Ivan Nourdin

BOCCONI & SPRINGER SERIES
Mathematics, Statistics, Finance and Economics
4

Selected Aspects of Fractional Brownian Motion

BOCCONI UNIVERSITY PRESS

Springer

To Lili, Juliette and Delphine

B&SS – Bocconi & Springer Series

Ivan Nourdin

Selected Aspects of Fractional Brownian Motion

Bocconi University Press

Springer

Ivan Nourdin
Université de Lorraine
Vandoeuvre-lès-Nancy
France

ISSN 2039-1471 ISSN 2039-148X (electronic)
B&SS – Bocconi & Springer Series
ISBN 978-88-470-2822-7 ISBN 978-88-470-2823-4 (eBook)
DOI 10.1007/978-88-470-2823-4
Springer Milan Heidelberg New York Dordrecht London

Library of Congress Control Number: 2012945544

Mathematics Subject Classification: 60G15, 60H07, 60F05, 46L54

9 8 7 6 5 4 3 2 1

Cover design: Beatrice B, Milano
Typesetting: PTP-Berlin, Protago TeX-Production GmbH, Germany (www.ptp-berlin.de)
Printing: Grafiche Porpora, Segrate (MI)

Printed on acid-free paper

Printed in Italy

Springer-Verlag Italia S.r.l., Via Decembrio 28, I-20137 Milano
Springer is a part of Springer Science+Business Media (www.springer.com)

Preface

As is well-known, the classical Brownian motion is a stochastic process which is selfsimilar of index 1/2 and has stationary increments. It is actually the only continuous Gaussian process (up to a constant factor) to have these two properties that are often observed in the ‘real life’, for instance in the movement of particles suspended in a fluid, or in the behavior of the logarithm of the price of a financial asset. More generally, it is natural to wonder whether there exists a stochastic process which would be at the same time Gaussian, with stationary increments and selfsimilar, but not necessarily with an index 1/2 as in the Brownian motion case. Such a process happens to exist, and was introduced by Kolmogorov [27] in the early 1940s for modeling turbulence in liquids. The name *fractional Brownian motion* (fBm in short), which is the terminology everyone uses nowadays, comes from the paper by Mandelbrot and Van Ness [29].

The law of fBm relies on a single parameter H between 0 and 1, the so-called Hurst parameter or selfsimilarity index. Fractional Brownian motion is interesting for modeling purposes, as it allows the modeler to adjust the value of H to be as close as possible to its observations. It is worthwhile noting at this stage, however, that the picture is not as rosy as it seems. Indeed, except when its selfsimilarity index is 1/2, fBm is neither a semimartingale, nor a Markov process. As a consequence, its toolbox is limited, so that solving problems involving fBm is often a non-trivial task. On the positive side, fBm offers new challenges for the specialists of stochastic calculus!

The goal of this book is to develop some aspects of fBm (as well as related topics), without seeking for completeness at all. To be comprehensive would have been an impossible task to fulfill anyway, given the huge amount of works that are nowadays dedicated to fBm[1]. Instead, my guiding thread was to develop the topics I found the most aesthetic (with all the subjectivity it implies!) by trying to avoid technicalities as much as possible, in order to show the reader that solving questions involving fBm may lead to beautiful mathematics. In fact, it was often an excuse for the development of a more general theory, for which the fBm then becomes a concrete and significant example.

[1] According to MathSciNet, at the time this book is written there is more than one thousand papers dealing with fBm.

The plan of the book, which is directed at an audience having a reasonable probability background (at the level of any of the standard texts) is as follows. Chapter 1 gathers the needed preliminary results; in particular, it recalls the exact definitions of Gaussian vectors, sequences and processes, as well as their basic properties. It also contains a proof of the existence of the fBm. Chapter 2 introduces the fBm and provides some of its main properties. Chapter 3 develops an integration theory with respect to fBm. When the Hurst parameter is greater than 1/2, it also gives a framework allowing to solve integral equations involving fBm. Chapter 4 is devoted to the study of the asymptotic behavior of the cumulative distribution function of the supremum of fBm. In passing, we prove that the supremum of any Gaussian process roughly behaves like a single Gaussian variable with variance equal to the largest variance achieved by the entire process. Chapter 5 contains all the definitions and results on Malliavin calculus that are relevant throughout the sequel. Chapter 6 gives a complete characterization of CLTs on the Wiener space in terms of "fourth moments conditions", by combining Stein's method with Malliavin calculus. The asymptotic behavior of the quadratic variation of fBm is then studied in detail. Chapter 7 shows how fBm (as well as another related process) arises naturally in the large limit of partial sums associated to time series with long-range dependence. Finally, Chapter 8 extends some of the results of Chapter 7 to the free probability context; in particular, it introduces the reader to a non-commutative counterpart of the fBm.

It is fair to mention that this book is not the only treatise devoted to fBm. Other references (focusing mainly on different aspects of fBm) include the books by Biagini, Hu, Øksendal and Zhang [3], Cohen and Istas [11], Embrechts and Maejima [17], Hu [22], Mishura [30], Nualart [45], Pipiras and Taqqu [49], Prakasa Rao [50] and Samorodnitsky and Taqqu [56]. Also, it is worthwhile noting that the present book complements in many respects the recent monograph [39] by the author with Peccati (and viceversa), that only tangentially deals with fBm.

Nancy and Paris, June 2012 Ivan Nourdin

Acknowledgements

This text stems from a postgraduate course I taught at the University of Luxembourg in 2011 and 2012. It is my great pleasure to thank Giovanni Peccati and Anton Thalmaier for their invitation, their hospitality and their generous support.

I would like to thank my friends and colleagues with whom I discussed various problems concerning fBm, especially Jean-Christophe Breton, Sébastien Darses, Mihai Gradinaru, Andreas Neuenkirch, David Nualart, Giovanni Peccati, Mark Podolskij, Anthony Réveillac, Francesco Russo, Thomas Simon, Jason Swanson, Murad Taqqu, Samy Tindel, Ciprian Tudor, Pierre Vallois and Frederi Viens. I also thank my PhD student Raghid Zeineddine for a careful reading and many useful comments.

I would like to express my gratitude to Lorenzo Peccati, one the editor of the Bocconi and Springer Series, for encouraging me to write this book. I am grateful to three anonymous referees for their helpful review, recommendations and suggestions.

I gratefully acknowledge a financial support from the Fondation des Sciences Mathématiques de Paris (which, among other things, allowed me to spend a sabbatical year in the University Paris 6 during the time this manuscript was prepared) and from the following two ANR Grants: *Ecru* ANR-09-BLAN-0114 and *Masterie* ANR-2010-BLAN-0121.

Finally, my last but not least thank goes to the three girls of my life, namely Lili, Juliette and Delphine, for their unbounded patience through the long hours while I was writing this book.

Contents

Chapter 1
Preliminaries

Our aim in this first chapter is to introduce the reader to the realm of Gaussian families (including vectors, sequences and processes). We start with some very elementary facts (generally stated without proofs) about Gaussian random variables and vectors, that are used throughout the book. We next deal with stochastic processes: first in a wide sense, then by focusing on the Gaussian case only. We conclude this chapter by proving the existence of the hero of this book, namely the fractional Brownian motion.

1.1 Gaussian Random Vectors

Here and throughout the book (except in Chapter 8), every random object is defined on an appropriate probability space $(\Omega, \mathscr{F}, P)$. The symbols 'E', 'Var' and 'Cov' denote, respectively, the expectation, the variance and the covariance associated with P.

