i+1}}\zeta_i^{p_i}| < \varepsilon\}. \end{aligned} \tag{6.2.5}$$

Note that if the radius of convergence of f is ∞, then $\Omega_i = U_i \times \mathbb{C}$ and $\Omega_i' = U_i' \times \mathbb{C}$.

Now letting Y be a subbranch, we shall construct a deformation atlas of length $e-1$ such that $\mathcal{H}_1$ is given by

$$w_1^{m_0-n_0}\eta_1^{m_1-n_1}\Big(w_1^{n_0}\eta_1^{n_1} + t^d f(\eta_1)\Big) - s = 0. \tag{6.2.6}$$

First define smooth hypersurfaces in $\Omega_i \times \Delta \times \Delta^\dagger$ and $\Omega_i' \times \Delta \times \Delta^\dagger$ $(i = 1, 2, \ldots, e-1)$ by

$$\mathcal{H}_i : \quad w_i^{m_{i-1}-n_{i-1}}\eta_i^{m_i-n_i}\Big(w_i^{n_{i-1}}\eta_i^{n_i} + t^d f(w_i^{p_{i-1}}\eta_i^{p_i})\Big) - s = 0,$$

$$\mathcal{H}_i' : \quad z_i^{m_{i+1}-n_{i+1}}\zeta_i^{m_i-n_i}\Big(z_i^{n_{i+1}}\zeta_i^{n_i} + t^d f(z_i^{p_{i+1}}\zeta_i^{p_i})\Big) - s = 0.$$

(Since $p_i \geq 0$ (6.2.4), $\mathcal{H}_i$ and $\mathcal{H}'_i$ are well-defined as hypersurfaces.) Evidently, by a coordinate change $(z_i, \zeta_i, s, t) = (\eta_{i+1}, w_{i+1}, s, t)$, $\mathcal{H}'_i$ becomes $\mathcal{H}_{i+1}$. We take g_i to be the transition function $z_i = 1/w_i,\ \zeta_i = w_i^{r_i}\eta_i$ of N_i. Then by Lemma 6.2.2, g_i transforms $\mathcal{H}_i$ to $\mathcal{H}'_i$, and therefore $DA_{e-1} := \{\mathcal{H}_i, \mathcal{H}'_i, g_i\}_{i=1,2,\dots,e-1}$ is a deformation atlas of length $e-1$ such that $\mathcal{H}_1$ coincides with the hypersurface (6.2.6).

Henceforth to simplify notation, we often omit subscripts of coordinates such as $w = w_i$. Also we set

$$f_i = f(w^{p_{i-1}}\eta^{p_i}), \qquad \widehat{f}_i = f(z^{p_{i+1}}\zeta^{p_i}). \tag{6.2.7}$$

Then DA_{e-1} is expressed as: for $i = 1, 2, \dots, e-1$,

$$\begin{cases} \mathcal{H}_i: & w^{m_{i-1}-n_{i-1}}\eta^{m_i-n_i}(w^{n_{i-1}}\eta^{n_i} + t^d f_i) - s = 0 \\ \mathcal{H}'_i: & z^{m_{i+1}-n_{i+1}}\zeta^{m_i-n_i}(z^{n_{i+1}}\zeta^{n_i} + t^d \widehat{f}_i) - s = 0 \\ g_i: & \text{the transition function } z = 1/w,\ \zeta = w^{r_i}\eta \text{ of } N_i. \end{cases} \tag{6.2.8}$$

Now Theorem 6.1.1 is generalized as follows:

Theorem 6.2.3 *If Y is tame, then DA_{e-1} given by* (6.2.8) *admits a complete propagation.*

Proof. The proof goes in the same way as that of Theorem 6.1.1 and we merely give the data of a complete propagation for the case $d = 1$. We shall separate into two cases: Case 1. $e = \lambda$ and Case 2. $e \leq \lambda - 1$. For Case 1, we set

$$\begin{cases} \mathcal{H}_\lambda: & w^{m_{\lambda-1}-n_{\lambda-1}}\eta^{m_\lambda-n_\lambda}\,(w^{n_{\lambda-1}}\eta^{n_\lambda} + t\, f_\lambda) - s = 0 \\ \mathcal{H}'_\lambda: & \zeta^{m_\lambda-n_\lambda}\,(\zeta^{n_\lambda} + t\, z^q\, \widehat{f}_\lambda) - s = 0 \\ g_\lambda: & \text{the transition function } z = 1/w,\ \zeta = w^{r_\lambda}\eta \text{ of } N_\lambda, \end{cases}$$

where $q := n_{\lambda-1} - r_\lambda n_\lambda$. This data gives a complete propagation of $DA_{\lambda-1}$. For Case 2, we define an e-th propagation of DA_{e-1} by

$$\begin{cases} \mathcal{H}_e: & w^{m_{e-1}-n_{e-1}}\eta^{m_e-n_e}\,(w^{n_{e-1}}\eta^{n_e} + t\, f_e) - s = 0 \\ \mathcal{H}'_e: & z^{m_{e+1}}\zeta^{m_e-n_e}\,(\zeta^{n_e} + t\, z^q\, \widehat{f}_e) - s = 0 \\ g_e: & \text{the transition function } z = 1/w,\ \zeta = w^{r_e}\eta \text{ of } N_e. \end{cases} \tag{6.2.9}$$

Then we define

$$\begin{cases} \mathcal{H}_{e+1}: & w^{m_e-n_e}\eta^{m_{e+1}}\,(w^{n_e} + t\,\eta^q\, f_{e+1}) - s = 0 \\ \mathcal{H}'_{e+1}: & z^{m_{e+2}}\zeta^{m_{e+1}}\,(1 + t\, z^{\,r_{e+1}\,q + n_e}\, \zeta^q\, \widehat{f}_{e+1}) - s = 0 \\ g_{e+1}: & \text{the transition function } z = 1/w,\ \zeta = w^{r_{e+1}}\eta \text{ of } N_{e+1}, \end{cases} \tag{6.2.10}$$

and for $i = e+2, e+3, \dots, \lambda$, we define

$$\begin{cases} \mathcal{H}_i: & w^{m_{i-1}}\eta^{m_i}\,(1 + t\, w^{\,a_{i-1}}\,\eta^{\,a_i}\, f_i) - s = 0 \\ \mathcal{H}'_i: & z^{m_{i+1}}\zeta^{m_i}\,(1 + t\, z^{\,a_{i+1}}\,\zeta^{\,a_i}\, \widehat{f}_i) - s = 0 \\ g_i: & \text{the transition function } z = 1/w,\ \zeta = w^{r_i}\eta \text{ of } N_i, \end{cases} \tag{6.2.11}$$

where f_i and $\widehat{f_i}$ are as in (6.2.7) and a_i are integers defined by (6.1.5), i.e.

$$\begin{cases} a_{e+1} := q, \quad a_{e+2} := r_{e+1}q + n_e \qquad \text{and} \\ a_{i+1} := r_i a_i - a_{i-1} \qquad \text{for} \quad i = e+2,\ e+3,\ \ldots,\ \lambda - 1. \end{cases} \tag{6.2.12}$$

As noted in the paragraph subsequent to (6.1.5), (1) $a_i \geq 0$, in fact, $a_\lambda > a_{\lambda-1} > \cdots > a_{e+1} \geq 0$ and (2) the transition function g_i of N_i transforms $w^{a_{i-1}}\eta^{a_i}$ to $z^{a_{i+1}}\zeta^{a_i}$. Thus (6.2.9), (6.2.10), (6.2.11) together give a complete propagation of DA_{e-1}. □

6.3 Proportional subbranches

In this section, we introduce "proportional subbranches", which play an important role in describing singularities associated with subbranches.

Lemma 6.3.1 *Let $Y = n_0\Delta_0 + n_1\Theta_1 + \cdots + n_e\Theta_e$ be a subbranch of a branch $X = m_0\Delta_0 + m_1\Theta_1 + \cdots + m_\lambda\Theta_\lambda$. Then the following conditions are equivalent:*

(1) $m_k n_{k-1} - m_{k-1} n_k = 0$ *for some integer* k $(1 \leq k \leq e)$.
(2) $m_i n_{i-1} - m_{i-1} n_i = 0$ *for* $i = 1, 2, \ldots, e$.
(3) $\frac{m_0}{n_0} = \frac{m_1}{n_1} = \cdots = \frac{m_e}{n_e}$. (Clearly this is restated as: There exist positive integers a and b satisfying $(am_0, am_1, \ldots, am_e) = (bn_0, bn_1, \ldots, bn_e)$).

Proof. (1) $\Longrightarrow$ (2): Since Y is a subbranch of a branch X,

$$\frac{m_{k+1} + m_{k-1}}{m_k} = \frac{n_{k+1} + n_{k-1}}{n_k} \ (= r_k). \tag{6.3.1}$$

By (1), we have $\frac{m_{k-1}}{m_k} = \frac{n_{k-1}}{n_k}$, and together with (6.3.1) we obtain

$$\frac{m_{k+1}}{m_k} = \frac{n_{k+1}}{n_k}.$$

This implies that $m_{k+1}n_k - m_{k+1}n_k = 0$. We repeat this argument for $k+1, k+2, \ldots, e$, and we obtain $m_i n_{i-1} - m_{i-1} n_i = 0$ for $k+1, k+2, \ldots, e$. Similarly, we may show that $m_i n_{i-1} - m_{i-1} n_i = 0$ for $k-1, k-2, \ldots, 1$. Thus (2) holds.

(2) $\Longrightarrow$ (3): (2) is rewritten as $\frac{m_i}{n_i} = \frac{m_{i-1}}{n_{i-1}}$ for $i = 1, 2, \ldots, e$, and thus $\frac{m_0}{n_0} = \frac{m_1}{n_1} = \cdots = \frac{m_e}{n_e}$.

(3) $\Longrightarrow$ (1): Trivial. Thus we established the statement. □

For a subbranch $Y = n_0\Delta_0 + n_1\Theta_1 + \cdots + n_e\Theta_e$ of a branch $X = m_0\Delta_0 + m_1\Theta_1 + \cdots + m_\lambda\Theta_\lambda$, if there exists positive integers a and b satisfying

$$(am_0, am_1, \ldots, am_e) = (bn_0, bn_1, \ldots, bn_e),$$

we say that Y is a *proportional* subbranch of X. As we have shown in Lemma 6.3.1, the proportionality of Y is equivalent to (1) $m_k n_{k-1} - m_{k-1} n_k = 0$ for some integer k ($1 \le k \le e$), or (2) $m_i n_{i-1} - m_{i-1} n_i = 0$ for $i = 1, 2, \dots, e$. For example, $X = 12\Delta_0 + 9\Theta_1 + 6\Theta_2 + 3\Theta_3$ and $Y = 8\Delta_0 + 6\Theta_1 + 4\Theta_2 + 2\Theta_3$ are proportional; $2X = 3Y$. In this example, X and Y have the same length 3. Actually, the following result holds.

Lemma 6.3.2 *Let $Y = n_0\Delta_0 + n_1\Theta_1 + \cdots + n_e\Theta_e$ be a proportional subbranch of $X = m_0\Delta_0 + m_1\Theta_1 + \cdots + m_\lambda\Theta_\lambda$.*

(1) *If Y is dominant, then Y and X have the same length; $e = \lambda$.*
(2) *If Y is not dominant, then the dominant subbranch Z containing Y is also proportional.* (Note: By (1), Z and X have the same length.)

Proof. (1): Supposing that $e < \lambda$, we will deduce a contradiction. As Y is proportional, $\frac{n_{e-1}}{n_e} = \frac{m_{e-1}}{m_e}$ and thus

$$\begin{aligned}\frac{n_{e-1}}{n_e} &= \frac{m_{e-1}}{m_e} = \frac{m_{e-1} + m_{e+1}}{m_e} - \frac{m_{e+1}}{m_e} = r_e - \frac{m_{e+1}}{m_e} \\ &< r_e,\end{aligned}$$

Therefore $\frac{n_{e-1}}{n_e} < r_e$. This implies that Y is not tame; so Y is wild (see Definition 5.5.3, p93 for "tame" and "wild"), i.e.

$$\frac{n_{e-1} + m_{e+1}}{n_e} < r_e.$$

On the other hand, since $\frac{m_{e-1} + m_{e+1}}{m_e} = r_e$, we have $\frac{n_{e-1} + m_{e+1}}{n_e} < \frac{m_{e-1} + m_{e+1}}{m_e}$. Using $\frac{n_{e-1}}{n_e} = \frac{m_{e-1}}{m_e}$ (the proportionality of Y), we deduce $\frac{m_{e+1}}{n_e} < \frac{m_{e+1}}{m_e}$ and thus $m_e < n_e$, giving a contradiction. Therefore $e = \lambda$.

Next we verify (2). We express $Z = n_0\Delta_0 + n_1\Theta_1 + \cdots + n_f\Theta_f$ where $e < f$. By an equivalent condition of proportionality, it suffices to show $m_k n_{k-1} - m_{k-1} n_k = 0$ for "some" k ($1 \le k \le f$). But from the proportionality of Y, $m_k n_{k-1} - m_{k-1} n_k = 0$ holds for $k = 1, 2, \dots, e$, and hence the assertion is verified. □

The proportionality of Y is related to other properties of Y.

Corollary 6.3.3 *Let $Y = n_0\Delta_0 + n_1\Theta_1 + \cdots + n_e\Theta_e$ be a subbranch of a branch $X = m_0\Delta_0 + m_1\Theta_1 + \cdots + m_\lambda\Theta_\lambda$.*

(1) *If Y is proportional, then Y is tame.*
(2) *If Y is wild, then Y is not proportional.*

Proof. Since (2) merely restates (1), it is enough to show (1). First we suppose that Y is dominant. Then by Lemma 6.3.2 (1), Y and X have the same length;

$e = \lambda$, and from $m_\lambda n_{\lambda-1} - m_{\lambda-1}n_\lambda = 0$, we have $\frac{n_{\lambda-1}}{n_\lambda} = \frac{m_{\lambda-1}}{m_\lambda}$. The right hand side is equal to r_λ, and so $\frac{n_{\lambda-1}}{n_\lambda} = r_\lambda$. This means that Y is tame (Definition 5.5.3, p93).

When Y is not dominant, we take the dominant subbranch Z containing Y. Then by Lemma 6.3.2 (2), Z is also proportional, and we may apply the above argument to dominant proportional Z, concluding that Z is tame, and by definition, Y is tame. This proves (1), and consequently (2). □

6.4 Singular fibers

This section is rather technical, and for the first reading, we recommend to skip. Let m, m', n, n' be positive integers such that $m - n > 0$ and $m - n' > 0$.

Lemma 6.4.1 *Let $C_{s,t} : z^{m'-n'}\zeta^{m-n}(z^{n'}\zeta^n + t) - s = 0$ be a family of curves, parameterized by s and t. Then $C_{s,t}$ is singular if and only if*

$$\begin{cases} (1)\ s = 0 & \textit{if} \quad mn' - m'n \neq 0 \\ (2)\ s = 0 \quad \textit{or} \quad \left(\frac{n-m}{n}s\right)^a = \left(\frac{n-m}{m}t\right)^b & \textit{if} \quad mn' - m'n = 0, \end{cases}$$

where a and b are the relatively prime positive integers[2] *satisfying $(am, am') = (bn, bn')$. In (2), for the case $\left(\frac{n-m}{n}s\right)^a = \left(\frac{n-m}{m}t\right)^b$, a point $(z, \zeta) \in \mathbb{C}^2$ is a singularity on $C_{s,t}$ precisely when*

$$z^{n'}\zeta^n = \frac{n-m}{m}t, \qquad z^{m'}\zeta^m = \frac{n-m}{n}s.$$

Proof. If $s = 0$, then clearly $C_{0,t} : z^{m'-n'}\zeta^{m-n}(z^{n'}\zeta^n + t) = 0$ is singular (normal crossing). We consider the case $s \neq 0$. Setting $F(z, \zeta, t) := z^{m'-n'}\zeta^{m-n}(z^{n'}\zeta^n + t)$, we express $C_{s,t} : F(z, \zeta, t) - s = 0$. Then

$$\begin{aligned} (z,\zeta) \in C_{s,t} \text{ is singular } &\iff \frac{\partial(F-s)}{\partial z}(z,\zeta) = \frac{\partial(F-s)}{\partial \zeta}(z,\zeta) = 0 \qquad (6.4.1) \\ &\iff \frac{\partial F}{\partial z}(z,\zeta) = \frac{\partial F}{\partial \zeta}(z,\zeta) = 0 \\ &\iff \frac{\partial \log F}{\partial z}(z,\zeta) = \frac{\partial \log F}{\partial \zeta}(z,\zeta) = 0. \end{aligned}$$

(As we assumed $s \neq 0$, from $F(z, \zeta, t) - s = 0$, we have $F(z, \zeta, t) \neq 0$ and so $\log F(z, \zeta, t)$ is well-defined. The choice of a branch of log is immaterial, because in the subsequent discussion, we are interested only in the derivative of log.) Since

$$\log F = (m' - n')\log z + (m - n)\log \zeta + \log(z^{n'}\zeta^n + t),$$

[2] If $mn' - m'n = 0$, then there exists such a pair a and b.

from $\dfrac{\partial \log(F)}{\partial z}(z,\zeta) = \dfrac{\partial \log(F)}{\partial \zeta}(z,\zeta) = 0$ (6.4.1), we have

$$\begin{cases} \dfrac{\partial \log(F)}{\partial z}(z,\zeta) = \dfrac{m'-n'}{z} + \dfrac{n' z^{n'-1}\zeta^n}{z^{n'}\zeta^n + t} = 0 \\[2ex] \dfrac{\partial \log(F)}{\partial \zeta}(z,\zeta) = \dfrac{m-n}{\zeta} + \dfrac{n z^{n'}\zeta^{n-1}}{z^{n'}\zeta^n + t} = 0, \end{cases}$$

and hence we deduce

$$z^{n'}\zeta^n = \frac{n'-m'}{m'}t, \qquad z^{n'}\zeta^n = \frac{n-m}{m}t. \tag{6.4.2}$$

In particular, $\dfrac{n'-m'}{m'}t = \dfrac{n-m}{m}t$, or equivalently

$$(mn' - m'n)t = 0. \tag{6.4.3}$$

We separate into two cases according to whether $mn' - m'n$ is zero or not.

Case 1 $mn' - m'n \neq 0$: From (6.4.3), we have $t = 0$, in which case, obviously $C_{s,0} : z^{m'}\zeta^m - s = 0$ is singular precisely when $s = 0$.

Case 2 $mn' - m'n = 0$: The equation (6.4.3) is vacuous, and it is easy to check that two equations (6.4.2) are equivalent. Hence $(z,\zeta) \in \mathbb{C}^2$ is a singularity on $C_{s,t}$ precisely when the following two equations are satisfied:

$$z^{n'}\zeta^n = \frac{n-m}{m}t \tag{6.4.4}$$

$$z^{m'}\zeta^m + t z^{m'-n'}\zeta^{m-n} - s = 0. \tag{6.4.5}$$

(The second equation is just the defining equation of $C_{s,t}$.) Substituting (6.4.4) into $z^{m'}\zeta^m + t\dfrac{z^{m'}\zeta^m}{z^{n'}\zeta^n} - s = 0$ (6.4.5), we obtain

$$z^{m'}\zeta^m + t z^{m'}\zeta^m \frac{1}{\dfrac{n-m}{m}t} - s = 0,$$

which yields $z^{m'}\zeta^m = \dfrac{n-m}{n}s$. Thus $(z,\zeta) \in \mathbb{C}^2$ is a singularity on $C_{s,t}$ precisely when

$$z^{n'}\zeta^n = \frac{n-m}{m}t, \qquad z^{m'}\zeta^m = \frac{n-m}{n}s. \tag{6.4.6}$$

It remains to derive the equation satisfied by s and t. Taking powers of (6.4.6), we have

$$z^{bn'}\zeta^{bn} = \left(\frac{n-m}{m}t\right)^b, \qquad z^{am'}\zeta^{am} = \left(\frac{n-m}{n}s\right)^a,$$

where a and b are the relatively prime positive integers satisfying $(am, am') = (bn, bn')$. Since $z^{bn'}\zeta^{bn} = z^{am'}\zeta^{am}$, we have $\left(\frac{n-m}{m}t\right)^b = \left(\frac{n-m}{n}s\right)^a$. This completes the proof of the assertion. $\square$

We proved that for the case $\left(\frac{n-m}{n}s\right)^a = \left(\frac{n-m}{m}t\right)^b$ in (2) of Lemma 6.4.1, a point $(z, \zeta) \in \mathbb{C}^2$ is a singularity on $C_{s,t}$ precisely when $z^{n'}\zeta^n = \frac{n-m}{m}t$ and $z^{m'}\zeta^m = \frac{n-m}{n}s$. We further show that the singular locus of $C_{s,t}$ is one-dimensional (so $C_{s,t}$ is non-reduced).

Lemma 6.4.2 *Suppose that $m'n - mn' = 0$. Fix s and t $(s, t \neq 0)$ such that $\left(\frac{n-m}{n}s\right)^a = \left(\frac{n-m}{m}t\right)^b$, where a and b are the relatively prime positive integers satisfying $(am, am') = (bn, bn')$. Then an algebraic set*

$$\left\{ (z, \zeta) \in \mathbb{C}^2 \;\middle|\; z^{n'}\zeta^n = \frac{n-m}{m}t, \quad z^{m'}\zeta^m = \frac{n-m}{n}s \right\} \tag{6.4.7}$$

is one-dimensional.

Proof. For brevity, we set $A = \frac{n-m}{m}t$ and $B = \frac{n-m}{n}s$; note that $A, B \neq 0$ because $m > n$ and $s, t \neq 0$, and also note that $A^b = B^a$ by assumption. We have to verify that the algebraic set which consists of $(z, \zeta) \in \mathbb{C}^2$ satisfying

$$\text{(i) } z^{n'}\zeta^n = A, \qquad \text{(ii) } z^{m'}\zeta^m = B$$

is one-dimensional. From (i), we deduce $\zeta = e^{2\pi i k/n}\left(\frac{A}{z^{n'}}\right)^{1/n}$, where k is an integer such that $1 \leq k \leq n$. Substituting this into (ii), we have

$$z^{m'}e^{2\pi i mk/n}\left(\frac{A}{z^{n'}}\right)^{m/n} = B,$$

that is, $z^{(m'n-mn')/n'}e^{2\pi i mk/n}A^{m/n} = B$. Since $m'n - mn' = 0$ by assumption, we have $e^{2\pi i mk/n}A^{m/n} = B$. Using $\frac{m}{n} = \frac{b}{a}$, we rewrite this equation as

$$e^{2\pi i bk/a}A^{b/a} = B. \tag{6.4.8}$$

Note that by the assumption $A^b = B^a$, we may choose k so that (6.4.8) holds; then z and $\zeta = e^{2\pi i k/n}\left(\frac{A}{z^{n'}}\right)^{1/n}$ (a solution of (i)) also satisfy (ii). Hence for any nonzero complex number z, the pair z and $\zeta = e^{2\pi i k/n}\left(\frac{A}{z^{n'}}\right)^{1/n}$ satisfies (i) and (ii). Consequently, the algebraic set (6.4.7) is one-dimensional. $\square$

Lemma 6.4.2 implies that any singularity in Lemma 6.4.1 is non-isolated. Accordingly, we may strengthen the statement of Lemma 6.4.1 as follows.

Proposition 6.4.3 *Let $C_{s,t} : z^{m'-n'}\zeta^{m-n}(z^{n'}\zeta^n + t) - s = 0$ be a family of curves, parameterized by s and t. Then $C_{s,t}$ is singular if and only if*

$$\begin{cases} (1)\ s = 0 & \text{if } mn' - m'n \neq 0 \\ (2)\ s = 0 \ \text{ or } \ \left(\frac{n-m}{n}s\right)^a = \left(\frac{n-m}{m}t\right)^b & \text{if } mn' - m'n = 0, \end{cases}$$

where a and b are the relatively prime positive integers satisfying $(am, am') = (bn, bn')$. In (2), for the case $\left(\frac{n-m}{n}s\right)^a = \left(\frac{n-m}{m}t\right)^b$, a point $(z,\zeta) \in \mathbb{C}^2$ is a singularity on $C_{s,t}$ precisely when

$$z^{n'}\zeta^n = \frac{n-m}{m}t, \qquad z^{m'}\zeta^m = \frac{n-m}{n}s,$$

and moreover any singularity is non-isolated (so $C_{s,t}$ is non-reduced).

Remark 6.4.4 *Using $am = bn$, we have*

$$\frac{n-m}{n} = \frac{n - \frac{b}{a}n}{n} = \frac{a-b}{a}, \qquad \frac{n-m}{m} = \frac{\frac{a}{b}m - m}{m} = \frac{a-b}{b},$$

and so the equation $\left(\frac{n-m}{n}s\right)^a = \left(\frac{n-m}{m}t\right)^b$ is rewritten as $\left(\frac{a-b}{a}s\right)^a = \left(\frac{a-b}{b}t\right)^b$, and likewise $z^{n'}\zeta^n = \frac{n-m}{m}t$ is rewritten as $z^{n'}\zeta^n = \frac{a-b}{b}t$.

Now we return to the discussion on deformation atlases. Let $Y = n_0\Delta_0 + n_1\Theta_1 + \cdots + n_e\Theta_e$ be a subbranch of a branch $X = m_0\Delta_0 + m_1\Theta_1 + \cdots + m_\lambda\Theta_\lambda$, and let $DA_{e-1}(Y,d)$ be the deformation atlas of weight d associated with Y: for $i = 1, 2, \ldots, e-1$,

$$\begin{cases} \mathcal{H}_i : & w^{m_{i-1}-n_{i-1}}\eta^{m_i-n_i}(w^{n_{i-1}}\eta^{n_i} + t^k f_i) - s = 0 \\ \mathcal{H}'_i : & z^{m_{i+1}-n_{i+1}}\zeta^{m_i-n_i}(z^{n_{i+1}}\zeta^{n_i} + t^k \widehat{f}_i) - s = 0 \\ g_i : & \text{the transition function } z = 1/w,\ \zeta = w^{r_i}\eta \text{ of } N_i, \end{cases}$$

where holomorphic functions $f_i(w,\eta) = f(w^{p_{i-1}}\eta^{p_i})$ and $\widehat{f}_i(z,\zeta) = f(z^{p_{i+1}}\zeta^{p_i})$ are as in (6.2.7). For the remainder of this section, we only treat the case where f_i and $\widehat{f}_i$ are constant, say $f_i = \widehat{f}_i = 1$; in §7.2, we will treat the nonconstant case, deducing a totally different result.

We shall construct a complex manifold $\mathcal{M}^{[e-1]}$ from $DA_{e-1}(Y,d)$. First construct complex 3-manifolds $\mathcal{M}_i$ $(i = 1, 2, \ldots, e-1)$ by patching $\mathcal{H}_i$ and $\mathcal{H}'_i$ via g_i. Next patch $\mathcal{M}_1, \mathcal{M}_2, \ldots, \mathcal{M}_{e-1}$ by 'plumbing'

$$(w_{i+1}, \eta_{i+1}, s, t) = (\zeta_i, z_i, s, t), \qquad i = 1, 2, \ldots, e-2,$$

which yields a complex 3-manifold $\mathcal{M}^{[e-1]}$. The natural projection $\Psi^{[e-1]} : \mathcal{M}^{[e-1]} \to \Delta \times \Delta^\dagger$ is called the *barking family of length $e-1$* obtained from $DA_{e-1}(Y,d)$.

The notion of proportionality is related to the description of singularities of a fiber $X_{s,t}^{[e-1]} = (\Psi^{[e-1]})^{-1}(s,t)$. (Recall Y is proportional if $\frac{m_0}{n_0} = \frac{m_1}{n_1} = \cdots = \frac{m_e}{n_e}$.) By Proposition 6.4.3, we have the following result.

Theorem 6.4.5 *Let $\Psi^{[e-1]} : \mathcal{M}^{[e-1]} \to \Delta \times \Delta^\dagger$ be the barking family of length $e-1$ obtained from $DA_{e-1}(Y,d)$. Set $X_{s,t}^{[e-1]} := (\Psi^{[e-1]})^{-1}(s,t)$, and then the following statements hold:*

(1) *If Y is not proportional, then $X_{s,t}^{[e-1]}$ is singular if and only if $s=0$.*

(2) *If Y is proportional, and f_i and $\widehat{f}_i$ are constant*[3]*, then $X_{s,t}^{[e-1]}$ is singular if and only if $s = 0$ or $\left(\frac{n-m}{n}s\right)^a = \left(\frac{n-m}{m}t^d\right)^b$, where*[4] *$m := m_0$, $n := n_0$, and a and b are the relatively prime positive integers satisfying $am = bn$. Moreover, for the case $\left(\frac{n-m}{n}s\right)^a = \left(\frac{n-m}{m}t^d\right)^b$, any singularity is non-isolated, that is, $X_{s,t}^{[e-1]}$ is non-reduced; in fact, a point (w_i, η_i)* (resp. (z_i, ζ_i)) *of $X_{s,t}^{[e-1]}$ is a singularity precisely when*

$$w_i^{m_{i-1}}\eta_i^{m_i} = \frac{n-m}{n}s, \qquad w_i^{n_{i-1}}\eta_i^{n_i} = \frac{n-m}{m}t^d$$
$$\left(\text{resp.} \quad z_i^{m_{i+1}}\zeta_i^{m_i} = \frac{n-m}{n}s, \qquad z_i^{n_{i+1}}\zeta_i^{n_i} = \frac{n-m}{m}t^d\right).$$

Corollary 6.4.6 *Let $Y = n_0\Delta_0 + n_1\Theta_1 + \cdots + n_e\Theta_e$ be a subbranch of a branch X, and let $DA_{e-1}(Y,d)$ be the deformation atlas of weight d associated with Y. Let $\Psi^{[e-1]} : \mathcal{M}^{[e-1]} \to \Delta \times \Delta^\dagger$ be the barking family of length $e-1$ obtained from $DA_{e-1}(Y,d)$. Set $X_{s,t}^{[e-1]} := (\Psi^{[e-1]})^{-1}(s,t)$. If Y is either* (i) *wild or* (ii) *tame and not proportional, then $X_{s,t}^{[e-1]}$ is singular if and only if $s = 0$.*

Proof. When Y is wild, Y is not proportional by Corollary 6.3.3 (2). Thus by Theorem 6.4.5 (1), $X_{s,t}^{[e-1]}$ is singular if and only if $s=0$. When Y is tame and not proportional, the assertion follows from Theorem 6.4.5 (1) again. □

Supplement: Technical results

In Proposition 6.4.3, we showed: *Let $C_{s,t} : z^{m'-n'}\zeta^{m-n}(z^{n'}\zeta^n + t) - s = 0$ be a family of curves, parameterized by s and t. Then $C_{s,t}$ is singular if and only if*

$$\begin{cases} (1)\ s = 0 & \text{if} \quad mn' - m'n \neq 0 \\ (2)\ s = 0 \quad \text{or} \quad \left(\frac{n-m}{n}s\right)^a = \left(\frac{n-m}{m}t\right)^b & \text{if} \quad mn' - m'n = 0, \end{cases}$$

[3] For nonconstant f_i and $\widehat{f}_i$, the statement totally differs. See Theorem 7.2.4 (1).

[4] From the proportionality, instead of $m = m_0$ and $n = n_0$, we may take arbitrary $m = m_i$ and $n = n_i$ $(0 \le i \le \lambda)$, as the relevant fractions are independent of the choice of i (Remark 6.4.4).

where a and b are the relatively prime positive integers satisfying $(am, am') = (bn, bn')$. In (2), for the case $\left(\frac{n-m}{n}s\right)^a = \left(\frac{n-m}{m}t\right)^b$, a point $(z, \zeta) \in \mathbb{C}^2$ is a singularity on $C_{s,t}$ precisely when

$$z^{m'}\zeta^m = \frac{n-m}{n}s, \qquad z^{n'}\zeta^n = \frac{n-m}{m}t, \tag{6.4.9}$$

and moreover any singularity is non-isolated, that is, $C_{s,t}$ is non-reduced.

We shall study a special case $a = 1$ in the above statement in more detail; we often encounter this case in our later application. To analyze this case, we need the subsequent technical lemmas.

Lemma 6.4.7 *A polynomial $P(X) = X^{b-1} + \sum_{i=1}^{b-1} \frac{c^i}{1-b} X^{b-1-i}$, where b ($b \geq 2$) is an integer, admits a factorization:*

$$P(X) = \frac{1}{b-1}\,(X-c)\left[\,\sum_{i=0}^{b-2}(i+1)c^{b-2-i}X^i\,\right].$$

Proof. To show this, we shall rewrite

$$P(X) = X^{b-1} + \frac{c}{1-b}X^{b-2} + \frac{c^2}{1-b}X^{b-3} + \frac{c^3}{1-b}X^{b-4} + \cdots + \frac{c^{b-1}}{1-b}.$$

First of all, we note

$$\begin{aligned}
&X^{b-1} + \frac{c}{1-b}X^{b-2} + \frac{c^2}{1-b}X^{b-3} + \frac{c^3}{1-b}X^{b-4} + \cdots + \frac{c^{b-1}}{1-b} \\
&= \frac{1}{b-1}\left[(b-1)X^{b-1} - cX^{b-2} - c^2X^{b-3} - c^3X^{b-4} - \cdots - c^{b-1}\right] \\
&= \frac{c^{b-1}}{b-1}\left[(b-1)\left(\frac{X}{c}\right)^{b-1} - \left(\frac{X}{c}\right)^{b-2} - \left(\frac{X}{c}\right)^{b-3} - \left(\frac{X}{c}\right)^{b-4} - \cdots - \left(\frac{X}{c}\right) - 1\right] \\
&= \frac{c^{b-1}}{b-1}\Big[(b-1)Y^{b-1} - \Big(Y^{b-2} + Y^{b-3} + Y^{b-4} + \cdots + Y + 1\Big)\Big],
\end{aligned}$$

where we set $Y = \frac{X}{c}$. Inside the brackets is rewritten as

$$\begin{aligned}
&(b-1)Y^{b-1} - \big(Y^{b-2} + Y^{b-3} + Y^{b-4} + \cdots + Y + 1\big) \\
&= \big(\underbrace{Y^{b-1} + Y^{b-1} + \cdots + Y^{b-1}}_{b-1}\big) - \big(Y^{b-2} + Y^{b-3} + Y^{b-4} + \cdots + Y + 1\big) \\
&= (Y^{b-1} - Y^{b-2}) + (Y^{b-1} - Y^{b-3}) + (Y^{b-1} - Y^{b-4}) + \cdots \\
&\qquad + (Y^{b-1} - Y) + (Y^{b-1} - 1),
\end{aligned}$$

and the last expression is further rewritten as follows:

$$
\begin{aligned}
&Y^{b-2}(Y-1)+Y^{b-3}(Y^2-1)+Y^{b-4}(Y^3-1)+\cdots\\
&\qquad +Y(Y^{b-2}-1)+(Y^{b-1}-1)\\
&=(Y-1)\Big[Y^{b-2}+Y^{b-3}\big(Y+1\big)+Y^{b-4}\big(Y^2+Y+1\big)+\\
&\qquad\qquad \cdots+Y\big(Y^{b-3}+Y^{b-4}+\cdots+Y+1\big)\\
&\qquad\qquad\qquad +\big(Y^{b-2}+Y^{b-3}+Y^{b-4}+\cdots+Y+1\big)\Big]\\
&=(Y-1)\Big[(b-1)Y^{b-2}+(b-2)Y^{b-3}+(b-3)Y^{b-4}+\cdots+2Y+1\Big].
\end{aligned}
$$

Thus we have an expression:

$$P(X)=\frac{c^{b-1}}{b-1}(Y-1)\Big[(b-1)Y^{b-2}+(b-2)Y^{b-3}+(b-3)Y^{b-4}+\cdots+2Y+1\Big].$$

Recalling that $Y=\dfrac{X}{c}$, we may derive the factorization in question as follows:

$$
\begin{aligned}
P(X)&=\frac{c^{b-1}}{b-1}\left(\frac{X}{c}-1\right)\left[(b-1)\left(\frac{X}{c}\right)^{b-2}+(b-2)\left(\frac{X}{c}\right)^{b-3}+\cdots+2\left(\frac{X}{c}\right)+1\right]\\
&=\frac{c^{b-2}}{b-1}(X-c)\left[(b-1)\left(\frac{X}{c}\right)^{b-2}+(b-2)\left(\frac{X}{c}\right)^{b-3}+\cdots+2\left(\frac{X}{c}\right)+1\right]\\
&=\frac{1}{b-1}\,(X-c)\Big[(b-1)X^{b-2}+(b-2)cX^{b-3}+\cdots+2c^{b-2}X+c^{b-2}\Big].
\end{aligned}
$$

□

We also need the following.

Lemma 6.4.8 *Suppose that b $(b\geq 2)$ is an integer. Set $c:=\dfrac{1-b}{b}t$, and then*

$$X^b+tX^{b-1}-\frac{c^b}{1-b}\;=\;(X-c)\left(X^{b-1}+\sum_{i=1}^{b-1}\frac{c^i}{1-b}X^{b-1-i}\right).$$

Proof. The right hand side equals

$$
\begin{aligned}
&X^b+\sum_{i=1}^{b-1}\frac{c^i}{1-b}X^{b-i}-cX^{b-1}-\sum_{i=1}^{b-1}\frac{c^{i+1}}{1-b}X^{b-1-i}\\
&=X^b+\left(\frac{c}{1-b}X^{b-1}+\sum_{i=2}^{b-1}\frac{c^i}{1-b}X^{b-i}\right)-cX^{b-1}-\left(\frac{c^b}{1-b}+\sum_{i=2}^{b-1}\frac{c^i}{1-b}X^{b-i}\right)\\
&=X^b+\frac{c}{1-b}X^{b-1}-cX^{b-1}-\frac{c^b}{1-b}.
\end{aligned}
$$

Since $\dfrac{c}{1-b}X^{b-1}-cX^{b-1}=tX^{b-1}$ by $c=\dfrac{1-b}{b}t$, the last expression is equal to

$$X^b+tX^{b-1}-\frac{c^b}{1-b}.$$

This completes the proof. □

Substituting the factorization shown in Lemma 6.4.7 into the right hand side of the equation in Lemma 6.4.8, we deduce a lemma useful for later application.

Lemma 6.4.9 *Suppose that* b $(b \geq 2)$ *is an integer. Set* $c := \frac{1-b}{b}t$*, and then*

$$X^b + tX^{b-1} - \frac{c^b}{1-b} = \frac{1}{b-1}\,(X-c)^2\left[\sum_{i=0}^{b-2}(i+1)c^{b-2-i}X^i\right].$$

Now we can show the following statement which is an elaborate form of Proposition 6.4.3 for the case $a = 1$.

Proposition 6.4.10 *Let* m, m', n, n' *be positive integers satisfying* $(m, m') = (bn, bn')$ *for some integer* $b \geq 2$*. Then a curve*

$$C_{s,t}: \quad z^{m'-n'}\zeta^{m-n}(z^{n'}\zeta^n + t) - s = 0$$

is singular if and only if (i) $s = 0$ *or* (ii) $s = \frac{(1-b)^{b-1}}{b^b}t^b$*. For the case* (ii)*, set* $c := \frac{1-b}{b}t$*, and then the curve* $C_{s,t}$ *is written as*

$$\frac{1}{b-1}\left(z^{n'}\zeta^n - c\right)^2\left[\sum_{i=0}^{b-2}(i+1)\,c^{b-2-i}\,(z^{n'}\zeta^n)^i\right] = 0.$$

(Note that this curve is non-reduced.)

Proof. By Proposition 6.4.3 (2) applied for $a = 1$, the curve $C_{s,t}$ is singular if and only if (i) $s = 0$ or (ii) $\frac{n-m}{n}s = \left(\frac{n-m}{m}t\right)^b$. As $m = bn$ by assumption, (ii) is rewritten as follows.

$$\begin{aligned} s &= \frac{n(n-m)^{b-1}}{m^b}t^b = \frac{n(n-bn)^{b-1}}{(bn)^b}t^b \\ &= \frac{(1-b)^{b-1}}{b^b}t^b. \end{aligned}$$

Thus $C_{s,t}$ is singular if and only if (i) $s = 0$ or (ii) $s = \frac{(1-b)^{b-1}}{b^b}t^b$. This proves the first half of the assertion. In the case (ii), we substitute $s = \frac{(1-b)^{b-1}}{b^b}t^b$ into $C_{s,t}: \ z^{m'}\zeta^m + tz^{m'-n'}\zeta^{m-n} - s = 0$, which yields

$$C_{s,t}: \quad z^{m'}\zeta^m + tz^{m'-n'}\zeta^{m-n} - \frac{(1-b)^{b-1}}{b^b}t^b = 0.$$

Using $m = bn$ and $m' = bn'$, this is rewritten as

$$C_{s,t}: \quad (z^{n'}\zeta^n)^b + t(z^{n'}\zeta^n)^{b-1} - \frac{(1-b)^{b-1}}{b^b}t^b = 0.$$

We set $X := z^{n'}\zeta^n$, and then

$$C_{s,t}: \quad X^b + tX^{b-1} - \frac{(1-b)^{b-1}}{b^b}t^b = 0.$$

From Lemma 6.4.9, the left hand side $X^b + tX^{b-1} - \dfrac{c^b}{1-b}$, where we set $c := \dfrac{1-b}{b}t$, admits a factorization:

$$\frac{1}{b-1}\,(X-c)^2 \left[\,\sum_{i=0}^{b-2}(i+1)c^{b-2-i}X^i\,\right],$$

and hence when $s = \dfrac{(1-b)^{b-1}}{b^b}t^b$,

$$C_{s,t}: \quad \frac{1}{b-1}\left(z^{n'}\zeta^n - c\right)^2 \left[\,\sum_{i=0}^{b-2}(i+1)\,c^{b-2-i}\,(z^{n'}\zeta^n)^i\,\right] = 0.$$

Therefore, when $s = \dfrac{(1-b)^{b-1}}{b^b}t^b$, the equation of $C_{s,t}$ admits the factorization described in the assertion. □

Example 6.4.11 Take $(m', m) = (9, 6)$ and $(n', n) = (3, 2)$. Then $(am', am) = (bn', bn)$ where $a = 1$ and $b = 3$. By Proposition 6.4.10, $C_{s,t}: \; z^6\zeta^4(z^3\zeta^2 + t) - s = 0$ is singular at (s, t) satisfying $s = \dfrac{4}{27}t^3$, in which case

$$C_{s,t}: \quad \left(z^3\zeta^2 + \frac{2}{3}t\right)^2 \left(z^3\zeta^2 - \frac{1}{3}t\right) = 0,$$

and so non-reduced.

7

Construction of Deformations of type A_l

7.1 Deformations of type A_l

Assume that $X = m_0\Delta_0 + m_1\Theta_1 + \cdots + m_\lambda\Theta_\lambda$ is a branch, so $\Delta_0 = \mathbb{C}$ and $\Theta_i = \mathbb{P}^1$ for $i = 1, 2, \ldots, \lambda$. As before, we set

$$r_i := \frac{m_{i-1} + m_{i+1}}{m_i}, \quad (i = 1, 2 \ldots, \lambda - 1), \qquad r_\lambda := \frac{m_{\lambda-1}}{m_\lambda},$$

and adopt the convention $m_{\lambda+1} = 0$. Recall that r_i $(i = 1, 2, \ldots, \lambda)$ are positive integers satisfying $r_i \geq 2$. Now let $Y = n_0\Delta_0 + n_1\Theta_1 + \cdots + n_e\Theta_e$ be a subbranch of X, so

$$\frac{n_{i-1} + n_{i+1}}{n_i} = r_i, \qquad (i = 1, 2, \ldots, e - 1),$$

and let l be a positive integer satisfying $lY \leq X$, i.e. $ln_i \leq m_i$ for $i = 0, 1, \ldots, e$. We note that Y itself is possibly multiple, that is, $\gcd(n_0, n_1, \ldots, n_e)$ may not be 1. To a multiple subbranch lY, we shall associate a special deformation atlas. We prepare notation; let d be a positive integer, and let $f(z)$ be a non-vanishing holomorphic function on a domain $\{z \in \mathbb{C} : |z| < \varepsilon\}$. Next define a sequence of integers $p_0, p_1, \ldots, p_{\lambda+1}$ inductively by

$$\begin{cases} p_0 = 0, \quad p_1 = 1 \quad \text{and} \\ p_{i+1} = r_i p_i - p_{i-1} \quad \text{for } i = 1, 2, \ldots, \lambda, \end{cases}$$

where remember (6.2.4):

$$p_{\lambda+1} > p_\lambda > \cdots > p_1 > p_0 = 0.$$

We then set $f_i = f(w^{p_{i-1}}\eta^{p_i})$ and $\widehat{f}_i = f(z^{p_{i+1}}\zeta^{p_i})$ (see (6.2.7)).

Lemma 7.1.1 *The following data gives a deformation atlas of length $e - 1$: for $i = 1, 2, \ldots, e - 1$,*

$$\begin{cases} \mathcal{H}_i: & w^{m_{i-1}-ln_{i-1}}\eta^{m_i-ln_i}(w^{n_{i-1}}\eta^{n_i}+t^d f_i)^l - s = 0 \\ \mathcal{H}'_i: & z^{m_{i+1}-ln_{i+1}}\zeta^{m_i-ln_i}(z^{n_{i+1}}\zeta^{n_i}+t^d \widehat{f}_i)^l - s = 0 \\ g_i: & \textit{the transition function } z = 1/w,\ \zeta = w^{r_i}\eta \textit{ of } N_i. \end{cases} \tag{7.1.1}$$

Proof. Clearly, by a coordinate change $(z,\zeta) = (\eta, w)$, $\mathcal{H}'_i$ becomes $\mathcal{H}_{i+1}$, and so it is sufficient to show that g_i transforms $\mathcal{H}_i$ to $\mathcal{H}'_i$. Recall that if a, b, c, and r are positive integers satisfying

$$\frac{a+c}{b} = r,$$

then $g: z = 1/w,\ \zeta = w^r\eta$ transforms $w^a\eta^b$ to $z^c\zeta^b$ (Lemma 5.3.2, p90). To apply this, we note

$$\frac{(m_{i-1}-ln_{i-1})+(m_{i+1}-ln_{i+1})}{m_i - ln_i} = r_i. \tag{7.1.2}$$

Indeed,

$$\begin{aligned} \frac{(m_{i-1}-ln_{i-1})+(m_{i+1}-ln_{i+1})}{m_i - ln_i} &= \frac{(m_{i-1}+m_{i+1})-l(n_{i-1}+n_{i+1})}{m_i - ln_i} \\ &= \frac{(r_i m_i) - l(r_i n_i)}{m_i - ln_i} \\ &= r_i, \end{aligned}$$

where in the second equality we used

$$\frac{m_{i-1}+m_{i+1}}{m_i} = r_i \quad \text{and} \quad \frac{n_{i-1}+n_{i+1}}{n_i} = r_i.$$

Thus by Lemma 5.3.2, the transition function $g_i: z = 1/w,\ \zeta = w^{r_i}\eta$ of N_i transforms $w^{m_{i-1}-ln_{i-1}}\eta^{m_i-ln_i}$ to $z^{m_{i+1}-ln_{i+1}}\zeta^{m_i-ln_i}$. Similarly, since

$$\frac{n_{i-1}+n_{i+1}}{n_i} = r_i \quad \text{and} \quad \frac{p_{i-1}+p_{i+1}}{p_i} = r_i,$$

the map g_i transforms $w^{n_{i-1}}\eta^{n_i}$ to $z^{n_{i+1}}\zeta^{n_i}$, and $w^{p_{i-1}}\eta^{p_i}$ to $z^{p_{i+1}}\zeta^{p_i}$ respectively. Therefore g_i transforms $\mathcal{H}_i$ to $\mathcal{H}'_i$. This completes the proof. □

We denote by $DA_{e-1}(lY, d)$ the deformation atlas in the above lemma, and we refer to the positive integer d as the *weight* of $DA_{e-1}(lY, d)$. The arithmetic property of the multiplicity sequence $(n_0, n_1, \dots, n_e)$ of the subbranch Y is deeply related to the propagatability of $DA_{e-1}(lY, d)$. To discuss it, we introduce an important class of subbranches. We say that a subbranch $Y = n_0\Delta_0 + n_1\Theta_1 + \cdots + n_e\Theta_e$ is of *type* A_l if $n_{e-1}/n_e \geq r_e$ and $lY \leq X$. The positive integer l is called the *barking multiplicity.*

Lemma 7.1.2 *Suppose that Y is a subbranch of a branch X such that $lY \leq X$. Then Y is of type A_l if and only if Y is dominant and tame.*

Proof. $\Longrightarrow$: We formally set $n_{e+1} := r_e n_e - n_{e-1}$. If Y is of type A_l, then $n_{e+1} \leq 0$. This implies that Y is dominant, and from Lemma 5.5.7 (1.a), p94, Y is tame.

$\Longleftarrow$: If Y is dominant and tame, then by definition, $n_{e-1}/n_e \geq r_e$, and so Y is of type A_l. □

Proposition 7.1.3 *Let Y be a subbranch of type A_l. Then the deformation atlas $DA_{e-1}(lY, d)$ admits a complete propagation.*

Proof. The proof is similar to that of Theorem 6.1.1, p99, and so we merely give a sketch for the case $d = 1$. Letting $q := n_{e-1} - r_e n_e$ (the slant of Y), we set

$$\begin{cases} \mathcal{H}_e: & w^{m_{e-1}-ln_{e-1}}\eta^{m_e-ln_e}\Big(w^{n_{e-1}}\eta^{n_e} + t\, f_e\Big)^l - s = 0 \\ \mathcal{H}'_e: & z^{m_{e+1}}\zeta^{m_e-ln_e}\Big(\zeta^{n_e} + t\, z^q\, \widehat{f}_e\Big)^l - s = 0 \\ g_e: & \text{the transition function } z = 1/w,\ \zeta = w^{r_e}\eta \text{ of } N_e. \end{cases} \tag{7.1.3}$$

It is easy to check that this data gives an e-th propagation. Next, we set

$$\begin{cases} \mathcal{H}_{e+1}: & w^{m_e-ln_e}\eta^{m_{e+1}}\Big(w^{n_e} + t\,\eta^q\, f_{e+1}\Big)^l - s = 0 \\ \mathcal{H}'_{e+1}: & z^{m_{e+2}}\zeta^{m_{e+1}}\Big(1 + t\, z^{r_{e+1}\,q + n_e}\,\zeta^q\,\widehat{f}_{e+1}\Big)^l - s = 0 \\ g_{e+1}: & \text{the transition function } z = 1/w,\ \zeta = w^{r_{e+1}}\eta \text{ of } N_{e+1}. \end{cases} \tag{7.1.4}$$

We claim that this data gives an $(e+1)$-st propagation. To see that g_{e+1} transforms $\mathcal{H}_{e+1}$ to $\mathcal{H}'_{e+1}$, we rewrite $\mathcal{H}_{e+1}$ as

$$\begin{aligned} \mathcal{H}_{e+1}: \quad & w^{m_e - l\,n_e}\,\frac{1}{w^{\,r_{e+1}\,m_{e+1}}}\,\big(w^{\,r_{e+1}}\,\eta\big)^{\,m_{e+1}} \\ & \times \left(w^{n_e} + t\,\frac{1}{w^{\,r_{e+1}\,q}}\,(w^{\,r_{e+1}}\eta)^q\, f_{e+1}\right)^l - s = 0. \end{aligned}$$

Recall that $\dfrac{m_e + m_{e+2}}{m_{e+1}} = r_{e+1}$, that is, $r_{e+1}m_{e+1} - m_e = m_{e+2}$. Thus

$$w^{m_e - l\,n_e}\,\frac{1}{w^{\,r_{e+1}\,m_{e+1}}} = \frac{1}{w^{\,r_{e+1}m_{e+1}\,-m_e\,+l\,n_e}} = \frac{1}{w^{\,m_{e+2}\,+l\,n_e}},$$

and so

$$\begin{aligned} \mathcal{H}_{e+1}: \quad & \frac{1}{w^{\,m_{e+2}\,+l\,n_e}}\,\big(w^{\,r_{e+1}}\,\eta\big)^{\,m_{e+1}}\,\left(w^{n_e} + t\,\frac{1}{w^{\,r_{e+1}\,q}}\,(\,w^{\,r_{e+1}}\eta\,)^q\, f_{e+1}\right)^l - s \\ & = 0. \end{aligned}$$

Therefore by $g_{e+1}: z = 1/w,\ \zeta = w^{r_{e+1}}\eta$, the hypersurface $\mathcal{H}_{e+1}$ is transformed to

$$z^{\,m_{e+2}\,+l\,n_e}\,\zeta^{\,m_{e+1}}\,\left(\frac{1}{z^{n_e}} + t\,z^{\,r_{e+1}\,q}\,\zeta^q\,\widehat{f}_{e+1}\right)^l - s = 0,$$

namely $z^{m_{e+2}}\,\zeta^{m_{e+1}}\,(1+t\,z^{r_{e+1}\,q+n_e}\,\zeta^q\,\widehat{f}_{e+1})^l - s = 0$. This is nothing but the equation of $\mathcal{H}'_{e+1}$, and thus g_{e+1} transforms $\mathcal{H}_{e+1}$ to $\mathcal{H}'_{e+1}$.

To construct further propagations, we define integers a_i ($i = e+1, e+2, \dots, \lambda+1$) inductively by setting $a_{e+1} := q$ and $a_{e+2} := r_{e+1}q + n_e$ and then by a recursive formula

$$a_{i+1} = r_i a_i - a_{i-1} \quad \text{for} \quad i = e+2, e+3, \dots, \lambda$$

(See the paragraph subsequent to (6.1.5) for the property of this sequence.) Then for $i = e+2, e+3, \dots, \lambda$, we set

$$\begin{cases} \mathcal{H}_i: & w^{m_{i-1}}\eta^{m_i}\Big(1+t\,w^{a_{i-1}}\,\eta^{a_i}\,f_i\Big)^l - s = 0 \\ \mathcal{H}'_i: & z^{m_{i+1}}\zeta^{m_i}\Big(1+t\,z^{a_{i+1}}\,\zeta^{a_i}\,\widehat{f}_i\Big)^l - s = 0 \\ g_i: & \text{the transition function } z = 1/w,\ \zeta = w^{r_i}\eta \text{ of } N_i, \end{cases} \tag{7.1.5}$$

where $m_{\lambda+1} = 0$ by convention. It is easy to check that (7.1.3), (7.1.4), (7.1.5) together give a complete propagation of $DA_{e-1}(lY, d)$. □

Description of deformation (type A_l)

Let $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$ be the barking family (specifically called *of type* A_l) obtained by patching the above complete deformation atlas. According to the values of $q = n_{e-1} - r_e n_e$ and e, the deformation from X to $X_{0,t} := \Psi^{-1}(0,t)$ is described in Figures 7.1.1, 7.1.2, 7.1.3, and 7.1.4.

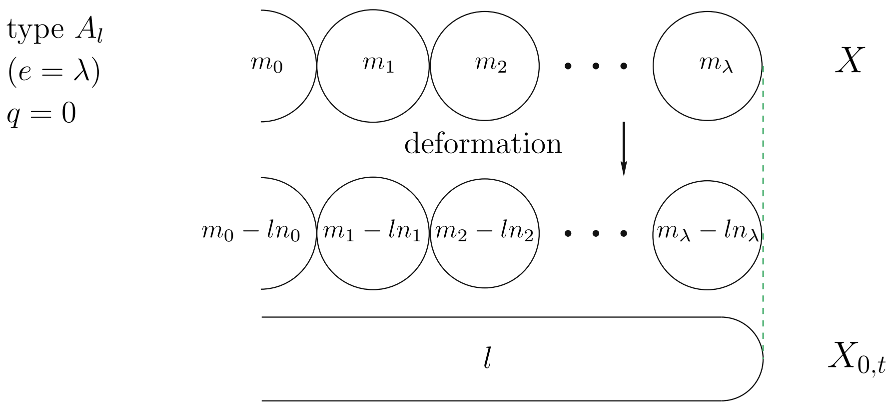

Fig. 7.1.1. This figure is for the case $n_\lambda = 1$. When $n_\lambda \geq 2$, there are disjoint n_λ connected components of multiplicity l.

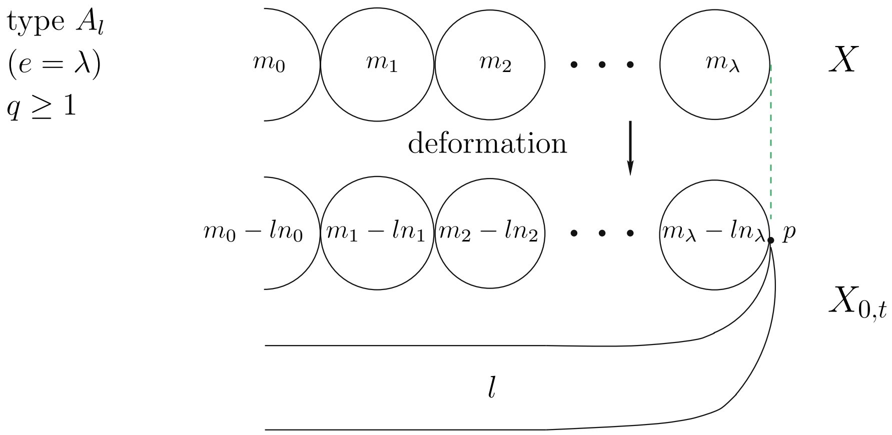

Fig. 7.1.2. This figure is for the case $n_\lambda = 1$. When $n_\lambda \geq 2$, there are n_λ connected components of multiplicity l, which intersect at one point $(z_\lambda, \zeta_\lambda) = (0, 0)$ denoted by p in the figure.

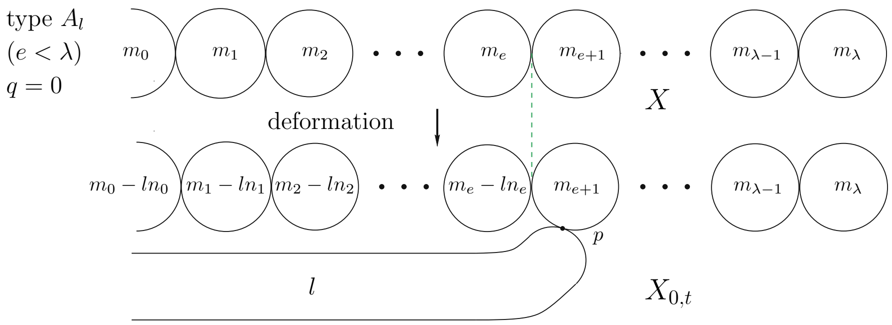

Fig. 7.1.3. This figure is for the case $n_e = 1$. When $n_e \geq 2$, there are disjoint n_e connected components of multiplicity l; each connected component intersects Θ_{e+1} at one point (when $n_e = 1$, at one point p in the figure).

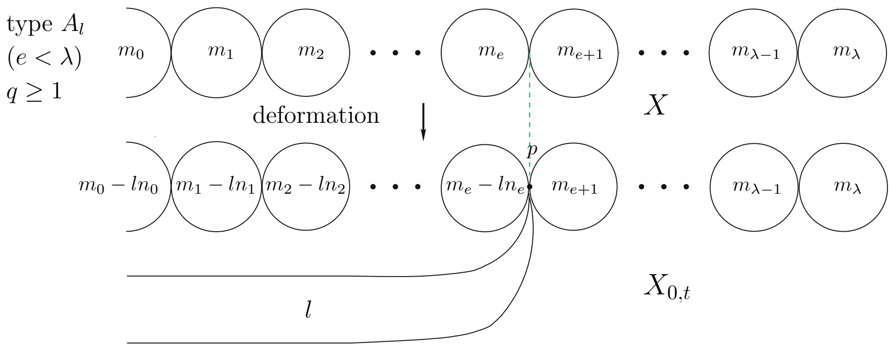

Fig. 7.1.4. This figure is for the case $n_e = 1$. When $n_e \geq 2$, there are n_e connected components of multiplicity l, which intersect at one point $(z_e, \zeta_e) = (0, 0)$ denoted by p in the figure.

7.2 Singular fibers

Let $Y = n_0\Delta_0 + n_1\Theta_1 + \cdots + n_e\Theta_e$ be a subbranch of a branch $X = m_0\Delta_0 + m_1\Theta_1 + \cdots + m_\lambda\Theta_\lambda$, and let l be a positive integer satisfying $lY \le X$, i.e. $ln_i \le m_i$ for $i = 0, 1, \ldots, e$. For a non-vanishing holomorphic function $f(x)$ on a domain $\{x \in \mathbb{C} : |x| < \varepsilon\}$, we introduce a sequence of holomorphic functions; first define a sequence of integers $p_0, p_1, \ldots, p_{\lambda+1}$ inductively by

$$\begin{cases} p_0 = 0, \quad p_1 = 1 \quad \text{and} \\ p_{i+1} = r_i p_i - p_{i-1} \quad \text{for } i = 1, 2, \ldots, \lambda, \end{cases}$$

and then set $f_i = f(w^{p_{i-1}}\eta^{p_i})$ and $\widehat{f_i} = f(z^{p_{i+1}}\zeta^{p_i})$. Denote by $DA_{e-1}(lY, d)$ the deformation atlas of weight d associated with lY and f (*Thus far we simply said "associated with lY", but in this section, f also plays an important role, and to emphasize it, we say "associated with lY and f"*): For $i = 1, 2, \ldots, e-1$,

$$\begin{cases} \mathcal{H}_i : & w^{m_{i-1}-ln_{i-1}}\eta^{m_i-ln_i}(w^{n_{i-1}}\eta^{n_i} + t^d f_i)^l - s = 0 \\ \mathcal{H}'_i : & z^{m_{i+1}-ln_{i+1}}\zeta^{m_i-ln_i}(z^{n_{i+1}}\zeta^{n_i} + t^d \widehat{f_i})^l - s = 0 \\ g_i : & \text{the transition function } z = 1/w, \ \zeta = w^{r_i}\eta \text{ of } N_i. \end{cases}$$

Next we construct complex 3-manifolds $\mathcal{M}_i$ $(i = 1, 2, \ldots, e-1)$ by patching $\mathcal{H}_i$ and $\mathcal{H}'_i$ via g_i. We then patch $\mathcal{M}_1, \mathcal{M}_2, \ldots, \mathcal{M}_{e-1}$ by 'plumbing'

$$(w_{i+1}, \eta_{i+1}, s, t) = (\zeta_i, z_i, s, t), \qquad i = 1, 2, \ldots, e-2,$$

which yields a complex 3-manifold $\mathcal{M}^{[e-1]}$. We say that the natural projection $\Psi^{[e-1]} : \mathcal{M}^{[e-1]} \to \Delta \times \Delta^\dagger$ is a *barking family of length* $e-1$. We investigate when a fiber $(\Psi^{[e-1]})^{-1}(s,t)$ is singular. For simplicity we set

$$m = m_i, \qquad m' = m_{i+1}, \qquad n = n_i, \qquad n' = n_{i+1},$$

and then $\mathcal{H}'_i : \ z^{m'-ln'}\zeta^{m-ln}(z^{n'}\zeta^n + t^d\widehat{f_i})^l - s = 0$. In what follows, we only consider the case $d = 1$ (we may apply the same argument for the case $d \ge 2$). We begin with investigation for the case where $\widehat{f_i}$ is constant (equivalently f is constant), say, $\widehat{f_i} \equiv 1$.

Proposition 7.2.1 *Let $C_{s,t} : z^{m'-ln'}\zeta^{m-ln}(z^{n'}\zeta^n + t)^l - s = 0$ be a family of curves, parameterized by s and t. Then $C_{s,t}$ is singular if and only if*

$$\begin{cases} (1)\ s = 0 & \text{if } \ mn' - m'n \neq 0 \\ (2)\ s = 0 \ \text{ or } \ \left(\dfrac{ln-m}{ln}\right)^{al} s^a = \left(\dfrac{ln-m}{m}\right)^b t^b & \text{if } \ mn' - m'n = 0, \end{cases}$$

where a and b are the relatively prime positive integers[1] *satisfying $(am, am') = (bn, bn')$. In (2), for the case $\left(\dfrac{ln-m}{ln}\right)^{al} s^a = \left(\dfrac{ln-m}{m}\right)^b t^b$, a point $(z,\zeta) \in \mathbb{C}^2$*

[1] Since $mn' - m'n = 0$, there exists such a pair a and b.

is a singularity on $C_{s,t}$ precisely when the following equations are satisfied:

$$z^{m'}\zeta^m = \left(\frac{ln-m}{ln}\right)^l s, \qquad z^{n'}\zeta^n = \frac{ln-m}{m}t.$$

Moreover, such a singularity is non-isolated; so the curve $C_{s,t}$ is non-reduced.

Proof. If $s = 0$, then $C_{s,t}$ is normal crossing (and non-reduced if $l \geq 2$) and so singular. We consider the case $s \neq 0$. Setting $F(z,\zeta,t) := z^{m'-ln'}\zeta^{m-ln}(z^{n'}\zeta^n + t)^l$, we write $C_{s,t} : F(z,\zeta,t) - s = 0$. Then $(z,\zeta) \in C_{s,t}$ is a singularity if and only if $\frac{\partial F}{\partial z}(z,\zeta) = \frac{\partial F}{\partial \zeta}(z,\zeta) = 0$. As noted in (6.4.1), this condition is equivalent to

$$\frac{\partial \log F}{\partial z}(z,\zeta) = \frac{\partial \log F}{\partial \zeta}(z,\zeta) = 0.$$

Since

$$\log F = (m' - ln')\log z + (m - ln)\log\zeta + l\log(z^{n'}\zeta^n + t),$$

we have

$$\frac{\partial \log F}{\partial z} = \frac{m'-ln'}{z} + \frac{ln' z^{n'-1}\zeta^n}{z^{n'}\zeta^n + t} = 0, \qquad \frac{\partial \log F}{\partial \zeta} = \frac{m-ln}{\zeta} + \frac{lnz^{n'}\zeta^{n-1}}{z^{n'}\zeta^n + t} = 0,$$

which yield two equations:

$$z^{n'}\zeta^n = \frac{ln'-m'}{m'}t, \qquad z^{n'}\zeta^n = \frac{ln-m}{m}t. \tag{7.2.1}$$

In particular, we obtain $\frac{ln'-m'}{m'}t = \frac{ln-m}{m}t$, that is,

$$(mn' - m'n)t = 0. \tag{7.2.2}$$

We separate into two cases according to whether $mn' - m'n$ is nonzero or not.

Case 1 $mn' - m'n \neq 0$: From (7.2.2), we have $t = 0$. Clearly $C_{s,0} : z^{m'}\zeta^m - s = 0$ is singular precisely when $s = 0$.

Case 2 $mn' - m'n = 0$: Then (7.2.2) is vacuous, and it is easy to check that two equations (7.2.1) are equivalent. So $(z,\zeta) \in \mathbb{C}^2$ is a singularity on $C_{s,t}$ if and only if

$$z^{n'}\zeta^n = \frac{ln-m}{m}t \tag{7.2.3}$$

$$z^{m'-ln'}\zeta^{m-ln}(z^{n'}\zeta^n + t)^l - s = 0. \tag{7.2.4}$$

(The second equation is just the defining equation of $C_{s,t}$.) Substituting (7.2.3) into $\frac{z^{m'}\zeta^m}{(z^{n'}\zeta^n)^l}(z^{n'}\zeta^n + t)^l - s = 0$ (7.2.4), we obtain

$$\frac{z^{m'}\zeta^m}{\left(\dfrac{ln-m}{m}t\right)^l}\left(\frac{ln-m}{m}t + t\right)^l - s = 0,$$

and thus $z^{m'}\zeta^m = \left(\frac{ln-m}{ln}\right)^l s$. Therefore $(z,\zeta) \in \mathbb{C}^2$ is a singularity on $C_{s,t}$ precisely when

$$z^{n'}\zeta^n = \frac{ln-m}{m}t, \qquad z^{m'}\zeta^m = \left(\frac{ln-m}{ln}\right)^l s. \tag{7.2.5}$$

By the same argument as the proof of Lemma 6.4.2, p111, the algebraic set in $\mathbb{C}^2$ defined by (7.2.5) is one-dimensional; hence any singularity of $C_{s,t}$ is non-isolated.

Finally we deduce the equation satisfied by s and t. From (7.2.5),

$$z^{bn'}\zeta^{bn} = \left(\frac{ln-m}{m}t\right)^b, \qquad z^{am'}\zeta^{am} = \left(\frac{ln-m}{ln}\right)^{al} s^a,$$

where a and b are the relatively prime positive integers satisfying $(am, am') = (bn, bn')$. Since $z^{bn'}\zeta^{bn} = z^{am'}\zeta^{am}$, we have $\left(\frac{ln-m}{m}t\right)^b = \left(\frac{ln-m}{ln}\right)^{al} s^a$. This completes the proof of our assertion. □

Next we treat the general case $\mathcal{H}_i' : \ z^{m'-ln'}\zeta^{m-ln}(z^{n'}\zeta^n + t^d\widehat{f_i})^l - s = 0$ where $\widehat{f_i}(z,\zeta)$ is *not* constant; recall that $\widehat{f_i} := f(z^{p_{i+1}}\zeta^{p_i})$, and $f = f(x)$ is a non-vanishing holomorphic function near $x = 0$. We write this family of complex curves as

$$C_{s,t} : \quad z^{m'-ln'}\zeta^{m-ln}(z^{n'}\zeta^n + th)^l - s = 0,$$

where $h = \widehat{f_i}(z,\zeta) = f(z^{p_{i+1}}\zeta^{p_i})$. We note that for the case $mn' - m'n \neq 0$, after some coordinate change, we may always assume that $h \equiv 1$ by Simplification Lemma (Lemma 4.1.1, p58). In contrast, for the case $mn' - m'n = 0$, the singularities of $C_{s,t}$ $(s \neq 0)$ differ according to whether h is constant or not; we already investigated the constant case in Proposition 7.2.1, and showed that any singularity is non-isolated. On the other hand, we will show that if h is not constant, then (i) $C_{s,t}$ $(s \neq 0)$ is smooth if $m' - ln' > 0$ and (ii) $C_{s,t}$ $(s \neq 0)$ has only isolated singularities if $m' - ln' = 0$.

For simplicity, we set $p' = p_{i+1}$ and $p = p_i$ $(p' > p)$. Writing

$$f(x) = c_0 + c_1x + c_2x^2 + \cdots, \qquad (c_0 \neq 0),$$

then we have $h(z,\zeta) := f(z^{p'}\zeta^p) = c_0 + c_1(z^{p'}\zeta^p) + c_2(z^{p'}\zeta^p)^2 + \cdots$.

Lemma 7.2.2 *Let* $C_{s,t} : z^{m'-ln'}\zeta^{m-ln}(z^{n'}\zeta^n + th)^l - s = 0$ *be a family of complex curves, parameterized by* s *and* t*, such that* $h(z,\zeta)$ *is not constant. Then* $C_{s,t}$ *is singular if and only if* $s = 0$.

The proof of this lemma is rather technical and we postpone it to §7.3 (see Proposition 7.3.2). Next we consider the case $i = \lambda$:

$$\mathcal{H}'_\lambda : \zeta^{m_\lambda - ln_\lambda}(\zeta^{n_\lambda} + t\widehat{f}_\lambda)^l - s = 0.$$

As usual, we express this family of complex curves as $C_{s,t} : \zeta^{m-ln}(\zeta^n + th)^l - s = 0$ where $h = \widehat{f}_\lambda(z,\zeta) = f(z^{p_{\lambda+1}}\zeta^{p_\lambda})$.

Lemma 7.2.3 *Let* p' *and* p $(p' > p)$ *be positive integers. Consider a family of complex curves* $C_{s,t} : \zeta^{m-ln}(\zeta^n + th)^l - s = 0$ *such that the holomorphic function* $h(z,\zeta) = \sum_{i\geq 0} c_i(z^{p'}\zeta^p)^i$, $(c_0 \neq 0)$ *is not constant. Then* $C_{s,t}$ *is singular if and only if* $s = 0$ *or* $\left(\dfrac{ln-m}{ln}\right)^{al} s^a = \left(\dfrac{ln-m}{m}\right)^b (c_0 t)^b$. *For the latter pair of* s *and* t, $C_{s,t}$ *has* n *singularities. In fact,* $(z,\zeta) \in C_{s,t}$ *is a singularity exactly when* $z = 0$ *and* ζ *satisfies* $\zeta^n = \dfrac{ln-m}{m}c_0 t$*; moreover such* (z,ζ) *is an* $A_{kp'-1}$*-singularity.*

We will give a proof of this lemma also in §7.3 (see Proposition 7.3.5).

We return to the discussion on deformations of branches; we apply the above results to them. Let $Y = n_0\Delta_0 + n_1\Theta_1 + \cdots + n_e\Theta_e$ be a subbranch of a branch $X = m_0\Delta_0 + m_1\Theta_1 + \cdots + m_\lambda\Theta_\lambda$, and let l be a positive integer satisfying $lY \leq X$, i.e. $ln_i \leq m_i$ for $i = 0, 1, \ldots, e$. For a non-vanishing holomorphic function $f(x)$ on a domain $\{x \in \mathbb{C} : |x| < \varepsilon\}$, we introduce a sequence of holomorphic functions; first define a sequence of integers $p_0, p_1, \ldots, p_{\lambda+1}$ inductively by

$$\begin{cases} p_0 = 0, \quad p_1 = 1 \quad \text{and} \\ p_{i+1} = r_i p_i - p_{i-1} \quad \text{for } i = 1, 2, \ldots, \lambda, \end{cases}$$

and then set $f_i = f(w^{p_{i-1}}\eta^{p_i})$ and $\widehat{f}_i = f(z^{p_{i+1}}\zeta^{p_i})$. Denote by $DA_{e-1}(lY, d)$ the deformation atlas of weight d associated with lY and f: For $i = 1, 2, \ldots, e-1$,

$$\begin{cases} \mathcal{H}_i : & w^{m_{i-1}-ln_{i-1}}\eta^{m_i - ln_i}(w^{n_{i-1}}\eta^{n_i} + t^d f_i)^l - s = 0 \\ \mathcal{H}'_i : & z^{m_{i+1}-ln_{i+1}}\zeta^{m_i-ln_i}(z^{n_{i+1}}\zeta^{n_i} + t^d \widehat{f}_i)^l - s = 0 \\ g_i : & \text{the transition function } z = 1/w, \ \zeta = w^{r_i}\eta \text{ of } N_i. \end{cases}$$

Next we construct complex 3-manifolds $\mathcal{M}_i$ $(i = 1, 2, \ldots, e-1)$ by patching $\mathcal{H}_i$ and $\mathcal{H}'_i$ via g_i. We then patch $\mathcal{M}_1, \mathcal{M}_2, \ldots, \mathcal{M}_{e-1}$ by 'plumbing'

$$(w_{i+1}, \eta_{i+1}, s, t) = (\zeta_i, z_i, s, t), \qquad i = 1, 2, \ldots, e-2$$

to obtain a complex 3-manifold $\mathcal{M}^{[e-1]}$. The projection $\Psi^{[e-1]} : \mathcal{M}^{[e-1]} \to \Delta \times \Delta^\dagger$ is called the *barking family of length* $e-1$ obtained from $DA_{e-1}(lY, d)$. The next theorem is important. (Remember that Y is proportional if $\dfrac{m_0}{n_0} = \dfrac{m_1}{n_1} = \cdots = \dfrac{m_e}{n_e}$.)

Theorem 7.2.4 *Let $\Psi^{[e-1]} : \mathcal{M}^{[e-1]} \to \Delta \times \Delta^\dagger$ be the barking family obtained from a deformation atlas $DA_{e-1}(lY, d)$. Set $X_{s,t}^{[e-1]} := (\Psi^{[e-1]})^{-1}(s,t)$, and then the following holds:*

(1) *If Y is* (i) *not proportional or* (ii) *proportional and f is not constant, then $X_{s,t}^{[e-1]}$ is singular if and only if $s = 0$.*

(2) *If Y is proportional and f is constant, then $X_{s,t}^{[e-1]}$ is singular if and only if*[2]

$$s = 0 \quad \text{or} \quad \left(\frac{ln-m}{ln}\right)^{al} s^a = \left(\frac{ln-m}{m}\right)^b (t^d)^b,$$

where $m := m_0$ and $n := n_0$, and a and b are the relatively prime positive integers satisfying $am = bn$. For the case $\left(\frac{ln-m}{ln}\right)^{al} s^a = \left(\frac{ln-m}{m}\right)^b (t^d)^b$, a point (w_i, η_i) (resp. (z_i, ζ_i)) *of $X_{s,t}^{[e-1]}$ is a singularity precisely when*

$$w_i^{m_{i-1}} \eta_i^{m_i} = \left(\frac{ln-m}{ln}\right)^l s, \qquad w_i^{n_{i-1}} \eta_i^{n_i} = \frac{ln-m}{m} t^d$$

$$\left(\text{resp.} \quad z_i^{m_{i+1}} \zeta_i^{m_i} = \left(\frac{ln-m}{ln}\right)^l s, \qquad z_i^{n_{i+1}} \zeta_i^{n_i} = \frac{ln-m}{m} t^d.\right).$$

Proof. Case (ii) in (1) is a consequence of Lemma 7.2.2. The remainder of the statement follows from Proposition 7.2.1. Here notice that instead of $m = m_0$ and $n = n_0$, we may take arbitrary $m = m_i$ and $n = n_i$ $(0 \le i \le \lambda)$; indeed from the proportionality, we have $am = bn$, i.e. $m = \frac{b}{a} n$, and so

$$\frac{ln-m}{ln} = \frac{ln - \frac{b}{a}n}{ln} = \frac{la-b}{la}, \qquad \frac{ln-m}{m} = \frac{ln - \frac{b}{a}n}{\frac{b}{a}n} = \frac{la-b}{b}.$$

Observe that $\frac{ln-m}{ln}$ and $\frac{ln-m}{m}$ are independent of the choice of $m = m_i$ and $n = n_i$ $(0 \le i \le \lambda)$. □

It is immediate to derive, as in the proof of Corollary 6.4.6, the following result from the above theorem.

Corollary 7.2.5 *Let $Y = n_0\Delta_0 + n_1\Theta_1 + \cdots + n_e\Theta_e$ be a subbranch of a branch X, and let l be a positive integer satisfying $lY \le X$. Denote by $DA_{e-1}(lY, d)$ the deformation atlas of weight d associated with lY and a non-vanishing holomorphic function $f(x)$ near $x = 0$. Let $\Psi^{[e-1]} : \mathcal{M}^{[e-1]} \to \Delta \times \Delta^\dagger$ be the*

[2] Instead of t, we use t^d because in the present situation, the weight of the deformation atlas $DA_{e-1}(lY, d)$ is d.

barking family obtained from $DA_{e-1}(lY, d)$ as in Theorem 7.2.4. *If Y is* (i) *wild,* (ii) *tame and not proportional, or* (iii) *tame, proportional, and f is not constant, then $X_{s,t}^{[e-1]} := (\Psi^{[e-1]})^{-1}(s,t)$ is singular if and only if $s = 0$.*

The above results are concerned with barking families of length $e-1$. We next, when Y is of type A_l, give a result concerning with barking families of full length. From Theorem 7.2.4 and Lemma 7.2.3, we have

Proposition 7.2.6 *Let $Y = n_0\Delta_0 + n_1\Theta_1 + \cdots + n_e\Theta_e$ be a subbranch of type A_l of a branch $X = m_0\Delta_0 + m_1\Theta_1 + \cdots + m_\lambda\Theta_\lambda$. Denote by $DA_{e-1}(lY, d)$ the deformation atlas of weight d associated with lY and a non-vanishing holomorphic function $f(x)$ near $x = 0$. For the barking family $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$ obtained from the complete propagation of $DA_{e-1}(lY, d)$, the following statements hold:*

(1) *If Y is not proportional, then $X_{s,t} = \Psi^{-1}(s,t)$ is singular if and only if $s = 0$.*
(2) *If Y is proportional and f is not constant, then $X_{s,t}$ is singular if and only if $s = 0$ or $\left(\dfrac{ln-m}{ln}\right)^{al} s^a = \left(\dfrac{ln-m}{m}\right)^b (c_0t^d)^b$ where*[3] *$m := m_0$, $n := n_0$, $c_0 = f(0)\,(\neq 0)$, and a and b are the relatively prime positive integers satisfying $am = bn$.*
For the latter pair of s and t, $X_{s,t}$ has only $A_{kp_{\lambda+1}-1}$-singularities near the edge of the branch X, where k is the minimal positive integer such that $c_k \neq 0$ in the expansion $f(x) = c_0 + c_1x + c_2x^2 + \cdots$. These singularities are nodes (A_1-singularities) exactly when[4] *$\lambda = 1$, $\Theta_1 \cdot \Theta_1 = -2$* (i.e. $r_1 = 2$), *and $c_1 \neq 0$.*
(3) *If Y is proportional and f is constant, then $X_{s,t}$ is singular if and only if $s = 0$ or $\left(\dfrac{ln-m}{ln}\right)^{al} s^a = \left(\dfrac{ln-m}{m}\right)^b (t^d)^b$.*

7.3 Supplement: Singularities of certain curves

This section is devoted to the verification of the technical results stated without proof in the previous section.

Lemma 7.3.1 *Let $f(x) = c_0 + c_1x + c_2x^2 + \cdots$, $(c_0 \neq 0)$ be a non-vanishing holomorphic function defined near $x = 0$. Given positive integers p' and p $(p' > p)$, set $h(z,\zeta) := f(z^{p'}\zeta^p)$. If $m'n - mn' = 0$, then $C_{s,t} : z^{m'-ln'}\zeta^{m-ln}(z^{n'}\zeta^n + th)^l - s = 0$, $(s \neq 0)$ is smooth.*

[3] From the proportionality, we may actually take arbitrary m_i and n_i instead of m_0 and n_0, as noted in the proof of Theorem 7.2.4.

[4] This will be proved in Lemma 7.3.7 below. The positive integer p' in Lemma 7.3.7 corresponds to $p_{\lambda+1}$ in the present situation, and we have $p_{\lambda+1} = 2$ exactly when $\lambda = 1$ and $r_1 = 2$.

Proof. Set $F(z,\zeta,t) := z^{m'-ln'}\zeta^{m-ln}(z^{n'}\zeta^n+th)^l$, and write $C_{s,t} : F(z,\zeta,t)-s=0$. Assuming that $C_{s,t}$, $(s\neq 0)$ has a singularity, we deduce a contradiction. Since

$$\log F = (m'-ln')\log z + (m-ln)\log\zeta + l\log(z^{n'}\zeta^n+th),$$

a point $(z,\zeta)\in C_{s,t}$ $(s\neq 0)$ is a singularity if and only if

$$\frac{\partial \log F}{\partial z} = \frac{m'-ln'}{z} + l\frac{n'z^{n'-1}\zeta^n+th_z}{z^{n'}\zeta^n+th} = 0,$$

$$\frac{\partial \log F}{\partial \zeta} = \frac{m-ln}{\zeta} + l\frac{nz^{n'}\zeta^{n-1}+th_\zeta}{z^{n'}\zeta^n+th} = 0,$$

or equivalently

$$m'z^{n'}\zeta^n + (m'-ln')th + tzh_z = 0, \qquad mz^{n'}\zeta^n + (m-ln)th + t\zeta h_\zeta = 0.$$

Deleting $z^{n'}\zeta^n$ from the above two equations, we derive

$$l(m'n-mn')h + mzh_z = m'\zeta h_\zeta.$$

Since $m'n-mn'=0$ by assumption, we have

$$mzh_z = m'\zeta h_\zeta. \tag{7.3.1}$$

Since

$$h(z,\zeta) = c_0 + c_1(z^{p'}\zeta^p) + c_2(z^{p'}\zeta^p)^2 + \cdots + c_i(z^{p'}\zeta^p)^i + \cdots,$$

the equation $mzh_z = m'\zeta h_\zeta$ (7.3.1) is concretely given by

$$m\Big(p'c_1(z^{p'}\zeta^p) + 2p'c_2(z^{p'}\zeta^p)^2 + \cdots + ip'c_i(z^{p'}\zeta^p)^i + \cdots\Big) \tag{7.3.2}$$
$$= m'\Big(pc_1(z^{p'}\zeta^p) + 2pc_2(z^{p'}\zeta^p)^2 + \cdots + ipc_i(z^{p'}\zeta^p)^i + \cdots\Big).$$

Namely, $(mp'-m'p)\sum_{i\geq 1} ic_i(z^{p'}\zeta^p)^i = 0$. Note $mp'-m'p>0$ by the assumption $m>m'$ and $p'>p$, and so this equation is equivalent to $\sum_{i\geq 1} ic_i(z^{p'}\zeta^p)^i = 0$. Thus the equation $mzh_z = m'\zeta h_\zeta$ (7.3.1) is equivalent to

$$\alpha f'(\alpha) = 0, \tag{7.3.3}$$

where we set $\alpha = z^{p'}\zeta^p$, $f(x) = c_0 + c_1x + c_2x^2 + \cdots$, $(c_0\neq 0)$, and $f'(x) := df/dx$. However, the holomorphic function $xf'(x)$ does not vanish near the origin unless $x=0$. Indeed, let k be the minimal positive integer such that $c_k\neq 0$ (there is such k, because $f(x)$ is not constant by assumption), and then $f(x) = c_0 + c_k x^k + c_{k+1}x^{k+1} + \cdots$. So

$$xf'(x) = x\Big(kc_k x^{k-1} + (k+1)c_{k+1}x^k + (k+2)c_{k+2}x^{k+1} + \cdots\Big)$$
$$= x^k\Big(kc_k + (k+1)c_{k+1}x + (k+2)c_{k+2}x^2 + \cdots\Big),$$

which, by $c_k \neq 0$, does not vanish near the origin unless $x = 0$. Thus (7.3.3) holds if and only if $\alpha = 0$, that is, $z = 0$ or $\zeta = 0$. However, substituting $z = 0$ or $\zeta = 0$ in the defining equation $C_{s,t} : z^{m'-ln'}\zeta^{m-ln}(z^{n'}\zeta^n + th)^l - s = 0$, we have $s = 0$. This gives a contradiction. □

Hence we have

Proposition 7.3.2 *Let $f(x) = c_0 + c_1 x + c_2 x^2 + \cdots$, $(c_0 \neq 0)$ be a non-vanishing holomorphic function defined near $x = 0$. Given positive integers p' and p $(p' > p)$, set $h(z,\zeta) := f(z^{p'}\zeta^p)$. Then $C_{s,t} : z^{m'-ln'}\zeta^{m-ln}(z^{n'}\zeta^n + th)^l - s = 0$ is singular if and only if $s = 0$.*

Next we study the singularities of a curve $C_{s,t} : \zeta^{m-ln}(\zeta^n + th)^l - s = 0$; the result is quite different from the case $C_{s,t} : z^{m'-ln'}\zeta^{m-ln}(z^{n'}\zeta^n + th)^l - s = 0$.

Lemma 7.3.3 *Let $f(x) = c_0 + c_1 x + c_2 x^2 + \cdots$, $(c_0 \neq 0)$ be a non-vanishing holomorphic function defined near $x = 0$. Given positive integers p' and p $(p' > p)$, set $h(z,\zeta) := f(z^{p'}\zeta^p)$, and consider a family of curves $C_{s,t} : \zeta^{m-ln}(\zeta^n + th)^l - s = 0$. Then $(z_0,\zeta_0) \in C_{s,t}$ $(s,t \neq 0)$ is a singularity precisely when*

$$z_0 = 0, \qquad \zeta_0^n = \frac{ln - m}{m} tc_0. \tag{7.3.4}$$

Proof. We set $F = \zeta^{m-ln}(\zeta^n + th)^l$, and then $C_{s,t} : F(z,\zeta,t) - s = 0$. Since

$$\log F = (m - ln)\log\zeta + l\log(\zeta^n + th),$$

a point $(z,\zeta) \in C_{s,t}$ $(s,t \neq 0)$ is a singularity if and only if

$$\frac{\partial \log F}{\partial z} = l\frac{th_z}{\zeta^n + th} = 0, \qquad \frac{\partial \log F}{\partial \zeta} = \frac{m - ln}{\zeta} + l\frac{n\zeta^{n-1} + th_\zeta}{\zeta^n + th} = 0,$$

or equivalently (by $t \neq 0$)

$$h_z = 0, \qquad m\zeta^n + (m - ln)th + lt\zeta h_\zeta = 0. \tag{7.3.5}$$

First we demonstrate that the solution of $h_z(z,\zeta) = 0$ is $z = 0$ and $\zeta = 0$. Since $h(z,\zeta) = \sum_{i\geq 0} c_i(z^{p'}\zeta^p)^i$, we have $h_z(z,\zeta) = p'\Big(\sum_{i\geq 0} ic_i(z^{ip'-1}\zeta^{ip})\Big)$. Let k be the smallest positive integer such that $c_k \neq 0$, and then

$$h_z(z,\zeta) = p' z^{kp'-1}\zeta^{kp}$$
$$\times\Big(kc_k + (k+1)c_{k+1}(z^{p'}\zeta^p) + (k+2)c_{k+2}(z^{p'}\zeta^p)^2 + \cdots\Big)$$

(Here note $kp' - 1 \geq 1$, because $p' \geq 2$ by $p' > p = 1$.) As $c_k \neq 0$,

$$kc_k + (k+1)c_{k+1}(z^{p'}\zeta^p) + (k+2)c_{k+2}(z^{p'}\zeta^p)^2 + \cdots$$

is nonzero for sufficiently small $|z^{p'}\zeta^p|$, and hence $h_z(z,\zeta) = 0$ holds exactly when $z^{kp'-1}\zeta^{kp} = 0$, so the solutions of $h_z(z,\zeta) = 0$ are just $z = 0$ and $\zeta = 0$. However, $\zeta = 0$ fails to be a solution of (7.3.5); when we put $\zeta = 0$, there is no z satisfying the equation on the right hand side of (7.3.5). To see this, we set

$$f(z,\zeta,t) = m\zeta^n + (m - ln)th + lt\zeta h_\zeta,$$

and then $f(z,0,t) = (m - ln)\,t\,h(z,0)$. Since $m - ln > 0$, $t \neq 0$, and $h(z,\zeta)$ is non-vanishing by assumption, $f(z,0,t)$ cannot be zero. This confirms that $\zeta = 0$ is not a solution of (7.3.5). On the other hand, another solution $z = 0$ of $h_z(z,\zeta) = 0$ indeed gives a solution of (7.3.5), which is seen as follows. First, set $z = 0$ in the equation on the right hand side of (7.3.5):

$$m\zeta^n + (m - ln)th(0,\zeta) + lt\zeta h_\zeta(0,\zeta) = 0. \tag{7.3.6}$$

From the expansion $h(z,\zeta) = c_0 + c_1(z^{p'}\zeta^p) + c_2(z^{p'}\zeta^p)^2 + \cdots$, we have

$$h(0,\zeta) = h(0,0)\ (\neq 0) \qquad \text{and} \qquad h_\zeta(0,\zeta) = 0, \tag{7.3.7}$$

and so (7.3.6) is $m\zeta^n + (m - ln)th(0,0) = 0$, that is, $\zeta^n = \dfrac{ln - m}{m} th(0,0)$. Thus the solution of (7.3.5) is $z = 0$ and $\zeta = \left(\dfrac{ln - m}{m}\, t\, h(0,0)\right)^{1/n}$. □

Next for the complex curve $C_{s,t}$ (in Lemma 7.3.3) with the singularity at

$$(z_0, \zeta_0) = \left(0,\ \left(\frac{ln - m}{m} tc_0\right)^{1/n}\right),$$

we derive the equation fulfilled by s and t. Since (z_0, ζ_0) satisfies the equation $C_{s,t} : \zeta^{m-ln}(\zeta^n + th)^l - s = 0$, substituting $z_0 = 0$ into it, we have

$$\zeta_0^{m-ln}\big(\zeta_0^n + th(0,\zeta_0)\big)^l - s = 0.$$

By $h(0,\zeta) = h(0,0) = c_0$ (7.3.7), we obtain $\zeta_0^{m-ln}(\zeta_0^n + tc_0)^l - s = 0$. Next, substituting $\dfrac{m}{ln - m}\zeta_0^n = tc_0$ (7.3.4) into this equation, we deduce $\zeta_0^m \left(\dfrac{ln}{ln - m}\right)^l - s = 0$, or

$$\zeta_0^m = \left(\frac{ln - m}{ln}\right)^l s.$$

Put $\zeta_0 = \left(\dfrac{ln - m}{m} tc_0\right)^{1/n}$ into this equation, which yields

$$\left(\frac{ln-m}{m}tc_0\right)^{m/n} = \left(\frac{ln-m}{ln}\right)^l s.$$

Now write $m/n = b/a$ where a and b be the relatively prime positive integers, and then $\left(\frac{ln-m}{m}tc_0\right)^{b/a} = \left(\frac{ln-m}{ln}\right)^l s$. Therefore $\left(\frac{ln-m}{m}tc_0\right)^b = \left(\frac{ln-m}{ln}\right)^{al} s^a$.

Further we shall show that the singularity $(z_0, \zeta_0) \in C_{s,t}$ in Lemma 7.3.3 is an A-singularity (recall that an A_μ-singularity is a singularity analytically equivalent to $y^2 = x^{\mu+1}$). First we claim that a projection $(z, \zeta) \in C_{s,t} \mapsto z \in \mathbb{C}$ restricted to a neighborhood of (z_0, ζ_0) is a double covering with the ramification point (z_0, ζ_0), in other words, $F(z_0, \zeta, t) - s = 0$ as a polynomial in ζ has a double root ζ_0. To see this, it suffices to show that $\frac{\partial F(z, \zeta, t) - s}{\partial \zeta}|_{z=z_0}$, as a polynomial in ζ, has only simple zeros. Since $\frac{\partial F(z, \zeta, t) - s}{\partial \zeta} = 0$ is explicitly given by

$$m\zeta^n + (m - ln)th + lt\zeta h_\zeta = 0, \qquad \text{(see (7.3.5))},$$

setting $z = z_0$ $(= 0)$, then from $h(0, \zeta) = c_0$ and $h_\zeta(0, \zeta) = 0$, we have

$$m\zeta^n + (m - ln)tc_0 = 0, \tag{7.3.8}$$

which as a polynomial in ζ has only simple roots; hence $F(z_0, \zeta, t) - s = 0$ (a polynomial in ζ) has at most double roots. Note that ζ_0 is indeed one of the double roots, because ζ_0 satisfies (7.3.8). Therefore the projection $(z, \zeta) \in C_{s,t} \mapsto z \in \mathbb{C}$ restricted to a neighborhood of (z_0, ζ_0) is a double covering with the ramification point (z_0, ζ_0). Then (z_0, ζ_0) is shown to be an A-singularity (by using the Newton polygon of this singularity (see §7.4, p137), or in the same way as the proofs of Lemma 21.6.1, p406 and Proposition 21.6.3, p408 of §21.6). Thus we have

Lemma 7.3.4 *An singularity* $(z_0, \zeta_0) = \left(0, \left(\frac{ln-m}{m}tc_0\right)^{1/n}\right)$ *of* $C_{s,t}$ $(s, t \neq 0)$ *in Lemma* 7.3.3 *is an A-singularity, and s and t satisfy* $\left(\frac{ln-m}{m}tc_0\right)^{m/n} = \left(\frac{ln-m}{ln}\right)^l s$.

Actually we may completely determine this A-singularity. Using a *Newton polygon*, it is not so difficult to check that (z_0, ζ_0) is an $A_{kp'-1}$-singularity; we postpone the proof to §7.4 (Proposition 7.4.4), but immediately after we summarize the above results as a proposition, for a special case we give a proof which uses Hesse matrices.

Proposition 7.3.5 *Let* $f(x) = c_0 + c_1 x + c_2 x^2 + \cdots$, $(c_0 \neq 0)$ *be a non-vanishing holomorphic function defined near* $x = 0$. *Given positive integers* p' *and* p $(p' > p)$, *set* $h(z, \zeta) := f(z^{p'}\zeta^p)$, *and consider a family of curves*

$C_{s,t} : \zeta^{m-ln}(\zeta^n + th)^l - s = 0$. *Then* $C_{s,t}$ *is singular if and only if* $s = 0$ *or* $\left(\frac{ln-m}{ln}\right)^{al} s^a = \left(\frac{ln-m}{m}\right)^b (c_0 t)^b$, *where* a *and* b *are the relatively prime integers such that* $am = bn$.

For the latter pair of s *and* t, $C_{s,t}$ *has* n *singularities. In fact,* $(z_0, \zeta_0) \in C_{s,t}$ *is a singularity exactly when* $z_0 = 0$ *and* $\zeta_0^n = \frac{ln-m}{m} c_0 t$; *moreover such* (z_0, ζ_0) *is an* $A_{kp'-1}$*-singularity, where* k *is the minimal positive integer such that* $c_k \neq 0$.

Computation of Hesse matrices

We investigate when the singularity $(z_0, \zeta_0) = \left(0, \left(\frac{ln-m}{m} tc_0\right)^{1/n}\right)$ in Lemma 7.3.3 is an A_1-singularity (a node). We note that for a singularity $(x, y) = (a, b)$ of a plane curve $G(x, y) = 0$, we can determine whether (a, b) is a node or not, in terms of the quadratic terms of the Taylor expansion

$$\begin{aligned} G(x,y) = c_{00} + c_{10}(x-a) + c_{01}(y-b) + c_{20}(x-a)^2 \\ + 2c_{11}(x-a)(y-b) + c_{02}(y-b)^2 + \cdots . \end{aligned}$$

(Actually, $c_{10} = c_{01} = 0$ because (a, b) is a singularity.) We shall explain this. The *Hesse matrix* of $G(x, y)$ at (a, b) is given by

$$H(a,b) := 2\begin{pmatrix} c_{20} & c_{11} \\ c_{11} & c_{02} \end{pmatrix} = \begin{pmatrix} \frac{\partial^2 G}{\partial x^2}(a,b) & \frac{\partial^2 G}{\partial x \partial y}(a,b) \\ \frac{\partial^2 G}{\partial y \partial x}(a,b) & \frac{\partial^2 G}{\partial y^2}(a,b) \end{pmatrix}.$$

We say that the singularity (a, b) is *non-degenerate* provided that $H(a, b)$ is invertible, or equivalently $\det(H(a, b)) \neq 0$ ([Mil], Chapter 7).

Theorem 7.3.6 (Milnor) *The following conditions are equivalent:* (1) (a, b) *is a non-degenerate singularity,* (2) (a, b) *is a node, and* (3) $\mu = 1$ *where* μ *is the Milnor number*[5] *of* (a, b).

Proof. (1) $\iff$ (3): By Lemma B.1 and Problem 2 in Appendix B of [Mil]. (1) $\Longrightarrow$ (2): Just the Morse Lemma. (2) $\Longleftarrow$ (1): Obvious. □

After the above preparation, we now prove

Lemma 7.3.7 *Let* $f(x) = c_0 + c_1 x + c_2 x^2 + \cdots$, $(c_0 \neq 0)$ *be a non-vanishing holomorphic function defined near* $x = 0$. *Given positive integers* p' *and* p $(p' > p)$, *set* $h(z, \zeta) := f(z^{p'} \zeta^p)$, *and consider a family of curves* $C_{s,t}$: $\zeta^{m-ln}(\zeta^n + th)^l - s = 0$. *Then the singularity* $(z_0, \zeta_0) = \left(0, \left(\frac{ln-m}{m} tc_0\right)^{1/n}\right)$ *in Lemma* 7.3.3 *is a node if and only if* $p' = 2$ *and* $c_1 \neq 0$.

[5] See [Di], [Mil]. The A_μ-singularity $y^2 = x^{\mu+1}$ has the Milnor number μ.

Proof. Setting $G(z,\zeta) := \zeta^{m-ln}\big(\zeta^n + th(z,\zeta)\big)^l - s$, we first note that

$$\frac{\partial G}{\partial z} = \zeta^{m-ln}\, t\, h_z\, l\big(\zeta^n + th\big)^{l-1},$$

$$\frac{\partial G}{\partial \zeta} = \zeta^{m-ln-1}\big(\zeta^n + th\big)^{l-1}\Big[m\zeta^n + (m-ln)th + lt\zeta h_\zeta\Big].$$

We now compute entries $\dfrac{\partial^2 G}{\partial\zeta\partial z}, \dfrac{\partial^2 G}{\partial\zeta^2}, \dfrac{\partial^2 G}{\partial z^2}$ of the Hesse matrix of G at the singular point $(z_0,\zeta_0) = \left(0,\ \left(\frac{ln-m}{m}tc_0\right)^{1/n}\right)$. First of all,

$$\frac{\partial^2 G}{\partial\zeta\partial z} = (m-ln)\zeta^{m-ln-1}\, t\, h_z\, l\big(\zeta^n + th\big)^{l-1} + \zeta^{m-ln}\, t\, h_{z\zeta}\, l\big(\zeta^n + th\big)^{l-1}$$
$$+ \zeta^{m-ln}\, t\, h_z\, l\big(l-1\big)\big(\zeta^n + th\big)^{l-2}\big(n\zeta^{n-1} + th_\zeta\big).$$

Since $p' \geq 2$ and $h(z,\zeta) = c_0 + c_1(z^{p'}\zeta^p) + c_2(z^{p'}\zeta^p)^2 + \cdots$, we have $h_z(0,\zeta) = 0$ and $h_{z\zeta}(0,\zeta) = 0$. So regardless of the value of ζ, we have $\dfrac{\partial^2 G}{\partial\zeta\partial z}(0,\zeta) = 0$. Next we note

$$\begin{aligned}
\frac{\partial^2 G}{\partial\zeta^2} = {} & \big(m-ln-1\big)\zeta^{m-ln-2}\big(\zeta^n + t\,h\big)^{l-1} \\
& \times \Big[m\,\zeta^n + (m-ln)\,t\,h + l\,t\,\zeta\,h_\zeta\Big] \\
& + \zeta^{m-ln-1}\big(l-1\big)\big(\zeta^n + t\,h\big)^{l-2}\big(n\,\zeta^{n-1} + t\,h_\zeta\big) \\
& \times \Big[m\,\zeta^n + (m-ln)\,t\,h + l\,t\,\zeta\,h_\zeta\Big] \\
& + \zeta^{m-ln-1}\big(\zeta^n + t\,h\big)^{l-1} \\
& \times \Big[m\,n\,\zeta^{n-1} + (m-ln)\,t\,h_\zeta + l\,t\,(h_\zeta + \zeta\,h_{\zeta\zeta})\Big].
\end{aligned}$$

From $h_\zeta(0,\zeta) = h_{\zeta\zeta}(0,\zeta) = 0$ and $h(0,0) = h(0,\zeta) = c_0$, we have

$$\begin{aligned}
\frac{\partial^2 G}{\partial\zeta^2}(0,\zeta) = {} & \big(m-ln-1\big)\zeta^{m-ln-2}\big(\zeta^n + tc_0\big)^{l-1}\Big[m\zeta^n + (m-ln)tc_0\Big] \\
& + \zeta^{m-ln-1}\big(l-1\big)\big(\zeta^n\big)^{l-2}\big(n\zeta^{n-1}\big)\Big[m\zeta^n + (m-ln)tc_0\Big] \\
& + \zeta^{m-ln-1}\big(\zeta^n + tc_0\big)^{l-1}\Big[mn\zeta^{n-1}\Big].
\end{aligned}$$

Evaluate this at $\zeta = \zeta_0$, and then since $m\zeta_0^n = (ln-m)tc_0$, the first and second terms vanish:

$$\frac{\partial^2 G}{\partial\zeta^2}(0,\zeta_0) = \zeta_0^{m-ln-1}\big(\zeta_0^n + tc_0\big)^{l-1}\big[mn\zeta_0^{n-1}\big].$$

Again, using $\zeta_0^n = \frac{ln - m}{m} tc_0$, we obtain

$$\frac{\partial^2 G}{\partial \zeta^2}(0, \zeta_0) = \zeta_0^{m-ln-1} \left(\frac{ln - m}{m} tc_0 + tc_0 \right)^{l-1} \left[mn\zeta_0^{n-1} \right]$$
$$= mn \left(tc_0 \frac{ln}{m} \right)^{l-1} \zeta_0^{m-ln+n-2},$$

which is clearly nonzero. Finally, we compute $\frac{\partial^2 G}{\partial z^2}(0, \zeta)$. Note that

$$\frac{\partial^2 G}{\partial z^2} = \zeta^{m-ln}\, t\, l \Big[h_{zz}(\zeta^n + th)^{l-1} + t\, h_z^2\, (l-1)(\zeta^n + th)^{l-2} \Big].$$

By $h_z(0, \zeta) = 0$ and $h(0, \zeta) = c_0$, we have

$$\frac{\partial^2 G}{\partial z^2}(0, \zeta) = \zeta^{m-ln}\, t\, l \left[h_{zz}(0, \zeta) \big(\zeta^n + tc_0 \big)^{l-1} \right].$$

Here from $h(z, \zeta) = c_0 + c_1(z^{p'}\zeta^p) + c_2(z^{p'}\zeta^p)^2 + \cdots$, it is easy to check

$$h_{zz}(0, \zeta) = \begin{cases} 0, & p' \geq 3, \\ 2c_1\zeta^p, & p' = 2. \end{cases} \tag{7.3.9}$$

Thus if $p' \geq 3$, then $\frac{\partial^2 G}{\partial z^2}(0, \zeta) = 0$, while if $p' = 2$, then

$$\frac{\partial^2 G}{\partial z^2}(0, \zeta) = \zeta^{m-ln}\, t\, l \left[h_{zz}(0, \zeta) \Big(\zeta^n + t\, c_0 \Big)^{l-1} \right]$$
$$= \zeta^{m-ln}\, t\, l \Big[2\, c_1\, \zeta^p (\zeta^n + t\, c_0)^{l-1} \Big]$$
$$= 2\, t\, l\, c_1\, \zeta^{m-ln+p}\, (\zeta^n + tc_0)^{l-1}.$$

Therefore if $p' \geq 3$, then by $\frac{\partial^2 G}{\partial z^2}(0, \zeta_0) = 0$, the Hesse matrix is

$$H = \begin{pmatrix} 0 & \frac{\partial^2 G}{\partial z \partial \zeta}(0, \zeta_0) \\ \frac{\partial^2 G}{\partial \zeta \partial z}(0, \zeta_0) & \frac{\partial^2 G}{\partial \zeta^2}(0, \zeta_0) \end{pmatrix},$$

which is clearly not invertible. On the other hand, for the case $p' = 2$, the Hesse matrix is

$$H = \begin{pmatrix} \frac{\partial^2 G}{\partial z^2}(0, \zeta_0) & \frac{\partial^2 G}{\partial z \partial \zeta}(0, \zeta_0) \\ \frac{\partial^2 G}{\partial \zeta \partial z}(0, \zeta_0) & \frac{\partial^2 G}{\partial \zeta^2}(0, \zeta_0) \end{pmatrix} = \begin{pmatrix} 2\, t\, l\, c_1\, \zeta_0^{m-ln+p} (\zeta_0^n + tc_0)^{l-1} & 0 \\ 0 & d \end{pmatrix},$$

where $d := mn\left(tc_0\frac{ln}{m}\right)^{l-1}\zeta_0^{m-ln+n-2}$. We note that

$$\zeta_0 \neq 0 \qquad \text{and} \qquad \zeta_0^n + tc_0 \neq 0. \tag{7.3.10}$$

To see this, recall that $\zeta_0^n = \frac{ln-m}{m}tc_0$. Since $c_0 \neq 0$, we have $\zeta_0 \neq 0$; in particular $d \neq 0$. Also $\zeta_0^n + tc_0 = \frac{ln-m}{m}tc_0 + tc_0 = \frac{ln}{m}tc_0 \neq 0$, and hence $\zeta_0^n + tc_0 \neq 0$. Therefore if $p' = 2$, then from

$$\det(H) = 2\,t\,l\,c_1\,\zeta_0^{m-ln+p}(\zeta_0^n + tc_0)^{l-1}\,d,$$

it follows that the Hesse matrix H is invertible exactly when $c_1 \neq 0$. We conclude that the Hesse matrix H is invertible if and only if $p' = 2$ and $c_1 \neq 0$; by Theorem 7.3.6, this is precisely the case where the singularity (z_0, ζ_0) is non-degenerate, that is, a node. This completes the proof of our assertion. □

7.4 Newton polygons and singularities

Let $f(x) = c_0 + c_1x + c_2x^2 + \cdots$, $(c_0 \neq 0)$ be a non-vanishing holomorphic function defined near $x = 0$. Given positive integers p' and p such that $p' > p$, we set $h(z,\zeta) := f(z^{p'}\zeta^p)$ and we consider a family of curves

$$C_{s,t} : \zeta^{m-ln}(\zeta^n + th)^l - s = 0.$$

By Lemma 7.3.4, for fixed $s, t \neq 0$ satisfying $\left(\frac{ln-m}{ln}\right)^{al} s^a = \left(\frac{ln-m}{m}\right)^b (c_0t)^b$, the curve $C_{s,t}$ has an A-singularity $(z_0, \zeta_0) = \left(0, \left(\frac{ln-m}{m}tc_0\right)^{1/n}\right)$. In this section, using a Newton polygon, we determine this singularity to be an $A_{kp'-1}$-singularity, where k is the least positive integer such that $c_k \neq 0$.

Before we give a proof, we review Newton polygons (see [BK] p376, [Oka2] for details). Let $f(x,y) = \sum_{i,j} a_{ij}x^iy^j$ be a polynomial (more generally, a convergent power series) in two variables x and y such that $f(0,0) = 0$. We assume that the plane curve $f(x,y) = 0$ has a singularity at the origin $(0,0)$. Then we consider the convex hull $\Gamma_+(f)$ of a set $\bigcup_{i,j}\left((i,j)+\mathbb{R}^2_+\right)$ in $\mathbb{R}^2$, where

(i,j) runs over the exponents of all monomials of f,
$\mathbb{R}^2_+ := \{\,(x,y) \in \mathbb{R}^2 \;:\; x \geq 0,\ y \geq 0\,\}$, and
"$(i,j) + \mathbb{R}^2_+$" stands for the translation of $\mathbb{R}^2_+$ by (i,j).

We write the boundary of the convex set $\Gamma_+(f)$ as $H_x \cup K \cup H_y$ (see Figure 7.4.1), where (i) H_x (resp. H_y) is a half line parallel to the x-axis (resp. y-axis) and (ii) K consists of segments of negative slopes. We denote K by $\Gamma(f)$, and we call it the *Newton polygon* (or *Newton boundary*) of f: We assume that $\Gamma(f)$ contains the end points (so $\Gamma(f)$ is closed). Each edge of the Newton polygon $\Gamma(f)$ joins two integral points; they are *vertices* of $\Gamma(f)$. Conventionally, an edge of the Newton polygon $\Gamma(f)$ is referred to as a *face*.

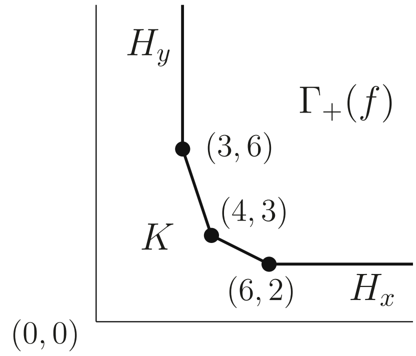

Fig. 7.4.1. The boundary of $\Gamma_+(f)$ is $H_x \cup K \cup H_y$, where $f = x^3y^6 + x^4y^3 + x^6y^2 + x^3y^7 + x^8y^3$.

The Newton polygon of f does not necessarily determine the topological type of the singularity $f(x, y) = 0$, as the following example indicates: $f(x, y) = x^2 + y^2$ defines a node, whereas $g(x, y) = y^2 + 2xy + x^2 = (x + y)^2$ defines a double line. However both f and g have the same Newton polygon, that is, a line segment joining $(0, 2)$ and $(2, 0)$. Motivated by this example, we introduce the concept of *Newton degenerate/nondegenerate* (see [Oka2] for details). For each face I of the Newton polygon $\Gamma(f)$, we associate a partial sum of f:

$$f_I = \sum_{(i,j)\in I} a_{ij} x^i y^j,$$

which is called a *face function*. We say that f is *Newton nondegenerate* if for each face I of $\Gamma(f)$, the face function f_I does not admit a factorization with a multiple factor; otherwise f is *Newton degenerate*. The notion of Newton degenerate/nondegenerate depends on the choice of coordinates x and y (see [LeOk]).

The following theorem is due to Mutsuo Oka ([Oka1], Theorem 2.1, p436); in fact, this theorem holds for arbitrary dimension, but we merely state it for the two dimensional case.

Theorem 7.4.1 (Oka) *Suppose that $f(x, y) = 0$ has an isolated singularity at the origin and that f is Newton nondegenerate. Then the Milnor fibration of this singularity is determined by the Newton polygon $\Gamma(f)$.*

(Note: From the Newton polygon, we can canonically construct an embedded resolution of the Newton nondegenerate singularity $f = 0$ [BK]. In particular, if two Newton nondegenerate singularities $f = 0$ and $g = 0$ have the same Newton polygon, then they admit the same embedded resolution. From this fact, the topological equivalence of the two singularities $f = 0$ and $g = 0$ follows.)

Now we return to discuss the curve $C_{s,t} : \zeta^{m-ln}(\zeta^n + th)^l - s = 0$, where

$$h(z, \zeta) = c_0 + c_1 z^{p'} \zeta^p + c_2 z^{2p'} \zeta^{2p} + \cdots, \qquad (c_0 \neq 0).$$

We fix s and t $(s, t \neq 0)$ such that $\left(\frac{ln-m}{ln}\right)^{al} s^a = \left(\frac{ln-m}{m}\right)^b (c_0 t)^b$; then by Lemma 7.3.4, $C_{s,t}$ has an A-singularity $(z_0, \zeta_0) = \left(0, \left(\frac{ln-m}{m} t c_0\right)^{1/n}\right)$. We now demonstrate that $(z_0, \zeta_0) \in C_{s,t}$ is an $A_{kp'-1}$-singularity, where k is the least positive integer such that $c_k \neq 0$. For the convenience of the discussion, we write $p_i = ip$ and $p_i' = ip'$; moreover if k, the least positive integer such that $c_k \neq 0$, is greater than 1, we renumber so that $c_k, c_{k+1}, c_{k+2}, \ldots$ is replaced by $c_1, c_2, c_3, \ldots$. Then

$$h(z, \zeta) = c_0 + c_1 z^{p_1'} \zeta^{p_1} + c_2 z^{p_2'} \zeta^{p_2} + \cdots, \qquad (c_0 \neq 0, \ \ c_1 \neq 0),$$

and our goal is, in new notation, to verify that the singularity (z_0, ζ_0) of the curve $C_{s,t}$ is an $A_{p_1'-1}$-singularity. As $z_0 = 0$ while $\zeta_0 \neq 0$ (Lemma 7.3.3), in order to apply the theory of Newton polygons, we need to translate the function $\zeta^{m-ln}(\zeta^n + th)^l - s$ by $\zeta \longmapsto \zeta + \zeta_0$ so that (z_0, ζ_0) is transformed to the origin $(0, 0)$; the resulting function is

$$(\zeta + \zeta_0)^{m-ln} \Big[(\zeta + \zeta_0)^n + th\big(z, \zeta + \zeta_0\big) \Big]^l - s.$$

Lemma 7.4.2 *Set* $f(z, \zeta) = \big(\zeta + \zeta_0\big)^{m-ln} \Big[(\zeta + \zeta_0)^n + th\big(z, \zeta + \zeta_0\big) \Big]^l - s$, *where* s *and* t $(s, t \neq 0)$ *are fixed and they satisfy* $\left(\frac{ln-m}{ln}\right)^{al} s^a = \left(\frac{ln-m}{m}\right)^b (c_0 t)^b$. *Then the Newton polygon* $\Gamma(f)$ *is the line segment joining* $(0, 2)$ *and* $(p_1', 0)$*; so* $\Gamma(f)$ *has only one face.*

Proof. Explicitly, f is given by

$$f(z, \zeta) = \big(\zeta + \zeta_0\big)^{m-ln} \Big[\big(\zeta + \zeta_0\big)^n + t\Big(c_0 + c_1 z^{p_1'} \big(\zeta + \zeta_0\big)^{p_1} + c_2 z^{p_2'} \big(\zeta + \zeta_0\big)^{p_2} + \cdots \Big) \Big]^l - s.$$

Then $f(0, \zeta) = a\zeta^2 +$ (higher terms), where a is a nonzero complex number; indeed, as we saw in the proof of Lemma 7.3.4 — in the present case, we translated (z_0, ζ_0) to the origin —, $f(0, \zeta)$ has a double root $\zeta = 0$. Thus the first term in $f(0, \zeta)$ is $a\zeta^2$. (Note: $f(0, 0) = 0$, and so $f(0, \zeta)$ has no constant term. Also, as the origin $(0, 0)$ is a singularity, $f(0, \zeta)$ has no linear term; otherwise by the implicit function theorem, $f(z, \zeta)$ is smooth at the origin.) We next investigate the first term in

$$f(z, 0) = \big(\zeta_0\big)^{m-ln} \Big[\big(\zeta_0\big)^n + t\Big(c_0 + c_1\, z^{p_1'} \big(\zeta_0\big)^{p_1} + c_2\, z^{p_2'} \big(\zeta_0\big)^{p_2} + \cdots \Big) \Big]^l - s.$$

Rewrite this expression as

$$f(z, 0) = \zeta_0^{m-ln} \Big[\big(\zeta_0^n + tc_0\big) + \Big(tc_1 z^{p_1'} \zeta_0^{p_1} + tc_2 z^{p_2'} \zeta_0^{p_2} + \cdots \Big) \Big]^l - s. \quad (7.4.1)$$

Here by assumption, $f(0,0) = \zeta_0^{m-ln}(\zeta_0^n + tc_0)^l - s = 0$, and so $s = \zeta_0^{m-ln}(\zeta_0^n + tc_0)^l$. Substitute this into (7.4.1), which yields:

$$\begin{aligned} f(z,0) &= \zeta_0^{m-ln}\Big[\,(\zeta_0^n + tc_0) + \Big(tc_1 z^{p_1'}\zeta_0^{p_1} + tc_2 z^{p_2'}\zeta_0^{p_2} + \cdots\Big)\Big]^l \\ &\qquad - \zeta_0^{m-ln}(\zeta_0^n + tc_0)^l \\ &= \zeta_0^{m-ln}\Big\{\Big[(\zeta_0^n + tc_0) + \Big(tc_1 z^{p_1'}\zeta_0^{p_1} + tc_2 z^{p_2'}\zeta_0^{p_2} + \cdots\Big)\Big]^l - (\zeta_0^n + tc_0)^l\Big\} \\ &= \zeta_0^{m-ln}\Big\{{}_l\mathrm{C}_{l-1}\,\zeta_0^{m-ln}\,(\zeta_0^n + tc_0)^{l-1}\,t\,c_1\,z^{p_1'}\zeta_0^{p_1} + (\text{higher terms in } z)\Big\}. \end{aligned}$$

Thus the first term in $f(z,0)$ is

$$ {}_l\mathrm{C}_{l-1}\,\zeta_0^{m-ln}\,(\zeta_0^n + tc_0)^{l-1}\,t\,c_1\,z^{p_1'}\zeta_0^{p_1}. $$

(Note: The coefficient ${}_l\mathrm{C}_{l-1}\,\zeta_0^{m-ln}\,(\zeta_0^n + tc_0)^{l-1}\,t\,c_1\,\zeta_0^{p_1}$ of $z^{p_1'}$ is nonzero; because c_0 and c_1 are nonzero by assumption, and also ζ_0 and $\zeta_0^n + tc_0$ are nonzero by (7.3.10).)

We also note that as $p_1' < p_2' < p_3' < \cdots$ and $p_1 < p_2 < p_3 < \cdots$, there is no monomial $z^i\zeta^j$ in $f(z,\zeta)$ such that (i,j) is below the line segment joining $(0,2)$ and $(p_1',0)$. Therefore, the Newton polygon $\Gamma(f)$ is the line segment joining two points $(0,2)$ and $(p_1',0)$. □

We next prove

Lemma 7.4.3 *Set* $f(z,\zeta) = \big(\zeta + \zeta_0\big)^{m-ln}\Big[\big(\zeta + \zeta_0\big)^n + th\big(z, \zeta + \zeta_0\big)\Big]^l - s$ *where* s *and* t $(s, t \neq 0)$ *satisfy* $\Big(\dfrac{ln-m}{ln}\Big)^{al} s^a = \Big(\dfrac{ln-m}{m}\Big)^b (c_0 t)^b$. *Then* f *is Newton nondegenerate.*

Proof. If p_1' is odd, the integral points on the Newton polygon $\Gamma(f)$ are $(0,2)$ and $(p_1',0)$. On the other hand, if p_1' is even, the integral points on $\Gamma(f)$ are $(0,2)$, $(\frac{p_1'}{2},1)$, and $(p_1',0)$; however since $p_1' < p_2' < p_3' < \cdots$ and $p_1 < p_2 < p_3 < \cdots$, the function f does not contain a monomial $z^{p_1'/2}\zeta$. Therefore irrespective of whether p_1' is odd or even, the face function for $I = \Gamma(f)$ has the form $f_I = a\zeta^2 + bz^{p_1'}$ where a and b are nonzero complex numbers. Evidently, f_I does not admit a factorization with a multiple factor. Therefore f is Newton nondegenerate. □

Now we can show the main result of this section.

Proposition 7.4.4 *Consider a holomorphic function*

$$ h(z,\zeta) = c_0 + c_1 z^{p_1'}\zeta^{p_1} + c_2 z^{p_2'}\zeta^{p_2} + \cdots, \qquad (c_0 \neq 0,\ \ c_1 \neq 0), $$

where $p_1' < p_2' < p_3' < \cdots$ *and* $p_1 < p_2 < p_3 < \cdots$. *Let* $C_{s,t} : \zeta^{m-ln}(\zeta^n + th)^l - s = 0$ *be a complex curve, where* s *and* t $(s, t \neq 0)$ *satisfy* $\Big(\dfrac{ln-m}{ln}\Big)^{al} s^a =$

$\left(\frac{ln-m}{m}\right)^b (c_0 t)^b$. *Then the singularity* $(z_0, \zeta_0) = \left(0, \left(\frac{ln-m}{m} t c_0\right)^{1/n}\right)$ *of* $C_{s,t}$ *in Lemma* 7.3.4 *is an* $A_{p_1'-1}$*-singularity.*

Proof. We set $f(z,\zeta) = \left(\zeta+\zeta_0\right)^{m-ln}\left[\left(\zeta+\zeta_0\right)^n + th\left(z,\zeta+\zeta_0\right)\right]^l - s$. By Lemma 7.4.3, f is Newton nondegenerate, and so from Oka's theorem (Theorem 7.4.1), the Milnor fiber of $f(z,\zeta) = 0$ is determined by the Newton polygon $\Gamma(f)$. In the present case, by Lemma 7.4.2, $\Gamma(f)$ is the same as the Newton polygon of an $A_{p_1'-1}$-singularity, and consequently the Milnor fiber of $f(z,\zeta) = 0$ is the same as that of an $A_{p_1'-1}$-singularity. But we already know that (z_0, ζ_0) is an A-singularity (Lemma 7.3.4), and furthermore, an A-singularity is completely determined by its Milnor fiber; in fact, the Milnor fiber of an A_n-singularity for odd n is a smooth complex curve of genus $\frac{n-1}{2}$ with two holes, while that for even n is a smooth complex curve of genus $\frac{n}{2}$ with one hole (Lemma 22.2.1, p427). Hence we conclude that the origin of $f(z,\zeta) = 0$ (accordingly (z_0, ζ_0) of $C_{s,t}$) is an $A_{p_1'-1}$-singularity. □

8

Construction of Deformations by Wild Subbranches

In this chapter, we construct barking deformations by using certain wild subbranches. We prepare notation: $X = m_0\Delta_0 + m_1\Theta_1 + \cdots + m_\lambda\Theta_\lambda$ is a branch and we set

$$r_i := \frac{m_{i-1} + m_{i+1}}{m_i}, \quad (i = 1, 2 \ldots, \lambda - 1), \qquad r_\lambda := \frac{m_{\lambda-1}}{m_\lambda}.$$

Let $Y = n_0\Delta_0 + n_1\Theta_1 + \cdots + n_e\Theta_e$ be a subbranch of X, and let l be a positive integer such that $lY \leq X$. We recall a deformation atlas $DA_{e-1}(lY, d)$. We first define a sequence of integers $p_0, p_1, \ldots, p_{\lambda+1}$ inductively by

$$\begin{cases} p_0 = 0, \quad p_1 = 1 \quad \text{and} \\ p_{i+1} = r_i p_i - p_{i-1} \quad \text{for } i = 1, 2, \ldots, \lambda, \end{cases}$$

where we remember (6.2.4):

$$p_{\lambda+1} > p_\lambda > \cdots > p_1 > p_0 = 0.$$

Let $f(z)$ be a non-vanishing holomorphic function on a domain $\{z \in \mathbb{C} : |z| < \varepsilon\}$, and we set $f_i = f(w^{p_{i-1}}\eta^{p_i})$ and $\widehat{f}_i = f(z^{p_{i+1}}\zeta^{p_i})$. Then the deformation atlas $DA_{e-1}(lY, d)$ is given as follows: for $i = 1, 2, \ldots, e-1$,

$$\begin{cases} \mathcal{H}_i : & w^{m_{i-1}-ln_{i-1}}\eta^{m_i-ln_i}(w^{n_{i-1}}\eta^{n_i} + t^d f_i)^l - s = 0 \\ \mathcal{H}'_i : & z^{m_{i+1}-ln_{i+1}}\zeta^{m_i-ln_i}(z^{n_{i+1}}\zeta^{n_i} + t^d \widehat{f}_i)^l - s = 0 \\ g_i : & \text{the transition function } z = 1/w,\ \zeta = w^{r_i}\eta \text{ of } N_i. \end{cases} \tag{8.1}$$

(See Lemma 7.1.1, p119.)

Proposition 8.1 *If Y is either* (1) *wild or* (2) *tame and not proportional, then after coordinate change, $f_i \equiv 1$ and $\widehat{f}_i \equiv 1$,* i.e. *$DA_{e-1}(lY, d)$ has the* *following form: for* $i = 1, 2, \ldots, e-1$,

$$\begin{cases} \mathcal{H}_i : & w^{m_{i-1}-ln_{i-1}}\eta^{m_i-ln_i}(w^{n_{i-1}}\eta^{n_i} + t^d)^l - s = 0 \\ \mathcal{H}'_i : & z^{m_{i+1}-ln_{i+1}}\zeta^{m_i-ln_i}(z^{n_{i+1}}\zeta^{n_i} + t^d)^l - s = 0 \\ g_i : & \text{the transition function } z = 1/w,\ \zeta = w^{r_i}\eta \text{ of } N_i. \end{cases} \tag{8.2}$$

Proof. By assumption, Y is not proportional (recall that any wild subbranch is not proportional by Corollary 6.3.3, p108). Thus $m_i n_{i-1} - m_{i-1} n_i \neq 0$ for $i = 1, 2, \ldots, e$, and so we may apply Simplification Lemma for the hypersurfaces $\mathcal{H}_i$ in (8.1); after some coordinate change, we may assume $f_i \equiv 1$, that is,

$$\mathcal{H}_i : \quad w^{m_{i-1}-ln_{i-1}}\eta^{m_i-ln_i}(w^{n_{i-1}}\eta^{n_i} + t^d)^l - s = 0,$$

and likewise we may assume $\widehat{f_i} \equiv 1$:

$$\mathcal{H}'_i : \quad z^{m_{i+1}-ln_{i+1}}\zeta^{m_i-ln_i}(z^{n_{i+1}}\zeta^{n_i} + t^d)^l - s = 0.$$

Therefore after some coordinate change, (8.1) is written as (8.2). □

8.1 Deformations of ripple type

A projective line Θ in a complex surface is called a (-2)-*curve* if its self-intersection number is -2; $\Theta \cdot \Theta = -2$. For a branch $X = m_0\Delta_0 + m_1\Theta_1 + \cdots + m_\lambda\Theta_\lambda$, if $\Theta_i \cdot \Theta_i = -2$ for $a \le i \le b$, we say that $\Theta_a + \Theta_{a+1} + \cdots + \Theta_b$ is a *chain of* (-2)-*curves*. In this section, for branches containing chains of (-2)-curves, we define a certain class of wild subbranches, from which we will construct complete deformation atlases. For this purpose, we first introduce special polynomials called *descending polynomials*, and then study how they are transformed under some maps.

Descending polynomials

We say that a polynomial $P = P(w, \eta, t)$ is a *descending polynomial* of *length* n if it is of the form $w\sum_{i=0}^{n} t^{n-i}a_i(w\eta)^i$, that is,

$$P(w, \eta, t) = w\Big(a_n w^n\eta^n + t a_{n-1}w^{n-1}\eta^{n-1} + t^2 a_{n-2}w^{n-2}\eta^{n-2} + \cdots + t^n a_0\Big)$$

where $a_i \in \mathbb{C}$ $(i = 0, 1, \ldots, n)$ and $a_n \neq 0$. For simplicity, unless otherwise mentioned, we assume that $a_n = 1$. Then a descending polynomial is rewritten as

$$P(w, \eta, t) = w\prod_{j=1}^{n}(w\eta + t\beta_j),$$

where $-\beta_j$ $(j = 1, 2, \ldots, n)$ are the solutions of an equation

$$x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_0 = 0.$$

Alternatively, we may define a descending polynomial as a polynomial which admits a factorization of the form $w\prod_{j=1}^{n}(w\eta + t\beta_j)$, where $\beta_j \in \mathbb{C}$.

Lemma 8.1.1 *Let $P = w\prod_{j=1}^{n}(w\eta + t\beta_j)$ be a descending polynomial of length n. Then a map $g:\ z = 1/w,\ \zeta = w^2\eta + t\beta_n w$ transforms P to a polynomial*

$$P' = \zeta \prod_{j=1}^{n-1}\Big[z\zeta + t(\beta_j - \beta_n)\Big].$$

Note: By a coordinate change $(z,\zeta) = (w,\eta)$, the polynomial P' becomes

$$w\prod_{j=1}^{n-1}\Big[w\eta + t(\beta_j - \beta_n)\Big],$$

which is a descending polynomial of length $n-1$. So the length is reduced by 1. The name "descending" comes from this property.

Proof. Since $P(w,\eta,t) = w\prod_{j=1}^{n}\left(\frac{1}{w}w^2\eta + t\beta_j\right)$, the map g transforms P to

$$\begin{aligned}\frac{1}{z}\prod_{j=1}^{n}\left[z\left(\zeta - t\beta_n\frac{1}{z}\right) + t\beta_j\right] &= \frac{1}{z}\prod_{j=1}^{n}\Big[z\zeta + t(\beta_j - \beta_n)\Big]\\ &= \frac{1}{z}\prod_{j=1}^{n-1}\Big[z\zeta + t(\beta_j - \beta_n)\Big]\cdot\Big[z\zeta + t(\beta_n - \beta_n)\Big]\\ &= \frac{1}{z}\prod_{j=1}^{n-1}\Big[z\zeta + t(\beta_j - \beta_n)\Big]\cdot z\zeta\\ &= \zeta\prod_{j=1}^{n-1}\Big[z\zeta + t(\beta_j - \beta_n)\Big].\end{aligned}$$

Hence g transforms $P(w,\eta,t)$ to $P'(\zeta,z,t)$. □

Remark 8.1.2 The following observation is useful when we later describe the deformation constructed from this polynomial: if $\beta_1,\beta_2,\ldots,\beta_n$ are distinct, then $\beta_1-\beta_n,\ \beta_2-\beta_n,\ldots,\ \beta_{n-1}-\beta_n$ are also distinct. If furthermore $\beta_1,\beta_2,\ldots,\beta_n$ are nonzero, then $\beta_1-\beta_n,\ \beta_2-\beta_n,\ldots,\ \beta_{n-1}-\beta_n$ are also nonzero.

The following lemma will play an essential role for the construction of complete deformation atlases by means of descending polynomials.

Lemma 8.1.3 *Given a descending polynomial $P = w\prod_{j=1}^{n}(w\eta+\beta_j)$ of length n, there exist*

(1) *a sequence of descending polynomials $P_1 = P, P_2, \ldots, P_{n+1}$ such that P_i has length $n+1-i$ (so $P_{n+1} = w$), and*
(2) *a sequence of maps $g_1, g_2, \ldots, g_n$ of the form $g_i : z = 1/w,\ \zeta = w^2\eta + t\alpha_i w$ for some $\alpha_i \in \mathbb{C}$ such that g_i transforms $P_i(w,\eta,t)$ to $P_{i+1}(\zeta,z,t)$.*

Proof. First of all, from the 'roots' $\beta_1, \beta_2, \dots, \beta_n$ of P, we define a sequence of complex numbers; for each i ($i = 0, 1, \dots, n+1$), we define $n+1-i$ complex numbers

$$\beta_{i,1},\ \beta_{i,2},\ \dots,\ \beta_{i,n+1-i}$$

inductively as follows. For $i = 0$, we set $\beta_{0,1} := \beta_1$, $\beta_{0,2} := \beta_2$, $\dots, \beta_{0,n+1} := \beta_{n+1}$. Assuming that we have already defined $\beta_{i,1}, \beta_{i,2}, \dots, \beta_{i,n+1-i}$, we next define $\beta_{i+1,1}, \beta_{i+1,2}, \dots, \beta_{i+1,n-i}$ recursively by

$$\beta_{i+1,j} = \beta_{i,j} - \beta_{i,n+1-i}, \qquad (\ j = 1, 2, \dots, n-i\ ).$$

Using the above sequence of complex numbers, we define descending polynomials and maps:

$$P_i = w \prod_{j=1}^{n+1-i} (w\eta + t\beta_{i,j}), \quad \text{and} \quad g_i : \quad z = \frac{1}{w}, \quad \zeta = w^2\eta + t\beta_{i,n+1-i}w.$$

Then the length of P_i is $n+1-i$, and by Lemma 8.1.1, g_i transforms $P_i(w, \eta, t)$ to $P_{i+1}(\zeta, z, t)$. Hence our assertion is confirmed. □

We say that the set $P_1, P_2, \dots, P_{n+1}$ and $g_1, g_2, \dots, g_n$ is a *descending sequence* associated with P.

Subbranches of ripple type

Next we construct complete deformation atlases by using descending polynomials. Assume that $X = m_0\Delta_0 + m_1\Theta_1 + \cdots + m_\lambda\Theta_\lambda$ is a branch such that for some e ($0 < e < \lambda$),

$$m_i = (\lambda + 1) - i \qquad \text{for} \quad i = e-1,\ e,\ \dots,\ \lambda. \tag{8.1.1}$$

This condition is equivalent to

$$\text{(i)} \quad m_\lambda = 1 \quad \text{and} \quad \text{(ii)} \quad r_e = r_{e+1} = \cdots = r_\lambda = 2. \tag{8.1.2}$$

Note that (ii) implies $\Theta_i \cdot \Theta_i = -2$ for $i = e, e+1, \dots, \lambda$, namely $\Theta_e + \Theta_{e+1} + \cdots + \Theta_\lambda$ is a *chain of* (-2)*-curves.*

A subbranch $Y = n_0\Delta_0 + n_1\Theta_1 + \cdots + n_e\Theta_e$ of X is called *of ripple type* if it satisfies

$$n_{e-1} = n_e = m_e. \tag{8.1.3}$$

Lemma 8.1.4 *Suppose that $Y = n_0\Delta_0 + n_1\Theta_1 + \cdots + n_e\Theta_e$ is a subbranch of ripple type. Then n_e divides all n_i ($i = 0, 1, \dots, e$).* (In particular, when $n_e \geq 2$, Y is a multiple subdivisor of X.)

Proof. We demonstrate this by induction on i ($i = e-1, e-2, \dots, 0$). From the definition of ripple type, $n_e = n_{e-1}$ and so it is trivial that n_e divides n_{e-1}.

Next we show that n_e divides n_{e-2}. Since Y is a subbranch,

$$n_{i-1} = r_i n_i - n_{i+1}, \qquad \text{for} \quad i = 1, 2, \ldots, e-1. \tag{8.1.4}$$

In particular $n_{e-2} = r_{e-1}n_{e-1} - n_e$; since n_e divides n_{e-1}, the right hand side is divisible by n_e, so n_e divides n_{e-2}. Next, by (8.1.4) for $i = e-2$, we have $n_{e-3} = r_{e-2}n_{e-2} - n_{e-1}$. Since n_e divides both n_{e-2} and n_{e-1}, it follows that n_e divides n_{e-3}. Repeating this argument, we conclude that n_e divides n_i $(i = e-1, e-2, \ldots, 0)$. □

Lemma 8.1.5 *A subbranch Y of ripple type is dominant and wild.*

Proof. We write $Y = n_0\Delta_0 + n_1\Theta_1 + \cdots + n_e\Theta_e$, and we set $n_{e+1} := r_e n_e - n_{e-1}$ formally. Recall that a subbranch is dominant if $n_{e+1} > m_e$ or $n_{e+1} \leq 0$. When Y is of ripple type, we have

$$\begin{aligned} n_{e+1} &= 2n_e - n_{e-1} && \text{by } r_e = 2 \quad (8.1.2) \\ &= 2m_e - m_e && \text{by } n_e = m_e \quad (8.1.3) \\ &= m_e \\ &> m_{e+1}, \end{aligned}$$

where the last inequality follows from the fact that $m_0, m_1, \ldots, m_\lambda$ strictly decrease. Thus $n_{e+1} > m_{e+1}$, and so Y is dominant. Next, recall that a subbranch Y is wild provided that (1) $e \leq \lambda - 1$ and $\frac{n_{e-1} + m_{e+1}}{n_e} < r_e$ or (2) $e = \lambda$ and $\frac{n_{\lambda-1}}{n_\lambda} < r_\lambda$ (Definition 5.5.3, p93). In the ripple case, if $e \leq \lambda - 1$, then we have

$$\begin{aligned} r_e &:= \frac{m_{e-1} + m_{e+1}}{m_e} \\ &> \frac{m_e + m_{e+1}}{n_e} && \text{by } m_{e-1} > m_e \text{ and } m_e = n_e \\ &= \frac{n_{e-1} + m_{e+1}}{n_e} && \text{by } n_{e-1} = m_e \quad (8.1.3). \end{aligned}$$

Thus $r_e > \frac{n_{e-1} + m_{e+1}}{n_e}$. This means that Y is wild. Next if $e = \lambda$, then from the definition of ripple type, $n_{\lambda-1} = n_\lambda$, and so $\frac{n_{\lambda-1}}{n_\lambda} = 1 < r_\lambda$ (note $r_\lambda = 2$ (8.1.2)). Therefore Y is wild, and so the assertion is confirmed. □

Example 8.1.6 If $\mathbf{m}$ is an arithmetic progression with difference 1, then there are many choices of $\mathbf{n}$ which is of ripple type. For example, if $\mathbf{m} = (6, 5, 4, 3, 2, 1)$, then all of the following sequences are multiplicity sequences of ripple type:

(1) $\mathbf{n} = (1, 1, 1, 1, 1, 1)$
(2) $\mathbf{n} = (2, 2, 2, 2, 2)$
(3) $\mathbf{n} = (3, 3, 3, 3)$
(4) $\mathbf{n} = (4, 4, 4)$
(4) $\mathbf{n} = (5, 5)$.

Actually, as seen from the lemma just below, *if* $\mathbf{m}$ *is an arithmetic progression with difference 1, then the sequences* $\mathbf{n}$ *as in the above examples exhaust all ripple types.* (Note: This is not true for general $\mathbf{m}$; in the definition of ripple type, we merely assumed that $(m_{e-1}, m_e, \dots, m_\lambda)$ is an arithmetic progression with difference 1, whereas the whole sequence $\mathbf{m} = (m_0, m_1, \dots, m_\lambda)$ is not necessarily an arithmetic progression.)

Lemma 8.1.7 *Let* $X = m_0\Delta_0 + m_1\Theta_1 + \cdots + m_\lambda\Theta_\lambda$ *be a branch such that*

$$m_i = \lambda + 1 - i, \qquad (i = 0, 1, \dots, \lambda). \tag{8.1.5}$$

Then a subbranch $Y = n_0\Delta_0 + n_1\Theta_1 + \cdots + n_e\Theta_e$ *is of ripple type if and only if*

$$n_0 = n_1 = \cdots = n_e = m_e.$$

Proof. Clearly, "$n_0 = n_1 = \cdots = n_e = m_e$" implies that Y is of ripple type. We show the converse. From the assumption (8.1.5), $r_i = 2$ for $i = 1, 2, \dots, \lambda$. Since Y is a subbranch, $n_{i+1} = r_i n_i - n_{i-1}$, and so $n_{i+1} = 2n_i - n_{i-1}$. It follows that we have

$$n_{i+1} - n_i = n_i - n_{i-1} \qquad \text{for } i = 1, 2, \dots, e-1.$$

This means that $\mathbf{n} = (n_0, n_1, \dots, n_e)$ is an arithmetic progression. Its difference $n_e - n_{e-1}$ is equal to zero by the definition of ripple type. Therefore $n_0 = n_1 = n_2 = \cdots = n_e$. Noting that $n_e = m_e$ by the definition of ripple type, we complete the proof. □

Complete propagation for ripple type

Suppose that $X = m_0\Delta_0 + m_1\Theta_1 + \cdots + m_\lambda\Theta_\lambda$ is a branch, and $Y = n_0\Delta_0 + n_1\Theta_1 + \cdots + n_e\Theta_e$ is its subbranch. As before, we set

$$r_i := \frac{m_{i-1} + m_{i+1}}{m_i}, \quad (i = 1, 2 \dots, \lambda - 1), \qquad r_\lambda := \frac{m_{\lambda-1}}{m_\lambda}.$$

We recall the deformation atlas $DA_{e-1}(Y, d)$ associated with Y. First we define a sequence of integers p_i $(i = 0, 1, \dots, \lambda + 1)$ inductively by

$$\begin{cases} p_0 = 0, \quad p_1 = 1 \\ p_{i+1} = r_i p_i - p_{i-1} \quad \text{for } i = 1, 2, \dots, \lambda. \end{cases}$$

Then $p_{\lambda+1} > p_\lambda > \cdots > p_1 > p_0 = 0$ (6.2.4). Let $f(z)$ be a non-vanishing holomorphic function defined around $z = 0$, and we set $f_i = f(w^{p_{i-1}}\eta^{p_i})$ and $\widehat{f_i} = f(z^{p_{i+1}}\zeta^{p_i})$ (see (6.2.7)). Then $DA_{e-1}(Y, d)$ is given by the following data: for $i = 1, 2, \dots, e-1$,

$$\begin{cases} \mathcal{H}_i : & w^{m_{i-1}-n_{i-1}}\eta^{m_i-n_i}(w^{n_{i-1}}\eta^{n_i} + t^d f_i) - s = 0 \\ \mathcal{H}'_i : & z^{m_{i+1}-n_{i+1}}\zeta^{m_i-n_i}(z^{n_{i+1}}\zeta^{n_i} + t^d \widehat{f_i}) - s = 0 \\ g_i : & \text{the transition function } z = 1/w,\ \zeta = w^{r_i}\eta \text{ of } N_i. \end{cases} \tag{8.1.6}$$

Theorem 8.1.8 *Assume that* $X = m_0\Delta_0 + m_1\Theta_1 + \cdots + m_\lambda\Theta_\lambda$ *is a branch such that for some* e $(0 < e < \lambda)$,

$$m_i = (\lambda + 1) - i \qquad \text{for} \quad i = e-1,\, e,\, \ldots,\, \lambda. \tag{8.1.7}$$

Let $Y = n_0\Delta_0 + n_1\Theta_1 + \cdots + n_e\Theta_e$ *be its subbranch of ripple type. If the weight* d *of* $DA_{e-1}(Y, d)$ *is divisible by* m_e*, then* $DA_{e-1}(Y, d)$ *admits a complete propagation.*

Proof. For simplicity, we first consider the case $d = m_e$ (obviously m_e divides d). Then

$$\mathcal{H}_e : \quad w^{m_{e-1}-n_{e-1}}\eta^{m_e-n_e}(w^{n_{e-1}}\eta^{n_e} + t^{m_e} f_e) - s = 0.$$

By the assumption (8.1.7), we have $m_{e-1} = m_e + 1$ and so

$$\mathcal{H}_e : \quad w^{m_e+1-n_{e-1}}\eta^{m_e-n_e}(w^{n_{e-1}}\eta^{n_e} + t^{m_e} f_e) - s = 0.$$

Since Y is of ripple type, $n_{e-1} = n_e = m_e$ and hence

$$\mathcal{H}_e : \quad w(w^{m_e}\eta^{m_e} + t^{m_e} f_e) - s = 0.$$

By Simplification Lemma, after some coordinate change, we may assume $f_e \equiv 1$:

$$\mathcal{H}_e : \quad w(w^{m_e}\eta^{m_e} + t^{m_e}) - s = 0.$$

Next, we set $P = w(w^{m_e}\eta^{m_e} + t^{m_e})$ and write $\mathcal{H}_e : P(w, \eta, t) - s = 0$. Noting that P is a descending polynomial of length m_e, we take a descending sequence $P_e, P_{e+1}, \ldots, P_{\lambda+1}$ and $g_e, g_{e+1}, \ldots, g_\lambda$ associated with $P = P_e$. (Note: In the definition of descending sequences, a descending sequence starts from P_1 and g_1, but for the present discussion, we reset the subscripts so that the first elements are P_e and g_e.) Then for $i = e, e+1, \ldots, \lambda$, we set

$$\begin{cases} \mathcal{H}_i : & P_i(w, \eta, t) - s = 0 \\ \mathcal{H}'_i : & P_{i+1}(\zeta, z, t) - s = 0 \\ g_i : & z = \dfrac{1}{w}, \quad \zeta = w^2\eta - \alpha_i t w. \end{cases}$$

From Lemma 8.1.3, this data gives a complete propagation of $DA_{e-1}(Y, d)$.

Next, we consider the general case; d is an arbitrary positive integer divisible by m_e. Then we write $d = m_e q$ where q is a positive integer. In the above discussion, replacing t by t^q, we obtain a complete propagation of $DA_{e-1}(lY, d)$. Hence we establish our assertion. □

Remark 8.1.9 In the above construction, the parameter of deformation is not t but t^d where d is a positive integer divisible by m_e. This is related to *simultaneous resolution* of cyclic quotient singularities. Let V be a cyclic quotient singularity. Suppose that $\mathcal{V} \to B$ is a deformation of V, which is contained in the *Artin component*. Then we need a base change of B for the simultaneous resolution of $\mathcal{V}$ (see Brieskorn [Br1] and [BR2]). This base change corresponds to our choice of the parameter t^d.

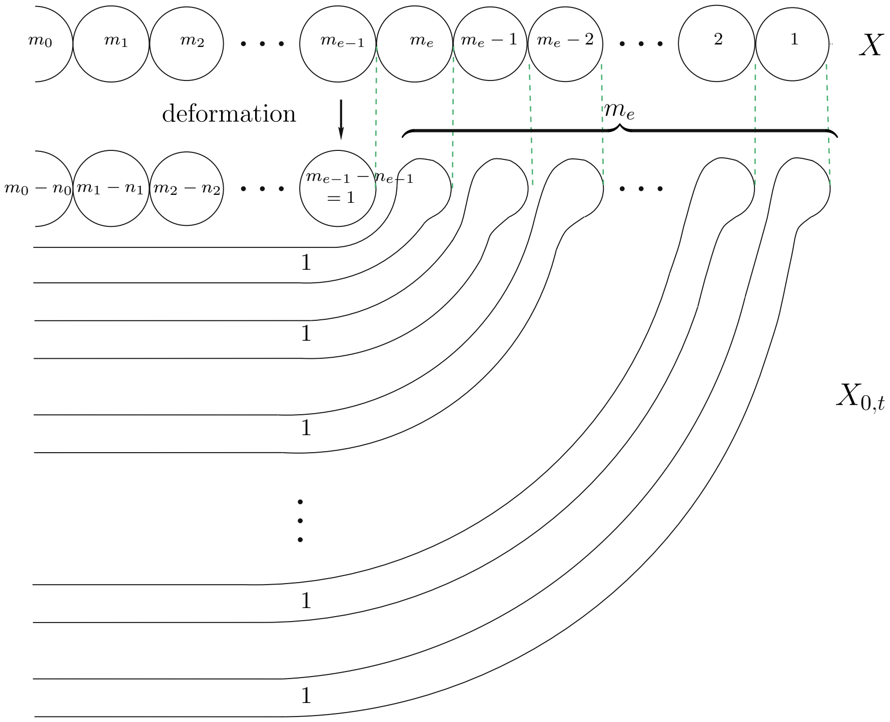

Fig. 8.1.1. Deformation of ripple type.

Let $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$ be the barking family obtained by patching the complete deformation atlas in Theorem 8.1.8. We shall describe the deformation from X to $X_{0,t} := \Psi^{-1}(0, t)$. For simplicity, we consider the case $d = m_e$. Then

$$\mathcal{H}_e|_{s=0} : \quad w \prod_{k=1}^{m_e} (w\eta + te^{2\pi i k/m_e}) = 0$$

is a disjoint union of one disk $\{w = 0\}$ and m_e annuli $\{w\eta + te^{2\pi i k/m_e} = 0\}$ where $k = 1, 2, \ldots, m_e$. Likewise, $\mathcal{H}_{e+i}|_{s=0}$ for $i = 0, 1, \ldots, \lambda - e$ is a disjoint union of one disk and $m_e - i$ annuli. From this observation, it is easy to draw the singular fiber $X_{0,t}$ (Figure 8.1.1).

8.2 Singular fibers

For a descending polynomial $P = w \prod_{j=1}^{n} (w\eta + t\beta_j)$ where $\beta_1, \beta_2, \ldots, \beta_n$ are *not* necessarily distinct, we consider a family of (non-compact) curves, parameterized by s and t:

$$C_{s,t} : \quad P(w, \eta, t) - s = 0.$$

Proposition 8.2.1 *The curve* $C_{s,t} : P(w,\eta,t) - s = 0$ *is singular precisely when*

$$\begin{cases} (1) \quad (s,t) = (0,0) & \text{if } \beta_1, \beta_2, \ldots, \beta_n \text{ are distinct} \\ (2) \quad s = 0 & \text{otherwise.} \end{cases}$$

Proof. First, we show that if $s \neq 0$, then $C_{s,t}$ is smooth, regardless of whether $\beta_1, \beta_2, \ldots, \beta_m$ are distinct or not. Note that if $s \neq 0$, then $P(w,\eta,t)$ does not vanish, because $P(w,\eta,t) = s$. Thus when $s \neq 0$, the following equivalence holds (see (6.4.1)):

$$(w,\eta) \in C_{s,t} \text{ is a singularity} \iff \frac{\partial \log P}{\partial w}(w,\eta) = \frac{\partial \log P}{\partial \eta}(w,\eta) = 0.$$

From $\log P = \log w + \sum_{j=1}^{n} \log(w\eta + t\beta_j)$, we derive

$$\frac{\partial \log P}{\partial w} = \frac{1}{w} + \sum_{j=1}^{n} \frac{\eta}{w\eta + t\beta_j}, \qquad \frac{\partial \log P}{\partial \eta} = \sum_{j=1}^{n} \frac{w}{w\eta + t\beta_j}.$$

Thus $(w,\eta) \in C_{s,t}$ is a singularity if and only if

$$\frac{1}{w} + \eta \left(\sum_{j=1}^{n} \frac{1}{w\eta + t\beta_j} \right) = 0, \qquad w \left(\sum_{j=1}^{n} \frac{1}{w\eta + t\beta_j} \right) = 0. \tag{8.2.1}$$

From the second equation, we have either $w = 0$ or $\sum_{j=1}^{n} \frac{1}{w\eta + t\beta_j} = 0$. But $w = 0$ does not satisfy the first equation of (8.2.1). On the other hand, if

$$\sum_{j=1}^{n} \frac{1}{w\eta + t\beta_j} = 0,$$

then from the first equation of (8.2.1), we have $1/w = 0$ — which is absurd! Thus if $s \neq 0$, then the curve $C_{s,t}$ is smooth.

Next, we consider the case $s = 0$; we investigate when $C_{0,t}$ is singular. We slightly change the expression of the factorization; we write $P = w \prod_{j=1}^{k} (w\eta + t\gamma_j)^{a_j}$, where each a_j is a positive integer and $\gamma_1, \gamma_2, \ldots, \gamma_k \in \mathbb{C}$ are *distinct.* Then the curve $C_{0,t} : P(w,\eta,t) = 0$ is a disjoint union of $w = 0$ (a disk) and $(w\eta + t\gamma_j)^{a_j} = 0$ (a multiple annulus of multiplicity a_j). Thus $C_{0,t}$ is smooth if and only if $a_1 = a_2 = \cdots = a_k = 1$, in other words $\beta_1, \beta_2, \ldots, \beta_n$ are distinct. This proves our assertion. □

We next deduce the following result.

Corollary 8.2.2 *Let* Y *be a subbranch of ripple type, and let* $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$ *be the barking family obtained by patching the complete propagation of* $DA_{e-1}(Y,d)$ *in Theorem* 8.1.8. *Then* $X_{s,t} := \Psi^{-1}(s,t)$ *is singular if and only if* $s = 0$.

Proof. We verify the following two claims, which together imply the statement above:

Claim 1: For $i = 1, 2, \ldots, e-1$, a curve $(\Psi|_{\mathcal{H}_i})^{-1}(s,t)$ (resp. $(\Psi|_{\mathcal{H}_i})^{-1}(s,t)$) is singular if and only if $s = 0$.

Claim 2: For $i = e, e+1, \ldots, \lambda$, a curve $(\Psi|_{\mathcal{H}_i})^{-1}(s,t)$ (resp. $(\Psi|_{\mathcal{H}_i})^{-1}(s,t)$) is singular if and only if $s = t = 0$.

We now show these claims. By Lemma 8.1.5, Y is dominant wild, and applying Corollary 6.4.6, p113, we see that Claim 1 is valid. Next we show Claim 2. Note that a descending polynomial of the form $P = w(w^n\eta^n + t^n)$ factorizes as $w\prod_{k=1}^{n}(w\eta + te^{2\pi \mathrm{i}k/n})$ and the roots $-e^{2\pi \mathrm{i}k/n}$ $(k = 1, 2, \ldots, n)$ are distinct; thus applying (1) of Proposition 8.2.1, we conclude that Claim 2 is valid. □

9

Subbranches of Types A_l, B_l, C_l

In this chapter, we introduce important notions "subbranches of types A_l, B_l, and C_l". In the first section, we summarize their properties often without proof, and the subsequent sections are devoted to the proofs of these properties. The proofs are routine and technical in nature. For the first reading, we recommend the reader to read only the first section (and assuming it) to skip to the next chapter.

9.1 Subbranches of types A_l, B_l, C_l

Let lY be a subbranch of a branch X where l is a positive integer and Y is a subbranch of X. Here Y itself is possibly multiple. We express $X = m_0\Delta_0 + m_1\Theta_1 + \cdots + m_\lambda\Theta_\lambda$ and $Y = n_0\Delta_0 + n_1\Theta_1 + \cdots + n_e\Theta_e$ ($e \leq \lambda$), and then set

$$r_i := \frac{m_{i-1} + m_{i+1}}{m_i}, \quad (i = 1, 2 \dots, \lambda - 1), \qquad r_\lambda := \frac{m_{\lambda-1}}{m_\lambda}.$$

Recall that r_i ($i = 1, 2, \dots, \lambda$) are positive integers satisfying $r_i \geq 2$. Next we recall the deformation atlas $DA_{e-1}(lY, k)$ associated with lY. First we define a sequence of integers p_i ($i = 0, 1, \dots, \lambda + 1$) inductively by

$$\begin{cases} p_0 = 0, \quad p_1 = 1 \\ p_{i+1} = r_i p_i - p_{i-1} \quad \text{for } i = 1, 2, \dots, \lambda. \end{cases}$$

Then $p_{\lambda+1} > p_\lambda > \cdots > p_1 > p_0 = 0$ (6.2.4). Let $f(z)$ be a non-vanishing holomorphic function defined around $z = 0$, and we set $f_i = f(w^{p_{i-1}}\eta^{p_i})$ and $\widehat{f}_i = f(z^{p_{i+1}}\zeta^{p_i})$ (see (6.2.7)). Then $DA_{e-1}(lY, d)$ is given by the following data (see Lemma 7.1.1): for $i = 1, 2, \dots, e - 1$,

$$\begin{cases} \mathcal{H}_i : & w^{m_{i-1} - ln_{i-1}}\eta^{m_i - ln_i}(w^{n_{i-1}}\eta^{n_i} + t^k f_i)^l - s = 0 \\ \mathcal{H}'_i : & z^{m_{i+1} - ln_{i+1}}\zeta^{m_i - ln_i}(z^{n_{i+1}}\zeta^{n_i} + t^k \widehat{f}_i)^l - s = 0 \\ g_i : & \text{the transition function } z = 1/w, \ \zeta = w^{r_i}\eta \text{ of } N_i. \end{cases}$$

Then we ask:

Problem When does $DA_{e-1}(lY, k)$ admit a complete propagation?

As we will show later, there are exactly three types of Y for which $DA_{e-1}(lY, k)$ admits a complete propagation (Theorem 13.1.1). Now we introduce these three types. Below, the notation $lY \le X$ means $ln_i \le m_i$ for $i = 0, 1, \dots, e$.

Definition 9.1.1 Let l be a positive integer and let X be a branch.

Type A_l A subbranch Y of X is of *type* A_l if one of the following conditions holds: (In fact, these conditions are equivalent. See Lemma 9.2.3.)

- **(A.1)** $lY \le X$ and $\frac{n_{e-1}}{n_e} \ge r_e$.
- **(A.2)** $lY \le X$ and lY is dominant tame.
- **(A.3)** $lY \le X$ and Y is dominant tame.

Type B_l A subbranch Y of X is of *type* B_l if $lY \le X$, $m_e = l$, and $n_e = 1$

Type C_l A subbranch Y of X is of *type* C_l if one of the following conditions holds: (In fact, these conditions are equivalent. See Lemma 9.4.2.)

- **(C.1)** $lY \le X$, n_e divides n_{e-1}, and $\frac{n_{e-1}}{n_e} < r_e$, and u divides l where $u := (m_{e-1} - ln_{e-1}) - (r_e - 1)(m_e - ln_e)$. (As in (C.3) "Note", $u > 0$.)
- **(C.2)** $lY \le X$, $n_e = r_e n_e - n_{e-1}$, and u divides l where u is in (C.1).
- **(C.3)** $lY \le X$, $n_e = r_e n_e - n_{e-1}$, and $m_e - m_{e+1}$ divides l.

  (Note: $\lambda \ge e+1$ holds for type C_l. See Corollary 9.4.4. Also note that by Lemma 9.1.5 below, $m_e - m_{e+1}$ is equal to u in (C.1); so $u > 0$.)

We provide respective examples of types A_l, B_l, C_l:

Example A_l $l = 2$, $\mathbf{m} = (12, 9, 6, 3)$ and $\mathbf{n} = (3, 2, 1)$.
Example B_l $l = 2$, $\mathbf{m} = (12, 7, 2, 1)$ and $\mathbf{n} = (3, 2, 1)$.
Example C_l $l = 5$, $\mathbf{m} = (30, 25, 20, 15, 10, 5)$ and $\mathbf{n} = (3, 3, 3, 3)$.
(In Example C_l, $m_e - ln_e = 0$ and so $u = 5$.)

Note: Take $l = 7$, $\mathbf{m} = (57, 16, 7, 5, 3, 1)$, and $\mathbf{n} = (7, 2, 1)$. Then Y satisfies the conditions of type C_l except that u divides l. Indeed $u = 2$, and so u does *not* divide $l = 7$. Consequently Y is not of type C_l.

Recall that a subbranch $Y = n_0\Delta_0 + n_1\Theta_1 + \cdots + n_e\Theta_e$ ($e \le \lambda$) of a branch $X = m_0\Delta_0 + m_1\Theta_1 + \cdots + m_\lambda\Theta_\lambda$ is proportional if $\frac{m_0}{n_0} = \frac{m_1}{n_1} = \cdots = \frac{m_e}{n_e}$.

Lemma 9.1.2 *Any subbranch Y of type C_l is "not" proportional.*

Proof. In fact, when $e = \lambda$, from a condition in (C.1), we have $\frac{n_{\lambda-1}}{n_\lambda} < r_\lambda = \frac{m_{\lambda-1}}{m_\lambda}$ and so $\frac{m_{\lambda-1}}{n_{\lambda-1}} > \frac{m_\lambda}{n_\lambda}$; this confirms the non-proportionality of Y. When $e < \lambda$, we show the non-proportionality of Y by contradiction; if Y is proportional, then $(m_{e-1}, m_e) = (cn_{e-1}, cn_e)$ for some rational number c. By (C.3), $n_e = r_e n_e - n_{e-1}$, and hence $cn_e = r_e cn_e - cn_{e-1}$, that is,

$m_e = r_e m_e - m_{e-1}$. Thus we have

$$\frac{m_e + m_{e-1}}{m_e} = r_e.$$

However, from the definition of a branch,

$$\frac{m_{e-1} + m_{e+1}}{m_e} = r_e,$$

and the comparison of the above two equations gives $m_{e+1} = m_e$. This is a contradiction. Therefore any subbranch of type C_l is not proportional. □

On the other hand, types A_l and B_l may be proportional. For instance if $X = lY$;

$$\mathbf{m} = (ln_0, ln_1, \dots, ln_\lambda), \quad \mathbf{n} = (n_0, n_1, \dots, n_\lambda),$$

then Y is of proportional type A_l; for a special case $n_\lambda = 1$ and $m_\lambda = l$, this is of proportional type B_l at the same time. A subbranch both of type A_l and B_l is simply referred to as of *type* AB_l.

Lemma 9.1.3 *Suppose that Y is a dominant subbranch of a branch X. Then Y is of type AB_l if and only if Y is of proportional type B_l.*

Proof. $\Longrightarrow$: Trivial.

$\Longleftarrow$: By proportionality, $\frac{m_0}{n_0} = \frac{m_1}{n_1} = \cdots = \frac{m_e}{n_e}$. Since $m_e = l$ and $n_e = 1$ (type B_l), these common fractions are equal to l. Namely

$$\big(m_0, m_1, \dots, m_e\big) = l\big(n_0, n_1, \dots, n_e\,(=1)\big). \tag{9.1.1}$$

Next we insist that $e = \lambda$; assuming $e < \lambda$, we derive a contradiction. Note that (9.1.1) with the equations $m_{i+1} = r_i m_i - m_{i-1}$ ($i = 1, 2, \dots, \lambda - 1$) implies that l divides all m_i ($i = 0, 1, \dots, \lambda$). We "define" $n_{e+1}, n_{e+2}, \dots, n_\lambda$ by $n_i := \frac{m_i}{l}$ ($i = e+1, e+2, \dots, \lambda$). Then $(m_0, m_1, \dots, m_\lambda) = l(n_0, n_1, \dots, n_\lambda)$. In particular the sequence $\mathbf{n} = (n_0, n_1, \dots, n_e)$ is contained in a dominant sequence $\mathbf{n}' = (n_0, n_1, \dots, n_\lambda)$, and so Y is not dominant (a contradiction!). Thus $e = \lambda$ and

$$\big(m_0, m_1, \dots, m_\lambda\big) = l\big(n_0, n_1, \dots, n_\lambda\,(=1)\big).$$

This shows that Y is of type AB_l. □

From this lemma, type AB_l coincides with proportional type B_l; so the arithmetic property of the latter is the same as that of type A_l — dominant tame. Thus as long as we are concerned with the arithmetic property of type B_l, it is enough to investigate that of non-proportional one. We remark that when we later construct deformations from subbranches of types A_l, B_l, and C_l, a subbranch of proportional type B_l (i.e. type AB_l) produces two different deformations according to the application of the respective constructions for types A_l and B_l.

We point out that a subbranch Y both of type B_l and C_l also exists; $l = 2$, $\mathbf{m} = (4, 3, 2, 1)$ and $\mathbf{n} = (1, 1, 1)$ is such an example. As we will see later, a subbranch Y both of type B_l and C_l produces the same deformation regardless of the application of the respective constructions for type B_l and C_l, and thus there is no reason to distinguish them; we adopt the following convention.

Convention 9.1.4 To avoid overlapping of type C_l with type B_l, we exclude the case $m_e = l$ and $n_e = 1$ from type C_l.

Now we give several comments on (C.1), (C.2), and (C.3) in the definition of type C_l.

Lemma 9.1.5 *The integer* $u := (m_{e-1} - ln_{e-1}) - (r_e - 1)(m_e - ln_e)$ *in* (C.1) *is equal to* $m_e - m_{e+1}$ *in* (C.3). (Note: since $m_e > m_{e+1}$, we have $u = m_e - m_{e+1} > 0$.)

Proof. In fact, we may write

$$\begin{aligned} u &= (m_{e-1} - r_e m_e) + m_e + l(r_e n_e - n_{e-1} - n_e) \\ &= (m_{e-1} - r_e m_e) + m_e \\ &= m_e - m_{e+1}. \end{aligned}$$

where the second and third equalities respectively follows from $n_e = r_e n_e - n_{e-1}$ (a condition in (C.2)) and $\dfrac{m_{e-1} + m_{e+1}}{m_e} = r_e$. □

By the above lemma, $u = m_e - m_{e+1} > 0$. We remark that "$lY \leq X$, n_e divides n_{e-1}, and $\frac{n_{e-1}}{n_e} < r_e$" (cf. (C.1)) implies $u > 0$ (Proposition 9.4.8). However, if we drop "$\frac{n_{e-1}}{n_e} < r_e$", then $u > 0$ fails; for example,

$l = 1$, $\mathbf{m} = (6, 5, 4, 3, 2, 1)$ and $\mathbf{n} = (5, 3, 1)$. Then n_e divides n_{e-1}, but $\frac{n_{e-1}}{n_e} = 3 > r_e = 2$. In this case $u = -1$. (Actually $\mathbf{n}$ is of type A_1.)

$l = 1$, $\mathbf{m} = (4, 3, 2, 1)$ and $\mathbf{n} = (3, 2, 1)$. Then n_e divides n_{e-1}, but $\frac{n_{e-1}}{n_e} = 2 = r_e = 2$. In this case $u = 0$. (Actually $\mathbf{n}$ is of type A_1.)

Secondly we point out that the condition (C.1) (or all other conditions) of type C_l implies that

(C′) $lY \leq X$, n_e divides n_{e-1}, and $\frac{n_{e-1}}{n_e} < r_e$, and $m_e - m_{e+1}$ divides l.

But the converse is *not* true; namely (C.1) is not equivalent to (C′). In fact, under the condition (C'), $m_e - m_{e+1}$ does not necessarily equal u in (C.1). (cf. Lemma 9.1.5.) For instance, $l = 1$, $\mathbf{m} = (13, 4, 3, 2, 1)$ and $\mathbf{n} = (2, 1)$, which satisfies all conditions of (C'). However $m_e - m_{e+1} = 1$, while $u = 2$. In particular, $m_e - m_{e+1}$ divides l, while u does not, and thus this example is not of type C_l.

Remark 9.1.6 *For type C_l, from the condition that n_e divides n_{e-1} and Y is a subbranch, it is easy to deduce that n_e divides n_i $(i = 0, 1, \ldots, e-1)$. Namely, when $n_e \geq 2$, a subbranch Y of type C_l itself is multiple. See the proof of Lemma* 8.1.4.

It is worth pointing out the following property (type $B_l^\sharp$ means non-proportional type B_l):

Type A_l	lY is dominant tame
Type $B_l^\sharp$	lY is dominant wild (Proposition 9.3.2)
Type C_l	lY is wild (Proposition 9.4.11)

As we explained above, proportional type B_l (i.e. type AB_l) is dominant tame. We also note that type $B_l^\sharp$ (non-proportional type B_l) and type C_l are wild, but in contrast with type $B_l^\sharp$, type C_l is in general *not* dominant, e.g.

$$l = 1, \quad \mathbf{m} = (6, 5, 4, 3, 2, 1) \quad \text{and} \quad \mathbf{n} = (1, 1, 1).$$

This is not dominant; $\mathbf{n}$ is contained in a dominant sequence $\mathbf{n}' = (1, 1, 1, 1, 1, 1)$. (Interesting enough, $\mathbf{n}'$ is not of type C_l but of type B_l where $l = 1$.) A more complicated example is the following:

$$l = 10, \quad \mathbf{m} = (40, 26, 12, 10, 8, 6, 4, 2) \quad \text{and} \quad \mathbf{n} = (3, 2, 1).$$
$$\text{(In this case } u = 2.)$$

This example is also of type C_l but not dominant; $\mathbf{n}$ is contained in $\mathbf{n}' := (3, 2, 1, 1)$ (type B_l where $l = 10$). Another curious example is: $l = 2$, $\mathbf{m} = (6, 5, 4, 3, 2, 1)$ and $\mathbf{n} = (2, 2)$. Then $\mathbf{n}$ is of type C_l contained in $\mathbf{n}' = (2, 2, 2)$, which is again of type C_l. See also Remark 20.2.4, p357 for this example.

Remark 9.1.7 If $n_{e-1} < n_e$, then Y is none of types A_l, B_l and C_l. (1) Y is not type A_l: In fact, $n_{e-1} < n_e$ implies $\frac{n_{e-1}}{n_e} < 1$, and so $\frac{n_{e-1}}{n_e} < r_e$ because $r_e \geq 2$. Thus Y does not fulfill (A.1). (2) Noting that $1 \leq n_{e-1} < n_e$, we have $1 < n_e$, and so Y is not of type B_l. (3) As $n_{e-1} < n_e$, the integer n_e does not divide n_{e-1}, and hence Y is not of type C_l.

Let $Y = n_0\Delta_0 + n_1\Theta_1 + \cdots + n_e\Theta_e$ be a subbranch of a branch $X = m_0\Delta_0 + m_1\Theta_1 + \cdots + m_\lambda\Theta_\lambda$. If Y is of type C_l, then $\lambda \geq e + 1$ by Corollary 9.4.4 below. On the other hand, this is not necessarily true for types A_l and B_l. It may occur that $\lambda = e$; for example,

Example A_l $\quad l = 1$, $\mathbf{m} = (9, 6, 3)$ and $\mathbf{n} = (3, 2, 1)$.
Example B_l $\quad l = 3$, $\mathbf{m} = (9, 6, 3)$ and $\mathbf{n} = (1, 1, 1)$.

Now setting

$$a := m_{e-1} - ln_{e-1}, \quad b := m_e - ln_e, \quad c := n_{e-1}, \quad d := n_e,$$

we restate the definitions of types A_l, B_l, and C_l as follows:

Type A_l A subbranch Y is of *type* A_l if $lY \leq X$ and $\frac{c}{d} \geq r_e$.
Type B_l A subbranch Y is of *type* B_l if $lY \leq X$, $b = 0$ and $d = 1$.
Type C_l A subbranch Y is of *type* C_l if one of the following conditions holds:

(C.1) $lY \leq X$, d divides c, and $\frac{c}{d} < r_e$, and u divides l where $u := a - (r_e - 1)b$.

(C.2) $lY \leq X$, $d = r_e d - c$, and u divides l where $u := a - (r_e - 1)b$.

We next summarize signs for some quantities concerning with types A_l, $B_l^\sharp$ and C_l, where type $B_l^\sharp$ means non-proportional type B_l.

Type A_l	$a \geq 0$	$b \geq 0$	$c > 0$	$d > 0$
Type $B_l^\sharp$	$a > 0$	$b = 0$	$c > 0$	$d = 1$
Type C_l	$a > 0$	$b \geq 0$	$c > 0$	$d > 0$

Here note that for any type, $c = n_{e-1} > 0$ and $d = n_e > 0$, and for type B_l, $d = 1$. In general $a \geq 0$ and $b \geq 0$ hold; the strict inequality $a > 0$ is valid for types $B_l^\sharp$ and C_l, which will be proved in Proposition 9.3.2 and Proposition 9.4.11 respectively. On the other hand, $b > 0$ is not true for type B_l because $b = m_e - ln_e = l - l = 0$. We also remark that for type A_l, $a = 0$ if and only if $b = 0$. Moreover $a = 0$ (equivalently $b = 0$) occurs precisely when $X = lY$ and in this case, Y is proportional (Corollary 9.2.8).

Next we provide the table for the signs of quantities $a - r_e b$, $r_e d - c$ and $u := a-(r_e-1)b$; this table are useful for our later construction of deformations associated with subbranches of types A_l, B_l, and C_l. In the table, type $B_l^\sharp$ means non-proportional type B_l, and for type A_l, if $e = \lambda$, then we formally set $m_{e+1} := 0$. For a subbranch Y of type C_l such that lY is not dominant, we formally set $n_{e+1} := r_e n_e - n_{e-1}$; then $m_{e+1} \geq ln_{e+1}$ by non-dominance, and hence $ln_{e+1} - m_{e+1} \leq 0$.

Table 9.1.8

Type A_l	$a - r_e b \leq -m_{e+1} < 0$	$r_e d - c \leq 0$	$u \leq b - m_{e+1}$
Type $B_l^\sharp$	$a - r_e b > 0$	$r_e d - c > 0$	$u > 0$
Type C_l	$a - r_e b > 0$ if lY is dominant $a - r_e b = ln_{e+1} - m_{e+1} \leq 0$ if lY is not dominant	$r_e d - c = d > 0$	$u = m_e - m_{e+1} > 0$

(The inequalities in the above table will be shown in Proposition 9.2.5, Proposition 9.3.2, and Proposition 9.4.11 for types A_l, $B_l^\sharp$, and C_l respectively.)

The following table for type C_l, to be proved in Lemma 9.4.10, will be used later in the construction of deformations.

Table 9.1.9

Type C_l	$u > b$ if lY is dominant $u \leq b$ if lY is not dominant

For a subbranch $Y = n_0\Delta_0 + n_1\Theta_1 + \cdots + m_e\Theta_e$ of a branch $X = m_0\Delta_0 + m_1\Theta_1 + \cdots + m_\lambda\Theta_\lambda$, recall that Θ_i is a (-2)-*curve* if the self-intersection number $\Theta_i \cdot \Theta_i = -2$; a *chain of* (-2)-*curve* is a set of (-2)-curves of the form $\Theta_a + \Theta_{a+1} + \cdots + \Theta_b$ where $a \leq b$. If Y is of type C_l, then in most cases the complement of Y in X contains a chain of (-2)-curves, where by the "complement of Y in X", we mean $\Theta_{e+1} + \Theta_{e+2} + \cdots + \Theta_\lambda$ (note $\lambda \geq e+1$ for type C_l by Corollary 9.4.4). To explain this result, we set $u := (m_{e-1} - ln_{e-1}) - (r_e - 1)(m_e - ln_e)$, and then u divides l by the definition of type C_l, and so we write $l = Nu$ where N is a positive integer. Next we set $b := m_e - ln_e$ and $d := n_e$, and if $u \leq b$, considering the division of b by u, we let v be the integer such that $b - vu \geq 0$ and $b - (v+1)u < 0$. According to whether $u > b$ or $u \leq b$, we have the following information about chains of (-2)-curves in the complement of Y in X. (Note: $r_i = 2$ is equivalent to Θ_i being a (-2)-curve.)

Table 9.1.10 (Type C_l) Refer Proposition 9.4.12 for the proof.

$b = 0$	$r_{e+1} = r_{e+2} = \cdots = r_\lambda = 2, \quad \lambda = e + Nd - 1$
$b \geq 1, \ u > b$	$r_{e+1} = r_{e+2} = \cdots = r_{\lambda-1} = 2, \quad \lambda = e + Nd$
$b \geq 1, \ u \leq b$, u does not divide v	$r_{e+1} = r_{e+2} = \cdots = r_{\lambda-1} = 2, \quad \lambda = e + Nd + v$
$b \geq 1, \ u \leq b$, u divides v	$r_{e+1} = r_{e+2} = \cdots = r_\lambda = 2, \quad \lambda = e + Nd + v - 1$

Example 9.1.11 (Exceptional example) In the above table, for the case $b \geq 1$ and $u > b$, if $\lambda = e + 1$, then the complement of Y in X may *not* contain a chain of (-2)-curves at all; $l = 2$, $\mathbf{m} = (5, 3, 1)$ and $\mathbf{n} = (1, 1)$ is such an example of type C_l, in which case $u = 2$ and so $N = 1$ in $l = Nu$, and consequently $\lambda = e + 1 = e + Nd = 2$. Then $r_\lambda \,(= 3) \neq 2$ and hence the complement Θ_λ of Y in X is not a (-2)-curve.

The following criterion for Y to be of type C_1 is useful.

Lemma 9.1.12 *When $l = 1$, a dominant subbranch Y is of type C_l if and only if the following conditions are fulfilled:* (1) *n_e divides n_{e-1}, and $\frac{n_{e-1}}{n_e} < r_e$,* (2) *$m_{e-1} - n_{e-1} = 1$, and* (3) *$m_e = n_e$.*

Proof. $\Longrightarrow$: Suppose that Y is of type C_1. Then by definition, (1) is satisfied. To show (2) and (3), we set $u := (m_{e-1} - ln_{e-1}) - (r_e - 1)(m_e - ln_e)$, and then u divides l by the definition of type C_l. In the present case, $l = 1$ and so $u = \pm 1$. As $u = m_e - m_{e+1} > 0$ for type C_l (Lemma 9.1.5), we have $u = 1$.

Now for simplicity, we set $a := m_{e-1} - n_{e-1}$ and $b := m_e - n_e$. Of course $b \geq 0$. Since $Y\,(= lY)$ is dominant and of type C_l, we have $a - r_e b \geq 1$ by Table 9.1.8. Thus

$$b \geq 0, \qquad a - r_e b \geq 1. \tag{9.1.2}$$

Our goal is to show that (2) $a = 1$ and (3) $b = 0$. Note that since $l = 1$,

$$u = (m_{e-1} - n_{e-1}) - (r_e - 1)(m_e - n_e) = a - (r_e - 1)b = (a - r_e b) + b.$$

On the other hand, as we saw above, $u = 1$ and thus $(a - r_e b) + b = 1$. Taking (9.1.2) into consideration, this equation holds exactly when $a - r_e b = 1$ and $b = 0$, that is, $a = 1$ and $b = 0$. Hence (2) and (3) hold.

$\Longleftarrow$: Taking into account (1), we only have to show that $u = a - (r_e - 1)b$ divides $l = 1$. But $a = 1$ (2) and $b = 0$ (3), and so $u = 1$, which obviously divides l. Therefore the condition (C.1) of type C_l is satisfied; so Y is of type C_l. □

Remark 9.1.13 Recall that a subbranch Y is of ripple type if $n_{e-1} = n_e = m_e$ (see (8.1.3), p146); then Y is dominant by Lemma 8.1.5. Clearly the three conditions of the above lemma are fulfilled and thus Y is of type C_1. However, the converse is not true; even if the conditions (1), (2) and (3) of Lemma 9.1.12 are fulfilled, it does *not* imply that Y is of ripple type. The following examples are of type C_1 but not of ripple type because $n_{e-1} \neq n_e$.

(1) $l = 1$, $\mathbf{m} = (15, 11, 7, 3, 2, 1)$ and $\mathbf{n} = (12, 9, 6, 3)$.

(2) $l = 1$, $\mathbf{m} = (22, 17, 12, 7, 2, 1)$ and $\mathbf{n} = (18, 14, 10, 6, 2)$.

A recipe to produce subbranches of type C_l

We close this section by giving a recipe to produce examples of subbranches of type C_l. Given $\mathbf{m} = (m_0, m_1, \ldots, m_\lambda)$, take two positive integers $n_e := 1$ and $n_{e-1} := r_e - 1$ where $r_e = \dfrac{m_{e-1} + m_{e+1}}{m_e}$. Then clearly $n_e = r_e n_e - n_{e-1}$, and hence the first condition in (C.3) of Definition 9.1.1 is fulfilled. Thus Y is of type C_l precisely when $m_e - m_{e+1}$ divides l. For example, (1) $m_e - m_{e+1} = 1$ or (2) $m_e - m_{e+1} = 2$ and l is even. If this is the case, we define $n_{e-2}, n_{e-3}, \ldots, n_\lambda$ inductively by $n_{i-1} := r_i n_i - n_{i+1}$ for $i = e-1, e-2, \ldots, 1$. This yields a sequence $\mathbf{n} = (n_0, n_1, \ldots, n_e)$ of type C_l.

9.2 Demonstration of properties of type A_l

In this section, we demonstrate the properties of type A_l. We begin by recalling the definition of dominance. Let $Y = n_0\Delta_0 + n_1\Theta_1 + \cdots + m_e\Theta_e$ be a subbranch of a branch $X = m_0\Delta_0 + m_1\Theta_1 + \cdots + m_\lambda\Theta_\lambda$. We set $n_{e+1} := r_e n_e - n_{e-1}$ formally, where $r_e := \dfrac{m_{e-1} + m_{e+1}}{m_e}$. Then Y is said to be dominant if either

(i) $n_{e+1} \leq 0$ or (ii) $n_{e+1} > m_{e+1}$ holds. According to (i) or (ii), Y is called tame or wild respectively. The condition (i) is rewritten as $\frac{n_{e-1}}{n_e} \geq r_e$, and so we have

Lemma 9.2.1 *A subbranch Y is dominant tame if and only if $\frac{n_{e-1}}{n_e} \geq r_e$ holds.*

We then note

Lemma 9.2.2 *Let lY be a subbranch of a branch $X = m_0\Delta_0 + m_1\Theta_1 + \cdots + m_\lambda\Theta_\lambda$. Then the following statements hold:*

(1) *Y is dominant $\Longrightarrow$ lY is dominant.*
(2) *Y is dominant tame $\Longleftrightarrow$ lY is dominant tame.*

Proof. We write $Y = n_0\Delta_0 + n_1\Theta_1 + \cdots + n_e\Theta_e$. First we show (1) by contradiction. Suppose that lY is not dominant. Then there exists an integer k_{e+1} $(0 < k_{e+1} \leq m_{e+1})$ satisfying

$$\frac{ln_{e-1} + k_{e+1}}{ln_e} = r_e. \tag{9.2.1}$$

Thus $ln_{e-1} + k_{e+1} = ln_e r_e$. It follows that l divides k_{e+1}. We write $k_{e+1} = ln_{e+1}$ where n_{e+1} is a positive integer, and then (9.2.1) is

$$\frac{ln_{e-1} + ln_{e+1}}{ln_e} = r_e.$$

Thus $\frac{n_{e-1} + n_{e+1}}{n_e} = r_e$. This implies that Y is not dominant, because the sequence $\mathbf{n} = (n_0, n_1, \ldots, n_e)$ is contained in a longer sequence $(n_0, n_1, \ldots, n_{e+1})$. This contradicts that Y is dominant. Hence lY is dominant, and so (1) is confirmed.

Next we show (2). Remember that a subbranch is dominant tame if and only if $\frac{n_{e-1}}{n_e} \geq r_e$ (Lemma 9.2.1). Obviously $\frac{n_{e-1}}{n_e} \geq r_e$ is equivalent to $\frac{ln_{e-1}}{ln_e} \geq r_e$, and so we confirm the equivalence in (2). □

We remark that in (1) of the above lemma, the converse is *not* true in general. For instance, $l = 4$, $\mathbf{m} = (6, 5, 4, 3, 2, 1)$ and $\mathbf{n} = (1, 1, 1)$. Then $l\mathbf{n} = (4, 4, 4)$ is dominant wild, whereas $\mathbf{n}$ is not dominant; indeed, $\mathbf{n}$ is contained in a dominant sequence $(1, 1, 1, 1, 1, 1)$.

Next we show the equivalence of conditions of type A_l.

Lemma 9.2.3 *The following conditions are equivalent:*

(A1) *$lY \leq X$ and $\frac{n_{e-1}}{n_e} \geq r_e$.*
(A2) *$lY \leq X$ and lY is dominant tame.*
(A3) *$lY \leq X$ and Y is dominant tame.*

Proof. The equivalence of (A1) and (A.2) follows from Lemma 9.2.2, while that of (A.2) and (A.3) follows from Lemma 9.2.2 (2). □

Next we derive a formula needed for later use.

Lemma 9.2.4 *Let lY be a subbranch* (note: Y is not assumed to be of type A_l). *Set $a := m_{e-1} - ln_{e-1}$ and $b := m_e - ln_e$. Then*

$$a - r_e b = -m_{e+1} - l(n_{e-1} - r_e n_e).$$

In fact,

$$\begin{aligned} a - r_e b &= (m_{e-1} - ln_{e-1}) - r_e(m_e - ln_e) \\ &= m_{e-1} - r_e m_e - ln_{e-1} + r_e ln_e \\ &= -m_{e+1} - ln_{e-1} + r_e ln_e, \end{aligned}$$

where the last equality follows from $\dfrac{m_{e-1} + m_{e+1}}{m_e} = r_e$.

Proposition 9.2.5 *Let Y be a subbranch of type A_l, and set*

$$a := m_{e-1} - ln_{e-1}, \quad b := m_e - ln_e, \quad c := n_{e-1} \quad \text{and} \quad d := n_e.$$

Then the following inequalities hold:

(1) $a - r_e b \le -m_{e+1}$ (2) $r_e d - c \le 0$ (3) $u \le b - m_{e+1}$, *where* $u := a - (r_e - 1)b$.

Proof. (1): If Y is of type A_l, then $\dfrac{n_{e-1}}{n_e} \ge r_e$, i.e.

$$n_{e-1} - r_e n_e \ge 0. \tag{9.2.2}$$

By Lemma 9.2.4, $a - r_e b = -m_{e+1} - l(n_{e-1} - r_e n_e)$, and so from (9.2.2), we derive $a - r_e b \le -m_{e+1}$. This proves (1).
(2): As $d = n_e$ and $c = n_{e-1}$, (2) is nothing but (9.2.2).
(3): By (1), $a - r_e b \le -m_{e+1}$, and hence together with $b \ge 0$, we have

$$u = b + (a - r_e b) \le b - m_{e+1}.$$

This proves (3). □

We gather several basic lemmas for subbranches (not necessarily of type A_l):

Lemma 9.2.6 *Let lY be a subbranch with the multiplicities $l\mathbf{n} = (ln_0, ln_1, \dots, ln_e)$. Let Z be the dominant subbranch containing lY, and write its multiplicities as*

$$\mathbf{k} = (ln_1, ln_2, \dots, ln_e, k_{e+1}, k_{e+2}, \dots, k_f).$$

Then l divides k_i for $i = e+1, e+2, \dots, f$. (In particular, "defining" n_i ($i = e+1, e+2, \dots, f$) by $n_i := \dfrac{k_i}{l}$, then $Z = lY'$ where $Y' = n_0\Delta_0 + n_1\Theta_1 + \cdots + n_f\Theta_f$.)

Proof. Since Z is a subbranch, we have

$$\frac{ln_{e-1} + k_{e+1}}{ln_e} = r_e, \tag{9.2.3}$$

$$\frac{ln_e + k_{e+2}}{k_{e+1}} = r_{e+1}, \tag{9.2.4}$$

$$\frac{k_{i-1} + k_{i+1}}{lk_i} = r_i, \qquad i = e+2, e+3, \ldots, f-1. \tag{9.2.5}$$

By (9.2.3), we have $n_{e-1} + \frac{k_{e+1}}{l} = r_e n_e$, and hence l divides k_{e+1}. Set $n_{e+1} := \frac{k_{e+1}}{l}$, i.e. $k_{e+1} = ln_{e+1}$ which we substitute into (9.2.4): $n_e + \frac{k_{e+2}}{l} = r_e n_{e+1}$. Hence l divides k_{e+2}. Repeating this argument, we see that l divides k_i for $i = e+1, e+2, \ldots, f$. □

Lemma 9.2.7 *Suppose that* $X = m_0\Delta_0 + m_1\Theta_1 + \cdots + m_\lambda\Theta_\lambda$ *is a branch. Let* lY *be a subbranch with the multiplicities* $l\mathbf{n} = (ln_0, ln_1, \ldots, ln_e)$*, and let* Z *be the dominant subbranch containing* lY (note: $Z = lY'$ for some Y' by Lemma 9.2.6)*. Set* $a := m_{e-1} - ln_{e-1}$ *and* $b := m_e - ln_e$*. Then the following statements hold:*

(I) *If* $a = 0$*, then* $Z = X$ (and so Z is "trivially" dominant tame.)
(II) *If* Y *is dominant tame, then the following equivalences hold:*

$$a = 0 \iff b = 0 \iff X = lY.$$

(Note: If lY is dominant but not tame, then (II) is *not* valid. For example, $l = 4$, $\mathbf{m} = (6, 5, 4, 3, 2, 1)$ and $\mathbf{n} = (1, 1, 1)$. Then $a = 0$ but $b \neq 0$.)

Proof. (I): By Lemma 9.2.6, we may express $Z = lY'$ where $Y' = n_0\Delta_0 + n_1\Theta_1 + \cdots + n_f\Theta_f$ $(e \leq f)$. It is enough to show that $m_i = ln_i$ for $i = 0, 1, \ldots, f$; in fact, once this is shown, we have $Z = m_0\Delta_0 + m_1\Theta_1 + \cdots + m_f\Theta_f$, and Z of this form is dominant precisely when $f = \lambda$, i.e. $Z = X$. Now we show that $m_i = ln_i$ firstly for $i = 0, 1, \ldots, e$. From the definition of a subbranch,

$$\frac{ln_{e-2} + ln_e}{ln_{e-1}} = \frac{m_{e-2} + m_e}{m_{e-1}} \ (= r_{e-1}).$$

In particular if $a = 0$, i.e. $m_{e-1} = ln_{e-1}$, then

$$ln_{e-2} + ln_e = m_{e-2} + m_e. \tag{9.2.6}$$

Taking into account $ln_{e-2} \leq m_{e-2}$ and $ln_e \leq m_e$, (9.2.6) implies that $ln_{e-2} = m_{e-2}$ and $ln_e = m_e$. Next, again from the definition of a subbranch,

$$\frac{ln_{e-3} + ln_{e-1}}{ln_{e-2}} = \frac{m_{e-3} + m_{e-1}}{m_{e-2}} \ (= r_{e-2}).$$

From $ln_{e-2} = m_{e-2}$ (we showed this just above), we have

$$ln_{e-3} + ln_{e-1} = m_{e-3} + m_{e-1}. \tag{9.2.7}$$

Taking into account $ln_{e-3} \le m_{e-3}$ and $ln_{e-1} \le m_{e-1}$, (9.2.7) implies that $ln_{e-3} = m_{e-3}$ and $ln_{e-1} = m_{e-1}$. Repeating this argument, we deduce $m_i = ln_i$ for $i = 0, 1, \ldots, e$. Similarly, we can show that $m_i = ln_i$ for $i = e+1, e+2, \ldots, f$. Therefore $m_i = ln_i$ holds for $i = 0, 1, \ldots, f$. This proves (I).

(II): The equivalence "$a = 0 \iff b = 0$" is already shown in Lemma 6.3.1, p107. To show the equivalence "$a = 0 \iff X = lY$", we note that if Y is dominant tame, then lY is also dominant tame by Lemma 9.2.3; thus the dominant subbranch Z containing lY is lY itself; $Z = lY$. We now show "$a = 0 \iff X = lY$".