Definition 1.1. *Let $\mu \in \mathbb{R}$ and $\sigma^2 \geqslant 0$. A real-valued random variable G is said to have a Gaussian distribution with mean μ and variance σ^2, written $G \sim \mathcal{N}(\mu, \sigma^2)$, if its characteristic function is given by*

$$E\left[e^{itG}\right] = e^{it\mu - t^2\sigma^2/2}, \quad \forall t \in \mathbb{R}.$$

Alternatively, we say that G is a Gaussian random variable with mean μ and variance σ^2. When $G \sim \mathcal{N}(0,1)$, we simply say that G is a standard Gaussian random variable.

The following properties hold.

Proposition 1.1. 1. *If $G \sim \mathcal{N}(\mu, 0)$, then $G = \mu$ with probability one.*
2. *If $\sigma \neq 0$, then $G \sim \mathcal{N}(\mu, \sigma^2)$ has a density f with support equal to $\mathbb{R}$, given by*

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/(2\sigma^2)}, \quad x \in \mathbb{R}.$$

Nourdin I.: *Selected Aspects of Fractional Brownian Motion*, B&SS,
DOI 10.1007/978-88-470-2823-4_1, © Springer-Verlag Italia 2012

3. *If* $G \sim \mathcal{N}(\mu, \sigma^2)$*, then* $E[e^{tG}] = e^{t\mu + t^2\sigma^2/2} < \infty$ *for all* $t \in \mathbb{R}$.
4. *If* $G_1 \sim \mathcal{N}(\mu_1, \sigma_1^2)$ *and* $G_2 \sim \mathcal{N}(\mu_2, \sigma_2^2)$ *are independent, then* $G_1 + G_2 \sim \mathcal{N}(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$.
5. *If* $a, b \in \mathbb{R}$ *and* $G \sim \mathcal{N}(\mu, \sigma^2)$*, then* $aG + b \sim \mathcal{N}(a\mu + b, a^2\sigma^2)$.

Let us now consider the situation where more than one Gaussian random variable is involved.

Definition 1.2. 1. *Let* $d \geqslant 2$*. A random vector* $G = (G_1, \ldots, G_d)$ *is said to have a* d*-dimensional Gaussian distribution if, for every* $t_1, \ldots, t_d \in \mathbb{R}$*, the random variable* $\sum_{k=1}^d t_k G_k$ *has a one-dimensional Gaussian distribution. When* $(G_1, \ldots, G_d)$ *has a* d*-dimensional Gaussian distribution, we say that the random variables* $G_1, \ldots, G_d$ *are jointly Gaussian or, alternatively, that* $(G_1, \ldots, G_d)$ *is a Gaussian random vector.*
2. *Let* I *be an arbitrary set. A Gaussian family indexed by* I *is a collection of random variables* $(G_i)_{i \in I}$ *such that, for every* $d \geqslant 2$ *and every* $i_1, \ldots, i_d \in I$*, the vector* $(G_{i_1}, \ldots, G_{i_d})$ *has a* d*-dimensional Gaussian distribution.*

It is immediate to check that the distribution of any d-dimensional Gaussian vector $(G_1, \ldots, G_d)$ is uniquely determined by its mean $\mu = (\mu_1, \ldots, \mu_d)$, where

$$\mu_k = E[G_k], \quad k = 1, \ldots, d,$$

and its covariance matrix $C = (C_{k,l})_{1 \leqslant k,l \leqslant d}$, given by

$$C_{k,l} = \mathrm{Cov}(G_k, G_l), \quad k, l = 1, \ldots, d.$$

One has indeed that, for all $t_1, \ldots, t_d \in \mathbb{R}$,

$$E\left[\exp\left(i \sum_{k=1}^d t_k G_k\right)\right] = \exp\left\{i \sum_{k=1}^d t_k \mu_k - \frac{1}{2} \sum_{k,l=1}^d t_k t_l C_{k,l}\right\}. \tag{1.1}$$

When $G = (G_1, \ldots, G_d)$ verifies (1.1), we write $G \sim \mathcal{N}_d(\mu, C)$. Thanks to (1.1), we get an easy-to-check criterion for independence.

Corollary 1.1. *Let* $G = (G_1, \ldots, G_d)$ *be a Gaussian random vector. Let* $J \subset \{1, \ldots, d\}$ *and set* $J^c = \{1, \ldots, d\} \setminus J$*. Then,* $\{G_j\}_{j \in J}$ *and* $\{G_k\}_{k \in J^c}$ *are independent if and only if* $\mathrm{Cov}(G_j, G_k) = 0$ *for all* $j \in J$ *and* $k \in J^c$.

Since a variance is positive, the covariance matrix C of a Gaussian random vector $(G_1, \ldots, G_d)$ is necessarily such that

$$\sum_{k,l=1}^d t_k t_l C_{k,l} = \mathrm{Var}\left(\sum_{k=1}^d t_k G_k\right) \geqslant 0, \quad t_1, \ldots, t_d \in \mathbb{R};$$

i.e., C is positive in the sense of symmetric matrices. It is a remarkable fact that the converse implication holds as well.

Theorem 1.1. *Let $C = (C_{k,l})_{1 \leqslant k,l \leqslant d}$ be a real-valued $d \times d$ symmetric positive matrix. Then, there exists a centered Gaussian random vector $G = (G_1, \ldots, G_d)$ admitting C as covariance matrix (that is, such that $E[G_k] = 0$ and $\mathrm{Cov}(G_k, G_l) = C_{k,l}$ for all $k, l = 1, \ldots, d$).*

Due to Theorem 1.1, we understand the importance to have criterions for a given symmetric matrix to be positive. This is indeed a crucial task when working in the Gaussian realm, and we defer this analysis in the subsequent Section 1.3.

1.2 Hermite Polynomials

We introduce a family of polynomials, namely the Hermite polynomials, that allow to do effective calculations on expectations involving Gaussian random vectors. Let us start with an auxiliary definition.

Definition 1.3. *The linear operator $\delta : \mathcal{C}^1 \to \mathcal{C}^0$ is defined as*

$$(\delta\varphi)(x) = x\varphi(x) - \varphi'(x), \quad x \in \mathbb{R}. \tag{1.2}$$

We then have the following useful duality formula.

Proposition 1.2. *Let $G \sim \mathcal{N}(0,1)$ and let $\varphi, \psi : \mathbb{R} \to \mathbb{R}$ be of class $\mathcal{C}^1$ and have no more than an exponential growth together with their derivatives. Then*

$$E[\psi'(G)\varphi(G)] = E[\psi(G)(\delta\varphi)(G)]. \tag{1.3}$$

Proof. An integration by parts (the bracket term is easily shown to vanish) immediately gives the desired conclusion; indeed,

$$\begin{aligned} E[\psi'(G)\varphi(G)] &= \int_{-\infty}^{\infty} \psi'(x)\varphi(x)\frac{1}{\sqrt{2\pi}}e^{-x^2/2}dx \\ &= \int_{-\infty}^{\infty} \psi(x)(\delta\varphi)(x)\frac{1}{\sqrt{2\pi}}e^{-x^2/2}dx = E[\psi(G)(\delta\varphi)(G)]. \quad \square \end{aligned}$$

The family of Hermite polynomials is defined as the orbit of 1 (viewed as a constant function) under the action of δ.

Definition 1.4. *For any integer $k \geqslant 1$, the kth Hermite polynomial is defined as $H_k = \delta^k 1$, where 1 indicates the function constantly equal to one and δ is defined by (1.2). By convention, we also set $H_{-1} = 0$ and $H_0 = 1$.*

The first few Hermite polynomials are $H_1 = X$, $H_2 = X^2 - 1$ and $H_3 = X^3 - 3X$. The next proposition gathers the main properties of these polynomials.

Proposition 1.3. *The family $(H_k)_{k \in \mathbb{N}} \subset \mathbb{R}[X]$ of Hermite polynomials has the following properties.*

1. *$H_k' = kH_{k-1}$ and $H_{k+1} = XH_k - kH_{k-1}$ for all $k \in \mathbb{N}$.*

2. *The family* $(\frac{1}{\sqrt{k!}} H_k)_{k \in \mathbb{N}}$ *is an orthonormal basis of* $L^2(\mathbb{R}, \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx)$.
3. *Let* (U, V) *be a Gaussian vector with* $U, V \sim \mathcal{N}(0, 1)$. *Then, for all* $k, l \in \mathbb{N}$,

$$E[H_k(U)H_l(V)] = \begin{cases} k!E[UV]^k & \text{if } k = l \\ 0 & \text{otherwise.} \end{cases}$$

Proof. 1. Let $D : \mathbb{R}[X] \to \mathbb{R}[X]$ be the differentiation operator (i.e., $DP = P'$), and recall the definition (1.2) of δ. It is readily checked that $D\delta - \delta D$ is the identity operator on $\mathbb{R}[X]$, that is, $(D\delta - \delta D)P = P$ for all $P \in \mathbb{R}[X]$. More generally, an easy induction (on k) leads to

$$D\delta^k - \delta^k D = k\delta^{k-1}, \quad k \geqslant 1.$$

We deduce that

$$H_k' = D\delta^k 1 = k\delta^{k-1} 1 + \delta^k D1 = kH_{k-1},$$

as well as

$$H_{k+1} = \delta^{k+1} 1 = \delta\delta^k 1 = \delta H_k = XH_k - H_k' = XH_k - kH_{k-1}.$$

2. Let $G \sim \mathcal{N}(0, 1)$. If $k \geqslant l \geqslant 1$ (recall the convention $H_{-1} = 0$), we can write

$$\begin{aligned} E[H_k(G)H_l(G)] &= E[H_k(G)(\delta^l 1)(G)] \\ &= E[H_k'(G)(\delta^{l-1} 1)(G)] \quad \text{by the duality formula (1.3)} \\ &= kE[H_{k-1}(G)H_{l-1}(G)] \quad \text{by point (1).} \end{aligned}$$

By continuing the procedure, we get that the family $(\frac{1}{\sqrt{k!}} H_k)_{k \in \mathbb{N}}$ is orthonormal in $L^2(\mathbb{R}, \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx)$. On the other hand, it is immediate to prove (e.g. by induction through the second equality in (1)) that the polynomial H_k has degree k for any $k \in \mathbb{N}$. Hence, to prove the claim at (2) it remains to show that the monomials X^k, $k \in \mathbb{N}$, generate a dense subspace of $L^2(\mathbb{R}, \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx)$. For this purpose, it is sufficient to prove that, if $f \in L^2(\mathbb{R}, \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx)$ satisfies $E[G^k f(G)] = 0$ for all $k \in \mathbb{N}$, then f is equal to zero. For $z \in \mathbb{C}$, let

$$\phi(z) = E[f(G)e^{izG}].$$

Using the dominated convergence theorem, we immediately see that ϕ is an entire function, with $\phi^{(k)}(z) = i^k E[G^k f(G)e^{izG}]$, $k \in \mathbb{N}$, $z \in \mathbb{C}$. Hence, $\phi^{(k)}(0) = 0$ for all $k \in \mathbb{N}$, that is, $\phi \equiv 0$. By the uniqueness of Fourier transforms, this implies $f \equiv 0$.

3. For all $c \in \mathbb{R}$, the function $x \mapsto e^{cx}$ belongs to $L^2(\mathbb{R}, \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx)$. Therefore, by point (2),

$$e^{cx} = \sum_{k=0}^{\infty} \frac{1}{k!} E[e^{cG} H_k(G)]\, H_k(x).$$

By applying (1.3) repeatedly, we get

$$E[e^{cG}H_k(G)] = E[e^{cG}(\delta^k 1)(G)] = c^k E[e^{cG}] = c^k e^{\frac{c^2}{2}},$$

so that

$$e^{cx-\frac{c^2}{2}} = \sum_{k=0}^{\infty} \frac{c^k}{k!} H_k(x).$$

For all $s, t \in \mathbb{R}$, the previous identity yields

$$\begin{aligned}\sum_{k,l=0}^{\infty} \frac{s^k t^l}{k!l!} E[H_k(U)H_l(V)] &= E\left[\sum_{k=0}^{\infty} \frac{s^k}{k!} H_k(U) \sum_{l=0}^{\infty} \frac{t^l}{l!} H_l(V)\right] \\ &= E\left[e^{sU-\frac{s^2}{2}} e^{tV-\frac{t^2}{2}}\right] = e^{-\frac{s^2+t^2}{2}} E[e^{sU+tV}] \\ &= e^{-\frac{s^2+t^2}{2}} e^{\frac{1}{2}E[(sU+tV)^2]} = e^{stE[UV]} \\ &= \sum_{k=0}^{\infty} \frac{s^k t^k}{k!} E[UV]^k.\end{aligned}$$

By identifying the coefficients in these two series expansions, we get the formula in (3). □

1.3 Gaussian Processes

We now extend the results of Section 1.1 in the level of stochastic processes. Let $\mathbb{T}$ be a given set (viewed as a set of 'times'); in this book, we will always consider $\mathbb{T} = [0, T]$ with $T < \infty$, $\mathbb{T} = [0, \infty)$ or $\mathbb{T} = \mathbb{R}$. We recall that a (real-valued) *stochastic process* $X = (X_t)_{t\in\mathbb{T}}$ is merely a collection, indexed by $\mathbb{T}$, of real-valued random variables defined on the same probability space $(\Omega, \mathscr{F}, P)$.

Definition 1.5. *Let $X = (X_t)_{t\in\mathbb{T}}$ and $Y = (Y_t)_{t\in\mathbb{T}}$ be two stochastic processes defined on the same probability space $(\Omega, \mathscr{F}, P)$. If $P(X_t = Y_t) = 1$ for all $t \in \mathbb{T}$, we say that X and Y are modifications of each other.*

Remark 1.1. Let X and Y be modifications of each other. In general, we do not have that

$$P(\forall t \in \mathbb{T} : X_t = Y_t) = 1. \tag{1.4}$$

(Here is an explicit counterexample: for $\mathbb{T} = [0, \infty)$, consider $X_t = 0$ and $Y_t = \mathbf{1}_{\{\xi=t\}}$, with ξ a positive random variable having a density; then $P(X_t = Y_t) = 1$ for all $t \geqslant 0$, but $P(\forall t \geqslant 0 : X_t = Y_t) = 0$.) We however that

$$P(\forall t \in \mathbb{T} \cap \mathbb{Q} : X_t = Y_t) = 1,$$

from which we deduce that (1.4) holds true whenever X and Y are further *continuous*. □

Definition 1.6. *Let $X = (X_t)_{t\in\mathbb{T}}$ and $Y = (Y_t)_{t\in\mathbb{T}}$ be two stochastic processes, possibly defined on two different probability spaces. We say that X and Y have the same law, and we write $X \overset{\text{law}}{=} Y$, to indicate that $(X_{t_1},\dots,X_{t_d})$ and $(Y_{t_1},\dots,Y_{t_d})$ have the same law for all $d \geqslant 1$ and all $t_1,\dots,t_d \in \mathbb{T}$.*

Remark 1.2. It is worthwhile noting that two modifications X and Y necessarily have the same law. Indeed, from Definition 1.5 we deduce that $P(X_{t_1} = Y_{t_1},\dots,X_{t_d} = Y_{t_d}) = 1$ for all $d \geqslant 1$ and all $t_1,\dots,t_d \in \mathbb{T}$, meaning in particular that $X \overset{\text{law}}{=} Y$. □

Definition 1.7. *A stochastic process $X = (X_t)_{t\in\mathbb{T}}$ is said to be Gaussian if, for all $d \geqslant 1$ and all $t_1,\dots,t_d \in \mathbb{T}$, $(X_{t_1},\dots,X_{t_d})$ is a Gaussian random vector. The mean of X is then the function $m : \mathbb{T} \to \mathbb{R}$ given by $m(t) = E[X_t]$, while the covariance of X is the function $\Gamma : \mathbb{T}^2 \to \mathbb{R}$ given by $\Gamma(s,t) = \mathrm{Cov}(X_s, X_t)$. When $m \equiv 0$, we say that X is centered.*

Exactly as for Gaussian random vectors, we have the following uniqueness in law result.