$\Longrightarrow$: If $a = 0$, then $X = Z$ by the assertion (I). Since $Z = lY$, we have $X = lY$.

$\Longleftarrow$: Trivial. This completes the proof of the assertion (II). □

As a corollary, we have the following result.

Corollary 9.2.8 *Let Y be a subbranch of type A_l, and set $a := m_{e-1} - ln_{e-1}$ and $b := m_e - ln_e$. Then the following equivalences hold: $a = 0 \iff b = 0 \iff X = lY$.*

Proof. By Lemma 9.2.3, if Y is of type A_l, then Y is dominant tame, and so the assertion follows from the above lemma. □

9.3 Demonstration of properties of type B_l

We begin with the following lemma for subbranches not necessarily of type B_l.

Lemma 9.3.1 *Let l be a positive integer and let $Y = n_0\Delta_0 + n_1\Theta_1 + \cdots + n_e\Theta_e$ be a subbranch of a branch $X = m_0\Delta_0 + m_1\Theta_1 + \cdots + m_\lambda\Theta_\lambda$ such that lY is a dominant wild subbranch of X. Set $a := m_{e-1} - ln_{e-1}$, $b := m_e - ln_e$, $c := n_{e-1}$, $d := n_e$ and $u := a - (r_e - 1)b$. Then the following inequalities hold:*

(1) $a, c, d > 0$, (2) $b \ge 0$, (3) $a - r_e b > 0$, (4) $r_e d - c > 0$, (5) $u > 0$.

Proof. We first verify (1) and (2). Since lY is a subbranch of X, we have $m_{e-1} \ge ln_{e-1}$ and $m_e \ge ln_e$, and so $a, b \ge 0$. Since Y is a subbranch of X, we have $n_{e-1}, n_e > 0$, and so $c, d > 0$. Hence to prove (1) and (2), it remains to show $a > 0$, which is carried out by contradiction. Suppose that $a = 0$, namely $m_{e-1} = ln_{e-1}$. Then

$$\begin{aligned} r_e &> \frac{ln_{e-1} + m_{e+1}}{ln_e} && \text{because } lY \text{ is wild} \\ &= \frac{m_{e-1} + m_{e+1}}{ln_e} && \text{by } m_{e-1} = ln_{e-1} \end{aligned}$$

$$\begin{aligned}&\geq \frac{m_{e-1}+m_{e+1}}{m_e} && \text{by } m_e \geq ln_e\\ &= r_e,\end{aligned}$$

and thus $r_e > r_e$, giving a contradiction. This proves $a > 0$. To show (3), we first note that

$$\begin{aligned}a - r_e b &:= (m_{e-1} - ln_{e-1}) - r_e(m_e - ln_e)\\ &= (m_{e-1} - r_e m_e) - ln_e + r_e ln_e,\end{aligned}$$

where $m_{e-1} - r_e m_e = -m_{e+1}$ by $\frac{m_{e-1}+m_{e+1}}{m_e} = r_e$, and therefore

$$a - r_e b = -m_{e+1} - ln_e + r_e ln_e.$$

Since lY is wild, we have $r_e > \frac{ln_{e-1}+m_{e+1}}{ln_e}$, and so

$$\begin{aligned}a - r_e b &= -m_{e+1} - ln_{e-1} + r_e ln_e\\ &> -m_{e+1} - ln_{e-1} + \frac{ln_{e-1}+m_{e+1}}{ln_e}\, ln_e\\ &= 0.\end{aligned}$$

Thus $a - r_e b > 0$, and (3) is proved. Similarly, (4) is shown as follows:

$$r_e d - c = r_e n_e - n_{e-1} > \frac{ln_{e-1}+m_{e+1}}{ln_e}\, n_e - n_{e-1} = \frac{m_{e+1}}{l} > 0.$$

Finally, it is immediate to show (5). Indeed, $a - r_e b > 0$ by (3) and $b \geq 0$ by (2), and so we have $u = a - (r_e - 1)b = (a - r_e b) + b > 0$. Thus (5) is proved.□

Recall that a subbranch $Y = n_0\Delta_0 + n_1\Theta_1 + \cdots + n_e\Theta_e$ of a branch $X = m_0\Delta_0 + m_1\Theta_1 + \cdots + m_\lambda\Theta_\lambda$ is proportional if $\frac{m_0}{n_0} = \frac{m_1}{n_1} = \cdots = \frac{m_e}{n_e}$. By Lemma 9.1.3, a dominant subbranch Y is both of type A_l and of type B_l (i.e. type AB_l) if and only if Y is of proportional type B_l; explicitly this is the case

$$\mathbf{m} = (ln_0, ln_1, \ldots, ln_\lambda), \quad \mathbf{n} = (n_0, n_1, \ldots, n_\lambda), \quad \text{and} \quad n_\lambda = 1.$$

The arithmetic property of proportional type B_l is the same as that of type A_l, namely dominant tame. Next we investigate the arithmetic property of non-proportional type B_l; remember that a subbranch $Y = n_0\Delta_0 + n_1\Theta_1 + \cdots + n_e\Theta_e$ is of type B_l provided that $m_e = l$ and $n_e = 1$.

Proposition 9.3.2 *Let $X = m_0\Delta_0 + m_1\Theta_1 + \cdots + m_\lambda\Theta_\lambda$ be a branch, and suppose that $Y = n_0\Delta_0 + n_1\Theta_1 + \cdots + m_e\Theta_e$ is a subbranch of non-proportional type B_l of X. Set $a := m_{e-1} - ln_{e-1}$, $b := m_e - ln_e$, $c := n_{e-1}$, $d := n_e\ (= 1)$ and $u := a - (r_e - 1)b$. Then*

(1) *lY is dominant wild, and*
(2) *$a > 0$, $\quad a - r_e b > 0$, $\quad r_e d - c > 0$, $\quad u > 0$.*

Proof. The proof of (1) consists of two steps:
Step 1 We demonstrate that lY is dominant by contradiction. Suppose that lY is not dominant. Then there exists an integer k_{e+1} $(0 < k_{e+1} \le m_{e+1})$ satisfying

$$\frac{ln_{e-1} + k_{e+1}}{ln_e} = r_e, \tag{9.3.1}$$

and so

$$\frac{ln_{e-1} + k_{e+1}}{ln_e} = \frac{m_{e-1} + m_{e+1}}{m_e} \ (= r_e).$$

Since $m_e = ln_e \, (= l)$ by the definition of type B_l, we have

$$ln_{e-1} + k_{e+1} = m_{e-1} + m_{e+1}.$$

As $ln_{e-1} \le m_{e-1}$ and $k_{e+1} \le m_{e+1}$, this holds exactly when

$$ln_{e-1} = m_{e-1}, \qquad k_{e+1} = m_{e+1}. \tag{9.3.2}$$

Note that from (9.3.1), we have $n_{e-1} + \frac{k_{e+1}}{l} = r_e n_e$. So l divides k_{e+1}, and in particular, $l \le k_{e+1}$. Namely $m_e \le m_{e+1}$ by $m_e = l$ (the definition of type B_l) and $m_{e+1} = k_{e+1}$ (9.3.2). This yields a contradiction because the sequence $m_0, m_1, \dots, m_\lambda$ is strictly decreasing. Therefore lY is dominant.
Step 2 We next show that lY is wild, that is, $\frac{ln_{e-1} + m_{e+1}}{ln_e} < r_e$ as follows:

$$\begin{aligned} \frac{ln_{e-1} + m_{e+1}}{ln_e} &< \frac{m_{e-1} + m_{e+1}}{ln_e} && \text{by } ln_{e-1} < m_{e-1} \\ &= \frac{m_{e-1} + m_{e+1}}{m_e} && \text{by } m_e = ln_e \, (= l) \\ &= r_e. \end{aligned} \tag{9.3.3}$$

Thus lY is dominant wild, and so (1) is confirmed. The assertion (2) follows immediately from Lemma 9.3.1 because lY is dominant wild. (Note: In (9.3.3), "$ln_{e-1} < m_{e-1}$" is *not* valid for proportional type B_l, as $ln_{e-1} = m_{e-1}$.) □

9.4 Demonstration of properties of type C_l

Let $Y = n_0\Delta_0 + n_1\Theta_1 + \cdots + n_e\Theta_e$ be a subbranch of a branch $X = m_0\Delta_0 + m_1\Theta_1 + \cdots + m_\lambda\Theta_\lambda$, where we set

$$r_i := \frac{m_{i-1} + m_{i+1}}{m_i}, \quad (i = 1, 2 \dots, \lambda - 1), \qquad r_\lambda := \frac{m_{\lambda-1}}{m_\lambda}.$$

For a while, we do *not* assume that Y is of type C_l; Y is an arbitrary subbranch.

Lemma 9.4.1 *Set* $u := (m_{e-1} - ln_{e-1}) - (r_e - 1)(m_e - ln_e)$. *Then*

$$u = m_e - m_{e+1} + l(r_e n_e - n_{e-1} - n_e).$$

(In particular, if $n_e = r_e n_e - n_{e-1}$, then $u = m_e - m_{e+1}$.)

Proof. In fact,

$$\begin{aligned} u &= (m_{e-1} - ln_{e-1}) - (r_e - 1)(m_e - ln_e) \\ &= m_e + m_{e-1} - r_e m_e + l(r_e n_e - n_{e-1} - n_e) \\ &= m_e - m_{e+1} + l(r_e n_e - n_{e-1} - n_e), \end{aligned}$$

where in the last equality we used $r_e m_e = m_{e-1} + m_{e+1}$. □

Next we show the equivalence of three conditions of type C_l.

Lemma 9.4.2 *The following conditions are equivalent:*

(C.1) $lY \le X$, *n_e divides n_{e-1}, and* $\frac{n_{e-1}}{n_e} < r_e$, *and u divides l where* $u := (m_{e-1} - ln_{e-1}) - (r_e - 1)(m_e - ln_e)$.
(C.2) $lY \le X$, $n_e = r_e n_e - n_{e-1}$, *and u divides l where u is in* (C.1).
(C.3) $lY \le X$, $n_e = r_e n_e - n_{e-1}$, *and* $m_e - m_{e+1}$ *divides l, where by convention,* $m_{e+1} = 0$ *if* $\lambda = e$.

(Note: By Lemma 9.4.1, $m_e - m_{e+1}$ equals u in (C.1) and (C.2). In (C.3), actually $m_{e+1} = 0$ does *not* occur as we will see in Corollary 9.4.4 below.)

Proof. We first show that (C.2) is equivalent to (C.1).
(C.2) $\Longrightarrow$ (C.1): This is easy. If $n_e = r_e n_e - n_{e-1}$, then n_e divides n_{e-1}, and $\frac{n_{e-1}}{n_e} = r_e - 1 < r_e$, hence (C.1) holds.
(C.2) $\Longleftarrow$ (C.1): Under the assumption that n_e divides n_{e-1} and $\frac{n_{e-1}}{n_e} < r_e$, it suffices to prove that $n_e = r_e n_e - n_{e-1}$, that is, $r_e - \frac{n_{e-1}}{n_e} = 1$ holds; setting $q := r_e - \frac{n_{e-1}}{n_e}$, we show $q = 1$. We first note that (i) q is an integer because n_e divides n_{e-1}, and (ii) q is positive because $\frac{n_{e-1}}{n_e} < r_e$. Therefore q is a positive integer. We then prove $q = 1$ by contradiction. Suppose that

$$q \ge 2. \tag{9.4.1}$$

We note

$$\begin{aligned} u &= (m_e - m_{e+1}) + l(r_e n_e - n_{e-1} - n_e) \qquad \text{by Lemma 9.4.1} \\ &= (m_e - m_{e+1}) + ln_e \left(r_e - \frac{n_{e-1}}{n_e} - 1 \right) \\ &= (m_e - m_{e+1}) + ln_e(q - 1). \end{aligned}$$

Here $m_e - m_{e+1} > 0$ because the sequence $m_0, m_1, \ldots, m_\lambda$ strictly decreases. On the other hand, $n_e \geq 1$ and $q - 1 \geq 1$ (9.4.1). Hence

$$u = (m_e - m_{e+1}) + ln_e(q-1) > l.$$

But u divides l by assumption, and so $l \geq u$. This is a contradiction. Therefore $q = 1$, and the claim is confirmed.

Finally, we show that (C.2) is equivalent to (C.3). This is evident. Indeed, $u = (m_e - m_{e+1}) + l(r_e n_e - n_{e-1} - n_e)$ by Lemma 9.4.1, and hence if $n_e = r_e n_e - n_{e-1}$, then $u = m_e - m_{e+1}$. □

Recall that a subbranch Y is of type C_l provided that Y satisfies one of the equivalent conditions of Lemma 9.4.2.

Corollary 9.4.3 *Let $Y = n_0\Delta_0 + n_1\Theta_1 + \cdots + n_e\Theta_e$ be a subbranch of a branch $X = m_0\Delta_0 + m_1\Theta_1 + \cdots + m_\lambda\Theta_\lambda$. Set $b := m_e - ln_e$ and $d := n_e$, and then*

(1) $m_e = ld + b$, *and*
(2) *if furthermore Y is of type C_l, then $m_{e+1} = ld + b - u$.*

Proof. From $d = n_e$ and $b = m_e - ln_e$, we have $m_e = ld + b$, and so (1) is confirmed. Next we show (2). If Y is of type C_l, we have $u = m_e - m_{e+1}$ (Lemma 9.1.5). Substituting (1) $m_e = ld + b$ into $u = m_e - m_{e+1}$, we obtain $u = ld + b - m_{e+1}$. This confirms (2). □

We also note the following.

Corollary 9.4.4 *Let $Y = n_0\Delta_0 + n_1\Theta_1 + \cdots + n_e\Theta_e$ be a subbranch of a branch $X = m_0\Delta_0 + m_1\Theta_1 + \cdots + m_\lambda\Theta_\lambda$. If Y is of type C_l, then $e + 1 \leq \lambda$.*

Proof. We show this by contradiction. Suppose that $m_{e+1}(:= r_e m_e - m_{e-1}) = 0$. We first claim that m_e divides both l and $m_e - ln_e$. In fact, since $m_e - m_{e+1}$ divides l by the definition (C.3) of type C_l and $m_{e+1} = 0$ by assumption, we see that m_e divides l; clearly this also assures that m_e divides $m_e - ln_e$. Next setting $b := m_e - ln_e$, we write $l = l'm_e$ and $b = b'm_e$ where l' (resp. b') is a positive (resp. nonnegative) integer. Then

$$b' = \frac{b}{m_e} = \frac{m_e - ln_e}{m_e} = \frac{m_e - l'm_e n_e}{m_e} = 1 - l'n_e.$$

Namely

$$b' = 1 - l'n_e. \tag{9.4.2}$$

From $l' \geq 1$ and $n_e \geq 1$, we have $b' \leq 0$. Since b' is nonnegative, we obtain $b' = 0$ (and so $b = 0$), and then by (9.4.2), $l' = n_e = 1$. Here note that $b = 0$ implies that $m_e = ln_e$, and thus together with $n_e = 1$, we have $m_e = l$. This means that Y is not only of type C_l but also of type B_l. But this contradicts Convention 9.1.4 (we excluded this case from type C_l). □

We collect several lemmas needed for later discussion.

Lemma 9.4.5 *Let lY be a subbranch with the multiplicities $l\mathbf{n} = (ln_0, ln_1, \ldots, ln_e)$. Let Z be the dominant subbranch containing lY, and write its multiplicities as*

$$\mathbf{k} = (ln_1, ln_2, \ldots, ln_e, k_{e+1}, k_{e+2}, \ldots, k_f).$$

Then

(I) *l divides k_i $(i = e+1, e+2, \ldots, f)$*. (So "defining" n_i $(i = e+1, e+2, \ldots, f)$ by $n_i := \frac{k_i}{l}$, then $Z = lY'$ and $\mathbf{k} = l\mathbf{n}'$ where $Y' := n_0\Delta_0 + n_1\Theta_1 + \cdots + n_f\Theta_f$ and $\mathbf{n}' := (n_0, n_1, \ldots, n_f)$.)

(II) *if n_e divides n_{e-1}, then n_e also divides n_i $(i = e+1, e+2, \ldots, f)$, and moreover $n_e \le n_{e+1} \le n_{e+2} \le \cdots \le n_f$.*

Proof. (I) is nothing but Lemma 9.2.6. We show (II); we first prove that n_e divides n_{e+1}. Since $Z = ln_0\Delta_0 + ln_1\Theta_1 + \cdots + ln_f\Theta_f$ is a subbranch, we have $ln_{i+1} = r_i ln_i - ln_{i-1}$, so $n_{i+1} = r_i n_i - n_{i-1}$ for $i = 1, 2, \ldots, f$. In particular $n_{e+1} = r_e n_e - n_{e-1}$. Hence n_e divides n_{e+1} (note: by assumption, n_e divides n_{e-1}), and consequently $n_e \le n_{e+1}$. Next since n_e divides n_{e+1}, from $n_{e+2} = r_{e+1}n_{e+1} - n_e$, it follows that n_e also divides n_{e+2}. Furthermore

$$\begin{aligned} n_{e+2} &= r_{e+1}n_{e+1} - n_e \\ &\ge 2n_{e+1} - n_e && \text{by } r_{e+1} \ge 2 \\ &= n_{e+1} + (n_{e+1} - n_e) \\ &\ge n_{e+1} && \text{by } n_{e+1} \ge n_e. \end{aligned} \tag{9.4.3}$$

Namely, $n_{e+1} \le n_{e+2}$. Then using the fact (as shown above) that n_e divides both n_{e+1} and n_{e+2}, it follows from $n_{e+3} = r_{e+2}n_{e+2} - n_{e+1}$ that n_e divides n_{e+3}. Also we can show $n_{e+2} \le n_{e+3}$ as in (9.4.3). Repeat this argument, and then (II) is shown. □

Lemma 9.4.6 *Let Y be a subbranch of type C_l. Then Y and lY are* (*not necessarily dominant*) *wild.*

Proof. We first verify the wildness of lY. We separate into two cases according to whether lY is dominant or not.

Case 1 lY is dominant: Since Y is of type C_l, $\frac{n_{e-1}}{n_e} < r_e$ and so $\frac{ln_{e-1}}{ln_e} < r_e$, which means that lY is wild.

Case 2 lY is not dominant: Let Z be the dominant subbranch containing lY. Then by Lemma 9.4.5 (I), the multiplicities of Z are of the form:

$$\mathbf{k} = (ln_1, ln_2, \ldots, ln_f).$$

Since Y is of type C_l, n_e divides n_{e-1} and so by Lemma 9.4.5 (II), we have

$$n_e \le n_{e+1} \le n_{e+2} \le \cdots \le n_f.$$

In particular $\frac{n_{f-1}}{n_f} \leq 1$. Since $r_f \geq 2$, we have $\frac{n_{f-1}}{n_f} < r_f$, and so $\frac{ln_{f-1}}{ln_f} < r_f$ This implies that Z is wild, and consequently (by definition) lY is wild. Similarly we can show the wildness of Y. □

For subsequent discussion, we need some result on Y not necessarily of type C_l.

Lemma 9.4.7 *Let lY be a subbranch which is not dominant. Set*

$$a := m_{e-1} - ln_{e-1}, \quad b := m_e - ln_e \quad \textit{and} \quad u := a - (r_e - 1)b.$$

If n_e divides n_{e-1}, then

(I) $a - r_e b = ln_{e+1} - m_{e+1}$ *where* $n_{e+1} := r_e n_e - n_{e-1}$. (In particular, from $ln_{e+1} \leq m_{e+1}$, we have $a - r_e b \leq 0$).
(II) $u > 0$ *and* $a > 0$.

Proof. The statement (I) is derived as follows:

$$\begin{aligned} a - r_e b &= m_{e-1} - ln_{e-1} - r_e(m_e - ln_e) \\ &= (m_{e-1} - r_e m_e) - ln_{e-1} + r_e ln_e \\ &= (-m_{e+1}) - ln_{e-1} + \frac{ln_{e-1} + ln_{e+1}}{ln_e} ln_e \\ &= -m_{e+1} + ln_{e+1}, \end{aligned}$$

where in the third equality we used $m_{e-1} - r_e m_e = -m_{e+1}$ and $r_e = \frac{ln_{e-1} + ln_{e+1}}{ln_e}$ (note that by assumption, lY is not dominant and so $0 < ln_{e+1} \leq m_{e+1}$).

Next we show (II). We first prove $u > 0$. By assumption, n_e divides n_{e-1} and thus by Lemma 9.4.5, we have

$$n_{e+1} \geq n_e. \tag{9.4.4}$$

Then

$$\begin{aligned} u &= b + (a - r_e b) \\ &= (m_e - ln_e) + (ln_{e+1} - m_{e+1}) && \text{by (I)} \\ &= (m_e - m_{e+1}) + l(n_{e+1} - n_e) \\ &> 0 && \text{by } m_e > m_{e+1} \text{ and (9.4.4).} \end{aligned}$$

This proves $u > 0$. Finally, we show $a > 0$. We divide into two cases according to whether $b = 0$ or $b > 0$.
Case $b = 0$: In this case we have $u = a - (r_e - 1)b = a$. Thus $a > 0$ because $u > 0$ as shown above.
Case $b > 0$: Noting that $a \geq 0$, we assume that $a = 0$ and deduce a contradiction. If $a = 0$, then we have $u = a - (r_e - 1)b = -(r_e - 1)b$. Since $r_e \geq 2$, together with $b > 0$, we obtain $u < 0$. This contradicts $u > 0$, and we conclude that $a > 0$. □

Proposition 9.4.8 *Let lY be a subbranch such that* (i) n_e *divides* n_{e-1} *and* (ii) $\frac{n_{e-1}}{n_e} < r_e$. *Then* $u > 0$ *where* $u := (m_{e-1} - ln_{e-1}) - (r_e - 1)(m_e - ln_e)$.

Proof. According to whether lY is dominant or not, we separate into two cases. If lY is dominant, then from the assumption (ii), we have $\frac{ln_{e-1}}{ln_e} < r_e$ which means that lY is (dominant) wild and hence by Lemma 9.3.1, we have $u > 0$. (Notice that in this case we do not need the assumption (i).) Next if lY is not dominant, together with (i), we conclude that $u > 0$ by Lemma 9.4.7 (II). (Notice that in this case we do not need (ii).) □

Remark 9.4.9 As is clear from the proof, $u > 0$ holds under a weaker assumption: (1) lY is dominant wild or (2) n_e divides n_{e-1}.

The above proposition will be often used later (e.g. for the proofs of Lemma 13.3.5, p242 and Lemma 13.4.5, p247). For $u := a - (r_e - 1)b$ where $a := m_{e-1} - ln_{e-1}$ and $b := m_e - ln_e$, the following inequalities are also valid.

Lemma 9.4.10 *Let Y be a subbranch of type C_l. Then*

$$\begin{cases} u > b & \text{if } lY \text{ is dominant} \\ u \leq b & \text{if } lY \text{ is not dominant.} \end{cases}$$

Proof. If lY is dominant, then (noting that type C_l is wild by Lemma 9.4.6), we have $a - r_e b > 0$ by Lemma 9.3.1, and so $u = b + (a - r_e b) > b$.

If lY is not dominant, then $a - r_e b \leq 0$ by Lemma 9.4.7 (I), and thus $u = b + (a - r_e b) \leq b$. □

We provide examples for the respective cases of the above lemma.

Example $(u > b)$
$l = 1$, $\mathbf{m} = (6, 5, 4, 3, 2, 1)$ and $\mathbf{n} = (4, 4, 4)$.
Then Y is of type C_l and lY is dominant; in this case $u = 1 > b = 0$.

Example $(u \leq b)$
$l = 1$, $\mathbf{m} = (6, 5, 4, 3, 2, 1)$ and $\mathbf{n} = (1, 1, 1)$.
Then Y is of type C_l and lY is not dominant; in this case $u = 1 < b = 3$.

Now we summarize the properties of type C_l obtained thus far.

Proposition 9.4.11 *Let $Y = n_0\Delta_0 + n_1\Theta_1 + \cdots + n_e\Theta_e$ be a subbranch of a branch $X = m_0\Delta_0 + m_1\Theta_1 + \cdots + m_\lambda\Theta_\lambda$. Set*

$a := m_{e-1} - ln_{e-1}$, $b := m_e - ln_e$, $c := n_{e-1}$, $d := n_e$ *and*
$u := a - (r_e - 1)b$.

Suppose that Y is of type C_l. Then the following statements hold:

(1) *Y and lY are (not necessarily dominant) wild* (Lemma 9.4.6).
(2) $a > 0$ (Lemma 9.4.7 (II)).

(3) $a - r_e b > 0$ *if* lY *is dominant* (Lemma[1] 9.3.1), *and*
$a - r_e b = l n_{e+1} - m_{e+1} \leq 0$ *if* lY *is not dominant* (Lemma 9.4.7 (I)).
(4) $r_e d - c = d > 0$ (the definition (C.2) or (C.3) of type C_l).
(5) $u = m_e - m_{e+1} > 0$ (Lemma 9.1.5).
(6) $u > b$ *if* lY *is dominant, and* $u \leq b$ *if* lY *is not dominant* (Lemma 9.4.10).
(7) $e + 1 \leq \lambda$ (Corollary 9.4.4).

Let $Y = n_0\Delta_0 + n_1\Theta_1 + \cdots + m_e\Theta_e$ be a subbranch of of a branch $X = m_0\Delta_0 + m_1\Theta_1 + \cdots + m_\lambda\Theta_\lambda$; recall that Θ_i is a (-2)-*curve* if the self-intersection number $\Theta_i \cdot \Theta_i = -2$, and a *chain of* (-2)-*curve* is a set of (-2)-curves of the form $\Theta_a + \Theta_{a+1} + \cdots + \Theta_b$ where $a \leq b$. (*It is valuable to keep in mind that the existence of a chain of* (-2)-*curves often implies the existence of various deformations.*)

We shall show that if Y is of type C_l, then in most cases, the complement of a subbranch Y of X contains a chain of (-2)-curves where the "complement" is $\Theta_{e+1} + \Theta_{e+2} + \cdots + \Theta_\lambda$ (note that $e + 1 \leq \lambda$ for type C_l as shown in Corollary 9.4.4). cf. Example 9.1.11 for an exceptional case where the complement contains no (-2)-curves.

Now we give the information on chains of (-2)-curves in the complement of Y in X. Below we note that $r_i = 2$ is equivalent to Θ_i being a (-2)-curve.

Proposition 9.4.12 *Let* $Y = n_0\Delta_0 + n_1\Theta_1 + \cdots + n_e\Theta_e$ *be a subbranch of type* C_l. *Set*

$$a := m_{e-1} - l n_{e-1}, \quad b := m_e - l n_e, \quad c := n_{e-1}, \quad d := n_e \quad \text{and}$$
$$u := a - (r_e - 1)b,$$

and (noting that u *divides* l *by the definition of type* C_l*), write* $l = Nu$ *where* N *is a positive integer, and if* $u \leq b$*, then (considering the division of* b *by* u*), let* v *be the positive integer such that* $b - vu \geq 0$ *and* $b - (v+1)u < 0$. *Then the following holds:*

(I) *if* $b = 0$, *then* $\lambda = e + Nd - 1$ *and*
$r_{e+1} = r_{e+2} = \cdots = r_\lambda = 2, \quad m_{\lambda-1} = 2u, \quad$ *and* $\quad m_\lambda = 2,$
(II) *if* $b \geq 1$ *and* $u > b$, *then* $\lambda = e + Nd$ *and*
$r_{e+1} = r_{e+2} = \cdots = r_{\lambda-1} = 2, \quad m_{\lambda-1} = b + u, \quad$ *and* $\quad m_\lambda = b,$
(Note: $r_\lambda := \dfrac{m_{\lambda-1}}{m_\lambda} = \dfrac{b+u}{b}$ is an integer and so in this case, b divides u.)
(III) *if* $b \geq 1$, $u \leq b$, *and* u *does not divide* b, *then* $\lambda = e + Nd + v$ *and*
$r_{e+1} = r_{e+2} = \cdots = r_{\lambda-1} = 2, \quad m_{\lambda-1} = b - (v-1)u, \quad$ *and* $\quad m_\lambda = b - vu.$
(IV) *if* $b \geq 1$, $u \leq b$, *and* u *divides* b (so $b = vu$), *then* $\lambda = e + Nd + v - 1$ *and* $r_{e+1} = r_{e+2} = \cdots = r_\lambda = 2, \quad m_{\lambda-1} = 2u, \quad$ *and* $\quad m_\lambda = u.$

[1] If Y is of type C_l, then lY is wild by (1), and so we can apply Lemma 9.3.1 for dominant wild Y.

Remark 9.4.13 Note that $u > b$ if lY is dominant, and $u \leq b$ otherwise. See Proposition 9.4.11 (6).

Proof. (I): By Corollary 9.4.3 applied for $b = 0$, we have

$$m_e = ld, \qquad m_{e+1} = ld - u, \qquad \text{where } d := n_e. \tag{9.4.5}$$

Since a sequence $(m_e, m_{e+1}, \ldots, m_\lambda)$ is inductively determined from m_e and m_{e+1} by the division algorithm, this sequence is uniquely characterized by the following properties:

(a) $m_e > m_{e+1} > \cdots > m_\lambda > 0$,
(b) $\dfrac{m_{e+i-1} + m_{e+i+1}}{m_{e+i}}$, $(i = 0, 1, \ldots, \lambda - e - 1)$ is an integer, and m_λ divides $m_{\lambda-1}$.

Therefore (9.4.5) implies that $m_{e+i} = ld - iu$ $(i = 0, 1, \ldots, \lambda - e - 1)$, an arithmetic progression. Note that

$$\begin{aligned} m_{e+(Nd)} &= ld - (Nd)u = ld - ld \qquad \text{by } l = Nu \\ &= 0, \end{aligned}$$

whereas $m_{e+(Nd-1)} = u$. Thus we conclude that $\lambda = e + (Nd - 1)$ and

$$(m_e, m_{e+1}, \ldots, m_\lambda) = (ld,\, ld - u,\, ld - 2u,\, \ldots\,,\, 2u,\, u),$$

from which we deduce $r_{e+1} = r_{e+2} = \cdots = r_\lambda = 2$. This proves the assertion (I).

(II) $b \geq 1$ and $u > b$: The proof is essentially the same as that for (I). By Corollary 9.4.3, we have

$$m_e = ld + b, \qquad m_{e+1} = ld + b - u. \tag{9.4.6}$$

As in (I), this implies that $m_{e+i} = ld + b - iu$, an arithmetic progression. When $i = Nd - 1$, we have

$$\begin{aligned} m_{e+(Nd-1)} &= ld + b - (Nd - 1)u \\ &= ld + b - ld + u \qquad \text{by } l = Nu \\ &= b + u, \end{aligned}$$

and likewise $m_{e+(Nd)} = b$. On the other hand, we have $m_{e+(Nd+1)} = b - u < 0$ (note $u > b$ by assumption), and thus $\lambda = e + Nd$. Therefore

$$m_{e+i} = \begin{cases} ld + b - iu, & i = 0, 1, \ldots, Nd - 1 \\ b, & i = Nd, \end{cases} \tag{9.4.7}$$

and $r_{e+1} = r_{e+2} = \cdots = r_{\lambda-1} = 2$.

(III) $b \geq 1$, $u \leq b$, and u does not divide b: The proof is similar to that of (II); by Corollary 9.4.3, we have

$$m_e = ld + b, \qquad m_{e+1} = ld + b - u, \tag{9.4.8}$$

which implies that $m_{e+i} = ld + b - iu$, an arithmetic progression. Let v be the positive integer such that $b - vu \geq 0$ and $b - (v+1)u < 0$. Then for $i = Nd + v - 1$, we have

$$\begin{aligned} m_{e+(Nd+v-1)} &= ld + b - (Nd + v - 1)u \\ &= ld + b - ld - vu + u \qquad \text{by } l = Nu \\ &= b - vu + u, \end{aligned}$$

and likewise $m_{e+(Nd+v)} = b - vu$. Similarly we obtain $m_{e+(Nd+v+1)} = b - (v+1)u$. As we took the positive integer v in such a way that $b - vu \geq 0$ and $b - (v+1)u < 0$, we have $m_{e+(Nd+v)} = b - vu > 0$ and $m_{e+(Nd+v+1)} < 0$; hence $\lambda = e + (Nd + v)$. We thus conclude that

$$m_{e+i} = \begin{cases} ld + b - iu, & i = 0, 1, \ldots, Nd + v - 1 \\ b - vu, & i = Nd + v, \end{cases}$$

from which we derive $r_{e+1} = r_{e+2} = \cdots = r_{\lambda-1} = 2$. This proves the assertion.

(IV) $b \geq 1$, $u \leq b$, and u divides b (i.e. $b = vu$): Using the computation of (III), we have $m_{e+(Nd+v)} = b - vu = 0$ in the present case, and thus $\lambda = e + (Nd + v - 1)$, and $m_{\lambda-1} = 2u$ and $m_\lambda = u$. The remaining statement follows from the same argument as in (III). □

Supplement

In the proof of Proposition 9.4.12, we only used the fact "u divides ld". The reader may wonder that in the definition of type C_l, we can replace "u divides l" by a weaker condition "u divides ld". Unfortunately this is not true, because in that case the deformation atlas associated with lY does *not* necessarily admit a complete propagation. This is confirmed by the following example, which illustrates the essential role of the condition "u divides l" in type C_l.

Example 9.4.14 Let $X = 32\Delta_0 + 24\Theta_1 + 16\Theta_2 + 8\Theta_3$. We take $Y = 2\Delta_0 + 2\Theta_1$ and $l = 12$. Then lY satisfies the condition of type C_l except "u divides l"; indeed $u = 8$ and $d\,(:= n_e) = 2$, hence u does not divide l but divides $ld = 24$.

We show that the deformation atlas associated with lY does *not* admit a complete propagation. First note that

$$\mathcal{H}_1 : \quad w^8(w^2\eta^2 + t^2)^{12} - s = 0.$$

(The exponent 2 of t^2 is necessary for making a first propagation possible. See the second equality of (9.4.9) below.) We take $g_1 :\ z = 1/w,\ \zeta = w^2\eta -$

$\sqrt{-1}tw$, where we note that there is no other choice of g_1 which transforms $\mathcal{H}_1$ to a hypersurface. Since

$$w^8(w^2\eta^2+t^2)^{12} = w^8\left[\frac{1}{w^2}(w^2\eta)^2+t^2\right]^{12},$$

the map g_1 transforms the polynomial $w^8(w^2\eta^2+t^2)^{12}$ to

$$\begin{aligned}\frac{1}{z^8}\left[z^2\left(\zeta+t\sqrt{-1}\frac{1}{z}\right)^2+t^2\right]^{12} &= \frac{1}{z^8}\Big[\Big(z^2\zeta^2+2\sqrt{-1}tz\zeta-t^2\Big)+t^2\Big]^{12}\\ &= \frac{1}{z^8}\Big[z^2\zeta^2+2\sqrt{-1}tz\zeta\Big]^{12}\\ &= z^4[\,z\zeta^2+2\sqrt{-1}t\zeta\,]^{12}.\end{aligned} \tag{9.4.9}$$

Thus the following data gives a first propagation.

$$\begin{cases}\mathcal{H}_1: & w^8(w^2\eta^2+t^2)^{12}-s=0\\ \mathcal{H}_1': & z^4(z\zeta^2+2\sqrt{-1}t\zeta)^{12}-s=0\\ g_1: & z=1/w,\ \zeta=w^2\eta-\sqrt{-1}tw.\end{cases}$$

Similarly, we can construct a second propagation as follows: Noting that

$$\mathcal{H}_2: \quad \eta^4\,(w^2\eta+2\sqrt{-1}tw)^{12}-s=0,$$

we take $g_2:\ z=1/w,\ \zeta=w^2\eta+2\sqrt{-1}tw$. (Note: there is no other choice of g_2 which transforms $\mathcal{H}_2$ to a hypersurface. See the second equality of (9.4.10) below.) Since

$$\eta^4(w^2\eta+2\sqrt{-1}tw)^{12} = \frac{1}{w^8}(w^2\eta)^4\left[(w^2\eta)+2\sqrt{-1}tw\right]^{12},$$

the map g_2 transforms a polynomial $\eta^4\,(\,w^2\eta+2\sqrt{-1}tw\,)^{12}$ to

$$\begin{aligned}z^8\left(\zeta-2\sqrt{-1}t\frac{1}{z}\right)^4\left[\left(\zeta-2\sqrt{-1}t\frac{1}{z}\right)+2\sqrt{-1}t\frac{1}{z}\right]^{12} &= z^8\left(\zeta-2\sqrt{-1}t\frac{1}{z}\right)^4\zeta^{12}\\ &= z^4\zeta^{12}(z\zeta-2\sqrt{-1}t)^4.\end{aligned} \tag{9.4.10}$$

Hence the following data gives a second propagation:

$$\begin{cases}\mathcal{H}_2: & \eta^4(\,w^2\eta+2\sqrt{-1}tw\,)^{12}-s=0\\ \mathcal{H}_2': & z^4\zeta^{12}(z\zeta-2\sqrt{-1}t)^4-s=0\\ g_2: & z=1/w,\ \zeta=w^2\eta+2\sqrt{-1}w.\end{cases}$$

It remains to construct a third propagation. However this is impossible, which is seen as follows. Note that $\mathcal{H}_3:\ w^{12}\eta^4(w\eta-2\sqrt{-1}t)^4-s=0$, and a standard form of a deformation g_3 of $z = 1/w$, $\zeta = w^2\eta$ is given by $z = 1/w$, $\zeta = w^2\eta + f(t)w$ where $f(t)$ is a holomorphic function in t. For brevity, we only consider the case $f(t) = \alpha t^k$ where $\alpha \in \mathbb{C}$ and k is a positive integer (the discussion below is valid for general $f(t)$). We claim that for any α and k, the map g_3 cannot transform

$$\mathcal{H}_3: \quad w^{12}\eta^4(w\eta - 2\sqrt{-1}t)^4 - s = 0$$

to a hypersurface. In fact, since

$$w^{12}\eta^4(w\eta - 2\sqrt{-1}t)^4 = w^4(w^2\eta)^4 \left(\frac{1}{w}(w^2\eta) - 2\sqrt{-1}t\right)^4,$$

the map g_3 transforms $\mathcal{H}_3$ to

$$\frac{1}{z^4}\left(\zeta - \alpha t^k \frac{1}{z}\right)^4 \left(z\zeta - \alpha t^k - 2\sqrt{-1}t\right)^4 - s = 0.$$

Clearly for any choice of $\alpha \in \mathbb{C}$ and a positive integer k, the left hand side, after expansion, contains a fractional term. So a further propagation is impossible, and consequently the deformation atlas associated with lY does not admit a complete propagation. (For a non-standard form of g_3 containing higher or lower order terms, the argument is essentially the same though the computation becomes complicated. cf. Example 5.5.12, p96.)

10

Construction of Deformations of Type B_l

Assume that $Y = n_0\Delta_0 + n_1\Theta_1 + \cdots + n_e\Theta_e$ is a subbranch of a branch $X = m_0\Delta_0 + m_1\Theta_1 + \cdots + m_\lambda\Theta_\lambda$, and l is a positive integer satisfying $lY \leq X$ (that is, $ln_i \leq m_i$ for $i = 0, 1, \ldots, e$). As before, we set

$$r_i := \frac{m_{i-1} + m_{i+1}}{m_i}, \quad (i = 1, 2 \ldots, \lambda - 1), \qquad r_\lambda := \frac{m_{\lambda-1}}{m_\lambda}.$$

Recall that r_i $(i = 1, 2, \ldots, \lambda)$ are positive integers satisfying $r_i \geq 2$.

The deformation atlas $DA_{e-1}(lY, k)$ associated with lY was defined as follows: First define a sequence of integers p_i $(i = 0, 1, \ldots, \lambda + 1)$ inductively by

$$\begin{cases} p_0 = 0, \quad p_1 = 1 \quad \text{and} \\ p_{i+1} = r_i p_i - p_{i-1} \quad \text{for } i = 1, 2, \ldots, \lambda. \end{cases}$$

Then $p_{\lambda+1} > p_\lambda > \cdots > p_1 > p_0 = 0$ (6.2.4). Let $f(z)$ be a non-vanishing holomorphic function defined around $z = 0$, and set $f_i = f(w^{p_{i-1}}\eta^{p_i})$ and $\widehat{f_i} = f(z^{p_{i+1}}\zeta^{p_i})$ (see (6.2.7)). Then the deformation atlas $DA_{e-1}(lY, k)$ is given as follows: for $i = 1, 2, \ldots, e - 1$,

$$\begin{cases} \mathcal{H}_i : & w^{m_{i-1} - ln_{i-1}}\eta^{m_i - ln_i}(w^{n_{i-1}}\eta^{n_i} + t^k f_i)^l - s = 0 \\ \mathcal{H}'_i : & z^{m_{i+1} - ln_{i+1}}\zeta^{m_i - ln_i}(z^{n_{i+1}}\zeta^{n_i} + t^k \widehat{f_i})^l - s = 0 \\ g_i : & \text{the transition function } z = 1/w, \ \zeta = w^{r_i}\eta \text{ of } N_i. \end{cases}$$

The following remarkable result holds.

Theorem 10.0.15 *Let $Y = n_0\Delta_0 + n_1\Theta_1 + \cdots + n_e\Theta_e$ be of type A_l, B_l, or C_l. Assume that k is a positive integer such that if Y is of type C_l, then it is divisible by n_e. Then $DA_{e-1}(lY, k)$ admits a complete propagation.*

Remark 10.0.16 If Y has length zero, i.e. $Y = n_0\Delta_0$, then we regard Y as type A_l; this convention is consistent in that we can construct a complete propagation of $\mathcal{H}_1$ (note that $DA_{e-1}(lY, k)$ in this case consists only of $\mathcal{H}_1$) by applying the construction of complete propagations for type A_l.

We have already shown the statement of Theorem 10.0.15 for type A_l (Proposition 7.1.3, p121). It remains to show this for types B_l and C_l. This chapter is devoted to verifying this statement for type B_l.

10.1 Deformations of type B_l

Suppose that $Y = n_0\Delta_0 + n_1\Theta_1 + \cdots + n_e\Theta_e$ is of type B_l, that is, $m_e = l$ and $n_e = 1$.

Remark 10.1.1 If the sequence of multiplicities of a branch X is an arithmetic progression with difference 1 and $m_\lambda = 1$, that is, $\mathbf{m} = (m_0, m_0 - 1, \ldots, 2, 1)$, then there are many choices of subbranches of type B_l. Indeed, for any e $(0 < e < \lambda)$, a subbranch $Y := \Delta_0 + \Theta_1 + \cdots + \Theta_e$ is of type B_l where we take $l = e$.

Now we shall construct a complete propagation of $DA_{e-1}(lY, k)$. Firstly, we set

$$\begin{cases} \mathcal{H}_e: & w^{m_{e-1}-ln_{e-1}}(w^{n_{e-1}}\eta + t^k f_e)^l - s = 0 \\ \mathcal{H}'_e: & z^{m_{e+1}}\zeta^{m_e} - s = 0 \\ g_e: & z = \dfrac{1}{w}, \quad \zeta = w^{r_e}\eta + t^k w^{r_e - n_{e-1}} f_e. \end{cases}$$

Note that by Proposition 9.3.2, $m_{e-1} - ln_{e-1} > 0$, and so all exponents of the terms in the defining equation of $\mathcal{H}_e$ are positive. Hence $\mathcal{H}_e$ is well-defined as a hypersurface. We now show that g_e transforms $\mathcal{H}_e$ to $\mathcal{H}'_e$. Since

$$w^{m_{e-1}-ln_{e-1}}(w^{n_{e-1}}\eta + t^k f_e)^l = w^{m_{e-1}-ln_{e-1}}\left[w^{n_{e-1}-r_e}(w^{r_e}\eta) + t^k f_e\right]^l,$$

the map g_e transforms $\mathcal{H}_e$ to

$$\begin{aligned} & z^{ln_{e-1}-m_{e-1}}\left[z^{r_e-n_{e-1}}\left(\zeta - t^k\frac{f_e}{z^{r_e-n_{e-1}}}\right) + t^k f_e\right]^l - s \\ &= z^{ln_{e-1}-m_{e-1}}\left[z^{r_e-n_{e-1}}\zeta\right]^l - s \\ &= z^{lr_e - m_{e-1}}\zeta^l - s \\ &= z^{m_{e+1}}\zeta^{m_e} - s, \end{aligned}$$

where in the last equality, we used $l = m_e$ and $m_e r_e = m_{e-1} + m_{e+1}$. Therefore g_e transforms $\mathcal{H}_e$ to $\mathcal{H}'_e$. We thus obtain an e-th propagation. We further propagate this 'trivially'. Namely for $i = e+1, e+2, \ldots, \lambda$, we set

$$\begin{cases} \mathcal{H}_i: & w^{m_{i-1}}\eta^{m_i} - s = 0 \\ \mathcal{H}'_i: & z^{m_{i+1}}\zeta^{m_i} - s = 0 \\ g_i: & \text{the transition function } z = 1/w, \ \zeta = w^{r_i}\eta \text{ of } N_i, \end{cases}$$

where $m_{\lambda+1} = 0$ by convention. Evidently g_i ($i = e+1, e+2, \dots, \lambda$) transforms $\mathcal{H}_i$ to $\mathcal{H}'_i$, and thus we obtain a complete propagation of $DA_{e-1}(lY, k)$. This completes the proof of Theorem 10.0.15 for type B_l.

Remark 10.1.2 The complete deformation atlas above is trivial around irreducible components Θ_i ($i = e+1, e+2, \dots, \lambda$). The same holds for type A_l (Remark 6.1.2, p103).

It may be worth pointing out the following:

Remark 10.1.3 The condition of type B_l is equivalent to $lY \le X$, $m_e = ln_e$ and $n_e = 1$. If we modify this condition to $lY \le X$, $m_e = ln_e$ and $n_e \ge 2$, then the above construction does *not* work. In fact, if $n_e \ge 2$, then the map g_e in the above proof does not transform $w^{m_{e-1}-ln_{e-1}}(w^{n_{e-1}}\eta + t^k f_e)^l$ to $z^{m_{e+1}}\zeta^{m_e}$.

Let $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$ be the barking family (specifically called *of type* B_l) obtained by patching the complete deformation atlas above. Then it is easy to describe the deformation from X to $X_{0,t} := \Psi(0,t)$; for instance,

$$\mathcal{H}_e|_{s=0} : \; w^{m_{e-1}-ln_{e-1}}(w^{m_{e-1}}\eta + t^k f_e)^l = 0$$

is a disjoint union of

$$\begin{cases} w^{m_{e-1}-ln_{e-1}} = 0 & \text{(a multiple disk of multiplicity } m_{e-1} - ln_{e-1}\text{) and} \\ (w^{m_{e-1}}\eta + t^k f_e)^l = 0 & \text{(a multiple annulus of multiplicity } l \, (= m_e)\text{).} \end{cases}$$

According to $e < \lambda$ or $e = \lambda$, $X_{0,t}$ is illustrated in Figure 10.1.1 or Figure 10.1.2 respectively.

Remark 10.1.4 A subbranch Y of type B_l is possibly of type A_l simultaneously (i.e. type AB_l). This is exactly the case where Y is of proportional type B_l (Lemma 9.1.3); explicitly $e = \lambda$, $n_\lambda = 1$, and $X = lY$ (i.e. $m_i = ln_i$ for $i = 0, 1, \dots, \lambda$). Then $\pi : M \to \Delta$ admits two different barking families $\Psi_A : \mathcal{M}_A \to \Delta \times \Delta^\dagger$ and $\Psi_B : \mathcal{M}_B \to \Delta \times \Delta^\dagger$, resulting from the respective

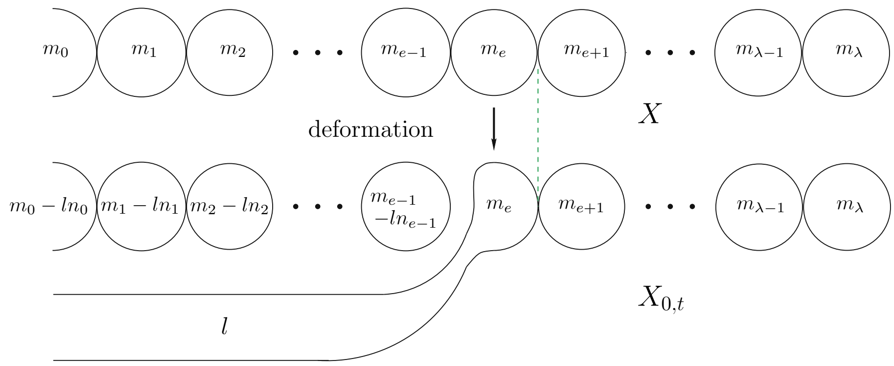

Fig. 10.1.1. Deformation of type B_l when $e < \lambda$.

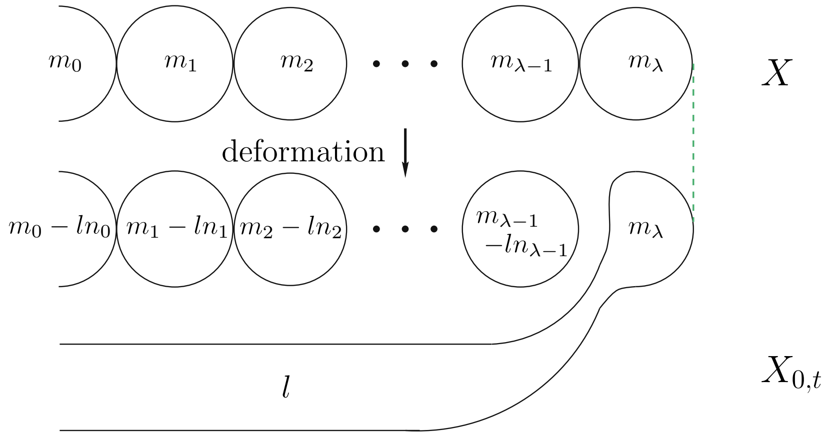

Fig. 10.1.2. Deformation of type B_l for $e = \lambda$.

constructions for type A_l and type B_l. It is curious that the singular fibers $\Psi_A^{-1}(0,t)$ and $\Psi_B^{-1}(0,t)$ are (topologically) the same; the irreducible component marked by "$m_i - ln_i$" on Figure 7.1.1, p122 and Figure 10.1.2 is vacuous if $m_i - ln_i = 0$, and so $X_{0,t}$ in Figure 7.1.1 is the same as $X_{0,t}$ in Figure 10.1.2, although the ambient spaces $\mathcal{M}_A$ and $\mathcal{M}_B$ are different because the gluing maps around Θ_λ are different.

10.2 Singular fibers

We state the result on the singular fibers of $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$.

Proposition 10.2.1 *Let $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$ be a barking family of non-proportional type B_l. Then $X_{s,t} := \Psi^{-1}(s,t)$ is singular if and only if $s = 0$.* (This is not valid for proportional type B_l by Theorem 7.2.4 (2), p128.)

To show this, we shall investigate the restriction of Ψ to $\mathcal{H}_i$ and $\mathcal{H}'_i$ $(i = 1, 2, \ldots, \lambda)$ separately (notation for restriction: $\Psi|_{\mathcal{H}_i}$ and $\Psi|_{\mathcal{H}'_i}$); we prove the following claim which is clearly equivalent to the above proposition.

Claim 10.2.2 $(\mathrm{A})_i$ *A fiber $(\Psi|_{\mathcal{H}_i})^{-1}(s,t)$ is singular if and only if $s = 0$.*
$(\mathrm{B})_i$ *A fiber $(\Psi|_{\mathcal{H}'_i})^{-1}(s,t)$ is singular if and only if $s = 0$.*

Proof. By construction, the deformations $\mathcal{H}'_e$, $\mathcal{H}_i$ and $\mathcal{H}'_i$ $(i = e+1, e+2, \ldots, \lambda)$ are respectively trivial deformations of H'_e, H_i and H'_i $(i = e+1, e+2, \ldots, \lambda)$. Consequently, the claims $(\mathrm{B})_e$, $(\mathrm{A})_i$, and $(\mathrm{B})_i$ for $i = e+1, e+2, \ldots, \lambda$ are valid. Note that a subbranch Y of non-proportional type B_l is wild, because lY is wild by Proposition 9.3.2, p165. Thus we may apply Corollary 7.2.5 (i), p128 to see that $(\mathrm{A})_i$, and $(\mathrm{B})_i$ for $i = 1, 2, \ldots, e-1$ are also valid.

To show the remaining claim $(\mathrm{A})_e$, we shall rewrite

$$\mathcal{H}_e : \ w^{m_{e-1}-ln_{e-1}}(w^{m_{e-1}}\eta + t^k f_e)^l - s = 0.$$

By the condition $m_e = l$ and $n_e = 1$ of type B_l, we have $m_e n_{e-1} - m_{e-1} n_e = ln_{e-1} - m_{e-1} < 0$, where the inequality is from Proposition 9.3.2 (2), p165 for non-proportional type B_l. Thus we may apply Simplification Lemma (Lemma 4.1.1); after some coordinate change, we may assume $f_e \equiv 1$ and then setting $m := m_{e-1}$ and $n := n_{e-1}$, we have

$$\mathcal{H}_e : \ w^{m-ln}(w^n\eta + t^k)^l - s = 0.$$

We now show the claim $(\mathrm{A})_e$: "$(\Psi|_{\mathcal{H}_e})^{-1}(s,t)$ is singular $\Longleftrightarrow s = 0$".

$\Longleftarrow$: Trivial, because $X_{0,t}$ is singular.
$\Longrightarrow$: It is enough to show that for (s,t) where $s \neq 0$, the fiber $(\Psi|_{\mathcal{H}_e})^{-1}(s,t)$ is smooth. We prove this by contradiction. Suppose that $(\Psi|_{\mathcal{H}_e})^{-1}(s,t)$, where $s \neq 0$, has a singularity (w,η). Setting $F = w^{m-ln}(w^n\eta + t^k)^l$ where $m = m_e$ and $n = n_e$, we note (see (6.4.1), p109)

$$(w,\eta) \in \mathcal{H}_e \text{ is a singularity } \Longleftrightarrow \frac{\partial \log F(w,\eta)}{\partial w} = \frac{\partial \log F(w,\eta)}{\partial \eta} = 0.$$

Since $\log F = (m - ln)\log w + l\log(w^n\eta + t^k)$, we obtain

$$\frac{\partial \log F}{\partial w} = \frac{m-ln}{w} + \frac{ln\, w^{n-1}\eta}{w^n\eta + t^k} = 0, \qquad \frac{\partial \log F}{\partial \eta} = \frac{lw^n}{w^n\eta + t^k} = 0,$$

and hence

$$(1)\ \ m\, w^n\eta + (m-ln)t^k = 0, \qquad (2)\ \ lw^n = 0.$$

From (2), we have $w = 0$, and substituting this into (1), we obtain $(m - ln)$ $t^k = 0$. Since

$$m - ln = m_{e-1} - ln_{e-1} > 0 \qquad \text{(Proposition 9.3.2)},$$

we deduce $t = 0$. Therefore if $(\Psi|_{\mathcal{H}_e})^{-1}(s,t)$ is singular, then $t = 0$. Next we investigate the values of s such that $(\Psi|_{\mathcal{H}_e})^{-1}(s,0)$ is singular. Since $(\Psi|_{\mathcal{H}_e})^{-1}(s,0)$ is defined by $w^{m_{e-1}}\eta^{m_e} - s = 0$, a fiber $(\Psi|_{\mathcal{H}_e})^{-1}(s,0)$ is singular precisely when $s = 0$. But we supposed $s \neq 0$, and so this case is excluded. Thus $(\Psi|_{\mathcal{H}_e})^{-1}(s,t)$ is smooth for $s \neq 0$. This completes the proof of our claim. □

11

Construction of Deformations of Type C_l

In this chapter, we shall construct complete propagations of the deformation atlases for subbranches of type C_l. We point out that when $l \geq 2$, a complete propagation is *not* unique. cf. Remark 10.1.4, p179; for a subbranch Y of type AB_l, i.e. both of type A_l and B_l, the deformation atlas $DA_{e-1}(lY, k)$ admits two different complete propagations.

11.1 Waving polynomials

We first note the following.

Lemma 11.1.1 *Let $\alpha, \beta \in \mathbb{C}$. Then a map $g : z = 1/w,\ \zeta = w^2\eta + t\alpha w$ transforms a polynomial $w\eta + t\beta$ to $z\zeta + t(\beta - \alpha)$.*

Proof. Since $w\eta + t\beta = \frac{1}{w}(w^2\eta) + t\beta$, the map g transforms $w\eta + t\beta$ to

$$z\left(\zeta - t\alpha\frac{1}{z}\right) + t\beta,$$

which is equal to $z\zeta + t(\beta - \alpha)$. □

Next we introduce a special class of polynomials, which will play a central role in constructing complete deformation atlases from subbranches of type C_l. A polynomial $A(w, \eta, t)$ is called a *waving polynomial* provided that it has the form

$$A(w, \eta, t) = w^{\,u}\, P_1(w, \eta, t)^{\,e_1}\, P_2(w, \eta, t)^{\,e_2}\ \cdots\ P_n(w, \eta, t)^{\,e_n}$$

where

(i) u and $e_1, e_2, \ldots, e_n$ are positive integers,
(ii) $P_i(w, \eta, t) = \prod_{j=1}^{\ell_i}(w\eta + t\beta_j^{(i)}),\ (\beta_j^{(i)} \in \mathbb{C})$ is a polynomial such that $\beta_j^{(i)}$ $(j = 1, 2, \ldots, \ell_i,\ i = 1, 2, \ldots, n)$ are all distinct.

In the above definition, the reader may instead prefer to assume that $e_1, e_2, \ldots, e_n$ are distinct; if they are not distinct, say, $e_{n-1} = e_n$, then setting $P'_{n-1} := P_{n-1}P_n$, we may write $A(w, \eta, t) = w^u\, P_1(w, \eta, t)^{e_1}\, P_2(w, \eta, t)^{e_2} \cdots P'_{n-1}(w, \eta, t)^{e_{n-1}}$. However, the definition as it is stated turns out to be a more convenient formulation when we apply it in the proof of lemmas below.

For a waving polynomial $A(w, \eta, t)$, we say that $A(\zeta, z, t)$ is an *interchanged waving polynomial*. Later, waving polynomials will be used to construct such deformations which look like "waving" (see Figure 12.3.1, p221).

Remark 11.1.2 When $u = 1$, $n = 1$ and $e_1 = 1$, a waving polynomial $A = wP_1(w, \eta, t)$ is nothing but a descending polynomial we introduced in §8.1, p144.

The following property of waving polynomials is very important.

Lemma 11.1.3 *Let $A(w, \eta, t) = w^u\, P_1^{e_1}\, P_2^{e_2} \cdots P_n^{e_n}$ be a waving polynomial where $P_i = \prod_{j=1}^{\ell_i}(w\eta + t\beta_j^{(i)})$. Then the following holds:*

(i) *There exists a map g of the form $z = 1/w$, $\zeta = w^2\eta + t\alpha$, $(\alpha \in \mathbb{C})$ such that it transforms $A(w, \eta, t)$ to some polynomial if and only if $e_k - u \geq 0$ for some k $(1 \leq k \leq n)$.*

(ii) *In* (i), *$\alpha = \beta_{j_\circ}^{(k)}$ for some $j_\circ$, and $A(w, \eta, t)$ is transformed to an interchanged waving polynomial*

$$A'(\zeta, z, t) = \zeta^u\, {P'_1}^{e'_1}\, {P'_2}^{e'_2} \cdots {P'_{n+1}}^{e'_{n+1}}, \tag{11.1.1}$$

where $$e'_i := \begin{cases} e_i & i = 1, 2, \ldots, n \\ e_k - u & i = n+1, \end{cases}$$ *and*

$$P'_i := \begin{cases} \prod_{j=1}^{\ell_i}[z\zeta + t(\beta_j^{(i)} - \alpha)] & i = 1, 2, \ldots, k-1, k+1, \ldots, n \\ \dfrac{\prod_{j=1}^{\ell_k}[z\zeta + t(\beta_j^{(k)} - \alpha)]}{z\zeta} & i = k \\ z\zeta & i = n+1. \end{cases}$$

(Actually $P'_k = \dfrac{\prod_{j=1}^{\ell_k}[z\zeta + t(\beta_j^{(k)} - \alpha)]}{z\zeta}$ is a polynomial, because $\beta_{j_\circ}^{(k)} - a = 0$ for some $j_\circ$, and so the numerator factorizes as $\prod_{j=1,\ j\neq j_\circ}^{\ell_k}[z\zeta + t(\beta_j^{(k)} - \alpha)] \cdot (z\zeta)$. Therefore $P'_k = \prod_{j=1, j\neq j_\circ}^{\ell_k}[z\zeta + t(\beta_j^{(k)} - \alpha)]$.)

Remark 11.1.4 If $\ell_k = 1$ or $e_k = u$, then the factor ${P'_k}^{e'_k}$ or ${P'_{n+1}}^{e'_{n+1}}$ of the interchanged waving polynomial $A'(\zeta, z, t)$ in (11.1.1) is respectively vacuous.

Proof. By Lemma 11.1.1, a map g of the from $z = 1/w$, $\zeta = w^2\eta + t\alpha w$ transforms A to

$$A' := \frac{1}{z^u}\, Q_1^{e_1}\, Q_2^{e_2} \cdots Q_n^{e_n},$$

where we set $Q_i := \prod_{j=1}^{\ell_i}[z\zeta + t(\beta_j^{(i)} - \alpha)]$. Note that $\beta_j^{(i)}$ are all distinct by the definition of waving polynomials, and accordingly $\beta_j^{(i)} - \alpha$ are also all distinct. Therefore

$$\begin{aligned} A' \text{ is a polynomial } &\iff z^u \text{ divides } Q_k^{e_k} \text{ for some } k \ (1 \le k \le n) \\ &\iff e_k \ge u \text{ and } \beta_{j_\circ}^{(k)} - \alpha = 0 \text{ for some } j_\circ. \end{aligned}$$

If this is the case, we may write $Q_k = \prod_{j=1,\ j\neq j_\circ}^{\ell_k} [z\zeta + t(\beta_j^{(k)} - \alpha)] \cdot (z\zeta)$, and consider polynomials Q_i' $(i = 1, 2, \ldots, n)$ defined by

$$Q_i' := \begin{cases} \dfrac{Q_k}{z\zeta} = \prod_{j=1,\ j\neq j_\circ}^{\ell_k} [z\zeta + t(\beta_j^{(k)} - \alpha)] & i = k \\ Q_i & i = 1, 2, \ldots, k-1, \\ & \quad k+1, \ldots, n. \end{cases}$$

Then we have

$$\begin{aligned} A' &= \frac{1}{z^u}\, Q_1^{e_1}\, Q_2^{e_2} \, \cdots \, Q_n^{e_n} \\ &= \frac{1}{z^u}\, {Q_1'}^{e_1}\, {Q_2'}^{e_2} \, \cdots \, {Q_{k-1}'}^{e_{k-1}} \left[Q_k' \cdot (z\zeta) \right]^{e_k} {Q_{k+1}'}^{e_{k+1}} \, \cdots \, {Q_n'}^{e_n} \\ &= \frac{1}{z^u}\, (z\zeta)^{e_k}\, {Q_1'}^{e_1}\, {Q_2'}^{e_2} \, \cdots \, {Q_n'}^{e_n} \\ &= \zeta^u\, (z\zeta)^{e_k - u}\, {Q_1'}^{e_1}\, {Q_2'}^{e_2} \, \cdots \, {Q_n'}^{e_n}. \end{aligned}$$

The last expression is a polynomial precisely when $e_k - u \ge 0$, and if this is the case, rewriting

$$P_i' := \begin{cases} Q_i' & i = 1, 2, \ldots, n \\ z\zeta & i = n+1, \end{cases} \qquad e_i' := \begin{cases} e_i & i = 1, 2, \ldots, n \\ e_k - u & i = n+1, \end{cases}$$

we have $A' = \zeta^u\, {P_1'}^{e_1'}\, {P_2'}^{e_2'} \, \cdots \, {P_{n+1}'}^{e_{n+1}'}$, completing the proof of the assertion. □

Let $A(w, \eta, t) = w^u\, P_1^{e_1}\, P_2^{e_2} \, \cdots \, P_n^{e_n}$ be a waving polynomial. Suppose that $e_k - u \ge 0$ for some k $(1 \le k \le n)$. Then by Lemma 11.1.3 (1), there exist (i) a map g of the form $z = 1/w,\ \zeta = w^2\eta + t\alpha w$ $(\alpha \in \mathbb{C})$ and (ii) an interchanged waving polynomial A' such that g transforms A to A', and A' is of the form

$$A'(\zeta, z, t) = \zeta^u\, {P_1'}^{e_1'}\, {P_2'}^{e_2'} \, \cdots \, {P_{n+1}'}^{e_{n+1}'},$$

where

$$e_i' := \begin{cases} e_i & i = 1, 2, \ldots, n \\ e_k - u & i = n+1, \end{cases} \tag{11.1.2}$$

and

$$P'_i := \begin{cases} \prod_{j=1}^{\ell_i}[z\zeta + t(\beta_j^{(i)} - \alpha)] & i = 1, 2, \ldots, k-1, k+1, \ldots, n \\ \dfrac{\prod_{j=1}^{\ell_k}[z\zeta + t(\beta_j^{(k)} - \alpha)]}{z\zeta} & i = k \\ z\zeta & i = n+1. \end{cases} \tag{11.1.3}$$

In this situation, we say that a *compression* occurs at the k-th factor $P_k^{e_k}$ of A, producing A'; notice that the number of the factors in P'_k is $\ell_k - u$, 'compressed' by u from the number ℓ_k of the factors in P_k.

For a waving polynomial $A(w, \eta, t) = w^u\, P_1^{e_1}\, P_2^{e_2}\, \cdots\, P_n^{e_n}$ where

$$P_i = \prod_{j=1}^{\ell_i}(w\eta + t\beta_j^{(i)}),$$

we define the *length* $\ell(A)$ of A by

$$\ell(A) := e_1\ell_1 + e_2\ell_2 + \cdots + e_n\ell_n. \tag{11.1.4}$$

For the interchanged waving polynomial $A(\zeta, z, t)$, this integer is also called the *length* of $A(\zeta, z, t)$.

Lemma 11.1.5 *Let $A(w, \eta, t) = w^u\, P_1^{e_1}\, P_2^{e_2}\, \cdots\, P_n^{e_n}$ be a waving polynomial, and let $A' = \zeta^u\, {P'_1}^{e'_1}\, {P'_2}^{e'_2}\, \cdots\, {P'_{n+1}}^{e'_{n+1}}$ be the interchanged waving polynomial in* Lemma 11.1.3*. Then $\ell(A') = \ell(A) - u$.*

Proof. By (11.1.2) and (11.1.3),

$$(e'_i,\, \ell'_i) = \begin{cases} (e_i,\, \ell_i) & i = 1, 2, \ldots, k-1, k+1, \ldots, n \\ (e_k,\, \ell_k - 1) & i = k \\ (e_k - u, 1) & i = n+1, \end{cases}$$

and hence we have

$$\begin{aligned} \ell(A') &:= e'_1\ell'_1 + e'_2\ell'_2 + \cdots + e'_{n+1}\ell'_{n+1} \\ &= \Big(\sum_{\substack{1 \le i \le n \\ i \ne k}} e_i\ell_i\Big) + e_k(\ell_k - 1) + (e_k - u) \\ &= (e_1\ell_1 + e_2\ell_2 + \cdots + e_n\ell_n) - u \\ &= \ell(A) - u. \end{aligned}$$

□

11.2 Waving sequences

For the application to the construction of complete deformation atlases, we are interested in particular waving polynomials which are of the form

$$A(w,\eta,t) = w^{\,u}\, P_1(w,\eta,t)^{\,l-a_1 u}\, P_2(w,\eta,t)^{\,l-a_2 u}\,\cdots\, P_n(w,\eta,t)^{\,l-a_n u}\, Q(w,\eta,t)^{\,b},$$

where

(1) l and u are positive integers and $a_1, a_2, \ldots, a_n$ are nonnegative integers such that $l - a_i u \ge 0$,
(2) b is a nonnegative integer, and
(3) Q is a polynomial of the form $w\eta + t\gamma$ for some $\gamma \in \mathbb{C}$.

Then from $A(w,\eta,t)$, we will construct a certain sequence of waving polynomials. This sequence will play a prominent role for constructing some complete deformation atlases. To avoid complicated notation and to clarify the idea, we consider three cases separately: Case I ($b = 0$), Case II ($b \ge 1$ and $u > b$), and Case III ($b \ge 1$ and $u \le b$).

Case I: $b = 0$

Proposition 11.2.1 *Let l and u be positive integers such that u divides l. Let $A = w^{\,u}\, P_1^{\,l-a_1 u}\, P_2^{\,l-a_2 u}\,\cdots\, P_n^{\,l-a_n u}$ be a waving polynomial where a_j ($j = 1,2,\ldots,n$) is a nonnegative integer satisfying $l - a_j u > 0$. Then there exist* (1) *a sequence of waving polynomials $A_1 = A, A_2, \ldots, A_f$ and* (2) *a sequence of maps $g_1, g_2, \ldots, g_{f-1}$ of the form $g_i : z = 1/w,\ \zeta = w^2\eta + t\alpha_i w$ ($\alpha_i \in \mathbb{C}$) such that*

(i) *g_i transforms $A_i(w,\eta,t)$ to $A_{i+1}(\zeta, z, t)$,*
(ii) *$A_f(w,\eta,t) = w^u$, and*
(iii) *$f = \dfrac{\ell(A)}{u} + 1$ where $\ell(A)$ is the length of A.*

Remark 11.2.2 In (iii), $\dfrac{\ell(A)}{u} + 1$ is an integer. In fact, set $e_j := l - a_i u$, and then

$$\frac{\ell(A)}{u} = \frac{e_1\ell_1 + e_2\ell_2 + \cdots + e_n\ell_n}{u}.$$

Since u divides l, u also divides e_j ($= l - a_j u$) for $j = 1,2,\ldots,n$. Consequently, u divides $e_1\ell_1 + e_2\ell_2 + \cdots + e_n\ell_n$.

Proof. We shall construct the sequence $A_1, A_2, \ldots, A_f$ inductively. Suppose that we have constructed an i-th waving polynomial of the form

$$A_i = w^{\,u}\, P_{i,1}^{\,l-a_{i,1} u}\, P_{i,2}^{\,l-a_{i,2} u}\,\cdots\, P_{i,n_i}^{\,l-a_{i,n_i} u} \tag{11.2.1}$$

where $a_{i,j}$ is a nonnegative integer satisfying $l - a_{i,j}u > 0$. Then

Claim 11.2.3 *Set $e_{i,j} := l - a_{i,j}u\,(>0)$, and then $e_{i,j} - u \ge 0$.*

To see this, noting that u divides l, we express $l = Nu$. Accordingly we rewrite the inequality $l - a_{i,j}u > 0$ as $(N - a_{i,j})u > 0$; hence $N > a_{i,j}$, that is, $N \geq a_{i,j} + 1$. Then our claim is confirmed as follows:

$$e_{i,j} - u = l - (a_{i,j} + 1)u = Nu - (a_{i,j} + 1)u > 0 \qquad \text{by } N \geq a_{i,j} + 1.$$

By the above claim, the waving polynomial (11.2.1) satisfies the assumption of Lemma 11.1.3, and thus there exists a map of the form $g_i: \ z = 1/w, \ \zeta = w^2 + t\alpha_i w$ which transforms $A_i(w, \eta, t)$ to an interchanged waving polynomial $A_i'(\zeta, z, t)$. Set $A_{i+1}(w, \eta, t) := A_i'(w, \eta, t)$; then A_{i+1} is of the form

$$A_{i+1} = w^u P_{i+1,1}^{l - a_{i+1,1} u} P_{i+1,2}^{l - a_{i+1,2} u} \cdots P_{i+1,n_{i+1}}^{l - a_{i+1,n_{i+1}} u}.$$

(The exponents of A_{i+1} are necessarily of the above form, because by Lemma 11.1.3, under the transformation of g_i, any exponent of A_i either (1) remains the same, i.e. $l - a_{i,j}u$ or (2) is subtracted by u, i.e. $l - (a_{i,j} + 1)u$.)

We may repeat the above inductive process until $i = f$ such that $A_f = w^u$. Finally we compute f. Notice that $\ell(A_f) = \ell(A_1) - (f-1)u$. Indeed, recursive application of Lemma 11.1.5 yields a formula $\ell(A_i) = \ell(A_1) - (i-1)u$ (for example, $\ell(A_3) = \ell(A_2) - u = \ell(A_1) - 2u$, and $\ell(A_4) = \ell(A_3) - u = (\ell(A_1) - 2u) - u = \ell(A_1) - 3u$). In particular, $\ell(A_f) = \ell(A_1) - (f-1)u$. On the other hand, $\ell(A_f) = 0$ because $A_f = w^u$. Thus $0 = \ell(A_1) - (f-1)u$, from which we deduce $f = \dfrac{\ell(A_1)}{u} + 1$. □

Remark 11.2.4 If we omit the condition that u divides l, then we finally reach to a waving polynomial, which is different from w^u. In fact, if u does not divide l, there exists a unique integer N satisfying $l - Nu > 0$ and $l - (N+1)u < 0$. Then the above inductive process terminates at a waving polynomial A_i of the form

$$w^u P_{i,1}^{l-Nu} P_{i,2}^{l-Nu} \cdots P_{i,n_i}^{l-Nu}. \tag{11.2.2}$$

Since $l - (N+1)u < 0$, it follows from Lemma 11.1.3, there exists no map of the form $g_i: \ z = 1/w, \ \zeta = w^2\eta + t\alpha_i w$ which transforms (11.2.2) to some polynomial.

Case II: $b \geq 1$ **and** $u > b$

Next we consider the second case and show the following result.

Proposition 11.2.5 *Let b, l, and u be positive integers such that u divides l and $u > b$. Let $A = w^u P_1^{l-a_1 u} P_2^{l-a_2 u} \cdots P_n^{l-a_n u} Q^b$ be a waving polynomial where $Q = w\eta + t\gamma$ ($\gamma \in \mathbb{C}$) and a_j is a nonnegative integer satisfying $l - a_j u > 0$. Then there exist* (1) *a sequence of waving polynomials $A_1 = A, A_2, \ldots, A_f$ and* (2) *a sequence of maps $g_1, g_2, \ldots, g_{f-1}$ of the form $g_i: z = 1/w, \ \zeta = w^2\eta + t\alpha_i w$ such that*

(i) *g_i transforms $A_i(w, \eta, t)$ to $A_{i+1}(\zeta, z, t)$,*

(ii) *A_i is of the form*

$$A_i = w^{\,u}\,P_{i,\,1}^{\,l\,-\,a_{i,1}\,u}\,P_{i,\,2}^{\,l\,-\,a_{i,2}\,u}\,\cdots\,P_{i,\,n_i}^{\,l\,-\,a_{i,n_i}\,u}\,Q_i^b$$

where $Q_i = w\eta + t\gamma_i$ ($\gamma_i \in \mathbb{C}$) and $A_f(w,\eta,t) = w^u Q_f^b = w^u(w\eta + t\gamma_f)^b$, and

(iii)

$$f = \frac{\ell(P_1^{\,l\,-\,a_1\,u}\,P_2^{\,l\,-\,a_2\,u}\,\cdots\,P_n^{\,l\,-\,a_n\,u})}{u} + 1 = \frac{\ell(A) - b}{u} + 1.$$

(Recall that $\ell(A)$ stands for the length of A. It is easy to check that f is an integer. See Remark 11.2.2.)

Proof. We first note

Claim 11.2.6 *A map $g:\ z = 1/w,\ \zeta = w^2\eta + t\alpha w$ transforms a polynomial $(w\eta + t\gamma)^b$ to $\big(z\zeta + t(\gamma - \alpha)\big)^b$.*

The proof of this claim is the same as that of Lemma 11.1.1.

Now we construct the desired sequences $A_1, A_2, \ldots, A_f$ and $g_1, g_2, \ldots, g_{f-1}$ inductively. Suppose that we have constructed an i-th waving polynomial of the form

$$A_i = w^{\,u}P_{i,\,1}^{\,l\,-\,a_{i,1}\,u}\,P_{i,\,2}^{\,l\,-\,a_{i,2}\,u}\,\cdots\,P_{i,\,n_i}^{\,l\,-\,a_{i,n_i}\,u}\,Q_i^b$$

where $Q_i = w\eta + t\gamma_i$ ($\gamma_i \in \mathbb{C}$) and $a_{i,j}$ is a nonnegative integer satisfying $l - a_{i,j}u > 0$. Then by Lemma 11.1.3, there exists a map $g_i:\ z = 1/w,\ \zeta = w^2 + t\alpha_i w$ ($\alpha_i \in \mathbb{C}$) which transforms A_i to an interchanged waving polynomial A_i' such that a compression occurs at some k ($1 \le k \le n_i$); since $b - u < 0$ by assumption, this compression can *not* occur at the factor Q_i^b. Set $A_{i+1}(w,\eta,t) := A_i'(w,\eta,t)$; then A_{i+1} is of the form

$$A_{i+1} = w^{\,u}\,P_{i+1,\,1}^{\,l\,-\,a_{i+1,1}\,u}\,P_{i+1,\,2}^{\,l\,-\,a_{i+1,2}\,u}\,\cdots\,P_{i+1,\,n_{i+1}}^{\,l\,-\,a_{i+1,n_{i+1}}\,u}\,Q_{i+1}^b$$

where by Claim 11.2.6, $Q_{i+1} = w\eta + t\gamma_{i+1}$ for some $\gamma_{i+1} \in \mathbb{C}$. We can repeat this inductive process until $i = f$ such that $n_f = 0$, that is, $A_f = w^u Q_f^b = w^u(w\eta + t\gamma_f)^b$.

Finally we show that $f = \dfrac{\ell(A) - b}{u} + 1$. As in the proof of Proposition 11.2.1,

$$\ell(A_f) = \ell(A) - (f-1)u. \tag{11.2.3}$$

Here $\ell(A_f) = b$ and

$$\begin{aligned}\ell(A) &= \ell(P_1^{\,l\,-\,a_1\,u}\,P_2^{\,l\,-\,a_2\,u}\,\cdots\,P_n^{\,l\,-\,a_n\,u}\,Q^{\,b})\\ &= \ell(P_1^{\,l\,-\,a_1\,u}\,P_2^{\,l\,-\,a_2\,u}\,\cdots\,P_n^{\,l\,-\,a_n\,u}) + b.\end{aligned}$$

Substituting them into (11.2.3), we have

$$b = \ell(P_1^{\,l\,-\,a_1\,u}\,P_2^{\,l\,-\,a_2\,u}\,\cdots\,P_n^{\,l\,-\,a_n\,u}) + b - (f-1)u.$$

Thus

$$f = \frac{\ell(P_1^{\,l\,-\,a_1\,u}\,P_2^{\,l\,-\,a_2\,u}\,\cdots\,P_n^{\,l\,-\,a_n\,u})}{u} + 1 = \frac{\ell(A) - b}{u} + 1.$$

This completes the proof of the assertion. □

Case III: $b \geq 1$ **and** $u \leq b$

For the remaining case, the following statement holds.

Proposition 11.2.7 *Let b, l, and u be positive integers such that u divides l and $u \leq b$. Let $A = w^u P_1^{l-a_1 u} P_2^{l-a_2 u} \cdots P_n^{l-a_n u} Q^b$ be a waving polynomial where $Q = w\eta + t\gamma$ ($\gamma \in \mathbb{C}$) and a_j is a nonnegative integer satisfying $l - a_j u > 0$. Then there exist (1) a sequence of waving polynomials $A_1 = A, A_2, \ldots, A_f$ and (2) a sequence of maps $g_1, g_2, \ldots, g_{f-1}$ of the form $g_i : z = 1/w,\ \zeta = w^2\eta + t\alpha_i w$ such that*

(i) *g_i transforms $A_i(w, \eta, t)$ to $A_{i+1}(\zeta, z, t)$,*
(ii) *A_i is of the form*

$$A_i = w^u P_{i,1}^{l-a_{i,1} u} P_{i,2}^{l-a_{i,2} u} \cdots P_{i,n_i}^{l-a_{i,n_i} u} Q_i^{b-v_i u},$$

where $Q_i = w\eta + t\gamma_i$ ($\gamma_i \in \mathbb{C}$) and $A_f(w, \eta, t) = w^u Q_f^{b-vu} = w^u(w\eta + t\gamma_f)^{b-vu}$ where (considering the division of b by u), $v = v_f$ is the positive integer such that $b - vu \geq 0$ and $b - (v+1)u < 0$, and

(iii)

$$f = \frac{\ell(P_1^{l-a_1 u} P_2^{l-a_2 u} \cdots P_n^{l-a_n u})}{u} + 1 + v = \frac{\ell(A) - b}{u} + 1 + v.$$

(It is easy to see that f is an integer by an argument similar to Remark 11.2.2.)

Remark 11.2.8 In (ii), when u divides b (so $b = vu$), we have $A_f(w, \eta, t) = w^u$.

Proof. The proof is essentially the same as Case II. (The only difference is that we may compress A_i at the factor Q_i in the process of the inductive construction of waving polynomials.) Namely, by induction, we can construct an i-th waving polynomial of the form

$$A_i = w^u P_{i,1}^{l-a_{i,1} u} P_{i,2}^{l-a_{i,2} u} \cdots P_{i,n_i}^{l-a_{i,n_i} u} Q_i^{b-v_i u}$$

where $Q_i = w\eta + t\gamma_i$ ($\gamma_i \in \mathbb{C}$), and $a_{i,j}$ and v_i are nonnegative integers such that $l - a_{i,j}u > 0$ and $b - v_i u > 0$. Repeat the inductive process as in Case II, which in this case, terminates at $i = f$ such that A_f is of the form

$$A_f = w^u Q_f^{b-vu} = w^u(w\eta + t\gamma_f)^{b-vu},$$

where v is the positive integer satisfying $b - vu \geq 0$ and $b-(v+1)u < 0$. Finally by the same argument as in Case II, we verify that $f = \dfrac{\ell(A) - b}{u} + 1 + v$. As in the proof of Proposition 11.2.1, we have

$$\ell(A_f) = \ell(A) - (f-1)u. \tag{11.2.4}$$

Here $\ell(A_f) = b - vu$ and

$$\begin{aligned}\ell(A) &= \ell(P_1^{l-a_1 u} P_2^{l-a_2 u} \cdots P_n^{l-a_n u} Q^b) \\ &= \ell(P_1^{l-a_1 u} P_2^{l-a_2 u} \cdots P_n^{l-a_n u}) + b.\end{aligned}$$

Substituting them into (11.2.4), we have

$$b - vu = \ell(P_1^{l-a_1 u} P_2^{l-a_2 u} \cdots P_n^{l-a_n u}) + b - (f-1)u.$$

Thus

$$f = \frac{\ell(P_1^{l-a_1 u} P_2^{l-a_2 u} \cdots P_n^{l-a_n u})}{u} + 1 + v = \frac{\ell(A) - b}{u} + 1 + v.$$

This completes the proof. □

Waving sequences

In Propositions 11.2.1, 11.2.5 and 11.2.7, given a waving polynomial A, we have constructed a sequence of waving polynomials $A_1 = A, A_2, \ldots, A_f$ together with a sequence of maps $g_1, g_2, \ldots, g_{f-1}$. It is called a *waving sequence* associated with A. (In general, a waving sequence is not uniquely determined by A.) As we will see below, waving sequences play a prominent role in constructing complete deformation atlases.

11.3 Deformations of type C_l

In this section we shall construct a complete propagation of the deformation atlas $DA_{e-1}(lY, k)$ for Y of type C_l (we will soon recall terminologies). In contrast to the constructions for types A_l and B_l, the construction for type C_l is quite involved, and moreover it turns out that when $l \geq 2$, in most cases, a complete propagation of $DA_{e-1}(lY, k)$ is *not* unique. (cf. Remark 10.1.4, p179; for a subbranch Y of type AB_l, i.e. both of type A_l and B_l, the deformation atlas $DA_{e-1}(lY, k)$ admits two different complete propagations.) Among all complete propagations, there are two distinguished ones in which cases the deformation from X to $X_{0,t}$ has an interesting 'periodicity', which we will explicitly describe in the next chapter.

Now let $X = m_0\Delta_0 + m_1\Theta_1 + \cdots + m_\lambda\Theta_\lambda$ be a branch, and we set

$$r_i := \frac{m_{i-1} + m_{i+1}}{m_i} \quad (i = 1, 2, \ldots, \lambda - 1), \qquad r_\lambda := \frac{m_{\lambda-1}}{m_\lambda}.$$

Assume that $Y = n_0\Delta_0 + n_1\Theta_1 + \cdots + n_e\Theta_e$ is a subbranch of X and l is a positive integer such that $lY \leq X$, i.e. $ln_i \leq m_i$ for $i = 0, 1, \ldots, e$. We recall a deformation atlas $DA_{e-1}(lY, k)$; let $f(z)$ be a non-vanishing holomorphic function on a domain $\{z \in \mathbb{C} : |z| < \varepsilon\}$. Define a sequence of integers p_i $(i = 0, 1, \ldots, \lambda+1)$ inductively by

$$\begin{cases} p_0 = 0, \quad p_1 = 1 \quad \text{and} \\ p_{i+1} = r_i p_i - p_{i-1} \quad \text{for } i = 1,\, 2,\, \ldots,\, \lambda. \end{cases}$$

Then $p_{\lambda+1} > p_\lambda > \cdots > p_1 > p_0 = 0$ (6.2.4), p105 and set $f_i = f(w^{p_{i-1}}\eta^{p_i})$ and $\widehat{f_i} = f(z^{p_{i+1}}\zeta^{p_i})$. We define a deformation atlas $DA_{e-1}(lY, k)$: for $i = 1, 2, \ldots, e-1$,

$$\begin{cases} \mathcal{H}_i: & w^{m_{i-1}-ln_{i-1}}\eta^{m_i-ln_i}(w^{n_{i-1}}\eta^{n_i} + t^k f_i)^l - s = 0 \\ \mathcal{H}_i': & z^{m_{i+1}-ln_{i+1}}\zeta^{m_i-ln_i}(z^{n_{i+1}}\zeta^{n_i} + t^k \widehat{f_i})^l - s = 0 \\ g_i: & \text{the transition function } z = 1/w,\ \zeta = w^{r_i}\eta \text{ of } N_i. \end{cases}$$

Next recall that a subbranch Y of X is of type C_l (Definition 9.1.1) provided that $lY \le X$, $n_e = r_e n_e - n_{e-1}$, and u divides l where

$$u = (m_{e-1} - ln_{e-1}) - (r_e - 1)(m_e - ln_e).$$

Theorem 11.3.1 *Let Y be a subbranch of type C_l, and let $DA_{e-1}(lY, k)$ be the deformation atlas associated with lY. If the positive integer k is divisible by n_e, then $DA_{e-1}(lY, k)$ admits a complete propagation.*

This theorem completes the proof of Theorem 10.0.15, p177, which insists: *If a subbranch Y is of type A_l, B_l, or C_l such that if Y is of type C_l the positive integer k is divisible by n_e, then $DA_{e-1}(lY, k)$ admits a complete propagation.* (We have already proved Theorem 10.0.15 for types A_l and B_l.)

Henceforth, we only consider the case $f_e \equiv 1$; since Y of type C_l is wild by Lemma 9.4.6, p169, after some coordinate change, we may always assume $f_e \equiv 1$ by Proposition 8.1, p143.

Step 1 (e-th propagation)

In order to construct an e-th propagation of $DA_{e-1}(lY, k)$, we need the following lemma.

Lemma 11.3.2 *Let l and r be positive integers, and let a, b, c, d be nonnegative integers such that $d = dr - c$ and $l - (a + b - rb) \ge 0$. Take $\alpha \in \mathbb{C}$ satisfying $\alpha^d + 1 = 0$. Then a map*

$$g:\ z = \frac{1}{w}, \quad \zeta = w^r\eta - t\alpha w$$

transforms a polynomial $P = w^a\,\eta^b\,(w^c\eta^d + t^d)^l$ to a polynomial

$$P'(z,\zeta,t) = z^{l-(a+b-rb)}\,(z\zeta + t\alpha)^b \left(z^{d-1}\zeta^d + \sum_{i=1}^{d-1} {}_dC_i\, t^i\, \alpha^i\, z^{d-i-1}\,\zeta^{d-i} \right)^l.$$

Proof. Since $P = w^a \eta^b (w^c\eta^d + t^d)^l = w^{a-br} (w^r\eta)^b \left[w^{c-dr}(w^r\eta)^d + t^d\right]^l$, the map $g:\ z = 1/w,\ \zeta = w^r\eta - t\alpha w$ transforms P to

$$z^{br-a}\left(\zeta + t\alpha\frac{1}{z}\right)^b \left[z^{dr-c}\left(\zeta + t\alpha\frac{1}{z}\right)^d + t^d\right]^l. \tag{11.3.1}$$

The inside the brackets is

$$\begin{aligned} z^{dr-c}\left(\zeta + t\alpha\frac{1}{z}\right)^d + t^d &= z^d\left(\zeta + t\alpha\frac{1}{z}\right)^d + t^d \qquad \text{by } \ d = dr - c \\ &= (z\zeta + t\alpha)^d + t^d \\ &= (z\zeta)^d + {}_d\mathrm{C}_1(z\zeta)^{d-1}(t\alpha) + \cdots \\ &\qquad + {}_d\mathrm{C}_{d-1}(z\zeta)(t\alpha)^{d-1} + (t\alpha)^d + t^d \\ &= (z\zeta)^d + {}_d\mathrm{C}_1(z\zeta)^{d-1}(t\alpha) + \cdots + {}_d\mathrm{C}_{d-1}(z\zeta)(t\alpha)^{d-1}, \end{aligned}$$

where in the last equality we used $\alpha^d + 1 = 0$. Thus

$$z^{dr-c}\left(\zeta + t\alpha\frac{1}{z}\right)^d + t^d = z\left(\sum_{i=0}^{d-1} {}_d\mathrm{C}_i\, t^i\, \alpha^i\, z^{d-i-1}\, \zeta^{d-i}\right).$$

Using this equation, we rewrite (11.3.1) as follows:

$$\begin{aligned} &z^{br-a}\left(\zeta + t\alpha\frac{1}{z}\right)^b \left[z^{dr-c}\left(\zeta + t\alpha\frac{1}{z}\right)^d + t^d\right]^l \\ &= z^{br-a}\left(\zeta + t\alpha\frac{1}{z}\right)^b \left[z\left(\sum_{i=0}^{d} {}_d\mathrm{C}_i\, t^i\, \alpha^i\, z^{d-i-1}\, \zeta^{d-i}\right)\right]^l \\ &= z^{br-a+l}\left(\zeta + t\alpha\frac{1}{z}\right)^b \left[\sum_{i=0}^{d} {}_d\mathrm{C}_i\, t^i\, \alpha^i\, z^{d-i-1}\, \zeta^{d-i}\right]^l \\ &= z^{br-a+l-b}\,(z\zeta + t\alpha)^b \left[\sum_{i=0}^{d} {}_d\mathrm{C}_i\, t^i\, \alpha^i\, z^{d-i-1}\, \zeta^{d-i}\right]^l, \end{aligned}$$

where the last expression equals P' in the assertion. This completes the proof.
□

Now we return to construct a complete propagation of $DA_{e-1}(lY, k)$, where for $i = 1, 2, \dots, e-1$,

$$\begin{cases} \mathcal{H}_i: & w^{m_{i-1}-ln_{i-1}}\eta^{m_i-ln_i}(w^{n_{i-1}}\eta^{n_i} + t^k)^l - s = 0 \\ \mathcal{H}'_i: & z^{m_{i+1}-ln_{i+1}}\zeta^{m_i-ln_i}(z^{n_{i+1}}\zeta^{n_i} + t^k)^l - s = 0 \\ g_i: & \text{the transition function of } N_i, \end{cases}$$

and n_e divides k by assumption. We give the construction only for the case $k = n_e$. (For the general case, we only have to replace t by t^{k/n_e} in the argument below.) To apply Lemma 11.3.2, we set $r = r_e$ and

$$a = m_{e-1} - l n_{e-1}, \qquad b = m_e - l n_e, \qquad c = n_{e-1}, \qquad d = n_e.$$

Then $\mathcal{H}_e :\ w^a \eta^b (w^c \eta^d + t^d)^l - s = 0$. From the definition of type C_l, it follows that (1) $d = dr - c$ and (2) setting $u := a - (r-1)b$, then $l \geq u$ (because u divides l). Thus all the assumptions of Lemma 11.3.2 are fulfilled. Taking $\alpha \in \mathbb{C}$ satisfying $\alpha^d + 1 = 0$, we define an e-th propagation of $DA_{e-1}(lY, k)$ by

$$\begin{cases} \mathcal{H}_e : & w^a \eta^b (w^c \eta^d + t^d)^l - s = 0 \\ \mathcal{H}'_e : & z^{l-u} (z\zeta + t\alpha)^b \left(z^{d-1}\zeta^d + \sum_{i=1}^{d-1} {}_d\mathrm{C}_i\, t^i\, \alpha^i\, z^{d-i-1}\, \zeta^{d-i} \right)^l - s = 0 \\ g_e : & z = \dfrac{1}{w}, \quad \zeta = w^r \eta - t\alpha w. \end{cases} \tag{11.3.2}$$

Step 2 (i-th propagation where $i \geq e+1$)

We proceed to construct an $(e+1)$-st propagation with

$$\mathcal{H}_{e+1} : \quad \eta^{l-u} (w\eta + t\alpha)^b \left(w^d \eta^{d-1} + \sum_{i=1}^{d-1} {}_d\mathrm{C}_i\, t^i\, \alpha^i\, w^{d-i}\, \eta^{d-i-1} \right)^l - s = 0.$$

To avoid complicated notation, we reset the subscript $e+1$ as 1; we rewrite $\mathcal{H}_{e+1}$ as $\mathcal{H}_1$, and setting

$$A := \eta^{l-u} (w\eta + t\alpha)^b \left(w^d \eta^{d-1} + \sum_{i=1}^{d-1} {}_d\mathrm{C}_i\, t^i\, \alpha^i\, w^{d-i}\, \eta^{d-i-1} \right)^l,$$

we express $\mathcal{H}_1 :\ A(w, \eta, t) - s = 0$. Consider the factorization inside the big brackets:

$$w^d \eta^{d-1} + \sum_{i=1}^{d-1} {}_d\mathrm{C}_i\, t^i\, \alpha^i\, w^{d-i}\, \eta^{d-i-1} = w \prod_{i=1}^{d-1} (w\eta + t\beta_i).$$

We claim that

$$\beta_1, \beta_2, \ldots, \beta_{d-1} \text{ are all distinct.} \tag{11.3.3}$$

To see this, we note that the roots of a polynomial $f(X) = (X+1)^d - 1$ are all distinct. In fact, if α is a multiple root, then $f(\alpha) = f'(\alpha) = 0$, and so $(\alpha+1)^d - 1 = 0$ and $d(\alpha+1)^{d-1} = 0$. From the second equation, we have $(\alpha+1)^{d-1} = 0$. Substituting this into the first equation, we have $-1 = 0$ —

which is absurd! Thus $f(X)$ has no multiple roots. Using this fact, we can show (11.3.3) as follows. Note that

$$(w\eta + t\alpha)^d - (t\alpha)^d = w\eta \prod_{i=1}^{d-1} (w\eta + t\beta_i).$$

Divide the both sides by $(t\alpha)^d$, which yields:

$$\left(\frac{w\eta}{t\alpha} + 1\right)^d - 1 = \frac{w\eta}{t\alpha} \prod_{i=1}^{d-1} \left(\frac{w\eta}{t\alpha} + \frac{\beta_i}{\alpha}\right).$$

Setting $X = \frac{w\eta}{t\alpha}$, we then have $(X+1)^d - 1 = X \prod_{i=1}^{d-1} \left(X + \frac{\beta_i}{\alpha}\right)$. As shown above, the left hand side does not have multiple roots, and therefore $\frac{\beta_i}{\alpha}$ $(i = 1, 2, \ldots, d-1)$ are all distinct; consequently $\beta_1, \beta_2, \ldots, \beta_{d-1}$ are all distinct. Thus (11.3.3) is confirmed.

We return to the construction of an $(e+1)$-st propagation of $DA_{e-1}(lY, k)$; we set

$$P_1 := w\eta, \qquad P_2 := \prod_{j=1}^{d-1} (w\eta + t\beta_j), \qquad Q := (w\eta + t\gamma).$$

Then

$$A = w^u P_1^{l-u} P_2^l Q^b, \tag{11.3.4}$$

and so A is a waving polynomial (the 'roots' $\beta_1, \beta_2, \ldots, \beta_{d-1}$ of P_2 are all distinct by (11.3.3)). For subsequent discussion, it is convenient to divide into three cases: Case I: $b = 0$, Case II: $b \geq 1$ and $u > b$, and Case III: $b \geq 1$ and $u \leq b$.

Case I: $b = 0$

In this case $A = w^u P_1^{l-u} P_2^l$, and hence A is a waving polynomial satisfying the assumption of Proposition 11.2.1. Thus there exists a waving sequence associated with A:

$$\begin{cases} A_1 = A, A_2, \ldots, A_f = w^u \\ g_1, g_2, \ldots, g_{f-1}. \end{cases}$$

Using this sequence, we define a complete propagation of $\mathcal{H}_1$ as follows (for consistency with the subscripts of the waving polynomials A_i, we reset the subscripts of $\mathcal{H}_i$; we rewrite $\mathcal{H}_{e+i}$ as $\mathcal{H}_i$.): for $i = 1, 2, \ldots, f-1$,

$$\begin{cases} \mathcal{H}_i: & A_i(w, \eta, t) - s = 0 \\ \mathcal{H}'_i: & A_{i+1}(\zeta, z, t) - s = 0 \\ g_i: & \text{the map in the waving sequence,} \end{cases}$$

where we note that $\mathcal{H}'_{f-1}:\ \zeta^u - s = 0$, because $A_f(\zeta, z, t) = \zeta^u$.

Remark 11.3.3 Since u divides l by the definition of type C_l, we may write $l = Nu$ where N is a positive integer. We claim that $f = Nd$. Indeed, the length of A (see (11.1.4)) is

$$\ell(A) = (l-u)\cdot 1 + l\cdot(d-1) = ld - u,$$

and thus by Proposition 11.2.1 (iii), $f = \dfrac{\ell(A)}{u} + 1 = \dfrac{ld-u}{u} + 1 = \dfrac{ld}{u} = Nd.$

Case II: $b \geq 1$ **and** $u > b$

In this case $A = w^u\, P_1^{l-u}\, P_2^l\, Q^b$, and A is a waving polynomial satisfying all the assumption of Proposition 11.2.5. Thus there exists a waving sequence associated with A:

$$\begin{cases} A_1 = A, A_2, \dots, A_f = w^u\,(w\eta + t\,\gamma_f)^b \\ g_1, g_2, \dots, g_{f-1}. \end{cases}$$

Using this sequence, we construct propagations of $\mathcal{H}_1 :\ A(w,\eta,t) - s = 0$ up to an $(f-1)$-st propagation as follows: for $i = 1, 2, \dots, f-1$,

$$\begin{cases} \mathcal{H}_i : & A_i(w,\eta,t) - s = 0 \\ \mathcal{H}_i' : & A_{i+1}(\zeta, z, t) - s = 0 \\ g_i : & \text{the map in the waving sequence.} \end{cases}$$

Noting that $\mathcal{H}_f :\ w^u\,(w\eta + t\,\gamma_f)^b - s = 0$, we next define an f-th propagation by

$$\begin{cases} \mathcal{H}_f : & w^u\,(w\eta + t\,\gamma_f)^b - s = 0 \\ \mathcal{H}_f' : & z^{(r_f-1)\,b-u}\zeta^b - s = 0 \\ g_f : & z = \dfrac{1}{w}, \quad \zeta = w^{r_f}\eta + t\,\gamma_f\, w^{r_f-1}. \end{cases}$$

This is well-defined. In fact, g_f transforms $\mathcal{H}_f$ to $\mathcal{H}_f'$; since the equation of $\mathcal{H}_f$ is written as

$$w^u\left[\frac{1}{w^{r_f-1}}\,(w^{r_f}\,\eta) + t\,\gamma_f\right]^b - s,$$

the map g_f transforms $\mathcal{H}_f$ to

$$\begin{aligned} \frac{1}{z^u}\left[z^{r_f-1}\left(\zeta - t\,\gamma_f\,\frac{1}{z^{r_f-1}}\right) + t\,\gamma_f\right]^b - s &= \frac{1}{z^u}\Big[\,(z^{r_f-1}\,\zeta - t\,\gamma_f) + t\,\gamma_f\Big]^b - s \\ &= \frac{1}{z^u}\,[\,z^{r_f-1}\,\zeta\,]^b - s \\ &= z^{(r_f-1)\,b-u}\,\zeta^b - s, \end{aligned}$$

which is equal to $\mathcal{H}_f'$. (According to Table 9.1.10, $m_f = b$ where $f = Nd$ in the present case corresponds to $e + Nd$ in the table; recall that we reset the subscripts.) Notice that $\mathcal{H}'_f$ is the trivial family of H'_f, and we may 'trivially' define further propagations. Namely for $i = f+1, f+2, \ldots, \lambda$,

$$\begin{cases} \mathcal{H}_i : & w^{m_{i-1}}\eta^{m_i} - s = 0 \\ \mathcal{H}'_i : & z^{m_{i+1}}\zeta^{m_i} - s = 0 \\ g_i : & \text{the transition function of } N_i. \end{cases}$$

This completes the construction of a complete propagation for Case II.

Remark 11.3.4 Writing $l = Nu$, then we have $f = Nd$. Indeed, since

$$\ell(A) = (l-u)\cdot 1 + l\cdot(d-1) + b = ld - u + b, \quad \text{(Proposition 11.2.5 (iii))},$$

we have $f = \dfrac{\ell(A) - b}{u} + 1 = \dfrac{(ld - u + b) - b}{u} + 1 = \dfrac{ld}{u} = Nd.$

Case III: $b \geq 1$ **and** $u \leq b$

The construction is similar to Case II, and we merely give an outline. Take a waving sequence associated with $A = A_1$ in Proposition 11.2.7:

$$\begin{cases} A_1 = A, A_2, \ldots, A_f = w^u(w\eta + t\,\gamma_f)^{b-vu} \\ g_1, g_2, \ldots, g_{f-1}, \end{cases}$$

where v is the integer satisfying $l - vu \geq 0$ and $l - (v+1)u < 0$. Using this waving sequence, we define propagations of $\mathcal{H}_1 : \ A(w,\eta,t) - s = 0$ up to an $(f-1)$-st propagation: for $i = 1, 2, \ldots, f-1$,

$$\begin{cases} \mathcal{H}_i : & A_i(w,\eta,t) - s = 0 \\ \mathcal{H}'_i : & A_{i+1}(\zeta, z, t) - s = 0 \\ g_i : & \text{the map in the waving sequence.} \end{cases}$$

Noting that $\mathcal{H}_f : \ w^u(w\eta + t\gamma_f)^{b-vu} - s = 0$, we next define an f-th propagation by

$$\begin{cases} \mathcal{H}_f : & w^u\,(w\eta + t\,\gamma_f)^{\,b-vu} - s = 0 \\ \mathcal{H}'_f : & z^{\,(r_f-1)\,(b-vu)-u}\,\zeta^{\,b-vu} - s = 0 \\ g_f : & z = \dfrac{1}{w}, \quad \zeta = w^{\,r_f}\,\eta + t\,\gamma_f\,w^{\,r_f-1}. \end{cases}$$

Since $\mathcal{H}'_f$ is the trivial family of H'_f, we may trivially define further propagations. That is, for $i = f+1, f+2, \ldots, \lambda$,

$$\begin{cases} \mathcal{H}_i : & w^{m_{i-1}}\eta^{m_i} - s = 0 \\ \mathcal{H}'_i : & z^{m_{i+1}}\zeta^{m_i} - s = 0 \\ g_i : & \text{the transition function of } N_i, \end{cases}$$

where $m_{\lambda+1} = 0$ by convention. This completes the construction of a complete propagation.

Remark 11.3.5 Writing $l = Nu$ where N is a positive integer, then we have $f = Nd + v$. In fact, since

$$\ell(A) = (l - u) \cdot 1 + l \cdot (d - 1) + b = ld - u + b,$$

together with Proposition 11.2.7 (iii),

$$f = \frac{\ell(A) - b}{u} + 1 + v = \frac{(ld - u + b) - b}{u} + 1 + v = \frac{ld}{u} + v = Nd + v.$$

Propagation of deformations

By patching the complete deformation atlas constructed above, we obtain a barking family (specifically, called *of type* C_l). Barking families of type C_l are quite different from those of types A_l and B_l. Namely, a barking family of type A_l or B_l propagates trivially beyond Θ_e (see Remark 10.1.2, p179), that is, the deformation around $m_{e+1}\Theta_{e+1} + m_{e+2}\Theta_{e+2} + \cdots + m_\lambda\Theta_\lambda$ is the trivial family of the original degeneration. In contrast, a barking family of type C_l propagates non-trivially beyond Θ_e. More precisely, write $l = Nu$, and in the case $b \geq 1$ and $u \leq b$, we let v be the integer such that $b - vu \geq 0$ and $b - (v+1)u < 0$. Then the barking family of type C_l propagates non-trivially beyond Θ_e until Θ_f, where

$$f = \begin{cases} e + Nd - 1 & \text{(I) } b = 0\text{, or (II) } b \geq 1 \text{ and } u > b \\ e + Nd + v - 1 & \text{(III) } b \geq 1 \text{ and } u \leq b. \end{cases} \tag{11.3.5}$$

See Figure 12.3.1, p221, Figure 12.3.2, p223, and Figure 12.3.3, p224.

Remark 11.3.6 When $l \geq 2$, in most cases, a complete propagation of $DA_{e-1}(lY, k)$ for a subbranch Y of type C_l is *not* unique. This is the case precisely when there are at least two (nonzero) exponents among exponents $l - u$, l, and b of the waving polynomial $A = w^u P_1^{l-u} P_2^l Q^b$ in (11.3.4). More explicitly, except for the following three cases, a complete propagation of $DA_{e-1}(lY, k)$ is not unique: (1) $l = u$ and $b = 0$, (2) $l = u = b\,(\neq 0)$, and (3) $l = l - u = b\,(\neq 0)$.

11.4 Singular fibers

Let us consider a waving polynomial

$$A(w, \eta, t) = w^u\, P_1^{e_1}\, P_2^{e_2}\, \cdots\, P_n^{e_n},$$

where $P_i = \prod_{j=1}^{\ell_i}(w\eta + t\beta_j^{(i)})$. By definition, $\beta_j^{(i)} \in \mathbb{C}$ $(j = 1, 2, \ldots, \ell_i,\ i = 1, 2, \ldots, n)$ are all distinct. We shall investigate the values (s, t) such that a curve

$$C_{s,t}:\ A(w, \eta, t) - s = 0$$

has a singularity. Clearly for the case $s = 0$, regardless of the value of t, the curve $C_{0,t}$ is (1) smooth if $u = e_1 = e_2 = \cdots = e_n = 1$, and (2) singular (in fact, non-reduced) otherwise. We next consider the case $s \neq 0$. Then

$$\begin{aligned}
(w,\eta) \in C_{s,t} \text{ is a singularity } &\iff \frac{\partial(A-s)}{\partial w}(w,\eta) = \frac{\partial(A-s)}{\partial \eta}(w,\eta) = 0 \\
&\iff \frac{\partial A}{\partial w}(w,\eta) = \frac{\partial A}{\partial \eta}(w,\eta) = 0 \\
&\iff \frac{\partial \log A}{\partial w}(w,\eta) = \frac{\partial \log A}{\partial \eta}(w,\eta) = 0.
\end{aligned}$$

(As we assumed $s \neq 0$, it follows from $A(w,\eta,t) - s = 0$ that $A(w,\eta,t) \neq 0$, and so $\log A$ is well-defined.) Since $A = w^u P_1^{e_1} P_2^{e_2} \cdots P_n^{e_n}$ where $P_i = \prod_{j=1}^{\ell_i}(w\eta + t\beta_j^{(i)})$, we have

$$\begin{aligned}
\frac{\partial \log A}{\partial w} &= \frac{\partial \log w^u}{\partial w} + e_1 \frac{\partial \log P_1}{\partial w} + e_2 \frac{\partial \log P_2}{\partial w} + \cdots + e_n \frac{\partial \log P_n}{\partial w} \\
&= u\frac{1}{w} + \eta \left[e_1 \sum_{j=1}^{\ell_1} \frac{1}{w\eta + t\beta_j^{(1)}} + e_2 \sum_{j=1}^{\ell_2} \frac{1}{w\eta + t\beta_j^{(2)}} + \cdots + e_n \sum_{j=1}^{\ell_n} \frac{1}{w\eta + t\beta_j^{(n)}} \right].
\end{aligned}$$

Similarly, we have

$$\frac{\partial \log A}{\partial \eta} = w \left[e_1 \sum_{j=1}^{\ell_1} \frac{1}{w\eta + t\beta_j^{(1)}} + e_2 \sum_{j=1}^{\ell_2} \frac{1}{w\eta + t\beta_j^{(2)}} + \cdots + e_n \sum_{j=1}^{\ell_n} \frac{1}{w\eta + t\beta_j^{(n)}} \right].$$

For brevity, we set

$$h(w,\eta,t) := e_1 \sum_{j=1}^{\ell_1} \frac{1}{w\eta + t\beta_j^{(1)}} + e_2 \sum_{j=1}^{\ell_2} \frac{1}{w\eta + t\beta_j^{(2)}} + \cdots + e_n \sum_{j=1}^{\ell_n} \frac{1}{w\eta + t\beta_j^{(n)}},$$

and then

$$\frac{\partial \log A}{\partial w} = u\frac{1}{w} + \eta h, \qquad \frac{\partial \log A}{\partial \eta} = wh.$$

So $C_{s,t}$ $(s \neq 0)$ is singular if and only if (a) $u\frac{1}{w} + \eta h = 0$ and (b) $wh = 0$. The equation (b) implies that $w = 0$ or $h = 0$. But by (a), w cannot be zero, and therefore $h = 0$. In this case, from (a), we obtain $u\frac{1}{w} = 0$, and so $u = 0$. However since $u \geq 1$, this does not hold. Hence for $s \neq 0$, the curve $C_{s,t}$ is smooth. We thus obtain the following result.

Proposition 11.4.1 *Let $C_{s,t}$ be a curve defined by $A(w,\eta,t) - s = 0$ where*

$$A(w,\eta,t) = w^u P_1(w,\eta,t)^{e_1} P_2(w,\eta,t)^{e_2} \cdots P_n(w,\eta,t)^{e_n},$$

is a waving polynomial. Then (1) if $u = e_1 = e_2 = \cdots = e_n = 1$, the curve $C_{s,t}$ is smooth for any (s,t), and (2) otherwise, $C_{s,t}$ is singular if and only if $s = 0$.

We then show the following important result.

Proposition 11.4.2 *Let $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$ be a barking family obtained from a complete propagation of $DA_{e-1}(lY, k)$ where Y is a subbranch of type C_l and the positive integer k is divisible by n_e* (see Theorem 11.3.1). *Then $X_{s,t} := \Psi^{-1}(s,t)$ is singular if and only if $s = 0$.*

Proof. We regard $\mathcal{H}_i$ (and also $\mathcal{H}'_i$) as a family of curves parameterized by s and t. (In this proof, we again reset the subscripts; $\mathcal{H}_i$ in the above discussion corresponds to $\mathcal{H}_{e+i}$.) By Proposition 11.4.1, there are two cases about singularities of $\mathcal{H}_i$ and $\mathcal{H}'_i$ for $i = e, e+1, \dots, f$: (1) $\mathcal{H}_i$ and $\mathcal{H}'_i$ are smooth regardless of the values of s and t, or (2) they are singular precisely when $s = 0$. On the other hand, $\mathcal{H}_i$ and $\mathcal{H}'_i$ for $i = 1, 2, \dots, e-1$ are singular precisely when $s = 0$ (Corollary 7.2.5, p128). For $i = f+1, f+2, \dots, \lambda$, $\mathcal{H}_i$ and $\mathcal{H}'_i$ are the trivial families of the original H_i and H'_i respectively (see the paragraph above (11.3.5)):

$$\mathcal{H}_i : \ w^{m_{i-1}}\eta^{m_i} - s = 0, \qquad \mathcal{H}'_i : \ z^{m_{i+1}}\zeta^{m_i} - s = 0,$$

where $m_{\lambda+1} = 0$ by convention, and so $\mathcal{H}_i$ and $\mathcal{H}'_i$ for $i = f+1, f+2, \dots, \lambda$ are singular precisely when $s = 0$. From these observations, we conclude that $X_{s,t}$ is singular precisely when $s = 0$. □

11.5 Supplement: The condition that u divides l

The aim of this section is (i) to clarify the role of the condition "u divides l" in the definition of type C_l where $u = (m_{e-1} - ln_{e-1}) - (r_e - 1)(m_e - ln_e)$, and (ii) to prove some results which will be used in a later chapter. Consider a map of the form

$$g : \ z = \frac{1}{w}, \quad \zeta = w^r\eta + \alpha_1(t)w + \alpha_2(t)w^2 + \cdots + \alpha_{r-1}(t)w^{r-1}$$

where (1) r is an integer satisfying $r \geq 2$ and (2) $\alpha_i(t)$ is holomorphic in t with $\alpha_i(0) = 0$. Namely, g is a deformation of a transition function $z = 1/w$, $\zeta = w^r\eta$.

Lemma 11.5.1 *Let $A(w,\eta,t) = w^u P_1^{e_1} P_2^{e_2} \cdots P_n^{e_n} Q^c$ be a waving polynomial such that $P_1 = w\eta$, $Q = w\eta + t\gamma$ ($\gamma \in \mathbb{C}$), and $e_1 = e_2 = \cdots = e_n$. Suppose that there exists a map*

$$g : \ z = \frac{1}{w}, \quad \zeta = w^r\eta + \alpha_1(t)w + \alpha_2(t)w^2 + \cdots + \alpha_{r-1}(t)w^{r-1}$$

($r \geq 2$, $\alpha_i(0) = 0$) which transforms $A(w,\eta,t)$ to a polynomial $B(z,\zeta,t)$ satisfying

$$\deg_z B(z,\zeta,0) < \deg_\zeta B(z,\zeta,0). \tag{11.5.1}$$

Then $r = 2$. ($\deg_z B(z,\zeta,0)$ stands for the degree of $B(z,\zeta,0)$ in the variable z.)

Remark 11.5.2 *The condition* (11.5.1) *is intended for later application to deformation atlases; consider a deformation atlas*

$$\begin{cases} \mathcal{H}_i: & A(w,\eta,t) - s = 0 \\ \mathcal{H}_i': & B(z,\zeta,t) - s = 0 \\ g_i: & z = 1/w, \quad \zeta = w^r\eta + \alpha_1(t)w + \alpha_2(t)w^2 + \cdots + \alpha_{r-1}(t)w^{r-1}. \end{cases}$$

Since

$$\mathcal{H}_i|_{t=0}: \ w^{m_{i-1}}\eta^{m_i} - s = 0, \qquad \mathcal{H}_i'|_{t=0}: \ z^{m_{i+1}}\zeta^{m_i} - s = 0,$$

the condition " $\deg_z B(z,\zeta,0) < \deg_\zeta B(z,\zeta,0)$ " *is a restatement of* $m_{i+1} < m_i$ *where recall that a sequence of multiplicities* $m_0, m_1, \ldots, m_\lambda$ *is strictly decreasing.*

The proof of Lemma 11.5.1 is rather technical and we leave it later (see §11.5.1, p203). For a moment, assuming Lemma 11.5.1, we shall deduce several consequences.

Corollary 11.5.3 *Let* $A(w,\eta,t) = w^u P_1^{e_1} P_2^{e_2} \cdots P_n^{e_n} Q^c$ *be a waving polynomial such that* $P_1 = w\eta$, $Q = w\eta + t\gamma$ $(\gamma \in \mathbb{C})$, *and* $e_1 = e_2 = \cdots = e_n$ $(= e)$. *If* $e - u < 0$ *and* $c - u < 0$, *then there exists no map of the form*

$$g: \ z = \frac{1}{w}, \quad \zeta = w^r\eta + \alpha_1(t)w + \alpha_2(t)w^2 + \cdots + \alpha_{r-1}(t)w^{r-1} \qquad (11.5.2)$$

$(r \geq 2, \ \alpha_i(0) = 0)$ *which transforms* $A(w,\eta,t)$ *to a polynomial* $B(z,\zeta,t)$ *satisfying*

$$\deg_z B(z,\zeta,0) < \deg_\zeta B(z,\zeta,0). \qquad (11.5.3)$$

Proof. We show this by contradiction. Suppose that there exists a map g with the property in the assertion. Since $e_1 = e_2 = \cdots = e_n$, we have $r = 2$ by Lemma 11.5.1, and thus g must be of the form $z = 1/w, \ \zeta = w^2\eta + \alpha_1(t)w$ (set $r = 2$ in (11.5.2)). However since $e - u < 0$ and $c - u < 0$, it follows from Lemma 11.1.3, p184 that there exists no map[1] of the form $z = 1/w, \ \zeta = w^2\eta + \alpha_1(t)w$ which transforms $A(w,\eta,t)$ to some polynomial. This is a contradiction. □

In §11.2, under the assumption "u divides l", we constructed waving sequences. Also for the case where u does not divide l, we can perform an analogous construction. Namely, as long as $l - (a_{i,j}+1)u > 0$ in the i-th step, we can proceed to an $(i+1)$-st step, and when we reach to the step such that $l - (a_{i,j}+1)u \leq 0$, we stop and obtain a 'waving sequence' in the wider sense; below we treat such sequences.

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[1] We may show this by the argument of the proof of Lemma 11.1.3. (Actually, in Lemma 11.1.3, we treated the case $\alpha_1(t) = \alpha t$ where $\alpha \in \mathbb{C}$. In the present case, we need to consider the Taylor expansion of $\alpha_1(t)$.)

Proposition 11.5.4 *Assume that l and u are positive integers such that $l \geq u$ and u does not divide l. Let $A(w,\eta,t) = w^u P_1^{l-a_1 u} P_2^{l-a_2 u} \cdots P_n^{l-a_n u} Q^b$ be a waving polynomial such that $P_1 = w\eta$ and $Q = w\eta + t\gamma$ ($\gamma \in \mathbb{C}$). If the waving sequence associated with $A(w,\eta,t)$ terminates at a step f, then the following statements hold:*

(1) *A_f is of the form*

$$w^u P_{f,1}^{l-Nu} P_{f,2}^{l-Nu} \cdots P_{f,n_f}^{l-Nu} Q_f^{b-vu},$$

where

(i) (*considering the division of l by u*), *N is the nonnegative integer such that $l - Nu > 0$ and $l-(N+1)u < 0$, and*

(ii) *v is an integer defined as follows: if $u > b$, set $v = 0$, and if $u \leq b$, v is the nonnegative integer such that $b - vu \geq 0$ and $b-(v+1)u < 0$.*

(2) *There exists no map of the form*

$$g: \; z = \frac{1}{w}, \quad \zeta = w^r\eta + \alpha_1(t)w + \alpha_2(t)w^2 + \cdots + \alpha_{r-1}(t)w^{r-1}$$

($r \geq 2$, $\alpha_i(0) = 0$) *which transforms A_f to a polynomial B_f satisfying*

$$\deg_z B_f(z,\zeta,0) < \deg_\zeta B_f(z,\zeta,0).$$

Proof. (1) is clear from the construction of waving sequences (see §11.2). To show (2), we set $e := l - Nu$ and $c := b - vu$, and then

$$A_f = w^u P_{f,1}^e P_{f,2}^e \cdots P_{f,n_f}^e Q_f^c.$$

We may assume that $P_{f,1} = w\eta$; in fact, in each step of the construction of the waving sequence, we may assume that $P_{i,1} = w\eta$, because the initial waving polynomial A has a factor $P_1 = w\eta$ (see the proof of Lemma 11.1.3). Therefore A_f fulfills all assumptions of Corollary 11.5.3, and so there exists no map g satisfying the condition of (2). □

We next deduce the following important result, which clarifies the role of the condition "u divides l" in the definition of type C_l.

Theorem 11.5.5 *Assume that l and u are positive integers and*

$$A(w,\eta,t) = w^u P_1^{l-a_1 u} P_2^{l-a_2 u} \cdots P_n^{l-a_n u} Q^b$$

is a waving polynomial such that $P_1 = w\eta$ and $Q = w\eta + t\gamma$ ($\gamma \in \mathbb{C}$). Then

$$\mathcal{H}_1: \; A(w,\eta,t) - s = 0$$

admits a complete propagation if and only if u divides l.

Proof. $\Longrightarrow$: If u does not divide l, then by Proposition 11.5.4 (1), the waving sequence terminates at a waving polynomial of the form

$$A_f = w^u P_{f,1}^{l-Nu} P_{f,2}^{l-Nu} \cdots P_{f,n_f}^{l-Nu} Q^{b-vu},$$

and a further propagation is impossible by Proposition 11.5.4 (2).

$\Longleftarrow$: We already showed this in the proof of Theorem 11.3.1. □

11.5.1 Proof of Lemma 11.5.1

Now we shall prove a technical lemma (Lemma 11.5.1), which insists:

Let $A(w,\eta,t) = w^u P_1^{e_1} P_2^{e_2} \cdots P_n^{e_n} Q^c$ *be a waving polynomial such that* $P_1 = w\eta$, $Q = w\eta + t\gamma$ $(\gamma \in \mathbb{C})$, *and* $e_1 = e_2 = \cdots = e_n$. *Suppose that there exists a map*

$$g: \; z = \frac{1}{w}, \quad \zeta = w^r\eta + \alpha_1(t)w + \alpha_2(t)w^2 + \cdots + \alpha_{r-1}(t)w^{r-1}$$

$(r \geq 2, \; \alpha_i(0) = 0)$ *such that* g *transforms* $A(w,\eta,t)$ *to a polynomial* $B(z,\zeta,t)$ *satisfying*

$$\deg_z B(z,\zeta,0) < \deg_\zeta B(z,\zeta,0). \tag{11.5.4}$$

Then $r = 2$.

We begin with preparation for a proof. By assumption, $e_1 = e_2 = \cdots = e_n$. We denote these common numbers by e, and then $A = P_1^e P_2^e \cdots P_n^e Q^c$. Without loss of generality, it is sufficient to show the statement for $n = 2$ (in fact, rewrite $P_2 P_3 \cdots P_n$ by P_2, and then $A = P_1^e P_2^e Q^c$). To avoid complicated notation, we first show the statement for the case $c = 0$. In this case $A = P_1^e P_2^e$; we explicitly write $P_2 = \prod_{j=1}^d (w\eta + t\beta_j)$ while $P_1 = w\eta$ by definition, and so

$$A = w^u (w\eta)^e \left[\prod_{j=1}^d (w\eta + t\beta_j) \right]^e. \tag{11.5.5}$$

From the definition of waving polynomials, the 'roots' of P_1 and P_2 are distinct (by a 'root' of P_2, we indicate β_j, and the 'root' of P_1 is zero). Thus none of the 'roots' of P_2 is zero:

$$\beta_j \neq 0. \tag{11.5.6}$$

We now verify Lemma 11.5.1; supposing that $r \geq 3$, we shall deduce a contradiction. First of all, we give a proof for a map g of the form $z = 1/w, \; \zeta = w^r\eta + t\alpha w^q$ where $\alpha \in \mathbb{C}$ and $1 \leq q \leq r-1$. We separate into two cases according to whether $\alpha = 0$ or not.

Case 1. $\alpha = 0$

In this case $g: \; z = 1/w, \; \zeta = w^r\eta$. Since (11.5.5) is written as

$$A = w^u \left(\frac{1}{w^{r-1}} \cdot w^r\eta \right)^e \left[\prod_{j=1}^d \left(\frac{1}{w^{r-1}} \cdot w^r\eta + t\beta_j \right) \right]^e,$$

the map g transforms A to

$$B = \frac{1}{z^u} (z^{r-1}\zeta)^e \left[\prod_{j=1}^d (z^{r-1}\zeta + t\beta_j) \right]^e.$$

Recall that β_j are nonzero (11.5.6), and hence

$$\left[\prod_{j=1}^{d}(z^{r-1}\zeta+t\beta_j)\right]^e \tag{11.5.7}$$

has a nonzero constant term; therefore in B, the factor z^u can not divide (11.5.7). Since B is a polynomial, z^u must divide $(z^{r-1}\zeta)^e$; namely z^u must divide $z^{(r-1)e}$, and thus

$$(r-1)e \geq u. \tag{11.5.8}$$

Then $B(z,\zeta,t)=z^{(r-1)e-u}\,\zeta^e\left[\prod_{j=1}^{d}(z^{r-1}\zeta+t\beta_j)\right]^e$, and we have

$$\begin{aligned}B(z,\zeta,0) &= z^{(r-1)e-u}\,\zeta^e\,\big[\underbrace{(z^{r-1}\zeta)(z^{r-1}\zeta)\cdots(z^{r-1}\zeta)}_{d}\big]^e\\ &= z^{(r-1)e-u}\,\zeta^e\,(z^{r-1}\zeta)^{ed}\\ &= z^{(r-1)(e+ed)-u}\,\zeta^{e+ed}.\end{aligned}$$

Since $\deg_z B(z,\zeta,0)<\deg_\zeta B(z,\zeta,0)$, we have

$$(r-1)(e+ed)-u<e+ed,$$

that is, $(r-2)(e+ed)<u$. Combined this with (11.5.8), we derive

$$(r-2)(e+ed)<(r-1)e.$$

Since $e>0$, we have $(r-2)(1+d)<(r-1)$, that is, $(r-2)d<1$. However since $d\geq 1$, if $r\geq 3$, then $(r-2)d\geq 1$, which yields a contradiction. Therefore we conclude that $r=2$. This proves the assertion for Case 1.

Case 2. $\alpha\neq 0$

Next we consider the case $g:\ z=1/w,\ \zeta=w^r\eta+t\alpha w^q$ where $\alpha\neq 0$ and $1\leq q\leq r-1$. Since

$$A=w^u\left(\frac{1}{w^{r-1}}\cdot w^r\eta\right)^e\left[\prod_{j=1}^{d}\left(\frac{1}{w^{r-1}}\cdot w^r\eta+t\beta_j\right)\right]^e,$$

the map g transforms A to

$$\begin{aligned}B &= \frac{1}{z^u}\left(z^{r-1}\cdot\left(\zeta-\frac{t\alpha}{z^q}\right)\right)^e\left[\prod_{j=1}^{d}\left(z^{r-1}\cdot\left(\zeta-\frac{t\alpha}{z^q}\right)+t\beta_j\right)\right]^e\\ &= \frac{1}{z^u}\left(z^{r-1}\zeta-t\,\alpha\,z^{r-1-q}\right)^e\left[\prod_{j=1}^{d}\left(z^{r-1}\zeta-t\,\alpha\,z^{r-1-q}+t\,\beta_j\right)\right]^e\\ &= z^{(r-1-q)e-u}\left(z^q\zeta-t\alpha\right)^e\left[\prod_{j=1}^{d}\left(z^{r-1}\zeta+t\left(\beta_j-\alpha z^{r-1-q}\right)\right)\right]^e.\end{aligned} \tag{11.5.9}$$

Our goal is to show that unless $r = 2$, $\deg_z B(z,\zeta,0) < \deg_\zeta B(z,\zeta,0)$ is not valid. To show this, we write $B = z^{(r-1-q)e-u} f(z,\zeta,t)$ where

$$f := (z^q\zeta - t\alpha)^e \left[\prod_{j=1}^{d} \Big(z^{r-1}\zeta + t\big(\beta_j - \alpha z^{r-1-q}\big)\Big) \right]^e .$$

Note that f has no constant term if and only if a 'root' of f is zero (the 'roots' of f are $-\alpha$ and $\beta_j - \alpha z^{r-1-q}$). Since $\alpha \neq 0$ and $\beta_j \neq 0$ (11.5.6), the following equivalences hold:

$$\begin{aligned} f \text{ has no constant term} &\iff \beta_j - \alpha z^{r-1-q} = 0 \\ &\qquad \text{identically for some } j \ (1 \le j \le d) \\ &\iff r - 1 = q \quad \text{and} \quad \beta_j - \alpha = 0 \\ &\qquad \text{for some } j \ (1 \le j \le d). \end{aligned} \tag{11.5.10}$$

With this preparation, we demonstrate that if $r \geq 3$, then the inequality

$$\deg_z B(z,\zeta,0) < \deg_\zeta B(z,\zeta,0)$$

fails; the verification of this statement divides into two cases, depending on whether or not f contains a constant term.

Case 2.1 f has a constant term: As above we write $B = z^{(r-1-q)e-u} f(z,\zeta,t)$, where

$$f = (z^q\zeta - t\alpha)^e \left[\prod_{j=1}^{d} \Big(z^{r-1}\zeta + t\big(\beta_j - \alpha z^{r-1-q}\big)\Big) \right]^e .$$

Since (i) f has a constant term and (ii) B is a polynomial, the exponent of the first factor $z^{(r-1-q)e-u}$ of B must be nonngetaive (otherwise B has a fractional term). Therefore

$$(r-1-q)e - u \geq 0. \tag{11.5.11}$$

Next we consider

$$B(z,\zeta,0) = z^{(r-1-q)e-u} \ (z^q\zeta)^e \ \big[\underbrace{(z^{r-1}\zeta)\cdots(z^{r-1}\zeta)}_{d} \big]^e .$$

Then

$$\deg_z B(z,\zeta,0) = \big[(r-1-q)e-u\big] + qe + ed(r-1), \qquad \deg_\zeta B(z,\zeta,0) = e + ed.$$

Here $\big[(r-1-q)e-u\big] \geq 0$ (11.5.11), and $qe \geq e$ by $q \geq 1$. Moreover, assuming that $r \geq 3$, then $ed(r-1) \geq ed$, and so $\deg_z B(z,\zeta,0) \geq \deg_\zeta B(z,\zeta,0)$. This is a contradiction, and thus we conclude that $r = 2$.

Remark 11.5.6 *The proof for Case* 1 *and Case* 2.1 *also works under a weaker assumption* $e_1 \geq e_2$ *instead of* $e_1 = e_2$*. However the proof for Case* 2.2 *below essentially uses the equality* $e_1 = e_2$*, and therefore the condition* $e_1 = e_2$ *cannot be dropped.*

Case 2.2 f has no constant term: In this case, as we saw in (11.5.10), $q = r - 1$ holds, and B (see (11.5.9)) takes a simple form:

$$\begin{aligned} B &= z^{(r-1-q)e-u}\big(z^q\zeta - t\alpha\big)^e \left[\prod_{j=1}^{d}\Big(z^{r-1}\zeta + t\big(\beta_j - \alpha z^{r-1-q}\big)\Big)\right]^e \\ &= \frac{1}{z^u}\big(z^{r-1}\zeta - t\alpha\big)^e \left[\prod_{j=1}^{d}\Big(z^{r-1}\zeta + t\big(\beta_j - \alpha\big)\Big)\right]^e. \end{aligned}$$

For simplicity we set $\beta'_j := \beta_j - \alpha$. Then

$$B = \frac{1}{z^u}\big(z^{r-1}\zeta - t\alpha\big)^e \left[\prod_{j=1}^{d}(z^{r-1}\zeta + t\beta'_j)\right]^e.$$

Again by (11.5.10), $\beta'_j = 0$ holds for some j $(1 \leq j \leq d)$. Without loss of generality, we assume $\beta'_d = 0$. Then

$$\begin{aligned} B &= \frac{1}{z^u}\big(z^{r-1}\zeta - t\alpha\big)^e \left[\prod_{j=1}^{d-1}(z^{r-1}\zeta + t\beta'_j)\cdot(z^{r-1}\zeta)\right]^e \\ &= z^{(r-1)e-u}\,\zeta^e\,\big(z^{r-1}\zeta - t\alpha\big)^e \left[\prod_{j=1}^{d-1}(z^{r-1}\zeta + t\beta'_j)\right]^e. \qquad (11.5.12) \end{aligned}$$

We note that β'_j $(1 \leq j \leq d-1)$ are nonzero. In fact β_j $(1 \leq j \leq d)$ are distinct, and accordingly $\beta'_j := \beta_j - \alpha$ $(1 \leq j \leq d)$ are also distinct; since $\beta'_d = 0$, the others β'_j $(1 \leq j \leq d-1)$ are nonzero. Therefore in (11.5.12), the factor

$$\big(z^{r-1}\zeta - t\alpha\big)^e \left[\prod_{j=1}^{d-1}(z^{r-1}\zeta + t\beta'_j)\right]^e$$

has a nonzero constant term. Since B is a polynomial, this implies that the exponent of the first factor $z^{(r-1)e-u}$ in (11.5.12) must be nonnegative (otherwise B has a fractional term), and so

$$(r-1)e \geq u. \qquad (11.5.13)$$

Next from (11.5.12),

$$\begin{aligned} B(z,\zeta,0) &= z^{(r-1)e-u}\,\zeta^e\,(z^{r-1}\zeta)^e\,\big[\underbrace{(z^{r-1}\zeta)\cdots(z^{r-1}\zeta)}_{d-1}\big]^e \\ &= z^{(r-1)e-u}\,\zeta^e\,\big(z^{r-1}\zeta\big)^{ed} \\ &= z^{(r-1)e-u+(r-1)ed}\,\zeta^{e+ed}. \end{aligned}$$

Since $\deg_z B(z, \zeta, 0) < \deg_\zeta B(z, \zeta, 0)$, we have

$$(r-1)e - u + (r-1)ed < e + ed,$$

that is, $(r-2)(e+ed) < u$. Combined this with (11.5.13), we obtain

$$(r-2)(e+ed) < (r-1)e.$$

Since $e > 0$, we have $(r-2)(1+d) < (r-1)$, and therefore

$$(r-2)d < 1.$$

However $d \geq 1$, and if we assume $r \geq 3$, then $(r-2)d \geq 1$, yielding a contradiction. Hence we conclude that $r = 2$. This completes the proof of Lemma 11.5.1 for the case where $c = 0$ in $A = w^u P_1^e P_2^e \cdots P_n^e Q^c$ and the map g is of the form $z = 1/w,\ \zeta = w^r \eta + t\alpha w^q$.

For a general waving polynomial $A = w^u P_1^e P_2^e \cdots P_n^e Q^c$ where $P_1 = w\eta$ and $Q = w\eta + t\gamma$ ($\gamma \in \mathbb{C}$), if g has a form $z = 1/w,\ \zeta = w^r\eta + t\alpha w^q$, then noting that g transforms $Q = w\eta + \gamma$ to $z^{r-1}\zeta - t\alpha z^{r-1-k} + t\gamma$, a similar argument to the above case yields the conclusion $r = 2$. The proof for a general map

$$g: \ z = \frac{1}{w}, \quad \zeta = w^r\eta + \alpha_1(t)w + \alpha_2(t)w^2 + \cdots + \alpha_{r-1}(t)w^{r-1}$$

is essentially the same; we only have to apply the above argument for the lowest degree term in z of $B(z, \zeta, t)$, and then we deduce $r = 2$. This completes the proof of Lemma 11.5.1.

12

Recursive Construction of Deformations of Type C_l

Let $DA_{e-1}(lY, k)$ be the deformation atlas associated with a subbranch Y of type C_l. In the previous chapter we constructed complete propagations of $DA_{e-1}(lY, k)$. Recall that if $l \geq 2$, then in most cases, a complete propagation of $DA_{e-1}(lY, k)$ is *not* unique; see Remark 11.3.6, p198. (The complete propagation of $DA_{e-1}(lY, k)$ for Y of type A_l or B_l is unique, unless Y is type AB_l, i.e. both of type A_l and B_l. See Remark 10.1.4, p179.) In this chapter we give two particular constructions of complete propagations of $DA_{e-1}(lY, k)$ for Y of type C_l. The resulting deformations have distinguished properties; they possess some interesting "periodicity".

12.1 Ascending, descending, and stable polynomials

We first introduce three kinds of polynomials: *ascending, descending, and stable polynomials*. (Actually we already defined descending polynomials, but for convenience we also include it here.) They are respectively polynomials of the following forms:

Ascending polynomial $$P(w, \eta, t) = \eta \prod_{j=1}^{n} (w\eta + t\beta_j), \quad \beta_j \in \mathbb{C}$$

Descending polynomial $$Q(w, \eta, t) = w \prod_{j=1}^{n} (w\eta + t\gamma_j), \quad \gamma_j \in \mathbb{C}$$

Stable polynomial $$R(w, \eta, t) = \prod_{j=1}^{n} (w\eta + t\delta_j), \quad \delta_j \in \mathbb{C}.$$

By abuse of terminology, we say that β_j (resp. γ_j, δ_j) is a 'root' of P (resp. Q, R). The positive integer n is called the *length* of respective polynomials. We say that $P(\zeta, z, t)$ is an *interchanged ascending polynomial*. Likewise $Q(\zeta, z, t)$ and $R(\zeta, z, t)$ are respectively called *interchanged descending polynomial* and *interchanged stable polynomial.* These polynomials have the following important property.

Lemma 12.1.1 (1) *A map $g: z = 1/w,\ \zeta = w^2\eta + t\alpha w$, where $\alpha \in \mathbb{C}$ is arbitrary, transforms P to an interchanged ascending polynomial P' of length $n+1$:*

$$P'(\zeta, z, t) = z \prod_{j=1}^{n+1} (z\zeta + t\beta'_j),$$

where we set $\beta'_j = \beta_j - \alpha$ $(j = 1, 2, \ldots, n)$ and $\beta'_{n+1} = -\alpha$.

(2) *A map $g: z = 1/w,\ \zeta = w^2\eta + t\gamma_n w$, where γ_n is the 'root' of the n-th factor of Q, transforms Q to an interchanged descending polynomial Q' of length $n-1$:*

$$Q'(\zeta, z, t) = \zeta \prod_{j=1}^{n-1} (z\zeta + t\gamma'_j),$$

where we set $\gamma'_j = \gamma_j - \gamma_n$.

(3) *A map $g: z = 1/w,\ \zeta = w^2\eta + t\alpha w$, where $\alpha \in \mathbb{C}$ is arbitrary, transforms R to an interchanged stable polynomial R' of length n:*

$$R'(\zeta, z, t) = \prod_{j=1}^{n} (z\zeta + t\delta'_j),$$

where we set $\delta'_j = \delta_j - \alpha$ $(j = 1, 2, \ldots, n)$.

Proof. We already showed (2) in Lemma 8.1.1, p145. We confirm (1) as follows:

$$\begin{aligned}
P(w, \eta, t) &= \frac{1}{w^2}\, w^2\eta \prod_{j=1}^{n} \left(\frac{1}{w} w^2\eta + t\beta_j \right) \\
&= z^2 \left(\zeta - t\alpha \frac{1}{z} \right) \prod_{j=1}^{n} \left[z \left(\zeta - t\alpha \frac{1}{z} \right) + t\beta_j \right] \\
&= z(z\zeta - t\alpha) \prod_{j=1}^{n} \Big[z\zeta + t(\beta_j - \alpha) \Big].
\end{aligned}$$

Similarly, we confirm (3):

$$\begin{aligned}
R(w, \eta, t) = \prod_{j=1}^{n} \left(\frac{1}{w} w^2\eta + t\delta_j \right) &= \prod_{j=1}^{n} \left[z \left(\zeta - t\alpha \frac{1}{z} \right) + t\delta_j \right] \\
= \prod_{j=1}^{n} \Big[z\zeta + t(\delta_j - \alpha) \Big]&.
\end{aligned}$$

□

As a consequence we have the following result.

Lemma 12.1.2 *Consider ascending, descending and stable polynomials:*

$$P(w,\eta,t) = \eta \prod_{j=1}^{n} (w\eta + t\beta_j), \quad Q(w,\eta,t) = w \prod_{j=1}^{n} (w\eta + t\gamma_j),$$
$$R(w,\eta,t) = \prod_{j=1}^{n} (w\eta + t\delta_j).$$

Then the following statements hold:

(A) *There exist* (1) *a sequence of descending polynomials* $Q_1 = Q, Q_2, \ldots, Q_{n+1}$ *with length*$(Q_i) = n+1-i$ *and* (2) *a sequence of maps* $g_1, g_2, \ldots, g_n$ *of the form* $g_i : z = 1/w, \zeta = w^2\eta + t\alpha_i w$ ($\alpha_i \in \mathbb{C}$ *is a 'root' of* Q_i) *such that* g_i *transforms* $Q_i(w,\eta,t)$ *to* $Q_{i+1}(\zeta,z,t)$.

(B) *Let* $g_1, g_2, \ldots, g_n$ *be a sequence of maps of the form* $g_i : z = 1/w, \zeta = w^2\eta + t\alpha_i w$, *where* $\alpha_i \in \mathbb{C}$ *is arbitrary. Then*

(B.1) *there exists a sequence of ascending polynomials* $P_1 = P, P_2, \ldots, P_{n+1}$ *with length*$(P_i) = n-1+i$ *such that* g_i *transforms* $P_i(w,\eta,t)$ *to* $P_{i+1}(\zeta,z,t)$, *and*

(B.2) *there exists a sequence of stable polynomials* $R_1 = R, R_2, \ldots, R_{n+1}$ *with length*$(R_i) = n$ *(independent of* i*) such that* g_i *transforms* $R_i(w,\eta,t)$ *to* $R_{i+1}(\zeta,z,t)$.

Proof. We already showed (A) in Lemma 8.1.3, p145. We shall show (B.1). By Lemma 12.1.1 (1), g transforms $P = P_1(w,\eta,t)$ to a polynomial $P_2(\zeta,z,t)$ such that $P_2(w,\eta,t)$ is an ascending polynomial. Next, again by Lemma 12.1.1 (1), g_2 transforms $P_2(w,\eta,t)$ to a polynomial $P_3(\zeta,z,t)$ such that $P_3(w,\eta,t)$ is an ascending polynomial. Repeating this process, we obtain a sequence of ascending polynomials $P_1, P_2, \ldots, P_{n+1}$ in (B.1). Similarly, we can show (B.2) by using Lemma 12.1.1 (3). □

The sequence $Q_1, Q_2, \ldots, Q_{n+1}$ together with $g_1, g_2, \ldots, g_n$ in Lemma 12.1.2 (A) is called a *descending sequence* associated with the descending polynomial $Q = Q_1$. Similarly, the sequence $P_1, P_2, \ldots, P_{n+1}$ in (B.1) (resp. $R_1, R_2, \ldots, R_{n+1}$ in (B.2)) is called an *ascending sequence* (resp. *stable sequence*) associated with the ascending polynomial $P = P_1$ (resp. stable polynomial $R = R_1$) and the maps $g_1, g_2, \ldots, g_n$. From these sequences, we may construct 'deformation atlases' $DA_{P,n}, DA_{Q,n}$ and $DA_{R,n}$ of length n as follows.

(i) $DA_{P,n}$: for $i = 1, 2, \ldots, n$,

$$\begin{cases} \mathcal{H}_i : & P_i(w,\eta,t) - s = 0 \\ \mathcal{H}_i : & P_{i+1}(\zeta,z,t) - s = 0 \\ g_i : & \text{the map in the ascending sequence.} \end{cases}$$

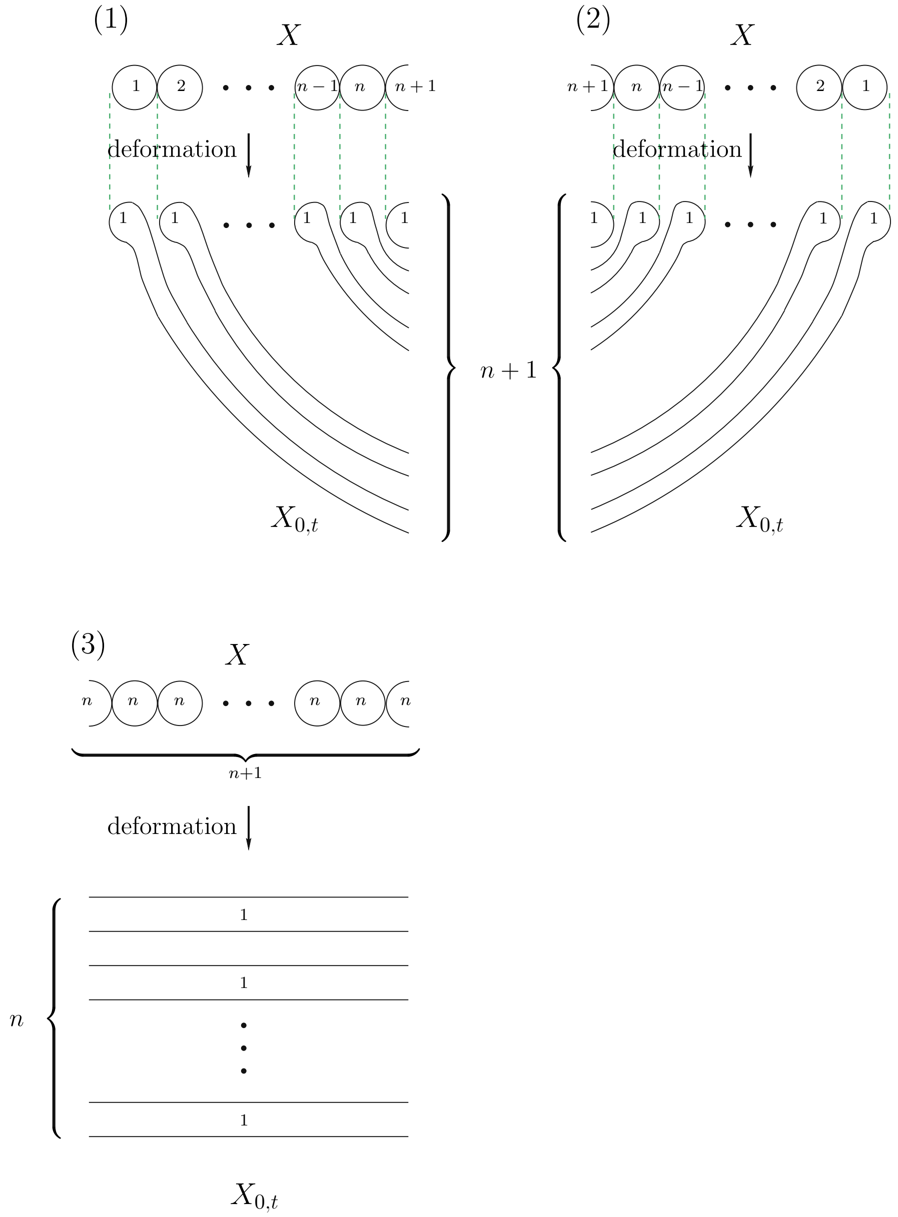


Fig. 12.1.1. Deformations (1), (2), (3) are respectively obtained from ascending, descending, stable sequences of length n.

(ii) $DA_{Q,n}$: for $i = 1, 2, \ldots, n$,

$$\begin{cases} \mathcal{H}_i : & Q_i(w, \eta, t) - s = 0 \\ \mathcal{H}_i : & Q_{i+1}(\zeta, z, t) - s = 0 \\ g_i : & \text{the map in the decending sequence.} \end{cases}$$

(iii) $DA_{R,n}$: for $i = 1, 2, \ldots, n$,

$$\begin{cases} \mathcal{H}_i : & R_i(w, \eta, t) - s = 0 \\ \mathcal{H}_i : & R_{i+1}(\zeta, z, t) - s = 0 \\ g_i : & \text{the map in the stable sequence.} \end{cases}$$

Let $\Psi_P : \mathcal{M}_P \to \Delta \times \Delta^\dagger$ be the barking family obtained from the deformation atlas $DA_{P,n}$. Then the deformation from $X := \Psi_P^{-1}(0, 0)$ to $X_{0,t} := \Psi_P^{-1}(0, t)$ is described in Figure 12.1.1 (1). Similarly, let $\Psi_Q : \mathcal{M}_Q \to \Delta \times \Delta^\dagger$ and $\Psi_R : \mathcal{M}_R \to \Delta \times \Delta^\dagger$ be the barking families obtained from the deformation atlases $DA_{Q,n}$ and $DA_{R,n}$ respectively. Then the deformation from $X := \Psi_Q^{-1}(0, 0)$ to $X_{0,t} := \Psi_Q^{-1}(0, t)$ is described in Figure 12.1.1 (2), and similarly the deformation from $X := \Psi_R^{-1}(0, 0)$ to $X_{0,t} := \Psi_R^{-1}(0, t)$ is described in Figure 12.1.1 (3).

12.2 Technical preparation I

In this section, we prove some more results on ascending, descending, and stable polynomials; we shall construct certain sequence of polynomials $A_i^{[\mathsf{k}]}$ and maps $g_i^{[\mathsf{k}]}$ which constitute a 'long sequence', starting from some particular polynomial A:

$$A = A_1^{[1]} \xmapsto{g_1^{[1]}} A_2^{[1]} \xmapsto{g_2^{[1]}} \cdots \xmapsto{g_n^{[1]}} A_{n+1}^{[1]} \xmapsto{g_{n+1}^{[1]}} A_1^{[2]} \xmapsto{g_1^{[2]}} \cdots$$

$$\cdots$$

$$\cdots \xmapsto{g_n^{[\mathsf{k}-1]}} A_{n+1}^{[\mathsf{k}-1]} \xmapsto{g_{n+1}^{[\mathsf{k}-1]}} A_1^{[\mathsf{k}]} \xmapsto{g_1^{[\mathsf{k}]}} A_2^{[\mathsf{k}]} \xmapsto{g_2^{[\mathsf{k}]}} \cdots \xmapsto{g_n^{[\mathsf{k}]}} A_{n+1}^{[\mathsf{k}]} \xmapsto{g_{n+1}^{[\mathsf{k}]}} A_1^{[\mathsf{k}+1]} \xmapsto{g_1^{[\mathsf{k}+1]}} \cdots$$

$$\cdots$$

$$\cdots \xmapsto{g_n^{[N-1]}} A_{n+1}^{[N-1]} \xmapsto{g_{n+1}^{[N-1]}} A_1^{[N]} \xmapsto{g_1^{[N]}} A_2^{[N]} \xmapsto{g_2^{[N]}} \cdots \xmapsto{g_n^{[N]}} A_{n+1}^{[N]} = w^u.$$

Proposition 12.2.1 *Let l and u be positive integers such that u divides l, and write $l = Nu$. Consider a polynomial $A(w, \eta, t) = P^{l-u} Q^l$ where $P = \eta$ and $Q = w \prod_{j=1}^n (w\eta + t\gamma_j)$, $\gamma_j \in \mathbb{C}$. Then for each k ($\mathsf{k} = 1, 2, \ldots, N$), there exist*

(i) *a sequence of ascending polynomials $P_1^{[\mathsf{k}]}, P_2^{[\mathsf{k}]}, \ldots, P_{n+1}^{[\mathsf{k}]}$ with $P_1^{[1]} = P$,*

(ii) *a sequence of descending polynomials $Q_1^{[\mathsf{k}]}, Q_2^{[\mathsf{k}]}, \ldots, Q_{n+1}^{[\mathsf{k}]}$ with $Q_1^{[1]} = Q$,*

(iii) *a sequence of maps* $g_1^{[k]}, g_2^{[k]}, \dots, g_n^{[k]}$ *of the form*

$$g_i^{[k]}: \quad z = \frac{1}{w}, \quad \zeta = w^2\eta + t\alpha_i^{[k]} w \quad \textit{for some } \alpha_i^{[k]} \in \mathbb{C},$$

and a sequence of maps $g_{n+1}^{[k]}: \ z = \frac{1}{w}, \ \zeta = w^2\eta \quad (k = 1, 2, \dots, N-1)$

such that setting

$$A_i^{[k]} := (P_i^{[k]})^{l-ku} (Q_i^{[k]})^{l-(k-1)u}, \qquad (k = 1, 2, \dots, N, \quad i = 1, 2, \dots, n+1),$$

then

(1) *for* $i = 1, 2, \dots, n$, *the map* $g_i^{[k]}$ *transforms* $A_i^{[k]}(w, \eta, t)$ *to* $A_{i+1}^{[k]}(\zeta, z, t)$, *whereas* $g_{n+1}^{[k]}$ *transforms* $A_{n+1}^{[k]}(w, \eta, t)$ *to* $A_1^{[k+1]}(\zeta, z, t)$, *and*
(2) $A_{n+1}^{[N]} = w^u$.

Proof. We carry out the construction of sequences in (i), (ii), and (iii) inductively. First of all, we construct sequences for $k = 1$.

Step 1 Take a descending sequence $Q_1, Q_2, \dots, Q_{n+1}$ and $g_1, g_2, \dots, g_n$ associated with $Q_1 = Q$. Next, take an ascending sequence $P_1, P_2, \dots, P_{n+1}$ associated with $P_1 := P$ and $g_1, g_2, \dots, g_n$. Then P_i and Q_i are respectively of the forms

$$P_i = \eta \prod_{j=1}^{i-1} (w\eta + t\beta_{i,j}), \qquad Q_i = w \prod_{j=1}^{n+1-i} (w\eta + t\gamma_{i,j}) \qquad \text{where} \quad \beta_{i,j},\, \gamma_{i,j} \in \mathbb{C}.$$

Setting $A_i := P_i^{l-u} Q_i^l$, we have

$$\begin{aligned} A_{n+1} = P_{n+1}^{l-u} Q_{n+1}^l &= \left[\eta \prod_{j=1}^{n} (w\eta + t\beta_{n+1,j}) \right]^{l-u} w^l \\ &= w^l \eta^{l-u} \left[\prod_{j=1}^{n} (w\eta + t\beta_{n+1,j}) \right]^{l-u}. \end{aligned} \qquad (12.2.1)$$

We then take a map $g_{n+1}: z = 1/w, \ \zeta = w^2\eta$ (see Remark 12.2.2 below for another choice of g_{n+1}). Since

$$\begin{aligned} &w^l \eta^{l-u} \left[\prod_{j=1}^{n} (w\eta + t\beta_{n+1,j}) \right]^{l-u} \\ &= \frac{1}{w^{l-2u}} (w^2\eta)^{l-u} \left[\prod_{j=1}^{n} \left(\frac{1}{w}(w^2\eta) + t\beta_{n+1,j} \right) \right]^{l-u}, \end{aligned}$$

the map g_{n+1} transforms A_{n+1} to $z^{l-2u}\,\zeta^{l-u}\left[\prod_{j=1}^{n}(z\zeta+t\beta_{n+1,\,j})\right]^{l-u}$, that is

$$z^{l-2u}\left[\zeta\prod_{j=1}^{n}(z\zeta+t\beta_{n+1,\,j})\right]^{l-u}. \tag{12.2.2}$$

Now we slightly change the notation to emphasize $\mathsf{k}=1$ (Step "1"). Instead of P_i, Q_i and g_i, we write $P_i^{[1]}, Q_i^{[1]}$ and $g_i^{[1]}$ respectively. Also A_i is denoted by $A_i^{[1]}$.

Step 2 We set $P_1^{[2]} := \eta$ and $Q_1^{[2]} := w\prod_{j=1}^{n}(w\eta+t\beta_{n+1,\,j})$, where we note that $P_1^{[2]}$ is an ascending polynomial of length 0 and $Q_1^{[2]}$ is a descending polynomial of length n. Next we set $A_1^{[2]} := (P_1^{[2]})^{l-2u}(Q_1^{[2]})^{l-u}$, that is

$$A_1^{[2]}(\zeta,z,t) = z^{l-2u}\left[\zeta\prod_{j=1}^{n}(z\zeta+t\beta_{n+1,\,j})\right]^{l-u}.$$

Note that $A_1^{[2]}$ coincides with (12.2.2), and hence as we saw in Step 1, the map $g_{n+1}^{[1]}\,(=g_{n+1}):\ z=1/w,\ \zeta=w^2\eta$ transforms $A_{n+1}^{[1]}(w,\eta,t)$ to $A_1^{[2]}(\zeta,z,t)$. We then repeat the process in Step 1. Namely, first we take the descending sequence $Q_1^{[2]}, Q_2^{[2]}, \dots, Q_{n+1}^{[2]}$ and $g_1^{[2]}, g_2^{[2]}, \dots, g_n^{[2]}$ associated with $Q_1^{[2]}$, and then we take the ascending sequence $P_1^{[2]}, P_2^{[2]}, \dots, P_{n+1}^{[2]}$ associated with $P_1^{[2]}$ and $g_1^{[2]}, g_2^{[2]}, \dots, g_n^{[2]}$. Setting

$$A_i^{[2]} := (P_i^{[2]})^{l-2u}\,(Q_i^{[2]})^{l-u},$$

we have

$$A_{n+1}^{[2]} = (P_{n+1}^{[2]})^{l-2u}\,(Q_{n+1}^{[2]})^{l-u} = \left[\eta\prod_{j=1}^{n}(w\eta+t\beta_{n+1,\,j}^{[2]})\right]^{l-2u} w^{l-u}$$

$$= w^{l-u}\,\eta^{l-2u}\left[\prod_{j=1}^{n}(w\eta+t\beta_{n+1,\,j}^{[2]})\right]^{l-2u}.$$

Next we take $g_{n+1}^{[2]} : z=1/w,\ \zeta=w^2\eta$. Since

$$w^{l-u}\,\eta^{l-2u}\left[\prod_{j=1}^{n}(w\eta+t\beta_{n+1,\,j}^{[2]})\right]^{l-2u}$$

$$= \frac{1}{w^{l-3u}}\,(w^2\eta)^{l-2u}\left[\prod_{j=1}^{n}\left(\frac{1}{w}(w^2\eta)+t\beta_{n+1,\,j}^{[2]}\right)\right]^{l-2u},$$

the map $g_{n+1}^{[2]}$ transforms the polynomial $A_{n+1}^{[2]}$ to $z^{l-3u}\zeta^{l-2u}\Big[\prod_{j=1}^{n}(z\zeta + t\beta_{n+1,\,j}^{[2]})\Big]^{l-2u}$, that is

$$z^{l-3u}\left[\zeta\prod_{j=1}^{n}(z\zeta+t\beta_{n+1,\,j}^{[2]})\right]^{l-2u}. \tag{12.2.3}$$

For Step 3, we begin with

$$P_1^{[3]} := \eta, \qquad Q_1^{[3]} := w\prod_{j=1}^{n}(w\eta+t\beta_{n+1,\,j}^{[2]}) \quad \text{and} \quad A_1^{[3]} := (P_1^{[3]})^{l-3u}(Q_1^{[3]})^{l-2u}.$$

Note that (12.2.3) coincides with $A_1^{[3]}$, and as we showed above, $g_{n+1}^{[2]}:\ z = 1/w,\ \zeta = w^2\eta$ transforms $A_{n+1}^{[2]}(w,\eta,t)$ to $A_1^{[3]}(\zeta,z,t)$.

Step k Repeating this process, in Step k where k satisfies $l-\mathsf{k}u \geq 0$, we have a polynomial:

$$A_i^{[\mathsf{k}]} := (P_i^{[\mathsf{k}]})^{l-\mathsf{k}u}(Q_i^{[\mathsf{k}]})^{l-(\mathsf{k}-1)u}, \tag{12.2.4}$$

where we note that $P_{n+1}^{[\mathsf{k}]}$ is an ascending polynomial of length 0 and $Q_{n+1}^{[\mathsf{k}]}$ is a descending polynomial of length n.

Step N (final step) By assumption u divides l, and we write $l = Nu$. Substituting $i = n+1$, $\mathsf{k} = N$ and $l = Nu$ in (12.2.4), we obtain the terminal polynomial

$$A_{n+1}^{[N]} = (Q_{n+1}^{[N]})^u = w^u.$$

Thus the sequences constructed above fulfill the desired properties, establishing Proposition 12.2.1. □

Remark 12.2.2 (Another choice of maps) *In Step* k*, we took a map* $g_{n+1}^{[\mathsf{k}]}:\ z = 1/w,\ \zeta = w^2\eta$*. Instead, we may take another map* $g_{n+1}^{[\mathsf{k}]'}:\ z = 1/w,\ \zeta = w^2\eta + t\beta_{n+1,\,n}^{[\mathsf{k}]}w$ *to construct a similar sequence to that in* Proposition 12.2.1*. We shall explain this for the case* $\mathsf{k}=1$*. Since*

$$\begin{aligned} A_{n+1}^{[1]} &= w^l\eta^{l-u}\prod_{j=1}^{n}(w\eta+t\beta_j)^{l-u} \\ &= \frac{1}{w^{l-2u}}(w^2\eta)^{l-u}\prod_{j=1}^{n}\left[\frac{1}{w}(w^2\eta)+t\beta_j\right]^{l-u} \ (12.2.1), \end{aligned}$$

(*where for brevity, we set* $\beta_j := \beta_{n+1,\,j}^{[1]}$)*, the map* $g_{n+1}^{[1]'}:\ z = 1/w,\ \zeta = w^2\eta + t\beta_n w$ *transforms* $A_{n+1}^{[1]}$ *to*

$$
\begin{aligned}
&z^{l-2u}\left(\zeta - t\beta_n\frac{1}{z}\right)^{l-u}\prod_{j=1}^{n}\left[\ z\left(\zeta - t\beta_n\frac{1}{z}\right)+t\beta_j\ \right]^{l-u}\\
&\quad = z^{l-2u}\left(\zeta - t\beta_n\frac{1}{z}\right)^{l-u}\prod_{j=1}^{n}(z\zeta - t\beta_n + t\beta_j)^{l-u}\\
&\quad = z^{l-2u}\left(\zeta - t\beta_n\frac{1}{z}\right)^{l-u}\prod_{j=1}^{n-1}(z\zeta - t\beta_n + t\beta_j)^{l-u}\cdot(z\zeta)^{l-u}\\
&\quad = z^{l-2u}\zeta^{l-u}\,(z\zeta - t\beta_n)^{l-u}\prod_{j=1}^{n-1}\Big(z\zeta + t(\beta_j-\beta_n)\Big)^{l-u}\\
&\quad = z^{l-2u}\left[\zeta(z\zeta - t\beta_n)\prod_{j=1}^{n-1}\Big(z\zeta + t(\beta_j-\beta_n)\Big)\right]^{l-u}.
\end{aligned}
$$

We rewrite the last expression; setting

$$
P_1^{[2]'} := z \qquad \text{and} \qquad Q_1^{[2]'} := \zeta\prod_{j=1}^{n}(z\zeta + t\beta'_j),
$$

where $\beta'_j := \beta_j - \beta_n$ $(j = 1, 2, \ldots, n-1)$ *and* $\beta'_n := -\beta_n$*, then the last expression is written as* $(P_1^{[2]'})^{l-2u}(Q_1^{[2]'})^{l-u}$*. Hence we may repeat the process Step* 2*, Step* 3*,… in the proof of Proposition* 12.2.1*.*

Now we slightly generalize Proposition 12.2.1; we start from more general polynomial A, and we construct certain sequence of polynomials $A_i^{[\mathsf{k}]}$ and maps $g_i^{[\mathsf{k}]}$ which constitute a 'long sequence' terminating at $A_{n+1}^{[N]} = w^u\,(w^r\eta + t\delta_{n+1}^{[N]})^b$, $\delta_{n+1}^{[N]} \in \mathbb{C}$:

$$
A = A_1^{[1]} \xmapsto{g_1^{[1]}} A_2^{[1]} \xmapsto{g_2^{[1]}} \cdots \xmapsto{g_n^{[1]}} A_{n+1}^{[1]} \xmapsto{g_{n+1}^{[1]}} A_1^{[2]} \xmapsto{g_1^{[2]}} \cdots
$$

$$
\cdots
$$

$$
\cdots \xmapsto{g_n^{[\mathsf{k}-1]}} A_{n+1}^{[\mathsf{k}-1]} \xmapsto{g_{n+1}^{[\mathsf{k}-1]}} A_1^{[\mathsf{k}]} \xmapsto{g_1^{[\mathsf{k}]}} A_2^{[\mathsf{k}]} \xmapsto{g_2^{[\mathsf{k}]}} \cdots \xmapsto{g_n^{[\mathsf{k}]}} A_{n+1}^{[\mathsf{k}]} \xmapsto{g_{n+1}^{[\mathsf{k}]}} A_1^{[\mathsf{k}+1]} \xmapsto{g_1^{[\mathsf{k}+1]}} \cdots
$$

$$
\cdots
$$

$$
\cdots \xmapsto{g_n^{[N-1]}} A_{n+1}^{[N-1]} \xmapsto{g_{n+1}^{[N-1]}} A_1^{[N]} \xmapsto{g_1^{[N]}} A_2^{[N]} \xmapsto{g_2^{[N]}} \cdots \xmapsto{g_n^{[N]}} A_{n+1}^{[N]} = w^u\,(w^r\eta + t\delta_{n+1}^{[N]})^b.
$$

A generalization of Proposition 12.2.1 is given as follows (when $b = 0$, the following proposition reduces to Proposition 12.2.1).

Proposition 12.2.3 *Let* b*,* l*, and* u *be positive integers such that* u *divides* l*, and write* $l = Nu$*. Consider a polynomial* $A(w, \eta, t) = P^{l-u}Q^lR^b$*, where* $P = \eta$*,* $Q = w\prod_{j=1}^{n}(w\eta + t\gamma_j)$*,* $(\gamma_j \in \mathbb{C})$*, and* $R = w\eta + t\delta$*,* $(\delta \in \mathbb{C})$*. Then for each* k $(\mathsf{k} = 1, 2, \ldots, N)$*, there exist*

(i) *a sequence of ascending polynomials* $P_1^{[k]}, P_2^{[k]}, \ldots, P_{n+1}^{[k]}$ *with* $P_1^{[1]} = P$,
(ii) *a sequence of descending polynomials* $Q_1^{[k]}, Q_2^{[k]}, \ldots, Q_{n+1}^{[k]}$ *with* $Q_1^{[1]} = Q$,
(iii) *a sequence of stable polynomials* $R_1^{[k]}, R_2^{[k]}, \ldots, R_{n+1}^{[k]}$ *with* $R_1^{[1]} = R$,
(iv) *a sequence of maps* $g_1^{[k]}, g_2^{[k]}, \ldots, g_n^{[k]}$ *of the form*

$$g_i^{[k]}: \quad z = \frac{1}{w}, \quad \zeta = w^2\eta + t\alpha_i^{[k]} w \quad \text{for some } \alpha_i^{[k]} \in \mathbb{C},$$

and $g_{n+1}^{[k]}: \ z = \frac{1}{w}, \ \zeta = w^2\eta \quad (\mathsf{k} = 1, 2, \ldots, N-1)$

such that setting

$$A_i^{[k]} := (P_i^{[k]})^{l-\mathsf{k}u} (Q_i^{[k]})^{l-(\mathsf{k}-1)u} (R_i^{[k]})^b, \qquad (\mathsf{k} = 1, 2, \ldots, N, \quad i = 1, 2, \ldots, n+1),$$

then

(1) *for* $i = 1, 2, \ldots, n$, *the map* $g_i^{[k]}$ *transforms* $A_i^{[k]}(w, \eta, t)$ *to* $A_{i+1}^{[k]}(\zeta, z, t)$, *whereas* $g_{n+1}^{[k]}$ *transforms* $A_{n+1}^{[k]}(w, \eta, t)$ *to* $A_1^{[k+1]}(\zeta, z, t)$, *and*
(2) *writing* $R_i^{[k]} = w\eta + t\delta_i^{[k]}$ *where* $\delta_i^{[k]} \in \mathbb{C}$, *then* $A_{n+1}^{[N]} = w^u (w^r\eta + t\delta_{n+1}^{[N]})^b$.

Proof. First we take the sequences in Proposition 12.2.1: for $\mathsf{k} = 1, 2, \ldots, N$,

- a sequence of ascending polynomials $P_1^{[k]}, P_2^{[k]}, \ldots, P_{n+1}^{[k]}$ with $P_1^{[1]} = P$,
- a sequence of descending polynomials $Q_1^{[k]}, Q_2^{[k]}, \ldots, Q_{n+1}^{[k]}$ with $Q_1^{[1]} = Q$,
- a sequence of maps $g_1^{[k]}, g_2^{[k]}, \ldots, g_n^{[k]}$, and $g_{n+1}^{[1]}, g_{n+1}^{[2]}, \ldots, g_{n+1}^{[N-1]}$.

Next, let $R_1^{[k]}, R_2^{[k]}, \ldots, R_{n+1}^{[k]}$ be a sequence of stable polynomials (a stable sequence) associated with $R_1^{[1]} := R$ and maps $g_1^{[k]}, g_2^{[k]}, \ldots, g_{n+1}^{[k]}$. Since $R_1^{[1]} = R$ has length 1, the stable polynomial $R_i^{[k]}$ also has length 1 by Lemma 12.1.1 (3), and so we may write $R_i^{[k]} = w\eta + t\delta_i^{[k]}$ for some $\delta_i^{[k]} \in \mathbb{C}$. We then set

$$A_i^{[k]} = (P_i^{[k]})^{l-\mathsf{k}u} (Q_i^{[k]})^{l-(\mathsf{k}-1)u} (R_i^{[k]})^b, \quad \mathsf{k} = 1, 2, \ldots, N, \ i = 1, 2, \ldots, n+1. \tag{12.2.5}$$

By construction, $A_i^{[k]}$ satisfies the property (1) in the assertion. We next show (2). Substitute $i = n+1$, $\mathsf{k} = N$ and $l = Nd$ in (12.2.5), which yields

$$A_{n+1}^{[N]} = (Q_{n+1}^{[N]})^u (R_{n+1}^{[N]})^b.$$

Since $Q_{n+1}^{[N]} = w$ and $R_{n+1}^{[N]} = w\eta + t\delta_{n+1}^{[N]}$, we have $A_{n+1}^{[N]} = w^u(w\eta + t\delta_{n+1}^{[N]})^b$, confirming the property (2). □

12.3 Recursive construction I

In this section we apply the result in the previous section to give a 'recursive' construction of a complete propagation of $DA_{e-1}(lY, k)$ where Y is of type

C_l and n_e divides k. For the sake of brevity, we assume that $k = n_e$ (the construction below is easily carried over to the general case by replacing the parameter t with t^{k/n_e}). We set

$$a = m_{e-1} - ln_{e-1}, \qquad b = m_e - ln_e, \qquad c = n_{e-1}, \qquad d = n_e$$

(hence $k = d$), and take $\alpha \in \mathbb{C}$ such that $\alpha^d + 1 = 0$. Also we set $u := a - (r_e - 1)b$. Then an e-th propagation of $DA_{e-1}(lY, k)$ is given by (11.3.2), p194, that is,

$$\begin{cases} \mathcal{H}_e : & w^a \eta^b (w^c \eta^d + t^d)^l - s = 0 \\ \mathcal{H}'_e : & z^{l-u}(z\zeta + t\alpha)^b \left(z^{d-1}\zeta^d + \sum_{i=1}^{d-1} {}_d\mathrm{C}_i\, t^i\, \alpha^i z^{d-i-1}\zeta^{d-i} \right)^l - s = 0 \\ g_e : & z = \dfrac{1}{w}, \quad \zeta = w^{r_e}\eta - t\alpha w. \end{cases} \tag{12.3.1}$$

and so

$$\mathcal{H}_{e+1} : \quad \eta^{l-u}(w\eta + t\alpha)^b \left(w^d \eta^{d-1} + \sum_{i=1}^{d-1} {}_d\mathrm{C}_i\, t^i\, \alpha^i w^{d-i} \eta^{d-i-1} \right)^l - s = 0.$$

For the subsequent discussion, it is essential to express

$$\mathcal{H}_{e+1} : \; P^l Q^{l-u} R^b - s = 0,$$

where we set

$$\begin{aligned} &P := \eta, \quad R := w\eta + t\alpha, \\ &Q := w^d \eta^{d-1} + \sum_{i=1}^{d-1} {}_d\mathrm{C}_i\, t^i\, w^{n-i}\eta^{n-1-i} = w \prod_{j=1}^{d-1} (w\eta + t\beta_j), \end{aligned}$$

and $-\beta_1, -\beta_2, \ldots, -\beta_{d-1}$ are the solutions of $x^{d-1} + \sum_{i=1}^{d-1} {}_d\mathrm{C}_i\, x^{d-1-i} = 0$. Note that P, Q, and R are respectively ascending, descending, and stable polynomials, and all the assumptions of Proposition 12.2.3 are fulfilled.

With the above preparation, we now construct further propagations. To avoid complicated notation, we first treat the case $b = 0$, and then the case $b \geq 1$.

Case 1: $b = 0$

We apply Proposition 12.2.1 to construct a complete propagation of $\mathcal{H}_{e+1}$ as follows. Write $l = Nu$, and for each $\mathsf{k} = 1, 2, \ldots, N$ and $i = 1, 2, \ldots, d$, we define

$$\begin{cases} \mathcal{H}_i^{[\mathsf{k}]}: & (P_i^{[\mathsf{k}]})^{l-\mathsf{k}u}\,(Q_i^{[\mathsf{k}]})^{l-(\mathsf{k}-1)u} - s = 0 \\ \mathcal{H}_i^{[\mathsf{k}]\prime}: & (P_{i+1}^{[\mathsf{k}]})^{l-\mathsf{k}u}\,(Q_{i+1}^{[\mathsf{k}]})^{l-(\mathsf{k}-1)u} - s = 0 \\ g_i^{[\mathsf{k}]}: & \text{the map defined in Proposition 12.2.1,} \end{cases} \tag{12.3.2}$$

where

(i) for consistency with the notations of ascending and descending sequences, we write $\mathcal{H}_1^{[1]}$ instead of $\mathcal{H}_{e+1}$ *etc.* (this expression is also useful to grasp the "periodicity" of the resulting deformation),

(ii) in the definition of $\mathcal{H}_i^{[\mathsf{k}]}$, $P_i^{[\mathsf{k}]}$ means $P_i^{[\mathsf{k}]}(w,\eta,t)$, while in the definition of $\mathcal{H}_i^{[\mathsf{k}]\prime}$, $P_{i+1}^{[\mathsf{k}]}$ means $P_{i+1}^{[\mathsf{k}]}(\zeta,z,t)$, and

(iii) note that $Q = Q_1^{[1]}$ has length $d-1$, and the integer n in Proposition 12.2.1 corresponds to $d-1$ in the present situation.

By Proposition 12.2.1 (1) and (2), the data (12.3.2) provides a complete propagation of the deformation atlas $DA_{e-1}(lY,k)$ such that $\mathcal{H}_d^{[N]\prime}: \zeta^u - s = 0$.

Remark 12.3.1 *When $l = u = 1$, this construction is nothing other than the construction for the ripple type which we gave in* §8.1, p144.

Finally, let $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$ be the barking family obtained by patching the complete propagation above, and then the deformation from X to $X_{0,t} := \Psi^{-1}(0,t)$ is described in Figure 12.3.1.

Remark 12.3.2 *When $b = 0$ in* (12.3.1), *we have $\mathcal{H}_e|_{s=0}: \; w^a(w^c\eta^d + t^d)^l = 0$, a union of multiple curves $w^a = 0$ and $(w^c\eta^d + t^d)^l = 0$. The latter curve admits a further factorization as follows: From the definition of type C_l, the integer $d\,(= n_e)$ divides $c\,(= n_{e-1})$, and so we may write $c = d\,d'$ for some positive integer d'. Then*

$$\begin{aligned} (w^c\eta^d + t^d)^l &= \Big((w^{d'}\eta)^d + t^d\Big)^l \\ &= \prod_{j=1}^{d} \Big(w^{d'}\eta + te^{2\pi\sqrt{-1}j/d}\Big)^l. \end{aligned}$$

Note that $w^{d'}\eta + te^{2\pi\sqrt{-1}j/d} = 0$ $(t \neq 0)$ *is smooth. Therefore $\mathcal{H}_e|_{s=0}$ consists of one irreducible component of multiplicity a and d irreducible components of multiplicity l. See* Figure 12.3.1.

Case 2: $b \geq 1$

In this case, we apply Proposition 12.2.3 to construct a complete propagation of the deformation atlas $DA_{e-1}(lY,k)$ as follows. Recall that u divides l by

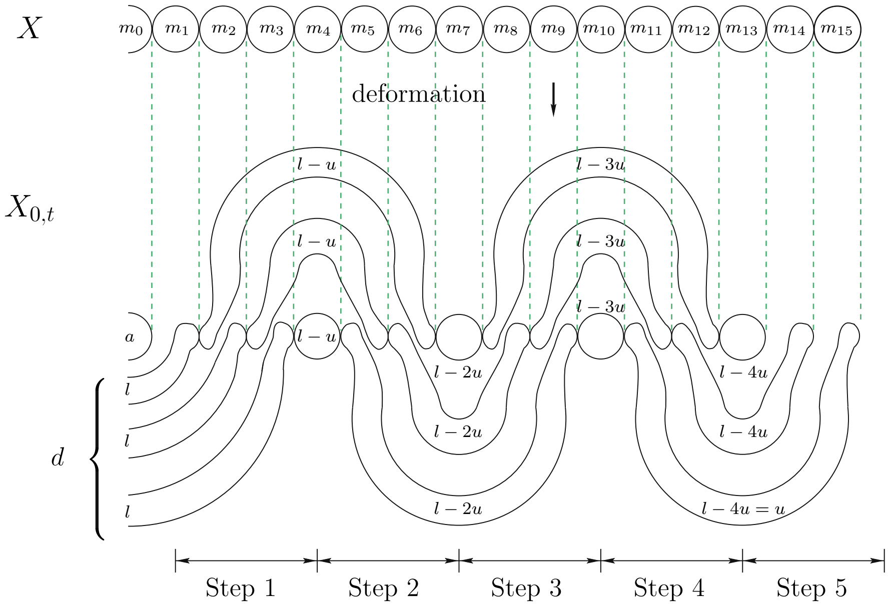


Fig. 12.3.1. Recursive Construction I ($b = 0$): Deformation from X to $X_{0,t}$ for the case $e = 0$, $d = 3$ and $l = 5u$. (Recall $a = m_{e-1} - ln_{e-1}$, $b = m_e - ln_e$, $d = n_e$ and $u = a + b - r_e b$.) In Step k ($\mathsf{k} = 1, 2, 3, 4, 5$), there are exactly d irreducible components of multiplicity $l - \mathsf{k}u$ in $X_{0,t}$.

the definition of type C_l, and we write $l = Nu$. For $\mathsf{k} = 1, 2, \ldots, N$ and $i = 1, 2, \ldots, d$, we define (see also the explanation subsequent to (12.3.2))

$$\begin{cases} \mathcal{H}_i^{[\mathsf{k}]}: & (P_i^{[\mathsf{k}]})^{l-\mathsf{k}u}\,(Q_i^{[\mathsf{k}]})^{l-(\mathsf{k}-1)u}\,(R_i^{[\mathsf{k}]})^b - s = 0 \\ \mathcal{H}_i^{[\mathsf{k}]\prime}: & (P_{i+1}^{[\mathsf{k}]})^{l-\mathsf{k}u}\,(Q_{i+1}^{[\mathsf{k}]})^{l-(\mathsf{k}-1)u}\,(R_{i+1}^{[\mathsf{k}]})^b - s = 0 \\ g_i^{[\mathsf{k}]}: & \text{the map defined in Proposition 12.2.3.} \end{cases} \tag{12.3.3}$$

However (12.3.3) does not yet give a complete propagation; indeed by Proposition 12.2.3 (2),

$$\mathcal{H}_d^{[N]}: \quad \zeta^u(z\zeta + t\delta)^b - s = 0 \quad \text{where } \delta := \delta_d^{[N]} \in \mathbb{C}.$$

To construct further propagations, we set $r := r_1^{[N+1]}$, and define

$$\begin{cases} \mathcal{H}_1^{[N+1]}: & w^u(w\eta + \delta)^b - s = 0 \\ \mathcal{H}_1^{[N+1]\prime}: & z^{(r-1)b-u}\zeta^b - s = 0 \\ g_1^{[N+1]}: & z = \dfrac{1}{w}, \quad \zeta = w^r\eta + \delta w^{r-1}. \end{cases}$$

We assert that $g_1^{[N+1]}$ transforms $\mathcal{H}_1^{[N+1]}$ to $\mathcal{H}_1^{[N+1]'}$. In fact, since

$$w^u(w\eta + t\delta)^b = w^u\left(\frac{1}{w^{r-1}}(w^r\eta) + t\delta\right)^b,$$

the map $g_1^{[N+1]}$ transforms $\mathcal{H}_1^{[N+1]}$ to

$$\begin{aligned}\frac{1}{z^u}\left(z^{r-1}\left(\zeta - t\delta\frac{1}{z^{r-1}}\right) + t\delta\right)^b - s &= \frac{1}{z^u}(z^{r-1}\zeta)^b - s \\ &= z^{b(r-1)-u}\zeta^b - s,\end{aligned}$$

which is nothing but the equation of $\mathcal{H}_1^{[N+1]'}$. (This computation is the same as that in the construction of a complete propagation of type B_l.) Further propagations of $DA_{e-1}(lY,k)$ are easy to construct; since $\mathcal{H}_1^{[N+1]'}$ is the trivial family of $H_1^{[N+1]'}$, we may propagate it trivially, and accomplish a complete propagation of $DA_{e-1}(lY,k)$.

Remark 12.3.3 *When $b \geq 1$ and $u \leq b$, we may compress $A_i^{[\mathsf{k}]}$ at the factor $Q_i^{[\mathsf{k}]}$ in the construction of waving sequences, which yields a non-recursive construction of deformations; the resulting deformation does not possess periodicity.*

Finally, let $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$ be the barking family obtained by patching the complete propagation above, and then the deformation from X to $X_{0,t} := \Psi^{-1}(0,t)$ is described in Figure 12.3.2 (when b does not divide u) and Figure 12.3.3 (when b divides u; see also Remark 12.3.4 below).

Remark 12.3.4 *If b divides u, we write $u = bb'$. Then we may express $\mathcal{H}_1^{[N+1]}$ as $(w^{b'+1}\eta + t\delta w^{b'})^b - s = 0$. In fact,*

$$\begin{aligned}w^u(w\eta + t\delta)^b &= (w^{b'})^b(w\eta + t\delta)^b \\ &= (w^{b'+1}\eta + t\delta w^{b'})^b.\end{aligned}$$

Therefore

$$\begin{cases} \mathcal{H}_1^{[N+1]} : & (w^{b'+1}\eta + t\delta w^{b'})^b - s = 0 \\ \mathcal{H}_1^{[N+1]'} : & \zeta^b - s = 0 \\ g_1^{[N+1]} : & z = \dfrac{1}{w}, \quad \zeta = w^{b'+1}\eta + t\delta w^{b'}. \end{cases}$$

Example 12.3.5 (The simplest example) Let $X = m_0\Delta_0 + m_1\Theta_1 + \cdots + m_\lambda\Theta_\lambda$ be a branch such that $m_{\lambda-2} = 2l+1$, $m_{\lambda-1} = l+1$, and $m_\lambda = 1$. In this case, $r_{\lambda-1} = 2$ and $r_\lambda = l+1$. Let lY be a subbranch of X such that

$$Y = n_0\Delta_0 + n_1\Theta_1 + \cdots + n_{\lambda-1}\Theta_{\lambda-1} \quad \text{and} \quad n_{\lambda-2} = n_{\lambda-1} = 1.$$

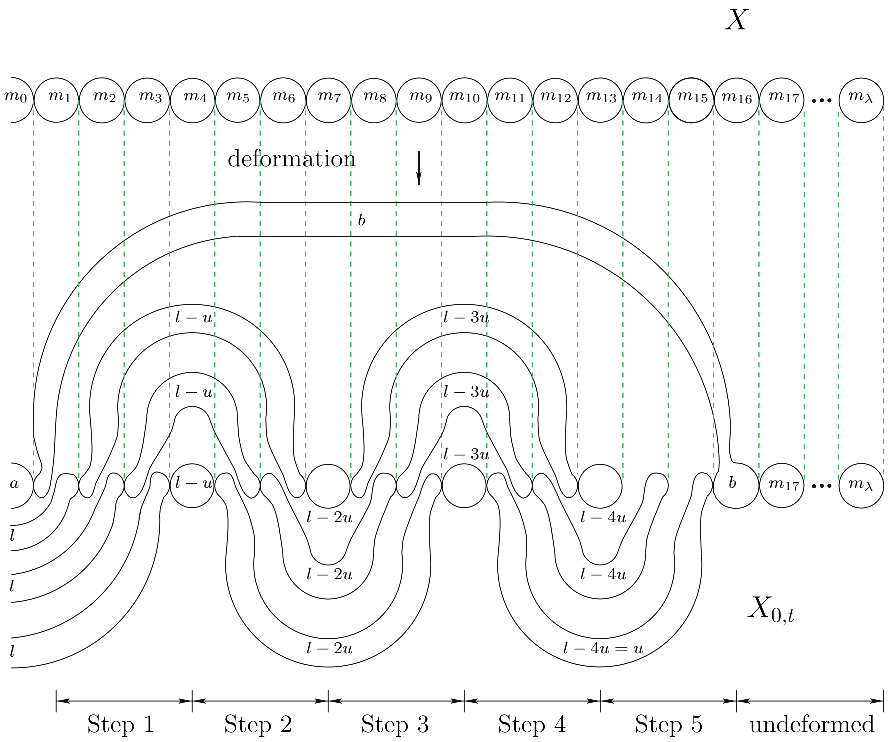


Fig. 12.3.2. Recursive Construction I ($b \geq 1$ and b does not divide u): Deformation from X to $X_{0,t}$ for the case $e = 0$, $d = 3$ and $l = 5u$. (Recall $a = m_{e-1} - ln_{e-1}$, $b = m_e - ln_e$, $d = n_e$ and $u = a + b - r_e b$.) In Step k ($\mathsf{k} = 1, 2, 3, 4, 5$), there are exactly d irreducible components of multiplicity $l - \mathsf{k}u$ in $X_{0,t}$.

(For instance, $X = 5\Delta_0 + 3\Theta_1 + \Theta_2$, $\;Y = \Delta_0 + \Theta_1\;$ and $\;l = 2$.) It is easy to check that Y is of type C_l. A complete propagation of $DA_{\lambda-2}(lY, k)$ is given by the following data:

$$\begin{cases} \mathcal{H}_{\lambda-1}: & w^{l+1}\eta(w\eta + t^k)^l - s = 0 \\ \mathcal{H}'_{\lambda-1}: & (z\zeta - t^k)\zeta^l - s = 0 \\ g_{\lambda-1}: & z = \dfrac{1}{w}, \quad \zeta = w^2\eta + t^k w, \end{cases}$$

$$\begin{cases} \mathcal{H}_{\lambda}: & (w\eta - t^k)w^l - s = 0 \\ \mathcal{H}'_{\lambda}: & \zeta - s = 0 \\ g_{\lambda}: & z = \dfrac{1}{w}, \quad \zeta = w^{l+1}\eta - t^k w^l. \end{cases}$$

(Note: For type C_l, in order to construct a complete propagation, we require that n_e divides k. In the present case, k may be an arbitrary positive integer, because $n_e(= n_{\lambda-1} = 1)$ always divides k.) The deformation from X to $X_{0,t}$ is shown in Figure 12.3.4.

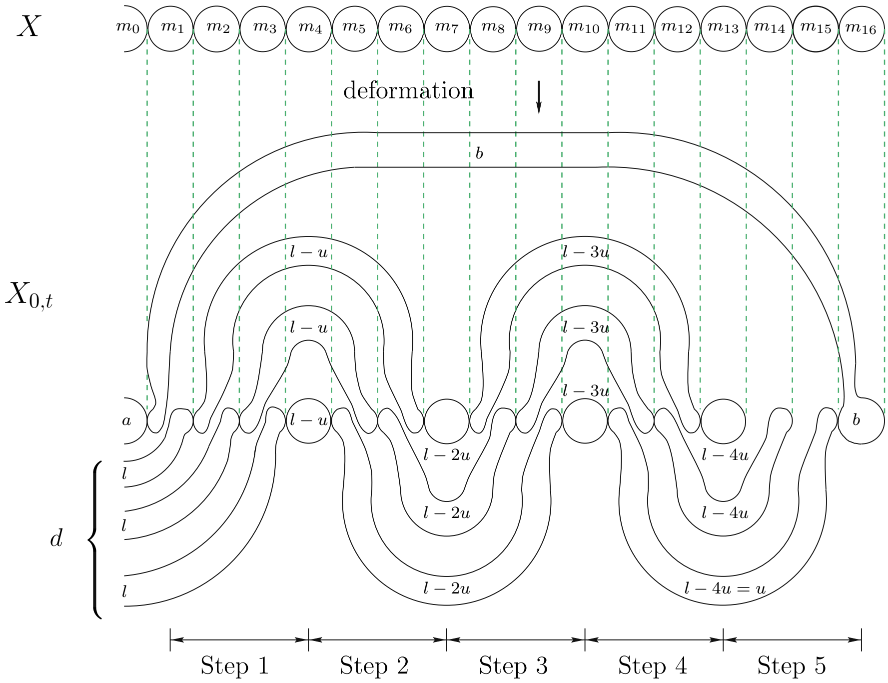


Fig. 12.3.3. Recursive Construction I ($b \geq 1$ and b divides u): Deformation from X to $X_{0,t}$ for the case $e = 0$, $d = 3$ and $l = 5u$. (Recall $a = m_{e-1} - ln_{e-1}$, $b = m_e - ln_e$, $d = n_e$ and $u = a + b - r_e b$.) See also Remark 12.3.4. In Step k ($\mathsf{k} = 1, 2, 3, 4, 5$), there are exactly d irreducible components of multiplicity $l - \mathsf{k}u$ in $X_{0,t}$.

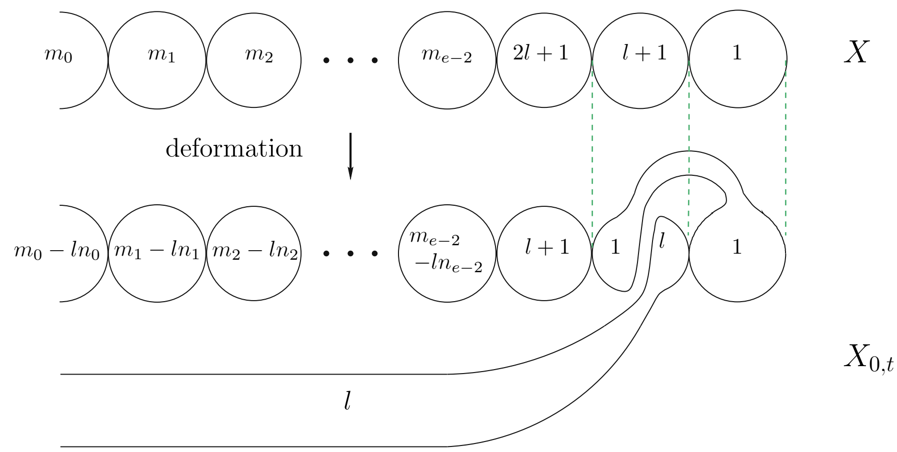


Fig. 12.3.4. The simplest deformation of type C_l

12.4 Technical preparation II

In this section we will give another recursive construction of a complete propagation of $DA_{e-1}(lY,k)$ where Y is of type C_l and n_e divides k. We will use two kinds of special polynomials; the first one is a stable polynomial

$$R(w,\eta,t)=\prod_{j=1}^{n}(w\eta+t\delta_j),\quad \delta_j\in\mathbb{C}$$

and another one is defined as follows. Fix positive integers l and u such that $l\geq u$. For a positive integer i satisfying $l-iu\geq 0$, a polynomial $S_i(w,\eta)=w^{l-(i-1)u}\eta^{l-iu}$ is called a *shrinking polynomial*, and we say that $S_i(\zeta,z)$ is an *interchanged shrinking polynomial*.

Lemma 12.4.1 (1) *A map* $g: z=1/w,\ \zeta=w^2\eta$ *transforms a shrinking polynomial* $S_i(w,\eta)=w^{l-(i-1)u}\eta^{l-iu}$ *to an interchanged shrinking polynomial*

$$S_{i+1}(\zeta,z)=z^{l-(i+1)u}\zeta^{l-iu}.$$

(2) *When* u *divides* l, *write* $l=Nu$ *where* N *is a positive integer. Then* $S_{N-1}(w,\eta)=w^{2u}\eta^u$ *and* $S_N(w,\eta)=w^u$.

Proof. First we show (1). Since

$$w^{l-(i-1)u}\eta^{l-iu}=\frac{1}{w^{2(l-iu)-l+(i-1)u}}(w^2\eta)^{l-iu},$$

the map g transforms $S_i(w,\eta)=w^{l-(i-1)u}\eta^{l-iu}$ to $z^{2(l-iu)-l+(i-1)u}\zeta^{l-iu}$ which is equal to $S_{i+1}(\zeta,z)=z^{l-(i+1)u}\zeta^{l-iu}$. The assertion (2) is clear from the definition of S_{N-1} and S_N. □

We next construct a long sequence of certain polynomials $A_i^{[\mathsf{k}]}$ together with certain maps $g_i^{[\mathsf{k}]}$, starting from a particular polynomial $A=A_1^{[1]}$:

$$A=A_1^{[1]}\xmapsto{g_1^{[1]}}A_2^{[1]}\xmapsto{g_2^{[1]}}\cdots\xmapsto{g_{N-1}^{[1]}}A_N^{[1]}\xmapsto{g_N^{[1]}}A_1^{[2]}\xmapsto{g_1^{[2]}}\cdots$$

$$\cdots$$

$$\cdots\xmapsto{g_{N-1}^{[\mathsf{k}-1]}}A_N^{[\mathsf{k}-1]}\xmapsto{g_N^{[\mathsf{k}-1]}}A_1^{[\mathsf{k}]}\xmapsto{g_1^{[\mathsf{k}]}}A_2^{[\mathsf{k}]}\xmapsto{g_2^{[\mathsf{k}]}}\cdots\xmapsto{g_{N-1}^{[\mathsf{k}]}}A_N^{[\mathsf{k}]}\xmapsto{g_N^{[\mathsf{k}]}}A_1^{[\mathsf{k}+1]}\xmapsto{g_1^{[\mathsf{k}+1]}}\cdots$$

$$\cdots$$

$$\cdots\xmapsto{g_{N-1}^{[d-1]}}A_N^{[d-1]}\xmapsto{g_N^{[d-1]}}A_1^{[d]}\xmapsto{g_1^{[d]}}A_2^{[d]}\xmapsto{g_2^{[d]}}\cdots\xmapsto{g_{N-1}^{[d]}}A_N^{[d]}=w^u.$$

Proposition 12.4.2 *Let l and u be positive integers such that u divides l, and write $l = Nu$. Consider a polynomial $A(w, \eta, t) = S\,T^l$, where $S := w^l \eta^{l-u}$ is a shrinking polynomial and $T = \prod_{j=1}^{d}(w\eta + t\beta_j)$, ($\beta_j \in \mathbb{C}$) is a stable polynomial. Then there exist*

(i) *a sequence of shrinking polynomials $S_1 = S, S_2, \ldots, S_N$,*
(ii) *a sequence of stable polynomials $T^{[1]} = T, T^{[2]}, \ldots, T^{[d]}$,*
(iii) *a sequence of maps $g_1^{[\mathsf{k}]}, g_2^{[\mathsf{k}]}, \ldots, g_{N-1}^{[\mathsf{k}]}$ ($\mathsf{k} = 1, 2, \ldots, d$); all of them are $z = 1/w,\ \zeta = w^2\eta$, and a sequence of maps $g_N^{[\mathsf{k}]}$ ($\mathsf{k} = 1, 2, \ldots, d-1$) where $g_N^{[\mathsf{k}]} : z = 1/w,\ \zeta = w^2\eta + t\alpha_N^{[\mathsf{k}]} w$ for some $\alpha_N^{[\mathsf{k}]} \in \mathbb{C}$*

such that setting $A_i^{[\mathsf{k}]} := S_i\,(T^{[\mathsf{k}]})^l$, ($\mathsf{k} = 1, 2, \ldots, d$, $i = 1, 2, \ldots, N$), then

(1) *for $i = 1, 2, \ldots, N-1$, the map $g_i^{[\mathsf{k}]}$ transforms $A_i^{[\mathsf{k}]}(w, \eta, t)$ to $A_{i+1}^{[\mathsf{k}]}(\zeta, z, t)$, whereas $g_N^{[\mathsf{k}]}$ transforms $A_N^{[\mathsf{k}]}(w, \eta, t)$ to $A_1^{[\mathsf{k}+1]}(\zeta, z, t)$, and*
(2) *$A_N^{[d]} = w^u$.*

Proof. We set $S_i = w^{l-(i-1)u}\eta^{l-iu}$, ($i = 1, 2, \ldots, N$) and

$$T^{[\mathsf{k}]} = \prod_{j=1}^{d+1-\mathsf{k}} (w\eta + t\beta_j^{[\mathsf{k}]}), \qquad (\mathsf{k} = 1, 2, \ldots, d),$$

where we define $\beta_j^{[\mathsf{k}]} \in \mathbb{C}$ inductively by setting $\beta_j^{[1]} := \beta_j$, and then by a recursive formula

$$\beta_j^{[\mathsf{k}+1]} := \beta_j^{[\mathsf{k}]} - \beta_{d+1-\mathsf{k}}^{[\mathsf{k}]}, \qquad j = 1, 2, \ldots, d - \mathsf{k}.$$

Also we set

$$g_i^{[\mathsf{k}]} :\ z = 1/w,\ \zeta = w^2\eta \quad (\mathsf{k} = 1, 2 \ldots, d,\ i = 1, 2, \ldots, N-1), \quad \text{and}$$

$$g_N^{[\mathsf{k}]} :\ z = 1/w,\ \zeta = w^2\eta + t\beta_{d+1-\mathsf{k}}^{[\mathsf{k}]} w \quad (\mathsf{k} = 1, 2 \ldots, d-1).$$

We then consider polynomials $A_i^{[\mathsf{k}]} := S_i\,(T^{[\mathsf{k}]})^l$, ($\mathsf{k} = 1, 2, \ldots, d$, $i = 1, 2, \ldots, N$), that is,

$$A_i^{[\mathsf{k}]} = w^{l-(i-1)u}\,\eta^{l-iu} \left[\prod_{j=1}^{d+1-\mathsf{k}} (w\eta + t\beta_j^{[\mathsf{k}]})\right]^l.$$

Since $S_N = w^u$ (Lemma 12.4.1) and $T^{[d]} = 1$, we have $A_N^{[d]} = w^u$, confirming (2).

We next show (1). Note that $g_i^{[\mathsf{k}]} : z = 1/w,\ \zeta = w^2\eta$ ($i = 1, 2, \ldots, N-1$) transforms (a) $S_i(w, \eta)$ to $S_{i+1}(\zeta, z)$ by Lemma 12.4.1, and (b) $T^{[\mathsf{k}]}(w, \eta, t)$ to $T^{[\mathsf{k}]}(\zeta, z, t)$ because $T^{[\mathsf{k}]}$ is a stable polynomial. Thus $g_i^{[\mathsf{k}]}$ ($i = 1, 2, \ldots, N-1$) transforms $A_i^{[\mathsf{k}]}(w, \eta, t)$ to $A_{i+1}^{[\mathsf{k}]}(\zeta, z, t)$. Finally we show the remaining part of (1) which insists that the map $g_N^{[\mathsf{k}]} : z = 1/w,\ \zeta = w^2\eta + t\beta_{d+1-\mathsf{k}}^{[\mathsf{k}]} w$ transforms

$A_N^{[\mathsf{k}]}(w,\eta,t)$ to $A_1^{[\mathsf{k}+1]}(\zeta,z,t)$. For brevity, we show this only for $\mathsf{k}=1$ (the other case is shown by a similar computation). Since

$$A_N^{[1]}(w,\eta,t) := S_N\,(T^{[1]})^l = w^u \prod_{j=1}^{d}(w\eta + t\beta_j^{[1]})^l = w^u \prod_{j=1}^{d}\left(\frac{1}{w}w^2\eta + t\beta_j^{[1]}\right)^l,$$

the map $g_N^{[1]}:\ z = 1/w,\ \zeta = w^2\eta + t\beta_d^{[1]}w$ transforms $A_N^{[1]}$ to

$$\begin{aligned}\frac{1}{z^u}\prod_{j=1}^{d}\left[z\left(\zeta - t\frac{\beta_d^{[1]}}{z}\right) + t\beta_j^{[1]}\right]^l &= \frac{1}{z^u}\prod_{j=1}^{d}\Big[z\zeta + t(\beta_j^{[1]} - \beta_d^{[1]})\Big]^l \\ &= \frac{1}{z^u}\prod_{j=1}^{d-1}\Big[z\zeta + t(\beta_j^{[1]} - \beta_d^{[1]})\Big]^l \cdot (z\zeta)^l \\ &= z^{l-u}\zeta^l\prod_{j=1}^{d-1}\Big[z\zeta + t(\beta_j^{[1]} - \beta_d^{[1]})\Big]^l.\end{aligned}$$

We set $\beta_j^{[2]} := \beta_j^{[1]} - \beta_d^{[1]}$, and then the last expression is written as

$$z^{l-u}\zeta^l \prod_{j=1}^{d-1}(z\zeta + t\beta_j^{[2]})^l,$$

which equals $A_1^{[2]}(\zeta,z,t) = S_1\,(T^{[2]})^l$ where

$$S_1(\zeta,z) = z^{l-u}\zeta^l \qquad \text{and} \qquad T^{[2]}(\zeta,z,t) = \prod_{j=1}^{d-1}(z\zeta + t\beta_j^{[2]}).$$

This confirms that $g_N^{[1]}$ transforms $A_N^{[1]}$ to $A_1^{[2]}$, and we establish the proposition. □

We slightly generalize Proposition 12.4.2; starting from more general polynomial $A = A_1^{[1]}$, we construct a long sequence of certain polynomials $A_i^{[\mathsf{k}]}$ and certain maps $g_i^{[\mathsf{k}]}$, which terminates at $A_N^{[d]} = w^u(w\eta + t\delta^{[d]})^b$, $\delta^{[d]} \in \mathbb{C}$:

$$A = A_1^{[1]} \xmapsto{g_1^{[1]}} A_2^{[1]} \xmapsto{g_2^{[1]}} \cdots \xmapsto{g_{N-1}^{[1]}} A_N^{[1]} \xmapsto{g_N^{[1]}} A_1^{[2]} \xmapsto{g_1^{[2]}} \cdots$$

$$\cdots$$

$$\cdots \xmapsto{g_{N-1}^{[\mathsf{k}-1]}} A_N^{[\mathsf{k}-1]} \xmapsto{g_N^{[\mathsf{k}-1]}} A_1^{[\mathsf{k}]} \xmapsto{g_1^{[\mathsf{k}]}} A_2^{[\mathsf{k}]} \xmapsto{g_2^{[\mathsf{k}]}} \cdots \xmapsto{g_{N-1}^{[\mathsf{k}]}} A_N^{[\mathsf{k}]} \xmapsto{g_N^{[\mathsf{k}]}} A_1^{[\mathsf{k}+1]} \xmapsto{g_1^{[\mathsf{k}+1]}} \cdots$$

$$\cdots$$

$$\cdots \xmapsto{g_{N-1}^{[d-1]}} A_N^{[d-1]} \xmapsto{g_N^{[d-1]}} A_1^{[d]} \xmapsto{g_1^{[d]}} A_2^{[d]} \xmapsto{g_2^{[d]}} \cdots \xmapsto{g_{N-1}^{[d]}} A_N^{[d]} = w^u(w\eta + t\delta^{[d]})^b.$$

When $b = 0$, the following proposition reduces to Proposition 12.4.2.

Proposition 12.4.3 *Let b, l, and u be positive integers such that u divides l. Write $l = Nu$ where N is a positive integer. Consider a polynomial $A(w, \eta, t) = S\,T^l\,R^b$, where $S := w^l \eta^{l-u}$, $T = \prod_{j=1}^{d}(w\eta + t\beta_j)$, $(\beta_j \in \mathbb{C})$, and $R := w\eta + t\delta$, $(\delta \in \mathbb{C})$. Then there exist*

(i) *a sequence of shrinking polynomials $S_1 = S, S_2, \ldots, S_N$,*
(ii) *a sequence of stable polynomials $T^{[1]} = T, T^{[2]}, \ldots, T^{[d]}$,*
(iii) *a sequence of stable polynomials $R^{[1]} = R, R^{[2]}, \ldots, R^{[d]}$,*
(iv) *a sequence of maps $g_1^{[\mathsf{k}]}, g_2^{[\mathsf{k}]}, \ldots, g_{N-1}^{[\mathsf{k}]}$ $(\mathsf{k} = 1, 2, \ldots, d)$; all of them are $z = 1/w,\ \zeta = w^2\eta$, and a sequence of maps $g_N^{[\mathsf{k}]}$ $(\mathsf{k} = 1, 2, \ldots, d-1)$ where $g_N^{[\mathsf{k}]} : z = 1/w,\ \zeta = w^2\eta + t\alpha_N^{[\mathsf{k}]} w$ for some $\alpha_N^{[\mathsf{k}]} \in \mathbb{C}$*

such that setting $A_i^{[\mathsf{k}]} := S_i\,(T^{[\mathsf{k}]})^l (R^{[\mathsf{k}]})^b$, $(\mathsf{k} = 1, 2, \ldots, d,\ i = 1, 2, \ldots, N)$, then

(1) *for $i = 1, 2, \ldots, N-1$, $g_i^{[\mathsf{k}]}$ transforms $A_i^{[\mathsf{k}]}(w, \eta, t)$ to $A_{i+1}^{[\mathsf{k}]}(\zeta, z, t)$, whereas $g_N^{[\mathsf{k}]}$ transforms $A_N^{[\mathsf{k}]}(w, \eta, t)$ to $A_1^{[\mathsf{k}+1]}(\zeta, z, t)$, and*
(2) *writing $R^{[\mathsf{k}]} = w\eta + t\delta^{[\mathsf{k}]}$ where $\delta^{[\mathsf{k}]} \in \mathbb{C}$, then $A_N^{[d]} = w^u(w\eta + t\delta^{[d]})^b$.*

Proof. First take the sequences in Proposition 12.4.2:

- a sequence of shrinking polynomials $S_1 = S, S_2, \ldots, S_N$,
- a sequence of stable polynomials $T^{[1]} = T, T^{[2]}, \ldots, T^{[d]}$,
- a sequence of maps $g_1^{[\mathsf{k}]}, g_2^{[\mathsf{k}]}, \ldots, g_{N-1}^{[\mathsf{k}]}$ $(\mathsf{k} = 1, 2, \ldots, d)$, and $g_N^{[\mathsf{k}]}$ $(\mathsf{k} = 1, 2, \ldots, d-1)$.

Recall that $T^{[\mathsf{k}]} = \prod_{j=1}^{d+1-\mathsf{k}}(w\eta + t\beta_j^{[\mathsf{k}]})$; we defined $\beta_j^{[\mathsf{k}]} \in \mathbb{C}$ inductively by setting $\beta_j^{[1]} := \beta_j$, and then by a recursive formula

$$\beta_j^{[\mathsf{k}+1]} := \beta_j^{[\mathsf{k}]} - \beta_{d+1-\mathsf{k}}^{[\mathsf{k}]}, \qquad j = 1, 2, \ldots, d - \mathsf{k}.$$

We next set $R^{[\mathsf{k}]} := w\eta + t\gamma^{[\mathsf{k}]}$, where $\gamma^{[\mathsf{k}]} \in \mathbb{C}$ is inductively defined by setting $\gamma^{[1]} := \gamma$, and then by a recursive formula

$$\gamma^{[\mathsf{k}+1]} := \gamma^{[\mathsf{k}]} - \beta_{d+1-\mathsf{k}}^{[\mathsf{k}]}, \qquad j = 1, 2, \ldots, d - \mathsf{k}.$$

Then it is easy to check that polynomials

$$A_i^{[\mathsf{k}]} := S_i\,(T^{[\mathsf{k}]})^l (R^{[\mathsf{k}]})^b \qquad (\mathsf{k} = 1, 2, \ldots, d, \quad i = 1, 2, \ldots, N)$$

satisfy the desired properties. □

12.5 Recursive construction II

We now explain another recursive construction of a complete propagation of the deformation atlas $DA_{e-1}(lY, k)$ where Y is of type C_l and n_e divides k. For the sake of brevity, we assume $k = n_e$; the construction below is easily carried over to the general case by replacing the parameter t by t^{k/n_e}. As before, we set

$$a = m_{e-1} - ln_{e-1}, \qquad b = m_e - ln_e, \qquad c = n_{e-1}, \qquad d = n_e$$

(hence $k = d$), and take $\alpha \in \mathbb{C}$ such that $\alpha^d + 1 = 0$. Also we set $u := a + b - r_e b$. Then as in (11.3.2), p194, an e-th propagation of $DA_{e-1}(lY, k)$ is given by

$$\begin{cases} \mathcal{H}_e : & w^a \eta^b (w^c \eta^d + t^d)^l - s = 0 \\ \mathcal{H}'_e : & z^{l-u}(z\zeta + t\alpha)^b \left(z^{d-1}\zeta^d + \sum_{i=1}^{d-1} {}_d\mathrm{C}_i \, t^i \, \alpha^i \, z^{d-i-1} \, \zeta^{d-i} \right)^l - s = 0 \\ g_e : & z = \dfrac{1}{w}, \quad \zeta = w^{r_e}\eta - t\alpha w. \end{cases}$$

Thus

$$\mathcal{H}_{e+1} : \quad \eta^{l-u}(w\eta + t\alpha)^b \left(w^d \eta^{d-1} + \sum_{i=1}^{d-1} {}_d\mathrm{C}_i \, t^i \, \alpha^i \, w^{d-i} \, \eta^{d-i-1} \right)^l - s = 0.$$

In the previous recursive construction, we expressed $\mathcal{H}_{e+1} : \ A(w, \eta, t) - s = 0$ where we set $A := P^{l-u} Q^l R^b$ and

$$P = \eta, \qquad Q = w^d \eta^{d-1} + \sum_{i=1}^{d-1} {}_d\mathrm{C}_i \, t^i \, \alpha^i \, w^{d-i} \, \eta^{d-i-1}, \qquad R = w\eta + t\alpha.$$

They are respectively ascending, descending, and stable polynomials; we note that $Q = w \prod_{j=1}^{d-1} (w\eta + t\beta_j)$ where β_j $(j = 1, 2, \ldots, d-1)$ are the solutions of

$$X^{d-1} + \sum_{i=1}^{d-1} {}_d\mathrm{C}_i \, t^i \, \alpha^i \, X^{d-i-1} = 0.$$

The factorization $A = P^{l-u} \, Q^l \, R^b$ played an essential role in the previous recursive construction. For another recursive construction, it is essential to choose another factorization $A = S\,T^l R^b$ where $S := w^l \eta^{l-u}$ and $T := \prod_{j=1}^{d} (w\eta + t\beta_j)$, while R is the same as above. Note that S and T are respectively shrinking and stable polynomials, and of course R is a stable polynomial.

Now we go back to the construction of propagations of $DA_{e-1}(lY, k)$; we have already constructed its e-th propagation. We give further propagations; we separate into two cases: $b = 0$ and $b \geq 1$.

Case 1: $b = 0$

In this case $A = S\,T^l$, and by Proposition 12.4.2, the following data gives a complete propagation: for $\mathsf{k} = 1, 2, \ldots, d$ and $i = 1, 2, \ldots, N-1$,

$$\begin{cases} \mathcal{H}_i^{[\mathsf{k}]} : & S_i(w, \eta) \, T^{[\mathsf{k}]}(w, \eta, t)^l - s = 0 \\ \mathcal{H}_i^{[\mathsf{k}]\prime} : & S_{i+1}(\zeta, z) \, T^{[\mathsf{k}]}(\zeta, z, t)^l - s = 0 \\ g_i^{[\mathsf{k}]} : & \text{the map in Proposition 12.4.2,} \end{cases}$$

where for consistency with the notation in Proposition 12.4.2, we write $\mathcal{H}_1^{[1]}$ instead of $\mathcal{H}_{e+1}$, and so on (cf. the explanation following (12.3.2)). This expression is also useful to grasp the "periodicity" of deformations. We note that in the final step d, by Proposition 12.4.2 (2), we have

$$\mathcal{H}_{N-1}^{[d]'} : \ \zeta^u - s = 0.$$

Remark 12.5.1 *Although* $\Theta_1^{[1]}$ *is a wild component of* lY, $g_1^{[1]}$ *is just the transition function (no deformation). This peculiar phenomenon occurs for the case* $l \geq 2$, *while it does not occur for* $l = 1$.

Let $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$ be the barking family obtained from the complete propagation above: the deformation from X to $X_{0,t} := \Psi^{-1}(0, t)$ is described in Figure 12.5.1.

Case 2: $b \geq 1$

In this case $A = S\,T^l R^b$, and the construction of a complete propagation uses the sequences in Proposition 12.4.3: for $\mathsf{k} = 1, 2, \ldots, d$ and $i = 1, 2, \ldots, N-1$, we set

$$\begin{cases} \mathcal{H}_i^{[\mathsf{k}]} : & S_i(w, \eta)\, T^{[\mathsf{k}]}(w, \eta, t)^l\, R^{[\mathsf{k}]}(w, \eta, t)^b - s = 0 \\ \mathcal{H}_i^{[\mathsf{k}]'} : & S_{i+1}(\zeta, z)\, T^{[\mathsf{k}]}(\zeta, z, t)^l\, R^{[\mathsf{k}]}(\zeta, z, t)^b - s = 0 \\ g_i^{[\mathsf{k}]} : & \text{the map in Proposition 12.4.3.} \end{cases}$$

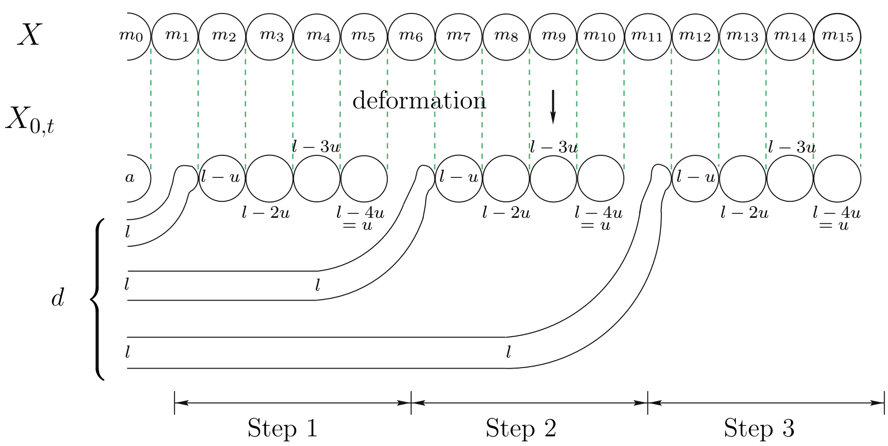


Fig. 12.5.1. Recursive Construction II ($b = 0$): Deformation from X to $X_{0,t}$ for the case $e = 0$, $d = 3$ and $l = 5u$. (Recall $a = m_{e-1} - ln_{e-1}$, $b = m_e - ln_e$, $d = n_e$ and $u = a + b - r_e b$.) In Step k ($\mathsf{k} = 1, 2, 3$), there are $d - \mathsf{k}$ irreducible components of multiplicity l in $X_{0,t}$.

By Proposition 12.4.3 (1) and (2), the map $g_i^{[\mathsf{k}]}$ transforms $\mathcal{H}_i^{[\mathsf{k}]}$ to $\mathcal{H}_i^{[\mathsf{k}]'}$, and

$$\mathcal{H}_{N-1}^{[d]'}: \quad \zeta^u(z\zeta + t\delta^{[d]})^b - s = 0.$$

Further propagations are constructed as follows: for simplicity we set $r := r_N^{[d]}$, and define

$$\begin{cases} \mathcal{H}_N^{[d]}: & w^u(w\eta + t\delta^{[d]})^b - s = 0 \\ \mathcal{H}_N^{[d]'}: & z^{(r-1)b-u}\zeta^b - s = 0 \\ g_N^{[d]}: & z = \dfrac{1}{w}, \quad \zeta = w^r\eta + t\delta^{[d]}w^{r-1}. \end{cases}$$

Since $\mathcal{H}_N^{[d]'}: \ z^{(r-1)b-u}\zeta^b - s = 0$ is the trivial family of $H_N^{[d]'}$, we can trivially propagate it to achieve a complete propagation. (This is exactly the situation we encountered in the construction of a complete propagation for type B_l.)

Finally, let $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$ be the barking family obtained from the complete propagation above, and then the deformation from X to $X_{0,t} := \Psi^{-1}(0,t)$ is described in Figure 12.5.2 (when b does not divide u) and Figure 12.5.3 (when b divides u; see also Remark 12.5.2 below).

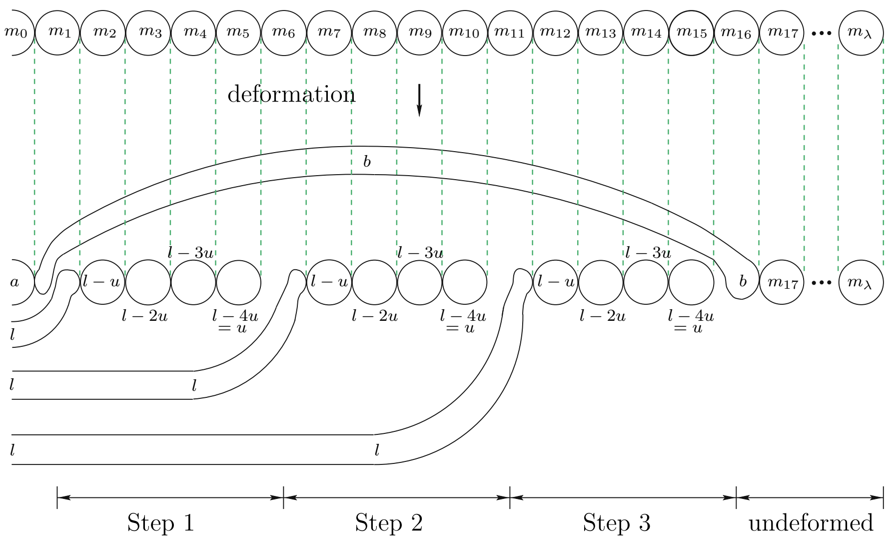


Fig. 12.5.2. Recursive Construction II ($b \geq 1$ and b does not divide u): Deformation from X to $X_{0,t}$ for the case $e = 0$, $d = 3$ and $l = 5u$. (Recall $a = m_{e-1} - ln_{e-1}$, $b = m_e - ln_e$, $d = n_e$ and $u = a + b - r_e b$.) In Step k ($\mathsf{k} = 1, 2, 3$), there are $d - \mathsf{k}$ irreducible components of multiplicity l in $X_{0,t}$.

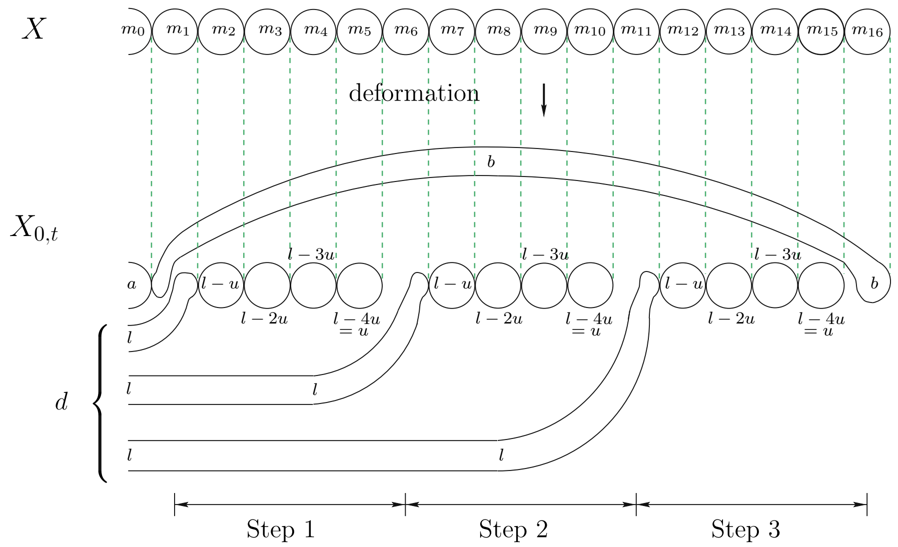


Fig. 12.5.3. Recursive Construction I ($b \geq 1$ and b divides u): Deformation from X to $X_{0,t}$ for the case $e = 0$, $d = 3$ and $l = 5u$. (Recall $a = m_{e-1} - ln_{e-1}$, $b = m_e - ln_e$, $d = n_e$ and $u = a + b - r_e b$.) See also Remark 12.5.2. In Step k ($\mathsf{k} = 1, 2, 3$), there are $d - \mathsf{k}$ irreducible components of multiplicity l in $X_{0,t}$.

Remark 12.5.2 *If b divides u, we write $u = b\,b'$. Then* (see Remark 12.3.4, p222)

$$\begin{cases} \mathcal{H}_N^{[d]} : & (w^{b'+1}\eta + t\delta^{[d]} w^{b'})^b - s = 0 \\ \mathcal{H}_N^{[d]\prime} : & \zeta^b - s = 0 \\ g_N^{[d]} : & z = \dfrac{1}{w}, \quad \zeta = w^{b'+1}\eta + t\delta^{[d]} w^{b'}. \end{cases}$$

12.6 Examples of non-recursive deformations of type C_l

Let Y be a subbranch of type C_l. We take a complete propagation of the deformation atlas $DA_{e-1}(lY, k)$, obtained by a *non*-recursive construction (namely, by the general method in Chapter 11). Let $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$ be the barking family obtained by patching this complete propagation; then the deformation from X to $X_{0,t} := \Psi^{-1}(0, t)$ does *not* possess periodicity, as seen from Figure 12.6.1 ($b = 0$), Figure 12.6.2 (when b does not divide u) and Figure 12.6.3 (when b divides u).

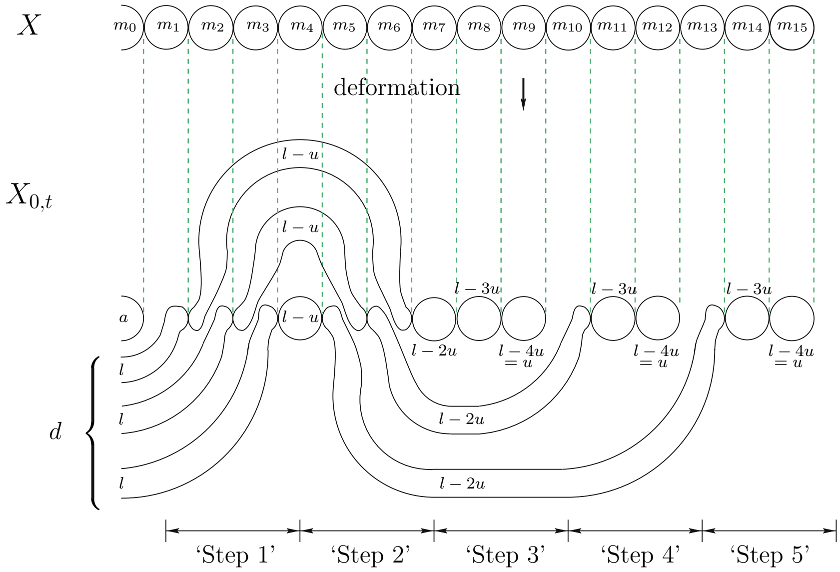


Fig. 12.6.1. Non-recursive Construction ($b = 0$): Deformation from X to $X_{0,t}$ for the case $e = 0$, $d = 3$ and $l = 5u$. (Recall $a = m_{e-1} - ln_{e-1}$, $b = m_e - ln_e$, $d = n_e$ and $u = a + b - r_e b$.) 'Steps 1, 2' are obtained by the recursive construction I, while 'Steps 3, 4, 5' are obtained by the recursive construction II.

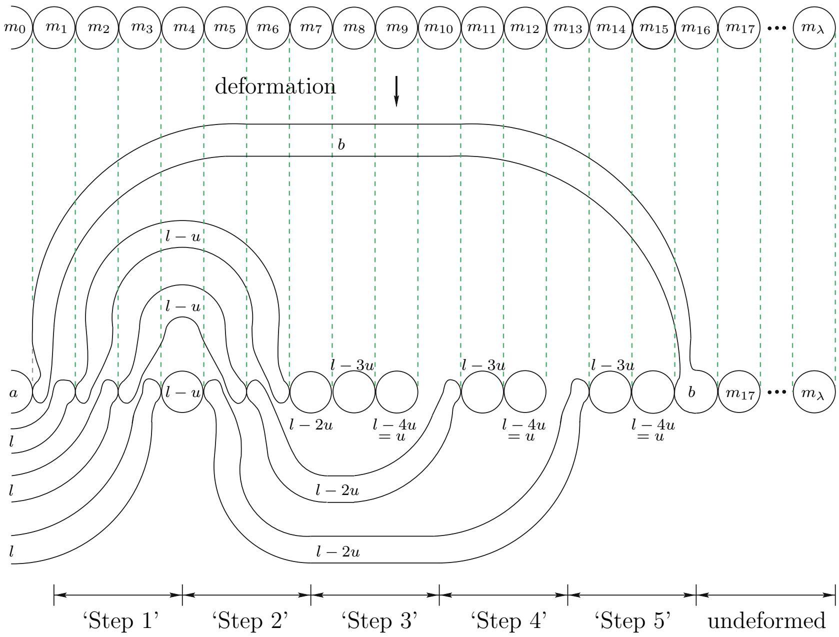


Fig. 12.6.2. Non-recursive Construction ($b \geq 1$ and b does not divide u): Deformation from X to $X_{0,t}$ for the case $e = 0$, $d = 3$ and $l = 5u$. (Recall $a = m_{e-1} - ln_{e-1}$, $b = m_e - ln_e$, $d = n_e$ and $u = a + b - r_e b$.) 'Steps 1, 2' are obtained by the recursive construction I, while 'Steps 3, 4, 5' are obtained by the recursive construction II.

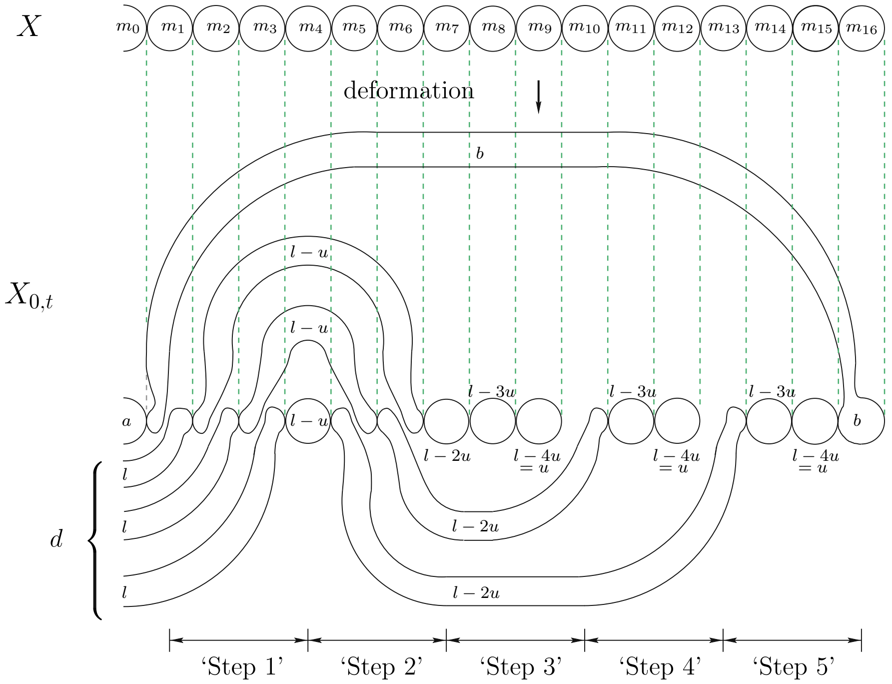


Fig. 12.6.3. Non-recursive Construction ($b \geq 1$ and b divides u): Deformation from X to $X_{0,t}$ for the case $e = 0$, $d = 3$ and $l = 5u$. (Recall $a = m_{e-1} - ln_{e-1}$, $b = m_e - ln_e$, $d = n_e$ and $u = a + b - r_e b$.) 'Steps 1, 2' are obtained by the recursive construction I, while 'Steps 3, 4, 5' are obtained by the recursive construction II.

13

Types A_l, B_l, and C_l Exhaust all Cases

13.1 Results

Let $X = m_0\Delta_0 + m_1\Theta_1 + \cdots + m_\lambda\Theta_\lambda$ be a branch. Assume that $Y = n_0\Delta_0 + n_1\Theta_1 + \cdots + n_e\Theta_e$ is a subbranch of X, and l is a positive integer satisfying $lY \le X$, i.e. $ln_i \le m_i$ for $i = 0, 1, \dots, e$. We recall a deformation atlas $DA_{e-1}(lY, k)$. We first define a sequence of integers p_i $(i = 0, 1, \dots, \lambda + 1)$ inductively by

$$\begin{cases} p_0 = 0, \quad p_1 = 1 \quad \text{and} \\ p_{i+1} = r_i p_i - p_{i-1} \quad \text{for } i = 1,\, 2,\, \dots,\, \lambda. \end{cases}$$

Then $p_{\lambda+1} > p_\lambda > \cdots > p_1 > p_0 = 0$ (6.2.4), p105. Let $f(z)$ be a non-vanishing holomorphic function on a domain $\{z \in \mathbb{C} : |z| < \varepsilon\}$, and we set

$$f_i = f(w^{p_{i-1}}\eta^{p_i}) \qquad \text{and} \qquad \widehat{f}_i = f(z^{p_{i+1}}\zeta^{p_i}) \qquad \text{(see (6.2.7), p106).} \tag{13.1.1}$$

With these notations, the deformation atlas $DA_{e-1}(lY, k)$ is given as follows: for $i = 1, 2, \dots, e-1$,

$$\begin{cases} \mathcal{H}_i : & w^{m_{i-1}-ln_{i-1}}\eta^{m_i-ln_i}(w^{n_{i-1}}\eta^{n_i} + t^k f_i)^l - s = 0 \\ \mathcal{H}'_i : & z^{m_{i+1}-ln_{i+1}}\zeta^{m_i-ln_i}(z^{n_{i+1}}\zeta^{n_i} + t^k \widehat{f}_i)^l - s = 0 \\ g_i : & \text{the transition function } z = 1/w,\ \zeta = w^{r_i}\eta \text{ of } N_i. \end{cases}$$

In Theorem 10.0.15, we showed that if Y is of type A_l, B_l, or C_l such that if Y is of type C_l the positive integer k is divisible by n_e, then $DA_{e-1}(lY, k)$ admits a complete propagation. Remember that Y is of type A_l, B_l, or C_l according to

Type A_l $lY \le X$ and Y is dominant tame (i.e. $\frac{n_{e-1}}{n_e} \ge r_e$),

Type B_l $lY \le X$, $m_e = l$ and $n_e = 1$, or

Type C_l $lY \le X$, $n_e = r_e n_e - n_{e-1}$, and u divides l where

$$u := (m_{e-1} - ln_{e-1}) - (r_e - 1)(m_e - ln_e).$$

At first glance, it seems that besides these types, there may be another type of a subbranch Y such that for some k, $DA_{e-1}(lY, k)$ admits a complete propagation. However this is false. In fact we will show the following result.

Theorem 13.1.1 *Let $Y = n_0\Delta_0 + n_1\Theta_1 + \cdots + n_e\Theta_e$ be a subbranch of a branch X, and let l be a positive integer satisfying $lY \leq X$. Then $DA_{e-1}(lY, k)$ admits a complete propagation if and only if Y is of type A_l, B_l, or C_l such that if Y is of type C_l the positive integer k is divisible by n_e.*

13.2 Preparation

The remainder of this chapter is devoted to the proof of Theorem 13.1.1. It is enough to demonstrate that if $DA_{e-1}(lY, k)$ admits a complete propagation, then Y must be one of types A_l, B_l, and C_l such that if Y is of type C_l the positive integer k is divisible by n_e; we already showed the converse in Theorem 10.0.15. First of all, we investigate when $DA_{e-1}(lY, k)$ admits an e-th propagation with $g_e : z = 1/w, \ \zeta = w^{r_e}\eta$ (the transition function of N_e).

Lemma 13.2.1 *Suppose that Y is dominant. If the transition function $g_e : z = 1/w, \ \zeta = w^{r_e}\eta$ transforms $\mathcal{H}_e$ to some hypersurface, then Y is of type A_l, and the e-th propagation of $DA_{e-1}(lY, k)$ is given by*

$$\begin{cases} \mathcal{H}_e : & w^{m_{e-1}-ln_{e-1}}\eta^{m_e-ln_e}(w^{n_{e-1}}\eta^{n_e} + t^k f_e)^l - s = 0 \\ \mathcal{H}_e' : & z^{m_{e+1}-ln_{e+1}}\zeta^{m_e-ln_e}(z^{n_{e+1}}\zeta^{n_e} + t^k \widehat{f}_e)^l - s = 0 \\ g_e : & \textit{the transition function } z = 1/w, \ \zeta = w^{r_e}\eta \textit{ of } N_e, \end{cases} \tag{13.2.1}$$

where f_e and $\widehat{f}_e$ are non-vanishing holomorphic functions defined by (13.1.1).

Proof. The equation $\mathcal{H}_e : \ w^{m_{e-1}-ln_{e-1}}\eta^{m_e-ln_e}(w^{n_{e-1}}\eta^{n_e} + t^k f_e)^l - s = 0$ is rewritten as

$$\frac{1}{w^{r_e(m_e - l n_e) - (m_{e-1} - l n_{e-1})}} \left(w^{r_e}\eta\right)^{m_e - l n_e} \times \left[\frac{1}{w^{r_e n_e - n_{e-1}}} \left(w^{r_e}\eta\right)^{n_e} + t^k f_e \right]^l - s = 0.$$

From $m_{e+1} = r_e m_e - m_{e-1}$, we have

$$r_e(m_e - ln_e) - (m_{e-1} - ln_{e-1}) = m_{e+1} + l\,(n_{e-1} - r_e n_e),$$

and so

$$\mathcal{H}_e : \quad \frac{1}{w^{m_{e+1} + l\,(n_{e-1} - r_e n_e)}} \left(w^{r_e}\eta\right)^{m_e - l n_e} \times \left[\frac{1}{w^{r_e n_e - n_{e-1}}} \left(w^{r_e}\eta\right)^{n_e} + t^k f_e \right]^l - s = 0.$$

Therefore the map $g_e : z = 1/w,\ \zeta = w^{r_e}$ transforms $\mathcal{H}_e$ to

$$z^{m_{e+1}+l(n_{e-1}-r_e n_e)}\zeta^{m_e-l n_e}\left[z^{r_e n_e - n_{e-1}}\zeta^{n_e}+t^k\widehat{f}_e\right]^l - s = 0. \quad (13.2.2)$$

We set $q := n_{e-1} - r_e n_e$ (the slant of Y), and then the left hand side of (13.2.2) is

$$z^{m_{e+1}+lq}\zeta^{m_e-l n_e}\left[\frac{1}{z^q}\zeta^{n_e}+t^k\widehat{f}_e\right]^l - s,$$

that is,

$$z^{m_{e+1}}\zeta^{m_e-l n_e}\left[\zeta^{n_e}+t^k z^q\widehat{f}_e\right]^l - s. \quad (13.2.3)$$

When $q \geq 0$, this is clearly a polynomial (with no fractional terms). On the other hand, when $m_{e+1}+lq < 0$, the expansion of (13.2.3) contains a fractional term

$$t^{kl} z^{m_{e+1}+lq}\zeta^{m_e-ln_e}\widehat{f}_e^{\,l}$$

where note that $\widehat{f}_e$ is non-vanishing. Thus in the case $m_{e+1}+lq < 0$, (13.2.3) cannot define a hypersurface, and consequently if $DA_{e-1}(lY,k)$ admits an e-th propagation with $g_e:\ z = 1/w,\ \zeta = w^{r_e}\eta$, then Y must be tame, where recall that (see Lemma 5.5.7, p94)

"Y is tame $\iff$ $q \geq 0$" and "Y is wild $\iff$ $0 > m_{e+1}+q$".

Moreover since Y is dominant by assumption, Y is dominant tame, equivalently of type A_l. The remainder of the assertion — the e-th propagation is given by (13.2.1) — is clear. □

Consequently, we obtain the following.

Corollary 13.2.2 *Suppose that Y is dominant. If $DA_{e-1}(lY,k)$ admits an e-th propagation such that g_e is the transition function $z = 1/w,\ \zeta = w^{r_e}\eta$ of N_e, then Y is of type A_l.*

The converse is also valid.

Lemma 13.2.3 *If Y is of type A_l, then $DA_{e-1}(lY,k)$ does not admit such an e-th propagation that g_e is a nontrivial deformation of the transition function $z = 1/w,\ \zeta = w^{r_e}\eta$ of N_e.*

Proof. We show this by contradiction. Suppose that there exists an e-th propagation of $DA_{e-1}(lY,k)$ such that g_e is a nontrivial deformation of $z = 1/w,\ \zeta = w^{r_e}\eta$. First we consider the case where f_e is a constant function, say $f_e \equiv 1$, and g_e is of the standard form:

$$g_e:\quad z = \frac{1}{w},\quad \zeta = w^{r_e}\eta + \alpha_1(t)w + \alpha_2(t)w^2 + \cdots + \alpha_{r_e-1}(t)w^{r_e-1},$$

where $\alpha_i(t)$ is holomorphic in t with $\alpha_i(0) = 0$ (see §5.5.1, p98 for the "standard form"). By a similar computation to the proof of Lemma 13.2.1,

g_e transforms $F_e = w^{m_{e-1}-ln_{e-1}}\eta^{m_e-n_e}(w^{n_{e-1}}\eta^{n_e}+t^k)^l$ to

$$G_e = z^{m_{e+1}}\left(\zeta - \alpha_1(t)\frac{1}{z} - \alpha_2(t)\frac{1}{z^2} - \cdots - \alpha_{r_e-1}(t)\frac{1}{z^{r_e-1}}\right)^{m_e-ln_e}$$
$$\times\left[\left(\zeta - \alpha_1(t)\frac{1}{z} - \alpha_2(t)\frac{1}{z^2} - \cdots - \alpha_{r_e-1}(t)\frac{1}{z^{r_e-1}}\right)^{n_e} + t^k z^q\right]^l,$$

where $q := n_{e-1} - r_e n_e$ (note that $q \geq 0$, because Y is of type A_l). Then it is easy to see that G_e, after expansion, contains a fractional term (a contradiction!). Similarly, we can deduce a contradiction for the case where f_e is not a constant function and g_e is not of the standard form. (The argument is essentially the same although the computations become messy.) □

In terms of Lemma 13.2.3, when we investigate whether or not $DA_{e-1}(lY,k)$ admits an e-th propagation such that g_e is a nontrivial deformation of $z = 1/w,\ \zeta = w^{r_e}\eta$, we may assume that Y is not of type A_l, in other words, either (1) Y is wild or (2) Y is tame and not dominant (recall that type A_l is dominant tame).

To simplify the subsequent discussion, we assume that $f_e \equiv 1$; the argument below works without assuming $f_e \equiv 1$, but the existence of higher terms in f_e causes complication of computation. We remark that when Y is (i) wild or (ii) tame and not proportional (i.e. $m_i n_{i-1} - m_{i-1} n_i \neq 0$ for $1 \leq i \leq e$), it follows from Proposition 8.1, p143 that after coordinate change, we have $f_e \equiv 1$.

Now we shall investigate when $DA_{e-1}(lY,k)$ admits an e-th propagation with a nontrivial deformation g_e of $z = 1/w,\ \zeta = w^{r_e}\eta$. Setting

$$a = m_{e-1} - ln_{e-1}, \qquad b = m_e - ln_e, \qquad c = n_{e-1}, \qquad d = n_e,$$

we express $\mathcal{H}_e :\ w^a\eta^b(w^c\eta^d + t^k)^l - s = 0$. We divide into two cases $b = 0$ and $b \geq 1$.

13.3 Case 1: $b = 0$

In this case, the following result holds.

Proposition 13.3.1 *Suppose that $b = 0$. If $DA_{e-1}(lY,k)$ admits an e-th propagation such that g_e is a nontrivial deformation of $z = 1/w,\ \zeta = w^{r_e}\eta$, then it is given by*

$$\begin{cases} \mathcal{H}_e : & w^a(w^c\eta^d + t^k)^l - s = 0 \\ \mathcal{H}'_e : & z^{ql-a}\left[\zeta\prod_{j=1}^{d-1}(z^q\zeta + t^h\beta_j)\right]^l - s = 0 \\ g_e : & z = \dfrac{1}{w}, \quad \zeta = w^{r_e}\eta - t^h\alpha w^q \end{cases}$$

where $\alpha \in \mathbb{C}$ *satisfies* $\alpha^d + 1 = 0$, *and* $-\beta_1, -\beta_2, \ldots, -\beta_{d-1}$ *are the solutions of an equation*

$$x^{d-1} + {}_dC_1\,\alpha\,x^{d-2} + \cdots + {}_dC_i\,\alpha^i\,x^{d-i-1} + \cdots + {}_dC_{d-1}\,\alpha^{d-1} = 0,$$

and q *and* h *are positive integers satisfying* $1 \le q \le r_e - 1$ *and*

(1) $qd = r_e d - c, \qquad k = dh, \qquad ql - a \ge 0$ *and*

(2) $\begin{cases} ql - a < l & \text{if } d = 1 \\ q = 1 & \text{if } d \ge 2. \end{cases}$

If furthermore DA_{e-1} *admits a complete propagation, then* (i) $d = 1$ *or* (ii) $d \ge 2$ *and* a *divides* l.

For a moment, assuming this proposition, we derive an important consequence. First recall notation:

$$a = m_{e-1} - ln_{e-1}, \qquad b = m_e - ln_e, \qquad c = n_{e-1}, \qquad d = n_e,$$

Then in the above statement, according to whether (i) $d = 1$ or (ii) $d \ge 2$ and a divides l, the subbranch Y is of type B_l or C_l. In fact, when (i) $d = 1$, together with $b = 0$ (the assumption of Case 1), we have $n_e = 1$ and $m_e = l$. So Y is of type B_l. On the other hand, when (ii) $d \ge 2$ and a divides l, Y is of type C_l, which is seen as follows. Since $q = 1$ by (2) of Proposition 13.3.1, we have $d = r_e d - c$ by (1) of Proposition 13.3.1. Setting $u := a - (r_e - 1)b$, then from $b = 0$, we have $u = a$ (that is, a is u in the definition of type C_l). By assumption, a divides l, and hence u divides l. Therefore all the conditions of type C_l are fulfilled, and so Y is of type C_l. Finally we note that by $k = dh$ of (1) of Proposition 13.3.1, the integer $n_e(= d)$ divides k.

Together with Corollary 13.2.2, we obtain the following result.

Corollary 13.3.2 *Suppose that* $b = 0$. *If* $DA_{e-1}(lY, k)$ *admits a complete propagation, then* Y *is of type* A_l, B_l, *or* C_l *such that if* Y *is of type* C_l *the positive integer* k *is divisible by* n_e.

Now we return to Proposition 13.3.1; to prove it, we require some preparation.

Step 1. When does g_e transform F_e to some polynomial?

We need the following technical lemma.

Lemma 13.3.3 *Suppose that* $b = 0$. *A nontrivial deformation* g_e *of the transition function* $z = 1/w,\ \zeta = w^{r_e}\eta$ *transforms* $F_e = w^a(w^c\eta^d + t^k)^l$ *to some polynomial if and only if* g_e *is of the form* $z = 1/w,\ \zeta = w^{r_e}\eta - t^h\alpha w^q$ *such that* $\alpha \in \mathbb{C}$ *satisfies* $\alpha^d + 1 = 0$ *and positive integers* q *and* h *satisfy*

$$qd = r_e d - c, \qquad k = dh, \qquad ql - a \ge 0. \tag{13.3.1}$$

In this case, g_e transforms F_e to a polynomial $G_e = z^{\,ql-a}\left[\zeta \prod_{j=1}^{d-1}\left(z^q\,\zeta + t^h\beta_j\right)\right]^l$, *where* $-\beta_1, -\beta_2, \dots, -\beta_{d-1}$ *are the solutions of an equation*

$$x^{d-1} + {}_dC_1\,\alpha\,x^{d-2} + \cdots + {}_dC_i\,\alpha^i\,x^{d-i-1} + \cdots + {}_dC_{d-1}\,\alpha^{d-1} = 0.$$

Proof. For simplicity we set $r = r_e$. First we show the assertion for g_e of the standard form:

$$g_e: \quad z = \frac{1}{w}, \quad \zeta = w^r\eta + \alpha_1(t)w + \alpha_2(t)w^2 + \cdots + \alpha_{r-1}(t)w^{r-1},$$

where $\alpha_i(t)$ is holomorphic in t with $\alpha_i(0) = 0$. Rewriting

$$F_e = w^a(w^c\eta^d + t^k)^l = w^a\left(\frac{1}{w^{rd-c}}(w^r\eta)^d + t^k\right)^l, \tag{13.3.2}$$

we see that g_e transforms F_e to

$$G_e = \frac{1}{z^a}\left[z^{rd-c}\left(\zeta - \alpha_1(t)\frac{1}{z} - \alpha_2(t)\frac{1}{z} - \cdots - \alpha_{r-1}(t)\frac{1}{z^{r-1}}\right)^d + t^k\right]^l.$$

It is easy to check that as a necessary condition that the expansion of G_e is a polynomial (with no fractional terms), the map g_e can contain only one term of deformation, say, $\alpha_q(t)w^q$ and moreover the Taylor expansion of $\alpha_q(t)$ must consist of a single term; so g_e is of the form $z = 1/w,\ \zeta = w^r\eta - t^h\alpha w^q$ where $\alpha \in \mathbb{C}$. Then g_e transforms F_e to

$$G_e = \frac{1}{z^a}\left[z^{rd-c}\left(\zeta + \alpha t^h\frac{1}{z^q}\right)^d + t^k\right]^l.$$

Here we note that $rd - c = qd$ must hold (otherwise the expansion of G_e contains a fractional term t^{kl}/z^a). When $rd - c = qd$ holds, we have

$$G_e = \frac{1}{z^a}\left[z^{qd}\left(\zeta + \alpha t^h\frac{1}{z^q}\right)^d + t^k\right]^l.$$

However the expansion of G_e still contains a fractional term $\frac{1}{z^a}(t^{dh}\alpha^d + t^k)^l$. This term must vanish, from which we derive a condition that $dh = k$ and $\alpha \in \mathbb{C}$ satisfies $\alpha^d + 1 = 0$. If this is the case, we may further rewrite:

$$\begin{aligned}G_e &= \frac{1}{z^a}\left[z^{qd}\left(\zeta + \alpha t^h\frac{1}{z^q}\right)^d + t^k\right]^l \\ &= \frac{1}{z^a}\Big[\Big(z^{qd}\zeta^d + \cdots + t^{hi}\alpha^i{}_d\mathrm{C}_i z^{q(d-i)}\zeta^{d-i} + \cdots \\ &\qquad + t^{h(d-1)}\alpha^{d-1}{}_d\mathrm{C}_{d-1}z^q\zeta + t^{dh}\alpha^d\Big) + t^k\Big]^l\end{aligned}$$

$$= \frac{1}{z^a}\Big[z^{qd}\zeta^d + \cdots + t^{hi}\alpha^i{}_d\mathrm{C}_i z^{q(d-i)}\zeta^{d-i} + \cdots$$
$$+ t^{h(d-1)}\alpha^{d-1}{}_d\mathrm{C}_{d-1}z^q\zeta + \Big(t^{dh}\alpha^d + t^k\Big)\Big]^l$$
$$= \frac{1}{z^a}\Big[z^{qd}\zeta^d + \cdots + t^{hi}\alpha^i{}_d\mathrm{C}_i z^{q(d-i)}\zeta^{d-i} + \cdots + t^{h(d-1)}\alpha^{d-1}{}_d\mathrm{C}_{d-1}z^q\zeta\Big]^l$$
$$= z^{ql-a}\Big[z^{q(d-1)}\zeta^d + \cdots + t^{hi}\alpha^i{}_d\mathrm{C}_i z^{q(d-i-1)}\zeta^{d-i} + \cdots + t^{h(d-1)}\alpha^{d-1}{}_d\mathrm{C}_{d-1}\zeta\Big]^l.$$

(In the fourth equality we used "$dh = k$ and $\alpha^d + 1 = 0$".) Finally we simplify the last expression. Write

$$z^{q(d-1)}\zeta^d + \cdots + t^{hi}\alpha^i{}_d\mathrm{C}_i z^{q(d-i-1)}\zeta^{d-i} + \cdots + t^{h(d-1)}\alpha^{d-1}{}_d\mathrm{C}_{d-1}\zeta$$
$$= \zeta\prod_{j=1}^{d-1}(z^q\zeta + t^h\beta_j),$$

where $-\beta_1, -\beta_2, \ldots, -\beta_{d-1}$ are the solutions of an equation

$$x^{d-1} + {}_d\mathrm{C}_1\,\alpha\,x^{d-2} + \cdots + {}_d\mathrm{C}_i\,\alpha^i\,x^{d-i-1} + \cdots + {}_d\mathrm{C}_{d-1}\,\alpha^{d-1} = 0.$$

Then we have

$$G_e = z^{ql-a}\left[\zeta\prod_{j=1}^{d-1}\Big(z^q\zeta + t^h\beta_j\Big)\right]^l,$$

which is a polynomial precisely when $ql - a \geq 0$. Therefore G_e is a polynomial if and only if (13.3.1) holds:

$$qd = r_e d - c, \qquad k = dh, \qquad ql - a \geq 0.$$

This proves the assertion. (Similarly we can show the assertion for g_e of a non-standard form; letting g_e transform F_e to G_e, investigate the condition where the fractional terms of G_e vanish). □

From Lemma 13.3.3, if $DA_{e-1}(lY, k)$ admits an e-th propagation such that g_e is a nontrivial deformation of $z = 1/w,\ \zeta = w^r\eta$, then it must be of the form

$$\begin{cases} \mathcal{H}_e: & w^a(w^c\eta^d + t^k)^l - s = 0 \\ \mathcal{H}'_e: & G_e(z,\zeta,t) - s = 0 \\ g_e: & z = 1/w, \quad \zeta = w^r\eta - t^h\alpha w^q, \end{cases} \tag{13.3.3}$$

where $G_e = z^{ql-a}[\zeta\prod_{j=1}^{d-1}(z^q\zeta + t^h\beta_j)]^l$. We point out that (13.3.3) is merely a "candidate" of an e-th propagation, and does not necessarily give an e-th propagation. To be sufficient, an inequality $\deg_z G_e(z,\zeta,0) < \deg_\zeta G_e(z,\zeta,0)$ must hold because $G_e(z,\zeta,0) = z^{m_{e+1}}\zeta^{m_e}$ and $m_{e+1} < m_e$ (recall that the sequence of multiplicities strictly decreases).

Step 2. When does $\deg_z G_e(z,\zeta,0) < \deg_\zeta G_e(z,\zeta,0)$ hold?

For the polynomial $G_e = z^{ql-a}\left[\zeta \prod_{j=1}^{d-1}(z^q\zeta + t^h\beta_j)\right]^l$, we shall investigate when $\deg_z G_e(z,\zeta,0) < \deg_\zeta G_e(z,\zeta,0)$ holds, equivalently when (13.3.3) indeed gives an e-th propagation of $DA_{e-1}(lY,k)$. We separate into two cases $d=1$ and $d \geq 2$.

Case 1.1 $d = 1$

In this case, $G_e(z,\zeta,t) = z^{ql-a}\zeta^l$ (no terms in t), so $G_e(z,\zeta,0) = z^{ql-a}\zeta^l$. Thus we have $(m_{e+1}, m_e) = (ql-a, l)$, and the following result holds.

Lemma 13.3.4 *Suppose that $d=1$. If $ql-a<l$, then $DA_{e-1}(lY,k)$ admits an e-th propagation, and it is given by* (13.3.3) :

$$\begin{cases} \mathcal{H}_e: & w^a\eta^b(w^c\eta + t^k)^l - s = 0 \\ \mathcal{H}'_e: & z^{ql-a}\zeta^l - s = 0 \\ g_e: & z = 1/w, \quad \zeta = w^r\eta - t^h\alpha w^q. \end{cases}$$

Case 1.2 $d \geq 2$

We further separate into two subcases $q=1$ and $q \geq 2$.

$q = 1$: **Lemma 13.3.5** *In this case, $DA_{e-1}(lY,k)$ admits an e-th propagation and it is given by* (13.3.3)*:*

$$\begin{cases} \mathcal{H}_e: & w^a\eta^b(w^c\eta^d + t^k)^l - s = 0 \\ \mathcal{H}'_e: & G_e(z,\zeta,t) - s = 0 \\ g_e: & z = 1/w, \quad \zeta = w^r\eta - t^h\alpha w, \end{cases}$$

where $G_e = z^{l-a}\left[\zeta \prod_{j=1}^{d-1}(z\,\zeta + t^h\beta_j)\right]^l$. *Moreover $DA_{e-1}(lY,k)$ admits a complete propagation if and only if a divides l.*

Proof. Since $G_e(z,\zeta,0) = z^{l-a+l(d-1)}\zeta^{ld} = z^{ld-a}\zeta^{ld}$, we have $(m_{e+1}, m_e) = (ld-a\,,\,ld)$. Firstly we shall confirm that $m_{e+1} < m_e$, i.e. $ld-a < ld$. Note that the condition $d = r_e d - c$ in (13.3.1), where $q=1$ in the present case, implies that (i) c divides d and (ii) $c/d < r_e$. Hence by Proposition 9.4.8, p171, we have $a > 0$. (Note $u := a - (r_e-1)b$ in that proposition is equal to a in the present case by the assumption $b=0$.) In particular $ld - a < ld$, and so $m_{e+1} < m_e$, proving the first half of the statement. Next we set $F_{e+1} := w^a P_1^{l-a} P_2^l$, where $P_1 := w\eta$ and $P_2 := \prod_{j=1}^{d-1}(w\eta + t^h\beta_j)$. Then $\mathcal{H}_{e+1}:\ F_{e+1}(w,\eta,t) - s = 0$. Since F_{e+1} is a waving polynomial satisfying the assumption of Theorem 11.5.5, p202, the deformation atlas $DA_{e-1}(lY,k)$ admits a complete propagation if and only if a divides l. □

$q \geq 2$: **Lemma 13.3.6** *In this case, $DA_{e-1}(lY, k)$ does "not" admit an e-th propagation.*

Proof. Supposing that $DA_{e-1}(lY, k)$ admits an e-th propagation, we deduce a contradiction. By Lemma 13.3.3, an e-th propagation must be of the form

$$\begin{cases} \mathcal{H}_e: & w^a \eta^b (w^c \eta^d + t^k)^l - s = 0 \\ \mathcal{H}'_e: & G_e(z, \zeta, t) - s = 0 \\ g_e: & z = 1/w, \quad \zeta = w^r \eta - t^h \alpha w^q, \end{cases}$$

where $G_e = z^{ql-a} \left[\zeta \prod_{j=1}^{d-1} \left(z^q \zeta + t^h \beta_j \right) \right]^l$. Substituting $t = 0$ in G_e, we have

$$G_e(z, \zeta, 0) = z^{ql-a} \left[\zeta \prod_{j=1}^{d-1} (z^q \zeta) \right]^l = z^{\, ql-a+ql(d-1)} \zeta^{\, ld}.$$

Thus $(m_{e+1}, m_e) = \big(ql - a + ql(d-1)\,,\, ld \big)$. To deduce a contradiction, we note

$$\begin{aligned} ql(d-1) &\geq 2l(d-1) && \text{by } q \geq 2 \\ &= ld + l(d-2) \\ &\geq ld && \text{by } d \geq 2. \end{aligned}$$

Hence

$$ql(d-1) \geq ld. \tag{13.3.4}$$

On the other hand, $ql - a \geq 0$ by (13.3.1). Combined with (13.3.4), we have

$$m_{e+1} = (ql - a) + ql(d-1) \geq ld = m_e.$$

This contradicts that $m_{e+1} < m_e$. Thus $DA_{e-1}(lY, k)$ does not admit an e-th propagation. □

The lemmas in Case 1.1 and Case 1.2 together confirm Proposition 13.3.1.

13.4 Case 2: $b \geq 1$

In this case we show the following result.

Proposition 13.4.1 *Suppose that $b \geq 1$. According to whether $d = 1$ or $d \geq 2$, the following statements hold:*

- $d = 1$: *$DA_{e-1}(lY, k)$ admits "at most" an e-th propagation, and does not admit a complete propagation.* (As we will see below, if $DA_{e-1}(lY, k)$ admits an e-th propagation, then $\lambda \geq e + 1$.)

- $d \geq 2$: *If $DA_{e-1}(lY, k)$ admits an e-th propagation, then it is given by*

$$\begin{cases} \mathcal{H}_e: & w^a\,\eta^b\,(w^c\,\eta^d + t^k)^l - s = 0 \\ \mathcal{H}'_e: & z^{\,l-(a+b-r_e b)}\,(z\,\zeta + t^h\alpha)^b \left[\zeta\,\prod_{j=1}^{d-1}(z\,\zeta + t^h\beta_j)\right]^l - s = 0 \\ g_e: & z = 1/w, \quad \zeta = w^{r_e}\eta - t^h\alpha w, \end{cases} \tag{13.4.1}$$

where $\alpha \in \mathbb{C}$ satisfies $\alpha^d + 1 = 0$, and $-\beta_1, -\beta_2, \ldots, -\beta_{d-1}$ are the solutions of an equation

$$x^{d-1} + {}_dC_1\,\alpha\,x^{d-2} + \cdots + {}_dC_i\,\alpha^i\,x^{d-i-1} + \cdots + {}_dC_{d-1}\,\alpha^{d-1} = 0,$$

and the following relations hold

$$d = r_e d - c, \qquad k = dh, \qquad l \geq a + b - r_e b. \tag{13.4.2}$$

If moreover $DA_{e-1}(lY, k)$ admits a complete propagation, then $u := a+b-r_e b$ divides l. (Note: In this case, together with $d = r_e d - c$, the subbranch Y is of type C_l. Also note that by $k = dh$ of (13.4.2), the integer $n_e(= d)$ divides k.)

Together with Corollary 13.2.2, this proposition yields the following result.

Corollary 13.4.2 *Suppose that $b \geq 1$. If $DA_{e-1}(lY, k)$ admits a complete propagation, then Y is of type A_l or C_l such that if Y is of type C_l the positive integer k is divisible by n_e* (see "Note" in the above proposition).

Step 1. When does g_e transform F_e to some polynomial G_e?

To prove Proposition 13.4.1, we need the following lemma.

Lemma 13.4.3 *Suppose that $b \geq 1$. A nontrivial deformation g_e of the transition function $z = 1/w$, $\zeta = w^{r_e}\eta$ transforms a polynomial $F_e = w^a\eta^b(w^c\eta^d + t^k)^l$ to some polynomial if and only if g_e is of the form $z = 1/w$, $\zeta = w^{r_e}\eta - t^h\alpha w^q$ such that $\alpha \in \mathbb{C}$ satisfies $\alpha^d + 1 = 0$ and the positive integers q and h satisfy*

$$qd = r_e d - c, \qquad k = dh, \qquad ql - qb - a + r_e b \geq 0. \tag{13.4.3}$$

If this is the case, g_e transforms F_e to a polynomial

$$G_e = z^{\,q\,l - q\,b - a + r_e\,b}\,(z^q\zeta + t^h\alpha)^b \left[\zeta\prod_{j=1}^{d-1}(z^q\,\zeta + t^h\beta_j)\right]^l,$$

where $-\beta_1, -\beta_2, \ldots, -\beta_{d-1}$ are the solutions of an equation

$$x^{d-1} + {}_dC_1\,\alpha\,x^{d-2} + \cdots + {}_dC_i\,\alpha^i\,x^{d-i-1} + \cdots + {}_dC_{d-1}\,\alpha^{d-1} = 0.$$

Proof. As in the proof of Lemma 13.3.3, if g_e transforms F_e to some polynomial, it is necessary that g_e has the form $z = 1/w,\ \zeta = w^{r_e}\eta - t^h\alpha w^q$. For brevity we set $r := r_e$. Since

$$F_e = w^a\eta^b(w^c\eta^d + t^k)^l = w^{a-rb}(w^r\eta)^b\left(\frac{1}{w^{rd-c}}(w^r\eta)^d + t^k\right)^l,$$

the map $g_e:\ z = 1/w,\ \zeta = w^r\eta - t^h\alpha w^q$ transforms F_e to

$$G_e(z,\zeta,t) = \frac{1}{z^{a-rb}}\left(\zeta + t^h\alpha\frac{1}{z^q}\right)^b\left[z^{rd-c}\left(\zeta + \alpha t^h\frac{1}{z^q}\right)^d + t^k\right]^l.$$

By the same argument as in the proof of Lemma 13.3.3, a necessary condition for G_e being a polynomial is given by $rd - c = qd,\quad k = dh\quad$ and $\quad \alpha^d + 1 = 0$. If this is the case, we have

$$G_e = \frac{1}{z^{a-rb}}\left(\zeta + t^h\alpha\frac{1}{z^q}\right)^b\left[z^{qd}\left(\zeta + t^h\alpha\frac{1}{z^q}\right)^d + t^k\right]^l.$$

The inside the brackets admits the following expansion:

$$\begin{aligned}
&z^{qd}\left(\zeta + t^h\alpha\frac{1}{z^q}\right)^d + t^k \\
&= z^{qd}\left\{\zeta^d + \cdots + {}_d\mathrm{C}_i\,\zeta^{d-i}\left(t^h\alpha\frac{1}{z^q}\right)^i + \cdots + \left(t^h\alpha\frac{1}{z^q}\right)^d\right\} + t^k \\
&= \Big\{z^{qd}\,\zeta^d + \cdots + t^{hi}\,{}_d\mathrm{C}_i\,\alpha^i z^{q(d-i)}\,\zeta^{d-i} + \cdots \\
&\qquad + t^{h(d-1)}\,{}_d\mathrm{C}_{d-1}\,\alpha^{d-1}\,z^q\,\zeta + t^{hd}\,\alpha^d\Big\} + t^k \\
&= z^{qd}\,\zeta^d + \cdots + t^{hi}\,{}_d\mathrm{C}_i\,\alpha^i\,z^{q(d-i)}\,\zeta^{d-i} + \cdots + t^{h(d-1)}\,{}_d\mathrm{C}_{d-1}\,\alpha^{d-1}\,z^q\,\zeta,
\end{aligned}$$

where in the last equality, we used $t^{dh}\alpha^d + t^k = 0$ following from $k = dh$ and $\alpha^d + 1 = 0$. Thus G_e equals

$$\begin{aligned}
&\frac{1}{z^{a-rb}}\left(\zeta + t^h\alpha\frac{1}{z^q}\right)^b \\
&\qquad\times\left[z^{qd}\zeta^d + \cdots + t^{hi}\,{}_d\mathrm{C}_i\,\alpha^i\,z^{q(d-i)}\,\zeta^{d-i} + \cdots + t^{h(d-1)}\,{}_d\mathrm{C}_{d-1}\,\alpha^{d-1}\,z^q\zeta\right]^l \\
&= \frac{1}{z^{a-rb}}\left(\zeta + t^h\alpha\frac{1}{z^q}\right)^b z^{ql} \\
&\quad\times\left[z^{q(d-1)}\zeta^d + \cdots + t^{hi}{}_d\mathrm{C}_i\alpha^i z^{q(d-i-1)}\zeta^{d-i} + \cdots + t^{h(d-1)}{}_d\mathrm{C}_{d-1}\alpha^{d-1}\zeta\right]^l \\
&= z^{(ql-qb)-(a-rb)}\,(z^q\zeta + t^h\alpha)^b \\
&\quad\times\left[z^{q(d-1)}\,\zeta^d + \cdots + t^{hi}\,{}_d\mathrm{C}_i\,\alpha^i\,z^{q(d-i-1)}\zeta^{d-i} + \cdots + t^{h(d-1)}\,{}_d\mathrm{C}_{d-1}\,\alpha^{d-1}\,\zeta\right]^l.
\end{aligned}$$

So G_e is a polynomial precisely when $(ql - qb) - (a - rb) \geq 0$. Therefore g_e transforms $F_e = w^a \eta^b (w^c \eta^d + t^k)^l$ to some polynomial G_e if and only if g_e has the form $z = 1/w,\ \zeta = w^r \eta - t^h \alpha w^q$ such that $\alpha \in \mathbb{C}$ satisfies $\alpha^d + 1 = 0$ and the positive integers q and h satisfy (13.4.3):

$$qd = rd - c, \qquad k = dh, \qquad (ql - qb) - (a - rb) \geq 0.$$

Finally, we write

$$\begin{aligned} &z^{q(d-1)} \zeta^d + \cdots + t^{hi}\, {}_d\mathrm{C}_i\, \alpha^i\, z^{q(d-i-1)}\, \zeta^{d-i} + \cdots + t^{h(d-1)}\, {}_d\mathrm{C}_{d-1}\, \alpha^{d-1}\, \zeta \\ &= \zeta \prod_{j=1}^{d-1} (z^q \zeta + t^h \beta_j), \end{aligned}$$

where $-\beta_1, -\beta_2, \ldots, -\beta_{d-1}$ are the solutions of an equation

$$x^{d-1} + {}_d\mathrm{C}_1\, \alpha\, x^{d-2} + \cdots + {}_d\mathrm{C}_i\, \alpha^i\, x^{d-i-1} + \cdots + {}_d\mathrm{C}_{d-1}\, \alpha^{d-1} = 0.$$

Then g_e transforms F_e to

$$G_e = z^{ql - qb - a + rb}\, (z^q \zeta + t^h \alpha)^b \left[\zeta \prod_{j=1}^{d-1} (z^q\, \zeta + t^h \beta_j) \right]^l.$$

This completes the proof of our assertion. □

Step 2. When does $\deg_z G_e(z, \zeta, 0) < \deg_\zeta G_e(z, \zeta, 0)$ hold?

By Lemma 13.4.3, if $DA_{e-1}(lY, k)$ admits an e-th propagation, then it must be of the form

$$\begin{cases} \mathcal{H}_e : & w^a \eta^b (w^c \eta^d + t^k)^l - s = 0 \\ \mathcal{H}'_e : & G_e(z, \zeta, t) - s = 0 \\ g_e : & z = 1/w, \quad \zeta = w^{r_e} \eta - t^h \alpha w^q, \end{cases}$$

where $G_e = z^{ql - qb - a + r_e b}\, (z^q \zeta + t^h \alpha)^b \left[\zeta \prod_{j=1}^{d-1} (z^q\, \zeta + t^h \beta_j) \right]^l$. However this does *not* necessarily gives an e-th propagation. In order that the above data gives an e-th propagation, noting that $G_e(z, \zeta, 0) = z^{m_{e+1}} \zeta^{m_e}$, an inequality $\deg_z G_e(z, \zeta, 0) < \deg_\zeta G_e(z, \zeta, 0)$ must hold (recall that the sequence of multiplicities strictly decreases). We now investigate when this holds, and moreover when a complete propagation of $DA_{e-1}(lY, k)$ is possible, separately for two cases $d = 1$ and $d \geq 2$.

Case 2.1 $d = 1$

In this case we show that $DA_{e-1}(lY, k)$ does not admit a complete propagation. The proof is quite involved; $DA_{e-1}(lY, k)$ possibly admits an e-th propagation, however, in which case no $(e+1)$-st propagation is possible.

When $d = 1$,

$$G_e = z^{\,q\,l\,-\,q\,b\,-\,a\,+\,r_e\,b}\,(z^q\zeta + t^h\alpha)^b\,\zeta^l = z^{\,q\,l\,-\,q\,b\,-\,a\,+\,r_e\,b}\zeta^l\,(z^q\zeta + t^h\alpha)^b,$$

where $ql - qb - a + r_eb \geq 0$ (a condition that G_e is a polynomial). Putting $t = 0$ in G_e, we have $G_e(z, \zeta, 0) = z^{ql-a+r_eb}\,\zeta^{l+b}$. Hence $m_{e+1} = ql - a + r_eb$ and $m_e = l + b$. Thus if $ql - a + r_eb < l + b$, then $DA_{e-1}(lY, k)$ admits an e-th propagation, given by

$$\begin{cases} \mathcal{H}_e: & w^a\eta^b\,(w^c\eta^d + t^k)^l - s = 0 \\ \mathcal{H}'_e: & z^{\,q\,l\,-\,q\,b\,-\,a\,+\,r_e\,b}\,\zeta^l\,(z^q\zeta + t^h\alpha)^b - s = 0 \\ g_e: & z = 1/w, \quad \zeta = w^{r_e}\eta - t^h\alpha w^q. \end{cases} \tag{13.4.4}$$

However we claim (in what follows, λ is the length of the branch X):

Lemma 13.4.4 (1) $\lambda \geq e + 1$ *and* (2) *no* $(e + 1)$*-st propagation is possible* (*and thus* $DA_{e-1}(lY, k)$ *does not admit a complete propagation*).

The proof of this lemma is rather technical, and we leave it to §13.6, p249.

Case 2.2. $d \geq 2$

In this case

$$G_e = z^{\,(ql-qb)-(a-r_eb)}\,(z^q\zeta + t^h\alpha)^b\left[\zeta\prod_{j=1}^{d-1}(z^q\zeta + t^h\beta_j)\right]^l.$$

We investigate when an inequality $\deg_z G_e(z, \zeta, 0) < \deg_\zeta G_e(z, \zeta, 0)$ holds, to determine when an e-th propagation of $DA_{e-1}(lY, k)$ is possible. We separate into two cases $q = 1$ and $q \geq 2$.

$q = 1$: **Lemma 13.4.5** *In this case,* $DA_{e-1}(lY, k)$ *admits an* e*-th propagation and it is given by*

$$\begin{cases} \mathcal{H}_e: & w^a\eta^b(w^c\eta^d + t^k)^l - s = 0 \\ \mathcal{H}'_e: & G_e(z, \zeta, t) - s = 0 \\ g_e: & z = 1/w, \quad \zeta = w^{r_e}\eta - t^h\alpha w, \end{cases} \tag{13.4.5}$$

where $G_e(z, \zeta, t) = z^{\,l-(a+b-r_eb)}\,(z\zeta + t^h\alpha)^b\left[\zeta\prod_{j=1}^{d-1}(z\zeta + t^h\beta_j)\right]^l$. *Furthermore* $DA_{e-1}(lY, k)$ *admits a complete propagation if and only if* $u := a + b - r_eb$ *divides* l.

Proof. Since

$$G_e(z, \zeta, 0) = z^{\,l-(a+b-r_eb)+b+l(d-1)}\,\zeta^{\,b+ld} = z^{\,r_eb+ld-a}\,\zeta^{\,b+ld},$$

we have $m_{e+1} = r_e b + ld - a$ and $m_e = b + ld$, and thus

$$m_e - m_{e+1} = a - r_e b + b. \tag{13.4.6}$$

We claim that $a - r_e b + b > 0$. In fact, recall that $d = r_e d - c$ (13.4.3), where $q = 1$ in the present case; in particular (i) c divides d and (ii) $c/d < r_e$. Hence by Proposition 9.4.8, p171, setting $u := a + b - r_e b$, we have $u > 0$. Thus the right hand side of (13.4.6) is positive; $m_e > m_{e+1}$. Therefore (13.4.5) gives an e-th propagation of $DA_{e-1}(lY, k)$.

Next we write $F_{e+1} = w^u P_1^{l-u} P_2^l Q^b$, where $P_1 = w\eta$, $P_2 = \prod_{j=1}^{d-1}(w\eta + t^h\beta_j)$, and $Q = w\eta + t^h\alpha$. Then $\mathcal{H}_{e+1} : \ F_{e+1}(w, \eta, t) - s = 0$. Since F_{e+1} is a waving polynomial satisfying the assumption of Theorem 11.5.5, p202, the deformation atlas $DA_{e-1}(lY, k)$ admits a complete propagation if and only if u divides l. □

$q \geq 2$: **Lemma 13.4.6** *In this case, $DA_{e-1}(lY, k)$ does "not" admit an e-th propagation.*

Proof. We verify this by contradiction. Suppose that $DA_{e-1}(lY, k)$ admits an e-th propagation. Then by Lemma 13.4.3, it must be of the form

$$\begin{cases} \mathcal{H}_e : & w^a \eta^b (w^c \eta^d + t^k)^l - s = 0 \\ \mathcal{H}'_e : & G_e(z, \zeta, t) - s = 0 \\ g_e : & z = 1/w, \quad \zeta = w^{r_e}\eta - t^h \alpha w^q, \end{cases}$$

where $G_e = z^{(q\,l - q\,b) - (a - r_e\,b)} (z^q\zeta + t^h\alpha)^b \left[\zeta \prod_{j=1}^{d-1} (z^q\,\zeta + t^h\beta_j)\right]^l$. Putting $t = 0$, we have

$$\begin{aligned} G_e(z, \zeta, 0) &= z^{\,q\,l - q\,b - (a - r_e\,b)} (z^q\zeta)^b \left[\zeta\, z^{q(d-1)}\, \zeta^{d-1}\right]^l \\ &= z^{\,(q\,l - q\,b - a + r_e\,b) + q\,b + l\,q\,(d-1)}\, \zeta^{\,b + l\,d}. \end{aligned}$$

Thus $m_{e+1} = (ql - qb - a + r_e b) + qb + lq(d-1)$ and $m_e = b + ld$. We then claim $m_{e+1} > m_e$. To see this, we note

$$\begin{aligned} lq(d-1) &\geq 2l(d-1) && \text{by } q \geq 2 \\ &= ld + (ld - 2l) \\ &\geq ld && \text{by } d \geq 2. \end{aligned}$$

Hence

$$lq(d-1) \geq ld. \tag{13.4.7}$$

On the other hand, we have $ql - qb - a + r_e b \geq 0$ (a condition in Lemma 13.4.3 ensuring that G_e is a polynomial) and $qb > b$ (by $q \geq 2$). Hence, with $lq(d-1) \geq ld$ (13.4.7), we obtain

$$m_{e+1} = (ql - qb - a + r_e b) + qb + lq(d-1) > b + ld = m_e,$$

that is, $m_{e+1} > m_e$. This is a contradiction, and therefore for the case $q \geq 2$, $DA_{e-1}(lY, k)$ does not admit an e-th propagation. □

The lemmas in Case 2.1 and Case 2.2 together give Proposition 13.4.1.

13.5 Conclusion

Corollary 13.3.2, p239 and Corollary 13.4.2, p244 together yield the following result.

Proposition 13.5.1 *If $DA_{e-1}(lY, k)$ admits a complete propagation, then Y is of type A_l, B_l, or C_l such that if Y is of type C_l the positive integer k is divisible by n_e.*

We already showed the converse (Theorem 10.0.15, p177), and thus we establish the main result of this chapter:

Theorem 13.5.2 *Let lY be a subbranch of a branch X, and write $Y = n_0\Delta_0 + n_1\Theta_1 + \cdots + n_e\Theta_e$. Then $DA_{e-1}(lY, k)$ admits a complete propagation if and only if Y is of type A_l, B_l, or C_l such that if Y is of type C_l the positive integer k is divisible by n_e.*

13.6 Supplement: Proof of Lemma 13.4.4

Let $X = m_0\Delta_0 + m_1\Theta_1 + \cdots + m_\lambda\Theta_\lambda$ be a branch. Assume that $Y = n_0\Delta_0 + n_1\Theta_1 + \cdots + n_e\Theta_e$ is a subbranch of X, and l is a positive integer satisfying $lY \leq X$, i.e. $ln_i \leq m_i$ for $i = 0, 1, \ldots, e$. Suppose that the deformation atlas $DA_{e-1}(lY, k)$ admits an e-th propagation. We set

$$a = m_{e-1} - ln_{e-1}, \qquad b = m_e - ln_e, \qquad c = n_{e-1}, \qquad d = n_e,$$

and then the e-th propagation must be of the form (13.4.4):

$$\begin{cases} \mathcal{H}_e : & w^a\eta^b(w^c\eta^d + t^k)^l - s = 0 \\ \mathcal{H}'_e : & G_e(z,\zeta,t) - s = 0 \\ g_e : & z = 1/w, \quad \zeta = w^{r_e}\eta - t^h\alpha w^q, \end{cases}$$

where $G_e := z^{\,q\,l\,-\,q\,b\,-\,a\,+\,r_e\,b}\,\zeta^l\,(z^q\,\zeta + t^h\,\alpha)^b$ and

$$ql - qb - a + r_e b \geq 0. \tag{13.6.1}$$

(The last inequality is a condition that G_e is a polynomial.) However, it turns out that further propagations fail; the aim of this section is to give a proof of Lemma 13.4.4 which insists: (In what follows, λ is the length of the branch X.)

Suppose that $b \geq 1$ and $d = 1$. Then (1) $\lambda \geq e + 1$ and (2) an $(e+1)$-st propagation of $DA_{e-1}(lY, k)$ fails (and thus $DA_{e-1}(lY, k)$ does not admit a complete propagation).

We first show (1). Since $G_e(z, \zeta, 0) = z^{ql-a+r_e b}\,\zeta^{l+b}$, we have $m_{e+1} = ql - a + r_e b$. We claim that $m_{e+1} > 0$. Indeed, from (13.6.1), we have $m_{e+1} =$

$ql - a + r_e b \geq bq > 0$. In particular $\lambda \geq e+1$, proving (1). We next show (2), which requires some preparation. We set

$$F_{e+1}(w, \eta, t) := G_e(\eta, w, t) = w^l\, \eta^{\,q\,l - q\,b - a + r_e\,b}\,(w\,\eta^q + t^h\alpha)^b.$$

For brevity we write $b' := ql - qb - a + r_e b$; then $F_{e+1} = w^l \eta^{b'}(w\eta^q + t^h\alpha)^b$, and the inequality (13.6.1) is simply written as

$$b' \geq 0. \tag{13.6.2}$$

We need the following technical lemma.

Lemma 13.6.1 *If a (possibly trivial) deformation g_{e+1} of the transition function $z = 1/w$, $\zeta = w^{r'}\eta$, where $r' := r_{e+1}$, transforms $F_{e+1} = w^l\eta^{b'}(w\eta^q + t^h\alpha)^b$ to some polynomial, then g_{e+1} is of the form $z = 1/w$, $\zeta = w^{r'}\eta - \alpha' t^{h'} w^{q'}$ where $\alpha' \in \mathbb{C}$ and q' is an integer satisfying*

$$1 \leq q' \leq r' - 1, \qquad (r' - q')b' - l \geq 0. \tag{13.6.3}$$

In this case, g_{e+1} transforms F_{e+1} to

$$G_{e+1} = z^{(r'-q')b'-l}\,(z^{q'}\zeta + t^{h'}\alpha')^{b'}\left[z^{q(r'-q')-1}(z^{q'}\zeta + t^{h'}\alpha')^q + t^{h'}\alpha\right]^b.$$

Proof. By a similar argument to the proof of Lemma 13.3.3, we see that g_e must be of the form $z = 1/w$, $\zeta = w^{r'}\eta - \alpha' t^{h'} w^{q'}$ for some positive integer h' and q' $(1 \leq q' \leq r' - 1)$. Since

$$F_{e+1} = \frac{1}{w^{r'b'-l}}(w^{r'}\eta)^{b'}\left[\frac{1}{w^{qr'-1}}(w^{r'}\eta)^q + t^h\alpha\right]^b,$$

the map $g_{e+1}:\ z = 1/w,\ \zeta = w^{r'}\eta - \alpha' t^{h'} w^{q'}$ transforms F_{e+1} to

$$\begin{aligned} G_{e+1} &= z^{r'b'-l}\left(\zeta + t^{h'}\alpha'\frac{1}{z^{q'}}\right)^{b'}\left[z^{qr'-1}\left(\zeta + t^{h'}\alpha'\frac{1}{z^{q'}}\right)^q + t^h\alpha\right]^b \\ &= z^{(r'-q')b'-l}\left(z^{q'}\zeta + t^{h'}\alpha'\right)^{b'}\left[z^{q(r'-q')-1}\left(z^{q'}\zeta + t^{h'}\alpha'\right)^q + t^h\alpha\right]^b. \end{aligned}$$

Here we note that the factor $\left(z^{q'}\zeta + t^{h'}\alpha'\right)^{b'}\left[z^{q(r'-q')-1}\left(z^{q'}\zeta + t^{h'}\alpha'\right)^q + t^h\alpha\right]^b$, after expansion, does not contain fractional terms because

$$\begin{array}{ll} q' \geq 1, \quad b' \geq 0 & \text{by (13.6.2)} \\ q(r'-q') - 1 & \text{by } q \geq 1 \text{ and } r' - q' \geq 1 \\ b \geq 1. & \end{array}$$

Therefore noting that $\alpha \neq 0$,

$$G_{e+1} = z^{(r'-q')b'-l}\left(z^{q'}\zeta + t^{h'}\alpha'\right)^{b'}\left[z^{q(r'-q')-1}(z^{q'}\zeta + t^{h'}\alpha')^q + t^h\alpha\right]^b$$

is a polynomial precisely when the exponent of the first factor is nonnegative, i.e. $(r'-q')b'-l \geq 0$. Therefore g_e transforms F_{e+1} to some polynomial G_{e+1} if and only if (13.6.3) is satisfied. This proves the assertion. □

Now we can show (2) of Lemma 13.4.4: $DA_{e-1}(lY,k)$ *does not admit an* $(e+1)$*-st propagation.* By Lemma 13.6.1, if $DA_{e-1}(lY,k)$ admits an $(e+1)$-st propagation, then it must be of the form

$$\begin{cases} \mathcal{H}_{e+1}: & w^l\eta^{b'}(w\eta^q + t^h\alpha)^b - s = 0 \\ \mathcal{H}'_{e+1}: & G_{e+1}(z,\zeta,t) - s = 0 \\ g_{e+1}: & z = 1/w, \quad \zeta = w^{r'}\eta - t^{h'}\alpha' w^{q'}, \end{cases} \tag{13.6.4}$$

where $G_{e+1} = z^{(r'-q')b'-l}\left(z^{q'}\zeta + t^{h'}\alpha'\right)^{b'}\left[z^{q(r'-q')-1}(z^{q'}\zeta + t^{h'}\alpha')^q + t^h\alpha\right]^b$ and

$$1 \leq q' \leq r'-1, \qquad (r'-q')b'-l \geq 0. \tag{13.6.5}$$

However we claim: *The data* (13.6.4) *does* "*not*" *give an* $(e+1)$*-st propagation of the deformation atlas* $DA_{e-1}(lY,k)$*.* We prove this by contradiction. Setting $G_{e+1}(z,\zeta,0) = z^{m_{e+2}}\zeta^{m_{e+1}}$, we shall show that $m_{e+2} \geq m_{e+1}$ which contradicts that the sequence of multiplicities strictly decreases. To see this, put $t=0$ in G_{e+1}, then we have

$$m_{e+2} = \Big[(r'-q')b'-l\Big] + q'b' + b\Big[q(r'-q')-1\Big] + bqq', \qquad m_{e+1} = b' + qb.$$

Hence

$$m_{e+2} - m_{e+1} = \Big[(r'-q')b'-l\Big] + (q'-1)b' + b\Big[q(r'-q')-1\Big] + bq(q'-1).$$

Note that $m_{e+2} - m_{e+1} \geq 0$ because each term in $m_{e+2} - m_{e+1}$ is nonnegative:

$$\begin{array}{ll} \big[(r'-q')b'-l\big] \geq 0 & \text{by (13.6.5)} \\ (q'-1)b' \geq 0 & \text{by } q' \geq 1 \text{ and } b' \geq 0 \quad (13.6.2) \\ b\big[q(r'-q')-1\big] \geq 0 & \text{by } b \geq 1,\ q \geq 1, \text{ and } r'-q' \geq 1 \quad (13.6.3) \\ bq(q'-1) \geq 0 & \text{by } b \geq 1,\ q \geq 1, \text{ and } q' \geq 1. \end{array}$$

Thus $m_{e+2} \geq m_{e+1}$. But this contradicts that $m_{e+2} < m_{e+1}$.

14

Construction of Deformations by Bunches of Subbranches

Thus far, we treated a deformation atlas associated with one subbranch. In this chapter we consider a deformation atlas associated with a set of subbranches.

14.1 Propagation sequences

Let $X = m_0\Delta_0 + m_1\Theta_1 + \cdots + m_\lambda\Theta_\lambda$ be a branch. Recall that the sequence $\mathbf{m} := (m_0, m_1, \ldots, m_\lambda)$ satisfies

(1) $m_0 > m_1 > \cdots > m_\lambda > 0$, and
(2) $r_i := \dfrac{m_{i-1} + m_{i+1}}{m_i}$ $(i = 1, 2, \ldots, \lambda - 1)$ and $r_\lambda := \dfrac{m_{\lambda-1}}{m_\lambda}$ are integers greater than 1.

For a subbranch $Y = n_0\Delta_0 + n_1\Theta_1 + \cdots + n_e\Theta_e$, we associated a deformation atlas $DA_{e-1}(Y, d)$ as follows. We first define a sequence of integers p_i $(i = 0, 1, \ldots, \lambda + 1)$ inductively by

$$\begin{cases} p_0 = 0, \quad p_1 = 1 \quad \text{and} \\ p_{i+1} = r_i p_i - p_{i-1} \quad \text{for } i = 1,\, 2,\, \ldots,\, \lambda. \end{cases}$$

Then $p_{\lambda+1} > p_\lambda > \cdots > p_1 > p_0 = 0$ (6.2.4), p105. Let $f(z)$ be a non-vanishing holomorphic function on a domain $\{z \in \mathbb{C} : |z| < \varepsilon\}$, and we set

$$f_i = f(w^{p_{i-1}}\eta^{p_i}) \qquad \text{and} \qquad \widehat{f_i} = f(z^{p_{i+1}}\zeta^{p_i}) \qquad \text{(see (6.2.7), p106).} \tag{14.1.1}$$

Then the deformation atlas $DA_{e-1}(Y, d)$ is given as follows: for $i = 1, 2, \ldots, e - 1$,

$$\begin{cases} \mathcal{H}_i: & w^{m_{i-1}-n_{i-1}}\eta^{m_i-n_i}(w^{n_{i-1}}\eta^{n_i} + t^d f_i) - s = 0 \\ \mathcal{H}'_i: & z^{m_{i+1}-n_{i+1}}\zeta^{m_i-n_i}(z^{n_{i+1}}\zeta^{n_i} + t^d \widehat{f_i}) - s = 0 \\ g_i: & \text{the transition function } z = 1/w,\ \zeta = w^{r_i}\eta \text{ of } N_i. \end{cases}$$

Before proceeding, we recall terminology; the multiplicity sequence $\mathbf{n} = (n_0, n_1, \dots, n_e)$ of Y satisfies

$$\frac{n_{i-1} + n_{i+1}}{n_i} = r_i \quad (i = 1, 2, \dots, e-1),$$

where for $e = 0$ or 1, this condition is empty. For a subbranch $Y' = n_0\Delta_0 + n_1\Theta_1 + \cdots + n_{e'}\Theta_{e'}$, a subbranch $Y = n_0\Delta_0 + n_1\Theta_1 + \cdots + n_e\Theta_e$ $(e < e')$ is said to *be contained in* Y'. Accordingly, $\mathbf{n} = (n_0, n_1, \dots, n_e)$ is said to *be contained in* $\mathbf{n}' = (n_0, n_1, \dots, n_{e'})$. By a *dominant sequence* containing a multiplicity sequence $\mathbf{n}$, we mean a multiplicity sequence which has the maximal length among all multiplicity sequences containing $\mathbf{n}$. A subbranch with a dominant (multiplicity) sequence is also called *dominant.* We classified dominant subbranches into two classes "tame" and "wild". As we showed in Lemma 5.5.7 p94, setting $q = n_{e-1} - r_e n_e$ (the *slant* of Y), then we have the following equivalences:

$$Y \text{ is tame} \iff q \geq 0,$$
$$Y \text{ is wild} \iff 0 > m_{e+1} + q,$$

Here if $e = \lambda$, we set $m_{\lambda+1} = 0$ by convention. More generally, a (not necessarily dominant) subbranch Y is called *tame* (resp. *wild*) if the dominant subbranch containing Y is tame (resp. wild).

Now we suppose that Y is dominant tame. By Theorem 6.1.1, p99, the deformation atlas $DA_{e-1}(Y, d)$ of arbitrary weight d admits a complete propagation, which is explicitly given as follows: The e-th propagation of $DA_{e-1}(Y, d)$ is

$$\begin{cases} \mathcal{H}_e: & w^{m_{e-1}-n_{e-1}}\eta^{m_e-n_e}\,(w^{n_{e-1}}\eta^{n_e} + t^d\, f_e) - s = 0 \\ \mathcal{H}'_e: & z^{m_{e+1}}\zeta^{m_e-n_e}\,(\zeta^{n_e} + t^d\, z^q\, \widehat{f}_e) - s = 0 \\ g_e: & \text{the transition function } z = 1/w,\ \zeta = w^{r_e}\eta \text{ of } N_e, \end{cases} \tag{14.1.2}$$

where f_e and $\widehat{f}_e$ are as in (14.1.1). Next the $(e+1)$-st propagation is

$$\begin{cases} \mathcal{H}_{e+1}: & w^{m_e-n_e}\eta^{m_{e+1}}\,(w^{n_e} + t^d\,\eta^q\, f_{e+1}) - s = 0 \\ \mathcal{H}'_{e+1}: & z^{m_{e+2}}\zeta^{m_{e+1}}\,(1 + t^d\, z^{r_{e+1}\,q + n_e}\,\zeta^q\, \widehat{f}_{e+1}) - s = 0 \\ g_{e+1}: & \text{the transition function } z = 1/w,\ \zeta = w^{r_{e+1}}\eta \text{ of } N_{e+1}. \end{cases} \tag{14.1.3}$$

For $i = e+2, e+3, \dots, \lambda$, the i-th propagation is given by

$$\begin{cases} \mathcal{H}_i: & w^{m_{i-1}}\eta^{m_i}\,(1 + t^d\, w^{a_{i-1}}\,\eta^{a_i}\, f_i) - s = 0 \\ \mathcal{H}'_i: & z^{m_{i+1}}\zeta^{m_i}\,(1 + t^d\, z^{a_{i+1}}\,\zeta^{a_i}\, \widehat{f}_i) - s = 0 \\ g_i: & \text{the transition function } z = 1/w,\ \zeta = w^{r_i}\eta \text{ of } N_i, \end{cases} \tag{14.1.4}$$

where $m_{\lambda+1} = 0$ by convention, and $a_{e+1}, a_{e+2}, \dots, a_{\lambda+1}$ are nonnegative integers defined by

$$\begin{cases} a_{e+1} := q, \quad a_{e+2} := r_{e+1}q + n_e & \text{and} \\ a_{i+1} := r_i a_i - a_{i-1} & \text{for} \quad i = e+2,\ e+3,\ \dots,\ \lambda. \end{cases} \tag{14.1.5}$$

As noted in the paragraph subsequent to (6.1.5), p102, (1) $a_i \geq 0$, in fact, $a_{\lambda+1} > a_\lambda > \cdots > a_{e+1} \geq 0$ and (2) the transition function g_i of N_i transforms $w^{a_{i-1}}\eta^{a_i}$ to $z^{a_{i+1}}\zeta^{a_i}$. In Theorem 6.1.1, we showed that (14.1.2), (14.1.3), (14.1.4) together give a complete propagation of $DA_{e-1}(Y, d)$.

For later application, it is convenient to express 'uniformly' the deformation atlas $DA_{e-1}(Y, d)$ and its complete propagation. For this purpose, we introduce a sequence of nonnegative integers $b_0, b_1, \dots, b_{\lambda+1}$:

$$b_i = \begin{cases} m_i - n_i, & i = 0, 1, \dots, e \\ m_i + a_i, & i = e+1, e+2, \dots, \lambda+1, \end{cases}$$

where $a_{e+1}, a_{e+2}, \dots, a_\lambda$ are in (14.1.5), and then the deformation atlas $DA_{e-1}(Y, d)$ and its complete propagation are uniformly expressed as follows: for $i = 1, 2, \dots, \lambda$,

$$\begin{cases} \mathcal{H}_i: & w^{m_{i-1}}\eta^{m_i} - s + t^d w^{b_{i-1}}\eta^{b_i} f_i = 0, \\ \mathcal{H}'_i: & z^{m_{i+1}}\zeta^{m_i} - s + t^d z^{b_{i+1}}\zeta^{b_i} \widehat{f}_i = 0, \\ g_i: & \text{the transition function } z = 1/w,\ \zeta = w^{r_i}\eta \text{ of } N_i. \end{cases} \tag{14.1.6}$$

We say that the sequence $b_0, b_1, \dots, b_{\lambda+1}$ is the *propagation sequence* for the dominant tame subbranch Y; the propagation sequence for a non-dominant tame subbranch Y' is defined as that for the dominant tame subbranch Y containing Y'.

14.2 Bunches of subbranches

We take a finite set $\mathbf{Y} = \{Y_1, Y_2, \dots, Y_l\}$ of subbranches, where

$$Y_k = n_{k,0}\Delta_0 + n_{k,1}\Theta_1 + \cdots + n_{k,e_k}\Theta_{e_k} \quad (k = 1, 2 \dots, l).$$

For individual Y_k, we already defined a deformation atlas $DA_{e_k-1}(Y_k, d_k)$: for $i = 1, 2, \dots, e_k - 1$,

$$\begin{cases} \mathcal{H}_i: & w^{m_{i-1}}\eta^{m_i} - s + t^{d_k} w^{m_{i-1}-n_{k,\,i-1}}\eta^{m_i - n_{k,\,i}} f_{k,i} = 0 \\ \mathcal{H}'_i: & z^{m_{i+1}}\zeta^{m_i} - s + t^{d_k} z^{m_{i+1}-n_{k,\,i+1}}\zeta^{m_i - n_{k,\,i}} \widehat{f}_{k,i} = 0 \\ g_i: & \text{the transition function } z = 1/w,\ \zeta = w^{r_i}\eta \text{ of } N_i. \end{cases} \tag{14.2.1}$$

Next we consider the problem of the association of a deformation atlas to the finite set $\mathbf{Y} = \{Y_1, Y_2, \dots, Y_l\}$ of subbranches. In contrast to the association of a deformation atlas to one subbranch, this is a subtle problem, unless all $Y_1, Y_2, \dots, Y_l$ have the same length (that is, $e_1 = e_2 = \cdots = e_l$). Actually, if all $Y_1, Y_2, \dots, Y_l$ have the same length, then we may easily construct a deformation atlas of length $e-1$ associated with $\mathbf{Y}$, where $e := e_1\,(= e_2 = \cdots = e_l)$. Specifically it is given as follows: for $i = 1, 2, \dots, e-1$,

$$\begin{cases} \mathcal{H}_i : & w^{m_{i-1}}\eta^{m_i} - s + \sum_{k=1}^{l} t^{d_k} w^{m_{i-1}-n_{k,\,i-1}}\eta^{m_i-n_{k,\,i}} f_{k,i} = 0 \\ \mathcal{H}'_i : & z^{m_{i+1}}\zeta^{m_i} - s + \sum_{k=1}^{l} t^{d_k} z^{m_{i+1}-n_{k,\,i+1}}\zeta^{m_i-n_{k,\,i}} \widehat{f}_{k,i} = 0 \\ g_i : & \text{the transition function } z = 1/w,\ \zeta = w^{r_i}\eta \text{ of } N_i. \end{cases} \tag{14.2.2}$$

However if the lengths of $Y_1, Y_2, \dots, Y_l$ are not equal, then (14.2.2) is no longer well-defined; it is merely defined for $i = 1, 2, \dots, e_{\min} - 1$ where $e_{\min} := \min\{e_1, e_2, \dots, e_l\}$. Nevertheless it is possible to modify (14.2.2) to define a deformation atlas of length $e_{\max} - 1$, where $e_{\max} := \max\{e_1, e_2, \dots, e_l\}$, provided that any $Y_k \in \mathbf{Y}$ with $e_k < e_{\max}$ is tame. We shall explain this procedure. For convenience, we set $e := e_{\max}$ and we say that Y_k ($k = 1, 2, \dots, l$) is *short* or *long* according to whether $e_k < e$ or $e_k = e$.

Claim 14.2.1 *Let $\mathbf{Y} = \{Y_1, Y_2, \dots, Y_l\}$ be a set of subbranches such that any short $Y_k \in \mathbf{Y}$ is tame. Then we may associate a deformation atlas of length $e-1$, where $e := \max\{e_1, e_2, \dots, e_l\}$.*

To show this, we first note that under the above assumption, for each subbranch Y_k ($k = 1, 2, \dots, l$), we may associate a deformation atlas of length $e-1$; according to whether Y_k is long or short, it is given as follows:

(1) If Y_k is long (i.e. $e_k = e$), the deformation atlas $DA_{e-1}(Y_k, d_k)$ is the desired one: for $i = 1, 2, \dots, e-1$,

$$\begin{cases} \mathcal{H}_i : & w^{m_{i-1}}\eta^{m_i} - s + t^{d_k} w^{m_{i-1}-n_{k,\,i-1}}\eta^{m_i-n_{k,\,i}} f_{k,i} = 0 \\ \mathcal{H}'_i : & z^{m_{i+1}}\zeta^{m_i} - s + t^{d_k} z^{m_{i+1}-n_{k,\,i+1}}\zeta^{m_i-n_{k,\,i}} \widehat{f}_{k,i} = 0 \\ g_i : & \text{the transition function } z = 1/w,\ \zeta = w^{r_i}\eta \text{ of } N_i. \end{cases} \tag{14.2.3}$$

(2) If Y_k is short (i.e. $e_k < e$), then Y_k is tame by assumption, and thus $DA_{e_k-1}(Y_k, d_k)$ admits a complete propagation. (Note: Any tame subbranch Y is contained in a dominant tame subbranch Y', for which the associated deformation atlas always admits a complete propagation, and consequently the deformation atlas associated with Y also admits a complete propagation.) Extract a deformation atlas of length $e-1$ from this complete deformation atlas: for $i = 1, 2, \dots, e-1$,

$$\begin{cases} \mathcal{H}_i : & w^{m_{i-1}}\eta^{m_i} - s + t^{d_k} w^{b_{k,i-1}}\eta^{b_{k,i}} f_{k,i} = 0, \\ \mathcal{H}'_i : & z^{m_{i+1}}\zeta^{m_i} - s + t^{d_k} z^{b_{k,i+1}}\zeta^{b_{k,i}} \widehat{f}_{k,i} = 0, \\ g_i : & \text{the transition function } z = 1/w,\ \zeta = w^{r_i}\eta \text{ of } N_i, \end{cases} \tag{14.2.4}$$

where $b_{k,0}, b_{k,1}, \ldots, b_{k,\lambda+1}$ is the propagation sequence for Y_k. (14.2.4) is the desired deformation atlas of length $e-1$.

Therefore regardless of whether Y_k is short or long, we have a deformation atlas of length $e-1$ associated with it. Based upon this observation, we now construct a deformation atlas of length $e-1$ associated with the set $\mathbf{Y} = \{Y_1, Y_2, \ldots, Y_l\}$ of subbranches satisfying a condition: *Any short Y_k is tame.* Firstly, divide $\mathbf{Y}$ into two sets: $\mathbf{Y} = \{Y_k\}_{k\in S} \cup \{Y_k\}_{k\in L}$ where $\{Y_k\}_{k\in S}$ is the set of short subbranches and $\{Y_k\}_{k\in L}$ is the set of long subbranches: $S = \{k : e_k < e\}$ and $L = \{k : e_k = e\}$. Then for $i = 1, 2, \ldots, e-1$, the following data gives a deformation atlas of length $e-1$:

$$\begin{cases} \mathcal{H}_i : & w^{m_{i-1}}\eta^{m_i} - s + \sum\limits_{k\in S} t^{d_k} w^{b_{k,i-1}}\eta^{b_{k,i}} f_{k,i} \\ & \qquad + \sum\limits_{k\in L} t^{d_k} w^{m_{i-1}-n_{k,i-1}}\eta^{m_i - n_{k,i}} f_{k,i} = 0, \\ \mathcal{H}'_i : & z^{m_{i+1}}\zeta^{m_i} - s + \sum\limits_{k\in S} t^{d_k} z^{b_{k,i+1}}\zeta^{b_{k,i}} \widehat{f}_{k,i} \\ & \qquad + \sum\limits_{k\in L} t^{d_k} z^{m_{i+1}-n_{k,i+1}}\zeta^{m_i - n_{k,i}} \widehat{f}_{k,i} = 0, \\ g_i : & \text{the transition function } z = 1/w,\ \zeta = w^{r_i}\eta \text{ of } N_i. \end{cases}$$

In fact, g_i transforms

$$t^{d_k} w^{b_{k,i-1}}\eta^{b_{k,i}} f_{k,i} \quad \text{to} \quad t^{d_k} z^{b_{k,i+1}}\zeta^{b_{k,i}} \widehat{f}_{k,i}, \quad \text{and}$$
$$t^{d_k} w^{m_{i-1}-n_{k,i-1}}\eta^{m_i-n_{k,i}} f_{k,i} \quad \text{to} \quad t^{d_k} z^{m_{i+1}-n_{k,i+1}}\zeta^{m_i-n_{k,i}} \widehat{f}_{k,i} \quad \text{respectively,}$$

and therefore g_i transforms $\mathcal{H}_i$ to $\mathcal{H}'_i$. Moreover by a coordinate change $(w, \eta) = (\zeta, z)$, $\mathcal{H}'_i$ becomes $\mathcal{H}_{i+1}$. Hence we obtained a deformation atlas of length $e-1$ associated with a finite set of subbranches $\mathbf{Y} = \{Y_1, Y_2, \ldots, Y_l\}$ under the assumption that any short $Y_k \in \mathbf{Y}$ is tame (recall: Y_k is short if $e_k < e := \max\{e_1, e_2, \ldots, e_l\}$). Motivated by this result, we now introduce the following concept:

Definition 14.2.2 A finite set $\mathbf{Y} = \{Y_1, Y_2, \ldots, Y_l\}$ of subbranches is called a *bunch* (*of subbranches*) provided that any short subbranch Y_k of $\mathbf{Y}$ is tame. The positive integer $e := \max\{e_1, e_2, \ldots, e_l\}$ is called the *length* of the bunch $\mathbf{Y}$, where e_k is the length of Y_k.

Using this terminology, the above result is summarized as follows:

Lemma 14.2.3 *Let $\mathbf{Y} = \{Y_1, Y_2, \ldots, Y_l\}$ be a bunch of subbranches where $Y_k = n_{k,0}\Delta_0 + n_{k,1}\Theta_1 + \cdots + n_{k,e_k}\Theta_{e_k}$ $(k = 1, 2, \ldots, l)$, and set $e = \max\{e_1, e_2, \ldots, e_l\}$. Divide $\mathbf{Y}$ into two sets respectively consisting of short and long subbranches: $\mathbf{Y} = \{Y_k\}_{k\in S} \cup \{Y_k\}_{k\in L}$ where $S = \{k : e_k < e\}$ and $L = \{k : e_k = e\}$. Then for a set of arbitrary positive integers $\mathbf{d} = \{d_1, d_2, \ldots, d_l\}$, the following data gives a deformation atlas of length*

$e-1$*: for* $i=1,2,\dots,e-1$,

$$\begin{cases} \mathcal{H}_i : & w^{m_{i-1}}\eta^{m_i} - s + \sum_{k\in S} t^{d_k} w^{b_{k,i-1}}\eta^{b_{k,i}} f_{k,i} \\ & \qquad + \sum_{k\in L} t^{d_k} w^{m_{i-1}-n_{k,i-1}}\eta^{m_i-n_{k,i}} f_{k,i} = 0, \\ \mathcal{H}'_i : & z^{m_{i+1}}\zeta^{m_i} - s + \sum_{k\in S} t^{d_k} z^{b_{k,i+1}}\zeta^{b_{k,i}} \widehat{f}_{k,i} \\ & \qquad + \sum_{k\in L} t^{d_k} z^{m_{i+1}-n_{k,i+1}}\zeta^{m_i-n_{k,i}} \widehat{f}_{k,i} = 0, \\ g_i : & \textit{the transition function } z = 1/w,\ \zeta = w^{r_i}\eta \textit{ of } N_i, \end{cases}$$

where $b_{k,0}, b_{k,1}, \dots, b_{k,\lambda+1}$ *is the propagation sequence for a short subbranch* Y_k $(k \in S)$.

We say that the deformation atlas in the above lemma, denoted by DA_{e-1} $(\mathbf{Y}, \mathbf{d})$, *is associated with* a bunch $\mathbf{Y} = \{Y_1, Y_2, \dots, Y_l\}$ and the set of positive integers $\mathbf{d} = \{d_1, d_2, \dots, d_l\}$ is called its *weight*.

Remark 14.2.4 *For a subbranch* Y *of type* A_l, B_l, *or* C_l, *we associated the deformation atlas* $DA_{e-1}(lY, d)$ (Theorem 10.0.15, p177)*: For* $i = 1, 2, \dots, e-1$,

$$\begin{cases} \mathcal{H}_i : & w^{m_{i-1}-ln_{i-1}}\eta^{m_i-ln_i}(w^{n_{i-1}}\eta^{n_i} + t^d f_i)^l - s = 0 \\ \mathcal{H}'_i : & z^{m_{i+1}-ln_{i+1}}\zeta^{m_i-ln_i}(z^{n_{i+1}}\zeta^{n_i} + t^d \widehat{f}_i)^l - s = 0 \\ g_i : & \textit{the transition function } z = 1/w,\ \zeta = w^{r_i}\eta \textit{ of } N_i. \end{cases}$$

After expansion, we have

$$\begin{cases} \mathcal{H}_i : & w^{m_{i-1}}\eta^{m_i} - s + \sum_{k=1}^{l} {}_lC_k\, t^{kd}\, w^{m_{i-1}-kn_{i-1}}\eta^{m_i-kn_i} f_i^k = 0, \\ \mathcal{H}'_i : & z^{m_{i+1}}\zeta^{m_i} - s + \sum_{k=1}^{l} {}_lC_k\, t^{kd}\, z^{m_{i+1}-kn_{i+1}}\zeta^{m_i-kn_i} \widehat{f}_i^k = 0, \\ g_i : & \textit{the transition function } z = 1/w,\ \zeta = w^{r_i}\eta \textit{ of } N_i. \end{cases}$$

In the terminology of this section, this is nothing other than the deformation atlas of weight $\mathbf{d} = \{d, 2d, \dots, ld\}$ *associated with a bunch* $\mathbf{Y} = \{Y, 2Y, \dots, lY\}$.

Recall that we have introduced important notions "tame" and "wild" for subbranches, which are related to propagatability of deformation atlases. We shall generalize these notions to those for a set of subbranches.

Definition 14.2.5 Suppose that $\mathbf{Y} = \{Y_1, Y_2, \dots, Y_l\}$ is a bunch, that is, any short Y_k in $\mathbf{Y}$ is tame. Then $\mathbf{Y}$ is called (1) *tame* if all $Y_k \in \mathbf{Y}$ are tame, and (2) *wild* if some $Y_k \in \mathbf{Y}$ is wild. (When all Y_k $(k = 1, 2, \dots, l)$ have length zero, i.e. $Y_k = n_{k,0}\Delta_0$, we consider $\mathbf{Y}$ to be tame. cf. Remark 10.0.16, p177.)

Remember that for a subbranch $Y' = n_0\Delta_0 + n_1\Theta_1 + \cdots + n_{e'}\Theta_{e'}$, a subbranch $Y = n_0\Delta_0 + n_1\Theta_1 + \cdots + n_e\Theta_e$ $(e < e')$ is said to *be contained in* Y'. For two bunches $\mathbf{Y} = \{Y_1, Y_2, \dots, Y_l\}$ and $\mathbf{Y}' = \{Y'_1, Y'_2, \dots, Y'_l\}$, we say that $\mathbf{Y}$ *is contained in* $\mathbf{Y}'$ if possibly after reordering, Y_k $(k = 1, 2, \dots, l)$

is contained in Y'_k. A bunch $\mathbf{Y}'$ is called *dominant* provided that there is no bunch containing $\mathbf{Y}'$. Given a bunch $\mathbf{Y}$, there is a unique dominant bunch $\mathbf{Y}'$ containing $\mathbf{Y}$. The existence of $\mathbf{Y}'$ is seen as follows. For each $Y_k \in \mathbf{Y}$, we first take a dominant subbranch Z_k containing it, and express $Z_k = n_{k,0}\Delta_0 + n_{k,1}\Theta_1 + \cdots + n_{k,e'_k}\Theta_{e'_k}$. We then consider a set $\mathbf{Z} = \{Z_1, Z_2, \ldots, Z_l\}$ which in general is *not* a bunch. Let $\{Z_k\}_{k\in W}$ be the subset of $\mathbf{Z}$ consisting of wild subbranches, and we set $e' := \min\{e'_k\}_{k\in W}$. For each k ($k = 1, 2, \ldots, l$), we define a subbranch Y'_k as follows:

$$Y'_k = \begin{cases} Z_k & \text{if } e'_k < e' \\ n_{k,0}\Delta_0 + n_{k,1}\Theta_1 + \cdots + n_{k,e'}\Theta_{e'} & \text{otherwise.} \end{cases} \tag{14.2.5}$$

We claim that $\mathbf{Y}' := \{Y'_1, Y'_2, \ldots, Y'_l\}$ is a unique dominant bunch containing $\mathbf{Y}$. Firstly, we show that $\mathbf{Y}'$ is a bunch: If $e'_k < e'$, then (i) $Y'_k = Z_k$ by (14.2.5), and (ii) Z_k (and hence Y'_k) is tame because $e'_k = \text{length}(Z_k) < e'$ where e' is the minimum of the lengths of the wild subbranches in $\mathbf{Z}$. From (i) and (ii), $\mathbf{Y}'$ is a bunch. Further, $\mathbf{Y}'$ is dominant: This is easily seen by the same argument in Example 14.2.6 below. Thus $\mathbf{Y}'$ is a dominant bunch containing $\mathbf{Y}$. We leave the reader to check the uniqueness of $\mathbf{Y}'$ with this property.

We note that if $\mathbf{Y}$ is tame, i.e. all $Y_k \in \mathbf{Y}$ are tame, then the above construction reduces to very simple one; for each $Y_k \in \mathbf{Y}$, take the dominant subbranch Y'_k containing it, where note that Y'_k is tame. Then $\mathbf{Y}' := \{Y'_1, Y'_2, \ldots, Y'_l\}$ is the dominant (tame) bunch containing $\mathbf{Y}$. cf. Example 14.2.6 below.

Finally we remark that letting $\mathbf{Y}'$ be the dominant bunch containing a bunch $\mathbf{Y}$, then we have the following equivalences:

$$\begin{array}{ccc} \mathbf{Y}' \text{ is tame} & \Longleftrightarrow & \mathbf{Y} \text{ is tame,} \\ \mathbf{Y}' \text{ is wild} & \Longleftrightarrow & \mathbf{Y} \text{ is wild.} \end{array}$$

Example 14.2.6 Let $X = 6\Delta_0 + 5\Theta_1 + 4\Theta_2 + 3\Theta_3 + 2\Theta_4 + \Theta_5$ be a branch, and take a set $\mathbf{Z}$ of dominant wild subbranches Z_1, Z_2, and Z_3:

$$\mathbf{Z} = \left\{ \begin{array}{c} Z_1 = \Delta_0 + \Theta_1 + \Theta_2 + \Theta_3 + \Theta_4 + \Theta_5 \\ Z_2 = 2\Delta_0 + 2\Theta_1 + 2\Theta_2 + 2\Theta_3 + 2\Theta_4 \\ Z_3 = 3\Delta_0 + 3\Theta_1 + 3\Theta_2 + 3\Theta_3 \end{array} \right\},$$

where we note that $\mathbf{Z}$ is not a bunch but merely a set of subbranches. Then we consider a bunch of length 3:

$$\mathbf{Y}' = \left\{ \begin{array}{c} Y'_1 = \Delta_0 + \Theta_1 + \Theta_2 + \Theta_3 \\ Y'_2 = 2\Delta_0 + 2\Theta_1 + 2\Theta_2 + 2\Theta_3 \\ Y'_3 = 3\Delta_0 + 3\Theta_1 + 3\Theta_2 + 3\Theta_3 \end{array} \right\},$$

where we note that Y_1' and Y_2' are *not* dominant while $Y_3' = Z_3$ is dominant. We claim that $\mathbf{Y}'$ is a dominant bunch. In fact, if there exists a bunch $\mathbf{Y}''$ containing $\mathbf{Y}'$, then Y_3' belongs to $\mathbf{Y}''$ because Y_3' is dominant. Since $\mathrm{length}\,(\mathbf{Y}'') > 3$, Y_3' is a short subbranch of $\mathbf{Y}''$. However Y_3' is wild; this contradicts that any short subbranch of a bunch is tame. Therefore the bunch $\mathbf{Y}'$ is dominant (wild).

Next we discuss the propagation problem of deformation atlases associated with bunches.

Theorem 14.2.7 *Assume that* $DA_{e-1}(\mathbf{Y}, \mathbf{d})$ *is the deformation atlas of weight* $\mathbf{d} = \{d_1, d_2, \dots, d_l\}$ *associated with a bunch* $\mathbf{Y} = \{Y_1, Y_2, \dots, Y_l\}$ (see Lemma 14.2.3)*. If* $\mathbf{Y}$ *is tame, then* $DA_{e-1}(\mathbf{Y}, \mathbf{d})$ *admits a complete propagation.*

Proof. Recall that $\mathbf{Y}$ is tame if all $Y_k \in \mathbf{Y}$ are tame, and so we may take the propagation sequence $b_{k,0}, b_{k,1}, \dots, b_{k,\lambda+1}$ for each $Y_k \in \mathbf{Y}$. Then a complete propagation of $DA_{e-1}(\mathbf{Y}, \mathbf{d})$ is simply given as follows: For $i = 1, 2, \dots, \lambda$,

$$\begin{cases} \mathcal{H}_i : & w^{m_{i-1}}\eta^{m_i} - s + \sum_{k=1}^{l} t^{d_k}\, w^{b_{k,i-1}}\, \eta^{b_{k,i}} f_{k,i} = 0, \\ \mathcal{H}_i' : & z^{m_{i+1}}\zeta^{m_i} - s + \sum_{k=1}^{l} t^{d_k}\, z^{b_{k,i+1}}\, \zeta^{b_{k,i}} \widehat{f}_{k,i} = 0, \\ g_i : & \text{the transition function } z = 1/w,\ \zeta = w^{r_i}\eta \text{ of } N_i, \end{cases} \tag{14.2.6}$$

where $m_{\lambda+1} = 0$ by convention. □

For (not necessarily tame) bunches, we have the following result.

Theorem 14.2.8 *Let* $X = m_0\Delta_0 + m_1\Theta_1 + \cdots + m_\lambda\Theta_\lambda$ *be a branch such that* $m_\lambda = 1$*. Assume that* $\mathbf{Y} = \{Y_1, Y_2, \dots, Y_l\}$ *is a* (*possibly wild*) *bunch such that* $\lambda = \mathrm{length}(\mathbf{Y})$*,* i.e. $\lambda = \max\{\mathrm{length}(Y_1), \mathrm{length}(Y_2), \dots, \mathrm{length}(Y_l)\}$*. Then* $DA_{\lambda-1}(\mathbf{Y}, \mathbf{d})$ *admits a complete propagation.*

Proof. Since $m_\lambda = 1$ and $DA_{\lambda-1}(\mathbf{Y}, \mathbf{d})$ has length $\lambda - 1$, it follows from Propagation Lemma (Lemma 5.2.2) that $DA_{\lambda-1}(\mathbf{Y}, \mathbf{d})$ admits a complete propagation. □

14.3 Example of a deformation by a wild bunch

In this section we provide an example of a wild bunch such that the associated deformation atlas admits a complete propagation. (This example will be used later, when we construct a splitting family of a degeneration of elliptic curves. See §17.3, p306.) We consider a branch $X = 6\Delta_0 + 4\Theta_1 + 2\Theta_2$, and let $\mathbf{Y} = \{Y_1, Y_2, Y_3\}$ be a bunch of subbranches of X, where

$$Y_1 = \Delta_0 + \Theta_1 + \Theta_2, \qquad Y_2 = 2\Delta_0 + 2\Theta_1 + 2\Theta_2, \qquad Y_3 = 4\Delta_0 + 3\Theta_1 + 2\Theta_2.$$

Then $\mathbf{Y}$ is wild; indeed, all of Y_1, Y_2, and Y_3 are wild. Taking $\mathbf{d} = (1, 2, 4)$, we define $DA_1(\mathbf{Y}, \mathbf{d})$ by

$$\begin{cases} \mathcal{H}_1 : & w^6\eta^4 - s + a\,t\,w^5\,\eta^3\,(1-\eta)^{1/2} + b\,t^2\,w^4\,\eta^2 + c\,t^4\,w^2\,\eta = 0 \\ \mathcal{H}_1' : & z^2\zeta^4 - s + a\,t\,z\,\zeta^3\,(1-z^2\zeta)^{1/2} + b\,t^2\,\zeta^2 + c\,t^4\,\zeta = 0 \\ g_1 : & z = \frac{1}{w}, \quad \zeta = w^2\eta, \end{cases}$$

where $a, b, c \in \mathbb{C}$ will be determined in the course of the subsequent construction. We note that the term with a coefficient a in $\mathcal{H}_1$ (resp. $\mathcal{H}_1'$) contains a factor $(1-\eta)^{1/2}$ (resp. $(1-z^2\zeta)^{1/2}$), and when we try to construct a complete propagation of $DA_1(\mathbf{Y}, \mathbf{d})$, the existence of this term forces us some computation to check whether or not fractional terms in certain equation vanish. We shall elucidate this by an explicit computation. First we note

$$\mathcal{H}_2 : \quad w^4\eta^2 - s + atw^3\eta\,(1-w\eta^2)^{1/2} + bt^2w^2 + ct^4w = 0.$$

Letting $g_2 :\ z = 1/w,\ \zeta = w^2\eta + t\alpha w$ where $\alpha \in \mathbb{C}$, we shall investigate whether we may find a, b, c so that g_2 transforms $\mathcal{H}_2$ to some hypersurface. Since

$$\begin{aligned} &w^4\eta^2 - s + atw^3\eta\,(1-w\eta^2)^{1/2} + bt^2w^2 + ct^4w \\ &\qquad = (w^2\eta)^2 - s + atw(w^2\eta)\left(1 - \frac{1}{w^3}(w^2\eta)^2\right)^{1/2} + bt^2w^2 + ct^4w, \end{aligned}$$

the map $g_2 :\ z = 1/w,\ \zeta = w^2\eta + t\alpha w$ transforms the left hand side of $\mathcal{H}_2$ to

$$\left(\zeta - t\alpha\frac{1}{z}\right)^2 - s + at\frac{1}{z}\left(\zeta - t\alpha\frac{1}{z}\right)\left[1 - z^3\left(\zeta - t\alpha\frac{1}{z}\right)^2\right]^{1/2} + bt^2\frac{1}{z^2} + ct^4\frac{1}{z}. \tag{14.3.1}$$

Using the Taylor expansion

$$(1-z)^{1/2} = 1 - \frac{1}{2}z - \frac{1}{8}z^2 - \frac{1}{16}z^3 - \cdots,$$

we see that the middle term of (14.3.1)

$$\frac{1}{z}\left(\zeta - t\alpha\frac{1}{z}\right)\left[1 - z^3\left(\zeta - t\alpha\frac{1}{z}\right)^2\right]^{1/2}$$

admits an expansion:

$$\frac{1}{z}\left(\zeta - t\alpha\frac{1}{z}\right) - \frac{1}{2}z^2\left(\zeta - t\alpha\frac{1}{z}\right)^3 - \frac{1}{8}z^5\left(\zeta - t\alpha\frac{1}{z}\right)^5 - \frac{1}{16}z^8\left(\zeta - t\alpha\frac{1}{z}\right)^7 - \cdots,$$

which contains just three fractional terms:

$$\frac{\zeta}{z} - t\alpha\frac{1}{z^2} + \frac{1}{2}t^3\alpha^3\frac{1}{z}.$$

Therefore the fractional terms contained in the expansion of (14.3.1) are

$$-2t\alpha\frac{\zeta}{z} + t^2\alpha^2\frac{1}{z^2} + at\left(\frac{\zeta}{z} - t\alpha\frac{1}{z^2} + \frac{1}{2}t^3\alpha^3\frac{1}{z}\right) + bt^2\frac{1}{z^2} + ct^4\frac{1}{z}.$$

Reordering these terms, we have

$$t\frac{\zeta}{z}(-2\alpha + a) + t^2\frac{1}{z^2}(\alpha^2 - a\alpha + b) + t^4\frac{1}{z}\left(\frac{1}{2}a\alpha^3 + c\right).$$

We choose $a, b, c \in \mathbb{C}$ such that

$$-2\alpha + a = 0, \qquad \alpha^2 - a\alpha + b = 0, \qquad \frac{1}{2}a\alpha^3 + c = 0,$$

that is $a = 2\alpha$, $b = \alpha^2$, and $c = -\alpha^4$. Then all fractional terms in (14.3.1) vanish, and $g_2 : \ z = 1/w,\ \zeta = w^2\eta + \alpha t w$ transforms $\mathcal{H}_2$ to a hypersurface $\mathcal{H}_2'$ defined by

$$\left(\zeta - t\alpha\frac{1}{z}\right)^2 - s + 2\alpha t\frac{1}{z}\left(\zeta - t\alpha\frac{1}{z}\right)\left[1 - z^3\left(\zeta - t\alpha\frac{1}{z}\right)^2\right]^{1/2} + \alpha^2 t^2\frac{1}{z^2} - \alpha^4 t^4\frac{1}{z} = 0,$$

which, after expansion, becomes a polynomial (without fractional terms). Therefore the following data gives a complete propagation of $DA_1(\mathbf{Y}, \mathbf{d})$: (for simplicity, we take $\alpha = 1$ and so $a = 2$, $b = 1$, and $c = -1$)

$$\begin{cases} \mathcal{H}_2 : & w^4\eta^2 - s + 2tw^3\eta(1 - w\eta^2)^{1/2} + t^2w^2 - t^4w = 0 \\ \mathcal{H}_2' : & \left(\zeta - t\frac{1}{z}\right)^2 - s + 2t\frac{1}{z}\left(\zeta - t\frac{1}{z}\right)\left[1 - z^3\left(\zeta - t\frac{1}{z}\right)^2\right]^{1/2} \\ & \qquad + t^2\frac{1}{z^2} - t^4\frac{1}{z} = 0 \\ g_1 : & z = \dfrac{1}{w}, \quad \zeta = w^2\eta + tw. \end{cases}$$

Part III

Barking Deformations of Degenerations

15

Construction of Barking Deformations (Stellar Case)

For the remainder of this book, unless otherwise mentioned, $\pi : M \to \Delta$ is a degeneration of *connected compact* complex curves.

15.1 Linear degenerations

In this section, after introducing some notation, we review the notion of a linear degeneration, originally defined in [Ta,II]. For a given degeneration $\pi : M \to \Delta$, we will express the singular fiber X as a sum of irreducible components: $X = \sum_i m_i \Theta_i$, where Θ_i is an irreducible component of multiplicity m_i. We call $X_{\text{red}} := \sum_i \Theta_i$ the *underlying reduced curve* of X.

Recall that a *node* is a singularity that is isomorphic to some neighborhood of the origin in $\{(x, y) \in \mathbb{C}^2 : xy = 0\}$. We say that X_{red} has at most *normal crossings* if all of the singularities of X_{red} are nodes. If, furthermore, all the irreducible components Θ_i are smooth (that is, not self-intersecting), then we say that X_{red} has at most *simple normal crossings.*

Now suppose that X_{red} has at most simple normal crossings. For an irreducible component Θ_i, if $\Theta_i \cap \Theta_j \neq \emptyset$, then we write

$$\Theta_i \cap \Theta_j = \{p_1^{(ij)}, p_2^{(ij)}, \dots, p_k^{(ij)}\},$$

where $k = k(i, j) := \#(\Theta_i \cap \Theta_j)$ is the number of points of intersection between Θ_i and Θ_j. Using this notation, we define a divisor $P_j^{(i)}$ on Θ_i, by

$$P_j^{(i)} := p_1^{(ij)} + p_2^{(ij)} + \cdots + p_k^{(ij)}, \tag{15.1.1}$$

and, setting N_i as the normal bundle of Θ_i in M, we have the following useful lemma.

Lemma 15.1.1 $N_i^{\otimes m_i} \cong \mathcal{O}_{\Theta_i}(-\sum_j m_j P_j^{(i)})$, *where the sum runs over all subscripts* j *such that* $\Theta_i \cap \Theta_j \neq \emptyset$.

Proof. Since the singular fiber X is linearly equivalent to a smooth fiber $X_s := \pi^{-1}(s)$, we have $\mathcal{O}_M(X) \cong \mathcal{O}_M(X_s)$. The restriction of this isomorphism to Θ_i gives

$$\mathcal{O}_M(X)|_{\Theta_i} \cong \mathcal{O}_M(X_s)|_{\Theta_i} \cong \mathcal{O}_{\Theta_i}, \tag{15.1.2}$$

where the last isomorphism follows from $X_s \cap \Theta_i = \emptyset$.

On the other hand, from the adjunction formula $\mathcal{O}_M(\Theta_i)|_{\Theta_i} = N_i$, we have

$$\begin{aligned}\mathcal{O}_M(X)|_{\Theta_i} &\cong \mathcal{O}_M(m_i\Theta_i)|_{\Theta_i} \otimes \mathcal{O}_M(\sum_{j\neq i} m_j\Theta_j)|_{\Theta_i} \\ &\cong N_i^{\otimes m_i} \otimes \mathcal{O}_{\Theta_i}(\sum_j m_j P_j^{(i)}).\end{aligned} \tag{15.1.3}$$

From (15.1.2) and (15.1.3), we obtain $N_i^{\otimes m_i} \cong \mathcal{O}_{\Theta_i}(-\sum_j m_j P_j^{(i)})$. □

By the above lemma, we have

$$\deg(N_i) = -\frac{1}{m_i}\sum_j \#(\Theta_i \cap \Theta_j)\cdot m_j, \tag{15.1.4}$$

where $\deg(N_i)$ is the degree of the line bundle N_i, and again the sum runs over all subscripts j such that $\Theta_i \cap \Theta_j \neq \emptyset$. By definition, $\deg(N_i)$ equals the self-intersection number $\Theta_i \cdot \Theta_i$ of Θ_i in M.

We next review the concept of a linear degeneration. Initially, we shall maintain the assumption that $X_{\mathrm{red}} = \sum_i \Theta_i$, the underlying reduced curve of the singular fiber $X = \sum_i m_i\Theta_i$, has at most simple normal crossings; so any irreducible component Θ_i is smooth. For the moment, we fix one irreducible component Θ_i, and take an open covering $\Theta_i = \bigcup_\alpha U_\alpha$ such that each $U_\alpha \times \mathbb{C}$ is a local trivialization of N_i with coordinates $(z_\alpha, \zeta_\alpha) \in U_\alpha \times \mathbb{C}$, and we let $\{g_{\alpha\beta}\}$ be the transition functions of N_i (that is, $\zeta_\alpha = g_{\alpha\beta}\zeta_\beta$). From Lemma 15.1.1, there exists a holomorphic section σ_i of $N_i^{\otimes(-m_i)}$ with

$$\mathrm{div}(\sigma_i) = \sum_j m_j P_j^{(i)},$$

where $\mathrm{div}(\sigma_i)$ stands for the *divisor* defined by σ_i. (Note: If τ is a meromorphic section of a line bundle on a complex curve C, then by $\mathrm{div}(\tau) = \sum_i a_i p_i - \sum_j b_j q_j$ where a_i, b_j are positive integers and $p_i, q_j \in C$, we mean that τ has a zero of order a_i at p_i and a pole of order b_j at q_j.)

Note that σ_i is uniquely determined up to scalar multiplication. We say that σ_i is the *standard section* of the line bundle $N_i^{\otimes(-m_i)}$ on Θ_i. Setting $\sigma_i = \{\sigma_{i,\alpha}\}$ as the local expression of σ_i, we have $\sigma_{i,\alpha} = g_{\alpha\beta}^{-m_i}\sigma_{i,\beta}$. Next, we let $\pi_{i,\alpha} : U_\alpha \times \mathbb{C} \to \mathbb{C}$ be a holomorphic function defined by

$$\pi_{i,\alpha}(z_\alpha, \zeta_\alpha) := \sigma_{i,\alpha}(z_\alpha)\zeta_\alpha^{m_i}.$$

Since $\sigma_{i,\alpha}\zeta_\alpha^{m_i} = (g_{\alpha\beta}^{-m_i}\sigma_{i,\beta})(g_{\alpha\beta}\zeta_\beta)^{m_i} = \sigma_{i,\beta}\zeta_\beta^{m_i}$, we have the following.

Lemma 15.1.2 *The set of holomorphic functions* $\{\pi_{i,\alpha}\}$ *defines a global holomorphic function* π_i *on* N_i.

Now we introduce an important concept.

Definition 15.1.3 A degeneration $\pi : M \to \Delta$ is *linear* if for each i,

(1) a tubular neighborhood $N(\Theta_i)$ of Θ_i in M is biholomorphic to a tubular neighborhood of the zero-section of N_i, and
(2) under the identification of the biholomorphism in (1), the followings hold: (2.1) $\pi|_{N(\Theta_i)} = \pi_i$ and (2.2) if $p \in \Theta_i \cap \Theta_j$, then there exist trivializations of N_i and N_j around p such that

$$\pi|_{N(\Theta_i)}(z_i, \zeta_i) = z_i^{m_j} \zeta_i^{m_i}, \qquad \pi|_{N(\Theta_j)}(z_j, \zeta_j) = z_j^{m_i} \zeta_j^{m_j},$$

and also that $N(\Theta_i)$ and $N(\Theta_j)$ are glued by $(z_i, \zeta_i) = (\zeta_j, z_j)$. (These trivializations are called *arranged*.)

When the underlying reduced curve $X_{\rm red}$ of the singular fiber X has at most normal crossings but not simple normal crossings (so X has self-intersecting irreducible components), we define the notion of a linear degeneration as follows. First, let M' be the complex surface obtained by blowing up M at all the self-intersection points of $X_{\rm red}$. Then we obtain a degeneration $\pi' : M' \to \Delta$ such that $X'_{\rm red}$ has at most simple normal crossings. We then say that $\pi : M \to \Delta$ is a *linear* degeneration provided that $\pi' : M' \to \Delta$ is a linear degeneration in the sense of Definition 15.1.3.

15.2 Deformation atlas

Throughout this section, $\pi : M \to \Delta$ is a linear degeneration such that the singular fiber X is *stellar* (star-shaped). Namely, X has a central irreducible component called a *core*, and branches emanate from the core, where a *branch* is a chain of projective lines. See Figure 4.2.1, p61 for example. We express

$$X = m_0\Theta_0 + \sum_{j=1}^{N} \mathrm{Br}^{(j)},$$

where Θ_0 is the core and $\mathrm{Br}^{(j)} = m_1^{(j)}\Theta_1^{(j)} + m_2^{(j)}\Theta_2^{(j)} + \cdots + m_{\lambda_j}^{(j)}\Theta_{\lambda_j}^{(j)}$ is a branch; note that $\Theta_i^{(j)}$ is a projective line, whereas Θ_0 is not necessarily a projective line. Precisely speaking, $\mathrm{Br}^{(j)}$ is an *unfringed branch* — let $\Delta_0^{(j)} \subset \Theta_0$ be a small open disk around the intersection point of Θ_0 and $\mathrm{Br}^{(j)}$, and then

$$\overline{\mathrm{Br}}^{(j)} = m_0\Delta_0^{(j)} + m_1^{(j)}\Theta_1^{(j)} + m_2^{(j)}\Theta_2^{(j)} + \cdots + m_{\lambda_j}^{(j)}\Theta_{\lambda_j}^{(j)}$$

is a *fringed branch* ($m_0\Delta_0^{(j)}$ is its *fringe*). For brevity we often refer to both unfringed branches and fringed branches simply as branches.

The genus of a smooth fiber of $\pi : M \to \Delta$ is computable from the quantities $g_0 := \mathrm{genus}(\Theta_0)$, m_0, and $m_1^{(j)}$ by the genus formula, which we shall explain. For a branched covering $f : \Sigma \to \Sigma'$ between complex curves Σ and Σ', recall that $v \in \Sigma$ is a *ramification point* if f is not a local homeomorphism around v; then $v' := f(v) \in \Sigma'$ is a *branch point*. In this case we may take a local coordinate z around v in Σ such that $f(z) = z^r$ for some integer r ($r \geq 2$). We say that r is the *ramification index* of v (and also[1] that of v', if $f : \Sigma \to \Sigma'$ is a Galois covering).

Theorem 15.2.1 *Let $\pi : M \to \Delta$ be a degeneration of complex curves of genus g with a stellar singular fiber $X = m_0\Theta_0 + \sum_{j=1}^{N} \mathrm{Br}^{(j)}$ where $\mathrm{Br}^{(j)} = m_1^{(j)}\Theta_1^{(j)} + m_1^{(j)}\Theta_1^{(j)} + \cdots + m_{\lambda_j}^{(j)}\Theta_{\lambda_j}^{(j)}$ is a branch. Setting $c_j := \gcd(m_0, m_1^{(j)})$, write $m_0 = c_j r_j$ and $m_1^{(j)} = c_j q_j$, and let a_j $(0 < a_j < r_j)$ be the integer satisfying $a_j q_j \equiv 1 \mod r_j$. Then*

(1) *The following "genus formula" holds:*

$$2 - 2g = m_0(2 - 2g_0) - \sum_{j=1}^{N} \frac{m_0}{r_j}(r_j - 1),$$

where $g_0 := \mathrm{genus}(\Theta_0)$.

(2) *Denote by γ the topological monodromy of $\pi : M \to \Delta$. Then the action of γ on a smooth fiber Σ is such that for each j, γ^{c_j} is a rotation by angle $2\pi a_j/r_j$ around each ramification point $v_j \in \Sigma$ — in particular, r_j is the ramification index of v_j — over a branch point v_j' of the cyclic covering $\Sigma \to \Sigma/\langle\gamma\rangle$; here $\Sigma/\langle\gamma\rangle$ denotes the quotient space of Σ under the γ-action.*

See [Ta,II] for the proof. (By applying the Riemann–Hurewitz formula, the assertion (1) easily follows.)

Now we consider a stellar singular fiber $X = m_0\Theta_0 + \sum_{j=1}^{N} \mathrm{Br}^{(j)}$ where Θ_0 is the core and $\mathrm{Br}^{(j)} = m_1^{(j)}\Theta_1^{(j)} + m_2^{(j)}\Theta_2^{(j)} + \cdots + m_{\lambda_j}^{(j)}\Theta_{\lambda_j}^{(j)}$ is a branch. Set

$$\begin{cases} r_0 := \dfrac{m_1^{(1)} + m_1^{(2)} + \cdots + m_1^{(N)}}{m_0} \\[2ex] r_i^{(j)} := \dfrac{m_{i-1}^{(j)} + m_{i+1}^{(j)}}{m_i^{(j)}} \ (i = 1, 2, \ldots, \lambda_j - 1), \text{ and } r_{\lambda_j}^{(j)} := \dfrac{m_{\lambda_j - 1}^{(j)}}{m_{\lambda_j}^{(j)}}, \end{cases}$$

and then r_0 and $r_i^{(j)}$ are positive integers, and the self-intersection number $\Theta_0 \cdot \Theta_0 = -r_0$ and $\Theta_i^{(j)} \cdot \Theta_i^{(j)} = -r_i^{(j)}$. We note that $r_i^{(j)} \geq 2$, while it may occur that $r_0 = 1$.

[1] If $f : \Sigma \to \Sigma'$ is a Galois covering, then r does not depend on the choice of $v \in f^{-1}(v')$.

We use the following notation:

- $p_1^{(j)} \in \Theta_0$: the intersection point of the core Θ_0 and a branch $\mathrm{Br}^{(j)}$
- $\Theta_i^{(j)} = U_i^{(j)} \cup V_i^{(j)}$: the standard open covering of the projective line $\Theta_i^{(j)}$ by two complex lines; so $w \in U_i^{(j)}$ and $z \in V_i^{(j)}$ satisfy $z = 1/w$
- N_0: the normal bundle of Θ_0 in M
- $N_i^{(j)}$: the normal bundle of $\Theta_i^{(j)}$ in M, which is obtained by identifying $(w, \eta) \in U_i^{(j)} \times \mathbb{C}$ with $(z, \zeta) \in V_i^{(j)} \times \mathbb{C}$ by $z = 1/w$, $\zeta = w^{r_i^{(j)}}\eta$. (Whenever we need to emphasize i and j, we write $w = w_i^{(j)}$ and $\eta = \eta_i^{(j)}$.)
- $W^{(j)}$: a tubular neighborhood of a branch $\mathrm{Br}^{(j)}$, which is obtained by plumbing normal bundles $N_i^{(j)}$ $(i = 1, 2, \ldots, \lambda_j)$, that is, identifying $N_i^{(j)}$ with $N_{i+1}^{(j)}$ $(i = 1, 2, \ldots, \lambda_j - 1)$ by $(z_i^{(j)}, \zeta_i^{(j)}) = (\eta_{i+1}^{(j)}, w_{i+1}^{(j)})$.
  (Note: $\mathrm{Br}^{(j)}$ is the exceptional set of the minimal resolution of a cyclic quotient singularity [Ta,II]. Since any cyclic quotient singularity is taut [La2], *the complex structure on* $W^{(j)}$ *is unique.*)

From the definition of a linear degeneration, we may regard N_0 (resp. $N_i^{(j)}$) — precisely speaking, a tubular neighborhood of the zero section of N_0, but for simplicity we are sloppy here — as a tubular neighborhood of Θ_0 (resp. $\Theta_i^{(j)}$) in M, and under this identification,

(i) $\pi|_{N_0}(z, \zeta) = \sigma(z)\zeta^{m_0}$ where σ is the standard section, that is, a holomorphic section of $N_0^{\otimes(-m_0)}$ such that $\mathrm{div}(\sigma) = \sum m_1^{(j)} p_1^{(j)}$,

(ii) $\pi|_{N_i^{(j)}}(w, \eta) = w^{m_{i-1}^{(j)}}\eta^{m_i^{(j)}}$ and $\pi|_{N_i^{(j)}}(z, \zeta) = z^{m_{i+1}^{(j)}}\zeta^{m_i^{(j)}}$.

For the subsequent construction of deformations of $\pi : M \to \Delta$, it is convenient to think of M as the graph of π:

$$\mathrm{Graph}(\pi) = \{(x, s) \in M \times \Delta \,|\, \pi(x) - s = 0\}.$$

Via the canonical isomorphism $\mathrm{Graph}(\pi) \cong M$ given by $(x, s) \mapsto x$, we identify M with $\mathrm{Graph}(\pi)$. We also set $W_0 := \mathrm{Graph}(\pi|_{N_0})$ and $W_i^{(j)} := \mathrm{Graph}(\pi|_{N_i^{(j)}})$. Namely, $W_0 : \sigma(z)\zeta^{m_0} - s = 0$ and

$$W_i^{(j)} = \begin{cases} w^{m_{i-1}^{(j)}}\eta^{m_i^{(j)}} - s = 0, & (w, \eta) \in U_i^{(j)} \times \mathbb{C}, \\ z^{m_{i+1}^{(j)}}\zeta^{m_i^{(j)}} - s = 0, & (z, \zeta) \in V_i^{(j)} \times \mathbb{C}. \end{cases}$$

Deformation atlas

We keep the above notation: λ_j is the length of the branch $\mathrm{Br}^{(j)}$. Let $\mathbf{e} = \{e_1, e_2, \ldots, e_N\}$ be a set of integers satisfying

$$1 \le e_j \le \lambda_j, \quad j = 1, 2, \ldots, N.$$

Consider a set $DA_{\mathbf{e}} = \{\mathcal{W}_0\,,\, DA^{(j)}_{e_j}\}_{j=1,2,\ldots,N}$ where

(i) $\mathcal{W}_0$ is a deformation, parameterized by $t \in \Delta^\dagger$, of a tubular neighborhood W_0 of the core Θ_0, and

(ii) $DA^{(j)}_{e_j} = \{\mathcal{H}^{(j)}_i\,,\, \mathcal{H}^{(j)\prime}_i\; g^{(j)}_i\}_{i=1,2,\ldots,e_j}$ is a deformation atlas of length e_j for the branch $\mathrm{Br}^{(j)}$ such that under a coordinate change $(z_0, \zeta_0) = (\eta^{(j)}_1, w^{(j)}_1)$ around $p^{(j)}_1$, the equation of $\mathcal{W}_0$ becomes that of $\mathcal{H}^{(j)}_1$.

We say that $DA_{\mathbf{e}}$ is a *deformation atlas* of *size* $\mathbf{e} = \{e_1, e_2, \ldots, e_N\}$ for the singular fiber X. For the special case where $\mathbf{e}$ is $\boldsymbol{\lambda} = \{\lambda_1, \lambda_2, \ldots, \lambda_N\}$, we call $DA_{\boldsymbol{\lambda}}$ a *complete deformation atlas*, from which we may construct a deformation family of $\pi : M \to \Delta$ as follows:

Step 1: First let $\mathcal{W}^{(j)}_i$ be the complex 3-manifold obtained by patching $\mathcal{H}^{(j)}_i$ and $\mathcal{H}^{(j)\prime}_i$ via the map $g^{(j)}_i$. Secondly for each $j = 1, 2, \ldots, N$, we patch $\mathcal{W}^{(j)}_i$ and $\mathcal{W}^{(j)}_{i+1}$ $(i = 1, 2 \ldots, \lambda_j - 1)$ by plumbing $(z^{(j)}_i, \zeta^{(j)}_i) = (\eta^{(j)}_{i+1}, w^{(j)}_{i+1})$; this yields a complex 3-manifold $\mathcal{W}^{(j)}$ which is a deformation of a tubular neighborhood $W^{(j)}$ of the branch $\mathrm{Br}^{(j)}$.

Step 2: Next we patch the complex 3-manifolds $\mathcal{W}_0$ and $\mathcal{W}^{(j)}$, $(j = 1, 2, \ldots, N)$ by plumbing $(w^{(j)}_1, \eta^{(j)}_1) = (\zeta_0, z_0)$ around $p^{(j)}_1$. Since $\mathcal{W}_0$ is a deformation of the tubular neighborhood W_0 of the core Θ_0, and $\mathcal{W}^{(j)}$ is a deformation of the tubular neighborhood $W^{(j)}$ of the branch $\mathrm{Br}^{(j)}$, the resulting complex 3-manifold $\mathcal{M}$ obtained by patching $\mathcal{W}_0$ and $\mathcal{W}^{(j)}$, $(j = 1, 2, \ldots, N)$ is a deformation of M. The natural projection $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$ is a deformation family of $\pi : M \to \Delta$, associated with the complete deformation atlas $DA_{\boldsymbol{\lambda}}$.

In the above construction, if $\mathcal{W}_0$ is 'realized' as a smooth hypersurface in $N_0 \times \Delta \times \Delta^\dagger$ (detailed account will be given later in §15.4), then $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$ is referred to as a *barking family* of $\pi : M \to \Delta$.

We next introduce an order on sets of N nonnegative integers: For two sets $\mathbf{e} = \{e_1, e_2, \ldots, e_N\}$ and $\mathbf{e}' = \{e'_1, e'_2, \ldots, e'_N\}$, we write $\mathbf{e} \geq \mathbf{e}'$ when

$$e_1 \geq e'_1, \quad e_2 \geq e'_2, \quad \ldots, \quad e_N \geq e'_N.$$

Now suppose that we are given a deformation atlas $DA_{\mathbf{e}}=\{\mathcal{W}_0, DA^{(j)}_{e_j}\}_{j=1,2,\ldots,N}$, where

$$DA^{(j)}_{e_j} = \{\mathcal{H}^{(j)}_i, \mathcal{H}^{(j)\prime}_i, g^{(j)}_i\}, \qquad i = 1, 2, \ldots, e_j.$$

For $\mathbf{e}'$ satisfying $\mathbf{e} \geq \mathbf{e}'$, we may define a 'smaller' deformation atlas $DA_{\mathbf{e}'}$ of size $\mathbf{e}'$ by restriction, that is, $\{\mathcal{W}_0, DA^{(j)}_{e'_j}\}_{j=1,2,\ldots,N}$, where

$$DA^{(j)}_{e'_j} = \{\mathcal{H}^{(j)}_i, \mathcal{H}^{(j)\prime}_i, g^{(j)}_i\}, \qquad i = 1, 2, \ldots, e'_j.$$

In this situation, we say that $DA_{\mathbf{e}}$ is an $\mathbf{e}$-*th propagation* of $DA_{\mathbf{e}'}$; for a particular case where $\mathbf{e}$ is $\boldsymbol{\lambda} = \{\lambda_1, \lambda_2, \ldots, \lambda_N\}$, then $DA_{\boldsymbol{\lambda}}$ is called a *complete propagation* of $DA_{\mathbf{e}'}$.

From here and onward, we mainly consider such deformation atlases $DA_{\mathbf{e}} = \{\mathcal{W}_0, DA^{(j)}_{e_j}\}_{j=1,2,\dots,N}$ as $\mathcal{W}_0$ is a smooth hypersurface in $N_0 \times \Delta \times \Delta^\dagger$. We refer the reader to §20.4, p368 for other cases.

15.3 Crusts

We keep the assumption that X is a stellar singular fiber, and we express $X = m_0\Theta_0 + \sum_{i=1}^N \mathrm{Br}^{(j)}$ where $\mathrm{Br}^{(j)} = m_1^{(j)}\Theta_1^{(j)} + m_2^{(j)}\Theta_2^{(j)} + \cdots + m_{\lambda_j}^{(j)}\Theta_{\lambda_j}^{(j)}$ is an (unfringed) branch; as before

$$\overline{\mathrm{Br}}^{(j)} = m_0\Delta_0^{(j)} + m_1^{(j)}\Theta_1^{(j)} + m_2^{(j)}\Theta_2^{(j)} + \cdots + m_{\lambda_j}^{(j)}\Theta_{\lambda_j}^{(j)}$$

is a fringed branch, where $\Delta_0^{(j)} \subset \Theta_0$ is a small open disk around the intersection point $p_1^{(j)} = \Theta_0 \cap \mathrm{Br}^{(j)}$ (for simplicity, we often refer to both unfringed branches and fringed branches simply as branches). For a subdivisor $\mathrm{br}^{(j)} = n_1^{(j)}\Theta_1^{(j)} + n_2^{(j)}\Theta_2^{(j)} + \cdots + n_{e_j}^{(j)}\Theta_{e_j}^{(j)}$ of $\mathrm{Br}^{(j)}$, we write

$$\overline{\mathrm{br}}^{(j)} = n_0\Delta_0^{(j)} + n_1^{(j)}\Theta_1^{(j)} + n_2^{(j)}\Theta_2^{(j)} + \cdots + n_{e_j}^{(j)}\Theta_{e_j}^{(j)},$$

and if $\overline{\mathrm{br}}^{(j)}$ is a subbranch of $\overline{\mathrm{Br}}^{(j)}$ (the notion of a "subbranch" is originally defined for fringed one), we conventionally say that $\mathrm{br}^{(j)}$ is a subbranch of $\mathrm{Br}^{(j)}$; this is the case (i) $e_j = 0$ or 1, or (ii) $e_j \geq 2$ and $\dfrac{n_{i-1}^{(j)} + n_{i+1}^{(j)}}{n_i^{(j)}} = r_i^{(j)}$ for $i = 1, 2, \dots, e_j - 1$. (Note: $e_j = 0$ is the case $\mathrm{br}^{(j)} = \emptyset$ and $\overline{\mathrm{br}}^{(j)} = n_0\Delta_0^{(j)}$)

Definition 15.3.1 Let $Y = n_0\Theta_0 + \sum_{j=1}^N \mathrm{br}^{(j)}$ be a connected subdivisor of X such that $0 < n_0 < m_0$ and $\mathrm{br}^{(j)} = n_1^{(j)}\Theta_1^{(j)} + \cdots + n_{e_j}^{(j)}\Theta_{e_j}^{(j)}$ is a subbranch of $\mathrm{Br}^{(j)}$ where possibly $\mathrm{br}^{(j)} = \emptyset$ for some j. Then Y is called a *crust* if the line bundle $N_0^{\otimes n_0}$ on Θ_0, where N_0 is the normal bundle of Θ_0 in M, has a meromorphic section τ which has a pole of order $n_1^{(j)}$ at such points $p_1^{(j)}$ as $\mathrm{br}^{(j)} \neq \emptyset$, and is holomorphic outside them.

We say that τ is a *core section* of the crust Y. Note that τ is not uniquely determined by Y; in general we may vary the positions of the zeros of τ (possibly collision of some zeros may occur) to produce a new core section of the crust Y. When τ has zeros, say, of order a_i at $q_i \in \Theta_0$ $(i = 1, 2, \dots, k)$, we may write $\mathrm{div}(\tau) = -\sum_{j=1}^N n_1^{(j)}p_1^{(j)} + D$ where $D = \sum_{i=1}^k a_iq_i$ is called an *auxiliary divisor* and q_i are *auxiliary points.*

A pair (Y, d) of a crust Y and a positive integer d is called a *weighted crust.* We often write a set of weighted crusts $\{(Y_1, d_1), (Y_2, d_2), \dots, (Y_l, d_l)\}$ as $(\mathbf{Y}, \mathbf{d})$.

A core section gives rise to a deformation around the core Θ_0 as follows.

Lemma 15.3.2 *Let (Y_k, d_k) $(k = 1, 2, \dots, l)$ be a set of weighted crusts where d_k is an arbitrary positive number, and denote by τ_k a core section of Y_k. Then the following equation defines a smooth hypersurface in $N_0 \times \Delta \times \Delta^\dagger$:*

$$\mathcal{W}: \quad \sigma\zeta^{m_0} - s + \sum_{k=1}^{l} t^{d_k}\,\sigma\,\tau_k\,\zeta^{m_0 - n_{k,0}} = 0,$$

where $n_{k,0}$ is the multiplicity of Θ_0 in Y_k.

Proof. Take an open covering $\Theta_0 = \bigcup_\alpha U_\alpha$ and let $(z_\alpha, \zeta_\alpha) \in U_\alpha \times \mathbb{C}$ be coordinates of local trivialization of N_0. We denote by $g_{\alpha\beta}$ the transition functions of N_0; so $\zeta_\alpha = g_{\alpha\beta}\zeta_\beta$ holds. Letting $\sigma = \{\sigma_\alpha\}$ and $\tau_k = \{\tau_{k,\alpha}\}$ be local expressions, since σ and τ are sections of $N_0^{\otimes(-m_0)}$ and $N_0^{\otimes n_{k,0}}$ respectively, we have

$$\sigma_\alpha = g_{\alpha\beta}^{-m_0}\sigma_\beta, \qquad \tau_\alpha = g_{\alpha\beta}^{n_{k,0}}\tau_\beta.$$

Using these transformation rules and $\zeta_\alpha = g_{\alpha\beta}\zeta_\beta$, we see that the transition function $g_{\alpha\beta}$ of N_0 transforms

$$\mathcal{H}_\alpha: \quad \sigma_\alpha\zeta_\alpha^{m_0} - s + \sum_{k=1}^{l} t^{d_k}\,\sigma_\alpha\,\tau_{k,\alpha}\,\zeta_\alpha^{m_0 - n_{k,0}} = 0$$

to

$$\mathcal{H}_\beta: \quad \sigma_\beta\zeta_\beta^{m_0} - s + \sum_{k=1}^{l} t^{d_k}\,\sigma_\beta\,\tau_{k,\beta}\,\zeta_\beta^{m_0 - n_{k,0}} = 0.$$

Therefore $\mathcal{W}$ is well-defined as a hypersurface in $N_0 \times \Delta \times \Delta^\dagger$. Further from $\dfrac{\partial \mathcal{W}}{\partial s} = -1 \neq 0$, the hypersurface $\mathcal{W}$ is smooth. □

We remark that when Θ_0 is a projective line, existence of τ in Definition 15.3.1 is equivalent to the following numerical condition (see Proposition 3.4.3, p52):

$$\frac{n_1^{(1)} + n_1^{(2)} + \cdots + n_1^{(N)}}{n_0} \geq \frac{m_1^{(1)} + m_1^{(2)} + \cdots + m_1^{(N)}}{m_0}.$$

For the case genus$(\Theta_0) \geq 1$, the condition of Definition 15.3.1 gives rise to a certain linear equivalence relation on divisors on Θ_0; recall that two divisors D_1 and D_2 on a curve C is *linearly equivalent* (notation: $D_1 \sim D_2$) if there exists a meromorphic function f on C such that $\mathrm{div}(f) = D_1 - D_2$.

Lemma 15.3.3 *Let Y be a crust with a core section τ such that*

$$\mathrm{div}(\tau) = -n_1^{(1)}p_1^{(1)} - n_1^{(2)}p_1^{(2)} \cdots - n_1^{(N)}p_1^{(N)} + D,$$

where D is a nonnegative divisor (an auxiliary divisor of τ) on Θ_0. Let a and b be the relatively prime positive integers satisfying $am_0 = bn_0$, and then the *following linear equivalence of divisors on* Θ_0 *holds:*

$$\Big(\sum_{j=1}^{N}(am_1^{(j)} - bn_1^{(j)})p_1^{(j)} + bD\Big) \sim 0.$$

Proof. Set $L = am_0 = bn_0$, and then (i) σ^a is a holomorphic section of $N_0^{\otimes(-am_0)} = N_0^{\otimes(-L)}$, and (ii) τ^b is a meromorphic section of $N_0^{\otimes bn_0} = N_0^{\otimes L}$. Therefore $\sigma^a\tau^b$ is a meromorphic section of $N_0^{\otimes(-L)} \otimes N_0^{\otimes L}$. From $N_0^{\otimes(-L)} \otimes N_0^{\otimes L} \cong \mathcal{O}_{\Theta_0}$, it follows that $\sigma^a\tau^b$ is a meromorphic *function* on Θ_0, and so $\operatorname{div}(\sigma^a\tau^b) \sim 0$. This linear equivalence, written explicitly by using

$$\operatorname{div}(\sigma) = \sum_{j=1}^{N} m_1^{(j)}p_1^{(j)} \quad \text{and} \quad \operatorname{div}(\tau) = -\sum_{j=1}^{N} n_1^{(j)}p_1^{(j)} + D,$$

yields the desired equivalence: $\Big(\sum_{j=1}^{N} am_1^{(j)}p_1^{(j)} - \sum_{j=1}^{N} bn_1^{(j)}p_1^{(j)} + bD\Big) \sim 0.$ □

15.4 Deformation atlas associated with one crust

Let $Y = n_0\Theta_0 + \sum_{j=1}^{N} \mathrm{br}^{(j)}$ be a crust of X, and denote the length of $\mathrm{br}^{(j)}$ by e_j (when $\mathrm{br}^{(j)} = \emptyset$, we set $e_j = 1$). We shall associate Y with a deformation atlas of size $\mathbf{e} - \mathbf{1}$, where

$$\mathbf{e} - \mathbf{1} = \{e_1 - 1,\ e_2 - 1,\ \ldots,\ e_N - 1\}.$$

The construction proceeds as follows.
Step 1. We first give the construction of $\mathcal{W}_0$. Let τ be a core section of the crust Y, that is, τ is a meromorphic section of the line bundle $N_0^{\otimes n_0}$ such that

$$\operatorname{div}(\tau) = -n_1^{(1)}p_1^{(1)} - n_1^{(2)}p_1^{(2)} \cdots - n_1^{(N)}p_1^{(N)} + D,$$

where D is a nonnegative divisor (an auxiliary divisor of τ) on Θ_0. Taking a positive integer d, we define a smooth hypersurface in $N_0 \times \Delta \times \Delta^\dagger$ as in Lemma 15.3.2:

$$\mathcal{W}_0 : \quad \sigma\zeta^{m_0} - s + t^d\sigma\tau\zeta^{m_0-n_0} = 0,$$

where σ is the standard section on the core Θ_0, i.e. a holomorphic section of $N_0^{\otimes(-m_0)}$ satisfying

$$\operatorname{div}(\sigma) = m_1^{(1)}p_1^{(1)} + m_1^{(2)}p_1^{(2)} + \cdots + m_1^{(N)}p_1^{(N)}.$$

Next we shall explicitly write the equation of $\mathcal{W}_0$ around the intersection point $p_1^{(j)}$ of the core Θ_0 and a branch $\mathrm{Br}^{(j)}$. For simplicity, we fix j and often omit it, such as $m_i^{(j)} = m_i$ and $\Theta_i^{(j)} = \Theta_i$. Taking local coordinates of Θ_0 around p_1

such that p_1 is the origin, we express $\sigma = z^{m_1} g(z)$ and $\tau = h(z)/z^{n_0}$ around p_1, where $g(z)$ and $h(z)$ are non-vanishing holomorphic functions; then $\mathcal{W}_0$ is

$$z^{m_1}\zeta^{m_0} g - s + t^d z^{m_1-n_1}\zeta^{m_0-n_0} gh = 0 \quad \text{around} \;\; p_1.$$

Rewriting $\zeta g^{1/m_0}$ by ζ', we make this equation into a simpler form:

$$z^{m_1}(\zeta')^{m_0} - s + t^d z^{m_1-n_1}(\zeta')^{m_0-n_0} f = 0 \quad \text{around} \;\; p_1, \tag{15.4.1}$$

where we set

$$f := g^{n_0/m_0} h. \tag{15.4.2}$$

Note that f is a non-vanishing holomorphic function on some domain, say $|z| < \varepsilon$.

Now to clarify the subsequent discussion, we use notation with subscripts: $z = z_0$ and $\zeta' = \zeta_0$ etc. Then (15.4.1) is

$$z_0^{m_1}\zeta_0^{m_0} - s + t^d z_0^{m_1-n_1}\zeta_0^{m_0-n_0} f(z_0) = 0 \quad \text{around} \;\; p_1.$$

Under a coordinate change $(z_0, \zeta_0) = (\eta_1, w_1)$, this equation becomes

$$w_1^{m_0}{\eta_1}^{m_1} - s + t^d w_1^{m_0-n_0}{\eta_1}^{m_1-n_1} f(\eta_1) = 0. \tag{15.4.3}$$

Step 2. Next we construct a deformation atlas $DA^{(j)}_{e_j-1}$ for each branch $\mathrm{Br}^{(j)}$. First we define a sequence of integers p_i $(i = 0, 1, \ldots, \lambda_j + 1)$ inductively by

$$\begin{cases} p_0 = 0, \quad p_1 = 1 \quad \text{and} \\ p_{i+1} = r_i p_i - p_{i-1} \quad \text{for } i = 1,\, 2,\, \ldots,\, \lambda_j. \end{cases}$$

Then $p_{\lambda_j+1} > p_{\lambda_j} > \cdots > p_1 > p_0 = 0$ (6.2.4), p105. From the holomorphic function f in (15.4.2), we construct a sequence of holomorphic functions as follows:

$$f_i = f(w^{p_{i-1}}\eta^{p_i}) \qquad \text{and} \qquad \widehat{f_i} = f(z^{p_{i+1}}\zeta^{p_i}) \qquad\qquad \text{(see (6.2.7), p106)}.$$

Using these holomorphic functions,a deformation atlas $DA^{(j)}_{e_j-1} = DA^{(j)}_{e_j-1}$ $(\mathrm{br}^{(j)}, d)$ for the branch $\mathrm{Br}^{(j)}$ is defined as follows (see p106): for $i = 1, 2, \ldots, e_j - 1$,

$$\begin{cases} \mathcal{H}_i: & w^{m_{i-1}-n_{i-1}}\eta^{m_i-n_i}(w^{n_{i-1}}\eta^{n_i} + t^d f_i) - s = 0 \\ \mathcal{H}'_i: & z^{m_{i+1}-n_{i+1}}\zeta^{m_i-n_i}(z^{n_{i-1}}\zeta^{n_i} + t^d \widehat{f_i}) - s = 0 \\ g_i: & \text{the transition function } z = 1/w,\ \zeta = w^{r_i}\eta \text{ of } N_i, \end{cases}$$

and then

$$DA_{\mathbf{e}-1}(Y, d) := \{\mathcal{W}_0, DA^{(j)}_{e_j-1}\}_{j=1,2,\ldots,N}$$

is a deformation atlas of size $\mathbf{e} - \mathbf{1}$ for the singular fiber X. We say that $DA_{\mathbf{e}-1}(Y, d)$ is the *deformation atlas associated with the crust* Y. For the sake of brevity, we often omit the subscript $\mathbf{e} - \mathbf{1}$ to write $DA(Y, d)$, when it is clear from the context.

15.5 Reduced barking

Proposition 15.5.1 *Let Y be a crust such that any subbranch* $\mathrm{br}^{(j)}$ $(\neq \emptyset)$ *satisfies either* (1) $n^{(j)}_{e_j-1}/n^{(j)}_{e_j} \geq r^{(j)}_{e_j}$ *or* (2) $\mathrm{length}(\mathrm{br}^{(j)}) = \mathrm{length}(\mathrm{Br}^{(j)})$ *and* $m^{(j)}_{\lambda_j} = n^{(j)}_{\lambda_j} = 1$. *Then* $\pi : M \to \Delta$ *admits a barking family.*

Proof. Let $DA(Y,d) = \{\mathcal{H}_0, DA^{(j)}_{e_j-1}\}_{j=1,2,\ldots,N}$ be the deformation atlas associated with Y, where the weight d is arbitrary. According to (1) or (2), $\mathrm{br}^{(j)}$ is of type A_1 or B_1 (Definition 9.1.1, p154), and hence $DA^{(j)}_{e_j-1}$ admits a complete propagation along $\mathrm{Br}^{(j)}$ (Theorem 10.0.15, p177) where for j such that $\mathrm{br}^{(j)} = \emptyset$, we apply the construction for type A_1 (Remark 10.0.16, p177). Therefore $DA_{\mathbf{e}-1}(Y,d)$ admits a complete propagation, which yields a barking family of $\pi : M \to \Delta$. □

See Figure 15.5.1 for example, where Y has a core section τ such that $\mathrm{div}(\tau) = -p^{(1)}_1 - p^{(2)}_1 + q$, where $q \in \Theta_0$: In the notation of Lemma 15.3.3,

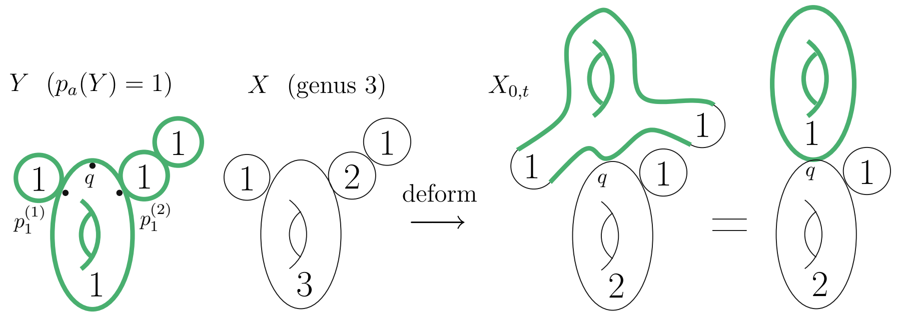


Fig. 15.5.1. Two subbranches of Y are of type B_1. The topological monodromy of X is shown in Figure 15.5.2. (Note: $p_a(Y)$ is the arithmetic genus of Y; see [GH].)

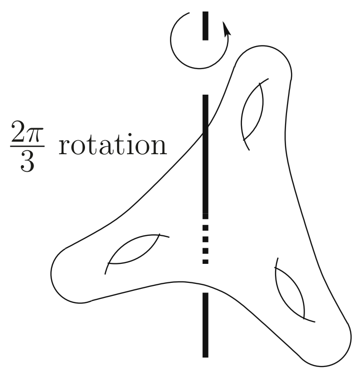


Fig. 15.5.2. The topological monodromy of X in Figure 15.5.1 is periodic of order 3

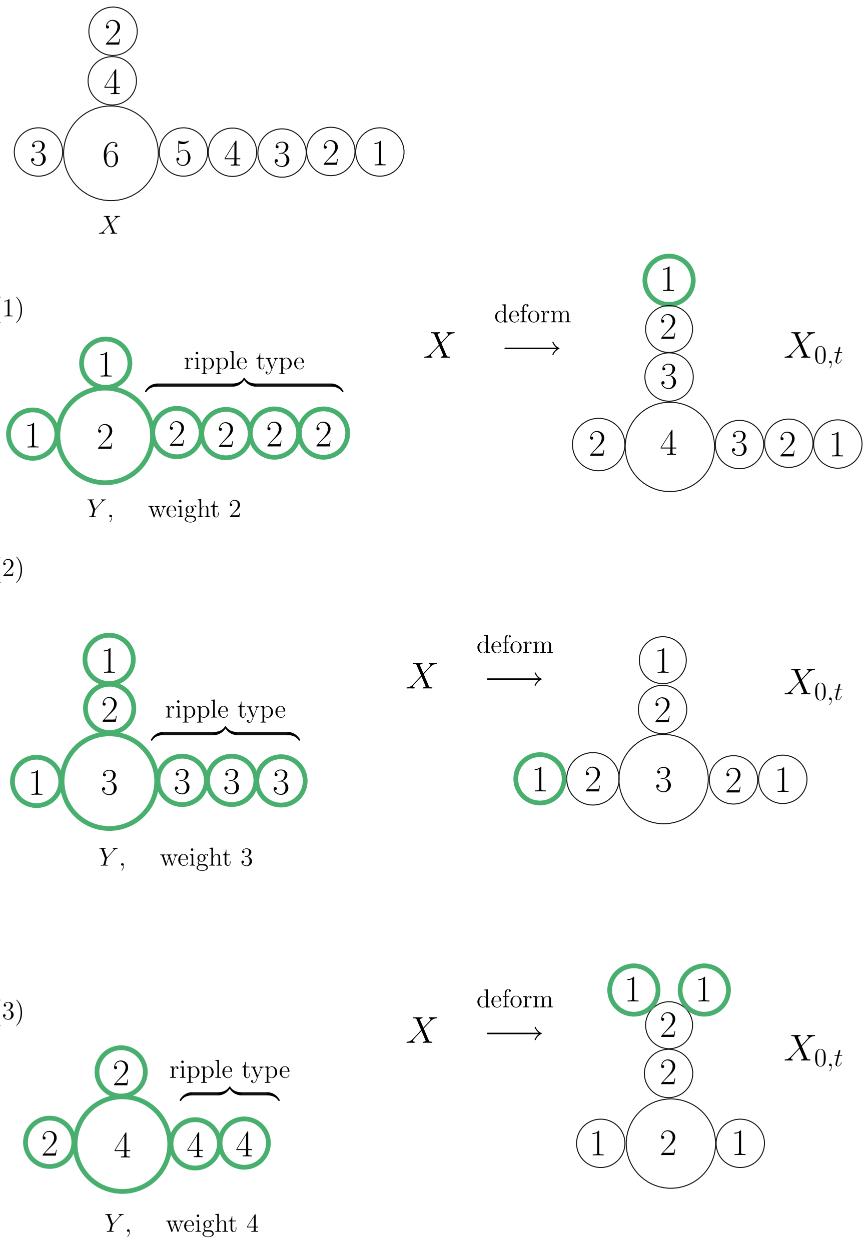


Fig. 15.5.3. Each crust Y above has exactly one subbranch of ripple type; other subbranches are tame. In (3), Y is a multiple subdivisor of multiplicity "2", and Y becomes "two" projective lines after the barking deformation. For (2), see also Figure 15.5.4.

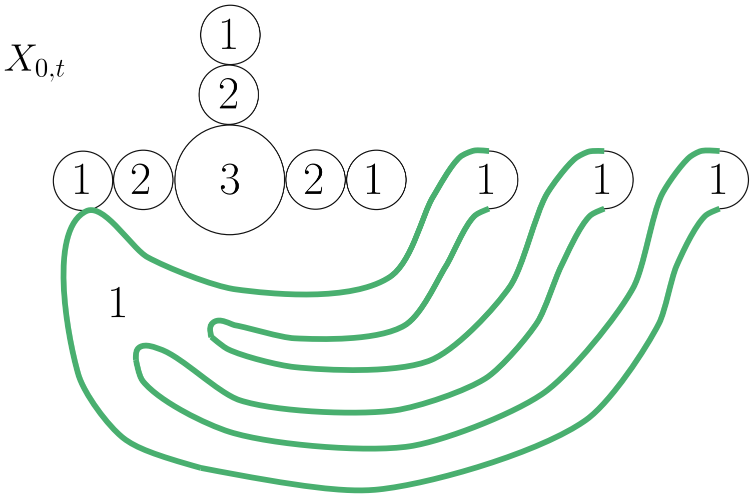


Fig. 15.5.4. A more geometrically precise figure of $X_{0,t}$ in Figure 15.5.3 (2). cf. Figure 8.1.1, p150.

$D = q$ (the divisor defined by q), and the three points $p_1^{(1)}$, $p_1^{(2)}$, and q satisfy a linear equivalence $2p_1^{(1)} + p_1^{(2)} \sim 3q$ (Lemma 15.3.3).

Before providing another construction of barking families, we recall

Definition 15.5.2 Let $\mathrm{Br} = m_1\Theta_1 + m_2\Theta_2 + \cdots + m_\lambda\Theta_\lambda$ be a branch such that for some e $(0 < e < \lambda)$,

$$m_i = (\lambda + 1) - i \qquad \text{for} \quad i = e-1,\, e,\, \ldots,\, \lambda. \tag{15.5.1}$$

Then a subbranch $\mathrm{br} = n_1\Theta_1 + n_2\Theta_2 + \cdots + n_e\Theta_e$ of Br is called *of ripple type* if it satisfies $n_{e-1} = n_e = m_e$. (Note: (15.5.1) is equivalent to (i) $m_\lambda = 1$ and (ii) $r_e = r_{e+1} = \cdots = r_\lambda = 2$. Here, (ii) implies $\Theta_i \cdot \Theta_i = -2$ for $i = e,\, e+1,\, \ldots,\, \lambda$, that is, $\Theta_e + \Theta_{e+1} + \cdots + \Theta_\lambda$ is a chain of (-2)-curves.)

Now we slightly generalize Proposition 15.5.1.

Proposition 15.5.3 *Let Y be a crust such that any subbranch $\mathrm{br}^{(j)}$ $(\neq \emptyset)$ satisfies one of the following conditions:*

(1) $n^{(j)}_{e_j-1}/n^{(j)}_{e_j} \geq r^{(j)}_{e_j}$,
(2) $\mathrm{length}(\mathrm{br}^{(j)}) = \mathrm{length}(\mathrm{Br}^{(j)})$ *and* $m^{(j)}_{\lambda_j} = n^{(j)}_{\lambda_j} = 1$,
(3) $\mathrm{br}^{(j)}$ *is of ripple type.*

Then $\pi : M \to \Delta$ admits a barking family.

Proof. According to (1), (2) or (3), the subbranch $\mathrm{br}^{(j)}$ is of type A_1, B_1, or C_1 (Definition 9.1.1, p154). Set $a := \mathrm{lcm}\,\{n_{e_j} : \mathrm{br}^{(j)} \text{ is of ripple type}\}$, and take such a positive integer d as is divisible by a. Then by Theorem 10.0.15, p177, the deformation atlas $DA_{\mathbf{e-1}}(Y, d)$ associated with the crust Y admits a complete propagation, from which we can construct a barking family of $\pi : M \to \Delta$. □

Remark 15.5.4 Notice that in Proposition 15.5.1, the weight d of $DA_{\mathbf{e-1}}$ is arbitrary, whereas in Proposition 15.5.3, we must choose such d as in the above proof.

In Proposition 15.5.3, (1) (resp. (2)) is "equivalent" to that $\mathrm{br}^{(j)}$ is of type A_1 (resp. B_1). In contrast, (3) implies that $\mathrm{br}^{(j)}$ is of type C_1, but the converse is *not* true (Remark 9.1.13, p160).

16

Simple Crusts (Stellar Case)

In this chapter we will establish an important theorem which gives a very powerful criterion for splittability of singular fibers.

16.1 Deformation atlases associated with multiple crusts

Assume that $Y = n_0\Theta_0 + \sum_{j=1}^{N} \mathrm{br}^{(j)}$ is a crust of a stellar singular fiber X, and l is a positive integer such that $lY \leq X$. For a pair (lY, d) where d is a positive integer, we shall associate a deformation atlas

$$DA(lY, d) = \{\mathcal{W}_0, DA^{(j)}_{e_j-1}\}_{j=1,2,\dots,N}, \tag{16.1.1}$$

where e_j is the length of the subbranch $\mathrm{br}^{(j)}$. (By convention, if $\mathrm{br}^{(j)} = \emptyset$, we set $e_j = 1$.) The construction of $DA(lY, d)$ proceeds as follows:

Step 1. Construction of $\mathcal{W}_0$
Let τ be a core section of the crust Y: It is a meromorphic section of $N_0^{\otimes n_0}$ such that

$$\mathrm{div}(\tau) = -n_1^{(1)} p_1^{(1)} - n_1^{(2)} p_1^{(2)} - \cdots - n_1^{(N)} p_1^{(N)} + D,$$

where D is a nonnegative divisor (an auxiliary divisor of τ) on Θ_0. We then define a smooth hypersurface in $N_0 \times \Delta \times \Delta^\dagger$ by

$$\mathcal{W}_0: \quad \sigma\zeta^{m_0} - s + \sum_{k=1}^{l} {}_l\mathrm{C}_k t^{kd} \sigma \tau^k \zeta^{m_0 - kn_0} = 0, \quad \text{(see Lemma 15.3.2, p272).}$$

Step 2. Construction of $DA^{(j)}_{e_j-1}$
Next we construct a deformation atlas $DA^{(j)}_{e_j-1}$ for each branch $\mathrm{Br}^{(j)}$. First of all, we shall explicitly write the equation of $\mathcal{W}_0$ around the intersection

point $p_1^{(j)}$ of the core Θ_0 and a branch $\mathrm{Br}^{(j)}$. We fix j, and we often omit the superscripts (j). Taking local coordinates of Θ_0 around p_1 such that p_1 is the origin, we express $\sigma = g(z)z^{m_1}$ and $\tau = h(z)/z^{n_1}$ around p_1, where $g(z)$ and $h(z)$ are non-vanishing holomorphic functions. Then $\mathcal{W}_0$ is locally

$$gz^{m_1}\zeta^{m_0} - s + \sum_{k=1}^{l} {}_l\mathrm{C}_k t^{kd} g h^k z^{m_1-kn_1}\zeta^{m_0-kn_0} = 0 \qquad \text{around } \ p_1.$$

Set $\zeta' = g^{1/m_0}\zeta$, and then this equation is rewritten as

$$z^{m_1}(\zeta')^{m_0} - s + \sum_{k=1}^{l} {}_l\mathrm{C}_k t^{kd} (g^{n_0/m_0}h)^k z^{m_1-kn_1}(\zeta')^{m_0-kn_0} = 0 \qquad \text{around } \ p_1.$$

For simplicity, we set $f := g^{n_0/m_0}h$ and then

$$z^{m_1}(\zeta')^{m_0} - s + \sum_{k=1}^{l} {}_l\mathrm{C}_k t^{kd} f^k z^{m_1-kn_1}(\zeta')^{m_0-kn_0} = 0 \qquad \text{around } \ p_1, \tag{16.1.2}$$

which 'factorizes' as

$$z^{m_1-ln_1}(\zeta')^{m_0-ln_0}\Big(z^{n_1}(\zeta')^{n_0} + t^d f\Big)^l - s = 0 \qquad \text{around } \ p_1.$$

By a coordinate change $(z, \zeta') = (\eta, w)$, this equation becomes

$$w^{m_0-ln_0}\eta^{m_1-ln_1}(w^{n_0}\eta^{n_1} + t^d f)^l - s = 0 \qquad \text{around } \ p_1. \tag{16.1.3}$$

Next for the branch $\mathrm{Br}^{(j)}$, we define such a deformation atlas $DA_{e_j-1}^{(j)}$ as $\mathcal{H}_1^{(j)}$ is given by (16.1.3). First, we define a sequence of integers p_i $(i = 0, 1, \ldots, \lambda_j + 1)$ inductively by

$$\begin{cases} p_0 = 0, \quad p_1 = 1 \quad \text{and} \\ p_{i+1} = r_i p_i - p_{i-1} \quad \text{for } i = 1,\, 2,\, \ldots,\, \lambda_j. \end{cases}$$

Then $p_{\lambda_j+1} > p_{\lambda_j} > \cdots > p_1 > p_0 = 0$ (6.2.4), p105. For the holomorphic function f in (16.1.3), we set

$$f_i = f(w^{p_{i-1}}\eta^{p_i}) \qquad \text{and} \qquad \widehat{f_i} = f(z^{p_{i+1}}\zeta^{p_i}) \qquad \text{(see (6.2.7), p106).}$$

Then we define a deformation atlas $DA_{e_j-1}^{(j)}$ for the branch $\mathrm{Br}^{(j)}$ as follows: For $i = 1, 2, \ldots, e_j - 1$,

$$\begin{cases} \mathcal{H}_i^{(j)}: & w^{m_{i-1}-ln_{i-1}}\eta^{m_i-ln_i}(w^{n_{i-1}}\eta^{n_i} + t^d f_i)^l - s = 0 \\ \mathcal{H}_i^{(j)\prime}: & z^{m_{i+1}-ln_{i+1}}\zeta^{m_i-ln_i}(z^{n_{i+1}}\zeta^{n_i} + t^d \widehat{f_i})^l - s = 0 \\ g_i: & \text{the transition function } z = 1/w,\ \zeta = w^{r_i}\eta \text{ of } N_i. \end{cases} \tag{16.1.4}$$

(Note: If $e_j = 0$ or 1, then $DA_{e_j-1}^{(j)}$ consists only of $\mathcal{H}_1^{(j)}$.)

16.2 Multiple barking

For a moment, instead of unfringed objects such as $\mathrm{Br}^{(j)}$, we consider fringed objects such as $\overline{\mathrm{Br}}^{(j)}$; if $\mathrm{br}^{(j)} = n_1\Theta_1 + n_2\Theta_2 + \cdots + n_e\Theta_e$ is an unfringed subbranch of Y contained in $\mathrm{Br}^{(j)}$, then we set $\overline{\mathrm{br}}^{(j)} := n_0\Delta_0 + n_1\Theta_1 + n_2\Theta_2 + \cdots + n_e\Theta_e$, a fringed subbranch, where $\Delta_0 = \Delta_0^{(j)}$ is a small open disc (a *fringe*) around the intersection point $p_1^{(j)} = \Theta_0 \cap \mathrm{Br}^{(j)}$. Herein for brevity we often refer to both $\mathrm{br}^{(j)}$ and $\overline{\mathrm{br}}^{(j)}$ simply as a subbranch. Letting l be a positive integer, recall that a subbranch $\overline{\mathrm{br}}^{(j)}$ of a branch $\overline{\mathrm{Br}}^{(j)}$ is called *of type* A_l, B_l, *or* C_l (Definition 9.1.1, p154) provided that

Type A_l $\quad l \cdot \overline{\mathrm{br}}^{(j)} \le \overline{\mathrm{Br}}^{(j)}$ and $\frac{n_{e-1}}{n_e} \ge r_e$,

Type B_l $\quad l \cdot \overline{\mathrm{br}}^{(j)} \le \overline{\mathrm{Br}}^{(j)}$, $m_e = l$, and $n_e = 1$,

Type C_l $\quad l \cdot \overline{\mathrm{br}}^{(j)} \le \overline{\mathrm{Br}}^{(j)}$, n_e divides n_{e-1}, $\frac{n_{e-1}}{n_e} < r_e$ and u divides l where $u := (m_{e-1} - ln_{e-1}) - (r_e - 1)(m_e - ln_e)$.

According to the above types of $\overline{\mathrm{br}}^{(j)}$, we also say that $\mathrm{br}^{(j)}$ is of type A_l, B_l, or C_l: By convention, if $\overline{\mathrm{br}}^{(j)} = n_0\Delta_0^{(j)}$ (so $\mathrm{br}^{(j)} = \emptyset$) and $ln_0 \le m_0$, then $\overline{\mathrm{br}}^{(j)}$ is of type A_l.

The reader may wonder why we do not directly define the concept "types A_l, B_l, and C_l" for unfringed $\mathrm{br}^{(j)}$. This is because in order to define Types A_l and C_l (when $e = 1$), we require "n_0". Also already in the very definition of a subbranch, we posed a condition $\dfrac{n_{i-1} + n_{i+1}}{n_i}$ $(i = 1, 2, \ldots, e-1)$, which includes "n_0" for $i = 1$.

We now introduce a very important concept.

Definition 16.2.1 Let $Y = n_0\Theta_0 + \sum_{j=1}^{N} \mathrm{br}^{(j)}$ be a crust and let l be a positive integer such that (1) $lY \le X$, i.e. lY is a subdivisor of X and (2) any subbranch $\mathrm{br}^{(j)}$ is either of type A_l, B_l, or C_l. Then Y is called a *simple crust* and l is called the *barking multiplicity* of Y. (Note: Y itself may be multiple. For example, see Figure 15.5.3 (3) where $l = 1$.)

The importance of simple crusts will be apparent from the following result.