Proposition 1.4. *Two Gaussian processes have the same law if and only if they have the same mean and the same covariance.*

Proof. See, e.g., [25, Lemma 11.1]. □

It is more difficult to cope with the existence problem. We start with a relevant definition.

Definition 1.8. *A symmetric function $\Gamma : \mathbb{T}^2 \to \mathbb{R}$ is of positive type if*

$$\sum_{k,l=1}^{d} a_k a_l \Gamma(t_k, t_l) \geqslant 0$$

for all $d \geqslant 1$, $t_1,\dots,t_d \in \mathbb{T}$ and $a_1,\dots,a_d \in \mathbb{R}$.

For example, the symmetric function

$$\Gamma : [0,\infty)^2 \to \mathbb{R}, \quad (s,t) \mapsto s \wedge t = \inf(s,t) \tag{1.5}$$

is of positive type; indeed, if $d \geqslant 1$, $t_1,\dots,t_d \in [0,\infty)$ and $a_1,\dots,a_d \in \mathbb{R}$ are given, then

$$\begin{aligned} \sum_{k,l=1}^{d} a_k a_l \Gamma(t_k, t_l) &= \sum_{k,l=1}^{d} a_k a_l \int_0^\infty \mathbf{1}_{[0,t_k]}(x)\mathbf{1}_{[0,t_l]}(x)dx \\ &= \int_0^\infty \left(\sum_{k=1}^{d} a_k \mathbf{1}_{[0,t_k]}(x) \right)^2 dx \geqslant 0. \end{aligned} \tag{1.6}$$

The next statement emphasizes three stability properties of symmetric functions of positive type.

Proposition 1.5. *Let $\Gamma_1, \Gamma_2 : \mathbb{T}^2 \to \mathbb{R}$ be two symmetric functions of positive type. Then, the sum $\Gamma_1 + \Gamma_2$, the product $\Gamma_1\Gamma_2$ and the function $e^{\Gamma_1} - 1$ are of positive type.*

Proof. The statement for the sum is obvious. Let us consider the product. Let $d \geqslant 1$, $t_1, \ldots, t_d \in \mathbb{T}$ and $a_1, \ldots, a_d \in \mathbb{R}$. We have to show that

$$\sum_{k,l=1}^{d} \Gamma_1(t_k, t_l)\Gamma_2(t_k, t_l)a_k a_l \geqslant 0. \tag{1.7}$$

Let $M_i = (m^i_{k,l})_{1\leqslant k,l\leqslant d} \in \mathcal{M}_d(\mathbb{R})$, $i = 1, 2$, be the two symmetric matrices defined by $m^i_{k,l} = \Gamma_i(t_k, t_l)$. It is a well-known fact from Linear Algebra that there exists an orthogonal matrix $P = (p_{k,l})_{1\leqslant k,l\leqslant d} \in \mathcal{M}_d(\mathbb{R})$, as well as a diagonal matrix $D = (d_{k,l})_{1\leqslant k,l\leqslant d} \in \mathcal{M}_d(\mathbb{R})$ with non-negative entries, such that $M_1 = PD\,^tP$, where $^t\cdot$ stands for the transpose operator. We then have

$$\sum_{k,l=1}^{d} \Gamma_1(t_k, t_l)\Gamma_2(t_k, t_l)a_k a_l = \sum_{j=1}^{d} d_{j,j} \sum_{k,l=1}^{d} \Gamma_2(t_k, t_l)\, p_{k,j}a_k\, p_{l,j}a_l,$$

from which we immediately get that (1.7) holds true, since

$$\sum_{k,l=1}^{d} \Gamma_2(t_k, t_l)\, p_{k,j}a_k\, p_{l,j}a_l \geqslant 0$$

by positivity of Γ_2. Finally, because of the previous stability with respect to sum and product, the function $\sum_{k=1}^{N} \frac{(\Gamma_1)^k}{k!}$ is of positive type for all $N \geqslant 1$, and so is its pointwise limit $e^{\Gamma_1} - 1$ as well. □

The next result explains why the class of symmetric functions of positive type is of particular interest for Gaussian processes. It represents the exact extension of Theorem 1.1 to the level of processes.

Theorem 1.2 (Kolmogorov). *Consider a symmetric function $\Gamma : \mathbb{T}^2 \to \mathbb{R}$. Then, there exists a centered Gaussian process $X = (X_t)_{t\in\mathbb{T}}$ having Γ for covariance function (that is, such that $E[X_t] = 0$ and $E[X_sX_t] = \Gamma(s,t)$ for all $s, t \in \mathbb{T}$) if and only if Γ is of positive type.*

Proof. See, e.g., [16, Theorem 12.1.3]. □

Going back to the function Γ given by (1.5), we deduce that there exists a centered Gaussian process $W = (W_t)_{t\geqslant 0}$ such that $E[W_sW_t] = s \wedge t$ for all $s, t \geqslant 0$: this is indeed the classical Brownian motion. Using Corollary 1.1, it is easily checked that W has independent increments, that is, $W(t_1), W(t_2) - W(t_1), \ldots, W(t_d) - W(t_{d-1})$ are independent for all $d \geqslant 1$ and $t_d > \ldots > t_1 \geqslant 0$. For further use, we also introduce the two-sided classical Brownian motion $W = \{W_t\}_{t\in\mathbb{R}}$ as

$$W_t = \begin{cases} W^1_t & \text{if } t \geqslant 0 \\ W^2_{-t} & \text{if } t < 0 \end{cases}, \tag{1.8}$$

where W^1 and W^2 are two independent (one-sided) classical Brownian motions. Equivalently, any centered Gaussian process admitting $\Gamma : \mathbb{R}^2 \to \mathbb{R}$, $(s,t) \mapsto \frac{1}{2}(|t|+|s|-|t-s|)$ for covariance function is a two-sided classical Brownian motion.

1.4 Continuity

Generally, when dealing with a Gaussian process for modeling purposes, we impose not only a covariance structure, but we also seek for *continuous* sample paths. The following lemma is the ideal tool to check this further property.

Lemma 1.1 (Kolmogorov–Čentsov). *Fix a compact interval $\mathbb{T} = [0,T] \subset \mathbb{R}_+$, and let $X = (X_t)_{t\in\mathbb{T}}$ be a centered Gaussian process. Suppose that there exists $C, \eta > 0$ such that, for all $s,t \in \mathbb{T}$,*

$$E[(X_t - X_s)^2] \leqslant C|t-s|^\eta. \tag{1.9}$$

Then, for all $\alpha \in \left(0, \frac{\eta}{2}\right)$, there exists a modification Y of X with α-Hölder continuous paths. In particular, X admits a continuous modification.

Proof. Fix $t > s$. Since X is Gaussian and centered, we have that

$$X_t - X_s \overset{\text{law}}{=} \sqrt{E[(X_t - X_s)^2]}G,$$

where $G \sim \mathcal{N}(0,1)$. We deduce from (1.9) that, for all $p \geqslant 1$,

$$E[|X_t - X_s|^p] \leqslant C^{p/2}E[|G|^p]\,|t-s|^{\eta p/2}.$$

Therefore, the general version of the classical Kolmogorov–Čentsov lemma (see, e.g., [25, Theorem 2.23]) applies and gives the desired result. □

Remark 1.3. If the process X in Lemma 1.1 is already known to be continuous, then the conclusion of Lemma 1.1 can be reformulated as: "for all $\alpha \in \left(0, \frac{\eta}{2}\right)$, the paths of X are α-Hölder continuous on $[0,T]$".