Proposition 16.2.2 *Let $\pi : M \to \Delta$ be a linear degeneration with a stellar singular fiber X.*

(1) *Suppose that X contains a simple crust Y of barking multiplicity l. Then $\pi : M \to \Delta$ admits a barking family $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$ such that in the deformation from X to $X_{0,t} = \Psi^{-1}(0, t)$, the subdivisor lY is barked off from X.*

(2) *In (1), if furthermore* (i) $\dfrac{n_1^{(1)} + n_1^{(2)} + \cdots + n_1^{(N)}}{n_0} = r_0$ *and* (ii) *a subbranch of type A_l in Y, if any, has zero slant (that is, $q := n_{e-1} - r_e n_e = 0$), then $X_{0,t} = \Psi^{-1}(0, t)$ is normally minimal.*

Proof. We write $Y = n_0\Theta_0 + \sum_{j=1}^{N} \mathrm{br}^{(j)}$, where

$$\mathrm{br}^{(j)} = n_1^{(j)}\Theta_1^{(j)} + n_2^{(j)}\Theta_2^{(j)} + \cdots + n_{e_j}^{(j)}\Theta_{e_j}^{(j)}.$$

For each subbranch $\mathrm{br}^{(j)}$, we associate a positive integer $a^{(j)}$ as follows:

$$a^{(j)} = \begin{cases} n_{e_j}^{(j)} & \mathrm{br}^{(j)} \text{ is of type } C_l, \\ 1 & \text{otherwise,} \end{cases}$$

and then set $d := \mathrm{lcm}\,(a^{(1)}, a^{(2)}, \ldots, a^{(N)})$; actually, for the subsequent discussion it suffices to take such a positive integer d as is divisible by $\mathrm{lcm}\,(a^{(1)}, a^{(2)}, \ldots, a^{(N)})$. Now we consider a deformation atlas

$$DA(lY, d) = \{\mathcal{W}_0, DA_{e_j-1}^{(j)}\}_{j=1,2,\ldots,N}.$$

By Theorem 10.0.15, p177 — when $\mathrm{br}^{(j)} = \emptyset$, we apply the construction for type A_l (see Remark 10.0.16 p177) —, each $DA_{e_j-1}^{(j)}$ admits a complete propagation. Consequently, $DA(lY, d)$ admits a complete propagation. Let $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$ be the barking family associated with this complete propagation. We next assert that in the deformation from X to $X_{0,t}$, the subdivisor lY is barked off from X. This immediately follows from the factorizations below:

$$\mathcal{W}_0|_{s=0} : \begin{cases} \sigma\tau^l\zeta^{m-ln}\left(\frac{1}{\tau}\zeta^n + t^d\right)^l = 0, & \text{around } p_1^{(j)} \quad (j = 1, 2, \ldots, N) \\ \sigma\zeta^{m-ln}(\zeta^n + t^d\tau)^l = 0, & \text{otherwise,} \end{cases}$$

and for $i = 1, 2, \ldots, e_j - 1$,

$$\begin{cases} \mathcal{H}_i^{(j)}|_{s=0} : & w^{m_{i-1}-ln_{i-1}}\eta^{m_i-ln_i}(w^{n_{i-1}}\eta^{n_i} + t^d f_i)^l = 0 \\ \mathcal{H}_i^{(j)\prime}|_{s=0} : & z_i^{m_{i+1}-ln_{i+1}}\zeta_i^{m_i-ln_i}(z_i^{n_{i+1}}\zeta_i^{n_i} + t^d \widehat{f}_i)^l = 0. \end{cases}$$

Thus we proved (1). The assertion (2) is derived from

(i) the description of $X_{0,t}$ around the core; see Lemma 3.2.2, p48, and
(ii) the description of $X_{0,t}$ around branches; see Figures 7.1.1, p122 and 7.1.3, p123 for type A_l with zero slant (that is, $q := n_{e-1} - r_e n_e = 0$), and see for type B_l, Figures 10.1.1, p179 and 10.1.2, p180, and for type C_l, §11.3, p191.

This completes the proof. □

Theorem 10.0.15, p177 gives the converse of (1) of Proposition 16.2.2:

Proposition 16.2.3 *Let $\pi : M \to \Delta$ be a linear degeneration with a stellar singular fiber X. If $\pi : M \to \Delta$ admits a barking family such that a subdivisor lY is barked off from X in the deformation from X to $X_{0,t}$, then Y is a simple crust of barking multiplicity l.*

We summarize Proposition 16.2.2 and Proposition 16.2.3 as follows:

Theorem 16.2.4 *Let $\pi : M \to \Delta$ be a linear degeneration with a stellar singular fiber X.*

(A) *If X contains a simple crust Y of barking multiplicity l, then $\pi : M \to \Delta$ admits a barking family $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$ such that lY is barked off from X in the deformation from X to $X_{0,t}$. In this case, if furthermore* (i) $\dfrac{n_1^{(1)} + n_1^{(2)} + \cdots + n_1^{(N)}}{n_0} = r_0$ *and* (ii) *each subbranch of type A_l of Y, if any, has zero slant ($q := n_{e-1} - r_e n_e = 0$), then $X_{0,t} := \Psi^{-1}(0,t)$ is normally minimal.*

(B) *Conversely, if $\pi : M \to \Delta$ admits a barking family such that a subdivisor lY is barked off from X in the deformation from X to $X_{0,t}$, then Y is a simple crust of barking multiplicity l.* (Note: As indicated by Example 18.4.2, p322, this is not true for a constellar singular fiber X — we will later define the notion of a simple crust for a constellar singular fiber, and construct a barking family associated with it.)

See Figures 16.3.1 and 16.3.2 for examples.

We note that from "one" simple crust, it is sometimes possible to construct "several" barking families. Indeed,

Proposition 16.2.5 *Let Y be a simple crust of barking multiplicity l. Then a barking family associated with it is generally "not" unique; in fact,*

(i) *If* $\dfrac{n_1^{(1)} + n_1^{(2)} + \cdots + n_1^{(N)}}{n_0} > \dfrac{m_1^{(1)} + m_1^{(2)} + \cdots + m_1^{(N)}}{m_0}$, *then the choice of a core section τ is generally not unique* (Remark 3.4.4, p53, and also §4.3, p74).

(ii) *If $l \geq 2$ and some subbranch $\mathrm{br}^{(j)}$ of Y is of type C_l, then in most cases, a complete propagation of $DA^{(j)}_{e_j-1}$ along the branch $\mathrm{Br}^{(j)}$ is not unique* (Remark 11.3.6, p198, and also Chapter 12, p209).

(For other cases, a barking family associated with Y is unique.)

The reader may conceive that if $l \geq 2$ and some subbranch $\mathrm{br}^{(j)}$ of Y is both of type A_l and B_l (type AB_l):

$$\mathbf{m} = (ln_0, ln_1, \ldots, ln_\lambda), \quad \mathbf{n} = (n_0, n_1, \ldots, n_\lambda), \quad \text{and} \quad n_\lambda = 1,$$

then $DA^{(j)}_{e_j-1}$ admits two different complete propagations resulting from the constructions of types A_l and B_l (Remark 10.1.4, p179); accordingly we obtain two different barking families from Y. However this is not the case, because as we show below in Corollary 16.7.4, any simple crust of a stellar singular fiber does *not* have a subbranch of type AB_l. (For a constellar singular fiber, the situation is different; a simple crust may have a subbranch of type AB_l. See Example 19.3.4, p337.)

Let $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$ be a barking family. For fixed $t \neq 0$, a singular fiber $X_{s,t} = \Psi^{-1}(s,t)$ is called the *main fiber* if $s = 0$, and is called a *subordinate fiber* if $s \neq 0$. We may completely describe the main fiber by using the results of §16.4 (the description of deformations of types A_l, B_l, and C_l) and §16.6. For subordinate fibers, we have the following result.

Proposition 16.2.6 *Let $\pi : M \to \Delta$ be a degeneration with a stellar singular fiber X. Suppose that X contains a simple crust Y of barking multiplicity l, and let $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$ be a barking family associated with it. Then any subordinate fiber of Ψ is a reduced curve only with A-singularities; these singularities are* (i) *near the core of X and* (ii) *near the edge of each proportional*[1] *subbranch.*

Proof. This result immediately follows by combining the statements about barking families restricted to the neighborhoods of the core (Proposition 21.6.3, p408) and branches (Proposition 7.2.6 (1), (2), p129; in the present case, (3) is excluded). □

For a special case where the core Θ_0 is the projective line and none of subbranches of Y is proportional, Ikuko Awata [Aw] described subordinate fibers.

We close this section by mentioning an interesting phenomenon: As we change the position of the branches (i.e. we move the attachment points of the branches to the core), the topological type of subordinate fibers often changes, while the topological type of the main fiber remains unchanged.

16.3 Criteria for splittability

Next we deduce a powerful criterion for the splittability of a singular fiber.

Criterion 16.3.1 *Let $\pi : M \to \Delta$ be a degeneration with a stellar singular fiber $X = m_0\Theta_0 + \sum_{j=1}^N \mathrm{Br}^{(j)}$. Then the following statements hold:*

(1) *Suppose that the core Θ_0 is an exceptional curve* (i.e. *Θ_0 is a projective line such that $\Theta_0 \cdot \Theta_0 = -1$*). *Then $\pi : M \to \Delta$ admits a splitting family.*
(2) *Suppose that the core Θ_0 is not an exceptional curve. If X contains a simple crust Y, then $\pi : M \to \Delta$ admits a splitting family.*

(The splitting families in (1) and (2) can be explicitly constructed.)

Proof. We show (1). Take a subdivisor $Y = \Theta_0 + \Theta_1^{(1)}$ and set $l = m_1^{(1)}$. Then Y is a simple crust of barking multiplicity l; indeed, $\mathrm{br}^{(1)} := \Theta_1^{(1)}$ is a subbranch of type B_l, and the existence of a core section τ is clear (Proposition 3.4.3, p52). Therefore by Theorem 16.2.4, there exists a barking family Ψ :

[1] If none of subbranches of Y is proportional, then the A-singularities are only near the core of X. Note that by Lemma 9.1.2, p154, any subbranch of type C_l is not proportional.

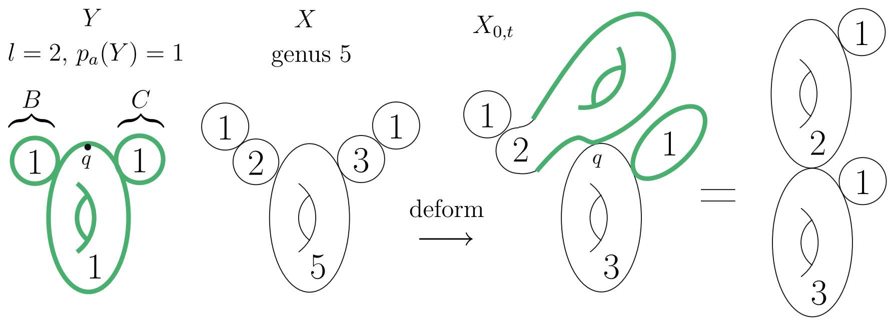


Fig. 16.3.1. The deformation of the subbranch of type C_l in Y is illustrated in greater detail in Figure 12.3.4, p224. (Note: $p_a(Y)$ is the arithmetic genus of Y; see [GH].)

$\mathcal{M} \to \Delta \times \Delta^{\dagger}$ associated with Y such that $X_{0,t}$ is normally minimal. In particular, $X_{0,t}$ has a nontrivial topological monodromy (different from that of X), and so apart from $X_{0,t}$, there must be another singular fiber in $\pi_t : M_t \to \Delta$ which also has a nontrivial topological monodromy (different from that of X). Namely, $\pi_t : M_t \to \Delta$ has (at least) two singular fibers with nontrivial topological monodromies — they are necessarily non-fake[2] singular fibers. This assures that $\Psi : \mathcal{M} \to \Delta \times \Delta^{\dagger}$ is a splitting family. (Note: The above discussion works not only for $j = 1$ but also for arbitrary j by taking $Y = \Theta_0 + \Theta_1^{(j)}$ and $l = m_1^{(j)}$).

Next we show (2). By assumption, $\pi : M \to \Delta$ is relatively minimal, and so the barking family associated with Y is a splitting family (by definition).□

We derive some more criteria for splittability. In what follows, for simplicity we often set

$$r = \frac{m_1^{(1)} + m_1^{(2)} + \cdots + m_1^{(N)}}{m_0}.$$

Note that r is a positive integer, because $-r = \Theta_0 \cdot \Theta_0$ (the self-intersection number of Θ_0).

Criterion 16.3.2 *Let $\pi : M \to \Delta$ be a degeneration with a stellar singular fiber $X = m_0\Theta_0 + \sum_{j=1}^N \mathrm{Br}^{(j)}$, and set $r = \dfrac{m_1^{(1)} + m_1^{(2)} + \cdots + m_1^{(N)}}{m_0}$. Suppose that*

(A) $N_0 \cong \mathcal{O}_{\Theta_0}(-p_1^{(1)} - p_1^{(2)} - \cdots - p_1^{(r)})$ *where N_0 is the normal bundle of Θ_0 in M, and*

(B) *there are r branches among all branches of X, say, $\mathrm{Br}^{(1)}, \mathrm{Br}^{(2)}, \ldots, \mathrm{Br}^{(r)}$ satisfying*

 (B1) *for $j = 1, 2, \ldots, r$, there exists an integer e_j where $1 \le e_j \le \lambda_j$ such that $m_{e_1}^{(1)} = m_{e_2}^{(2)} = \cdots = m_{e_r}^{(r)}$,*

[2] A singular fiber is *fake* if it becomes a smooth fiber after blowing down.

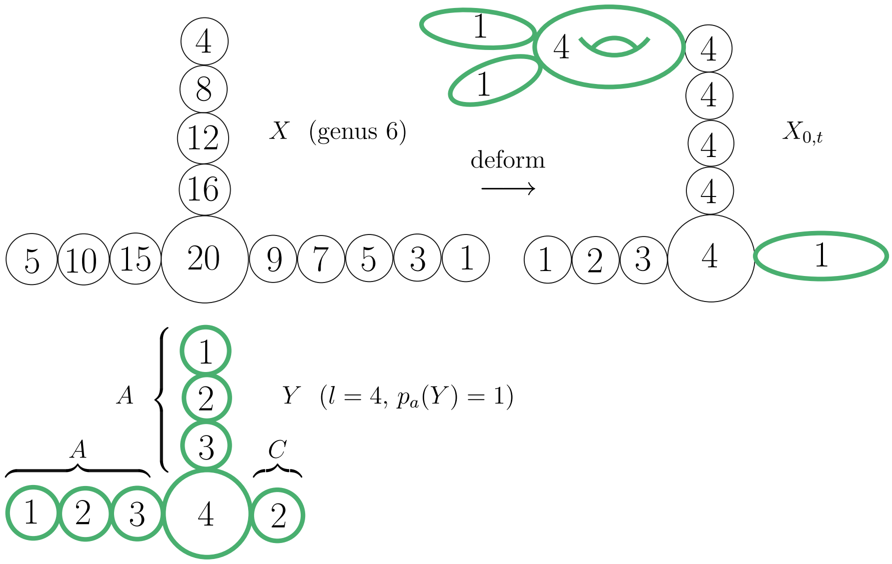


Fig. 16.3.2. $X_{0,t}$ contains three projective lines of multiplicity 1, denoted by ellipses, which were barked off from X.

(B2) $\Theta_1^{(j)} + \Theta_2^{(j)} + \cdots + \Theta_{e_j-1}^{(j)}$ $(j = 1, 2, \ldots, r)$ *is a chain of* (-2)*-curves,* i.e. $\Theta_i^{(j)}$ $(i = 1, 2, \ldots, e_j - 1)$ *has the self-intersection number* -2 (this condition is vacuous for j such that $e_j = 1$).

Then $\pi : M \to \Delta$ *admits a splitting family which is explicitly constructible from the above data.* (Note: (A) is an analytic condition, while (B) is a numerical one.)

Proof. We consider a subdivisor $Y = \Theta_0 + \sum_{j=1}^{r} \mathrm{br}^{(j)}$ of X, where

$$\mathrm{br}^{(j)} := \Theta_1^{(j)} + \Theta_2^{(j)} + \cdots + \Theta_{e_j}^{(j)}.$$

By (A) and (B2), Y is a crust; (A) guarantees the existence of a core section τ, while (B2) ensures that $\mathrm{br}^{(j)}$ is a subbranch. Set $l := m_{e_1}$ $(= m_{e_2} = \cdots = m_{e_r})$. Then by (B1), any subbranch $\mathrm{br}^{(j)}$ is of type B_l, so that Y is a simple crust of barking multiplicity l. Therefore we can apply Criterion 16.3.1. □

When Θ_0 is the projective line, the above criterion takes a simpler form:

Criterion 16.3.3 *Let* $\pi : M \to \Delta$ *be a degeneration with a stellar singular fiber* $X = m_0\Theta_0 + \sum_{j=1}^{N} \mathrm{Br}^{(j)}$ *such that the core* Θ_0 *is the projective line; for*

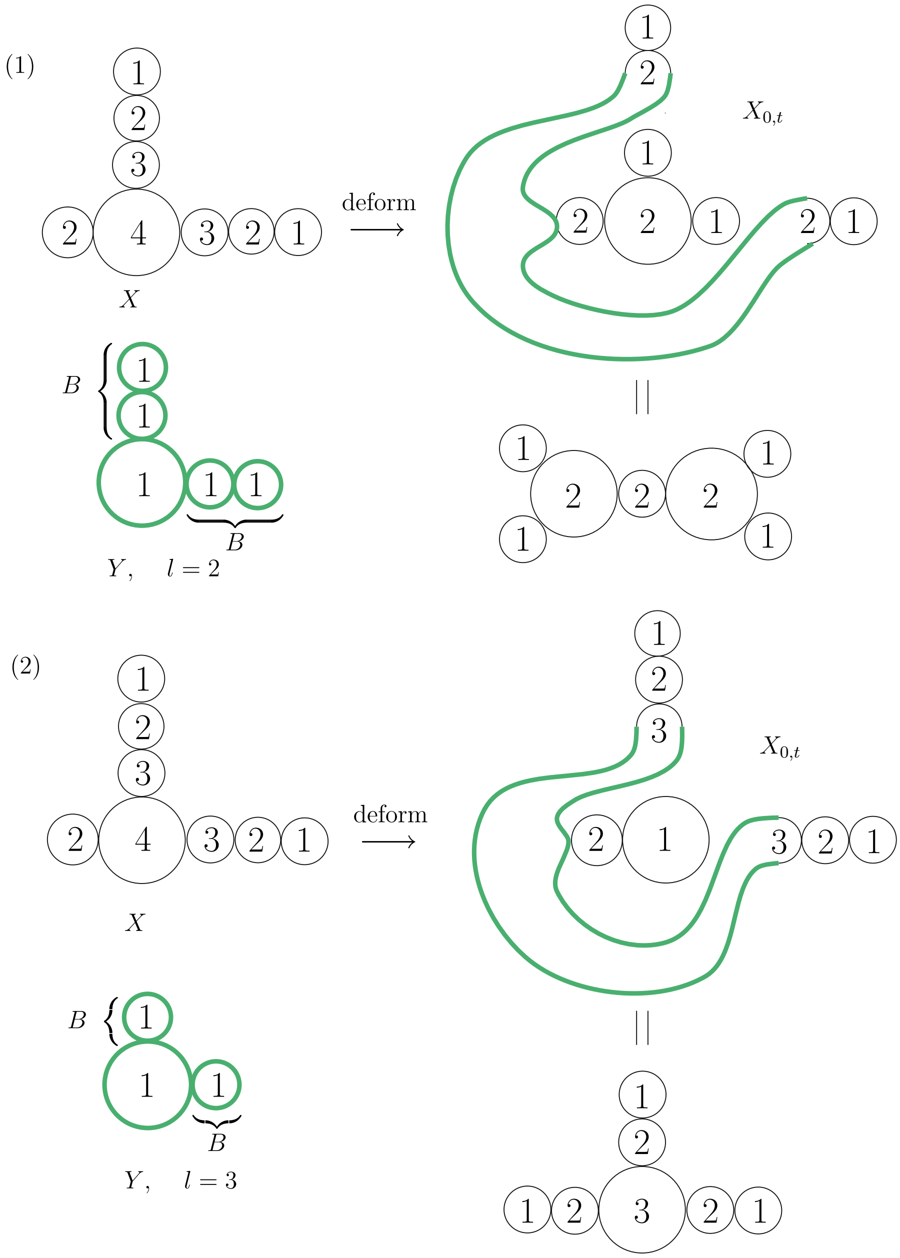


Fig. 16.3.3. Examples of simple crusts and the deformations associated with them

brevity, set[3] $r := \dfrac{m_1^{(1)} + m_1^{(2)} + \cdots + m_1^{(N)}}{m_0}$. *Suppose that there are* r *branches among all branches of* X, *say,* $\mathrm{Br}^{(1)}, \mathrm{Br}^{(2)}, \ldots, \mathrm{Br}^{(r)}$ *satisfying*

(B1) *for* $j = 1, 2, \ldots, r$, *there exists an integer* e_j *where* $1 \le e_j \le \lambda_j$ *such that* $m_{e_1}^{(1)} = m_{e_2}^{(2)} = \cdots = m_{e_r}^{(r)}$, *and*

(B2) $\Theta_1^{(j)} + \Theta_2^{(j)} + \cdots + \Theta_{e_j-1}^{(j)}$ $(j = 1, 2, \ldots, r)$ *is a chain of* (-2)*-curves,* i.e. $\Theta_i^{(j)}$ $(i = 1, 2, \ldots, e_j - 1)$ *has the self-intersection number* -2 (this condition is vacuous for j such that $e_j = 1$).

Then $\pi : M \to \Delta$ *admits a splitting family.* (See Figure 16.3.3 for example.)

Proof. We show the assertion by applying Criterion 16.3.2. It is enough to check that the condition (A) of Criterion 16.3.2 is fulfilled. This is evident; since $\deg N_0 = -r$ and Θ_0 is the projective line, we have

$$N_0 \cong \mathcal{O}_{\Theta_0}(-p_1^{(1)} - p_1^{(2)} - \cdots - p_1^{(r)}).$$

□

16.4 Singularities of fibers

In this section, we suppose that $\pi : M \to \Delta$ is a degeneration with a stellar singular fiber $X = m_0\Theta_0 + \sum_{j=1}^{N} \mathrm{Br}^{(j)}$. Given a simple crust Y of barking multiplicity l of X, we let $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$ be a barking family associated with Y (Proposition 16.2.2), that is, obtained from a complete propagation of a deformation atlas $DA(lY, d)$. The primary aim of this section is, for the convenience of later discussion, to summarize the description of a singular fiber $X_{s,t} := \Psi^{-1}(s, t)$ around the core and the branches of X.

We first give the description of singularities of $X_{s,t}$ near the core Θ_0 (see Chapter 21, p383 for details). By construction, the restriction of $X_{s,t}$ around the core is given by $\sigma(z)\zeta^{m_0 - ln_0}\left(\zeta^{n_0} + t^d\tau(z)\right)^l - s = 0$. Among the singularities of $X_{s,t}$, the description of those near the core Θ_0 notably applies the "plot function"

$$K(z) := n_0 \frac{d\sigma(z)}{dz}\tau(z) + m_0\sigma(z)\frac{d\tau(z)}{dz},$$

which is defined on Θ_0 (we often simply write $K(z) = n_0\sigma_z\tau + m_0\sigma\tau_z$). Since $m_0 > ln_0$ (see Lemma 16.7.1, p299), the following result holds by Proposition 21.8.3 (1), p418 — we note that t in that proposition is replaced by t^d, because in the present case, d (the weight of the deformation atlas $DA(lY, d)$) is not assumed to be 1.

[3] Recall that r is a positive integer, because $-r = \Theta_0 \cdot \Theta_0$ (the self-intersection number of Θ_0).

Lemma 16.4.1 *Let $C_{s,t}$ be the restriction of $X_{s,t}$ around the core Θ_0:*

$$C_{s,t}: \qquad \sigma(z)\zeta^{m_0-ln_0}\left(\zeta^{n_0}+t^d\tau(z)\right)^l - s = 0.$$

Let $K(z) = n_0\sigma_z\tau + m_0\sigma\tau_z$ be the plot function on C. Then $(\alpha,\beta) \in C_{s,t}$ $(s,t \neq 0)$ is a singularity if and only if α and β satisfy

(a) $K(\alpha) = 0, \ \ \sigma(\alpha) \neq 0, \ \ \tau(\alpha) \neq 0$ *and* (b) $\beta^{n_0} = \dfrac{ln_0 - m_0}{m_0} t^d\tau(\alpha).$

Futhermore, s and t satisfy $\left(\dfrac{ln_0-m_0}{ln_0}\right)^{al} s^a = \left(\dfrac{ln_0-m_0}{m_0}\right)^b (t^d)^b\sigma(\alpha)^a\tau(\alpha)^b$, where a and b are the relatively prime positive integers such that $am_0 = bn_0$; see (21.2.7), p395.

In particular, if $K(z)$ is identically zero, then $(\alpha,\beta) \in C_{s,t}$ $(s,t \neq 0)$ is a singularity precisely when $\sigma(\alpha) \neq 0$, $\tau(\alpha) \neq 0$, and $\beta^{n_0} = \dfrac{ln_0-m_0}{m_0}t^d\tau(\alpha)$. Observe that the set of (α,β) satisfying this condition is one-dimensional; hence $C_{s,t}$ has only non-isolated singularities. On the other hand, if $K(z)$ is not identically zero, then the following elaborate result holds: *If α has order r as a zero of $K(z)$, then the singularity (α,β) is an A_r-singularity* (Theorem 21.6.7, p410). We remark that if the core Θ_0 is the projective line, then $K(z)$ is not identically zero (Proposition 16.7.6, p301).

Next, we describe $X_{s,t}$ around a branch $\mathrm{Br}^{(j)}$ of the singular fiber X. We say that the restriction of the barking family $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$ to a neighborhood of $\mathrm{Br}^{(j)}$ is a deformation of *type* A_l, B_l, or C_l according to the type of the subbranch $\mathrm{br}^{(j)}$ of Y. In what follows, for simplicity we often omit (j) of $\Theta_i^{(j)}$, such as $\mathrm{Br}^{(j)} = m_1\Theta_1 + m_2\Theta_2 + \cdots + m_\lambda\Theta_\lambda$ and $\mathrm{br}^{(j)} = n_1\Theta_1 + n_2\Theta_2 + \cdots + n_e\Theta_e$.

Deformation of type A_l

(1) This deformation is trivial 'beyond' Θ_{e+1}, that is, trivial around irreducible components $\Theta_{e+2}, \Theta_{e+3}, \ldots, \Theta_\lambda$ (see Figure 6.1.4, p103).

(2) For any i $(1 \leq i \leq \lambda)$, the transition function $z = 1/w, \ \zeta = w^{r_i}\eta$ of N_i is not deformed.

(3) First suppose that $\mathrm{br}^{(j)}$ is proportional, i.e. there exist relatively prime positive integers a and b satisfying $(am_0, am_1, \ldots, am_\lambda) = (bn_0, bn_1, \ldots, bn_\lambda)$. Let $f(z)$ be the holomorphic function in the definition of the deformation atlas $DA^{(j)}_{e_j-1}$; see (16.1.3), p280. Then a fiber $X_{s,t}$ is singular near the branch $\mathrm{Br}^{(j)}$ if and only if either Case 1 or Case 2 below holds:

Case 1. $f(z)$ is constant[4], and

$s = 0$ or $\left(\dfrac{ln_0-m_0}{ln_0}\right)^{al} s^a = \left(\dfrac{ln_0-m_0}{m_0}\right)^b (t^d)^b$,

Case 2. $f(z)$ is not constant, and

$s = 0$ or $\left(\dfrac{ln_0-m_0}{ln_0}\right)^{al} s^a = \left(\dfrac{ln_0-m_0}{m_0}\right)^b (c_0t^d)^b$, where $c_0 = f(0)$.

[4] Actually, this does not occur for a branch of a stellar singular fiber, but may occur for a branch of a constellar one.

In Case 2, for the pair of s and t satisfying the equation on the right hand side, $X_{s,t}$ has only $A_{kp_{\lambda+1}-1}$-singularities near the edge of the branch $\mathrm{Br}^{(j)}$ (Proposition 7.2.6, p129), where k is the minimal positive integer such that $c_k \neq 0$ in the expansion $f(z) = c_0 + c_1 z + c_2 z^2 + \cdots$, and $p_{\lambda+1}$ is inductively defined via

$$\begin{cases} p_0 = 0, \quad p_1 = 1 \quad \text{and} \\ p_{i+1} = r_i p_i - p_{i-1} \quad \text{for } i = 1, 2, \ldots, \lambda. \end{cases}$$

Next suppose that $\mathrm{br}^{(j)}$ is not proportional. Then $X_{s,t}$ is singular near the branch $\mathrm{Br}^{(j)}$ if and only if $s = 0$ (Proposition 7.2.6, p129).

Deformation of type B_l

(1) This deformation is trivial beyond Θ_e, that is, trivial around irreducible components $\Theta_{e+1}, \Theta_{e+2}, \ldots, \Theta_\lambda$ (see Figure 10.1.1, p179).
(2) A deformation of the transition function $z = 1/w,\ \zeta = w^{r_i}\eta$ of N_i occurs only at $i = e$.
(3) $X_{s,t}$ is singular near the branch $\mathrm{Br}^{(j)}$ if and only if $s = 0$ (Proposition 10.2.1, p180).

Deformation of type C_l

(1) To be consistent with the notation in §11.3, p191, instead of d, we use k for the weight of a deformation atlas for type C_l; we assume that k is divisible by n_e. For a subbranch $\mathrm{br}^{(j)}$ of type C_l, the construction of a complete propagation of the deformation atlas $DA_{e-1}(l \cdot \mathrm{br}^{(j)}, k)$ was carried out separately for three cases (Cases I, II, III in §11.3, p191); the resulting deformation for each case is non-trivial around $\Theta_e, \Theta_{e+1}, \ldots, \Theta_f$, and trivial around $\Theta_{f+1}, \Theta_{f+2}, \ldots, \Theta_\lambda$ where

$$f = \begin{cases} e + Nn_e - 1 & \text{Case I. } b = 0 \quad \text{or} \quad \text{Case II. } b \geq 1 \text{ and } u > b \\ e + Nn_e + v - 1 & \text{Case III. } b \geq 1 \text{ and } u \leq b, \end{cases} \tag{16.4.1}$$

(see (11.3.5), p198 where we set $d := n_e$). The integers N and v are defined as follows: First, set

$$u := (m_{e-1} - ln_{e-1}) - (r_e - 1)(m_e - ln_e),$$

and then u divides l by the definition of type C_l; we set $N := \frac{l}{u}$. Next we set $b := m_e - ln_e$, and when $b \geq 1$ and $u \leq b$, let v be the integer such that $l - vu \geq 0$ and $l - (v+1)u < 0$.
We remark that when $l \geq 2$, a complete propagation of $DA_{e-1}(l \cdot \mathrm{br}^{(j)}, k)$ is *not* unique (Chapter 12, p209); accordingly we obtain several deformations from one subbranch of type C_l.
(2) Deformations of transition functions $z = 1/w,\ \zeta = w^{r_i}\eta$ occur at “many” irreducible components between Θ_e and Θ_f. The exact number

for "many" depends on the construction of a complete propagation of $DA_{e-1}(l \cdot \mathrm{br}^{(j)}, k)$.
(3) $X_{s,t}$ is singular near the branch $\mathrm{Br}^{(j)}$ if and only if $s = 0$ (Proposition 11.4.2, p200).

Taking into consideration the above property (1) for each type, we introduce the *propagation number* $\rho(\mathrm{br}^{(j)})$ of a subbranch $\mathrm{br}^{(j)}$ of type A_l, B_l, or C_l as follows:

$$\rho(\mathrm{br}^{(j)}) = \begin{cases} e+1 & \text{if } \mathrm{br}^{(j)} \text{ is of type } A_l \\ e & \text{if } \mathrm{br}^{(j)} \text{ is of type } B_l \\ f & \text{if } \mathrm{br}^{(j)} \text{ is of type } C_l, \end{cases} \tag{16.4.2}$$

where e is the length of $\mathrm{br}^{(j)}$, and f is as in (16.4.1). Then from the above property (1) for types A_l, B_l, and C_l, we have

Lemma 16.4.2 *Let $\pi : M \to \Delta$ be a degeneration with a stellar singular fiber X, and let $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$ be a barking family associated with a simple crust Y of X. Take a subbranch $\mathrm{br}^{(j)}$ of Y, and denote by $\rho = \rho(\mathrm{br}^{(j)})$ its propagation number. Then the barking family $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$, restricted to a neighborhood of a branch $\mathrm{Br}^{(j)}$ of X, is trivial beyond an irreducible component Θ_ρ; in other words, trivial around $\Theta_{\rho+1}, \Theta_{\rho+2}, \ldots, \Theta_\lambda$.*

The irreducible component Θ_ρ in the above lemma is called *semi-rigid*, and Θ_i $(i \geq \rho+1)$ are called *rigid*.
We give a peculiar example of subbranches.

Example 16.4.3 It may happen that a subbranch of type C_l is 'contained' in a subbranch of type B_l. For instance,
(1) type B_l $\quad l = 10, \quad \mathbf{m} = (40, 26, 12, 10, 8, 6, 4, 2)$ and $\mathbf{n} = (3, 2, 1, 1)$.
(2) type C_l $\quad l = 10, \quad \mathbf{m} = (40, 26, 12, 10, 8, 6, 4, 2)$ and $\mathbf{n} = (3, 2, 1)$.
(In (2), $m_e > ln_e$ and $u = 2$.)

From these two subbranches, we may construct two distinct deformations of the branch, which coincide around irreducible components $\Theta_0, \Theta_1, \Theta_2$, but are totally different around irreducible components $\Theta_3, \Theta_4, \Theta_5, \Theta_6, \Theta_7$.

We close this section by summarizing as a theorem the above descriptions of barking families restricted to neighborhoods of the core and branches. To that end, we first recall some terminology. Let $\pi : M \to \Delta$ be a degeneration of compact complex curves, and let $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$ be a baking family obtained from a simple crust Y of barking multiplicity l. For fixed t $(t \neq 0)$, a singular fiber $X_{s,t} := \Psi^{-1}(s, t)$ is called the *main fiber* if $s = 0$, and it is called a *subordinate fiber* if $s \neq 0$: The original singular fiber $X := \pi^{-1}(0)$ splits into one main fiber and several subordinate fibers. The main fiber $X_{0,t}$ is described essentially in terms of the factorization of its defining equation. For the description of the subordinate fibers, we have the following result.

Theorem 16.4.4 *Let $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$ be a barking family obtained from a simple crust Y of barking multiplicity l. Then the singularities of a subordinate fiber $X_{s,t}$ are as follows:*

(1) *If the plot function $K(z) = n_0\sigma_z\tau + m_0\sigma\tau_z$ is not identically zero, then $X_{s,t}$ has A-singularities near the core Θ_0* (Theorem 21.6.7, p410)*, whereas if $K(z)$ is identically zero, then $X_{s,t}$ has non-isolated singularities near the core* (Proposition 21.8.3 (1), p418)*. For the former case, the following inequality holds* (Corollary 21.4.4, p403)*:*

$$(\textit{the number of the A-singularities near the core}) \\ \leq \gcd(m_0, n_0) \cdot \Big[N - v + k + (2g_0 - 2) - \sum_{j \in J_0} \mathrm{ord}_{p_j}(\omega)\Big],$$

where
N is the number of the branches of X,
v is the number of the proportional subbranches of Y,
k is the number of the zeros of τ,
g_0 is the genus of the core Θ_0,
J_0 is the set of indices j such that $n_0m_1^{(j)} - m_0n_1^{(j)} = 0$ (there are v such indices), and
$\mathrm{ord}_{p_j}(\omega)$ is the order of a meromorphic 1-form $\omega(z) := d\log\big(\sigma^{n_0}\tau^{m_0}\big)$.
In the generic case[5]*, $X_{s,t}$ has only nodes (A_1-singularities) near the core Θ_0.*

(2) *If Y has proportional subbranches, then $X_{s,t}$ has A-singularities near the edge of each proportional subbranch* (Proposition 7.2.6, p129).

Actually, for a subordinate fiber $X_{s,t}$, (a) the number of A-singularities of $X_{s,t}$ near the core Θ_0 and (b) the 'complexity' of A-singularities of $X_{s,t}$ near the edge of each proportional subbranch of Y are closely related. This is so-called the *seesaw phenomenon* of the singularities of a subordinate fiber. See §21.7, p413 for details.

16.5 Application to a constellar case

In some cases, the construction of barking families for a stellar singular fiber is generalized to that for a constellar one. Recall that a singular fiber is *constellar* (constellation-shaped) if it is obtained by bonding stellar singular fibers. Here "bonding" is an operation, originally introduced by Matsumoto and Montesinos (see [MM2] and also [Ta,II]), which produces a new singular fiber from given stellar singular fibers. Let X_1 (resp. X_2) be stellar singular fibers of $\pi_1 : M_1 \to \Delta$ (resp. $\pi_2 : M_2 \to \Delta$), and $\overline{\mathrm{Br}}_1$ (resp. $\overline{\mathrm{Br}}_2$) be a fringed branch of X_1 (resp. X_2), and we express

$$\overline{\mathrm{Br}}_1 = m_0\Delta_0 + m_1\Theta_1 + \cdots + m_\lambda\Theta_\lambda, \qquad \overline{\mathrm{Br}}_2 = m_0'\Delta_0' + m_1'\Theta_1' + \cdots + m_\nu'\Theta_\nu.$$

[5] The case where any zero α of the plot function $K(z)$ such that $\sigma(\alpha) \neq 0$ and $\tau(\alpha) \neq 0$ is simple (i.e. of order 1).

We pose an assumption: $m_\lambda = m'_\nu$, and let κ ($\kappa \geq -1$) be an integer such that if $\kappa = -1$, then the following condition is satisfied: There exists a pair of integers λ_0 and ν_0 ($0 \leq \lambda_0 < \lambda,\ \ 0 \leq \nu_0 < \nu$) satisfying $m_{\lambda_0+1} + m'_{\nu_0+1} = m_{\lambda_0} = m'_{\nu_0}$.

Under this assumption, by bonding the branches $\overline{\mathrm{Br}}_1$ and $\overline{\mathrm{Br}}_2$ of X_1 and X_2, we may construct a singular fiber $X = X(\kappa)$ (a κ-bonding of X_1 and X_2) of some degeneration: Two fringed branches $\overline{\mathrm{Br}}_1$ and $\overline{\mathrm{Br}}_2$ are joined to become a (fringed) κ-trunk $\overline{\mathrm{Tk}}$ of the new singular fiber X. Specifically, we set $m := m_\lambda = m'_\nu$, and then the trunk $\overline{\mathrm{Tk}}$ is a chain of projective lines, given as follows ($m_0\Delta_0$ and $m'_0\Delta'_0$ are its fringes):

if $\kappa \geq 0$,

$$\begin{aligned}\overline{\mathrm{Tk}} = m_0\Delta_0 + m_1\Theta_1 + \cdots + m_{\lambda-1}\Theta_{\lambda-1} + m\Theta_\lambda + m\Theta_{\lambda+1} + \cdots + m\Theta_{\lambda+\kappa} \\ + m'_\nu\Theta'_\nu + m'_{\nu-1}\Theta'_{\nu-1} + \cdots + m'_0\Delta'_0,\end{aligned}$$

if $\kappa = -1$,

$$\overline{\mathrm{Tk}} = m_0\Delta_0 + m_1\Theta_1 + \cdots + m_{\lambda_0}\Theta_{\lambda_0} + m'_{\nu_0-1}\Theta'_{\nu_0-1} + m'_{\nu_0-2}\Theta'_{\nu_0-2} + \cdots + m'_0\Delta'_0.$$

Now we return to discuss barking families. Let X be the constellar singular fiber obtained as above by bonding two stellar singular fibers X_1 and X_2. Suppose that X_1 contains a simple crust Y_1, and let br_1 be the subbranch of Y_1 contained in the branch Br_1 of X_1. By Lemma 16.4.2, p291, any barking family associated with Y_1 is, when restricted to a neighborhood of the branch Br_1, trivial beyond an irreducible component Θ_ρ (semi-rigid component), where $\rho := \rho(\mathrm{br}_1)$ is the propagation number (16.4.2) of the subbranch br_1. If $\rho(\mathrm{br}_1) + 1 \leq \mathrm{length}(\mathrm{Tk})$ (this is always satisfied if $\kappa \geq 0$), we can trivially propagate this barking family of $\pi_1 : M_1 \to \Delta$ to that of $\pi : M \to \Delta$. Therefore we obtain the following criterion.

Criterion 16.5.1 (Trivial Propagation Criterion) *Let* X_1 (resp. X_2) *be a stellar singular fiber of* $\pi_1 : M_1 \to \Delta$ (resp. $\pi_2 : M_2 \to \Delta$)*, and let* Br_1 (resp. Br_2) *be a branch of* X_1 (resp. X_2)*. Suppose that* X *is a constellar singular fiber of* $\pi : M \to \Delta$*, obtained from* X_1 *and* X_2 *by* κ*-bonding of* Br_1 *and* Br_2 (*recall that* κ ($\kappa \geq -1$) *is an integer*)*. If* X_1 *contains a simple crust* Y_1 *such that in the case* $\kappa = -1$,

$$\rho(\mathrm{br}_1) + 1 \leq \mathrm{length}(\mathrm{Tk}), \tag{16.5.1}$$

where $\rho(\mathrm{br}_1)$ *is the propagation number of the subbranch* br_1 *of* Y_1 *contained in* Br_1*, then any barking family of* $\pi_1 : M_1 \to \Delta$ *associated with* Y_1 *trivially propagates to that of* $\pi : M \to \Delta$*.* (See Figure 16.5.1 for an example for $\kappa = -1$.)

This criterion is easily generalized to the case where a constellar singular fiber X is obtained from the bonding of an arbitrary number of stellar singular fibers.

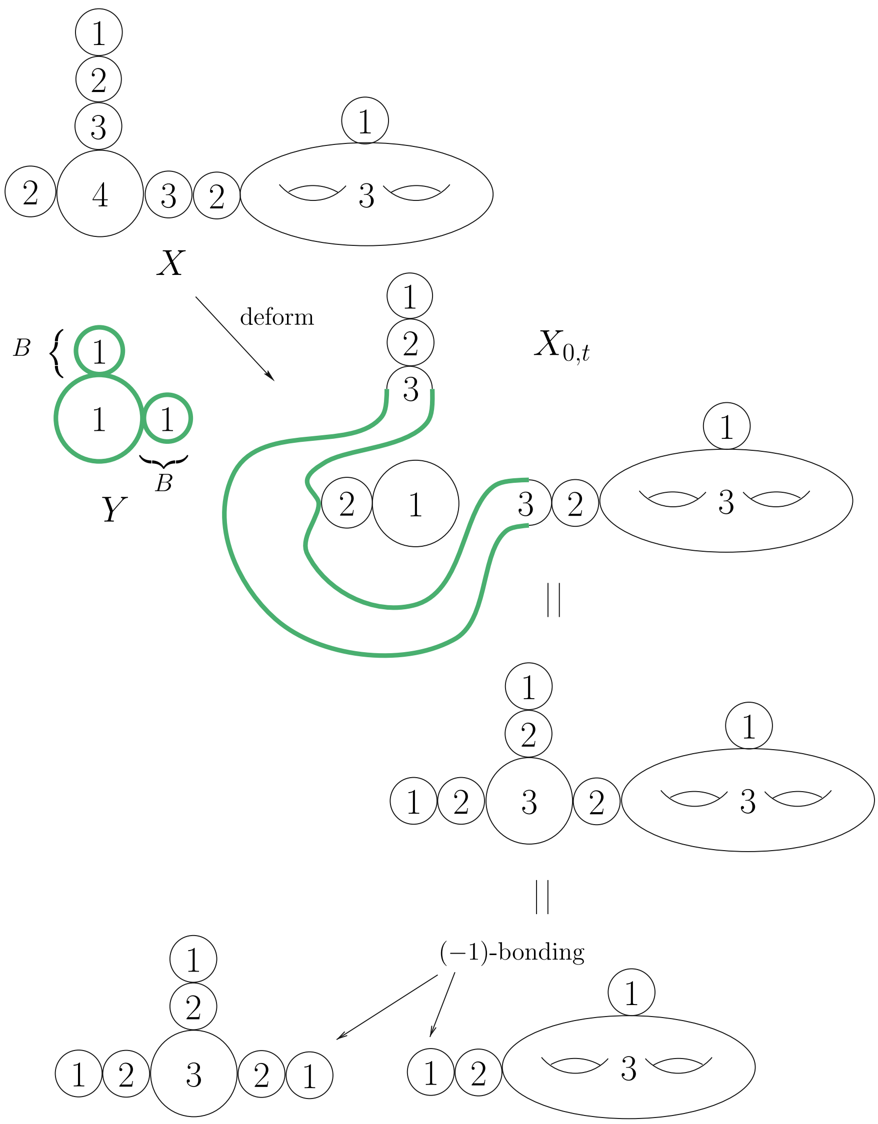


Fig. 16.5.1. genus$(X) = 7$, barking multiplicity $l = 3$. This deformation is a 'generalization' of the deformation in Figure 16.3.3 (2), p287.

Criterion 16.5.2 (Exceptional Curve Criterion) *Let $\pi : M \to \Delta$ be a normally minimal degeneration with a (stellar or constellar) singular fiber X which contains an exceptional curve, say, Θ_0. If any irreducible component intersecting Θ_0 is a projective line, then $\pi : M \to \Delta$ admits a splitting family.*

(Note: If X is stellar, then Θ_0 must be the core, and this assumption is always satisfied.)

Proof. If X is stellar, then Θ_0 is the core, and the assertion follows from Criterion 16.3.1, p284. We next consider the case where X is constellar. Let Θ_k $(k = 1, 2, \ldots, N)$ be the set of irreducible components of X intersecting Θ_0. Then $m_k < m_0$, because

$$\Theta_0 \cdot \Theta_0 = -\frac{\sum_{k=1}^{N} m_k}{m_0} = -1.$$

Now we take one irreducible component, say Θ_1, among Θ_k $(k = 1, 2, \ldots, N)$, and we consider a subdivisor $Y = \Theta_0 + \Theta_1$. We set $l := m_1$ (note that $l < m_0$ by the above argument). Then Y is a simple crust of barking multiplicity l, where Θ_0 is its core and Θ_1 is a subbranch of type B_l. From Y, we can construct a barking family $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$ of $\pi : M \to \Delta$ such that it is trivial beyond $\Theta_1, \Theta_2, \ldots, \Theta_N$ (Proposition 16.2.2 (2), p281). Moreover, since the subbranch of Y is of type B_l, the singular fiber $X_{0,t}$ is normally minimal (Theorem 16.2.4 (A)), and so $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$ is a splitting family by Lemma 1.1.2, p28. □

16.6 Barking genus

Let $Y = n_0\Theta_0 + \sum_{j=1}^{N} \mathrm{br}^{(j)}$ be a simple crust of barking multiplicity l of a stellar singular fiber X; so $\mathrm{br}^{(j)} = n_1^{(j)}\Theta_1^{(j)} + n_2^{(j)}\Theta_2^{(j)} + \cdots + n_{e_j}^{(j)}\Theta_{e_j}^{(j)}$ is a subbranch of type A_l, B_l, or C_l. We denote by $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$ a barking family associated with Y. In the process of deformation from X to $X_{0,t} = \Psi^{-1}(0, t)$, the subdivisor lY is barked off from X to become a multiple curve lY_t (being one part of $X_{0,t}$). Note that Y_t is a smoothing of Y away from several points (the zeros of τ, and such points p as in Figure 7.1.2, p123 and Figure 7.1.4, p123 if Y has a subbranch of type A_l). The genus of the reduced curve Y_t $(t \neq 0)$ — the genus of one component of Y_t unless Y_t is connected — is called the *barking genus* of Y, and denoted by $g_b(Y)$; note that if $d \geq 2$ where d is the weight of the deformation atlas $DA(lY, d)$ in (16.1.1), then Y_t consists of d isomorphic components. We also note that if Y has a subbranch of type C_l, then Y_t, $(t \neq 0)$ has boundary: Each boundary component (a circle) corresponds to a subbranch of type C_l, and in the process of 'reverse' deformation from Y_t to Y, each boundary component is pinched to a point on the irreducible component $\Theta_{e_j}^{(j)}$ of the subbranch $\mathrm{br}^{(j)}$ of type C_l. For example, the circle l_2 in Figure 3.3.2, p51 is pinched to the point p_2 in that figure. For more complicated examples, see Figure 8.1.1, p150 and Figure 12.3.1, p221. We remark that if none of subbranches of Y is of type C_l, then Y_t $(t \neq 0)$ has no boundary.

Remark 16.6.1 Let $g_a(Y)$ be the *arithmetic genus* of the simple crust Y as a subdivisor in M (see [GH]):

$$g_a(Y) = 1 + \frac{K_M \cdot Y + Y \cdot Y}{2},$$

where K_M is the canonical bundle of M. Since Y_t is linearly equivalent to Y in $\mathcal{M}$, we have $\text{genus}(Y_t) = g_a(Y)$, so that $g_a(Y)$ equals the barking genus $g_b(Y)$ of Y.

Next, we ask whether there exists a degeneration whose singular fiber is the simple crust Y. If the answer is positive, then the barking genus of Y may be simply defined as the genus of (a connected component of) a smooth fiber of this degeneration. Unfortunately the answer is negative. In fact, if there exists a degeneration whose singular fiber is Y, then each subbranch of Y must have the properties of branches: e.g. the ratio n_{e-1}/n_e for a subbranch $\text{br}^{(j)} = n_1\Theta_1 + n_2\Theta_2 + \cdots + n_e\Theta_e$ is an integer at least 2. However this is not always the case: for example,

(1) If $\text{br}^{(j)}$ is of type A_l, then $n_{e-1}/n_e \geq r_e$, and so $n_{e-1}/n_e \geq 2$. But n_{e-1}/n_e is not necessarily an integer. For instance, $\mathbf{m} = (9, 6, 3)$ and $\mathbf{n} = (8, 5, 2)$, and then $n_{e-1}/n_e = 5/2$.
(2) If $\text{br}^{(j)}$ is of type B_l or C_l, then n_{e-1}/n_e is a positive integer. But "$n_{e-1}/n_e \geq 2$" does not necessarily hold. For instance, (i) $l = 4$, $\mathbf{m} = (6, 5, 4, 3, 2, 1)$, and $\mathbf{n} = (1, 1, 1)$ (type B_4) and (ii) $l = 1$, $\mathbf{m} = (6, 5, 4, 3, 2, 1)$, and $\mathbf{n} = (4, 4, 4)$ (type C_1).

Taking (2) into consideration, *for the rest of this section we allow a branch of a singular fiber to contain an exceptional curve* (*a projective line with the self-intersection number* -1). Then after we 'modify' Y, we are able to realize it as a singular fiber of some 'degeneration' — a degeneration in the above wider sense. Specifically, instead of $Y = n_0\Theta_0 + \sum_j \text{br}^{(j)}$ we consider $\dot{Y} = n_0\Theta_0 + \sum_j \dot{\text{br}}^{(j)}$, where $\dot{\text{br}}^{(j)}$ (the *enlargement* of the subbranch $\text{br}^{(j)}$) is defined as follows: (1) if $\text{br}^{(j)}$ is of type B_l or C_l, we simply take $\text{br}^{(j)}$ itself as $\dot{\text{br}}^{(j)}$, and (2) for type A_l, we make $\text{br}^{(j)}$ longer to construct $\dot{\text{br}}^{(j)}$ in the following way: Noting that $n_{e-1}/n_e \geq r_e$, we first define a sequence of integers

$$n_{e+1} > n_{e+2} > \cdots > n_f > n_{f+1} = 0$$

by the division algorithm; that is, $n_{e-1} = r_e n_e - n_{e+1}$ $(0 \leq n_{e+1} < n_e)$ and then for $i = e+1, e+2, \ldots, f$, inductively by $n_{i-1} = r'_i n_i - n_{i+1}$ $(0 \leq n_{i+1} < n_i)$, where r'_i is an integer at least 2. Using this sequence, we define the enlargement $\dot{\text{br}}^{(j)}$ of the subbranch $\text{br}^{(j)}$ of type A_l by

$$\dot{\text{br}}^{(j)} := \text{br}^{(j)} + n_{e+1}\Theta_{e+1} + n_{e+2}\Theta_{e+2} + \cdots + n_f\Theta_f.$$

Next we construct an ambient space of $\dot{\mathrm{br}}^{(j)}$ for a subbranch $\mathrm{br}^{(j)}$ of type A_l. (*A priori*, $\dot{\mathrm{br}}^{(j)}$ is not a branch of a singular fiber; it is not yet embedded in a complex surface.) Noting that

$$\frac{n_{i-1}+n_{i+1}}{n_i} = r'_i \quad (i = e+1, e+2, \dots f-1) \quad \text{and} \quad \frac{n_{f-1}}{n_f} = r'_f, \quad (16.6.1)$$

we take line bundles L_i on Θ_i $(i = 1, 2, \dots, f)$, obtained by patching $(w_i, \eta_i) \in U_i \times \mathbb{C}$ and $(z_i, \zeta_i) \in V_i \times \mathbb{C}$ via

$$\begin{cases} z_i = \dfrac{1}{w_i}, \quad \zeta_i = w_i^{r_i}\eta_i & \text{if } 1 \le i \le e \ \ \text{(in this case } L_i = N_i\text{)}, \\ z_i = \dfrac{1}{w_i}, \quad \zeta_i = w_i^{r'_i}\eta_i & \text{if } e+1 \le i \le f. \end{cases}$$

Then we patch L_i with L_{i+1} $(i = 1, 2, \dots, f-1)$ by plumbing $(z_i, \zeta_i) = (\eta_{i+1}, w_{i+1})$. This yields a smooth complex surface $T^{(j)}$ which is an ambient space of the enlargement $\dot{\mathrm{br}}^{(j)}$ for a subbranch $\mathrm{br}^{(j)}$ of type A_l.

Next recall that for a subbranch $\mathrm{br}^{(j)}$ of type B_l or C_l, we took $\mathrm{br}^{(j)}$ itself as $\dot{\mathrm{br}}^{(j)}$. We then construct an ambient space $T^{(j)}$ of $\dot{\mathrm{br}}^{(j)}$ as follows (we note that $T^{(j)}$ constructed below is generally different from a tubular neighborhood of $\mathrm{br}^{(j)}$ "in M"). First we define a line bundle L_i on Θ_i for $i = 1, 2, \dots, e$: For $i = 1, 2, \dots, e-1$, we simply take $L_i := N_i$, obtained by gluing $U \times \mathbb{C}$ with $V \times \mathbb{C}$ by $z = 1/w,\ \zeta = w^{r_i}\eta$, while setting $r'_e := n_{e-1}/n_e$, we define a line bundle L_e by gluing $U \times \mathbb{C}$ with $V \times \mathbb{C}$ by $z = 1/w,\ \zeta = w^{r'_e}\eta$. Then we plumb L_i and L_{i+1} $(i = 1, 2, \dots, e-1)$ to construct a smooth complex surface $T^{(j)}$; this surface is an ambient space of $\dot{\mathrm{br}}^{(j)} (= \mathrm{br}^{(j)})$ for a subbranch $\mathrm{br}^{(j)}$ of type B_l or C_l.

Remark 16.6.2 In general, $T^{(j)}$ for type B_l or C_l does *not* coincide with a tubular neighborhood of $\dot{\mathrm{br}}^{(j)}$ $(= \mathrm{br}^{(j)})$ "in M". In fact, the self-intersection number of Θ_e in $T^{(j)}$ may be -1 (this occurs precisely when $n_{e-1}/n_e = 1$), whereas the self-intersection number of Θ_e "in M" is never -1.

Next, recall that N_0 is a tubular neighborhood of the core Θ_0 in M. By plumbing, we glue N_0 with each $T^{(j)}$ (the ambient space of the enlargement $\dot{\mathrm{br}}^{(j)}$) around the intersection point $p_1^{(j)}$ of Θ_0 and $\mathrm{Br}^{(j)}$. We denote the resulting smooth complex surface by S. We shall realize $\dot{Y} := n_0\Theta_0 + \sum_j \dot{\mathrm{br}}^{(j)}$, the *enlargement* of the simple crust Y, as a 'singular fiber' of a family $\{\dot{Y}_t\}_{t\in\Delta^\dagger}$ in $S \times \Delta^\dagger$. First, we consider the following equations in $N_0 \times \Delta \times \Delta^\dagger$:

$$\mathcal{W}_0 : \begin{cases} \frac{1}{\tau}\zeta^{n_0} + t^d = 0 & \text{on } z \in \Theta_0^\times \\ \zeta^{n_0} + t^d\tau = 0 & \text{on } z \in D \end{cases} \quad \text{(see Lemma 15.3.2, p272)}, \quad (16.6.2)$$

where $\Theta_0^\times = \Theta_0 \setminus \{\text{the zeros of } \tau\}$ and $D := \bigcup_i D_i$: Each D_i is a small disk around a zero $q_i \in \Theta_0$ of τ. Next we consider the following equations in

$T^{(j)} \times \Delta \times \Delta^\dagger$: for $i = 1, 2, \ldots, \text{length}(\dot{\text{br}}^{(j)})$,

$$\begin{cases} w^{n_{i-1}}\eta^{n_i} + t^d f_i - s = 0 \\ z^{n_{i+1}}\zeta^{n_i} + t^d \widehat{f}_i - s = 0. \end{cases} \tag{16.6.3}$$

The families defined by (16.6.2) and (16.6.3) are patched together to constitute a global family $\{\dot{Y}_t\}_{t\in\Delta^\dagger}$ in $S \times \Delta^\dagger$ such that $\dot{Y}_0 = \dot{Y}$. Here we have to be careful; if τ has zeros, this family cannot be the graph of a degeneration. In fact, if it does, say, it is realized as the graph of a degeneration $\rho : S \to \Delta^\dagger$, then $\rho^{-1}(t_1) \cap \rho^{-1}(t_2) = \emptyset$ must hold for distinct t_1 and t_2 (any two fibers must be disjoint!). But this is not the case in the present situation, because regardless of the value t, the curve $\dot{Y}_t$ always passes through the zeros $q_i \in \Theta_0$ of τ; indeed, the curve $\zeta^{n_0} + t^d\tau = 0$ in (16.6.2) always passes through a point $(z, \zeta, t) = (q_i, 0, 0)$.

To remedy this situation, we delete the zeros of τ from the core Θ_0; let $N_0^\times$ be the restriction of the line bundle N_0 to $\Theta_0^\times = \Theta_0 \setminus \{\text{the zeros of } \tau\}$, and we consider a smooth complex surface $S^\times$, obtained by plumbing $N_0^\times$ with each $T^{(j)}$ around $p_1^{(j)}$ (the intersection point of Θ_0 and $\text{Br}^{(j)}$). For the complex surface $S^\times$, we may define a holomorphic map $\rho : S^\times \to \Delta^\dagger$: For the case $d = 1$ where d is the weight of $DA(lY, d)$, the map ρ is given around the core by $\rho(z, \zeta) = -\dfrac{1}{\tau(z)}\zeta^{n_0}$, and ρ is given around each enlargement $\dot{\text{br}}^{(j)}$ by

$$\begin{cases} \rho(w, \eta) = -\dfrac{w^{n_{i-1}}\eta^{n_i}}{f_i} \\ \rho(z, \zeta) = -\dfrac{z^{n_{i+1}}\zeta^{n_i}}{\widehat{f}_i}. \end{cases}$$

For the case $d \geq 2$, instead of ρ, we only have to take the d-th power of ρ, that is, $\rho^d : S^\times \to \Delta^\dagger$; hereafter, we rewrite ρ^d by ρ. We then obtain a degeneration $\rho : S^\times \to \Delta^\dagger$ whose singular fiber $\rho^{-1}(0)$ is $\dot{Y}^\times = n_0\Theta_0^\times + \sum_j \dot{\text{br}}^{(j)}$, where $\dot{Y}^\times$ is obtained from $\dot{Y}$ by replacing the core Θ_0 by $\Theta_0^\times = \Theta_0 \setminus \{\text{the zeros of } \tau\}$. Of course if the core section τ has no zeros, then $\dot{Y}^\times$ is $\dot{Y}$ itself. (Note: When τ has zeros, all fibers of ρ are non-compact, and so $\rho : S^\times \to \Delta^\dagger$ is not proper.)

We summarize the above discussion.

Proposition 16.6.3 *Given a simple crust $Y = n_0\Theta_0 + \sum_j \text{br}^{(j)}$, there exists a degeneration $\rho : S^\times \to \Delta^\dagger$ whose singular fiber $\rho^{-1}(0)$ is $\dot{Y}^\times = n_0\Theta_0^\times + \sum_j \dot{\text{br}}^{(j)}$, where $\Theta_0^\times = \Theta_0 \setminus \{$the zeros of $\tau\}$ and $\dot{\text{br}}^{(j)}$ is the enlargement of* $\text{br}^{(j)}$.

Note: (a) If the core section τ has no zeros, then $\dot{Y}^\times$ is $\dot{Y}$ itself. (b) By construction, the genus of a connected component of a smooth fiber of $\rho : S^\times \to \Delta$ coincides with the barking genus $g_b(Y)$ of Y.

16.7 Constraints on simple crusts

In this section, we will deduce several constraints on simple crusts.

Lemma 16.7.1 *Let* $Y = n_0\Theta_0 + \sum_{j=i}^{N} \mathrm{br}^{(j)}$ *be a simple crust of barking multiplicity* l *of a stellar singular fiber* $X = m_0\Theta_0 + \sum_{j=i}^{N} \mathrm{Br}^{(j)}$. *Then* $ln_0 < m_0$.

Proof. We show this by contradiction. Supposing that $ln_0 = m_0$, then we have $X = lY$ (see Claim 16.7.2 below for a proof). Next let τ be a core section of Y, that is, a meromorphic section of $N_0^{\otimes n_0}$ such that $\mathrm{div}(\tau) = -\sum_j n_1^{(j)} p_1^{(j)} + D$ where $p_1^{(j)}$ is the intersection point of a branch $\mathrm{Br}^{(j)}$ and the core Θ_0, and D is a nonnegative divisor on Θ_0. We claim that $D = 0$. To see this, we first note that τ^l is a section of $N_0^{\otimes(ln_0)}$; indeed, since $m_0 = ln_0$, we have $N_0^{\otimes(ln_0)} = N^{\otimes m_0}$. From this with $\deg N_0 = \dfrac{\sum_j m_1^{(j)}}{m_0}$, we derive an equation of degrees:

$$\sum_j ln_1^{(j)} + l \deg D = \sum_j m_1^{(j)}.$$

Since $ln_1^{(j)} = m_1^{(j)}$ (from $X = lY$), we have $l \deg D = 0$, and so $\deg D = 0$. Since D is a nonnegative divisor, this implies that $D = 0$. Thus we have $\mathrm{div}(\tau) = -\sum_j n_1^{(j)} p_1^{(j)}$, so that

$$\mathrm{div}(\tau^{-l}) = \sum_j ln_1^{(j)} p_1^{(j)} = \sum_j m_1^{(j)} p_1^{(j)}.$$

Consequently $\sigma := \tau^{-l}$ is the standard section of X (i.e. σ is a holomorphic section of the line bundle $N_0^{\otimes(-m_0)}$ on Θ_0 with a zero of order $m_1^{(j)}$ at each $p_1^{(j)}$). However, "$X = lY$ ($l \geq 2$) and $\sigma = \tau^{-l}$" implies that any smooth fiber of $\pi : M \to \Delta$ is disconnected. In fact, from $\sigma = \tau^{-l}$ and $m_0 = ln_0$, the equation $\sigma\zeta^{m_0} - s = 0$ (the defining equation of the original degeneration around the core) is written as $\tau^{-l}\zeta^{ln_0} - s = 0$, which admits a factorization:

$$\prod_{k=1}^{l} \left(\tau^{-1}\zeta^{n_0} - \sqrt[l]{s} e^{2\pi \mathrm{i} k/l}\right) = 0.$$

This shows that any smooth fiber of $\pi : M \to \Delta$, when restricted to a neighborhood of the core Θ_0, is a disjoint union of l curves; from which it follows immediately that any smooth fiber of $\pi : M \to \Delta$ consists of l disjoint curves. This contradicts that any smooth fiber of $\pi : M \to \Delta$ is connected. Hence we conclude that $ln_0 < m_0$. □

We now give a proof of the claim used in the proof of the above lemma.

Claim 16.7.2 *Let* $Y = n_0\Theta_0 + \sum_{j=i}^{N} \mathrm{br}^{(j)}$ *be a simple crust of barking multiplicity* l *of a stellar singular fiber* $X = m_0\Theta_0 + \sum_{j=i}^{N} \mathrm{Br}^{(j)}$. *If* $ln_0 = m_0$, *then* $X = lY$.

Proof. Recall the inequality in Lemma 3.4.1, p52:

$$\frac{n_1^{(1)} + n_1^{(2)} + \cdots + n_1^{(N)}}{n_0} \geq \frac{m_1^{(1)} + m_1^{(2)} + \cdots + m_1^{(N)}}{m_0} \ (= r_0).$$

So

$$\frac{ln_1^{(1)} + ln_1^{(2)} + \cdots + ln_1^{(N)}}{ln_0} \geq \frac{m_1^{(1)} + m_1^{(2)} + \cdots + m_1^{(N)}}{m_0}.$$

If $ln_0 = m_0$, then we have

$$ln_1^{(1)} + ln_1^{(2)} + \cdots + ln_1^{(N)} \geq m_1^{(1)} + m_1^{(2)} + \cdots + m_1^{(N)}. \tag{16.7.1}$$

On the other hand, $ln_1^{(j)} \leq m_1^{(j)}$ holds for $j = 1, 2, \ldots, N$; because lY is a subdivisor of X. Hence the inequality of (16.7.1) is actually an equality, and $ln_1^{(j)} = m_1^{(j)}$ holds for $j = 1, 2, \ldots, N$. Now for each j, two equations $ln_1^{(j)} = m_1^{(j)}$ and $ln_0 = m_0$ together with

$$\frac{n_{i-1}^{(j)} + n_{i+1}^{(j)}}{n_i^{(j)}} = \frac{m_{i-1}^{(j)} + m_{i+1}^{(j)}}{m_i^{(j)}} \ (= r_i^{(j)}), \qquad i = 1, 2, \ldots, \lambda_j - 1$$

imply that $ln_i^{(j)} = m_i^{(j)}$ $(j = 1, 2 \ldots, \lambda_j)$, and so $l \cdot \mathrm{br}^{(j)} = \mathrm{Br}^{(j)}$. From this with the assumption $ln_0 = m_0$, we conclude that $X = lY$. □

Next we note

Lemma 16.7.3 *Let* $Y = n_0\Theta_0 + \sum_{j=i}^{N} \mathrm{br}^{(j)}$ *be a simple crust of barking multiplicity* l *of a stellar singular fiber* $X = m_0\Theta_0 + \sum_{j=i}^{N} \mathrm{Br}^{(j)}$. *Then* Y *does not have such a subbranch* $\mathrm{br}^{(j)}$ *as* $l \cdot \mathrm{br}^{(j)} = \mathrm{Br}^{(j)}$.

In fact, from Lemma 16.7.1, we have $ln_0 < m_0$, and so $l \cdot \mathrm{br}^{(j)} \neq \mathrm{Br}^{(j)}$.

Next recall that a subbranch $\mathrm{br}^{(j)}$ of type AB_l satisfies $l \cdot \mathrm{br}^{(j)} = \mathrm{Br}^{(j)}$. Thus, as a consequence of the above lemma, we have the following result.

Corollary 16.7.4 *Let* $Y = n_0\Theta_0 + \sum_{j=i}^{N} \mathrm{br}^{(j)}$ *be a simple crust of barking multiplicity* l *of a stellar singular fiber* $X = m_0\Theta_0 + \sum_{j=i}^{N} \mathrm{Br}^{(j)}$. *Then none of subbranches of* Y *is of type* AB_l.

In contrast, if X is constellar, Y may have a subbranch of type AB_l. See Example 19.3.4, p337.

Now let $X = m_0\Theta_0 + \sum_{j=i}^{N} \mathrm{Br}^{(j)}$ be a stellar singular fiber, and let σ be the standard section of X, that is, σ is a holomorphic section of the line bundle $N_0^{\otimes(-m_0)}$ on the core Θ_0 with a zero of order $m_1^{(j)}$ at each $p_1^{(j)}$, where $p_1^{(j)}$ is the intersection point of a branch $\mathrm{Br}^{(j)}$ and the core Θ_0. Assume that $Y = n_0\Theta_0 + \sum_{j=i}^{N} \mathrm{br}^{(j)}$ is a simple crust of barking multiplicity l of X, and let τ be a core section of Y, that is, τ is a meromorphic section of the line bundle $N_0^{\otimes n_0}$ on Θ_0 with a pole of order $n_1^{(j)}$ at each $p_1^{(j)}$. We denote by $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$ a barking family associated with Y. Among the singularities of a singular fiber $X_{s,t} := \Psi^{-1}(s,t)$, we will later describe those near the core Θ_0 (Chapter 21, p383); the plot function $K(z) := n_0 \dfrac{d\sigma(z)}{dz}\tau(z) + m_0\sigma(z)\dfrac{d\tau(z)}{dz}$, defined on Θ_0, will play an essential role (we often simply write $K(z) = n_0\sigma_z\tau + m_0\sigma\tau_z$). We now deduce some constraint related to the plot function $K(z)$.

Lemma 16.7.5 *Let $Y = n_0\Theta_0 + \sum_{j=i}^{N} \mathrm{br}^{(j)}$ be a simple crust of barking multiplicity l of a stellar singular fiber $X = m_0\Theta_0 + \sum_{j=i}^{N} \mathrm{Br}^{(j)}$. Let σ be the standard section of X, and let τ be a core section of Y. If the plot function $K(z) = n_0\sigma_z\tau + m_0\sigma\tau_z$ is identically zero, then $aX = bY$, where a and b are the relatively prime positive integers satisfying $am_0 = bn_0$.* (In particular, all subbranches of Y are proportional.)

Proof. We note that

$$n_0\sigma_z\tau + m_0\sigma\tau_z = 0 \iff n_0\frac{\sigma_z}{\sigma} + m_0\frac{\tau_z}{\tau} = 0 \iff \frac{d\log(\sigma^{n_0}\tau^{m_0})}{dz} = 0$$
$$\iff \log(\sigma^{n_0}\tau^{m_0}) \text{ is constant.} \qquad (16.7.2)$$

Thus $\sigma^{n_0}\tau^{m_0}$ is constant and clearly nonzero. Let $p_1^{(j)}$ be the intersection point of a branch $\mathrm{Br}^{(j)}$ and the core Θ_0. Then $\sigma^{n_0}\tau^{m_0}$ is locally of the form $(z^{m_1^{(j)}})^{n_0}\left(\dfrac{1}{z^{n_1^{(j)}}}\right)^{m_0} h$, that is, $z^{m_1^{(j)}n_0 - n_1^{(j)}m_0}h$ around $p_1^{(j)}$, where $h = h(z)$ is a non-vanishing holomorphic function. Since $\sigma^{n_0}\tau^{m_0}$ is constant, we deduce that $m_1^{(j)}n_0 - n_1^{(j)}m_0 = 0$ and h is constant. Hence $\dfrac{m_1^{(j)}}{n_1^{(j)}} = \dfrac{m_0}{n_0}$ for $j = 1, 2, \ldots, N$. Let a and b be the relatively prime positive integers satisfying $\dfrac{m_0}{n_0} = \dfrac{b}{a}$. Then (i) $am_0 = bn_0$, and with the above equation $\dfrac{m_1^{(j)}}{n_1^{(j)}} = \dfrac{m_0}{n_0}$, we have (ii) $am_1^{(j)} = bn_1^{(j)}$ for $j = 1, 2, \ldots, N$. Further, (ii) implies that (iii) $a \cdot \mathrm{Br}^{(j)} = b \cdot \mathrm{br}^{(j)}$ for $j = 1, 2, \ldots, N$. From (i) and (iii), we conclude that $aX = bY$. □

We also have the following result.

Lemma 16.7.6 *Let $Y = n_0\Theta_0 + \sum_{j=i}^{N} \mathrm{br}^{(j)}$ be a simple crust of barking multiplicity l of a stellar singular fiber $X = m_0\Theta_0 + \sum_{j=i}^{N} \mathrm{Br}^{(j)}$. Let σ be the standard section of X, and let τ be a core section of Y. If the core Θ_0 is the projective line, then the plot function $K(z) = n_0\sigma_z\tau + m_0\sigma\tau_z$ is never identically zero.* (In general, this statement is not valid for $\mathrm{genus}(\Theta_0) \geq 1$.)

Proof. By contradiction. If $K(z) = n_0\sigma_z\tau + m_0\sigma\tau_z$ is identically zero, then by Lemma 16.7.5, we have $aX = bY$, and so $X = \frac{b}{a}Y$ where a and b are the relatively prime positive integers satisfying $am_0 = bn_0$. We separate into two cases: (1) $a = 1$ and (2) $a \geq 2$. If $a = 1$, then $X = bY$ ($b \geq 2$). But this implies, as we saw in the proof of Lemma 16.7.1, that any smooth fiber of $\pi : M \to \Delta$ is disconnected (a contradiction!). Next we consider the case $a \geq 2$. Since a and b are relatively prime, from $X = \frac{b}{a}Y$, the positive integer a must divide all multiplicities of Y. We write $n_0 = an_0'$ and $n_i^{(j)} = a{n_i^{(j)}}'$. Then the isomorphism $N^{\otimes n_0} \cong \mathcal{O}_{\Theta_0}(-\sum_{j=1}^N n_1^{(j)}p_1^{(j)})$ is rewritten as

$$(N^{\otimes n_0'})^a \cong \mathcal{O}_{\Theta_0}\big(-a\sum_{j=1}^N {n_1^{(j)}}'p_1^{(j)}\big).$$

From this, we deduce $N^{\otimes n_0'} \cong \mathcal{O}_{\Theta_0}(-\sum_{j=1}^N {n_1^{(j)}}'p_1^{(j)})$; because any line bundle on the projective line Θ_0 is uniquely determined by its degree. We now take an a-th root $\tau^{1/a}$ of τ: It is a meromorphic section of $N^{\otimes n_0'}$ with a pole of order ${n_1^{(j)}}'$ at each $p_1^{(j)}$. Since $K(z) = n_0\sigma_z\tau + m_0\sigma\tau_z$ is identically zero by assumption, we have $\sigma^{n_0}\tau^{m_0} = c$ (constant) as in (16.7.2), and so $\sigma = c^{1/n_0}\tau^{-m_0/n_0} = c^{1/n_0}\tau^{-b/a}$. Thus $\sigma = \text{const}\cdot\mu^b$, where we set $\mu := \tau^{-1/a}$. However, again as in the proof of Lemma 16.7.1, this implies that any smooth fiber of $\pi : M \to \Delta$ is disconnected, consisting of b disjoint curves (a contradiction!). Therefore, if the core Θ_0 is the projective line, then the plot function $K(z) = n_0\sigma_z\tau + m_0\sigma\tau_z$ is never identically zero. □

In the above proof, we used the fact that two line bundles L_1 and L_2 on Θ_0 satisfying $L_1^{\otimes a} \cong L_2^{\otimes a}$ (a is a positive integer) are isomorphic: $L_1 \cong L_2$. Unless Θ_0 is the projective line, this is no longer true, and the above lemma fails; $m_0\sigma\tau_z + n_0\sigma_z\tau$ may be identically zero. This occurs, for example, when the core Θ_0 is of genus ≥ 1, and

$$X = 15\Theta_0 + \textstyle\sum_{j=1}^6 5\Theta_1^{(j)}, \quad Y = 6\Theta_0 + \sum_{j=1}^6 2\Theta_1^{(j)}, \quad \text{and} \quad l = 1 \text{ or } 2.$$

In this case, $a = 2$, $b = 5$, and $2X = 5Y$.

17

Compound barking (Stellar Case)

17.1 Crustal sets

Let $\pi : M \to \Delta$ be a degeneration with a stellar singular fiber

$$X = m_0\Theta_0 + \sum_{j=1}^{N} \mathrm{Br}^{(j)},$$

where Θ_0 is the core and $\mathrm{Br}^{(j)} = m_1^{(j)}\Theta_1^{(j)} + m_2^{(j)}\Theta_2^{(j)} + \cdots + m_{\lambda_j}^{(j)}\Theta_{\lambda_j}^{(j)}$ is a branch. We consider a set of crusts $\mathbf{Y} = \{Y_1, Y_2, \ldots, Y_l\}$ of X, where

$$Y_k = n_{k,0}\Theta_0 + \sum_{j=1}^{N} \mathrm{br}_k^{(j)} \qquad \text{for } k = 1, 2, \ldots, l.$$

Then $\mathbf{Y}$ is called a *crustal set* if for each j ($j = 1, 2, \ldots, N$), the set $\mathbf{br}^{(j)} = \{\mathrm{br}_1^{(j)}, \mathrm{br}_2^{(j)}, \ldots, \mathrm{br}_l^{(j)}\}$ of subbranches is a bunch — this means that if $\mathrm{br}_k^{(j)} \in \mathbf{br}^{(j)}$ satisfies $\mathrm{length}(\mathrm{br}_k^{(j)}) < \mathrm{length}(\mathrm{Br}_k^{(j)})$, then $\mathrm{br}_k^{(j)}$ is tame (Definition 14.2.2, p257). A crustal set $\mathbf{Y}$ is *dominant* if for each j, the bunch $\mathbf{br}^{(j)}$ is dominant; recall that a bunch is dominant if there is no bunch containing it.

For a moment, we fix j and omit it. Suppose that $\mathbf{br} = \{\mathrm{br}_1, \mathrm{br}_2, \ldots, \mathrm{br}_l\}$ is a set of dominant subbranches ($\mathbf{br}$ is *not* necessarily a bunch). We shall introduce an operation, which associates the set $\mathbf{br}$ with a bunch. Firstly we write $\mathrm{br}_k \in \mathbf{br}$ as

$$\mathrm{br}_k = n_{k,1}\Theta_1 + n_{k,2}\Theta_2 + \cdots + n_{k,e_k}\Theta_{e_k},$$

and put $e := \min\{e_k : \mathrm{br}_k \text{ is wild}\}$. Take a shorter subbranch

$$\mathrm{br}'_k = n_{k,1}\Theta_1 + n_{k,2}\Theta_2 + \cdots + n_{k,e}\Theta_e,$$

and then it is easy to see that a set $\mathbf{br}' = \{\mathrm{br}'_1, \mathrm{br}'_2, \ldots, \mathrm{br}'_l\}$ is a dominant bunch. This functor, which associates $\mathbf{br}$ with $\mathbf{br}'$, is called the *cut-off operation*.

In order to generalize this operation to that for a set of crusts, we need some preparation. Letting $Y = n_0\Theta_0 + \sum_{j=1}^{N} \mathrm{br}^{(j)}$ be a crust, if all $\mathrm{br}^{(j)}$ $(\neq \emptyset)$ are dominant, we say that Y is a *dominant crust*.

Lemma 17.1.1 *Any crust is contained in a unique dominant crust.*

Proof. In fact, for a crust $Y = n_0\Theta_0 + \sum_{j=1}^{N} \mathrm{br}^{(j)}$, we let $\mathrm{br}^{(j)}_{\mathrm{dom}}$ be the unique dominant subbranch containing $\mathrm{br}^{(j)}$ (Proposition 5.4.5, p92). Then $Y_{\mathrm{dom}} := n_0\Theta_0 + \sum_{j=1}^{N} \mathrm{br}^{(j)}_{\mathrm{dom}}$ is a unique dominant crust containing Y. □

Now suppose that $\mathbf{Y} = \{Y_1, Y_2, \ldots, Y_l\}$ is a set of dominant crusts, and we write $Y_k = n_{k,0}\Theta_{k,0} + \sum_{j=1}^{N} \mathrm{br}^{(j)}$. By definition, each $\mathbf{br}^{(j)} := \{\mathrm{br}^{(j)}_1, \mathrm{br}^{(j)}_2, \ldots, \mathrm{br}^{(j)}_l\}$ is a set of dominant subbranches. We then apply the cut-off operation to $\mathbf{br}^{(j)}$ to obtain a dominant bunch

$$\mathbf{br}^{(j)\prime} = \{\mathrm{br}^{(j)\prime}_1, \mathrm{br}^{(j)\prime}_2, \ldots, \mathrm{br}^{(j)\prime}_l\}.$$

Set $Y'_k := n_{k,0}\Theta_{k,0} + \sum_{j=1}^{N} \mathrm{br}^{(j)\prime}$, and then $\mathbf{Y}' := \{Y'_1, Y'_2, \ldots, Y'_l\}$ is a dominant crustal set. The functor which associates a set $\mathbf{Y}$ of dominant crusts with a dominant crustal set $\mathbf{Y}'$ is also called a *cut-off operation*. We summarize the functors:

$$\begin{aligned} \text{a set of crusts} \quad &\overset{(1)}{\longmapsto} \quad \text{a set of dominant crusts} \\ &\overset{(2)}{\longmapsto} \quad \text{a dominant crustal set,} \end{aligned}$$

where (1) is given by Lemma 17.1.1 and (2) is the cut-off operation.

17.2 Deformation atlas associated with a crustal set

Suppose that $\mathbf{Y} = \{Y_1, Y_2, \ldots, Y_l\}$ is a crustal set, and $\mathbf{d} = \{d_1, d_2, \ldots, d_l\}$ is a set of arbitrary positive integers. Then we say that

$$(\mathbf{Y}, \mathbf{d}) := \{(Y_1, d_1), (Y_2, d_2), \ldots, (Y_l, d_l)\}$$

is a *weighted crustal set* and $\mathbf{d}$ is its *weight*. We will associate $(\mathbf{Y}, \mathbf{d})$ with a deformation atlas $DA_{\mathbf{e-1}}(\mathbf{Y}, \mathbf{d})$ of size $\mathbf{e} - \mathbf{1} = (e_1 - 1, e_2 - 1, \ldots, e_l - 1)$, where e_j is the length of the bunch $\mathbf{br}^{(j)}$;

$$e_j := \max\{\, \mathrm{length}(\mathrm{br}^{(j)}_1),\ \mathrm{length}(\mathrm{br}^{(j)}_2),\ \ldots,\ \mathrm{length}(\mathrm{br}^{(j)}_l)\,\}.$$

The construction of $DA_{\mathbf{e-1}}(\mathbf{Y}, \mathbf{d})$ proceeds as follows:
Step 1. Let τ_k be a core section of the crust Y_k $(k = 1, 2, \ldots, l)$, i.e. τ_k is a meromorphic section of the line bundle $N_0^{\otimes n_{k,0}}$ on Θ_0 with a pole of order

$n_{k,1}$ at $p_1^{(j)}$, and is holomorphic outside $\{p_1^{(j)}\}$. We then define a hypersurface in $N_0 \times \Delta \times \Delta^\dagger$ by

$$\mathcal{W}_0 : \quad \sigma\zeta^{m_0} - s + \sum_{k=1}^{l} t^{d_k}\, \sigma\, \tau_k\, \zeta^{m_0 - n_{k,0}} = 0, \quad \text{(Lemma 15.3.2, p272)}, \tag{17.2.1}$$

where σ is the standard section of the singular fiber X — a holomorphic section of $N_0^{\otimes(-m_0)}$ such that

$$\operatorname{div}(\sigma) = m_1^{(1)} p_1^{(1)} + m_1^{(2)} p_1^{(2)} \cdots + m_1^{(N)} p_1^{(N)}.$$

Step 2. Next we construct a deformation atlas associated with each bunch $\mathbf{br}^{(j)} = \{\mathrm{br}_1^{(j)}, \mathrm{br}_2^{(j)}, \dots, \mathrm{br}_l^{(j)}\}$. For simplicity, fix j and omit the superscripts (j). We express

$$\sigma = z^{m_1} g \quad \text{and} \quad \tau_k = \frac{h_k}{z^{n_{k,1}}} \qquad \text{around} \quad p_1,$$

where g and h_k are non-vanishing holomorphic functions, and then the equation of $\mathcal{W}_0$ is locally

$$z^{m_1}\, \zeta^{m_0}\, g - s + \sum_{k=1}^{l} t^{d_k}\, z^{m_1 - n_1}\, \zeta^{m_0 - n_{k,0}}\, g\, h_k \qquad \text{around} \;\; p_1.$$

We simplify this equation by a coordinate change; replacing $\zeta g^{1/m_0}$ with ζ', then

$$z^{m_1}\, (\zeta')^{m_0} - s + \sum_{k=1}^{l} t^{d_k}\, z^{m_1 - n_1}\, (\zeta')^{m_0 - n_{k,0}}\, f_k, \qquad \text{where} \;\; f_k := g^{n_{k,0}/m_0}\, h_k.$$

To clarify the subsequent discussion, we now put the subscripts, such as $z = z_0$ and $\zeta' = \zeta_0$:

$$z_0^{m_1}\, \zeta_0^{m_0} - s + \sum_{k=1}^{l} t^{d_k}\, z_0^{m_1 - n_1}\, \zeta_0^{m_0 - n_{k,0}}\, f_k(z_0).$$

By a coordinate change $(z_0, \zeta_0) = (\eta_1, w_1)$, this equation becomes

$$w_1^{m_0}\, \eta_1^{m_1} - s + \sum_{k=1}^{l} t^{d_k}\, w_1^{m_0 - n_0}\, \eta_1^{m_1 - n_{k,1}}\, f_k(\eta_0). \tag{17.2.2}$$

Next define a sequence of integers p_i $(i = 0, 1, \dots, \lambda_j + 1)$ inductively by

$$\begin{cases} p_0 = 0, \quad p_1 = 1 \quad \text{and} \\ p_{i+1} = r_i p_i - p_{i-1} \quad \text{for } i = 1,\, 2,\, \dots,\, \lambda_j. \end{cases}$$

Then $p_{\lambda_j+1} > p_{\lambda_j} > \cdots > p_1 > p_0 = 0$ (6.2.4), p105, and for the holomorphic function f_k in (17.2.2), we consider a sequence of holomorphic functions:

$$f_{k,i} = f_k(w^{p_{i-1}}\eta^{p_i}) \qquad \text{and} \qquad \widehat{f}_{k,i} = f_k(z^{p_{i+1}}\zeta^{p_i}) \qquad \text{(see (6.2.7), p106).}$$

We then define: for $i = 1, 2, \ldots, e_j - 1$,

$$\begin{cases} \mathcal{H}_i^{(j)}: & w^{m_{i-1}}\,\eta^{m_i} - s + \sum_{k=1}^{l} t^{d_k}\, w^{m_{i-1}-n_{k,i-1}}\,\eta^{m_i-n_{k,i}}\, f_{k,i} = 0 \\ \mathcal{H}_i^{(j)\prime}: & z^{m_{i+1}}\,\zeta^{m_i} - s + \sum_{k=1}^{l} t^{d_k}\, z^{m_{i+1}-n_{k,i+1}}\,\zeta^{m_i-n_{k,i}}\, \widehat{f}_{k,i} = 0 \\ g_i^{(j)}: & \text{the transition function } z = 1/w,\ \zeta = w^{r_i}\eta \text{ of } N_i. \end{cases} \tag{17.2.3}$$

By Lemma 14.2.3, p257, this data gives a deformation atlas $DA^{(j)}_{e_j-1}$ for the branch $\mathrm{Br}^{(j)}$ such that $\mathcal{H}_1^{(j)}$ is given by (17.2.2). Thus $\mathcal{W}_0$ and $DA^{(j)}_{e_j-1}$ ($j = 1, 2 \ldots, N$) together constitute a deformation atlas $DA_{\mathbf{e-1}}(\mathbf{Y}, \mathbf{d})$, which is referred to as a *deformation atlas associated with a crustal set* $\mathbf{Y}$.

Theorem 17.2.1 *Let $\pi : M \to \Delta$ be a degeneration with a stellar singular fiber X, and let $DA_{\mathbf{e-1}}(\mathbf{Y}, \mathbf{d})$ be the deformation atlas associated with a crustal set $\mathbf{Y} = \{Y_1, Y_2, \ldots, Y_l\}$ of X; the weight $\mathbf{d}$ is arbitrary. If $\mathbf{br}^{(j)} := \{\mathrm{br}_1^{(j)}, \mathrm{br}_2^{(j)}, \ldots \mathrm{br}_l^{(j)}\}$ satisfies*

(1) $\mathbf{br}^{(j)}$ *is tame* (i.e. *all* $\mathrm{br}_k^{(j)}$ $(k = 1, 2, \ldots, l)$ *are tame*), *or*
(2) $\mathrm{length}(\mathbf{br}^{(j)}) = \mathrm{length}(\mathrm{Br}^{(j)})$ *and* $m^{(j)}_{\lambda_j} = 1$,

then $DA_{\mathbf{e-1}}(\mathbf{Y}, \mathbf{d})$ admits a complete propagation.

Proof. According to whether (1) or (2) holds, applying Theorem 14.2.7, p260 or Theorem 14.2.8, p260, we see that $DA^{(j)}_{e_j-1}$, the deformation atlas given by (17.2.3), admits a complete propagation along the branch $\mathrm{Br}^{(j)}$. Thus $DA_{\mathbf{e-1}}(\mathbf{Y}, \mathbf{d})$ admits a complete propagation. □

17.3 Example of a crustal set

We consider a degeneration of elliptic curves with the singular fiber X described in Figure 17.3.1 (X is II^* in Kodaira's notation), and we take three points on Θ_0: $p_1 = 1$, $p_2 = 0$ and $p_3 = \infty$.

Let us take a weighted crustal set $(\mathbf{Y}, \mathbf{d}) = \{(Y_1, 1), (Y_2, 2), (Y_3, 4)\}$ as in Figure 17.3.2; obviously, $\mathbf{Y}$ is dominant. For $a, b, c \in \mathbb{C}$, we define a hypersurface $\mathcal{W}_0$ by

$$\begin{cases} \mathcal{H}_0: & w^5(w-1)^3\eta^6 - s + atw^4(w-1)^3\eta^5 + bt^2w^4(w-1)^2\eta^4 \\ & \qquad + ct^4w^2(w-1)\eta^2 = 0 \\ \mathcal{H}_0': & z^4(1-z)^3\zeta^6 - s + atz^3(1-z)^3\zeta^5 + bt^2z^2(1-z)^2\zeta^4 \\ & \qquad + ct^4z(1-z)\zeta^2 = 0 \\ g_0: & z = \dfrac{1}{w}, \quad \zeta = w^2\eta. \end{cases}$$

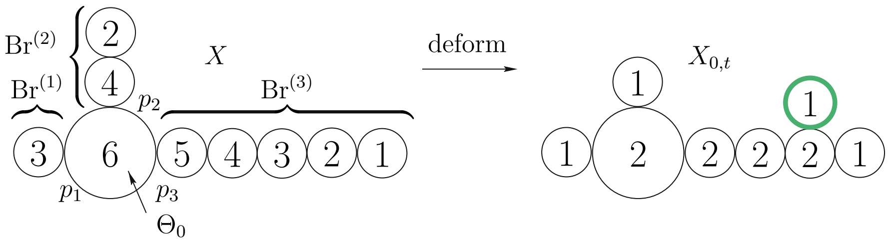


Fig. 17.3.1.

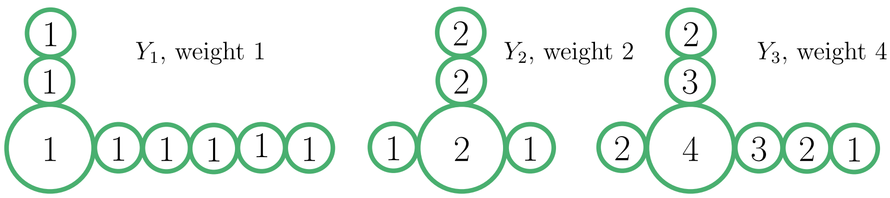


Fig. 17.3.2.

Notice the following:

(i) A bunch $\mathbf{br}^{(1)} = \{\mathrm{br}_2^{(1)}, \mathrm{br}_3^{(1)}\}$ (note $\mathrm{br}_1^{(1)} = \emptyset$) is tame, and so $DA(\mathbf{Y}, \mathbf{d})$ admits a complete propagation along the branch $\mathrm{Br}^{(1)}$ by Theorem 14.2.7, p260.

(ii) A bunch $\mathbf{br}^{(3)} = \{\mathrm{br}_1^{(3)}, \mathrm{br}_2^{(3)}, \mathrm{br}_3^{(3)}\}$ is wild. However $m_5^{(3)} = 1$ and the length of $\mathbf{br}^{(3)}$ is equal to the length of $\mathrm{Br}^{(3)}$, and thus from Theorem 14.2.8, p260 it follows that $DA(\mathbf{Y}, \mathbf{d})$ admits a complete propagation along the branch $\mathrm{Br}^{(3)}$.

Thus it remains to construct a complete propagation along the branch $\mathrm{Br}^{(2)}$. (Note: $\mathbf{br}^{(2)}$ satisfies neither (1) nor (2) of Theorem 17.2.1, and we cannot apply Theorem 17.2.1.) In a new coordinate $\zeta = (1-z)^{-1/2}\zeta'$ near $z = 0$, the hypersurface $\mathcal{H}_0'$ is written as

$$\mathcal{H}_0' : \quad z^4(\zeta')^6 - s + atz^3(\zeta')^5(1-z)^{1/2} + bt^2z^2(\zeta')^4 + ct^4z(\zeta')^2 = 0,$$

and by a coordinate change $(z, \zeta') = (\eta, w)$, we have

$$\mathcal{H}_1^{(2)} : \quad w^6\eta^4 - s + atw^5\eta^3(1-\eta)^{1/2} + bt^2w^4\eta^2 + ct^4w^2\eta = 0.$$

We take $a = 2\alpha$, $b = \alpha^2$ and $c = -\alpha^4$, and then $\mathcal{H}_1^{(2)}$ (and so $DA(\mathbf{Y}, \mathbf{d})$) admits a complete propagation along the branch $\mathrm{Br}^{(2)}$ (see §14.3, p260).

18

Deformations of Tubular Neighborhoods of Trunks

In this chapter, we introduce a degeneration such that its singular fiber is a "trunk" and a smooth fiber is a disjoint union of $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$ (hence non-compact), and then we will construct its various deformations.

18.1 Trunks

Suppose that a sequence of positive integers $\mathbf{m} = (m_0, m_1, \dots, m_{\lambda+1})$ satisfies:

$$r_i := \frac{m_{i-1} + m_{i+1}}{m_i} \quad (i = 1, 2, \dots, \lambda) \quad \text{is an integer greater than 1.}$$

We then construct a degeneration of "non-compact" complex curves. Take λ copies $\Theta_1, \Theta_2, \dots, \Theta_\lambda$ of the projective line. Let $\Theta_i = U_i \cup U_i'$ $(i = 1, 2, \dots, \lambda)$ be the standard open covering by two complex lines with coordinates $z_i \in U_i'$ and $w_i \in U_i$ such that $z_i = 1/w_i$ on $U_i \cap U_i'$, and we consider a line bundle $N_i = \mathcal{O}_{\Theta_i}(-r_i)$ on Θ_i, which is obtained by gluing $(z_i, \zeta_i) \in U_i' \times \mathbb{C}$ with $(w_i, \eta_i) \in U_i \times \mathbb{C}$ by

$$z_i = \frac{1}{w_i}, \quad \zeta_i = w_i^{r_i} \eta_i.$$

We construct a complex surface M from $N_1, N_2, \dots, N_\lambda$ by plumbing, that is, identify N_i with N_{i+1} $(i = 1, 2, \dots, \lambda - 1)$ by $(z_i, \zeta_i) = (\eta_{i+1}, w_{i+1})$. Next we define a holomorphic map $\pi : M \to \Delta\, (= \mathbb{C})$ as follows: For $i = 1, 2, \dots, \lambda$,

$$\begin{cases} \pi(w_i, \eta_i) = w_i^{m_{i-1}} \eta_i^{m_i} \\ \pi(z_i, \zeta_i) = z_i^{m_{i+1}} \zeta_i^{m_i}. \end{cases}$$

It is easy to check that π is well-defined; it is compatible with the patchings of M. Then a smooth fiber of $\pi : M \to \Delta$ consists of k annuli, where $k = \gcd(m_0, m_1, \dots, m_\lambda)$. On the other hand, the singular fiber is, as shown in Figure 18.1.1,

$$X = \pi^{-1}(0) = m_0 \Delta_0 + m_1 \Theta_1 + m_2 \Theta_2 + \cdots + m_\lambda \Theta_\lambda + m_{\lambda+1} \Delta_{\lambda+1},$$

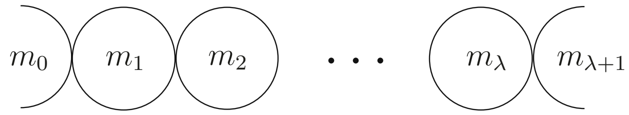


Fig. 18.1.1. A trunk X

where $\Delta_0 = \mathbb{C}$ and $\Delta_{\lambda+1} = \mathbb{C}$, while Θ_i $(i = 1, 2, \ldots, \lambda)$ are projective lines such that Θ_i and Θ_{i+1} (resp. Δ_0 and Θ_1, and also $\Delta_{\lambda+1}$ and Θ_λ) intersect transversely at one point (in later chapters we will sometimes shrink M so that Δ_0 and $\Delta_{\lambda+1}$ become small open disks.) Note that Δ_0 (resp. $\Delta_{\lambda+1}$) is a fiber of the line bundle N_1 (resp. N_λ) over $0 \in U_1$ (resp. $0 \in U'_\lambda$). We say that X is a *trunk*.

Remark 18.1.1 More precisely, X is a *fringed trunk* where $m_0\Delta_0$ and $m_{\lambda+1}\Delta_{\lambda+1}$ are *fringes*, whereas $m_1\Theta_1 + m_2\Theta_2 + \cdots + m_\lambda\Theta_\lambda$ is an *unfringed trunk*. However for brevity, if no fear of confusion, an unfringed/fringed trunk is simply called a trunk. Whenever we need to distinguish unfringed and fringed trunks, we use the notations Tk and $\overline{\text{Tk}}$ respectively for them. Possibly, $\lambda = 0$ in which case $\overline{\text{Tk}} = m_0\Delta_0 + m_1\Delta_1$ and Tk is just one point (the intersection of Δ_0 and Δ_1).

For the remainder of this chapter, $\pi : M \to \Delta$ is assumed to be a degeneration whose singular fiber is a trunk. We shall define the notion of a complete deformation atlas for the trunk. It consists of the following data: For $i = 1, 2, \ldots, \lambda$,

$$\begin{cases} \mathcal{H}_i : & \text{a deformation of } H_i \ : \ w^{m_{i-1}}\eta^{m_i} - s = 0, \\ \mathcal{H}'_i : & \text{a deformation of } H'_i \ : \ z^{m_{i+1}}\zeta^{m_i} - s = 0, \text{ and} \\ g_i : & \text{a deformation of the transition function } z = 1/w, \ \zeta = w^{r_i}\eta \text{ of } N_i \end{cases}$$

such that (1) g_i transforms $\mathcal{H}_i$ to $\mathcal{H}'_i$ and (2) (possibly after expressing $\mathcal{H}_i$ and $\mathcal{H}'_i$ in some coordinates), $\mathcal{H}'_i$ becomes $\mathcal{H}_{i+1}$ by a coordinate change $(z_i, \zeta_i) = (\eta_{i+1}, w_{i+1})$. Given a complete deformation atlas, we can construct a *barking family* of $\pi : M \to \Delta$ by patching $\mathcal{H}_i$ and $\mathcal{H}'_i$ by g_i for $i = 1, 2, \ldots, \lambda$ and then by patching $\mathcal{H}'_i$ and $\mathcal{H}_{i+1}$ by plumbing $(z_i, \zeta_i) = (\eta_{i+1}, w_{i+1})$ for $i = 1, 2, \ldots, \lambda - 1$.

Let l be a positive integer such that $m_i - ln_i \geq 0$ for $i = 0, 1, \lambda, \lambda + 1$, and let $f = f(z)$ and $h = h(z)$ be holomorphic functions near the origin. Then we define smooth hypersurfaces in $U_1 \times \mathbb{C} \times \Delta \times \Delta^\dagger$ and $U'_\lambda \times \mathbb{C} \times \Delta \times \Delta^\dagger$ respectively by

$$\mathcal{H}_1(f) : \quad w^{m_0 - ln_0}\eta^{m_1 - ln_1}\left(w^{n_0}\eta^{n_1} + t^d f(\eta)\right)^l - s = 0 \tag{18.1.1}$$

$$\mathcal{H}'_\lambda(h) : \quad z^{m_{\lambda+1} - ln_{\lambda+1}}\zeta^{m_\lambda - ln_\lambda}\left(z^{n_{\lambda+1}}\zeta^{n_\lambda} + t^d h(\zeta)\right)^l - s = 0, \tag{18.1.2}$$

where d is a positive integer. We would like to consider an "initial deformation problem":

Problem 18.1.2 *Given holomorphic functions f and h, when does there exist a complete deformation atlas $\{\mathcal{H}_i, \mathcal{H}'_i, g_i\}_{i=1,2,\ldots,\lambda}$ such that $\mathcal{H}_1 = \mathcal{H}_1(f)$* (18.1.1) *and $\mathcal{H}'_\lambda = \mathcal{H}'_\lambda(h)$* (18.1.2) ?

18.2 Subtrunks, I

Let $X = m_0\Delta_0 + m_1\Theta_1 + \cdots + m_\lambda\Theta_\lambda + m_{\lambda+1}\Delta_{\lambda+1}$ be a trunk, so that

$$r_i := \frac{m_{i-1} + m_{i+1}}{m_i}, \qquad i = 1, 2, \ldots, \lambda$$

is an integer greater than 1. We say that the *length* of X is $\lambda + 1$. We suppose that $Y = n_0\Delta_0 + n_1\Theta_1 + \cdots + n_l\Theta_l$ $(l < \lambda+1)$ is a subdivisor of the trunk X; so $0 < n_i \le m_i$ holds for each i. Then l is the *length* of Y; on the other hand, when $Y = n_0\Delta_0 + n_1\Theta_1 + \cdots + n_\lambda\Theta_\lambda + n_{\lambda+1}\Delta_{\lambda+1}$, the length of Y is $\lambda + 1$.

Using some special subdivisors of the trunk X, we intend to construct complete deformation atlases for X. In this section, we exclusively consider subdivisors Y of X such that length(Y) = length(X); another case length$(Y) <$ length(X) will be treated in the next section.

Definition 18.2.1 A subdivisor $Y = n_0\Delta_0 + n_1\Theta_1 + \cdots + n_\lambda\Theta_\lambda + n_{\lambda+1}\Delta_{\lambda+1}$ of a trunk $X = m_0\Delta_0 + m_1\Theta_1 + \cdots + m_\lambda\Theta_\lambda + m_{\lambda+1}\Delta_{\lambda+1}$ is called a *subtrunk* if for some integer e $(1 \le e \le \lambda)$,

$$\frac{n_{i-1} + n_{i+1}}{n_i} = r_i, \qquad i = 1,\ 2, \ldots,\ e-1,\ e+1, \ldots, \lambda.$$

(Note: It may occur that $\dfrac{n_{e-1} + n_{e+1}}{n_e} = r_e$.)

Now we introduce a deformation atlas associated with a subtrunk $Y = n_0\Delta_0 + n_1\Theta_1 + \cdots + n_\lambda\Theta_\lambda + n_{\lambda+1}\Delta_{\lambda+1}$. First we define a sequence of integers p_i $(i = 0, 1, \ldots, \lambda+1)$ inductively by

$$\begin{cases} p_0 = 0, \quad p_1 = 1 \quad \text{and} \\ p_{i+1} = r_i p_i - p_{i-1} \quad \text{for } i = 1,\ 2,\ \ldots,\ \lambda. \end{cases}$$

Then $p_{\lambda+1} > p_\lambda > \cdots > p_1 > p_0 = 0$ (6.2.4), p105. Letting $f = f(z)$ and $h = h(z)$ be non-vanishing holomorphic functions defined on a domain $\{z \in \mathbb{C} : |z| < \varepsilon\}$, we then construct a sequence of holomorphic functions: For $i = 1, 2, \ldots, \lambda$,

$$\begin{cases} f_i(w,\eta) := f(\, w^{p_{i-1}}\, \eta^{p_i}\,) & \text{on } \Omega_i \\ \widehat{f}_i(z,\zeta) := f(\, z^{p_{i+1}}\, \zeta^{p_i}\,) & \text{on } \widehat{\Omega}_i \\ h_i(w,\eta) := h(\, w^{p_{\lambda+2-i}}\, \eta^{p_{\lambda+1-i}}\,) & \text{on } \Gamma_i \\ \widehat{h}_i(z,\zeta) := h(\, z^{p_{\lambda-i}}\, \zeta^{p_{\lambda+1-i}}\,) & \text{on } \widehat{\Gamma}_i, \end{cases} \tag{18.2.1}$$

where domains Ω_i, Ω_i', Γ_i, and Γ_i' are respectively given by

$$\Omega_i := \{(w,\eta) \in U_i \,:\, |\, w^{p_{i-1}}\eta^{p_i} \,| < \varepsilon\}, \quad \widehat{\Omega}_i := \{(z,\zeta) \in U_i' \,:\, |\, z^{p_{i+1}}\zeta^{p_i} \,| < \varepsilon\}$$
$$\Gamma_i := \{(w,\eta) \in U_i \,:\, |\, w^{p_{\lambda+2-i}}\, \eta^{p_{\lambda+1-i}} \,| < \varepsilon\},$$
$$\widehat{\Gamma}_i := \{(z,\zeta) \in U_i' \,:\, |\, z^{p_{\lambda-i}}\, \zeta^{p_{\lambda+1-i}} \,| < \varepsilon\}.$$

Let l be a positive integer such that $lY \leq X$, and let k be an arbitrary positive integer. We define a deformation atlas $DA(lY, k)$ with

$$\begin{cases} \mathcal{H}_1 = \mathcal{H}_1(f): & w^{m_0 - ln_0}\eta^{m_1 - ln_1}\big(w^{n_0}\eta^{n_1} + t^k f(\eta)\big)^l - s = 0 \\ \mathcal{H}_\lambda' = \mathcal{H}_\lambda'(h): & z^{m_{\lambda+1} - ln_{\lambda+1}}\zeta^{m_\lambda - ln_\lambda}\big(z^{n_{\lambda+1}}\zeta^{n_\lambda} + t^k h(\zeta)\big)^l - s = 0 \end{cases}$$

as follows (in the below, $f_i, \widehat{f}_i, h_i, \widehat{h}_i$ are as in (18.2.1)): For $i = 1, 2, \ldots, e-1$,

$$\begin{cases} \mathcal{H}_i: & w^{m_{i-1} - ln_{i-1}}\eta^{m_i - ln_i}(w^{n_{i-1}}\eta^{n_i} + t^k f_i)^l - s = 0 \\ \mathcal{H}_i': & z^{m_{i+1} - ln_{i+1}}\zeta^{m_i - ln_i}(z^{n_{i+1}}\zeta^{n_i} + t^k \widehat{f}_i)^l - s = 0 \\ g_i: & \text{the transition function } z = 1/w,\ \zeta = w^{r_i}\eta \text{ of } N_i, \end{cases} \tag{18.2.2}$$

and for $i = e+1, e+2, \ldots, \lambda$,

$$\begin{cases} \mathcal{H}_i: & w^{m_{i-1} - ln_{i-1}}\eta^{m_i - ln_i}(w^{n_{i-1}}\eta^{n_i} + t^k h_i)^l - s = 0 \\ \mathcal{H}_i': & z^{m_{i+1} - ln_{i+1}}\zeta^{m_i - ln_i}(z^{n_{i+1}}\zeta^{n_i} + t^k \widehat{h}_i)^l - s = 0 \\ g_i: & \text{the transition function } z = 1/w,\ \zeta = w^{r_i}\eta \text{ of } N_i. \end{cases} \tag{18.2.3}$$

It is easy to check that g_i $(i = 1, 2, \ldots, e-1, e+1, \ldots \lambda)$ transforms $\mathcal{H}_i$ to $\mathcal{H}_i'$, and so $DA(lY, k)$ is well-defined, however, we note that a priori $DA(lY, k)$ is *not* complete; it is not defined for $i = e$.

Next we introduce important notions "subtrunks of types A_l and B_l", for which $DA(lY, k)$ admits an e-th (and hence a complete) propagation.

Definition 18.2.2 Let $Y = n_0\Delta_0 + n_1\Theta_1 + \cdots + n_\lambda\Theta_\lambda + n_{\lambda+1}\Delta_{\lambda+1}$ be a subtrunk of a trunk X, and so for some e $(1 \leq e \leq \lambda)$,

$$\frac{n_{i-1} + n_{i+1}}{n_i} = r_i, \qquad i = 1,\, 2, \ldots,\, e-1,\, e+1, \ldots, \lambda.$$

Let l be a positive integer.

Type A_l The subtrunk Y is of *type* A_l if $lY \leq X$ and $\dfrac{n_{e-1} + n_{e+1}}{n_e} = r_e$ (hence $\dfrac{n_{i-1} + n_{i+1}}{n_i} = r_i$ for all $i = 1, 2, \ldots, \lambda$).

Type B_l The subtrunk Y is of *type* B_l if $lY \leq X$, $m_e = l$ and $n_e = 1$.

For a trunk $X = m_0\Delta_0 + m_1\Delta_1$ $(\Delta_0 = \Delta_1 = \mathbb{C})$ of length 1, we adopt the convention that a subdivisor $Y = n_0\Delta_0 + n_1\Delta_1$ is *of type* A_l if $lY \leq X$ (that is, $ln_0 \leq m_0$ and $ln_1 \leq m_1$).

Now we give respective examples of types A_l and B_l.

Example (type A_l) $\mathbf{m} = (14, 13, 12, 11, 10)$. Then $\mathbf{n} = (2, 2, 2, 2, 2)$ is of type A_l where $l = 2, 3, 4$ or 5.

Example (type B_l) $\mathbf{m} = (6, 5, 4, 3, 2, 3, 4)$ and $\mathbf{n} = (1, 1, 1, 1, 1, 1, 1)$ where $e = 4$. Then $\mathbf{n}$ is of type B_2, because $m_4 = 2$ and $n_4 = 1$.

We note that there is a subtrunk both of type A_l and B_l (called of *type* AB_l).

Example (type AB_l) $l = 2$, $\mathbf{m} = (8, 6, 4, 2, 4, 6)$ and $\mathbf{n} = (4, 3, 2, 1, 2, 3)$, where $e = 3$.

A subtrunk Y is of type AB_l exactly when $X = lY$ and $n_e = 1$; such Y is of the form

$$Y = n_0\Delta_0 + n_1\Theta_1 + \cdots + n_{a-1}\Theta_{a-1} + \Theta_a + \Theta_{a+1} + \cdots + \Theta_b \\ + n_{b+1}\Theta_{b+1} + \cdots + n_{\lambda+1}\Delta_{\lambda+1},$$

where the multiplicity of Θ_i ($a \le i \le b$) is 1, and we may take as e an arbitrary integer between a and b.

We now return to the construction of complete deformation atlases. We say that a subtrunk Y of a trunk X is *proportional* if

$$\frac{m_0}{n_0} = \frac{m_1}{n_1} = \cdots = \frac{m_{\lambda+1}}{n_{\lambda+1}}.$$

Lemma 18.2.3 *Let $Y = n_0\Delta_0 + n_1\Theta_1 + \cdots + n_\lambda\Theta_\lambda + n_{\lambda+1}\Delta_{\lambda+1}$ be a subtrunk of a trunk $X = m_0\Delta_0 + m_1\Theta_1 + \cdots + m_\lambda\Theta_\lambda + m_{\lambda+1}\Delta_{\lambda+1}$, and let l be a positive integer such that $lY \le X$. For non-vanishing holomorphic functions $f = f(z)$ and $h = h(z)$, define a deformation atlas $DA(lY, k)$ by* (18.2.2) *and* (18.2.3). *If Y is of type A_l such that in the proportional case, the following "fringe condition" on f and h holds:*

$$\widehat{f}_\lambda = \widehat{h}_\lambda, \tag{18.2.4}$$

then $DA(lY, k)$ for arbitrary k admits a complete propagation.

Remark 18.2.4 *The condition $\widehat{f}_\lambda = \widehat{h}_\lambda$ is satisfied, for example, if* (1) $f = h$ *identically and* (2) $m_i = m_{\lambda-i}$ ($i = 1, 2, \ldots, [\lambda/2]$) *where* $[\lambda/2]$ *is the greatest integer not exceeding* $\lambda/2$. *See Figure* 19.3.3, p339 *for example.*

Proof. We note

$$\mathcal{H}_e: \quad w^{m_{e-1}-ln_{e-1}}\eta^{m_e-ln_e}(w^{n_{e-1}}\eta^{n_e} + t^k f_e)^l - s = 0$$

$$\mathcal{H}'_e: \quad w^{m_{e-1}-ln_{e-1}}\eta^{m_e-ln_e}(w^{n_{e-1}}\eta^{n_e} + t^k \widehat{h}_e)^l - s = 0.$$

If Y is not proportional, then we may simplify these equations. In fact, since $m_{e-1}n_e - m_e n_{e-1} \ne 0$ and $m_e n_{e+1} - m_{e+1}n_e \ne 0$, we can apply the Simplification Lemma (Lemma 4.1.1, p58); after some coordinate change we may assume that $f_e = \widehat{h}_e = 1$:

$$\begin{aligned} \mathcal{H}_e: &\quad w^{m_{e-1}-ln_{e-1}}\eta^{m_e-ln_e}(w^{n_{e-1}}\eta^{n_e}+t^k)^l - s = 0 \\ \mathcal{H}'_e: &\quad w^{m_{e-1}-ln_{e-1}}\eta^{m_e-ln_e}(w^{n_{e-1}}\eta^{n_e}+t^k)^l - s = 0. \end{aligned}$$

Now take g_e to be the transition function $z = 1/w,\ \zeta = w^{r_e}\eta$ of N_e. Then clearly g_e transforms $\mathcal{H}_e$ to $\mathcal{H}_e$, and we obtain a complete propagation of $DA(lY, k)$.

If Y is proportional, then $\widehat{f}_\lambda = \widehat{h}_\lambda$ by assumption and the following data gives a complete deformation atlas: For $i = 1, 2, \dots, \lambda$,

$$\begin{cases} \mathcal{H}_i: & w^{m_{i-1}-ln_{i-1}}\eta^{m_i-ln_i}(w^{n_{i-1}}\eta^{n_i}+t^k h_i)^l - s = 0 \\ \mathcal{H}'_i: & z^{m_{i+1}-ln_{i+1}}\zeta^{m_i-ln_i}(z^{n_{i+1}}\zeta^{n_i}+t^k \widehat{h}_i)^l - s = 0 \\ g_i: & \text{the transition function } z = 1/w,\ \zeta = w^{r_i}\eta \text{ of } N_i. \end{cases}$$

□

We next consider type B_l.

Lemma 18.2.5 *Let $Y = n_0\Delta_0 + n_1\Theta_1 + \cdots + n_\lambda\Theta_\lambda + n_{\lambda+1}\Delta_{\lambda+1}$ be a subtrunk of a trunk $X = m_0\Delta_0 + m_1\Theta_1 + \cdots + m_\lambda\Theta_\lambda + m_{\lambda+1}\Delta_{\lambda+1}$, and let l be a positive integer such that $lY \le X$. For non-vanishing holomorphic functions $f = f(z)$ and $h = h(z)$, define a deformation atlas $DA(lY, k)$ by* (18.2.2) *and* (18.2.3). *If Y is of type B_l, then $DA(lY, k)$ for arbitrary k admits a complete propagation.*

Proof. From the condition of type B_l, we have $m_e - ln_e = 0$, and so

$$\begin{aligned} \mathcal{H}_e: &\quad w^{m_{e-1}-ln_{e-1}}(w^{n_{e-1}}\eta^{n_e}+t^k f_e)^l - s = 0 \\ \mathcal{H}'_e: &\quad w^{m_{e-1}-ln_{e-1}}(w^{n_{e-1}}\eta^{n_e}+t^k \widehat{h}_e)^l - s = 0. \end{aligned}$$

For simplicity, we write $r = r_e$ and set $a = m_{e-1}$, $b = n_{e-1}$, $c = m_{e+1}$, and $d = n_{e+1}$. Then an equation $m_e r_e = m_{e-1} + m_{e+1}$ is written as

$$lr = a + c, \tag{18.2.5}$$

where we used $m_e = l$ (because Y is of type B_l). We also have

$$\mathcal{H}_e: \quad w^{a-lb}(w^b\eta + t^k f_e)^l - s = 0, \qquad \mathcal{H}'_e: \quad z^{c-ld}(z^d\zeta + t^k\widehat{h}_e)^l - s = 0.$$

We claim that a map[1] $g_e:\ z = 1/w,\ \zeta = w^r\eta - t^k\, w^d\, \widehat{h}_e + t^k\, w^{r-b}\, f_e$ transforms $\mathcal{H}_e$ to $\mathcal{H}'_e$. To see this, rewrite the left hand side of $\mathcal{H}_e$ as

$$w^{a-lb}\left[\frac{1}{w^{r-b}}(w^r\eta) + t^k f_e\right]^l - s,$$

[1] Observe that g_e contains $\widehat{h}_e = \widehat{h}_e(z,\zeta)$, but we can express g_e only in variables w, η and t by the implicit function theorem.

which is transformed by the map g_e to

$$\begin{aligned}
&\frac{1}{z^{a-lb}}\left[z^{r-b}\left(\zeta+t^k\frac{1}{z^d}\,\widehat{h}_e-t^k\frac{1}{z^{r-b}}\,f_e\right)+t^kf_e\right]^l-s\\
&=\frac{1}{z^{a-lb}}\left[z^{r-b}\zeta+t^kz^{r-b-d}\,\widehat{h}_e\right]^l-s\\
&=z^{\,l\,(r-b-d)\,-\,(a-l\,b)}\left(z^d\zeta+t^k\,\widehat{h}_e\right)^l-s\\
&=z^{c-ld}\left(z^d\zeta+t^k\,\widehat{h}_e\right)^l-s,
\end{aligned}$$

where in the last equality, we used

$$\begin{aligned}
l(r-b-d)-(a-lb)&=(lr-a)-ld\\
&=c-ld \qquad\qquad \text{by (18.2.5).}
\end{aligned}$$

This shows that g_e transforms $\mathcal{H}_e$ to $\mathcal{H}'_e$, and we obtain a complete propagation of $DA(lY,k)$. $\square$

In the above lemma, if Y is of type B_l but not of type A_l (so Y is *not* of type AB_l), then we may slightly simplify the proof. In fact, in this case, Y is not proportional and we have $m_{e-1}n_e-m_en_{e-1}\neq 0$ and $m_en_{e+1}-m_{e+1}n_e\neq 0$. Thus we may apply Simplification Lemma (Lemma 4.1.1); after some coordinate change, we may assume that $f_e\equiv 1$ and $\widehat{h}_e\equiv 1$, enabling us to simplify the computation in the above proof.

We summarize Lemma 18.2.3 and Lemma 18.2.5 as follows.

Proposition 18.2.6 *Let $Y=n_0\Delta_0+n_1\Theta_1+\cdots+n_\lambda\Theta_\lambda+n_{\lambda+1}\Delta_{\lambda+1}$ be a subtrunk of a trunk $X=m_0\Delta_0+m_1\Theta_1+\cdots+m_\lambda\Theta_\lambda+m_{\lambda+1}\Delta_{\lambda+1}$, and let l be a positive integer such that $lY\leq X$. For non-vanishing holomorphic functions $f=f(z)$ and $h=h(z)$, define a deformation atlas $DA(lY,k)$ by* (18.2.2) *and* (18.2.3)*. Suppose that Y is of type A_l or B_l such that in the case of proportional type A_l, the fringe condition $\widehat{f}_\lambda=\widehat{h}_\lambda$* (18.2.4) *is satisfied. Then $DA(lY,k)$ for arbitrary k admits a complete propagation.*

If Y is both of type A_l and B_l (type AB_l), then we can construct two different complete propagations of $DA(lY,k)$ by applying the constructions of types A_l and B_l. Accordingly, we obtain two different barking families $\Psi_A:\mathcal{M}_A\to\Delta\times\Delta^\dagger$ and $\Psi_B:\mathcal{M}_B\to\Delta\times\Delta^\dagger$. It is curious that the singular fibers $\Psi_A^{-1}(0,t)$ and $\Psi_B^{-1}(0,t)$ are topologically the same; an irreducible component marked by "m_i-ln_i" in Figure 18.4.2 (3) and Figure 18.4.3 (2) is vacuous if $m_i-ln_i=0$, and hence $X_{0,t}$ in Figure 18.4.2 (3) is topologically the same as $X_{0,t}$ in Figure 18.4.3 (2). But the gluing maps of $\mathcal{M}_A$ and $\mathcal{M}_B$ around Θ_e are different, and so the ambient spaces $\mathcal{M}_A$ and $\mathcal{M}_B$ are different.

18.3 Subtrunks, II

Let $X = m_0\Delta_0 + m_1\Theta_1 + \cdots + m_\lambda\Theta_\lambda + m_{\lambda+1}\Delta_{\lambda+1}$ be a trunk, so that

$$r_i := \frac{m_{i-1} + m_{i+1}}{m_i}, \qquad i = 1, 2, \ldots, \lambda$$

is an integer greater than 1. In this section, we treat a subdivisor Y of X such that $\mathrm{length}(Y) < \mathrm{length}(X)$.

Definition 18.3.1 A subdivisor $Y = n_0\Delta_0 + n_1\Theta_1 + \cdots + n_e\Theta_e$ $(e < \lambda + 1)$ of a trunk X is called a *subtrunk* provided that

$$\frac{n_{i-1} + n_{i+1}}{n_i} = r_i, \qquad i = 1, 2, \ldots, e - 1.$$

Later we will further consider a "disconnected" subtrunk Y consisting of two connected components:

$$Z = m_0\Delta_0 + m_1\Theta_1 + \cdots + m_e\Theta_e \text{ and } Z' = m_f\Theta_f + m_{f+1}\Theta_{f+1} + \cdots + m_{\lambda+1}\Delta_{\lambda+1},$$

where $e < f$ and the following equations hold:

$$\frac{n_{i-1} + n_{i+1}}{n_i} = r_i, \qquad i = 1, 2, \ldots, e - 1, f + 1, f + 2, \ldots, \lambda.$$

However for the moment, to simplify the discussion, we only consider connected Y.

We now introduce important notions "subtrunks of types A_l, B_l, and C_l", which are exactly corresponding to subbranches of types A_l, B_l, and C_l respectively.

Definition 18.3.2 Let $Y = n_0\Delta_0 + n_1\Theta_1 + \cdots + n_e\Theta_e$ be a subtrunk of a trunk $X = m_0\Delta_0 + m_1\Theta_1 + \cdots + m_\lambda\Theta_\lambda + m_{\lambda+1}\Delta_{\lambda+1}$ $(e < \lambda + 1)$, so that Y satisfies

$$\frac{n_{i-1} + n_{i+1}}{n_i} = r_i, \qquad i = 1, 2, \ldots, e - 1.$$

Let l be a positive integer.

Type A_l $\quad Y$ is of *type* A_l if $lY \le X$ and $\frac{n_{e-1}}{n_e} \ge r_e$.
Type B_l $\quad Y$ is of *type* B_l if $lY \le X$, $m_e = l$ and $n_e = 1$.
Type C_l $\quad Y$ is of *type* C_l if $lY \le X$, n_e divides n_{e-1}, $\frac{n_{e-1}}{n_e} < r_e$, and u divides l where $u := (m_{e-1} - ln_{e-1}) - (r_e - 1)(m_e - ln_e)$.

Also we adopt the following convention: A subdivisor Y of length zero ($Y = n_0\Delta_0$) is *of type* A_l if $lY \le X$ (that is, $ln_0 \le m_0$). In this case $DA(lY, k)$ consists only of $\mathcal{H}_1$.

Next we introduce several quantities related to the above types. First, we define the *propagation number* $\rho(Y)$ of a subtrunk Y of type A_l, B_l, or C_l in the same way as that for a subbranch of type A_l, B_l, or C_l (see (16.4.2), p291):

$$\rho(Y) = \begin{cases} e+1 & \text{if } Y \text{ is of type } A_l \\ e & \text{if } Y \text{ is of type } B_l \\ f & \text{if } Y \text{ is of type } C_l. \end{cases} \tag{18.3.1}$$

Here the positive integer f is defined as follows (see also (16.4.1), p290). Recall that for a subbranch $Y = n_0\Delta_0 + n_1\Theta_1 + \cdots + n_e\Theta_e$ of type C_l, the construction of a complete propagation of the deformation atlas $DA_{e-1}(lY, k)$, where k is divisible by n_e, was carried out separately for three cases (Cases I, II, III in §11.3, p191); the resulting deformation is non-trivial around $\Theta_e, \Theta_{e+1}, \ldots, \Theta_f$, and trivial around $\Theta_{f+1}, \Theta_{f+2}, \ldots, \Theta_\lambda$ where the positive integer f in question is given by

$$f = \begin{cases} e + Nn_e - 1 & \text{Case I. } b = 0 \quad \text{or} \quad \text{Case II. } b \geq 1 \text{ and } u > b \\ e + Nn_e + v - 1 & \text{Case III. } b \geq 1 \text{ and } u \leq b. \end{cases} \tag{18.3.2}$$

Here the integers N and v are defined as follows: First, set

$$u := (m_{e-1} - ln_{e-1}) - (r_e - 1)(m_e - ln_e),$$

and then u divides l by the definition of type C_l, and write $l = Nu$, so N is a positive integer. Next we set $b := m_e - ln_e$, and when $b \geq 1$ and $u \leq b$, let v be the integer such that $l - vu \geq 0$ and $l - (v+1)u < 0$.

Another important quantity associated with a subtrunk Y is a slant; when $Y = n_0\Delta_0 + n_1\Theta_1 + \cdots + n_e\Theta_e$ is of type A_l, the integer $q(Y) := n_{e-1} - r_e n_e$ is referred to as the *slant* of the subtrunk Y. From the condition $n_{e-1}/n_e \geq r_e$ of type A_l, we have $q(Y) \geq 0$.

For a moment, let $Y = n_0\Delta_0 + n_1\Theta_1 + \cdots + n_e\Theta_e$ be an arbitrary subtrunk (not necessarily of type A_l, B_l, or C_l) of a trunk $X = m_0\Delta_0 + m_1\Theta_1 + \cdots + m_\lambda\Theta_\lambda + m_{\lambda+1}\Delta_{\lambda+1}$. Letting l be a positive integer satisfying $lY \leq X$, we shall introduce a deformation atlas $DA(lY, k)$, where k is an arbitrary positive integer. First we define a sequence of integers p_i $(i = 0, 1, \ldots, \lambda+1)$ inductively by

$$\begin{cases} p_0 = 0, \quad p_1 = 1 \quad \text{and} \\ p_{i+1} = r_i p_i - p_{i-1} \quad \text{for } i = 1,\, 2,\, \ldots,\, \lambda. \end{cases}$$

Then $p_{\lambda+1} > p_\lambda > \cdots > p_1 > p_0 = 0$ (6.2.4), p105. Letting $f = f(z)$ be a non-vanishing holomorphic function defined on a domain $\{z \in \mathbb{C} : |z| < \varepsilon\}$, we then construct a sequence of holomorphic functions: for $i = 1, 2, \ldots, \lambda$,

$$f_i(w, \eta) := f(\, w^{p_{i-1}}\, \eta^{p_i}\,), \qquad \widehat{f}_i(z, \zeta) := f(\, z^{p_{i+1}}\, \zeta^{p_i}\,). \tag{18.3.3}$$

Then the deformation atlas $DA(lY, k)$ is given as follows: for $i = 1, 2, \ldots, e-1$,

$$\begin{cases} \mathcal{H}_i : & w^{m_{i-1} - ln_{i-1}} \eta^{m_i - ln_i} (w^{n_{i-1}} \eta^{n_i} + t^k f_i)^l - s = 0 \\ \mathcal{H}_i' : & z^{m_{i+1} - ln_{i+1}} \zeta^{m_i - ln_i} (z^{n_{i+1}} \zeta^{n_i} + t^k \widehat{f}_i)^l - s = 0 \\ g_i : & \text{the transition function } z = 1/w, \ \zeta = w^{r_i}\eta \text{ of } N_i. \end{cases} \tag{18.3.4}$$

Proposition 18.3.3 *Let $Y = n_0\Delta_0 + n_1\Theta_1 + \cdots + n_e\Theta_e$ be a subtrunk of a trunk $X = m_0\Delta_0 + m_1\Theta_1 + \cdots + m_\lambda\Theta_\lambda + m_{\lambda+1}\Delta_{\lambda+1}$ $(e < \lambda + 1)$, and let l be a positive integer such that $lY \leq X$. For a non-vanishing holomorphic function $f = f(z)$, define a deformation atlas $DA(lY, k)$ by* (18.3.4). *Then the following statements hold:*

(1) *Suppose that Y is of type A_l, B_l, or C_l such that* (i) $\rho(Y) \leq \text{length}(X) - 1$ *where $\rho(Y)$ is the propagation number of Y and* (ii) *if Y is type C_l the positive integer k is divisible by n_e. Then $DA(lY, k)$ admits a complete propagation with*

$$\begin{cases} \mathcal{H}_1 = \mathcal{H}_1(f): & w^{m_0 - ln_0}\eta^{m_1 - ln_1}\big(w^{n_0}\eta^{n_1} + t^k f(\eta)\big)^l - s = 0 \\ \mathcal{H}'_\lambda: & z^{m_{\lambda+1}}\zeta^{m_\lambda} - s = 0. \end{cases}$$

(2) *Suppose that Y is of type A_l such that* $\text{length}(Y) = \text{length}(X) - 1$. *Then $DA(lY, k)$ admits a complete propagation with*

$$\begin{cases} \mathcal{H}_1 = \mathcal{H}_1(f): & w^{m_0 - ln_0}\eta^{m_1 - ln_1}\big(w^{n_0}\eta^{n_1} + t^k f(\eta)\big)^l - s = 0 \\ \mathcal{H}'_\lambda: & z^{m_{\lambda+1}}\zeta^{m_\lambda} - s + \sum_{i=0}^{l} {}_l\mathrm{C}_i\, t^{ki}\, z^{m_{\lambda+1}+iq}\,\zeta^{m_\lambda - in_\lambda} = 0, \end{cases}$$

where $q = n_{\lambda-1} - r_\lambda n_\lambda$ is the slant of Y. (Note: If $q = 0$, then $\mathcal{H}'_\lambda$ admits a 'factorization' $z^{m_{\lambda+1}}\zeta^{m_\lambda - ln_\lambda}(\zeta^{n_\lambda} + t^k)^l - s = 0$.)

Proof. First we show (1). According to the type (A_l, B_l, or C_l) of Y, we apply the construction of a complete propagation for a subbranch of the corresponding type (A_l, B_l, or C_l), and then we obtain a ρ-th propagation of $DA(lY, k)$ with

$$\mathcal{H}'_\rho: \quad z^{m_{\rho+1}}\zeta^{m_\rho} - s = 0.$$

Clearly $\mathcal{H}'_\rho$ admits a further propagation in the trivial way, and so we obtain a complete propagation with the desired property. Next we show (2) (in this case, the propagation number $\rho(Y)$ equals $\text{length}(X)$). Firstly we set

$$\begin{cases} \mathcal{H}_\lambda: & w^{m_{\lambda-1}}\eta^{m_\lambda} - s + \sum_{i=1}^{l} {}_l\mathrm{C}_i\, t^{ik}\, w^{m_{\lambda-1} - in_{\lambda-1}}\, \eta^{m_\lambda - in_\lambda}\, f_\lambda^i = 0 \\ \mathcal{H}'_\lambda: & z^{m_{\lambda+1}}\zeta^{m_\lambda} - s + \sum_{i=0}^{l} {}_l\mathrm{C}_i\, t^{ik}\, z^{m_{\lambda+1}+iq}\,\zeta^{m_\lambda - in_\lambda}\, \widehat{f}_\lambda^i = 0 \\ g_\lambda: & \text{the transition function } z = 1/w,\ \zeta = w^{r_\lambda}\eta \text{ of } N_\lambda. \end{cases}$$

Since $m_{\lambda+1}n_\lambda - m_\lambda(-q) = m_{\lambda+1}n_\lambda + m_\lambda q > 0$ by $q \geq 0$, we may apply Simplification Lemma (Lemma 4.1.1, p58); after some coordinate change, $\mathcal{H}'_\lambda$ becomes

$$z^{m_{\lambda+1}}\zeta^{m_\lambda} - s + \sum_{i=0}^{l} {}_l C_i\, t^{ki}\, z^{m_{\lambda+1}+iq}\,\zeta^{m_\lambda - in_\lambda} = 0,$$

and therefore $\{\mathcal{H}_i, \mathcal{H}_i, g_i\}_{i=1,2,\ldots,\lambda}$ gives a complete deformation atlas with the desired property. □

With relation to (1) of Proposition 18.3.3, we introduce some terminologies. For a subtrunk Y of type A_l, B_l, or C_l satisfying the assumption of (1), letting $\rho = \rho(Y)$ be the propagation number of Y, we say that the irreducible component Θ_ρ of the trunk X is *semi-rigid* and that Θ_i $(i \geq \rho+1)$ is *rigid* under the barking deformation, which is constructed from the complete deformation atlas in Proposition 18.3.3 (1). We remark that Lemma 16.4.2, p291 indicates the geometric significance of these terminologies.

From the results we thus far obtained in this chapter, we have already proved most of the statements in the following theorem (below, Y is connected except for (3)).

Theorem 18.3.4 *Let Y be a subtrunk of a trunk $X = m_0\Delta_0 + m_1\Theta_1 + \cdots + m_\lambda\Theta_\lambda + m_{\lambda+1}\Delta_{\lambda+1}$, and let l be a positive integer such that $lY \leq X$. Suppose that $f = f(z)$ and $h = h(z)$ are non-vanishing holomorphic functions. According to the length of Y, the following statements hold:*

(1) $\mathrm{length}(Y) = \mathrm{length}(X)$: *Suppose that Y is of type A_l or B_l such that in the case of proportional type A_l, the fringe condition $\widehat{f}_\lambda = \widehat{h}_\lambda$ (18.2.4) on f and h is satisfied. Then $DA(lY,k)$ admits a complete propagation with*

$$\begin{cases} \mathcal{H}_1 = \mathcal{H}_1(f): & w^{m_0 - ln_0}\eta^{m_1 - ln_1}\left(w^{n_0}\eta^{n_1} + t^d f(\eta)\right)^l - s = 0 \\ \mathcal{H}'_\lambda = \mathcal{H}'_\lambda(h): & z^{m_{\lambda+1} - ln_{\lambda+1}}\zeta^{m_\lambda - ln_\lambda}\left(z^{n_{\lambda+1}}\zeta^{n_\lambda} + t^d h(\zeta)\right)^l - s = 0. \end{cases}$$

(2) $\mathrm{length}(Y) < \mathrm{length}(X)$:

(2a) *Suppose that Y is of type A_l, B_l, or C_l such that* (i) $\rho(Y) \leq \mathrm{length}(X) - 1$ *where $\rho(Y)$ is the propagation number of Y and* (ii) *if Y is of type C_l the positive integer k is divisible by n_e. Then $DA(lY,k)$ admits a complete propagation with*

$$\begin{cases} \mathcal{H}_1 = \mathcal{H}_1(f): & w^{m_0 - ln_0}\eta^{m_1 - ln_1}\left(w^{n_0}\eta^{n_1} + t^k f(\eta)\right)^l - s = 0 \\ \mathcal{H}'_\lambda: & z^{m_{\lambda+1}}\zeta^{m_\lambda} - s = 0. \end{cases}$$

(2b) *Suppose that Y is of type A_l and* $\mathrm{length}(Y) = \mathrm{length}(X) - 1$. *Then $DA(lY,k)$ admits a complete propagation with*

$$\begin{cases} \mathcal{H}_1 = \mathcal{H}_1(f): & w^{m_0 - ln_0}\eta^{m_1 - ln_1}\left(w^{n_0}\eta^{n_1} + t^k f(\eta)\right)^l - s = 0 \\ \mathcal{H}'_\lambda: & z^{m_{\lambda+1}}\zeta^{m_\lambda} - s + \sum_{i=0}^{l} {}_l\mathrm{C}_i\, t^{ki}\, z^{m_{\lambda+1} + iq}\, \zeta^{m_\lambda - in_\lambda} = 0, \end{cases}$$

where $q = n_{\lambda-1} - r_\lambda n_\lambda$ is the slant of Y.

(3) *Suppose that Y consists of two connected components Z and Z', where*

$$Z = n_0\Delta_0 + n_1\Theta_1 + \cdots + n_e\Theta_e, \quad Z' = n_f\Theta_f + n_{f+1}\Theta_{f+1} + \cdots + n_{\lambda+1}\Delta_{\lambda+1},$$

$(e < f)$ and $lZ \leq X$ and $l'Z' \leq X$ for positive integers l and l'. If Z (resp. Z') is of type A_l, B_l, or C_l (resp. $A_{l'}$, $B_{l'}$, or $C_{l'}$) and $\rho(Z) < \rho(Z')$, then

the deformation atlas consisting of $DA(lZ,k)$ and $DA(l'Z',k')$ admits a complete propagation with

$$\begin{cases} \mathcal{H}_1 = \mathcal{H}_1(f): \quad w^{m_0 - l n_0}\eta^{m_1 - l n_1}\big(w^{n_0}\eta^{n_1} + t^k f(\eta)\big)^l - s = 0 \\ \mathcal{H}'_\lambda = \mathcal{H}'_\lambda(h): \quad z^{m_{\lambda+1} - l' n_{\lambda+1}}\zeta^{m_\lambda - l' n_\lambda}\big(z^{n_{\lambda+1}}\zeta^{n_\lambda} + t^{k'} h(\zeta)\big)^{l'} - s = 0, \end{cases}$$

where if Z (resp. Z') is of type C_l (resp. $C_{l'}$), we take such k (resp. k') as is divisible by n_e (resp. n'_e).

Proof. (1) and (2) are respectively Proposition 18.2.6, p315 and Proposition 18.3.3, p318. We show (3). First, as in the proof of Proposition 18.3.3, we construct a $\rho(Z)$-th propagation of $DA(lZ,k)$ such that $\mathcal{H}_\rho$ ($\rho := \rho(Z)$) is trivial. Similarly, we construct a $\rho(Z')$-th propagation of $DA(l'Z',k')$ such that $\mathcal{H}_{\rho'}$ ($\rho' := \rho(Z')$) is trivial. Then we propagate these $\rho(Z)$-th and $\rho(Z')$-th propagations trivially to achieve a complete deformation atlas. (For examples of (1) and (2), see Figure 18.4.2, p323, Figure 18.4.3, p324, and Figure 18.4.4, p325. For an example of (3), see Figure 19.3.5, p341.) □

Remark 18.3.5 In (3), the following extreme case is important: $Z = n_0\Delta_0$ and $Z' = n_{\lambda+1}\Delta_{\lambda+1}$ ($\lambda \geq 2$) such that $lZ \leq X$ and $l'Z' \leq X$; in this case Z and Z' are respectively of type A_l and of type $A_{l'}$. Note that $\rho(Z) = 1$ and $\rho(Z') = \lambda$, and since $\lambda \geq 2$, the condition $\rho(Z) < \rho(Z')$ in (3) is fulfilled.

18.4 Other constructions of deformations

From the viewpoint of deformations, trunks are quite different from branches: *In general, a trunk has much more deformations than a branch.* Specifically, let l be a positive integer and let Y be a subbranch of a branch X satisfying $lY \leq X$. Then a deformation atlas $DA(lY,k)$ admits a complete propagation if and only if Y is of type A_l, B_l, or C_l such that if Y is of type C_l the positive integer k is divisible by n_e (Theorem 13.1.1, p236). In contrast, this is *not* the case for trunks; they further admit another type of deformations, as we will see in this section. The difference stems from the fact that the multiplicity sequence of a trunk has different properties from that of a branch. For a branch $X = m_0\Delta_0 + m_1\Theta_1 + \cdots + m_\lambda\Theta_\lambda$, its multiplicity sequence $\mathbf{m} = (m_0, m_1, \ldots, m_\lambda)$ is determined by the division algorithm, and necessarily $m_0 > m_1 > \cdots > m_\lambda$. However, this is not the case for a trunk $X = m_0\Delta_0 + m_1\Theta_1 + \cdots + m_\lambda\Theta_\lambda + m_{\lambda+1}\Delta_{\lambda+1}$; the multiplicity sequence $\mathbf{m} = (m_0, m_1, \ldots, m_{\lambda+1})$ is in general *not* decreasing. For instance, $\mathbf{m} = (6,5,4,3,2,3,4)$. Moreover the same integers may be adjacent in the multiplicity sequence, such as

$$\mathbf{m} = (6,5,4,3,2,2,2,2,3,4) \quad \text{or} \quad (4,3,2,1,1,1).$$

Also the multiplicity sequence of some trunk may be obtained by giving up the division algorithm "halfway." For example, $\mathbf{m} = (32,24,16)$ is contained in $(32,24,16,8)$, where the latter sequence is obtained by the "full" division algorithm, while the former is not. We also note that if $\mathbf{m}$ is the multiplicity sequence of a branch, it is also the multiplicity sequence of some trunk. This means that in some sense the set of trunks is much 'bigger' than that of branches.

Since the construction of deformations of trunks is much affected by the property of their multiplicity sequences, it is conceivable that a similar statement to Proposition 13.3.1, p238 for branches — if the deformation atlas $DA(lY, k)$ admits a complete propagation, then the subbranch Y must be of type A_l, B_l, or C_l — does *not* hold for trunks; because in that proof, inequalities $m_0 > m_1 > \cdots > m_\lambda$ was essentially used.

From the above fact, we expect deformations of trunks, which branches do not have. Actually, they do exist; before we present them, we recall an operation (*Matsumoto–Montesinos bonding*) which yields a trunk from two branches, and conversely any trunk is obtained in this way. Consider two (fringed) branches

$$\overline{\mathrm{Br}}_1 = m_0\Delta_0 + m_1\Theta_1 + \cdots + m_\lambda\Theta_\lambda, \qquad \overline{\mathrm{Br}}_2 = m_0'\Delta_0' + m_1'\Theta_1' + \cdots + m_\nu'\Theta_\nu'.$$

Let κ ($\kappa \geq -1$) be an integer. When $\kappa = -1$, we assume that the following condition on the branches $\overline{\mathrm{Br}}_1$ and $\overline{\mathrm{Br}}_2$ is satisfied: There exists a pair of integers λ_0 and ν_0 ($0 \leq \lambda_0 < \lambda,\ \ 0 \leq \nu_0 < \nu$) satisfying

$$m_{\lambda_0+1} + m'_{\nu_0+1} = m_{\lambda_0} = m'_{\nu_0}. \tag{18.4.1}$$

Then the κ-bonding of the two branches Br_1 and Br_2 yields a trunk $X = X(\kappa)$ (in what follows, we set $m := m_\lambda = m'_\nu$):

If $\kappa \geq 0$,

$$\begin{aligned} X = m_0\Delta_0 + m_1\Theta_1 + \cdots + m_{\lambda-1}\Theta_{\lambda-1} + m\Theta_\lambda + m\Theta_{\lambda+1} + \cdots + m\Theta_{\lambda+\kappa} \\ + m'_{\nu-1}\Theta'_{\nu-1} + m'_{\nu-2}\Theta'_{\nu-2} + \cdots + m'_0\Delta'_0. \end{aligned}$$

If $\kappa = -1$,

$$X = m_0\Delta_0 + m_1\Theta_1 + \cdots + m_{\lambda_0}\Theta_{\lambda_0} + m'_{\nu_0-1}\Theta'_{\nu_0-1} + m'_{\nu_0-2}\Theta'_{\nu_0-2} + \cdots + m'_0\Delta'_0.$$

Observe that if $\kappa \geq 1$, then $\Theta_{\lambda+1} + \Theta_{\lambda+2} + \cdots + \Theta_{\lambda+\kappa-1}$ is a chain of (-2)-curves[2]. For $\kappa = 0, -1$, we have the following result.

Lemma 18.4.1 *Let X be a trunk defined as above for $\kappa = 0$ or -1, and set $r_i := -\Theta_i \cdot \Theta_i \left(= \dfrac{m_{i-1} + m_{i+1}}{m_i}\right)$ where $\Theta_i \cdot \Theta_i$ is the self-intersection number of an irreducible component Θ_i of X.*

(1) *If $\kappa = 0$, then $r_\lambda \geq 3$.* (Hence $\Theta_\lambda \cdot \Theta_\lambda \neq -2$.)
(2) *If $\kappa = -1$, then $r_{\lambda_0} \geq 3$.* (Hence $\Theta_{\lambda_0} \cdot \Theta_{\lambda_0} \neq -2$.)

[2] A (-2)-curve is a projective line with the self-intersection number -2.

Proof. We first show (2). Since $m_{\lambda_0 - 1} > m_{\lambda_0}$ and $m'_{\nu_0 - 1} > m'_{\nu_0} = m_{\lambda_0}$ (18.4.1), we have

$$r_{\lambda_0} := \frac{m_{\lambda_0 - 1} + m'_{\nu_0 - 1}}{m_{\lambda_0}} > \frac{m_{\lambda_0} + m_{\lambda_0}}{m_{\lambda_0}} = 2.$$

Similarly, we can show (1). □

Therefore if $\kappa = 0$ (resp. -1), then any chain of (-2)-curves in the trunk X does not contain the "middle" irreducible component Θ_λ (resp. Θ_{λ_0}) of X; the reason why we care about the existence/position of a chain of (-2)-curves is that the chain will be used for constructing deformations of the trunk at the end in this section.

We remark that for the following cases, we can apply the construction of deformations developed in [Ta,I] to obtain deformations of trunks.

(i) $\kappa \geq 0$ and $m = 1$: In this case, the trunk X contains an irreducible component Θ_e $(\lambda \leq e \leq \lambda + \kappa)$ of multiplicity 1 and $X_{\mathrm{red}} \setminus \Theta_e$ is topologically disconnected, where X_{red} is the underlying reduced curve of X.
(ii) $\kappa \geq 1$: In this case, the trunk X contains adjacent irreducible components Θ_i and Θ_{i+1} with the same multiplicity (i.e. $m_\lambda = m'_\nu$); so X has a multiple node — a *multiple node* (of multiplicity m) is defined by $\{(x, y) \in \mathbb{C}^2 : x^m y^m = 0\}$.

The next example demonstrates that for a trunk X, the barking families associated with subtrunks of types A_l, B_l, and C_l do *not* exhaust such deformations as some subdivisor lY is barked off from X.

Example 18.4.2 In Example 9.4.14, p174, for a branch $X = 32\Delta_0 + 24\Theta_1 + 16\Theta_2 + 8\Theta_3$, we showed that no matter how we choose a weight k, the deformation atlas $DA(lY, k)$, where $Y = 2\Delta_0 + 2\Theta_1$ is a subbranch of X and $l = 12$, admits a second propagation but not a third propagation; so it does not admit a complete propagation. Nevertheless for a trunk $X' = 32\Delta_0 + 24\Theta_1 + 16\Delta_2$, we can use this example to construct a complete propagation of a deformation atlas $DA(lY', k)$ where $Y' = 2\Delta_0 + 2\Theta_1$ is a subtrunk of X' and $l = 12$; indeed, the second propagation of $DA(lY, k)$ gives a complete propagation of $DA(lY', k)$. Note that the subtrunk Y' is none of types A_l, B_l, and C_l.

Next, using "unfringed subtrunks" we provide a curious construction of deformations of a trunk $X = X(\kappa)$, where $\kappa \geq 1$ and

$$\begin{aligned} X = m_0\Delta_0 + m_1\Theta_1 + \cdots + m_{\lambda-1}\Theta_{\lambda-1} + m\Theta_\lambda + m\Theta_{\lambda+1} + \cdots + m\Theta_{\lambda+\kappa} \\ + m'_{\nu-1}\Theta'_{\nu-1} + m'_{\nu-2}\Theta'_{\nu-2} + \cdots + m'_0\Delta'_0. \end{aligned} \tag{18.4.2}$$

First of all, we explain a terminology. A subdivisor $Y = \Theta_d + \Theta_{d+1} + \cdots + \Theta_e$ $(\lambda \leq d < e \leq \lambda + \kappa)$ of the trunk X is called an *unfringed subtrunk* (of *type* B_m); we note that $\Theta_d, \Theta_{d+1}, \ldots, \Theta_e$ are (-2)-curves. The terminology

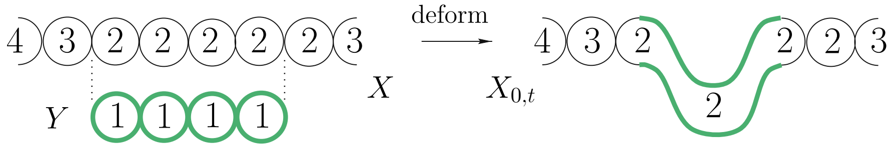


Fig. 18.4.1. Deformation associated with an unfringed subtrunk Y of type B_2

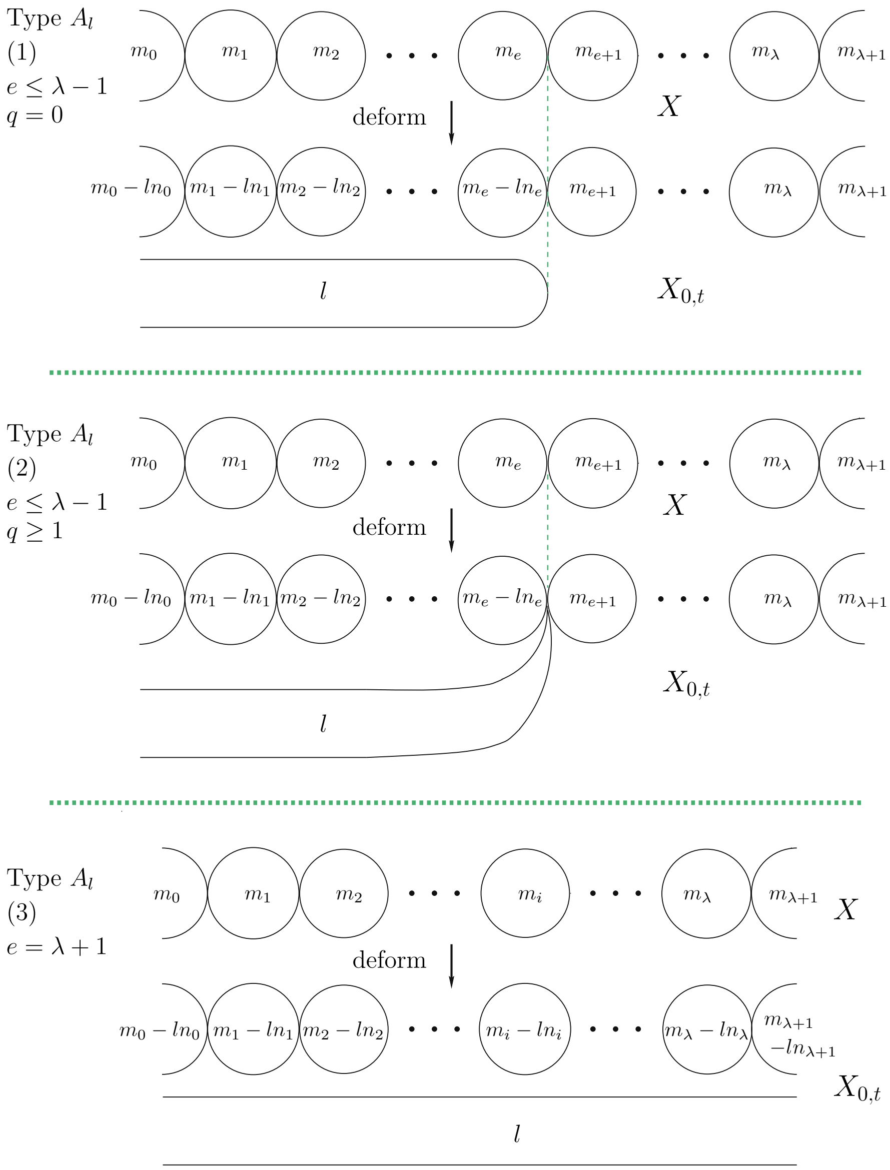


Fig. 18.4.2. Deformation of type A_l.

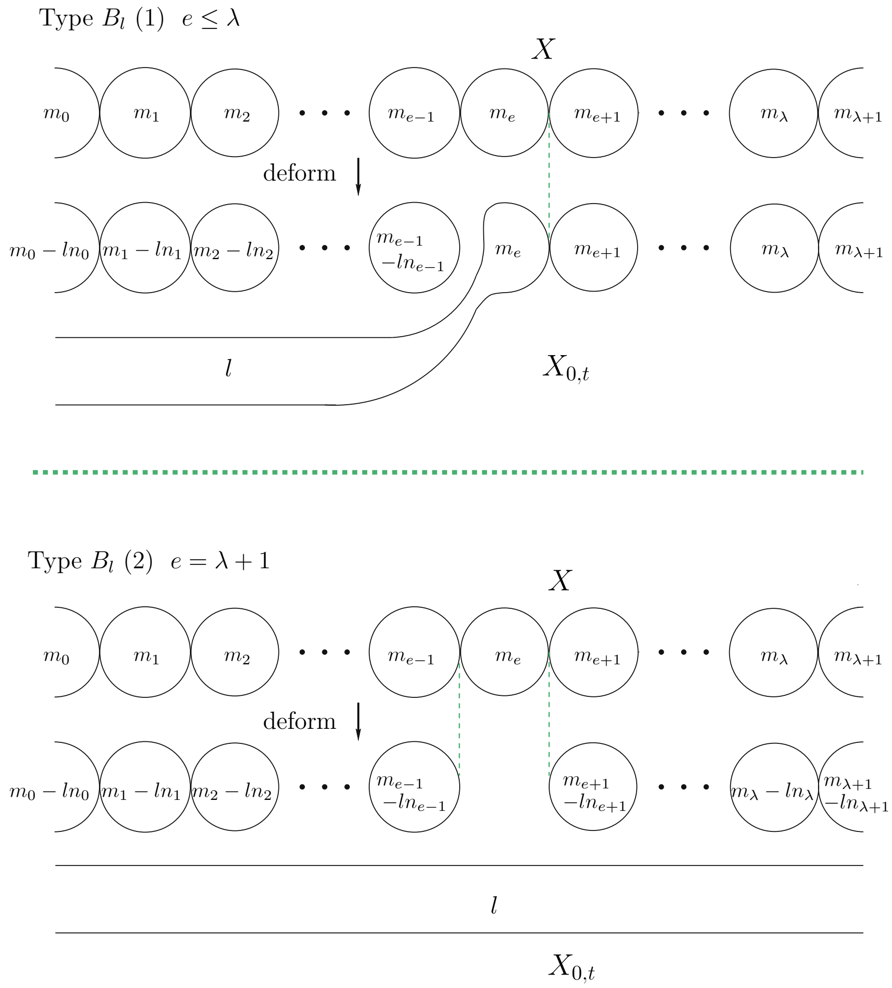


Fig. 18.4.3. Deformation of type B_l.

"type B_m" is derived from the fact that the subsequent construction of a deformation associated with Y is essentially the same as that associated with a subbranch of type B_m.

We require the following lemma.

Lemma 18.4.3 *Consider a map $g : z = 1/w,\ \zeta = w^r\eta + t^k w^{r-1} f$, where k and r are integers satisfying $k \geq 1$ and $r \geq 2$, and $f = f(w, \eta)$ is a holomorphic function. Then g transforms a polynomial $P = (w\eta + t^k f)^m$ to a polynomial $Q = z^{m(r-1)}\zeta^m$.*

Proof. Since

$$P = (w\eta + t^k f)^m = \left[\frac{1}{w^{r-1}}(w^r\eta) + t^k f\right]^m,$$

Type C_l

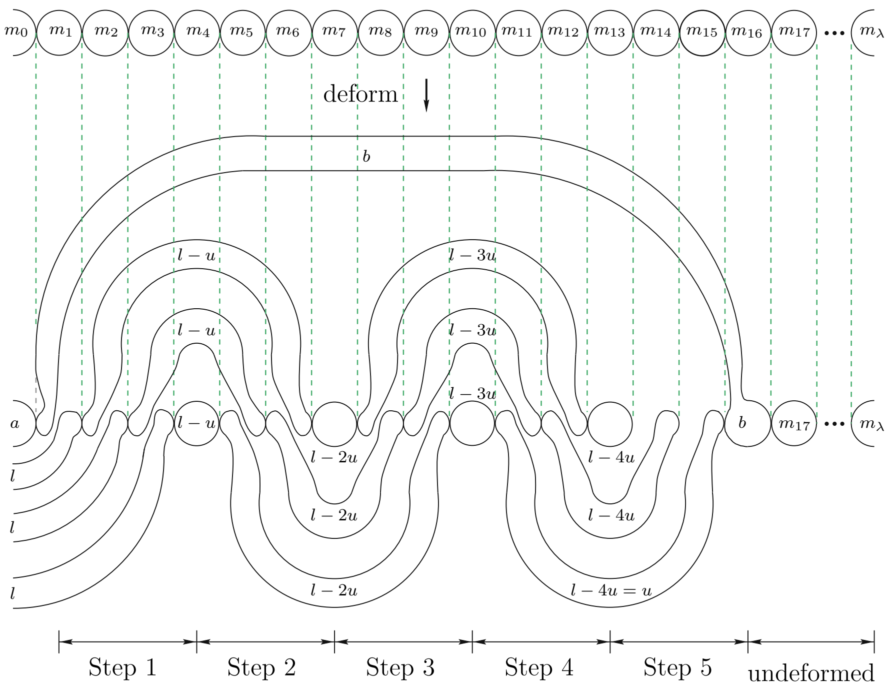


Fig. 18.4.4. Deformation of type C_l. cf. Figure 12.3.2, p223.

the map g transforms P to

$$\left[z^{r-1}\left(\zeta - t^k \frac{1}{z^{r-1}} f\right) + t^k f\right]^m,$$

which is equal to $Q = z^{m(r-1)}\zeta^m$. □

Now letting $Y = \Theta_d + \Theta_{d+1} + \cdots + \Theta_e$ be an unfringed subtrunk of type B_m of the trunk X in (18.4.2), we define a deformation atlas $DA(mY, k)$ as follows: For $i = d+1, d+2, \ldots, e-1$,

$$\begin{cases} \mathcal{H}_i: & (w\eta + t^k f_i)^m - s = 0 \\ \mathcal{H}'_i: & (z\zeta + t^k \widehat{f}_i)^m - s = 0 \\ g_i: & \text{the transition function } z = 1/w,\ \zeta = w^2\eta \text{ of } N_i, \end{cases} \tag{18.4.3}$$

where holomorphic functions f_i and $\widehat{f}_i$ are as in (18.3.3).

Proposition 18.4.4 *Let $Y = \Theta_d + \Theta_{d+1} + \cdots + \Theta_e$ $(d < e)$ be an unfringed subtrunk of type B_m of the trunk X in* (18.4.2)*. Then the deformation atlas given by* (18.4.3) *admits a complete propagation.*

Proof. By definition, $\mathcal{H}_e : (w\eta + t^k f_e)^m - s = 0$, and by Lemma 18.4.3, the following data gives an e-th propagation of $DA(mY, k)$:

$$\begin{cases} \mathcal{H}_e : & (w\eta + t^k f_e)^m - s = 0 \\ \mathcal{H}'_e : & z^{m(r_e-1)}\zeta^m - s = 0 \\ g_i : & z = \dfrac{1}{w}, \quad \zeta = w^{r_e}\eta + t^k w^{r_e-1} f_e. \end{cases}$$

Since $\mathcal{H}'_e$ is the trivial family of H'_e, we can further propagate it trivially for $i = e+1, e+2, \ldots, \lambda+1$. Likewise we can propagate $DA(mY, k)$ trivially for $i = d, d-1, \ldots, 1$. Thus we obtain a complete propagation of $DA(mY, k)$. □

Let $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$ be the barking family obtained from the complete deformation atlas in the above proposition. The deformation from X to $X_{0,t} = \Psi^{-1}(0, t)$ is, for example, shown in Figure 18.4.1.

19

Construction of Barking Deformations (Constellar Case)

The aim of this chapter is to generalize the notions and results for stellar singular fibers to those for constellar singular fibers.

19.1 Notation

Let $\pi : M \to \Delta$ be a normally minimal degeneration of compact complex curves of genus g ($g \geq 1$) with a singular fiber $X = \sum m_i \Theta_i$. Here "normally minimal" means that (1) any singularity of the underlying reduced curve $X_{\mathrm{red}} = \sum_i \Theta_i$ of X is a node (namely, X_{red} has at most normal crossings) and (2) if an irreducible component of X is an exceptional curve (a projective line with the self-intersection number -1), then it intersects other irreducible components at at least three points. By Matsumoto–Montesinos' Theorem [MM2], a degeneration of complex curves becomes normally minimal possibly after successive blow up or down, and moreover a normally minimal degeneration is uniquely determined by the original degeneration.

The topological monodromy γ of a degeneration is either periodic (i.e. $\gamma^n = \mathrm{id}$ for some positive integer n) or pseudo-periodic (i.e. γ^n for some positive integer n is generated by Dehn twists). According to whether γ is periodic or pseudo-periodic, the singular fiber of a normally minimal degeneration is *stellar* (star-shaped) or *constellar* (constellation-shaped). A constellar singular fiber is obtained by bonding stellar ones along their branches — this operation is called *Matsumoto–Montesinos bonding*: See §16.5, p292 and §18.4, p320. We refer to [MM2] and [Ta,II] for more details.

When we study the splittability problem of a degeneration, without loss of generality we may assume that the degeneration is normally minimal, as noted in §1.1, p23. The advantage to treat a normally minimal degeneration is that its singular fiber X is 'inductively' described; if a normally minimal singular fiber X is constellar, then X is obtained from stellar singular fibers of lower genera by means of bonding.

In [Ta,I], we showed the following result.

Criterion 19.1.1 *Let $\pi : M \to \Delta$ be a normally minimal degeneration such that the singular fiber X contains a multiple node, where a multiple node* (*of multiplicity m*) *is defined by $x^m y^m = 0$. Then $\pi : M \to \Delta$ is atomic if and only if X is a reduced curve with one node* (i.e. *Lefschetz fiber*)*.*

If an irreducible component of X is self-intersecting, then its self-intersection point is a multiple node, and so by means of the above criterion, X admits a splitting. For this reason, *for our subsequent discussion, we only consider such singular fibers as all irreducible components Θ_i are smooth.*

We prepare some notation. We denote by N_i the normal bundle of Θ_i in M. For an irreducible component Θ_i of X, if $\Theta_i \cap \Theta_j \neq \emptyset$, then we write

$$\Theta_i \cap \Theta_j = \{p_1^{(ij)}, p_2^{(ij)}, \dots, p_k^{(ij)}\},$$

where $k = k(i,j) := \#(\Theta_i \cap \Theta_j)$ is the number of points of intersection between Θ_i and Θ_j. We next define a divisor $P_j^{(i)}$ on Θ_i by

$$P_j^{(i)} := p_1^{(ij)} + p_2^{(ij)} + \cdots + p_k^{(ij)}. \tag{19.1.1}$$

By Lemma 15.1.1, p265, we have $N_i^{\otimes m_i} \cong \mathcal{O}_{\Theta_i}(-\sum_j m_j P_j^{(i)})$, where the sum runs over all indices j such that $\Theta_i \cap \Theta_j \neq \emptyset$. Thus $N_i^{\otimes(-m_i)}$ has a holomorphic section σ_i such that $\mathrm{div}(\sigma_i) = \sum_j m_j P_j^{(i)}$.

Now we assume that $\pi : M \to \Delta$ is a linear degeneration; so a tubular neighborhood of Θ_i in M is biholomorphic to that of the zero section in N_i. Consider a smooth hypersurface $W_i := \mathrm{Graph}(\pi|_{N_i})$ in $N_i \times \Delta$:

$$W_i : \quad \sigma_i \zeta^{m_i} - s = 0. \tag{19.1.2}$$

We note that if Θ_i intersects Θ_j, then possibly after some coordinate change, W_i becomes a simpler equation $z^{m_j}\zeta^{m_i} - s = 0$ around each intersection point of Θ_i and Θ_j. We then patch hypersurfaces $\{W_i\}$ by plumbings — we glue W_i with W_j by $(z_i, \zeta_i) = (\zeta_j, z_j)$ around each intersection point. This yields a complex 3-manifold:

$$\mathrm{Graph}(\pi) = \{(x,s) \in M \times \Delta \,:\, \pi(x) - s = 0\}.$$

We often identify $\mathrm{Graph}(\pi)$ with M under the projection $(x,s) \in \mathrm{Graph}(\pi) \mapsto x \in M$.

Definition 19.1.2 Let $\pi : M \to \Delta$ be a degeneration with a singular fiber $X = \sum m_i \Theta_i$. A *complete deformation atlas* of X is a set $\{\mathcal{W}_i\}$, where $\mathcal{W}_i$ is a deformation of W_i (see (19.1.2)) parameterized by $\Delta \times \Delta^\dagger$ such that if $\Theta_i \cap \Theta_j \neq \emptyset$, then the equation of $\mathcal{W}_i$ becomes that of $\mathcal{W}_j$ under a coordinate change $(z_i, \zeta_i) = (\zeta_j, z_j)$ around each intersection point of Θ_i and Θ_j.

Given a complete deformation atlas, we may construct a deformation family of the degeneration $\pi : M \to \Delta$ by patching $\{\mathcal{W}_i\}$: When $\Theta_i \cap \Theta_j \neq \emptyset$, we patch $\mathcal{W}_i$ and $\mathcal{W}_j$ by plumbing $(z_i, \zeta_i) = (\zeta_j, z_j)$ around each intersection point of Θ_i and Θ_j. We denote by $\mathcal{M}$ the resulting complex 3-manifold; then the natural projection $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$ is a deformation family of $\pi : M \to \Delta$.

We recall that the "core" of a stellar singular fiber X is the central irreducible component of X, from which branches emanate. We may also define cores — in general, there may be many — for a constellar singular fiber. That is, if a constellar X is obtained by bonding stellar singular fibers $X_1, X_2, \ldots, X_n$, then the core of X_i $(i = 1, 2, \ldots, n)$ becomes, after bonding, a *core* in X (we will give more detailed account in the next section).

We are interested in a deformation family $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$ such that for any core Θ_i of X, the deformation $\mathcal{W}_i$ of W_i is realized as a smooth hypersurface in $N_i \times \Delta \times \Delta^\dagger$. In this case we say that $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$ is a *barking family* of the degeneration $\pi : M \to \Delta$.

19.2 Tensor condition

For the rest of this chapter, unless otherwise stated, $\pi : M \to \Delta$ is a "normally minimal" degeneration; so the singular fiber X is either stellar or constellar. As we already constructed barking families for the stellar case, we will construct those for the constellar case. A constellar singular fiber X is obtained from stellar singular fibers, say $X_1, X_2, \ldots, X_n$, by Matsumoto–Montesinos bonding. In the bonding, some pair of branches of X_i and X_j (or two branches of X_i) is joined to become a *trunk* of X (see §18.4, p320). A branch of X_i not used for making a trunk becomes, after bonding, a *branch* of X. The irreducible component of X corresponding to the core of X_i is referred to as a *core* of X.

We denote an unfringed branch and a fringed branch respectively by Br and $\overline{\mathrm{Br}}$:

$$\begin{aligned} \mathrm{Br} &= m_1\Theta_1 + m_2\Theta_2 + \cdots + m_\lambda\Theta_\lambda, \\ \overline{\mathrm{Br}} &= m_0\Delta_0 + m_1\Theta_1 + m_2\Theta_2 + \cdots + m_\lambda\Theta_\lambda, \end{aligned}$$

where $m_0\Delta_0$ is a fringe; Δ_0 is a small open disk (in a core) around the point at which the branch Br is attached. Likewise an unfringed trunk and a fringed trunk are respectively denoted by Tk and $\overline{\mathrm{Tk}}$:

$$\begin{aligned} \mathrm{Tk} &= m_1\Theta_1 + m_2\Theta_2 + \cdots + m_\lambda\Theta_\lambda, \\ \overline{\mathrm{Tk}} &= m_0\Delta_0 + m_1\Theta_1 + m_2\Theta_2 + \cdots + m_\lambda\Theta_\lambda + m_\lambda\Delta_{\lambda+1}, \end{aligned}$$

where $m_0\Delta_0$ and $m_{\lambda+1}\Delta_{\lambda+1}$ are fringes; Δ_0 and $\Delta_{\lambda+1}$ are small open disks in cores around the points at which the trunk Tk is attached. *We conventionally refer to both unfringed and fringed branches/trunks simply as branches/trunks, if there is no fear of confusion.*

Now we consider a connected subdivisor Y of a constellar singular fiber X — "connectedness" is not essential, but to simplify discussion, we assume it — with the following properties:

(I) At least one irreducible component of Y is a core of X (which is also called a core of Y).
(II) For a branch $\overline{\mathrm{Br}}$ of X, the intersection $\overline{\mathrm{br}} := Y \cap \overline{\mathrm{Br}}$, if non-empty, is a subbranch of $\overline{\mathrm{Br}}$.
(III) For a trunk $\overline{\mathrm{Tk}}$ of X, the intersection $\overline{\mathrm{tk}} := Y \cap \overline{\mathrm{Tk}}$, if non-empty, is a subtrunk of $\overline{\mathrm{Tk}}$.

Now we express $X = \sum m_i\Theta_i$ $(0 < m_i)$ and $Y = \sum n_i\Theta_i$ $(0 \le n_i \le m_i)$. Suppose that an irreducible component Θ_i is a core of X but not that of Y. If Θ_i intersects Y — this is precisely when Θ_i intersects a subtrunk tk of Y at some point —, then we say that Θ_i is *adjacent* to Y. (In this case, $\mathrm{length}(\overline{\mathrm{tk}}) = \mathrm{length}(\overline{\mathrm{Tk}}) - 1$ holds.)

We adopt the following notations:

(i) $\mathrm{Core}(X) = \{\text{cores of } X\}$
(ii) $\mathrm{Core}(Y) = \{\text{cores of } Y\}$
(iii) $\mathrm{Adja}(Y) = \{\Theta_i \in \mathrm{Core}(X) \setminus \mathrm{Core}(Y) \text{ such that } \Theta_i \text{ is adjacent to } Y\}$
(iv) For $\Theta_i \in \mathrm{Adja}(Y)$, if $\mathrm{tk}^{(j)}$ is a subtrunk intersecting Θ_i, then we denote the slant of $\mathrm{tk}^{(j)}$ by $q_i^{(j)}$ $(:= n_{e-1} - r_e n_e)$. See p317.

Recall that for a divisor $D = \sum_k a_k p_k$ on a curve Θ where a_k is an integer and $p_k \in \Theta$, its *support* $\mathrm{Supp}(D)$ is a point set $\{p_k\}$.

Definition 19.2.1 Suppose that $\Theta_i \in \mathrm{Core}(Y)$ or $\mathrm{Adja}(Y)$. Then we say that Θ_i satisfies the *tensor condition* provided that the following conditions are fulfilled:

(T1) If $\Theta_i \in \mathrm{Adja}(Y)$, then any slant $q_i^{(j)}$ in (iv) above does not depend on j. (We then write $q_i = q_i^{(j)}$; it is called the *slant at* $\Theta_i \in \mathrm{Adja}(Y)$.)
(T2) There is a nonnegative divisor D_i on Θ_i such that $\mathrm{Supp}(D_i) \subset \Theta_i \setminus \bigcup_j(\Theta_i \cap \Theta_j)$, where the union runs over all j satisfying $\Theta_j \subset Y$ and $\Theta_i \cap \Theta_j \neq \emptyset$, and

$$\text{(T2.1)} \qquad N_i^{\otimes n_i} \cong \mathcal{O}_{\Theta_i}\Big(-\sum_j n_j P_j^{(i)} + D_i\Big), \qquad \text{if } \Theta_i \in \mathrm{Core}(Y)$$

$$\text{(T2.2)} \qquad N_i^{\otimes(-q_i)} \cong \mathcal{O}_{\Theta_i}\Big(-\sum_j n_j P_j^{(i)} + D_i\Big), \qquad \text{if } \Theta_i \in \mathrm{Adja}(Y),$$

where the divisor $P_j^{(i)}$ is as in (19.1.1). (We call D_i an *auxiliary divisor.*)

Note that (T2.1) (resp. (T2.2)) is equivalent to the existence of a meromorphic section τ_i of $N_i^{\otimes n_i}$ (resp. $N_i^{\otimes(-q_i)}$) such that $\mathrm{div}(\tau_i) = -\sum_j n_j P_j^{(i)} + D_i$. We say that τ_i is a *core section* on Θ_i. (If $q = 0$, then $N_i^{\otimes(-q)} \cong \mathcal{O}_{\Theta_i}$, so that τ_i is a meromorphic *function* on Θ_i.)

Remark 19.2.2 Suppose that $\Theta_i \in \mathrm{Adja}(Y)$ and $q_i = 0$. Then (T2.2) is restated as "$\sum_j n_j P_j^{(i)}$ is linearly equivalent to some nonnegative divisor D_i such that $\mathrm{Supp}(D_i) \subset \Theta_i \setminus \bigcup_j (\Theta_i \cap \Theta_j)$". Observe that if Θ_i is the projective line, this condition is always satisfied.

We say that a subdivisor Y *satisfies the tensor condition* if any $\Theta_i \in \mathrm{Core}(Y)$ or $\mathrm{Adja}(Y)$ satisfies the tensor condition.

Definition 19.2.3 Let Y be a connected subdivisor of a constellar singular fiber X such that at least one irreducible component of Y is a core of X. Then Y is called a *crust* if

(1) Y satisfies the tensor condition,
(2) for a branch $\overline{\mathrm{Br}}$ of X, if $\overline{\mathrm{br}} := Y \cap \overline{\mathrm{Br}} \neq \emptyset$, then $\overline{\mathrm{br}}$ is a subbranch of $\overline{\mathrm{Br}}$,
(3) for a trunk $\overline{\mathrm{Tk}}$ of X, if $\overline{\mathrm{tk}} := Y \cap \overline{\mathrm{Tk}} \neq \emptyset$, then $\overline{\mathrm{tk}}$ is a subtrunk of $\overline{\mathrm{Tk}}$.

Note that (1) is an analytic condition, whereas (2) and (3) are numerical ones.

We need the following technical lemma for our later discussion.

Lemma 19.2.4 *Let q be a nonnegative integer. Suppose that τ_i is a meromorphic section of a line bundle $N_i^{\otimes(-q)}$ on Θ_i such that τ_i has a pole of order n_j $(n_j \geq 0)$ at each intersection point, if any, of Θ_i and Θ_j. Define a smooth hypersurface in $N_i \times \Delta \times \Delta^\dagger$ by*

$$\mathcal{W}_i: \quad \sigma_i \zeta_i^{m_i} - s + \sum_{k=1}^{l} {}_lC_k\, t^{kd}\, \sigma_i\, \tau_i^k\, \zeta_i^{m_i+kq} = 0.$$

Let $p_j^{(i)} \in \Theta_i$ be an intersection point of Θ_i and Θ_j, and then

(1) *after some coordinate change, $\mathcal{W}_i$ is locally given by*

$$z^{m_j} \zeta^{m_i} - s + \sum_{k=1}^{l} {}_lC_k\, t^{kd}\, z^{m_j - n_j}\, \zeta^{m_i+kq} = 0 \qquad \text{around } p_j^{(i)},$$

(2) *if moreover $n_j = 0$, then after some coordinate change, $\mathcal{W}_i$ is locally given by*

$$z^{m_j} \zeta^{m_i} - s = 0 \quad \text{around } p_j^{(i)}.$$

Proof. To show (1), we only have to apply the argument in the proof of the Simplification Lemma (Lemma 4.1.1, p58); since

$$m_i n_j - m_j(-q) = m_i n_j + m_j q > 0,$$

after some coordinate change, $\mathcal{W}_i$ is locally of the form in (1). Next we show (2). If $n_j = 0$, we may write $\mathcal{W}_i$ in (1) as

$$z^{m_j} \zeta^{m_i} \Big(1 + \sum_{k=1}^{l} {}_l\mathrm{C}_k\, t^{kd}\, \zeta^{kq}\Big) - s = 0.$$

By a coordinate change $(z', \zeta') = \left(z\,,\, \zeta\Big(1 + \sum_{k=1}^{l} {}_l\mathrm{C}_k\, t^{kd}\, \zeta^{kq}\Big)^{1/m_i}\right),$ this equation becomes $(z')^{m_j}(\zeta')^{m_i} - s = 0$. Thus (2) is proved. □

19.3 Multiple barking (constellar case)

We introduce a special class of the crusts defined in Definition 19.2.3; this class generalizes the notion of the simple crust (of a stellar singular fiber) to that of a constellar singular fiber.

Definition 19.3.1 Let Y be a crust of a constellar singular fiber X and let l be a positive integer. Then Y is said to be a *simple crust* of *barking multiplicity* l if the following conditions are satisfied:

(1) $lY \leq X$, i.e. lY is a subdivisor of X.
(2) any subbranch of Y is of type A_l, B_l, or C_l.
(3) any subtrunk of Y is of type A_l, B_l, or C_l.
(4) if cores Θ_i and Θ_j of Y (possibly $\Theta_i = \Theta_j$) are joined by a trunk Tk containing *no* subtrunk of Y, then $\mathrm{length}(\overline{\mathrm{Tk}}) \geq 3$ (i.e. the chain of the projective lines of the trunk Tk has length at least 2).

For the reason to pose the condition (4), we refer to Remark 18.3.5, p320.

To simplify our subsequent discussion, we always assume that a simple crust Y is connected. (Actually for many cases, we do not need this assumption; moreover, connected components may have distinct barking multiplicities. See Figure 19.3.5, p341, and also Theorem 18.3.4 (3), p319.)

The main result of this chapter is stated as:

Theorem 19.3.2 *Let $\pi : M \to \Delta$ be a linear degeneration. If the singular fiber X contains a simple crust Y of barking multiplicity l, then $\pi : M \to \Delta$ admits a barking family $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$ such that in the process of the deformation from X to $X_{0,t}$, the subdivisor lY is barked off from X.* (Note: The converse is not true. See §18.4, p320, and in particular Example 18.4.2, p322.)

In order to clarify the argument, we verify Theorem 19.3.2 initially for the case where X is a bonding of *two* stellar singular fibers along one pair of branches, and after that, we will give a proof for the general case: X is a bonding of an arbitrary number of stellar singular fibers. In either case, the main steps in the construction of the barking family in question are

Step 1. For each core of X, construct an "initial deformation" around it,
Step 2. Propagate the initial deformations along branches of X,
Step 3. Propagate the initial deformations along trunks of X.

Proof of Theorem 19.3.2 (Special case)

Suppose that X is a bonding of two stellar singular fibers. We express $X = X_1 + \mathrm{Tk} + X_2$ where Tk is the trunk connecting X_1 and X_2. See Figure 19.3.1 for example. We then express

$$X_1 = m_0\Theta_0 + \sum_i \mathrm{Br}^{(i)}, \qquad X_2 = m_0'\Theta_0' + \sum_j \mathrm{Br}^{(j)\prime},$$

where Θ_0 and Θ_0' are cores, and $\mathrm{Br}^{(i)}$ and $\mathrm{Br}^{(j)\prime}$ are branches of X_1 and X_2 respectively. We now carry out the construction of a complete deformation atlas of X; we separate into two cases according to the number (2 or 1) of the cores of Y.

Case 1. Y has two cores (see the figure on the left in Figure 19.3.1).
We express $Y = Y_1 + \mathrm{tk} + Y_2$, where $Y_1 \le X_1$, $Y_2 \le X_2$, and $\mathrm{tk} \le \mathrm{Tk}$. (Here "$A \le B$" means that A is a subdivisor of B.) We concretely write a subbranch $\mathrm{br}^{(i)}$ of Y_1 as

$$\mathrm{br}^{(i)} = n_1^{(i)}\Theta_1^{(i)} + n_2^{(i)}\Theta_2^{(i)} + \cdots + n_{e_i}^{(i)}\Theta_{e_i}^{(i)},$$

and write a subbranch $\mathrm{br}^{(j)\prime}$ of Y_2 as

$$\mathrm{br}^{(j)\prime} = n_1^{(j)\prime}\Theta_1^{(j)\prime} + n_2^{(j)\prime}\Theta_2^{(j)\prime} + \cdots + n_{e_j}^{(j)\prime}\Theta_{e_j}^{(j)\prime}.$$

For subbranches $\mathrm{br}^{(i)}$ and $\mathrm{br}^{(j)\prime}$, we define positive integers $a(\mathrm{br}^{(i)})$ and $a(\mathrm{br}^{(j)\prime})$ respectively by

$$a(\mathrm{br}^{(i)}) = \begin{cases} n_{e_i}, & \mathrm{br}^{(i)} \text{ is of type } C_l \\ 1, & \text{otherwise} \end{cases}, \quad a(\mathrm{br}^{(j)\prime}) = \begin{cases} n'_{e_j}, & \mathrm{br}^{(j)\prime} \text{ is of type } C_l \\ 1, & \text{otherwise.} \end{cases}$$

We then set $d := \mathrm{lcm}\{a(\mathrm{br}^{(i)}), a(\mathrm{br}^{(j)\prime}\}$, where $\mathrm{br}^{(i)}$ (resp. $\mathrm{br}^{(j)\prime}$) runs over all subbranches of Y_1 (resp. Y_2).

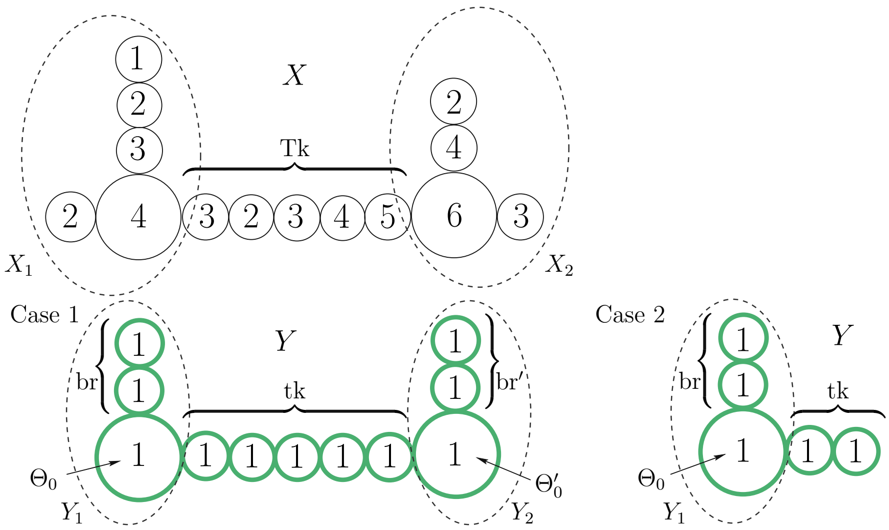


Fig. 19.3.1. genus$(X) = 2$, and the barking multiplicity $l = 3$ for Case 1 and $l = 2$ for Case 2.

Next we define smooth hypersurfaces (*initial deformations*) in $N_0 \times \Delta \times \Delta^\dagger$ and $N_0' \times \Delta \times \Delta^\dagger$ respectively by

$$\mathcal{W}_0 : \quad \sigma \zeta^{m_0} - s + \sum_{k=1}^{l} {}_l\mathrm{C}_k \, t^{kd} \, \sigma \, \tau^k \, \zeta^{m_0 - kn_0} = 0,$$

$$\mathcal{W}_0' : \quad \sigma' \zeta^{m_0} - s + \sum_{k=1}^{l} {}_l\mathrm{C}_k \, t^{kd} \, \sigma' \, (\tau')^k \, \zeta^{m_0' - kn_0'} = 0,$$

where τ and τ' are core sections, that is, meromorphic sections of $N_0^{\otimes n_0}$ and $(N_0')^{\otimes n_0'}$ respectively in the tensor condition (Definition 19.2.1). By Theorem 10.0.15, p177, we can completely propagate $\mathcal{W}_0$ (resp. $\mathcal{W}_0'$) along each branch of X_1 (resp. X_2): if $\mathrm{Br}^{(i)}$ does not contain a subbranch of Y, we apply the construction for subbranches of type A_l. Next by Theorem 18.3.4 (1), p319, we can completely propagate $\mathcal{W}_0$ and $\mathcal{W}_0'$ along the trunk Tk. Therefore we achieve a complete deformation atlas of X.

Case 2. Y **has only one core** (see the figure on the right in Figure 19.3.1). The core of Y is either Θ_0 or Θ_0'. Without loss of generality, we may assume that the core of Y is Θ_0, and we express $Y = Y_1 + \mathrm{tk}$, where $Y_1 \le X_1$ and $\mathrm{tk} \le \mathrm{Tk}$. (Note: $\mathrm{length}(\mathrm{tk}) \le \mathrm{length}(\mathrm{Tk}) - 1$.) Now we express

$$\mathrm{tk} = n_1\Theta_1 + n_2\Theta_2 + \cdots + n_e\Theta_e, \qquad (e < \mathrm{length}(\overline{\mathrm{Tk}})).$$

As in Case 1, we define a positive integer $a(\mathrm{br}^{(i)})$ for each subbranch $\mathrm{br}^{(i)}$ of Y_1 by

$$a(\mathrm{br}^{(i)}) := \begin{cases} n_{e_i}, & \mathrm{br}^{(i)} \text{ is of type } C_l \\ 1, & \text{otherwise,} \end{cases}$$

while we set $a(\mathrm{tk}) := n_e$ for the subtrunk tk. We then set $d := \mathrm{lcm}\{a(\mathrm{br}^{(i)}), a(\mathrm{tk})\}$ where $\mathrm{br}^{(i)}$ runs over all subbranches of Y.

Next we define a smooth hypersurface (an initial deformation) $\mathcal{W}_0$ in $N_0 \times \Delta \times \Delta^\dagger$ by

$$\mathcal{W}_0 : \quad \sigma \zeta^{m_0} - s + \sum_{k=1}^{l} {}_l\mathrm{C}_k \, t^{kd} \, \sigma \, \tau^k \, \zeta^{m_0 - kn_0} = 0,$$

where τ is a core section on Θ_0, that is, the meromorphic section of $N_0^{\otimes n_0}$ in the tensor condition. We also define a smooth hypersurface (an initial deformation) $\mathcal{W}_0'$ in $N_0' \times \Delta \times \Delta^\dagger$ by

$$\mathcal{W}_0' : \begin{cases} \sigma' \zeta^{m_0'} - s + \sum_{k=1}^{l} {}_l\mathrm{C}_k \, t^{kd} \, \sigma' \, \tau^k \, \zeta^{m_i + kq} = 0 & \text{if } \mathrm{length}(\overline{\mathrm{tk}}) = \mathrm{length}(\overline{\mathrm{Tk}}) - 1 \\ \sigma' \zeta^{m_0'} - s = 0 & \text{if } \mathrm{length}(\overline{\mathrm{tk}}) \le \mathrm{length}(\overline{\mathrm{Tk}}) - 2, \end{cases}$$

where $q := n_{e-1} - r_e n_e$ is the slant of tk, and τ' is a core section on Θ'_0. By Theorem 10.0.15, p177, we can completely propagate $\mathcal{W}_0$ along each branch of X_1. Further, by applying the construction for subbranches of type A_l, we can also completely propagate $\mathcal{W}'_0$ along each branch of X_2. It remains to completely propagate $\mathcal{W}_0$ and $\mathcal{W}'_0$ along the trunk Tk. We carry out this as follows: By Lemma 19.2.4, p331, according to the length of tk, possibly after some coordinate change, $\mathcal{W}'_0$ is of the following form around the intersection point $p' = \mathrm{Tk} \cap \Theta'_0$:

$$\begin{cases} z^{m_\lambda}\zeta^{m'_0} - s + \sum_{k=1}^{l} {}_l\mathrm{C}_k \, t^{kd} \, z^{m_\lambda - n_\lambda} \, \zeta^{m'_0 + kq} = 0 & \text{if } \mathrm{length}(\overline{\mathrm{tk}}) = \mathrm{length}(\overline{\mathrm{Tk}}) - 1 \\ z^{m_\lambda}\zeta^{m'_0} - s = 0 & \text{if } \mathrm{length}(\overline{\mathrm{tk}}) \leq \mathrm{length}(\overline{\mathrm{Tk}}) - 2. \end{cases}$$

Then we apply Theorem 18.3.4 (2), p319 to achieve a complete propagation of $\mathcal{W}_0$ and $\mathcal{W}'_0$ along the trunk Tk. This completes the construction of a complete deformation atlas of X for Case 2.

Proof of Theorem 19.3.2 (General case)

We now give a proof of Theorem 19.3.2, p332 for the general case: a constellar singular fiber X is a bonding of an *arbitrary* number of stellar singular fibers. The proof is essentially the same as that in the special case. To avoid inessential complication of argument, we assume that for any trunk Tk of X, the intersection $\mathrm{Tk} \cap Y$ is connected — for the disconnected case, we further need to apply Theorem 18.3.4 (3), p319.

First of all, for a subbranch $\mathrm{br}^{(i)} = n_1\Theta_1 + n_2\Theta_2 + \cdots + n_{e_i}\Theta_{e_i}$ and a subtrunk $\mathrm{tk}^{(j)} = n_1\Theta_1 + n_2\Theta_2 + \cdots + n_{e_j}\Theta_{e_j}$ of the simple crust Y, we define positive integers $a(\mathrm{br}^{(i)})$ and $a(\mathrm{tk}^{(j)})$ respectively by

$$a(\mathrm{br}^{(i)}) = \begin{cases} n_{e_i}, & \mathrm{br}^{(i)} \text{ is of type } C_l, \\ 1, & \text{otherwise,} \end{cases} \qquad a(\mathrm{tk}^{(j)}) = \begin{cases} n_{e_j}, & \mathrm{tk}^{(j)} \text{ is of type } C_l, \\ 1, & \text{otherwise.} \end{cases}$$

We then set $d := \mathrm{lcm}\{a(\mathrm{br}^{(i)}),\, a(\mathrm{tk}^{(j)})\}$, where $\mathrm{br}^{(i)}$ and $\mathrm{tk}^{(j)}$ respectively run over all subbranches and subtrunks of Y.

Suppose that Θ_i is a core of X but not that of Y. We recall that if Θ_i intersects Y — this is exactly when Θ_i intersects a subtrunk tk of Y at some point —, then we say that Θ_i is adjacent to Y. We use the following notations:

(i) $\mathrm{Core}(X) = \{\text{cores of } X\}$
(ii) $\mathrm{Core}(Y) = \{\text{cores of } Y\}$
(iii) $\mathrm{Adja}(Y) = \{\Theta_i \in \mathrm{Core}(X) \setminus \mathrm{Core}(Y) \text{ such that } \Theta_i \text{ is adjacent to } Y\}$
(iv) q_i: the slant of a subtrunk intersecting $\Theta_i \in \mathrm{Adja}(Y)$. (By the tensor condition, q_i does not depend on the choice of subtrunks of Y intersecting Θ_i.)

We say that q_i in (iv) is the *slant at* $\Theta_i \in \mathrm{Adja}(Y)$.

Now we carry out the construction of a complete deformation atlas of X in three steps:

Step 1. For each core Θ_i of X, we define a smooth hypersurface (an initial deformation) in $N_i \times \Delta \times \Delta^\dagger$ by

$$\mathcal{W}_i : \quad \begin{cases} \sigma_i \zeta_i^{m_i} - s + \sum_{k=1}^{l} {}_l\mathrm{C}_k t^{kd}\, \sigma_i\, \tau_i^k\, \zeta_i^{m_i - kn_i} = 0, & \Theta_i \in \mathrm{Core}(Y), \\ \sigma_i \zeta_i^{m_i} - s + \sum_{k=1}^{l} {}_l\mathrm{C}_k t^{kd}\, \sigma_i\, \tau_i^k\, \zeta_i^{m_i + kq_i} = 0, & \Theta_i \in \mathrm{Adja}(Y), \\ \sigma_i \zeta_i^{m_i} - s = 0, & \text{otherwise}, \end{cases}$$

where $d = \mathrm{lcm}\{a(\mathrm{br}^{(i)}),\, a(\mathrm{tk}^{(j)})\}$, and the nonnegative integer q_i is the slant at Θ_i, and τ_i is a core section in the tensor condition, i.e. a meromorphic section of (i) $N_i^{\otimes n_i}$ if $\Theta_i \in \mathrm{Core}(Y)$ and of (ii) $N_i^{\otimes(-q_i)}$ if $\Theta_i \in \mathrm{Adja}(Y)$.

Step 2. We propagate the initial deformations $\{\mathcal{W}_i\}$ along all branches of X. Let $\mathrm{Br}^{(j)}$ be a branch of X. If $\mathrm{Br}^{(j)}$ emanates from a core Θ_i in $\mathrm{Core}(Y)$, then by Theorem 10.0.15 p177, $\mathcal{W}_i$ admits a complete propagation along $\mathrm{Br}^{(j)}$. Similarly if $\mathrm{Br}^{(j)}$ emanates from a core in $\mathrm{Adja}(Y)$, then $\mathcal{W}_i$ admits a complete propagation along $\mathrm{Br}^{(j)}$ by applying the construction for subbranches of type A_l. The remaining case is easy; if $\mathrm{Br}^{(j)}$ emanates from a core Θ_i neither in $\mathrm{Core}(Y)$ nor $\mathrm{Adja}(Y)$, then $\mathcal{W}_i$ admits a complete propagation along $\mathrm{Br}^{(j)}$ in the trivial way.

Step 3. Finally, we propagate the initial deformations $\{\mathcal{W}_i\}$ along all trunks of X. Let $\mathrm{Tk}^{(j)}$ be a trunk of X connecting two cores Θ_i and Θ_k of X (possibly, $\Theta_i = \Theta_k$ and $\mathrm{Tk}^{(j)}$ connects two points of Θ_i). If at least one of Θ_i and Θ_k is a core of Y, then $\mathcal{W}_i$ and $\mathcal{W}_k$ admit a complete propagation along $\mathrm{Tk}^{(j)}$ by Theorem 18.3.4, p319. For other cases, $\mathcal{W}_i$ and $\mathcal{W}_k$ admit a complete propagation along $\mathrm{Tk}^{(j)}$ in the trivial way. Thus we obtain a complete deformation atlas of X, establishing Theorem 19.3.2, p332.

Remark 19.3.3 As is clear from the above construction, we may weaken the analytic assumption that $\pi : M \to \Delta$ is a linear degeneration; we need the assumption of linearity only around irreducible components of Y.

For examples of Theorem 19.3.2, see Figures 19.3.2, 19.3.3, 19.3.4, and 19.3.5 below. It is straightforward to generalize the notion of the "compound barking" for a stellar singular fiber (see Chapter 17, p303) to that for a constellar singular fiber. The definition is almost the same as that for the stellar case, except that we also need to take into account the compatibility along trunks.

We next give a comment on subbranches of simple crusts. Recall that a subbranch $\overline{\mathrm{br}} = n_0\Delta_0 + n_1\Theta_1 + \cdots + n_e\Theta_e$ of a branch $\overline{\mathrm{Br}} = m_0\Delta_0 + m_1\Theta_1 + \cdots + m_\lambda\Theta_\lambda$ is of *type* AB_l if $e = \lambda$, $n_\lambda = 1$, and $l \cdot \overline{\mathrm{br}} = \overline{\mathrm{Br}}$. We showed that any simple crust of a stellar singular fiber does *not* have a subbranch of type

AB_l (Corollary 16.7.4, p300). In contrast, for a constellar singular fiber X, a simple crust of X may have a subbranch of type AB_l, for instance,

Example 19.3.4 Consider two stellar singular fibers X_1 and X_2:

$$X_1 = 6\Theta_0 + \mathrm{Br}^{(1)} + \mathrm{Br}^{(2)} + \mathrm{Br}^{(3)},$$

where $\mathrm{Br}^{(j)} = 4\Theta_1^{(j)} + 2\Theta_2^{(j)}$ for each $j = 1, 2, 3$, and

$$X_2 = 6\Theta_0' + \mathrm{Br}^{(1)'} + \mathrm{Br}^{(2)'} + \mathrm{Br}^{(3)'},$$

where $\mathrm{Br}^{(1)'} = 4\Theta_1^{(1)'} + 2\Theta_2^{(1)'}$, $\mathrm{Br}^{(2)'} = 3\Theta_1^{(2)'}$ and $\mathrm{Br}^{(3)'} = 5\Theta_1^{(3)'} + 4\Theta_2^{(3)'} + 3\Theta_3^{(3)'} + 2\Theta_4^{(3)'} + \Theta_5^{(3)'}$. Note that X_1 is the singular fiber of a 'degeneration' whose smooth fiber consists of *two* (disjoint) elliptic curves, while X_2 is the singular fiber of a degeneration whose smooth fiber is an elliptic curve.

Now we construct a constellar singular fiber by bonding X_1 and X_2 along the branches $\mathrm{Br}^{(1)}$ and $\mathrm{Br}^{(1)'}$: The identification of $2\Theta_1^{(1)}$ of X_1 and $2\Theta_1^{(1)'}$ of X_2 yields a constellar singular fiber X — the singular fiber of a degeneration of complex curves of genus 3. (Note: All smooth fibers of this degeneration are connected, because X contains an irreducible component $\Theta_5^{(3)'}$ of multiplicity 1.) We then take a simple crust Y of X:

$$Y = 3\Theta_0 + \mathrm{br}^{(1)} + \mathrm{br}^{(2)} + \mathrm{br}^{(3)},$$

where $\mathrm{br}^{(j)} = 2\Theta_1^{(j)} + \Theta_2^{(j)}$ for each $j = 1, 2, 3$. Observe that Y is contained in the "X_1-part" of X. Moreover $2Y = X_1$, and *all subbranches of Y are of type AB_l*.

We close this section by stating a theorem about singularities of singular fibers of barking families. To that end, we first recall some terminology. Let $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$ be a baking family obtained from a simple crust Y of barking multiplicity l. For fixed t ($t \neq 0$), a singular fiber $X_{s,t}$ is called the *main fiber* if $s = 0$, and it is called a *subordinate fiber* if $s \neq 0$: The original singular fiber X splits into one main fiber and several subordinate fibers. The main fiber $X_{0,t}$ may be described essentially in terms of the factorization of its defining equation. For the subordinate fibers, we have the following theorem (we leave the reader to check that singularities do not appear near the trunks of X) which generalizes Theorem 16.4.4, p292.

Theorem 19.3.5 *Let $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$ be a barking family associated with a simple crust Y of barking multiplicity l. Then the singularities of a subordinate fiber $X_{s,t}$ are as follows:*

(1) *For each core Θ_i of X, if the plot function $K_i(z) = n_i \dfrac{d\sigma_i}{dz}\tau_i + m_i\sigma_i\dfrac{d\tau_i}{dz}$ on Θ_i is not identically zero, then $X_{s,t}$ has A-singularities near the core Θ_i* (Theorem 21.6.7, p410)*; whereas if $K_i(z)$ is identically zero, then $X_{s,t}$*

has non-isolated singularities near Θ_i (Proposition 21.8.3 (1), p418). *For the former case, the following inequality holds* (Corollary 21.4.4, p403)*:*

$$\begin{aligned}&(\textit{the number of the } A\textit{-singularities near the core } \Theta_i)\\ &\qquad \le \gcd(m_i, n_i)\cdot\Big[N_i - v_i + k_i + (2g_i - 2) - \sum_{j\in J_i}\operatorname{ord}_{p_j}(\omega)\Big],\end{aligned}$$

where

N_i *is the number of the intersection points of* Θ_i *with other irreducible components of* X*,*

v_i *is the number of the intersection points of* Θ_i *with the proportional subbranches/subtrunks of* Y*,*

k_i *is the number of the zeros of* τ_i*,*

g_i *is the genus of the core* Θ_i*,*

J_i *is the set of the indices* j *of the intersection points* p_j *of* Θ_i *with the proportional subbranches/subtrunks of* Y *(there are* v_i *such indices), and*

$\operatorname{ord}_{p_j}(\omega_i)$ *is the order of a meromorphic* 1*-form* $\omega_i(z) := d\log(\sigma_i^{n_i}\tau_i^{m_i})$ *at* p_j*.*

In the generic case[1], $X_{s,t}$ *has only nodes* (A_1*-singularities*) *near the core* Θ_i*.*

(2) *If* Y *has proportional subbranches, then* $X_{s,t}$ *has A-singularities near the edge of each proportional subbranch* (Proposition 7.2.6, p129).

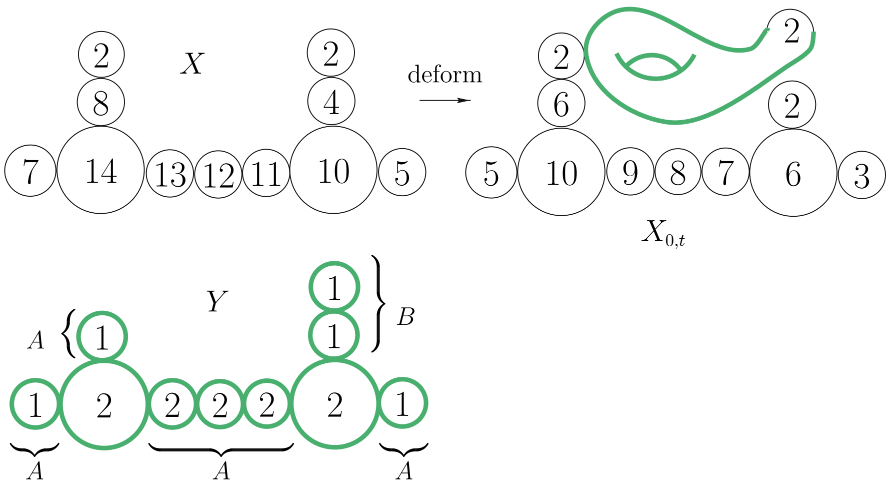


Fig. 19.3.2. genus$(X) = 5$, barking multiplicity $l = 2$, and baking genus $g_b(Y) = 1$. The singular fiber X is a (-1)-bonding of two stellar singular fibers X_1 (from the left) and X_2 (from the right). Note that X_1 and X_2 are respectively of genus 3 and 2.

[1] The case where any zero α of the plot function $K_i(z)$ such that $\sigma_i(\alpha) \neq 0$ and $\tau_i(\alpha) \neq 0$ is simple (i.e. of order 1).

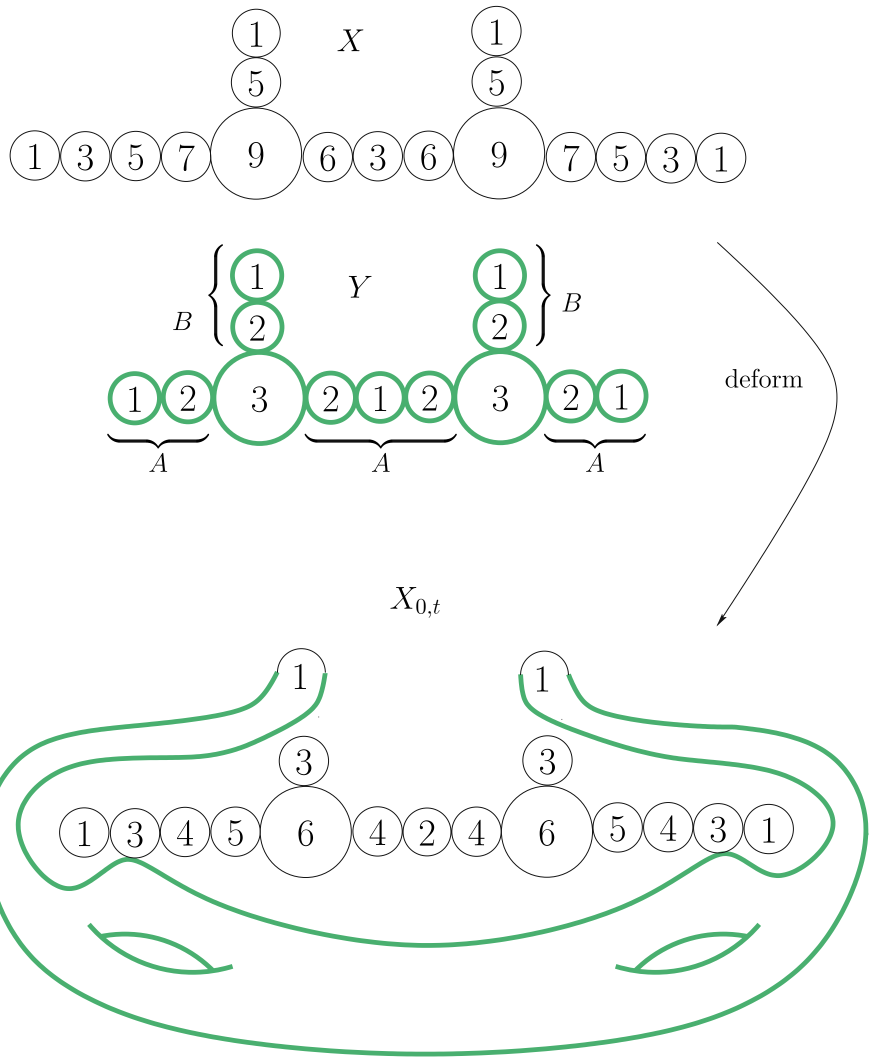


Fig. 19.3.3. genus$(X) = 6$, barking multiplicity $l = 1$, and baking genus $g_b(Y) = 2$. Since $\mathrm{Tk} = 3 \cdot \mathrm{tk}$, the subtrunk tk is proportional (Remark 18.2.4, p313). When (1) $s = 0$ and (2) $s = \frac{4}{27}t^3$, the fiber $X_{s,t}$ is singular. In the case (2), $X_{s,t}$ is non-reduced (see Example 6.4.11, p117).

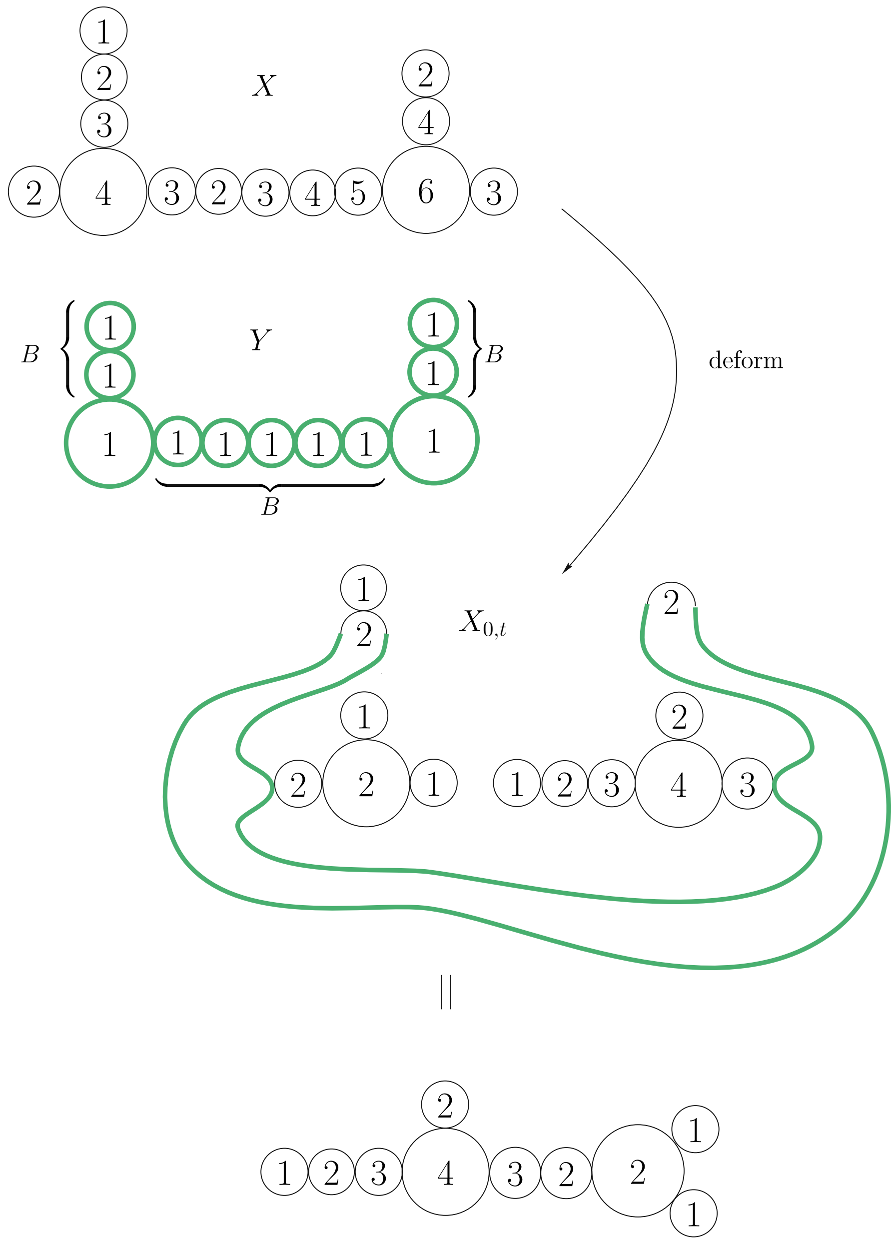


Fig. 19.3.4. genus$(X) = 5$, barking multiplicity $l = 2$, and barking genus $g_b(Y) = 0$.

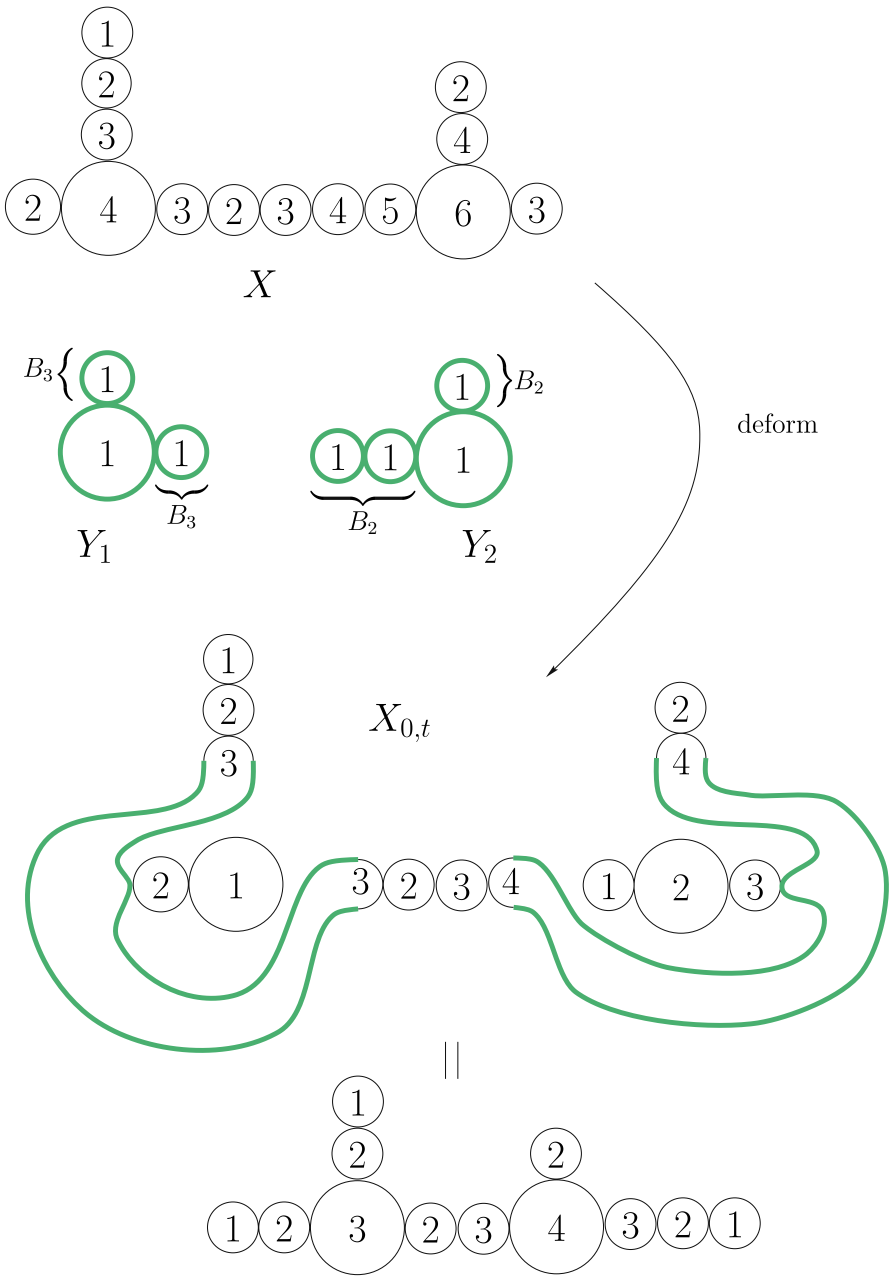


Fig. 19.3.5. genus$(X) = 7$, and Y_1 (resp. Y_2) is a simple crust of barking multiplicity 3 (resp. 2). This deformation barks $3Y_1$ and $2Y_2$ simultaneously.

19.4 Criteria for splittability

We shall give several criteria for the splittability of a singular fiber. Before we present them, we generalize the notion of the "barking genus" for a simple crust of a stellar singular fiber to the constellar case. We outline how to define it, and leave the details to the reader. First, let Y be a simple crust of barking multiplicity l of a constellar singular fiber X, and let $\overline{\mathrm{tk}}^{(j)} := \overline{\mathrm{Tk}}^{(j)} \cap Y$ be a (fringed) subtrunk of Y. We say that $\overline{\mathrm{tk}}^{(j)}$ (also unfringed $\mathrm{tk}^{(j)}$) is *long* (resp. *short*) if

$$\mathrm{length}(\overline{\mathrm{tk}}^{(j)}) = \mathrm{length}(\overline{\mathrm{Tk}}^{(j)}) \qquad \Big(\text{ resp. } \ \mathrm{length}(\overline{\mathrm{tk}}^{(j)}) < \mathrm{length}(\overline{\mathrm{Tk}}^{(j)}) \Big).$$

Now we define the *enlargement* $\dot{Y}$ of the simple crust Y in the following procedure:

(i) Replace a subbranch $\mathrm{br}^{(j)}$ of type A_l in Y by its enlargement $\dot{\mathrm{br}}^{(j)}$ (§16.6, p295).

(ii) For a short subtrunk $\mathrm{tk}^{(j)}$ of type A_l in Y, define its enlargement $\dot{\mathrm{tk}}^{(j)}$ in a similar way to that for a subbranch of type A_l, and then replace $\mathrm{tk}^{(j)}$ in Y by $\dot{\mathrm{tk}}^{(j)}$.

The resulting divisor $\dot{Y}$ is called the enlargement of Y. A similar construction to the stellar case (Proposition 16.6.3, p298) yields a degeneration with a singular fiber $\dot{Y}^{\times}$, where $\dot{Y}^{\times}$ is obtained from $\dot{Y}$ by deleting the zeros, if any, of the core sections:

$$\dot{Y}^{\times} = \dot{Y} \setminus \bigcup\nolimits_i \{ \text{the zeros of } \tau_i \},$$

where i runs over all indices such that Θ_i is a core of Y, and τ_i is the core section on Θ_i. In other words, replacing each core Θ_i of Y by $\Theta_i^{\times} = \Theta_i \setminus \{ \text{the zeros of } \tau_i \}$, we obtain $\dot{Y}^{\times}$. Of course, if the core section on any core of Y has no zeros, then $\dot{Y}^{\times}$ is $\dot{Y}$ itself. The genus of (a connected component of) a smooth fiber of the above degeneration (with the singular fiber $\dot{Y}^{\times}$) is called the *barking genus* of Y; we denote it by $g_b(Y)$. Clearly $g_b(Y) \le g$ where g is the genus of a smooth fiber of $\pi : M \to \Delta$.

From the descriptions of deformations of types A_l, B_l, and C_l together with those of deformations around the cores (§16.4, p288), the following result holds.

Proposition 19.4.1 *Let $\pi : M \to \Delta$ be a linear degeneration with a* (*stellar or constellar*) *singular fiber X. Assume that X contains a simple crust Y of barking multiplicity l, and let $\Psi : \mathcal{M} \to \Delta \times \Delta^{\dagger}$ be a barking family associated with Y* (Theorem 19.3.2). *Then $X_{0,t} = \Psi^{-1}(0,t)$ has at most normal crossings if and only if the following conditions are satisfied:*

(i) *If a subbranch of Y is of type A_l, then its slant is zero.*

(ii) *If a short subtrunk of* Y $\big($i.e. $\text{length}(\overline{\text{tk}}) < \text{length}(\overline{\text{Tk}})\big)$ *is of type* A_l*, then its slant is zero.*

(iii) *Let* τ_i *be the core section on a core* $\Theta_i \in \text{Core}(Y)$ *or* $\text{Adja}(Y)$*, and then* (1) *the order of any zero of* τ_i *is* 1*, and* (2) *any such zero is "not" an intersection point of* Θ_i *with a branch* (*resp. trunk*) *containing no subbranch* (*resp. subtrunk*) *of* Y*.*

Next, recall that a subdivisor $Y = \sum_i n_i\Theta_i$ of X is called *multiple* if $\gcd\{n_i\} = 1$. Also recall that a singular fiber is *fake* if it becomes a smooth fiber after successive blow up or down.

Lemma 19.4.2 *Let* $\pi : M \to \Delta$ *be a linear degeneration of complex curves of genus* g *with a* (*stellar or constellar*) *singular fiber* X*. Assume that* X *contains a simple crust* Y *of barking multiplicity* l*, and let* $\Psi : \mathcal{M} \to \Delta \times \Delta^{\dagger}$ *be a barking family associated with* Y (Theorem 19.3.2)*. Then the following statements hold:*

(1) *If the barking genus* $g_b(Y) = g$*, then* $l = 1$*,* Y *is not multiple, and* $X_{0,t} := \Psi^{-1}(0, t)$ *for* $t \neq 0$ *is a fake singular fiber.* (Figure 20.1.1, p350 for example.)

(2) *If the barking genus* $g_b(Y) < g$*, then* $X_{0,t}$ *is not a fake singular fiber.*

Proof. We show (1). Note that the singular fiber $X_{0,t}$ ($t \neq 0$) contains an irreducible component $lY_{0,t}$ where $Y_{0,t}$ is a deformation of Y. By the assumption $g_b(Y) = g$ (i.e. the genus of $Y_{0,t}$ is g), if $l \geq 2$, then a smooth fiber near $X_{0,t}$ has genus at least lg, which is greater than g (a contradiction!). Hence $l = 1$, and for the same reason, Y is not multiple. We then claim that $X_{0,t}$ is a fake singular fiber; that is, the normally minimal singular fiber $X'_{0,t}$ (obtained from $X_{0,t}$ by successive blowing up or down) is a smooth fiber. This is clear because $X'_{0,t}$ contains an irreducible component (corresponding to $Y_{0,t}$ in $X_{0,t}$) of genus g, and such a normally minimal fiber is necessarily a smooth fiber. This proves (1). The assertion (2) follows easily from the description of the configuration of $X_{0,t}$ and the assumption $g_b(Y) < g$. □

Now we give a very powerful criterion for the splittability of a singular fiber.

Criterion 19.4.3 *Let* $\pi : M \to \Delta$ *be a linear degeneration of complex curves of genus* g *with a* (*stellar or constellar*) *singular fiber* X*. Then* $\pi : M \to \Delta$ *admits a splitting family, if* (1)*,* (2)*, or* (3) *below holds:*

(1) *The singular fiber* X *contains a simple crust* Y *such that*
   - (1a) Y *contains no exceptional curve* (e.g. when X contains no exceptional curve)*, or*
   - (1b) *the barking genus* $g_b(Y) < g$*.*

(2) *The singular fiber* X *contains an exceptional curve* Θ_0 *such that*
   - (2a) *at least one irreducible component, say* Θ_1*, of* X *intersecting* Θ_0 *is a projective line, and*

(2b) *any irreducible component of* X *intersecting* Θ_0 *satisfies the tensor condition* (*Definition* 19.2.1) *with respect to a subdivisor* $Y := \Theta_0 + \Theta_1$ *where* Θ_1 *is in* (2a).

(3) *The singular fiber* X *contains an exceptional curve* Θ_0 *such that any irreducible component intersecting* Θ_0 *is a projective line.* (Note: If X is stellar, then Θ_0 is its core, and this condition is always satisfied.)

Proof. First of all, under the condition (1), we show the existence of a splitting family. Let $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$ be a barking family of $\pi : M \to \Delta$ associated with Y (Theorem 19.3.2). We claim that the topological monodromy around the main fiber $X_{0,t} := \Psi^{-1}(0,t)$, $(t \neq 0)$ is nontrivial. This is seen as follows: In case (1a), after blow up if necessary, $X_{0,t}$ becomes a normally minimal *singular* fiber. In case (1b), $X_{0,t}$ is not a fake singular fiber by Lemma 19.4.2 (2). In either case, $X_{0,t}$ has a nontrivial topological monodromy (different from that around the original singular fiber $X = \pi^{-1}(0)$). Therefore $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$ is indeed a splitting family (see the proof of Lemma 1.1.2, p28).

Next, under the condition (2), we show the existence of a splitting family. By assumption, $Y = \Theta_0 + \Theta_1$ is a simple crust of barking multiplicity $l := m_1$. Here Θ_0 is its core, and Θ_1 is a subbranch or a subtrunk (of length 1) which, in either case, is of type B_l. Let $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$ be a barking family associated with Y (Theorem 19.3.2). Since the subbranch/subtrunk Θ_1 is of type B_l, the main fiber $X_{0,t} := \Psi^{-1}(0,t)$ $(t \neq 0)$ is normally minimal by Proposition 16.2.2 (2), p281. In particular, the topological monodromy around $X_{0,t}$ is nontrivial, and hence $X_{0,t}$ is not a fake singular fiber. Therefore $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$ is indeed a splitting family. Finally for (3), the existence of a splitting family is nothing but Criterion 16.5.2, p294. □

Remark 19.4.4 If a constellar singular fiber X is a bonding of stellar singular fibers of "genus 1", then by a combinatorial argument, we can show that X satisfies a condition (1), (2), or (3) in the above criterion. Thus X admits a splitting.

Criterion 19.4.5 *Let* $\pi : M \to \Delta$ *be a linear degeneration with a constellar singular fiber* X *which is a bonding of two stellar singular fibers* X_1 *and* X_2*. Denote by* m_1 (*resp.* m_2) *the multiplicity of the core of* X_1 (*resp.* X_2)*. Then the following statements hold:*

(1) *If both cores of* X_1 *and* X_2 *are exceptional curves, then* X *admits a splitting.*
(2) *If the core of* X_1 *is an exceptional curve and* $m_1 \geq m_2$*, then* X *admits a splitting.*

Proof. In terms of Criterion 19.4.3 (3), it is enough to show that under the assumption of (1) or (2), at least one of the cores of X is an exceptional curve. For (1), if $m_1 \neq m_2$ (we assume $m_1 > m_2$), then the core of X_1 remains an exceptional curve after bonding, and if $m_1 = m_2$, then both cores of X_1 and X_2 remain exceptional curves after bonding; so the claim is confirmed. For

(2), from the assumption, the core of X_1 remains an exceptional curve after bonding, and so the claim is confirmed. □

19.5 Looped trunks

We bond two branches of "one" stellar singular fiber. Then the resulting trunk together with the core is called a *looped trunk*; see Figure 19.5.1. (Unfortunately, this terminology is slightly inconsistent in that a looped trunk is the union of a trunk and a core.) If a singular fiber possesses such a looped trunk as contains a chain of (-2)-curves[2], then it is often possible to construct a splitting family. We give such examples. Let X be a singular fiber with a looped trunk, which is specifically given below, and let Θ_0 be the core in the looped trunk.

Example 1: Consider a looped trunk Tk in Figure 19.5.1 — we did not write the multiplicities of irreducible components, because they are immaterial for our subsequent discussion. In the figure, r_e is a positive integer satisfying $r_e \geq 2$, and the negative number beside an irreducible component stands for its self-intersection number, e.g. $\Theta_e \cdot \Theta_e = -r_e$. We then take a simple crust Y, as in the figure, of barking multiplicity $l := m_e$ where m_e is the multiplicity of the irreducible component Θ_e. We note that the looped subtrunk — actually, Y itself — is of type B_l (the extreme case is shown in Figure 19.5.2). Let $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$ be the barking family associated with Y, and then the barking genus $g_b(Y) = 1$, and Y is deformed to a smooth elliptic curve $Y_{0,t}$ in $X_{0,t} = \Psi^{-1}(0, t)$. See Figure 19.5.3 for a concrete example.

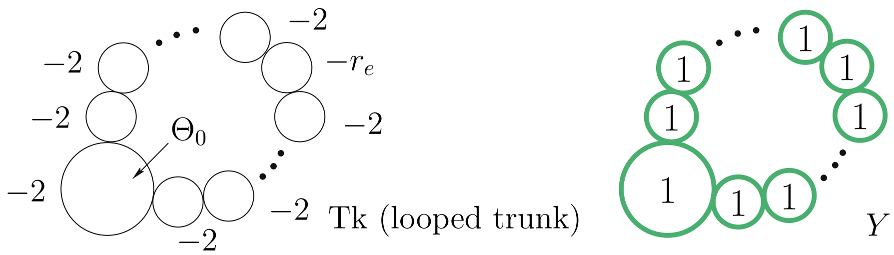


Fig. 19.5.1. Example 1

[2] A (-2)-curve is a projective line with the self-intersection number -2.

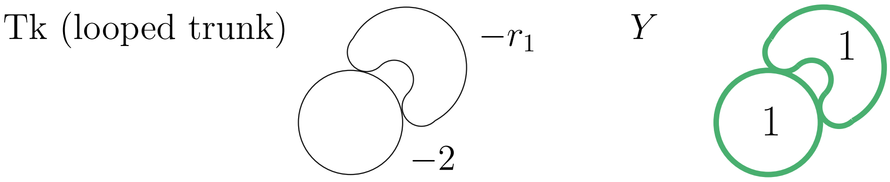


Fig. 19.5.2. The extreme case of Example 1

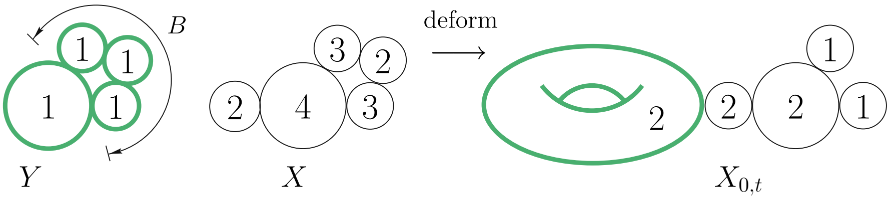


Fig. 19.5.3. Concrete example 1: genus$(X) = 2$, barking multiplicity $l = 2$, and barking genus $g_b(Y) = 1$.

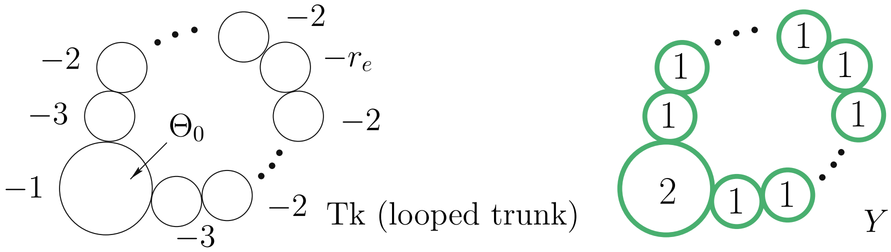


Fig. 19.5.4. Example 2: Each negative integer stands for the self-intersection number, and $-r_e \leq -2$.

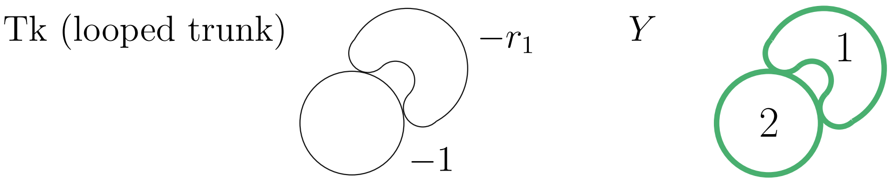


Fig. 19.5.5. The extreme case of Example 2

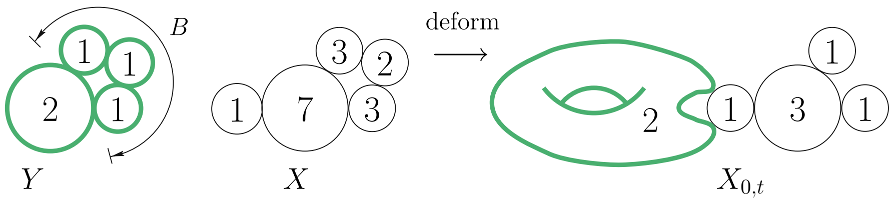


Fig. 19.5.6. Concrete example 2: genus(X) = 4, barking multiplicity $l = 2$, and barking genus $g_b(Y) = 1$.

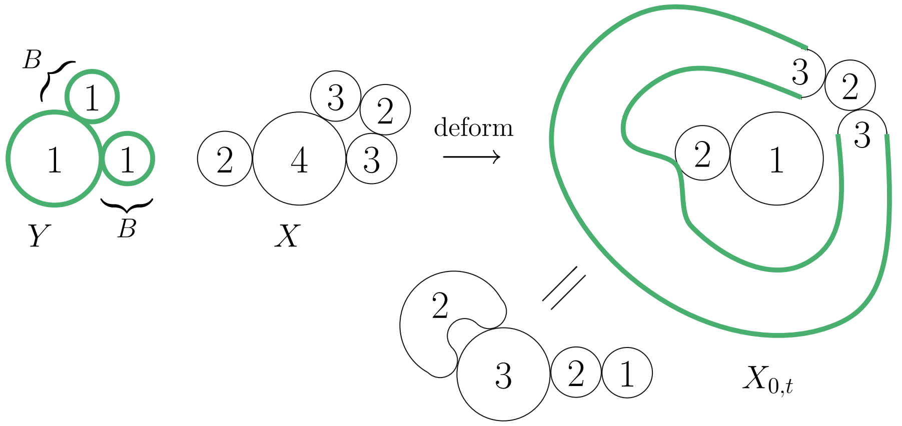


Fig. 19.5.7. The singular fiber X in Figure 19.5.3 contains another simple crust Y above, which has barking multiplicity $l = 3$ and barking genus $g_b(Y) = 0$.

Example 2: Consider a looped trunk Tk in Figure 19.5.4, and take a simple crust Y of barking multiplicity $l := m_e$ as in that figure; the looped subtrunk (Y itself) is also of type B_l as in Example 1 (the extreme case is shown in Figure 19.5.5). See Figure 19.5.6 for a concrete example. We point out a significant difference between Figure 19.5.3 and Figure 19.5.6: In Figure 19.5.3, the deformation $Y_{0,t}$ ($\subset X_{0,t}$) of Y intersects other irreducible components only at *one* point; while $Y_{0,t}$ in Figure 19.5.6 intersects other irreducible components at *two* points, because the multiplicity of the core of Y is 2.

20

Further Examples

In this chapter we gather miscellaneous but important examples and phenomena.

20.1 Fake singular fibers

We give an example of a barking family $\Psi : \mathcal{M} \to \Delta \times \Delta^{\dagger}$ such that $X_{0,t} = \Psi^{-1}(0, t)$ is a fake singular fiber (Figure 20.1.1), namely, after blow down, $X_{0,t}$ becomes a smooth fiber (Figure 20.1.2). We note that the barking genus $g_b(Y) = 2$, as explained in Figure 20.1.3.

20.2 Splitting families which give the same splitting

Assume that $\Psi : \mathcal{M} \to \Delta \times \Delta^{\dagger}$ and $\Psi' : \mathcal{M}' \to \Delta \times \Delta^{\dagger}$ are splitting families of a degeneration $\pi : M \to \Delta$. We say that Ψ and Ψ' are *topologically equivalent* if there exist orientation preserving homeomorphisms $H : \mathcal{M} \to \mathcal{M}'$ and $h : \Delta \times \Delta^{\dagger} \to \Delta \times \Delta^{\dagger}$ such that $h(0,0) = (0,0)$ and the following diagrams are commutative:

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\quad H \quad} & \mathcal{M}' \\ \Psi \downarrow & & \downarrow \Psi' \\ \Delta \times \Delta^{\dagger} & \xrightarrow{\quad h \quad} & \Delta \times \Delta^{\dagger}, \end{array} \qquad \begin{array}{ccc} M_t & \xrightarrow{\quad H_t \quad} & M'_t \\ \pi_t \downarrow & & \downarrow \pi'_t \\ \Delta \times \{t\} & \xrightarrow{\quad h_t \quad} & \Delta \times \{t\}, \end{array}$$

where $H_t := H|_{M_t}$ and $h_t := h|_{\Delta \times \{t\}}$ are restrictions of H and h respectively. (Note that if Ψ and Ψ' are topologically equivalent, then for each t, $\pi_t : M_t \to \Delta$ and $\pi'_t : M'_t \to \Delta$ are topologically equivalent. However the converse is *not* true.)

We are interested in two splitting families with the following properties: (i) Ψ and Ψ' are topologically different, nevertheless (ii) they gives the same splitting of the singular fiber X. In this section, we will give examples of

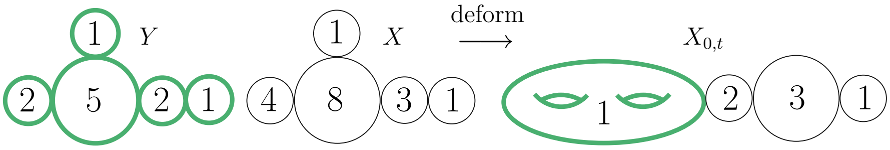


Fig. 20.1.1. genus$(X) = 2$, and the core of X is an exceptional curve.

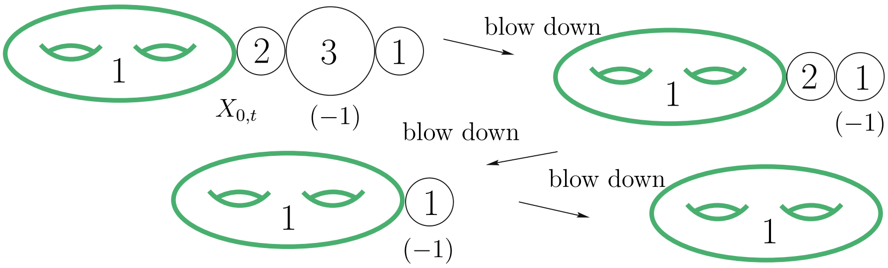


Fig. 20.1.2. After successive blow down, $X_{0,t}$ becomes a smooth curve. "(-1)" below an irreducible component means that the component is an exceptional curve.

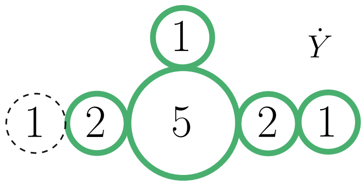


Fig. 20.1.3. $\dot{Y}$ is the enlargement of Y in Figure 20.1.1, where the dashed circle is an irreducible component attached to Y. Since $\dot{Y}$ is the singular fiber of a degeneration of curves of genus 2, the barking genus of Y is 2. See Proposition 16.6.3, p298.

such splitting families. Before proceeding, we require some preparation. Let $\chi(X)$ denote the (*topological*) *Euler characteristic* of the underlying topological space of X.

Lemma 20.2.1 *Let $\pi : M \to \Delta$ be a degeneration of complex curves of genus g with a singular fiber X. If a splitting family of $\pi : M \to \Delta$ splits X into $X_1, X_2, \ldots, X_n$, then*

$$\chi(X) - 2(1-g) = \sum_{i=1}^{n} \Big(\chi(X_i) - 2(1-g)\Big).$$

Proof. Since M_t is diffeomorphic to M, we have $\chi(M) = \chi(M_t)$. It is easy to check that $\chi(M) = \chi(X) - 2(1-g)$ and $\chi(M_t) = \sum_{i=1}^{n}\big(\chi(X_i) - 2(1-g)\big)$; see [BPV] p97. Hence the assertion follows. □

We apply this lemma to the case $g = 1$.

Lemma 20.2.2 *If $g = 1$, then the following statements hold:*

(1) *If X splits into $X_1, X_2, \ldots, X_n$, then $\chi(X) = \sum_{i=1}^{n} \chi(X_i)$.*
(2) *If furthermore $\chi(X) = \chi(X_1) + 1$, then $n = 2$ and X splits into X_1 and X_2 where X_2 is a projective line with one node* (i.e. *a Lefschetz fiber of genus* 1).

Proof. (1) is clear from the above lemma. (2) follows from the fact that if X is a singular fiber of a degeneration of elliptic curves, then $\chi(X) \geq 1$ where the equality holds precisely when X is a projective line with one node. □

From (2) of the above lemma, it is immediate to deduce the following result.

Proposition 20.2.3 *Let $\pi : M \to \Delta$ be a degeneration of elliptic curves. Assume that $\Psi : \mathcal{M} \to \Delta \times \Delta^{\dagger}$ is its splitting family such that $\chi(X) = \chi(X_{0,t}) + 1$. Then in this family, X splits into X_1 and X_2, where $X_1 := X_{0,t}$ and X_2 is a projective line with one node* (i.e. *a Lefschetz fiber of genus* 1).

Now we provide examples of topologically different splitting families which, however, give the same splitting of a singular fiber. In what follows, if a singular fiber X splits into singular fibers $X_1, X_2, \ldots, X_n$, we use expression $X \to X_1 + X_2 + \cdots + X_n$.

20.2.1 Example 1

Let us consider four baking families in Figure 15.5.3 (1), p276 and Figure 20.2.1. These families have the same $X_{0,t}$. Moreover, since $\chi(X) = 10$ and $\chi(X_{0,t}) = 9$, it follows from Proposition 20.2.3 that in these four examples, the singular fiber X splits into $X_1 := X_{0,t}$ and X_2, where X_2 is a projective line with one node. (In Kodaira's notation, $X = II^*$, $X_1 = III^*$, and $X_2 = I_1$.) *Nonetheless the topological types of these four barking families are different.* We first verify that the barking families $\Psi^{(1)}$ and $\Psi^{(2)}$ in (1) and (2) of Figure 20.2.1 are topologically different. The proof is done by contradiction. Assume that $\Psi^{(1)}$ and $\Psi^{(2)}$ are topologically equivalent. Then there exists a family of homeomorphisms $H_t : M_t^{(1)} \to M_t^{(2)}$ which make the following diagram commute:

$$\begin{array}{ccc} M_t^{(1)} & \xrightarrow{\;H_t\;} & M_t^{(2)} \\ \big\downarrow{\scriptstyle \pi_{1,t}} & & \big\downarrow{\scriptstyle \pi_{2,t}} \\ \Delta \times \{t\} & \xrightarrow{\;h_t\;} & \Delta \times \{t\}. \end{array}$$

To avoid confusion, let us write $X_{0,t}$ in Figure 20.2.1 (1) as $X_{0,t}^{(1)}$, and $X_{0,t}$ in (2) as $X_{0,t}^{(2)}$. Then H_t maps $X_{0,t}^{(1)}$ to $X_{0,t}^{(2)}$ homeomorphically, and so the

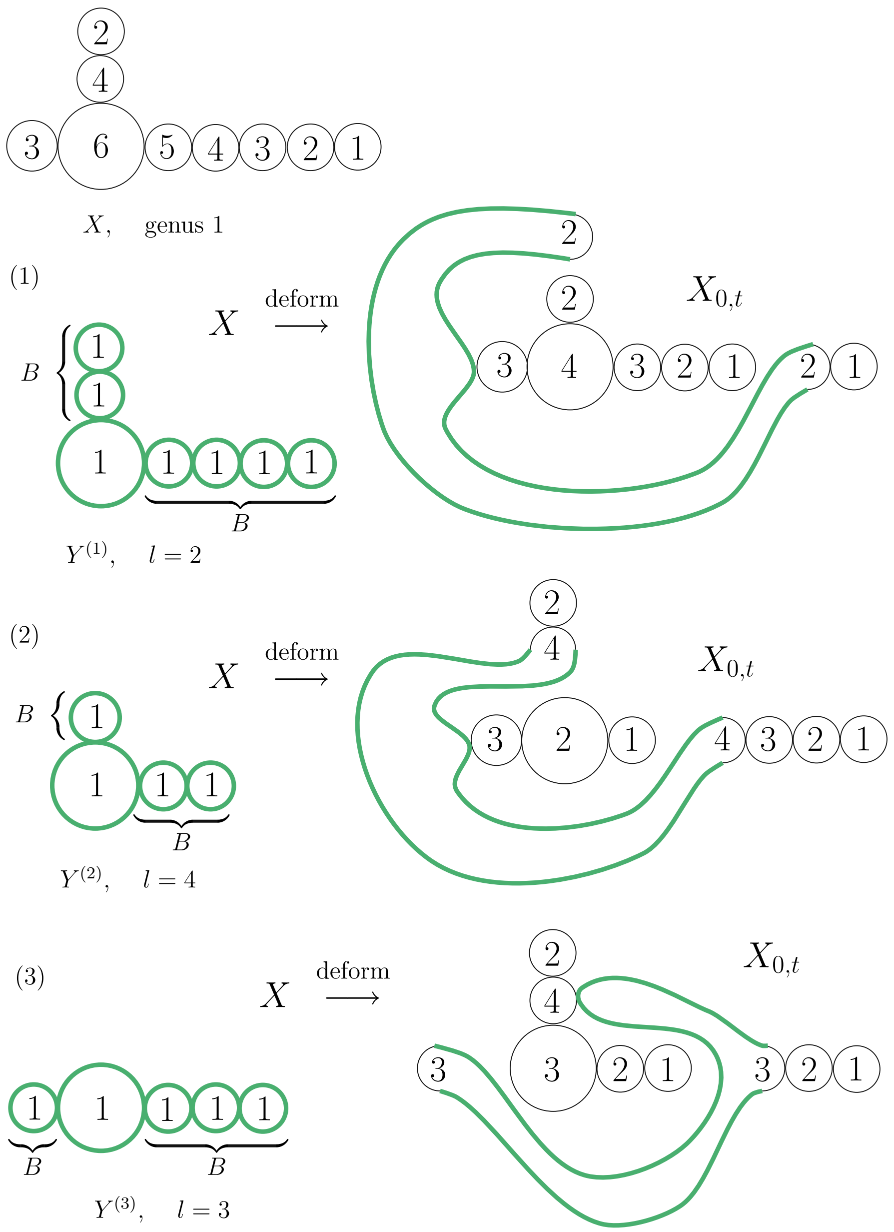


Fig. 20.2.1. Topologically different splitting families which give the same splitting $II^* \to III^* + I_1$, where $X = II^*$, $X_{0,t} = III^*$, and I_1 is a projective line with one node.

core $C_{0,t}^{(1)}$ of $X_{0,t}^{(1)}$ is mapped to the core $C_{0,t}^{(2)}$ of $X_{0,t}^{(2)}$ homeomorphically. Since $\mathcal{C}^{(1)} := \{C_{0,t}^{(1)}\}_{t\in\Delta'}$ and $\mathcal{C}^{(2)} := \{C_{0,t}^{(2)}\}_{t\in\Delta'}$ are closed sets, it follows from the continuity that in the limit $t \to 0$, the crust $Y^{(1)}$ is homeomorphic to $Y^{(2)}$. This is absurd, and therefore $\Psi^{(1)}$ and $\Psi^{(2)}$ are topologically different. For the other cases, topological difference is deduced from the same argument. (This result may be also obtained by showing that their discriminants (plane curve singularities in $\Delta \times \Delta^\dagger$) are different.)

20.2.2 Example 2

In Example 1, from four different simple crusts, we constructed four topologically different barking families which, however, give the same splitting $II^* \to III^* + I_1$. In this subsection, from *one* simple crust, we will construct three topologically different barking families which give the same splitting $II^* \to I_3^* + I_1$. The construction below is based on the non-uniqueness of a complete propagation of a deformation atlas of type C_l.

Leaving the explicit construction to the next paragraph, we consider three barking families which deform X to $X_{0,t}$ as in Figures 20.2.3, 20.2.4, and 20.2.5. Note that they all give the same splitting $II^* \to I_3^* + I_1$, where $II^* = X$, $I_3^* = X_{0,t}$ and I_1 is a projective line with one node. In fact, since $\chi(X) = 10$ and $\chi(X_{0,t}) = 9$, from Proposition 20.2.3, X splits into X_1 and X_2 where $X_1 = X_{0,t} = I_3^*$ and $X_2 = I_1$.

Now we give the explicit construction of these three barking families. Taking a simple crust Y of barking multiplicity $l = 2$ in Figure 20.2.2, we will construct a complete propagation of the deformation atlas $DA(2Y, k)$; in what follows, we take the weight $k = 2$. We only give the essential part of the construction, that is, three different complete propagations along the branch

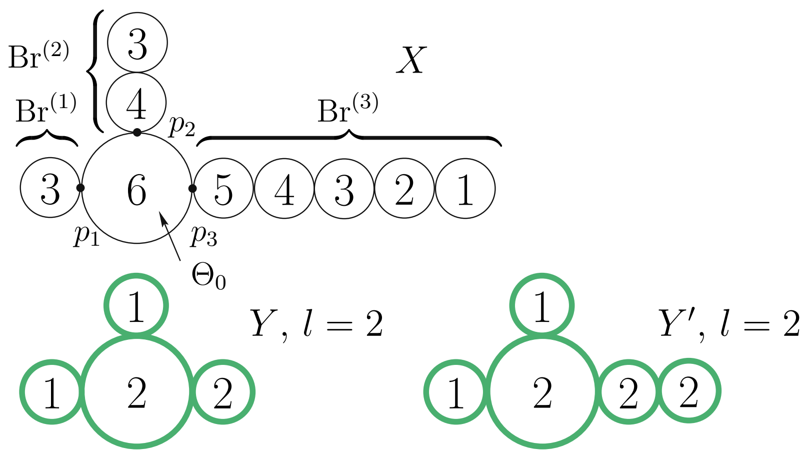


Fig. 20.2.2.

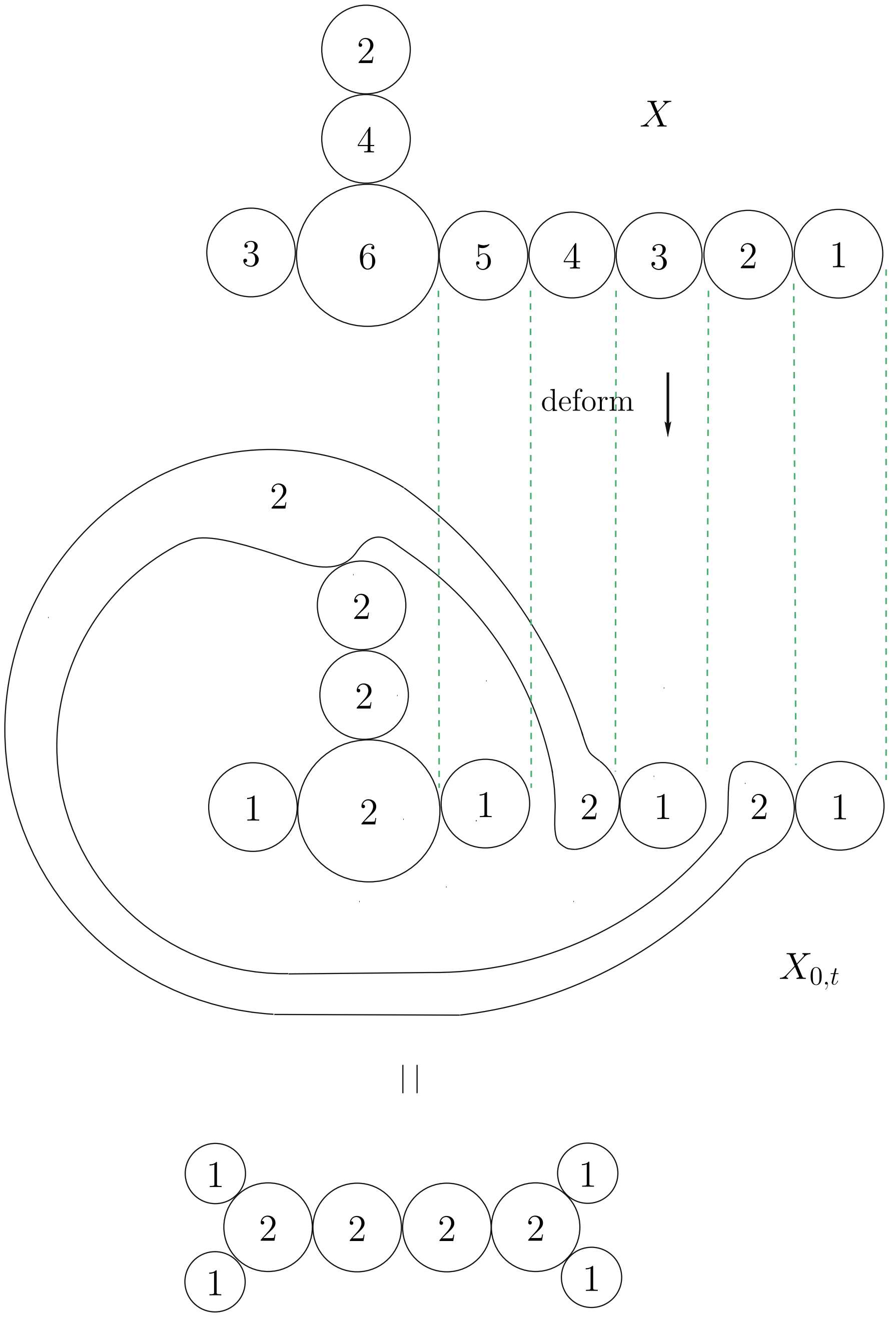


Fig. 20.2.3. Construction 1 gives a splitting $II^* \to I_3^* + I_1$, where $X = II^*$, $X_{0,t} = I_3^*$ and I_1 is a projective line with one node.

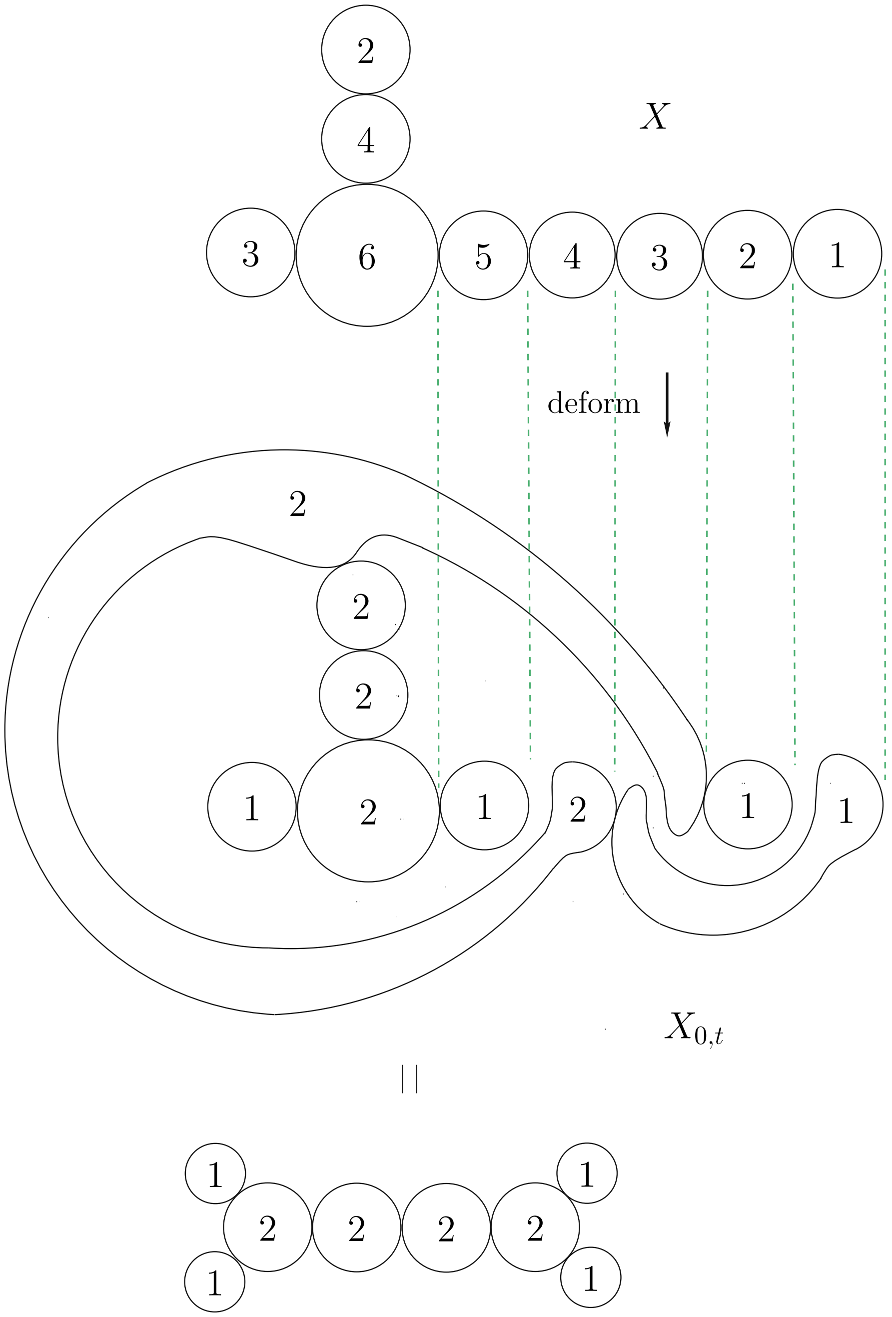


Fig. 20.2.4. Construction 2 gives a splitting $II^* \to I_3^* + I_1$, where $X = II^*$, $X_{0,t} = I_3^*$ and I_1 is a projective line with one node.

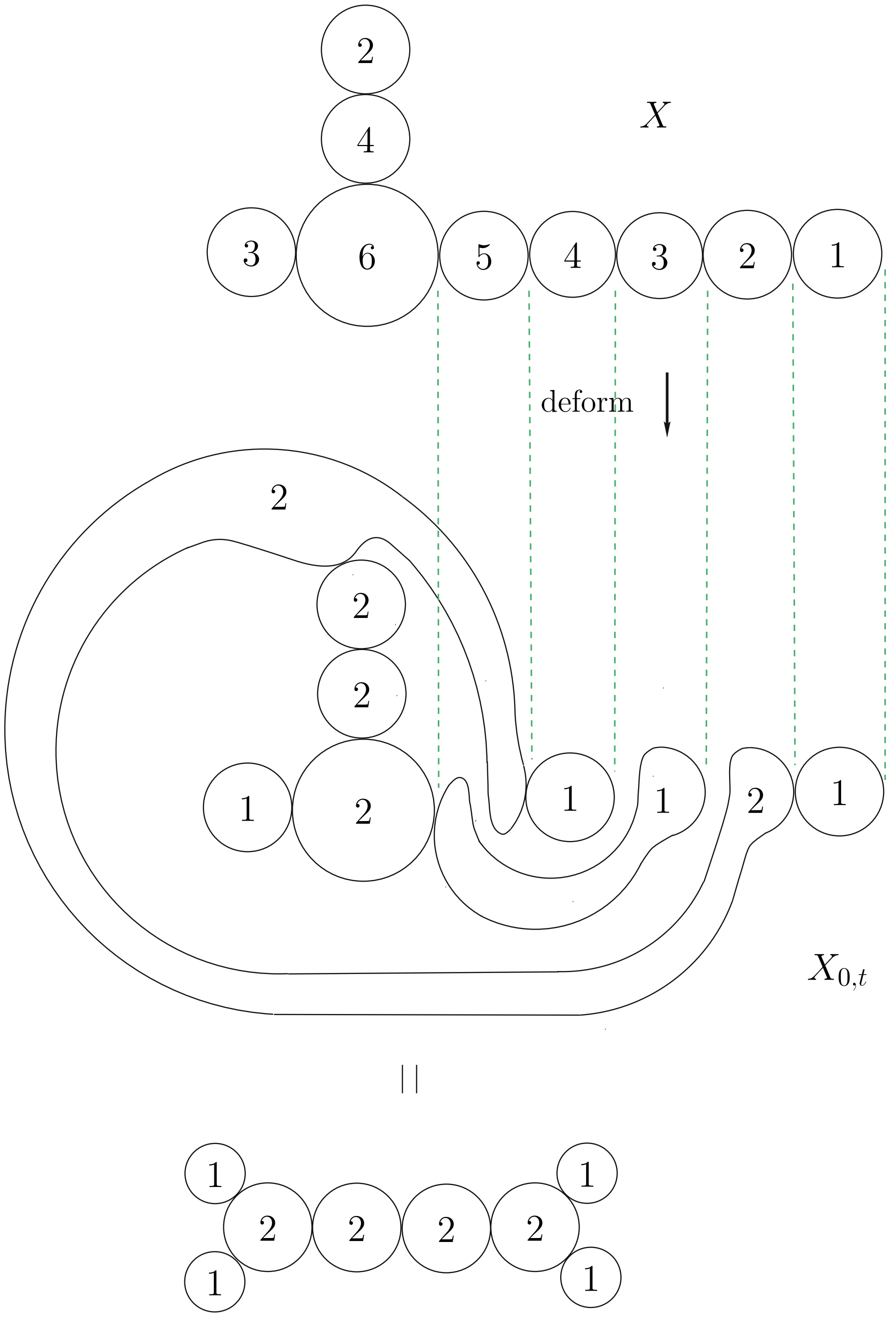


Fig. 20.2.5. Construction 3 gives a splitting $II^* \to I_3^* + I_1$, where $X = II^*$, $X_{0,t} = I_3^*$ and I_1 is a projective line with one node.

$\mathrm{Br}^{(3)}$; we note that the subbranch $\mathrm{br}^{(3)}$ of Y is of type C_l, while $\mathrm{br}^{(1)}$ and $\mathrm{br}^{(2)}$ are of type A_l, and complete propagations along $\mathrm{Br}^{(1)}$ and $\mathrm{Br}^{(2)}$ are easy to construct.

Remark 20.2.4 *In* Figure 20.2.2*, the simple crust Y is contained in another simple crust Y' of the same barking multiplicity* 2*. Observe that the subbranch* $\mathrm{br}^{(3)}$ *of Y is of type C_2, and the subbranch* $\mathrm{br}^{(3)'}$ *of Y' is also of type C_2. This example also confirms that a subbranch of type C_l is not necessarily dominant.*

The three different constructions of complete propagation along $\mathrm{Br}^{(3)}$ (Constructions 1, 2 and 3) are carried out in the following way. Since $\mathcal{H}_1$: $w^2\eta(w^2\eta^2 - t^2)^2 - s = 0$ where for brevity we omit superscripts such as $\mathcal{H}_1 = \mathcal{H}_1^{(3)}$ etc, Constructions 1 and 2 start from the following first propagation of $DA(2Y, 2)$:

$$\begin{cases} \mathcal{H}_1 : & w^2\eta(w^2\eta^2 - t^2)^2 - s = 0 \\ \mathcal{H}'_1 : & \zeta(z^2\zeta^2 - t^2)^2 - s = 0 \\ g_1 : & z = \dfrac{1}{w}, \quad \zeta = w^2\eta. \end{cases} \tag{20.2.1}$$

(Note that we may consider this to be the deformation atlas associated with $2Y'$, where Y' is in Figure 20.2.2.)

On the other hand, noting that $\mathcal{H}_1$ admits a 'factorization'

$$w^2\eta(w\eta + t)^2(w\eta - t)^2 - s = 0,$$

Construction 3 starts from the following first propagation of $DA(2Y, 2)$:

$$\begin{cases} \mathcal{H}_1 : & w^2\eta(w\eta + t)^2(w\eta - t)^2 - s = 0 \\ \mathcal{H}'_1 : & z\zeta^2(z\zeta - t)(z\zeta - 2t)^2 - s = 0 \\ g_1 : & z = \dfrac{1}{w}, \quad \zeta = w^2\eta + tw. \end{cases} \tag{20.2.2}$$

Further propagations of these atlases along the branch $\mathrm{Br}^{(3)}$ are explicitly given in §20.2.3, p357.

20.2.3 Three different complete propagations

We provide three different complete propagations of the deformation atlases (20.2.1), p357 and (20.2.2), p357 along the branch $\mathrm{Br}^{(3)}$ explicitly.

Construction 1. The first construction of a complete propagation of (20.2.1) is as follows:

$$\begin{cases} \mathcal{H}_2: & w(w\eta+t)^2(w\eta-t)^2-s=0 \\ \mathcal{H}_2': & z\zeta^2(z\zeta-2t)^2-s=0 \\ g_2: & z=\frac{1}{w}, \quad \zeta=w^2\eta+tw. \end{cases}$$

$$\begin{cases} \mathcal{H}_3: & w^2\eta(w\eta-2t)^2-s=0 \\ \mathcal{H}_3': & \zeta(z\zeta-2t)^2-s=0 \\ g_3: & z=\frac{1}{w}, \quad \zeta=w^2\eta. \end{cases}$$

$$\begin{cases} \mathcal{H}_4: & w(w\eta-2t)^2-s=0 \\ \mathcal{H}_4': & z\zeta^2-s=0 \\ g_4: & z=\frac{1}{w}, \quad \zeta=w^2\eta-2tw. \end{cases}$$

$$\begin{cases} \mathcal{H}_5: & w^2\eta-s=0 \\ \mathcal{H}_5': & \zeta-s=0 \\ g_5: & z=\frac{1}{w}, \quad \zeta=w^2\eta. \end{cases}$$

The deformation from X to $X_{0,t}$ around the branch $\mathrm{Br}^{(3)}$ is shown in Figure 20.2.6.

Construction 2. Next we give another construction of a complete propagation of (20.2.1).

$$\begin{cases} \mathcal{H}_2: & w(w\eta+t)^2(w\eta-t)^2-s=0 \\ \mathcal{H}_2': & z\zeta^2(z\zeta-2t)^2-s=0 \\ g_2: & z=\frac{1}{w}, \quad \zeta=w^2\eta+tw. \end{cases}$$

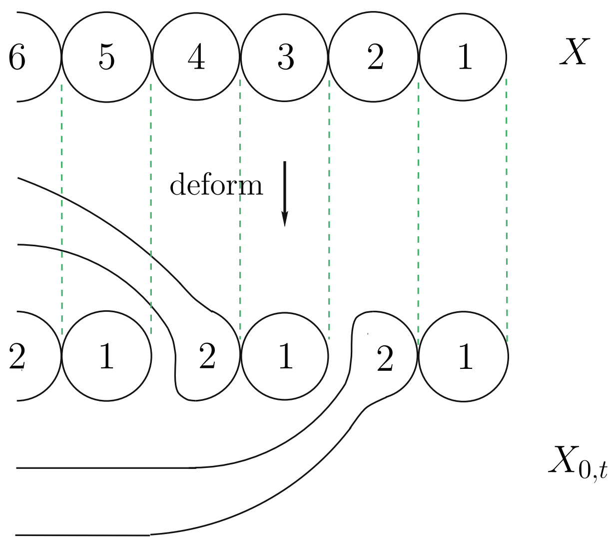


Fig. 20.2.6. Construction 1 (deformation of $\mathrm{Br}^{(3)}$)

$$\begin{cases} \mathcal{H}_3: & w^2\eta(w\eta-2t)^2-s=0 \\ \mathcal{H}_3': & z\zeta^2(z\zeta+2t)-s=0 \\ g_3: & z=\dfrac{1}{w}, \quad \zeta=w^2\eta-2tw. \end{cases}$$

$$\begin{cases} \mathcal{H}_4: & w^2\eta(w\eta+2t)-s=0 \\ \mathcal{H}_4': & \zeta(z\zeta+2t)-s=0 \\ g_4: & z=\dfrac{1}{w}, \quad \zeta=w^2\eta. \end{cases}$$

$$\begin{cases} \mathcal{H}_5: & w(w\eta+2t)-s=0 \\ \mathcal{H}_5': & \zeta-s=0 \\ g_5: & z=\dfrac{1}{w}, \quad \zeta=w^2\eta+2tw. \end{cases}$$

The deformation from X to $X_{0,t}$ around the branch $\mathrm{Br}^{(3)}$ is shown in Figure 20.2.7.

Construction 3. Finally, we give a complete propagation of (20.2.2), p357.

$$\begin{cases} \mathcal{H}_2: & w^2\eta(w\eta-t)(w\eta-2t)^2-s=0 \\ \mathcal{H}_2': & \zeta(z\zeta-t)(z\zeta-2t)^2-s=0 \\ g_2: & z=\dfrac{1}{w}, \quad \zeta=w^2\eta. \end{cases}$$

$$\begin{cases} \mathcal{H}_3: & w(w\eta-t)(w\eta-2t)^2-s=0 \\ \mathcal{H}_3': & \zeta(z\zeta-t)^2-s=0 \\ g_3: & z=\dfrac{1}{w}, \quad \zeta=w^2\eta-tw. \end{cases}$$

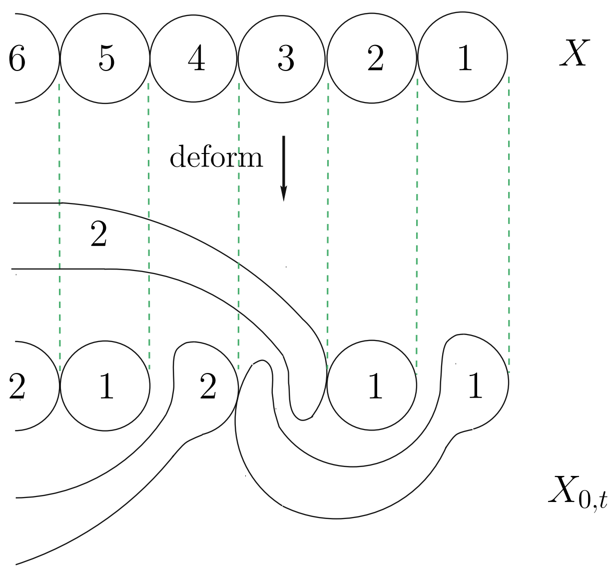


Fig. 20.2.7. Construction 2 (deformation of $\mathrm{Br}^{(3)}$)

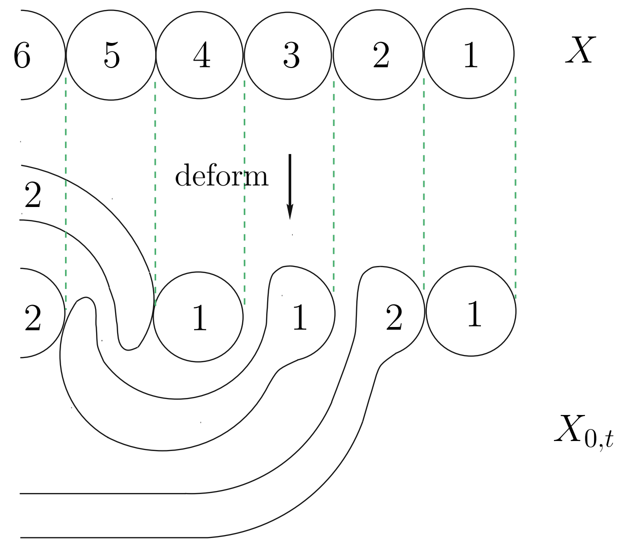


Fig. 20.2.8. Construction 3 (deformation of $\mathrm{Br}^{(3)}$)

$$\begin{cases} \mathcal{H}_4: & w(w\eta - t)^2 - s = 0 \\ \mathcal{H}'_4: & z\zeta^2 - s = 0 \\ g_4: & z = \frac{1}{w}, \quad \zeta = w^2\eta - tw. \end{cases}$$

$$\begin{cases} \mathcal{H}_5: & w^2\eta - s = 0 \\ \mathcal{H}'_5: & \zeta - s = 0 \\ g_5: & z = \frac{1}{w}, \quad \zeta = w^2\eta. \end{cases}$$

The deformation from X to $X_{0,t}$ around the branch $\mathrm{Br}^{(3)}$ is shown in Figure 20.2.8.

20.3 Example of a practical computation of a compound barking

In this section we provide an example which is heuristic to know how a practical computation to find a "barkable" crustal set goes on. Let us consider a degeneration of curves of genus 4 with the singular fiber X shown in Figure 20.3.1, where for simplicity, we take $p_1 = 1$, $p_2 = \infty$, $p_3 = 0$.

Take a set $\mathbf{Y}$ of dominant crusts in the singular fiber X as in Figure 20.3.2, and let $\mathbf{Y}'$ be the crustal set obtained from $\mathbf{Y}$ by the cut-off operation (see p304). We then consider a deformation atlas associated with $\mathbf{Y}$.

$$DA(\mathbf{Y}', \mathbf{d}) = \{\mathcal{W}_0,\ DA_0^{(1)},\ DA_3^{(2)},\ DA_5^{(3)}\}, \quad \text{(the weight } \mathbf{d} \text{ is as in Figure 20.3.2)},$$

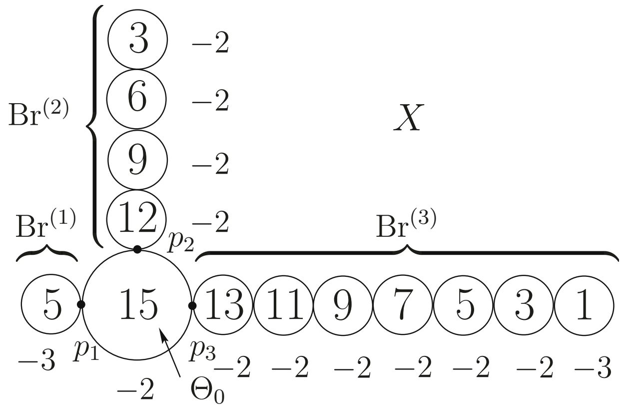


Fig. 20.3.1. genus(X) = 4. The negative integer near an irreducible component stands for the self-intersection number of that component.

where

(i) $DA_i^{(j)}$ stands for a deformation atlas of length i for the branch $\mathrm{Br}^{(j)}$,
(ii) $\mathcal{W}_0$ is a hypersurface in $N_0 \times \Delta \times \Delta^\dagger$ obtained by patching the following two hypersurfaces $\mathcal{H}_0$ and $\mathcal{H}'_0$ by a map $g_0 :\ z = 1/w,\ \zeta = w^2\eta$:

$$\begin{aligned}\mathcal{H}_0 :\quad & w^{13}(w-1)^5\eta^{15} - s + c_1tw^{12}(w-1)^5\eta^{14} + c_2t^2w^{11}(w-1)^5\eta^{13}\\ &+ c_3t^3w^{10}(w-1)^5\eta^{12} + c_4t^3w^{11}(w-1)^4\eta^{12} + c_5t^3w^{10}(w-1)^4\eta^{12}\\ &+ c_6t^6w^8(w-1)^3\eta^9 + c_7t^7w^7(w-1)^3\eta^8 + c_8t^9w^5(w-1)^2\eta^6 = 0,\end{aligned}$$

and

$$\begin{aligned}\mathcal{H}'_0 :\quad & z^{12}(1-z)^5\zeta^{15} - s + c_1tz^{11}(1-z)^5\zeta^{14} + c_2t^2z^{10}(1-z)^5\zeta^{13}\\ &+ c_3t^3z^9(1-z)^5\zeta^{12} + c_4t^3z^9(1-z)^4\zeta^{12} + c_5t^3z^{10}(1-z)^4\zeta^{12}\\ &+ c_6t^6z^7(1-z)^3\zeta^9 + c_7t^7z^6(1-z)^3\zeta^8 + c_8t^9z^5(1-z)^2\zeta^6 = 0.\end{aligned}$$

We claim that $DA(\mathbf{Y}', \mathbf{d})$ admits a complete propagation; since the set $\mathbf{br}^{(1)} = \{\mathrm{br}_1^{(1)}, \mathrm{br}_2^{(1)}, \dots, \mathrm{br}_8^{(1)}\}$ is tame, $DA_0^{(1)}$ admits a complete propagation along the branch $\mathrm{Br}^{(1)}$ by Theorem 14.2.7, p260. Thus it suffices to show that $DA_3^{(2)}$ and $DA_5^{(3)}$ admit complete propagations along the branches $\mathrm{Br}^{(2)}$ and $\mathrm{Br}^{(3)}$ respectively.

Complete propagation along the branch $\mathbf{Br}^{(2)}$

We construct a complete propagation of the deformation atlas $DA_3^{(2)}$ (the subscript means "length 3") along the branch $\mathrm{Br}^{(2)}$. We first change coordinates

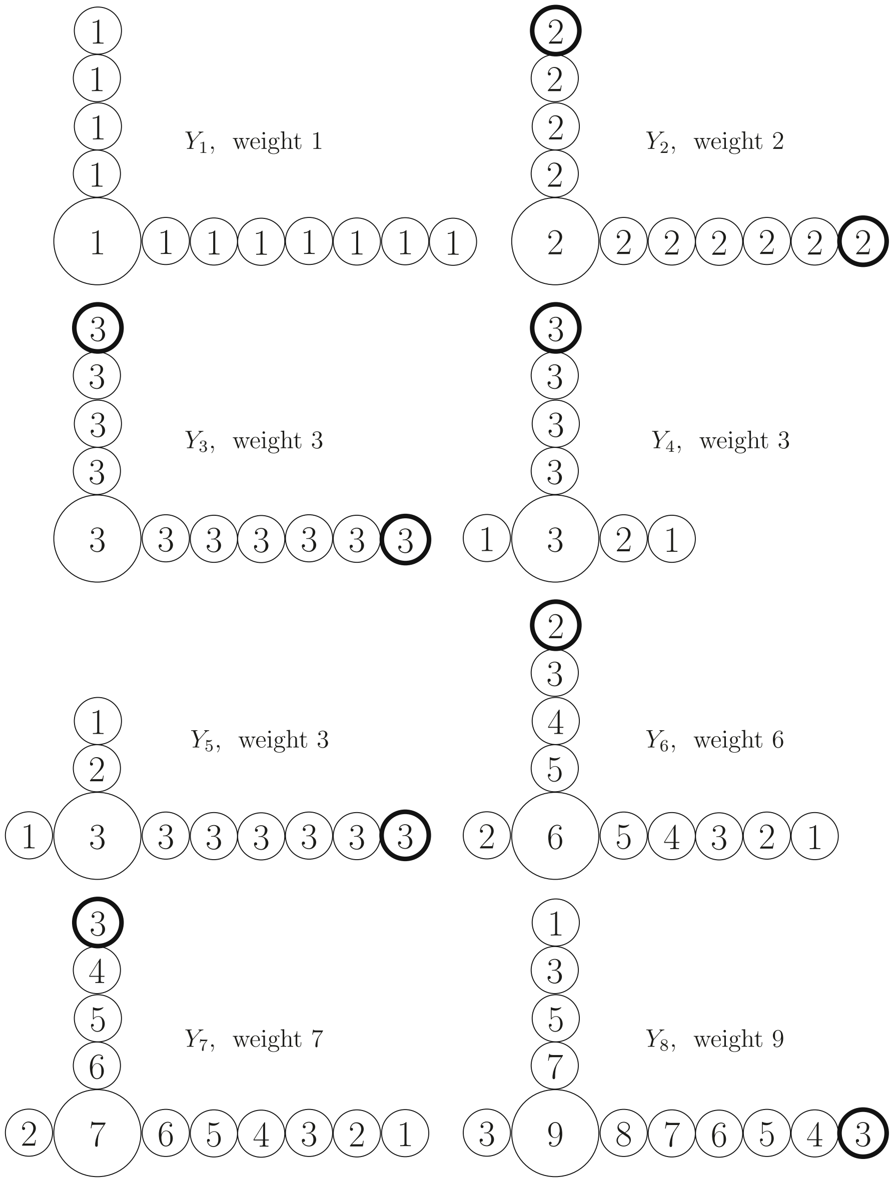


Fig. 20.3.2. A set of weighted dominant crusts for X in Figure 20.3.1. Each bold circle is a wild irreducible component.

around p_2; rewriting $(1-z)^{1/3}\zeta$ by ζ, we have

$$\begin{aligned}\mathcal{H}_0' :\quad & z^{12}\zeta^{15} - s + c_1 t z^{11}\zeta^{14}(1-z)^{1/3} + c_2 t^2 z^{10}\zeta^{13}(1-z)^{2/3}\\ & + c_3 t^3 z^9 \zeta^{12}(1-z) + c_4 t^3 z^9 \zeta^{12} + c_5 t^3 z^{10}\zeta^{12}\\ & + c_6 t^6 z^7 \zeta^9 + c_7 t^7 z^6 \zeta^8 (1-z)^{1/3} + c_8 t^9 z^5 \zeta^6 = 0.\end{aligned}$$

Under a coordinate change $(w, \eta) = (\zeta, z)$, we obtain

$$\begin{aligned}\mathcal{H}_1^{(2)} :\quad & w^{15}\eta^{12} - s + c_1 t w^{14}\eta^{11}(1-\eta)^{1/3} + c_2 t^2 w^{13}\eta^{10}(1-\eta)^{2/3}\\ & + c_3 t^3 w^{12}\eta^9(1-\eta) + c_4 t^3 w^{12}\eta^9 + c_5 t^3 w^{12}\eta^{10}\\ & + c_6 t^6 w^9 \eta^7 + c_7 t^7 w^8 \eta^6 (1-\eta)^{1/3} + c_8 t^9 w^6 \eta^5 = 0.\end{aligned}$$

Note that the deformation atlas $DA_3^{(2)}$ consists of the first, second, and third propagations of $\mathcal{H}_1^{(2)}$ along the branch $\mathrm{Br}^{(2)}$ such that g_i $(i = 1, 2, 3)$ is the transition function of N_i. We next construct a fourth propagation; since $\Theta_4^{(2)}$ is the wild component of the set $\{\mathrm{br}_1^{(2)}, \mathrm{br}_2^{(2)}, \ldots, \mathrm{br}_8^{(2)}\}$ (see Figure 20.3.3), we require careful consideration to construct it. Note that

$$\begin{aligned}\mathcal{H}_4^{(2)} :\quad & w^6\eta^3 - s + c_1 t w^5 \eta^2 (1 - w^3\eta^4)^{1/3} + c_2 t^2 w^4 \eta (1 - w^3 \eta^4)^{2/3}\\ & + c_3 t^3 w^3 (1 - w^3\eta^4) + c_4 t^3 w^3 + c_5 t^3 w^6 \eta^4\\ & + c_6 t^6 w^3 \eta + c_7 t^7 w^2 (1 - w^3\eta^4)^{1/3} + c_8 t^9 w^3 \eta^2 = 0.\end{aligned}$$

Set $g_4^{(2)} :\ z = 1/w,\ \zeta = w^2\eta + t\alpha w$, which transforms $\mathcal{H}_4^{(2)}$ to

$$\begin{aligned}\mathcal{H}_4^{(2)\prime} : & \left(\zeta - \frac{t\alpha}{z}\right)^3 - s + c_1 \frac{t}{z}\left(\zeta - \frac{t\alpha}{z}\right)^2 \left[1 - z^5\left(\zeta - \frac{t\alpha}{z}\right)^4\right]^{1/3}\\ & + c_2 \frac{t^2}{z^2}\left(\zeta - \frac{t\alpha}{z}\right)\left[1 - z^5\left(\zeta - \frac{t\alpha}{z}\right)^4\right]^{2/3} + c_3 \frac{t^3}{z^3}\left[1 - z^5\left(\zeta - \frac{t\alpha}{z}\right)^4\right]\\ & + c_4 \frac{t^3}{z^3} + c_5 t^3 z^2 \left(\zeta - \frac{t\alpha}{z}\right)^4 + c_6 \frac{t^6}{z}\left(\zeta - \frac{t\alpha}{z}\right)\\ & + c_7 \frac{t^7}{z^2}\left[1 - z^5\left(\zeta - \frac{t\alpha}{z}\right)^4\right]^{1/3} + c_8 t^9 z \left(\zeta - \frac{t\alpha}{z}\right)^2 = 0.\end{aligned}$$

The expansion of this expression has no fractional terms exactly when the coefficients of the following terms in the expansion are zero: $\frac{t\zeta^2}{z}$, $\frac{t^2\zeta}{z^2}$, $\frac{t^3}{z^3}$, $\frac{t^6\zeta}{z}$, $\frac{t^7}{z^2}$, $\frac{t^{11}}{z}$. This condition is given by the following equations:

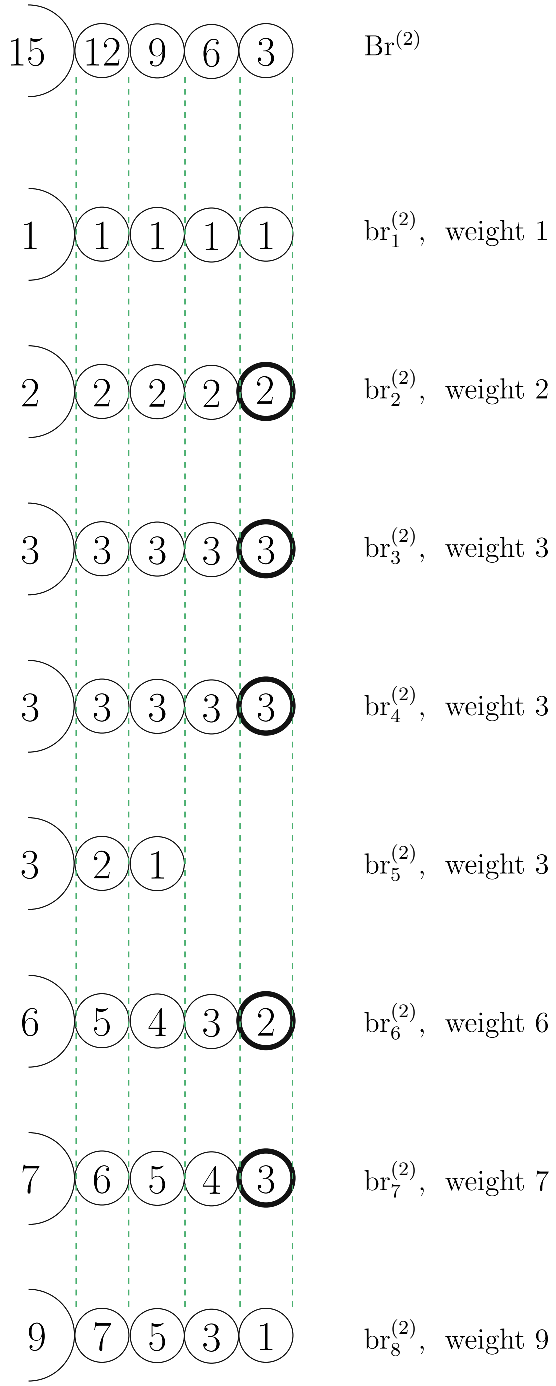


Fig. 20.3.3. The branch $\mathrm{Br}^{(2)}$ and the set of subbranches $\mathbf{br}^{(2)}$. A bold circle is a wild irreducible component.

$$-3\alpha + c_1 = 0 \tag{E1}$$
$$3\alpha^2 - 2c_1\alpha + c_2 = 0 \tag{E2}$$
$$-\alpha^3 + c_1\alpha^2 - c_2\alpha + c_3 + c_4 = 0 \tag{E3}$$
$$\frac{2}{3}c_1\alpha^5 + \frac{4}{3}c_1\alpha^5 - \frac{2}{3}c_2\alpha^4 + 4c_3\alpha^3 - 4c_5\alpha^3 + c_6 = 0 \tag{E4}$$
$$-\frac{1}{3}c_1\alpha^6 + \frac{2}{3}c_2\alpha^5 - c_3\alpha^4 + c_5\alpha^4 - c_6\alpha + c_7 = 0 \tag{E5}$$
$$-\frac{1}{9}c_1\alpha^{10} + \frac{1}{9}c_2\alpha^9 - \frac{c_7}{3}\alpha^4 + c_8\alpha^2 = 0. \tag{E6}$$

From (E1), (E2), ... ,(E6), using α, c_3, c_5 we can express other indeterminants:

$$\begin{cases} c_1 = 3\alpha, \quad c_2 = 3\alpha^2, \quad c_4 = \alpha^3 - c_3, \quad c_6 = -4c_3\alpha^3 + 4c_5\alpha^3 - 4\alpha^6 \\ c_7 = -3c_3\alpha^4 + 3c_5\alpha^4 - 5\alpha^7, \quad c_8 = -c_3\alpha^6 + c_5\alpha^6 - \frac{5}{3}\alpha^9. \end{cases} \tag{20.3.1}$$

Complete propagation along the branch $\mathbf{Br}^{(3)}$

Next we construct a complete propagation of the deformation atlas $DA_5^{(3)}$ (the subscript means "length 5") along the branch $\mathrm{Br}^{(3)}$. First, we slightly change notation; for consistency with the above discussion, in $\mathcal{H}_0$ we will write (z, ζ) instead of (w, η), and divide the equation by -1:

$$\begin{aligned}\mathcal{H}_0: \quad & z^{13}(1-z)^5\zeta^{15} + s + c_1tz^{12}(1-z)^5\zeta^{14} + c_2t^2z^{11}(1-z)^5\zeta^{13} \\ & + c_3t^3z^{10}(1-z)^5\zeta^{12} - c_4t^3z^{11}(1-z)^4\zeta^{12} - c_5t^3z^{10}(1-z)^4\zeta^{12} \\ & + c_6t^6z^8(1-z)^3\zeta^9 + c_7t^7z^7(1-z)^3\zeta^8 - c_8t^9z^5(1-z)^2\zeta^6 = 0.\end{aligned}$$

Then rewriting $(1-z)^{1/3}\zeta$ by ζ, we have

$$\begin{aligned}\mathcal{H}_0': \quad & z^{13}\zeta^{15} + s + c_1tz^{12}\zeta^{14}(1-z)^{1/3} + c_2t^2z^{11}\zeta^{13}(1-z)^{2/3} \\ & + c_3t^3z^{10}\zeta^{12}(1-z) - c_4t^3z^{11}\zeta^{12} - c_5t^3z^{10}\zeta^{12} \\ & + c_6t^6z^8\zeta^9 + c_7t^7z^7\zeta^8(1-z)^{1/3} - c_8t^9z^5\zeta^6 = 0.\end{aligned}$$

After a coordinate change $(w, \eta) = (\zeta, z)$, we obtain

$$\begin{aligned}\mathcal{H}_1^{(3)}: \quad & w^{15}\eta^{13} + s + c_1tw^{14}\eta^{12}(1-\eta)^{1/3} + c_2t^2w^{13}\eta^{11}(1-\eta)^{2/3} \\ & + c_3t^3w^{12}\eta^{10}(1-\eta) - c_4t^3w^{12}\eta^{11} - c_5t^3w^{12}\eta^{10} \\ & + c_6t^6w^9\eta^8 + c_7t^7w^8\eta^7(1-\eta)^{1/3} - c_8t^9w^6\eta^5 = 0.\end{aligned}$$

Note that the deformation atlas $DA_5^{(3)}$ consists of propagations from the first to the fifth, of $\mathcal{H}_1^{(3)}$ along the branch $\mathrm{Br}^{(3)}$ such that g_i $(i = 1, 2, 3, 4, 5)$ is the transition function of N_i. We proceed to construct a sixth propagation;

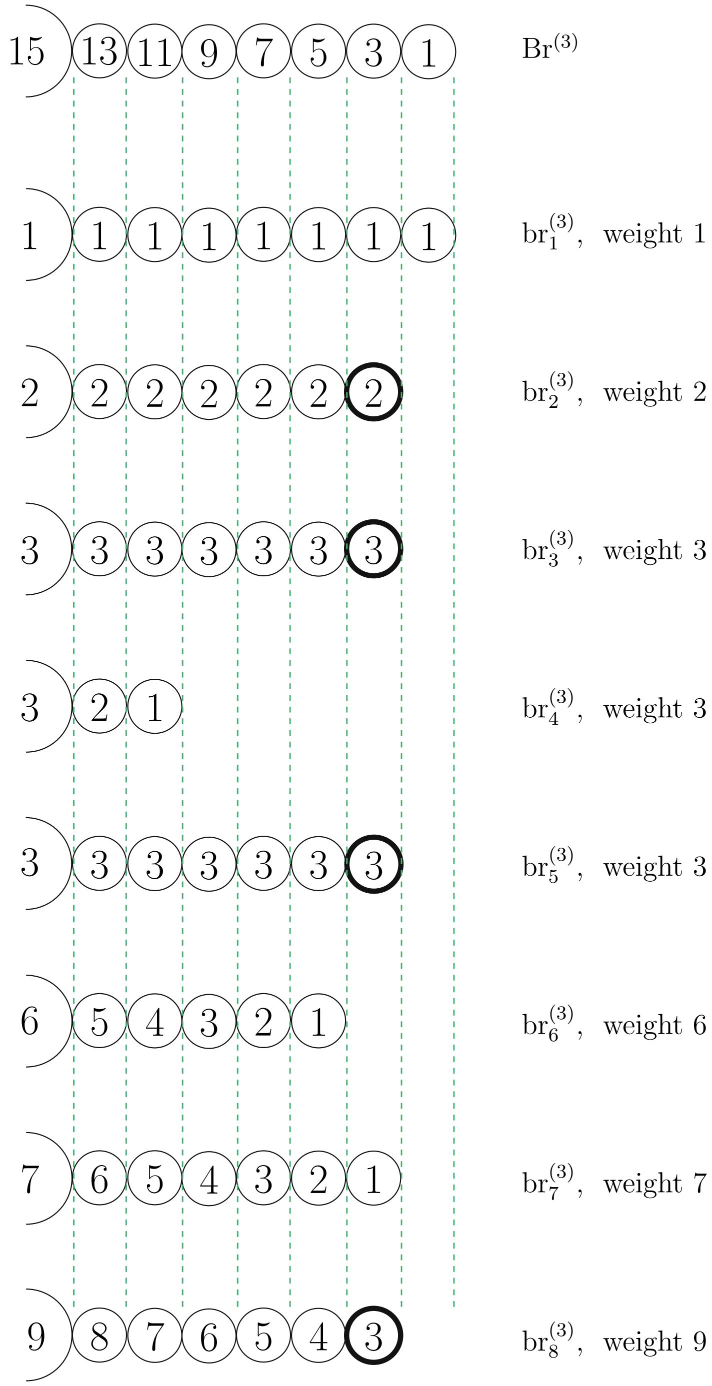


Fig. 20.3.4. The branch $\mathrm{Br}^{(3)}$ and the set of subbranches $\mathbf{br}^{(3)}$. A bold circle is a wild irreducible component.

since $\Theta_6^{(3)}$ is the wild component of the set $\{\mathrm{br}_1^{(3)}, \mathrm{br}_2^{(3)}, \dots, \mathrm{br}_8^{(3)}\}$ (see Figure 20.3.4), we need careful consideration in order to construct it. Note that

$$\begin{aligned}\mathcal{H}_6^{(3)}: \quad & w^5\eta^3 - s + c_1 t w^4\eta^2(1-w^5\eta^6)^{1/3} + c_2 t^2 w^3 \eta(1-w^5\eta^6)^{2/3} \\ & + c_3 t^3 w^2(1-w^5\eta^6) - c_4 t^3 w^7\eta^6 - c_5 t^3 w^2 \\ & + c_6 t^6 w^4 \eta^3 + c_7 t^7 w^3 \eta^2 (1-w^5\eta^6)^{1/3} - c_8 t^9 w = 0.\end{aligned}$$

Set $g_6^{(3)}: \ z = 1/w, \ \zeta = w^2\eta + t\beta w$, which transforms $\mathcal{H}_6^{(3)}$ to

$$\begin{aligned}\mathcal{H}_6^{(3)\prime}: & z\left(\zeta - \frac{t\beta}{z}\right)^3 + s + c_1 t\left(\zeta - \frac{t\beta}{z}\right)^2\left[1 - z^7\left(\zeta - \frac{t\beta}{z}\right)^6\right]^{1/3} \\ & + c_2\frac{t^2}{z}\left(\zeta - \frac{t\beta}{z}\right)\left[1 - z^7\left(\zeta - \frac{t\beta}{z}\right)^6\right]^{2/3} + c_3\frac{t^3}{z^2}\left[1 - z^7\left(\zeta - \frac{t\beta}{z}\right)^6\right] \\ & - c_4 t^3 z^5\left(\zeta - \frac{t\beta}{z}\right)^6 - c_5\frac{t^3}{z^2} + c_6 t^6 z^2\left(\zeta - \frac{t\beta}{z}\right)^3 \\ & + c_7 t^7 z\left(\zeta - \frac{t\beta}{z}\right)^2\left[1 - z^7\left(\zeta - \frac{t\beta}{z}\right)^6\right]^{1/3} - c_8\frac{t^9}{z} = 0.\end{aligned}$$

The expansion of this expression contains no fractional terms precisely when the coefficients of the following terms in the expansion are zero: $\frac{t^2\zeta}{z}, \frac{t^3}{z^2}, \frac{t^9}{z}$. This condition is given by the following equations:

$$3\beta^2 - 2c_1\beta + c_2 = 0 \tag{E7}$$

$$-\beta^3 + c_1\beta^2 - c_2\beta + c_3 - c_5 = 0 \tag{E8}$$

$$-\frac{1}{3}c_1\beta^8 + \frac{2}{3}c_2\beta^7 - c_3\beta^6 - c_4\beta^6 - c_6\beta^3 - c_7\beta^2 - c_8 = 0. \tag{E9}$$

Recall that $c_1 = 3\alpha$ and $c_2 = 3\alpha^2$ by (20.3.1). Putting them into the equation (E7), we have $3\beta^2 - 6\alpha\beta + 3\alpha^2 = 0$. So $\beta = \alpha$, and we substitute α for β. From (E8) together with $c_1 = 3\alpha$ and $c_2 = 3\alpha^2$ in (20.3.1), we have

$$-\alpha^3 + c_3 - c_5 = 0, \tag{$*$}$$

and likewise from (E9) together with equations in (20.3.1), we obtain

$$\frac{4}{3}\alpha^3 + c_3 - c_5 = 0. \tag{$**$}$$

Note that ($*$) and ($**$) imply $-\alpha^3 = \frac{4}{3}\alpha^3$ and consequently $\alpha = 0$ (so $\beta = 0$). Then from (E8), $c_3 = c_5$, and from (E9), $c_8 = 0$. Putting them into (20.3.1), we finally obtain

$$c_1 = 0, \quad c_2 = 0, \quad c_3 = -c_4 = c_5, \quad c_6 = 0, \quad c_7 = 0, \quad c_8 = 0.$$

Hence it suffices to take $\alpha = \beta = 0$ and $c_3 \neq 0$, and let $c_4 = -c_3$ and $c_5 = c_3$. As a result, for the construction of a complete deformation atlas in question, we only need three weighted crusts $(Y_3, 3), (Y_4, 3), (Y_5, 3)$ among the eight weighted crusts in Figure 20.3.2.

Remark 20.3.1 In the above construction, although $\Theta_4^{(2)}$ and $\Theta_6^{(3)}$ are wild, we did *not* deform their transition functions.

20.4 Wild cores

Let Θ_0 be a core of the singular fiber X of a degeneration $\pi : M \to \Delta$, and let W_0 be a tubular neighborhood of Θ_0 in M. Assume that $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$ is a barking family of the degeneration $\pi : M \to \Delta$. Then recall that $\mathcal{W}_0$ is a deformation of W_0 in $\mathcal{M}$. From the definition of a barking family, $\mathcal{W}_0$ is a smooth hypersurface in the trivial deformation $N_0 \times \Delta \times \Delta^\dagger$ of N_0, where N_0 is the normal bundle of Θ_0 in M. More generally, it is sometimes possible to construct such a deformation $\mathcal{W}_0$ as is a smooth hypersurface in a non-trival defromation $\mathcal{N}_0$ of N_0 (in this case, it is generally complicated to propagate $\mathcal{W}_0$ along each branch/trunk). The resulting family is not a barking family but is a more general family. We shall exhibit such an example.

Let us consider a degeneration of elliptic curves with the singular fiber X in Figure 20.4.1. We express $X = 3\Theta_0 + \mathrm{Br}^{(1)} + \mathrm{Br}^{(2)} + \mathrm{Br}^{(3)}$ where Θ_0 is the core and $\mathrm{Br}^{(j)} = 2\Theta_1^{(j)} + \Theta_2^{(j)}$ is a branch. For simplicity, we take the intersection points of Θ_0 and $\mathrm{Br}^{(j)}$ $(j = 1, 2, 3)$ as 0, 1, ∞.

We take an open covering $\Theta_0 = U \cup V$ by two complex lines with coordinates $w \in U$ and $z \in V$ such that $z = 1/w$ on $U \cap V$; the normal bundle N_0 of Θ_0 in M is obtained by patching $(w, \eta) \in U \times \mathbb{C}$ and $(z, \zeta) \in V \times \mathbb{C}$ by $z = 1/w$, $\zeta = w^2\eta$. We next take nonzero $a, \alpha \in \mathbb{C}$ satisfying $\alpha^3 + a = 0$, and define a deformation atlas around the core Θ_0 by

$$\begin{cases} \mathcal{H}_0 : & w^2(w-1)^2\eta^3 - s + at^3w = 0 \\ \mathcal{H}_0' : & z^2(1-z)^2\zeta^3 - s + 3t\alpha z(1-z)^2\zeta^2 \\ & \qquad + 3t^2\alpha^2(1-z)^2\zeta + t^3\alpha^3(z-2) = 0 \\ g_0 : & z = \dfrac{1}{w}, \quad \zeta = w^2\eta - t\alpha w. \end{cases}$$

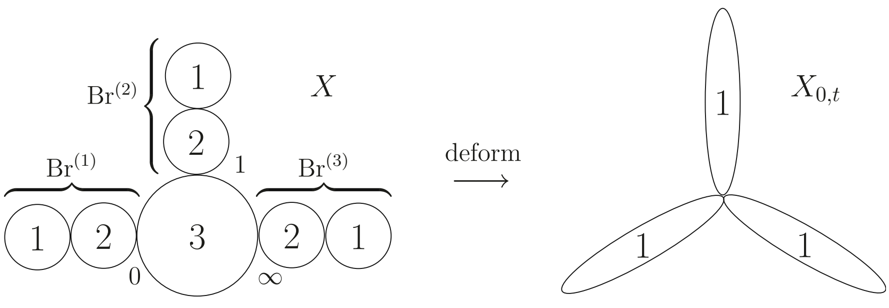


Fig. 20.4.1. $X_{0,t}$ consists of three projective lines intersecting at one point which is an ordinary triple point. In Kodaira's notation, X and $X_{0,t}$ are respectively denoted by IV^* and IV.

Using the condition $\alpha^3 = a$, it is easy to check that g_0 transforms $\mathcal{H}_0$ to $\mathcal{H}'_0$. The deformation defined by $\{\mathcal{H}_0, \mathcal{H}'_0, g_0\}$ is realized *not* in $N_0 \times \Delta \times \Delta^\dagger$, but in a nontrivial deformation $\mathcal{N}_0$ of N_0 obtained by patching $U \times \mathbb{C} \times \Delta \times \Delta^\dagger$ and $V \times \mathbb{C} \times \Delta \times \Delta^\dagger$ by g_0.

Now we set $\zeta' = (1-z)^{2/3}\zeta$, and then $\mathcal{H}'_0$ is written as

$$z^2(\zeta')^3 - s + 3t\alpha z(\zeta')^2(1-z)^{2/3} + 3t^2\alpha^2\zeta'(1-z)^{4/3} + t^3\alpha^3(z-2) = 0.$$

By a coordinate change $(w, \eta) = (\zeta', z)$, this becomes

$$\mathcal{H}_1^{(3)}: \quad w^3\eta^2 - s + 3t\,\alpha\, w^2\eta(1-\eta)^{2/3} + 3t^2\alpha^2 w(1-\eta)^{4/3} + t^3\alpha^3(\eta - 2) = 0.$$

We propagate $\mathcal{H}_1^{(3)}$ along the branch $\mathrm{Br}^{(3)}$ of X. Set $g_1^{(3)}: \; z = \frac{1}{w}$, $\zeta = w^2\eta - t\beta w$ where $\beta \in \mathbb{C}$, and then $g_1^{(3)}$ transforms $\mathcal{H}_1^{(3)}$ to

$$\begin{aligned} &z\left(\zeta + t\beta\frac{1}{z}\right)^2 - s + 3t\alpha\left(\zeta + t\beta\frac{1}{z}\right)\left\{1 - z^2\left(\zeta + t\beta\frac{1}{z}\right)\right\}^{2/3} \\ &\quad + 3t^2\alpha^2\frac{1}{z}\left\{1 - z^2\left(\zeta + t\beta\frac{1}{z}\right)\right\}^{4/3} + t^3\alpha^3\left\{z^2\left(\zeta + t\beta\frac{1}{z}\right) - 2\right\} = 0. \end{aligned}$$

The expansion of the left hand side has the fractional terms:

$$t^2\beta^2\frac{1}{z} + 3t^2\alpha\beta\frac{1}{z} + 3t^2\alpha^2\frac{1}{z}.$$

Take β such that $\beta^2 + 3\alpha\beta + 3\alpha^2 = 0$, and then $g_1^{(3)}$ transforms $\mathcal{H}_1^{(3)}$ to a hypersurface $\mathcal{H}_1^{(3)\prime}$, and $\{\mathcal{H}_1^{(3)}, \mathcal{H}_1^{(3)\prime}, g_1^{(3)}\}$ gives a first propagation. This admits a complete propagation, because the multiplicity of $\Theta_2^{(3)}$ is 1 and so we may apply Propagation Lemma 5.2.2, p88.

We next consider propagations of $\mathcal{H}_0$ along the branch $\mathrm{Br}^{(1)}$, where

$$\mathcal{H}_0: \quad w^2(w-1)^2\eta^3 - s + at^3w = 0$$

For consistency with the discussion above, we use (z, ζ)-coordinates instead of (w, η):

$$\mathcal{H}_0: \quad z^2(z-1)^2\zeta^3 - s + at^3z = 0$$

We simplify this equation; by a coordinate change $z' = z$ and $\zeta' = (z-1)^{2/3}\zeta$ around $z = 0$, the hypersurface $\mathcal{H}_0$ is locally given by

$$\mathcal{H}_0: \quad (z')^2(\zeta')^3 - s + at^3z' = 0 \quad \text{around } z' = 0.$$

We make a further coordinate change $(z', \zeta') = (\eta, w)$, under which this equation becomes $w^3\eta^2 - s + at^3\eta = 0$, being a hypersurface $\mathcal{H}_1^{(1)}$. Then it is easy to check that the following data gives a first propagation of $\mathcal{H}_1^{(1)}$ along the branch $\mathrm{Br}^{(1)}$.

$$\begin{cases} \mathcal{H}_1^{(1)}: & w^3\eta^2 - s + at^3\eta = 0 \\ \mathcal{H}_1^{(1)\prime}: & z\zeta^2 - s + at^3z^2\zeta = 0 \\ g_1^{(1)}: & z = \dfrac{1}{w}, \quad \zeta = w^2\eta. \end{cases}$$

Since the multiplicity of $\Theta_2^{(1)}$ is 1, Propagation Lemma (Lemma 5.2.2, p88) ensures its complete propagation.

Similarly, we propagate $\mathcal{H}_0: \ z^2(z-1)^2\zeta^3 - s + at^3z = 0$ along the branch $\mathrm{Br}^{(2)}$, where for consistency we use (z,ζ)-coordinates instead of (w,η). By a coordinate change $z' = z-1$ and then $\zeta' = (z'+1)^{2/3}\zeta$ around $z=1$, the hypersurface $\mathcal{H}_0$ is given by

$$(z')^2(\zeta')^3 - s + at^3(z'+1) = 0$$

which, under a coordinate change $(z',\zeta') = (\eta, w)$, becomes

$$\mathcal{H}_1^{(2)}: \quad w^3\eta^2 - s + at^3(\eta+1).$$

We set

$$\begin{cases} \mathcal{H}_1^{(2)}: & w^3\eta^2 - s + at^3(\eta+1) = 0 \\ \mathcal{H}_1^{(2)\prime}: & z\zeta^2 - s + at^3(z^2\zeta+1) = 0 \\ g_1^{(2)}: & z = \dfrac{1}{w}, \quad \zeta = w^2\eta. \end{cases}$$

This data gives a first propagation of $\mathcal{H}_0$ along $\mathrm{Br}^{(2)}$. Since the multiplicity of $\Theta_2^{(2)}$ is 1, Propagation Lemma (Lemma 5.2.2) again ensures its complete propagation. Consequently we obtain a complete deformation atlas of X, from which we can construct a splitting family: the deformation is illustrated in Figure 20.4.1.

20.5 Replacement and grafting

In this section, for singular fibers obtained by particular bondings (*replacement* and *grafting*), we construct their barking families. The construction is easy, but is useful for application.

Replacement

We consider a degeneration $\pi : M \to \Delta$ (with non-compact fibers) such that its singular fiber $X = m_0\Delta_0 + m_1\Theta_1 + \cdots + m_\lambda\Theta_\lambda$ is shown in Figure 20.5.1, where $\Delta_0 = \mathbb{C}$, and Θ_i $(i = 1, 2, \ldots, \lambda - 1)$ is a projective line, and $\mathrm{genus}(\Theta_\lambda) \geq 1$. As illustrated in Figure 20.5.1, the singular fiber X is obtained by bonding X_1 and X_2. For this particular bonding, we say that X is a *replacement* of the end component of X_1 by X_2. Note that X contains a trunk.

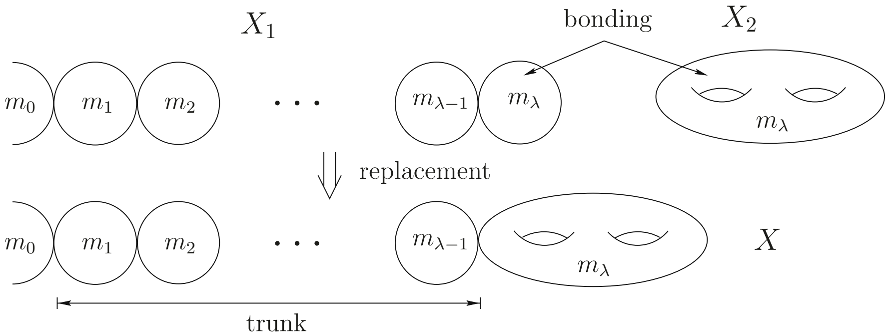


Fig. 20.5.1. X is a bonding ofX_1 and X_2.

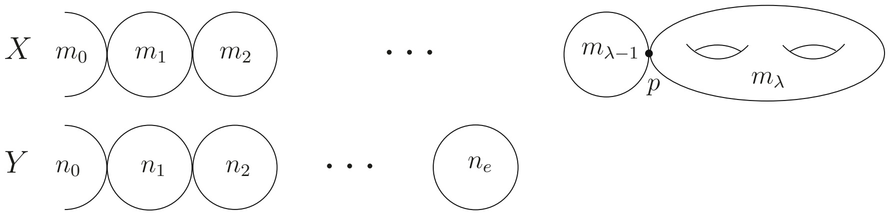


Fig. 20.5.2. Case 1: $e \leq \lambda - 1$

Let Y be the subtrunk of X shown in Figure 20.5.2 and let l be a positive integer such that $lY \leq X$. We then construct a deformation atlas $DA_{e-1}(lY, k)$. Firstly, we define a sequence of integers p_i $(i = 0, 1, \ldots, \lambda)$ inductively by

$$\begin{cases} p_0 = 0, \quad p_1 = 1 \quad \text{and} \\ p_{i+1} = r_i p_i - p_{i-1} \quad \text{for } i = 1,\, 2,\, \ldots,\, \lambda - 1. \end{cases}$$

Then $p_\lambda > p_{\lambda-1} > \cdots > p_1 > p_0 = 0$ (6.2.4), p105. Taking a non-vanishing holomorphic function $f = f(z)$, we set

$$f_i = f(w^{p_{i-1}} \eta^{p_i}), \qquad \widehat{f}_i = f(z^{p_{i+1}} \zeta^{p_i}),$$

and we define $DA_{e-1}(lY, k)$ as follows: For $i = 1, 2, \ldots, e-1$

$$\begin{cases} \mathcal{H}_i : & w^{m_{i-1} - l n_{i-1}} \eta^{m_i - l n_i} (w^{n_{i-1}} \eta^{n_i} + t^k f_i)^l - s = 0 \\ \mathcal{H}'_i : & z^{m_{i+1} - l n_{i+1}} \zeta^{m_i - l n_i} (z^{n_{i+1}} \zeta^{n_i} + t^k \widehat{f}_i)^l - s = 0 \\ g_i : & \text{the transition function } z = 1/w,\ \zeta = w^{r_i} \eta \text{ of } N_i. \end{cases} \tag{20.5.1}$$

We shall investigate when the deformation atlas $DA_{e-1}(lY, k)$ admits a complete propagation. We divide into two cases according to the length e of Y.

Case 1 $e \leq \lambda - 1$: From Proposition 18.3.3 (1), p318, it is easy to deduce the following.

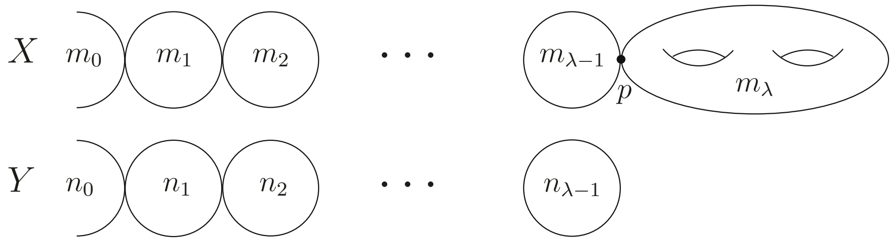


Fig. 20.5.3. Special Case 1: $e = \lambda - 1$

Criterion 20.5.1 *Suppose that the subtrunk* Y *is of type* A_l, B_l, *or* C_l *such that* (i) $\rho(Y) \leq \text{length}(X) - 1$ *where* $\rho(Y)$ *is the propagation number of* Y *(see* (16.4.2), p291) *and* (ii) *if* Y *is of type* C_l *the positive integer* k *is divisible by* n_e. *Then* $DA_{e-1}(lY, k)$ *admits a complete propagation.*

For a special case $e = \lambda - 1$, if Y is of type B_l, then by Criterion 20.5.1, $DA_{\lambda-2}(lY, k)$ admits a complete propagation. On the other hand, if Y is not of type B_l, we have the following result.

Criterion 20.5.2 *Suppose that* $l = 1$, $\frac{n_{\lambda-2}}{n_{\lambda-1}} = r_{\lambda-1}$, *and there exists an effective divisor* D *on* Θ_λ *such that* (i) *the support of* D *does not contain the intersection point* p *of* $\Theta_{\lambda-1}$ *and* Θ_λ, *and* (ii) D *is linearly equivalent to* $n_{\lambda-1}p$. *Then* $DA_{\lambda-2}(Y, k)$ *admits a complete propagation.*

Proof. Since $n_{\lambda-2}/n_{\lambda-1} = r_{\lambda-1}$, the subtrunk Y is dominant tame, and so we may construct a $(\lambda - 1)$-st propagation $DA_{\lambda-1}(Y, k)$ of $DA_{\lambda-2}(Y, k)$ as follows:

$$\begin{cases} \mathcal{H}_{\lambda-1} : & w^{m_{\lambda-2}}\eta^{m_{\lambda-1}} - s + t^k w^{m_{\lambda-2}-ln_{\lambda-1}}\eta^{m_{\lambda-1}-ln_{\lambda-1}} f_{\lambda-1} = 0 \\ \mathcal{H}'_{\lambda-1} : & z^{m_\lambda}\zeta^{m_{\lambda-1}} - s + t^k z^{m_\lambda - ln_\lambda}\zeta^{m_{\lambda-1}-ln_{\lambda-1}} \widehat{f}_{\lambda-1} = 0 \\ g_{\lambda-1} : & \text{the transition function } z = 1/w,\ \zeta = w^{r_{\lambda-1}}\eta \text{ of } N_{\lambda-1}. \end{cases}$$

By the assumption $n_\lambda = 0$, we have

$$m_\lambda n_{\lambda-1} - m_{\lambda-1} n_\lambda = m_\lambda n_{\lambda-1} \neq 0. \qquad (20.5.2)$$

Thus we may apply Simplification Lemma (Lemma 4.1.1, p58); after some coordinate change, $\mathcal{H}'_{\lambda-1}$ is of the form

$$w^{m_\lambda}\eta^{m_{\lambda-1}} - s + t^k w^{m_\lambda}\eta^{m_{\lambda-1}-n_{\lambda-1}} = 0 \qquad \text{around } p. \qquad (20.5.3)$$

To achieve a complete propagation, it remains to propagate $\mathcal{H}'_{\lambda-1}$ to a deformation atlas around Θ_λ. Note that the equation (the graph) of the degeneration $\pi : M \to \Delta$ restricted to a neighborhood of Θ_λ is given by

$$W_\lambda : \ \sigma\zeta^{m_\lambda} - s = 0,$$

where σ is the standard section on Θ_λ (that is, a holomorphic section of $N_\lambda^{\otimes(-m_\lambda)}$ with $\mathrm{div}(\sigma) = m_{\lambda-1}p$, where the point p is the intersection of $\Theta_{\lambda-1}$ and Θ_λ). By the assumption (ii), there exists a meromorphic function τ on Θ_λ such that $\mathrm{div}(\tau) = -n_{\lambda-1}p + D$. We then define a smooth hypersurface $\mathcal{W}_\lambda$ (a deformation of W_λ) in $N_\lambda \times \Delta \times \Delta^\dagger$ by

$$\mathcal{W}_\lambda : \quad \sigma\eta^{m_\lambda} - s + ct^d\sigma\tau\eta^{m_\lambda} = 0.$$

Again applying Simplification Lemma (Lemma 4.1.1), after some coordinate change, we may assume that $\mathcal{W}_\lambda$ has the form

$$z^{m_{\lambda-1}}\zeta^{m_\lambda} - s + t^d z^{m_{\lambda-1}-n_{\lambda-1}}\zeta^{m_\lambda} = 0 \qquad \text{around } p.$$

By a coordinate change $(z, \zeta) = (\eta, w)$, this equation becomes the equation (20.5.3) of $\mathcal{H}'_{\lambda-1}$, and therefore $DA_{\lambda-1}(lY, k)$ and $\mathcal{W}_\lambda$ together constitute a complete propagation of $DA_{\lambda-2}(Y, k)$. □

We next derive several consequences of Criterion 20.5.2.

Corollary 20.5.3 *Suppose that* $l = 1$, $\frac{n_{\lambda-2}}{n_{\lambda-1}} = r_{\lambda-1}$, *and* Θ_λ *is an elliptic curve. If* $n_{\lambda-1} \geq 2$, *then* $DA_{\lambda-2}(Y, k)$ *admits a complete propagation.*

Proof. Since Θ_λ is an elliptic curve and $n_{\lambda-1} \geq 2$, there exists an effective divisor D on Θ_λ such that (i) the support of D does not contain the intersection point p of $\Theta_{\lambda-1}$ and Θ_λ, and (ii) D is linearly equivalent to $n_{\lambda-1}p$. For instance, take $D = n_{\lambda-1}q$ where $q \in \Theta_\lambda$ is chosen so that $q - p$ is an n_λ-torsion in $\mathrm{Pic}(\Theta_\lambda)$, that is, $n_\lambda(q-p) \sim 0$ (linearly equivalent). Then apply Criterion 20.5.2. □

Corollary 20.5.4 *Suppose that* $l = 1$, $\frac{n_{\lambda-2}}{n_{\lambda-1}} = r_{\lambda-1}$, *and* Θ_λ *is a hyperelliptic curve. If* $n_{\lambda-1}$ *is even, and the intersection point* p *of* $\Theta_{\lambda-1}$ *and* Θ_λ *is a Weierstrass point of* Θ_λ, *then* $DA_{\lambda-2}(Y, k)$ *admits a complete propagation.*

Proof. We write $n_{\lambda-1} = 2a$ where a is a positive integer. Since p is a Weierstrass point, there are two points q_1 and q_2 on Θ_λ such that $q_1 + q_2 \sim 2p$ (linearly equivalent). In particular, $aq_1 + aq_2 \sim 2ap = n_{\lambda-1}p$. Thus $D := aq_1 + aq_2$ is an effective divisor on Θ_λ satisfying the assumption of Criterion 20.5.2, and hence the assertion follows. □

Next we discuss another case.

Case 2 $e = \lambda$:
In this case we have

Criterion 20.5.5 *Consider a singular fiber* X *and its subdivisor* Y *described in Figure* 20.5.4, *and let* $DA_{\lambda-1}$ *be a deformation atlas given as follows*[1]*: For*

[1] cf. (20.5.1). In the present case, we assume $f_i = \widehat{f}_i = 1$.

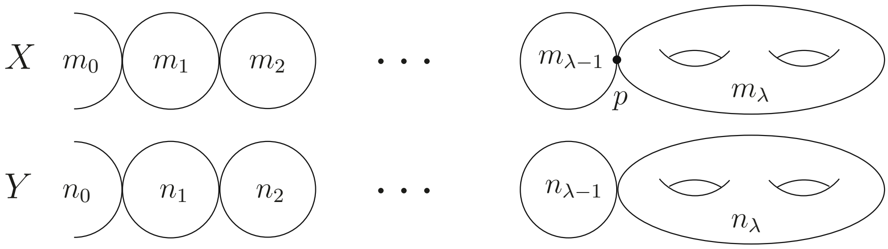


Fig. 20.5.4. Case 2: $e = \lambda$

$i = 1, 2, \ldots, \lambda - 1,$

$$\begin{cases} \mathcal{H}_i : & w^{m_{i-1}-ln_{i-1}}\eta^{m_i-ln_i}(w^{n_{i-1}}\eta^{n_i} + t^k)^l - s = 0 \\ \mathcal{H}'_i : & z^{m_{i+1}-ln_{i+1}}\zeta^{m_i-ln_i}(z^{n_{i+1}}\zeta^{n_i} + t^k)^l - s = 0 \\ g_i : & \textit{the transition function } z = 1/w,\ \zeta = w^{r_i}\eta \textit{ of } N_i. \end{cases}$$

If (i) $(m_0, m_1, \ldots, m_\lambda) = l(n_0, n_1, \ldots, n_\lambda)$ *and* (ii) $N_\lambda^{\otimes n_\lambda} \cong \mathcal{O}_{\Theta_\lambda}(-n_{\lambda-1}p)$ *where* p *is the intersection point of* $\Theta_{\lambda-1}$ *and* Θ_λ, *then* $DA_{\lambda-1}$ *admits a complete propagation.*

Proof. By (ii), there exists a meromorphic section τ of $N_\lambda^{\otimes n_\lambda}$ such that

$$\operatorname{div}(\tau) = -n_{\lambda-1}p, \quad \text{where } p := \Theta_{\lambda-1} \cap \Theta_\lambda. \tag{20.5.4}$$

From $m_\lambda = ln_\lambda$, it follows that $\sigma := 1/\tau^l$ is a holomorphic section of $N_\lambda^{\otimes(-m_\lambda)}$ such that $\operatorname{div}(\sigma) = m_{\lambda-1}p$ (that is, σ is the standard section on Θ_λ). We then define a smooth hypersurface in $N_\lambda \times \Delta \times \Delta^\dagger$ by

$$\mathcal{W}_\lambda : \quad \sigma\zeta^{m_\lambda} - s + t^k\sigma\tau\zeta^{m_\lambda - n_\lambda} = 0,$$

which is a deformation of $W_\lambda : \sigma\zeta^{m_\lambda} - s = 0$. Using $\sigma = 1/\tau^l$, we rewrite this as

$$\mathcal{W}_\lambda : \quad \frac{1}{\tau^l}\zeta^{m_\lambda} - s + t^k\frac{1}{\tau^{l-1}}\zeta^{m_\lambda - n_\lambda} = 0.$$

Since $m_\lambda - n_\lambda = (l-1)n_\lambda$ by $m_\lambda = ln_\lambda$, the above equation is further rewritten as

$$\mathcal{W}_\lambda : \quad \frac{1}{\tau^l}\zeta^{ln_\lambda} - s + ct^k\frac{1}{\tau^{l-1}}\zeta^{(l-1)n_\lambda}. \tag{20.5.5}$$

Next we express $\tau = h(z)/z^{n_{\lambda-1}}$ around p, where h is a non-vanishing holomorphic function, and then (20.5.5) is locally given by

$$z^{ln_{\lambda-1}}\zeta^{ln_\lambda}h^l - s + t^k z^{(l-1)n_{\lambda-1}}\zeta^{(l-1)n_\lambda}h^{l-1} \quad \text{around } p.$$

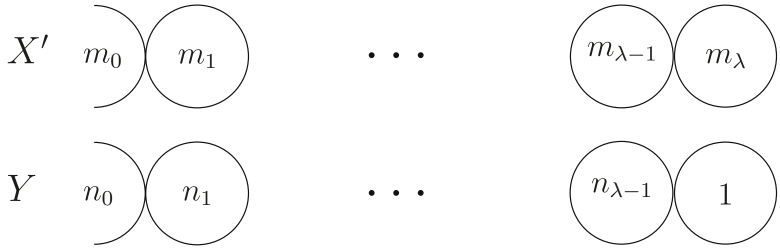


Fig. 20.5.5.

By a coordinate change[2] $(z', \zeta') = (z, \zeta h^{1/n_\lambda})$ or $(z', \zeta') = (zh^{1/n_\lambda}, \zeta)$, this equation becomes

$$(z')^{ln_{\lambda-1}}(\zeta')^{ln_\lambda} - s + t^k (z')^{(l-1)n_{\lambda-1}}(\zeta')^{(l-1)n_\lambda} = 0.$$

Using $ln_{\lambda-1} = m_{\lambda-1}$ and $ln_\lambda = m_\lambda$, we have

$$(z')^{m_{\lambda-1}}(\zeta')^{m_\lambda} - s + t^k (z')^{m_{\lambda-1}-n_{\lambda-1}}(\zeta')^{m_\lambda - n_\lambda}. \qquad (20.5.6)$$

Thus $\mathcal{W}_\lambda$ is locally given by (20.5.6) around p. On the other hand,

$$\mathcal{H}'_{\lambda-1}: \quad w^{m_\lambda}\eta^{m_{\lambda-1}} - s + t^k w^{m_\lambda - n_\lambda}\eta^{m_{\lambda-1}-n_{\lambda-1}} = 0,$$

and hence $\mathcal{H}'_{\lambda-1}$ becomes $\mathcal{W}_\lambda$ under a coordinate change $(w, \eta) = (\zeta, z)$. Therefore $DA_{\lambda-1}$ and $\mathcal{W}_\lambda$ together give a complete deformation atlas of X.□

Grafting

We consider a branch X' and its subbranch Y described in Figure 20.5.5; we assume that $m_\lambda \geq 2$ and $n_{\lambda-1} = r'_\lambda + 1$, where $-r'_\lambda$ is the self-intersection number of Θ_λ in the "ambient complex surface" (tubular neighborhood) M' of X'.

Next, from X', we construct a singular fiber X in Figure 20.5.6 by bonding two singular fibers, as illustrated in Figure 20.5.7 (while we keep Y as above). This particular bonding is called *grafting*. We note that the self-intersection number $-r'_\lambda$ of Θ_λ in M' is different from the self-intersection number $-r_\lambda$ of Θ_λ in M (the tubular neighborhood of X). In fact,

$$r_\lambda = \frac{m_{\lambda-1} + 1 + (m_\lambda - 1)}{m_\lambda} = \frac{m_{\lambda-1}}{m_\lambda} + 1,$$

and so $r_\lambda = r'_\lambda + 1$.

[2] Since $m_\lambda = ln_\lambda$ and $m_{\lambda-1} = ln_{\lambda-1}$, we have $m_{\lambda-1}n_\lambda - m_\lambda n_{\lambda-1} = 0$, and so we cannot apply Simplification Lemma (but Lemma 4.1.5, p59 may be applicable).

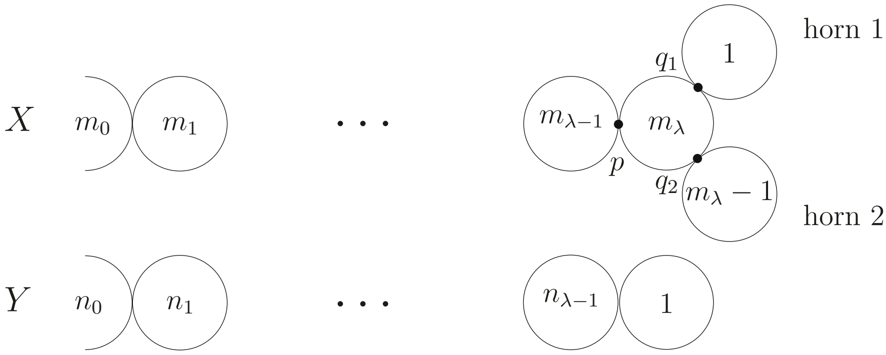


Fig. 20.5.6. X is obtained by grafting, as illustrated in Figure 20.5.7.

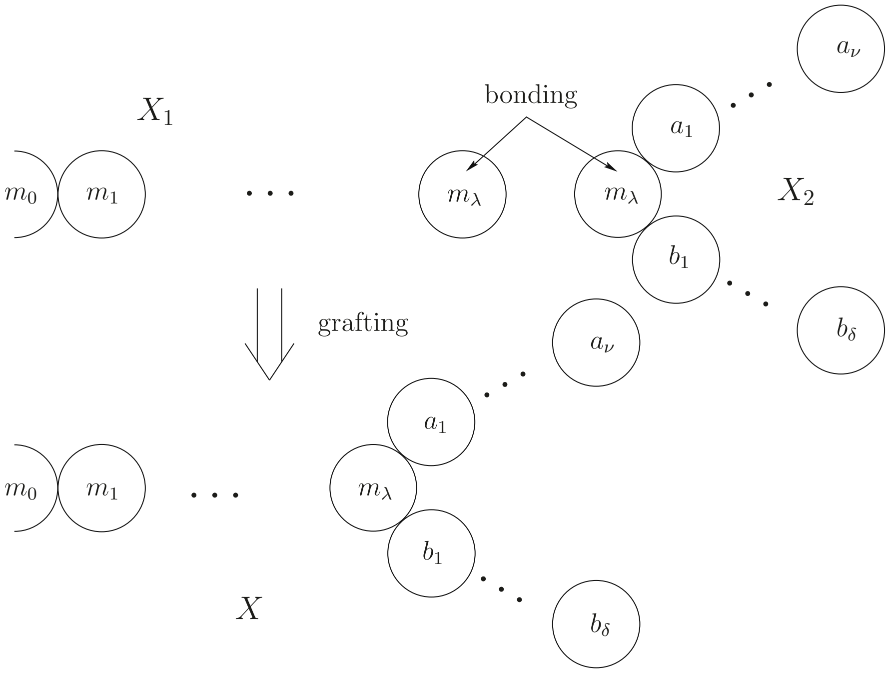


Fig. 20.5.7. X is a bonding of a branch X_1 and a singular fiber X_2 — specifically, X_2 is the singular fiber of a degeneration of projective lines such that its monodromy is a rotation of the projective line with two fixed points; the rotation angle is $2\pi a_1/m_\lambda$ around one fixed point and $2\pi b_1/m_\lambda$ around another (we note that $a_1 + b_1 = m_\lambda$ holds). For this particular bonding, X is called a *grafting* at the end component of X_1 by X_2.

Lemma 20.5.6 *Consider a singular fiber X and its subbranch Y described in* Figure 20.5.6*. Then $DA_{\lambda-1}(Y,k)$ admits a complete propagation.*

Proof. Taking three intersection points p, q_1, q_2 as in Figure 20.5.6, we define a deformation atlas around Θ_λ as follows:

$$\begin{cases} \mathcal{H}_\lambda: & (w-q_1)(w-q_2)^{m_\lambda-1}(w-p)^{m_{\lambda-1}-n_{\lambda-1}}\eta^{m_\lambda-1} \\ & \qquad \times\Big[(w-p)^{n_{\lambda-1}}\eta+t^k\Big]-s=0 \\ \mathcal{H}'_\lambda: & (1-q_1z)(1-q_2z)^{m_\lambda-1}(1-pz)^{m_{\lambda-1}-n_{\lambda-1}}\zeta^{m_\lambda-1} \\ & \qquad \times\Big[(1-pz)^{n_{\lambda-1}}\zeta+t^k\Big]-s=0 \\ g_\lambda: & \text{the transition function } z=1/w,\ \zeta=w^{r_\lambda}\eta \text{ of } N_\lambda. \end{cases}$$

By assumption, we have $n_{\lambda-1}=r'_\lambda+1$, $n_\lambda=1$, and $m_{\lambda-1}=r_\lambda m_\lambda$. So

$$m_\lambda n_{\lambda-1}-m_{\lambda-1}n_\lambda=m_\lambda(r'_\lambda+1)-r'_\lambda m_\lambda=m_\lambda\neq 0.$$

Thus we may apply Simplification Lemma (Lemma 4.1.1, p58) to $\mathcal{H}_\lambda$; under some coordinate change, we have

$$\mathcal{H}_\lambda: \quad w^{m_{\lambda-1}}\eta^{m_\lambda}-s+t^k w^{m_{\lambda-1}-n_{\lambda-1}}\eta^{m_\lambda-1}=0 \qquad \text{around } p.$$

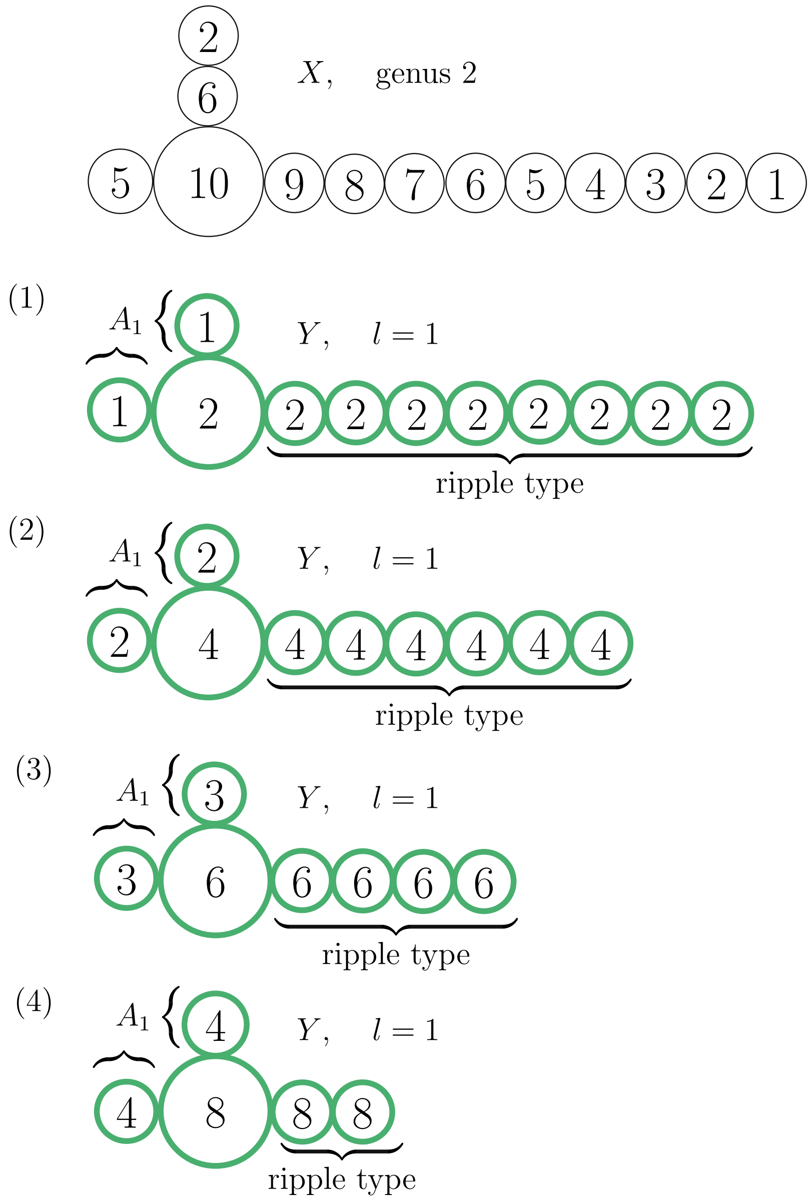
Likewise under some coordinate change, we have

$$\mathcal{H}'_\lambda: \quad z^{m_\lambda}\zeta^{m_{\lambda-1}}-s+t^k z^{m_\lambda-1}\zeta^{m_{\lambda-1}-n_{\lambda-1}}=0 \qquad \text{around } p.$$

By a coordinate change $(w,\eta)=(\zeta,z)$, $\mathcal{H}'_\lambda$ becomes $\mathcal{H}_\lambda$; so we obtain a λ-th propagation of $DA_{\lambda-1}(Y,k)$. Further propagations along two “horns” (horn 1 and horn 2 in Figure 20.5.6) are easily carried out; along horn 1 by Propagation Lemma (Lemma 5.2.2, p88), and along horn 2 by applying the construction for subbranches of type A_l. We thus accomplish the construction of a complete deformation atlas of X. □

20.6 Increasing multiplicities of simple crusts

In the barking family associated with a simple crust Y of barking multiplicity l, the subdivisor lY is barked off from X. We point out that the simple crust Y itself may be multiple (the *multiplicity* of a subdivisor $Y=\sum_i n_i\Theta_i$ is $\gcd\{n_i\}$, and if this number is greater than 1, then Y is called *multiple*). In some interesting cases, we can ‘increase’ the multiplicity of Y. See four examples of Y in Figure 20.6.1, where the barking multiplicity l is 1 for all cases, and as we increase the multiplicity of Y, the number of the connected components of barked parts $Y_{0,t}$ (gray projective lines in the figure) in $X_{0,t}$ also increases, as described in Figure 20.6.2. Note that the multiplicity of Y equals the number of the connected components of the barked parts $Y_{0,t}$.



X, genus 2
2
6
5
10
9
8
7
6
5
4
3
2
1
(1)
A_1
1
Y, $l = 1$
1
2
2 2 2 2 2 2 2 2
ripple type
(2)
A_1
2
Y, $l = 1$
2
4
4 4 4 4 4 4
ripple type
(3)
A_1
3
Y, $l = 1$
3
6
6 6 6 6
ripple type
(4)
A_1
4
Y, $l = 1$
4
8
8 8
ripple type

Fig. 20.6.1.

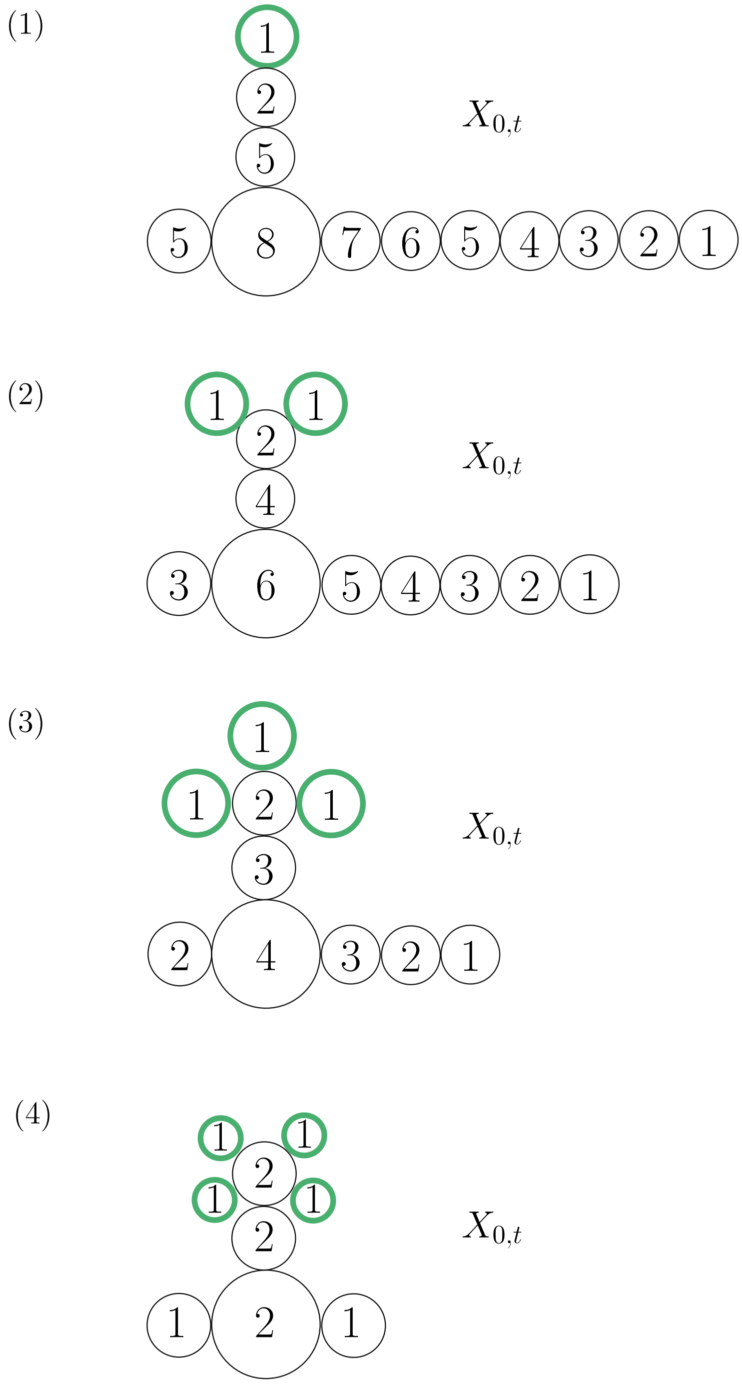


Fig. 20.6.2. The figures of $X_{0,t}$ for the barking families associated with the simple crusts in Figure 20.6.2 where in each case, $Y_{0,t}$ is expressed by gray projective lines.

Part IV

Singularities of Subordinate Fibers near Cores

21

Singularities of Fibers around Cores

In this chapter, we complete the description of the singularities of subordinate fibers of barking families (recall that a singular fiber $X_{s,t}$ is called a *subordinate fiber* if $s, t \neq 0$). For any barking family, we already described such singularities of a subordinate fiber as appear near the branches/trunks of the original singular fiber X; so it remains to describe the singularities appearing near each core of X. Unless otherwise mentioned, throughout this chapter, $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$ denotes the "restriction" of a barking family around a core C.

Let l, m, n, m_j $(j = 1, 2, \ldots, h)$ be positive integers, and let n_j $(j = 1, 2, \ldots, h)$ be a nonnegative integer such that $m > ln$ and $m_j \geq ln_j$ (instead of m_0 and n_0, we simply write m and n). Assume that N is a line bundle on C such that

(i) $N^{\otimes(-m)}$ has a holomorphic section σ with a zero of order m_j at p_j $(j = 1, 2, \ldots, h)$, and
(ii) $N^{\otimes n}$ has a meromorphic section τ with a pole of order n_j at p_j $(j = 1, 2, \ldots, h)$ and with a zero of order a_i at q_i $(i = 1, 2, \ldots, k)$.

Then we consider a barking family such that a fiber $X_{s,t} := \Psi^{-1}(s, t)$ is given by

$$X_{s,t} : \qquad \sigma(z)\zeta^{m-ln}\big(\zeta^n + t\tau(z)\big)^l - s = 0.$$

In this chapter, we derive a formula of the number of the subordinate fibers in $\pi_t : M_t \to \Delta$. We also give a formula of the number of the singularities of each subordinate fiber. Further, we demonstrate that all the singularities of a subordinate fiber $X_{s,t}$ $(s, t \neq 0)$ are A-singularities (whereas $X_{0,t}$ is generally non-reduced, so that it has non-isolated singularities). To deduce these results, the *plot function*

$$K(z) := n\frac{d\sigma(z)}{dz}\tau(z) + m\sigma(z)\frac{d\tau(z)}{dz}$$

on the core C (actually, $K(z)$ is a meromorphic section of some line bundle on C) plays a crucial role. By Lemma 21.2.1 below, a point $(z, \zeta) = (\alpha, \beta) \in X_{s,t}$,

$(s, t \neq 0)$ is a singularity precisely when α and β satisfy (a) $K(\alpha) = 0$, $\sigma(\alpha) \neq 0$, $\tau(\alpha) \neq 0$ and (b) $\beta^n = \frac{ln - m}{m} t\tau(\alpha)$. As indicated by this, the zeros of $K(z)$ play an important role in describing the singularities of a subordinate fiber. For instance, if α is a zero of order r of $K(z)$, then (α, β) is an A_r-singularity of $X_{s,t}$.

Main results of this chapter

(i) A point $(z, \zeta) = (\alpha, \beta)$ of a subordinate fiber $X_{s,t}$ $(s, t \neq 0)$ is a singularity exactly when (a) $K(\alpha) = 0$, $\sigma(\alpha) \neq 0$, $\tau(\alpha) \neq 0$ and (b) $\beta^n = \frac{ln - m}{m} t\tau(\alpha)$. In this case, (α, β) is an A-singularity. More precisely, if α has order r as a zero of $K(z)$, then the singularity (α, β) is an A_r-singularity. See Theorem 21.6.7, p410.

(ii) Let $\{\alpha_1, \alpha_2, \dots, \alpha_\nu\}$ be the set of zeros α of $K(z)$ such that $\sigma(\alpha) \neq 0$ and $\tau(\alpha) \neq 0$. Setting $\mu(z) := \sigma(z)^{n'}\tau(z)^{m'}$ where $m' := m/\gcd(m, n)$ and $n' := n/\gcd(m, n)$, consider a set

$$S = \{\, \mu(\alpha_1),\ \mu(\alpha_2),\ \dots,\ \mu(\alpha_\nu)\, \}.$$

Let b be the number of the distinct elements in this set. Then the number of the subordinate fibers of $\pi_t : M_t \to \Delta$ $(t \neq 0$ is fixed) is bn'. See Theorem 21.4.3, p403.

(iii) In (ii), the distinct elements are in one to one correspondence with "tassels" (a tassel is a disjoint union of certain n' subordinate fibers). Suppose that a subordinate fiber $X_{s,t}$ belongs to a tassel corresponding to an element, say $\lambda\,(= \lambda(X_{s,t}))$. Among zeros $\alpha_1, \alpha_2, \dots, \alpha_\nu$ of $K(z)$, if there are $c\,(= c(X_{s,t}))$ zeros of $K(z)$ which attain the value λ:

$$\lambda = \mu(\alpha_{i_1}) = \mu(\alpha_{i_2}) = \cdots = \mu(\alpha_{i_c}),$$

then the number of the singularities of $X_{s,t}$ is dc, where $d := \gcd(m, n)$. See Theorem 21.4.3, p403.

To state more results, we use the following notation:

h (resp. k): the number of the zeros of σ (resp. τ),
v: the number[1] of indices j such that $nm_j - mn_j = 0$,
g: the genus of the core C, and
$\mathrm{ord}_{p_j}(\omega)$: the order of the zero or pole of $\omega(z)$ at p_j, where $\omega(z) := d\log\big(\sigma^n\tau^m\big)$ is a meromorphic 1-form on C.

We then set

$$u := h - v + k + (2g - 2) - \sum_{j \in J_0} \mathrm{ord}_{p_j}(\omega),$$

[1] In other words, h is the number of the branches and v is the number of the proportional subbranches.

where J_0 is the set of indices j such that $nm_j - mn_j = 0$. (Actually, ω is holomorphic at p_j ($j \in J_0$) by Lemma 21.3.3 (2), p396.) By Corollary 21.4.4, p403, in $\pi_t : M_t \to \Delta$ where $t \neq 0$ is fixed,

$$\Big(\text{the number of the subordinate fibers}\Big) \leq n'u,$$
$$\Big(\text{the number of the singularities of a subordinate fiber}\Big) \leq du,$$

where $n' := n/d$ and $d := \gcd(m, n)$. Also by Proposition 21.3.6, p400,

$$\Big(\text{the number of the singularities of all subordinate fibers}\Big) \leq nu,$$

where the equality holds precisely when any zero α of $K(z)$ satisfying $\sigma(\alpha) \neq 0$ and $\tau(\alpha) \neq 0$ is of order 1. In this case, all singularities are nodes by (i) above.

21.1 Branched coverings and ramification points

First of all, we explain the relationship between singularities and branched coverings. Consider a plane curve V in $\mathbb{C}^2$ defined by a polynomial equation $F(x, y) = 0$. We let $\mathrm{pr} : V \to \mathbb{C}$ be a projection given by $(x, y) \mapsto y$. It is a branched covering, and a point $p = (x_0, y_0) \in \mathbb{C}^2$ is a ramification point of $\mathrm{pr} : V \to \mathbb{C}$ precisely when

$$F(x_0, y_0) = 0, \qquad \frac{\partial F}{\partial x}(x_0, y_0) = 0.$$

Algebraically these two equations mean that $F(x, y_0)$, regarded as a polynomial in a single variable x, has a multiple root x_0.

Next, a point $p = (x_0, y_0) \in \mathbb{C}^2$ is a singularity on V precisely when

$$F(x_0, y_0) = 0, \qquad \frac{\partial F}{\partial x}(x_0, y_0) = 0, \qquad \frac{\partial F}{\partial y}(x_0, y_0) = 0.$$

In particular, a singularity $p = (x_0, y_0)$ is necessarily a ramification point of the branched covering $\mathrm{pr} : V \to \mathbb{C}$. Note that two equations $F(x_0, y_0) = 0$ and $\frac{\partial F}{\partial x}(x_0, y_0) = 0$ do *not* imply that $p = (x_0, y_0)$ is a singularity: If $\frac{\partial F}{\partial y}(x_0, y_0) \neq 0$, then by the implicit function theorem, $V : F(x, y) = 0$ is smooth at p.

We summarize the above explanation:

Let $V : F(x, y) = 0$ be a plane curve. Suppose that a point (x_0, y_0) satisfies $F(x_0, y_0) = 0$ and $\frac{\partial F}{\partial x}(x_0, y_0) = 0$. Then (x_0, y_0) is a ramification point of the branched covering $\mathrm{pr} : (x, y) \in V \mapsto y \in \mathbb{C}$. *Moreover, according to $\frac{\partial F}{\partial y}(x_0, y_0) \neq 0$ or $= 0$, the point (x_0, y_0) is a smooth point or a singularity of V.*

In this chapter, we often regard a singularity as a ramification point of a branched covering, and then we deduce important results.

Remark 21.1.1 In the above, we assumed that $F(x, y)$ is a polynomial. Actually, also for an analytic function $F(x, y)$, as long as we focus our attention on a sufficiently small neighborhood (germ) of $p = (x_0, y_0)$, we may assume that it is defined by a polynomial. In fact, if p is an isolated singularity of V, then by Artin's Approximation Theorem [Art2], the germ of p in V is isomorphic to an *algebraic* singularity — a singularity defined by a polynomial.

Now we turn to the discussion of barking families around a core. Throughout this chapter, by a barking family, we mean its restriction to a neighborhood of a core. Unless otherwise mentioned, $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$ stands for a barking family restricted to a neighborhood of a core C. We prepare notations. Let l, m, n, m_j be positive integers ($j = 1, 2, \ldots, h$), and let n_j ($j = 1, 2, \ldots, h$) be a nonnegative integer such that[2] $m > ln$ and $m_j \geq ln_j$. Assume that N is a line bundle on C such that

(i) $N^{\otimes(-m)}$ has a holomorphic section σ with a zero of order m_j at p_j ($j = 1, 2, \ldots, h$), and
(ii) $N^{\otimes n}$ has a meromorphic section τ with a pole of order n_j at p_j ($j = 1, 2, \ldots, h$) and with a zero of order a_i at q_i ($i = 1, 2, \ldots, k$).

Then we consider a barking family such that a fiber $X_{s,t} := \Psi^{-1}(s, t)$ is given by

$$X_{s,t}: \quad \sigma(z)\zeta^{m-ln}\big(\zeta^n + t\tau(z)\big)^l - s = 0.$$

For the time being, let $X_{s,t}$ ($s, t \neq 0$) be an arbitrary fiber (singular or smooth).

Lemma 21.1.2 *Let $(z, \zeta) = (\alpha, \beta)$ be a point of $X_{s,t}$. Then $s \neq 0$ if and only if $\sigma(\alpha) \neq 0$, $\beta \neq 0$, and $\beta^n + t\tau(\alpha) \neq 0$.*

Proof. This follows immediately from $s \neq 0$ and $\sigma(\alpha)\beta^{m-ln}\big(\beta^n + t\tau(\alpha)\big)^l - s = 0$. □

Now we consider a branched covering $\mathrm{pr} : X_{s,t} \to C$ given by $(z, \zeta) \mapsto z$. We inspect its ramification/branch points. (Recall that for a branched covering $\phi : \widetilde{M} \to M$, a point $p \in \widetilde{M}$ is a *ramification point* if ϕ is not one to one around it, and $\phi(p) \in M$ is a *branch point*.) Setting

$$F_{s,t}(z, \zeta) = \sigma(z)\zeta^{m-ln}\big(\zeta^n + t\tau(z)\big)^l - s,$$

we write $X_{s,t} : F_{s,t}(z, \zeta) = 0$. A point (z, ζ) is a ramification point of the branched covering $\mathrm{pr} : X_{s,t} \to C$ precisely when z and ζ satisfy

[2] Although we may also define $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$ for $m = ln$, here we only treat the case $m > ln$. The case $m = ln$, dealt with in §21.8 Supplement p417, is not important for the application to degenerations of *compact* complex curves (Lemma 16.7.1, p299).

$$F_{s,t}(z,\zeta) = 0, \qquad \frac{\partial F_{s,t}}{\partial \zeta}(z,\zeta) = 0.$$

Algebraically, these two equations imply that $F_{s,t}(z,\zeta)$, as a polynomial in ζ, has multiple roots, and so the discriminant of the polynomial $F_{s,t}(z,\zeta)$ in ζ vanishes.

Lemma 21.1.3 *Suppose that $s \neq 0$. For $(\alpha,\beta) \in X_{s,t}$, the following equivalence holds:*

$$\frac{\partial F_{s,t}}{\partial \zeta}(\alpha,\beta) = 0 \quad \Longleftrightarrow \quad \beta^n = \frac{ln-m}{m} t\tau(\alpha).$$

Proof. Since

$$\begin{aligned}
&\frac{\partial F_{s,t}}{\partial \zeta}(z,\zeta) \\
&= \sigma(z)(m-ln)\zeta^{m-ln-1}\Big(\zeta^n + t\tau(z)\Big)^l + \sigma(z)\zeta^{m-ln} l \Big(\zeta^n + t\tau(z)\Big)^{l-1} n\zeta^{n-1} \\
&= \sigma(z)\zeta^{m-ln-1}\Big(\zeta^n + t\tau(z)\Big)^{l-1}\Big(m\zeta^n + (m-ln)t\tau(z)\Big),
\end{aligned}$$

the condition $\frac{\partial F_{s,t}}{\partial \zeta}(\alpha,\beta) = 0$ is explicitly given by

$$\sigma(\alpha)\beta^{m-ln-1}\Big(\beta^n + t\tau(\alpha)\Big)^{l-1}\Big(m\beta^n + (m-ln)t\tau(\alpha)\Big) = 0.$$

By the assumption $s \neq 0$, we have $\sigma(\alpha) \neq 0$, $\beta \neq 0$, and $\beta^n + t\tau(\alpha) \neq 0$ (see Lemma 21.1.2). Thus the above equation is equivalent to $m\beta^n + (m-ln)t\tau(\alpha) = 0$, that is, $\beta^n = \frac{ln-m}{m} t\tau(\alpha)$. □

Next we show

Lemma 21.1.4 *A point $(\alpha,\beta) \in X_{s,t}$ $(s,t \neq 0)$ is a ramification point of the branched covering $\mathrm{pr} : X_{s,t} \to C$, $(z,\zeta) \mapsto z$ if and only if α and β satisfy $\beta^n = \frac{ln-m}{m} t\tau(\alpha)$. Moreover in this case, s is written by α and β: $s = \left(\frac{ln}{ln-m}\right)^l \sigma(\alpha)\beta^m$.*

Proof. Since $s \neq 0$, the equation $\frac{\partial F_{s,t}}{\partial \zeta}(\alpha,\beta) = 0$ is equivalent to $\beta^n = \frac{ln-m}{m} t\tau(\alpha)$ by Lemma 21.1.3. Thus $(\alpha,\beta) \in X_{s,t}$ is a ramification point of $\mathrm{pr} : X_{s,t} \to C$ if and only if $\beta^n = \frac{ln-m}{m} t\tau(\alpha)$.

Next from $\beta^n = \frac{ln-m}{m} t\tau(\alpha)$, we have $t\tau(\alpha) = \frac{m}{ln-m}\beta^n$. Substituting this into the defining equation of $X_{s,t}$:

$$F_{s,t}(\alpha,\beta) = \sigma(\alpha)\beta^{m-ln}\Big(\beta^n + t\tau(\alpha)\Big)^l - s = 0,$$

we then obtain

$$\sigma(\alpha)\beta^{m-ln}\Big(\beta^n + \frac{m}{ln-m}\beta^n\Big)^l - s = 0,$$

and so $s = \Big(\dfrac{ln}{ln-m}\Big)^l \sigma(\alpha)\beta^m$. This completes the proof of our assertion. □

An important restatement of this lemma is

Proposition 21.1.5 *Fix $t \neq 0$. Then $(z,\zeta) = (\alpha,\beta) \in N$ is a ramification point of* pr : $X_{s,t} \to C$ *for "some" s if and only if α and β satisfy $\beta^n = \frac{ln-m}{m}t\tau(\alpha)$. Moreover, this fiber $X_{s,t}$ is determined by α and β as $s = \Big(\dfrac{ln}{ln-m}\Big)^l \sigma(\alpha)\beta^m$.*

Proof. $\Longrightarrow$: If (α,β) is a ramification point of $\mathrm{pr} : X_{s,t} \to C$, then by Lemma 21.1.4, $\beta^n = \frac{ln-m}{m}t\tau(\alpha)$ and $s = \Big(\dfrac{ln}{ln-m}\Big)^l \sigma(\alpha)\beta^m$.

$\Longleftarrow$: Setting $s = \Big(\dfrac{ln}{ln-m}\Big)^l \sigma(\alpha)\beta^m$, we claim that (α,β) lies on $X_{s,t}$. In fact,

$$\begin{aligned}
F_{s,t}(\alpha,\beta) &= \sigma(\alpha)\beta^{m-ln}\Big(\beta^n + t\tau(\alpha)\Big)^l - s \\
&= \sigma(\alpha)\beta^{m-ln}\left(\beta^n + \frac{m}{ln-m}\beta^n\right)^l - s \qquad \text{by } \beta^n = \frac{ln-m}{m}t\tau(\alpha) \\
&= \left(\frac{m}{ln-m}\right)^l \sigma(\alpha)\beta^m - s \\
&= 0 \qquad \text{by } s = \left(\frac{ln}{ln-m}\right)^l \sigma(\alpha)\beta^m.
\end{aligned}$$

Moreover, since α and β satisfy $\beta^n = \frac{ln-m}{m}t\tau(\alpha)$, we have $\frac{\partial F_{s,t}}{\partial \zeta}(\alpha,\beta) = 0$ by Lemma 21.1.3. So, $(z,\zeta) = (\alpha,\beta) \in N$ is a ramification point of $\mathrm{pr} : X_{s,t} \to C$. □

Supposing that $(\alpha,\beta) \in X_{s,t}$ $(s,t \neq 0)$ is a ramification point of $\mathrm{pr} : X_{s,t} \to C$, we shall deduce an equation fulfilled by α. As we showed in Lemma 21.1.4,

$$\beta^n = \frac{ln-m}{m}t\tau(\alpha) \qquad \text{and} \qquad s = \left(\frac{ln}{ln-m}\right)^l \sigma(\alpha)\beta^m.$$

Eliminating β from these equations, we may derive an equation satisfied by α as follows. Let m' and n' be the relatively prime positive integers satisfying $\frac{m}{n} = \frac{m'}{n'}$, that is, $m' = m/\gcd(m,n)$ and $n' = n/\gcd(m,n)$. We then consider powers of the above equations:

$$\beta^{m'n} = \left(\frac{ln-m}{m}\right)^{m'} t^{m'}\tau(\alpha)^{m'} \qquad \text{and} \qquad s^{n'} = \left(\frac{ln}{ln-m}\right)^{n'l} \sigma(\alpha)^{n'}\beta^{n'm}.$$

Since $m'n = n'm$, we have $\beta^{m'n} = \beta^{n'm}$, and so the comparison of the above two equations yields

$$\left(\frac{ln-m}{m}\right)^{m'} t^{m'}\tau(\alpha)^{m'} = \left(\frac{ln-m}{ln}\right)^{n'l} \frac{s^{n'}}{\sigma(\alpha)^{n'}},$$

or

$$\sigma(\alpha)^{n'}\tau(\alpha)^{m'} - \left(\frac{ln-m}{ln}\right)^{n'l} \left(\frac{m}{ln-m}\right)^{m'} \frac{s^{n'}}{t^{m'}} = 0. \tag{21.1.1}$$

Therefore a branch point $\alpha \in C$ is a zero of the following function on C:

$$D_{s,t}(z) := \sigma(z)^{n'}\tau(z)^{m'} - \lambda_{s,t}, \tag{21.1.2}$$

where $m' = m/\gcd(m,n)$, $n' = n/\gcd(m,n)$, and $\lambda_{s,t} = \left(\frac{ln-m}{ln}\right)^{n'l} \times \left(\frac{m}{ln-m}\right)^{m'} \frac{s^{n'}}{t^{m'}}$. We say that $D_{s,t}(z)$ is a *discriminant function* on the core C. In terms of $D_{s,t}(z)$, the above result on ramification points is summarized as follows:

Proposition 21.1.6 *If* $(\alpha,\beta) \in N$ *is a ramification point of* $\mathrm{pr} : X_{s,t} \to C$, *then*

$$\text{(a)} \quad D_{s,t}(\alpha) = 0 \quad \textit{and} \quad \text{(b)} \quad \beta^n = \frac{ln-m}{m} t\tau(\alpha).$$

(Note: When $s \neq 0$, the condition $D_{s,t}(\alpha) = 0$ (that is, $\sigma(\alpha)^{n'}\tau(\alpha)^{m'} - \lambda_{s,t} = 0$) implies that $\sigma(\alpha) \neq 0$ and $\tau(\alpha) \neq 0$.)

Note that $(\alpha,\beta) \in N$ is a singularity of $X_{s,t}$ precisely when

$$F_{s,t}(\alpha,\beta) = 0, \qquad \frac{\partial F_{s,t}}{\partial z}(\alpha,\beta) = 0, \qquad \frac{\partial F_{s,t}}{\partial \zeta}(\alpha,\beta) = 0.$$

Among these three equations, the two equations $F_{s,t}(\alpha,\beta) = 0$ and $\frac{\partial F_{s,t}}{\partial \zeta}(\alpha,\beta) = 0$ imply that (α,β) is a ramification point of $\mathrm{pr} : X_{s,t} \to C$, $(z,\zeta) \mapsto z$. Thus, as a corollary of the above proposition, we have

Corollary 21.1.7 *If* $(\alpha,\beta) \in N$ *is a singularity of* $X_{s,t}$, *then*

$$\text{(a)} \quad D_{s,t}(\alpha) = 0 \quad \textit{and} \quad \text{(b)} \quad \beta^n = \frac{ln-m}{m} t\tau(\alpha).$$

Before proceeding, we note the following properties of the discriminant function $D_{s,t}(z)$.

Lemma 21.1.8 (1) *$D_{s,t}(z)$ is a meromorphic function on the core C.*
(2) *Assume that $s,t \neq 0$. If $D_{s,t}(\alpha) = 0$ and $\sigma(\alpha) \neq 0$, then $\tau(\alpha) \neq 0$.*
(3) *Assume that $s,t \neq 0$. Then* (i) *$D_{s,t}(z)$ has a pole of order $-(n'm_j - m'n_j)$ at p_j for j such that $n'm_j - m'n_j < 0$, and is holomorphic outside them, and* (ii) *$D_{s,t}(z)$ has no zeros at p_j and q_i for j such that $n'm_j - m'n_j > 0$ and for arbitrary i.* (Note: $D_{s,t}(z)$ may have a zero at p_j for j such that $n'm_j - m'n_j = 0$. See Remark 21.1.9 below.)

Proof. First of all, note that $\sigma^{n'}\tau^{m'}$ is a meromorphic *function* on C, because $\sigma^{n'}\tau^{m'}$ is a section of $(N^{\otimes -m})^{n'} \otimes (N^{\otimes n})^{m'} \cong \mathcal{O}_C$ (the trivial bundle). Also note that for fixed $s,t \neq 0$, $\lambda_{s,t}$ is a constant function on C. Therefore $D_{s,t}(z) = \sigma^{n'}\tau^{m'} - \lambda_{s,t}$ is a meromorphic function on C. This verifies (1). The assertion (2) is easy. When $s,t \neq 0$, we have $\lambda_{s,t} \neq 0$. On the other hand, if α is a zero of $D_{s,t}(z)$, then $\sigma(\alpha)^{n'}\tau(\alpha)^{m'} = \lambda_{s,t}$. Hence if $\sigma(\alpha) \neq 0$, then we have $\tau(\alpha) \neq 0$. Finally we show (3). Recall that $\sigma(z)$ has a zero of order m_j at p_j; while $\tau(z)$ has a pole of order n_j at p_j and has a zero of order a_i at q_i. Thus $\sigma^{n'}\tau^{m'}$ has a pole of order $-(n'm_j - m'n_j)$ at p_j if $n'm_j - m'n_j < 0$. Since the poles of $D_{s,t}(z) = \sigma^{n'}\tau^{m'} - \lambda_{s,t}$ coincide with the poles of $\sigma^{n'}\tau^{m'}$, the assertion (i) of (3) follows. Next, we note that $\sigma^{n'}\tau^{m'}$ has a zero of order $n'm_j - m'n_j$ at p_j if $n'm_j - m'n_j > 0$, and has a zero of order $m'a_i$ at each q_i. However, since $\lambda_{s,t}$ is a nonzero constant, the zeros of $\sigma^{n'}\tau^{m'}$ are *not* zeros of $D_{s,t}(z) = \sigma^{n'}\tau^{m'} - \lambda_{s,t}$. This proves (ii) of (3). □

Remark 21.1.9 For j such that $nm_j - mn_j = 0$, the point p_j may be a zero of $D_{s,t}(z)$. To see this, write $\sigma(z) = (z-p_j)^{m_j} g_j(z)$ and $\tau(z) = \dfrac{1}{(z-p_j)^{n_j}} h_j(z)$ around p_j, where $g_j(z)$ and $h_j(z)$ are non-vanishing holomorphic functions. Then

$$\begin{aligned}\sigma(z)^{n'}\tau(z)^{m'} &= (z-p_j)^{n'm_j} g_j(z)^{n'} \cdot \frac{1}{(z-p_j)^{m'n_j}} h_j(z)^{m'} \\ &= g_j(z)^{n'} h_j(z)^{m'} \qquad \text{around} \quad p_j.\end{aligned}$$

So $D_{s,t}(p_j) = g_j(p_j)^{n'} h_j(p_j)^{m'} - \lambda_{s,t}$, which may be zero.

The converse of Proposition 21.1.6 is not valid: Even if α and β satisfy

$$\text{(a)} \quad D_{s,t}(\alpha) = 0 \quad \text{and} \quad \text{(b)} \quad \beta^n = \frac{ln - m}{m} t\tau(\alpha),$$

it does not imply that (α, β) lies on $X_{s,t}$. In fact, from the two equations (a) and (b), what we can deduce is an equation

$$s^{n'} = \left(\frac{ln}{ln - m}\right)^{n'l} \sigma(\alpha)^{n'} \beta^{n'm}. \tag{21.1.3}$$

But in general, $s \neq \left(\dfrac{ln}{ln-m}\right)^{l} \sigma(\alpha)\beta^m$; in this case by Lemma 21.1.4, (α,β) does not lie on $X_{s,t}$. Actually, (α,β) lies on $X_{s',t}$ where we set $s' := \left(\dfrac{ln}{ln-m}\right)^{l} \sigma(\alpha)\beta^m$, and (α,β) is a ramification point of $\mathrm{pr} : X_{s',t} \to C$.

For our subsequent discussion, it is convenient to introduce the notion of a tassel.

Definition 21.1.10 Let $m' := m/\gcd(m,n)$ and $n' := n/\gcd(m,n)$. Fix $s \neq 0$, and set $s_j := e^{2\pi \mathrm{i} jm'/n'} s$ for each $j = 1, 2, \ldots, n'$. Then a disjoint union of

n' fibers

$$T = X_{s_1,t} \amalg X_{s_2,t} \amalg \cdots \amalg X_{s_{n'},t}$$

is referred to as a *tassel*. (When $n' = 1$, a tassel consists of a single fiber $X_{s,t}$.)

We note that if α is a zero of $D_{s,t}(z)$, then from (21.1.1), we have

$$s^{n'} = t^{m'} \left(\frac{ln-m}{m}\right)^{m'} \left(\frac{ln}{ln-m}\right)^{n'l} \sigma(\alpha)^{n'}\tau(\alpha)^{m'}.$$

For the subsequent discussion, we choose the following n'-th root:

$$s = t^{m'/n'} \left(\frac{ln-m}{m}\right)^{m'/n'} \left(\frac{ln}{ln-m}\right)^{l} \sigma(\alpha)\tau(\alpha)^{m'/n'}. \qquad (21.1.4)$$

Now let $T = X_{s_1,t} \amalg X_{s_2,t} \amalg \cdots \amalg X_{s_{n'},t}$ be a tassel, and we consider the discriminant function $D_{s_j,t}(z)$ for each $X_{s_j,t}$ $(j = 1, 2, \dots, n')$. Since $s_j^{n'} = s^{n'}$ and

$$\lambda_{s_j,t} = \left(\frac{ln-m}{ln}\right)^{n'l} \left(\frac{m}{ln-m}\right)^{m'} \frac{s_j^{n'}}{t^{m'}},$$

it follows that $\lambda_{s_1,t} = \lambda_{s_2,t} = \cdots = \lambda_{s_{n'},t}\,(= \lambda_{s,t})$. Hence the discriminant functions for the fibers $X_{s_1,t}, X_{s_2,t}, \dots, X_{s_{n'},t}$ coincide:

$$D_{s_1,t}(z) = D_{s_2,t}(z) = \cdots = D_{s_{n'},t}(z).$$

This common function is called the *discriminant function associated with the tassel* T, and we write it as $D_T(z)$. The significance of tassels is manifest in the following theorem.

Theorem 21.1.11 *Fix* $s \neq 0$, *and set* $s_j := e^{2\pi i j m'/n'} s$ $(j = 1, 2, \dots, n')$. *Consider a tassel*

$$T = X_{s_1,t} \amalg X_{s_2,t} \amalg \cdots \amalg X_{s_{n'},t}$$

and a projection $\mathrm{pr} : T \to C$, $(z, \zeta) \mapsto z$. *Then a point* $(\alpha, \beta) \in N$ *is a ramification point of* $\mathrm{pr} : T \to C$ *if and only if* α *and* β *satisfy*

$$\text{(a)} \quad D_T(\alpha) = 0 \quad \textit{and} \quad \text{(b)} \quad \beta^n = \frac{ln-m}{m} t\tau(\alpha).$$

Proof. $\Longrightarrow$: By Proposition 21.1.6.

$\Longleftarrow$: Suppose that α and β satisfy (a) and (b). Set $s' := \left(\frac{ln}{ln-m}\right)^l \sigma(\alpha)\beta^m$, and then by Proposition 21.1.6, (α, β) lies on $X_{s',t}$, and it is a ramification point of $\mathrm{pr} : X_{s',t} \to C$. The comparison of s' with (21.1.3) yields $(s')^{n'} = s^{n'}$, and hence s' is equal to s_j for some j; so (α, β) is a ramification point of $\mathrm{pr} : X_{s_j,t} \to C$. Consequently, (α, β) is a ramification point of $\mathrm{pr} : T \to C$. $\square$

Using this theorem, we may determine the number of the ramification points of $\mathrm{pr} : X_{s,t} \to C$. First, note that given a zero α of $D_{s,t}(z)$, there are n solutions $\beta_1, \beta_2, \ldots, \beta_n$ of $\beta^n = \frac{ln - m}{m} t\tau(\alpha)$:

$$\beta_k := e^{2\pi \mathrm{i} k/n} \left(\frac{ln - m}{m} t\tau(\alpha) \right)^{1/n}, \qquad k = 1, 2, \ldots, n.$$

Thus $\mathrm{pr} : T \to C$ has n ramification points with the z-coordinate α:

$$(\alpha, \beta_1),\ (\alpha, \beta_2),\ \ldots,\ (\alpha, \beta_n). \tag{21.1.5}$$

By Proposition 21.1.5, (α, β_k) lies on a fiber $X_{s_j,t}$ (where $s_j := e^{2\pi \mathrm{i} jm'/n'} s$) precisely when

$$s_j = \left(\frac{ln}{ln - m} \right)^l \sigma(\alpha) \beta_k^m.$$

Here we may rewrite the right hand side as follows:

$$\begin{aligned}
\left(\frac{ln}{ln - m} \right)^l \sigma(\alpha) \beta_k^m &= \left(\frac{ln}{ln-m} \right)^l \sigma(\alpha) e^{2\pi \mathrm{i} km/n} \left(\frac{ln-m}{m} t\tau(\alpha) \right)^{m/n} \\
&= e^{2\pi \mathrm{i} km/n} t^{m/n} \left(\frac{ln - m}{m} \right)^{m/n} \left(\frac{ln}{ln-m} \right)^l \sigma(\alpha) \tau(\alpha)^{m/n} \\
&= e^{2\pi \mathrm{i} km'/n'} t^{m'/n'} \left(\frac{ln-m}{m} \right)^{m'/n'} \left(\frac{ln}{ln-m} \right)^l \sigma(\alpha) \tau(\alpha)^{m'/n'} \\
&= e^{2\pi \mathrm{i} km'/n'} s \qquad \text{by (21.1.4).}
\end{aligned}$$

Thus a ramification point (α, β_k) lies on a fiber $X_{s_j,t}$ precisely when $s_j = e^{2\pi \mathrm{i} km'/n'} s$, that is, $k \equiv j \bmod n'$. Therefore, among the n ramification points (21.1.5), only d points (where $d := \gcd(m, n)$)

$$(\alpha, \beta_j),\ (\alpha, \beta_{j+n'}),\ \ldots,\ (\alpha, \beta_{j+(d-1)n'})$$

lie on one fiber $X_{s_j,t}$. Thus we obtain

Proposition 21.1.12 *Let $m' := m/\gcd(m, n)$ and $n' := n/\gcd(m, n)$. Fix $s \neq 0$, and set $s_j := e^{2\pi \mathrm{i} jm'/n'} s$ $(j = 1, 2, \ldots, n')$. For a tassel*

$$T = X_{s_1,t} \amalg X_{s_2,t} \amalg \cdots \amalg X_{s_{n'},t},$$

consider the discriminant function $D_T(z) = \sigma(z)^{n'} \tau(z)^{m'} - \lambda_T$ associated with T, where $\lambda_T = \left(\frac{ln - m}{ln} \right)^{n'l} \left(\frac{m}{ln - m} \right)^{m'} \frac{s^{n'}}{t^{m'}}$, and let $z = \alpha_1, \alpha_2, \ldots, \alpha_\nu$ be the zeros of $D_T(z)$. Then the following holds:

(i) *For each $\alpha = \alpha_i$ $(i = 1, 2, \ldots, \nu)$, the tassel T has n ramification points with the z-coordinate α:*

$$(\alpha, \beta_1), \quad (\alpha, \beta_2), \quad \ldots, \quad (\alpha, \beta_n),$$

where $\beta_k := e^{2\pi \mathrm{i} k/n} \left(\frac{ln - m}{m} t\tau(\alpha) \right)^{1/n}$. (So T has $n\nu$ ramification points.)

(ii) *Among the n ramification points in* (i)*, d of them* (*where* $d := \gcd(m, n)$) *lie on one fiber. Specifically,*

$$(\alpha, \beta_j),\ (\alpha, \beta_{j+n'}),\ \ldots,\ (\alpha, \beta_{j+(d-1)n'})$$

lie on $X_{s_j, t}$ for each $j = 1, 2, \ldots, n'$. (Hence $\mathrm{pr} : X_{s_j,t} \to C$ has $d\nu$ ramification points.)

21.2 Singularities of fibers

We shall investigate the singularities of a fiber $X_{s,t} = \Psi^{-1}(s,t)$ of a barking family $\Psi : \mathcal{M} \to \Delta \times \Delta^{\dagger}$. Recall that

$$X_{s,t}: \quad \sigma(z)\zeta^{m-ln}\big(\zeta^n + t\tau(z)\big)^l - s = 0,$$

where

σ is a holomorphic section of a line bundle $N^{\otimes(-m)}$ on the compact curve (the core) C with a zero of order m_j at p_j ($j = 1, 2, \ldots, h$), and
τ is a meromorphic section of a line bundle $N^{\otimes n}$ on C with a pole of order n_j at p_j ($j = 1, 2, \ldots, h$) and with a zero of order a_i at q_i ($i = 1, 2, \ldots, k$).

For fixed $t \neq 0$, we say that a singular fiber $X_{s,t}$ is a *subordinate fiber* if $s \neq 0$, while $X_{0,t}$ is the *main fiber*. In a previous chapter, we already described the main fiber; see §16.4, p288. We shall describe subordinate fibers. For simplicity, setting $F_{s,t}(z,\zeta) = \sigma(z)\zeta^{m-ln}\big(\zeta^n + t\tau(z)\big)^l - s$, we express $X_{s,t} : F_{s,t}(z,\zeta) = 0$. By definition, $(\alpha,\beta) \in X_{s,t}$ is a singularity precisely when

$$\frac{\partial F_{s,t}}{\partial z}(\alpha,\beta) = \frac{\partial F_{s,t}}{\partial \zeta}(\alpha,\beta) = 0,$$

Explicitly, these equations are respectively given by

$$\beta^{m-ln}\big(\beta^n + t\tau(\alpha)\big)^{l-1}\Big[\sigma_z(\alpha)\beta^n + t\big(\sigma_z(\alpha)\tau(\alpha) + l\sigma(\alpha)\tau_z(\alpha)\big)\Big] = 0, \tag{21.2.1}$$

$$\sigma(\alpha)\beta^{m-ln-1}\big(\beta^n + t\tau(\alpha)\big)^{l-1}\Big[m\beta^n + (m - ln)t\tau(\alpha)\Big] = 0, \tag{21.2.2}$$

where we set $\sigma_z := \dfrac{d\sigma}{dz}$ and $\tau_z := \dfrac{d\tau}{dz}$.

Lemma 21.2.1 *A point $(\alpha,\beta) \in X_{s,t}$ $(s, t \neq 0)$ is a singularity if and only if $\sigma(\alpha) \neq 0$, $\tau(\alpha) \neq 0$, and α and β satisfy*

$$n\sigma_z(\alpha)\tau(\alpha) + m\sigma(\alpha)\tau_z(\alpha) = 0 \tag{21.2.3}$$

$$\beta^n = \frac{ln - m}{m}t\tau(\alpha), \tag{21.2.4}$$

where $\sigma_z := \dfrac{d\sigma}{dz}$ and $\tau_z := \dfrac{d\tau}{dz}$. (Note: β depends on t, while α does not; the equation (21.2.3) does not contain t.)

Proof. $\Longrightarrow$: Assume that (α, β) is a singularity of $X_{s,t}$ $(s, t \neq 0)$. By Lemma 21.1.2, $\sigma(\alpha) \neq 0$, $\beta \neq 0$, and $\beta^n + t\tau(\alpha) \neq 0$. So the equations (21.2.1) and (21.2.2) are respectively equivalent to

$$\sigma_z(\alpha)\beta^n = -t\Big[\sigma_z(\alpha)\tau(\alpha) + l\sigma(\alpha)\tau_z(\alpha)\Big], \tag{21.2.5}$$

$$\beta^n = \frac{ln - m}{m} t\tau(\alpha). \tag{21.2.6}$$

(Note: (21.2.6) ensures that $\tau(\alpha) \neq 0$. In fact, if $\tau(\alpha) = 0$, then (21.2.6) implies that $\beta = 0$, but then by Lemma 21.1.2, $s = 0$; this contradicts our assumption.) We now verify that (21.2.5) is equivalent to (21.2.3). First, we show that (21.2.5) implies (21.2.3). Multiplying (21.2.6) by $\sigma_z(\alpha)$, we have

$$\sigma_z(\alpha)\beta^n = \frac{ln - m}{m} t\sigma_z(\alpha)\tau(\alpha).$$

This equation with (21.2.5) yields

$$\frac{ln - m}{m} t\sigma_z(\alpha)\tau(\alpha) = -t\Big[\sigma_z(\alpha)\tau(\alpha) + l\sigma(\alpha)\tau_z(\alpha)\Big].$$

Since $t \neq 0$ by assumption, we deduce $m\sigma(\alpha)\tau_z(\alpha) + n\sigma_z(\alpha)\tau(\alpha) = 0$. Hence we obtain the equivalence of (21.2.5) and (21.2.3). Therefore if (α, β) is a singularity of $X_{s,t}$ $(s, t \neq 0)$, then $\sigma(\alpha) \neq 0$, $\tau(\alpha) \neq 0$, and

$$n\sigma_z(\alpha)\tau(\alpha) + m\sigma(\alpha)\tau_z(\alpha) = 0, \qquad \beta^n = \frac{ln - m}{m} t\tau(\alpha).$$

$\Longleftarrow$: We can easily show this by reversing the above argument. □

Moreover we have the following result.

Lemma 21.2.2 *In Lemma* 21.2.1*, s is determined by α and β:*

$$s = \left(\frac{ln}{ln - m}\right)^l \sigma(\alpha)\beta^m.$$

Proof. The defining equation of $X_{s,t}$ evaluated at (α, β) is

$$\sigma(\alpha)\beta^{m-ln}\big(\beta^n + t\tau(\alpha)\big)^l - s = 0.$$

Substituting $t\tau(\alpha) = \frac{m}{ln - m}\beta^n$ (21.2.4) into this equation, we obtain

$$\sigma(\alpha)\beta^{m-ln}\left(\beta^n + \frac{m}{ln - m}\beta^n\right)^l - s = 0,$$

and so $s = \left(\frac{ln}{ln - m}\right)^l \sigma(\alpha)\beta^m$. □

We set $K(z) := n\dfrac{d\sigma(z)}{dz}\tau(z) + m\sigma\dfrac{d\tau(z)}{dz}(z)$, where $z \in C$. We say that $K(z)$ is the *plot function* (on the complex curve C); we often write $K(z) = n\sigma_z\tau + m\sigma\tau_z$.

Now we give an important restatement of Lemma 21.2.1 and Lemma 21.2.2.

Theorem 21.2.3 *Fix $t \neq 0$. Then a point $(z,\zeta) = (\alpha,\beta) \in N$ is a singularity of "some" subordinate fiber if and only if α and β satisfy*

$$\text{(a)}\quad K(\alpha) = 0,\ \sigma(\alpha) \neq 0,\ \text{and}\ \tau(\alpha) \neq 0 \qquad \text{and} \qquad \text{(b)}\quad \beta^n = \frac{ln - m}{m} t\tau(\alpha).$$

In this case, this subordinate fiber is $X_{s,t}$ where s is given by $s := \left(\dfrac{ln}{ln - m}\right)^l \times \sigma(\alpha)\beta^m$.

Proof. By Lemma 21.2.1, if (α,β) is a singularity of some subordinate fiber, then α and β satisfy (a) and (b). We show the converse. Suppose that α and β satisfy (a) and (b). Setting $s := \left(\dfrac{ln}{ln - m}\right)^l \sigma(\alpha)\beta^m$, then (α,β) is a point on the subordinate fiber $X_{s,t}$. In fact,

$$\begin{aligned} F_{s,t}(\alpha,\beta) &= \sigma(\alpha)\beta^{m-ln}\big(\beta^n + t\tau(\alpha)\big)^l - s \\ &= \sigma(\alpha)\beta^{m-ln}\left(\beta^n + \frac{m}{ln-m}\beta^n\right)^l - s \qquad \text{by} \quad \text{(b)} \\ &= \left(\frac{m}{ln-m}\right)^l \sigma(\alpha)\beta^m - s \\ &= 0 \qquad\qquad \text{by} \quad s = \left(\frac{ln}{ln-m}\right)^l \sigma(\alpha)\beta^m. \end{aligned}$$

Thus (α,β) is a point on $X_{s,t}$. By assumption, α and β satisfy (a) and (b), and hence by Lemma 21.2.1, (α,β) is a singularity of $X_{s,t}$. This completes the proof of our assertion. □

For the subordinate fiber $X_{s,t}$ in Theorem 21.2.3, we further derive the equation fulfilled by s and t. In the equations of Theorem 21.2.3, substituting $\beta = \left(\dfrac{ln-m}{m}t\tau(\alpha)\right)^{1/n}$ into $s = \left(\dfrac{ln}{ln-m}\right)^l \sigma(\alpha)\beta^m$, we obtain

$$s = \left(\frac{ln}{ln-m}\right)^l \sigma(\alpha)\left(\frac{ln-m}{m}t\tau(\alpha)\right)^{m/n}.$$

Let m' and n' be the relatively prime positive integers such that $\frac{m}{n} = \frac{m'}{n'}$, that is, $m' = m/\gcd(m,n)$ and $n' = n/\gcd(m,n)$. Then

$$s = \left(\frac{ln}{ln-m}\right)^l \sigma(\alpha)\left(\frac{ln-m}{m}t\tau(\alpha)\right)^{m'/n'}.$$

Thus s and t satisfy

$$s^{n'} = t^{m'}\left(\frac{ln}{ln-m}\right)^{n'l}\left(\frac{ln-m}{m}\right)^{m'} \sigma(\alpha)^{n'}\tau(\alpha)^{m'}. \qquad (21.2.7)$$

21.3 Zeros of the plot function

Let $K(z) = n\sigma_z\tau + m\sigma\tau_z$ be the plot function on the core C. In this section, we study the zeros of $K(z)$.

Lemma 21.3.1 (1) *The plot function $K(z) = n\sigma_z\tau + m\sigma\tau_z$ is a meromorphic section of a line bundle $N^{\otimes(n-m)} \otimes \Omega^1_C$ on C, where Ω^1_C is the cotangent bundle of C.* (Note: If $\deg(N) = -r$, then $\deg(N^{\otimes(n-m)} \otimes \Omega^1_C) = r(m-n) + (2g-2)$.)