1.5 Existence of the Fractional Brownian Motion

In this section, we show the existence of the fractional Brownian motion. Further properties of fractional Brownian motion are discussed at length in the next Chapter 2.

Proposition 1.6. *Let $H > 0$ be a real parameter. Then, there exists a continuous centered Gaussian process $B^H = (B_t^H)_{t\geqslant 0}$ with covariance function given by*

$$\Gamma_H(s,t) = \frac{1}{2}\left(s^{2H} + t^{2H} - |t-s|^{2H}\right), \quad s,t \geqslant 0, \tag{1.10}$$

if and only if $H \leqslant 1$. In this case, the sample paths of B^H are, for any $\alpha \in (0,H)$, α-Hölder continuous on each compact set.

Proof. According to Kolmogorov Theorem 1.2, to get our first claim (except for the continuity property, see below for this), we must show that Γ_H is of positive type if and only if $H \leqslant 1$.

Assume first that $H > 1$. When $t_1 = 1, t_2 = 2, a_1 = -2$ and $a_2 = 1$, we have

$$a_1^2\Gamma_H(t_1, t_1) + 2a_1a_2\Gamma_H(t_1, t_2) + a_2^2\Gamma_H(t_2, t_2) = 4 - 2^{2H} < 0.$$

As a consequence, Γ_H is not of positive type when $H > 1$.

The function Γ_1 is of positive type; indeed, $\Gamma_1(s, t) = st$ so that, for all $d \geqslant 1$, $t_1, \dots, t_d \geqslant 0$ and $a_1, \dots, a_d \in \mathbb{R}$,

$$\sum_{k,l=1}^{d} \Gamma_1(t_k, t_l)a_k a_l = \left(\sum_{k=1}^{d} t_k a_k\right)^2 \geqslant 0.$$

Consider now the case $H \in (0, 1)$. For any $x \in \mathbb{R}$, the change of variable $v = u|x|$ (whenever $x \neq 0$) leads to the representation

$$|x|^{2H} = \frac{1}{c_H}\int_0^\infty \frac{1 - e^{-u^2x^2}}{u^{1+2H}}du, \tag{1.11}$$

where $c_H = \int_0^\infty (1 - e^{-u^2})u^{-1-2H}du < \infty$. Therefore, for any $s, t \geqslant 0$, we have

$$\begin{aligned}
&s^{2H} + t^{2H} - |t - s|^{2H}\\
&= \frac{1}{c_H}\int_0^\infty \frac{(1 - e^{-u^2t^2})(1 - e^{-u^2s^2})}{u^{1+2H}}du\\
&\qquad + \frac{1}{c_H}\int_0^\infty \frac{e^{-u^2t^2}(e^{2u^2ts} - 1)e^{-u^2s^2}}{u^{1+2H}}du\\
&= \frac{1}{c_H}\int_0^\infty \frac{(1 - e^{-u^2t^2})(1 - e^{-u^2s^2})}{u^{1+2H}}du\\
&\qquad + \frac{1}{c_H}\sum_{n=1}^{\infty}\frac{2^n}{n!}\int_0^\infty \frac{t^n e^{-u^2t^2}s^n e^{-u^2s^2}}{u^{1-2n+2H}}du,
\end{aligned}$$

so that, for all $d \geqslant 1$, $t_1, \dots, t_d \geqslant 0$ and $a_1, \dots, a_d \in \mathbb{R}$,

$$\begin{aligned}
&\sum_{k,l=1}^{d} \frac{1}{2}\left(t_k^{2H} + t_l^{2H} - |t_k - t_l|^{2H}\right)a_k a_l\\
&= \frac{1}{2c_H}\int_0^\infty \frac{\left(\sum_{k=1}^{d}(1 - e^{-u^2t_k^2})a_k\right)^2}{u^{1+2H}}du\\
&\qquad + \frac{1}{2c_H}\sum_{n=1}^{\infty}\frac{2^n}{n!}\int_0^\infty \frac{\left(\sum_{k=1}^{d} t_k^n e^{-u^2t_k^2}a_k\right)^2}{u^{1-2n+2H}}du \geqslant 0.
\end{aligned}$$

That is, Γ_H is of positive type when $H \in (0, 1)$.

To conclude the proof of Proposition 1.6, suppose that $H \in (0, 1]$, and consider a centered Gaussian process $B^H = (B_t^H)_{t \geqslant 0}$ with covariance function given by (1.10). We then have

$$E[(B_t^H - B_s^H)^2] = |t-s|^{2H}, \quad s, t \geqslant 0,$$

so that Kolmogorov–Čentsov Lemma 1.1 (see also Remark 1.3) applies and shows the second claim of Proposition 1.6 (see also Remark 1.2). □

Chapter 2
Fractional Brownian Motion

Fractional Brownian motion is a stochastic process which deviates significantly from Brownian motion and semimartingales, and others classically used in probability theory. As a centered Gaussian process, it is characterized by the stationarity of its increments and a medium- or long-memory property which is in sharp contrast with martingales and Markov processes. The aim of this chapter is to introduce this process and to provide some of its main and basic properties.

2.1 Definition

The fractional Brownian motion, which was introduced by Kolmogorov in [27] and further developed by Mandelbrot and Van Ness in [29], is defined as follows.

Definition 2.1. *Let $H \in (0, 1]$. A fractional Brownian motion (fBm in short) of Hurst parameter H is a centered continuous Gaussian process $B^H = (B_t^H)_{t \geqslant 0}$ with covariance function*

$$E[B_t^H B_s^H] = \frac{1}{2}\big(t^{2H} + s^{2H} - |t-s|^{2H}\big). \tag{2.1}$$

According to Proposition 1.6, fractional Brownian motion well exists and has Hölder continuous paths. When $H > \frac{1}{2}$, it is readily checked that its covariance function verifies

$$\frac{1}{2}\big(t^{2H} + s^{2H} - |t-s|^{2H}\big) = H(2H-1)\int_0^t du \int_0^s dv\, |v-u|^{2H-2}. \tag{2.2}$$

Unfortunately, the useful identity (2.2) is not valid when $H \leqslant \frac{1}{2}$ since, in this case, the kernel $|v-u|^{2H-2}$ is not integrable.

The following proposition emphasizes two particular values of H, and shows that the case $H = 1$ is somehow trivial. As a consequence, from now on, we will always assume that $0 < H < 1$.

Nourdin I.: *Selected Aspects of Fractional Brownian Motion*, B&SS,
DOI 10.1007/978-88-470-2823-4_2,

Proposition 2.1. *Let B^H be a fractional Brownian motion of Hurst index $H \in (0,1]$.*

1. *If $H = \frac{1}{2}$ then fBm is nothing but a classical Brownian motion.*
2. *If $H = 1$ then $B_t^H = tB_1^H$ almost surely for all $t \geqslant 0$.*

Proof. 1. We immediately see that the covariance of $B^{\frac{1}{2}}$ reduces to $(s,t) \mapsto s \wedge t$, so that $B^{\frac{1}{2}}$ is a classical Brownian motion.