(2) *The number of the zeros of $K(z)$ is finite.*

Proof. (1): We write $K(z) = \sigma\tau\dfrac{d\log(\sigma^n\tau^m)}{dz}$. Setting $f := \sigma^n\tau^m$, then f is a section of $(N^{\otimes(-m)})^{\otimes n} \otimes (N^{\otimes n})^{\otimes m} \cong \mathcal{O}_C$ (the trivial bundle on C). Hence f is a meromorphic "function" on C, and accordingly its derivative f' (or precisely $f'(z)dz$) is a meromorphic section of the cotangent bundle Ω^1_C. Therefore the ratio $\dfrac{f'}{f}\left(= \dfrac{d\log(f)}{dz}\right)$ is a meromorphic section of Ω^1_C. On the other hand, $\sigma\tau$ is a meromorphic section of $N^{\otimes(n-m)}$. Thus $K(z) = \sigma\tau\dfrac{d\log(\sigma^n\tau^m)}{dz}$ is a meromorphic section of $N^{\otimes(n-m)} \otimes \Omega^1_C$. (2): By (1), $K(z)$ is a meromorphic section of a line bundle on the *compact* complex curve C, and hence the number of the zeros of $K(z)$ is finite. □

Remark 21.3.2 $K(z)$ is not necessarily holomorphic. For instance, $m = 2$, $n = 1$, and

σ is a holomorphic section of $N^{\otimes(-2)}$ with zeros of order 1 at p_1 and p_2, and

τ is a meromorphic section of N with poles of order 1 at p_1 and p_2 and with a zero of order 1 at q_1.

In this case, $K(z) = \sigma_z\tau + 2\sigma\tau_z$ has poles of order 1 at p_1 and p_2.

Next we express $K(z) = \sigma\tau\omega$, where we set $\omega(z) := \dfrac{d\log\bigl(\sigma^n\tau^m\bigr)}{dz}$.

Lemma 21.3.3 *For $\omega(z) := \dfrac{d\log\bigl(\sigma^n\tau^m\bigr)}{dz}$, the following holds:*

(1) *$\omega(z)$ is a meromorphic section of the cotangent bundle Ω^1_C* (i.e. *$\omega(z)$ is a meromorphic 1-form*[3] *on the core C*).

(2) *$\omega(z)$ has a simple pole at p_j for such j as satisfies $nm_j - mn_j \neq 0$, whereas $\omega(z)$ is holomorphic at p_j for such j as satisfies $nm_j - mn_j = 0$.*

(3) *$\omega(z)$ has a simple pole at q_i for all $i = 1, 2, \ldots, k$.*

[3] Actually, it is standard to write $\omega = \left(\dfrac{d\log\bigl(\sigma^n\tau^m\bigr)}{dz}\right)dz$ or $d\log\bigl(\sigma^n\tau^m\bigr)$.

Proof. The assertion (1) is clear from the proof of Lemma 21.3.1 (1). We show (2). We write $\sigma = (z - p_j)^{m_j} g_j(z)$ and $\tau = \dfrac{1}{(z-p_j)^{n_j}} h_j(z)$ around $z = p_j$, where $g_j(z)$ and $h_j(z)$ are non-vanishing holomorphic functions. Then $\sigma^n \tau^m = (z - p_j)^{nm_j - mn_j} f_j(z)$ around p_j, where we set $f_j(z) := g_j(z)^n h_j(z)^m$. So

$$\frac{d\log\bigl(\sigma^n\tau^m\bigr)}{dz} = \frac{nm_j - mn_j}{z - p_j} + (\text{higher terms}) \qquad \text{around } z = p_j.$$

Therefore, if $nm_j - mn_j \neq 0$, then $\omega(z) := \dfrac{d\log\bigl(\sigma^n\tau^m\bigr)}{dz}$ has a simple pole at $z = p_j$, whereas if $nm_j - mn_j = 0$, then $\omega(z)$ is holomorphic at $z = p_j$. Finally we show (3). As in the proof of (2), we may express

$$\omega(z) = \frac{ma_i}{z - q_i} + (\text{higher terms}) \qquad \text{around } z = q_i.$$

Thus $\omega(z)$ has a simple pole at $z = q_i$. □

We let J_0 be the set of indices j such that $nm_j - mn_j = 0$. By the above lemma, p_j $(j \notin J_0)$ and q_i $(i = 1, 2, \ldots, k)$ are poles of ω, whereas p_j $(j \in J_0)$ may be a zero of ω. This fact, with the expression $K(z) = \sigma\tau\omega$, implies

Lemma 21.3.4 *For $\alpha \in C$, the following equivalence holds:*

$$K(\alpha) = 0,\ \sigma(\alpha) \neq 0 \text{ and } \tau(\alpha) \neq 0 \quad \Longleftrightarrow \quad \omega(\alpha) = 0 \text{ and } \alpha \neq p_j \ (j \in J_0).$$

Further, the order of α in $K(z)$ coincides with that of α in $\omega(z)$.

Now we can show

Lemma 21.3.5 *Let h (resp. k) be the number of the zeros of σ (resp. τ), and let v be the number of indices j such that $nm_j - n_j m = 0$, and let g be the genus of the core C. Set $\omega := \dfrac{d\log\bigl(\sigma^n\tau^m\bigr)}{dz}$, and then the following equation holds:*

$$\Bigl(\textit{the sum of the orders of zeros } \alpha \textit{ of } K(z) \textit{ such that } \sigma(\alpha) \neq 0 \textit{ and } \tau(\alpha) \neq 0\Bigr)$$
$$= \Bigl(h - v + k + (2g-2) - \sum_{j \in J_0} \operatorname{ord}_{p_j}(\omega)\Bigr),$$

where J_0 is the set of indices j such that $nm_j - mn_j = 0$, and $\operatorname{ord}_{p_j}(\omega)$ stands for the order of the zero of ω at p_j. (Note: By Lemma 21.3.3 (2), ω is holomorphic at p_j $(j \in J_0)$.)

Proof. In terms of Lemma 21.3.4, it suffices to show that

$$\Bigl(\text{the sum of the orders of zeros } \alpha \text{ of } \omega(z) \text{ such that } \alpha \neq p_j \ (j \in J_0)\Bigr)$$
$$= \Bigl(h - v + k + (2g-2) - \sum_{j \in J_0} \operatorname{ord}_{p_j}(\omega)\Bigr).$$

First of all, we note that the degree of the cotangent bundle Ω^1_C of the complex curve C of genus g is $2g-2$. Since $\omega(z)$ is a section of Ω^1_C, we have $\deg(\omega) = 2g-2$, and thus

$$2g-2 = \Big(\text{the sum of the orders of the zeros of } \omega\Big) - \Big(\text{the sum of the orders of the poles of } \omega\Big). \tag{21.3.1}$$

Here by Lemma 21.3.3, $\omega(z)$ has a simple pole at q_i for each $i = 1, 2, \ldots, k$, and at p_j for j such that $nm_j - mn_j \neq 0$ (there are $h - v$ such indices j). Thus

$$\Big(\text{the sum of the orders of the poles of } \omega\Big) = (h-v) + k.$$

On the other hand, $\omega(z)$ may have a zero at p_j for j such that $nm_j - mn_j = 0$ (Lemma 21.3.3). Hence

$$\begin{aligned}&\Big(\text{the sum of the orders of the zeros of } \omega\Big)\\ &= \Big(\text{the sum of the orders of zeros } \alpha \text{ of } \omega \text{ such that } \alpha \neq p_j \ (j \in J_0)\Big)\\ &\quad + \sum_{j \in J_0} \operatorname{ord}_{p_j}(\omega),\end{aligned}$$

where J_0 is the set of indices j such that $nm_j - mn_j = 0$. The above two equations with (21.3.1) yields the equation in the assertion. □

When C is the projective line $\mathbb{P}^1$, it may be heuristic to give the explicit form of ω (and that of the plot function $K(z)$). First we take the standard open covering $\mathbb{P}^1 = U \cup V$ by two complex lines U and V, where $z \in U$ is identified with $w \in V$ via $z = 1/w$. Let N be a line bundle on $\mathbb{P}^1$ obtained by patching $(z, \zeta) \in U \times \mathbb{C}$ with $(w, \eta) \in V \times \mathbb{C}$ via $z = 1/w$ and $\zeta = w^r \eta$, where

$$r := \frac{m_1 + m_2 + \cdots + m_h}{m}.$$

We then take σ (a holomorphic section of $N^{\otimes(-m)}$) and τ (a meromorphic section of $N^{\otimes n}$) as follows:

$$\sigma = \begin{cases} (z-p_1)^{m_1}(z-p_2)^{m_2}\ldots(z-p_h)^{m_h} & \text{on } U \\ (1-p_1w)^{m_1}(1-p_2w)^{m_2}\ldots(1-p_hw)^{m_h} & \text{on } V, \end{cases}$$

and

$$\tau = \begin{cases} \dfrac{(z-q_1)^{a_1}(z-q_2)^{a_2}\cdots(z-q_k)^{a_k}}{(z-p_1)^{n_1}(z-p_2)^{n_2}\cdots(z-p_h)^{n_h}} & \text{on } U \\ \dfrac{(1-q_1w)^{a_1}(1-q_2w)^{a_2}\cdots(1-q_kw)^{a_k}}{(1-p_1w)^{n_1}(1-p_2w)^{n_2}\cdots(1-p_hw)^{n_h}} & \text{on } V. \end{cases}$$

Since

$$\log(\sigma^n\tau^m) = \log\left[\prod_{j=1}^{h}(z-p_j)^{nm_j-mn_j}\cdot\prod_{i=1}^{k}(z-q_i)^{ma_i}\right]$$
$$= \sum_{j=1}^{h}(nm_j-mn_j)\log(z-p_j) + \sum_{i=1}^{k} ma_i\log(z-q_i),$$

we have

$$\omega = \frac{d\log(\sigma^n\tau^m)}{dz} = \sum_{j=1}^{h}\frac{nm_j-mn_j}{z-p_j} + \sum_{i=1}^{k}\frac{ma_i}{z-q_i},$$

or (from $nm_j - mn_j = 0$ for $j \in J_0$)

$$\omega = \sum_{j\notin J_0}\frac{nm_j-mn_j}{z-p_j} + \sum_{i=1}^{k}\frac{ma_i}{z-q_i}.$$

Thus ω vanishes at p_{j_0} $(j_0 \in J_0)$ precisely when

$$\sum_{j\notin J_0}\frac{nm_j-mn_j}{p_{j_0}-p_j} + \sum_{i=1}^{k}\frac{ma_i}{p_{j_0}-q_i} = 0.$$

Next we explicitly express the plot function $K(z) = \sigma\tau\dfrac{d\log\bigl(\sigma^n\tau^m\bigr)}{dz}$ $(= \sigma\tau\omega)$ on the projective line $\mathbb{P}^1$:

$$K(z) = (z-p_1)^{m_1-n_1}(z-p_2)^{m_2-n_2}\cdots(z-p_h)^{m_h-n_h}$$
$$\times(z-q_1)^{a_1}(z-q_2)^{a_2}\cdots(z-q_k)^{a_k}\Bigl(\sum_{j\notin J_0}\frac{nm_j-mn_j}{z-p_j} + \sum_{i=1}^{k}\frac{ma_i}{z-q_i}\Bigr),$$

which, after expansion, becomes a polynomial.

Now we return to the discussion on the plot function $K(z)$ for a complex curve C of arbitrary genus. We consider an obvious inequality

$$\bigl(\text{the number of the zeros of } K(z)\bigr)$$
$$\le \bigl(\text{the sum of the orders of the zeros of } K(z)\bigr).$$

In particular,

$$\bigl(\text{the number of zeros } \alpha \text{ of } K(z) \text{ such that } \sigma(\alpha)\neq 0 \text{ and } \tau(\alpha)\neq 0\bigr)$$
$$\le \bigl(\text{the sum of the orders of zeros } \alpha \text{ of } K(z) \text{ such that}$$
$$\sigma(\alpha)\neq 0 \text{ and } \tau(\alpha)\neq 0\bigr).$$

By Lemma 21.3.5, the right hand side equals $h - v + k + (2g-2) - \sum_{j\in J_0} \mathrm{ord}_{p_j}(\omega)$, and hence

$$\Big(\text{the number of zeros } \alpha \text{ of } K(z) \text{ such that } \sigma(\alpha)\neq 0 \text{ and } \tau(\alpha)\neq 0\Big) \\ \leq \Big(h - v + k + (2g-2) - \sum_{j\in J_0} \mathrm{ord}_{p_j}(\omega)\Big). \tag{21.3.2}$$

The equality holds precisely when all zeros α of $K(z)$ such that $\sigma(\alpha) \neq 0$ and $\tau(\alpha) \neq 0$ have order 1. We may apply this inequality to deduce a sharp upper bound of the number of the singularities of all subordinate fibers in $\pi_t : M_t \to \Delta$. By Theorem 21.2.3, a point $(\alpha, \beta) \in X_{s,t}$ $(s, t \neq 0)$ is a singularity of "some" subordinate fiber if and only if α and β satisfy

$$\text{(a)}\quad K(\alpha) = 0,\ \sigma(\alpha) \neq 0,\ \tau(\alpha) \neq 0 \quad \text{and} \quad \text{(b)}\quad \beta^n = \frac{ln - m}{m} t\tau(\alpha). \tag{21.3.3}$$

For each zero α of $K(z)$ satisfying $\sigma(\alpha) \neq 0$ and $\tau(\alpha) \neq 0$, there are n solutions β of the equation $\beta^n = \frac{ln - m}{m} t\tau(\alpha)$. Therefore with (21.3.2), we obtain

Proposition 21.3.6 *In $\pi_t : M_t \to \Delta$ where $t \neq 0$ is fixed,*

$$\Big(\textit{the number of the singularities of all subordinate fibers}\Big) \\ \leq n\Big(h - v + k + (2g-2) - \sum_{j\in J_0} \mathrm{ord}_{p_j}(\omega)\Big),$$

where J_0 is the set of indices j such that $nm_j - mn_j = 0$. The equality holds precisely when all zeros α of $K(z)$ such that $\sigma(\alpha) \neq 0$ and $\tau(\alpha) \neq 0$ have order 1.

21.4 The number of subordinate fibers and singularities

We may further determine the number of the subordinate fibers in $\pi_t : M_t \to \Delta$, and moreover for each subordinate fiber, we may determine the number of all singularities on it. For each α satisfying (a) in (21.3.3), the equation (b) has n solutions:

$$\beta_k = e^{2\pi i k/n}\left(\frac{ln - m}{m} t\tau(\alpha)\right)^{1/n}, \qquad k = 1, 2, \ldots, n, \tag{21.4.1}$$

and hence there exist n singularities with the z-coordinate α:

$$(\alpha, \beta_1),\quad (\alpha, \beta_2),\quad \ldots,\quad (\alpha, \beta_n). \tag{21.4.2}$$

However, these singularities are generally not on the same subordinate fiber. Indeed, by Theorem 21.2.3, (α, β_k) lies on the subordinate fiber $X_{s_k,t}$, where

$$s_k = \left(\frac{ln}{ln-m}\right)^l \sigma(\alpha)\beta_k^m = \left(\frac{ln}{ln-m}\right)^l \sigma(\alpha) e^{2\pi i k m/n} \left(\frac{ln-m}{m} t\tau(\alpha)\right)^{m/n}$$
$$= e^{2\pi i k m/n} t^{m/n} \left(\frac{ln-m}{m}\right)^{m/n} \left(\frac{ln}{ln-m}\right)^l \sigma(\alpha)\tau(\alpha)^{m/n}.$$

Or, set $m' := m/\gcd(m,n)$ and $n' := n/\gcd(m,n)$, and then a singularity (α, β_k) lies on the subordinate fiber $X_{s_k,t}$ where

$$s_k = e^{2\pi i k m'/n'} t^{m'/n'} \left(\frac{ln-m}{m}\right)^{m'/n'} \left(\frac{ln}{ln-m}\right)^l \sigma(\alpha)\tau(\alpha)^{m'/n'}, \quad k = 1, 2, \ldots, n.$$

Note that for $k = 1, 2, \ldots, n$, we have only n' distinct values of s_k. In fact, $s_k = s_{k'}$ if and only if $k \equiv k' \bmod n'$.

To clarify the subsequent argument, we use a subscript $j = 1, 2, \ldots, n'$; that is,

$$s_j = e^{2\pi i j m'/n'} t^{m'/n'} \left(\frac{ln-m}{m}\right)^{m'/n'} \left(\frac{ln}{ln-m}\right)^l \sigma(\alpha)\tau(\alpha)^{m'/n'}, \quad j = 1, 2, \ldots, n'. \tag{21.4.3}$$

Then a singularity (α, β_k) lies on a subordinate fiber $X_{s_j,t}$ precisely when $k \equiv j \bmod n'$. Therefore, among the n singularities (21.4.2), only d singularities ($d := \gcd(m,n)$)

$$(\alpha, \beta_j), \quad (\alpha, \beta_{j+n'}), \quad \ldots, \quad (\alpha, \beta_{j+(d-1)n'})$$

lie on one subordinate fiber $X_{s_j,t}$. Thus we obtain

Proposition 21.4.1 *Let α be a zero of the plot function $K(z)$ such that $\sigma(\alpha) \neq 0$ and $\tau(\alpha) \neq 0$. Then*

(i) *There are n singularities with the z-coordinate α: (α, β_1), (α, β_2), $\ldots$, (α, β_n), where $\beta_k = e^{2\pi i k/n} \left(\frac{ln-m}{m} t\tau(\alpha)\right)^{1/n}$, $\quad k = 1, 2, \ldots, n$.*

(ii) *The n singularities in (i) lie on n' subordinate fibers $X_{s_1,t}$, $X_{s_2,t}$, $\ldots$, $X_{s_{n'},t}$, where*

$$s_j = e^{2\pi i j m'/n'} t^{m'/n'} \left(\frac{ln-m}{m}\right)^{m'/n'} \left(\frac{ln}{ln-m}\right)^l \sigma(\alpha)\tau(\alpha)^{m'/n'}, \quad j = 1, 2, \ldots, n'.$$

Among these n *singularities,* d *of them* ($where$ $d := \gcd(m, n)$) *lie on one subordinate fiber. Specifically,* (α, β_j), $(\alpha, \beta_{j+n'})$, $\ldots$, $(\alpha, \beta_{j+(d-1)n'})$ *lie on* $X_{s_j,t}$ *for each* $j = 1, 2, \ldots, n'$.

We say that a zero α of $K(z)$ is *essential* if $\sigma(\alpha) \neq 0$ and $\tau(\alpha) \neq 0$. Given an essential zero α of $K(z)$, the disjoint union of n' subordinate fibers in (ii):

$$T = X_{s_1,t} \amalg X_{s_2,t} \amalg \cdots \amalg X_{s_{n'},t}$$

is nothing but a tassel, which we previously defined in Definition 21.1.10. We say that a tassel is *smooth* (resp. *singular*) if all fibers in the tassel are smooth (resp. singular). All fibers in the above tassel T are singular, and so T is a singular tassel. (Actually, if one fiber in a tassel is smooth (resp. singular), then all fibers in the tassel are necessarily smooth (resp. singular). See Proposition 21.4.1 (ii).)

We point out that two essential zeros α and α' of $K(z)$ may determine the same tassel. To explain this clearly, for s_j in Proposition 21.4.1 (ii), we write $s_j(\alpha)$ for α, and $s_j(\alpha')$ for α'. Then α and α' determine the same tassel precisely when

$$\{ s_1(\alpha), s_2(\alpha), \ldots, s_{n'}(\alpha) \} = \{ s_1(\alpha'), s_2(\alpha'), \ldots, s_{n'}(\alpha') \} \quad \text{as a set.}$$

Equivalently, $s_j(\alpha) = \big(\text{an } n'\text{-th root of the unity}\big) \cdot s_j(\alpha')$ holds for some j $(1 \leq j \leq n')$. In terms of (21.4.3), this condition is given by

$$\sigma(\alpha)\tau(\alpha)^{m'/n'} = \big(\text{an } n'\text{-th root of the unity}\big) \cdot \sigma(\alpha')\tau(\alpha')^{m'/n'}.$$

Taking the n'-th power, we may rewrite this as

$$\sigma(\alpha)^{n'}\tau(\alpha)^{m'} = \sigma(\alpha')^{n'}\tau(\alpha')^{m'}.$$

We thus verified

Lemma 21.4.2 *Two essential zeros* α *and* α' *of* $K(z)$ *determine the same tassel if and only if* $\sigma(\alpha)^{n'}\tau(\alpha)^{m'} = \sigma(\alpha')^{n'}\tau(\alpha')^{m'}$.

Let $\alpha_1, \alpha_2, \ldots, \alpha_\nu$ be the essential zeros of $K(z)$; so none of them is a zero of $\sigma(z)$ or $\tau(z)$. (Note: By Lemma 21.3.1 (2), $K(z)$ has at most a finite number of zeros.) Setting $\mu(z) := \sigma(z)^{n'}\tau(z)^{m'}$, we consider a set

$$S = \{ \mu(\alpha_1), \mu(\alpha_2), \ldots, \mu(\alpha_\nu) \}.$$

By Lemma 21.4.2, the number of the distinct elements in this set is equal to the number of the tassels. Taking this fact into account, we use a new notation $\lambda^{(i)}$ $(i = 1, 2, \ldots, b)$ to denote the distinct elements in S, and accordingly we write $\alpha_1^{(i)}, \alpha_2^{(i)}, \ldots, \alpha_{c^{(i)}}^{(i)}$ for the essential zeros of $K(z)$ which attain the value $\lambda^{(i)}$:

$$\lambda^{(i)} = \mu(\alpha_1^{(i)}) = \mu(\alpha_2^{(i)}) = \cdots = \mu(\alpha_{c^{(i)}}^{(i)}).$$

Then $\alpha_1^{(i)}, \alpha_2^{(i)}, \ldots, \alpha_{c^{(i)}}^{(i)}$ determine the same tassel (the "$\lambda^{(i)}$-tassel").

Theorem 21.4.3 (1) *Let* $\{\alpha_1, \alpha_2, \ldots, \alpha_\nu\}$ *be the set of the essential zeros of* $K(z)$ (i.e. *none of them is a zero of* $\sigma(z)$ *or* $\tau(z)$). *Set* $\mu(z) := \sigma(z)^{n'}\tau(z)^{m'}$, *where* $m' := m/\gcd(m,n)$ *and* $n' := n/\gcd(m,n)$, *and let* b *be the number of the distinct elements in the set* $\{\,\mu(\alpha_1),\ \mu(\alpha_2), \ldots, \mu(\alpha_\nu)\,\}$. *Then the number of the subordinate fibers in* $\pi_t : M_t \to \Delta$ *(where* $t \neq 0$ *is fixed) is* bn'.

(2) *If a subordinate fiber* $X_{s,t}$ *belongs to a* $\lambda^{(i)}$*-tassel, then the number of the singularities of* $X_{s,t}$ *is* $c^{(i)}d$, *where* $d = \gcd(m,n)$ *and* $c^{(i)}$ *is the number of the essential zeros of* $K(z)$ *which attain the value* $\lambda^{(i)}$*:*

$$\lambda^{(i)} = \mu(\alpha_1^{(i)}) = \mu(\alpha_2^{(i)}) = \cdots = \mu(\alpha_{c^{(i)}}^{(i)}).$$

Proof. (1): From Lemma 21.4.2, $\pi_t : M_t \to \Delta$ has b tassels. Each tassel consists of n' subordinate fibers, and thus $\pi_t : M_t \to \Delta$ has $n'b$ subordinate fibers. The assertion (2) is clear, because for each $\alpha_j^{(i)}$ ($j = 1, 2, \ldots, c^{(i)}$), there correspond d singularities on $X_{s,t}$ (Proposition 21.4.1 (ii)). □

We recall the notation: (i) h (resp. k) is the number of the zeros of σ (resp. τ), and (ii) v is the number of indices j such that $nm_j - mn_j = 0$, and (iii) g is the genus of the core C. Also we recall (21.3.2):

$$\Big(\text{the number of the essential zeros of } K(z)\Big) \leq \Big(h - v + k + (2g-2) - \sum_{j \in J_0} \mathrm{ord}_{p_j}(\omega)\Big),$$

where $\omega := \dfrac{d\log(\sigma^n\tau^m)}{dz}$ is a meromorphic 1-form on C, and J_0 stands for the set of indices j such that $nm_j - mn_j = 0$. The equality holds precisely when all essential zeros $\alpha_1, \alpha_2, \ldots, \alpha_\nu$ have order 1. Consequently, with Theorem 21.4.3, we have

Corollary 21.4.4 *Let* h, k, v, g *be as above, and set* $\omega(z) := \dfrac{d\log(\sigma^n\tau^m)}{dz}$. *Then*

(1) *In* $\pi_t : M_t \to \Delta$ *(where* $t \neq 0$ *is fixed),*

$$\Big(\textit{the number of the subordinate fibers}\Big) \leq n'\Big(h - v + k + (2g-2) - \sum_{j \in J_0} \mathrm{ord}_{p_j}(\omega)\Big),$$

where $n' := n/\gcd(m,n)$ *and* J_0 *is the set of indices* j *such that* $nm_j - mn_j = 0$. *The equality holds precisely when*[4] (A) *all essential zeros* $\alpha_1, \alpha_2, \ldots, \alpha_\nu$ *of* $K(z)$ *have order* 1 *and* (B) $\mu(\alpha_1)$, $\mu(\alpha_2)$, $\ldots$, $\mu(\alpha_\nu)$ *are all distinct, where* $\mu(z) := \sigma(z)^{n'}\tau(z)^{m'}$.

[4] (B) means that $\pi_t : M_t \to \Delta$ has ν tassels, while (A) with Theorem 21.6.7, p410 implies that each subordinate fiber is a complex curve with only one node, i.e. a Lefschetz fiber.

(2) *Moreover the following inequality holds:*

$$\Big(\textit{the number of the singularities of a subordinate fiber}\Big) \leq d\Big(h - v + k + (2g-2) - \sum_{j \in J_0} \mathrm{ord}_{p_j}(\omega)\Big),$$

where $d := \gcd(m,n)$*. The equality holds precisely when*[5] (C) *all essential zeros* $\alpha_1, \alpha_2, \dots, \alpha_\nu$ *of* $K(z)$ *have order* 1 *and* (D) $\mu(\alpha_1) = \mu(\alpha_2) = \cdots = \mu(\alpha_\nu)$*.*

Comment on the exponent of the parameter *t*

We close this section by giving a comment on the effect of a change of the exponent of t. So far we have treated a barking family:

$$X_{s,t}: \qquad \sigma(z)\zeta^{m-ln}\Big(\zeta^n + t\tau(z)\Big)^l - s = 0.$$

If we replace t by t^b:

$$X_{s,t}: \qquad \sigma(z)\zeta^{m-ln}\Big(\zeta^n + t^b\tau(z)\Big)^l - s = 0,$$

then the reader may wonder that the number of the singularities of a subordinate fiber $X_{s,t}$ (and also the number of the subordinate fibers in $\pi_t : M_t \to \Delta$) increases. But this is not the case. In fact, by Lemma 21.2.1, a point $(\alpha, \beta) \in X_{s,t}$ is a singularity if and only if (a) $K(\alpha) = 0$, $\sigma(\alpha) \neq 0$, $\tau(\alpha) \neq 0$ and (b) $\beta^n = \frac{ln-m}{m} t^b \tau(\alpha)$. Notice that for each zero α of $K(z)$, regardless of the value of the exponent b of t^b, the number of the solutions β satisfying $\beta^n = \frac{ln-m}{m} t^b \tau(\alpha)$ is always n.

21.5 Discriminant functions and tassels

In the previous section, we gave a formula of the number of the subordinate fibers as well as a formula of the number of the singularities of a subordinate fiber. We may express these formulas in terms of the discriminant function. First of all, we recall Theorem 21.4.3 (1):

Let $\{\alpha_1, \alpha_2, \dots, \alpha_\nu\}$ *be the set of the essential zeros of* $K(z)$ (i.e. *none of them is a zero of* σ *or* τ). *Set* $\mu(z) := \sigma(z)^{n'}\tau(z)^{m'}$*, and let* b *be the number of the distinct elements in the set*

$$S = \{\, \mu(\alpha_1),\ \mu(\alpha_2), \dots, \mu(\alpha_\nu)\,\}.$$

Then the number of the subordinate fibers in $\pi_t : M_t \to \Delta$ *is* bn'*.*

[5] (D) means that $\pi_t : M_t \to \Delta$ has only one tassel, while (C) with Theorem 21.6.7, p410 implies that each subordinate fiber is a complex curve only with nodes.

The distinct elements in S, which we denote by $\lambda^{(i)}$ $(i = 1, 2, \ldots, b)$, are in one to one correspondence with singular tassels. Recall that a singular tassel is a disjoint union of n' subordinate fibers; the tassel corresponding to an element $\lambda^{(i)}$ is called the $\lambda^{(i)}$-tassel $T^{(i)}$:

$$\lambda^{(i)}\text{-tassel } T^{(i)} = X_{s_1^{(i)},t} \amalg X_{s_2^{(i)},t} \amalg \cdots \amalg X_{s_{n'}^{(i)},t}$$

where for $j = 1, 2, \ldots, n'$,

$$s_j^{(i)} := e^{2\pi \mathrm{i} jm'/n'} t^{m'/n'} \left(\frac{ln - m}{m}\right)^{m'/n'} \left(\frac{ln}{ln - m}\right)^{l} \sigma(\alpha)\tau(\alpha)^{m'/n'}.$$

Next we recall Theorem 21.4.3 (2):

If a subordinate fiber $X_{s,t}$ belongs to a $\lambda^{(i)}$-tassel $T^{(i)}$, then the number of the singularities on $X_{s,t}$ is $c^{(i)}d$, where $d = \gcd(m, n)$ and $c^{(i)}$ is the number of the essential zeros of $K(z)$ which attain the value $\lambda^{(i)}$:

$$\lambda^{(i)} = \mu(\alpha_1^{(i)}) = \mu(\alpha_2^{(i)}) = \cdots = \mu(\alpha_{c^{(i)}}^{(i)}), \quad \textit{where } \mu(z) := \sigma(z)^{n'}\tau(z)^{m'}. \tag{$*$}$$

In terms of the discriminant function $D_{T^{(i)}}(z) = \sigma(z)^{n'}\tau(z)^{m'} - \lambda^{(i)}$ associated with the tassel $T^{(i)}$, the condition $(*)$ is restated as follows: $\alpha_1^{(i)}, \alpha_2^{(i)}, \ldots, \alpha_{c^{(i)}}^{(i)}$ are zeros of $D_{T^{(i)}}(z)$. Therefore we obtain

Theorem 21.5.1 (1) *Let b be the number of the singular tassels in $\pi_t : M_t \to \Delta$ (where $t \neq 0$ is fixed). Then the number of the subordinate fibers in $\pi_t : M_t \to \Delta$ is bn', where $n' := n/\gcd(m, n)$.*

(2) *Denote by $T^{(1)}, T^{(2)}, \ldots, T^{(b)}$ the singular tassels in $\pi_t : M_t \to \Delta$. Suppose that a subordinate fiber $X_{s,t}$ $(s, t \neq 0)$ belongs to a $\lambda^{(i)}$-tassel $T^{(i)}$. Let $c^{(i)}$ be the number of such essential zeros of $K(z)$ as are also zeros of the discriminant function*

$$D_{T^{(i)}}(z) = \sigma(z)^{n'}\tau(z)^{m'} - \lambda^{(i)}.$$

Then the number of the singularities of $X_{s,t}$ is $c^{(i)}d$, where $d = \gcd(m, n)$.

21.6 Determination of the singularities

As before, $X_{s,t} := \Psi^{-1}(s, t)$ is a fiber of a barking family $\Psi : \mathcal{M} \to \Delta \times \Delta^{\dagger}$; that is, $X_{s,t} : F_{s,t}(z, \zeta) = 0$ where

$$F_{s,t}(z, \zeta) := \sigma(z)\zeta^{m-ln}\big(\zeta^n + t\tau(z)\big)^l - s.$$

For the time being, we let $X_{s,t}$ $(s, t \neq 0)$ be an arbitrary fiber (singular or smooth). We shall demonstrate that the branched covering $\mathrm{pr} : X_{s,t} \to C$, $(z, \zeta) \mapsto z$ restricted to a neighborhood of each ramification point is a double covering. This fact plays an essential role in showing that any singularity of a subordinate fiber is an A-singularity.

Lemma 21.6.1 *Suppose that $s, t \neq 0$. For $\alpha \in C$ such that $\sigma(\alpha) \neq 0$ and $\tau(\alpha) \neq 0$, any root $\zeta = \beta$ of the polynomial $F_{s,t}(\alpha, \zeta)$ in ζ is of multiplicity either* 1 *or* 2.

Proof. It is enough to show that any multiple root $\zeta = \beta$ of the polynomial $F_{s,t}(\alpha, \zeta)$ in ζ is a double root (a root of multiplicity 2). First of all, we note that $F_{s,t}(\alpha, \zeta)$ has a root $\zeta = \beta$ of multiplicity k ($k \geq 2$) precisely when $\dfrac{\partial F_{s,t}}{\partial \zeta}(\alpha, \zeta)$ has a root β of multiplicity $k-1$. So we only have to show that $\dfrac{\partial F_{s,t}}{\partial \zeta}(\alpha, \zeta)$ has only simple roots (that is, roots of multiplicity 1). Since $F_{s,t}(z, \zeta) = \sigma(z)\zeta^{m-ln}\big(\zeta^n + t\tau(z)\big)^l - s$, we have

$$\begin{aligned}\frac{\partial F_{s,t}}{\partial \zeta}(z, \zeta) &= \sigma(z)(m - ln)\zeta^{m-ln-1}\Big(\zeta^n + t\tau(z)\Big)^l \\ &\qquad + \sigma(z)\zeta^{m-ln} l \Big(\zeta^n + t\tau(z)\Big)^{l-1} n\zeta^{n-1} \\ &= \sigma(z)\zeta^{m-ln-1}\Big(\zeta^n + t\tau(z)\Big)^{l-1}\Big[m\zeta^n + \big(m - ln\big)t\tau(z)\Big].\end{aligned}$$

Hence

$$\frac{\partial F_{s,t}}{\partial \zeta}(\alpha, \zeta) = \sigma(\alpha)\zeta^{m-ln-1}\Big(\zeta^n + t\tau(\alpha)\Big)^{l-1}\Big[m\zeta^n + \big(m - ln\big)t\tau(\alpha)\Big].$$

Noting the assumption $\sigma(\alpha) \neq 0$ and $\tau(\alpha) \neq 0$, the solutions of $\dfrac{\partial F_{s,t}}{\partial \zeta}(\alpha, \zeta) = 0$ are (1) $\zeta = 0$ (a root of multiplicity $m - ln - 1$), (2) $\sqrt[n]{-\tau(\alpha)t}$ (roots of multiplicity $l - 1$), and (3) $\sqrt[n]{\dfrac{ln - m}{m}\tau(\alpha)t}$ (simple roots). However the condition $s \neq 0$ excludes the multiple roots $\zeta = 0$, $\sqrt[n]{-\tau(\alpha)t}$. Indeed, setting $\zeta = 0$ or $\sqrt[n]{-\tau(\alpha)t}$ in the equation $F_{s,t}(\alpha, \zeta) = 0$, we have $-s = 0$. Hence any root $\zeta = \beta$ of $\dfrac{\partial F_{s,t}}{\partial \zeta}(\alpha, \zeta)$ is a simple root, and so any multiple root of $F_{s,t}(\alpha, \zeta)$ has multiplicity 2. □

We keep the above notations: For a barking family $X_{s,t} : F_{s,t}(z, \zeta) = 0$ where

$$F_{s,t}(z, \zeta) := \sigma(z)\zeta^{m-ln}\big(\zeta^n + t\tau(z)\big)^l - s,$$

let $D_{s,t}(z) = \sigma(z)^{n'}\tau(z)^{m'} - \lambda_{s,t}$ be the discriminant function (21.1.2) on C, where $m' := m/\gcd(m, n)$, $n' := n/\gcd(m, n)$, and $\lambda_{s,t} := \Big(\dfrac{ln - m}{ln}\Big)^{n'l} \times \Big(\dfrac{m}{ln - m}\Big)^{m'} \dfrac{s^{n'}}{t^{m'}}$.

Corollary 21.6.2 *Suppose that $s, t \neq 0$. Let $z = \alpha$ be a zero of $D_{s,t}(z)$ and take a root $\zeta = \beta$ of $F_{s,t}(\alpha, \zeta)$. Then*

(1) *The multiplicity of the root* β *is* 1 *or* 2. (Geometrically, depending on whether the multiplicity of β is 1 or 2, the branched covering $\mathrm{pr} : X_{s,t} \to C$, $(z, \zeta) \mapsto z$ is locally a homeomorphism or a double covering around $(\alpha, \beta) \in X_{s,t}$.)

(2) *If* $(\alpha, \beta) \in X_{s,t}$ *is a singularity, then the multiplicity of the root* β *is* 2. (Geometrically, $\mathrm{pr} : X_{s,t} \to C$ is a double covering around (α, β).)

Proof. We show (1). From $F_{s,t}(\alpha, \beta) = 0$, we have $\sigma(\alpha)\beta^{m-ln}\big(\beta^n + t\tau(\alpha)\big)^l - s = 0$. Since $s \neq 0$, we deduce $\sigma(\alpha) \neq 0$. On the other hand, $D_{s,t}(\alpha) = 0$ by assumption. So we may apply Lemma 21.1.8 (2), p389 to obtain $\tau(\alpha) \neq 0$. Thus the assumption $\sigma(\alpha) \neq 0$ and $\tau(\alpha) \neq 0$ of Lemma 21.6.1 is fulfilled, and so the multiplicity of β is either 1 or 2. We show (2). If $(\alpha, \beta) \in X_{s,t}$ is a singularity, then $\mathrm{pr} : X_{s,t} \to C$ cannot be a homeomorphism around (α, β); otherwise (α, β) is a smooth point of $X_{s,t}$. Then by (1), $\mathrm{pr} : X_{s,t} \to C$ is a double covering around (α, β), and the root $\zeta = \beta$ of $F_{s,t}(\alpha, \zeta)$ has multiplicity 2. □

We shall determine the singularities of a subordinate fiber $X_{s,t}$ $(s, t \neq 0)$. Let $(\alpha, \beta) \in X_{s,t}$ be a singularity. Then from Corollary 21.6.2 (2), $\mathrm{pr} : X_{s,t} \to C$, $(z, \zeta) \mapsto z$ is a double covering around (α, β), so that the singularity (α, β) may be defined by a pseudo-polynomial of degree 2:

$$a_{s,t}(z)\zeta^2 + b_{s,t}(z)\zeta + c_{s,t}(z) = 0. \tag{21.6.1}$$

In what follows, for brevity we omit s, t to write $a(z)$ etc. Notice that $a(z)$ is non-vanishing near $z = \alpha$. Indeed, if $a(\alpha) = 0$, then the equation (21.6.1) for $z = \alpha$ reduces to an equation of degree 1: $b(\alpha)\zeta + c(\alpha) = 0$. This has only one root $\beta = -\dfrac{c(\alpha)}{b(\alpha)}$ (of multiplicity 1). But since $(\alpha, \beta) \in X_{s,t}$ is a singularity, this contradicts that β is a double root (Corollary 21.6.2 (2)). So $a(z)$ is non-vanishing near $z = \alpha$. Dividing the both sides of (21.6.1) by $a(z)$, we may assume that $a(z) = 1$, so that the left hand side of (21.6.1) is a Weierstrass polynomial $f(z, \zeta) = \zeta^2 + b(z)\zeta + c(z)$. Now we set

$$\zeta' = \zeta + \frac{b(z)}{2}, \qquad d(z) = \frac{b(z)^2}{4} - c(z).$$

(Note: $d(z)$ is the discriminant of $f(z, \zeta)$ viewed as a polynomial in ζ.) We then write $\zeta^2 + b(z)\zeta + c(z) = 0$ simply as $(\zeta')^2 = d(z)$. Here note that $d(z)$ is not identically zero. Otherwise $(\zeta')^2 = 0$, which defines a non-isolated singularity — this is a contradiction; see Proposition 21.4.1. Now we express $d(z) = (z - \alpha)^r e(z)$ where $e(z)$ is a non-vanishing holomorphic function near $z = \alpha$, and we set $\zeta'' = \dfrac{\zeta'}{e(z)^{1/2}}$. Then $(\zeta')^2 = d(z)$ is rewritten as $(\zeta'')^2 = (z - \alpha)^r$, which defines an A_{r-1}-singularity $(0, \alpha)$ where $\zeta'' = 0$ corresponds to $\zeta = \beta$. Therefore $(\alpha, \beta) \in X_{s,t}$ is an A_{r-1}-singularity. We summarize the above discussion as follows.

Proposition 21.6.3 *Assume that* (α, β) *is a singularity of a subordinate fiber* $X_{s,t}$ $(s, t \neq 0)$*. Let* $\zeta^2 + b_{s,t}(z)\zeta + c_{s,t}(z) = 0$ *be a local equation of* $X_{s,t}$ *around* (α, β)*, and denote its discriminant by* $d_{s,t}(z)$ $\left(= \dfrac{b_{s,t}(z)^2}{4} - c_{s,t}(z)\right)$*. Let* r *be the order of a zero*[6] α *of* $d_{s,t}(z)$*. Then the singularity* $(\alpha, \beta) \in X_{s,t}$ *is an* A_{r-1}*-singularity (that is, analytically equivalent to* $y^2 = x^r$*).*

We next derive a relationship between the plot function $K(z) = n\sigma_z\tau + m\sigma\tau_z$ and the discriminant function $D_{s,t}(z) = \sigma(z)^{n'}\tau(z)^{m'} - \lambda_{s,t}$, where $m' := m/\gcd(m, n)$, $n' := n/\gcd(m, n)$, and $\lambda_{s,t} := \left(\dfrac{ln - m}{ln}\right)^{n'l}\left(\dfrac{m}{ln - m}\right)^{m'}\dfrac{s^{n'}}{t^{m'}}$. We note that

$$\begin{aligned}\frac{dD_{s,t}(z)}{dz} &= n'\sigma_z\sigma^{n'-1}\tau^{m'} + m'\sigma^{n'}\tau_z\tau^{m'-1} = \sigma^{n'-1}\tau^{m'-1}\Big(n'\sigma_z\tau + m'\sigma\tau_z\Big)\\ &= \frac{\sigma^{n'-1}\tau^{m'-1}}{d}K(z),\end{aligned}$$

where $d := \gcd(m, n)$. Thus we obtain a formula

$$\frac{dD_{s,t}(z)}{dz} = \frac{\sigma^{n'-1}\tau^{m'-1}}{d}K(z). \tag{21.6.2}$$

We recall that given a singularity $(\alpha, \beta) \in X_{s,t}$ $(s, t \neq 0)$, we have $D_{s,t}(\alpha) = 0$ (Corollary 21.1.7) and $K(\alpha) = 0$ (Lemma 21.2.1). Moreover, the following is valid.

Lemma 21.6.4 *Let* $K(z)$ *be the plot function and let* $D_{s,t}(z)$ *be the discriminant function on the core* C*. For a singularity* (α, β) *of a subordinate fiber* $X_{s,t}$ $(s, t \neq 0)$*, the following holds:*

(1) *The order of* α *as a zero of* $D_{s,t}(z)$ *is* k $(k \geq 2)$ *if and only if the order of* α *as a zero of* $K(z)$ *is* $k - 1$*.*
(2) *Any zero of order* 1 *of* $D_{s,t}(z)$ *is not a zero of* $K(z)$*.*

Proof. $\Longrightarrow$ of (1): Suppose that α has order k as a zero of $D_{s,t}(z)$; we write $D_{s,t}(z) = (z - \alpha)^k g(z)$ where $g(\alpha) \neq 0$. Then

$$\begin{aligned}\frac{dD_{s,t}(z)}{dz} &= k(z - \alpha)^{k-1}g(z) + (z - \alpha)^k g'(z)\\ &= (z - \alpha)^{k-1}\Big(kg(z) + (z - \alpha)g'(z)\Big).\end{aligned}$$

By (21.6.2), $K(z) = \dfrac{d}{\sigma^{n'-1}\tau^{m'-1}} \cdot \dfrac{dD_{s,t}(z)}{dz}$ where $d := \gcd(m, n)$, and so we have

$$K(z) = (z - \alpha)^{k-1} \cdot \frac{d}{\sigma^{n'-1}\tau^{m'-1}} \cdot \Big(kg(z) + (z - \alpha)g'(z)\Big).$$

[6] Since $\zeta^2 + b_{s,t}(\alpha)\zeta + c_{s,t}(\alpha) = 0$ has a multiple (double) root $\zeta = \beta$, we have $d_{s,t}(\alpha) = 0$.

Noting that $\frac{d}{\sigma^{n'-1}\tau^{m'-1}} \cdot \Big(kg(z) + (z-\alpha)g'(z)\Big)$ does not vanish at $z = \alpha$, we conclude that α has order $k-1$ as a zero of $K(z)$.

$\Longleftarrow$ of (1): Suppose that α has order $k-1$ $(k \geq 2)$ as a zero of $K(z)$. Denote by l the order of α as a zero of $D_{s,t}(z)$. Then we may write $D_{s,t}(z) = (z-\alpha)^l h(z)$ where $h(\alpha) \neq 0$. Applying the argument of "$\Longrightarrow$", we see that α, as a zero of $K(z)$, has order $l-1$. By assumption, α has order $k-1$ as a zero of $K(z)$, and so $l-1 = k-1$, that is, $l = k$. Therefore α has order k as a zero of $D_{s,t}(z)$, and hence the assertion is confirmed.

The assertion (2) is clear by applying the argument of (1) for $k = 1$. □

Now we can show an important lemma.

Lemma 21.6.5 *Let (α, β) be a singularity of a subordinate fiber $X_{s,t}$ $(s, t \neq 0)$. Let $\zeta^2 + b_{s,t}(z)\zeta + c_{s,t}(z) = 0$ be a local equation of $X_{s,t}$ around (α, β)* (see Proposition 21.6.3)*, and denote its discriminant by $d_{s,t}(z)$ $\Big(= \frac{b_{s,t}(z)^2}{4} - c_{s,t}(z)\Big)$. Then the order of α as a zero of $d_{s,t}(z)$ is equal to the order of α as a zero of the discriminant function $D_{s,t}(z)$.*

Proof. Suppose that the order of α in $D_{s,t}(z)$ (resp. $d_{s,t}(z)$) is R (resp. r); then we may write $D_{s,t}(z) = (z-\alpha)^R H_{s,t}(z)$ and $d_{s,t}(z) = (z-\alpha)^r h_{s,t}(z)$, where $H_{s,t}(\alpha) \neq 0$ and $h_{s,t}(\alpha) \neq 0$. Now let $X_{s',t}$ be a smooth fiber near $X_{s,t}$ (i.e. s' is near s). Then the zero α of $D_{s,t}(z)$ decomposes into R simple zeros (zeros of order 1) of $D_{s',t}(z)$:

$$D_{s',t}(z) = (z-\alpha_1)(z-\alpha_2)\cdots(z-\alpha_R)\, H_{s',t}(z)$$

In fact, $D_{s',t}(z)$ does not have a multiple zero. If $D_{s',t}(z)$ has a multiple zero, say α', then by Lemma 21.6.4, α' is a zero of $K(z)$. But this contradicts that $X_{s',t}$ is a smooth fiber (see Lemma 21.2.1). So α decomposes into R simple zeros of $D_{s',t}(z)$. Accordingly, the singularity (α, β) splits into R smooth points on the (smooth) fiber $X_{s',t}$: By Lemma 21.2.1, we may write $\beta = \Big(\frac{ln-m}{m} t\tau(\alpha)\Big)^{1/n}$, and so $(\alpha, \beta) \in X_{s,t}$ splits into R smooth points

$$(\alpha_1, \beta_1),\ (\alpha_2, \beta_2),\ \ldots,\ (\alpha_R, \beta_R) \in X_{s',t},$$

where $\beta_i := \Big(\frac{ln-m}{m} t\tau(\alpha_i)\Big)^{1/n}$.

On the other hand, as we showed in Proposition 21.6.3, $(\alpha, \beta) \in X_{s,t}$ is an A_{r-1}-singularity. Under a generic deformation, an A_{r-1}-singularity splits into r smooth points (see Remark 21.6.6 below). Hence we conclude that $r = R$.□

Remark 21.6.6 Let $V : y^2 = x^r$ be an A_{r-1}-singularity. Then the versal deformation family of V is given by $y^2 = x^r + a_{r-1}x^{r-1} + a_{r-2}x^{r-2} + \cdots + a_0$, parameterized by $(a_{r-1}, a_{r-2}, \ldots, a_0) \in \mathbb{C}^r$. See [AGV], p80. For generic $(a_{r-1}, a_{r-2}, \ldots, a_0)$, the right hand side of the above equation has no multiple roots. Geometrically, this means that under a generic deformation, the A_{r-1}-singularity V splits into r smooth points.

Let $X_{s,t} : F_{s,t}(z,\zeta) = 0$ be a barking family, where

$$F_{s,t}(z,\zeta) := \sigma(z)\zeta^{m-ln}\big(\zeta^n + t\tau(z)\big)^l - s.$$

For a singularity $(\alpha,\beta) \in X_{s,t}$, we consider a local equation $f_{s,t}(z,\zeta) = 0$ of $X_{s,t}$ around (α,β); then we have a local factorization

$$F_{s,t}(z,\zeta) = f_{s,t}(z,\zeta)\, g_{s,t}(z,\zeta) \qquad \text{for } (z,\zeta) \text{ near } (\alpha,\beta),$$

where $g_{s,t}(\alpha,\beta) \neq 0$. We denote by $\Delta_{s,t}(z)$ (resp. $d_{s,t}$) the discriminant of $F_{s,t}(z,\zeta)$ (resp. $f_{s,t}(z,\zeta)$) viewed as a polynomial in ζ. Then the order of $z = \alpha$ in $\Delta_{s,t}(z)$ is generally different from the order of $z = \alpha$ in $d_{s,t}(z)$; see Lemma 21.6.9 below. Moreover, this fact with Lemma 21.6.5 implies that *the order of $z = \alpha$ in $\Delta_{s,t}(z)$ is generally different from the order of $z = \alpha$ in the discriminant function $D_{s,t}(z)$.*

Now we return to the discussion on the singularities of a subordinate fiber.

Theorem 21.6.7 *Let (α,β) be a singularity of a subordinate fiber $X_{s,t}$ $(s,t \neq 0)$, and let $D_{s,t}(z) = \sigma(z)^{n'}\tau(z)^{m'} - \lambda_{s,t}$ be the discriminant function* (21.1.2) *on C. Then*

(1) $D_{s,t}(\alpha) = 0$ *and* $\beta^n = \dfrac{ln-m}{m} t\tau(\alpha)$.
(2) *Denote by r the order of α in $D_{s,t}(z)$ (equivalently, the order of α in $K(z)$ is $r-1$; see Lemma* 21.6.4*). Then $r \geq 2$, and $(\alpha,\beta) \in X_{s,t}$ is an A_{r-1}-singularity.* (Note: If $r = 1$, then $(\alpha,\beta) \in X_{s,t}$ is a smooth ramification point of the branched covering $\mathrm{pr} : X_{s,t} \to C$, $(z,\zeta) \mapsto z$.)

Proof. We previously showed (1) in Corollary 21.1.7, p389. The assertion (2) follows from Proposition 21.6.3 with Lemma 21.6.5. □

We give an important consequence.

Theorem 21.6.8 *Suppose that $X_{s,t}$ $(s,t \neq 0)$ is a subordinate fiber. Let $\alpha_1, \alpha_2, \dots, \alpha_\mu$ be the zeros of $D_{s,t}(z)$, and denote by r_i the order of α_i. Then the following holds.*

(1) *Set $m' := m/\gcd(m,n)$ and $n' := n/\gcd(m,n)$, and then*

$$r_1 + r_2 + \cdots + r_\mu = -\sum_{j \in J_-} (n'm_j - m'n_j),$$

where J_- is the set of indices j such that $n'm_j - m'n_j < 0$.
(2) *For α_i, if $r_i \geq 2$, then $X_{s,t}$ has d singularities with the z-coordinate α_i where $d := \gcd(m,n)$, and they are A_{r_i-1}-singularities. If $r_i = 1$, then $X_{s,t}$ has d smooth points with the z-coordinate α_i, and they are ramification points of* $\mathrm{pr} : X_{s,t} \to C$.
(3) *The number of the singularities of $X_{s,t}$ are $d\kappa$, where κ is the number of the zeros of order ≥ 2 of $D_{s,t}(z)$. Specifically, the singularities of $X_{s,t}$ consists of d-tuples of A_{r_i-1}-singularities for i such that $r_i \geq 2$.*

Proof. Recall that $\sigma(z)$ has a zero of order m_j at p_j, and $\tau(z)$ has a pole of order n_j at p_j. Thus the function $\sigma(z)^{n'}\tau(z)^{m'}$ is locally of the form $z^{n'm_j - m'n_j} h_j(z)$ around p_j, where $h_j(z)$ is a non-vanishing holomorphic function. So $D_{s,t}(z) = \sigma(z)^{n'}\tau(z)^{m'} - \lambda_{s,t}$ is locally

$$z^{n'm_j - m'n_j} h_j(z) - \lambda_{s,t} \qquad \text{around} \quad p_j.$$

In particular, $D_{s,t}(z)$ has a pole of order $-(n'm_j - m'n_j)$ at each p_j such that $n'm_j - mn_j < 0$. Therefore we have

$$\big(\text{the sum of the orders of the poles of } D_{s,t}(z)\big) = -\sum_{j\in J_-}(n'm_j - m'n_j).$$

Since $D_{s,t}(z)$ is a meromorphic function on C, the left hand side is equal to the sum of the orders of the zeros of $D_{s,t}(z)$. Thus

$$\big(\text{the sum of the orders of the zeros of } D_{s,t}(z)\big) = -\sum_{j\in J_-}(n'm_j - m'n_j).$$

This proves (1). Next we show (2). If $r_i \geq 2$, then by Proposition 21.4.1, p401, $X_{s,t}$ has d singularities with the z-coordinate α_i, where $d = \gcd(m,n)$: By Theorem 21.6.7, they are A_{r_i-1}-singularities. On the other hand, if $r_i = 1$, then by Proposition 21.1.12, p392, $X_{s,t}$ has d smooth points with the z-coordinate α_i, and they are ramification points of $\mathrm{pr} : X_{s,t} \to C$. This verifies (2). The assertion (3) is apparent from (2). □

To clarify the discussion up to this point in this chapter, we briefly summarize the essential roles of the plot function $K(z)$ and the discriminant function $D_{s,t}(z)$. Firstly, by Theorem 21.2.3, p395,

> $(z,\zeta) = (\alpha,\beta)$ *is a singularity of "some" subordinate fiber in* $\pi_t : M_t \to \Delta$ *if and only if* $K(\alpha) = 0$ *and* $\beta^n = \dfrac{ln-m}{m} t\tau(\alpha)$.

Fix $s \neq 0$, and set $s_j := e^{2\pi \mathrm{i} jm'/n'} s$ for each $j = 1, 2, \ldots, n'$. Then a disjoint union of n' fibers

$$T = X_{s_1,t} \amalg X_{s_2,t} \amalg \cdots \amalg X_{s_{n'},t}$$

is called a tassel. By Theorem 21.1.11, p391,

> $(z,\zeta) = (\alpha,\beta)$ *is a ramification point of* $\mathrm{pr} : T \to C$, $(z,\zeta) \mapsto z$ *if and only if* $D_{s,t}(\alpha) = 0$ *and* $\beta^n = \dfrac{ln-m}{m} t\tau(\alpha)$.

Moreover, if this is the case, the following statement holds:

> *Let* r *be the order of* α *in* $D_{s,t}(z)$ *(equivalently, the order of* α *in* $K(z)$ *is* $r-1$*; see Lemma* 21.6.4*). Then if* $r \geq 2$*, the point* (α,β) *is an* A_{r-1}*-singularity of the tassel* T*, whereas if* $r = 1$*, then* (α,β) *is a smooth point of* T*. See Theorem* 21.6.8.

Supplement: Relationship between discriminants

Consider two polynomials in ζ whose coefficients $a_i(z)$ and $b_j(z)$ are holomorphic functions in z:

$$f(z,\zeta) = \zeta^m + a_{m-1}(z)\zeta^{m-1} + \cdots + a_0(z),$$
$$g(z,\zeta) = \zeta^n + b_{n-1}(z)\zeta^{n-1} + \cdots + b_0(z).$$

We denote by $\Delta_f(z)$ (resp. $\Delta_g(z)$) the discriminant of f (resp. g) viewed as a polynomial in ζ. Also, $\Delta_{fg}(z)$ denotes the discriminant of the product fg viewed as a polynomial in ζ. Let $\alpha_1(z), \alpha_2(z), \ldots, \alpha_m(z)$ be the roots of f as a polynomial in ζ, and let $\beta_1(z), \beta_2(z), \ldots, \beta_n(z)$ be the roots of g as a polynomial in ζ. By definition,

$$\Delta_{fg}(z) = \prod_{i<j}\big(\alpha_i(z) - \alpha_j(z)\big)^2 \cdot \prod_{k<l}\big(\beta_k(z) - \beta_l(z)\big)^2 \cdot \prod_{i,\,k}\big(\alpha_i(z) - \beta_k(z)\big)^2.$$

Since $\Delta_f(z) = \prod_{i<j}\big(\alpha_i(z) - \alpha_j(z)\big)^2$ and $\Delta_g(z) = \prod_{k<l}\big(\beta_k(z) - \beta_l(z)\big)^2$, we may write

$$\Delta_{fg}(z) = \Delta_f(z)\Delta_g(z)E(z),$$

where $E(z) := \prod_{i,k}\big(\alpha_i(z) - \beta_k(z)\big)^2$. We claim that $E(z)$ is holomorphic in z (note: in general, $\alpha_i = \alpha_i(z)$ and $\beta_k = \beta_k(z)$ are not holomorphic). To see this, we note that from $g(z,\zeta) = \prod_{k=1}^n(\zeta - \beta_k)$, we have $g(z,\alpha_i) = \prod_{k=1}^n(\alpha_i - \beta_k)$, and hence

$$E(z) = g(z,\alpha_1)^2 g(z,\alpha_2)^2 \cdots g(z,\alpha_m)^2. \tag{21.6.3}$$

Clearly the right hand side is a symmetric polynomial in $\alpha_1, \alpha_2, \ldots, \alpha_n$. Since any symmetric polynomial is expressible in terms of elementary symmetric polynomials, from the relationship between the roots and the coefficients of the polynomial f, it follows that $E(z)$ is expressible in terms of the coefficients $a_0(z), a_1(z), \ldots, a_{m-1}(z)$ of f. Thus $E(z)$ is holomorphic in z, and our claim is confirmed.

Now we consider the case where f is such that $f(0,\zeta) = \zeta^m$. Then $\zeta = 0$ is a multiple root of $f(0,\zeta)$, so that $z = 0$ is a root of the discriminant $\Delta_f(z)$. Moreover, the following result holds.

Lemma 21.6.9 *Consider two polynomials:*

$$f(z,\zeta) = \zeta^m + a_{m-1}(z)\zeta^{m-1} + \cdots + a_0(z),$$
$$g(z,\zeta) = \zeta^n + b_{n-1}(z)\zeta^{n-1} + \cdots + b_0(z)$$

such that $f(0,\zeta) = \zeta^m$ and $g(0,0) \neq 0$. Denote by r_f (resp. r_g and r_{fg}) the multiplicity of a factor z in $\Delta_f(z)$ (resp. $\Delta_g(z)$ and $\Delta_{fg}(z)$). Then $r_f + r_g = r_{fg}$ holds. (Hence $r_f \leq r_{fg}$, and the equality holds if and only if $r_g = 0$, that is, $g(0,\zeta)$ has no multiple roots.)

Proof. Let $\alpha_1(z), \alpha_2(z), \ldots, \alpha_m(z)$ be the roots of $f(z,\zeta)$ as a polynomial in ζ. By the assumption $f(0,\zeta) = \zeta^m$, we have $\alpha_i(0) = 0$ for $i = 1, 2, \ldots, m$. On the other hand, letting $\beta_1(z), \beta_2(z), \ldots, \beta_n(z)$ be the roots of the polynomial

$g(z,\zeta)$ in ζ, then by the assumption $g(0,0)\neq 0$, we have $\beta_i(0)\neq 0$ for $i=1,2,\dots,n$. Setting $z=0$ in $E(z)$ (21.6.3), we obtain

$$E(0)=\prod_{i,k}\bigl(\alpha_i(0)-\beta_k(0)\bigr)^2=\prod_{i,k}\bigl(-\beta_k(0)\bigr)^2\neq 0.$$

Next we write $\Delta_f = z^{r_f}h_f(z)$ and $\Delta_g = z^{r_g}h_g(z)$, where $h_f(0)\neq 0$ and $h_g(0)\neq 0$. Then the equation $\Delta_{fg}(z)=\Delta_f(z)\Delta_g(z)E(z)$ is written as

$$\Delta_{fg}(z)=z^{r_f+r_g}\,h_f(z)\,h_g(z)\,E(z).$$

Since $h_f(0)\neq 0$, $h_g(0)\neq 0$, and $E(0)\neq 0$, we have $h_f(0)h_g(0)E(0)\neq 0$. Thus r_f+r_g equals the multiplicity r_{fg} of the factor z in $\Delta_{fg}(z)$; so $r_f+r_g=r_{fg}$. □

21.7 Seesaw phenomenon

We consider a barking family $X_{s,t}:\ \sigma(z)\zeta^{m-ln}\bigl(\zeta^n+t\tau(z)\bigr)^l-s=0$. We set $\omega:=\dfrac{d\log(\sigma^n\tau^m)}{dz}$. In Proposition 21.3.6, p400, we showed: *In $\pi_t: M_t\to\Delta$,*

$$\Bigl(\textit{the number of the singularities of all subordinate fibers}\Bigr) \le n\Bigl(h-v+k+(2g-2)-\sum_{j\in J_0}\mathrm{ord}_{p_j}(\omega)\Bigr), \tag{21.7.1}$$

where J_0 is the set of indices j such that $nm_j-mn_j=0$. The equality holds precisely when all zeros α of $K(z)$ such that $\sigma(\alpha)\neq 0$ and $\tau(\alpha)\neq 0$ have order 1.

We shall compute $\sum_{j\in J_0}\mathrm{ord}_{p_j}(\omega)$ explicitly. First of all, we locally write

$$\sigma(z)=(z-p_j)^{m_j}g_j(z)\quad\text{and}\quad \tau(z)=\frac{1}{(z-p_j)^{n_j}}\,h_j(z)\quad\text{around}\quad p_j,$$

where $g_j(z)$ and $h_j(z)$ are non-vanishing holomorphic functions. Taking a new coordinate $\zeta'=g_j(z)^{1/m}\zeta$, and setting $f_j(z):=g_j(z)^{n/m}h_j(z)$, we may simply write

$$\sigma(z)=(z-p_j)^{m_j}\quad\text{and}\quad \tau(z)=\frac{1}{(z-p_j)^{n_j}}\,f_j(z)\quad\text{around}\quad p_j.$$

(This rewriting is the same as that in (16.1.2), p280.) Then locally

$$\begin{aligned}\omega=\frac{d\log\bigl(\sigma^n\tau^m\bigr)}{dz}&=\frac{d\log\Bigl((z-p_j)^{nm_j-mn_j}f_j(z)^m\Bigr)}{dz}\\ &=\frac{nm_j-mn_j}{z-p_j}+m\frac{f_j'(z)}{f_j(z)}\qquad\text{around}\quad p_j.\end{aligned}$$

Now we assume that $j \in J_0$ (so, $nm_j - mn_j = 0$ holds). Then we have $\omega = m\dfrac{f_j'(z)}{f_j(z)}$ around p_j. For brevity, we take such a coordinate as p_j is the origin, and we expand $f_j(z)$ around p_j:

$$f_j(z) = c_0 + c_1 z + c_2 z^2 + \cdots, \qquad (c_0 \neq 0).$$

Let e be the smallest "positive" integer such that $c_e \neq 0$; so

$$f_j(z) = c_0 + c_e z^e + c_{e+1} z^{e+1} + \cdots. \tag{21.7.2}$$

Then

$$\begin{aligned} m\,\frac{f_j'(z)}{f_j(z)} &= m\,\frac{c_e e z^{e-1} + c_{e+1}(e+1)z^e + \cdots}{c_0 + c_e z^e + c_{e+1}z^{e+1} + \cdots} \\ &= m z^{e-1}\,\frac{c_e e + c_{e+1}(e+1)z + \cdots}{c_0 + c_e z^e + c_{e+1}z^{e+1} + \cdots}, \end{aligned}$$

and hence $m\dfrac{f_j'(z)}{f_j(z)}$ has a zero of order $e-1$ at p_j. (As e depends on j, we hereafter write e_j.) Since $\omega = m\dfrac{f_j'(z)}{f_j(z)}$, we obtain $\mathrm{ord}_{p_j}(\omega) = e_j - 1$. Therefore (21.7.1) is[7]

$$\begin{aligned} &\Big(\text{the number of the singularities of all subordinate fibers}\Big) \\ &\qquad \leq n\Big(h - v + k + (2g-2) - \sum_{j \in J_0} (e_j - 1)\Big), \end{aligned} \tag{21.7.3}$$

where e_j is the smallest positive integer such that $c_{e_j} \neq 0$ in the expansion of $f_j(z)$ (see (21.7.2)). Observe that as e_j becomes larger, the number of the singularities near the core decreases. (Note: "Barking family" in this chapter is the restriction of a global barking family to a neighborhood of a core, and so **(21.7.3) is actually an inequality concerning the number of the singularities near the core.**)

Let us turn to the global situation; that is, $\pi : M \to \Delta$ is a degeneration of "compact" complex curves. For simplicity, we assume that its singular fiber $X := \pi^{-1}(0)$ is stellar. We write $X = m_0 C + \sum_{j=1}^h \mathrm{Br}^{(j)}$, where C is the core and $\mathrm{Br}^{(j)} = m_1^{(j)}\Theta_1^{(j)} + m_2^{(j)}\Theta_2^{(j)} + \cdots + m_{\lambda_j}^{(j)}\Theta_{\lambda_j}^{(j)}$ is a branch. Now let $Y = n_0 C + \sum_{j=1}^h \mathrm{br}^{(j)}$ be a simple crust of barking multiplicity l of X, where $\mathrm{br}^{(j)} = n_1^{(j)}\Theta_1^{(j)} + n_2^{(j)}\Theta_2^{(j)} + \cdots + n_{a_j}^{(j)}\Theta_{a_j}^{(j)}$ is a subbranch — in the notation

[7] The right hand side below is also written as $n\Big(h + k + (2g-2) - \sum_{j \in J_0} e_j\Big)$.

used in this chapter, $m = m_0$, $n = n_0$, $m_j = m_1^{(j)}$, and $n_j = n_1^{(j)}$. Let $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$ be a barking family associated with Y. Setting

$$u := h - v + k + (2g - 2) - \sum_{j \in J_0} \big(e_j - 1\big),$$

we say that u is the *core invariant* — it is a nonnegative integer (see the equation in Lemma 21.3.5, p397) — of the barking family $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$. As we noted above, (21.7.3) is actually an inequality concerning the number of the singularities near the core. Namely,

$$\big(\text{the number of the singularities near the core } C \text{ of all subordinate fibers}\big) \le nu. \qquad (21.7.4)$$

In particular, if $u = 0$, then any subordinate fiber has no singularities near the core C.

Now we recall that a singularity of a subordinate fiber is either (1) near the core or (2) near the edge of such a branch $\mathrm{Br}^{(j)}$ as contains a proportional subbranch of the simple crust Y. We review case (2). First, we note that $j \in J_0$ (that is, $n_0 m_1^{(j)} - m_0 n_1^{(j)} = 0$ holds) precisely when $\mathrm{br}^{(j)}$ is a proportional subbranch. Assuming that Y has at least one proportional subbranch, we concentrate our attention on one proportional subbranch, say $\mathrm{br}^{(j)}$; note that the length a_j of the proportional subbranch $\mathrm{br}^{(j)}$ equals the length λ_j of the branch $\mathrm{Br}^{(j)}$. By Proposition 7.3.5, p133, each subordinate fiber has $n_{\lambda_j}^{(j)}$ $A_{e_j u_j - 1}$-singularities near the edge of $\mathrm{Br}^{(j)}$. Here the positive integer u_j is defined as follows: First, define a sequence of nonnegative integers $b_{\lambda_j+1} > b_{\lambda_j} > \cdots > b_1 > b_0 = 0$ inductively via

$$\begin{cases} b_0 = 0, \quad b_1 = 1 \quad \text{and} \\ b_{i+1} = r_i b_i - b_{i-1} \quad \text{for } i = 1, 2, \ldots, \lambda_j, \end{cases}$$

where $r_i := -\Theta_i \cdot \Theta_i$, that is, $-r_i$ is the self-intersection number of an irreducible component Θ_i of the branch $\mathrm{Br}^{(j)}$. We then set $u_j := b_{\lambda_j+1}$.

From the above discussion, we see that the singularities of a subordinate fiber have the following property: *Let* $\mathrm{br}^{(j)}$ *be a proportional subbranch. Then as* e_j *becomes larger, the number of the singularities near the core decreases*[8]*; whereas each* $A_{e_j u_j - 1}$*-singularity near the edge of the branch* $\mathrm{Br}^{(j)}$ *becomes complicated.* This phenomenon is called the *seesaw phenomenon.*

We shall consider a special case where the singular fiber X has three branches, and the core C is the projective line $\mathbb{P}^1$; so $h = 3$ and $g = 0$. Suppose that Y is a simple crust (of barking multiplicity l) of X such that Y has at least one proportional subbranch (i.e. $v \ge 1$), and the core section τ

[8] Also note that as e_j becomes larger, the function $f_j(z)$ (21.7.2) approaches a constant function $F(z) = c_0$.

has no zeros (i.e. $k = 0$). Then (21.7.4) reads

$$\Big(\text{the number of the singularities near } C \text{ of all subordinate fibers}\Big) \le n\Big(1 - v - \sum_{j\in J_0}\big(e_j - 1\big)\Big). \tag{21.7.5}$$

Since $1 - v - \sum_{j\in J_0}(e_j - 1) \geq 0$ (see the equation in Lemma 21.3.5, p397), we have $v = 1$ and $e_j = 1$; thus the simple crust Y has only one proportional subbranch, say $\mathrm{br}^{(j_0)}$. In this case, the right hand side of the inequality (21.7.5) is 0, and so any subordinate fiber has no singularities near the core C. On the other hand, (noting that $e_{j_0} = 1$) each subordinate fiber has $n^{(j_0)}_{\lambda_{j_0}}$ $A_{u_{j_0}-1}$-singularities near the edge of the branch $\mathrm{Br}^{(j_0)}$.

Remark 21.7.1 If furthermore the length λ_{j_0} of the branch $\mathrm{Br}^{(j_0)}$ is 1, and $r_1 = 2$, then $u_{j_0} = 2$, so that each subordinate fiber has $n^{(j_0)}_{\lambda_{j_0}}$ A_1-singularities (nodes) near the edge of the branch $\mathrm{Br}^{(j_0)}$.

The next result immediately follows from the above discussion.

Proposition 21.7.2 *Let $\pi : M \to \Delta$ be a degeneration with a stellar singular fiber X such that X has three branches, and the core C is the projective line $\mathbb{P}^1$. Suppose that Y is a simple crust (of barking multiplicity l) of X such that Y has at least one proportional subbranch and the core section τ has no zeros. Let $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$ be a barking family associated with the simple crust Y. Then the following holds:*

(1) *Y has only one proportional subbranch (say, $\mathrm{br}^{(j_0)}$).*
(2) *Each subordinate fiber has no singularities near the core C, but has $n^{(j_0)}_{\lambda_{j_0}}$ $A_{u_{j_0}-1}$-singularities near the edge of the branch $\mathrm{Br}^{(j_0)}$.*
(3) *Set $m := m^{(j_0)}_{\lambda_{j_0}}$ and $n := n^{(j_0)}_{\lambda_{j_0}}$, and write $\frac{m}{n} = \frac{m'}{n'}$, where m' and n' are relatively prime positive integers. Then $\pi_t : M \to \Delta$ $(t \neq 0)$ has n' subordinate fibers.* (Note: If n divides m, then $n' = 1$, and $\pi_t : M_t \to \Delta$ has only one subordinate fiber; that is, X splits into two singular fibers, one main fiber and one subordinate fiber.)

Proof. We already showed (1) and (2) in the above discussion. We show (3). Fixing $t \neq 0$, we set

$$s = \left(\frac{ln}{ln - m}\right)^l \left(\frac{ln - m}{m} tc_0\right)^{m'/n'},$$

where $c_0 := f_{j_0}(0)$ (see (21.7.2)) and we fix an n'-th root of $\left(\frac{ln - m}{m} tc_0\right)^{m'}$. Then by Lemma 7.3.4, p133, the subordinate fibers in $\pi_t : M_t \to \Delta$ are

$$X_{s_1,t}, X_{s_2,t}, \ldots, X_{s_{n'},t},$$

where $s_k = e^{2\pi i k/n'} s$, $(k = 1, 2, \dots, n')$. Hence, $\pi_t : M_t \to \Delta$ has n' subordinate fibers. □

Example 21.7.3 Consider a stellar singular fiber of genus 1 with three branches:

$$X = 6C + \mathrm{Br}^{(1)} + \mathrm{Br}^{(2)} + \mathrm{Br}^{(3)},$$

where $\mathrm{Br}^{(1)} = 3\Theta_1^{(1)}$, $\mathrm{Br}^{(2)} = 4\Theta_1^{(2)} + 2\Theta_2^{(2)}$, and $\mathrm{Br}^{(3)} = 5\Theta_1^{(3)} + 4\Theta_2^{(3)} + 3\Theta_3^{(3)} + 2\Theta_4^{(3)} + \Theta_5^{(3)}$ are branches. (Note: X is II^* in Kodaira's notation.) We take a simple crust (of barking multiplicity 1) of X:

$$Y = 2C + \mathrm{br}^{(1)} + \mathrm{br}^{(2)} + \mathrm{br}^{(3)},$$

where $\mathrm{br}^{(1)} = \Theta_1^{(1)}$, $\mathrm{br}^{(2)} = \Theta_1^{(2)}$, and $\mathrm{br}^{(3)} = 2\Theta_1^{(3)} + 2\Theta_2^{(3)} + 2\Theta_3^{(3)} + 2\Theta_4^{(3)}$ are subbranches: $\mathrm{br}^{(1)}$ and $\mathrm{br}^{(2)}$ are of type A_1, and $\mathrm{br}^{(3)}$ is of type C_1. Note that Y has only one proportional subbranch $\mathrm{br}^{(1)}$. Also, note that the core section τ has no zeros. Indeed, let N be the normal bundle of the core C in M, and then τ is a meromorphic section of $N^{\otimes 2}$ with poles of order 1, 1, and 2 respectively at points p_1, p_2, and p_3, where $p_j := C \cap \mathrm{Br}^{(j)}$. Since the line bundle $N^{\otimes 2}$ has degree -4 (and $-4 = (-1) + (-1) + (-2)$), the core section τ has no zeros.

21.8 Supplement: The case $m = ln$

We consider a barking family around the core C:

$$X_{s,t} :\ \sigma(z)\zeta^{m-ln}\big(\zeta^n + t\tau(z)\big)^l - s = 0,$$

where

(i) σ is a holomorphic section of the line bundle $N^{\otimes(-m)}$ on the complex curve (the core) C with a zero of order m_j at p_j $(j = 1, 2, \dots, h)$, and
(ii) τ is a meromorphic section of the line bundle $N^{\otimes n}$ on C with a pole of order n_j at p_j $(j = 1, 2, \dots, h)$ and with a zero of order a_i at q_i $(i = 1, 2, \dots, k)$.

So far in this chapter, we have described the singularities of subordinate fibers for the case $m > ln$. In this section, we give several results on subordinate fibers for the case $m = ln$, and compare these with those for the case $m > ln$. We first note

Lemma 21.8.1 *$m = ln$ if and only if $m_j = ln_j$ for all $j = 1, 2, \dots, h$. Further if this is the case, then $X = lY$, and τ has no zeros.*

Proof. As we saw in Lemma 3.4.1, p52, $\frac{n_1+n_2+\cdots+n_h}{n} \geq \frac{m_1+m_2+\cdots+m_h}{m}$, and thus

$$\frac{ln_1 + ln_2 + \cdots + ln_h}{ln} \geq \frac{m_1 + m_2 + \cdots + m_h}{m}. \tag{21.8.1}$$

Since $m = ln$, we have $ln_1 + ln_2 + \cdots + ln_h \geq m_1 + m_2 + \cdots + m_h$. However since $m_j \geq ln_j$ $(j = 1, 2, \ldots, h)$, the above inequality is actually an equality, and consequently the equalities $m_j = ln_j$ $(j = 1, 2, \ldots, h)$ hold. Again by Lemma 3.4.1, the equality of (21.8.1) implies that τ has no zeros. Finally, the assertion $X = lY$ easily follows from $m = ln$ and $m_j = ln_j$ $(j = 1, 2, \ldots, h)$.□

We also note

Lemma 21.8.2 *Let (α, β) be a point of $X_{s,t}$: $\sigma(z)\zeta^{m-ln}\big(\zeta^n + t\tau(z)\big)^l - s = 0$.*

(1) *Suppose that $m > ln$. Then $s \neq 0$ if and only if $\sigma(\alpha) \neq 0$, $\beta \neq 0$, and $\beta^n + t\tau(\alpha) \neq 0$.*
(2) *Suppose that $m = ln$. Then $s \neq 0$ if and only if $\sigma(\alpha) \neq 0$ and $\beta^n + t\tau(\alpha) \neq 0$.*

Proof. This follows immediately from $\sigma(\alpha)\beta^{m-ln}\big(\beta^n + t\tau(\alpha)\big)^l - s = 0$. □

Now we discuss the singularities of a subordinate fiber $X_{s,t}$ $(s, t \neq 0)$ from the viewpoint of the plot function $K(z) = n\sigma_z\tau + m\sigma\tau_z$.

(1) Case $m > ln$: Remember that by Lemma 21.2.1, $(\alpha, \beta) \in X_{s,t}$ $(s, t \neq 0)$ is a singularity if and only if α and β satisfy

$$\text{(a)} \quad K(\alpha) = 0, \quad \sigma(\alpha) \neq 0, \quad \tau(\alpha) \neq 0 \quad \text{and} \quad \text{(b)} \quad \beta^n = \frac{ln - m}{m} t\tau(\alpha).$$

If $K(z)$ is not identically zero, there are at most a finite number of α satisfying $K(\alpha) = 0$ (see Lemma 21.3.1). For each α, there are n solutions β satisfying $\beta^n = \frac{ln - m}{m} t\tau(\alpha)$. Hence $X_{s,t}$ has a finite number of singularities; in particular, $X_{s,t}$ has only isolated singularities. On the other hand, if $K(z)$ is identically zero, the condition for (α, β) being a singularity is simply given by $\sigma(\alpha) \neq 0$, $\tau(\alpha) \neq 0$, and $\beta^n = \frac{ln - m}{m} t\tau(\alpha)$. For each α satisfying $\sigma(\alpha) \neq 0$ and $\tau(\alpha) \neq 0$ (such α form a one-dimensional set), there are n solutions β satisfying $\beta^n = \frac{ln - m}{m} t\tau(\alpha)$. Thus the singular locus is one-dimensional, so that (α, β) is a non-isolated singularity and $X_{s,t}$ $(s, t \neq 0)$ is non-reduced.

(2) Case $m = ln$: The same statement as Lemma 21.2.1 holds. But in this case, the condition (b): $\beta^n = \frac{ln - m}{m} t\tau(\alpha)$ is simply given by $\beta^n = 0$ (so $\beta = 0$). Thus a point $(\alpha, \beta) \in X_{s,t}$ $(s, t \neq 0)$ is a singularity if and only if α and β satisfy (a) $K(\alpha) = 0$, $\sigma(\alpha) \neq 0$, $\tau(\alpha) \neq 0$, and (b) $\beta = 0$. Therefore, if $K(z)$ is not identically zero, then $X_{s,t}$ has only isolated singularities, whereas if $K(z)$ is identically zero, then any singularity of $X_{s,t}$ is non-isolated.

We summarize the above discussion.

Proposition 21.8.3 (1) *Suppose that $m > ln$. Then $(\alpha, \beta) \in X_{s,t}$ $(s, t \neq 0)$ is a singularity if and only if $K(\alpha) = 0$, $\sigma(\alpha) \neq 0$, $\tau(\alpha) \neq 0$, and $\beta^n = \frac{ln - m}{m} t\tau(\alpha)$, where $K(z) = n\sigma_z\tau + m\sigma\tau_z$. Furthermore, if $K(z)$ is not identically zero, then $X_{s,t}$ has only isolated singularities. Otherwise, any singularity of $X_{s,t}$ is non-isolated; so $X_{s,t}$ is non-reduced.*

(2) *Suppose that $m = ln$. Then $(\alpha, \beta) \in X_{s,t}$ $(s, t \neq 0)$ is a singularity if and only if $K(\alpha) = 0$, $\sigma(\alpha) \neq 0$, $\tau(\alpha) \neq 0$, and $\beta = 0$. Furthermore, if $K(z)$ is not identically zero, then $X_{s,t}$ has only isolated singularities. Otherwise, any singularity of $X_{s,t}$ is non-isolated; so $X_{s,t}$ is non-reduced.*

Finally we inspect when $K(z) = n\sigma_z\tau + m\sigma\tau_z$ is identically zero. Note that

$$n\sigma_z\tau + m\sigma\tau_z = 0 \iff n\frac{\sigma_z}{\sigma} + m\frac{\tau_z}{\tau} = 0 \iff \frac{d\log(\sigma^n\tau^m)}{dz} = 0$$
$$\iff \log(\sigma^n\tau^m) \text{ is constant.}$$

Thus $\sigma^n\tau^m$ is constant, and clearly nonzero. We also note that $\sigma^n\tau^m$ is locally of the form $(z^{m_j})^n\left(\frac{1}{z^{n_j}}\right)^m h_j(z)$, that is, $z^{nm_j - mn_j}h_j(z)$ around p_j where $h_j(z)$ is a non-vanishing holomorphic function. Since $\sigma^n\tau^m$ is constant, its local expression $z^{nm_j - mn_j}h_j(z)$ is also constant, and so we have $nm_j - mn_j = 0$, that is $\frac{m_j}{n_j} = \frac{m}{n}$ $(j = 1, 2, \ldots, h)$. Now let a and b be the relatively prime positive integers satisfying $\frac{b}{a} = \frac{m}{n}\left(= \frac{m_j}{n_j}\right)$, and then $am = bn$ and $am_j = bn_j$ $(j = 1, 2, \ldots, h)$. Thus $aX = bY$, and we obtain the following.

Lemma 21.8.4 *If $K(z) = n\sigma_z\tau + m\sigma\tau_z$ is identically zero, then $aX = bY$ holds, where a and b are the relatively prime positive integers satisfying $am = bn$.*

22

Arrangement Functions and Singularities, I

We retain the notation of the previous chapter. Let l, m, n, m_j be positive integers $(j = 1, 2, \ldots, h)$, and let n_j $(j = 1, 2, \ldots, h)$ be a nonnegative integer such that $m > ln$ and $m_j \geq ln_j$. Then we consider a barking family (restricted to a neighborhood of a core C), that is, $X_{s,t}:\ \sigma(z)\zeta^{m-ln}\big(\zeta^n + t\tau(z)\big)^l - s = 0$, where

(i) σ is a holomorphic section of a line bundle $N^{\otimes(-m)}$ on the complex curve (the core) C with a zero of order m_j at p_j $(j = 1, 2, \ldots, h)$, and
(ii) τ is a meromorphic section of a line bundle $N^{\otimes n}$ on C with a pole of order n_j at p_j $(j = 1, 2, \ldots, h)$ and with a zero of order a_i at q_i $(i = 1, 2, \ldots, k)$.

In this and next chapter, we introduce the "arrangement function" $J(z)$ defined on the complex curve (the core) C. Roughly speaking, $J(z)$ is obtained from the plot function $K(z) = n\sigma_z\tau + m\sigma\tau_z$ by deleting such zeros as are unnecessary from the viewpoint of the description of the singularities of subordinate fibers. In a sense, $J(z)$ is the most concise function whose zeros have enough information to describe the singularities (near the core C) of subordinate fibers.

In this chapter we introduce the arrangement function $J(z)$ for the case $C = \mathbb{P}^1$ (the projective line), and in §23.2 of the next chapter for genus$(C) = 1$, and in §23.4 for genus$(C) \geq 2$. When $C = \mathbb{P}^1$, the arrangement function is that which was originally introduced by Awata [Aw] (see also [AhAw]), although we define it here in a slightly different way. The arrangement function $J(z)$ for genus$(C) \geq 1$ is a certain polynomial expressed in terms of the (Riemann) theta function — when $C = \mathbb{P}^1$, instead of the theta function, we simply take a polynomial of the form $z - a$. As such, when genus$(C) \geq 1$, it is generally hard to compute the zeros of $J(z)$ explicitly. However we demonstrate that $J(z)$ is a holomorphic section of some line bundle L on C (Lemma 23.4.2, p458). Consequently we may determine the sum of the orders of the zeros of $J(z)$, as it is equal to the degree of L. Using this result, we reprove some important results obtained in the previous chapter, such as the sharp upper bound for the number of the singularities of a subordinate fiber.

22.1 Arrangement polynomials

For the remainder of this chapter, we suppose that $C = \mathbb{P}^1$ (the projective line). Take the standard open covering $\mathbb{P}^1 = U \cup V$ by two complex lines U and V, where $z \in U$ is identified with $w \in V$ via $z = 1/w$, and let N be a line bundle on $\mathbb{P}^1$ obtained by patching $(z, \zeta) \in U \times \mathbb{C}$ with $(w, \eta) \in V \times \mathbb{C}$ via $z = 1/w$ and $\zeta = w^r \eta$, where

$$r := \frac{m_1 + m_2 + \cdots + m_h}{m}.$$

In what follows, we assume that $h \geq 3$. This is the case we often encounter; note that if the core of a stellar singular fiber X (of a degeneration of *compact* complex curves) is the projective line $\mathbb{P}^1$, then X has at least three branches.

For brevity, we take $p_h = \infty$, and then take σ (a holomorphic section of $N^{\otimes(-m)}$) and τ (a meromorphic section of $N^{\otimes n}$) as follows:

$$\sigma = \begin{cases} (z - p_1)^{m_1}(z - p_2)^{m_2} \cdots (z - p_{h-1})^{m_{h-1}} & \text{on } U \\ (1 - p_1 w)^{m_1}(1 - p_2 w)^{m_2} \cdots (1 - p_{h-1} w)^{m_{h-1}} w^{m_h} & \text{on } V, \end{cases}$$

and

$$\tau = \begin{cases} \dfrac{(z - q_1)^{a_1}(z - q_2)^{a_2} \cdots (z - q_k)^{a_k}}{(z - p_1)^{n_1}(z - p_2)^{n_2} \cdots (z - p_{h-1})^{n_{h-1}}} & \text{on } U \\ \dfrac{(1 - q_1 w)^{a_1}(1 - q_2 w)^{a_2} \cdots (1 - q_k w)^{a_k}}{(1 - p_1 w)^{n_1}(1 - p_2 w)^{n_2} \cdots (1 - p_{h-1} w)^{n_{h-1}} w^{n_h}} & \text{on } V. \end{cases}$$

Hereafter we carry out the computation only on U, as that on V is the same.

Remark 22.1.1 We may choose $p_1, p_2, \ldots, p_h$ such that none of them is ∞. Unfortunately, such a choice may result in the appearance of a singularity $(\alpha, \beta) \in X_{s,t}$ with $\alpha = \infty$. However, by taking $p_h = \infty$, we may insure this does not occur; as we showed in Lemma 21.2.1, α, the z-coordinate of a singularity, is not equal to p_j $(j = 1, 2, \ldots, h)$.

Modifying the plot function $K(z) = \dfrac{d \log(\sigma^n \tau^m)}{dz}$, we shall introduce an arrangement polynomial. We first note

$$\begin{aligned} \log(\sigma^n \tau^m) &= \log \left[\prod_{j=1}^{h-1} (z - p_j)^{nm_j - mn_j} \cdot \prod_{i=1}^{k} (z - q_i)^{ma_i} \right] \\ &= \sum_{j=1}^{h-1} (nm_j - mn_j) \log(z - p_j) + \sum_{i=1}^{k} ma_i \log(z - q_i). \end{aligned} \tag{22.1.1}$$

(The reader may wonder about which branch of log we choose, however this is immaterial since we are only interested in the logarithmic derivative

$\dfrac{d\log(f)}{dz} = \dfrac{f'}{f}$.) Then

$$\frac{d\log(\sigma^n\tau^m)}{dz} = \sum_{j=1}^{h-1}\frac{nm_j - mn_j}{z-p_j} + \sum_{i=1}^{k}\frac{ma_i}{z-q_i}. \tag{22.1.2}$$

At first we assume that $nm_j - mn_j \neq 0$ for all $j = 1, 2, \ldots, h$, and we introduce an arrangement polynomial. We set $P(z) := (z-p_1)(z-p_2)\cdots(z-p_{h-1})$ and

$$P_j(z) := \frac{P(z)}{z-p_j} = (z-p_1)(z-p_2)\cdots(z-p_{j-1})(z-p_{j+1})\cdots(z-p_{h-1}).$$

Likewise, we set $Q(z) := (z-q_1)(z-q_2)\cdots(z-q_k)$ and

$$Q_i(z) := \frac{Q(z)}{z-q_i} = (z-q_1)(z-q_2)\cdots(z-q_{i-1})(z-q_{i+1})\cdots(z-q_k).$$

We then define an *arrangement polynomial* $J(z)$ by

$$\begin{aligned} J(z) &:= PQ\frac{d\log(\sigma^n\tau^m)}{dz} \\ &= Q\sum_{j=1}^{h-1}(nm_j - mn_j)P_j + P\sum_{i=1}^{k} ma_iQ_i. \end{aligned} \tag{22.1.3}$$

Remark 22.1.2 Since arbitrary three points on $\mathbb{P}^1$ are transformed to $0, 1, \infty$ by some automorphism of $\mathbb{P}^1$, it is convenient for the practical computation of $J(z)$ to choose $p_{h-2} = 0$, $p_{h-1} = 1$, and $p_h = \infty$. However, in our later discussion, we vary the points $p_1, p_2, \ldots, p_{h-1}$ so that $J(z)$ has only one root. For this reason, we do not fix three points in the present context.

We next define $J(z)$, when $nm_j - mn_j = 0$ for some j, say, $nm_j - mn_j = 0$ for $j = h-v, h-v+1, \ldots, h-1$. In this case we note (cf. (22.1.1))

$$\log(\sigma^n\tau^m) = \sum_{j=1}^{h-v-1}(nm_j - mn_j)\log(z-p_j) + \sum_{i=1}^{k} ma_i\log(z-q_i).$$

Accordingly instead of $P(z)$ and $P_j(z)$, we consider polynomials

$$\widehat{P}(z) := (z-p_1)(z-p_2)\cdots(z-p_{h-v-1}),$$

$$\widehat{P}_j(z) := (z-p_1)(z-p_2)\cdots(z-p_{j-1})(z-p_{j+1})\cdots(z-p_{h-v-1}).$$

Then we define an *arrangement polynomial* by

$$\begin{aligned} J(z) &:= \widehat{P}(z)Q(z)\frac{d\log(\sigma^n\tau^m)}{dz} \\ &= Q\sum_{j=1}^{h-1-v}(nm_j - mn_j)\widehat{P}_j + \widehat{P}\sum_{i=1}^{k} ma_iQ_i. \end{aligned} \tag{22.1.4}$$

Note that if $nm_j - mn_j \neq 0$ for all $j = 1, 2, \ldots, h$, then $J(z)$ coincides with that in (22.1.3).

Next we need

Lemma 22.1.3 *Let $K(z) = n\sigma_z\tau + m\sigma\tau_z$ be the plot function. Then for a complex number α, the following equivalence holds:*

$$K(\alpha) = 0, \quad \sigma(\alpha) \neq 0, \quad \textit{and} \quad \tau(\alpha) \neq 0 \quad \Longleftrightarrow \quad J(\alpha) = 0 \quad \textit{and} \quad \alpha \neq p_j \quad (j \in J_0),$$

where J_0 is the set of indices j such that $nm_j - mn_j = 0$. Moreover, the multiplicity of α in $K(z)$ is equal to that of α in $J(z)$.

(Note that "$\sigma(\alpha) \neq 0$ and $\tau(\alpha) \neq 0$" is equivalent to "α is none of $p_1, p_2, \ldots, p_h$ and $q_1, q_2, \ldots, q_k$".)

Proof. We first prove this for the case where $nm_j - mn_j \neq 0$ for all j. It suffices to show the following equivalence:

$$K(\alpha) = 0 \quad \text{and} \quad \alpha \neq p_1, p_2, \ldots, p_h, q_1, q_2, \ldots, q_k \quad \Longleftrightarrow \quad J(\alpha) = 0.$$

From the expressions:

$$J(z) = P(z)Q(z)\frac{d\log(\sigma^n\tau^m)}{dz} = \prod_{j=1}^{h}(z-p_j)\prod_{i=1}^{k}(z-q_i)\frac{d\log(\sigma^n\tau^m)}{dz}$$

$$K(z) = \sigma\tau\frac{d\log(\sigma^n\tau^m)}{dz} = \prod_{j=1}^{h}(z-p_j)^{m_j-n_j}\prod_{i=1}^{k}(z-q_i)^{ma_i}\frac{d\log(\sigma^n\tau^m)}{dz},$$

we derive an equation

$$K(z) = \prod_{j=1}^{h}(z-p_j)^{m_j-n_j-1}\prod_{i=1}^{k}(z-q_i)^{ma_i-1}\cdot J(z).$$

Thus the roots of $K(z)$ (excluding $p_1, p_2, \ldots, p_h, q_1, q_2, \ldots, q_k$) are in one to one correspondence with the zeros of $J(z)$, and moreover the corresponding roots have the same multiplicity.

For the general case, the proof is almost the same as above. The only difference is that $J(z)$ may have a zero at p_{j_0} for $j_0 \in J_0$, because the right hand side of

$$J(p_{j_0}) = Q(p_{j_0})\left(\sum_{j=1}^{h-v}(nm_j - mn_j)\widehat{P}_j(p_{j_0})\right) + \widehat{P}(p_{j_0})\left(\sum_{i=1}^{k} ma_iQ_i(p_{j_0})\right)$$

is possibly zero. We leave the details to the reader. □

Now we recall Lemma 21.2.1, p393: *A point $(\alpha, \beta) \in X_{s,t}$ $(s, t \neq 0)$ is a singularity if and only if α and β satisfy*

$$\text{(a)} \quad K(\alpha) = 0, \quad \sigma(\alpha) \neq 0, \quad \tau(\alpha) \neq 0 \quad \textit{and} \quad \text{(b)} \quad \beta^n = \frac{ln-m}{m}t\tau(\alpha),$$

where $K(z) = n\sigma_z\tau + m\sigma\tau_z$ *is the plot function on* C. In terms of the equivalence of Lemma 22.1.3, we may restate this as

Proposition 22.1.4 *A point* $(\alpha, \beta) \in X_{s,t}$ $(s, t \neq 0)$ *is a singularity if and only if* α *and* β *satisfy*

$$\text{(a)} \quad J(\alpha) = 0, \quad \alpha \neq p_j \ (j \in J_0) \quad \textit{and} \quad \text{(b)} \quad \beta^n = \frac{ln - m}{m} t\tau(\alpha),$$

where J_0 *is the set of indices* j *such that* $nm_j - mn_j = 0$.

Next we note

Lemma 22.1.5 *Let* v *be the number of indices* j *such that* $nm_j - mn_j = 0$. *Then the degree of the arrangement polynomial* $J(z)$ *is* $h - v + k - 2$.

Proof. Without loss of generality, we may assume that $nm_j - mn_j = 0$ for $j = h - v, h - v + 1, \ldots, h - 1$. We set

$$c_j = \begin{cases} nm_j - mn_j, & j = 1, 2, \ldots, h - v - 1, \\ ma_{j-h+v+1}, & j = h - v, h - v + 1, \ldots, h - v + k - 1, \end{cases} \tag{22.1.5}$$

and then the coefficient of the highest degree term $z^{h-v+k-2}$ in $J(z)$ is $c_1 + c_2 + \cdots + c_{h-v+k-1}$, which, we have to verify, is nonzero. Letting a_i be the order of the zero of τ at q_i, then as we saw in (3.4.2), p52,

$$\frac{m_1 + m_2 + \cdots + m_h}{m} = \frac{n_1 + n_2 + \cdots + n_h - (a_1 + a_2 + \cdots + a_k)}{n},$$

and so

$$\sum_{j=1}^{h} (nm_j - mn_j) + \sum_{i=1}^{k} ma_i = 0.$$

Using (22.1.5), we obtain

$$c_1 + c_2 + \cdots + c_{h-v+k-1} = \sum_{j=1}^{h-v-1} (nm_j - mn_j) + \sum_{i=1}^{k} ma_i = mn_h - nm_h. \tag{22.1.6}$$

Recall that we assumed that $nm_j - mn_j = 0$ for $j = h - v, h - v + 1, \ldots, h - 1$. In particular, $nm_h - n_h m \neq 0$ and therefore (22.1.6) assures that $c_1 + c_2 + \cdots + c_{h-v+k-1} \neq 0$. This confirms that the coefficient of $z^{h-v+k-2}$ is nonzero. □

Since the degree of $J(z)$ is $h - v + k - 2$, the sum of the orders of the zeros of $J(z)$ is equal to $h - v + k - 2$. In particular,

$$\Big(\text{the sum of the orders of zeros } \alpha \text{ of } J(z) \text{ such that } \alpha \neq p_j \ (j \in J_0)\Big)$$
$$= \Big(h - v + k - 2 - \sum_{j \in J_0} \operatorname{ord}_{p_j}\big(J(z)\big)\Big),$$

where J_0 is the set of indices j such that $nm_j - mn_j = 0$, and $\mathrm{ord}_{p_j}\big(J(z)\big)$ denotes the order of the zero of $J(z)$ at p_j ($j \in J_0$). Now setting $\omega := d\log\big(\sigma^n\tau^m\big)$ (a meromorphic 1-form on C), we write $J(z) = \widehat{P}(z)Q(z)\,\omega$. Note that the polynomial $\widehat{P}(z)Q(z)$ does not vanish at p_j ($j \in J_0$), and thus $\mathrm{ord}_{p_j}\big(J(z)\big) = \mathrm{ord}_{p_j}(\omega)$ holds. Hence the above equation is rewritten as

$$\Big(\text{the sum of the orders of zeros } \alpha \text{ of } J(z) \text{ such that } \alpha \neq p_j\ (j \in J_0)\Big)$$
$$= \Big(h - v + k - 2 - \sum_{j\in J_0} \mathrm{ord}_{p_j}(\omega)\Big).$$

By the equivalence in Lemma 22.1.3, we may rewrite this as

$$\Big(\text{the sum of the orders of zeros } \alpha \text{ of } K(z) \text{ such that } \sigma(\alpha) \neq 0 \text{ and } \tau(\alpha) \neq 0\Big)$$
$$= \Big(h - v + k - 2 - \sum_{j\in J_0} \mathrm{ord}_{p_j}(\omega)\Big).$$

Therefore we derive

$$\Big(\text{the number of zeros } \alpha \text{ of } K(z) \text{ such that } \sigma(\alpha) \neq 0 \text{ and } \tau(\alpha) \neq 0\Big)$$
$$\leq \Big(h - v + k - 2 - \sum_{j\in J_0} \mathrm{ord}_{p_j}(\omega)\Big).$$

Once we obtain this inequality, it is straightforward to recover the inequalities in Proposition 21.3.6, p400 and Corollary 21.4.4, p403 (concerning the number of the subordinate fibers in $\pi_t : M_t \to \Delta$ and the number of the singularities of a subordinate fiber).

Finally, we remark that the number of the roots of the arrangement polynomial

$$J(z) = J(z, p_1, p_2, \ldots, p_{h-v-1}, q_1, q_2, \ldots, q_k)$$

varies as $p_1, p_2, \ldots, p_{h-v-1}$ and $q_1, q_2, \ldots, q_k$ vary (here we ignore the multiplicity of a root). Accordingly the number of the singularities of a singular fiber $X_{s,t}$ ($s, t \neq 0$) varies. We say that $p_1, p_2, \ldots, p_{h-v-1}$ and $q_1, q_2, \ldots, q_k$ are *in J-generic position* if the discriminant of the arrangement polynomial $J(z)$ is nonzero:

$$D(p_1, p_2, \ldots, p_{h-v-1}, q_1, q_2, \ldots, q_k) \neq 0.$$

In the generic case, the number of the roots of $J(z)$ is $h - v + k - 2$, and these roots are simple.

For a special case $v = 0$ (i.e. all $nm_j - mn_j \neq 0$) and $k = 0$ (τ has no zeros), "generic position" essentially describes the situation for $h \geq 4$; when $h = 3$, three points p_1, p_2, p_3 are always in generic position because $J(z)$ in this case is of degree 1 (a linear polynomial).

Remark 22.1.6 The reader may wonder that the discriminant of an arrangement polynomial $J(z)$ may be identically zero regardless of the positions of p_j and q_i, in which case generic position does not make sense. Fortunately this is not the case; the discriminant of an arrangement polynomial is never identically zero as we will show in Corollary 22.3.3, p432.

22.2 Vanishing cycles

The Milnor fiber of an A_N-singularity $V : y^2 = x^{N+1}$ is a smooth fiber V_t near $V = V_0$ in a smoothing family $\{V_t\}$ of V; see [Di]. For instance, $y^2 = x^{N+1} + t$ for each $t \neq 0$ is a Milnor fiber. Note that the map $(x, y) \in$ Milnor fiber $\mapsto x \in \mathbb{C}$ realizes the Milnor fiber as a double covering $\widetilde{D}_N$ of a disk D branched over $N + 1$ points $x = \alpha_1, \alpha_2, \ldots, \alpha_{N+1}$ (the roots of $x^{N+1} + t$). More precisely,

Lemma 22.2.1 (1) *If $N = 2n + 1$, then $\widetilde{D}_N$ is a smooth complex curve of genus n with two holes* (Figure 22.2.3). (2) *If $N = 2n$, then $\widetilde{D}_N$ is a smooth complex curve of genus n with one hole* (Figure 22.2.4).

Proof. (1) $N = 2n+1$: Consider a hyperelliptic covering $p : H \to \mathbb{P}^1$ branched over $N + 1$ points $\alpha_1, \alpha_2, \ldots, \alpha_{N+1} \in \mathbb{P}^1$, where the hyperelliptic curve H has genus $\dfrac{(N+1)-2}{2} = n$. For a small disk $E \subset \mathbb{P}^1 \setminus \{\alpha_1, \alpha_2, \ldots, \alpha_{N+1}\}$, the inverse image $p^{-1}(E)$ consists of two disjoint disks. Hence $\widetilde{D}_N = H \setminus p^{-1}(E)$ is a complex curve of genus n with two holes (see Figure 22.2.1).

(2) $N = 2n$: Setting $N' := N + 1 = 2n + 1$, we apply the argument in the odd case (1). Namely, consider a hyperelliptic covering $p : H \to \mathbb{P}^1$ branched over $N' + 1$ points $\alpha_1, \alpha_2, \ldots, \alpha_{N'+1} \in \mathbb{P}^1$; then H is a hyperelliptic

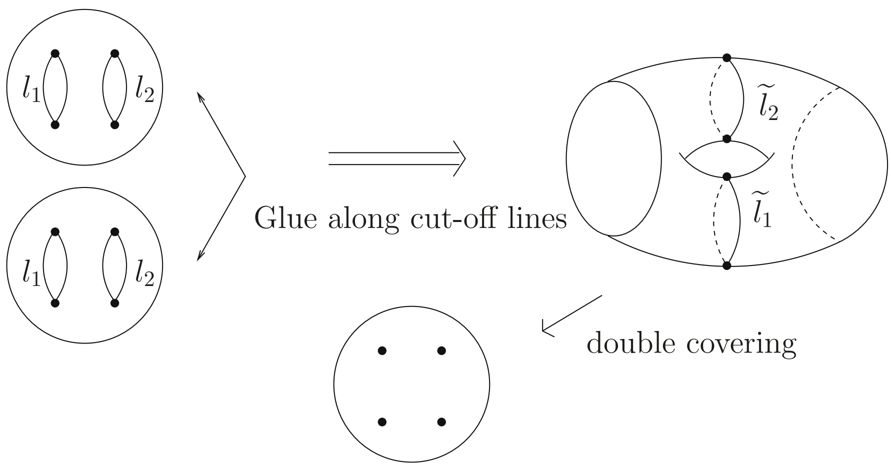


Fig. 22.2.1. $N = 3$: Even (four) number of branched points

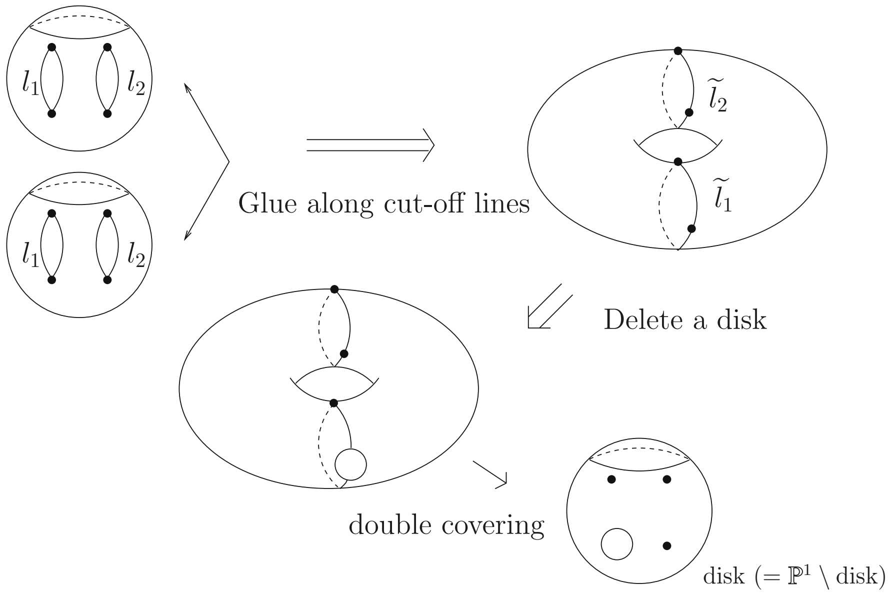


Fig. 22.2.2. $N = 2$: Odd (three) number of branched points

curve of genus n. Take a small disk E around $\alpha_{N'+1}$ such that E does not contain $\alpha_1, \alpha_2, \ldots, \alpha_{N'}$. We consider the complements $D = \mathbb{P}^1 \setminus E$ and $\widetilde{D}_N = H \setminus p^{-1}(E)$. Then $p^{-1}(E)$ is connected (indeed, a disk) because E contains a branch point $\alpha_{N'+1}$. Thus $\widetilde{D}_N$ is a complex curve of genus n with one hole (see Figure 22.2.2), and $p : \widetilde{D}_N \to D$ is a double covering. This verifies (2). □

For example, if $N = 1$, then $\widetilde{D}_N$ is a projective line with two holes — an annulus. In this case, $\widetilde{D}_N$ is the Milnor fiber of a node (A_1-singularity). If $N = 2$, then $\widetilde{D}_N$ is a complex curve of genus 1 with one hole. In this case, $\widetilde{D}_N$ is the Milnor fiber of a cusp (A_2-singularity).

In what follows, by a loop, we mean a simple closed "real" curve. Recall that a *vanishing cycle* is a loop on the Milnor fiber V_t such that it is pinched to the singularity in the limit $t \to 0$. See Figure 22.2.3 and Figure 22.2.4.

We now return to discuss a barking family $X_{s,t} :\ \sigma(z)\zeta^{m-ln}\big(\zeta^n + t\tau(z)\big)^l - s = 0$. We consider the plot function $K(z) = n\sigma_z\tau + m\sigma\tau_z$ and the arrangement polynomial $J(z)$. As we saw in Lemma 22.1.3, for $\alpha \in C$, the following equivalence holds:

$$K(\alpha) = 0,\ \sigma(\alpha) \neq 0 \text{ and } \tau(\alpha) \neq 0 \quad \Longleftrightarrow \quad J(\alpha) = 0 \text{ and } \alpha \neq p_j\ (j \in J_0),$$

where J_0 is the set of indices j such that $nm_j - mn_j = 0$. Further, the order of α in $K(z)$ coincides with that of α in $J(z)$. Thus in terms of $J(z)$, Theorem 21.6.8 implies: *Let* (α, β) *be a singularity of a subordinate fiber* $X_{s,t}$, $(s, t \neq 0)$.

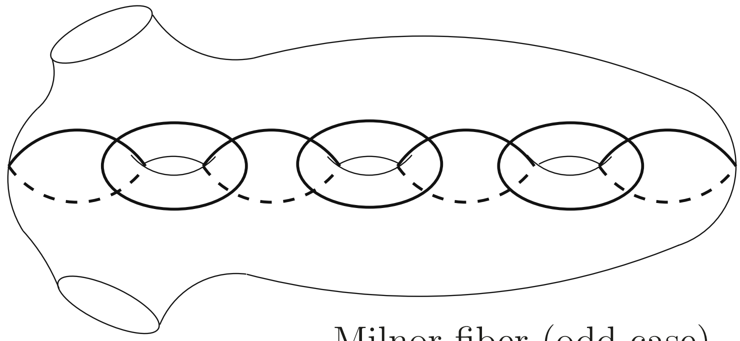


Fig. 22.2.3. Vanishing cycles (bold loops) on the Milnor fiber of the A_N-singularity for odd N (the figure for $N = 7$). The Milnor fiber has two boundary components.

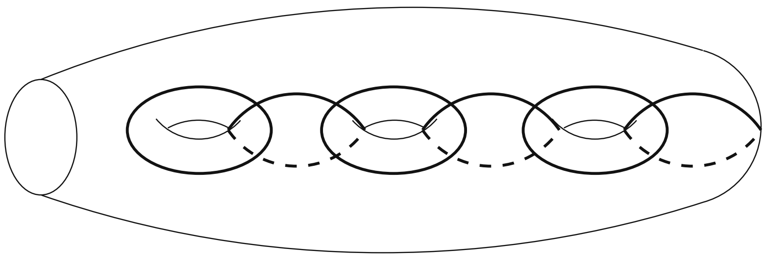


Fig. 22.2.4. Vanishing cycles (bold loops) on the Milnor fiber of the A_N-singularity for even N (the figure for $N = 6$). The Milnor fiber has one boundary component.

Then (1) α *is a zero of* $J(z)$, *and* (2) *denote by* q *the order of* α *as a zero of* $J(z)$, *and then* $(\alpha, \beta) \in X_{s,t}$ *is an* A_q*-singularity.* In particular, for J-generic case, i.e. when $J(z)$ has only simple roots (then there are $h - v + k - 2$ simple roots), any subordinate fiber $X_{s,t}$ $(s, t \neq 0)$ has only nodes. In contrast, if $J(z)$ has only one root (of multiplicity $h - v + k - 2$), then any subordinate fiber $X_{s,t}$ $(s, t \neq 0)$ has only A_N-singularities, where we set $N := h - v + k - 2$.

As described in Figure 22.2.3 and Figure 22.2.4, the configuration of the vanishing cycles of an A_N-singularity is a chain of loops. In particular, if $N \geq 2$, each vanishing cycle is non-separating. (A loop l on a smooth complex curve S is called *non-separating* if $S \setminus l$ is connected as a topological space). As a consequence, we have

Theorem 22.2.2 *Suppose that a node of a subordinate fiber* $X_{s,t}$ $(s, t \neq 0)$ *arises via the Morsification of some* A_N*-singularity* $(N \geq 2)$*. Then the vanishing cycle of that node is non-separating.*

This confirms the validity of a special case of Ahara's conjecture: Ahara conjectured that the vanishing cycle of any node of a subordinate fiber is non-separating.

22.3 Discriminants of arrangement polynomials

The reader may wonder that the discriminant D of an arrangement polynomial $J(z)$ may be identically zero, regardless of the positions of $p_1, p_2, \ldots, p_{h-v-1}$ and $q_1, q_2, \ldots, q_k$, in which case "generic position" does not make sense. Fortunately this is not the case; the discriminant of an arrangement polynomial is never identically zero. We demonstrate this for more general polynomials — polynomials of arrangement type. We fix nonzero complex numbers $c_1, c_2, \ldots, c_n$ ($n \geq 3$) such that $c_1 + c_2 + \cdots + c_n \neq 0$. Next let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be distinct complex numbers, and set

$$B_j(z) = (z - \lambda_1)(z - \lambda_2) \cdots (z - \lambda_{j-1})(z - \lambda_{j+1}) \cdots (z - \lambda_n).$$

Then we define a polynomial of arrangement type as follows:

$$A(z) = \sum_{j=1}^{n} c_j B_j(z).$$

(By the assumption $c_1 + c_2 + \cdots + c_n \neq 0$, the degree of $A(z)$ is $n - 1$.) The discriminant D of $A(z)$ is a polynomial in $\lambda_1, \lambda_2, \ldots, \lambda_n$. We assert that for generic values of $\lambda_1, \lambda_2, \ldots, \lambda_n$, the discriminant $D = D(\lambda_1, \lambda_2, \ldots, \lambda_n)$ is nonzero. To show this, we require some preparation. In the subsequent discussion, we regard $A(z)$ as a polynomial in variables z and $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_n)$, and we write $A(z, \boldsymbol{\lambda})$ instead of $A(z)$.

Lemma 22.3.1 *Consider an algebraic variety*

$$V = \{ (z, \boldsymbol{\lambda}) \in \mathbb{C} \times (\mathbb{C}^n \setminus \Delta) \ : \ A(z, \boldsymbol{\lambda}) = 0 \},$$

where $\Delta := \{\boldsymbol{\lambda} \in \mathbb{C}^n \ : \ \prod_{i<j} (\lambda_j - \lambda_i) = 0\}$ *(that is,* Δ *is the locus where some of* $\lambda_1, \lambda_2, \ldots, \lambda_n$ *coincide). Then* V *is smooth.*

Proof. We show this only for $n = 4$; the argument below works for arbitrary $n \geq 3$. For brevity we set $\boldsymbol{\lambda} = (\alpha, \beta, \gamma, \delta)$, that is, $\alpha = \lambda_1$, $\beta = \lambda_2$, $\gamma = \lambda_3$, $\delta = \lambda_4$. Also we set $a = c_1$, $b = c_2$, $c = c_3$, $d = c_4$. Then

$$\begin{aligned} A(z, \boldsymbol{\lambda}) = {} & a(z - \beta)(z - \gamma)(z - \delta) + b(z - \alpha)(z - \gamma)(z - \delta) \\ & + c(z - \alpha)(z - \beta)(z - \delta) + d(z - \alpha)(z - \beta)(z - \gamma). \end{aligned}$$

Supposing that V has a singularity, we shall deduce a contradiction. Let $(z_0, \boldsymbol{\lambda}_0) \in V$ be a singularity. Then

$$\frac{\partial A}{\partial z}(z_0, \boldsymbol{\lambda}_0) = \frac{\partial A}{\partial \alpha}(z_0, \boldsymbol{\lambda}_0) = \frac{\partial A}{\partial \beta}(z_0, \boldsymbol{\lambda}_0) = \frac{\partial A}{\partial \gamma}(z_0, \boldsymbol{\lambda}_0) = \frac{\partial A}{\partial \delta}(z_0, \boldsymbol{\lambda}_0) = 0.$$

Since

$$\frac{\partial A}{\partial \alpha} = -b(z-\gamma)(z-\delta) - c(z-\beta)(z-\delta) - d(z-\beta)(z-\gamma),$$

a condition $\frac{\partial A}{\partial \alpha}(z_0, \boldsymbol{\lambda}_0) = 0$ is explicitly given by

$$-b(z_0-\gamma_0)(z_0-\delta_0) - c(z_0-\beta_0)(z_0-\delta_0) - d(z_0-\beta_0)(z_0-\gamma_0) = 0. \quad (22.3.1)$$

On the other hand, the equation of V evaluated at $(z_0, \boldsymbol{\lambda}_0)$ is

$$a(z_0-\beta_0)(z_0-\gamma_0)(z_0-\delta_0) + b(z_0-\alpha_0)(z_0-\gamma_0)(z_0-\delta_0) \\ + c(z_0-\alpha_0)(z_0-\beta_0)(z-\delta_0) + d(z_0-\alpha_0)(z_0-\beta_0)(z_0-\gamma_0) = 0. \quad (22.3.2)$$

Substituting (22.3.1) into this equation, we obtain $a(z_0-\beta_0)(z_0-\gamma_0)(z_0-\delta_0) = 0$. Since $a \neq 0$ by assumption, we have $z_0 = \beta_0$, γ_0, or δ_0. But this generates a contradiction. For instance, if $z_0 = \beta_0$, then by (22.3.2),

$$b(\beta_0-\alpha_0)(\beta_0-\gamma_0)(\beta_0-\delta_0) = 0.$$

However as $\alpha_0, \beta_0, \gamma_0, \delta_0$ are distinct and $b \neq 0$ by assumption, the left hand side cannot be zero (a contradiction!). Therefore V is smooth. □

Now it is straightforward to see that the discriminant $D(\boldsymbol{\lambda})$ of $A(z, \boldsymbol{\lambda})$ is nonzero for generic $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \dots, \lambda_n)$. Consider a projection $f : V \to \mathbb{C}^n$ given by

$$f(z, \lambda_1, \lambda_2, \dots, \lambda_n) = (\lambda_1, \lambda_2, \dots, \lambda_n).$$

Then f is a proper surjective holomorphic map. Next note the following equivalences on the complex manifold[1] $V = \{(z, \boldsymbol{\lambda}) \in \mathbb{C} \times \mathbb{C}^n \,:\, A(z, \boldsymbol{\lambda}) = 0\}$:

$$\begin{aligned} (z, \boldsymbol{\lambda}) \text{ is a critical point of } f &\Longleftrightarrow \frac{\partial A}{\partial z}(z, \boldsymbol{\lambda}) = 0 \\ &\Longleftrightarrow A(z, \boldsymbol{\lambda}), \text{ as a polynomial in } z, \text{ has multiple roots} \\ &\Longleftrightarrow \text{the discriminant of } A(z, \boldsymbol{\lambda}) \text{ vanishes: } D(\boldsymbol{\lambda}) = 0. \end{aligned}$$

Therefore $\boldsymbol{\lambda} \in \operatorname{crit}(f)$ (i.e. $\boldsymbol{\lambda}$ is a critical value of f) if and only if $D(\boldsymbol{\lambda}) = 0$; so

$$\operatorname{crit}(f) = \{\boldsymbol{\lambda} \in \mathbb{C}^n \,:\, D(\boldsymbol{\lambda}) = 0\}. \quad (22.3.3)$$

On the other hand, since V is smooth (Lemma 22.3.1), the critical locus $\operatorname{crit}(f)$ is a Zariski closed subset[2] of $\mathbb{C}^n$. Hence the set (22.3.3) is also a Zariski closed subset of $\mathbb{C}^n$; this means that $D(\boldsymbol{\lambda})$ is nonzero for generic $\boldsymbol{\lambda}$. Thus we obtain

[1] By Lemma 22.3.1, V is a complex manifold.

[2] See Corollary 10.7 "generic smoothness" of [Hart], p272; actually for our purpose it is enough to know that $\operatorname{crit}(f) \subset \mathbb{C}^n$ is a set with measure zero — which follows from Sard Lemma.

Lemma 22.3.2 *Fix nonzero complex numbers $c_1, c_2, \ldots, c_n$ satisfying $c_1 + c_2 + \cdots + c_n \neq 0$, and consider a polynomial $A(z) = \sum_{j=1}^{n} c_j B_j(z)$, where*

$$B_j(z) = (z - \lambda_1)(z - \lambda_2) \cdots (z - \lambda_{j-1})(z - \lambda_{j+1}) \cdots (z - \lambda_n)$$

and $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{C}$ are all distinct. Then the discriminant $D(\boldsymbol{\lambda})$ of $A(z, \boldsymbol{\lambda})$, viewed as a polynomial in z, is not identically zero. (Hence for a generic value of $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_n)$, the roots of $A(z, \boldsymbol{\lambda})$ are all distinct.)

We apply this result to the arrangement polynomial $J(z)$. For brevity, we only consider the case where all $nm_j - mn_j$ $(j = 1, 2, \ldots, h)$ are nonzero. Also we take $p_h = \infty$. Then

$$J(z) = Q(z) \sum_{j=1}^{h-1} (nm_j - mn_j) P_j(z) + P(z) \sum_{i=1}^{k} ma_i Q_i(z),$$

where $P(z) = (z - p_1)(z - p_2) \cdots (z - p_{h-1})$ and $Q(z) = (z - q_1)(z - q_2) \cdots (z - q_k)$, and

$$P_j(z) = (z - p_1)(z - p_2) \cdots (z - p_{j-1})(z - p_{j+1}) \cdots (z - p_{h-1})$$
$$Q_i(z) = (z - q_1)(z - q_2) \cdots (z - q_{i-1})(z - q_{i+1}) \cdots (z - q_k).$$

We set $n = h + k - 1$, and we define $c_1, c_2, \ldots, c_n$ and $\lambda_1, \lambda_2, \ldots, \lambda_n$ by

$$c_j = \begin{cases} nm_j - mn_j, \\ ma_{j-h+1}, \end{cases} \qquad \lambda_j = \begin{cases} p_j, & j = 1, 2, \ldots, h-1, \\ q_{j-h+1}, & j = h, h+1, \ldots, h+k-1. \end{cases}$$

Then we may express $J(z) = \sum_{j=1}^{n} c_j B_j(z)$, where

$$B_j(z) = (z - \lambda_1)(z - \lambda_2) \cdots (z - \lambda_{j-1})(z - \lambda_{j+1}) \cdots (z - \lambda_n).$$

So $J(z)$ is a polynomial of arrangement type. Now we check that $c_1, c_2, \ldots, c_n$ fulfill the conditions of Lemma 22.3.2: $c_j \neq 0$ $(j = 1, 2, \ldots, n)$ and $c_1 + c_2 + \cdots + c_n \neq 0$. The latter was already shown in the proof of Lemma 22.1.5. On the other hand, by assumption, $nm_j - mn_j \neq 0$ $(j = 1, 2, \ldots, h-1)$ and $ma_i > 0$ $(i = 1, 2, \ldots, k)$, and thus $c_j \neq 0$ $(j = 1, 2, \ldots, n)$. Therefore we may apply Lemma 22.3.2, and we conclude

Corollary 22.3.3 *The discriminant D of an arrangement polynomial $J(z)$ is not identically zero.*

22.4 The coefficients of arrangement polynomials take arbitrary values

Given two polynomials

$$f(z) = a_0 z^m + a_1 z^{m-1} + \cdots + a_m \qquad \text{and} \qquad g(z) = b_0 z^n + b_1 z^{n-1} + \cdots + b_n,$$

the *resultant* $R(f,g)$ of $f(z)$ and $g(z)$ is the determinant of an $(m+n)\times(m+n)$-matrix formed by their coefficients:

$$R(f,g) = \begin{vmatrix} a_0 & a_1 & \cdots & a_m & & & & \\ & a_0 & a_1 & \cdots & a_m & & 0 & \\ & & \ddots & \ddots & & \ddots & & \\ & 0 & & a_0 & a_1 & \cdots & a_m & \\ b_0 & b_1 & \cdots & & b_n & & & \\ & b_0 & b_1 & \cdots & & b_n & 0 & \\ & & \ddots & \ddots & & & \ddots & \\ & 0 & & b_0 & b_1 & \cdots & & b_n \end{vmatrix}.$$

As is well known, $f(z)$ and $g(z)$ have a common root if and only if $R(f,g)=0$ (see [CLO], [vdW]).

Now we return to discuss polynomials of arrangement type. Fix nonzero complex numbers $c_1, c_2, \ldots, c_n$ $(n \geq 2)$ such that $c_1+c_2+\cdots+c_n \neq 0$. Setting

$$B_j(z) = (z-\lambda_1)(z-\lambda_2)\cdots(z-\lambda_{j-1})(z-\lambda_{j+1})\cdots(z-\lambda_n),$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{C}$, we consider a polynomial of arrangement type:

$$A(z) = \sum_{j=1}^{n} c_j B_j(z).$$

(Note that $\deg A(z) = n-1$.) Hereafter instead of $A(z)$ and $B_j(z)$, we use notations $A(z,\boldsymbol{\lambda})$ and $B_j(z,\boldsymbol{\lambda})$ to emphasize $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_n)$.

We set $c := c_1+c_2+\cdots+c_n$ $(\neq 0)$. The aim of this section is to demonstrate that any polynomial $P(z)$ of degree $n-1$ with the highest degree term cz^{n-1} is 'expressed' by $A(z,\boldsymbol{\lambda})$; that is, $P(z) = A(z,\boldsymbol{\lambda})$ for some $\boldsymbol{\lambda}$. Moreover we show that we may take such $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ as $\lambda_1, \lambda_2, \ldots, \lambda_n$ are distinct. To be explicit, we write

$$P(z) = cz^{n-1} + a_{n-2}z^{n-2} + \cdots + a_0,$$

and let $e_d(x_1, x_2, \ldots, x_{n-1})$ be the elementary symmetric polynomial of degree d $(d = 1, 2, \ldots, n-1)$ in $x_1, x_2, \ldots, x_{n-1}$. Then "the equation of the coefficients" $P(z) = A(z,\boldsymbol{\lambda})$ is, by the comparison of the coefficients, equivalent to a system of $n-1$ equations:

$$\begin{cases} E_1: & a_{n-2} = \sum_{j=1}^{n} c_j\, e_1(\lambda_1, \lambda_2, \ldots, \lambda_{j-1}, \lambda_{j+1}, \ldots, \lambda_n), \\ E_2: & a_{n-3} = \sum_{j=1}^{n} c_j\, e_2(\lambda_1, \lambda_2, \ldots, \lambda_{j-1}, \lambda_{j+1}, \ldots, \lambda_n), \\ & \vdots \\ E_{n-1}: & a_0 = \sum_{j=1}^{n} c_j\, e_{n-1}(\lambda_1, \lambda_2, \ldots, \lambda_{j-1}, \lambda_{j+1}, \ldots, \lambda_n). \end{cases} \tag{22.4.1}$$

Note that we have $n-1$ equations with n indeterminants $\lambda_1, \lambda_2, \ldots, \lambda_n$. From elimination theory [CLO], [vdW], this system of equations has a common solution if and only if the resultant R of this system is either identically zero or is a nonconstant polynomial. We explain the actual procedure of elimination for $n = 4$; in the course of the explanation, we will also introduce the resultant R of the system of equations. For simplicity we write $\lambda_1 = \alpha$, $\lambda_2 = \beta$, $\lambda_3 = \gamma$, and $\lambda_4 = \delta$. The elimination procedure to find a common solution of the equations E_1, E_2, E_3, E_4 is as follows. First, note that E_1 and E_2 have a common solution if and only if $R_{12}(\alpha, \beta, \gamma) = 0$, where R_{12} is the resultant of $E_1(\alpha, \beta, \gamma, \delta)$ and $E_2(\alpha, \beta, \gamma, \delta)$ regarded as equations in δ. Likewise, E_1 and E_3 have a common solution if and only if $R_{13}(\alpha, \beta, \gamma) = 0$, where R_{13} is the resultant of $E_1(\alpha, \beta, \gamma, \delta)$ and $E_3(\alpha, \beta, \gamma, \delta)$ regarded as equations in δ. Secondly, $R_{12}(\alpha, \beta, \gamma) = 0$ and $R_{13}(\alpha, \beta, \gamma) = 0$ have a common solution if and only if $R_{123}(\alpha, \beta) = 0$, where $R_{123}(\alpha, \beta)$ is the resultant of $R_{12}(\alpha, \beta, \gamma)$ and $R_{13}(\alpha, \beta, \gamma)$ regarded as polynomials in γ. Therefore *there exist $\alpha, \beta, \gamma, \delta$ satisfying E_1, E_2, E_3, E_4 if and only if there exist α, β satisfying $R_{123}(\alpha, \beta) = 0$.* We say that $R = R_{123}(\alpha, \beta)$ is the resultant of the equations E_1, E_2, E_3, E_4.

For general n, we can similarly define the resultant $R = R(\lambda_1, \lambda_2)$ of the equations $E_1, E_2, \ldots, E_n$ in (22.4.1). Then these equations have a common solution if and only if $R(\lambda_1, \lambda_2)$ is either identically zero or is a nonconstant polynomial.

Proposition 22.4.1 *Fix nonzero complex numbers $c_1, c_2, \ldots, c_n$ such that $c := c_1 + c_2 + \cdots + c_n \neq 0$. Given a polynomial $P(z) = cz^{n-1} + a_{n-2}z^{n-2} + \cdots + a_0$, where $a_0, a_1, \ldots, a_{n-2} \in \mathbb{C}$ are arbitrary, there exist distinct complex numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$ such that $P(z)$ is expressed as*

$$P(z) = \sum_{j=1}^{n} c_j B_j(z, \boldsymbol{\lambda}), \tag{22.4.2}$$

where $B_j(z, \boldsymbol{\lambda}) := (z - \lambda_1)(z - \lambda_2) \cdots (z - \lambda_{j-1})(z - \lambda_{j+1}) \cdots (z - \lambda_n)$.

Proof. We set $A(z, \boldsymbol{\lambda}) := \sum_{j=1}^{n} c_j B_j(z, \boldsymbol{\lambda})$. To a polynomial $P(z) = cz^{n-1} + a_{n-2}z^{n-2} + \cdots + a_0$, we associate an algebraic set

$$W = \{\boldsymbol{\lambda} \in \mathbb{C}^n \,:\, P(z) = A(z, \boldsymbol{\lambda})\}.$$

("$P(z) = A(z, \boldsymbol{\lambda})$" is the equation of the coefficients of z^i for $i = 0, 1, 2, \ldots, n-2$.) Then W is the space of (not necessarily distinct) $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{C}$ satisfying (22.4.2). Note that W is non-empty; in fact, $\dim_{\mathbb{C}} W = 1$ by Lemma 22.4.2 (1) below — technical lemmas used in the proof are presented below. Next we set

$$\Delta = \{\boldsymbol{\lambda} \in \mathbb{C}^n \,:\, \prod_{i<j}(\lambda_j - \lambda_i) = 0\}.$$

Geometrically, Δ consists of such points $\lambda_1, \lambda_2, \ldots, \lambda_n$ as are "not" distinct. Hence $W \setminus (W \cap \Delta)$ is the space of $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ satisfying (22.4.2) such that $\lambda_1, \lambda_2, \ldots, \lambda_n$ are distinct. We note that $W \setminus (W \cap \Delta)$ is non-empty — actually, one-dimensional —, because $\dim_{\mathbb{C}} W = 1$ (Lemma 22.4.2 (1)) and $W \cap \Delta$ is a discrete set (Lemma 22.4.3). An arbitrary point $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ of $W \setminus (W \cap \Delta)$ satisfies the condition: $\lambda_1, \lambda_2, \ldots, \lambda_n$ are distinct and they fulfill (22.4.2). Therefore our assertion is confirmed. □

We remark that given a polynomial $P(z) = cz^{n-1} + a_{n-2}z^{n-2} + \cdots + a_0$, as long as we seek for "not necessarily distinct" $\lambda_1, \lambda_2, \ldots, \lambda_n$ such that $P(z) = A(z, \boldsymbol{\lambda})$, we generally need not to pose the conditions: $c := c_1 + c_2 + \cdots + c_n \neq 0$ and $c_1, c_2, \ldots, c_n$ are nonzero. However if we would like to find "distinct" $\lambda_1, \lambda_2, \ldots, \lambda_n$, then we require these conditions. We explain this by examples.

(I) When $n = 2$, if $c_1 + c_2 = 0$, then $A(z, \boldsymbol{\lambda}) := c_1(z - \lambda_1) + c_2(z - \lambda_2) = c_1(\lambda_1 - \lambda_2)$. Clearly, a polynomial $P(z) = 0 \cdot z + 0$ is "not" expressed by such $A(z, \boldsymbol{\lambda})$ as $\lambda_1 \neq \lambda_2$.

(II) When some of $c_1, c_2, \ldots, c_n$ are zero, in general we may not find "distinct" $\lambda_1, \lambda_2, \ldots, \lambda_n$ satisfying $P(z) = A(z, \boldsymbol{\lambda})$. For instance, if $c_1 \neq 0$ and $c_2 = c_3 = \cdots = c_n = 0$, then $A(z, \boldsymbol{\lambda}) := c_1 B_1(\boldsymbol{\lambda}) = c_1(z - \lambda_2)(z - \lambda_2) \cdots (z - \lambda_n)$. In this case, any polynomial $P(z) = c_1 z^{n-1} + a_{n-2}z^{n-2} + \cdots + a_0$, $(a_i \in \mathbb{C})$ is expressible as $A(z, \boldsymbol{\lambda})$ for some $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ (actually, λ_1 is arbitrary). However, in general we cannot find "distinct" $\lambda_1, \lambda_2, \ldots, \lambda_n$ satisfying $P(z) = A(z, \boldsymbol{\lambda})$. For instance, if $P(z) = c_1(z - \alpha)^n$, then $P(z) = A(z, \boldsymbol{\lambda})$ holds precisely when $\lambda_2 = \lambda_3 = \cdots = \lambda_n$ (not distinct!).

We also remark that if we fix two elements among $\lambda_1, \lambda_2, \ldots, \lambda_n$, say $\lambda_{n-1} = 0$ and $\lambda_n = 1$, then a polynomial $P(z) = c(z - \alpha)^{n-1}$ is expressible as $A(z, \boldsymbol{\lambda})$ only for *not* distinct $\lambda_1, \lambda_2, \ldots, \lambda_n$. For example, $n = 3$, $c_1 = -1$, $c_2 = 1$, $c_3 = 1$, and

$$A(z, \boldsymbol{\lambda}) := -(z - \lambda_2)(z - \lambda_3) + (z - \lambda_1)(z - \lambda_2) + (z - \lambda_1)(z - \lambda_2).$$

In this case, if we fix $\lambda_2 = 0$ and $\lambda_3 = 1$ (while λ_1 is arbitrary), then $A(z, \boldsymbol{\lambda}) = z^2 - 2\lambda_1 z + \lambda_1$, and its discriminant is $D = 4(\lambda_1^2 - \lambda_1)$. So $A(z, \boldsymbol{\lambda})$ is of the form $(z - \alpha)^2$ precisely when $\lambda_1 = 0$ or 1. In either case, λ_1, λ_2, and λ_3 are not distinct.

Technical Lemmas

We now verify technical lemmas used in the proof of Proposition 22.4.1. We fix nonzero complex numbers $c_1, c_2, \ldots, c_n$ such that $c := c_1 + c_2 + \cdots + c_n \neq 0$. For $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{C}^n$, we define a polynomial of arrangement type:

$$A(z) = \sum_{j=1}^{n} c_j B_j(z),$$

where $B_j(z, \boldsymbol{\lambda}) := (z - \lambda_1)(z - \lambda_2) \cdots (z - \lambda_{j-1})(z - \lambda_{j+1}) \cdots (z - \lambda_n)$.

Lemma 22.4.2 *Given a polynomial $P(z)$ of degree $n - 1$ with the highest degree term cz^{n-1}, consider an algebraic set*[3] $W = \{\boldsymbol{\lambda} \in \mathbb{C}^n \,:\, P(z) = A(z, \boldsymbol{\lambda})\}$.

[3] "$P(z) = A(z, \boldsymbol{\lambda})$" is the equation of the coefficients of z^i for $i = 0, 1, 2, \ldots, n-2$.

Then (1) $\dim_{\mathbb{C}} W = 1$ *and* (2) *fixing one of* $\lambda_1, \lambda_2, \ldots, \lambda_n$, *say* $\lambda_k = u$ (constant), *then* $W_{k,c} = \{\boldsymbol{\lambda} \in W : \lambda_k = u\}$ *is a finite set of points.*

Proof. We show this by induction on n ($n \geq 2$). The statement is trivial for $n = 2$. Assuming the validity of the statement for $n - 1$, we show the statement for n. To avoid complicated notation, we verify this only for $n = 4$; the argument below works for arbitrary $n \geq 3$.

For brevity we set $\boldsymbol{\lambda} = (\alpha, \beta, \gamma, \delta)$, that is, $\alpha = \lambda_1$, $\beta = \lambda_2$, $\gamma = \lambda_3$, $\delta = \lambda_4$. Also we set $a = c_1$, $b = c_2$, $c = c_3$, $d = c_4$. So

$$\begin{aligned}A(z, \boldsymbol{\lambda}) := &\, a(z-\beta)(z-\gamma)(z-\delta) + b(z-\alpha)(z-\gamma)(z-\delta) \\ &+ c(z-\alpha)(z-\beta)(z-\delta) + d(z-\alpha)(z-\beta)(z-\gamma).\end{aligned}$$

We then consider a polynomial of arrangement type for $n = 3$:

$$\widetilde{A}(z, \widetilde{\boldsymbol{\lambda}}) := a(z-\beta)(z-\gamma) + b(z-\alpha)(z-\gamma) + c'(z-\alpha)(z-\beta),$$

where we set $c' := c + d$ and $\widetilde{\boldsymbol{\lambda}} := (\alpha, \beta, \gamma)$. Next, taking a root ρ of $P(z)$, we consider a polynomial $\widetilde{P}(z) := \dfrac{P(z)}{z-\rho}$.

Now we apply the inductive hypothesis on $n = 3$ for the equation $\widetilde{P}(z) = \widetilde{A}(z, \widetilde{\boldsymbol{\lambda}})$; then the statements (1) and (2) hold. In particular, by (2), when $\gamma = \rho$ (constant), the equation $\widetilde{P}(z) = \widetilde{A}(z, \widetilde{\boldsymbol{\lambda}})$ has a finite number of solutions. We take one solution, say $\widetilde{\boldsymbol{\lambda}}_0 = (\alpha_0, \beta_0, \rho)$. Multiplying both sides of $\widetilde{P}(z) = \widetilde{A}(z, \widetilde{\boldsymbol{\lambda}}_0)$ by $z - \rho$, we have

$$(z-\rho)\widetilde{P}(z) = (z-\rho)\widetilde{A}(z, \widetilde{\boldsymbol{\lambda}}_0).$$

Here note that $(z-\rho)\widetilde{P}(z) = P(z)$ and

$$\begin{aligned}(z-\rho)\widetilde{A}(z, \widetilde{\boldsymbol{\lambda}}_0) = &\, a(z-\beta_0)(z-\rho)(z-\rho) + b(z-\alpha_0)(z-\rho)(z-\rho) \\ &+ c'(z-\alpha_0)(z-\beta_0)(z-\rho).\end{aligned}$$

Since $c' = c + d$, the equation $(z-\rho)\widetilde{P}(z) = (z-\rho)\widetilde{A}(z, \widetilde{\boldsymbol{\lambda}}_0)$ is rewritten as

$$\begin{aligned}P(z) = &\, a(z-\beta_0)(z-\rho)(z-\rho) + b(z-\alpha_0)(z-\rho)(z-\rho) \\ &+ c(z-\alpha_0)(z-\beta_0)(z-\rho) + d(z-\alpha_0)(z-\beta_0)(z-\rho).\end{aligned}$$

That is, $P(z) = A(z, \alpha_0, \beta_0, \rho, \rho)$. Thus $P(z) = A(z, \boldsymbol{\lambda})$ (or the equations E_1, E_2, E_3 of the coefficients of z^i for $i = 0, 1, 2$) has a solution $\boldsymbol{\lambda} = (\alpha_0, \beta_0, \rho, \rho)$. In particular, the resultant $R(\alpha, \beta)$ of the equations E_1, E_2, E_3 is either identically zero or is a nonconstant polynomial; otherwise E_1, E_2, E_3 has no common solution. It is easy to check that $R(\alpha, \beta)$ is not identically zero. Therefore the algebraic set $R(\alpha, \beta) = 0$ is one-dimensional, and consequently the algebraic set W is one-dimensional. This proves (1). The assertion (2) is clear; we set $\beta = u$ (constant), and then there are a finite number of complex numbers α which satisfy $R(\alpha, u) = 0$. □

We retain the above notations: $c_1, c_2, \ldots, c_n$ are nonzero complex numbers such that $c := c_1 + c_2 + \cdots + c_n \neq 0$. Taking $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{C}^n$, we define a polynomial of arrangement type:

$$A(z) = \sum_{j=1}^{n} c_j B_j(z),$$

where $B_j(z, \boldsymbol{\lambda}) := (z - \lambda_1)(z - \lambda_2) \cdots (z - \lambda_{j-1})(z - \lambda_{j+1}) \cdots (z - \lambda_n)$.

Lemma 22.4.3 *Given a polynomial $P(z)$ of degree $n - 1$ with the highest degree term cz^{n-1}, consider an algebraic set*[4] $W = \{\boldsymbol{\lambda} \in \mathbb{C}^n : P(z) = A(z, \boldsymbol{\lambda})\}$. *Then the intersection $W \cap \Delta$ is a finite set of points, where $\Delta := \{\boldsymbol{\lambda} \in \mathbb{C}^n : \prod_{i<j}(\lambda_j - \lambda_i) = 0\}$ (that is, Δ is the locus where some of $\lambda_1, \lambda_2, \ldots, \lambda_n$ coincide).*

Proof. We show this only for $n = 4$; the argument below works for arbitrary $n \geq 3$. For brevity we set $\boldsymbol{\lambda} = (\alpha, \beta, \gamma, \delta)$, that is, $\alpha = \lambda_1$, $\beta = \lambda_2$, $\gamma = \lambda_3$, $\delta = \lambda_4$. Also we set $a = c_1$, $b = c_2$, $c = c_3$, $d = c_4$. Then

$$\begin{aligned} A(z, \boldsymbol{\lambda}) := {} & a(z - \beta)(z - \gamma)(z - \delta) + b(z - \alpha)(z - \gamma)(z - \delta) \\ & + c(z - \alpha)(z - \beta)(z - \delta) + d(z - \alpha)(z - \beta)(z - \gamma). \end{aligned}$$

It suffices to show that the intersection of W with each irreducible component of Δ is a finite set of points. We first show this for an irreducible component

$$\Delta_{\gamma=\delta} := \Big\{(\alpha, \beta, \gamma, \delta) \in \mathbb{C}^4 \, : \, \gamma = \delta\Big\}.$$

(Note: Δ has six irreducible components $\Delta_{\alpha=\beta}$, $\Delta_{\beta=\gamma}$, $\Delta_{\gamma=\delta}$, $\Delta_{\alpha=\gamma}$, $\Delta_{\alpha=\delta}$, $\Delta_{\beta=\delta}$.) We note that $W \cap \Delta_{\gamma=\delta} = \Big\{(\alpha, \beta, \gamma) \in \mathbb{C}^3 \, : \, P(z) = A(z, \alpha, \beta, \gamma, \gamma)\Big\}$. Here

$$\begin{aligned} A(z, \alpha, \beta, \gamma, \gamma) = {} & a(z - \beta)(z - \gamma)^2 + b(z - \alpha)(z - \gamma)^2 \\ & + c(z - \alpha)(z - \beta)(z - \gamma) + d(z - \alpha)(z - \beta)(z - \gamma), \end{aligned}$$

and so $z = \gamma$ is a root of $A(z, \alpha, \beta, \gamma, \gamma)$. Substituting $z = \gamma$ into the equation

$$P(z) = A(z, \alpha, \beta, \gamma, \gamma), \tag{22.4.3}$$

we have $P(\gamma) = 0$, and thus γ is also a root of $P(z)$. So $\widetilde{P}(z) := \dfrac{P(z)}{z - \gamma}$ is a polynomial (of degree $n - 2$). Dividing the both sides of (22.4.3) by $z - \gamma$, we deduce an equation $\widetilde{P}(z) = \widetilde{A}(z, \widetilde{\boldsymbol{\lambda}})$, where we set $\widetilde{\boldsymbol{\lambda}} := (\alpha, \beta, \gamma)$ and

$$\widetilde{A}(z, \widetilde{\boldsymbol{\lambda}}) := a(z - \beta)(z - \gamma) + b(z - \alpha)(z - \gamma) + c(z - \alpha)(z - \beta) + d(z - \alpha)(z - \beta).$$

[4] "$P(z) = A(z, \boldsymbol{\lambda})$" is the equation of the coefficients of z^i for $i = 0, 1, \ldots, n - 2$.

Further we set $c' = c + d$, and then

$$\widetilde{A}(z, \widetilde{\boldsymbol{\lambda}}) = a(z-\beta)(z-\gamma) + b(z-\alpha)(z-\gamma) + c'(z-\alpha)(z-\beta).$$

Now applying Lemma 22.4.2 (2) to the equation $\widetilde{P}(z) = \widetilde{A}(z, \boldsymbol{\lambda})$, we see that for fixed γ, the set of $\widetilde{\boldsymbol{\lambda}} = (\alpha, \beta, \gamma)$ satisfying $\widetilde{P}(z) = \widetilde{A}(z, \boldsymbol{\lambda})$ is a finite set of points. Consequently, the intersection of W with the irreducible component $\Delta_{\gamma=\delta}$ of Δ is a finite set of points. Similarly, we can show this for any other irreducible component of Δ. Thus our assertion is confirmed. □

23

Arrangement Functions and Singularities, II

Subsequent to the previous chapter, in this chapter we introduce an "arrangement function" for the remaining case, that is, the case genus $(C) \geq 1$. The (Riemann) theta function plays an essential role in defining the arrangement function.

23.1 Theta function

Let C be an elliptic curve (a complex curve of genus 1). We express $C = \mathbb{C}/\Gamma$ where $\Gamma = \{a + b\beta \,:\, a, b \in \mathbb{Z}\}$ is a lattice and $\beta = u + \mathrm{i}v$ is a complex number such that $u > 0$ and $v > 0$. We recall a *theta function* $\vartheta(z)$ on $\mathbb{C}$ (traditionally, this function is often denoted by $\vartheta_3(z)$):

$$\vartheta(z) = \sum_{n=-\infty}^{\infty} e^{\pi \mathrm{i} n^2 \beta + 2\pi \mathrm{i} n z}. \tag{23.1.1}$$

It is a quasi doubly-periodic holomorphic function on $\mathbb{C}$. Namely,

$$\vartheta(z+1) = \vartheta(z), \qquad \vartheta(z+\beta) = e^{-\pi\mathrm{i} - 2\pi\mathrm{i}z}\vartheta(z). \tag{23.1.2}$$

The invariance of $\vartheta(z)$ under $z \mapsto z+1$ guarantees that $\vartheta(z)$ admits a Fourier expansion, which is nothing other than (23.1.1). As is well-known, $\vartheta(z)$ has a simple zero at each point of the lattice $\Gamma = \{a + b\beta \,:\, a, b \in \mathbb{Z}\}$, and no other zeros. In fact, $\vartheta(z)$ admits an infinite product expression, which gives the information on the position of the zeros. We set $q = e^{2\pi\mathrm{i}\beta}$, and then for $|q| < 1$,

$$\vartheta(z) = c \prod_{m=1}^{\infty} \left(1 + q^{2m-1} e^{2\pi\mathrm{i}z}\right)\left(1 + q^{2m-1} e^{-2\pi\mathrm{i}z}\right),$$

where $c := \prod_{m=1}^{\infty}(1 - q^{2m})$. See [Gros], p100 and [McMo], p135.

Geometrically, the theta function $\vartheta(z)$ is considered as a holomorphic section of a line bundle as follows. First, define a Γ-action on $\mathbb{C}^2$ generated by

$$(z, \zeta) \longmapsto (z+1, \zeta) \quad \text{and} \quad (z, \zeta) \longmapsto (z+\beta, e^{-\pi\mathrm{i}-2\pi\mathrm{i}z}\zeta),$$

and then the quotient space $L := (C \times \mathbb{C})/\Gamma$ is a line bundle of degree 1 on C. By (23.1.2), the theta function $\vartheta(z)$ descends to a holomorphic section of L, with a simple zero at one point (the image of the points of the lattice Γ in $C = \mathbb{C}/\Gamma$). The holomorphic section of L determined by $\vartheta(z)$ is essentially a unique holomorphic section of L. To see this, we apply the Riemann-Roch Theorem:

$$\dim H^0(C, L) - \dim H^1(C, L) = \deg(L) + (1 - g),$$

where in the present case, the genus g of C is 1 and $\deg(L) = 1$. By the Serre duality, $H^1(C, L) \cong H^0(C, L^{-1} \otimes \Omega^1_C)$ where Ω^1_C is the cotangent bundle of C. Since the cotangent bundle of the elliptic curve is trivial, we have $H^0(C, L^{-1} \otimes \Omega^1_C) = H^0(C, L^{-1})$. On the other hand, $H^0(C, L^{-1}) = 0$ because $\deg(L^{-1}) = -1 < 0$. Thus by the Serre duality, $H^1(C, L) = 0$. Therefore the Riemann–Roch Theorem implies that $\dim H^0(C, L) = 1$. This implies that a holomorphic section of L is unique up to scalar multiplication, and so the holomorphic section determined by $\vartheta(z)$ is essentially unique.

Next we set $\vartheta_p := \vartheta(z - p)$, a translated theta function which has simple zeros at the points $p + a + b\beta \in \mathbb{C}$ where $a, b \in \mathbb{Z}$. The transformation rule of $\vartheta_p(z)$ is given by

$$\begin{cases} \vartheta_p(z+1) := \vartheta(z - p + 1) = \vartheta(z - p) = \vartheta_p(z) \\ \vartheta_p(z+\beta) := \vartheta(z - p + \beta) = e^{-\pi\mathrm{i} - 2\pi\mathrm{i}(z-p)}\vartheta(z - p) = e^{-\pi\mathrm{i} - 2\pi\mathrm{i}(z-p)}\vartheta_p(z). \end{cases}$$

Namely

$$\vartheta_p(z+1) = \vartheta_p(z), \qquad \vartheta_p(z+\beta) = e^{-\pi\mathrm{i} - 2\pi\mathrm{i}(z-p)}\vartheta_p(z). \tag{23.1.3}$$

Let L_p be the line bundle on C with the transition functions "1" (for the gluing map $z \mapsto z + 1$ of C) and "$e^{-\pi\mathrm{i} - 2\pi\mathrm{i}(z-p)}$" (for the gluing map $z \mapsto z + \beta$ of C). Then the theta function $\vartheta_p(z)$ descends to a holomorphic section of L_p with one simple zero at the image of p in $C = \mathbb{C}/\Gamma$; hence the degree of L_p is 1. Hereafter, whenever there is no fear of confusion, the holomorphic section of L_p determined by ϑ_p is also denoted by ϑ_p, and the image of $p \in \mathbb{C}$ in $C = \mathbb{C}/\Gamma$ is also denoted by p. For the subsequent discussion, it is useful to keep in mind the following correspondence on functions on $\mathbb{C}$ and the elliptic curve C:

$$z - p \quad \longleftrightarrow \quad \vartheta_p(z).$$

Next we review some basic definitions concerning divisors. Two divisors $\sum_{j=1}^h a_j p_j$ and $\sum_{i=1}^k b_i q_i$ on C are *linearly equivalent* if there exists a meromorphic function f on C with a zero of order a_j at each p_j and with a pole of order b_i at each q_i; then we write $\mathrm{div}(f) = \sum_{j=1}^h a_j p_j - \sum_{i=1}^k b_i q_i$. By Abel's Theorem (Theorem 24.1.5, p467), the linear equivalence on an elliptic curve is simply expressed as an equality with respect to the standard addition on the elliptic curve: $\sum_{j=1}^h a_j p_j = \sum_{i=1}^k b_i q_i$. Given a divisor $\sum_{j=1}^h a_j p_j$ on the elliptic curve C, its $1/m$-division point, where m is a positive integer, is such a point $q \in C$ as $mq = \sum_{j=1}^{h} a_j p_j$; we express $q = (\sum_{j=1}^{h} a_j p_j)/m$. Note that q is not unique; indeed there are m^2 such points.

For a point p of the elliptic curve C, if there is no fear of confusion, a lift of p to the universal covering $\mathbb{C}$ is also denoted by p; accordingly we treat relevant equations on C as "equations on $\mathbb{C}$ mod Γ", such as $\sum_{j=1}^{h} a_j p_j \equiv \sum_{i=1}^{k} b_i q_i$ mod Γ.

Let m, n, m_j $(j = 1, 2, \ldots, h)$ be positive integers and let n_j $(j = 1, 2, \ldots, h)$ and a_i $(i = 1, 2, \ldots, k)$ be nonnegative integers. Suppose that N is a line bundle on C such that

(i) $N^{\otimes(-m)}$ has a holomorphic section σ with a zero of order m_j at a point p_j $(j = 1, 2, \ldots, h)$, and
(ii) $N^{\otimes n}$ has a meromorphic section τ with a pole of order n_j at a point p_j $(j = 1, 2, \ldots, h)$ and with a zero of order a_i at a point q_i $(i = 1, 2, \ldots, k)$.

Lemma 23.1.1 *The following equations hold:*

(1) $\dfrac{\sum_{j=1}^{h} m_j}{m} = \dfrac{\sum_{j=1}^{h} n_j - \sum_{i=1}^{k} a_i}{n}$ *and this number is a positive integer.*
(2) *Denote by r the positive integer in* (1)*. Then there exist integers c and d such that*

$$\frac{\sum_{j=1}^{h} m_j p_j}{rm} = \frac{\sum_{j=1}^{h} n_j p_j - \sum_{i=1}^{k} a_i q_i}{rn} + \frac{c}{rmn} + \frac{d}{rmn}\beta.$$

(*Recall that* 1 *and β is a basis of* $\Gamma = \{a + b\beta \,:\, a, b \in \mathbb{Z}\}$ *and* $C = \mathbb{C}/\Gamma$.)

Proof. (1) is clear, because the both sides of the equation are equal to $-\deg(N)$. We show (2). Note that $\sigma^n \tau^m$ is a meromorphic function on the elliptic curve C; indeed, it is a section of the trivial bundle $(N^{\otimes(-m)})^{\otimes n}(N^{\otimes n})^{\otimes m} \cong \mathcal{O}_C$. Since $\operatorname{div}(\sigma^n \tau^m) = \sum_{j=1}^{h}(nm_j - mn_j)p_j + \sum_{i=1}^{k} m a_i q_i$, it follows from Abel's Theorem that

$$\sum_{j=1}^{h}(nm_j - mn_j)p_j + \sum_{i=1}^{k} m a_i q_i \equiv 0 \qquad \text{mod} \;\; \Gamma.$$

That is,

$$n \sum_{j=1}^{h} m_j p_j \equiv m \left(\sum_{j=1}^{h} n_j p_j - \sum_{i=1}^{k} a_i q_i \right) \qquad \text{mod} \;\; \Gamma.$$

Or

$$n \sum_{j=1}^{h} m_j p_j = m \left(\sum_{j=1}^{h} n_j p_j - \sum_{i=1}^{k} a_i q_i \right) + c + d\beta, \qquad \text{where} \quad c, d \in \mathbb{Z}.$$

Dividing the both sides by rmn, we have

$$\frac{\sum_{j=1}^h m_j p_j}{rm} = \frac{\sum_{j=1}^h n_j p_j - \sum_{i=1}^k a_i q_i}{rn} + \frac{c}{rmn} + \frac{d}{rmn}\beta.$$

□

Taking into account (2) of Lemma 23.1.1, hereafter instead of $C = \mathbb{C}/\Gamma$, we consider an elliptic curve $\mathbb{C}/\frac{1}{rmn}\Gamma$; we rewrite this curve as C, and also we rewrite $\frac{1}{rmn}\Gamma$ as Γ. Then from (2) of Lemma 23.1.1,

$$\frac{\sum_{j=1}^h m_j p_j}{rm} \equiv \frac{\sum_{j=1}^h n_j p_j - \sum_{i=1}^k a_i q_i}{rn} \qquad \mathrm{mod}\ \ \Gamma, \tag{23.1.4}$$

where by (1) of Lemma 23.1.1,

$$r = \frac{\sum_{j=1}^h m_j}{m} = \frac{\sum_{j=1}^h n_j - \sum_{i=1}^k a_i}{n} \ \text{ is a positive integer.} \tag{23.1.5}$$

Conversely, suppose that positive integers m, n, m_j, and nonnegative integers n_j, a_i, and complex numbers p_j, q_i satisfy (23.1.4) and (23.1.5). Then we shall show that there exists a line bundle N on C such that

(i) $N^{\otimes(-m)}$ has a holomorphic section σ with a zero of order m_j at a point p_j $(j = 1, 2, \ldots, h)$ — by convention, the corresponding point on $C = \mathbb{C}/\Gamma$ to $p_j \in \mathbb{C}$ (resp. $q_i \in \mathbb{C}$) is also denoted by p_j (resp. q_i) —, and
(ii) $N^{\otimes n}$ has a meromorphic section τ with a pole of order n_j at a point p_j $(j = 1, 2, \ldots, h)$ and with a zero of order a_i at a point q_i $(i = 1, 2, \ldots, k)$.

First, let p be the point on $C = \mathbb{C}/\Gamma$ defined by the fractions in (23.1.4). We set $N := \mathcal{O}_C(-rp)$; this is a line bundle of degree $-r$ on C where r is a positive integer in (23.1.5). We claim that N satisfies (i) and (ii). To show this, let L_p be the line bundle with the transition functions "1" (for the gluing $z \mapsto z + 1$ of C) and "$g_p(z)$" (for the gluing $z \mapsto z + \beta$ of C), where we set $g_p(z) := e^{-\pi \mathrm{i} - 2\pi \mathrm{i}(z-p)}$. Then L_p has a holomorphic section $\vartheta_p(z)$, because the transformation rule (23.1.3) of $\vartheta_p(z)$ is compatible with the transition functions of L_p. Since $\vartheta_p(z)$ has a zero of order 1 at p, and $\vartheta_p(z)$ does not vanish outside p, we have $L_p \cong \mathcal{O}_C(p)$. As $N := \mathcal{O}_C(-rp)$, we obtain $N \cong L_p^{\otimes(-r)}$. Now we fix a trivialization of N such that the transition functions are "1" (for the gluing $z \mapsto z + 1$ of C) and "$1/g_p(z)^r$" (for the gluing $z \mapsto z + \beta$ of C). Accordingly the transition functions of $N^{\otimes(-m)}$ are "1" and "$g_p(z)^{rm}$", while the transition functions of $N^{\otimes n}$ are "1" and "$1/g_p(z)^{rn}$".

We claim that

$$N^{\otimes(-m)} = \mathcal{O}_C\Big(\sum_{j=1}^h m_j p_j\Big), \qquad N^{\otimes n} = \mathcal{O}_C\Big(-\sum_{j=1}^h n_j p_j + \sum_{i=1}^k a_i q_i\Big). \tag{23.1.6}$$

First we show the validity of the equation on the left. By definition, $p = \frac{\sum_{j=1}^h m_j p_j}{rm}$, and so $rmp = \sum_{j=1}^h m_j p_j$. Since $N = \mathcal{O}_C(-rp)$, we have $N^{\otimes(-m)} = \mathcal{O}_C(mrp) = \mathcal{O}_C(\sum_{j=1}^h m_j p_j)$. This confirms the validity of the equation on the left. Similarly, we may show the validity of the equation on the right.

From (23.1.6), the existence of σ and τ in (i) and (ii) above is clear. We may explicitly give them as follows:

$$\sigma(z) = \prod_{j=1}^{h} \vartheta_{p_j}(z)^{m_j}, \qquad \tau(z) = \frac{1}{e^{2\pi i r n b z}} \cdot \frac{\prod_{i=1}^{k} \vartheta_{q_i}(z)^{a_i}}{\prod_{j=1}^{h} \vartheta_{p_j}(z)^{n_j}}. \tag{23.1.7}$$

In fact, the knowledge of the location of the zeros of the theta function shows that σ has a zero of order m_j at p_j, while τ has a pole of order n_j at p_j and a zero of order a_i at q_i. Next we check that σ and τ are indeed sections of $N^{\otimes(-m)}$ and $N^{\otimes n}$ respectively. Firstly,

$$\begin{aligned}\sigma(z+\beta) &:= \prod_{j=1}^{h} \vartheta_{p_j}(z+\beta)^{m_j} = \prod_{j=1}^{h} g_{p_j}(z)^{m_j} \vartheta_{p_j}(z)^{m_j} \\ &= \prod_{j=1}^{h} g_{p_j}(z)^{m_j} \cdot \prod_{j=1}^{h} \vartheta_{p_j}(z)^{m_j} \\ &= \prod_{j=1}^{h} g_{p_j}(z)^{m_j} \cdot \sigma(z) \qquad \text{by (23.1.7).}\end{aligned}$$

Thus by (1) of Lemma 23.1.2 below, we have $\sigma(z+\beta) = g_p(z)^{rm} \cdot \sigma(z)$. On the other hand, clearly $\sigma(z+1) = \sigma(z)$. So σ is a section of $N^{\otimes(-m)}$. Similarly,

$$\begin{aligned}\tau(z+\beta) &:= \frac{1}{e^{2\pi i r n b(z+\beta)}} \cdot \frac{\prod_{i=1}^{k} \vartheta_{q_i}(z+\beta)^{a_i}}{\prod_{j=1}^{h} \vartheta_{p_j}(z+\beta)^{n_j}} \\ &= \frac{1}{e^{2\pi i r n b z}\, e^{2\pi i r n b \beta}} \cdot \frac{\prod_{i=1}^{k} g_{q_i}(z)^{a_i} \vartheta_{q_i}(z)^{a_i}}{\prod_{j=1}^{h} g_{p_j}(z)^{n_j} \vartheta_{p_j}(z)^{n_j}} \\ &= \frac{1}{e^{2\pi i r n b \beta}} \cdot \frac{\prod_{i=1}^{k} g_{q_i}(z)^{a_i}}{\prod_{j=1}^{h} g_{p_j}(z)^{n_j}} \cdot \frac{1}{e^{2\pi i r n b z}} \frac{\prod_{i=1}^{k} \vartheta_{q_i}(z)^{a_i}}{\prod_{j=1}^{h} \vartheta_{p_j}(z)^{n_j}} \\ &= \frac{1}{e^{2\pi i r n b \beta}} \cdot \frac{\prod_{i=1}^{k} g_{q_i}(z)^{a_i}}{\prod_{j=1}^{h} g_{p_j}(z)^{n_j}} \cdot \tau(z).\end{aligned}$$

By (2) of Lemma 23.1.2 below, we have $\tau(z+\beta) = \dfrac{1}{g_p(z)^{rn}}\, \tau(z)$. On the other hand, clearly $\tau(z+1) = \tau(z)$. This confirms that τ is a section of $N^{\otimes n}$.

Finally we give the proof of the technical lemma used in the above discussion.

Lemma 23.1.2 *Write the congruence* (23.1.4) *as an actual equation*

$$p = \frac{\sum_{j=1}^h m_j p_j}{rm} = \frac{\sum_{j=1}^h n_j p_j - \sum_{i=1}^k a_i q_i}{rn} + a + b\beta, \tag{23.1.8}$$

where a and b are integers. Set $g_p(z) = e^{2\pi \mathrm{i} p}\, e^{-\pi \mathrm{i} - 2\pi \mathrm{i} z}$ and $g_{p_j}(z) = e^{2\pi \mathrm{i} p_j} \times e^{-\pi \mathrm{i} - 2\pi \mathrm{i} z}$, and then

$$(1)\ \ g_p(z)^{rm} = \prod_{j=1}^h g_{p_j}(z)^{m_j} \qquad \text{and} \qquad (2)\ \ g_p(z)^{rn} = e^{2\pi \mathrm{i}\, r\, n\, b\, \beta} . \frac{\prod_{j=1}^h g_{p_j}(z)^{n_j}}{\prod_{i=1}^k g_{q_i}(z)^{a_i}}.$$

Proof. For brevity, we set $h(z) = e^{-\pi \mathrm{i} - 2\pi \mathrm{i} z}$; then

$$g_p(z) = e^{2\pi \mathrm{i} p} h(z) \qquad \text{and} \qquad g_{p_j}(z) = e^{2\pi \mathrm{i} p_j} h(z). \tag{23.1.9}$$

Using (23.1.8), we have $g_p(z) = e^{2\pi \mathrm{i} p} \cdot h(z) = \exp\Big(2\pi \mathrm{i} \frac{\sum_{j=1}^h m_j p_j}{rm}\Big) \cdot h(z)$, and so

$$\begin{aligned} g_p(z)^{rm} &= \exp\Big(2\pi \mathrm{i} \sum_{j=1}^h m_j p_j\Big) \cdot h(z)^{rm} \\ &= \prod_{j=1}^h \Big(\exp(2\pi \mathrm{i} p_j) \cdot h(z)\Big)^{m_j} \qquad \text{by (23.1.5)} \\ &= \prod_{j=1}^h g_{p_j}(z)^{m_j} \qquad \text{by (23.1.9).} \end{aligned}$$

This verifies (1). Next we show (2). Using (23.1.8),

$$g_p(z) = \exp\left[2\pi \mathrm{i} \left(\frac{\sum_{j=1}^h n_j p_j - \sum_{i=1}^k a_i q_i}{rn} + a + b\beta\right)\right] \cdot h(z),$$

and hence

$$\begin{aligned} g_p(z)^{rn} &= \exp\Big[2\pi \mathrm{i} \Big(\big(\textstyle\sum_{j=1}^h n_j p_j - \sum_{i=1}^k a_i q_i\big) + rn(a + b\beta)\Big)\Big] \cdot h(z)^{rn} \\ &= \exp\Big[2\pi \mathrm{i} rn(a + b\beta)\Big] \cdot \frac{\prod_{j=1}^h \Big(\exp(2\pi \mathrm{i} p_j) \cdot h(z)\Big)^{n_j}}{\prod_{i=1}^k \Big(\exp(2\pi \mathrm{i} q_i) \cdot h(z)\Big)^{a_i}} \\ &= \exp\Big[2\pi \mathrm{i} rn(a + b\beta)\Big] \cdot \frac{\prod_{j=1}^h g_{p_j}(z)^{n_j}}{\prod_{i=1}^k g_{q_i}(z)^{a_i}} \qquad \text{by (23.1.9),} \end{aligned}$$

where in the second equality we used $rn = \sum_{j=1}^h n_j - \sum_{i=1}^k a_i$ (23.1.5); so $h(z)^{rn} = \prod_{j=1}^h h(z)^{n_j} / \prod_{i=1}^k h(z)^{a_i}$. Since r, n, a are integers, we have $\exp\big[2\pi \mathrm{i} rn(a + b\beta)\big] = \exp(2\pi \mathrm{i} rnb\beta)$ and hence (2) is confirmed. □

23.2 Genus 1: Arrangement functions

We consider the plot function $K(z) = n\sigma_z\tau + m\sigma\tau_z$, where σ is a holomorphic section of a line bundle $N^{\otimes(-m)}$ on C with a zero of order m_j at p_j ($j = 1, 2, \ldots, h$), while τ is a meromorphic section of a line bundle $N^{\otimes n}$ on C with a pole of order n_j at p_j ($j = 1, 2, \ldots, h$) and a zero of order a_i at q_i ($i = 1, 2, \ldots, k$). Recall that $K(z)$ was used for describing the singularities of a singular fiber $X_{s,t}$ ($s, t \neq 0$) where

$$X_{s,t}: \quad \sigma(z)\zeta^{m-ln}\big(\zeta^n + t\tau(z)\big)^l - s = 0.$$

The z-coordinate of a singularity of a singular fiber $X_{s,t}$ ($s, t \neq 0$) is necessarily a zero of $K(z)$ (Proposition 21.8.3 (1), p418). However, $K(z)$ also has zeros p_j and q_i, none of which is the z-coordinate of a singularity of a singular fiber $X_{s,t}$ ($s, t \neq 0$); see Lemma 21.2.1. Taking this fact into consideration, as we done when C is the projective line, it is economical to consider an "arrangement function" $J(z)$ (to be given soon) instead of $K(z)$. First from (23.1.7), we have

$$\log(\sigma^n\tau^m) = n\Big(\sum_{j=1}^{h} m_j \log\vartheta_{p_j}(z)\Big) + m\Big(\sum_{i=1}^{k} a_i \log\vartheta_{q_i}(z) - \sum_{j=1}^{h} n_j \log\vartheta_{p_j}(z)\Big) - 2\pi\mathrm{i}rmnbz.$$

Here we are indifferent to the choice of a branch of $\log(f)$ (where $f := \sigma^n\tau^m$), since we are only interested in the logarithmic derivative $\dfrac{d\log(f)}{dz} = \dfrac{f'}{f}$.

Remark 23.2.1 Note that $f = \sigma^n\tau^m$ is a meromorphic function, because it is a section of the trivial bundle $(N^{\otimes(-m)})^{\otimes n} \otimes (N^{\otimes n})^{\otimes m} \cong \mathcal{O}_C$. In particular, f' (or precisely $f'(z)dz$) is a meromorphic section of the cotangent bundle Ω^1_C, and therefore the ratio $\dfrac{f'}{f}$ is a meromorphic section of Ω^1_C; in the present case (the genus of C is one), we have $\Omega^1_C \cong \mathcal{O}_C$, and thus $\dfrac{f'}{f}\left(= \dfrac{d\log(f)}{dz}\right)$ is a meromorphic function.

We point out that under different choices p'_i of p_i and q'_i of q_i such that $p'_i \equiv p_i$ and $q'_i \equiv q_i \mod \Gamma$, we may make the last term "$2\pi\mathrm{i}rmnbz$" in the above expression of $\log(\sigma^n\tau^m)$ vanish. In fact, choose p_i and q_i such that

$$\frac{\sum_{j=1}^h m_j p_j}{m} = \frac{\sum_{j=1}^h n_j p_j - \sum_{i=1}^k a_i q_i}{n},$$

then

$$p = \frac{\sum_{j=1}^h m_j p_j}{rm} = \frac{\sum_{j=1}^h n_j p_j - \sum_{i=1}^k a_i q_i}{rn},$$

and we have

$$\log(\sigma^n\tau^m) = n\Big(\sum_{j=1}^{h} m_j \log\vartheta_{p_j}(z)\Big) + m\Big(\sum_{i=1}^{k} a_i \log\vartheta_{q_i}(z) - \sum_{j=1}^{h} n_j \log\vartheta_{p_j}(z)\Big).$$

Differentiating this equation yields

$$\frac{d\log(\sigma^n\tau^m)}{dz} = n\left(\sum_{j=1}^{h} m_j \frac{\vartheta'_{p_j}(z)}{\vartheta_{p_j}(z)}\right) + m\left(\sum_{i=1}^{k} a_i \frac{\vartheta'_{q_i}(z)}{\vartheta_{q_i}(z)} - \sum_{j=1}^{h} n_j \frac{\vartheta'_{p_j}(z)}{\vartheta_{p_j}(z)}\right), \tag{23.2.1}$$

where $\vartheta'_{p_j}(z)$ stands for the derivative $\dfrac{d\vartheta_{p_j}(z)}{dz}$. To simplify the subsequent discussion, for the time being, we assume that $nm_j - mn_j \neq 0$ for all $j = 1, 2, \ldots, h$. We set $P(z) = \prod_{j=1}^{h} \vartheta_{p_j}(z)$ and $Q(z) = \prod_{i=1}^{k} \vartheta_{q_i}(z)$, and also we set

$$P_j(z) = \vartheta_{p_1}(z)\,\vartheta_{p_2}(z)\cdots\vartheta'_{p_j}(z)\cdots\vartheta_{p_h}(z),$$

$$Q_i(z) = \vartheta_{q_1}(z)\,\vartheta_{q_2}(z)\cdots\vartheta'_{q_i}(z)\cdots\vartheta_{q_k}(z),$$

where $\vartheta'_p(z) = \dfrac{d\vartheta_p(z)}{dz}$. Then $J(z) := P(z)Q(z)\dfrac{d\log(\sigma^n\tau^m)}{dz}$ is called an *arrangement function* on C. (cf. the plot function $K(z) = \sigma\tau\dfrac{d\log(\sigma^n\tau^m)}{dz}$.) Multiplying $P(z)Q(z)$ with the right hand side of (23.2.1), we may explicitly write it:

$$\begin{aligned} J(z) &= nQ(z)\Big(\sum_{j=1}^{h} m_j P_j(z)\Big) + mP(z)\Big(\sum_{i=1}^{k} a_i Q_i(z)\Big) - mQ(z)\Big(\sum_{j=1}^{h} n_j P_j(z)\Big) \\ &= Q(z)\Big(\sum_{j=1}^{h}(nm_j - mn_j)P_j(z)\Big) + P(z)\Big(\sum_{i=1}^{k} ma_i Q_i(z)\Big). \end{aligned} \tag{23.2.2}$$

Lemma 23.2.2 *Suppose that $nm_j - mn_j \neq 0$ for all j. Then for $\alpha \in C$, the following equivalence holds:*

$$K(\alpha) = 0,\quad \sigma(\alpha) \neq 0,\quad \textit{and}\quad \tau(\alpha) \neq 0 \quad\Longleftrightarrow\quad J(\alpha) = 0.$$

Moreover, the order of α in $K(z)$ is equal to that of α in $J(z)$.

Proof. As "$\sigma(\alpha) \neq 0$ and $\tau(\alpha) \neq 0$" is equivalent to "α is none of $p_1, p_2, \ldots, p_h$ and $q_1, q_2, \ldots, q_k$", we have to show the following equivalence:

$$K(\alpha) = 0 \quad\text{and}\quad \alpha \neq p_1, p_2, \ldots, p_h, q_1, q_2, \ldots, q_k \quad\Longleftrightarrow\quad J(\alpha) = 0.$$

We first demonstrate that $J(p_j) \neq 0$ and $J(q_i) \neq 0$. To see this, we set $R_j(z) = \vartheta_{p_1}(z)\,\vartheta_{p_2}(z)\cdots\vartheta_{p_{j-1}}(z)\,\vartheta_{p_{j+1}}(z)\cdots\vartheta_{p_h}(z)$ and write $P_j(z) = R_j(z)\,\vartheta'_{p_j}(z)$.

Then

$$J(p_j) = Q(p_j) \cdot (nm_j - mn_j) R_j(p_j) \vartheta'_{p_j}(p_j). \tag{23.2.3}$$

Clearly $Q(p_j) \neq 0$ and $R_j(p_j) \neq 0$. Further, $\vartheta'_{p_j}(p_j) \neq 0$. In fact, since p_j is a simple zero of ϑ_{p_j}, we may write $\vartheta_{p_j}(z) = (z - p_j)h(z)$ where $h(p_j) \neq 0$. Then $\vartheta'_{p_j}(z) = h(z) + (z - p_j)h'(z)$, and thus $\vartheta'_{p_j}(p_j) = h(p_j) \neq 0$. Since $Q(p_j) \neq 0$, $R_j(p_j) \neq 0$, and $\vartheta'_{p_j}(p_j) \neq 0$, we conclude from (23.2.3) that $J(p_j) \neq 0$. Similarly, we can show $J(q_i) \neq 0$.

On the other hand, from the expressions $J(z) = PQ\dfrac{d\log(\sigma^n\tau^m)}{dz}$ and $K(z) = \sigma\tau\dfrac{d\log(\sigma^n\tau^m)}{dz}$, we have $K(z) = \dfrac{\sigma\tau}{PQ}\, J(z)$. The equivalence in question immediately follows from this equation, because (i) the zeros of $\dfrac{\sigma\tau}{PQ}$ are p_j $(j = 1, 2, \ldots, h)$ and q_i $(i = 1, 2, \ldots, k)$ and (ii) $J(p_j) \neq 0$ and $J(q_i) \neq 0$.□

Next we consider the case where there are indices j such that $nm_j - mn_j = 0$; say, for v indices $j = h - v + 1, h - v + 2, \ldots, h$, we have $nm_j - mn_j = 0$. Then instead of $P(z)$ and $P_j(z)$ (while we keep $Q(z)$ and $Q_i(z)$), we introduce $\widehat{P}(z)$ and $\widehat{P}_j(z)$ $(j = 1, 2, \ldots, h - v)$ as follows:

$$\widehat{P}(z) = \prod_{j=1}^{h-v} \vartheta_{p_j}(z), \qquad \widehat{P}_j(z) = \vartheta_{p_1}(z)\,\vartheta_{p_2}(z) \cdots \vartheta'_{p_j}(z) \cdots \vartheta_{p_{h-v}}(z),$$

where $\vartheta'_p(z) = \dfrac{d\vartheta_p(z)}{dz}$. Multiplying $\widehat{P}(z)Q(z)$ with the right hand side of (23.2.1), we define an *arrangement function* by

$$J(z) = Q(z)\left(\sum_{j=1}^{h-v}(nm_j - mn_j)\widehat{P}_j(z)\right) + \widehat{P}(z)\left(\sum_{i=1}^{k} ma_i Q_i(z)\right). \tag{23.2.4}$$

We let J_0 be the set of indices j such that $nm_j - mn_j = 0$; in the present case, $J_0 = \{\, h - v + 1,\ h - v + 2,\ \ldots,\ h \,\}$. Then $J(z)$ may have a zero at p_{j_0} for $j_0 \in J_0$; indeed, the right hand side of

$$J(p_{j_0}) = Q(p_{j_0})\left(\sum_{j=1}^{h-v}(nm_j - mn_j)\widehat{P}_j(p_{j_0})\right) + \widehat{P}(p_{j_0})\left(\sum_{i=1}^{k} ma_i Q_i(p_{j_0})\right)$$

is possibly zero. Noting this, we may generalize Lemma 23.2.2 as follows:

Lemma 23.2.3 *For $\alpha \in C$, the following equivalence holds:*

$$K(\alpha) = 0, \quad \sigma(\alpha) \neq 0, \quad \text{and} \quad \tau(\alpha) \neq 0 \quad \Longleftrightarrow \quad J(\alpha) = 0 \quad \text{and} \quad \alpha \neq p_j \quad (j \in J_0),$$

where J_0 is a set of indices j such that $nm_j - mn_j = 0$. (If $J_0 = \emptyset$, the condition "$\alpha \neq p_j$ $(j \in J_0)$" is vacuous.) *Moreover, the order of α in $K(z)$ is equal to that of α in $J(z)$.*

Next we recall Lemma 21.2.1, p393: *A point* $(\alpha, \beta) \in X_{s,t}$ $(s, t \neq 0)$ *is a singularity if and only if* α *and* β *satisfy*

$$\text{(a)}\quad K(\alpha) = 0, \quad \sigma(\alpha) \neq 0, \quad \textit{and} \quad \tau(\alpha) \neq 0 \quad \textit{and}$$
$$\text{(b)}\quad \beta^n = \frac{ln - m}{m} t\tau(\alpha),$$

where $K(z) = n\sigma_z\tau + m\sigma\tau_z$ *is the plot function on* C. In terms of the equivalence in Lemma 23.2.3, we can restate this as follows:

Proposition 23.2.4 *A point* $(\alpha, \beta) \in X_{s,t}$ $(s, t \neq 0)$ *is a singularity if and only if* α *and* β *satisfy*

$$\text{(a)}\quad J(\alpha) = 0, \quad \alpha \neq p_j \ (j \in J_0) \quad \textit{and} \quad \text{(b)}\quad \beta^n = \frac{ln - m}{m} t\tau(\alpha),$$

where J_0 *is the set of indices* j *such that* $nm_j - mn_j = 0$.

Now for a point $p \in C$, recall that L_p is the line bundle on C with the transition functions "1" (for the gluing map $z \mapsto z + 1$ of C) and "$e^{-\pi i - 2\pi i(z-p)}$" (for the gluing map $z \mapsto z + \beta$ of C). The translated theta function $\vartheta_p(z) := \vartheta(z - p)$ descends to a holomorphic section of L_p with one simple zero at the image of p in $C = \mathbb{C}/\Gamma$; hence the degree of L_p is 1. The following lemma is important.

Lemma 23.2.5 *Suppose that* $nm_j - mn_j \neq 0$ *for each* $j = 1, 2, \dots, h - v$, *and* $nm_j - mn_j = 0$ *for each* $j = h - v + 1, h - v + 2, \dots, h$ *(possibly* $v = 0$*). Then* (1) *the arrangement function* $J(z)$ *is a holomorphic section of a line bundle* $L := \bigotimes_{j=1}^{h-v} L_{p_j} \otimes \bigotimes_{i=1}^{k} L_{q_i}$ *on* C, *and* (2) *the degree of* L *is* $h - v + k$.

Proof. Note that $\widehat{P}Q$ is a section of a line bundle $L = \bigotimes_{j=1}^{h-v} L_{p_j} \otimes \bigotimes_{i=1}^{k} L_{q_i}$, while $\dfrac{d\log(\sigma^n\tau^m)}{dz}$ is a meromorphic function (a section of the trivial bundle) by Remark 23.2.1. Therefore $J(z) = \widehat{P}Q\dfrac{d\log(\sigma^n\tau^m)}{dz}$ is a section of L. The rest of the statement is clear. From the second expression of $J(z)$ in (23.2.2), $J(z)$ is holomorphic. Since the line bundles L_{p_j} and L_{q_i} have degree 1, the degree of L is $h - v + k$. □

Remark 23.2.6 Since C is an elliptic curve, the cotangent bundle Ω^1_C is trivial. Thus by Lemma 21.3.1, p396, the plot function $K(z)$ is a meromorphic section of a line bundle $N^{\otimes(n-m)}$ (its degree is $r(m - n)$ where $\deg N = -r < 0$). Note that $K(z)$ is meromorphic, whereas $J(z)$ is holomorphic.

From Lemma 23.2.5, the sum of the orders of zeros of $J(z)$ is equal to $h - v + k$. We are interested in a 'special case', that is,

$$\big(\text{the sum of the orders of zeros } \alpha \text{ of } J(z) \text{ such that } \alpha \neq p_j \ (j \in J_0)\big)$$
$$= (h - v + k) - \sum_{j \in J_0} \operatorname{ord}_{p_j}\big(J(z)\big),$$

where J_0 is the set of indices j such that $nm_j - mn_j = 0$, and $\operatorname{ord}_{p_j}\big(J(z)\big)$ stands for the order of the zero of $J(z)$ at p_j $(j \in J_0)$. Now setting $\omega := d\log\big(\sigma^n\tau^m\big)$ (a meromorphic 1-form on C), we write $J(z) = \widehat{P}(z)Q(z)\,\omega$. Since $\widehat{P}(z)Q(z)$ does not vanish at p_j $(j \in J_0)$, we have $\operatorname{ord}_{p_j}(J(z)) = \operatorname{ord}_{p_j}(\omega)$. Hence the above equation is rewritten as

$$\big(\text{the sum of the orders of zeros } \alpha \text{ of } J(z) \text{ such that } \alpha \neq p_j \ (j \in J_0)\,\big) \\ = (h - v + k) - \sum_{j \in J_0} \operatorname{ord}_{p_j}(\omega).$$

Using the equivalence in Lemma 23.2.2, we may further rewrite this equation as

$$\big(\text{the sum of the orders of zeros } \alpha \text{ of } K(z) \text{ such that } \sigma(\alpha) \neq 0 \text{ and } \tau(\alpha) \neq 0\,\big) \\ = (h - v + k) - \sum_{j \in J_0} \operatorname{ord}_{p_j}(\omega).$$

In particular,

$$\big(\text{the number of zeros } \alpha \text{ of } K(z) \text{ such that } \sigma(\alpha) \neq 0 \text{ and } \tau(\alpha) \neq 0\,\big) \\ \leq (h - v + k) - \sum_{j \in J_0} \operatorname{ord}_{p_j}(\omega).$$

Once we obtain this inequality, it is straightforward to recover the inequalities in Proposition 21.3.6, p400 and Corollary 21.4.4, p403 (concerning the number of the subordinate fibers in $\pi_t : M_t \to \Delta$ and the number of the singularities of a subordinate fiber).

23.3 Riemann theta functions and Riemann factorization

When the complex curve (the core) C has genus ≥ 2, the "arrangement function" is defined in terms of the Riemann theta function composed with the Abel–Jacobi map — in the subsequent discussion, we often quote results from Supplement (§24.1, p461), where we summarized the results on the Riemann theta function and related topics.

Let $a_1, b_1, a_2, b_2, \ldots, a_g, b_g$ be loops (simple closed curves) on C which form the standard generators of the fundamental group of C:

$$\pi_1(C) = \langle\, a_1, b_1, a_2, b_2, \ldots, a_g, b_g \,:\, [a_1, b_1]\,[a_2, b_2]\,\cdots\,[a_g, b_g] = 1\,\rangle,$$

where $[a_i, b_i] := a_i b_i a_i^{-1} b_i^{-1}$. We take holomorphic 1-forms $\omega_1, \omega_2, \ldots, \omega_g$ on C which form a "normalized" basis of $H^0(C, \Omega^1_C)$, that is, $\int_{a_i} \omega_j = \delta_{ij}$ where $\delta_{ij} = 0$ for $i \neq j$ and 1 for $i = j$. We consider a lattice Λ in $\mathbb{C}^g$ generated by

$2g$ vectors $\mathbf{e}_i$ and $\boldsymbol{\beta}_i$ $(i = 1, 2, \dots, g)$:

$$\mathbf{e}_i = \begin{pmatrix} \int_{a_i} \omega_1 \\ \int_{a_i} \omega_2 \\ \vdots \\ \int_{a_i} \omega_g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \qquad \boldsymbol{\beta}_i = \begin{pmatrix} \int_{b_i} \omega_1 \\ \int_{b_i} \omega_2 \\ \vdots \\ \int_{b_i} \omega_g \end{pmatrix},$$

where "1" is in i-th place. Namely,

$$\Lambda = \mathbb{Z}\mathbf{e}_1 + \mathbb{Z}\mathbf{e}_2 + \cdots + \mathbb{Z}\mathbf{e}_g + \mathbb{Z}\boldsymbol{\beta}_1 + \mathbb{Z}\boldsymbol{\beta}_2 + \cdots + \mathbb{Z}\boldsymbol{\beta}_g. \tag{23.3.1}$$

A g-dimensional complex torus $\mathrm{Jac}(C) := \mathbb{C}^g/\Lambda$ (the quotient of $\mathbb{C}^g$ by Λ) is called the *Jacobian variety* of C.

The *Riemann theta function* $\vartheta(\mathbf{x})$ is a holomorphic function on $\mathbb{C}^g$, given by

$$\vartheta(\mathbf{x}) = \sum_{\mathbf{n} \in \mathbb{Z}^g} \exp\left[2\pi \mathrm{i} \left(\frac{1}{2}\langle \mathbf{n}, B\mathbf{n}\rangle + \langle \mathbf{n}, \mathbf{x}\rangle\right)\right],$$

where $\mathbf{n}$ and $\mathbf{x}$ are column vectors, $\langle \mathbf{x}, \mathbf{y}\rangle = x_1\overline{y}_1 + x_2\overline{y}_2 + \cdots + x_g\overline{y}_g$, and $B = \left(\int_{b_i} \omega_j\right)$ is a $g \times g$ matrix. (Note that $\vartheta(\mathbf{x})$ is a higher dimensional generalization of the "one-dimensional" theta function (23.1.1).) It possesses quasi doubly-periodicity:

$$\vartheta(\mathbf{x} + \mathbf{e}_i) = \vartheta(\mathbf{x}), \qquad \vartheta(\mathbf{x} + \boldsymbol{\beta}_i) = e^{-2\pi \mathrm{i}(x_i + \frac{1}{2}\beta_{ii})} \cdot \vartheta(\mathbf{x}),$$

where x_i and β_{ii} are respectively the i-th coordinate of $\mathbf{x}$ and $\boldsymbol{\beta}_i$ (so $\beta_{ii} = \int_{b_i} \omega_i$). Now we consider a Λ-action on $\mathbb{C}^g \times \mathbb{C}$ generated by

$$(\mathbf{x}, \xi) \longmapsto (\mathbf{x} + \mathbf{e}_i\, ,\, \xi), \qquad (\mathbf{x}, \xi) \longmapsto (\mathbf{x} + \boldsymbol{\beta}_i\, ,\, e^{-2\pi \mathrm{i}(x_i + \frac{1}{2}\beta_{ii})} \cdot \xi)$$

for $i = 1, 2, \dots, g$. Then the quotient

$$\mathcal{L} := (\mathbb{C}^g \times \mathbb{C})/\Lambda \tag{23.3.2}$$

is a line bundle on the Jacobian variety $\mathrm{Jac}(C)$, and the Riemann theta function $\vartheta(\mathbf{x})$ descends to a holomorphic section of $\mathcal{L}$, which we often denote also by ϑ. The complex variety $\mathrm{Zero}(\vartheta) = \{\, \vartheta(\mathbf{x}) = 0 \,\}$ in $\mathrm{Jac}(C)$ is called the *theta divisor*.

Next let $\mathbf{u} : C \to \mathrm{Jac}(C)$ be the *Abel–Jacobi map*: fix a base point $z_0 \in C$, and then it is given by

$$\mathbf{u}(z) = \begin{pmatrix} \int_{z_0}^{z} \omega_1 \\ \int_{z_0}^{z} \omega_2 \\ \vdots \\ \int_{z_0}^{z} \omega_g \end{pmatrix} \mod \Lambda.$$

As is well-known, if genus $(C) \geq 1$, the Abel–Jacobi map $\mathbf{u}$ is an embedding [ACGH]. Taking $\mathbf{d} \in \mathrm{Jac}(C)$, we define a translation map of $\mathrm{Jac}(C)$ by $\mathbf{x} \mapsto \mathbf{x} + \mathbf{d}$. Then we consider the following line bundle on $\mathrm{Jac}(C)$:

$$\mathcal{L}_{\mathbf{d}}: \quad \text{the pull-back of } \mathcal{L} \text{ under the translation } \mathbf{x} \mapsto \mathbf{x} + \mathbf{d}. \tag{23.3.3}$$

Note that $\vartheta\big(\mathbf{x} - \mathbf{d}\big)$ is a holomorphic section of $\mathcal{L}_{\mathbf{d}}$.

Using the Riemann theta function, we may explicitly express meromorphic functions on the complex curve C. To explain this, we prepare some notation. Fix $\mathbf{c} \in \mathrm{Jac}(C)$ such that $\vartheta(\mathbf{c}) = 0$. Also fixing $p \in C$, we consider a holomorphic section $\vartheta_p(z) := \vartheta\big(\mathbf{u}(z) - \mathbf{u}(p) - \mathbf{c}\big)$ of a line bundle L_p on C, where L_p is the pull-back of the line bundle $\mathcal{L}_{\mathbf{u}(p)+\mathbf{c}}$ (see (23.3.3)) by the Abel–Jacobi map $\mathbf{u} : C \to \mathrm{Jac}(C)$. We note that $\vartheta_p(z)$ may be identically zero. This occurs exactly when $\mathbf{u}(z) - \mathbf{u}(p) - \mathbf{c} \in \mathrm{Zero}(\vartheta)$ for all $z \in C$, that is, $\mathbf{c} \in -\mathrm{Zero}(\vartheta) + \mathbf{u}(C) - \mathbf{u}(p)$, where "$-\mathrm{Zero}(\vartheta) + \mathbf{u}(C) - \mathbf{u}(p)$" stands for the translation of the set $-\mathrm{Zero}(\vartheta)$ by $\mathbf{u}(C) - \mathbf{u}(p)$ in $\mathrm{Jac}(C)$. Thus $\vartheta_p(z)$ is identically zero precisely when

$$\mathbf{c} \in \mathrm{Zero}(\vartheta) \cap \Big(-\mathrm{Zero}(\vartheta) + \mathbf{u}(C) - \mathbf{u}(p)\Big).$$

Henceforth, we always take $\mathbf{c} \in \mathrm{Zero}(\vartheta)$ *such that* $\vartheta_p(z)$ *is not identically zero.* Then by Lemma 24.1.7, p471, the sum of the orders of the zeros of $\vartheta_p(z)$ equals g, so that $\vartheta_p(z)$ has g zeros "including multiplicities"; in other words, these g zeros are not necessarily distinct[1]. One of the g zeros is p. Indeed, since $\vartheta(\mathbf{x})$ is an even function, $\vartheta_p(p) = \vartheta(-\mathbf{c}) = \vartheta(\mathbf{c}) = 0$. The remaining zeros, denoted by $o_1, o_2, \ldots, o_{g-1} \in C$, are called the *surplus zeros* of $\vartheta_p(z)$. See §24.1 Supplement, Theorem 24.1.16, p478 for the position of surplus zeros (see also [Na3], p102, or [Si] II, Theorem 1, p175).

Now let $f(z)$ be a meromorphic function on the complex curve C (of genus $g \geq 1$), and we express its divisor as $\mathrm{div}(f) = \sum_{i=1}^{n} p_i - \sum_{i=1}^{n} q_i$ where the points $p_1, p_2, \ldots, p_n \in C$ (resp. $q_1, q_2, \ldots, q_n \in C$) are *not* necessarily distinct.

Claim 23.3.1 *There exists* $\mathbf{c} \in \mathrm{Zero}(\vartheta)$ *such that* (1) *the surplus zeros* $o_1, o_2, \ldots, o_{g-1}$ *of* $\vartheta_p(z) := \vartheta\big(\mathbf{u}(z) - \mathbf{u}(p) - \mathbf{c}\big)$ *are independent of the choice of* $p \in C$ *and* (2) *none of* $o_1, o_2, \ldots, o_{g-1}$ *is* p_i *or* q_i $(i = 1, 2, \ldots, n)$.

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[1] If we take 'generic' $\mathbf{c} \in \mathrm{Zero}(\vartheta)$, these zeros are distinct. See Remark 23.3.2 below.

Proof. We denote by $\mathrm{Sym}^i(C)$ the i-th symmetric product of the curve C, that is,

$$\mathrm{Sym}^i(C) = \underbrace{C \times C \times \cdots \times C}_{i} / \mathfrak{S}_i,$$

where the symmetric group $\mathfrak{S}_i$ acts on $C \times C \times \cdots \times C$ by

$$(z_1, z_2, \dots, z_i) \longmapsto (z_{\sigma(1)}, z_{\sigma(2)}, \dots, z_{\sigma(i)}), \qquad \sigma \in \mathfrak{S}_i.$$

We extend the Abel–Jacobi map $\mathbf{u} : C \to \mathrm{Jac}(C)$ to the map $\mathbf{u} : \mathrm{Sym}^i(C) \to \mathrm{Jac}(C)$ by

$$(p_1, p_2, \dots, p_i) \longmapsto \mathbf{u}(p_1) + \mathbf{u}(p_2) + \cdots + \mathbf{u}(p_i).$$

Now let D be the critical set of $\mathbf{u} : \mathrm{Sym}^{g-1}(C) \to \mathrm{Jac}(C)$, and we set $D' := \mathbf{u}^{-1}(D)$; so the restriction $\mathbf{u} : \mathrm{Sym}^{g-1}(C) \setminus D' \to W_{g-1} \setminus D$ is biholomoprhic. Next we take $(o_1, o_2, \dots, o_{g-1}) \in \mathrm{Sym}^{g-1}(C) \setminus D'$ such that none of $o_1, o_2, \dots, o_{g-1}$ is p_i or q_i $(i = 1, 2, \dots, n)$, and we set

$$\mathbf{c} := \mathbf{u}(o_1) + \mathbf{u}(o_2) + \cdots + \mathbf{u}(o_{g-1}) - \boldsymbol{\kappa}. \tag{23.3.4}$$

Then by Theorem 24.1.16 (2), p478, $\mathbf{c} \in \mathrm{Zero}(\vartheta)$, that is, $\vartheta(\mathbf{c}) = 0$. In particular, the 'even function' $\vartheta_p(z) = \vartheta\big(\mathbf{u}(z) - \mathbf{u}(p) - \mathbf{c}\big)$ has a zero p. Further by Theorem 24.1.16 (4), p478, $o_1, o_2, \dots, o_{g-1}$ are the surplus zeros of $\vartheta_p(z)$. Note that none of the surplus zeros $o_1, o_2, \dots, o_{g-1}$ of $\vartheta_p(z)$ is p_i or q_i; because we chose $(o_1, o_2, \dots, o_{g-1}) \in \mathrm{Sym}^{g-1}(C) \setminus D'$ such that none of them is p_i or q_i. Also since $\mathbf{c} + \boldsymbol{\kappa} \in W_{g-1} \setminus D$, the surplus zeros $o_1, o_2, \dots, o_{g-1}$ of $\vartheta_p(z)$ are independent of the choice of $p \in C$ (Theorem 24.1.16 (3)). This completes the proof of our claim. □

Remark 23.3.2 We may find $\mathbf{c} \in \mathrm{Zero}(\vartheta)$ such that $\vartheta_p(z) := \vartheta(\mathbf{u}(z) - \mathbf{u}(p) - \mathbf{c})$ has g "distinct" zeros. In fact, take $(o_1, o_2, \dots, o_{g-1}) \in \mathrm{Sym}^{g-1}(C) \setminus D'$ in the above proof such that $p, o_1, o_2, \dots, o_{g-1}$ are distinct. Then for the $\mathbf{c}$ defined by (23.3.4), $\vartheta_p(z)$ has g distinct zeros $p, o_1, o_2, \dots, o_{g-1}$. Actually, $\vartheta_p(z)$ for generic $\mathbf{c} \in \mathrm{Zero}(\vartheta)$ has g distinct zeros, as we may vary $o_1, o_2, \dots, o_{g-1}$.

Now we consider a meromorphic function $f(z)$ on the complex curve C; we express $\mathrm{div}(f) = \sum_{i=1}^n p_i - \sum_{i=1}^n q_i$, where the points $p_1, p_2, \dots, p_n \in C$ (resp. $q_1, q_2, \dots, q_n \in C$) are *not* necessarily distinct. Choosing $\mathbf{c} \in \mathrm{Zero}(\vartheta)$ as in Claim 23.3.1, we set

$$\vartheta_{p_i}(z) := \vartheta\big(\mathbf{u}(z) - \mathbf{u}(p_i) - \mathbf{c}\big), \qquad \vartheta_{q_i}(z) := \vartheta\big(\mathbf{u}(z) - \mathbf{u}(q_i) - \mathbf{c}\big).$$

Then the following factorization (Riemann Factorization Theorem; see [Si] II, p176, [Mu2] I, p158) holds:

$$f(z) = a\, \frac{\prod_{i=1}^{n} \vartheta_{p_i}(z)}{\prod_{i=1}^{n} \vartheta_{q_i}(z)}, \tag{23.3.5}$$

where a is some complex number. Notice that all the surplus zeros $o_1, o_2, \ldots, o_{g-1}$ in the numerator and denominator cancel. Also note a correspondence between functions on $\mathbb{C}$ and C:

$$z - p \quad \longleftrightarrow \quad \vartheta_p(z).$$

We remark that precisely speaking, in the expression of the Riemann factorization, we need to choose branches of ϑ_{p_i} and ϑ_{q_i}, as they are multi-valued (they are sections of line bundles). Working on the universal covering space $\widetilde{C}$ of C, we may clarify this procedure. Specifically, instead of the Abel–Jacobi map $\mathbf{u} : C \to \mathrm{Jac}(C)$, we consider its lift $\widetilde{\mathbf{u}} : \widetilde{C} \to \mathbb{C}^g$. Accordingly, we use the notation $\widetilde{\vartheta}$ for the Riemann theta function on $\mathbb{C}^g$ while ϑ stands for the holomorphic section of the line bundle $\mathcal{L}$ (see (23.3.2)) on $\mathrm{Jac}(C)$, determined by $\widetilde{\vartheta}$.

Fix $\widetilde{\mathbf{c}} \in \mathbb{C}^g$ such that $\widetilde{\vartheta}(\widetilde{\mathbf{c}}) = 0$. Taking $\widetilde{p} \in \widetilde{C}$, we consider a holomorphic function $\widetilde{\vartheta}_{\widetilde{p}}(z) := \vartheta\big(\widetilde{\mathbf{u}}(z) - \widetilde{\mathbf{u}}(\widetilde{p}) - \widetilde{\mathbf{c}}\big)$ on $\widetilde{C}$. Now let $f(z)$ be a meromorphic function on C, and we express

$$\mathrm{div}(f) = \sum_{i=1}^{n} p_i - \sum_{i=1}^{n} q_i, \qquad \text{(divisor expression)},$$

where the points $p_1, p_2, \ldots, p_n \in C$ (resp. $q_1, q_2, \ldots, q_n \in C$) are not necessarily distinct. Fixing lifts $\widetilde{p}_i \in \widetilde{C}$ of $p_i \in C$ and $\widetilde{q}_i \in \widetilde{C}$ of $q_i \in C$, we define a meromorphic function $\widetilde{f}$ on $\widetilde{C}$:

$$\widetilde{f}(\widetilde{z}) := a\, \frac{\prod_{i=1}^{n} \widetilde{\vartheta}_{\widetilde{p}_i}(\widetilde{z})}{\prod_{i=1}^{n} \widetilde{\vartheta}_{\widetilde{q}_i}(\widetilde{z})}, \tag{23.3.6}$$

where a is a complex number. We denote by Γ the covering transformation group of the universal covering $\widetilde{C} \to C$. Then by Lemma 23.3.3 (2) below, $\widetilde{f}(\widetilde{z})$ is Γ-invariant. Hence $\widetilde{f}(\widetilde{z})$ descends to the meromorphic function $f(z)$ on $C = \widetilde{C}/\Gamma$, and (23.3.6) corresponds to the Riemann factorization (23.3.5).

Lemma 23.3.3 *Let $a_1, b_1, a_2, b_2, \ldots, a_g, b_g$ be loops on C which form the standard generators of the fundamental group of C:*

$$\pi_1(C) = \langle\, a_1, b_1, a_2, b_2, \ldots, a_g, b_g \;:\; [a_1, b_1]\,[a_2, b_2] \cdots [a_g, b_g] = 1 \,\rangle,$$

where $[a_i, b_i] := a_i b_i a_i^{-1} b_i^{-1}$. Denote by A_k and B_k $(k = 1, 2, \ldots, g)$ the covering transformations of the universal covering $\widetilde{C} \to C$ respectively corresponding to the loops a_k and b_k. Then

(1) $\widetilde{f}(A_k \cdot \widetilde{z}) = \widetilde{f}(\widetilde{z})$ *and* $\widetilde{f}(B_k \cdot \widetilde{z}) = e^{2\pi \mathrm{i} \lambda_k} \cdot \widetilde{f}(\widetilde{z})$, *where* $\lambda_k := \sum_{i=1}^{n} \widetilde{\mathbf{u}}_k(\widetilde{p}_i) - \sum_{i=1}^{n} \widetilde{\mathbf{u}}_k(\widetilde{q}_i)$.

(2) *There exist lifts* $\widetilde{p}_j$ *of* p_j *and* $\widetilde{q}_j$ *of* q_j *such that* $\lambda_k = 0$. (Accordingly, $\widetilde{f}(A_k \cdot \widetilde{z}) = \widetilde{f}(\widetilde{z})$ and $\widetilde{f}(B_k \cdot \widetilde{z}) = \widetilde{f}(\widetilde{z})$, and so $\widetilde{f}$ is Γ-invariant.)

Proof. (1): Recall the transformation rule of the Riemann theta function:

$$\widetilde{\vartheta}_{\tilde{p}}\,(A_k \cdot \widetilde{z}) = \widetilde{\vartheta}_{\tilde{p}}\,(\widetilde{z}), \qquad \widetilde{\vartheta}_{\tilde{p}}\,(B_k \cdot \widetilde{z}) = \exp\left[-2\pi \mathrm{i}\left(\widetilde{u}(\widetilde{p}) + \frac{1}{2}\beta\right)\right] \cdot \widetilde{\vartheta}_{\tilde{p}}\,(\widetilde{z}). \tag{23.3.7}$$

Thus from (23.3.6), clearly $\widetilde{f}(A_k \cdot \widetilde{z}) = \widetilde{f}(\widetilde{z})$. On the other hand,

$$\begin{aligned}\widetilde{f}(B_k \cdot \widetilde{z}) &= \frac{\prod_{i=1}^{n} \widetilde{\vartheta}_{\tilde{p}_i}(B_k \cdot \widetilde{z})}{\prod_{i=1}^{n} \widetilde{\vartheta}_{\tilde{q}_i}(B_k \cdot \widetilde{z})} = \frac{\prod_{i=1}^{n} \exp\left[-2\pi \mathrm{i}\Big(\widetilde{\mathbf{u}}(\widetilde{p}_i) + \frac{1}{2}\beta\Big)\right] \cdot \widetilde{\vartheta}_{\tilde{p}_i}(\widetilde{z})}{\prod_{i=1}^{n} \exp\left[-2\pi \mathrm{i}\Big(\widetilde{\mathbf{u}}(\widetilde{q}_i) + \frac{1}{2}\beta\Big)\right] \cdot \widetilde{\vartheta}_{\tilde{q}_i}(\widetilde{z})} \\ &= \exp\left[2\pi \mathrm{i}\Big(\sum_{i=1}^{n} \widetilde{\mathbf{u}}_k(\widetilde{p}_i) - \sum_{i=1}^{n} \widetilde{\mathbf{u}}_k(\widetilde{q}_i)\Big)\right] \cdot \widetilde{f}(\widetilde{z}) \\ &= \exp(2\pi \mathrm{i} \lambda_k) \cdot \widetilde{f}(\widetilde{z}).\end{aligned}$$

(2): We show this only for $g = 1$ (the argument below works for $g \geq 2$). For simplicity we omit the subscript $1(= g)$; we denote λ_1 by λ etc. By Abel's Theorem p467,

$$\lambda := \sum_{i=1}^{n} \widetilde{\mathbf{u}}(\widetilde{p}_i) - \sum_{i=1}^{n} \widetilde{\mathbf{u}}(\widetilde{q}_i) \equiv 0 \qquad \mathrm{mod} \quad \Lambda,$$

where Λ is the lattice (23.3.1). So we may write $\lambda = m\mathbf{e} + n\boldsymbol{\beta}$ where $m,\, n \in \mathbb{Z}$, and $\mathbf{e}$ and $\boldsymbol{\beta}$ is the basis of Λ. Now we take a new lift of p_1. Instead of the original $\widetilde{p}_1$, we take $\widetilde{p}_1 - m\mathbf{e} - n\boldsymbol{\beta}$ as $\widetilde{p}_1$. Then $\lambda = \sum_{i=1}^{n} \widetilde{\mathbf{u}}(\widetilde{p}_i) - \sum_{i=1}^{n} \widetilde{\mathbf{u}}(\widetilde{q}_i) = 0$, and so the assertion is confirmed. □

Supplement: Another form of the Riemann factorization

We shall slightly modify the Riemann factorization. Let $f(z)$ be a meromorphic function on C. By Lemma 23.3.3 (1), $\widetilde{f}(A_k \cdot \widetilde{z}) = \widetilde{f}(\widetilde{z})$ and $\widetilde{f}(B_k \cdot \widetilde{z}) = e^{2\pi \mathrm{i} \lambda_k} \cdot \widetilde{f}(\widetilde{z})$. We take a holomorphic 1-form ω on C such that

$$\int_{a_k} \omega = 1, \qquad \int_{b_k} \omega = \lambda_k.$$

(As in the proof of Abel's Theorem p467, we may take such ω.) We fix a base point $z_0 \in C$. Then the integration $\mu(z) := \int_{z_0}^{z} \omega$ (a multi-valued function on C) lifts to a single-valued holomorphic function $\widetilde{\mu}(\widetilde{z})$ on the universal covering space $\widetilde{C}$ such that

$$\widetilde{\mu}(A_k \cdot \widetilde{z}) = \widetilde{\mu}(\widetilde{z}), \qquad \widetilde{\mu}(B_k \cdot \widetilde{z}) = \widetilde{\mu}(\widetilde{z}) + \lambda_k.$$

Now we set $g(\widetilde{z}) = e^{-2\pi \mathrm{i}\widetilde{\mu}(\widetilde{z})}\widetilde{f}(\widetilde{z})$, and then $g(\widetilde{z})$ is a Γ-invariant function on $\widetilde{C}$. So $g(\widetilde{z})$ descends to the meromorphic function $f(z)$ on C. Moreover, $g(\widetilde{z})$ induces the following factorization of $f(z)$:

$$f(z) = \exp\left(-2\pi \mathrm{i}\int_{z_0}^{z} \omega\right) \cdot \frac{\prod_{i=1}^{n} \vartheta_{p_i}(z)}{\prod_{i=1}^{n} \vartheta_{q_i}(z)}.$$

This is the expression in Theorem 5, p103 of [Na3].

23.4 Genus ≥ 2: Arrangement functions

In this section, we introduce an arrangement function for the case genus $(C) \geq 2$. Consider the restriction of a barking family around the core C:

$$X_{s,t}: \quad \sigma(z)\zeta^m - s + \sum_{k=1}^{l} {}_l\mathrm{C}_k t^k \sigma(z)\tau(z)^k \zeta^{m-kn} = 0,$$

where σ is a holomorphic section of a line bundle $N^{\otimes(-m)}$ on the complex curve C with a zero of order m_j at each p_j $(j = 1, 2, \dots, h)$, while τ is a meromorphic section of a line bundle $N^{\otimes n}$ on C with a pole of order n_j at each p_j $(j = 1, 2, \dots, h)$ and with a zero of order a_i at each q_i $(i = 1, 2, \dots, k)$.

We recall two functions, which were used for describing the singularities of a subordinate fiber (a singular fiber $X_{s,t}$ such that $s, t \neq 0$):

(i) the plot function $K(z) = n\sigma_z\tau + m\sigma\tau_z$: The z-coordinate of a singularity of a subordinate fiber is necessarily a zero of $K(z)$ (Theorem 21.2.3, p395).
(ii) the discriminant function $D_{s,t}(z) = \sigma(z)^{n'}\tau(z)^{m'} - \lambda_{s,t}$ where

$$\lambda_{s,t} := \left(\frac{ln - m}{ln}\right)^{n'l} \left(\frac{m}{ln - m}\right)^{m'} \frac{s^{n'}}{t^{m'}},$$

and $m' := m/\gcd(m, n)$ and $n' := n/\gcd(m, n)$; see (21.1.2). Let $X_{s,t}$ $(s, t \neq 0)$ be a (smooth or singular) fiber, and then the zeros of $D_{s,t}(z)$ are the branched points of the branched covering $(z, \zeta) \in X_{s,t} \longmapsto z \in C$.

We shall express $D_{s,t}(z)$ in terms of the Riemann theta function. Note that $\sigma^{n'}\tau^{m'}$ (where $m' := m/\gcd(m, n)$ and $n' := n/\gcd(m, n)$) is a meromorphic *function* on C, because $\sigma^{n'}\tau^{m'}$ is a section of $(N^{\otimes -m})^{n'} \otimes (N^{\otimes n})^{m'} \cong \mathcal{O}_C$ (the trivial bundle). Thus we may apply the Riemann Factorization Theorem to express

$$\sigma^{n'}\tau^{m'} = \gamma \,\frac{\prod_{i=1}^{k} \vartheta_{q_i}(z)^{m'a_i}}{\prod_{j=1}^{h} \vartheta_{p_j}(z)^{m'n_j - n'm_j}}, \qquad (\gamma\text{: a complex number}), \quad (23.4.1)$$

where we take $\mathbf{c} \in \mathrm{Jac}(C)$ as in Claim 23.3.1 and we set $\vartheta_p(z) := \vartheta\big(\mathbf{u}(z) - \mathbf{u}(p) - \mathbf{c}\big)$.

Remark 23.4.1 For the case $g \geq 2$ (where $g := \mathrm{genus}(C)$), σ and τ are not expressible in terms of the Riemann theta function. In contrast, they are expressible for $g = 1$. cf. (23.1.7). This difference is due to the fact that $\vartheta_p(z)$ has g zeros, and so if $g \geq 2$, then $\vartheta_p(z)$ has $g-1$ surplus zeros.

Using (23.4.1), we may express the discriminant function $D_{s,t}(z) = \sigma(z)^{n'} \times \tau(z)^{m'} - \lambda_{s,t}$ as follows:

$$D_{s,t}(z) = \gamma \,\frac{\prod_{i=1}^{k} \vartheta_{q_i}(z)^{m'a_i}}{\prod_{j=1}^{h} \vartheta_{p_j}(z)^{m'n_j - n'm_j}} - \lambda_{s,t}. \qquad (23.4.2)$$

Next recall that the plot function $K(z) = \sigma\tau \dfrac{d\log(\sigma^n\tau^m)}{dz}$ is a meromorphic section of a line bundle $N^{\otimes(n-m)} \otimes \Omega^1_C$ of degree $r(m-n) + (2g-2)$ on the curve C, where $r := -\deg(N)$. Also recall the role of $K(z)$: *A point* (α, β) *of a singular fiber* $X_{s,t}$ $(s, t \neq 0)$ *is a singularity if and only if* (i) α *is not a zero of* σ *or* τ *and* (ii) $K(\alpha) = 0$ *and* $\beta^n = \dfrac{ln-m}{m} t\tau(\beta)$ (see Lemma 21.2.1, p393). From the viewpoint of the description of the singularities of $X_{s,t}$, the plot function $K(z)$ has p_j $(j = 1, 2, \ldots, h)$ and q_i $(i = 1, 2, \ldots, k)$ as unnecessary zeros. For this reason, as in the case $g = 0, 1$ $(g := \mathrm{genus}(C))$, we introduce an arrangement function.

We set $d = \gcd(m, n)$. Taking the d-th powers of (23.4.1), we have

$$\sigma^n\tau^m = \gamma^d \,\frac{\prod_{i=1}^{k} \vartheta_{q_i}(z)^{ma_i}}{\prod_{j=1}^{h} \vartheta_{p_j}(z)^{mn_j - nm_j}}, \qquad (23.4.3)$$

and so

$$\log(\sigma^n\tau^m) = \log(\gamma^d) + \sum_{i=1}^{k} ma_i \log\vartheta_{q_i}(z) - \sum_{j=1}^{h} (mn_j - nm_j) \log\vartheta_{p_j}(z). \qquad (23.4.4)$$

Now we introduce an arrangement function. At first, we assume that $nm_j - mn_j \neq 0$ for all $j = 1, 2, \ldots, h$. We set $P(z) = \prod_{j=1}^{h} \vartheta_{p_j}(z)$ and $Q(z) = \prod_{i=1}^{k} \vartheta_{q_i}(z)$, and also we set

$$\begin{cases} P_j(z) = \vartheta_{p_1}(z)\,\vartheta_{p_2}(z)\cdots\vartheta'_{p_j}(z)\cdots\vartheta_{p_h}(z), \\ Q_i(z) = \vartheta_{q_1}(z)\,\vartheta_{q_2}(z)\cdots\vartheta'_{q_i}(z)\cdots\vartheta_{q_k}(z), \end{cases} \tag{23.4.5}$$

where $\vartheta'_p(z) = \dfrac{d\vartheta_p(z)}{dz}$. Then we define an arrangement function $J(z)$ by

$$\begin{aligned} J(z) &:= P(z)Q(z)\frac{d\log(\sigma^n\tau^m)}{dz} \\ &= P(z)Q(z)\left(\sum_{i=1}^{k} ma_i\frac{\vartheta'_{q_i}(z)}{\vartheta_{q_i}(z)} - \sum_{j=1}^{h}(mn_j - nm_j)\frac{\vartheta'_{p_j}(z)}{\vartheta_{p_j}(z)}\right) \\ &= Q(z)\Big(\sum_{j=1}^{h}(nm_j - mn_j)P_j(z)\Big) + P(z)\Big(\sum_{i=1}^{k} ma_iQ_i(z)\Big), \end{aligned} \tag{23.4.6}$$

where in the second equality we used (23.4.4). From the third expression, clearly $J(z)$ is holomorphic (whereas $K(z)$ is in general meromorphic by Lemma 21.3.1, p396). Before proceeding, we note that $\vartheta_p(z) := \vartheta\big(\mathbf{u}(z) - \mathbf{u}(p) - \mathbf{c}\big)$ is a holomorphic section of a line bundle L_p on C, where L_p is the pull-back of the line bundle $\mathcal{L}_{\mathbf{u}(p)+\mathbf{c}}$ (see (23.3.3)) by the Abel–Jacobi map $\mathbf{u} : C \to \mathrm{Jac}(C)$; by Lemma 24.1.7, p471, L_p has degree g.

Since $\dfrac{d\log(\sigma^n\tau^m)}{dz}$ is a meromorphic section of the cotangent bundle Ω^1_C (Remark 23.2.1) and PQ is a holomorphic section of a line bundle $L := \bigotimes_{j=1}^{h} L_{p_j} \otimes \bigotimes_{i=1}^{k} L_{q_i}$, it follows that $J(z) = PQ\dfrac{d\log(\sigma^n\tau^m)}{dz}$ is a holomorphic section of $L \otimes \Omega^1_C$; as L_{p_j} and L_{q_i} have the degree g, we have $\deg(L \otimes \Omega^1_C) = g(h+k) + (2g-2)$.

Observe that conversion from $K(z)$ to $J(z)$ amounts to the exclusion of the unnecessary zeros $p_1, p_2, \ldots, p_h, q_1, q_2, \ldots, q_k$; see Lemma 23.2.2. However, in the present case (the genus $g \geq 2$), $J(z)$ still possesses another type of unnecessary zeros, which $K(z)$ does not have. Indeed, since $\vartheta_{p_j}(z)$ and $\vartheta_{q_i}(z)$ have the surplus zeros $o_1, o_2, \ldots, o_{g-1}$, the factor $P(z)Q(z)$ in $J(z) = P(z)Q(z)\dfrac{d\log\sigma^n\tau^m}{dz}$ causes $J(z)$ to have zeros $o_1, o_2, \ldots, o_{g-1}$. Note that $o_1, o_2, \ldots, o_{g-1}$ are not zeros of $\dfrac{d\log\sigma^n\tau^m}{dz}$, because in the Riemann factorization (23.4.3), the surplus zeros in the numerator and denominator cancel.

Next, we note that since $J(z)$ is a holomorphic section of the line bundle $L \otimes \Omega^1_C$, we have

$$\big(\text{the sum of the orders of the zeros of } J(z)\big) = g(h+k) + (2g-2),$$

where the right hand side is the degree of $L \otimes \Omega^1_C$. We claim that

$$\begin{aligned} &(\text{the sum of the orders of zeros of } J(z) \text{ which are} \\ &\ \text{distinct from } o_1, o_2, \ldots, o_{g-1}) \\ &\qquad = (h+k) + (2g-2). \end{aligned} \tag{23.4.7}$$

To see this, first note that the sum of the orders of the surplus zeros $o_1, o_2, \ldots, o_{g-1}$ in $P(z)Q(z)$ is $(g-1)(h+k)$ (whereas, as we mentioned above, $o_1, o_2, \ldots, o_{g-1}$ are not zeros of $\frac{d\log\sigma^n\tau^m}{dz}$). Subtracting $(g-1)(h+k)$ from $g(h+k)+(2g-2)$, we see that the right hand side of (23.4.7) is equal to $(h+k)+(2g-2)$, confirming (23.4.7).

Next we introduce $J(z)$, when there are indices j such that $nm_j - mn_j = 0$; say, for v indices $j = h-v+1, h-v+2, \ldots, h$, we have $nm_j - mn_j = 0$. In this case, instead of $P(z)$ and $P_j(z)$, we take $\widehat{P}(z)$ and $\widehat{P}_j(z)$ $(j = 1, 2, \ldots, h-v)$:

$$\widehat{P}(z) = \prod_{j=1}^{h-v} \vartheta_{p_j}(z), \qquad \widehat{P}_j(z) = \vartheta_{p_1}(z)\,\vartheta_{p_2}(z)\cdots\vartheta'_{p_j}(z)\cdots\vartheta_{p_{h-v}}(z),$$

and we define an *arrangement function*:

$$\begin{aligned} J(z) &:= \widehat{P}(z)Q(z)\frac{d\log(\sigma^n\tau^m)}{dz} \\ &= Q(z)\Big(\sum_{j=1}^{h-v}(nm_j - mn_j)\widehat{P}_j(z)\Big) + \widehat{P}(z)\Big(\sum_{i=1}^{k} ma_iQ_i(z)\Big). \end{aligned}$$

Then $J(z)$ is a holomorphic section of a line bundle $L \otimes \Omega^1_C$ on the complex curve C, where $L := \bigotimes_{j=1}^{h-v} L_{p_j} \otimes \bigotimes_{i=1}^{k} L_{q_i}$. (Recall that L_p is the pull-back of the line bundle $\mathcal{L}_{\mathbf{u}(p)+\mathbf{c}}$ (see (23.3.3)) by the Abel–Jacobi map $\mathbf{u} : C \to \mathrm{Jac}(C)$, and L_p has a holomorphic section ϑ_p.) By Lemma 24.1.7, p471, the degrees of both L_{p_j} and L_{q_i} are g, and so the degree of L is $g(h-v+k)$. Thus we obtain

Lemma 23.4.2 *Suppose that $nm_j - mn_j \neq 0$ for $j = 1, 2, \ldots, h-v$ (possibly, $v = 0$), and $nm_j - mn_j = 0$ for $j = h-v+1, h-v+2, \ldots, h$. Then (1) $J(z)$ is a holomorphic section of a line bundle $L \otimes \Omega^1_C$ on C, where Ω^1_C is the cotangent bundle of C and $L := \bigotimes_{j=1}^{h-v} L_{p_j} \otimes \bigotimes_{i=1}^{k} L_{q_i}$, and (2) the degree of $L \otimes \Omega^1_C$ is $g(h-v+k)+(2g-2)$.*

By a similar argument to the proof of Lemma 23.2.2, it is immediate to show

Lemma 23.4.3 *Let $K(z) = n\sigma_z\tau + m\sigma\tau_z$ be the plot function on C. Let $o_1, o_2, \ldots, o_{g-1}$ be the surplus zeros of $\vartheta_{p_j}(z)$ (and also those of $\vartheta_{q_i}(z)$). Then for $\alpha \in C$, the following equivalence holds:*

$$\begin{aligned} &K(\alpha) = 0,\ \sigma(\alpha) \neq 0,\ \text{and}\ \tau(\alpha) \neq 0 \\ &\quad\Longleftrightarrow\quad J(\alpha) = 0,\ \alpha \neq p_j\ (j \in J_0),\ \text{and}\ \alpha\ \text{is none of}\ o_1, o_2, \ldots, o_{g-1}, \end{aligned}$$

where J_0 is the set of indices j such that $nm_j - mn_j = 0$. Moreover, the order of α in $K(z)$ is equal to that of α in $J(z)$.

Next recall Lemma 21.2.1, p393: *A point $(\alpha, \beta) \in X_{s,t}$ $(s, t \neq 0)$ is a singularity if and only if α and β satisfy*

$$\text{(a)}\quad K(\alpha) = 0,\quad \sigma(\alpha) \neq 0,\quad \tau(\alpha) \neq 0 \qquad \textit{and} \qquad \text{(b)}\quad \beta^n = \frac{ln-m}{m}t\tau(\alpha).$$

In terms of the equivalence in Lemma 23.4.3, we can restate this as follows:

Lemma 23.4.4 *Denote by* J_0 *the set of indices* j *such that* $nm_j - mn_j = 0$. *Let* $o_1, o_2, \ldots, o_{g-1}$ *be the surplus zeros of* $\vartheta_{p_j}(z)$ *(and also those of* $\vartheta_{q_i}(z)$*). Then a point* $(\alpha, \beta) \in X_{s,t}$ $(s, t \neq 0)$ *is a singularity if and only if* α *and* β *satisfy*

(a) $J(\alpha) = 0$, $\alpha \neq p_j$ $(j \in J_0)$, *and* α *is none of* $o_1, o_2, \ldots, o_{g-1}$ *and*
(b) $\beta^n = \dfrac{ln - m}{m} t\tau(\alpha)$.

Now from Lemma 23.4.2 (see also the proof of (23.4.7)), we can deduce

$$\begin{aligned}&\big(\text{the sum of the orders of zeros of } J(z) \text{ which are distinct}\\ &\quad \text{from } o_1, o_2, \ldots, o_{g-1}\big)\\ &\qquad = (h - v + k) + (2g - 2).\end{aligned}$$

In particular, we have

$$\begin{aligned}&\big(\text{the sum of the orders of zeros of } J(z) \text{ which are distinct}\\ &\quad \text{from } o_1, o_2, \ldots, o_{g-1} \text{ and } p_j \ (j \in J_0)\big)\\ &\qquad = (h - v + k) + (2g - 2) - \sum_{j \in J_0} \operatorname{ord}_{p_j}\big(J(z)\big),\end{aligned} \tag{23.4.8}$$

where J_0 is the set of indices j such that $nm_j - mn_j = 0$, and $\operatorname{ord}_{p_j}\big(J(z)\big)$ stands for the order of the zero of $J(z)$ at p_j $(j \in J_0)$. Now setting $\omega := d\log\big(\sigma^n \tau^m\big)$ (a meromorphic 1-form on C), we write $J(z) = \widehat{P}(z)Q(z)\,\omega$. Since $\widehat{P}(z)Q(z)$ does not vanish at p_j $(j \in J_0)$, we have $\operatorname{ord}_{p_j}(J(z)) = \operatorname{ord}_{p_j}(\omega)$. Using this equation with the equivalence in Lemma 23.4.3, we may rewrite (23.4.8) as

$$\begin{aligned}&\big(\text{the sum of the orders of zeros } \alpha \text{ of } K(z) \text{ such that } \sigma(\alpha) \neq 0 \text{ and } \tau(\alpha) \neq 0\big)\\ &\quad = (h - v + k) + (2g - 2) - \sum_{j \in J_0} \operatorname{ord}_{p_j}(\omega).\end{aligned}$$

In particular,

$$\begin{aligned}&\big(\text{the number of zeros } \alpha \text{ of } K(z) \text{ such that } \sigma(\alpha) \neq 0 \text{ and } \tau(\alpha) \neq 0\big)\\ &\quad \leq (h - v + k) + (2g - 2) - \sum_{j \in J_0} \operatorname{ord}_{p_j}(\omega).\end{aligned}$$

Once we obtain this inequality, it is straightforward to recover the inequalities in Proposition 21.3.6, p400 and Corollary 21.4.4, p403 (concerning the number of the subordinate fibers in $\pi_t : M_t \to \Delta$ and the number of the singularities of a subordinate fiber).

24

Supplement

This chapter is a brief introduction to the Riemann theta function and related topics, which we require for our discussion.

24.1 Riemann theta function and related topics

The residue theorem is a key ingredient to deduce various formulas explained in this section. Let $f(z)$ be a meromorphic function defined on a domain D in $\mathbb{C}$. We denote the zeros of f by $p_1, p_2, \ldots, p_\lambda$, where p_i has order m_i, and denote the poles of f by $q_1, q_2, \ldots, q_\nu$, where q_j has order n_j. Given a holomorphic function $h(z)$ on the domain D, the residue theorem gives the 'weighted sum' of the values of $h(z)$ at $p_1, p_2, \ldots, p_\lambda$ and $q_1, q_2, \ldots, q_\nu$. Namely,

$$\frac{1}{2\pi \mathrm{i}} \int_\Gamma h \cdot d \log f = \frac{1}{2\pi \mathrm{i}} \int_\Gamma h \cdot \frac{f'}{f} = \sum_{i=1}^{\lambda} m_i h(p_i) - \sum_{j=1}^{\nu} m_j h(q_j),$$

where the contour $\Gamma \subset D$ surrounds $p_1, p_2, \ldots, p_\lambda$ and $q_1, q_2, \ldots, q_\nu$. Choosing various $h(z)$, we later derive many important formulas from the above formula.

Now let C be a complex curve — Riemann surface — of genus $g \geq 1$. We take $2g$ loops (simple closed "real" curves) $a_1, b_1, a_2, b_2, \ldots, a_g, b_g$ on C such that they are the standard generators of the fundamental group of C:

$$\pi_1(C) = \langle\, a_1, b_1, a_2, b_2, \ldots, a_g, b_g \,:\, [a_1, b_1]\,[a_2, b_2]\, \cdots\, [a_g, b_g] = 1 \,\rangle, \quad (24.1.1)$$

where $[a_i, b_i] := a_i b_i a_i^{-1} b_i^{-1}$. Cutting C along these loops, we obtain $4g$-gon P; the oriented edges of the boundary ∂P are "somewhat informally" denoted by

$$a_1,\, b_1,\, a_1^{-1},\, b_1^{-1},\, a_2,\, b_2,\, a_2^{-1},\, b_2^{-1},\, \ldots,\, a_g,\, b_g,\, a_g^{-1},\, b_g^{-1}.$$

See Figure 24.1.1. When $g = 1$, we often omit the subscripts to write a, b, a^{-1}, b^{-1}.

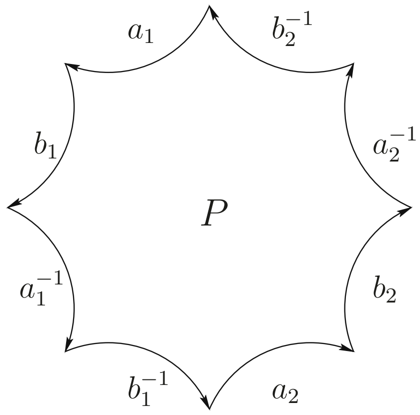


Fig. 24.1.1. The $4g$-gon P for $g=2$

Lemma 24.1.1 *Let $f(z)$ be a meromorphic function on a complex curve C, and let* $\operatorname{div}(f)=\sum_{i=1}^{\lambda} m_i p_i - \sum_{j=1}^{\nu} n_j q_j$ *be the divisor of f. Then* $\sum_{i=1}^{\lambda} m_i - \sum_{j=1}^{\nu} n_j = 0$. (Namely, the divisor defined by a meromorphic function has degree 0.)

Proof. By the residue theorem,

$$\frac{1}{2\pi \mathrm{i}} \int_{\partial P} d\log f = \sum_{i=1}^{\lambda} m_i - \sum_{j=1}^{\nu} n_j. \tag{24.1.2}$$

We shall compute the left hand side explicitly. First of all, we consider the case $g=1$; then ∂P consists of a, b, a^{-1}, b^{-1}, and

$$\begin{aligned}\int_{\partial P} d\log f &= \int_a d\log f + \int_b d\log f + \int_{a^{-1}} d\log f + \int_{b^{-1}} d\log f \\ &= \int_a \Big(d\log f(p_+) - d\log f(p_-)\Big) + \int_b \Big(d\log f(q_+) - d\log f(q_-)\Big),\end{aligned}$$

where p_+ and p_- (resp. q_+ and q_-) are respectively corresponding points on the edges a and a^{-1} (resp. b and b^{-1}). Since $d\log f(p_+) = d\log f(p_-)$ and $d\log f(q_+) = d\log f(q_-)$, we have

$$\int_{a_i} \Big(d\log f(p_+) - d\log f(p_-)\Big) = 0, \qquad \int_{b_i} \Big(d\log f(q_+) - d\log f(q_-)\Big) = 0.$$

Hence $\int_{\partial P} d\log f = 0$, and by (24.1.2), $\sum_{i=1}^{\lambda} m_i - \sum_{j=1}^{\nu} n_j = 0$. This proves the assertion for the case $g=1$. For general g, we note

$$\int_{\partial P} d\log f = \sum_{i=1}^{g} \left[\int_{a_i} d\log f + \int_{b_i} d\log f + \int_{a_i^{-1}} d\log f + \int_{b_i^{-1}} d\log f \right].$$

By the same computation as that for $g = 1$, the inside the brackets is 0, and therefore $\int_{\partial P} d\log f = 0$. With (24.1.2), we deduce $\sum_{i=1}^{\lambda} m_i - \sum_{j=1}^{\nu} n_j = 0$.□

We take a normalized basis $\omega_1, \omega_2, \ldots, \omega_g$ of the space $H^0(C, \Omega_C^1)$ of holomorphic 1-forms, so that $\int_{a_i} \omega_j = \delta_{ij}$ holds. We consider $2g$ vectors in $\mathbb{C}^g$: for $i = 1, 2, \ldots, g$,

$$\mathbf{e}_i = \begin{pmatrix} \int_{a_i} \omega_1 \\ \int_{a_i} \omega_2 \\ \vdots \\ \int_{a_i} \omega_g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \qquad \boldsymbol{\beta}_i = \begin{pmatrix} \int_{b_i} \omega_1 \\ \int_{b_i} \omega_2 \\ \vdots \\ \int_{b_i} \omega_g \end{pmatrix}, \tag{24.1.3}$$

where "1" in $\mathbf{e}_i$ is in i-th place; we denote the j-th coordinate of $\boldsymbol{\beta}_i$ by β_{ij} (that is, $\beta_{ij} = \int_{b_i} \omega_j$). We next define a lattice Λ generated by the above $2g$ vectors:

$$\Lambda = \mathbb{Z}\mathbf{e}_1 + \mathbb{Z}\mathbf{e}_2 + \cdots + \mathbb{Z}\mathbf{e}_g + \mathbb{Z}\boldsymbol{\beta}_1 + \mathbb{Z}\boldsymbol{\beta}_2 + \cdots + \mathbb{Z}\boldsymbol{\beta}_g.$$

Then a g-dimensional complex torus $\mathrm{Jac}(C) := \mathbb{C}^g/\Lambda$ (the quotient of $\mathbb{C}^g$ by Λ) is called the *Jacobian variety* of C. Now fixing a base point $z_0 \in C$, we define a holomorphic map $\mathbf{u} : C \to \mathbb{C}^g$ by

$$\mathbf{u}(z) = \begin{pmatrix} \int_{z_0}^{z} \omega_1 \\ \int_{z_0}^{z} \omega_2 \\ \vdots \\ \int_{z_0}^{z} \omega_g \end{pmatrix}. \tag{24.1.4}$$

Actually, $\mathbf{u}$ is multi-valued; the value $\mathbf{u}(z)$ depends on the choice of the path of integration from z_0 to z. However, $\mathbf{u} : C \to \mathbb{C}^g$ descends to a single-valued holomorphic map — we denote it also by $\mathbf{u}$, that is, $\mathbf{u} : C \to \mathrm{Jac}(C)$. This map is called the *Abel–Jacobi map.*

A *theta function of order* m ($m \geq 0$) is a holomorphic function θ on $\mathbb{C}^g$ satisfying

$$\theta(\mathbf{x} + \mathbf{e}_i) = \theta(\mathbf{x}), \qquad \theta(\mathbf{x} + \boldsymbol{\beta}_i) = e^{-2\pi m \mathrm{i}\,\left(x_i + \frac{1}{2}\beta_{ii}\right)}\theta(\mathbf{x}), \qquad i = 1, 2, \ldots, g,$$

where x_i and β_{ii} are respectively the i-th coordinate of $\mathbf{x}$ and $\boldsymbol{\beta}_i$ (so $\beta_{ii} = \int_{b_i} \omega_i$). Among theta functions, the most important one is the *Riemann theta* *function*:

$$\vartheta(\mathbf{x}) = \sum_{\mathbf{n}\in\mathbb{Z}^g} \exp\left[2\pi\mathrm{i}\left(\frac{1}{2}\langle\mathbf{n}, B\mathbf{n}\rangle + \langle\mathbf{n},\mathbf{x}\rangle\right)\right],$$

where $\mathbf{n}$ and $\mathbf{x}$ are column vectors, $\langle\mathbf{x},\mathbf{y}\rangle = x_1\overline{y}_1 + x_2\overline{y}_2 + \cdots + x_g\overline{y}_g$, and $B = (\int_{b_i}\omega_j)$ is a $g\times g$ matrix. The Riemann theta function possesses quasi doubly-periodicity:

$$\vartheta(\mathbf{x}+\mathbf{e}_i) = \vartheta(\mathbf{x}), \qquad \vartheta(\mathbf{x}+\boldsymbol{\beta}_i) = e^{-2\pi\mathrm{i}(x_i+\frac{1}{2}\beta_{ii})}\cdot\vartheta(\mathbf{x}). \tag{24.1.5}$$

Now we consider a Λ-action on $\mathbb{C}^g\times\mathbb{C}$ generated by

$$(\mathbf{x},\xi)\longmapsto(\mathbf{x}+\mathbf{e}_i\,,\,\xi), \qquad (\mathbf{x},\xi)\longmapsto(\mathbf{x}+\boldsymbol{\beta}_i\,,\,e^{-2\pi\mathrm{i}(x_i+\frac{1}{2}\beta_{ii})}\cdot\xi)$$

for $i = 1,2,\ldots,g$ (recall: x_i and β_{ii} are respectively the i-th coordinate of $\mathbf{x}$ and $\boldsymbol{\beta}_i$ (so $\beta_{ii} = \int_{b_i}\omega_i$)). Then the quotient $\mathcal{L} := (\mathbb{C}^g\times\mathbb{C})/\Lambda$ is a line bundle on the Jacobian variety $\mathrm{Jac}(C)$, and the theta function ϑ determines a holomorphic section of $\mathcal{L}$. Henceforth we also denote by ϑ the holomorphic section of $\mathcal{L}$ determined by ϑ.

We remark that most formulas given below are deduced as follows: For each formula to be shown, we choose a suitable contour integral and then compute it in two different ways. The first method applies the residue theorem and the second computes it using explicit parameterizations, using the fact that the contour is a $4g$-gon — the cancellation along each set of four edges $a_i, b_i, a_i^{-1}, b_i^{-1}$ is crucial.

Lemma 24.1.2 (Riemann bilinear relation) *Suppose that ω is a holomorphic 1-form and θ is a meromorphic 1-form on a complex curve C of genus $g\geq 1$. Let P be the $4g$-gon obtained by cutting C along the loops $a_1, b_1, a_2, b_2, \ldots, a_g, b_g$, and define a holomorphic function f on P by $f(z) = \int_{z_0}^{z}\omega$ where $z_0\in C$ is a base point. Then the following equation holds:*

$$\int_{\partial P} f\cdot\theta = \sum_{i=1}^{g}\left(\int_{a_i}\omega\int_{b_i}\theta - \int_{b_i}\omega\int_{a_i}\theta\right).$$

Proof. We first give the proof for the case $g = 1$; in this case $\partial P = aba^{-1}b^{-1}$, and

$$\begin{aligned}\int_{\partial P} f\cdot\theta &= \int_a f\cdot\theta + \int_b f\cdot\theta + \int_{a^{-1}} f\cdot\theta + \int_{b^{-1}} f\cdot\theta \\ &= \left(\int_a f\cdot\theta + \int_{a^{-1}} f\cdot\theta\right) + \left(\int_b f\cdot\theta + \int_{b^{-1}} f\cdot\theta\right) \end{aligned} \tag{24.1.6}$$

We shall rewrite the integrals in the last expression. We denote corresponding points on the edges a and a^{-1} by p_+ and p_-. Then

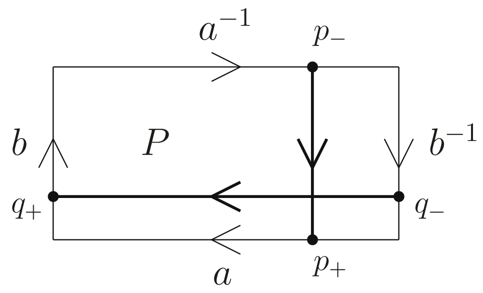


Fig. 24.1.2.

$$\int_a f\cdot\theta + \int_{a^{-1}} f\cdot\theta = \int_a \Big(f(p_+) - f(p_-)\Big)\cdot\theta = \int_a \left(\int_{b^{-1}}\omega\right)\theta = \int_{b^{-1}}\omega\int_a\theta$$
$$= -\int_b\omega\int_a\theta,$$

where the second equality follows from the fact that the path from p_- to p_+ is isotopic to the edge b^{-1} (see Figure 24.1.2). Likewise, we denote corresponding points on the edges b and b^{-1} by q_+ and q_-. Then

$$\int_b f\cdot\theta + \int_{b^{-1}} f\cdot\theta = \int_b \Big(f(q_+) - f(q_-)\Big)\theta = \int_b \left(\int_a\omega\right)\theta$$
$$= \int_a\omega\int_b\theta,$$

where the second equality follows from the fact that the path from q_- to q_+ is isotopic to the edge a (see Figure 24.1.2). With (24.1.6), we obtain the bilinear relation:

$$\int_{\partial P} f\cdot\theta = -\int_a\theta\int_b\omega + \int_a\omega\int_b\theta.$$

For general g, the boundary ∂P is $a_1b_1a_1^{-1}b_1^{-1}a_2b_2a_2^{-1}b_2^{-1}\cdots a_gb_ga_g^{-1}b_g^{-1}$, and so

$$\int_{\partial P} f\cdot\theta = \sum_{i=1}^{g}\left(\int_{a_i} f\cdot\theta + \int_{b_i} f\cdot\theta + \int_{a_i^{-1}} f\cdot\theta + \int_{b_i^{-1}} f\cdot\theta\right). \tag{24.1.7}$$

For each $i = 1, 2, \ldots, g$, by the same computation as for the case $g = 1$, we deduce

$$\int_{a_i} f\cdot\theta + \int_{b_i} f\cdot\theta + \int_{a_i^{-1}} f\cdot\theta + \int_{b_i^{-1}} f\cdot\theta = \int_{a_i}\omega\int_{b_i}\theta - \int_{b_i}\omega\int_{a_i}\theta.$$

This with (24.1.7) gives the bilinear relation. □

Remark 24.1.3 In the above proof, the computation of the integral $\int_{\partial P} f \cdot \theta$ essentially used the fact that the $4g$-gon P obtained by cutting the complex curve C has a boundary consisting of g quadruples of edges: $a_i, b_i, a_i^{-1}, b_i^{-1}$ (for $i = 1, 2, \ldots, g$). In contrast, the application of the residue theorem does not require a domain to have any particular boundary.

Next we derive the "reciprocity law", which will be effectively used for our later computations.

Lemma 24.1.4 (Reciprocity law) *Suppose that $\omega_1, \omega_2, \ldots, \omega_g$ is a normalized basis of $H^0(C, \Omega_C^1)$, so that $\int_{a_i} \omega_j = 0$ for $i \neq j$ and 1 for $i = j$. Let $\omega_{p,q}$ be an abelian differential of the third kind*[1] *such that*

(i) *$\omega_{p,q}$ has simple poles at p and q with the residue $+1$ at p and the residue -1 at q, and $\omega_{p,q}$ is holomorphic outside p and q, and*
(ii) *$\int_{a_j} \omega_{p,q} = 0$ for $j = 1, 2, \ldots, g$.*

Then $\int_{b_j} \omega_{p,q} = 2\pi \mathrm{i} \int_q^p \omega_j$ holds.

Proof. We take loops $a_1, b_1, a_2, b_2, \ldots, a_g, b_g$ on C that are the standard generators of the fundamental group of C (see (24.1.1)) — we choose these loops such that p and q do not lie on them. We denote by P the $4g$-gon obtained by cutting C along these loops. Fix a base point $z_0 \in C$, and for each ω_j ($j = 1, 2, \ldots, g$), we define a holomorphic function $h_j(z) := \int_{z_0}^z \omega_j$ on P. Since the residue of $h_j \, \omega_{p,q}$ at p (resp. q) is $h_j(p)$ (resp. $h_j(q)$), the residue theorem yields

$$\frac{1}{2\pi \mathrm{i}} \int_{\partial P} h_j \, \omega_{p,q} = h_j(p) - h_j(q) = \int_q^p \omega_j. \tag{24.1.8}$$

On the other hand, by Riemann's bilinear relation,

$$\int_{\partial P} h_j \, \omega_{p,q} = \sum_{i=1}^{g} \left(\int_{a_i} \omega_j \int_{b_i} \omega_{p,q} - \int_{b_i} \omega_j \int_{a_i} \omega_{p,q} \right).$$

Since $\int_{a_i} \omega_j = \delta_{ij}$ and $\int_{a_i} \omega_{p,q} = 0$, it follows that $\int_{\partial P} h_j \, \omega_{p,q} = \int_{b_j} \omega_{p,q}$, where by (24.1.8), the left hand side equals $2\pi \mathrm{i} \int_q^p \omega_j$. This shows the reciprocity law: $2\pi \mathrm{i} \int_q^p \omega_j = \int_{b_j} \omega_{p,q}$. □

Next for a meromorphic function f on a complex curve C, we express its divisor as follows:

$$\operatorname{div}(f) := \sum_{i=1}^{n} p_i - \sum_{i=1}^{n} q_i,$$

where $p_1, p_2, \ldots, p_n$ (resp. $q_1, q_2, \ldots, q_n$) need *not* to be distinct. We claim that

$$\sum_{i=1}^{n} \mathbf{u}(p_i) - \sum_{i=1}^{n} \mathbf{u}(q_i) \equiv 0 \mod \Lambda, \tag{24.1.9}$$

[1] A meromorphic 1-form whose poles are simple (i.e. of order 1)

where $\mathbf{u} : C \to \mathbb{C}^g$ is the 'Abel–Jacobi map' — instead of the usual Abel–Jacobi map $\mathbf{u} : C \to \mathrm{Jac}(C)\,(= \mathbb{C}^g/\Lambda)$, we here intend its lift, that is, a (multi-valued) holomorphic map $C \to \mathbb{C}^g$; see (24.1.4).

To avoid complicated notation, we show (24.1.9) only for $g = 2$. In this case, the Abel–Jacobi map is

$$\mathbf{u}(z) = \begin{pmatrix} u_1 \\ \\ u_2 \end{pmatrix},$$

where $u_1 = \int_{z_0}^{z} \omega_1$ and $u_2 = \int_{z_0}^{z} \omega_2$. We shall compute an integral $\int_{\partial P} \mathbf{u} \cdot d\log(f)$ in two different ways to verify (24.1.9). First, by the residue theorem,

$$\frac{1}{2\pi \mathrm{i}} \int_{\partial P} \mathbf{u} \cdot d\log(f) = \sum_{i=1}^{n} \mathbf{u}(p_i) - \sum_{i=1}^{n} \mathbf{u}(q_i). \tag{24.1.10}$$

Secondly, by Riemann's bilinear relation (Lemma 24.1.2), we have for each $j = 1, 2$,

$$\int_{\partial P} u_j \cdot d\log(f) = \left[\int_{a_1} \omega_j \int_{b_1} d\log(f) - \int_{b_1} \omega_j \int_{a_1} d\log(f) \right] + \left[\int_{a_2} \omega_j \int_{b_2} d\log(f) - \int_{b_2} \omega_j \int_{a_2} d\log(f) \right].$$

Since a contour integral of the logarithmic function takes a value in $2\pi\mathrm{i}\mathbb{Z}$, we may write

$$\int_{a_i} d\log(f) = 2\pi \mathrm{i} m_i, \qquad \int_{b_i} d\log(f) = 2\pi \mathrm{i} m_i',$$

where $m_i, m_i' \in \mathbb{Z}$. Recall that ω_i is normalized: $\int_{a_i} \omega_j = \delta_{ij}$. Therefore setting $\beta_{ij} := \int_{b_i} \omega_j$, we have

$$\begin{aligned}\frac{1}{2\pi \mathrm{i}} \int_{\partial P} \mathbf{u} \cdot d\log(f) &= \left[m_1' \begin{pmatrix} 1 \\ \\ 0 \end{pmatrix} - m_2' \begin{pmatrix} 0 \\ \\ 1 \end{pmatrix} \right] + \left[m_1 \begin{pmatrix} \beta_{11} \\ \\ \beta_{12} \end{pmatrix} - m_2 \begin{pmatrix} \beta_{21} \\ \\ \beta_{22} \end{pmatrix} \right] \\ &= m_1' \mathbf{e}_1 - m_2' \mathbf{e}_2 + m_1 \boldsymbol{\beta}_1 - m_2 \boldsymbol{\beta}_2.\end{aligned}$$

The comparison with (24.1.10) yields

$$\sum_{i=1}^{n} \mathbf{u}(p_i) - \sum_{i=1}^{n} \mathbf{u}(q_i) = m_1' \mathbf{e}_1 - m_2' \mathbf{e}_2 + m_1 \boldsymbol{\beta}_1 - m_2 \boldsymbol{\beta}_2,$$

and so $\sum_{i=1}^{n} \mathbf{u}(p_i) - \sum_{i=1}^{n} \mathbf{u}(q_i) \in \Lambda$. This verified the "only if" part of Abel's Theorem:

Theorem 24.1.5 (Abel's Theorem) *Let D be a divisor of degree 0 on a complex curve C of genus $g \geq 1$, and write $D = \sum_{i=1}^{n} p_i - \sum_{i=1}^{n} q_i$ where*

$p_1, p_2, \dots, p_n$ (resp. $q_1, q_2, \dots, q_n$) need "not" to be distinct. Then there exists a meromorphic function f on C satisfying $D = \mathrm{div}(f)$ if and only if $\sum_{i=1}^n \mathbf{u}(p_i) - \sum_{i=1}^n \mathbf{u}(p_i) \equiv 0 \bmod \Lambda$.

(Note: Abel's Theorem implies that (i) the Abel–Jacobi map $\mathbf{u} : C \to \mathrm{Jac}(C)$ is an embedding and (ii) when we extend $\mathbf{u}$ to a homomorphism $\mathbf{u} : \mathrm{Div}(C) \to \mathrm{Jac}(C)$ where $\mathrm{Div}(C)$ denotes the group of the divisors on C, the kernel of $\mathbf{u}$ consists of the divisors of the meromorphic functions on C.)

Proof. We already proved the "only if" part. We now show the "if" part. At first, we assume that $p_1, p_2, \dots, p_n$ (resp. $q_1, q_2, \dots, q_n$) are distinct. Let $\omega_1, \omega_2, \dots, \omega_g$ be a normalized basis of $H^0(C, \Omega_C^1)$, so that $\int_{a_i} \omega_j = \delta_{ij}$. For a pair of points $p, q \in C$, we denote by $\omega_{p,q}$ the meromorphic 1-form on C such that

(1) $\omega_{p,q}$ has simple poles at p and q with the residue $+1$ at p and the residue -1 at q, and $\omega_{p,q}$ is holomorphic outside p and q, and
(2) $\int_{a_i} \omega_{p,q} = 0$ for $i = 1, 2, \dots, g$.

Fixing a base point $z_0 \in C$, we define a meromorphic 1-form θ on C by

$$\theta := c_1\,\omega_1 + c_2\,\omega_2 + \cdots + c_g\,\omega_g + \sum_{i=1}^n \omega_{p_i,\,z_0} - \sum_{i=1}^n \omega_{q_i,\,z_0},$$

where the complex numbers c_i will be determined below. We note that θ has a simple pole at p_i (resp. q_i) with the residue $+1$ (resp. -1); whereas θ has no pole at z_0, because a pole z_0 of $\omega_{p_i,\,z_0}$ (with the residue -1) and a pole z_0 of $-\omega_{q_i,\,z_0}$ (with the residue $+1$) cancel.

Now we shall determine $c_i \in \mathbb{C}$ in such a way that $\int_{b_j} \theta \in 2\pi\mathrm{i}\mathbb{Z}$. First, since $\int_{a_i} \omega_{p,q} = 0$, we have

$$\int_{a_i} \theta = c_i. \tag{24.1.11}$$

On the other hand, since $\int_{b_j} \omega_i = \beta_{ji}$ (by definition; see (24.1.3)), we have

$$\int_{b_j} \theta = c_1\beta_{j1} + c_2\beta_{j2} + \cdots + c_g\beta_{jg} + \sum_{i=1}^n \int_{b_j} \omega_{p_i,\,z_0} - \sum_{i=1}^n \int_{b_j} \omega_{q_i,\,z_0}. \tag{24.1.12}$$

By the reciprocity law (Lemma 24.1.4),

$$\int_{b_j} \omega_{p_i,z_0} = 2\pi\mathrm{i} \int_{z_0}^{p_i} \omega_j, \qquad \int_{b_j} \omega_{q_i,z_0} = 2\pi\mathrm{i} \int_{z_0}^{q_i} \omega_j,$$

and so (24.1.12) is written as

$$\int_{b_j} \theta = c_1\beta_{j1} + c_2\beta_{j2} + \cdots + c_g\beta_{jg} + 2\pi\mathrm{i}\Big(\sum_{i=1}^n u_j(p_i) - \sum_{i=1}^n u_j(q_i) \Big).$$

By assumption, $\sum_{i=1}^n \mathbf{u}(p_i) - \sum_{i=1}^n \mathbf{u}(q_i) \in \Lambda$, and therefore we may choose such $c_1, c_2, \ldots, c_g \in 2\pi\mathrm{i}\mathbb{Z}$ as $\int_{b_j} \theta \in 2\pi\mathrm{i}\mathbb{Z}$ holds; by (24.1.11), $\int_{a_i} \theta \in 2\pi\mathrm{i}\mathbb{Z}$ also holds.

Next we define a meromorphic function $g(z) := \exp\left(-\int_{z_0}^{z} \theta\right)$ on the $4g$-gon P, where P is obtained by cutting C along the loops[2] $a_1, b_1, a_2, b_2, \ldots, a_g, b_g$ (we choose these loops such that $p_1, p_2, \ldots, p_n$ and $q_1, q_2, \ldots, q_n$ do not lie on them). Since $\int_{a_i} \theta \in 2\pi\mathrm{i}\mathbb{Z}$ and $\int_{b_j} \theta \in 2\pi\mathrm{i}\mathbb{Z}$ for $i, j = 1, 2, \ldots, g$, the function $g(z)$ determines a meromorphic function $f(z)$ on C. We claim that $\mathrm{div}(f) = \sum_{i=1}^n p_i - \sum_{i=1}^n q_i$, that is, f has a simple zero at p_i and a simple pole at q_i. In fact, since $\theta = \frac{1}{z}dz$ around p_i, we have $f(z) = \exp(-\log(1/z)) = z$ around p_i; while since $\theta = -\frac{1}{z}dz$ around q_i, we have $f(z) = \exp(\log(1/z)) = 1/z$ around q_i. For the case where $p_1, p_2, \ldots, p_n$ (resp. $p_1, p_2, \ldots, p_n$) are not distinct, we also have $\mathrm{div}(f) = \sum_{i=1}^n p_i - \sum_{i=1}^n q_i$. For instance, if $p_1 = p_2 = p_3$, then we have $f(z) = \exp(-3\log(1/z)) = z^3$ around p_1, so that f has a zero of order 3 at p_1. Also for instance, if $q_1 = q_2 = q_3 = q_4$, then $f(z) = \exp(4\log(1/z)) = 1/z^4$ around q_1, so that f has a pole of order 4 at q_1. Thus f is our desired function. This completes the proof of Abel's Theorem. □

Next we recall the Λ-action on $\mathbb{C}^g \times \mathbb{C}$, generated by

$$(\mathbf{x}, \xi) \longmapsto (\mathbf{x} + \mathbf{e}_i\,,\, \xi), \qquad (\mathbf{x}, \xi) \longmapsto (\mathbf{x} + \boldsymbol{\beta}_i\,,\, e^{-2\pi\mathrm{i}(x_i + \frac{1}{2}\beta_{ii})} \cdot \xi)$$

for $i = 1, 2, \ldots, g$, where x_i and β_{ii} are respectively the i-th coordinate of $\mathbf{x}$ and $\boldsymbol{\beta}_i$ (so $\beta_{ii} = \int_{b_i} \omega_i$); see (24.1.3) for $\mathbf{e}_i$ and $\boldsymbol{\beta}_i$. Then the quotient

$$\mathcal{L} := (\mathbb{C}^g \times \mathbb{C})/\Lambda \tag{24.1.13}$$

is a line bundle on the Jacobian variety $\mathrm{Jac}(C)$, and the Riemann theta function ϑ determines a holomorphic section of $\mathcal{L}$, which we also denote by ϑ. Taking $\mathbf{d} \in \mathrm{Jac}(C)$, we define a translation in $\mathrm{Jac}(C)$ by $\mathbf{x} \mapsto \mathbf{x} + \mathbf{d}$. Then we consider the following line bundle on $\mathrm{Jac}(C)$:

$$\mathcal{L}_{\mathbf{d}}: \quad \text{the pull-back of } \mathcal{L} \text{ under the translation } \mathbf{x} \mapsto \mathbf{x} + \mathbf{d}. \tag{24.1.14}$$

Note that $\vartheta\big(\mathbf{x} - \mathbf{d}\big)$ is a holomorphic section of $\mathcal{L}_{\mathbf{d}}$.

Now we take loops $a_1, b_1, a_2, b_2, \ldots, a_g, b_g$ on C which are the standard generators of the fundamental group of C:

$$\pi_1(C) = \langle\, a_1, b_1, a_2, b_2, \ldots, a_g, b_g \;:\; [a_1, b_1]\,[a_2, b_2] \,\cdots\, [a_g, b_g] = 1 \,\rangle. \tag{24.1.15}$$

Cutting C along these loops, we obtain a $4g$-gon P; the oriented edges of the boundary ∂P are "somewhat informally" denoted by

$$a_1,\, b_1,\, a_1^{-1},\, b_1^{-1},\, a_2,\, b_2,\, a_2^{-1},\, b_2^{-1},\, \ldots,\, a_g,\, b_g,\, a_g^{-1},\, b_g^{-1}.$$

When $g = 1$, we often omit the subscript: a, b, a^{-1}, b^{-1}.

[2] The standard generators of $\pi_1(C)$; see (24.1.1).

The following lemma is useful for our later discussion. (For (2) below, we choose the loops $a_1, b_1, a_2, b_2, \dots, a_g, b_g$ such that none of the zeros of $h(z)$ lies on them.)

Lemma 24.1.6 *Denote by $p_{i,+}$ and $p_{i,-}$ (resp. $q_{i,+}$ and $q_{i,-}$) corresponding points on the edges a_i and a_i^{-1} (resp. b_i and b_i^{-1}) of the $4g$-gon P. Then*

(1) *The Abel–Jacobi map* $\mathbf{u}$ *has the following transformation rule:*

$$\mathbf{u}(p_{i,+}) = \mathbf{u}(p_{i,-}) - \boldsymbol{\beta}_i, \qquad \mathbf{u}(q_{i,+}) = \mathbf{u}(q_{i,-}) + \mathbf{e}_i,$$

where $\boldsymbol{\beta}_i, \mathbf{e}_i \in \mathbb{C}^g$ are in (24.1.3).

(2) *Fixing a point* $\mathbf{d}$ *of the Jacobian variety* $\mathrm{Jac}(C)$, *define a holomorphic section $h(z) := \vartheta\big(\mathbf{u}(z) - \mathbf{d}\big)$ of a line bundle*[3] $\mathbf{u}^*\mathcal{L}_{\mathbf{d}}$ *on C. Then*

$$h(p_{i,+}) = \exp\Big[2\pi\mathrm{i}\big(u_i(p_{i,-}) - d_i\big) + \pi\mathrm{i}\beta_{ii}\Big] \cdot h(p_{i,-}), \qquad h(q_{i,+}) = h(q_{i,-}),$$

where u_i (resp. d_i and β_{ii}) is the i-th coordinate of $\mathbf{u}$ *(resp.* $\mathbf{d}$ *and* $\boldsymbol{\beta}_i$*).*

Proof. (1): We show this only for $g = 1$, as the proof is essentially the same for $g \geq 2$. For brevity we omit the subscript $i\,(= 1)$. We note that

$$\mathbf{u}(p_+) - \mathbf{u}(p_-) = \int_{z_0}^{p_+} \omega - \int_{z_0}^{p_-} \omega = \int_{p_-}^{p_+} \omega = \int_{b^{-1}} \omega = -\int_b \omega,$$

where the third equality follows from the fact that the path from p_- to p_+ is isotopic to the edge b^{-1} (see Figure 24.1.2, p465). By definition $\int_b \omega = \boldsymbol{\beta}$, and so we obtain $\mathbf{u}(p_+) - \mathbf{u}(p_-) = -\boldsymbol{\beta}$. Similarly,

$$\mathbf{u}(q_+) - \mathbf{u}(q_-) = \int_{z_0}^{q_+} \omega - \int_{z_0}^{q_-} \omega = \int_{q_-}^{q_+} \omega = \int_a \omega,$$

where the last equality follows from the fact that the path from q_- to q_+ is isotopic to the edge a (see Figure 24.1.2). By definition $\int_a \omega = \mathbf{e}\,(= 1)$, and so we obtain $\mathbf{u}(q_+) - \mathbf{u}(q_-) = \mathbf{e}$. This proves (1). The assertion (2) follows immediately from (1) with the transformation rule of the Riemann theta function:

$$\vartheta(\mathbf{x} + \mathbf{e}_i) = \vartheta(\mathbf{x}), \qquad \vartheta(\mathbf{x} + \boldsymbol{\beta}_i) = \exp\big(-2\pi\mathrm{i}x_i - \pi\mathrm{i}\beta_{ii}\big) \cdot \vartheta(\mathbf{x}).$$

□

We next study the zeros of the holomorphic section $h(z) := \vartheta\big(\mathbf{u}(z) - \mathbf{d}\big)$ in the above lemma.

[3] The pull-back of the line bundle $\mathcal{L}_{\mathbf{d}}$ (24.1.14) by the Abel–Jacobi map $\mathbf{u} : C \to \mathrm{Jac}(C)$

Lemma 24.1.7 *Let C be a complex curve of genus $g \geq 1$. Fixing a point $\mathbf{d}$ of the Jacobian variety* $\mathrm{Jac}(C)$*, define a holomorphic section* $h(z) := \vartheta\big(\mathbf{u}(z) - \mathbf{d}\big)$ *of the line bundle $\mathbf{u}^*\mathcal{L}_{\mathbf{d}}$ on C. Then the sum of the orders of the zeros of $h(z)$ is g (so the line bundle $\mathbf{u}^*\mathcal{L}_{\mathbf{d}}$ has degree g).*

Remark 24.1.8 (1) Precisely speaking, we must exclude the case where $h(z)$ is identically zero; this is exactly when $\mathbf{u}(z) - \mathbf{d} \in \mathrm{Zero}(\vartheta)$ for all $z \in C$, that is,

$$\mathbf{d} \in -\mathrm{Zero}(\vartheta) - \mathbf{u}(C),$$

where $\mathrm{Zero}(\vartheta) := \{\, \mathbf{x} \in \mathrm{Jac}(C) \,:\, \vartheta(\mathbf{x}) = 0 \,\}$ and "$-\mathrm{Zero}(\vartheta) - \mathbf{u}(C)$" stands for the translation of the set $-\mathrm{Zero}(\vartheta)$ by $-\mathbf{u}(C)$ in $\mathrm{Jac}(C)$. (2) For generic $\mathbf{d} \in \mathrm{Jac}(C)$, $h(z)$ has g distinct zeros. See Remark 23.3.2, p452.

Proof. Let P be the $4g$-gon obtained by cutting C along the loops $a_1, b_1, a_2, b_2, \dots, a_g, b_g$ (the standard generators of $\pi_1(C)$; see (24.1.15)), where we choose these loops such that none of the zeros of $h(z)$ lies on them. By the residue theorem, $\frac{1}{2\pi \mathrm{i}} \int_{\partial P} d\log h$ equals the sum of the orders of the zeros of $h(z)$. Thus it is enough show

$$\frac{1}{2\pi \mathrm{i}} \int_{\partial P} d\log h = g.$$

We explicitly compute the contour integral on the left hand side, first for $g = 1$. As $\partial P = aba^{-1}b^{-1}$, we have

$$\begin{aligned}
\int_{\partial P} d\log h &= \int_a d\log h + \int_b d\log h + \int_{a^{-1}} d\log h + \int_{b^{-1}} d\log h \\
&= \left(\int_a d\log h + \int_{a^{-1}} d\log h\right) + \left(\int_b d\log h + \int_{b^{-1}} d\log h\right) \\
&= \int_a \Big(d\log h(p_+) - d\log h(p_-)\Big) + \int_b \Big(d\log h(q_+) - d\log h(q_-)\Big),
\end{aligned}$$

where p_+ and p_- (resp. q_+ and q_-) denote corresponding points on the edges a and a^{-1} (resp. b and b^{-1}). Thus

$$\int_{\partial P} d\log h = \int_a d\log\left(\frac{h(p_+)}{h(p_-)}\right) + \int_b d\log\left(\frac{h(q_+)}{h(q_-)}\right). \tag{24.1.16}$$

We shall compute the integrals on the right hand side. By Lemma 24.1.6,

$$h(p_+) = e^{2\pi \mathrm{i}\big(\mathbf{u}(p_-) - \mathbf{d}\big) + \pi \mathrm{i}\beta} h(p_-),$$

where we set $\beta := \int_b \omega$. (Note: In the present case $g = 1$, $\mathbf{u}(p_-)$ and $\mathbf{d}$ are actually scalars, and this expression coincides with that in Lemma 24.1.6.)

Therefore $\log\left(\dfrac{h(p_+)}{h(p_-)}\right) = 2\pi\mathrm{i}\bigl(\mathbf{u}(p_-)-\mathbf{d}\bigr)+\pi\mathrm{i}\beta$ and hence

$$d\log\left(\frac{h(p_+)}{h(p_-)}\right) = 2\pi\mathrm{i}\omega. \tag{24.1.17}$$

On the other hand, by Lemma 24.1.6, we have $h(q_+) = h(q_-)$ and so

$$d\log\left(\frac{h(q_+)}{h(q_-)}\right) = 0. \tag{24.1.18}$$

Substituting (24.1.17) and (24.1.18) into (24.1.16), we obtain

$$\frac{1}{2\pi\mathrm{i}}\int_{\partial P} d\log h = \frac{1}{2\pi\mathrm{i}}\left[\int_a 2\pi\mathrm{i}\omega + \int_b 0\right] = 1.$$

Therefore by the residue theorem, the assertion for $g = 1$ is confirmed. Next we consider the case $g \geq 2$. In this case,

$$\begin{aligned}&\frac{1}{2\pi\mathrm{i}}\int_{\partial P} d\log h\\ &\quad= \frac{1}{2\pi\mathrm{i}}\sum_{i=1}^{g}\left[\int_{a_i} d\log h + \int_{b_i} d\log h + \int_{a_i^{-1}} d\log h + \int_{b_i^{-1}} d\log h\right].\end{aligned}$$

By the same computation as that for $g = 1$, the inside the brackets equals $2\pi\mathrm{i}$, and so

$$\frac{1}{2\pi\mathrm{i}}\int_{\partial P} d\log h = g.$$

Thus by the residue theorem, the sum of the orders of the zeros of $h(z)$ is g. □

We further investigate the relationship between the zeros of the holomorphic section $h(z) := \vartheta\bigl(\mathbf{u}(z)-\mathbf{d}\bigr)$ in the above lemma. For this purpose, we prepare several lemmas. In what follows, $\mathbf{u}^*\mathcal{L}_{\mathbf{d}}$ denotes the pull-back of the line bundle $\mathcal{L}_{\mathbf{d}}$ (24.1.14) via the Abel–Jacobi map $\mathbf{u} : C \to \mathrm{Jac}(C)$.

Lemma 24.1.9 *For the holomorphic section* $h(z) := \vartheta\bigl(\mathbf{u}(z)-\mathbf{d}\bigr)$ *of the line bundle* $\mathbf{u}^*\mathcal{L}_{\mathbf{d}}$ *on the complex curve* C*, the following integral* $\mathbf{A}_i$ $(i = 1, 2, \ldots, g)$ *is independent of* $\mathbf{d} \in \mathrm{Jac}(C)$*:*

$$\mathbf{A}_i := \int_{a_i} \mathbf{u}\cdot d\log h + \int_{a_i^{-1}} \mathbf{u}\cdot d\log h. \tag{24.1.19}$$

(Caution: Here, a_i and a_i^{-1} are "not" loops on C but edges of ∂P.)

Proof. We note that $\mathbf{A}_i$ is explicitly given by

$$\int_{a_i}\Bigl[\mathbf{u}(p_{i,+})\cdot d\log h(p_{i,+}) - \mathbf{u}(p_{i,-})\cdot d\log h(p_{i,-})\Bigr],$$

where $p_{i,+}$ and $p_{i,-}$ are corresponding points on the edges a_i and a_i^{-1}.

Now we show the assertion. We give the proof only for $g = 1$, as the proof is essentially the same for $g \geq 2$. For brevity we omit the subscript $i\,(= 1)$. By Lemma 24.1.6, we have $h(p_+) = e^{2\pi\mathrm{i}\left(\mathbf{u}(p_-)-\mathbf{d}\right)+\pi\mathrm{i}\beta}h(p_-)$ where $\beta := \int_b \omega$ (note that as $g = 1$, $\mathbf{u}(p_-)$ and $\mathbf{d}$ are actually scalars), and so

$$\log h(p_+) = \log h(p_-) + 2\pi\mathrm{i}\big(\mathbf{u}(p_-) - \mathbf{d}\big) + \pi\mathrm{i}\beta.$$

Hence (A): $d\log h(p_+) = d\log h(p_-) + 2\pi\mathrm{i}\omega(p_-)$. By Lemma 24.1.6, we also have (B): $\mathbf{u}(p_+) = \mathbf{u}(p_-) - \beta$. Using (A) and (B), we rewrite

$$\begin{aligned}
\mathbf{A} &= \int_a \Big[\mathbf{u}(p_+)\cdot d\log h(p_+) - \mathbf{u}(p_-)\cdot d\log h(p_-)\Big] \\
&= \int_a \Big[\Big(\mathbf{u}(p_-) - \beta\Big)\Big(d\log h(p_-) + 2\pi\mathrm{i}\,\omega(p_-)\Big) - \mathbf{u}(p_-)\cdot d\log h(p_-)\Big] \\
&= \int_a \Big[-\beta\cdot d\log h(p_-) + 2\pi\mathrm{i}\,\mathbf{u}(p_-)\,\omega(p_-) - 2\pi\mathrm{i}\,\beta\,\omega(p_-)\Big] \\
&= -\int_a \beta\cdot d\log h(p_-) + \int_a \Big[2\pi\mathrm{i}\,\mathbf{u}(p_-)\,\omega(p_-) - 2\pi\mathrm{i}\,\beta\,\omega(p_-)\Big]. \qquad (24.1.20)
\end{aligned}$$

In (24.1.20), the integral $\int_a \Big[2\pi\mathrm{i}\,\mathbf{u}(p_-)\,\omega(p_-) - 2\pi\mathrm{i}\,\beta\,\omega(p_-)\Big]$ is a constant independent of $\mathbf{d}$, because $2\pi\mathrm{i}\,\mathbf{u}(p_-)\,\omega(p_-) - 2\pi\mathrm{i}\,\beta\,\omega(p_-)$ does not contain $\mathbf{d}$. The integral $\int_a \beta\cdot d\log h(p_-)$ in (24.1.20) is also independent of $\mathbf{d}$. To see this, first note that

$$\int_a \beta\cdot d\log h(p_-) = \beta\log\frac{h(r)}{h(r')},$$

where r and r' are the ordered end points of the edge a. Here by Lemma 24.1.6, $h(r) = h(r')$ and so we have $\beta\log\dfrac{h(r)}{h(r')} \in 2\pi\mathrm{i}\beta\,\mathbb{Z}$. Therefore

$$\int_a \beta\cdot d\log h(p_-) \in 2\pi\mathrm{i}\beta\,\mathbb{Z}.$$

We note that (i) the integral $\int_a \beta\cdot d\log h(p_-)$ is continuous with respect to $\mathbf{d}$ and (ii) as we saw above, the value $\int_a \beta\cdot d\log h(p_-)$ lies in the discrete set $2\pi\mathrm{i}\beta\mathbb{Z}$. Hence the integral $\int_a \beta\cdot d\log h(p_-)$ must be a constant independent of $\mathbf{d}$. We thus conclude that the integral $\mathbf{A}$ is a constant independent of $\mathbf{d}$. □

Next we show

Lemma 24.1.10 *For the holomorphic section $h(z) := \vartheta\big(\mathbf{u}(z) - \mathbf{d}\big)$ of the line bundle $\mathbf{u}^*\mathcal{L}_\mathbf{d}$ on the complex curve C, the following equation holds:*

$$\int_{b_i} \mathbf{u}\cdot d\log h + \int_{b_i^{-1}} \mathbf{u}\cdot d\log h = 2\pi\mathrm{i}\,d_i\mathbf{e}_i + \mathbf{B}_i, \qquad (i = 1, 2, \ldots, g), \tag{24.1.21}$$

where

> d_i *is the* i*-th coordinate of the vector* $\mathbf{d} \in \mathbb{C}^g$*,*
> $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_g$ *is the standard basis of* $\mathbb{C}^g$*; see* (24.1.3)*, and*
> $\mathbf{B}_i \in \mathbb{C}^g$ *is a constant vector independent of* $\mathbf{d}$*.*

Proof. First we note that the left hand side of (24.1.21) is explicitly given by

$$\int_{b_i} \Big[\mathbf{u}(q_{i,+}) \cdot d \log h(q_{i,+}) - \mathbf{u}(q_{i,-}) \cdot d \log h(q_{i,-}) \Big],$$

where $q_{i,+}$ and $q_{i,-}$ are corresponding points on the edges b_i and b_i^{-1}.

Now we show the assertion. We give the proof only for $g = 1$, as the proof is essentially the same for $g \geq 2$. For simplicity we omit the subscript $i\,(= 1)$. By Lemma 24.1.6, $\mathbf{u}(q_+) = \mathbf{u}(q_-) + 1$ and $h(q_+) = h(q_-)$. Therefore

$$\begin{aligned}
\int_b \mathbf{u} \cdot d \log h + \int_{b^{-1}} \mathbf{u} \cdot d \log h \\
&= \int_b \Big[\mathbf{u}(q_+) \cdot d \log h(q_+) - \mathbf{u}(q_-) \cdot d \log h(q_-) \Big] \\
&= \int_b \Big[\Big(\mathbf{u}(q_-) + 1 \Big) d \log h(q_-) - \mathbf{u}(q_-) \cdot d \log h(q_-) \Big] \\
&= \int_b d \log h(q_-) \\
&= \log \frac{h(s)}{h(s')},
\end{aligned}$$

where s and s' are the ordered end points of the edge b. Here by Lemma 24.1.6, $h(s) = e^{-2\pi \mathrm{i}\left(\mathbf{u}(s') - \mathbf{d}\right) + \pi \mathrm{i} \beta} h(s')$; note that as $g = 1$, $\mathbf{u}(p_-)$ and $\mathbf{d}$ are actually scalars. Thus $\log \dfrac{h(s)}{h(s')} = -2\pi \mathrm{i}\big(\mathbf{u}(s') - \mathbf{d}\big) + \pi \mathrm{i} \beta$, and hence we have

$$\int_b \mathbf{u} \cdot d \log h + \int_{b^{-1}} \mathbf{u} \cdot d \log h = -2\pi \mathrm{i}\big(\mathbf{u}(s') - \mathbf{d}\big) + \pi \mathrm{i} \beta.$$

We may write the right hand side as $2\pi \mathrm{i}\mathbf{d} + \mathbf{B}$, where $\mathbf{B} := -2\pi \mathrm{i}\mathbf{u}(s') + \pi \mathrm{i} \beta$ is a constant independent of $\mathbf{d}$. This completes the proof. □

Taking $\mathbf{d} \in \mathrm{Jac}(C)$, we set $h(z) := \vartheta\big(\mathbf{u}(z) - \mathbf{d}\big)$ and then we consider integrals $\mathbf{A}_i$ and $\mathbf{B}_i$ respectively in (24.1.19) and (24.1.21). We define the *Riemann constant* $\boldsymbol{\kappa}$ (a point of $\mathrm{Jac}(C) := \mathbb{C}^g/\Lambda$) by

$$\boldsymbol{\kappa} = \frac{1}{2\pi \mathrm{i}} \sum_{i=1}^{g} (\mathbf{A}_i + \mathbf{B}_i) \mod \Lambda. \tag{24.1.22}$$

Since $\mathbf{A}_i$ and $\mathbf{B}_i$ do not depend on the choice of $\mathbf{d}$ (Lemma 24.1.9 and Lemma 24.1.10), *the Riemann constant* $\boldsymbol{\kappa}$ *is uniquely determined by the complex curve* C*.*

The significance of $\boldsymbol{\kappa}$ is manifest in the following additive formula.

Lemma 24.1.11 *Let C be a complex curve of genus $g \geq 1$. Fixing a point $\mathbf{d}$ of the Jacobian variety* $\mathrm{Jac}(C)$*, define a holomorphic section $h(z) := \vartheta\big(\mathbf{u}(z)-\mathbf{d}\big)$ of the line bundle*[4] $\mathbf{u}^*\mathcal{L}_{\mathbf{d}}$ *on C. Let $p_1, p_2, \ldots, p_g \in C$ be the zeros*[5] *of $h(z)$. Then*

$$\mathbf{u}(p_1) + \mathbf{u}(p_2) + \cdots + \mathbf{u}(p_g) = \mathbf{d} + \boldsymbol{\kappa}, \tag{24.1.23}$$

where $\boldsymbol{\kappa}$ is the Riemann constant (24.1.22).

Proof. Let P be the $4g$-gon obtained by cutting C along the loops $a_1, b_1, a_2, b_2, \ldots, a_g, b_g$ (the standard generators of $\pi_1(C)$, see (24.1.15)), where we choose these loops such that none of the zeros of $h(z)$ lies on them. By the residue theorem, we have

$$\frac{1}{2\pi \mathrm{i}} \int_{\partial P} \mathbf{u}(z) \cdot d\log h = \mathbf{u}(p_1) + \mathbf{u}(p_2) + \cdots + \mathbf{u}(p_g). \tag{24.1.24}$$

We rewrite $\int_{\partial P} \mathbf{u}(z) \cdot d\log h$ as follows:

$$\begin{aligned}
&\int_{\partial P} \mathbf{u}\cdot d\log h \\
&= \sum_{i=1}^{g} \left(\int_{a_i} \mathbf{u}\cdot d\log h + \int_{b_i} \mathbf{u}\cdot d\log h + \int_{a_i^{-1}} \mathbf{u}\cdot d\log h + \int_{b_i^{-1}} \mathbf{u}\cdot d\log h \right) \\
&= \sum_{i=1}^{g} \left(\int_{a_i} \mathbf{u}\cdot d\log h + \int_{a_i^{-1}} \mathbf{u}\cdot d\log h \right) + \sum_{i=1}^{g} \left(\int_{b_i} \mathbf{u}\cdot d\log h + \int_{b_i^{-1}} \mathbf{u}\cdot d\log h \right) \\
&= \sum_{i=1}^{g} \mathbf{A}_i + \sum_{i=1}^{g} (2\pi \mathrm{i}\, d_i \mathbf{e}_i + \mathbf{B}_i),
\end{aligned}$$

where in the last equality we used Lemmas 24.1.9 and 24.1.10. Since $\sum_{i=1}^{g} d_i \mathbf{e}_i = \mathbf{d}$, we have $\int_{\partial P} \mathbf{u} \cdot d\log h = \sum_{i=1}^{g} \mathbf{A}_i + (2\pi \mathrm{i}\, \mathbf{d} + \sum_{i=1}^{g} \mathbf{B}_i)$. Thus

$$\frac{1}{2\pi \mathrm{i}} \int_{\partial P} \mathbf{u} \cdot d\log h = \mathbf{d} + \boldsymbol{\kappa},$$

where $\boldsymbol{\kappa} \left(= \frac{1}{2\pi \mathrm{i}} \sum_{i=1}^{g} (\mathbf{A}_i + \mathbf{B}_i) \right)$ is the Riemann constant. With (24.1.24), we obtain

$$\mathbf{u}(p_1) + \mathbf{u}(p_2) + \cdots + \mathbf{u}(p_g) = \mathbf{d} + \boldsymbol{\kappa}. \qquad \square$$

We summarize Lemma 24.1.7 and Lemma 24.1.11 as follows:

Theorem 24.1.12 *Fixing a point $\mathbf{d}$ of* $\mathrm{Jac}(C)$*, define a holomorphic section $h(z) := \vartheta\big(\mathbf{u}(z)-\mathbf{d}\big)$ of the line bundle $\mathbf{u}^*\mathcal{L}_{\mathbf{d}}$ on C (where $\mathcal{L}_{\mathbf{d}}$ is the line bundle*

[4] The pull-back of the line bundle $\mathcal{L}_{\mathbf{d}}$ (24.1.14) via the Abel–Jacobi map $\mathbf{u} : C \to \mathrm{Jac}(C)$

[5] By convention, $p_1, p_2, \ldots, p_g$ are not necessarily distinct. See Lemma 24.1.7.

(24.1.14)). *Then* (1) $h(z)$ *has* g *zeros*[6]*; so the degree of the line bundle* $\mathbf{u}^*\mathcal{L}_{\mathbf{d}}$ *is* g*, and* (2) *let* $p_1, p_2, \ldots, p_g \in C$ *be the zeros of* $h(z)$*, and then*

$$\mathbf{u}(p_1) + \mathbf{u}(p_2) + \cdots + \mathbf{u}(p_g) = \mathbf{d} + \boldsymbol{\kappa}, \qquad (24.1.25)$$

where $\boldsymbol{\kappa} \in \mathrm{Jac}(C)$ *is the Riemann constant* (24.1.22).

Now we fix a point $\mathbf{c} \in \mathrm{Zero}(\vartheta)$, i.e. $\vartheta(\mathbf{c}) = 0$, and also fix a point $p \in C$. Let L_p be the line bundle on C that is the pull-back of the line bundle $\mathcal{L}_{\mathbf{u}(p)+\mathbf{c}}$ (24.1.14) by the Abel–Jacobi map $\mathbf{u} : C \to \mathrm{Jac}(C)$. Then we consider a holomorphic section $\vartheta_p(z) := \vartheta\big(\mathbf{u}(z) - \mathbf{u}(p) - \mathbf{c}\big)$ of L_p. Note that $\vartheta_p(z)$ may be identically zero — this happens precisely when $\mathbf{u}(z) - \mathbf{u}(p) - \mathbf{c} \in \mathrm{Zero}(\vartheta)$ holds for all $z \in C$, that is,

$$\mathbf{c} \in -\mathrm{Zero}(\vartheta) + \mathbf{u}(C) - \mathbf{u}(p),$$

where "$-\mathrm{Zero}(\vartheta) + \mathbf{u}(C) - \mathbf{u}(p)$" stands for the translation of the set $-\mathrm{Zero}(\vartheta)$ by $\mathbf{u}(C) - \mathbf{u}(p)$ in $\mathrm{Jac}(C)$. Thus $\vartheta_p(z)$ is identically zero precisely when

$$\mathbf{c} \in \mathrm{Zero}(\vartheta) \cap \Big(-\mathrm{Zero}(\vartheta) + \mathbf{u}(C) - \mathbf{u}(p)\Big).$$

Henceforth, we always take $\mathbf{c} \in \mathrm{Zero}(\vartheta)$ *such that* $\vartheta_p(z)$ *is not identically zero.* Then by applying Theorem 24.1.12 for $\mathbf{d} := \mathbf{u}(p) + \mathbf{c}$, we see that $\vartheta_p(z)$ has g zeros — by convention, these zeros are not necessarily distinct. Clearly, p is one of the zeros (since $\vartheta(\mathbf{c}) = 0$ by assumption and $\vartheta(\mathbf{x})$ is an even function, we have $\vartheta_p(p) = \vartheta(-\mathbf{c}) = \vartheta(\mathbf{c}) = 0$); while the remaining zeros, denoted by $o_1, o_2, \ldots, o_{g-1} \in C$, are called the *surplus zeros* of $\vartheta_p(z)$. So the zeros of $\vartheta_p(z)$ are $p, o_1, o_2, \ldots, o_{g-1}$. From the additive formula (24.1.25), we obtain

$$\mathbf{u}(p) + \mathbf{u}(o_1) + \mathbf{u}(o_2) + \cdots + \mathbf{u}(o_{g-1}) = \mathbf{u}(p) + \mathbf{c} + \boldsymbol{\kappa},$$

and thus $\mathbf{u}(o_1) + \mathbf{u}(o_2) + \cdots + \mathbf{u}(o_{g-1}) = \mathbf{c} + \boldsymbol{\kappa}$. This proves

Corollary 24.1.13 *For the holomorphic section* $\vartheta_p(z) := \vartheta\big(\mathbf{u}(z) - \mathbf{u}(p) - \mathbf{c}\big)$ *of the line bundle* L_p $(= u^*\mathcal{L}_{\mathbf{u(p)+c}})$ *on the complex curve* C*, the followings hold:* (1) $\vartheta_p(z)$ *has* g *zeros* (*one zero* p *and* $g-1$ *surplus zeros* $o_1, o_2, \ldots, o_{g-1}$) *and* (2) *the surplus zeros* $o_1, o_2, \ldots, o_{g-1}$ *satisfy*

$$\mathbf{u}(o_1) + \mathbf{u}(o_2) + \cdots + \mathbf{u}(o_{g-1}) = \mathbf{c} + \boldsymbol{\kappa}. \qquad (24.1.26)$$

We denote by $\mathrm{Sym}^i(C)$ the i-th symmetric product of the curve C, that is,

$$\mathrm{Sym}^i(C) = \underbrace{C \times C \times \cdots \times C}_{i} / \mathfrak{S}_i,$$

[6] By convention, they are not necessarily distinct.

where the symmetric group $\mathfrak{S}_i$ acts on $C \times C \times \cdots \times C$ by

$$(z_1, z_2, \ldots, z_i) \longmapsto (z_{\sigma(1)}, z_{\sigma(2)}, \ldots, z_{\sigma(i)}), \qquad \sigma \in \mathfrak{S}_i.$$

We extend the Abel–Jacobi map $\mathbf{u} : C \to \mathrm{Jac}(C)$ to the map $\mathbf{u} : \mathrm{Sym}^i(C) \to \mathrm{Jac}(C)$ by

$$(p_1, p_2, \ldots, p_i) \longmapsto \mathbf{u}(p_1) + \mathbf{u}(p_2) + \cdots + \mathbf{u}(p_i).$$

Lemma 24.1.14 *Given* $(p_1, p_2, \ldots, p_{g-1}) \in \mathrm{Sym}^{g-1}(C)$*, set*

$$\mathbf{c} := \mathbf{u}(p_1) + \mathbf{u}(p_2) + \cdots + \mathbf{u}(p_{g-1}) - \boldsymbol{\kappa}, \qquad (\boldsymbol{\kappa} : \textit{the Riemann constant}).$$

Then $\vartheta(\mathbf{c}) = 0$ *holds.*

Proof. First note that $\mathbf{u} : \mathrm{Sym}^g(C) \to \mathrm{Jac}(C)$ is birational ([Na3], Theorem 1, p87); hence there are analytic subsets E' and E of dim $< g$ such that the restriction $\mathbf{u} : \mathrm{Sym}^g(C) \setminus E' \to \mathrm{Jac}(C) \setminus E$ is biholomorphic. We say that $p_1, p_2, \ldots, p_{g-1} \in C$ are *in a generic position* if we may take $p_g \in C$ such that $(p_1, p_2, \ldots, p_g) \in \mathrm{Sym}^g(C) \setminus E'$.

We now show the assertion. It suffices to show it for $p_1, p_2, \ldots, p_{g-1} \in C$ in a generic position; indeed, once this is shown, by a continuity argument, for any $p_1, p_2, \ldots, p_{g-1} \in C$, we have

$$\vartheta\Big(\mathbf{u}(p_1) + \mathbf{u}(p_2) + \cdots + \mathbf{u}(p_{g-1}) - \boldsymbol{\kappa}\Big) = 0.$$

Now suppose that $p_1, p_2, \ldots, p_{g-1} \in C$ are in a generic position in the above sense, and we set

$$\mathbf{d} := \mathbf{u}(p_1) + \mathbf{u}(p_2) + \cdots + \mathbf{u}(p_g) - \boldsymbol{\kappa}.$$

Then we claim that $p_1, p_2, \ldots, p_g$ are the zeros of $h(z) := \vartheta\big(\mathbf{u}(z) - \mathbf{d}\big)$. In fact, if $q_1, q_2, \ldots, q_g \in C$ are the zeros of $h(z)$, then by the additive formula (24.1.25) we have

$$\mathbf{d} = \mathbf{u}(q_1) + \mathbf{u}(q_2) + \cdots + \mathbf{u}(q_g) - \boldsymbol{\kappa}.$$

Hence

$$\mathbf{u}(p_1) + \mathbf{u}(p_2) + \cdots + \mathbf{u}(p_g) = \mathbf{u}(q_1) + \mathbf{u}(q_2) + \cdots + \mathbf{u}(q_g).$$

Since we chose $(p_1, p_2, \ldots, p_g) \in \mathrm{Sym}^g(C) \setminus E'$ such that $\mathbf{u}$ is biholomorphic (thus, one to one) around it, we have

$$\{p_1, p_2, \ldots, p_g\} = \{q_1, q_2, \ldots, q_g\}.$$

Namely, $p_1, p_2, \ldots, p_g$ are the zeros of $h(z)$. In particular, p_g is a zero of $h(z)$; so

$$h(p_g) = \vartheta\big(\mathbf{u}(p_g) - \mathbf{d}\big) = 0. \qquad (24.1.27)$$

Finally, since $\mathbf{c} = \mathbf{d} - \mathbf{u}(p_g)$, we have

$$\begin{aligned}\vartheta(\mathbf{c}) &= \vartheta\big(\mathbf{d} - \mathbf{u}(p_g)\big) \\ &= \vartheta\big(\mathbf{u}(p_g) - \mathbf{d}\big) && \text{because } \vartheta \text{ is an even function} \\ &= 0 && \text{by (24.1.27).}\end{aligned}$$

This completes the proof of our assertion. □

Next we show

Proposition 24.1.15 *Consider two sets in the Jacobian variety* $\mathrm{Jac}(C)$*:*

$$W_{g-1} := \mathbf{u}(\mathrm{Sym}^{g-1}(C)) \quad \text{and} \quad \mathrm{Zero}(\vartheta) := \{\, \mathbf{x} \in \mathrm{Jac}(C) \,:\, \vartheta(\mathbf{x}) = 0 \,\}.$$

Then $W_{g-1} = \mathrm{Zero}(\vartheta) + \boldsymbol{\kappa}$ *holds (that is,* W_{g-1} *is the translation of* $\mathrm{Zero}(\vartheta)$ *by the Riemann constant* $\boldsymbol{\kappa}$*). In particular,* $\mathrm{Zero}(\vartheta)$ *and* W_{g-1} *are biholomorphic.*

Proof. We first show that if $\mathbf{c} \in \mathrm{Zero}(\vartheta)$, then $\mathbf{c} + \boldsymbol{\kappa} \in W_{g-1}$. Letting $p \in C$, we set $\vartheta_p(z) := \vartheta\big(\mathbf{u}(z) - \mathbf{u}(p) - \mathbf{c}\big)$, and we denote its surplus zeros by $o_1, o_2, \ldots, o_{g-1}$. By the additive formula (24.1.26),

$$\mathbf{u}(o_1) + \mathbf{u}(o_2) + \cdots + \mathbf{u}(o_{g-1}) = \mathbf{c} + \boldsymbol{\kappa}.$$

Hence $\mathbf{c} + \boldsymbol{\kappa} \in W_{g-1}$. Conversely, if $\mathbf{w} \in W_{g-1}$, then by Lemma 24.1.14, we have $\mathbf{w} - \boldsymbol{\kappa} \in \mathrm{Zero}(\vartheta)$. This implies that $W_{g-1} = \mathrm{Zero}(\vartheta) + \boldsymbol{\kappa}$. □

We summarize the above results.

Theorem 24.1.16 *Let* $D' \subset \mathrm{Sym}^{g-1}(C)$ *and* $D \subset W_{g-1}$ *be the critical sets of* $\mathbf{u}$*, so that the restriction* $\mathbf{u} : \mathrm{Sym}^{g-1}(C) \setminus D' \to W_{g-1} \setminus D$ *is biholomoprhic. Then the following holds.*

(1) *Fix* $\mathbf{c} \in \mathrm{Zero}(\vartheta)$*. For any* $p \in C$*, the surplus zeros of* $o_1, o_2, \ldots, o_{g-1}$ *of* $\vartheta_p(z) := \vartheta\big(\mathbf{u}(z) - \mathbf{u}(p) - \mathbf{c}\big)$ *satisfy* $\mathbf{u}(o_1) + \mathbf{u}(o_2) + \cdots + \mathbf{u}(o_{g-1}) = \mathbf{c} + \boldsymbol{\kappa}$*.*
(2) $W_{g-1} = \mathrm{Zero}(\vartheta) + \boldsymbol{\kappa}$ *where* $\boldsymbol{\kappa} \in \mathrm{Jac}(C)$ *is the Riemann constant.* (So if $\mathbf{c} \in \mathrm{Zero}(\vartheta)$, then $\mathbf{c} + \boldsymbol{\kappa} \in W_{g-1}$.)
(3) *For* $\mathbf{c} \in \mathrm{Zero}(\vartheta)$ *such that* $\mathbf{c} + \boldsymbol{\kappa} \in W_{g-1} \setminus D$*, the surplus zeros* $o_1, o_2, \ldots, o_{g-1}$ *of* $\vartheta_p(z)$ *are independent of the choice of* $p \in C$*.* (Hence for generic $\mathbf{c} \in \mathrm{Zero}(\vartheta)$, the surplus zeros of $\vartheta_p(z)$ are independent of $p \in C$.)
(4) *Given* $o_1, o_2, \ldots, o_{g-1} \in C$*, set* $\mathbf{c} := \mathbf{u}(o_1) + \mathbf{u}(o_2) + \cdots + \mathbf{u}(o_{g-1}) - \boldsymbol{\kappa}$*. If* $\mathbf{c} + \boldsymbol{\kappa} \in W_{g-1} \setminus D$*, then the surplus zeros of* $\vartheta_p(z) := \vartheta\big(\mathbf{u}(z) - \mathbf{u}(p) - \mathbf{c}\big)$ *are* $o_1, o_2, \ldots, o_{g-1}$*.*

Proof. The assertions (1) and (2) are respectively Corollary 24.1.13 and Proposition 24.1.15. We show (3). We fix $p \in C$, and we demonstrate that for arbitrary point $p' \in C$, the surplus zeros of $\vartheta_p(z)$ and $\vartheta_{p'}(z)$ coincide. Let $o_1, o_2, \ldots, o_{g-1}$ be the surplus zeros of $\vartheta_p(z)$. Then by (1),

$$\mathbf{u}(o_1) + \mathbf{u}(o_2) + \cdots + \mathbf{u}(o_{g-1}) = \mathbf{c} + \boldsymbol{\kappa}. \tag{24.1.28}$$

Similarly, the surplus zeros $o'_1, o'_2, \ldots, o'_{g-1}$ of $\vartheta_{p'}(z)$ satisfy

$$\mathbf{u}(o'_1) + \mathbf{u}(o'_2) + \cdots + \mathbf{u}(o'_{g-1}) = \mathbf{c} + \boldsymbol{\kappa}.$$

Thus

$$\mathbf{u}(o_1) + \mathbf{u}(o_2) + \cdots + \mathbf{u}(o_{g-1}) = \mathbf{u}(o'_1) + \mathbf{u}(o'_2) + \cdots + \mathbf{u}(o'_{g-1}). \quad (24.1.29)$$

Note that (24.1.28) with the assumption $\mathbf{c} + \boldsymbol{\kappa} \in W_{g-1} \setminus D$ implies that

$$(o_1, o_2, \ldots, o_{g-1}) \in \mathrm{Sym}^{g-1}(C) \setminus D'.$$

In particular, $\mathbf{u}$ is biholomorphic around $(o_1, o_2, \ldots, o_{g-1})$, and so from (24.1.29) we deduce

$$\{o_1, o_2, \ldots, o_{g-1}\} = \{o'_1, o'_2, \ldots, o'_{g-1}\}.$$

Therefore the surplus zeros of $\vartheta_p(z)$ and $\vartheta_{p'}(z)$ coincide; so (3) is confirmed. The assertion (4) is clear from the proof of (3). □

Part V

Classification of Atoms of Genus ≤ 5

25

Classification Theorem

Let $\pi : M \to \Delta$ be a degeneration of (compact) complex curves of genus g ($g \geq 1$); in the subsequent discussion, for simplicity we often say "a degeneration of genus g". Without loss of generality — if necessary, by blowing M up or down —, we may assume that its singular fiber $X = \pi^{-1}(0)$ is normally minimal. So X is either stellar (star-shaped) or constellar (constellation-shaped). Note that X is stellar precisely when the topological monodromy γ is periodic (i.e. $\gamma^n = \mathrm{id}$ for some positive integer n); while X is constellar precisely when γ is pseudo-periodic (i.e. γ^n for some positive integer n is generated by Dehn twists).

We say that a degeneration $\pi : M \to \Delta$ is *atomic* if it does not admit any splitting family. Moreover, $\pi : M \to \Delta$ is *absolutely atomic* if any degeneration, which is topologically equivalent to $\pi : M \to \Delta$, does not admit any splitting family.

Next recall that a barking deformation is constructed from a weighted crustal set — a finite set of weighted crusts satisfying a certain condition. In the end of this book, we provide a list of weighted crustal sets for a large class of singular fibers of genus from 1 to 5, including all stellar singular fibers. This list is not exhaustive, but enough to determine absolute atoms up to genus 5.

The classification of absolute atoms is based on induction on genus, which we shall explain. First of all, we recall the following conjecture we proposed in [Ta,I]:

Conjucture 25.1 *A degeneration is absolutely atomic if and only if its singular fiber is either a reduced curve with one node* (i.e. *Lefschetz fiber*) *or a multiple of a smooth curve.*

("If" is valid. See [Ta,I].) Supposing that Conjecture 25.1 is valid for genus $\leq g-1$, then by [Ta,I], to classify atomic degenerations of genus g, we only have to investigate the splittability for degenerations $\pi : M \to \Delta$ such that either

(A) $X = \pi^{-1}(0)$ is stellar, or

(B) X is constellar and (B.1) X has no multiple node and (B.2) if X has an irreducible component Θ_0 of multiplicity 1, then Θ_0 is a projective line, and intersects other irreducible components of X only at one point (hence Θ_0 intersects only one irreducible component).

To these cases, we apply the results of this book, mainly Criterion 19.4.3, p343: If a singular fiber X contains a simple crust and X satisfies some mild condition, then the degeneration admits a splitting — so we only have to find a simple crust (or more generally, a weighted crustal set) of the singular fiber X. Here, we note the following:

(i) If all irreducible components of X are projective lines — for example, this is the case when X is stellar with its core being a projective line —, then we can easily check the existence of a simple crust numerically.
(ii) Using the existence of simple crusts of some stellar singular fibers, it is immediate to see that most singular fibers in (B) admit splittings by Criterion 16.5.1, p293. **Essentially we only need to investigate the splittability of stellar singular fibers and some exceptional constellar singular fibers. Furthermore by Criterion 16.3.1 (1), p284, among stellar singular fibers, we only need to investigate the case where the core is not an exceptional curve.** According to Ashikaga and Ishizaka [AI], the number of degenerations of genus 3 is about 1600, and among them there are only about 50 degenerations with stellar singular fibers (and if the core is not an exceptional curve, there are much fewer than 50). Therefore the above criteria drastically reduce the number of the cases to be checked.
(iii) In (B), if X contains an exceptional curve, then in most cases the splittability follows immediately from Criterion 19.4.3 (3), p343 and Criterion 19.4.5, p344.

We now state our result on the classification of atomic fibers.

Theorem 25.2 *A degeneration of genus ≤ 5 is absolutely atomic if and only if its singular fiber is either a reduced curve with one node* (i.e. *Lefschetz fiber*) *or a multiple of a smooth curve.*

Proof. By [Ta,I], a reduced curve with one node, and a multiple of a smooth curve are atomic. Hence we must show that any other singular fiber splits. As mentioned above, it is enough to show that the singular fibers in (A) and (B) split. For genus ≤ 5, any singular fiber in (A) and (B) splits, because it contains a simple crust (or a weighted crustal set which admits complete propagation). See our list of weighted crustal sets for genus ≤ 5 in the next chapter. □

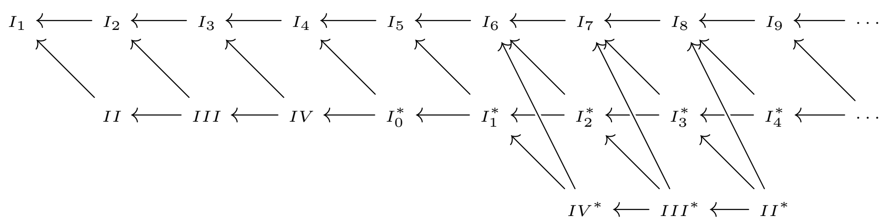
Remark 25.3 When we try to apply the above proof to the case genus ≥ 6, we encounter with technical difficulty; as the genus g grows, the number of the singular fibers of genus g in types (A) and (B) rapidly increases (though it is much less than the total number of all singular fibers of genus g), and it is labour-intensive to list such singular fibers and to find weighted crustal sets for them. Probably computer-aided research is promising.

Explicitly the list of the singular fibers of the absolutely atomic degenerations of genus ≤ 5 is as follows:

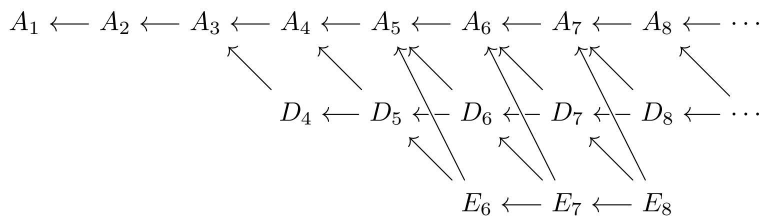
	absolute atoms
genus 1	$m\Theta$, where Θ is a smooth curve of genus 1, any reduced curve with one node (Lefschetz fiber)
genus 2	any reduced curve with one node (Lefschetz fiber)
genus 3	2Θ, where Θ is a smooth curve of genus 2, any reduced curve with one node (Lefschetz fiber)
genus 4	3Θ, where Θ is a smooth curve of genus 2, any reduced curve with one node (Lefschetz fiber)
genus 5	4Θ, where Θ is a smooth curve of genus 2, 2Θ, where Θ is a smooth curve of genus 3, any reduced curve with one node (Lefschetz fiber)

Supplement: Adjacency diagram

When a singular fiber X splits into singular fibers $X_1, X_2, \ldots, X_n$ in a splitting family, we say that X_i is *adjacent* to X (notation: $X \to X_i$). Using barking families (and result from [Ta,I]), we may describe the adjacencies among singular fibers of genus 1 (see also the list of simple crusts in §26.1, p492):



A similar diagram is known for simple singularities (or, *A*-*D*-*E*-singularities, Du Val singularities). See V. I. Arnold (ed.) [Ar2] p30:



26

List of Weighted Crustal Sets for Singular Fibers of Genus ≤ 5

In what follows, we explain the convention and notation used in the list, such as notation for singular fibers and crusts, arrangement of the list, and names of splittability criteria. For each genus g ($g = 1, 2, 3, 4, 5$), the list of crustal sets consists of the list for stellar singular fibers and that for constellar singular fibers, where stellar singular fibers are arranged according to the orders of their topological monodromies. (Note that the order of a periodic topological monodromy equals the multiplicity of the core.) We refer to genus formula p268 for how to compute the genus of a smooth fiber in terms of the data of the stellar singular fiber.

Notation for stellar singular fibers

- We express a stellar singular fiber X, for example as follows:

$$(-2), \quad X = 6A + 3B_1 + 4C_1 + 2C_2 + 5D_1 + 4D_2 + 3D_3 + 2D_4 + D_5$$

  where (-2) means that the self-intersection number[1] of the core A of X is -2, and $3B_1$, $4C_1 + 2C_2$ and $5D_1 + 4D_2 + 3D_3 + 2D_4 + D_5$ are branches emanating from A. These branches are respectively called a *B-branch*, a *C-branch*, and a *D-branch*.
- When the core A has genus ≥ 1, we give information on the normal bundle N_A of A:

$$X = 3A + B_1 + 2C_1 + C_2, \quad N_A^{\otimes 3} = \mathcal{O}(-b_1 - 2c_1)$$

  where b_1 and c_1 are respectively the intersection points of B-branch and C-branch with A.
- $X \to X'$: This means that X is deformed to $X' := X_{0,t}$.
- $p \sim q$: Two points p and q on the core A are *linearly equivalent*.

[1] We denote this only for the case where A is a projective line, because in this case the definition of crusts is numerical.

Notation for weighted crustal sets (WCSs)

A barking deformation is constructed from a finite set of weighted crusts satisfying a certain condition, that is, a weighted crustal set.

- WCS stands[2] for a weighted crustal set for X, and we write

$$WCS = (Y_1, d_1) + (Y_2, d_2) + \cdots + (Y_l, d_l),$$

  where Y_i is a crust and d_i is a weight. For consistency, for a simple crust Y of barking multiplicity l, we write $WCS = \sum_{i=1}^{l}(iY, id)$, where d is a weight.
- $X = \ldots$
  (i) $WCS = \ldots,$ semi-rigid: B_1, C_1, D_4
  This means that B_1, C_1 and D_4 are semi-rigid irreducible components (e.g. **[2] 10.3 type** in the list). For genus 1, 2, and 3 cases, the WCSs numbered (i), (ii), (iii), ... are equipped with information about semi-rigid irreducible components (while for the WCSs numbered (1), (2), (3), ..., such information is not provided). This information is useful for applying Criterion 16.5.1, p293 to constellar singular fibers.

Arrangement of the list for genus 1

For singular fibers of genus 1, we adopt Kodaira's notation I_n, II, III, IV, I_n^*, II^*, III^*, IV^* (see [Ko1] or the appendix of [Ta,II]), and we arrange them according to the order of their topological monodromies. For instance, the topological monodromies of II and II^* have the same order 6, and we put them in order as follows: (Below [1] stands for genus 1.)

Order 6

[1] II **type,** (-1) (the self-intersection number of the core is -1.)
$X = \ldots$

$WCS = (Y, 1)$ (this weighted crustal set consists of only one crust Y with weight 1. Two cases (1) and (2) below for the choices of Y.)

(1) : $II \to I_1$ (this means that $X = II$ is deformed to $X_{0,t} = I_1$ in the barking family associated with the weighted crustal set where Y is just below.)
$Y = \ldots$
(2) : $II \to I_1$
$Y = \ldots$

[1] II^* **type,** (-2)
$X = \ldots$
...

[2] In some cases, instead we write a weighted set of dominant crusts, from which we can easily obtain a weighted crustal set by the cut-off operation (see p304).

Arrangement of the list for genus ≥ 2

Let $\pi : M \to \Delta$ be a degeneration of complex curves of genus g whose singular fiber X is stellar. The periodic topological monodromy γ acts on a smooth fiber $\Sigma := \pi^{-1}(s)$, $(s \neq 0)$ as a homeomorphism of finite order. We then have a cyclic covering $\Sigma \to \Sigma/\langle\gamma\rangle$ with a covering transformation γ, where $\Sigma/\langle\gamma\rangle$ denotes the quotient space of Σ under the γ-action. If there are k branch points in $\Sigma/\langle\gamma\rangle$, say, with ramification indices $r_1, r_2, \ldots, r_k$ respectively, then $r = (r_1, r_2, \ldots, r_k)$ is called the *ramification data* of γ. Stellar singular fibers of genus g $(g \geq 2)$ with a ramification data r are put in order as follows: (Below **[g]** stands for genus g)

Order m $\quad r = (r_1, r_2, \ldots, r_k)$

[g] m.1 type
[g] m.2 type
[g] m.3 type
...

If some stellar singular fibers of genus g have topological monodromies of the same order m but with the different ramification data, say r and r', then they are put in order as follows:

Order m $\quad r = (r_1, r_2, \ldots, r_k)$

[g] m.1.1 type
[g] m.1.2 type
[g] m.1.3 type
...

Order m $\quad r' = (r'_1, r'_2, \ldots, r'_l)$

[g] m.2.1 type
[g] m.2.2 type
[g] m.2.3 type
...

Expression of constellar singular fibers: Welding and Connecting

A constellar singular fiber is obtained from stellar ones of lower genera by bonding their branches (*Matsumoto–Montesinos bonding*). Let X_1 and X_2 be stellar singular fibers, and let $\overline{\mathrm{Br}}_1$ (resp. $\overline{\mathrm{Br}}_2$) be a branch of X_1 (resp. X_2):

$$\overline{\mathrm{Br}}_1 = m_0\Theta_0 + m_1\Theta_1 + \cdots + m_\lambda\Theta_\lambda, \qquad \overline{\mathrm{Br}}_2 = m'_0\Theta'_0 + m'_1\Theta'_1 + \cdots + m'_\nu\Theta'_\nu.$$

Then $\gcd(m_0, m_1, \ldots, m_\lambda)$ is the *multiplicity* of $\overline{\mathrm{Br}}_1$; actually, it is equal to m_λ. If $m_\lambda \geq 2$, the branch $\overline{\mathrm{Br}}_1$ is called *multiple*. Now suppose that $\overline{\mathrm{Br}}_1$ has the same multiplicity as $\overline{\mathrm{Br}}_2$ (hence $m_\lambda = m'_\nu$). Given an integer κ ($\kappa \geq -1$), we can define κ-bonding of X_1 and X_2 by connecting $\overline{\mathrm{Br}}_1$ and $\overline{\mathrm{Br}}_2$ (see §16.5, p292); after bonding, $\overline{\mathrm{Br}}_1$ and $\overline{\mathrm{Br}}_2$ are joined to become a "κ-trunk" $\overline{\mathrm{Tk}}$ of a constellar singular fiber $X = X(\kappa)$. Set $m := m_\lambda = m'_\nu$ (the common multiplicity of $\overline{\mathrm{Br}}_1$ and $\overline{\mathrm{Br}}_2$), and then the κ-trunk is a chain of projective lines given as follows:

if $\kappa \geq 0$,

$$\overline{\mathrm{Tk}} = m_0 D_0 + m_1\Theta_1 + \cdots + m_{\lambda-1}\Theta_{\lambda-1} + m\Theta_\lambda + m\Theta_{\lambda+1} + \cdots + m\Theta_{\lambda+\kappa} + m'_\nu\Theta'_\nu + m'_{\nu-1}\Theta'_{\nu-1} + \cdots + m'_0 D'_0.$$

if $\kappa = -1$,

$$\overline{\mathrm{Tk}} = m_0 D_0 + m_1\Theta_1 + \cdots + m_{\lambda_0}\Theta_{\lambda_0} + m'_{\nu_0-1}\Theta'_{\nu_0-1} + m'_{\nu_0-2}\Theta'_{\nu_0-2} + \cdots + m'_0 D'_0.$$

It is easy to generalize bonding of two stellar singular fibers to that of an arbitrary number of stellar singular fibers.

For the splitting problem of singular fibers, we are only concerned with κ-bonding such that (1) $\kappa = -1$ or (2) $\kappa = 0$ and $m \geq 2$ where $m := m_\lambda = m'_\nu$, because for any other bonding, by Criterion 1.2.5, p30 or Criterion 1.2.6, p31, the singular fiber X always admits a splitting.

For simplicity, we call (-1)-bonding a *welding* and express it as $X = \mathrm{wd}(X_1 + X_2)$. For the case where X is obtained by (-1)-bonding of two branches of *one* singular fiber X_1, i.e. self-bonding, we call X a *self-welding* of X_1 and write $X = \mathrm{sw}(X_1)$. We call 0-bonding a *connecting* and we write $X = \mathrm{cn}(X_1 + X_2)$. (As mentioned above, we are concerned with such connecting as $m \geq 2$.) For the self-bonding case, we call X a *self-connecting* of X_1 and write $X = \mathrm{sc}(X_1)$. For instance,

- **sw(III^*) type**
  $X = 4A + 2B_1 + 3C_1 + 2C_2 + 3D_1 + 2D_2, \quad C_2 = D_2$
  X is a self-welding of[3] III^* obtained by identifying $2C_1$ and $2D_1$ of $4A + 2B_1 + 3C_1 + 2C_2 + 3D_1 + 2D_2$ (the cut-off of C_3 and D_3 from $III^* = 4A + 2B_1 + 3C_1 + 2C_2 + C_3 + 3D_1 + 2D_2 + D_3$).
- **sc(III^*) type**
  $X = 4A + 2B_1 + 3C_1 + 2C_2 + C_3 + 3D_1 + 2D_2 + D_3, \quad C_3 = D_3$
  X is a self-connecting of III^* obtained by identifying C_3 and D_3 of III^*.
- **wd($IV^* + III^*$) type**
  $X = X_1 + X_2, \quad B_1(X_1) = C_2(X_2)$
  $X_1 = 3A + 2B_1 + 2C_1 + C_2 + 2D_1 + D_2$
  $X_2 = 4A + 2B_1 + 3C_1 + 2C_2 + 3D_1 + 2D_2 + D_3.$

[3] III^* is Kodaira's notation for a singular fiber of genus 1. See [Ko1] or the appendix of [Ta,II].

X is a welding of IV^* and III^* obtained by identifying[4] $2B_1$ of X_1 with $2C_2$ of X_2. Here $IV^* = 3A + 2B_1 + B_2 + 2C_1 + C_2 + 2D_1 + D_2$. Then X_1 is the cut-off of B_2 from IV^*, and X_2 is the cut-off of C_3 from III^*.
If Y is a subdivisor of X, then we write $Y = Y(X_1) + Y(X_2)$ where $Y(X_1) := Y \cap X_1$ and $Y(X_2) := Y \cap X_2$.

- **cn($IV^* + III^*$) type**
$X = X_1 + X_2, \quad B_2(X_1) = C_3(X_2)$
$X_1 = 3A + 2B_1 + B_2 + 2C_1 + C_2 + 2D_1 + D_2$
$X_2 = 4A + 2B_1 + 3C_1 + 2C_2 + C_3 + 3D_1 + 2D_2 + D_3$.
X is a connecting of IV^* and III^* obtained by identifying B_2 of $X_1 = IV^*$ with C_3 of $X_2 = III^*$.

Names of splitting criteria

- Exceptional Curve Criterion: Criterion 16.5.2, p294.
- "Use a deformation of a plane curve singularity":
If the relatively minimal model $X_{\min}$ of X has a reduced plane curve singularity, then apply Criterion 1.2.4, p30.
- Multiple Criterion: Criterion 1.2.2, p30.
- Trivial Propagation Criterion: Criterion 16.5.1, p293.
- SR technique: Assume that X contains a set $S = \{\Theta_k\}$ of irreducible components of multiplicity 1 such that $X' = X \setminus S$ is the exceptional set of a rational singularity V. Let M' be a tubular neighborhood of X' in M. Then the contraction of M' at X' yields V, and $\pi|_{M'} : M' \to \Delta$ descends to a map $\pi' : V \to \Delta$. We take a deformation $f : \mathcal{V} \to \Delta^\dagger$ of V which admits a simultaneous resolution $r : \mathcal{M}' \to \mathcal{V}$; see Remark below. Then by Riemenschneider's Theorem [Ri1], the map $\pi' : V \to \Delta$ extends to a map $\Pi' : \mathcal{V} \to \Delta$. Hence we obtain a map $\Psi' := (\Pi, r \circ f) : \mathcal{V} \to \Delta \times \Delta^\dagger$. By assumption, the multiplicity of any irreducible component Θ_k in the set S is 1, and so by Criterion 1.2.6, p31 ([Ta,I] for more details), the composite map $r \circ \Psi' : \mathcal{M}' \to \Delta \times \Delta^\dagger$ extends to a deformation $\Psi : \mathcal{M} \to \Delta \times \Delta^\dagger$. This construction is called the *simultaneous resolution technique* (SR technique). (Actually, we may generalize this construction to the case where a connected component of X' is the exceptional set of a rational singularity.)

Remark on simultaneous resolution

The base space (i.e. the parameter space) of the versal family of a rational surface singularity V is generally not reducible, and there exists a unique irreducible component which admits a simultaneous resolution. This irreducible component is called the *Artin component*, which is known to be smooth.

[4] To distinguish B_1 of X_1 from B_1 of X_2 etc, we often write the former one as $B_1(X_1)$ and the latter one as $B_1(X_2)$ etc.

For example, see [BR2] p33. A *simultaneous resolution* of a deformation of the singularity V is a "fiberwise" resolution possibly after base change. Precisely speaking, a simultaneous resolution of a deformation $f : \mathcal{V} \to \Delta^\dagger$ is a surjective map $r : \mathcal{M}' \to \mathcal{V}$, together with a pair of maps f' and p, such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{M}' & \xrightarrow{\ r\ } & \mathcal{V} \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ \Delta^\dagger & \xrightarrow{\ p\ } & \Delta^\dagger, \end{array}$$

where (1) $f' : \mathcal{M}' \to \Delta^\dagger$ is a deformation of M', (2) for each $t \in \Delta^\dagger$ the restriction $r : (f')^{-1}(t) \to f^{-1}(t)$ is a resolution of singularities, and (3) p is a base change, i.e. $p(z) = z^n$ for some positive integer n.

We remark that the total space $\mathcal{M}'$ of the simultaneous resolution $f' : \mathcal{M}' \to \Delta^\dagger$ is smooth; this is a consequence of the following result. Let $g : \mathcal{W} \to T$ be a flat family of complex manifolds over a complex analytic space T. If the base space T is smooth and each fiber $g^{-1}(t)$ $(t \in T)$ is smooth, then the total space $\mathcal{W}$ itself is smooth; in fact, a local parameter of T together with a local parameter of a fiber $g^{-1}(t)$ constitutes a local parameter of $\mathcal{W}$. (See [Hart], III Theorem 10.2). This statement was ring-theoretically proved in EGA IV, Proposition (6.5.1):

Proposition 26.1 *Let $h : (A, \mathfrak{m}) \to (B, \mathfrak{n})$ be a flat homomorphism between local rings of finite type, where $\mathfrak{m}$ (resp. $\mathfrak{n}$) is the maximal ideal of A (resp. B). If both $(A, \mathfrak{m})$ and $\big(B/\mathfrak{m}B,\ (\mathfrak{n} + \mathfrak{m}B)/\mathfrak{m}B\big)$ are regular local rings, then $(B, \mathfrak{n})$ is a regular local ring.*

This is geometrically restated as: for the flat morphism $g : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ induced from the homomorphism h, if the base space $\mathrm{Spec}(A)$ is smooth and the fiber $\mathrm{Spec}(B/\mathfrak{m}B)$ over a point $\mathrm{Spec}(A/\mathfrak{n}A)$ is also smooth, then the total space $\mathrm{Spec}(B)$ is smooth.

Although EGA treats the algebraic case, "smoothness" for the analytic case is the regularity of an analytic local ring, and hence the proof for the algebraic case may be carried over to the analytic case.

26.1 Genus 1

26.1.1 Stellar singular fibers, $A = \mathbb{P}^1$

Order 6

[1] *II* **type,** (-1) (the self-intersection number of the core is -1.)
$X = 6A + B_1 + 2C_1 + 3D_1$

$WCS = (Y, 1)$ (this weighted bunch of crusts consists of only one crust Y with weight 1. Two cases (1) and (2) for the choices of Y.)

(1) : $II \to I_1$ (this means that $X = II$ is deformed to $X_{0,t} = I_1$ in the barking family associated with the weighted bunch of crusts where Y is just below.)

$Y = A + B_1$

$(2) : II \to I_1$

$Y = 2A + B_1 + D_1$

Or apply SR technique for a cyclic quotient singularity after contracting A. (See "Names of splitting criteria" p491 for the splitting criteria used in this list.)

Or use deformation of a plane curve singularity (cusp singularity) for the relatively minimal model.

[1] II^* **type,** (-2)

$X = 6A + 3B_1 + 4C_1 + 2C_2 + 5D_1 + 4D_2 + 3D_3 + 2D_4 + D_5$

$WCS = (Y, 2)$
$(1) : II^* \to III^*$
$Y = 2A + B_1 + C_1 + 2D_1 + 2D_2 + 2D_3 + 2D_4$

$WCS = (Y, 3)$
$(2) : II^* \to IV^*$
$Y = 3A + B_1 + 2C_1 + C_2 + 3D_1 + 3D_2 + 3D_3$

$WCS = (Y, 4)$
$(3) : II^* \to I_2^*$
$Y = 4A + 2B_1 + 2C_1 + 4D_1 + 4D_2$

$WCS = (Y, 5)$
$(4) : II^* \to I_5$
$Y = 5A + 2B_1 + 3C_1 + C_2 + 5D_1$

$WCS = (Y, 2) + (2Y, 4)$
$(5) : II^* \to I_3^*$
$Y = 2A + B_1 + C_1 + 2D_1 + 2D_2$

$WCS = (Y_1, 1) + (Y_2, 2) + (Y_3, 4)$
$(6) : II^* \to I_3^*$
$Y_1 = A + C_1 + C_2 + D_1 + D_2 + D_3 + D_4 + D_5,$
$Y_2 = 2A + B_1 + 2C_1 + 2C_2 + D_1,$
$Y_3 = 4A + 2B_1 + 3C_1 + 2C_2 + 3D_1 + 2D_2 + D_3$

$WCS = (Y_1, 1) + (Y_2, 2) + (Y_3, 3) + (Y_4, 3) + (Y_5, 4) + (Y_6, 5)$
$(7) : II^* \to I_8$
$Y_1 = A + B_1 + D_1 + D_2 + D_3 + D_4 + D_5,$
$Y_2 = 2A + 2B_1 + C_1 + D_1,$

$Y_3 = 3A + 3B_1 + C_1 + 2D_1 + D_2$,
$Y_4 = 3A + 3B_1 + 2C_1 + C_2 + D_1$,
$Y_5 = 4A + 3B_1 + 2C_1 + 3D_1 + 2D_2 + D_3$,
$Y_6 = 5A + 3B_1 + 3C_1 + C_2 + 4D_1 + 3D_2 + 2D_3 + D_4$

$WCS = (Y, 1) + (2Y, 2) + (3Y, 3)$, semi-rigid: B_1, C_2, D_3
$(8) : II^* \to III^*$
$Y = A + B_1 + D_1 + D_2 + D_3$

$WCS = (Y, 1) + (2Y, 2) + (3Y, 3) + (4Y, 4)$, semi-rigid: B_1, C_2, D_2
$(9) : II^* \to III^*$
$Y = A + C_1 + C_2 + D_1 + D_2$

Or apply SR technique for the E_8-singularity.

Order 4

[1] III **type,** (-1)
$X = 4A + B_1 + C_1 + 2D_1$

$WCS = (Y, 1)$
$(1) : III \to I_2$
$Y = A + B_1$
$(2) : III \to I_1$
$Y = 2A + B_1 + C_1$
$(3) : III \to I_2$
$Y = 2A + B_1 + D_1$

Or apply SR technique for A_1-singularity.

Or use deformation of a plane curve singularity for the relatively minimal model.

[1] III^* **type,** (-2)
$X = 4A + 2B_1 + 3C_1 + 2C_2 + C_3 + 3D_1 + 2D_2 + D_3$

$WCS = (Y, 1)$
$(1) : III^* \to IV^*$
$Y = A + C_1 + C_2 + C_3 + D_1 + D_2 + D_3$

$WCS = (Y, 2)$
$(2) : III^* \to I_1^*$
$Y = 2A + 2C_1 + 2C_2 + 2D_1 + 2D_2$
$(3) : III^* \to I_2^*$
$Y = 2A + B_1 + C_1 + 2D_1 + 2D_2$

$WCS = (Y, 3)$
$(4) : III^* \to I_0^*$
$Y = 3A + 3C_1 + 3D_1$
$(5) : III^* \to I_6$
$Y = 3A + B_1 + 2C_1 + C_2 + 3D_1$

$WCS = (Y, 1) + (2Y, 2)$
$(6) : III^* \to I_2^*$
$Y = A + C_1 + C_2 + D_1 + D_2,$

$WCS = (Y, 1) + (Y_2, 2) + (Y_3, 2) + (Y_4, 3)$
$(7) : III^* \to I_7$
$Y_1 = A + B_1 + C_1 + C_2 + C_3,$
$Y_2 = 2A + B_1 + C_1 + 2D_1 + 2D_2,$
$Y_3 = 2A + 2B_1 + 2D_1 + 2D_2,$
$Y_4 = 3A + 2B_1 + 2C_1 + C_2 + 2D_1 + D_2,$

$WCS = (Y_1, 1) + (Y_2, 1) + (Y_3, 2) + (Y_4, 3)$
$(8) : III^* \to I_6$
$Y_1 = A + B_1 + C_1 + C_2 + C_3,$
$Y_2 = A + B_1 + D_1 + D_2 + D_3,$
$Y_3 = 2A + 2B_1 + C_1 + D_1,$
$Y_4 = 3A + 2B_1 + 2C_1 + C_2 + 2D_1 + D_2,$

$WCS = (Y, 1) + (2Y, 2) + (3Y, 3)$, semi-rigid: B_1, C_2, D_2
$(9) : III^* \to IV^*$
$Y = A + C_1 + D_1$

Or apply SR technique for E_7-singularity.

Order 3

[1] IV **type,** (-1)
$X = 3A + B_1 + C_1 + D_1$

$WCS = (Y, 1)$
$(1) : IV \to I_3$
$Y = A + B_1$
$(2) : IV \to I_2$
$Y = 2A + B_1 + C_1$
$(3) : IV \to III$
$Y = A + B_1 + C_1$
$(4) : IV \to II$
$Y = A + B_1 + C_1 + D_1$

Or use deformation of a plane curve singularity (ordinary triple point) for the relatively minimal model.

[1] IV^* **type,** (-2)
$X = 3A + 2B_1 + B_2 + 2C_1 + C_2 + 2D_1 + D_2$

$WCS = (Y, 1)$
$(1) : IV^* \to I_1^*$
$Y = A + B_1 + B_2 + C_1 + C_2$

$WCS = (Y, 2)$
$(2) : IV^* \to I_0^*$
$Y = 2A + 2B_1 + 2C_1$
$(3) : IV^* \to I_6$
$Y = 2A + B_1 + C_1 + 2D_1$

$WCS = (Y, 1) + (2Y, 2)$, semi-rigid: B_1, C_1, D_1
$(4) : IV^* \to I_1^*$
$Y = A + B_1 + C_1$

Or apply SR technique for E_6-singularity.

Order 2

[1] I_0^* **type,** (-2)
$X = 2A + B_1 + C_1 + D_1 + E_1$

$WCS = (Y, 1)$
$(1) : I_0^* \to I_4$
$Y = A + B_1 + C_1$
$(2) : I_0^* \to IV$
$Y = A + B_1 + C_1 + D_1$

Or apply SR technique for a cyclic quotient singularity.

26.1.2 I_n^*

[1] I_n^* **type**
$X = B_1 + C_1 + 2A_0 + 2A_1 + \cdots + 2A_{n-1} + 2A_n + D_1 + E_1, \quad n \geq 1$
where $A_0 \cap B_1 = b_0$, $A_0 \cap C_1 = c_0$, $A_n \cap D_1 = d_0$ and $A_n \cap E_1 = e_0$, and $A_i \cap A_{i+1} = a_i$.

$WCS = (Y, 1)$
$(1) : I_n^* \to I_{n-1}^*$
$Y = A_0 + B_1 + C_1$
Or apply multiple criterion.
$(2) : I_n^* \to I_{n+4}$
$Y = B_1 + A_0 + A_1 + \cdots + A_{n-1} + A_n + D_1$

26.1.3 mI_n

[1] mI_n **type**
$X = mI_n$, $m \geq 2, n \geq 1$
$(1) : mI_n \to mI_{n-1}$
Apply multiple criterion.

$X = mI_0$ atom

26.2 Genus 2

26.2.1 Stellar singular fibers, $A = \mathbb{P}^1$

Order 10 $r = (2, 5, 10)$

[2] 10.1 type, (-1)
$X = 10A + 5B_1 + C_1 + 4D_1 + 2D_2$

Apply exceptional curve criterion.

$WCS = (Y, 1)$
(1) $Y = A + C_1$
(2) $Y = 2A + B_1 + C_1$
(3) $Y = 3A + B_1 + C_1 + D_1$
(4) $Y = 4A + 2B_1 + C_1 + D_1$
(5) $Y = 6A + 3B_1 + C_1 + 2D_1$

Or apply SR technique for a cyclic quotient singularity after contracting A.

[2] 10.2 type, (-1)
$X = 10A + 5B_1 + 2C_1 + 3D_1 + 2D_2 + D_3$

Apply exceptional curve criterion.

$WCS = (Y, 1)$
$Y = 6A + 3B_1 + C_1 + 2D_1 + 2D_2$

[2] 10.3 type, (-2)
$X = 10A + 5B_1 + 6C_1 + 2C_2 + 9D_1 + 8D_2 + 7D_3 + 6D_4 + 5D_5 + 4D_6 + 3D_7 + 2D_8 + D_9$

(i) $WCS = \sum_{i=1}^{6} (iY, i)$, semi-rigid[5]: B_1, C_1, D_4
$Y = A + C_1 + D_1 + D_2 + D_3 + D_4$

$WCS = (Y, 2)$
(1) $Y = 2A + B_1 + C_1 + 2D_1 + 2D_2 + 2D_3 + 2D_4 + 2D_5 + 2D_6 + 2D_7 + 2D_8$

$WCS = (Y, 4)$
(2) $Y = 4A + 2B_1 + 2C_1 + 4D_1 + 4D_2 + 4D_3 + 4D_4 + 4D_5 + 4D_6$

$WCS = (Y, 5)$
(3) $Y = 5A + 2B_1 + 3C_1 + C_2 + 5D_1 + 5D_2 + 5D_3 + 5D_4 + 5D_5$

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[5] This means that B_1, C_1 and D_4 are semi-rigid irreducible components. For genus 1, 2 and 3 case, the WCSs numbered by (i), (ii), (iii),... are equipped with information on semi-rigid irreducible components. (Otherwise we number them by (1), (2), (3),...) These WCSs have semi-rigid components "close" to the core, and this information is useful for applying Criterion 16.5.1, p293 to constellar singular fibers.

$WCS = (Y, 6)$
(4) $Y = 6A + 3B_1 + 3C_1 + 6D_1 + 6D_2 + 6D_3 + 6D_4$

$WCS = (Y, 7)$
(5) $Y = 7A + 3B_1 + 4C_1 + C_2 + 7D_1 + 7D_2 + 7D_3$

$WCS = (Y, 8)$
(6) $Y = 8A + 4B_1 + 4C_1 + 8D_1 + 8D_2$

$WCS = (Y, 9)$
(7) $Y = 9A + 4B_1 + 5C_1 + C_2 + 9D_1$

[2] 10.4 type, (-2)
$X = 10A + 5B_1 + 7C_1 + 4C_2 + C_3 + 8D_1 + 6D_2 + 4D_3 + 2D_4$

(i) $WCS = \sum_{i=1}^{4}(iY, i)$, semi-rigid: B_1, C_2, D_3
$Y = A + C_1 + C_2 + D_1 + D_2 + D_3$

$WCS = (Y, 1)$
(1) $Y = 4A + 2B_1 + 3C_1 + 2C_2 + C_3 + 3D_1 + 2D_2 + D_3$

$WCS = (Y_1, 1) + (Y_2, 2)$
(1) $Y_1 = A + B_1 + B_2 + B_3 + D_1 + D_2 + D_3 + D_4$
$Y_2 = 2A + B_1 + C_1 + 2D_1 + 2D_2 + 2D_3 + 2D_4$

$WCS = (Y_1, 1) + (Y_2, 2) + (2Y_1, 3)$
(2) $Y_1 = 2A + B_1 + 2C_1 + 2C_2 + D_1$,
$Y_2 = 3A + B_1 + 3C_1 + 3C_2 + 2D_1 + D_2$

Or apply SR technique for E_8-singularity.

Order 8 $r = (2, 8, 8)$

[2] 8.1 type, (-1)
$X = 8A + 4B_1 + C_1 + 3D_1 + D_2$

Apply exceptional curve criterion.

$WCS = (Y, 1)$
(1) $Y = A + C_1$
(2) $Y = 2A + C_1 + D_1$
(3) $Y = 2A + B_1 + D_1 + D_2$
(4) $Y = 2A + B_1 + C_1$
(5) $Y = 3A + B_1 + C_1 + D_1$
(6) $Y = 4A + 2B_1 + C_1 + D_1$
(7) $Y = 6A + B_1 + C_1 + 4D_1 + 2D_2$

[2] 8.2 type, (-2)
$X = 8A + 4B_1 + 5C_1 + 2C_2 + C_3 + 7D_1 + 6D_2 + 5D_3 + 4D_4 + 3D_5 + 2D_6 + D_7$

(i) $WCS = \sum_{i=1}^{5}(iY, i)$, semi-rigid: B_1, C_1, D_3
$Y = A + C_1 + D_1 + D_2 + D_3$

$WCS = (Y, 2)$
(1) $Y = 2A + B_1 + C_1 + 2D_1 + 2D_2 + 2D_3 + 2D_4 + 2D_5 + 2D_6$

$WCS = (Y, 3)$
(2) $Y = 3A + B_1 + 2C_1 + C_2 + C_3 + 3D_1 + 3D_2 + 3D_3 + 3D_4 + 3D_5$

$WCS = (Y, 4)$
(3) $Y = 4A + 2B_1 + 2C_1 + 4D_1 + 4D_2 + 4D_3 + 4D_4$

$WCS = (Y, 5)$
(4) $Y = 5A + 2B_1 + 3C_1 + C_2 + 5D_1 + 5D_2 + 5D_3$

$WCS = (Y, 6)$
(5) $Y = 6A + 3B_1 + 3C_1 + 6D_1 + 6D_2$

$WCS = (Y, 7)$
(6) $Y = 7A + 3B_1 + 4C_1 + C_2 + 7D_1$

Order 6 $\quad r = (3, 6, 6)$

[2] 6.1.1 type, (-1)
$X = 6A + B_1 + C_1 + 4D_1 + 2D_2$

Apply exceptional curve criterion.

$WCS = (Y, 1)$
(1) $Y = A + B_1$
(2) $Y = 2A + B_1 + C_1$
(3) $Y = 2A + B_1 + D_1$
(4) $Y = 3A + B_1 + 2D_1 + D_2$
(5) $Y = 4A + B_1 + C_1 + 2D_1$

[2] 6.1.2 type, (-2)
$X = 6A+2B_1+5C_1+4C_2+3C_3+2C_4+C_5+5D_1+4D_2+3D_3+2D_4+D_5$

(i) $WCS = \sum_{i=1}^{5}(iY, i)$, semi-rigid: B_1, C_1, D_1
$Y = A + C_1 + D_1$

$WCS = (Y, 1)$
(1) $Y = C_1 + C_2 + C_3 + C_4 + C_5 + D_1 + D_2 + D_3 + D_4 + D_5$

$WCS = (Y, 2)$
(2) $Y = 2A + 2C_1 + 2C_2 + 2C_3 + 2C_4 + 2D_1 + 2D_2 + 2D_3 + 2D_4$

$WCS = (Y, 3)$
(3) $Y = 3A + 3C_1 + 3C_2 + 3C_3 + 3D_1 + 3D_2 + 3D_3$
(4) $Y = 3A + B_1 + 2C_1 + C_2 + 3D_1 + 3D_2 + 3D_3$

$WCS = (Y, 4)$
(5) $Y = 4A + 4C_1 + 4C_2 + 4D_1 + 4D_2$
(6) $Y = 4A + B_1 + 3C_1 + 2C_2 + C_3 + 4D_1 + 4D_2$

$WCS = (Y, 5)$
(7) $Y = 5A + 5C_1 + 5D_1$
(8) $Y = 5A + B_1 + 4C_1 + 3C_2 + 2C_3 + C_4 + 5D_1$

Order 6 $r = (2, 2, 3, 3)$

[2] 6.2.1 type, (-2)
$X = 6A + 2B_1 + 3C_1 + 3D_1 + 4E_1 + 2E_2$

(i) $WCS = \sum_{i=1}^{3}(iY, i)$, semi-rigid: B_1, C_1, D_1, E_1
$Y = A + C_1 + D_1$

$WCS = (Y, 1) + (2Y, 2)$
(1) $Y = 2A + B_1 + C_1 + D_1 + E_1$

Order 5 $r = (5, 5, 5)$

[2] 5.1 type, (-1)
$X = 5A + B_1 + C_1 + 3D_1 + D_2$

Apply exceptional curve criterion.

$WCS = (Y, 1)$
(1) $Y = A + B_1$
(2) $Y = A + D_1 + D_2$
(3) $Y = 2A + B_1 + C_1$
(4) $Y = 2A + B_1 + D_1$
(5) $Y = 3A + B_1 + C_1 + D_1$
(6) $Y = 4A + B_1 + C_1 + 2D_1$

Or apply SR technique for A_1-singularity after contracting A.

[2] 5.2 type, (-1)
$X = 5A + B_1 + 2C_1 + C_2 + 2D_1 + D_2$

Apply exceptional curve criterion.

$WCS = (Y, 1)$
(1) $Y = A + B_1$
(2) $Y = 2A + B_1 + C_1 + C_2$
(3) $Y = 2A + C_1 + C_2 + D_1 + D_2$
(4) $Y = 3A + B_1 + C_1 + D_1$

$WCS = (Y, 2)$
(5) $Y = 4A + 2C_1 + 2D_1$
(6) $Y = 4A + B_1 + C_1 + 2D_1$

Or apply SR technique for a cyclic quotient singularity after contracting A.

[2] 5.3 type, (-2)
$X = 5A + 2B_1 + B_2 + 4C_1 + 3C_2 + 2C_3 + C_4 + 4D_1 + 3D_2 + 2D_3 + D_4$

(i) $WCS = \sum_{i=1}^{4}(iY, i)$, semi-rigid: B_1, C_1, D_1
$Y = A + C_1 + D_1$

$WCS = (Y, 1)$
(1) $Y = A + C_1 + C_2 + C_3 + C_4 + D_1 + D_2 + D_3 + D_4$

$WCS = (Y, 2)$
(2) $Y = 2A + 2C_1 + 2C_2 + 2C_3 + 2D_1 + 2D_2 + 2D_3$
(3) $Y = 2A + B_1 + B_2 + C_1 + 2D_1 + 2D_2 + 2D_3$

$WCS = (Y, 3)$
(4) $Y = 3A + 3C_1 + 3C_2 + 3D_1 + 3D_2$
(5) $Y = 3A + B_1 + 2C_1 + C_2 + 3D_1 + 3D_2$

$WCS = (Y, 4)$
(6) $Y = 4A + 4C_1 + 4D_1$
(7) $Y = 4A + B_1 + 3C_1 + 2C_2 + C_3 + 4D_1$

[2] 5.4 type, (-2)
$X = 5A + 3B_1 + B_2 + 3C_1 + C_2 + 4D_1 + 3D_2 + 2D_3 + D_4$

(i) $WCS = \sum_{i=1}^{3}(iY, i)$, semi-rigid: B_1, C_1, D_1
$Y = A + B_1 + C_1$

$WCS = (Y, 1)$
(1) $Y = A + B_1 + B_2 + C_1 + C_2$
(2) $Y = A + C_1 + C_2 + D_1 + D_2 + D_3 + D_4$
(3) $Y = 3A + 2B_1 + B_2 + 2C_1 + C_2 + 2D_1 + D_2$

$WCS = (Y, 2)$
(4) $Y = 2A + B_1 + C_1 + 2D_1 + 2D_2 + 2D_3$

$WCS = (Y, 3)$
(5) $Y = 3A + 2B_1 + B_2 + C_1 + 3D_1 + 3D_2$

$WCS = (Y, 4)$
(6) $Y = 4A + 2B_1 + 2C_1 + 4D_1$

Or apply SR technique for D_6-singularity.

Order 4 $r = (2, 2, 4, 4)$

[2] 4.1 type, (-2)
$X = 4A + B_1 + 2C_1 + 2D_1 + 3E_1 + 2E_2 + E_3$

(i) $WCS = \sum_{i=1}^{2}(iY, i)$, semi-rigid: B_1, C_1, D_1, E_1
$Y = A + C_1 + D_1$

(ii) $WCS = \sum_{i=1}^{2}(iY, i)$, semi-rigid: B_1, C_1, D_1, E_2
$Y = A + D_1 + E_1 + E_2$

$WCS = (Y, 1)$
(1) $Y = A + B_1 + E_1 + E_2 + E_3$
(2) $Y = 2A + B_1 + C_1 + D_1 + E_1$

$WCS = (Y, 2)$
(3) $Y = 2A + B_1 + C_1 + 2E_1 + 2E_2$
(4) $Y = 2A + C_1 + D_1 + 2E_1 + 2E_2$

$WCS = (Y, 3)$
(5) $Y = 3A + B_1 + C_1 + D_1 + 3E_1$

Or apply SR technique for D_5-singularity.

Order 3 $r = (3, 3, 3, 3)$

[2] 3.1 type, (-2)
$X = 3A + B_1 + C_1 + 2D_1 + D_2 + 2E_1 + E_2$

(i) $WCS = \sum_{i=1}^{2}(iY, i)$, semi-rigid: B_1, C_1, D_1, E_1
$Y = A + D_1 + E_1$

$WCS = (Y, 1)$
(1) $Y = A + B_1 + C_1$
(2) $Y = A + B_1 + D_1 + D_2$
(3) $Y = A + D_1 + D_2 + E_1 + E_2$
(4) $Y = 2A + B_1 + C_1 + D_1 + E_1$

$WCS = (Y, 2)$
(5) $Y = 2A + B_1 + C_1 + 2D_1$
(6) $Y = 2A + B_1 + D_1 + 2E_1$
(7) $Y = 2A + 2D_1 + 2E_1$

Or apply SR technique for A_3-singularity.

Order 2 $r = (2, 2, 2, 2, 2, 2)$

[2] 2.1 type, (-3)
$X = 2A + B_1 + C_1 + D_1 + E_1 + F_1 + G_1$

$WCS = (Y, 1)$
(1) $Y = A + B_1 + C_1 + D_1$

Or apply SR technique for a cyclic quotient singularity.

26.2.2 Stellar singular fibers, genus$(A) = 1$

[2] A1.1 type, (-1)
$X = 2A + B_1 + C_1$, $N_A^{\otimes 2} = \mathcal{O}(-b_1 - c_1)$

$WCS = (Y, 1)$
(1) $Y = A + B_1 + C_1, \quad N_A = \mathcal{O}(-b_1 - c_1 + q)$,
where $b_1 + c_1 \sim 2q$ (linearly equivalent), i.e. q is a Weierstrass point of A.

26.2.3 Self-welding of stellar singular fibers of genus 1

[2] sw(III^*) type
$X = 4A + 2B_1 + 3C_1 + 2C_2 + 3D_1 + 2D_2, \quad C_2 = D_2$

$WCS = (Y, 1) + (2Y, 2)$
(1) $Y = A + C_1 + C_2 + D_1 + D_2$

[2] sw(IV^*) type
$X = 3A + 2B_1 + 2C_1 + 2D_1 + D_2, \quad B_1 = C_1$

$WCS = (Y, 2)$
(1) $Y = 2A + B_1 + C_1 + 2D_1$

$WCS = (Y, 1) + (2Y, 2)$
(2) $Y = A + B_1 + C_1$

26.3 Genus 3

26.3.1 Stellar singular fibers, $A = \mathbb{P}^1$

Order 14 $\quad r = (2, 7, 14)$

[3] 14.1 type, (-1)
$X = 14A + 7B_1 + C_1 + 6D_1 + 4D_2 + 2D_3$

Apply exceptional curve criterion.

$WCS = (Y, 1)$
(1) $Y = A + C_1$
(2) $Y = 2A + B_1 + C_1$
(3) $Y = 3A + B_1 + C_1 + D_1$
(4) $Y = 4A + 2B_1 + C_1 + D_1$
(5) $Y = 5A + 2B_1 + C_1 + 2D_1 + D_2$
(6) $Y = 6A + 3B_1 + C_1 + 2D_1$

Or apply SR technique for a cyclic quotient singularity after contracting A.

[3] 14.2 type, (-1)
$X = 14A + 7B_1 + 2C_1 + 5D_1 + D_2$

Apply exceptional curve criterion.

$WCS = (Y, 1)$
(1) $Y = A + B_1 + D_1 + D_2$
(2) $Y = 8A + 4B_1 + C_1 + 3D_1 + D_2$

[3] 14.3 type,(-1)
$X = 14A + 7B_1 + 3C_1 + C_2 + 4D_1 + 2D_2$

Apply exceptional curve criterion.

$WCS = (Y, 1)$
(1) $Y = 4A + 2B_1 + C_1 + C_2 + D_1$

[3] 14.4 type, (-2)
$X = 14A + 7B_1 + 8C_1 + 2C_2 + 13D_1 + 12D_2 + 11D_3 + 10D_4 + 9D_5 + 8D_6 + 7D_7 + 6D_8 + 5D_9 + 4D_{10} + 3D_{11} + 2D_{12} + D_{13}$

(i) $WCS = \sum_{i=1}^{8}(iY, i)$, semi-rigid: B_1, C_1, D_6
$Y = A + C_1 + D_1 + D_2 + D_3 + D_4 + D_5 + D_6$

$WCS = (Y, 2)$
(1) $Y = 2A + B_1 + C_1 + 2D_1 + 2D_2 + 2D_3 + 2D_4 + 2D_5 + 2D_6 + 2D_7 + 2D_8 + 2D_9 + 2D_{10} + 2D_{11} + 2D_{12}$

$WCS = (Y, 4)$
(2) $Y = 4A + 2B_1 + 2C_1 + 4D_1 + 4D_2 + 4D_3 + 4D_4 + 4D_5 + 4D_6 + 4D_7 + 4D_8 + 4D_9 + 4D_{10}$

$WCS = (Y, 6)$
(3) $Y = 6A + 3B_1 + 3C_1 + 6D_1 + 6D_2 + 6D_3 + 6D_4 + 6D_5 + 6D_6 + 6D_7 + 6D_8$

$WCS = (Y, 8)$
(4) $Y = 8A + 4B_1 + 4C_1 + 8D_1 + 8D_2 + 8D_3 + 8D_4 + 8D_5 + 8D_6$

$WCS = (Y, 10)$
(5) $Y = 10A + 5B_1 + 5C_1 + 10D_1 + 10D_2 + 10D_3 + 10D_4$

$WCS = (Y, 12)$
(6) $Y = 12A + 6B_1 + 6C_1 + 12D_1 + 12D_2$

$WCS = (Y, 7)$
(7) $Y = 7A + 3B_1 + 4C_1 + C_2 + 7D_1 + 7D_2 + 7D_3 + 7D_4 + 7D_5 + 7D_6 + 7D_7$

$WCS = (Y, 9)$
(8) $Y = 9A + 4B_1 + 5C_1 + C_2 + 9D_1 + 9D_2 + 9D_3 + 9D_4 + 9D_5$

$WCS = (Y, 11)$
(9) $Y = 11A + 5B_1 + 6C_1 + C_2 + 11D_1 + 11D_2 + 11D_3$

$WCS = (Y, 13)$
(10) $Y = 13A + 6B_1 + 7C_1 + C_2 + 13D_1$

[3] 14.5 type, (-2)
$X = 14A + 7B_1 + 9C_1 + 4C_2 + 3C_3 + 2C_4 + C_5 + 12D_1 + 10D_2 + 8D_3 + 6D_4 + 4D_5 + 2D_6$

(i) $WCS = \sum_{i=1}^{4}(iY, i)$, semi-rigid: B_1, C_2, D_5
$Y = A + C_1 + C_2 + D_1 + D_2 + D_3 + D_4 + D_5$

$WCS = (Y, 2)$
(1) $Y = 6A + 3B_1 + 4C_1 + 2C_2 + 2C_3 + 2C_4 + 5D_1 + 4D_3 + 3D_4 + 2D_5 + D_6$

$WCS = (Y, 4)$
(2) $Y = 12A + 6B_1 + 8C_1 + 4C_2 + 10D_1 + 8D_2 + 6D_3 + 4D_4 + 2D_5$

[3] 14.6 type, (-2)
$X = 14A + 7B_1 + 10C_1 + 6C_2 + 2C_3 + 11D_1 + 8D_2 + 5D_3 + 2D_4 + D_5$

(i) $WCS = (Y, 1) + (2Y, 2)$, semi-rigid: B_1, C_3, D_4
$Y = A + C_1 + C_2 + C_3 + D_1 + D_2 + D_3 + D_4$

$WCS = (Y, 1) + (2Y, 2)$
(1) $Y = 4A + 2B_1 + 3C_1 + 2C_2 + C_3 + 3D_1 + 2D_2 + D_3$

Order 12 $r = (6, 12, 12)$

[3] 12.1 type, (-1)
$X = 12A + B_1 + 3C_1 + 8D_1 + 4D_2$

Apply exceptional curve criterion.

$WCS = (Y, 1)$
(1) $Y = A + B_1$
(2) $Y = 2A + B_1 + D_1$
(3) $Y = 4A + B_1 + C_1 + 2D_1$
(4) $Y = 5A + B_1 + C_1 + 3D_1 + D_2$
(5) $Y = 6A + B_1 + C_1 + 4D_1 + 2D_2$

Or apply SR technique for a cyclic quotient singularity after contracting A.

[3] 12.2 type, (-1)
$X = 12A + B_1 + 6C_1 + 5D_1 + 3D_2 + D_3$

Apply exceptional curve criterion.

$WCS = (Y, 1)$
(1) $Y = A + B_1$
(2) $Y = 2A + B_1 + C_1$
(3) $Y = 2A + B_1 + D_1 + D_2 + D_3$
(4) $Y = 2A + C_1 + D_1 + D_2 + D_3$
(5) $Y = 3A + B_1 + C_1 + D_1$
(6) $Y = 4A + B_1 + 2C_1 + D_1$
(7) $Y = 6A + B_1 + 3C_1 + 2D_1$

Or apply SR technique for a cyclic quotient singularity after contracting A.

[3] 12.3 type, (-1)
$X = 12A + 3B_1 + 4C_1 + 5D_1 + 3D_2 + D_3$

Apply exceptional curve criterion.

$WCS = (Y, 1) + (2Y, 2) + (3Y, 3)$
(1) $Y = 3A + B_1 + C_1 + D_1$

[3] 12.4 type, (-2)
$X = 12A + 6B_1 + 7C_1 + 2C_2 + C_3 + 11D_1 + 10D_2 + 9D_3 + 8D_4 + 7D_5 + 6D_6 + 5D_7 + 4D_8 + 3D_9 + 2D_{10} + D_{11}$

(i) $WCS = \sum_{i=1}^{7}(iY, i)$, semi-rigid: B_1, C_1, D_5
$Y = A + C_1 + D_1 + D_2 + D_3 + D_4 + D_5$

$WCS = (Y, 2)$
(1) $Y = 2A + B_1 + C_1 + 2D_1 + 2D_2 + 2D_3 + 2D_4 + 2D_5 + 2D_6 + 2D_7 + 2D_8 + 2D_9 + 2D_{10}$

$WCS = (Y, 4)$
(2) $Y = 4A + 2B_1 + 2C_1 + 4D_1 + 4D_2 + 4D_3 + 4D_4 + 4D_5 + 4D_6 + 4D_7 + 4D_8$

$WCS = (Y, 6)$
(3) $Y = 6A + 3B_1 + 3C_1 + 6D_1 + 6D_2 + 6D_3 + 6D_4 + 6D_5 + 6D_6$

$WCS = (Y, 8)$
(4) $Y = 8A + 4B_1 + 4C_1 + 8D_1 + 8D_2 + 8D_3 + 8D_4$

$WCS = (Y, 10)$
(5) $Y = 10A + 5B_1 + 5C_1 + 10D_1 + 10D_2$

$WCS = (Y, 5)$
(6) $Y = 5A + 2B_1 + 3C_1 + C_2 + C_3 + 5D_1 + 5D_2 + 5D_3 + 5D_4 + 5D_5 + 5D_6 + 5D_7$

$WCS = (Y, 7)$
(7) $Y = 7A + 3B_1 + 4C_1 + C_2 + 7D_1 + 7D_2 + 7D_3 + 7D_4 + 7D_5$

$WCS = (Y, 9)$
(8) $Y = 9A + 4B_1 + 5C_1 + C_2 + 9D_1 + 9D_2 + 9D_3$

$WCS = (Y, 11)$
(9) $Y = 11A + 5B_1 + 6C_1 + C_2 + 11D_1$

[3] 12.5 type, (-2)
$X = 12A + 4B_1 + 9C_1 + 6C_2 + 3C_3 + 11D_1 + 10D_2 + 9D_3 + 8D_4 + 7D_5 + 6D_6 + 5D_7 + 4D_8 + 3D_9 + 2D_{10} + D_{11}$

(i) $WCS = \sum_{i=1}^{9}(iY, i)$, semi-rigid: B_1, C_1, D_3
$Y = A + C_1 + D_1 + D_2 + D_3$

$WCS = (Y, 3)$
(1) $Y = 3A + B_1 + 2C_1 + C_2 + 3D_1 + 3D_2 + 3D_3 + 3D_4 + 3D_5 + 3D_6 + 3D_7 + 3D_8 + 3D_9$

$WCS = (Y, 6)$
(2) $Y = 6A + 2B_1 + 4C_1 + 2C_2 + 6D_1 + 6D_2 + 6D_3 + 6D_4 + 6D_5 + 6D_6$

$WCS = (Y, 9)$
(3) $Y = 9A + 3B_1 + 6C_1 + 3C_2 + 9D_1 + 9D_2 + 9D_3$

$WCS = (Y, 4)$
(4) $Y = 4A + B_1 + 3C_1 + 2C_2 + C_3 + 4D_1 + 4D_2 + 4D_3 + 4D_4 + 4D_5 + 4D_6 + 4D_7 + 4D_8$

$WCS = (Y, 7)$
(5) $Y = 7A + 2B_1 + 5C_1 + 3C_2 + C_3 + 7D_1 + 7D_2 + 7D_3 + 7D_4 + 7D_5$

[3] 12.6 type, (-2)
$X = 12A + 8B_1 + 4B_2 + 7C_1 + 2C_2 + C_3 + 9D_1 + 6D_2 + 3D_3$

(i) $WCS = (Y, 1) + (2Y, 2)$, semi-rigid: C_2, D_3
$Y = 3A + 2B_1 + B_2 + 2C_1 + C_2 + 2D_1 + D_2$

Order 9 $r = (3, 9, 9)$

[3] 9.1 type, (-1)
$X = 9A + B_1 + 3C_1 + 5D_1 + D_2$

Apply exceptional curve criterion.

$WCS = (Y, 1)$
(1) $Y = A + B_1$
(2) $Y = 2A + B_1 + D_1$
(3) $Y = 3A + B_1 + C_1 + D_1$
(4) $Y = 4A + B_1 + C_1 + 2D_1$
(5) $Y = 5A + B_1 + C_1 + 3D_1 + D_2$
(6) $Y = 6A + B_1 + 2C_1 + 3D_1$

Or apply SR technique for a cyclic quotient singularity after contracting A.

[3] 9.2 type, (-1)
$X = 9A + B_1 + 2C_1 + C_2 + 6D_1 + 3D_2$

Apply exceptional curve criterion.

$WCS = (Y, 1)$
(1) $Y = A + B_1$
(2) $Y = 2A + B_1 + D_1$

(3) $Y = 4A + B_1 + C_1 + C_2 + 2D_1$
(4) $Y = 5A + B_1 + C_1 + 3D_1 + D_2$
(5) $Y = 6A + B_1 + C_1 + 4D_1 + 2D_2$

Or apply SR technique for a cyclic quotient singularity after contracting A.

[3] 9.3 type, (-1)
$X = 9A + 3B_1 + 2C_1 + C_2 + 4D_1 + 3D_2 + 2D_3 + D_4$

Apply exceptional curve criterion.

$WCS = (Y, 2)$
(1) $Y = 4A + B_1 + C_1 + C_2 + 2D_1 + 2D_2 + 2D_3$

$WCS = (Y, 3)$
(2) $Y = 6A + 2B_1 + C_1 + 3D_1 + 3D_2$

[3] 9.4 type, (-2)
$X = 9A + 3B_1 + 7C_1 + 5C_2 + 3C_3 + C_4 + 8D_1 + 7D_2 + 6D_3 + 5D_4 + 4D_5 + 3D_6 + 2D_7 + D_8$

(i) $WCS = \sum_{i=1}^{7}(iY, i)$, semi-rigid: B_1, C_1, D_2
$Y = A + C_1 + D_1 + D_2$

$WCS = (Y, 1)$
(1) $Y = A + C_1 + C_2 + C_3 + C_4 + D_1 + D_2 + D_3 + D_4 + D_5 + D_6 + D_7 + D_8$

$WCS = (Y, 3)$
(2) $Y = 3A + B_1 + 2C_1 + C_2 + 3D_1 + 3D_2 + 3D_3 + 3D_4 + 3D_5 + 3D_6$

$WCS = (Y, 4)$
(3) $Y = 4A + B_1 + 3C_1 + 2C_2 + C_3 + 4D_1 + 4D_2 + 4D_3 + 4D_4 + 4D_5$

$WCS = (Y, 5)$
(4) $Y = 5A + B_1 + 4C_1 + 3C_2 + 2C_3 + C_4 + 5D_1 + 5D_2 + 5D_3 + 5D_4$

$WCS = (Y, 6)$
(5) $Y = 6A + 2B_1 + 4C_1 + 2C_2 + 6D_1 + 6D_2 + 6D_3$

$WCS = (Y, 7)$
(6) $Y = 7A + 2B_1 + 5C_1 + 3C_2 + C_3 + 7D_1 + 7D_2$

$WCS = (Y, 8)$
(7) $Y = 8A + 2B_1 + 6C_1 + 4C_2 + 2C_3 + 8D_1$

[3] 9.5 type, (-2)
$X = 9A + 6B_1 + 3B_2 + 4C_1 + 3C_2 + 2C_3 + C_4 + 8D_1 + 7D_2 + 6D_3 + 5D_4 + 4D_5 + 3D_6 + 2D_7 + D_8$

(i) $WCS = \sum_{i=1}^{6}(iY, i)$, semi-rigid: B_1, C_1, D_3
$Y = A + B_1 + D_1 + D_2 + D_3$

$WCS = (Y, 2)$
(1) $Y = 2A + B_1 + C_1 + C_2 + C_3 + C_4 + 2D_1 + 2D_2 + 2D_3 + 2D_4 + 2D_5 + 2D_6 + 2D_7$

$WCS = (Y, 3)$
(2) $Y = 3A + 2B_1 + B_2 + C_1 + 3D_1 + 3D_2 + 3D_3 + 3D_4 + 3D_5 + 3D_6$

$WCS = (Y, 5)$
(3) $Y = 5A + 3B_1 + B_2 + 2C_1 + C_2 + 5D_1 + 5D_2 + 5D_3 + 5D_4$

$WCS = (Y, 6)$
(4) $Y = 6A + 4B_1 + 2B_2 + 2C_1 + 6D_1 + 6D_2 + 6D_3$

$WCS = (Y, 7)$
(5) $Y = 7A + 4B_1 + B_2 + 3C_1 + 2C_2 + C_3 + 7D_1 + 7D_2$

[3] 9.6 type, (-2)
$X = 9A + 5B_1 + B_2 + 6C_1 + 3C_2 + 7D_1 + 5D_2 + 3D_3 + D_4$

(i) $WCS = \sum_{i=1}^{5}(iY, i)$, semi-rigid: B_1, C_1, D_2
$Y = A + B_1 + D_1 + D_2$

$WCS = (Y, 1)$
(1) $Y = A + B_1 + D_1 + D_2 + D_3 + D_4$
(2) $Y = 3A + 2B_1 + B_2 + 2C_1 + C_2 + 2D_1 + D_2$
(3) $Y = 5A + 3B_1 + B_2 + 3C_1 + C_2 + 4D_1 + 3D_2 + 2D_3 + D_4$

Or apply SR technique for E_7-singularity.

Order 8 $r = (4, 8, 8)$

[3] 8.1 type, (-1)
$X = 8A + B_1 + C_1 + 6D_1 + 4D_2 + 2D_3$

Apply exceptional curve criterion.

$WCS = (Y, 1)$
(1) $Y = A + B_1$
(2) $Y = 2A + B_1 + C_1$
(3) $Y = 2A + B_1 + D_1$
(4) $Y = 3A + B_1 + C_1 + D_1$
(5) $Y = 4A + B_1 + C_1 + 2D_1$
(6) $Y = 5A + B_1 + C_1 + 3D_1 + D_2$
(7) $Y = 6A + B_1 + C_1 + 4D_1 + 2D_2$

Or apply SR technique for a cyclic quotient singularity after contracting A.

[3] 8.2 type, (-1)
$X = 8A + B_1 + 2C_1 + 5D_1 + 2D_2 + D_3$

Apply exceptional curve criterion.

$WCS = (Y, 1)$
(1) $Y = A + B_1$
(2) $Y = 2A + B_1 + D_1$
(3) $Y = 3A + B_1 + 2D_1 + D_2 + D_3$
(4) $Y = 4A + B_1 + C_1 + 2D_1$
(5) $Y = 5A + B_1 + C_1 + 3D_1 + D_2$

$WCS = (Y, 2)$
(6) $Y = 6A + B_1 + C_1 + 4D_1 + 2D_2$

Or apply SR technique for a cyclic quotient singularity after contracting A.

[**3**] **8.3 type,** (-1)
$X = 8A + 2B_1 + 3C_1 + C_2 + 3D_1 + D_2$

Apply exceptional curve criterion.

$WCS = (Y, 1)$
(1) $Y = 2A + C_1 + C_2 + D_1 + D_2$
(2) $Y = 5A + B_1 + 2C_1 + C_2 + 2D_1 + D_2$

$WCS = (Y_1, 2) + (Y_2, 3)$
(3) $Y_1 = 4A + B_1 + C_1 + 2D_1,$
$Y_2 = 6A + B_1 + 2C_1 + 3D_1$

[**3**] **8.4 type,** (-2)
$X = 8A + 2B_1 + 7C_1 + 6C_2 + 5C_3 + 4C_4 + 3C_5 + 2C_6 + C_7 + 7D_1 + 6D_2 + 5D_3 + 4D_4 + 3D_5 + 2D_6 + D_7$

(i) $WCS = \sum_{i=1}^{7}(iY, i),$ semi-rigid: B_1, C_1, D_1
$Y = A + C_1 + D_1$

$WCS = (Y, 1)$
(1) $Y = 2A + C_1 + C_2 + C_3 + C_4 + C_5 + C_6 + C_7 + D_1 + D_2 + D_3 + D_4 + D_5 + D_6 + D_7$

$WCS = (Y, 2)$
(2) $Y = 2A + 2C_1 + 2C_2 + 2C_3 + 2C_4 + 2C_5 + 2C_6 + 2D_1 + 2D_2 + 2D_3 + 2D_4 + 2D_5 + 2D_6$

$WCS = (Y, 3)$
(3) $Y = 3A + 3C_1 + 3C_2 + 3C_3 + 3C_4 + 3C_5 + 3D_1 + 3D_2 + 3D_3 + 3D_4 + 3D_5$

$WCS = (Y, 4)$
(4) $Y = 4A + 4C_1 + 4C_2 + 4C_3 + 4C_4 + 4D_1 + 4D_2 + 4D_3 + 4D_4$

$WCS = (Y, 5)$
(5) $Y = 5A + 5C_1 + 5C_2 + 5C_3 + 5D_1 + 5D_2 + 5D_3$

$WCS = (Y, 6)$
(6) $Y = 6A + 6C_1 + 6C_2 + 6D_1 + 6D_2$

$WCS = (Y, 7)$
(7) $Y = 7A + 7C_1 + 7D_1$

$WCS = (Y, 4)$
(8) $Y = 4A + B_1 + 3C_1 + 2C_2 + C_3 + 4D_1 + 4D_2 + 4D_3 + 4D_4$

$WCS = (Y, 5)$
(9) $Y = 5A + B_1 + 4C_1 + 3C_2 + 2C_3 + C_4 + 5D_1 + 5D_2 + 5D_3$

$WCS = (Y, 6)$
(10) $Y = 6A + B_1 + 5C_1 + 4C_2 + 3D_3 + 2D_4 + D_5 + 6D_1 + 6D_2$

$WCS = (Y, 7)$
(11) $Y = 7A + B_1 + 6C_1 + 5C_2 + 4C_3 + 3C_4 + 2C_5 + C_6 + 7D_1$

[3] 8.5 type, (-2)
$X = 8A + 3B_1 + B_2 + 6C_1 + 4C_2 + 2C_3 + 7D_1 + 6D_2 + 5D_3 + 4D_4 + 3D_5 + 2D_6 + D_7$

(i) $WCS = \sum_{i=1}^{6}(iY, i)$, semi-rigid: B_1, C_1, D_2
$Y = A + C_1 + D_1 + D_2$

$WCS = (Y, 2)$
(1) $Y = 2A + B_1 + B_2 + C_1 + 2D_1 + 2D_2 + 2D_3 + 2D_4 + 2D_5 + 2D_6$

$WCS = (Y, 3)$
(2) $Y = 3A + B_1 + 2C_1 + C_2 + 3D_1 + 3D_2 + 3D_3 + 3D_4 + 3D_5$

$WCS = (Y, 4)$
(3) $Y = 4A + B_1 + 3C_1 + 2C_2 + C_3 + 4D_1 + 4D_2 + 4D_3 + 4D_4$

$WCS = (Y, 5)$
(4) $Y = 5A + 2B_1 + B_2 + 3C_1 + C_2 + 5D_1 + 5D_2 + 5D_3$

$WCS = (Y, 6)$
(5) $Y = 6A + 2B_1 + 4C_1 + 2C_2 + 6D_1 + 6D_2$

[3] 8.6 type, (-2)
$X = 8A + 5B_1 + 2B_2 + B_3 + 5C_1 + 2C_2 + C_3 + 6D_1 + 4D_2 + 2D_3$

(i) $WCS = \sum_{i=1}^{5}(iY, i)$, semi-rigid: B_1, C_1, D_1
$Y = A + B_1 + C_1$

$WCS = (Y, 1)$
(1) $Y = 3A + 2B_1 + B_2 + B_3 + 2C_1 + C_2 + C_3 + 2D_1 + D_2$

$WCS = (Y, 2)$
(2) $Y = 6A + 4B_1 + 2B_2 + 4C_1 + 2C_2 + 4D_1 + 2D_2$

Order 7 $r = (7, 7, 7)$

[3] 7.1 type, (-1)
$X = 7A + B_1 + C_1 + 5D_1 + 3D_2 + D_3$

Apply exceptional curve criterion.

$WCS = (Y, 1)$
(1) $Y = A + B_1$
(2) $Y = 2A + B_1 + C_1$

(3) $Y = 2A + B_1 + D_1$
(4) $Y = 3A + B_1 + C_1 + D_1$
(5) $Y = 4A + B_1 + C_1 + 2D_1$
(6) $Y = 5A + B_1 + C_1 + 3D_1 + D_2$
(7) $Y = 6A + B_1 + C_1 + 4D_1 + 2D_2$

Or apply SR technique for A_2-singularity after contracting A.

[3] 7.2 type, (-1)
$X = 7A + B_1 + 2C_1 + C_2 + 4D_2 + D_3$

Apply exceptional curve criterion.

$WCS = (Y, 1)$
(1) $Y = A + B_1$
(2) $Y = 2A + B_1 + D_1$
(3) $Y = 3A + B_1 + C_1 + C_2 + D_1$
(4) $Y = 4A + B_1 + C_1 + 2D_1$
(5) $Y = 5A + B_1 + C_1 + 3D_1 + D_2$

Or apply SR technique for a cyclic quotient singularity after contracting A.

[3] 7.3 type, (-1)
$X = 7A + B_1 + 3C_1 + 2C_2 + C_3 + 3D_2 + 2D_2 + D_3$

Apply exceptional curve criterion.

$WCS = (Y, 1)$
(1) $Y = A + B_1$
(2) $Y = 2A + B_1 + C_1 + C_2 + C_3$
(3) $Y = 2A + C_1 + C_2 + C_3 + D_1 + D_2 + D_3$
(4) $Y = 3A + B_1 + C_1 + D_1$
(5) $Y = 5A + B_1 + 2C_1 + C_2 + 2D_1 + D_2$

$WCS = (Y, 2)$
(6) $Y = 4A + 2C_1 + 2C_2 + 2D_1 + 2D_2$
(7) $Y = 4A + B_1 + C_1 + 2D_1 + 2D_2$

$WCS = (Y, 3)$
(8) $Y = 6A + 3C_1 + 3D_1$
(9) $Y = 6A + B_1 + 2C_1 + 3D_1$

Or apply SR technique for a cyclic quotient singularity after contracting A.

[3] 7.4 type, (-1)
$X = 7A + 2B_1 + B_2 + 2C_1 + C_2 + 3D_1 + 2D_2 + D_3$

Apply exceptional curve criterion.

$WCS = (Y, 1)$
(1) $Y = 3A + B_1 + B_2 + C_1 + C_2 + D_1$

$WCS = (Y, 2)$
(2) $Y = 4A + B_1 + C_1 + 2D_1 + 2D_2$
(3) $Y = 6A + 2B_1 + 2C_1 + 2D_1$

[3] 7.5 type, (-2)
$X = 7A + 2B_1 + B_2 + 6C_1 + 5C_2 + 4C_3 + 3C_4 + 2C_5 + C_6 + 6D_1 + 5D_2 + 4D_3 + 3D_4 + 2D_5 + D_6$

(i) $WCS = \sum_{i=1}^{6}(iY, i)$, semi-rigid: B_1, C_1, D_1
$Y = A + C_1 + D_1$

$WCS = (Y, 1)$
(1) $Y = A + C_1 + C_2 + C_3 + C_4 + C_5 + C_6 + D_1 + D_2 + D_3 + D_4 + D_5 + D_6$

$WCS = (Y, 2)$
(2) $Y = 2A + 2C_1 + 2C_2 + 2C_3 + 2C_4 + 2C_5 + 2D_1 + 2D_2 + 2D_3 + 2D_4 + 2D_5$

$WCS = (Y, 3)$
(3) $Y = 3A + 3C_1 + 3C_2 + 3C_3 + 3C_4 + 3D_1 + 3D_2 + 3D_3 + 3D_4$

$WCS = (Y, 4)$
(4) $Y = 4A + 4C_1 + 4C_2 + 4C_3 + 4D_1 + 4D_2 + 4D_3$

$WCS = (Y, 5)$
(5) $Y = 5A + 5C_1 + 5C_2 + 5D_1 + 5D_2$

$WCS = (Y, 6)$
(6) $Y = 6A + 6C_1 + 6D_1$

$WCS = (Y, 3)$
(7) $Y = 3A + B_1 + B_2 + 2C_1 + C_2 + 3D_1 + 3D_2 + 3D_3 + 3D_4$

$WCS = (Y, 4)$
(8) $Y = 4A + B_1 + 3C_1 + 2C_2 + C_3 + 4D_1 + 4D_2 + 4D_3$

$WCS = (Y, 5)$
(9) $Y = 5A + B_1 + 4C_1 + 3C_2 + 2C_3 + C_4 + 5D_1 + 5D_2$

$WCS = (Y, 6)$
(10) $Y = 6A + B_1 + 5C_1 + 4C_2 + 3D_3 + 2D_4 + D_5 + 6D_1$

[3] 7.6 type, (-2)
$X = 7A + 3B_1 + 2B_2 + B_3 + 5C_1 + 3C_2 + C_3 + 6D_1 + 5D_2 + 4D_3 + 3D_4 + 2D_5 + D_6$

(i) $WCS = \sum_{i=1}^{5}(iY, i)$, semi-rigid: B_1, C_1, D_2
$Y = A + C_1 + D_1 + D_2$

(ii) $WCS = \sum_{i=1}^{3}(iY, i)$, semi-rigid: B_1, C_2, D_1
$Y = A + B_1 + C_1 + C_2$

$WCS = (Y, 1)$
(1) $Y = 2A + C_1 + C_2 + C_3 + D_1 + D_2 + D_3 + D_4 + D_5 + D_6$

$WCS = (Y, 2)$
(2) $Y = 2A + B_1 + B_2 + B_3 + C_1 + 2D_1 + 2D_2 + 2D_3 + 2D_4 + 2D_5$
(3) $Y = 4A + 2B_1 + 2B_2 + 3C_1 + 2C_2 + C_3 + 3D_1 + 2D_2 + D_3$

$WCS = (Y, 3)$
(4) $Y = 3A + B_1 + 2C_1 + C_2 + 3D_1 + 3D_2 + 3D_3 + 3D_4$
(5) $Y = 6A + 3B_1 + 4C_1 + 2C_2 + 5D_1 + 4D_2 + 3D_3 + 2D_4 + D_5$

$WCS = (Y, 4)$
(6) $Y = 4A + B_1 + 3C_1 + 2C_2 + C_3 + 4D_1 + 4D_2 + 4D_3$

$WCS = (Y, 5)$
(7) $Y = 5A + 2B_1 + B_2 + 3C_1 + C_2 + 5D_1 + 5D_2$

$WCS = (Y, 6)$
(8) $Y = 6A + 2B_1 + 4C_1 + 2C_2 + 6D_1$

[3] 7.7 type, (-2)
$X = 7A + 4B_1 + B_2 + 4C_1 + C_2 + 6D_1 + 5D_2 + 4D_3 + 3D_4 + 2D_5 + D_6$

(i) $WCS = \sum_{i=1}^{4}(iY, i)$, semi-rigid: B_1, C_1, D_1
$Y = A + B_1 + C_1$

$WCS = (Y, 1)$
(1) $Y = A + B_1 + B_2 + C_1 + C_2$
(2) $Y = A + B_1 + B_2 + D_1 + D_2 + D_3 + D_4 + D_5 + D_6$
(3) $Y = 3A + 2B_1 + B_2 + 2C_1 + C_2 + 2D_1 + D_2$
(4) $Y = 5A + 3B_1 + B_2 + 3C_1 + C_2 + 4D_1 + 3D_2 + 2D_3 + D_4$

$WCS = (Y, 2)$
(5) $Y = 2A + B_1 + C_1 + 2D_1 + 2D_2 + 2D_3 + 2D_4 + 2D_5$

$WCS = (Y, 3)$
(6) $Y = 3A + B_1 + 2C_1 + C_2 + 3D_1 + 3D_2 + 3D_3 + 3D_4$

$WCS = (Y, 4)$
(7) $Y = 4A + 2B_1 + 2C_1 + 4D_1 + 4D_2 + 4D_3$

$WCS = (Y, 5)$
(8) $Y = 5A + 2B_1 + 3C_1 + C_2 + 5D_1 + 5D_2$

$WCS = (Y, 6)$
(9) $Y = 6A + 3B_1 + 3C_1 + 6D_1$

Or apply SR technique for D_8-singularity.

[3] 7.8 type, (-2)
$X = 7A + 4B_1 + B_2 + 5C_1 + 3C_2 + C_3 + 5D_1 + 3D_2 + D_3$

(i) $WCS = \sum_{i=1}^{5}(iY, i)$, semi-rigid: B_1, C_1, D_1
$Y = A + C_1 + D_1$

$WCS = (Y, 1)$
(1) $Y = A + B_1 + B_2 + C_1 + C_2 + C_3$
(2) $Y = A + C_1 + C_2 + C_3 + D_1 + D_2 + D_3$
(3) $Y = 3A + 2B_1 + B_2 + 2C_1 + C_2 + 2D_1 + D_2$
(4) $Y = 4A + 2B_1 + 3C_1 + 2C_2 + C_3 + 3D_1 + 2D_2 + D_3$

Or apply SR technique for E_6-singularity.

Order 6 $r = (2, 2, 6, 6)$

[3] 6.1.1 type, (-2)
$X = 6A + B_1 + 3C_1 + 3D_1 + 5E_1 + 4E_2 + 3E_3 + 2E_4 + E_5$

(i) $WCS = \sum_{i=1}^{3}(iY, i)$, semi-rigid: B_1, C_1, D_1, E_1
$Y = A + C_1 + D_1$

$WCS = (Y, 1)$
(1) $Y = A + B_1 + E_1 + E_2 + E_3 + E_4 + E_5$
(2) $Y = 2A + B_1 + C_1 + D_1 + E_1$
(3) $Y = 4A + B_1 + 2C_1 + 2D_1 + 3E_1 + 2E_2 + E_3$

$WCS = (Y, 2)$
(4) $Y = 2A + B_1 + C_1 + 2E_1 + 2E_2 + 2E_3 + 2E_4$
(5) $Y = 2A + C_1 + D_1 + 2E_1 + 2E_2 + 2E_3 + 2E_4$

$WCS = (Y, 3)$
(6) $Y = 3A + B_1 + C_1 + D_1 + 3E_1 + 3E_2 + 3E_3$

$WCS = (Y, 4)$
(7) $Y = 4A + B_1 + C_1 + 2D_1 + 4E_1 + 4E_2$

Order 6 $r = (2, 3, 3, 6)$

[3] 6.2.1 type, (-2)
$X = 6A + 2B_1 + 2C_1 + 3D_1 + 5E_1 + 4E_2 + 3E_3 + 2E_4 + E_5$

(i) $WCS = \sum_{i=1}^{3}(iY, i)$, semi-rigid: B_1, C_1, D_1, E_3
$Y = A + D_1 + E_1 + E_2 + E_3$

(ii) $WCS = \sum_{i=1}^{2}(iY, i)$, semi-rigid: B_1, C_1, D_1, E_1
$Y = A + B_1 + C_1$

$WCS = (Y, 3)$
(1) $Y = 3A + B_1 + C_1 + D_1 + 3E_1 + 3E_2 + 3E_3$

$WCS = (Y, 4)$
(2) $Y = 4A + B_1 + C_1 + 2D_1 + 4E_1 + 4E_2$

$WCS = (Y, 1) + (2Y, 2)$
(3) $Y = 2A_1 + B_1 + C_1 + D_1 + E_1$

[3] 6.2.2 type, (-2)
$X = 6A + B_1 + 3C_1 + 4D_1 + 2D_2 + 4E_1 + 2E_2$

(i) $WCS = \sum_{i=1}^{4}(iY, i)$, semi-rigid: B_1, C_1, D_1, E_1
$Y = A + D_1 + E_1$

$WCS = (Y, 1)$
(1) $Y = 2A + B_1 + C_1 + D_1 + E_1$
(2) $Y = 3A + B_1 + C_1 + 2D_1 + D_2 + 2E_1 + E_2$

Or apply SR technique for E_6-singularity.

Order 4 $r = (4, 4, 4, 4)$

[3] 4.1.1 type, (-1)
$X = 4A + B_1 + C_1 + D_1 + E_1$

$WCS = (Y, 1)$
(1) $Y = A + B_1$
(2) $Y = 2A + B_1 + C_1$
(3) $Y = 3A + B_1 + C_1 + D_1$

Or use deformation of a plane curve singularity after blow down of A.

[3] 4.1.2 type, (-2)
$X = 4A + B_1 + C_1 + 3D_1 + 2D_2 + D_3 + 3E_1 + 2E_2 + E_3$

(i) $WCS = \sum_{i=1}^{3}(iY, i)$, semi-rigid: B_1, C_1, D_1, E_1
$Y = A + D_1 + E_1$

$WCS = (Y, 1)$
(1) $Y = A + B_1 + C_1$
(2) $Y = A + B_1 + D_1 + D_2 + D_3$
(3) $Y = A + D_1 + D_2 + D_3 + E_1 + E_2 + E_3$
(4) $Y = 2A + B_1 + C_1 + D_1 + E_1$
(5) $Y = 3A + B_1 + C_1 + 2D_1 + D_2 + 2E_1 + E_2$

$WCS = (Y, 2)$
(6) $Y = 2A + B_1 + C_1 + 2D_1 + 2D_2$
(7) $Y = 2A + B_1 + D_1 + 2E_1 + 2E_2$
(8) $Y = 2A + 2D_1 + 2D_2 + 2E_1 + 2E_2$

$WCS = (Y, 3)$
(9) $Y = 3A + B_1 + C_1 + D_1 + 3E_1$
(10) $Y = 3A + 3D_1 + 3E_1$

Or apply SR technique for A_5-singularity.

[3] 4.1.3 type, (-3)
$X = 4A+3B_1+2B_2+B_3+3C_1+2C_2+C_3+3D_1+2D_2+D_3+3E_1+2E_2+E_3$

(i) $WCS = \sum_{i=1}^{3}(iY, i)$, semi-rigid: B_1, C_1, D_1, E_1
$Y = A + B_1 + C_1 + D_1$

$WCS = (Y, 1)$
(1) $Y = A + B_1 + B_2 + B_3 + C_1 + C_2 + C_3 + D_1 + D_2 + D_3$

$WCS = (Y, 2)$
(2) $Y = 2A + 2B_1 + 2B_2 + 2C_1 + 2C_2 + 2D_1 + 2D_2$
(3) $Y = 2A + B_1 + C_1 + 2D_1 + 2D_2 + 2E_1 + 2E_2$

$WCS = (Y, 3)$
(4) $Y = 3A + 3B_1 + 3C_1 + 3D_1$
(5) $Y = 3A + B_1 + 2C_1 + C_2 + 3D_1 + 3E_1$
(6) $Y = 3A + 2B_1 + B_2 + 2C_1 + C_2 + 2D_1 + D_2 + 3E_1$

Order 4 $r = (2, 2, 2, 4, 4)$

[3] 4.2.1 type, (-2)
$X = 4A + B_1 + C_1 + 2D_1 + 2E_1 + 2F_1$

(i) $WCS = \sum_{i=1}^{2}(iY, i)$, semi-rigid: B_1, C_1, D_1, E_1, F_1
$Y = A + D_1 + E_1$

$WCS = (Y, 1)$
(1) $Y = A + B_1 + C_1$
(2) $Y = 2A + B_1 + C_1 + D_1 + E_1$
(3) $Y = 2A + B_1 + D_1 + E_1 + F_1$

Or apply SR technique for D_4-singularity.

[3] 4.2.2 type, (-3)
$X = 4A + 2B_1 + 2C_1 + 2D_1 + 3E_1 + 2E_2 + E_3 + 3F_1 + 2F_2 + F_3$

(i) $WCS = \sum_{i=1}^{2}(iY, i)$, semi-rigid: B_1, C_1, D_1, E_1, F_1
$Y = A + B_1 + C_1 + D_1$

(ii) $WCS = \sum_{i=1}^{3}(iY, i)$, semi-rigid: B_1, C_1, D_1, E_2, F_2
$Y = A + B_1 + E_1 + E_2 + F_1 + F_2$

$WCS = (Y, 2)$
(1) $Y = 2A + B_1 + C_1 + D_1 + E_1 + 2F_1 + 2F_2$
(2) $Y = 2A + B_1 + C_1 + 2E_1 + 2E_2 + 2F_1 + 2F_2$

$WCS = (Y, 3)$
(3) $Y = 3A + B_1 + C_1 + D_1 + 3E_1 + 3F_1$

Order 3 $r = (3, 3, 3, 3, 3)$

[3] 3.1 type, (-2)
$X = 3A + B_1 + C_1 + D_1 + E_1 + 2F_1 + F_2$

$WCS = (Y, 1)$
(1) $Y = A + B_1 + C_1$
(2) $Y = A + B_1 + F_1 + F_2$
(3) $Y = 2A + B_1 + C_1 + D_1 + E_1$

$WCS = (Y, 2)$
(4) $Y = 2A + B_1 + C_1 + 2F_1$

Or apply SR technique for A_2-singularity.