2. When $H = 1$, we have, for all $t \geqslant 0$,

$$\begin{aligned} E[(B_t^H - tB_1^H)^2] &= E[(B_t^H)^2] - 2tE[B_t^H B_1^H] + t^2 E[(B_1^H)^2] \\ &= t^2 - t(t^2 + 1 - (1-t)^2) + t^2 = 0, \end{aligned}$$

that is, $B_t^H = tB_1^H$ almost surely. □

2.2 Basic Properties

Proposition 2.2. *Let B^H be a fractional Brownian motion of Hurst parameter $H \in (0,1)$. Then:*

1. [Selfsimilarity] *For all $a > 0$, $(a^{-H} B_{at}^H)_{t \geqslant 0} \stackrel{\text{law}}{=} (B_t^H)_{t \geqslant 0}$.*
2. [Stationarity of increments] *For all $h > 0$, $(B_{t+h}^H - B_h^H)_{t \geqslant 0} \stackrel{\text{law}}{=} (B_t^H)_{t \geqslant 0}$.*
3. [Time inversion] $(t^{2H} B_{1/t}^H)_{t>0} \stackrel{\text{law}}{=} (B_t^H)_{t>0}$.

Conversely, any continuous Gaussian process $B^H = (B_t^H)_{t \geqslant 0}$ with $B_0^H = 0$, $\mathrm{Var}(B_1^H) = 1$ and such that (1) and (2) hold, is a fractional Brownian motion of index H.

Proof. To prove that points (1), (2) and (3) hold, the way is the same: in these three cases, it is readily checked that the process in the left-hand side is centered, Gaussian and has (2.1) for covariance. Proposition 1.4 allows then to conclude. Conversely, let $B^H = (B_t^H)_{t \geqslant 0}$ be a continuous Gaussian process with $B_0^H = 0$ and $\mathrm{Var}(B_1^H) = 1$ that further verifies (1) and (2). We must show that B^H is centered and has (2.1) for covariance. From (2) with $t = h > 0$, we get that $E[B_{2t}^H] = 2E[B_t^H]$, whereas from (1) we infer that $E[B_{2t}^H] = 2^H E[B_t^H]$. Combining these two equalities gives $E[B_t^H] = 0$ for all $t > 0$. That is, B^H is centered. Now, let $s, t \geqslant 0$. We have

$$\begin{aligned} E[B_s^H B_t^H] &= \frac{1}{2}\big(E[(B_t^H)^2] + E[(B_s^H)^2] - E[(B_t^H - B_s^H)^2]\big) \\ &= \frac{1}{2}\big(E[(B_t^H)^2] + E[(B_s^H)^2] - E[(B_{|t-s|}^H)^2]\big) \quad \text{because of (2)} \\ &= \frac{1}{2}E[(B_1^H)^2]\big(t^{2H} + s^{2H} - |t-s|^{2H}\big) \quad \text{because of (1)} \\ &= \frac{1}{2}\big(t^{2H} + s^{2H} - |t-s|^{2H}\big). \end{aligned}$$

The proof of the proposition is done. □

2.3 Three Stochastic Representations

In this section, we show that fractional Brownian motion can be represented as a Wiener integral in (at least) three different ways.

Let us anticipate from Section 5.1 that the Wiener integral of a two-sided Brownian motion $W = (W_t)_{t\geqslant 0}$ is nothing but a suitable Gaussian family

$$\left\{\int_{\mathbb{R}} f(s)dW_s : f \in L^2(\mathbb{R})\right\},$$

whose law is characterized by the following features:

$$E\left[\int_{\mathbb{R}} f(u)dW_u\right] = 0 \tag{2.3}$$

$$E\left[\int_{\mathbb{R}} f(u)dW_u \int_{\mathbb{R}} g(u)dW_u\right] = \int_{\mathbb{R}} f(u)g(u)du. \tag{2.4}$$

The first representation of fBm (obtained in [29]) is the so-called *time representation*.

Proposition 2.3. *Let* $H \in \left(0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right)$*, set*

$$c_H = \sqrt{\frac{1}{2H} + \int_0^\infty \left((1+u)^{H-\frac{1}{2}} - u^{H-\frac{1}{2}}\right)^2 du} < \infty,$$

and let $W = (W_t)_{t\in\mathbb{R}}$ *be a two-sided classical Brownian motion, see (1.8). Then, (any continuous modification of) the process* $B^H = (B_t^H)_{t\geqslant 0}$*, defined as*

$$B_t^H = \frac{1}{c_H}\left(\int_{-\infty}^0 \left((t-u)^{H-\frac{1}{2}} - (-u)^{H-\frac{1}{2}}\right)dW_u + \int_0^t (t-u)^{H-\frac{1}{2}}dW_u\right), \tag{2.5}$$

is a fractional Brownian motion of Hurst parameter H.

Proof. We first check that the Wiener integral in (2.5) is well-defined. When $u \to -\infty$, one has, for all fixed $t > 0$,

$$\left((t-u)^{H-\frac{1}{2}} - (-u)^{H-\frac{1}{2}}\right)^2 \sim \left(H - \frac{1}{2}\right)^2 t^2(-u)^{2H-3}.$$

Hence, $u \mapsto \left((t-u)^{H-\frac{1}{2}} - (-u)^{H-\frac{1}{2}}\right)^2$ is integrable at $-\infty$ because $2H-3 < -1$. Since $2H-1 > -1$, it is integrable at $u \to 0^-$ as well. Therefore, the Wiener integral $\int_{-\infty}^0 \left((t-u)^{H-\frac{1}{2}} - (-u)^{H-\frac{1}{2}}\right)dW_u$ is well-defined. Similarly, one shows that the constant c_H is finite and that the Wiener integral $\int_0^t (t-u)^{H-\frac{1}{2}}dW_u$ is well-defined. Consequently, B_t^H is well-defined for any $t \geqslant 0$.

Now, let us show that (any continuous modification of) B^H is a fractional Brownian motion of Hurst parameter H. First, using (2.3), it is clear that B^H is centered and Gaussian. Second, observe that

$$B_t^H = \frac{1}{c_H}\int_{\mathbb{R}} \left((t-u)^{H-\frac{1}{2}}\mathbf{1}_{\{u<t\}} - (-u)^{H-\frac{1}{2}}\mathbf{1}_{\{u<0\}}\right) dW_u.$$

Fix $t > s \geqslant 0$. By using (2.4) and with $v = u - s = (t-s)w$, we can write

$$\begin{aligned} E[(B_t^H - B_s^H)^2] &= \frac{1}{c_H^2}\int_{\mathbb{R}} \left((t-u)^{H-\frac{1}{2}}\mathbf{1}_{\{u<t\}} - (s-u)^{H-\frac{1}{2}}\mathbf{1}_{\{u<s\}}\right)^2 du \\ &= \frac{1}{c_H^2}\int_{\mathbb{R}} \left((t-s-v)^{H-\frac{1}{2}}\mathbf{1}_{\{v<t-s\}} - (-v)^{H-\frac{1}{2}}\mathbf{1}_{\{v<0\}}\right)^2 dv \\ &= \frac{(t-s)^{2H}}{c_H^2}\int_{\mathbb{R}} \left((1-w)^{H-\frac{1}{2}}\mathbf{1}_{\{w<1\}} - (-w)^{H-\frac{1}{2}}\mathbf{1}_{\{w<0\}}\right)^2 dv \\ &= (t-s)^{2H}. \end{aligned}$$

Using moreover that $B_0^H = 0$, we get that, for any $t > s \geqslant 0$,

$$\begin{aligned} E[B_s^H B_t^H] &= \frac{1}{2}\left(E[(B_s^H - B_0^H)^2] + E[(B_t^H - B_0^H)^2] - E[(B_t^H - B_s^H)^2]\right) \\ &= \frac{1}{2}\left(s^{2H} + t^{2H} - (t-s)^{2H}\right), \end{aligned}$$

which leads to the desired conclusion. □

Remark 2.1. Proposition 2.3 provides an alternative proof for the fact that Γ_H given by (1.10) is of positive type when $H \in (0,1)$.

Our second representation of fBm is the *spectral representation* (also called *harmonizable representation*).