[3] 3.2 type, (-3)
$X = 3A + B_1 + 2C_1 + C_2 + 2D_1 + D_2 + 2E_1 + E_2 + 2F_1 + F_2$

(i) $WCS = \sum_{i=1}^{2}(iY, i)$, semi-rigid: B_1, C_1, D_1, E_1, F_1
$Y = A + C_1 + D_1 + E_1$

$WCS = (Y, 1)$
(1) $Y = A + C_1 + C_2 + D_1 + D_2 + E_1 + E_2$
(2) $Y = A + B_1 + C_1 + C_2 + D_1 + D_2$

$WCS = (Y, 2)$
(3) $Y = 2A + 2C_1 + 2D_1 + 2E_1$
(4) $Y = 2A + B_1 + C_1 + 2D_1 + 2E_1$
(5) $Y = 2A + C_1 + D_1 + 2E_1 + 2F_1$
(6) $Y = 2A + B_1 + C_1 + D_1 + E_1 + 2F_1$

Order 2 $r = (2, 2, 2, 2, 2, 2, 2, 2)$

[3] 2.1 type, (-4)
$X = 2A + B_1 + C_1 + D_1 + E_1 + F_1 + G_1 + H_1 + I_1$

$WCS = (Y, 1)$
(1) $Y = A + B_1 + C_1 + D_1 + E_1$

Or apply SR technique for a cyclic quotient singularity.

26.3.2 Stellar singular fibers, genus$(A) = 1, 2$

genus$(A) = 1$

[3] A1.1 type
$X = 4A + 2B_1 + 2C_1$, $N_A^{\otimes 4} = \mathcal{O}(-2b_1 - 2c_1)$

Take b_1, c_1 so that $b_1 - c_1$ is torsion of order 2 in $\mathrm{Pic}(A)$.
$WCS = (Y, 1) + (2Y, 2)$
(1) $Y = A + B_1$, $N_A = \mathcal{O}(-b_1)$,

[3] A1.2 type
$X = 3A + B_1 + 2C_1 + C_2$, $N_A^{\otimes 3} = \mathcal{O}(-b_1 - 2c_1)$

$WCS = (Y, 1)$
(1) $Y = A + B_1 + C_1 + C_2, \quad N_A = \mathcal{O}(-b_1 - c_1 + q)$,
where $2b_1 + c_1 \sim 3q$ (linearly equivalent).

[3] A1.3 type
$X = 2A + B_1 + C_1 + D_1 + E_1, \quad N_A^{\otimes 2} = \mathcal{O}(-b_1 - c_1 - d_1 - e_1)$

$WCS = (Y, 1)$
(1) $Y = A + B_1 + C_1, \quad N_A = \mathcal{O}(-b_1 - c_1)$,
where $b_1 + c_1 \sim d_1 + e_1$.

genus$(A) = 2$

[3] A2.1 type
$X = 2A$ atom

26.3.3 Self-welding of stellar singular fibers of genus 2

[3] sw([2]8.2) type
$X = 8A+4B_1+5C_1+2C_2+7D_1+6D_2+5D_3+4D_4+3D_5+2D_6, \quad C_2 = D_6$

$WCS = (Y, 1) + (2Y, 2)$
(1) $Y = A + C_1 + C_2 + D_1 + D_2 + D_3 + D_4 + D_5 + D_6$

[3] sw([2]6.1.2) type
$X = 6A+2B_1+5C_1+4C_2+3C_3+2C_4+5D_1+4D_2+3D_3+2D_4, \quad C_4 = D_4$

$WCS = (Y, 1) + (2Y, 2)$
(1) $Y = A + C_1 + C_2 + C_3 + C_4 + D_1 + D_2 + D_3 + D_4$

[3] sw([2]5.2) type
$X = 5A + B_1 + 2C_1 + 2D_1, \quad C_1 = D_1$

$WCS = (Y, 1)$
(1) $Y = A + B_1$
(2) $Y = 3A + B_1 + C_1 + D_1$
$WCS = (Y, 1) + (2Y, 2)$
(3) $Y = 2A + C_1 + D_1$

[3] sw([2]5.3(1)) type
$X = 5A + 2B_1 + 4C_1 + 3C_2 + 2C_3 + 4D_1 + 3D_2 + 2D_3 + D_4, \quad B_1 = C_3$

$WCS = (Y, 4)$
(1) $Y = 4A + B_1 + 3C_1 + 2C_2 + C_3 + 4D_1$

$WCS = (Y, 1) + (2Y, 2)$
(2) $Y = A + B_1 + C_1 + C_2 + C_3$

[3] sw([2]5.3(2)) type
$X = 5A + 2B_1 + B_2 + 4C_1 + 3C_2 + 2C_3 + 4D_1 + 3D_2 + 2D_3, \quad C_3 = D_3$

$WCS = (Y, 2)$
(1) $Y = 4A + 2B_1 + 3C_1 + 2C_2 + C_3 + 3D_1 + 2D_2 + D_3$

$WCS = (Y, 1) + (2Y, 2)$
(2) $Y = A + C_1 + C_2 + C_3 + D_1 + D_2 + D_3$

[3] sw([2]5.4) type
$X = 5A + 3B_1 + B_2 + 3C_1 + 4D_1 + 3D_2, \quad C_1 = D_2$

$WCS = (Y, 1) + (2Y, 2) + (3Y, 3)$
(1) $Y = A + C_1 + D_1 + D_2$

[3] sw([2]3.1) type
$X = 3A + B_1 + C_1 + 2D_1 + 2E_1, \quad D_1 = E_1$

$WCS = (Y, 1)$
(1) $Y = A + B_1 + C_1$
(2) $Y = 2A + B_1 + C_1 + D_1 + E_1$

$WCS = (Y, 1) + (2Y, 2)$
(3) $Y = A + D_1 + E_1$

26.3.4 Welding of stellar singular fibers of genus 2 and genus 1

[3] wd([2]8.2(2)+II) type
$X = X_1 + X_2, \quad D_2(X_1) = A(X_2)$
$X_1 = 8A + 4B_1 + 5C_1 + 2C_2 + C_3 + 7D_1 + 6D_2$
$X_2 = 6A + 2C_1 + 3D_1$

$WCS = (Y, 1) + (2Y, 2), \quad Y = Y(X_1) + Y(X_2)$
(1) $Y(X_1) = A + C_1 + C_2 + D_1 + D_2$
$Y(X_2) = A + C_1$
(2) $Y(X_1) = 2A + B_1 + C_1 + 2D_1 + 2D_2$
$Y(X_2) = 2A + C_1 + D_1$

[3] wd([2]5.1(1)+II^*) type
$X = X_1 + X_2, \quad A(X_1) = D_1(X_2)$
$X_1 = 5A + C_1 + 3D_1 + D_2$
$X_2 = 6A + 3B_1 + 4C_1 + 2C_2 + 5D_1$

$WCS = (Y, 1), \quad Y = Y(X_1) + Y(X_2)$
(1) $Y(X_1) = A + C_1 + D_1 + D_2$
$Y(X_2) = 0$
(2) $Y(X_1) = 2A + C_1 + D_1$
$Y(X_2) = 2A + B_1 + C_1 + 2D_1$

[3] wd([2]5.2(1)+II^*) type
$X = X_1 + X_2, \quad A(X_1) = D_1(X_2)$
$X_1 = 5A + 2C_1 + C_2 + 2D_1 + D_2$
$X_2 = 6A + 3B_1 + 4C_1 + 2C_2 + 5D_1$

$WCS = (Y, 1) + (2Y, 2), \quad Y = Y(X_1)$
(1) $Y = 2A + C_1 + D_1$

26.4 Genus 4

26.4.1 Stellar singular fibers, $A = \mathbb{P}^1$

Order 18 $\quad r = (2, 9, 18)$

[4] 18.1 type, (-1)
$X = 18A + 9B_1 + 2C_1 + 7D_1 + 3D_2 + 2D_3 + D_4$

$WCS = (Y, 2)$
(1) $Y = 10A + 5B_1 + C_1 + 4D_1 + 2D_2 + 2D_3$

$WCS = (Y, 1) + (2Y, 2)$
(2) $Y = 8A + 4B_1 + C_1 + 3D_1 + D_2$

[4] 18.2 type, (-1)
$X = 18A + 9B_1 + 4C_1 + 2C_2 + 5D_1 + 2D_2 + D_3$

$WCS = (Y, 2)$
(1) $Y = 14A + 7B_1 + 3C_1 + C_2 + 4D_1 + 2D_2$

$WCS = (Y, 1) + (2Y, 2)$
(2) $Y = 4A + 2B_1 + C_1 + C_2 + D_1$

[4] 18.3 type, (-1)
$X = 18A + 9B_1 + C_1 + 8D_1 + 6D_2 + 4D_3 + 2D_4$

$WCS = (Y, 1)$
(1) $Y = A + C_1$
(2) $Y = 2A + B_1 + C_1$
(3) $Y = 3A + B_1 + C_1 + D_1$
(4) $Y = 5A + 2B_1 + C_1 + 2D_1 + D_2$
(5) $Y = 6A + 3B_1 + C_1 + 2D_1$
(6) $Y = 8A + 4B_1 + C_1 + 3D_1 + D_2$
(7) $Y = 10A + 5B_1 + C_1 + 4D_1 + 2D_2$

$WCS = (Y, 1) + (2Y, 2)$
(8) $Y = 2A + B_1 + D_1 + D_2 + D_3 + D_4$

Or apply SR technique for a cyclic quotient singularity after contracting A.

[4] 18.4 type, (-2)

$X = 18A + 9B_1 + 10C_1 + 2C_2 + 17D_1 + 16D_2 + 15D_3 + 14D_4 + 13D_5 + 12D_6 + 11D_7 + 10D_8 + 9D_9 + 8D_{10} + 7D_{11} + 6D_{12} + 5D_{13} + 4D_{14} + 3D_{15} + 2D_{16} + D_{17}$

$WCS = (Y, 2)$
(1) $Y = 2A + B_1 + C_1 + 2D_1 + 2D_2 + 2D_3 + 2D_4 + 2D_5 + 2D_6 + 2D_7 + 2D_8 + 2D_9 + 2D_{10} + 2D_{11} + 2D_{12} + 2D_{13} + 2D_{14} + 2D_{15} + 2D_{16}$

$WCS = (Y, 4)$
(2) $Y = 4A + 2B_1 + 2C_1 + 4D_1 + 4D_2 + 4D_3 + 4D_4 + 4D_5 + 4D_6 + 4D_7 + 4D_8 + 4D_9 + 4D_{10} + 4D_{11} + 4D_{12} + 4D_{13} + 4D_{14}$

$WCS = (Y, 6)$
(3) $Y = 6A + 3B_1 + 3C_1 + 6D_1 + 6D_2 + 6D_3 + 6D_4 + 6D_5 + 6D_6 + 6D_7 + 6D_8 + 6D_9 + 6D_{10} + 6D_{11} + 6D_{12}$

$WCS = (Y, 8)$
(4) $Y = 8A + 4B_1 + 4C_1 + 8D_1 + 8D_2 + 8D_3 + 8D_4 + 8D_5 + 8D_6 + 8D_7 + 8D_8 + 8D_9 + 8D_{10}$

$WCS = (Y, 10)$
(5) $Y = 10A + 5B_1 + 5C_1 + 10D_1 + 10D_2 + 10D_3 + 10D_4 + 10D_5 + 10D_6 + 10D_7 + 10D_8$

$WCS = (Y, 12)$
(6) $Y = 12A + 6B_1 + 6C_1 + 12D_1 + 12D_2 + 12D_3 + 12D_4 + 12D_5 + 12D_6$

$WCS = (Y, 14)$
(7) $Y = 14A + 7B_1 + 7C_1 + 14D_1 + 14D_2 + 14D_3 + 14D_4$

$WCS = (Y, 16)$
(8) $Y = 16A + 8B_1 + 8C_1 + 16D_1 + 16D_2$

$WCS = (Y, 9)$
(9) $Y = 9A + 4B_1 + 5C_1 + C_2 + 9D_1 + 9D_2 + 9D_3 + 9D_4 + 9D_5 + 9D_6 + 9D_7 + 9D_8 + 9D_9$

$WCS = (Y, 11)$
(10) $Y = 11A + 5B_1 + 6C_1 + C_2 + 11D_1 + 11D_2 + 11D_3 + 11D_4 + 11D_5 + 11D_6 + 11D_7$

$WCS = (Y, 13)$
(11) $Y = 13A + 6B_1 + 7C_1 + C_2 + 13D_1 + 13D_2 + 13D_3 + 13D_4 + 13D_5$

$WCS = (Y, 15)$
(12) $Y = 15A + 7B_1 + 8C_1 + C_2 + 15D_1 + 15D_2 + 15D_3$

[4] 18.5 type, (-2)

$X = 18A + 9B_1 + 13C_1 + 8C_2 + 3C_3 + C_4 + 14D_1 + 10D_2 + 6D_3 + 2D_4$

$WCS = (Y, 1)$
(1) $Y = 4A + 2B_1 + 3C_1 + 2C_2 + C_3 + C_4 + 3D_1 + 2D_2 + D_3$

[4] 18.6 type, (-2)

$X = 18A + 9B_1 + 11C_1 + 4C_2 + C_3 + 16D_1 + 14D_2 + 12D_3 + 10D_4 + 8D_5 + 6D_6 + 4D_7 + 2D_8$

$WCS = (Y, 1)$

(1) $Y = 8A+4B_1+5C_1+2C_2+C_3+7D_1+6D_2+5D_3+4D_4+3D_5+2D_6+D_7$

Order 16 $r = (2, 16, 16)$

[4] 16.1 type, (-1)

$X = 16A + 8B_1 + C_1 + 7D_1 + 5D_2 + 3D_3 + D_4$

$WCS = (Y, 1)$

(1) $Y = A + C_1$
(2) $Y = 2A + B_1 + C_1$
(3) $Y = 2A + C_1 + D_1 + D_2 + D_3 + D_4$
(4) $Y = 2A + B_1 + D_1 + D_2 + D_3 + D_4$
(5) $Y = 3A + B_1 + C_1 + D_1$
(6) $Y = 4A + 2B_1 + C_1 + D_1$
(7) $Y = 5A + 2B_1 + C_1 + 2D_1 + D_2$
(8) $Y = 6A + 3B_1 + C_1 + 2D_1$
(9) $Y = 7A + 3B_1 + C_1 + 3D_1 + 2D_2 + D_3$
(10) $Y = 8A + 4B_1 + C_1 + 3D_1 + D_2$
(11) $Y = 10A + 5B_1 + C_1 + 4D_1 + 2D_2$
(12) $Y = 12A + 6B_1 + C_1 + 5D_1 + 3D_2 + D_3$
(13) $Y = 14A + 7B_1 + C_1 + 6D_1 + 4D_2 + 2D_3$

Or apply SR technique for a cyclic quotient singularity.

[4] 16.2 type, (-1)

$X = 16A + 8B_1 + 3C_1 + 2C_2 + C_3 + 5D_1 + 4D_2 + 3D_3 + 2D_4 + D_5$

$WCS = (Y, 2)$

(1) $Y = 6A + 3B_1 + C_1 + 2D_1 + 2D_2 + 2D_3 + 2D_4$
(2) $Y = 10A + 5B_1 + 2C_1 + 2C_2 + 3D_1 + 2D_2 + D_3$

[4] 16.3 type, (-2)

$X = 16A + 8B_1 + 9C_1 + 2C_2 + C_3 + 15D_1 + 14D_2 + 13D_3 + 12D_4 + 11D_5 + 10D_6 + 9D_7 + 8D_8 + 7D_9 + 6D_{10} + 5D_{11} + 4D_{12} + 3D_{13} + 2D_{14} + D_{15}$

$WCS = (Y, 2)$

(1) $Y = 2A + B_1 + C_1 + 2D_1 + 2D_2 + 2D_3 + 2D_4 + 2D_5 + 2D_6 + 2D_7 + 2D_8 + 2D_9 + 2D_{10} + 2D_{11} + 2D_{12} + 2D_{13} + 2D_{14}$

$WCS = (Y, 4)$

(2) $Y = 4A + 2B_1 + 2C_1 + 4D_1 + 4D_2 + 4D_3 + 4D_4 + 4D_5 + 4D_6 + 4D_7 + 4D_8 + 4D_9 + 4D_{10} + 4D_{11} + 4D_{12}$

$WCS = (Y, 6)$
(3) $Y = 6A + 3B_1 + 3C_1 + 6D_1 + 6D_2 + 6D_3 + 6D_4 + 6D_5 + 6D_6 + 6D_7 + 6D_8 + 6D_9 + 6D_{10}$

$WCS = (Y, 8)$
(4) $Y = 8A+4B_1+4C_1+8D_1+8D_2+8D_3+8D_4+8D_5+8D_6+8D_7+8D_8$

$WCS = (Y, 10)$
(5) $Y = 10A + 5B_1 + 5C_1 + 10D_1 + 10D_2 + 10D_3 + 10D_4 + 10D_5 + 10D_6$

$WCS = (Y, 12)$
(6) $Y = 12A + 6B_1 + 6C_1 + 12D_1 + 12D_2 + 12D_3 + 12D_4$

$WCS = (Y, 14)$
(7) $Y = 14A + 7B_1 + 7C_1 + 14D_1 + 14D_2$

$WCS = (Y, 7)$
(8) $Y = 7A + 3B_1 + 4C_1 + C_2 + C_3 + 7D_1 + 7D_2 + 7D_3 + 7D_4 + 7D_5 + 7D_6 + 7D_7 + 7D_8 + 7D_9$

$WCS = (Y, 9)$
(9) $Y = 9A+4B_1+5C_1+C_2+9D_1+9D_2+9D_3+9D_4+9D_5+9D_6+9D_7$

$WCS = (Y, 11)$
(10) $Y = 11A + 5B_1 + 6C_1 + C_2 + 11D_1 + 11D_2 + 11D_3 + 11D_4 + 11D_5$

$WCS = (Y, 13)$
(11) $Y = 13A + 5B_1 + 6C_1 + C_2 + 13D_1 + 13D_2 + 13D_3$

$WCS = (Y, 15)$
(12) $Y = 15A + 7B_1 + 8C_1 + C_2 + 15D_1$

[4] 16.4 type, (-2)
$X = 16A + 8B_1 + 11C_1 + 6C_2 + C_3 + 13D_1 + 10D_2 + 7D_3 + 4D_4 + D_5$

$WCS = (Y, 1)$
(1) $Y = A + C_1 + D_1 + D_2 + D_3 + D_4 + D_5$
(2) $Y = 4A + 2B_1 + 3C_1 + 2C_2 + C_3 + 3D_1 + 2D_2 + D_3$
(3) $Y = 6A + 3B_1 + 4C_1 + 2C_2 + 5D_1 + 4D_2 + 3D_3 + 2D_4 + D_5$

Or apply SR technique for the E_8-singularity.

Order 15 $\quad r = (3, 5, 15)$

[4] 15.1 type, (-1)
$X = 15A + 5B_1 + 3C_1 + 7D_1 + 6D_2 + 5D_3 + 4D_4 + 3D_5 + 2D_6 + D_7$

$WCS = (Y, 3)$
(1) $Y = 6A + 2B_1 + C_1 + 3D_1 + 3D_2 + 3D_3 + 3D_4 + 3D_5$

$WCS = (Y, 5)$
(2) $Y = 10A + 3B_1 + 2C_1 + 5D_1 + 5D_2 + 5D_3$

$WCS = (Y, 6)$
(3) $Y = 12A + 4B_1 + 2C_1 + 6D_1 + 6D_2$

[4] 15.2 type, (-1)
$X = 15A + 5B_1 + 6C_1 + 3C_2 + 4D_1 + D_2$

$WCS = (Y, 1)$
(1) $Y = 3A + B_1 + C_1 + D_1 + D_2$

[4] 15.3 type, (-1)
$X = 15A + 5B_1 + 6C_1 + 3C_2 + 4D_1 + D_2$

$WCS = (Y, 1)$
(1) $Y = A + D_1$
(2) $Y = 2A + C_1 + D_1$
(3) $Y = 3A + B_1 + C_1 + D_1$
(4) $Y = 4A + B_1 + 2C_1 + D_1$
(5) $Y = 5A + B_1 + 3C_1 + C_2 + D_1$
(6) $Y = 6A + 2B_1 + 3C_1 + D_1$
(7) $Y = 7A + 2B_1 + 4C_1 + C_2 + D_1$
(8) $Y = 9A + 3B_1 + 5C_1 + C_2 + D_1$

[4] 15.4 type, (-1)

$X = 15A + 3B_1 + 10C_1 + 5C_2 + 2D_1 + D_2$

$WCS = (Y, 1) + (2Y, 2)$
(1) $Y = 6A + B_1 + 4C_1 + 2C_2 + D_1 + D_2$

[4] 15.5 type, (-2)
$X = 15A + 5B_1 + 12C_1 + 9C_2 + 6C_3 + 3C_4 + 13D_1 + 11D_2 + 9D_3 + 7D_4 + 5D_5 + 3D_6 + D_7$

$WCS = (Y_1, 1)+(Y_2, 2)+(Y_3, 3)+(Y_4, 3)+(Y_5, 3)+(Y_6, 6)+(Y_7, 7)+(Y_8, 9)$
(1) $Y_1 = A + C_1 + C_2 + C_3 + C_4 + D_1 + D_2 + D_3 + D_4 + D_5 + D_6 + D_7$,
$Y_2 = 2A + 2C_1 + 2C_2 + 2C_3 + 2C_4 + 2D_1 + 2D_2 + 2D_3 + 2D_4 + 2D_5 + 2D_6$,
$Y_3 = 3A + 3C_1 + 3C_2 + 3C_3 + 3C_4 + 3D_1 + 3D_2 + 3D_3 + 3D_4 + 3D_5 + 3D_6$,
$Y_4 = 3A + B_1 + 3C_1 + 3C_2 + 3C_3 + 3C_4 + 2D_1 + D_2$,
$Y_5 = 3A + B_1 + 2C_1 + C_2 + 3D_1 + 3D_2 + 3D_3 + 3D_4 + 3D_5 + 3D_6$,
$Y_6 = 6A + 2B_1 + 5C_1 + 4C_2 + 3C_3 + 2C_4 + 5D_1 + 4D_2 + 3D_3 + 2D_4 + D_5$,
$Y_7 = 7A+2B_1+6C_1+5C_2+4C_3+3C_4+6D_1+5D_2+4D_3+3D_4+2D_5+D_6$,
$Y_8 = 9A+3B_1+7C_1+5C_2+3C_3+C_4+8D_1+7D_2+6D_3+5D_4+4D_5+3D_6$

$WCS = (Y, 1) + (2Y, 2) + (3Y, 3) + (4Y, 4) + (5Y, 5)$,
semi-rigid: $5B_1, 12C_1, 5D_5$
(2) $Y = A + B_1 + D_1 + D_2 + D_3 + D_4 + D_5$

$WCS = \sum_{i=1}^{9}(iY, i)$, semi-rigid: $5B_1, 9C_2, 9D_3$
(3) $Y = A + C_1 + C_2 + D_1 + D_2 + D_3$

[4] **15.6 type,** (-2)

$X = 15A + 6B_1 + 3B_2 + 10C_1 + 5C_2 + 14D_1 + 13D_2 + 12D_3 + 11D_4 + 10D_5 + 9D_6 + 8D_7 + 7D_8 + 6D_9 + 5D_{10} + 4D_{11} + 3D_{12} + 2D_{13} + D_{14}$

$WCS = (Y, 3)$

(1) $Y = 3A + B_1 + 2C_1 + C_2 + 3D_1 + 3D_2 + 3D_3 + 3D_4 + 3D_5 + 3D_6 + 3D_7 + 3D_8 + 3D_9 + 3D_{10} + 3D_{11} + 3D_{12}$

$WCS = (Y, 6)$

(2) $Y = 6A + 2B_1 + 4C_1 + 2C_2 + 6D_1 + 6D_2 + 6D_3 + 6D_4 + 6D_5 + 6D_6 + 6D_7 + 6D_8 + 6D_9$

$WCS = (Y, 9)$

(3) $Y = 9A + 3B_1 + 6C_1 + 3C_2 + 9D_1 + 9D_2 + 9D_3 + 9D_4 + 9D_5 + 9D_6$

$WCS = (Y, 12)$

(4) $Y = 12A + 4B_1 + 8C_1 + 4C_2 + 12D_1 + 12D_2 + 12D_3$

$WCS = (Y, 5)$

(5) $Y = 5A + 2B_1 + B_2 + 3C_1 + C_2 + 5D_1 + 5D_2 + 5D_3 + 5D_4 + 5D_5 + 5D_6 + 5D_7 + 5D_8 + 5D_9 + 5D_{10}$

$WCS = (Y, 8)$

(6) $Y = 8A + 3B_1 + B_2 + 5C_1 + 2C_2 + 8D_1 + 8D_2 + 8D_3 + 8D_4 + 8D_5 + 8D_6 + 8D_7$

[4] **15.7 type,** (-2)

$X = 15A + 9B_1 + 3B_2 + 10C_1 + 5C_2 + 11D_1 + 7D_2 + 3D_3 + 2D_4 + D_5$

$WCS = (Y, 3)$

(1) $Y = 12A + 7B_1 + 2B_2 + 8C_1 + 4C_2 + 9D_1 + 6D_2 + 3D_3$

[4] **15.8 type,** (-2)

$X = 15A + 8B_1 + B_2 + 10C_1 + 5C_2 + 12D_1 + 9D_2 + 6D_3 + 3D_4$

$WCS = (Y, 1)$

(1) $Y = 3A + 2B_1 + B_2 + 2C_1 + C_2 + 2D_1 + D_2$

(2) $Y = 5A + 3B_1 + B_2 + 3C_1 + C_2 + 4D_1 + 3D_2 + 2D_3 + D_4$

Or apply SR technique for E_8-singularity.

Order 12 $\quad r = (4, 6, 12)$

[4] **12.1.1 type,** (-1)

$X = 12A + 2B_1 + 3C_1 + 7D_1 + 2D_2 + D_3$

$WCS = (Y_1, 1) + (Y_2, 2)$

(1) $Y_1 = 4A + B_1 + C_1 + 2D_1,$

$Y_2 = 8A + 2B_1 + 2C_1 + 4D_1$

[4] 12.1.2 type, (-1)
$X = 12A + 2B_1 + C_1 + 9D_1 + 6D_2 + 3D_3$

$WCS = (Y, 1)$
(1) $Y = A + C_1$
(2) $Y = 2A + C_1 + D_1$
(3) $Y = 3A + C_1 + 2D_1 + D_2$
(4) $Y = 4A + C_1 + 3D_1 + 2D_2 + D_3$
(5) $Y = 6A + B_1 + C_1 + 4D_1 + 2D_2$
(6) $Y = 7A + B_1 + C_1 + 5D_1 + 3D_2 + D_3$
(7) $Y = 8A + B_1 + C_1 + 6D_1 + 4D_2 + 2D_3$

Or apply SR technique for a cyclic quotient singularity after contracting A.

[4] 12.1.3 type, (-2)
$X = 12A + 3B_1 + 10C_1 + 8C_2 + 6C_3 + 4C_4 + 2C_5 + 11D_1 + 10D_2 + 9D_3 + 8D_4 + 7D_5 + 6D_6 + 5D_7 + 4D_8 + 3D_9 + 2D_{10} + D_{11}$

$WCS = (Y, 4)$
(1) $Y = 4A + B_1 + 3C_1 + 2C_2 + C_3 + 4D_1 + 4D_2 + 4D_3 + 4D_4 + 4D_5 + 4D_6 + 4D_7 + 4D_8$

$WCS = (Y, 5)$
(2) $Y = 5A + B_1 + 4C_1 + 3C_2 + 2C_3 + C_4 + 5D_1 + 5D_2 + 5D_3 + 5D_4 + 5D_5 + 5D_6 + 5D_7$

$WCS = (Y, 6)$
(3) $Y = 6A + B_1 + 5C_1 + 4C_2 + 3C_3 + 2C_4 + C_5 + 6D_1 + 6D_2 + 6D_3 + 6D_4 + 6D_5 + 6D_6$

$WCS = (Y, 8)$
(4) $Y = 8A + 2B_1 + 6C_1 + 4C_2 + 2C_3 + 8D_1 + 8D_2 + 8D_3 + 8D_4$

$WCS = (Y, 9)$
(5) $Y = 9A + 2B_1 + 7C_1 + 5C_2 + 3C_3 + C_4 + 9D_1 + 9D_2 + 9D_3$

$WCS = (Y, 10)$
(6) $Y = 10A + 2B_1 + 8C_1 + 6C_2 + 4C_3 + 2C_4 + 10D_1 + 10D_2$

$WCS = (Y, 11)$
(7) $Y = 11A + 2B_1 + 9C_1 + 7C_2 + 5C_3 + 3C_4 + C_5 + 11D_1$

[4] 12.1.4 type, (-2)
$X = 12A+5B_1+3B_2+B_3+9C_1+6C_2+3C_3+10D_1+8D_2+6D_3+4D_4+2D_5$

$WCS = (Y_1, 2) + (Y_2, 3)$
(1) $Y_1 = 4A + 2B_1 + 2B_2 + 3C_1 + 2C_2 + C_3 + 3D_1 + 2D_2 + D_3$,
$Y_2 = 6A + 3B_1 + 3B_2 + 4C_1 + 2C_2 + 5D_1 + 4D_2 + 3D_3 + 2D_4 + D_5$

$WCS = (Y_1, 4) + (Y_2, 2) + (Y_3, 5)$
(2) $Y_1 = 4A + 2B_1 + 2B_2 + 3C_1 + 2C_2 + C_3 + 3D_1 + 2D_2 + D_3,$
$Y_2 = 8A + 4B_1 + 6C_1 + 4C_2 + 2C_3 + 6D_1 + 4D_2 + 2D_3,$
$Y_3 = 10A + 5B_1 + 7C_1 + 4C_2 + C_3 + 8D_1 + 6D_2 + 4D_3 + 2D_4$

Order 12 $\quad r = (3, 12, 12)$

[4] **12.2.1 type,** (-1)
$X = 12A + 4B_1 + C_1 + 7D_1 + 2D_2 + D_3$

$WCS = (Y, 1)$
(1) $Y = A + C_1$
(2) $Y = 2A + C_1 + D_1$
(3) $Y = 3A + B_1 + C_1 + D_1$
(4) $Y = 4A + B_1 + C_1 + 2D_1$
(5) $Y = 5A + B_1 + C_1 + 3D_1 + D_2 + D_3$
(6) $Y = 6A + 2B_1 + C_1 + 3D_1$

Or apply SR technique for a cyclic quotient singularity after contracting A.

[4] **12.2.2 type,** (-2)
$X = 12A + 5B_1 + 3B_2 + B_3 + 8C_1 + 4C_2 + 11D_1 + 10D_2 + 9D_3 + 8D_4 + 7D_5 + 6D_6 + 5D_7 + 4D_8 + 3D_9 + 2D_{10} + D_{11}$

$WCS = (Y, 2)$
(1) $Y = 2A + B_1 + B_2 + B_3 + C_1 + 2D_1 + 2D_2 + 2D_3 + 2D_4 + 2D_5 + 2D_6 + 2D_7 + 2D_8 + 2D_9 + 2D_{10}$

$WCS = (Y, 3)$
(2) $Y = 3A + B_1 + 2C_1 + C_2 + 3D_1 + 3D_2 + 3D_3 + 3D_4 + 3D_5 + 3D_6 + 3D_7 + 3D_8 + 3D_9$

$WCS = (Y, 5)$
(3) $Y = 5A+2B_1+B_2+3C_1+C_2+5D_1+5D_2+5D_3+5D_4+5D_5+5D_6+5D_7$

$WCS = (Y, 6)$
(4) $Y = 6A + 2B_1 + 4C_1 + 2C_2 + 6D_1 + 6D_2 + 6D_3 + 6D_4 + 6D_5 + 6D_6$

$WCS = (Y, 9)$
(5) $Y = 9A + 3B_1 + 6C_1 + 3C_2 + 9D_1 + 9D_2 + 9D_3$

$WCS = (Y, 10)$
(6) $Y = 10A + 4B_1 + 2B_2 + 6C_1 + 2C_2 + 10D_1 + 10D_2$

Order 10 $\quad r = (2, 2, 5, 5)$

[4] **10.1.1 type,** (-2)
$X = 10A + 5B_1 + 5C_1 + 4D_1 + 2D_2 + 6E_1 + 2E_2$

$WCS = (Y, 1) + (2Y, 2)$
(1) $Y = 2A + B_1 + C_1 + D_1 + D_2 + E_1$

Order 10 $r = (5, 10, 10)$

[4] 10.2.1 type, (-1)
$X = 10A + B_1 + 2C_1 + 7D_1 + 4D_2 + D_3$

$WCS = (Y, 1)$
(1) $Y = A + B_1$
(2) $Y = 2A + B_1 + D_1$
(3) $Y = 5A + B_1 + C_1 + 3D_1 + D_2$

Or apply SR technique for a cyclic quotient singularity after contracting A.

[4] 10.2.2 type, (-1)
$X = 10A + 4B_1 + 2B_2 + 3C_1 + 2C_2 + C_3 + 3D_1 + 2D_2 + D_3$

$WCS = (Y, 1)$
(1) $Y = 3A + B_1 + C_1 + C_2 + C_3 + D_1 + D_2 + D_3$

[4] 10.2.3 type, (-1)
$X = 10A + B_1 + 3C_1 + 2C_2 + C_3 + 6D_1 + 2D_2$

$WCS = (Y, 1)$
(1) $Y = A + B_1$
(2) $Y = 2A + B_1 + D_1$
(3) $Y = 3A + B_1 + C_1 + C_2 + C_3 + D_1$
(4) $Y = 4A + B_1 + C_1 + 2D_1$

Or apply SR technique for a cyclic quotient singularity after contracting A.

[4] 10.2.4 type, (-1)
$X = 10A + B_1 + C_1 + 8D_1 + 6D_2 + 4D_3 + 2D_4$

$WCS = (Y, 1)$
(1) $Y = A + B_1$
(2) $Y = 2A + B_1 + C_1$
(3) $Y = 2A + B_1 + D_1$
(4) $Y = 3A + B_1 + C_1 + D_1$
(5) $Y = 3A + B_1 + 2D_1 + D_2$
(6) $Y = 4A + B_1 + C_1 + 2D_1$
(7) $Y = 4A + B_1 + 3D_1 + 2D_2 + D_3$
(8) $Y = 5A + B_1 + C_1 + 3D_1 + D_2$
(9) $Y = 5A + B_1 + 4D_1 + 3D_2 + 2D_3 + D_4$
(10) $Y = 6A + B_1 + C_1 + 4D_1 + 2D_2$
(11) $Y = 7A + B_1 + C_1 + 5D_1 + 3D_2 + D_3$
(12) $Y = 8A + B_1 + C_1 + 6D_1 + 4D_2 + 2D_3$
(13) $Y = 9A + B_1 + C_1 + 7D_1 + 5D_2 + 3D_3 + D_4$

Or apply SR technique for A_4-singularity after contracting A.

[4] 10.2.5 type, (-2)

$X = 10A + 4B_1 + 2B_2 + 7C_1 + 4C_2 + C_3 + 9D_1 + 8D_2 + 7D_3 + 6D_4 + 5D_5 + 4D_6 + 3D_7 + 2D_8 + D_9$

$WCS = (Y, 3)$
(1) $Y = 3A + B_1 + 2C_1 + C_2 + 3D_1 + 3D_2 + 3D_3 + 3D_4 + 3D_5 + 3D_6 + 3D_7$

$WCS = (Y, 4)$
(2) $Y = 4A + B_1 + 3C_1 + 2C_2 + C_3 + 4D_1 + 4D_2 + 4D_3 + 4D_4 + 4D_5 + 4D_6$

$WCS = (Y, 5)$
(3) $Y = 5A + 2B_1 + B_2 + 3C_1 + C_2 + 5D_1 + 5D_2 + 5D_3 + 5D_4 + 5D_5$

$WCS = (Y, 6)$
(4) $Y = 6A + 2B_1 + 4C_1 + 2C_2 + 6D_1 + 6D_2 + 6D_3 + 6D_4$

$WCS = (Y, 7)$
(5) $Y = 7A + 2B_1 + 5C_1 + 3C_2 + C_3 + 7D_1 + 7D_2 + 7D_3$

$WCS = (Y, 9)$
(6) $Y = 9A + 3B_1 + 6C_1 + 3C_2 + 9D_1$

[4] 10.2.6 type, (-2)

$X = 10A + 6B_1 + 2B_2 + 7C_1 + 4C_2 + C_3 + 7D_1 + 4D_2 + D_3$

$WCS = (Y, 1)$
(1) $Y = A + C_1 + C_2 + C_3 + D_1 + D_2 + D_3$
(2) $Y = 4A + 2B_1 + 3C_1 + 2C_2 + C_3 + 3D_1 + 2D_2 + D_3$
(3) $Y = 7A + 4B_1 + B_2 + 5C_1 + 3C_2 + C_3 + 5D_1 + 3D_2 + D_3$

[4] 10.2.7 type, (-2)

$X = 10A + 3B_1 + 2B_2 + B_3 + 8C_1 + 6C_2 + 4C_3 + 2C_4 + 9D_1 + 8D_2 + 7D_3 + 6D_4 + 5D_5 + 4D_6 + 3D_7 + 2D_8 + D_9$

$WCS = (Y, 3)$
(1) $Y = 3A + B_1 + B_2 + B_3 + 2C_1 + C_2 + 3D_1 + 3D_2 + 3D_3 + 3D_4 + 3D_5 + 3D_6 + 3D_7$

$WCS = (Y, 4)$
(2) $Y = 4A + B_1 + 3C_1 + 2C_2 + C_3 + 4D_1 + 4D_2 + 4D_3 + 4D_4 + 4D_5 + 4D_6$

$WCS = (Y, 5)$
(3) $Y = 5A + B_1 + 5C_1 + 3C_2 + C_3 + 5D_1 + 5D_2 + 5D_3 + 5D_4 + 5D_5$

$WCS = (Y, 7)$
(4) $Y = 7A + 2B_1 + B_2 + 5C_1 + 3C_2 + C_3 + 7D_1 + 7D_2 + 7D_3$

$WCS = (Y, 8)$
(5) $Y = 8A + 2B_1 + 6C_1 + 4C_2 + 2C_3 + 8D_1 + 8D_2$

[4] 10.2.8 type, (-2)

$X = 10A + 2B_1 + 9C_1 + 8C_2 + 7C_3 + 6C_4 + 5C_5 + 4C_6 + 3C_7 + 2C_8 + C_9 + 9D_1 + 8D_2 + 7D_3 + 6D_4 + 5D_5 + 4D_6 + 3D_7 + 2D_8 + D_9$

$WCS = (Y, 1)$
(1) $Y = A + C_1 + C_2 + C_3 + C_4 + C_5 + C_6 + C_7 + C_8 + C_9 + D_1 + D_2 + D_3 + D_4 + D_5 + D_6 + D_7 + D_8 + D_9$

$WCS = (Y, 2)$
(2) $Y = 2A + 2C_1 + 2C_2 + 2C_3 + 2C_4 + 2C_5 + 2C_6 + 2C_7 + 2C_8 + 2D_1 + 2D_2 + 2D_3 + 2D_4 + 2D_5 + 2D_6 + 2D_7 + 2D_8$

$WCS = (Y, 3)$
(3) $Y = 3A + 3C_1 + 3C_2 + 3C_3 + 3C_4 + 3C_5 + 3C_6 + 3C_7 + 3D_1 + 3D_2 + 3D_3 + 3D_4 + 3D_5 + 3D_6 + 3D_7$

$WCS = (Y, 4)$
(4) $Y = 4A + 4C_1 + 4C_2 + 4C_3 + 4C_4 + 4C_5 + 4C_6 + 4D_1 + 4D_2 + 4D_3 + 4D_4 + 4D_5 + 4D_6$

$WCS = (Y, 5)$
(5) $Y = 5A + 5C_1 + 5C_2 + 5C_3 + 5C_4 + 5C_5 + 5D_1 + 5D_2 + 5D_3 + 5D_4 + 5D_5$

$WCS = (Y, 6)$
(6) $Y = 6A + 6C_1 + 6C_2 + 6C_3 + 6C_4 + 6D_1 + 6D_2 + 6D_3 + 6D_4$

$WCS = (Y, 7)$
(7) $Y = 7A + 7C_1 + 7C_2 + 7C_3 + 7D_1 + 7D_2 + 7D_3$

$WCS = (Y, 8)$
(8) $Y = 8A + 8C_1 + 8C_2 + 8D_1 + 8D_2$

$WCS = (Y, 9)$
(9) $Y = 9A + 9C_1 + 9D_1$

$WCS = (Y, 5)$
(10) $Y = 5A + B_1 + 4C_1 + 3C_2 + 2C_3 + C_4 + 5D_1 + 5D_2 + 5D_3 + 5D_4 + 5D_5$

$WCS = (Y, 6)$
(11) $Y = 6A + B_1 + 5C_1 + 4C_2 + 3C_3 + 2C_4 + C_5 + 6D_1 + 6D_2 + 6D_3 + 6D_4$

$WCS = (Y, 7)$
(12) $Y = 7A + B_1 + 6C_1 + 5C_2 + 4C_3 + 3C_4 + 2C_5 + C_6 + 7D_1 + 7D_2 + 7D_3$

$WCS = (Y, 8)$
(13) $Y = 8A + B_1 + 7C_1 + 6C_2 + 5C_3 + 4C_4 + 3C_5 + 2C_6 + C_7 + 8D_1 + 8D_2$

$WCS = (Y, 9)$
(14) $Y = 9A + B_1 + 8C_1 + 7C_2 + 6C_3 + 5C_4 + 4C_5 + 3C_6 + 2C_7 + C_8 + 9D_1$

Order 9 $r = (9, 9, 9)$

[4] **9.1 type,** (-1)
$X = 9A + B_1 + 4C_1 + 3C_2 + 2C_3 + C_4 + 4D_1 + 3D_2 + 2D_3 + D_4$

$WCS = (Y, 1)$
(1) $Y = A + B_1$
(2) $Y = 2A + B_1 + C_1 + C_2 + C_3 + C_4$
(3) $Y = 2A + C_1 + C_2 + C_3 + C_4 + D_1 + D_2 + D_3 + D_4$
(4) $Y = 3A + B_1 + C_1 + D_1$
(5) $Y = 5A + B_1 + 2C_1 + C_2 + 2D_1 + D_2$

$WCS = (Y, 2)$
(6) $Y = 4A + 2C_1 + 2C_2 + 2C_3 + 2D_1 + 2D_2 + 2D_3$

$WCS = (Y, 3)$
(7) $Y = 6A + 3C_1 + 3C_2 + 3D_1 + 3D_2$

$WCS = (Y, 4)$
(8) $Y = 8A + 4C_1 + 4D_1$

Or apply SR technique for a cyclic quotient singularity after contracting A.

[4] **9.2 type,** (-1)
$X = 9A + B_1 + C_1 + 7D_1 + 5D_2 + 3D_3 + D_4$

$WCS = (Y, 1)$
(1) $Y = A + B_1$
(2) $Y = 2A + B_1 + C_1$
(3) $Y = 2A + C_1 + D_1$
(4) $Y = 3A + B_1 + C_1 + D_1$
(5) $Y = 3A + B_1 + 2D_1 + D_2$
(6) $Y = 4A + B_1 + C_1 + 2D_1$
(7) $Y = 4A + B_1 + C_1 + 2D_1$
(8) $Y = 5A + B_1 + C_1 + 3D_1 + D_2$
(9) $Y = 5A + B_1 + 4D_1 + 3D_2 + 2D_3 + D_4$
(10) $Y = 6A + B_1 + C_1 + 4D_1 + 2D_2$
(11) $Y = 7A + B_1 + C_1 + 5D_1 + 3D_2 + D_3$
(12) $Y = 8A + B_1 + C_1 + 6D_1 + 4D_2 + 2D_3$

Or apply SR technique for a cyclic quotient singularity after contracting A.

[4] **9.3 type,** (-1)
$X = 9A + 2B_1 + B_2 + 2C_1 + C_2 + 5D_1 + D_2$

$WCS = (Y, 1)$
(1) $Y = A + D_1 + D_2 + D_3$
(2) $Y = 4A + B_1 + B_2 + C_1 + C_2 + 2D_1$
(3) $Y = 5A + B_1 + C_1 + 3D_1 + D_2$

[4] 9.4 type, (-2)

$$X = 9A + 2B_1 + B_2 + 8C_1 + 7C_2 + 6C_3 + 5C_4 + 4C_5 + 3C_6 + 2C_7 + C_8 + 8D_1 + 7D_2 + 6D_3 + 5D_4 + 4D_5 + 3D_6 + 2D_7 + D_8$$

$WCS = (Y, 1)$
(1) $Y = 2A + C_1 + C_2 + C_3 + C_4 + C_5 + C_6 + C_7 + C_8 + D_1 + D_2 + D_3 + D_4 + D_5 + D_6 + D_7 + D_8$

$WCS = (Y, 2)$
(2) $Y = 2A + 2C_1 + 2C_2 + 2C_3 + 2C_4 + 2C_5 + 2C_6 + 2C_7 + 2D_1 + 2D_2 + 2D_3 + 2D_4 + 2D_5 + 2D_6 + 2D_7$

$WCS = (Y, 3)$
(3) $Y = 3A + 3C_1 + 3C_2 + 3C_3 + 3C_4 + 3C_5 + 3C_6 + 3D_1 + 3D_2 + 3D_3 + 3D_4 + 3D_5 + 3D_6$

$WCS = (Y, 4)$
(4) $Y = 4A + 4C_1 + 4C_2 + 4C_3 + 4C_4 + 4C_5 + 4D_1 + 4D_2 + 4D_3 + 4D_4 + 4D_5$

$WCS = (Y, 5)$
(5) $Y = 5A + 5C_1 + 5C_2 + 5C_3 + 5C_4 + 5D_1 + 5D_2 + 5D_3 + 5D_4$

$WCS = (Y, 6)$
(6) $Y = 6A + 6C_1 + 6C_2 + 6C_3 + 6D_1 + 6D_2 + 6D_3$

$WCS = (Y, 7)$
(7) $Y = 7A + 7C_1 + 7C_2 + 7D_1 + 7D_2$

$WCS = (Y, 8)$
(8) $Y = 8A + 8C_1 + 8D_1$

$WCS = (Y, 4)$
(9) $Y = 4A + B_1 + B_2 + 3C_1 + 2C_2 + C_3 + 4D_1 + 4D_2 + 4D_3 + 4D_4 + 4D_5$

$WCS = (Y, 5)$
(10) $Y = 5A + B_1 + 4C_1 + 3C_2 + 2C_3 + C_4 + 5D_1 + 5D_2 + 5D_3 + 5D_4$

$WCS = (Y, 6)$
(11) $Y = 6A + B_1 + 5C_1 + 4C_2 + 3C_3 + 2C_4 + C_5 + 6D_1 + 6D_2 + 6D_3$

$WCS = (Y, 7)$
(12) $Y = 7A + B_1 + 6C_1 + 5C_2 + 4C_3 + 3C_4 + 2C_5 + C_6 + 7D_1 + 7D_2$

$WCS = (Y, 8)$
(13) $Y = 8A + B_1 + 7C_1 + 6C_2 + 5C_3 + 4C_4 + 3C_5 + 2C_6 + C_7 + 8D_1$

[4] 9.5 type, (-2)

$$X = 9A + 5B_1 + B_2 + 5C_1 + C_2 + 8D_1 + 7D_2 + 6D_3 + 5D_4 + 4D_5 + 3D_6 + 2D_7 + D_8$$

$WCS = (Y, 1)$
(1) $Y = A + B_1 + B_2 + C_1 + C_2$
(2) $Y = A + B_1 + B_2 + D_1 + D_2 + D_3 + D_4 + D_5 + D_6 + D_7 + D_8$
(3) $Y = 3A + 2B_1 + B_2 + 2C_1 + C_2 + 2D_1 + D_2$
(4) $Y = 5A + 3B_1 + B_2 + 3C_1 + C_2 + 4D_1 + 3D_2 + 2D_3 + D_4$
(5) $Y = 7A + 4B_1 + B_2 + 4C_1 + C_2 + 6D_1 + 5D_2 + 4D_3 + 3D_4 + 2D_5 + D_6$

$WCS = (Y, 2)$
(6) $Y = 2A + B_1 + C_1 + 2D_1 + 2D_2 + 2D_3 + 2D_4 + 2D_5 + 2D_6 + 2D_7$

$WCS = (Y, 3)$
(7) $Y = 3A + B_1 + 2C_1 + C_2 + 3D_1 + 3D_2 + 3D_3 + 3D_4 + 3D_5 + 3D_6$

$WCS = (Y, 4)$
(8) $Y = 4A + 2B_1 + 2C_1 + 4D_1 + 4D_2 + 4D_3 + 4D_4 + 4D_5$

$WCS = (Y, 5)$
(9) $Y = 5A + 3B_1 + B_2 + 2C_1 + 5D_1 + 5D_2 + 5D_3 + 5D_4$

$WCS = (Y, 6)$
(10) $Y = 6A + 3B_1 + 3C_1 + 6D_1 + 6D_2 + 6D_3$

$WCS = (Y, 7)$
(11) $Y = 7A + 4B_1 + B_2 + 3C_1 + 7D_1 + 7D_2$

$WCS = (Y, 8)$
(12) $Y = 8A + 4B_1 + 4C_1 + 8D_1$

Or apply SR technique for D_{10}-singularity.

Order 8 $r = (2, 2, 8, 8)$

[4] 8.1 type, (-2)
$X = 8A + 4B_1 + 4C_1 + D_1 + 7E_1 + 6E_2 + 5E_3 + 4E_4 + 3E_5 + 2E_6 + E_7$

$WCS = (Y, 1)$
(1) $Y = A + D_1 + E_1 + E_2 + E_3 + E_4 + E_5 + E_6 + E_7$
(2) $Y = 2A + B_1 + C_1 + D_1 + E_1$

$WCS = (Y, 2)$
(3) $Y = 2A + B_1 + C_1 + 2E_1 + 2E_2 + 2E_3 + 2E_4 + 2E_5 + 2E_6$
(4) $Y = 2A + B_1 + D_1 + 2E_1 + 2E_2 + 2E_3 + 2E_4 + 2E_5 + 2E_6$

$WCS = (Y, 3)$
(5) $Y = 3A + B_1 + C_1 + D_1 + 3E_1 + 3E_2 + 3E_3 + 3E_4 + 3E_5$

$WCS = (Y, 4)$
(6) $Y = 4A + 2B_1 + 2C_1 + 4E_1 + 4E_2 + 4E_3 + 4E_4$
(7) $Y = 4A + B_1 + 2C_1 + D_1 + 4E_1 + 4E_2 + 4E_3 + 4E_4$

$WCS = (Y, 5)$
(8) $Y = 5A + 2B_1 + 2C_1 + D_1 + 5E_1 + 5E_2 + 5E_3$

$WCS = (Y, 6)$
(9) $Y = 6A + 3B_1 + 3C_1 + 6E_1 + 6E_2$
(10) $Y = 6A + 2B_1 + 3C_1 + D_1 + 6E_1 + 6E_2$

$WCS = (Y, 7)$
(11) $Y = 7A + 3B_1 + 3C_1 + D_1 + 7E_1$

Or apply SR technique for D_9-singularity.

[4] **8.2 type,** (-2)
$X = 8A + 4B_1 + 4C_1 + 3D_1 + D_2 + 5E_1 + 2E_2 + E_3$

$WCS = (Y, 1)$
(1) $Y = 2A + B_1 + C_1 + D_1 + D_2 + E_1$

$WCS = (Y, 2)$
(2) $Y = 6A + 3B_1 + 3C_1 + 2D_1 + 4E_1 + 2E_2$

Order 6 $\quad r = (2, 2, 2, 3, 6)$

[4] **6.1.1 type,** (-2)
$X = 6A + B_1 + 2C_1 + 3D_1 + 3E_1 + 3F_1$

$WCS = (Y, 1)$
(1) $Y = 2A + B_1 + D_1 + E_1 + F_1$
(2) $Y = 4A + B_1 + C_1 + 2D_1 + 2E_1 + 2F_1$

[4] **6.1.2 type,** (-3)
$X = 6A + 3B_1 + 3C_1 + 3D_1 + 4E_1 + 2E_2 + 5F_1 + 4F_2 + 3F_3 + 2F_4 + F_5$

$WCS = (Y, 2)$
(1) $Y = 2A + B_1 + C_1 + D_1 + E_1 + 2F_1 + 2F_2 + 2F_3 + 2F_4$

$WCS = (Y, 4)$
(2) $Y = 4A + 2B_1 + 2C_1 + 2D_1 + 2E_1 + 4F_1 + 4F_2$

Order 6 $\quad r = (3, 3, 6, 6)$

[4] **6.2.1 type,** (-1)
$X = 6A + B_1 + C_1 + 2D_1 + 2E_1$

$WCS = (Y, 1)$
(1) $Y = A + B_1$
(2) $Y = 2A + B_1 + C_1$
(3) $Y = 3A + B_1 + C_1 + D_1$
(4) $Y = 3A + C_1 + D_1 + E_1$

Or apply SR technique for a cyclic quotient singularity after contracting A.

[4] **6.2.2 type,** (-2)
$X = 6A + B_1 + 2C_1 + 4D_1 + 2D_2 + 5E_1 + 4E_2 + 3E_3 + 2E_4 + E_5$

$WCS = (Y, 1)$
(1) $Y = 3A + B_1 + C_1 + 2D_1 + D_2 + 2E_1 + E_2$

$WCS = (Y, 2)$
(2) $Y = 2A + B_1 + D_1 + 2E_1 + 2E_2 + 2E_3 + 2E_4$

$WCS = (Y, 3)$
(3) $Y = 3A + B_1 + 2D_1 + D_2 + 3E_1 + 3E_2 + 3E_3$
(4) $Y = 3A + C_1 + 2D_1 + D_2 + 3E_1 + 3E_2 + 3E_3$
(5) $Y = 3A + B_1 + C_1 + D_1 + 3E_1 + 3E_2 + 3E_3$

$WCS = (Y, 4)$
(6) $Y = 4A + B_1 + C_1 + 2D_1 + 4E_1 + 4E_2$

$WCS = (Y, 5)$
(7) $Y = 5A + B_1 + C_1 + 3D_1 + D_2 + 5E_1$

[4] 6.2.3 type, (-3)
$X = 6A + 4B_1 + 2B_2 + 4C_1 + 2C_2 + 5D_1 + 4D_2 + 3D_3 + 2D_4 + D_5 + 5E_1 + 4E_2 + 3E_3 + 2E_4 + E_5$

$WCS = (Y, 2)$
(1) $Y = 2A + B_1 + C_1 + 2D_1 + 2D_2 + 2D_3 + 2D_4 + 2E_1 + 2E_2 + 2E_3 + 2E_4$

$WCS = (Y, 3)$
(2) $Y = 3A + B_1 + 2C_1 + C_2 + 3D_1 + 3D_2 + 3D_3 + 3E_1 + 3E_2 + 3E_3$
(3) $Y = 3A + 2B_1 + B_2 + 2C_1 + C_2 + 2D_1 + D_2 + 3E_1 + 3E_2 + 3E_3$

$WCS = (Y, 4)$
(4) $Y = 4A + 2B_1 + 2C_1 + 4D_1 + 4D_2 + 4E_1 + 4E_2$

$WCS = (Y, 5)$
(5) $Y = 5A + 2B_1 + 3C_1 + C_2 + 5D_1 + 5E_1$
(6) $Y = 5A + 3B_1 + B_2 + 3C_1 + C_2 + 4D_1 + 3D_2 + 2D_3 + D_4 + 5E_1$

Order 6 $\quad r = (2, 6, 6, 6)$

[4] 6.3.1 type, (-1)
$X = 6A + B_1 + C_1 + D_1 + 3E_1$

$WCS = (Y, 1)$
(1) $Y = A + B_1$
(2) $Y = 2A + B_1 + C_1$
(3) $Y = 2A + C_1 + D_1$
(4) $Y = 3A + B_1 + C_1 + D_1$
(5) $Y = 3A + C_1 + D_1 + E_1$
(6) $Y = 4A + C_1 + D_1 + 2E_1$

Or apply SR technique for A_1-singularity after contracting A.

[4] 6.3.2 type, (-3)
$X = 6A + 3B_1 + 5C_1 + 4C_2 + 3C_3 + 2C_4 + C_5 + 5D_1 + 4D_2 + 3D_3 + 2D_4 + D_5 + 5E_1 + 4E_2 + 3E_3 + 2E_4 + E_5$

$WCS = (Y, 1)$
(1) $Y = A + C_1 + C_2 + C_3 + C_4 + C_5 + D_1 + D_2 + D_3 + D_4 + D_5 + E_1 + E_2 + E_3 + E_4 + E_5$

$WCS = (Y, 2)$
(2) $Y = 2A + 2C_1 + 2C_2 + 2C_3 + 2C_4 + 2D_1 + 2D_2 + 2D_3 + 2D_4 + 2E_1 + 2E_2 + 2E_3 + 2E_4$
(3) $Y = 2A + B_1 + C_1 + 2D_1 + 2D_2 + 2D_3 + 2D_4 + 2E_1 + 2E_2 + 2E_3 + 2E_4$

$WCS = (Y, 3)$
(4) $Y = 3A + 3C_1 + 3C_2 + 3C_3 + 3D_1 + 3D_2 + 3D_3 + 3E_1 + 3E_2 + 3E_3$
(5) $Y = 3A + B_1 + 2C_1 + C_2 + 3D_1 + 3D_2 + 3D_3 + 3E_1 + 3E_2 + 3E_3$

$WCS = (Y, 4)$
(6) $Y = 4A + 4C_1 + 4C_2 + 4D_1 + 4D_2 + 4E_1 + 4E_2$
(7) $Y = 4A + 2B_1 + 2C_1 + 4D_1 + 4D_2 + 4E_1 + 4E_2$
(8) $Y = 4A + B_1 + 3C_1 + 2C_2 + C_3 + 4D_1 + 4D_2 + 4E_1 + 4E_2$

$WCS = (Y, 5)$
(9) $Y = 5A + 5C_1 + 5D_1 + 5E_1$
(10) $Y = 5A + B_1 + 4C_1 + 3C_2 + 2C_3 + C_4 + 5D_1 + 5E_1$

Order 6 $\quad r = (2, 2, 3, 3, 3)$

[4] 6.4.1 type, (-2)
$X = 6A + 2B_1 + 2C_1 + 2D_1 + 3E_1 + 3F_1$

$WCS = (Y, 1) + (2Y, 2)$
(1) $Y = 2A_1 + B_1 + C_1 + D_1 + E_1$

[4] 6.4.2 type, (-3)
$X = 6A + 3B_1 + 3C_1 + 4D_1 + 2D_2 + 4E_1 + 2E_2 + 4F_1 + 2F_2$

$WCS = (Y_1, 2) + (Y_2, 3)$
(1) $Y_1 = 2A + B_1 + C_1 + D_1 + E_1 + 2F_1 + 2F_2,$
$Y_2 = 3A + B_1 + C_1 + 2D_1 + D_2 + 2E_1 + E_2 + 3F_1$

Order 5 $\quad r = (5, 5, 5, 5)$

[4] 5.1 type, (-1)
$X = 5A + B_1 + C_1 + D_1 + 2E_1 + E_2$

$WCS = (Y, 1)$
(1) $Y = A + B_1$
(2) $Y = 2A + B_1 + C_1$
(3) $Y = 2A + D_1 + E_1 + E_2$

(4) $Y = 3A + C_1 + D_1 + E_1$
(5) $Y = 4A + B_1 + C_1 + D_1 + E_1$

$WCS = (Y, 2)$
(6) $Y = 4A + C_1 + D_1 + 2E_1 + E_2$

Or apply SR technique for a cyclic quotient singularity after contracting A.

[4] **5.2 type,** (-2)
$X = 5A + 2B_1 + B_2 + 2C_1 + C_2 + 2D_1 + D_2 + 4E_1 + 3E_2 + 2E_3 + E_4$

$WCS = (Y, 1)$
$Y = 2A + B_1 + B_2 + C_1 + C_2 + D_1 + D_2 + E_1$

$WCS = (Y, 2)$
(2) $Y = 2A + B_1 + B_2 + C_1 + C_2 + 2E_1 + 2E_2 + 2E_3$
(3) $Y = 4A + 2B_1 + 2C_1 + 2D_1 + 2E_1$
(4) $Y = 4A + B_1 + 2C_1 + 2D_1 + 3E_1 + 2E_2 + E_3$

$WCS = (Y, 3)$
(5) $Y = 3A + B_1 + C_1 + D_1 + 3E_1 + 3E_2$

[4] **5.3 type,** (-2)
$X = 5A + B_1 + 3C_1 + C_2 + 3D_1 + D_2 + 3E_1 + E_2$

$WCS = (Y, 1)$
(1) $Y = A + B_1 + C_1 + C_2$
(2) $Y = A + C_1 + C_2 + D_1 + D_2$
(3) $Y = 2A + B_1 + C_1 + D_1 + E_1$
(4) $Y = 3A + 2C_1 + C_2 + 2D_1 + D_2 + 2E_1 + E_2$

[4] **5.4 type,** (-2)
$X = 5A + 2B_1 + B_2 + 2C_1 + C_2 + 3D_1 + D_2 + 3E_1 + E_2$

$WCS = (Y, 1)$
(1) $Y = 2A + B_1 + B_2 + C_1 + C_2 + D_1 + E_1$
(2) $Y = 3A + B_1 + C_1 + 2D_1 + D_2 + 2E_1 + E_2$

$WCS = (Y, 2)$
(3) $Y = 4A + 2B_1 + 2C_1 + 2D_1 + 2E_1$

$WCS = (Y, 1) + (2Y, 2)$
(4) $Y = A + B_1 + D_1$

[4] **5.5 type,** (-2)
$X = 5A + B_1 + C_1 + 4D_1 + 3D_2 + 2D_3 + D_4 + 4E_1 + 3E_2 + 2E_3 + E_4$

$WCS = (Y, 1)$
(1) $Y = A + B_1 + C_1$
(2) $Y = A + B_1 + E_1 + E_2 + E_3 + E_4$

(3) $Y = A + D_1 + D_2 + D_3 + D_4 + E_1 + E_2 + E_3 + E_4$
(4) $Y = 2A + B_1 + C_1 + D_1 + E_1$
(5) $Y = 3A + B_1 + C_1 + 2D_1 + D_2 + 2E_1 + E_2$
(6) $Y = 4A + B_1 + C_1 + 3D_1 + 2D_2 + D_3 + 3E_1 + 2E_2 + E_3$

$WCS = (Y, 2)$
(7) $Y = 2A + 2D_1 + 2D_2 + 2D_3 + 2E_1 + 2E_2 + 2E_3$
(8) $Y = 2A + B_1 + C_1 + 2D_1 + 2D_2 + 2D_3$
(9) $Y = 2A + C_1 + D_1 + 2E_1 + 2E_2 + 2E_3$

$WCS = (Y, 3)$
(10) $Y = 3A + 3D_1 + 3D_2 + 3E_1 + 3E_2$
(11) $Y = 3A + B_1 + C_1 + D_1 + 3E_1 + 3E_2$

$WCS = (Y, 4)$
(12) $Y = 4A + 4D_1 + 4E_1$

[4] 5.6 type, (-3)
$X = 5A + 3B_1 + B_2 + 4C_1 + 3C_2 + 2C_3 + C_4 + 4D_1 + 3D_2 + 2D_3 + D_4$
$+ 4E_1 + 3E_2 + 2E_3 + E_4$

$WCS = (Y, 1)$
(1) $Y = A + B_1 + B_2 + C_1 + C_2 + C_3 + C_4 + D_1 + D_2 + D_3 + D_4$
(2) $Y = A + C_1 + C_2 + C_3 + C_4 + D_1 + D_2 + D_3 + D_4 + E_1 + E_2 + E_3 + E_4$

$WCS = (Y, 2)$
(3) $Y = 2A + 2C_1 + 2C_2 + 2C_3 + 2D_1 + 2D_2 + 2D_3 + 2E_1 + 2E_2 + 2E_3$
(4) $Y = 2A + B_1 + C_1 + 2D_1 + 2D_2 + 2D_3 + 2E_1 + 2E_2 + 2E_3$

$WCS = (Y, 3)$
(5) $Y = 3A + 3C_1 + 3C_2 + 3D_1 + 3D_2 + 3E_1 + 3E_2$
(6) $Y = 3A + B_1 + 2C_1 + 3C_2 + 3D_1 + 3D_2 + 3E_1 + 3E_2$
(7) $Y = 3A + 2B_1 + B_2 + C_1 + 3D_1 + 3D_2 + 3E_1 + 3E_2$

$WCS = (Y, 4)$
(8) $Y = 4A + 4C_1 + 4D_1 + 4E_1$
(9) $Y = 4A + B_1 + 3C_1 + 2C_2 + C_3 + 4D_1 + 4E_1$
(10) $Y = 4A + 2B_1 + 3C_1 + 2C_2 + C_3 + 3D_1 + 2D_2 + D_3 + 4E_1$

Order 4 $r = (2, 2, 2, 2, 4, 4)$

[4] 4.1.1 type, (-3)
$X = 4A + B_1 + 2C_1 + 2D_1 + 2E_1 + 2F_1 + 3G_1 + 2G_2 + G_3$

$WCS = (Y, 1)$
(1) $Y = 2A + B_1 + C_1 + D_1 + E_1 + F_1 + G_1$

$WCS = (Y, 2)$
(2) $Y = 2A + B_1 + C_1 + D_1 + E_1 + 2G_1 + 2G_2$
(3) $Y = 2A + C_1 + D_1 + E_1 + F_1 + 2G_1 + 2G_2$

Order 4 $\quad r = (2, 4, 4, 4, 4)$

[4] **4.2.1 type,** (-2)
$X = 4A + B_1 + C_1 + D_1 + 2E_1 + 3F_1 + 2F_2 + F_3$

$WCS = (Y, 1)$
(1) $Y = A + B_1 + C_1$
(2) $Y = A + B_1 + F_1 + F_2 + F_3$
(3) $Y = 2A + B_1 + C_1 + D_1 + E_1$
(4) $Y = 2A + B_1 + C_1 + D_1 + F_1$
(5) $Y = 2A + B_1 + C_1 + E_1 + F_1$
(6) $Y = 3A + B_1 + C_1 + D_1 + E_1 + 2F_1 + F_2$

$WCS = (Y, 2)$
(7) $Y = 2A + B_1 + C_1 + 2F_1 + 2F_2$
(8) $Y = 2A + B_1 + E_1 + 2F_1 + 2F_2$

$WCS = (Y, 3)$
(9) $Y = 3A + B_1 + C_1 + D_1 + 3F_1$
(10) $Y = 3A + B_1 + C_1 + E_1 + 3F_1$

Or apply SR technique for A_4-singularity.

[4] **4.2.2 type,** (-3)
$X = 4A + B_1 + 2C_1 + 3D_1 + 2D_2 + D_3 + 3E_1 + 2E_2 + E_3 + 3F_1 + 2F_2 + F_3$

$WCS = (Y, 1)$
(1) $Y = A + B_1 + D_1 + D_2 + D_3 + E_1 + E_2 + E_3$
(2) $Y = A + D_1 + D_2 + D_3 + E_1 + E_2 + E_3 + F_1 + F_2 + F_3$

$WCS = (Y, 2)$
(3) $Y = 2A + B_1 + C_1 + D_1 + E_1 + 2F_1 + 2F_2$
(4) $Y = 2A + 2D_1 + 2D_2 + 2E_1 + 2E_2 + 2F_1 + 2F_2$
(5) $Y = 2A + B_1 + C_1 + 2D_1 + 2D_2 + 2E_1 + 2E_2$
(6) $Y = 2A + B_1 + D_1 + 2E_1 + 2E_2 + 2F_1 + 2F_2$
(7) $Y = 2A + C_1 + D_1 + 2E_1 + 2E_2 + 2F_1 + 2F_2$

$WCS = (Y, 3)$
(8) $Y = 3A + B_1 + C_1 + 2D_1 + D_2 + 2E_1 + E_2 + 3F_1$
(9) $Y = 3A + 3D_1 + 3E_1 + 3F_1$
(10) $Y = 3A + B_1 + C_1 + D_1 + 3E_1 + 3F_1$

Order 3 $\quad r = (3, 3, 3, 3, 3, 3)$

[4] **3.1 type,** (-1)
$X = 3A + B_1 + C_1 + D_1 + E_1 + F_1 + G_1$

$WCS = (Y, 1)$
(1) $Y = A + B_1$

Or apply SR technique for A_1-singularity.

[4] 3.2 type, (-3)

$X = 3A + B_1 + C_1 + D_1 + 2E_1 + E_2 + 2F_1 + F_2 + 2G_1 + G_2$

$WCS = (Y, 1)$
(1) $Y = A + B_1 + C_1 + D_1$
(2) $Y = A + B_1 + C_1 + E_1 + E_2$
(3) $Y = A + B_1 + E_1 + E_2 + F_1 + F_2$
(4) $Y = A + E_1 + E_2 + F_1 + F_2 + G_1 + G_2$
(5) $Y = 2A + B_1 + C_1 + D_1 + E_1 + F_1 + G_1$

$WCS = (Y, 2)$
(6) $Y = 2A + B_1 + C_1 + D_1 + E_1 + 2F_1$
(7) $Y = 2A + B_1 + C_1 + 2E_1 + 2F_1$
(8) $Y = 2A + 2E_1 + 2F_1 + 2G_1$

Or apply SR technique for a quotient singularity.

[4] 3.3 type, (-4)

$X = 3A+2B_1+B_2+2C_1+C_2+2D_1+D_2+2E_1+E_2+2F_1+F_2+2G_1+G_2$

$WCS = (Y, 1)$
(1) $Y = A + B_1 + B_2 + C_1 + C_2 + D_1 + D_2 + E_1 + E_2$

$WCS = (Y, 2)$
(2) $Y = 2A + 2B_1 + 2C_1 + 2D_1 + 2E_1$
(3) $Y = 2A + B_1 + C_1 + 2D_1 + 2E_1 + 2F_1$
(4) $Y = 2A + B_1 + D_1 + 2E_1 + 2F_1 + 2G_1$

Order 2 $\quad r = (2, 2, 2, 2, 2, 2, 2, 2, 2, 2)$

[4] 2.1 type, (-5)

$X = 2A + B_1 + C_1 + D_1 + E_1 + F_1 + G_1 + H_1 + I_1 + J_1 + K_1$

$WCS = (Y, 1)$
(1) $Y = A + B_1 + C_1 + D_1 + E_1 + F_1$

Or apply SR technique for a cyclic quotient singularity.

26.4.2 Stellar singular fibers, genus(A) = 1, 2

genus$(A) = 1$

[4] A1.1 type

$X = 6A + 3B_1 + 3C_1, \quad N_A^{\otimes 6} = \mathcal{O}(-3b_1 - 3c_1)$

Take b_1, c_1 so that $b_1 - c_1$ is torsion of order 3 in Pic(A).
$WCS = (Y, 1) + (2Y, 2) + (Y, 3Y)$
(1) $Y = A + B_1, \quad N_A = \mathcal{O}(-b_1),$

[4] **A1.2 type**
$X = 4A + B_1 + 3C_1 + 2C_2 + C_3, \quad N_A^{\otimes 4} = \mathcal{O}(-b_1 - 3c_1)$

$WCS = (Y, 1)$
(1) $Y = A + B_1 + C_1 + C_2 + C_3, \quad N_A = \mathcal{O}(-b_1 - c_1 + q)$,
where $3b_1 + c_1 \sim 4q$.

[4] **A1.3 type**
$X = 3A + B_1 + C_1 + D_1, \quad N_A^{\otimes 3} = \mathcal{O}(-b_1 - c_1 - d_1)$

$WCS = (Y, 1)$
(1) $Y = A + B_1, \quad N_A = \mathcal{O}(-b_1)$,
where $c_1 + d_1 \sim 2b_1$, i.e. b_1 is a Weierstrass point on A.
(2) $Y = 2A + B_1 + C_1, \quad N_A^{\otimes 2} = \mathcal{O}(-b_1 - c_1)$,
where $b_1 + c_1 \sim 2d_1$, i.e. d_1 is a Weierstrass point on A.

[4] **A1.4 type**
$X = 3A + 2B_1 + B_2 + 2C_1 + C_2 + 2D_1 + D_2, \quad N_A^{\otimes 3} = \mathcal{O}(-2b_1 - 2c_1 - 2d_1)$

$WCS = (Y, 1)$
(1) $Y = A + B_1 + B_2 + C_1 + C_2, \quad N_A = \mathcal{O}(-b_1 - c_1)$,
where $b_1 + c_1 \sim 2d_1$, i.e. d_1 is a Weierstrass point on A.

$WCS = (Y, 2)$
(2) $Y = 2A + B_1 + B_2 + C_1 + C_2 + 2D_1, \quad N_A^{\otimes 2} = \mathcal{O}(-b_1 - c_1 - 2d_1)$,
where $b_1 + c_1 \sim 2d_1$, i.e. d_1 is a Weierstrass point on A.
(3) $Y = 2A + 2B_1 + 2C_1, \quad N_A^{\otimes 2} = \mathcal{O}(-2b_1 - 2c_1)$,
where $2b_1 + 2c_1 \sim 4d_1$, e.g. d_1 is a Weierstrass point on A.

[4] **A1.5 type**
$X = 2A + B_1 + C_1 + D_1 + E_1 + F_1 + G_1, \quad N_A^{\otimes 2} = \mathcal{O}(-b_1 - c_1 - d_1 - e_1 - f_1 - g_1)$

$WCS = (Y, 1)$
(1) $Y = A + B_1 + C_1 + D_1, \quad N_A = \mathcal{O}(-b_1 - c_1 - d_1)$,
where $b_1 + c_1 + d_1 \sim e_1 + f_1 + g_1$

genus$(A) = 2$

[4] **A2.1 type**
$X = 3A$ atom

[4] **A2.2 type**
Suppose A is a hyperelliptic curve.
$X = 2A + B_1 + C_1, \quad N_A^{\otimes 2} = \mathcal{O}(-b_1 - c_1)$

$WCS = (Y, 1)$
(1) $Y = A + B_1 + C_1, \quad N_A = \mathcal{O}(-b_1 - c_1 + q)$,
where $b_1 + c_1 \sim 2q$ (linearly equivalent), i.e. q is a Weierstrass point on A.

26.4.3 Self-welding and self-connecting of genus 3 or 2

[4] **sw3.1 type** sw([3] 12.4)
$X = 12A + 6B_1 + 7C_1 + 2C_2 + 11D_1 + 10D_2 + 9D_3 + 8D_4 + 7D_5 + 6D_6 + 5D_7 + 4D_8 + 3D_9 + 2D_{10}, \quad C_2 = D_{10}$

$WCS = (Y, 1) + (2Y, 2)$
(1) $Y = A+B_1+C_1+C_2+D_1+D_2+D_3+D_4+D_5+D_6+D_7+D_8+D_9+D_{10}$

$WCS = (Y, 2) + (2Y, 4) + (3Y, 6) + (4Y, 8) + (5Y, 10)$,
semi-rigid: $C_2(= D_{10}), D_7$
(2) $Y = 2A + B_1 + C_1 + D_1 + D_2$

$WCS = (Y, 1) + (2Y, 2) + (3Y, 3) + (4Y, 4) + (5Y, 5) + (6Y, 6)$
(3) $Y = A + B_1 + D_1 + D_2 + D_3 + D_4 + D_5 + D_6$,

$WCS = (Y, 1) + (2Y, 2) + (3Y, 3) + (4Y, 4) + (5Y, 5) + (6Y, 6) + (7Y, 7)$
(4) $Y = A + C_1 + D_1 + D_2 + D_3 + D_4 + D_5$

[4] **sw3.2 type** sw([3] 9.3)
$X = 9A + 3B_1 + 2C_1 + 4D_1 + 3D_2 + 2D_3, \quad C_1 = D_3$

$WCS = (Y, 1) + (2Y, 2)$
(1) $Y = 2A + C_1 + D_1 + D_2$

[4] **sw3.3 type** sw([3] 9.4)
$X = 9A + 3B_1 + 7C_1 + 5C_2 + 3C_3 + 8D_1 + 7D_2 + 6D_3 + 5D_4 + 4D_5 + 3D_6, \quad C_3 = D_6$

$WCS = (Y, 1) + (2Y, 2) + (3Y, 3)$
(1) $Y = A + C_1 + C_2 + C_3 + D_1 + D_2 + D_3 + D_4 + D_5 + D_6$

[4] **sw3.4 type** sw([3] 9.5)
$X = 9A + 6B_1 + 3B_2 + 4C_1 + 3C_2 + 2C_3 + 8D_1 + 7D_2 + 6D_3 + 5D_4 + 4D_5 + 3D_6 + 2D_7, \quad C_3 = D_7$

$WCS = (Y, 3) + (2Y, 6)$
(1) $Y = 3A + 2B_1 + B_2 + C_1 + 3D_1 + 3D_2 + 3D_3$

[4] **sw3.5 type** sw([3] 8.4)
$X = 8A + 2B_1 + 7C_1 + 6C_2 + 5C_3 + 4C_4 + 3C_5 + 2C_6 + 7D_1 + 6D_2 + 5D_3 + 4D_4 + 3D_5 + 2D_6, \quad C_6 = D_6$

$WCS = (Y, 1) + (2Y, 2)$, semi-rigid: B_1
(1) $Y = A + C_1 + C_2 + C_3 + C_4 + C_5 + C_6 + D_1 + D_2 + D_3 + D_4 + D_5 + D_6$

[4] **sw3.6 type** sw([3] 8.5)
$X = 8A+3B_1+6C_1+4C_2+2C_3+7D_1+6D_2+5D_3+4D_4+3D_5, \quad B_1 = D_5$

$WCS = (Y, 1) + (2Y, 2) + (3Y, 3)$, semi-rigid: C_1
(1) $Y = A + B_1 + D_1 + D_2 + D_3 + D_4 + D_5$

[4] **sw3.7 type** sw([3] 8.6)
$X = 8A + 5B_1 + 2B_2 + 5C_1 + 2C_2 + 6D_1 + 4D_2 + 2D_3, \quad B_2 = C_2$

$WCS = (Y, 1) + (2Y, 2)$, semi-rigid: D_1
(1) $Y = A + B_1 + B_2 + C_1 + C_2$

[4] **sw3.8 type** sw([3] 7.3)
$X = 7A + B_1 + 3C_1 + 2C_2 + 3D_1 + 2D_2, \quad C_2 = D_2$

$WCS = (Y, 1) + (2Y, 2)$
(1) $Y = 2A + C_1 + C_2 + D_1 + D_2$

[4] **sw3.9 type** sw([3] 7.4) case 1
$X = 7A + 2B_1 + 2C_1 + 3D_1 + 2D_2 + D_3, \quad B_1 = C_1$

$WCS = (Y, 2)$
(1) $Y = 4A + B_1 + C_1 + 2D_1 + 2D_2$

$WCS = (Y, 1) + (2Y, 2)$, semi-rigid: D_1
(2) $Y = 2A + B_1 + C_1$

[4] **sw3.10 type** sw([3] 7.4) case 2
$X = 7A + 2B_1 + B_2 + 2C_1 + 3D_1 + 2D_2, \quad C_1 = D_2$

$WCS = (Y, 1) + (2Y, 2)$, semi-rigid: B_1
(1) $Y = 2A + C_1 + D_1 + D_2$

[4] **sw3.11 type** sw([3] 7.5) case 1
$X = 7A + 2B_1 + 6C_1 + 5C_2 + 4C_3 + 3C_4 + 2C_5 + 6D_1 + 5D_2 + 4D_3 + 3D_4 + 2D_5 + D_6, \quad B_1 = C_5$

$WCS = (Y, 4)$
(1) $Y = 4A + B_1 + 3C_1 + 2C_2 + C_3 + C_4 + C_5 + 4D_1 + 4D_2 + 4D_3$

$WCS = (Y, 5)$
(2) $Y = 5A + B_1 + 4C_1 + 3C_2 + 2C_3 + C_4 + C_5 + 5D_1 + 5D_2$

$WCS = (Y, 6)$
(3) $Y = 6A + B_1 + 5C_1 + 4C_2 + 3C_3 + 2C_4 + C_5 + 6D_1$

$WCS = (Y, 1) + (2Y, 2)$, semi-rigid: D_1
(4) $Y = A + B_1 + C_1 + C_2 + C_3 + C_4 + C_5$

[4] **sw3.12 type** sw([3] 7.5) case 2
$X = 7A + 2B_1 + B_2 + 6C_1 + 5C_2 + 4C_3 + 3C_4 + 2C_5 + 6D_1 + 5D_2 + 4D_3 + 3D_4 + 2D_5, \quad C_5 = D_5$

$WCS = (Y, 1) + (2Y, 2)$, semi-rigid: B_1
(1) $Y = A + C_1 + C_2 + C_3 + C_4 + C_5 + D_1 + D_2 + D_3 + D_4 + D_5$

[4] sw3.13 type sw([3] 7.6) case 1
$X = 7A+3B_1+5C_1+3C_2+6D_1+5D_2+4D_3+3D_4+2D_5+D_6, \quad B_1 = C_2$

$WCS = (Y, 3)$
(1) $Y = 3A + B_1 + 2C_1 + C_2 + 3D_1 + 3D_2 + 3D_3 + 3D_4$

$WCS = (Y, 1) + (2Y, 2) + (3Y, 3)$, semi-rigid: D_1
(2) $Y = A + B_1 + C_1 + C_2$

[4] sw3.14 type sw([3] 7.6) case 2
$X = 7A + 3B_1 + 2B_2 + 5C_1 + 3C_2 + C_3 + 6D_1 + 5D_2 + 4D_3 + 3D_4 + 2D_5,$
$B_2 = D_5$

$WCS = (Y, 1) + (2Y, 2) + (3Y, 3)$, semi-rigid: C_1
(1) $Y = A + B_1 + D_1 + D_2 + D_3 + D_4$

[4] sw3.15 type sw([3] 7.6) case 3
$X = 7A+3B_1+2B_2+B_3+5C_1+3C_2+6D_1+5D_2+4D_3+3D_4, \quad C_2 = D_4$

$WCS = (Y, 1) + (2Y, 2) + (3Y, 3)$, semi-rigid: B_1
(1) $Y = A + C_1 + C_2 + D_1 + D_2 + D_3 + D_4$

[4] sw3.16 type sw([3] 7.7)
$X = 7A + 4B_1 + B_2 + 4C_1 + 6D_1 + 5D_2 + 4D_3, \quad C_1 = D_3$

$WCS = (Y, 1) + (2Y, 2) + (3Y, 3) + (4Y, 4)$, semi-rigid: B_1
(1) $Y = A + C_1 + D_1 + D_2 + D_3$

[4] sw3.17 type sw([3] 4.1.2)
$X = 4A + B_1 + C_1 + 3D_1 + 2D_2 + 3E_1 + 2E_2, \quad D_2 = E_2$

$WCS = (Y, 1)$
(1) $Y = A + B_1 + C_1$
(2) $Y = 2A + B_1 + C_1 + D_1 + E_1$

$WCS = (Y, 1) + (2Y, 2)$
(3) $Y = A + D_1 + D_2 + E_1 + E_2$

$WCS = (Y, 1) + (2Y, 2) + (3Y, 3)$
(4) $Y = A + D_1 + E_1$

[4] sw3.18 type sw([3] 4.1.3)
$X = 4A + 3B_1 + 2B_2 + 3C_1 + 2C_2 + 3D_1 + 2D_2 + D_3 + 3E_1 + 2E_2 + E_3,$
$B_2 = C_2$

$WCS = (Y, 2)$
(1) $Y = 2A + B_1 + C_1 + 2D_1 + 2D_2 + 2E_1 + 2E_2$

$WCS = (Y, 3)$
(2) $Y = 3A + 2B_1 + B_2 + 2C_1 + C_2 + 2D_1 + D_2 + 3E_1$

$WCS = (Y, 1) + (2Y, 2)$
(3) $Y = A + B_1 + B_2 + C_1 + C_2 + D_1 + D_2$

$WCS = (Y, 1) + (2Y, 2) + (3Y, 3)$, semi-rigid: E_1
(4) $Y = A + B_1 + C_1 + D_1$

[4] **sw3.19 type** sw([3] 4.2.2)
$X = 4A + 2B_1 + 2C_1 + 2D_1 + 3E_1 + 2E_2 + 3F_1 + 2F_2, \quad E_2 = F_2$

$WCS = (Y, 1) + (2Y, 2)$
(1) $Y = A + B_1 + E_1 + E_2 + F_1 + F_2$
(2) $Y = A + B_1 + C_1 + D_1$

[4] **sw3.20 type** sw([3] 3.2)
$X = 3A + B_1 + 2C_1 + 2D_1 + 2E_1 + E_2 + 2F_1 + F_2, \quad C_1 = D_1$

$WCS = (Y, 2)$
(1) $Y = 2A + B_1 + C_1 + D_1 + E_1 + E_2 + 2F_1$
(2) $Y = 2A + C_1 + D_1 + 2E_1 + 2F_1$

$WCS = (Y, 1) + (2Y, 2)$
(3) $Y = A + C_1 + D_1 + E_1$

[4] **sc2.1 type** sc([2] 4.1)
$X = 4A + B_1 + 2C_1 + 2D_1 + 3E_1 + 2E_2 + E_3, \quad C_1 = D_1$

$WCS = (Y, 1)$
(1) $Y = A + B_1 + E_1 + E_2 + E_3$

$WCS = (Y, 1) + (2Y, 2)$
(2) $Y = A + C_1 + D_1$

26.4.4 Welding of stellar singular fibers of genus 3 and genus 1

If X is a welding of a stellar singular fiber X_1 of genus 3 and a stellar singular fiber X_2 of genus 1, then the splittability of X follows from (1) exceptional curve criterion or (2) trivial extension criterion by using a simple crust of X_1 or X_2.

26.4.5 Welding of stellar singular fibers of genus 2 and genus 2

[4] **wd([2]10.3+[2]8.1) type**
$X = X_1 + X_2, \quad D_2(X_1) = A(X_2)$
$X_1 = 10A + 5B_1 + 6C_1 + 2C_2 + 9D_1 + 8D_2$
$X_2 = 8A + 4B_1 + 3D_1 + D_2$

$WCS = (Y, 1), \quad Y = Y(X_1) + Y(X_2)$
$Y(X_1) = 2A + B_1 + C_1 + 2D_1 + 2D_2$
$Y(X_2) = 2A + C_1 + D_1 + D_2$

[4] wd([2]8.2+[2]6.1.1) type

$X = X_1 + X_2, \quad D_2(X_1) = A(X_2)$
$X_1 = 8A + 4B_1 + 5C_1 + 2C_2 + C_3 + 7D_1 + 6D_2$
$X_2 = 6A + C_1 + 4D_1 + 2D_2$

$WCS = (Y, 1), \quad Y = Y(X_1) + Y(X_2)$
$Y(X_1) = 2A + B_1 + C_1 + 2D_1 + 2D_2$
$Y(X_2) = 2A + C_1 + D_1$

26.4.6 Welding of stellar singular fibers of genus 2, 1, and 1

[4] wd([2]5.1+II^*+II^*) type

$X = X_1 + X_2 + X_3, \quad A(X_1) = D_1(X_2) = D_1(X_3)$
$X_1 = 5A + 3D_1 + D_2$
$X_2 = 6A + 3B_1 + 4C_1 + 2C_2 + 5D_1$
$X_3 = 6A + 3B_1 + 4C_1 + 2C_2 + 5D_1$

$WCS = (Y, 1) + (2Y, 2) + (3Y, 3), \quad Y = Y(X_1) + Y(X_2) + Y(X_3)$
$Y(X_1) = A + D_1$
$Y(X_2) = A + B_1 + D_1$
$Y(X_3) = A + B_1 + D_1$

26.5 Genus 5

26.5.1 Stellar singular fibers, $A = \mathbb{P}^1$

Order 22 $\quad r = (2, 11, 22)$

[5] 22.1 type, (-1)

$X = 22A + 11B_1 + 2C_1 + 9D_1 + 5D_2 + D_3$

$WCS = (Y, 1)$
(1) $Y = 2A + B_1 + D_1 + D_2 + D_3$

[5] 22.2 type, (-1)

$X = 22A + 11B_1 + 4C_1 + 2C_2 + 7D_1 + 6D_2 + 5D_3 + 4D_4 + 3D_5 + 2D_6 + D_7$

$WCS = (Y, 2)$
(1) $Y = 6A + 3B_1 + C_1 + 2D_1 + 2D_2 + 2D_3 + 2D_4 + 2D_5 + 2D_6$

[5] 22.3 type, (-1)

$X = 22A + 11B_1 + 6C_1 + 2C_2 + 5D_1 + 3D_2 + D_3$

$WCS = (Y, 1)$
(1) $Y = 4A + 2B_1 + C_1 + D_1 + D_2 + D_3$

[5] 22.4 type, (-1)

$X = 22A + 11B_1 + 8C_1 + 2C_2 + 3D_1 + 2D_2 + D_3$

$WCS = (Y, 1) + (2Y, 2)$
(1) $Y = 8A + 4B_1 + 3C_1 + C_2 + D_1$

[5] 22.5 type, (-1)
$X = 22A + 11B_1 + C_1 + 10D_1 + 8D_2 + 6D_3 + 4D_4 + 2D_5$

$WCS = (Y, 1)$
(1) $Y = A + C_1$
(2) $Y = 2A + B_1 + C_1$
(3) $Y = 3A + B_1 + C_1 + D_1$
(4) $Y = 4A + 2B_1 + C_1 + D_1$
(5) $Y = 6A + 3B_1 + C_1 + 2D_1$

$WCS = (Y, 1) + (2Y, 2)$
(6) $Y = 2A + B_1 + D_1 + D_2 + D_3 + D_4 + D_5$

Or apply SR technique for a cyclic quotient singularity after contracting A.

[5] 22.6 type, (-2)
$X = 22A + 11B_1 + 12C_1 + 2C_2 + 21D_1 + 20D_2 + 19D_3 + 18D_4 + 17D_5 + 16D_6 + 15D_7 + 14D_8 + 13D_9 + 12D_{10} + 11D_{11} + 10D_{12} + 9D_{13} + 8D_{14} + 7D_{15} + 6D_{16} + 5D_{17} + 4D_{18} + 3D_{19} + 2D_{20} + D_{21}$

$WCS = (Y, 2)$
(1) $Y = 2A + B_1 + C_1 + 2D_1 + 2D_2 + 2D_3 + 2D_4 + 2D_5 + 2D_6 + 2D_7 + 2D_8 + 2D_9 + 2D_{10} + 2D_{11} + 2D_{12} + 2D_{13} + 2D_{14} + 2D_{15} + 2D_{16} + 2D_{17} + 2D_{18} + 2D_{19} + 2D_{20}$

$WCS = (Y, 4)$
(2) $Y = 4A + 2B_1 + 2C_1 + 4D_1 + 4D_2 + 4D_3 + 4D_4 + 4D_5 + 4D_6 + 4D_7 + 4D_8 + 4D_9 + 4D_{10} + 4D_{11} + 4D_{12} + 4D_{13} + 4D_{14} + 4D_{15} + 4D_{16} + 4D_{17} + 4D_{18}$

$WCS = (Y, 6)$
(3) $Y = 6A + 3B_1 + 3C_1 + 6D_1 + 6D_2 + 6D_3 + 6D_4 + 6D_5 + 6D_6 + 6D_7 + 6D_8 + 6D_9 + 6D_{10} + 6D_{11} + 6D_{12} + 6D_{13} + 6D_{14} + 6D_{15} + 6D_{16}$

$WCS = (Y, 8)$
(4) $Y = 8A + 4B_1 + 4C_1 + 8D_1 + 8D_2 + 8D_3 + 8D_4 + 8D_5 + 8D_6 + 8D_7 + 8D_8 + 8D_9 + 8D_{10} + 8D_{11} + 8D_{12} + 8D_{13} + 8D_{14}$

$WCS = (Y, 10)$
(5) $Y = 10A + 5B_1 + 5C_1 + 10D_1 + 10D_2 + 10D_3 + 10D_4 + 10D_5 + 10D_6 + 10D_7 + 10D_8 + 10D_9 + 10D_{10} + 10D_{11} + 10D_{12}$

$WCS = (Y, 12)$
(6) $Y = 12A + 6B_1 + 6C_1 + 12D_1 + 12D_2 + 12D_3 + 12D_4 + 12D_5 + 12D_6 + 12D_7 + 12D_8 + 12D_9 + 12D_{10}$

$WCS = (Y, 14)$
(7) $Y = 14A + 7B_1 + 7C_1 + 14D_1 + 14D_2 + 14D_3 + 14D_4 + 14D_5 + 14D_6 + 14D_7 + 14D_8$

$WCS = (Y, 16)$
(8) $Y = 16A + 8B_1 + 8C_1 + 16D_1 + 16D_2 + 16D_3 + 16D_4 + 16D_5 + 16D_6$

$WCS = (Y, 18)$
(9) $Y = 18A + 9B_1 + 9C_1 + 18D_1 + 18D_2 + 18D_3 + 18D_4$

$WCS = (Y, 20)$
(10) $Y = 20A + 10B_1 + 10C_1 + 20D_1 + 20D_2$

$WCS = (Y, 11)$
(11) $Y = 11A + 5B_1 + 6C_1 + C_2 + 11D_1 + 11D_2 + 11D_3 + 11D_4 + 11D_5 + 11D_6 + 11D_7 + 11D_8 + 11D_9 + 11D_{10} + 11D_{11}$

[5] 22.7 type, (-2)
$X = 22A + 11B_1 + 14C_1 + 6C_2 + 4C_3 + 2C_4 + 19D_1 + 16D_2 + 13D_3 + 10D_4 + 7D_5 + 4D_6 + D_7$

$WCS = (Y, 1)$
(1) $Y = 8A+4B_1+5C_1+2C_2+C_3+7D_1+6D_2+5D_3+4D_4+3D_5+2D_6+D_7$

$WCS = (Y, 2) + (2Y, 4)$
(2) $Y = 6A + 3B_1 + 4C_1 + 2C_2 + 2C_3 + 5D_1 + 4D_2 + 3D_3 + 2D_4 + D_5$

[5] 22.8 type, (-2)
$X = 22A+11B_1+16C_1+10C_2+4C_3+2C_4+17D_1+12D_2+7D_3+2D_4+D_5$

$WCS = (Y, 2)$
(1) $Y = 18A+9B_1+13C_1+8C_2+3C_3+C_4+14D_1+10D_2+6D_3+2D_4+D_5$

$WCS = (Y, 1) + (2Y, 2)$
(2) $Y = 4A + 2B_1 + 3C_1 + 2C_2 + C_3 + C_4 + 3D_1 + 2D_2 + D_3$

[5] 22.9 type, (-2)
$X = 22A + 11B_1 + 15C_1 + 8C_2 + C_3 + 18D_1 + 14D_2 + 10D_3 + 6D_4 + 2D_5$

$WCS = (Y, 1)$
(1) $Y = 4A + 2B_1 + 3C_1 + 2C_2 + C_3 + 3D_1 + 2D_2 + D_3$

$WCS = (Y, 1) + (2Y, 2)$
(2) $Y = 6A + 3B_1 + 4C_1 + 2C_2 + 5D_1 + 4D_2 + 3D_3 + 2D_4 + D_5$

[5] 22.10 type, (-2)
$X = 22A + 11B_1 + 13C_1 + 4C_2 + 3C_3 + 2C_4 + C_5 + 20D_1 + 18D_2 + 16D_3 + 14D_4 + 12D_5 + 10D_6 + 8D_7 + 6D_8 + 4D_9 + 2D_{10}$

$WCS = (Y, 2)$
(1) $Y = 10A + 5B_1 + 6C_1 + 2C_2 + 2C_3 + 2C_4 + 9D_1 + 8D_2 + 7D_3 + 6D_4 + 5D_5 + 4D_6 + 3D_7 + 2D_8 + D_9$

$WCS = (Y, 4)$
(2) $Y = 20A + 10B_1 + 12C_1 + 4C_2 + 18D_1 + 16D_2 + 14D_3 + 12D_4 + 10D_5 + 8D_6 + 6D_7 + 4D_8 + 2D_9$

Order 20 $\quad r = (2, 20, 20)$

[5] 20.1 type, (-1)
$X = 20A + 10B_1 + C_1 + 9D_1 + 7D_2 + 5D_3 + 3D_4 + D_5$

$WCS = (Y, 1)$
(1) $Y = A + C_1$
(2) $Y = 2A + B_1 + C_1$
(3) $Y = 2A + C_1 + D_1 + D_2 + D_3 + D_4 + D_5$
(4) $Y = 2A + B_1 + D_1 + D_2 + D_3 + D_4 + D_5$
(5) $Y = 3A + B_1 + C_1 + D_1$
(6) $Y = 4A + 2B_1 + C_1 + D_1$
(7) $Y = 6A + 3B_1 + C_1 + 2D_1$

Or apply SR technique for a cyclic quotient singularity after contracting A.

[5] 20.2 type, (-1)
$X = 20A + 10B_1 + 3C_1 + C_2 + 7D_1 + D_2$

$WCS = (Y, 1)$
(1) $Y = 2A + B_1 + D_1 + D_2$
(2) $Y = 6A + 3B_1 + C_1 + C_2 + 2D_1$

[5] 20.3 type, (-2)
$X = 20A + 10B_1 + 11C_1 + 2C_2 + C_3 + 19D_1 + 18D_2 + 17D_3 + 16D_4 + 15D_5 + 14D_6 + 13D_7 + 12D_8 + 11D_9 + 10D_{10} + 9D_{11} + 8D_{12} + 7D_{13} + 6D_{14} + 5D_{15} + 4D_{16} + 3D_{17} + 2D_{18} + D_{19}$

$WCS = (Y, 2)$
(1) $Y = 2A + B_1 + C_1 + 2D_1 + 2D_2 + 2D_3 + 2D_4 + 2D_5 + 2D_6 + 2D_7 + 2D_8 + 2D_9 + 2D_{10} + 2D_{11} + 2D_{12} + 2D_{13} + 2D_{14} + 2D_{15} + 2D_{16} + 2D_{17} + 2D_{18}$

$WCS = (Y, 4)$
(2) $Y = 4A + 2B_1 + 2C_1 + 4D_1 + 4D_2 + 4D_3 + 4D_4 + 4D_5 + 4D_6 + 4D_7 + 4D_8 + 4D_9 + 4D_{10} + 4D_{11} + 4D_{12} + 4D_{13} + 4D_{14} + 4D_{15} + 4D_{16}$

$WCS = (Y, 6)$
(3) $Y = 6A + 3B_1 + 3C_1 + 6D_1 + 6D_2 + 6D_3 + 6D_4 + 6D_5 + 6D_6 + 6D_7 + 6D_8 + 6D_9 + 6D_{10} + 6D_{11} + 6D_{12} + 6D_{13} + 6D_{14}$

$WCS = (Y, 8)$
(4) $Y = 8A + 4B_1 + 4C_1 + 8D_1 + 8D_2 + 8D_3 + 8D_4 + 8D_5 + 8D_6 + 8D_7 + 8D_8 + 8D_9 + 8D_{10} + 8D_{11} + 8D_{12}$

$WCS = (Y, 10)$
(5) $Y = 10A + 5B_1 + 5C_1 + 10D_1 + 10D_2 + 10D_3 + 10D_4 + 10D_5 + 10D_6 + 10D_7 + 10D_8 + 10D_9 + 10D_{10}$

$WCS = (Y, 12)$
(6) $Y = 12A + 6B_1 + 6C_1 + 12D_1 + 12D_2 + 12D_3 + 12D_4 + 12D_5 + 12D_6 + 12D_7 + 12D_8$

$WCS = (Y, 14)$
(7) $Y = 14A + 7B_1 + 7C_1 + 14D_1 + 14D_2 + 14D_3 + 14D_4 + 14D_5 + 14D_6$

$WCS = (Y, 16)$
(8) $Y = 16A + 8B_1 + 8C_1 + 16D_1 + 16D_2 + 16D_3 + 16D_4$

$WCS = (Y, 18)$
(9) $Y = 18A + 9B_1 + 9C_1 + 18D_1 + 18D_2$

$WCS = (Y, 9)$
(10) $Y = 9A + 4B_1 + 5C_1 + C_2 + C_3 + 9D_1 + 9D_2 + 9D_3 + 9D_4 + 9D_5 + 9D_6 + 9D_7 + 9D_8 + 9D_9 + 9D_{10} + 9D_{11}$

$WCS = (Y, 11)$
(11) $Y = 11A + 5B_1 + 6C_1 + C_2 + 11D_1 + 11D_2 + 11D_3 + 11D_4 + 11D_5 + 11D_6 + 11D_7 + 11D_8 + 11D_9$

[5] 20.4 type, (-2)
$X = 20A + 10B_1 + 13C_1 + 6C_2 + 5C_3 + 4C_4 + 3C_5 + 2C_6 + C_7 + 17D_1 + 14D_2 + 11D_3 + 8D_4 + 5D_5 + 2D_6 + D_7$

$WCS = (Y, 2)$
(1) $Y = 6A + 3B_1 + 4C_1 + 2C_2 + 2C_3 + 2C_4 + 2C_5 + 2C_6 + 5D_1 + 4D_2 + 3D_3 + 2D_4 + D_5$

$WCS = (Y, 4)$
(2) $Y = 12A + 6B_1 + 8C_1 + 4C_2 + 4C_3 + 4C_4 + 10D_1 + 8D_2 + 6D_3 + 4D_4 + 2D_5$

$WCS = (Y, 6)$
(3) $Y = 18A + 9B_1 + 12C_1 + 6C_2 + 15D_1 + 12D_2 + 9D_3 + 6D_4 + 3D_5$

Order 15 $\quad r = (3, 15, 15)$

[5] 15.1 type, (-1)
$X = 15A + 5B_1 + 2C_1 + C_2 + 8D_1 + D_2$

$WCS = (Y, 1)$
(1) $Y = A + D_1 + D_2$
(2) $Y = 3A + B_1 + 2D_1 + D_2$
(3) $Y = 7A + 2B_1 + C_1 + C_2 + 4D_1 + D_2$
(4) $Y = 9A + 3B_1 + C_1 + 5D_1 + D_2$

[5] 15.2 type, (-1)
$X = 15A + B_1 + 10C_1 + 5C_2 + 4D_1 + D_2$

$WCS = (Y, 1)$
(1) $Y = A + B_1$
(2) $Y = 3A + B_1 + 2C_1 + C_2$
(3) $Y = 3A + 2C_1 + C_2 + D_1 + D_2$
(4) $Y = 4A + B_1 + 2C_1 + D_1$
(5) $Y = 7A + B_1 + 4C_1 + C_2 + 2D_1 + D_2$

Or apply SR technique for a cyclic quotient singularity after contracting A.

[5] 15.3 type, (-2)

$X = 15A+5B_1+11C_1+7C_2+3C_3+2C_4+C_5+14D_1+13D_2+12D_3+11D_4 +10D_5+9D_6+8D_7+7D_8+6D_9+5D_{10}+4D_{11}+3D_{12}+2D_{13}+D_{14}$

$WCS = (Y, 3)$
(1) $Y = 3A + B_1 + 2C_1 + C_2 + 3D_1 + 3D_2 + 3D_3 + 3D_4 + 3D_5 + 3D_6 + 3D_7 + 3D_8 + 3D_9 + 3D_{10} + 3D_{11} + 3D_{12}$

$WCS = (Y, 6)$
(2) $Y = 6A + 2B_1 + 4C_1 + 2C_2 + 6D_1 + 6D_2 + 6D_3 + 6D_4 + 6D_5 + 6D_6 + 6D_7 + 6D_8 + 6D_9$

$WCS = (Y, 9)$
(3) $Y = 9A + 3B_1 + 6C_1 + 3C_2 + 9D_1 + 9D_2 + 9D_3 + 9D_4 + 9D_5 + 9D_6$

$WCS = (Y, 12)$
(4) $Y = 12A + 4B_1 + 8C_1 + 4C_2 + 12D_1 + 12D_2 + 12D_3$

$WCS = (Y, 4)$
(5) $Y = 4A+B_1+3C_1+2C_2+C_3+C_4+C_5+4D_1+4D_2+4D_3+4D_4+4D_5 + 4D_6 + 4D_7 + 4D_8 + 4D_9 + 4D_{10} + 4D_{11}$

$WCS = (Y, 7)$
(6) $Y = 7A + 2B_1 + 5C_1 + 3C_2 + C_3 + 7D_1 + 7D_2 + 7D_3 + 7D_4 + 7D_5 + 7D_6 + 7D_7 + 7D_8$

[5] 15.4 type, (-2)

$X = 15A + 10B_1 + 5B_2 + 7C_1 + 6C_2 + 5C_3 + 4C_4 + 3C_5 + 2C_6 + C_7 + 13D_1 + 11D_2 + 9D_3 + 7D_4 + 5D_5 + 3D_6 + D_7$

$WCS = (Y, 3)$
(1) $Y = 6A + 4B_1 + 2B_2 + 3C_1 + 3C_2 + 3C_3 + 3C_4 + 3C_5 + 5D_1 + 4D_2 + 3D_3 + 2D_4 + D_5$

$WCS = (Y, 6)$
(2) $Y = 12A + 8B_1 + 4B_2 + 6C_1 + 6C_2 + 10D_1 + 8D_2 + 6D_3 + 4D_4 + 2D_5$

Order 12 $r = (6, 12, 12)$

[5] 12.1 type, (-1)

$X = 12A + 2B_1 + 5C_1 + 3C_2 + C_3 + 5D_1 + 3D_2 + D_3$

$WCS = (Y, 1)$
(1) $Y = 2A + C_1 + C_2 + C_3 + D_1 + D_2 + D_3$

[5] 12.2 type, (-1)

$X = 12A + B_1 + C_1 + 10D_1 + 8D_2 + 6D_3 + 4D_4 + 2D_5$

$WCS = (Y, 1)$
(1) $Y = A + B_1$
(2) $Y = 2A + B_1 + C_1$

(3) $Y = 3A + B_1 + 2D_1 + D_2$
(4) $Y = 3A + B_1 + C_1 + D_1$
(5) $Y = 4A + B_1 + C_1 + 2D_1$
(6) $Y = 4A + B_1 + 3D_1 + 2D_2 + D_3$
(7) $Y = 5A + B_1 + C_1 + 3D_1 + D_2$
(8) $Y = 5A + B_1 + 4D_1 + 3D_2 + 2D_3 + D_4$
(9) $Y = 6A + B_1 + 5D_1 + 4D_2 + 3D_3 + 2D_4 + D_5$

Or apply SR technique for the A_5-singularity after contracting A.

[5] 12.3 type, (-2)

$X = 12A + 2B_1 + 11C_1 + 10C_2 + 9C_3 + 8C_4 + 7C_5 + 6C_6 + 5C_7 + 4C_8 + 3C_9 + 2C_{10} + C_{11} + 11D_1 + 10D_2 + 9D_3 + 8D_4 + 7D_5 + 6D_6 + 5D_7 + 4D_8 + 3D_9 + 2D_{10} + D_{11}$

$WCS = (Y, 1)$
(1) $Y = A + C_1 + C_2 + C_3 + C_4 + C_5 + C_6 + C_7 + C_8 + C_9 + C_{10} + C_{11} + D_1 + D_2 + D_3 + D_4 + D_5 + D_6 + D_7 + D_8 + D_9 + D_{10} + D_{11}$

$WCS = (Y, 2)$
(2) $Y = 2A + 2C_1 + 2C_2 + 2C_3 + 2C_4 + 2C_5 + 2C_6 + 2C_7 + 2C_8 + 2C_9 + 2C_{10} + 2D_1 + 2D_2 + 2D_3 + 2D_4 + 2D_5 + 2D_6 + 2D_7 + 2D_8 + 2D_9 + 2D_{10}$

$WCS = (Y, 3)$
(3) $Y = 3A + 3C_1 + 3C_2 + 3C_3 + 3C_4 + 3C_5 + 3C_6 + 3C_7 + 3C_8 + 3C_9 + 3D_1 + 3D_2 + 3D_3 + 3D_4 + 3D_5 + 3D_6 + 3D_7 + 3D_8 + 3D_9$

$WCS = (Y, 4)$
(4) $Y = 4A + 4C_1 + 4C_2 + 4C_3 + 4C_4 + 4C_5 + 4C_6 + 4C_7 + 4C_8 + 4D_1 + 4D_2 + 4D_3 + 4D_4 + 4D_5 + 4D_6 + 4D_7 + 4D_8$

$WCS = (Y, 5)$
(5) $Y = 5A + 5C_1 + 5C_2 + 5C_3 + 5C_4 + 5C_5 + 5C_6 + 5C_7 + 5D_1 + 5D_2 + 5D_3 + 5D_4 + 5D_5 + 5D_6 + 5D_7$

$WCS = (Y, 6)$
(6) $Y = 6A + 6C_1 + 6C_2 + 6C_3 + 6C_4 + 6C_5 + 6C_6 + 6D_1 + 6D_2 + 6D_3 + 6D_4 + 6D_5 + 6D_6$

$WCS = (Y, 7)$
(7) $Y = 7A + 7C_1 + 7C_2 + 7C_3 + 7C_4 + 7C_5 + 7D_1 + 7D_2 + 7D_3 + 7D_4 + 7D_5$

$WCS = (Y, 8)$
(8) $Y = 8A + 8C_1 + 8C_2 + 8C_3 + 8C_4 + 8D_1 + 8D_2 + 8D_3 + 8D_4$

$WCS = (Y, 9)$
(9) $Y = 9A + 9C_1 + 9C_2 + 9C_3 + 9D_1 + 9D_2 + 9D_3$

$WCS = (Y, 10)$
(10) $Y = 10A + 10C_1 + 10C_2 + 10D_1 + 10D_2$

$WCS = (Y, 11)$
(11) $Y = 11A + 11C_1 + 11D_1$

$WCS = (Y, 6)$
(12) $Y = 6A + B_1 + 5C_1 + 4C_2 + 3C_3 + 2C_4 + C_5 + 6D_1 + 6D_2 + 6D_3 + 6D_4 + 6D_5 + 6D_6$

[5] 12.4 type, (-2)
$X = 12A+7B_1+2B_2+B_3+7C_1+2C_2+C_3+10D_1+8D_2+6D_3+4D_4+2D_5$

$WCS = (Y, 1)$
(1) $Y = 5A + 3B_1 + B_2 + B_3 + 3C_1 + C_2 + C_3 + 4D_1 + 3D_2 + 2D_3 + D_4$

$WCS = (Y, 2)$
(2) $Y = 10A + 6B_1 + 2B_2 + 6C_1 + 2C_2 + 8D_1 + 6D_2 + 4D_3 + 2D_4$

Order 11 $r = (11, 11, 11)$

[5] 11.1 type, (-1)
$X = 11A + B_1 + C_1 + 9D_1 + 7D_2 + 5D_3 + 3D_4 + D_5$

$WCS = (Y, 1)$
(1) $Y = A + B_1$
(2) $Y = 2A + B_1 + C_1$
(3) $Y = 2A + B_1 + D_1$
(4) $Y = 3A + B_1 + C_1 + D_1$
(5) $Y = 3A + B_1 + 2D_1 + D_2$
(6) $Y = 4A + B_1 + C_1 + 2D_1$
(7) $Y = 4A + B_1 + 3D_1 + 2D_2 + D_3$
(8) $Y = 5A + B_1 + C_1 + 3D_1 + D_2$
(9) $Y = 5A + B_1 + 4D_1 + 3D_2 + 2D_3 + D_4$
(10) $Y = 6A + B_1 + C_1 + 4D_1 + 2D_2$
(11) $Y = 6A + B_1 + 5D_1 + 4D_2 + 3D_3 + 2D_4 + D_5$
(12) $Y = 7A + B_1 + C_1 + 5D_1 + 3D_2 + D_3$
(13) $Y = 8A + B_1 + C_1 + 6D_1 + 4D_2 + 2D_3$
(14) $Y = 9A + B_1 + C_1 + 7D_1 + 5D_2 + 3D_3 + D_4$
(15) $Y = 10A + B_1 + C_1 + 8D_1 + 6D_2 + 4D_3 + 2D_4$

Or apply SR technique for the A_4-singularity after contracting A.

[5] 11.2 type, (-1)
$X = 11A + B_1 + 2C_1 + C_2 + 8D_1 + 5D_2 + 2D_3 + D_4$

$WCS = (Y, 1)$
(1) $Y = A + B_1$
(2) $Y = 2A + B_1 + D_1$
(3) $Y = 5A + B_1 + C_1 + C_2 + 3D_1 + D_2$
(4) $Y = 6A + B_1 + C_1 + 4D_1 + 2D_2$
(5) $Y = 7A + B_1 + C_1 + 5D_1 + 3D_2 + D_3$

$WCS = (Y, 2)$
(6) $Y = 8A + B_1 + C_1 + 6D_1 + 4D_2 + 2D_3$

Or apply SR technique for a cyclic quotient singularity after contracting A.

[5] 11.3 type, (-1)
$X = 11A + B_1 + 3C_1 + C_2 + 7D_1 + 3D_2 + 2D_3 + D_4$

$WCS = (Y, 1)$
(1) $Y = A + B_1$
(2) $Y = 2A + B_1 + D_1$
(3) $Y = 3A + B_1 + C_1 + C_2 + D_1$
(4) $Y = 4A + B_1 + C_1 + 2D_1$
(5) $Y = 5A + B_1 + C_1 + 3D_1 + D_2$
(6) $Y = 7A + B_1 + 2C_1 + C_2 + 4D_1 + D_2$
(7) $Y = 8A + B_1 + 2C_1 + 5D_1 + 2D_2 + D_3$

Or apply SR technique for a cyclic quotient singularity after contracting A.

[5] 11.4 type, (-1)
$X = 11A + B_1 + 4C_1 + C_2 + 6D_1 + D_2$

$WCS = (Y, 1)$
(1) $Y = A + B_1$
(2) $Y = 2A + B_1 + D_1$
(3) $Y = 2A + B_1 + C_1 + C_2$
(4) $Y = 2A + C_1 + C_2 + D_1$
(5) $Y = 3A + B_1 + C_1 + D_1$
(6) $Y = 3A + C_1 + 2D_1 + D_2$
(7) $Y = 4A + B_1 + C_1 + 2D_1$
(8) $Y = 5A + B_1 + 2C_1 + C_2 + 2D_1$
(9) $Y = 5A + 2C_1 + C_2 + 3D_1 + D_2$
(10) $Y = 6A + B_1 + 2C_1 + 3D_1$
(11) $Y = 7A + B_1 + 2C_1 + 4D_1 + D_2$
(12) $Y = 8A + B_1 + 3C_1 + C_2 + 4D_1$
(13) $Y = 9A + B_1 + 3C_1 + 5D_1 + D_2$

Or apply SR technique for a cyclic quotient singularity after contracting A.

[5] 11.5 type, (-1)
$X = 11A + B_1 + 5C_1 + 4C_2 + 3C_3 + 2C_4 + C_5 + 5D_1 + 4D_2 + 3D_3 + 2D_4 + D_5$

$WCS = (Y, 1)$
(1) $Y = A + B_1$
(2) $Y = 2A + B_1 + C_1 + C_2 + C_3 + C_4 + C_5$
(3) $Y = 2A + C_1 + C_2 + C_3 + C_4 + C_5 + D_1 + D_2 + D_3 + D_4 + D_5$
(4) $Y = 3A + B_1 + C_1 + D_1$

(5) $Y = 5A + B_1 + 2C_1 + C_2 + 2D_1 + D_2$
(6) $Y = 7A + B_1 + 3C_1 + 2C_2 + C_3 + 3D_1 + 2D_2 + D_3$

Or apply SR technique for a cyclic quotient singularity after contracting A.

[5] **11.6 type,** (-1)
$X = 11A + 2B_1 + B_2 + 2C_1 + C_2 + 7D_1 + 3D_2 + 2D_3 + D_4$

$WCS = (Y, 1)$
(1) $Y = 5A + B_1 + B_2 + C_1 + C_2 + 3D_1 + D_2$

$WCS = (Y, 2)$
(2) $Y = 6A + B_1 + C_1 + 4D_1 + 2D_2 + 2D_3$

[5] **11.7 type,** (-1)
$X = 11A + 2B_1 + B_2 + 2C_1 + C_2 + 7D_1 + 3D_2 + 2D_3 + D_4$

$WCS = (Y, 1)$
(1) $Y = A + D_1 + D_2 + D_3$
(2) $Y = 3A + C_1 + C_2 + 2D_1 + D_2$
(3) $Y = 7A + B_1 + 2C_1 + C_2 + 4D_1 + D_2$

[5] **11.8 type,** (-1)
$X = 11A + 2B_1 + B_2 + 4C_1 + C_2 + 5D_1 + 4D_2 + 3D_3 + 2D_4 + D_5$

$WCS = (Y, 1)$
(1) $Y = 2A + C_1 + C_2 + D_1 + D_2 + D_3 + D_4 + D_5$
(2) $Y = 5A + B_1 + B_2 + 2C_1 + C_2 + 2D_1 + D_2$

$WCS = (Y, 3)$
(3) $Y = 6A + B_1 + 2C_1 + 3D_1 + 3D_2 + 3D_3$

$WCS = (Y, 4)$
(4) $Y = 8A + B_1 + 3C_1 + C_2 + 4D_1 + 4D_2$

[5] **11.9 type,** (-1)
$X = 11A + 3B_1 + B_2 + 3C_1 + C_2 + 5D_1 + 4D_2 + 3D_3 + 2D_4 + D_5$

$WCS = (Y, 1)$
(1) $Y = 3A + B_1 + B_2 + C_1 + C_2 + D_1$
(2) $Y = 7A + 2B_1 + B_2 + 2C_1 + C_2 + 3D_1 + 2D_2 + D_3$

$WCS = (Y, 2)$
(3) $Y = 4A + B_1 + C_1 + 2D_1 + 2D_2 + 2D_3 + 2D_4$

$WCS = (Y, 4)$
(4) $Y = 8A + 2B_1 + 2C_1 + 4D_1 + 4D_2$

[5] **11.10 type,** (-1)
$X = 11A + 3B_1 + B_2 + 4C_1 + C_2 + 4D_1 + D_2$

$WCS = (Y, 1)$
(1) $Y = 2A + C_1 + C_2 + D_1 + D_2$
(2) $Y = 3A + B_1 + B_2 + C_1 + D_1$
(3) $Y = 5A + B_1 + 2C_1 + C_2 + 2D_1 + D_2$
(4) $Y = 8A + B_1 + 3C_1 + C_2 + 3D_1 + D_2$

[5] 11.11 type, (-2)
$X = 11A+2B_1+B_2+10C_1+9C_2+8C_3+7C_4+6C_5+5C_6+4C_7+3C_8+2C_9 +C_{10}+10D_1+9D_2+8D_3+7D_4+6D_5+5D_6+4D_7+3D_8+2D_9+D_{10}$

$WCS = (Y, 1)$
(1) $Y = A + C_1 + C_2 + C_3 + C_4 + C_5 + C_6 + C_7 + C_8 + C_9 + C_{10} + D_1 + D_2 + D_3 + D_4 + D_5 + D_6 + D_7 + D_8 + D_9 + D_{10}$

$WCS = (Y, 2)$
(2) $Y = 2A + 2C_1 + 2C_2 + 2C_3 + 2C_4 + 2C_5 + 2C_6 + 2C_7 + 2C_8 + 2C_9 + 2D_1 + 2D_2 + 2D_3 + 2D_4 + 2D_5 + 2D_6 + 2D_7 + 2D_8 + 2D_9$

$WCS = (Y, 3)$
(3) $Y = 3A + 3C_1 + 3C_2 + 3C_3 + 3C_4 + 3C_5 + 3C_6 + 3C_7 + 3C_8 + 3D_1 + 3D_2 + 3D_3 + 3D_4 + 3D_5 + 3D_6 + 3D_7 + 3D_8$

$WCS = (Y, 4)$
(4) $Y = 4A + 4C_1 + 4C_2 + 4C_3 + 4C_4 + 4C_5 + 4C_6 + 4C_7 + 4D_1 + 4D_2 + 4D_3 + 4D_4 + 4D_5 + 4D_6 + 4D_7$

$WCS = (Y, 5)$
(5) $Y = 5A + 5C_1 + 5C_2 + 5C_3 + 5C_4 + 5C_5 + 5C_6 + 5D_1 + 5D_2 + 5D_3 + 5D_4 + 5D_5 + 5D_6$

$WCS = (Y, 6)$
(6) $Y = 6A+6C_1+6C_2+6C_3+6C_4+6C_5+6D_1+6D_2+6D_3+6D_4+6D_5$

$WCS = (Y, 7)$
(7) $Y = 7A + 7C_1 + 7C_2 + 7C_3 + 7C_4 + 7D_1 + 7D_2 + 7D_3 + 7D_4$

$WCS = (Y, 8)$
(8) $Y = 8A + 8C_1 + 8C_2 + 8C_3 + 8D_1 + 8D_2 + 8D_3$

$WCS = (Y, 9)$
(9) $Y = 9A + 9C_1 + 9C_2 + 9D_1 + 9D_2$

$WCS = (Y, 10)$
(10) $Y = 10A + 10C_1 + 10D_1$

$WCS = (Y, 5)$
(11) $Y = 5A + B_1 + B_2 + 4C_1 + 3C_2 + 2C_3 + C_4 + 5D_1 + 5D_2 + 5D_3 + 5D_4 + 5D_5 + 5D_6$

$WCS = (Y, 6)$
(12) $Y = 6A+B_1+5C_1+4C_2+3C_3+2C_4+C_5+6D_1+6D_2+6D_3+6D_4+6D_5$

$WCS = (Y, 7)$
(13) $Y = 7A + B_1 + 6C_1 + 5C_2 + 4C_3 + 3C_4 + 2C_5 + C_6 + 7D_1 + 7D_2 + 7D_3 + 7D_4$

$WCS = (Y, 8)$
(14) $Y = 8A + B_1 + 7C_1 + 6C_2 + 5C_3 + 4C_4 + 3C_5 + 2C_6 + C_7 + 8D_1 + 8D_2 + 8D_3$

$WCS = (Y, 9)$
(15) $Y = 9A + B_1 + 8C_1 + 7C_2 + 6C_3 + 5C_4 + 4C_5 + 3C_6 + 2C_7 + C_8 + 9D_1 + 9D_2$

$WCS = (Y, 10)$
(16) $Y = 10A + B_1 + 9C_1 + 8C_2 + 7C_3 + 6C_4 + 5C_5 + 4C_6 + 3C_7 + 2C_8 + C_9 + 10D_1$

[5] 11.12 type, (-2)
$X = 11A + 3B_1 + B_2 + 9C_1 + 7C_2 + 5C_3 + 3C_4 + C_5 + 10D_1 + 9D_2 + 8D_3 + 7D_4 + 6D_5 + 5D_6 + 4D_7 + 3D_8 + 2D_9 + D_{10}$

$WCS = (Y, 1)$
(1) $Y = A + C_1 + C_2 + C_3 + C_4 + C_5 + D_1 + D_2 + D_3 + D_4 + D_5 + D_6 + D_7 + D_8 + D_9 + D_{10}$

$WCS = (Y, 3)$
(2) $Y = 3A + B_1 + B_2 + 2C_1 + C_2 + 3D_1 + 3D_2 + 3D_3 + 3D_4 + 3D_5 + 3D_6 + 3D_7 + 3D_8$

$WCS = (Y, 4)$
(3) $Y = 4A + B_1 + 3C_1 + 2C_2 + C_3 + 4D_1 + 4D_2 + 4D_3 + 4D_4 + 4D_5 + 4D_6 + 4D_7 + 4D_8 + 4D_9 + 4D_{10}$

$WCS = (Y, 5)$
(4) $Y = 5A + B_1 + 4C_1 + 3C_2 + 2C_3 + C_4 + 5D_1 + 5D_2 + 5D_3 + 5D_4 + 5D_5 + 5D_6 + 5D_7 + 5D_8 + 5D_9 + 5D_{10}$

$WCS = (Y, 6)$
(5) $Y = 6A + B_1 + 5C_1 + 4C_2 + 3C_3 + 2C_4 + C_5 + 6D_1 + 6D_2 + 6D_3 + 6D_4 + 6D_5$

$WCS = (Y, 7)$
(6) $Y = 7A + 2B_1 + B_2 + 5C_1 + 3C_2 + C_3 + 7D_1 + 7D_2 + 7D_3 + 7D_4$

$WCS = (Y, 8)$
(7) $Y = 8A + 2B_1 + 6C_1 + 4C_2 + 2C_3 + 8D_1 + 8D_2 + 8D_3$

$WCS = (Y, 9)$
(8) $Y = 9A + 2B_1 + 7C_1 + 5C_2 + 3C_3 + C_4 + 9D_1 + 9D_2$

$WCS = (Y, 10)$
(9) $Y = 10A + 2B_1 + 8C_1 + 6C_2 + 4C_3 + 2C_4 + 10D_1$

[5] 11.13 type, (-2)
$X = 11A + 4B_1 + B_2 + 8C_1 + 5C_2 + 2C_3 + C_4 + 10D_1 + 9D_2 + 8D_3 + 7D_4 + 6D_5 + 5D_6 + 4D_7 + 3D_8 + 2D_9 + D_{10}$

$WCS = (Y, 2)$
(1) $Y = 2A + B_1 + B_2 + C_1 + 2D_1 + 2D_2 + 2D_3 + 2D_4 + 2D_5 + 2D_6 + 2D_7 + 2D_8 + 2D_9$

$WCS = (Y, 3)$
(2) $Y = 3A+B_1+2C_1+C_2+3D_1+3D_2+3D_3+3D_4+3D_5+3D_6+3D_7+3D_8$

$WCS = (Y, 4)$
(3) $Y = 4A + B_1 + 3C_1 + 2C_2 + C_3 + C_4 + 4D_1 + 4D_2 + 4D_3 + 4D_4 + 4D_5 + 4D_6 + 4D_7$

$WCS = (Y, 5)$
(4) $Y = 5A + 2B_1 + B_2 + 3C_1 + C_2 + 5D_1 + 5D_2 + 5D_3 + 5D_4 + 5D_5 + 5D_6$

$WCS = (Y, 6)$
(5) $Y = 6A + 2B_1 + 4C_1 + 2C_2 + 6D_1 + 6D_2 + 6D_3 + 6D_4 + 6D_5$

$WCS = (Y, 7)$
(6) $Y = 7A + 2B_1 + 5C_1 + 3C_2 + C_3 + 7D_1 + 7D_2 + 7D_3 + 7D_4$

$WCS = (Y, 8)$
(7) $Y = 8A + 3B_1 + B_2 + 5C_1 + 2C_2 + 8D_1 + 8D_2 + 8D_3$

$WCS = (Y, 9)$
(8) $Y = 9A + 3B_1 + 6C_1 + 3C_2 + 9D_1 + 9D_2$

[5] 11.14 type, (-2)
$X = 11A + 4B_1 + B_2 + 9C_1 + 7C_2 + 5C_3 + 3C_4 + C_5 + 9D_1 + 7D_2 + 5D_3 + 3D_4 + D_5$

$WCS = (Y, 1)$
(1) $Y = A + C_1 + C_2 + C_3 + C_4 + C_5 + D_1 + D_2 + D_3 + D_4 + D_5$
(2) $Y = 5A + 2B_1 + B_2 + 4C_1 + 3C_2 + 2C_3 + C_4 + 4D_1 + 3D_2 + 2D_3 + D_4$
(3) $Y = 6A+2B_1+5C_1+4C_2+3C_3+2C_4+C_5+5D_1+4D_2+3D_3+2D_4+D_5$

[5] 11.15 type, (-2)
$X = 11A + 7B_1 + 3B_2 + 2B_3 + B_4 + 5C_1 + 4C_2 + 3C_3 + 2C_4 + C_5 + 10D_1 + 9D_2 + 8D_3 + 7D_4 + 6D_5 + 5D_6 + 4D_7 + 3D_8 + 2D_9 + D_{10}$

$WCS = (Y, 2)$
(1) $Y = 2A + B_1 + C_1 + C_2 + C_3 + C_4 + C_5 + 2D_1 + 2D_2 + 2D_3 + 2D_4 + 2D_5 + 2D_6 + 2D_7 + 2D_8 + 2D_9$

$WCS = (Y, 3)$
(2) $Y = 3A + 2B_1 + B_2 + B_3 + B_4 + C_1 + 3D_1 + 3D_2 + 3D_3 + 3D_4 + 3D_5 + 3D_6 + 3D_7 + 3D_8$

$WCS = (Y, 5)$
(3) $Y = 5A + 3B_1 + B_2 + 2C_1 + C_2 + 5D_1 + 5D_2 + 5D_3 + 5D_4 + 5D_5 + 5D_6$

$WCS = (Y, 8)$
(4) $Y = 8A + 5B_1 + 2B_2 + B_3 + 3C_1 + C_2 + 8D_1 + 8D_2 + 8D_3$

$WCS = (Y, 9)$
(5) $Y = 9A + 5B_1 + B_2 + 4C_1 + 3C_2 + 2C_3 + C_4 + 9D_1 + 9D_2$

$WCS = (Y, 10)$
(6) $Y = 10A + 6B_1 + 2B_2 + 4C_1 + 2C_2 + 10D_1$

[5] 11.16 type, (-2)
$X = 11A + 8B_1 + 5B_2 + 2B_3 + B_4 + 5C_1 + 4C_2 + 3C_3 + 2C_4 + C_5 + 9D_1 + 7D_2 + 5D_3 + 3D_4 + D_5$

$WCS = (Y, 2)$
(1) $Y = 4A + 3B_1 + 2B_2 + B_3 + 2C_1 + 2C_2 + 2C_3 + 2C_4 + 3D_1 + 2D_2 + D_3$

$WCS = (Y, 3)$
(2) $Y = 6A + 4B_1 + 2B_2 + 3C_1 + 3C_2 + 3C_3 + 5D_1 + 4D_2 + 3D_3 + 2D_4 + D_5$

[5] 11.17 type, (-2)
$X = 11A + 6B_1 + B_2 + 6C_1 + C_2 + 10D_1 + 9D_2 + 8D_3 + 7D_4 + 6D_5 + 5D_6 + 4D_7 + 3D_8 + 2D_9 + D_{10}$

$WCS = (Y, 1)$
(1) $Y = A + B_1 + B_2 + C_1 + C_2$
(2) $Y = A + B_1 + B_2 + D_1 + D_2 + D_3 + D_4 + D_5 + D_6 + D_7 + D_8 + D_9 + D_{10}$
(3) $Y = 3A + 2B_1 + B_2 + 2C_1 + C_2 + 2D_1 + D_2$
(4) $Y = 5A + 3B_1 + B_2 + 3C_1 + C_2 + 4D_1 + 3D_2 + 2D_3 + D_4$
(5) $Y = 7A + 4B_1 + B_2 + 4C_1 + C_2 + 6D_1 + 5D_2 + 4D_3 + 3D_4 + 2D_5 + D_6$
(6) $Y = 9A + 5B_1 + B_2 + 5C_1 + C_2 + 8D_1 + 7D_2 + 6D_3 + 5D_4 + 4D_5 + 3D_6 + 2D_7 + D_8$

$WCS = (Y, 2)$
(7) $Y = 2A + B_1 + C_1 + 2D_1 + 2D_2 + 2D_3 + 2D_4 + 2D_5 + 2D_6 + 2D_7 + 2D_8 + 2D_9$

$WCS = (Y, 4)$
(8) $Y = 4A + 2B_1 + 2C_1 + 4D_1 + 4D_2 + 4D_3 + 4D_4 + 4D_5 + 4D_6 + 4D_7 + 4D_8 + 4D_9 + 4D_{10}$

$WCS = (Y, 6)$
(9) $Y = 6A + 3B_1 + 3C_1 + 6D_1 + 6D_2 + 6D_3 + 6D_4 + 6D_5$

$WCS = (Y, 8)$
(10) $Y = 8A + 4B_1 + 4C_1 + 8D_1 + 8D_2 + 8D_3$

$WCS = (Y, 10)$
(11) $Y = 10A + 5B_1 + 5C_1 + 10D_1$

Or apply SR technique for D_{12}-singularity.

[5] 11.18 type, (-2)
$X = 11A+6B_1+B_2+7C_1+3C_2+2C_3+C_4+9D_1+7D_2+5D_3+3D_4+D_5$

$WCS = (Y, 1)$
(1) $Y = A + B_1 + B_2 + D_1 + D_2 + D_3 + D_4 + D_5$
(2) $Y = 3A + 2B_1 + B_2 + 2C_1 + C_2 + C_3 + C_4 + 2D_1 + D_2$

[5] 11.19 type, (-2)
$X = 11A + 6B_1 + B_2 + 8C_1 + 5C_2 + 2C_3 + C_4 + 8D_1 + 5D_2 + 2D_3 + D_4$

$WCS = (Y, 1)$
(1) $Y = 3A + 2B_1 + B_2 + 2C_1 + C_2 + 2D_1 + D_2$
(2) $Y = 4A + 2B_1 + 3C_1 + 2C_2 + C_3 + C_4 + 3D_1 + 2D_2 + D_3 + D_4$

$WCS = (Y, 2)$
(3) $Y = 8A + 4B_1 + 6C_1 + 4C_2 + 2C_3 + 6D_1 + 4D_2 + 2D_3$

[5] 11.20 type, (-2)
$X = 11A + 7B_1 + 3B_2 + 2B_3 + B_4 + 7C_1 + 3C_2 + 2C_3 + C_4 + 8D_1 + 5D_2 + 2D_3 + D_4$

$WCS = (Y, 1)$
(1) $Y = 3A + 2B_1 + B_2 + B_3 + B_4 + 2C_1 + C_2 + 2D_1 + D_2$

$WCS = (Y, 2)$
(2) $Y = 6A + 4B_1 + 2B_2 + 2B_3 + 4C_1 + 2C_2 + 2C_3 + 4D_1 + 2D_2$
(3) $Y = 8A + 5B_1 + 2B_2 + B_3 + 5C_1 + 2C_2 + 2C_3 + 6D_1 + 4D_2 + 2D_3$

$WCS = (Y, 3)$
(4) $Y = 9A + 6B_1 + 3B_2 + 6C_1 + 3C_2 + 6D_1 + 3D_2$

Order 10 $\quad r = (2, 2, 10, 10)$

[5] 10.1 type, (-2)
$X = 10A + 5B_1 + 5C_1 + D_1 + 9E_1 + 8E_2 + 7E_3 + 6E_4 + 5E_5 + 4E_6 + 3E_7 + 2E_8 + E_9$

$WCS = (Y, 1)$
(1) $Y = A + D_1 + E_1 + E_2 + E_3 + E_4 + E_5 + E_6 + E_7 + E_8 + E_9$
(2) $Y = 2A + B_1 + C_1 + D_1 + E_1$
(3) $Y = 6A + 3B_1 + 3C_1 + D_1 + 5E_1 + 4E_2 + 3E_3 + 2E_4 + E_5$
(4) $Y = 8A + 4B_1 + 4C_1 + D_1 + 7E_1 + 6E_2 + 5E_3 + 4E_4 + 3E_5 + 2E_6 + E_7$

$WCS = (Y, 2)$
(5) $Y = 2A + B_1 + C_1 + 2E_1 + 2E_2 + 2E_3 + 2E_4 + 2E_5 + 2E_6 + 2E_7 + 2E_8$
(6) $Y = 2A + B_1 + D_1 + 2E_1 + 2E_2 + 2E_3 + 2E_4 + 2E_5 + 2E_6 + 2E_7 + 2E_8$

$WCS = (Y, 3)$
(7) $Y = 3A + B_1 + C_1 + D_1 + 3E_1 + 3E_2 + 3E_3 + 3E_4 + 3E_5 + 3E_6 + 3E_7$

$WCS = (Y, 4)$
(8) $Y = 4A + 2B_1 + 2C_1 + 4E_1 + 4E_2 + 4E_3 + 4E_4 + 4E_5 + 4E_6$
(9) $Y = 4A + B_1 + 2C_1 + D_1 + 4E_1 + 4E_2 + 4E_3 + 4E_4 + 4E_5 + 4E_6$

$WCS = (Y, 5)$
(10) $Y = 5A + 2B_1 + 2C_1 + D_1 + 5E_1 + 5E_2 + 5E_3 + 5E_4 + 5E_5$

$WCS = (Y, 6)$
(11) $Y = 6A + 3B_1 + 3C_1 + 6E_1 + 6E_2 + 6E_3 + 6E_4$
(12) $Y = 6A + 2B_1 + 3C_1 + D_1 + 6E_1 + 6E_2 + 6E_3 + 6E_4$

$WCS = (Y, 7)$
(13) $Y = 7A + 3B_1 + 3C_1 + D_1 + 7E_1 + 7E_2 + 7E_3$

$WCS = (Y, 8)$
(14) $Y = 8A + 4B_1 + 4C_1 + 8E_1 + 8E_2$
(15) $Y = 8A + 3B_1 + 4C_1 + D_1 + 8E_1 + 8E_2$

$WCS = (Y, 9)$
(16) $Y = 9A + 4B_1 + 4C_1 + D_1 + 9E_1$

Or apply SR technique for E_{11}-singularity.

[5] **10.2 type,** (-2)
$X = 10A + 5B_1 + 5C_1 + 3D_1 + 2D_2 + D_3 + 7E_1 + 4E_2 + E_3$

$WCS = (Y, 1)$
(1) $Y = 4A + 2B_1 + 2C_1 + D_1 + 3E_1 + 2E_2 + E_3$

$WCS = (Y, 2)$
(2) $Y = 6A + 3B_1 + 3C_1 + 2D_1 + 2D_2 + D_3 + 4E_1 + 2E_2$

Order 8 $\quad r = (2, 4, 8, 8)$

[5] **8.1 type,** (-1)
$X = 8A + B_1 + C_1 + 2D_1 + 4E_1$

$WCS = (Y, 1)$
(1) $Y = A + B_1$
(2) $Y = 2A + B_1 + C_1$
(3) $Y = 2A + B_1 + E_1$
(4) $Y = 3A + B_1 + C_1 + E_1$
(5) $Y = 4A + B_1 + C_1 + 2E_1$
(6) $Y = 4A + B_1 + D_1 + 2E_1$
(7) $Y = 6A + B_1 + C_1 + D_1 + 3E_1$

[5] **8.2 type,** (-2)
$X = 8A + 4B_1 + C_1 + 5D_1 + 2D_2 + D_3 + 6E_1 + 4E_2 + 2E_3$

$WCS = (Y, 1)$
(1) $Y = 2A + B_1 + C_1 + D_1 + E_1$
(2) $Y = 3A + B_1 + C_1 + 2D_1 + D_2 + D_3 + 2E_1 + E_2$
(3) $Y = 4A + 2B_1 + C_1 + 2D_1 + 3E_1 + 2E_2 + E_3$

$WCS = (Y, 2)$
(4) $Y = 6A + 3B_1 + C_1 + 4D_1 + 2D_2 + 4E_1 + 2E_2$

[5] 8.3 type, (-2)
$X = 8A + 4B_1 + 3C_1 + C_2 + 3D_1 + D_2 + 6E_1 + 4E_2 + 2E_3$

$WCS = (Y, 1)$
(1) $Y = 2A + B_1 + C_1 + C_2 + D_1 + D_2 + E_1$

$WCS = (Y_1, 2) + (Y_2, 3)$
(2) $Y_1 = 4A + 2B_1 + 2C_1 + 2D_1 + 2E_1,$
$Y_2 = 6A + 3B_1 + 3C_1 + 3D_1 + 3E_1$

[5] 8.4 type, (-2)
$X = 8A + 4B_1 + 2C_1 + 5D_1 + 2D_2 + D_3 + 5E_1 + 2E_2 + E_3$

$WCS = (Y, 2)$
(1) $Y = 6A + 3B_1 + C_1 + 4D_1 + 2D_2 + 4E_1 + 2E_2$

[5] 8.5 type, (-2)
$X = 8A + 4B_1 + 2C_1 + 3D_1 + D_2 + 7E_1 + 6E_2 + 5E_3 + 4E_4 + 3E_5 + 2E_6 + E_7$

$WCS = (Y, 2)$
(1) $Y = 2A + B_1 + D_1 + D_2 + 2E_1 + 2E_2 + 2E_3 + 2E_4 + 2E_5 + 2E_6 + E_7$

$WCS = (Y, 4)$
(2) $Y = 4A + 2B_1 + C_1 + D_1 + 4E_1 + 4E_2 + 4E_3 + 4E_4$

$WCS = (Y, 5)$
(3) $Y = 5A + 2B_1 + C_1 + 2D_1 + D_2 + 5E_1 + 5E_2 + 5E_3$

$WCS = (Y, 6)$
(4) $Y = 6A + 3B_1 + C_1 + 2D_1 + 6E_1 + 6E_2$

[5] 8.6 type, (-3)
$X = 8A + 4B_1 + 6C_1 + 4C_2 + 2C_3 + 7D_1 + 6D_2 + 5D_3 + 4D_4 + 3D_5$
$+ 2D_6 + D_7 + 7E_1 + 6E_2 + 5E_3 + 4E_4 + 3E_5 + 2E_6 + E_7$

$WCS = (Y, 2)$
(1) $Y = 2A + B_1 + C_1 + 2D_1 + 2D_2 + 2D_3 + 2D_4 + 2D_5 + 2D_6 + 2E_1$
$+ 2E_2 + 2E_3 + 2E_4 + 2E_5 + 2E_6$

$WCS = (Y, 3)$
(2) $Y = 3A + B_1 + C_1 + 3D_1 + 3D_2 + 3D_3 + 3D_4 + 3D_5 + 3E_1 + 3E_2$
$+ 3E_3 + 3E_4 + 3E_5$

$WCS = (Y, 4)$
(3) $Y = 4A+2B_1+2C_1+4D_1+4D_2+4D_3+4D_4+4E_1+4E_2+4E_3+4E_4$
(4) $Y = 4A+2B_1+3C_1+2C_2+C_3+3D_1+2D_2+D_3+4E_1+4E_2+4E_3+4E_4$

$WCS = (Y, 5)$
(5) $Y = 5A + 2B_1 + 3C_1 + C_2 + 5D_1 + 5D_2 + 5D_3 + 5E_1 + 5E_2 + 5E_3$

$WCS = (Y, 6)$
(6) $Y = 6A + 3B_1 + 3C_1 + 6D_1 + 6D_2 + 6E_1 + 6E_2$
(7) $Y = 6A+3B_1+4C_1+2C_2+5D_1+4D_2+3D_3+2D_4+D_5+6E_1+6E_2$

$WCS = (Y, 7)$
(8) $Y = 7A + 2B_1 + 5C_1 + 3C_2 + C_3 + 7D_1 + 7E_1$

Order 6 $r = (6, 6, 6, 6)$

[5] 6.1.1 type, (-2)
$X = 6A+B_1+C_1+5D_1+4D_2+3D_3+2D_4+D_5+5E_1+4E_2+3E_3+2E_4+E_5$

$WCS = (Y, 1)$
(1) $Y = A + D_1 + D_2 + D_3 + D_4 + D_5 + E_1 + E_2 + E_3 + E_4 + E_5$
(2) $Y = A + B_1 + C_1$
(3) $Y = A + B_1 + D_1 + D_2 + D_3 + D_4 + D_5$
(4) $Y = 2A + B_1 + C_1 + D_1 + E_1$
(5) $Y = 3A + B_1 + C_1 + 2D_1 + D_2 + 2E_1 + E_2$
(6) $Y = 5A + B_1 + C_1 + 4D_1 + 3D_2 + 2D_3 + D_4 + 4E_1 + 3E_2 + 2E_3 + E_4$

$WCS = (Y, 2)$
(7) $Y = 2A + 2D_1 + 2D_2 + 2D_3 + 2D_4 + 2E_1 + 2E_2 + 2E_3 + 2E_4$

$WCS = (Y, 3)$
(8) $Y = 3A + 3D_1 + 3D_2 + 3D_3 + 3E_1 + 3E_2 + 3E_3$
(9) $Y = 3A + B_1 + C_1 + C_3 + 3E_1 + 3E_2 + 3E_3$

$WCS = (Y, 4)$
(10) $Y = 4A + 4D_1 + 4D_2 + 4E_1 + 4E_2$
(11) $Y = 4A + B_1 + C_1 + 2D_1 + 4E_1 + 4E_2$

$WCS = (Y, 5)$
(12) $Y = 5A + 5D_1 + 5E_1$
(13) $Y = 5A + B_1 + C_1 + 3D_1 + D_2 + 5E_1$

Or apply SR technique for A_9-singularity.

Order 6 $r = (2, 3, 3, 3, 6)$

[5] 6.2.1 type, (-2)
$X = 6A + B_1 + 2C_1 + 2D_1 + 3E_1 + 4F_1 + 2F_2$

$WCS = (Y, 1)$
(1) $Y = 3A + B_1 + C_1 + D_1 + E_1 + 2F_1 + F_2$

[5] 6.2.2 type, (-3)
$X = 6A+2B_1+3C_1+4D_1+2D_2+4E_1+2E_2+5F_1+4F_2+3F_3+2F_4+F_5$

$WCS = (Y, 3)$
(1) $Y = 3A + B_1 + C_1 + 2D_1 + D_2 + 2E_1 + E_2 + 3F_1 + 3F_2 + 3F_3$

Order 6 $r = (2, 2, 3, 6, 6)$

[5] 6.3.1 type, (-2)
$X = 6A + B_1 + C_1 + 3D_1 + 3E_1 + 4F_1 + 2F_2$

$WCS = (Y, 1)$
(1) $Y = A + B_1 + C_1$
(2) $Y = 2A + B_1 + C_1 + D_1 + E_1$
(3) $Y = 2A + B_1 + C_1 + D_1 + F_1$
(4) $Y = 3A + B_1 + C_1 + D_1 + E_1 + 2F_1 + F_2$
(5) $Y = 4A + B_1 + C_1 + 2D_1 + 2E_1 + 2F_1$

Or apply SR technique for D_5-singularity.

[5] 6.3.2 type, (-3)
$X = 6A + 2B_1 + 3C_1 + 3D_1 + 5E_1 + 4E_2 + 3E_3 + 2E_4 + E_5 + 5F_1 + 4F_2 + 3F_3 + 2F_4 + F_5$

$WCS = (Y, 2)$
(1) $Y = 2A+B_1+C_1+D_1+2E_1+2E_2+2E_3+2E_4+2F_1+2F_2+2F_3+2F_4$

$WCS = (Y, 3)$
(2) $Y = 3A + B_1 + C_1 + D_1 + 3E_1 + 3E_2 + 3E_3 + 3F_1 + 3F_2 + 3F_3$

$WCS = (Y, 4)$
(3) $Y = 4A + 2C_1 + 2D_1 + 4E_1 + 4E_2 + 4F_1 + 4F_2$

Order 6 $r = (2, 2, 2, 2, 3, 3)$

[5] 6.4.1 type, (-3)
$X = 6A + 2B_1 + 3C_1 + 3D_1 + 3E_1 + 3F_1 + 4G_1 + 2G_2$

$WCS = (Y, 2) + (2Y, 4)$
(1) $Y = 2A + B_1 + C_1 + D_1 + E_1 + F_1 + G_1$

Order 4 $r = (2, 2, 4, 4, 4, 4)$

[5] 4.1.1 type, (-2)
$X = 4A + B_1 + C_1 + D_1 + E_1 + 2F_1 + 2G_1$

$WCS = (Y, 1)$
(1) $Y = A + B_1 + C_1$
(2) $Y = 2A + B_1 + C_1 + D_1 + E_1$
(3) $Y = 2A + B_1 + C_1 + D_1 + F_1$

(4) $Y = 2A + B_1 + C_1 + G_1 + F_1$
(5) $Y = 3A + B_1 + C_1 + D_1 + E_1 + F_1 + G_1$

Or apply SR technique for A_3-singularity.

[5] 4.1.2 type, (-3)
$X = 4A + B_1 + C_1 + 2D_1 + 2E_1 + 3F_1 + 2F_2 + F_3 + 3G_1 + 2G_2 + G_3$

$WCS = (Y, 1)$
(1) $Y = A + B_1 + F_1 + F_2 + F_3 + G_1 + G_2 + G_3$
(2) $Y = 2A + B_1 + C_1 + D_1 + E_1 + F_1 + G_1$

$WCS = (Y, 2)$
(3) $Y = 2A + B_1 + C_1 + 2F_1 + 2F_2 + 2G_1 + 2G_2$
(4) $Y = 2A + B_1 + D_1 + 2F_1 + 2F_2 + 2G_1 + 2G_2$
(5) $Y = 2A + B_1 + C_1 + D_1 + E_1 + 2F_1 + 2F_2$

$WCS = (Y, 3)$
(6) $Y = 3A + B_1 + C_1 + D_1 + 3F_1 + 3G_1$
(7) $Y = 3A + B_1 + D_1 + E_1 + 3F_1 + 3G_1$

[5] 4.1.3 type, (-4)
$X = 4A + 2B_1 + 2C_1 + 3D_1 + 2D_2 + D_3 + 3E_1 + 2E_2 + E_3 + 3F_1 + 2F_2 + F_3 + 3G_1 + 2G_2 + G_3$

$WCS = (Y, 1)$
(1) $Y = A + D_1 + D_2 + D_3 + E_1 + E_2 + E_3 + F_1 + F_2 + F_3 + G_1 + G_2 + G_3$

$WCS = (Y, 2)$
(2) $Y = 2A + 2D_1 + 2D_2 + 2E_1 + 2E_2 + 2F_1 + 2F_2 + 2G_1 + 2G_2$
(3) $Y = 2A + B_1 + C_1 + 2D_1 + 2D_2 + 2E_1 + 2E_2 + 2F_1 + 2F_2$
(4) $Y = 2A + B_1 + D_1 + 2E_1 + 2E_2 + 2F_1 + 2F_2 + 2G_1 + 2G_2$
(5) $Y = 2A + B_1 + C_1 + D_1 + E_1 + 2F_1 + 2F_2 + 2G_1 + 2G_2$

$WCS = (Y, 3)$
(6) $Y = 3A + 3D_1 + 3E_1 + 3F_1 + 3G_1$
(7) $Y = 3A + B_1 + C_1 + D_1 + 3E_1 + 3F_1 + 3G_1$

Order 4 $\quad r = (2, 2, 2, 2, 2, 4, 4)$

[5] 4.2.1 type, (-3)
$X = 4A + B_1 + C_1 + 2D_1 + 2E_1 + 2F_1 + 2G_1 + 2H_1$

$WCS = (Y, 1)$
(1) $Y = 2A + B_1 + C_1 + D_1 + E_1 + F_1 + G_1$
(2) $Y = 2A + B_1 + D_1 + E_1 + F_1 + G_1 + H_1$

[5] 4.2.2 type, (-4)
$X = 4A + 2B_1 + 2C_1 + 2D_1 + 2E_1 + 2F_1 + 3G_1 + 2G_2 + G_3 + 3H_1 + 2H_2 + H_3$

$WCS = (Y, 2)$
(1) $Y = 2A + B_1 + C_1 + D_1 + E_1 + 2G_1 + 2G_2 + 2H_1 + 2H_2$
(2) $Y = 2A + B_1 + C_1 + D_1 + E_1 + F_1 + G_1 + 2H_1 + 2H_2$

Order 3 $\quad r = (3, 3, 3, 3, 3, 3, 3)$

[5] 3.1 type, (-3)
$X = 3A + B_1 + C_1 + D_1 + E_1 + F_1 + 2G_1 + G_2 + 2H_1 + H_2$

[5] 3.2 type, (-4)
$X = 3A+B_1+C_1+2D_1+D_2+2E_1+E_2+2F_1+F_2+2G_1+G_2+2H_1+H_2$

$WCS = (Y, 1)$
(1) $Y = A + D_1 + D_2 + E_1 + E_2 + F_1 + F_2 + G_1 + G_2$
(2) $Y = A + B_1 + D_1 + D_2 + E_1 + E_2 + F_1 + F_2$
(3) $Y = A + B_1 + C_1 + D_1 + D_2 + E_1 + E_2$

$WCS = (Y, 2)$
(4) $Y = 2A + 2D_1 + 2E_1 + 2F_1 + 2G_1$
(5) $Y = 2A + B_1 + C_1 + 2D_1 + 2E_1 + 2F_1$
(6) $Y = 2A + B_1 + D_1 + 2E_1 + 2F_1 + 2G_1$
(7) $Y = 2A + B_1 + C_1 + D_1 + E_1 + 2F_1 + 2G_1$
(8) $Y = 2A + B_1 + C_1 + D_1 + E_1 + F_1 + G_1 + 2H_1$

Order 2 $\quad r = (2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2)$

[5] 2.1 type, (-6)
$X = 2A + B_1 + C_1 + D_1 + E_1 + F_1 + G_1 + H_1 + I_1 + J_1 + K_1 + L_1 + M_1$

$WCS = (Y, 1)$
(1) $Y = A + B_1 + C_1 + D_1 + E_1 + F_1 + G_1$

Or apply SR technique for a cyclic quotient singularity.

26.5.2 Stellar singular fibers, genus$(A) = 1, 2, 3$

genus$(A) = 1$

[5] A1.1 type
$X = 8A + 4B_1 + 4C_1, \qquad N_A^{\otimes 8} = \mathcal{O}(-4b_1 - 4c_1)$

Take b_1, c_1 so that $b_1 - c_1$ is torsion of order 4 in Pic(A).
$WCS = (Y, 1) + (2Y, 2) + (3Y, 3) + (4Y, 4)$
(1) $Y = A + B_1, \quad N_A = \mathcal{O}(-b_1)$,

[5] A1.2 type
$X = 6A + 2B_1 + 4C_1 + 2C_2, \quad N_A^{\otimes 6} = \mathcal{O}(-2b_1 - 4c_1)$

Take b_1, c_1 so that $b_1 - c_1$ is torsion of order 4 in $\mathrm{Pic}(A)$.
$WCS = (Y, 1) + (2Y, 2) + (3Y, 3) + (4Y, 4)$
(1) $Y = A + C_1, \quad N_A = \mathcal{O}(-c_1)$,

[5] A1.3 type
$X = 5A + B_1 + 4C_1 + 3C_2 + 2C_3 + C_4, \quad N_A^{\otimes 5} = \mathcal{O}(-b_1 - 4c_1)$

$WCS = (Y, 1)$
(1) $Y = A + B_1 + C_1 + C_2 + C_3 + C_4, \quad N_A = \mathcal{O}(-b_1 - c_1 + q)$,
where $4b_1 + c_1 \sim 5q$.

[5] A1.4 type
$X = 5A + 2B_1 + B_2 + 3C_1 + C_2, \quad N_A^{\otimes 5} = \mathcal{O}(-2b_1 - 3c_1)$

$WCS = (Y, 1) + (2Y, 2)$
(1) $Y = A + B_1 + C_1, \quad N_A = \mathcal{O}(-b_1 - c_1 + q)$,
where $3b_1 + 2c_1 \sim 5q$.

[5] A1.5 type
$X = 4A + B_1 + C_1 + 2D_1, \quad N_A^{\otimes 4} = \mathcal{O}(-b_1 - c_1 - 2d_1)$

$WCS = (Y, 1)$
(1) $Y = A + B_1, \quad N_A = \mathcal{O}(-b_1)$,
where $c_1 + 2d_1 \sim 3b_1$.
(2) $Y = 2A + B_1 + C_1, \quad N_A^{\otimes 2} = \mathcal{O}(-b_1 - c_1)$,
where $b_1 + c_1 \sim 2d_1$, i.e. d_1 is a Weierstrass point on A.
(3) $Y = 3A + B_1 + C_1 + D_1, \quad N_A^{\otimes 3} = \mathcal{O}(-b_1 - c_1 - d_1)$,
where $b_1 + c_1 \sim 2d_1$, i.e. d_1 is a Weierstrass point on A.

[5] A1.6 type
$X = 4A + 2B_1 + 3C_1 + 2C_2 + C_3 + 3D_1 + 2D_2 + D_3$,
$N_A^{\otimes 4} = \mathcal{O}(-2b_1 - 3c_1 - 3d_1)$
(1) $Y = A + C_1 + C_2 + C_3 + D_1 + D_2 + D_3, \quad N_A = \mathcal{O}(-c_1 - d_1)$,
where $c_1 + d_1 \sim 2b_1$, i.e. b_1 is a Weierstrass point on A.

$WCS = (Y, 2)$
(2) $Y = 2A + 2C_1 + 2C_2 + 2D_1 + 2D_2, \quad N_A^{\otimes 2} = \mathcal{O}(-2c_1 - 2d_1)$,
where $c_1 + d_1 \sim 2b_1$, i.e. b_1 is a Weierstrass point on A.

$WCS = (Y, 3)$
(3) $Y = 3A + 3C_1 + 3D_1, \quad N_A^{\otimes 3} = \mathcal{O}(-3c_1 - 3d_1)$,
where $3c_1 + 3d_1 \sim 6b_1$, e.g. b_1 is a Weierstrass point on A.
(4) $Y = 3A + B_1 + 2C_1 + C_2 + 3D_1, \quad N_A^{\otimes 3} = \mathcal{O}(-b_1 - 2c_1 - 3d_1)$,
where $2b_1 + c_1 \sim 3d_1$.

[5] A1.7 type
$X = 4A + 2B_1 + 2C_1 + 2D_1 + 2E_1, \quad N_A^{\otimes 4} = \mathcal{O}(-2b_1 - 2c_1 - 2d_1 - 2e_1)$

Take b_1, c_1, d_1 and e_1 so that $b_1 + c_1 - d_1 - e_1$ is torsion of order 2 in Pic(A).

$WCS = (Y, 1) + (2Y, 2)$
(1) $Y = A + B_1 + C_1, \quad N_A = \mathcal{O}(-b_1 - c_1)$,

[5] A1.8 type
$X = 3A+B_1+C_1+2D_1+D_2+2E_1+E_2, \quad N_A^{\otimes 3} = \mathcal{O}(-b_1-c_1-2d_1-2e_1)$

$WCS = (Y, 1)$
(1) $Y = A + B_1 + C_1, \quad N_A = \mathcal{O}(-b_1 - c_1)$,
where $2b_1 + 2c_1 \sim 2d_1 + 2e_1$.
(2) $Y = A + B_1 + D_1 + D_2, \quad N_A = \mathcal{O}(-b_1 - d_1)$,
where $2b_1 + d_1 \sim c_1 + 2e_1$.
(3) $Y = A + D_1 + D_2 + E_1 + E_2, \quad N_A = \mathcal{O}(-d_1 - e_1)$,
where $b_1 + c_1 \sim d_1 + e_1$.
(4) $Y = 2A + B_1 + C_1 + D_1 + E_1, \quad N_A^{\otimes 2} = \mathcal{O}(-b_1 - c_1 - d_1 - e_1)$,
where $b_1 + c_1 \sim d_1 + e_1$.

$WCS = (Y, 2)$
(5) $Y = 2A + B_1 + C_1 + 2D_1, \quad N_A^{\otimes 2} = \mathcal{O}(-b_1 - c_1 - 2d_1)$,
where $b_1 + c_1 + 2d_1 \sim 4e_1$.
(6) $Y = 2A + B_1 + C_1 + 2D_1, \quad N_A^{\otimes 2} = \mathcal{O}(-b_1 - d_1 - 2e_1)$,
where $b_1 + 2e_1 \sim 2c_1 + d_1$.
(7) $Y = 2A + 2D_1 + 2E_1, \quad N_A^{\otimes 2} = \mathcal{O}(-2d_1 - 2e_1)$,
where $2b_1 + 2c_1 \sim 2d_1 + 2e_1$.

[5] A1.9 type
$X = 2A + B_1 + C_1 + D_1 + E_1 + F_1 + G_1 + H_1 + I_1$,
$N_A^{\otimes 2} = \mathcal{O}(-b_1 - c_1 - d_1 - e_1 - f_1 - g_1 - h_1 - i_1)$

$WCS = (Y, 1)$
(1) $Y = A + B_1 + C_1 + D_1 + E_1, \quad N_A = \mathcal{O}(-b_1 - c_1 - d_1 - e_1)$,
where $b_1 + c_1 + d_1 + e_1 \sim f_1 + g_1 + h_1 + i_1$

genus$(A) = 2$

[5] A2.1 type
$X = 4A$ atom

[5] A2.2 type
Suppose A is a hyperelliptic curve.
$X = 2A + B_1 + C_1 + D_1 + E_1, \quad N_A^{\otimes 2} = \mathcal{O}(-b_1 - c_1 - d_1 - e_1)$

$WCS = (Y, 1)$
(1) $Y = A + B_1 + C_1, \quad N_A = \mathcal{O}(-b_1 - c_1)$,
where $b_1 + c_1 \sim d_1 + e_1$

genus$(A) = 3$

[5] A3.1 type
$X = 2A$ atom

26.5.3 Self-welding and self-connecting of genus 4 or 3

[5] sw4.1 type sw([4] 16.2)
$X = 16A + 8B_1 + 3C_1 + 2C_2 + 5D_1 + 4D_2 + 3D_3 + 2D_4, \quad C_2 = D_4$

$WCS = \sum_{i=1}^{8}(iY, i)$
(1) $Y = A + B_1$

[5] sw4.2 type sw([4] 16.3)
$X = 16A + 8B_1 + 9C_1 + 2C_2 + 15D_1 + 14D_2 + 13D_3 + 12D_4 + 11D_5 + 10D_6 + 9D_7 + 8D_8 + 7D_9 + 6D_{10} + 5D_{11} + 4D_{12} + 3D_{13} + 2D_{14},$
$C_2 = D_{14}$

$WCS = \sum_{i=1}^{8}(iY, i)$
(1) $Y = A + B_1 + D_1 + D_2 + D_3 + D_4 + D_5 + D_6 + D_7 + D_8$

[5] sw4.3 type sw([4] 10.2.2)
$X = 10A + 4B_1 + 2B_2 + 3C_1 + 2C_2 + 3D_1 + 2D_2, \quad C_2 = D_2$

$WCS = (Y, 1) + (2Y, 2)$
(1) $Y = 3A + B_1 + C_1 + C_2 + D_1 + D_2$
Note: A multiple curve of genus 2 is barked off.

[5] sw4.4 type sw([4] 10.2.5)
$X = 10A + 4B_1 + 2B_2 + 7C_1 + 4C_2 + 9D_1 + 8D_2 + 7D_3 + 6D_4 + 5D_5 + 4D_6, \quad C_2 = D_6$

$WCS = (Y, 1) + (2Y, 2) + (3Y, 3) + (4Y, 4)$
(1) $Y = A + C_1 + C_2 + D_1 + D_2 + D_3 + D_4 + D_5 + D_6$

[5] sw4.5 type sw([4] 10.2.7)
$X = 10A + 3B_1 + 2B_2 + B_3 + 8C_1 + 6C_2 + 4C_3 + 2C_4 + 9D_1 + 8D_2 + 7D_3 + 6D_4 + 5D_5 + 4D_6 + 3D_7 + 2D_8, \quad C_4 = D_8$

$WCS = \sum_{i=1}^{8}(iY, i)$
(1) $Y = A + C_1 + D_1 + D_2$

[5] sw4.6 type sw([4] 10.2.8)
$X = 10A + 2B_1 + 9C_1 + 8C_2 + 7C_3 + 6C_4 + 5C_5 + 4C_6 + 3C_7 + 2C_8 + 9D_1 + 8D_2 + 7D_3 + 6D_4 + 5D_5 + 4D_6 + 3D_7 + 2D_8, \quad C_8 = D_8$

$WCS = (Y, 1) + (2Y, 2)$
(1) $Y = A + C_1 + C_2 + C_3 + C_4 + C_5 + C_6 + C_7 + C_8 + D_1 + D_2 + D_3 + D_4 + D_5 + D_6 + D_7 + D_8$

[5] sw4.7 type sw([4] 9.1)
$X = 9A + B_1 + 4C_1 + 3C_2 + 2C_3 + 4D_1 + 3D_2 + 2D_3, \quad C_3 = D_3$

$WCS = (Y, 1) + (2Y, 2)$
(1) $Y = 2A + C_1 + C_2 + C_3 + D_1 + D_2 + D_3$

[5] sw4.8 type sw([4] 9.3)
$X = 9A + 2B_1 + 2C_1 + 5D_1 + D_2, \quad B_1 = C_1$

$WCS = (Y, 1) + (2Y, 2)$
(1) $Y = 2A + B_1 + C_1$

[5] sw4.9 type sw([4] 9.4) case 1
$X = 9A + 2B_1 + 8C_1 + 7C_2 + 6C_3 + 5C_4 + 4C_5 + 3C_6 + 2C_7 + 8D_1 + 7D_2 + 6D_3 + 5D_4 + 4D_5 + 3D_6 + 2D_7 + D_8, \quad B_1 = C_7$

$WCS = (Y, 5)$
(1) $Y = 5A + B_1 + 4C_1 + 3C_2 + 2C_3 + C_4 + C_5 + C_6 + C_7 + 5D_1 + 5D_2 + 5D_3 + 5D_4$

$WCS = (Y, 6)$
(2) $Y = 6A + B_1 + 5C_1 + 4C_2 + 3C_3 + 2C_4 + C_5 + C_6 + C_7 + 6D_1 + 6D_2 + 6D_3$

$WCS = (Y, 7)$
(3) $Y = 7A + B_1 + 6C_1 + 5C_2 + 4C_3 + 3C_4 + 2C_5 + C_6 + C_7 + 7D_1 + 7D_2$

$WCS = (Y, 8)$
(4) $Y = 8A + B_1 + 7C_1 + 6C_2 + 5C_3 + 4C_4 + 3C_5 + 2C_6 + C_7 + 8D_1$

$WCS = (Y, 1) + (2Y, 2)$
(5) $Y = A + B_1 + C_1 + C_2 + C_3 + C_4 + C_5 + C_6 + C_7$

[5] sw4.10 type sw([4] 9.4) case 2
$X = 9A + 2B_1 + B_2 + 8C_1 + 7C_2 + 6C_3 + 5C_4 + 4C_5 + 3C_6 + 2C_7 + 8D_1 + 7D_2 + 6D_3 + 5D_4 + 4D_5 + 3D_6 + 2D_7, \quad C_7 = D_7$

$WCS = (Y, 2)$
(1) $Y = 8A + 2B_1 + 7C_1 + 6C_2 + 5C_3 + 4C_4 + 3C_5 + 2C_6 + C_7 + 7D_1 + 6D_2 + 5D_3 + 4D_4 + 3D_5 + 2D_6 + D_7$

$WCS = (Y, 1) + (2Y, 2)$
(2) $Y = A + B_1 + C_1 + C_2 + C_3 + C_4 + C_5 + C_6 + C_7 + D_1 + D_2 + D_3 + D_4 + D_5 + D_6 + D_7$

[5] sw4.11 type sw([4] 9.5)
$X = 9A + 5B_1 + B_2 + 5C_1 + 8D_1 + 7D_2 + 6D_3 + 5D_4, \quad C_1 = D_4$

$WCS = (Y, 1) + (2Y, 2) + (3Y, 3) + (4Y, 4) + (5Y, 5)$
(1) $Y = A + C_1 + D_1 + D_2 + D_3 + D_4$

[5] sw4.12 type sw([4] 6.2.3)

$X = 6A + 4B_1 + 2B_2 + 4C_1 + 2C_2 + 5D_1 + 4D_2 + 3D_3 + 2D_4 + 5E_1 + 4E_2 + 3E_3 + 2E_4, \quad D_4 = E_4$

$WCS = (Y, 1) + (2Y, 2)$
(1) $Y = A + B_1 + B_2 + D_1 + D_2 + D_3 + D_4 + E_1 + E_2 + E_3 + E_4$

[5] sw4.13 type sw([4] 6.3.2)

$X = 6A + 3B_1 + 5C_1 + 4C_2 + 3C_3 + 2C_4 + 5D_1 + 4D_2 + 3D_3 + 2D_4 + 5E_1 + 4E_2 + 3E_3 + 2E_4 + E_5, \quad C_4 = D_4$

$WCS = (Y, 5)$
(1) $Y = 5A + 2B_1 + 4C_1 + 3C_2 + 2C_3 + C_4 + 4D_1 + 3D_2 + 2D_3 + D_4 + 5E_1$

$WCS = (Y, 1) + (2Y, 2)$
(2) $Y = A + C_1 + C_2 + C_3 + C_4 + D_1 + D_2 + D_3 + D_4 + E_1 + E_2 + E_3 + E_4$

[5] sw4.14 type sw([4] 5.2) case 1

$X = 5A + 2B_1 + 2C_1 + 2D_1 + D_2 + 4E_1 + 3E_2 + 2E_3 + E_4, \quad B_1 = C_1$

$WCS = (Y, 3)$
(1) $Y = 3A + B_1 + C_1 + D_1 + 3E_1 + 3E_2$

$WCS = (Y, 1) + (2Y, 2)$
(2) $Y = A + B_1 + C_1$

[5] sw4.15 type sw([4] 5.2) case 2

$X = 5A + 2B_1 + B_2 + 2C_1 + C_2 + 2D_1 + 4E_1 + 3E_2 + 2E_3, \quad D_1 = E_3$

$WCS = (Y, 1) + (2Y, 2)$
(1) $Y = A + D_1 + E_1 + E_2 + E_3$

[5] sw4.16 type sw([4] 5.4)

$X = 5A + 2B_1 + 2C_1 + 3D_1 + D_2 + 3E_1 + E_2, \quad B_1 = C_1$

$WCS = (Y, 1)$
(1) $Y = A + D_1 + D_2 + E_1 + E_2$
(2) $Y = 3A + B_1 + C_1 + 2D_1 + D_2 + 2E_1 + E_2$

$WCS = (Y, 1) + (2Y, 2)$
(3) $Y = A + B_1 + C_1$

[5] sw4.17 type sw([4] 5.5)

$X = 5A + B_1 + C_1 + 4D_1 + 3D_2 + 2D_3 + 4E_1 + 3E_2 + 2E_3, \quad D_3 = E_3$

$WCS = (Y, 1)$
(1) $Y = A + B_1 + C_1$
(2) $Y = 4A + B_1 + C_1 + 3D_1 + 2D_2 + D_3 + 3E_1 + 2E_2 + E_3$

$WCS = (Y, 1) + (2Y, 2)$
(3) $Y = A + D_1 + D_2 + D_3 + E_1 + E_2 + E_3$

[5] sw4.18 type sw([4] 5.6) case 1
$X = 5A + 3B_1 + 4C_1 + 3C_2 + 4D_1 + 3D_2 + 2D_3 + D_4 + 4E_1 + 3E_2 + 2E_3 + E_4, \quad B_1 = C_2$

$WCS = (Y, 1) + (2Y, 2) + (3Y, 3)$
(1) $Y = A + B_1 + C_1 + C_2 + D_1 + D_2$

[5] sw4.19 type sw([4] 5.6) case 2
$X = 5A + 3B_1 + B_2 + 4C_1 + 3C_2 + 2C_3 + 4D_1 + 3D_2 + 2D_3 + 4E_1 + 3E_2 + 2E_3 + E_4, \quad C_3 = D_3$

$WCS = (Y, 1) + (2Y, 2)$
(1) $Y = A + C_1 + C_2 + C_3 + D_1 + D_2 + D_3 + E_1 + E_2 + E_3$

[5] sw4.20 type sw([4] 4.2.2)
$X = 4A + B_1 + 2C_1 + 3D_1 + 2D_2 + 3E_1 + 2E_2 + 3F_1 + 2F_2 + F_3, \quad D_2 = E_2$

$WCS = (Y, 3)$
(1) $Y = 3A + B_1 + C_1 + 2D_1 + D_2 + 2E_1 + E_2 + 3F_1$

$WCS = (Y, 1) + (2Y, 2)$
(2) $Y = A + C_1 + D_1 + D_2 + E_1 + E_2$
(3) $Y = A + D_1 + D_2 + E_1 + E_2 + F_1 + F_2$

[5] sw4.21 type sw([4] 3.2)
$X = 3A + B_1 + C_1 + D_1 + 2E_1 + 2F_1 + 2G_1 + G_2, \quad E_1 = F_1$

$WCS = (Y, 1)$
(1) $Y = A + B_1 + C_1 + D_1$
(2) $Y = A + B_1 + G_1 + G_2$
(3) $Y = 2A + B_1 + C_1 + D_1 + E_1 + F_1 + G_1$

$WCS = (Y, 1) + (2Y, 2)$
(4) $Y = A + E_1 + F_1 + G_1$

[5] sw4.22 type sw([4] 3.3)
$X = 3A + 2B_1 + 2C_1 + 2D_1 + D_2 + 2E_1 + E_2 + 2F_1 + F_2 + 2G_1 + G_2, \quad B_1 = C_1$

$WCS = (Y, 1)$
(1) $Y = A + D_1 + D_2 + E_1 + E_2 + F_1 + F_2 + G_1 + G_2$

$WCS = (Y, 2)$
(2) $Y = 2A + B_1 + C_1 + D_1 + E_1 + 2F_1 + 2G_1$
(3) $Y = 2A + B_1 + C_1 + 2D_1 + 2E_1 + 2F_1$

$WCS = (Y,1) + (2Y,2)$
(4) $Y = A + B_1 + C_1 + D_1 + E_1$

[5] sw4.23 type sw([4] A1.4)
A: an elliptic curve
$X = 3A + 2B_1 + 2C_1 + 2D_1 + D_2, \quad B_1 = C_1 \quad N_A^{\otimes 3} = \mathcal{O}(-2b_1 - 2c_1 - 2d_1)$

$WCS = (Y,1) + (2Y,2)$
(1) $Y = A + B_1 + C_1, \quad N_A = \mathcal{O}(-b_1 - c_1),$
where $b_1 + c_1 \sim 2d_1$, i.e. d_1 is a Weierstrass point on A.

[5] sw3.1 type sw([3] 6.2.2)
$X = 6A + B_1 + 3C_1 + 4D_1 + 4E_1, \quad D_1 = E_1$

$WCS = (Y,1)$
(1) $Y = 2A + B_1 + C_1 + D_1 + E_1$

$WCS = (Y,1) + (2Y,2)$
(2) $Y = A + D_1 + E_1$

[5] sc3.1 type sc([3] 6.2.2)
$X = 6A + B_1 + 3C_1 + 4D_1 + 2D_2 + 4E_1 + 2E_2, \quad D_2 = E_2$

$WCS = (Y,1)$
(1) $Y = 2A + B_1 + C_1 + D_1 + E_1$
(2) $Y = 3A + B_1 + C_1 + 2D_1 + D_2 + 2E_1 + E_2$

$WCS = (Y,1) + (2Y,2)$
(3) $Y = A + D_1 + D_2 + E_1 + E_2$

[5] sw3.2 type sw([3] 4.1.3)
$X = 4A + 3B_1 + 2B_2 + 3C_1 + 2C_2 + 3D_1 + 2D_2 + 3E_1 + 2E_2,$
$B_2 = C_2,\ D_2 = E_2$

$WCS = (Y,1) + (2Y,2)$
(1) $Y = A + B_1 + B_2 + C_1 + C_2 + D_1 + D_2 + E_1 + E_2$

26.5.4 Welding of stellar singular fibers of genus 4 and genus 1

[5] wd([4]16.4(1)+III^*) type
$X = X_1 + X_2, \quad D_4(X_1) = A(X_2)$
$X_1 = 16A + 8B_1 + 11C_1 + 6C_2 + C_3 + 13D_1 + 10D_2 + 7D_3 + 4D_4$
$X_2 = 4A + 2B_1 + 3D_1 + 2D_2 + D_3$

$WCS = (Y,1), \quad Y = Y(X_1) + Y(X_2)$
$Y(X_1) = 10A + 5B_1 + 7C_1 + 4C_2 + C_3 + 8D_1 + 6D_2 + 4D_3 + 2D_4$
$Y(X_2) = 2A + B_1 + D_1$

[5] wd([4]16.4(2)+II^*) type

$X = X_1 + X_2, \quad C_2(X_1) = A(X_2)$
$X_1 = 16A + 8B_1 + 11C_1 + 6C_2 + 13D_1 + 10D_2 + 7D_3 + 4D_4 + D_5$
$X_2 = 6A + 3B_1 + 4C_1 + 2C_2$

$WCS = (Y, 1), \quad Y = Y(X_1) + Y(X_2)$
$Y(X_1) = 6A + 3B_1 + 4C_1 + 2C_2 + 5D_1 + 4D_2 + 3D_3 + 2D_4 + D_5$
$Y(X_2) = 2A + B_1 + C_1$

[5] wd([4]12.1.4+IV^*) type

$X = X_1 + X_2, \quad C_3(X_1) = A(X_2)$
$X_1 = 12A + 5B_1 + 3B_2 + 9C_1 + 6C_2 + 3C_3 + 10D_1 + 8D_2 + 6D_3 + 4D_4 + 2D_5$
$X_2 = 3A + 2C_1 + C_2 + 2D_1 + D_2$

$WCS = (Y, 2), \quad Y = Y(X_1) + Y(X_2)$
$Y(X_1) = 4A + 2B_1 + 2B_2 + 3C_1 + 2C_2 + C_3 + 3D_1 + 2D_2 + D_3$
$Y(X_2) = 2A + 2C_1 + 2D_1$

[5] wd([4]5.1+II^*) type

$X = X_1 + X_2, \quad A(X_1) = D_1(X_2)$
$X_1 = 5A + C_1 + D_1 + 2E_1 + E_2$
$X_2 = 6A + 3B_1 + 4C_1 + 2C_2 + 5D_1$

$WCS = (Y, 1), \quad Y = Y(X_1) + Y(X_2)$
$Y(X_1) = 2A + C_1 + D_1$
$Y(X_2) = 2A + B_1 + C_1 + 2D_1$

$WCS = (Y, 1), \quad Y = Y(X_1) + Y(X_2)$
$Y(X_1) = 2A + D_1 + E_1 + E_2$
$Y(X_2) = 2A + B_1 + C_1 + 2D_1$

$WCS = (Y, 1) + (2Y, 2), \quad Y = Y(X_1) + Y(X_2)$
$Y(X_1) = 2A + E_1$
$Y(X_2) = A + C_1 + C_2 + D_1$

26.5.5 Welding of stellar singular fibers of genus 3 and genus 2

[5] wd([3]14.4+[2]10.1) type

$X = X_1 + X_2, \quad D_4(X_1) = A(X_2)$
$X_1 = 14A + 7B_1 + 8C_1 + 2C_2 + 13D_1 + 12D_2 + 11D_3 + 10D_4$
$X_2 = 10A + 5B_1 + 4D_1 + 2D_2$

$WCS = (Y, 1) + (2Y, 2), \quad Y = Y(X_1) + Y(X_2)$
$Y(X_1) = 2A + B_1 + C_1 + 2D_1 + 2D_2 + 2D_3 + 2D_4$
$Y(X_2) = 2A + B_1 + C_1 + C_2$

[5] wd([3]14.6+[2]8.1) type

$X = X_1 + X_2, \quad D_2(X_1) = A(X_2)$
$X_1 = 14A + 7B_1 + 10C_1 + 6C_2 + 2C_3 + 11D_1 + 8D_2$
$X_2 = 8A + 4B_1 + C_1$

$WCS = (Y, 1), \quad Y = Y(X_1) + Y(X_2)$
$Y(X_1) = 2A + B_1 + C_1 + 2D_1 + 2D_2$
$Y(X_2) = 2A + B_1 + C_1$

[5] wd([3]14.6 + [2]5.1) type

$X = X_1 + X_2, \quad D_3(X_1) = A(X_2)$
$X_1 = 14A + 7B_1 + 10C_1 + 6C_2 + 2C_3 + 11D_1 + 8D_2 + 5D_3$
$X_2 = 5A + B_1 + C_1$

$WCS = (Y, 1), \quad Y = Y(X_2)$
$Y(X_2) = A + B_1 + C_1$

[5] wd([3]14.6 + [2]5.4) type

$X = X_1 + X_2, \quad D_3(X_1) = A(X_2)$
$X_1 = 14A + 7B_1 + 10C_1 + 6C_2 + 2C_3 + 11D_1 + 8D_2 + 5D_3$
$X_2 = 5A + 3C_1 + C_2 + 4D_1 + 3D_2 + 2D_3 + D_4$

$WCS = (Y, 1) + (2Y, 2), \quad Y = Y(X_1) + Y(X_2)$
$Y(X_1) = A + C_1 + C_2 + C_3 + D_1 + D_2 + D_3$
$Y(X_2) = A + C_1 + D_1 + D_2 + D_3$

[5] wd([3]12.4 + [2]10.1) type

$X = X_1 + X_2, \quad D_2(X_1) = A(X_2)$
$X_1 = 2A + 6B_1 + 7C_1 + 2C_2 + C_3 + 11D_1 + 10D_2$
$X_2 = 10A + 5B_1 + 4D_1 + 2D_2$

$WCS = (Y, 1) + (2Y, 2), \quad Y = Y(X_1) + Y(X_2)$
$Y(X_1) = 2A + B_1 + C_1 + 2D_1 + 2D_2$
$Y(X_2) = 2A + B_1 + D_1 + D_2$

[5] wd([3]12.4 + [2]8.1) type

$X = X_1 + X_2, \quad D_4(X_1) = A(X_2)$
$X_1 = 2A + 6B_1 + 7C_1 + 2C_2 + C_3 + 11D_1 + 10D_2 + 9D_3 + 8D_4$
$X_2 = 8A + 4B_1 + 3D_1 + D_2$

$WCS = (Y, 2), \quad Y = Y(X_1) + Y(X_2)$
$Y(X_1) = 10A + 5B_1 + 6C_1 + 2C_2 + 9D_1 + 8D_2 + 7D_3 + 6D_4$
$Y(X_2) = 6A + 3B_1 + 2D_1$

[5] wd([3]12.5 + [2]10.1) type

$X = X_1 + X_2, \quad D_2(X_1) = A(X_2)$
$X_1 = 12A + 4B_1 + 9C_1 + 6C_2 + 3C_3 + 11D_1 + 10D_2$
$X_2 = 10A + 5B_1 + 4D_1 + 2D_2$

$WCS = (Y,1) + (2Y,2) + (3Y,3) + (4Y,4), \quad Y = Y(X_1) + Y(X_2)$
$Y(X_1) = A + B_1 + D_1 + D_2$
$Y(X_2) = A + D_1$

[5] wd([3]9.1 + [2]10.3) type
$X = X_1 + X_2, \quad A(X_1) = D_1(X_2)$
$X_1 = 9A + 3C_1 + 5D_1 + D_2$
$X_2 = 10A + 5B_1 + 6C_1 + 2C_2 + 9D_1$

$WCS = (Y,1) + (2Y,2) + (3Y,3) + (4Y,4) + (5Y,5), \quad Y = Y(X_1) + Y(X_2)$
$Y(X_1) = A + D_1$
$Y(X_2) = A + B_1 + D_1$

[5] wd([3]9.2 + [2]10.3) type
$X = X_1 + X_2, \quad A(X_1) = D_1(X_2)$
$X_1 = 9A + 2C_1 + C_2 + 6D_1 + 3D_2$
$X_2 = 10A + 5B_1 + 6C_1 + 2C_2 + 9D_1$

(i) $WCS = (Y,1) + (2Y,2), \quad Y = Y(X_1) + Y(X_2)$
$Y(X_1) = A + C_1$
$Y(X_2) = A + C_1 + C_2 + D_1$

(ii) $WCS = \sum_{i=1}^{6}(iY,i), \quad Y = Y(X_1) + Y(X_2)$
$Y(X_1) = A + D_1$
$Y(X_2) = A + C_1 + D_1$

[5] wd([3]9.5 + [2]8.1) type
$X = X_1 + X_2, \quad D_1(X_1) = A(X_2)$
$X_1 = 9A + 6B_1 + 3B_2 + 4C_1 + 3C_2 + 2C_3 + C_4 + 8D_1$
$X_2 = 8A + 4B_1 + 3D_1 + D_2$

(i) $WCS = (Y,1) + (2Y,2) + (3Y,3), \quad Y = Y(X_1) + Y(X_2)$
$Y(X_1) = A + B_1 + B_2 + D_1$
$Y(X_2) = A + D_1$

(ii) $WCS = (Y,1) + (2Y,2) + (3Y,3) + (4Y,4), \quad Y = Y(X_1) + Y(X_2)$
$Y(X_1) = A + C_1 + D_1$
$Y(X_2) = A + B_1$

[5] wd([3]8.1 + [2]10.3) type
$X = X_1 + X_2, \quad A(X_1) = D_2(X_2)$
$X_1 = 8A + C_1 + 6D_1 + 4D_2 + 2D_3$
$X_2 = 10A + 5B_1 + 6C_1 + 2C_2 + 9D_1 + 8D_2$

(i) $WCS = (Y,1), \quad Y = Y(X_1) + Y(X_2)$
$Y(X_1) = 2A + C_1 + D_1$
$Y(X_2) = 2A + B_1 + C_1 + 2D_1 + 2D_2$

(ii) $WCS = (Y,1) + (2Y,2), \quad Y = Y(X_1) + Y(X_2)$
$Y(X_1) = A + D_1 + D_2 + D_3$
$Y(X_2) = A + C_1 + C_2 + D_1 + D_2$

[5] wd([3]8.2 + [2]10.3) type
$X = X_1 + X_2, \quad A(X_1) = D_2(X_2)$
$X_1 = 8A + 2C_1 + 5D_1 + 2D_2 + D_3$
$X_2 = 10A + 5B_1 + 6C_1 + 2C_2 + 9D_1 + 8D_2$

$WCS = (Y,1) + (2Y,2), \quad Y = Y(X_1)$
$Y(X_1) = A + C_1 + D_1 + D_2$

[5] wd([3]7.1 + [2]10.3) type
$X = X_1 + X_2, \quad A(X_1) = D_3(X_2)$
$X_1 = 7A + C_1 + 5D_1 + 3D_2 + D_3$
$X_2 = 10A + 5B_1 + 6C_1 + 2C_2 + 9D_1 + 8D_2 + 7D_3$

$WCS = (Y,1), \quad Y = Y(X_1)$
$Y(X_1) = A + C_1 + D_1 + D_2 + D_3$

[5] wd([3]7.1 + [2]8.2) type
$X = X_1 + X_2, \quad A(X_1) = D_1(X_2)$
$X_1 = 7A + C_1 + 5D_1 + 3D_2 + D_3$
$X_2 = 8A + 4B_1 + 5C_1 + 2C_2 + C_3 + 7D_1$

$WCS = (Y,1), \quad Y = Y(X_1)$
$Y(X_1) = A + C_1 + D_1 + D_2 + D_3$

[5] wd([3]7.2 + [2]10.3) type
$X = X_1 + X_2, \quad A(X_1) = D_3(X_2)$
$X_1 = 7A + 2C_1 + C_2 + 4D_2 + D_3$
$X_2 = 10A + 5B_1 + 6C_1 + 2C_2 + 9D_1 + 8D_2 + 7D_3$

$WCS = (Y,1) + (2Y,2), \quad Y = Y(X_1) + Y(X_2)$
$Y(X_1) = A + C_1$
$Y(X_2) = A + C_1 + C_2 + D_1 + D_2 + D_3$

[5] wd([3]7.2 + [2]8.2) type
$X = X_1 + X_2, \quad A(X_1) = D_1(X_2)$
$X_1 = 7A + 2C_1 + C_2 + 4D_2 + D_3$
$X_2 = 8A + 4B_1 + 5C_1 + 2C_2 + C_3 + 7D_1$

(i) $WCS = (Y,1) + (2Y,2), \quad Y = Y(X_1) + Y(X_2)$
$Y(X_1) = A + C_1$
$Y(X_2) = A + C_1 + C_2 + D_1$

(ii) $WCS = (Y,1) + (2Y,2) + (3Y,3) + (4Y,4), \quad Y = Y(X_1) + Y(X_2)$
$Y(X_1) = A + D_1$
$Y(X_2) = A + B_1 + D_1$

[5] wd([3]7.3 + [2]10.4) type
$X = X_1 + X_2, \quad A(X_1) = C_1(X_2)$
$X_1 = 7A + B_1 + 3D_1 + 2D_2 + D_3$
$X_2 = 10A + 5B_1 + 7C_1 + 8D_1 + 6D_2 + 4D_3 + 2D_4$

$WCS = (Y,1), \quad Y = Y(X_1) + Y(X_2)$
$Y(X_1) = 2A + B_1 + C_1 + D_1 + D_2 + D_3$
$Y(X_2) = 2A + B_1 + 2C_1 + D_1$

[5] wd([3]7.6 + [2]10.4) type
$X = X_1 + X_2, \quad A(X_1) = C_1(X_2)$
$X_1 = 7A + 5C_1 + 3C_2 + C_3 + 6D_1 + 5D_2 + 4D_3 + 3D_4 + 2D_5 + D_6$
$X_2 = 10A + 5B_1 + 7C_1 + 8D_1 + 6D_2 + 4D_3 + 2D_4$

$WCS = (Y,1) + (2Y,2) + (3Y,3) + (4Y,4) + (5Y,5), \quad Y = Y(X_1) + Y(X_2)$
$Y(X_1) = A + C_1 + D_1 + D_2$
$Y(X_2) = A + B_1 + C_1$

[5] wd([3]6.1.1 + [2]8.2) type
$X = X_1 + X_2, \quad A(X_1) = D_2(X_2)$
$X_1 = 6A + 3C_1 + 3D_1 + 5E_1 + 4E_2 + 3E_3 + 2E_4 + E_5$
$X_2 = 8A + 4B_1 + 5C_1 + 2C_2 + C_3 + 7D_1 + 6D_2$

$WCS = (Y,1) + (2Y,2) + (3Y,3), \quad Y = Y(X_1)$
$Y(X_1) = A + C_1 + D_1 + E_1 + E_2 + E_3$

[5] wd([3]6.2.2 + [2]8.2) type
$X = X_1 + X_2, \quad A(X_1) = D_2(X_2)$
$X_1 = 6A + 3C_1 + 4D_1 + 2D_2 + 4E_1 + 2E_2$
$X_2 = 8A + 4B_1 + 5C_1 + 2C_2 + C_3 + 7D_1 + 6D_2$

$WCS = (Y,1) + (2Y,2) + (3Y,3) + (4Y,4), \quad Y = Y(X_1) + Y(X_2)$
$Y(X_1) = A + D_1 + E_1$
$Y(X_2) = A + D_1 + D_2 + B_1$

[5] wd([3]A1.2 + [2]A1.1) type
$X = X_1 + X_2, \quad C_1(X_1) = A(X_2)$

- $N_{A(X_1)}^{\otimes 3} = \mathcal{O}(-p_1 - 2p_2)$
- $N_{A(X_2)}^{\otimes 2} = \mathcal{O}(-3p_2 - p_3)$

- $p_1 = A(X_1) \cap B_1(X_1), \quad p_2 = A(X_1) \cap C_1(X_1),$
- $p_3 = A(X_2) \cap C_1(X_2).$

$X_1 = 3A + B_1 + 2C_1$
$X_2 = 2A + C_1$

$WCS = (Y, 1), \quad Y = Y(X_1)$
$Y(X_1) = A + B_1$

Hence $N_{A(X_1)} = \mathcal{O}(-p_1)$, and $2p_1 \sim 2p_2$. Moreover, we choose p_2, p_3 so that there exists $q \in A(X_2)$ satisfying $q \neq p_2, p_3, \quad p_2 + p_3 \sim 2q$.

Note: $2p_1 \sim 2p_2$ does not imply $p_1 \sim p_2$.

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