Proposition 2.4. *Let* $H \in \left(0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right)$*, set*

$$d_H = \sqrt{2\int_0^\infty \frac{1-\cos u}{u^{2H+1}}du} < \infty,$$

and let $W = (W_t)_{t\in\mathbb{R}}$ *be a two-sided classical Brownian motion, see (1.8). Then, (any continuous modification of) the process* $B^H = (B_t^H)_{t\geqslant 0}$*, defined as*

$$B_t^H = \frac{1}{d_H}\left(\int_{-\infty}^0 \frac{1-\cos(ut)}{|u|^{H+\frac{1}{2}}}dW_u + \int_0^\infty \frac{\sin(ut)}{|u|^{H+\frac{1}{2}}}dW_u\right), \tag{2.6}$$

is a fractional Brownian motion of Hurst parameter H.

Proof. First, it is straightforward to check that the Wiener integral in (2.6) is well-defined. Next, let us show that (any continuous modification of) B^H is a fractional Brownian motion of Hurst parameter H. First, using (2.3), it is clear that B^H is centered and Gaussian. On the other hand, by (2.4) we have, for any $t > s$,

$$\begin{aligned}
&E[(B_t^H - B_s^H)^2] \\
&= \frac{1}{d_H^2}\int_{-\infty}^0 \frac{\big(\cos(ut)-\cos(us)\big)^2}{|u|^{2H+1}}du + \frac{1}{d_H^2}\int_0^\infty \frac{\big(\sin(ut)-\sin(us)\big)^2}{|u|^{2H+1}}du \\
&= \frac{1}{d_H^2}\int_0^\infty \frac{\big(\cos(ut)-\cos(us)\big)^2+\big(\sin(ut)-\sin(us)\big)^2}{u^{2H+1}}du \\
&= \frac{2}{d_H^2}\int_0^\infty \frac{1-\cos(u(t-s))}{u^{2H+1}}du \\
&= \frac{2(t-s)^{2H}}{d_H^2}\int_0^\infty \frac{1-\cos v}{v^{2H+1}}dv = (t-s)^{2H}.
\end{aligned}$$

Using moreover that $B_0^H = 0$, we get that, for any $t > s \geqslant 0$,

$$\begin{aligned}
E[B_s^H B_t^H] &= \frac{1}{2}\big(E[(B_s^H - B_0^H)^2] + E[(B_t^H - B_0^H)^2] - E[(B_t^H - B_s^H)^2]\big) \\
&= \frac{1}{2}\big(s^{2H} + t^{2H} - (t-s)^{2H}\big),
\end{aligned}$$

which leads to the desired conclusion. □

Finally, the following Proposition 2.5 (see [13, 34]) provides a third representation of fractional Brownian motion. It shows more precisely that fBm has the form of a Volterra process, that is, can be represented as $B_t^H = \int_0^t K_H(t,s)dW_s$, where $W = (W_t)_{t\geqslant 0}$ is a classical Brownian motion and K_H is an explicit square integrable kernel.

Proposition 2.5. *Let $H \in \big(0,\frac{1}{2}\big)\cup\big(\frac{1}{2},1\big)$ and, for $t > s > 0$, set*

$$K_H(t,s) = \begin{cases} \sqrt{\frac{H(2H-1)}{\int_0^1 (1-x)^{1-2H}x^{H-\frac{3}{2}}dx}}\, s^{\frac{1}{2}-H}\int_s^t (u-s)^{H-\frac{3}{2}}u^{H-\frac{1}{2}}du & \text{if } H > \frac{1}{2} \\ \sqrt{\frac{2H}{(1-2H)\int_0^1 (1-x)^{-2H}x^{H-\frac{1}{2}}dx}} \\ \quad\times\Big[\big(\frac{t}{s}\big)^{H-\frac{1}{2}}(t-s)^{H-\frac{1}{2}} - \big(H-\frac{1}{2}\big)s^{\frac{1}{2}-H}\int_s^t u^{H-\frac{3}{2}}(u-s)^{H-\frac{1}{2}}du\Big] & \text{if } H < \frac{1}{2} \end{cases}.$$

Let $W = (W_t)_{t\geqslant 0}$ be a classical Brownian motion, and define $B^H = (B_t^H)_{t\geqslant 0}$ by

$$B_t^H = \int_0^t K_H(t,s)dW_s. \tag{2.7}$$

Then, (any continuous modification of) B^H is a fractional Brownian motion of Hurst parameter H.

Proof. To limit the size of the book, we only do the proof for $H \in (\frac{1}{2}, 1)$. (The case $H \in (0, \frac{1}{2})$ is slightly more difficult to handle, and we refer to [45, Proposition 5.1.3] for the details.) So, assume that $H > \frac{1}{2}$. It is straightforward to check that $\int_0^t K_H(t,s)^2 ds < \infty$ for all $t > 0$. Thus, (2.7) is well-defined. On the other hand, using (2.3), it is clear that B^H is centered and Gaussian. Let us now compute $E[B_t^H B_s^H]$ for $t > s$. We have, using (2.4),

$$
\begin{aligned}
&E[B_t^H B_s^H] \qquad (2.8)\\
&= \int_0^s K_H(t,u) K_H(s,u) du\\
&= \frac{H(2H-1)}{\int_0^1 (1-x)^{1-2H} x^{H-\frac{3}{2}} dx} \int_0^s du\, u^{1-2H} \int_u^t dy (y-u)^{H-\frac{3}{2}} y^{H-\frac{1}{2}}\\
&\qquad \times \int_u^s dz (z-u)^{H-\frac{3}{2}} z^{H-\frac{1}{2}}\\
&= \frac{H(2H-1)}{\int_0^1 (1-x)^{1-2H} x^{H-\frac{3}{2}} dx} \int_0^t dy\, y^{H-\frac{1}{2}} \int_0^s dz\, z^{H-\frac{1}{2}}\\
&\qquad \times \int_0^{y\wedge z} du\, u^{1-2H} (y-u)^{H-\frac{3}{2}} (z-u)^{H-\frac{3}{2}}. \qquad (2.9)
\end{aligned}
$$

Setting $a = \frac{z\vee y - u}{z \wedge y - u}$ and then $x = \frac{y \vee z}{(y\wedge z)a}$ yields

$$
\begin{aligned}
&\int_0^{y\wedge z} u^{1-2H} (y-u)^{H-\frac{3}{2}} (z-u)^{H-\frac{3}{2}} du\\
&= |z-y|^{2H-2} \int_{\frac{z\vee y}{z\wedge y}}^{\infty} a^{H-\frac{3}{2}} [a(z\wedge y) - z \vee y]^{1-2H} da\\
&= |z-y|^{2H-2} (yz)^{\frac{1}{2}-H} \int_0^1 x^{H-\frac{3}{2}} (1-x)^{1-2H} dx. \qquad (2.10)
\end{aligned}
$$

By plugging (2.10) into (2.9) and by using (2.2) as well, we get

$$
\begin{aligned}
E[B_t^H B_s^H] &= H(2H-1) \int_0^t dy \int_0^s dz |z-y|^{2H-2}\\
&= \frac{1}{2}\big(t^{2H} + s^{2H} - |t-s|^{2H}\big).
\end{aligned}
$$

The proof of Proposition 2.5 is thus complete. □

2.4 Semimartingale Property

In this section, we study the asymptotic behavior of the p-variations of the fractional Brownian motion. As a byproduct, we will show that fBm is never a semimartingale except, of course, when it is the classical Brownian motion corresponding to $H = \frac{1}{2}$. (Recall that a real-valued process is called a semimartingale if it can be decomposed as the sum of a local martingale and a càdlàg adapted process of locally bounded variation. For a standard reference on the notion of semimartingale, we refer the reader to, e.g., the book [51] by Protter.) At this stage however, it is worth to mention the following surprising result proved in [9] by Cheridito. Suppose that B^H is a fBm with Hurst parameter $H \in (3/4, 1)$ and let W be an independent classical Brownian motion. Then $M_t = B_t^H + W_t$, $t \geqslant 0$, **is** a semimartingale! We refer to [9] for the details.

This being said, let us go back to the goal of this section, that is, showing that fBm is not a semimartingale except when its Hurst index is $1/2$. Using Hermite polynomials, we start with a general preliminary result, which may be viewed as a law of large numbers for fBm.

Theorem 2.1. *Let $G \sim \mathcal{N}(0,1)$ and let $f : \mathbb{R} \to \mathbb{R}$ be a measurable function such that $E[f^2(G)] < \infty$. Let B^H be a fractional Brownian motion of Hurst index $H \in (0,1)$. Then, as $n \to \infty$,*

$$\frac{1}{n}\sum_{k=1}^{n} f(B_k^H - B_{k-1}^H) \overset{L^2}{\to} E[f(G)]. \tag{2.11}$$

Remark 2.2. Using the selfsimilarity property of B^H (Proposition 2.2(2)), we immediately deduce that, under the assumptions of Theorem 2.1, we have equivalently that

$$\frac{1}{n}\sum_{k=1}^{n} f\left(n^H (B_{k/n}^H - B_{(k-1)/n}^H)\right) \overset{L^2}{\to} E[f(G)] \text{ as } n \to \infty. \tag{2.12}$$

Proof of Theorem 2.1. When $H = \frac{1}{2}$, the convergence (2.11) follows, of course, directly from the classical law of large numbers, due to the independence of increments in this case. Assume now that $H \neq \frac{1}{2}$. Since $E[f^2(G)] < \infty$, we can expand f in terms of Hermite polynomials (Proposition 1.3(2)), and write:

$$f(x) = \sum_{l=0}^{\infty} \frac{c_l}{\sqrt{l!}} H_l(x), \quad x \in \mathbb{R}. \tag{2.13}$$

The orthogonality property of Hermite polynomials implies that $\sum_{l=0}^{\infty} c_l^2 = E[f^2(G)]$ is finite. Also, choosing $x = G$ and taking the expectation in (2.13) leads to $c_0 = E[f(G)]$. Hence

$$-E[f(G)] + \frac{1}{n}\sum_{k=1}^{n} f(B_k^H - B_{k-1}^H) = \frac{1}{n}\sum_{k=1}^{n}\big(f(B_k^H - B_{k-1}^H) - E[f(G)]\big)$$
$$= \frac{1}{n}\sum_{l=1}^{\infty}\frac{c_l}{\sqrt{l!}}\sum_{k=1}^{n} H_l(B_k^H - B_{k-1}^H).$$

We deduce, by using among other Proposition 1.3(3) to go from the second to the third line,

$$E\bigg[\bigg(-E[f(G)] + \frac{1}{n}\sum_{k=1}^{n} f(B_k^H - B_{k-1}^H)\bigg)^2\bigg]$$
$$= \frac{1}{n^2}\sum_{l=1}^{\infty}\frac{c_l^2}{l!}\sum_{k,k'=1}^{n} E[H_l(B_k^H - B_{k-1}^H)H_l(B_{k'}^H - B_{k'-1}^H)]$$
$$= \frac{1}{n^2}\sum_{l=1}^{\infty} c_l^2 \sum_{k,k'=1}^{n} E[(B_k^H - B_{k-1}^H)(B_{k'}^H - B_{k'-1}^H)]^l$$
$$= \frac{1}{n^2}\sum_{l=1}^{\infty} c_l^2 \sum_{k,k'=1}^{n} \rho_H(k-k')^l,$$

with

$$\rho_H(x) = \rho_H(|x|) = \frac{1}{2}\big(|x+1|^{2H} + |x-1|^{2H} - 2|x|^{2H}\big), \quad x \in \mathbb{Z}.$$

Because $\rho_H(x) = E[B_1^H(B_{|x|+1}^H - B_{|x|}^H)]$, we have, by Cauchy-Schwarz, that

$$|\rho_H(x)| \leqslant \sqrt{E[(B_1^H)^2]}\sqrt{E[(B_{|x|+1}^H - B_{|x|}^H)^2]} = 1.$$

This leads to

$$E\bigg[\bigg(-E[f(G)] + \frac{1}{n}\sum_{k=1}^{n} f(B_k^H - B_{k-1}^H)\bigg)^2\bigg]$$
$$\leqslant \frac{1}{n^2}\sum_{l=1}^{\infty} c_l^2 \sum_{k,k'=1}^{n} |\rho_H(k-k')| = \mathrm{Var}(f(G))\frac{1}{n^2}\sum_{k,k'=1}^{n} |\rho_H(k-k')|$$
$$= \mathrm{Var}(f(G))\frac{1}{n^2}\sum_{k'=1}^{n}\sum_{k=1-k'}^{n-k'} |\rho_H(k)| \leqslant 2\,\mathrm{Var}(f(G))\frac{1}{n}\sum_{k=0}^{n-1}|\rho_H(k)|.$$

To conclude, it remains to study the asymptotic behavior of $\sum_{k=1}^{n-1}|\rho_H(k)|$. It is readily checked that $\rho_H(k) \sim H(2H-1)k^{2H-2}$ as $k \to \infty$. If $H < \frac{1}{2}$ then

$\sum_{k=1}^{n-1} |\rho_H(k)| \to \sum_{k=1}^{\infty} |\rho_H(k)| < \infty$ as $n \to \infty$, implying in turn that (2.11) holds. If $H > \frac{1}{2}$ then $\sum_{k=1}^{n-1} |\rho_H(k)| \sim H(2H-1) \sum_{k=1}^{n-1} k^{2H-2} \sim H n^{2H-1}$ as $n \to \infty$, and (2.11) holds as well, because $H < 1$. □

As a direct application of the previous proposition, we deduce the following result about the p-variations of the fBm.

Corollary 2.1. *Let B^H be a fractional Brownian motion of Hurst parameter $H \in (0,1)$, and let $p \in [1, +\infty)$. Then, in $L^2(\Omega)$ and as $n \to \infty$, one has*

$$\sum_{k=1}^{n} \left| B^H_{k/n} - B^H_{(k-1)/n} \right|^p \to \begin{cases} 0 & \text{if } p > \frac{1}{H}, \\ E[|G|^p] & \text{if } p = \frac{1}{H}, \text{ with } G \sim \mathcal{N}(0,1), \\ +\infty & \text{if } p < \frac{1}{H}. \end{cases}$$

Proof. Just apply (2.12) with $f(x) = |x|^p$. □

We are now ready to prove that fractional Brownian motion is not a semimartingale, except when its Hurst parameter is $\frac{1}{2}$. This explains why integrating with respect to it is an interesting and non-trivial problem, see Chapter 3.

Theorem 2.2 (Rogers [53]). *Let B^H be a fractional Brownian motion of Hurst index $H \in (0, 1/2) \cup (1/2, 1)$. Then B^H is not a semimartingale.*

Proof. By the selfsimilarity property of B^H (Proposition 2.2(1)), it is sufficient to consider the time interval [0, 1]. Let us recall two main features of semimartingales on [0, 1]. If S denotes such a semimartingale, then:

1. $\sum_{k=1}^{n} (S_{k/n} - S_{(k-1)/n})^2 \to \langle S \rangle_1 < \infty$ in probability as $n \to \infty$;
2. if we have, moreover, that $\langle S \rangle_1 = 0$, then S has bounded variations; in particular, with probability one, $\sup_{n \geqslant 1} \sum_{k=1}^{n} \left| S_{k/n} - S_{(k-1)/n} \right| < \infty$.

The proof is now divided into two parts, according to the value of H with respect to $\frac{1}{2}$.

- If $H < \frac{1}{2}$, Corollary 2.1 yields that $\sum_{k=1}^{n} (B^H_{k/n} - B^H_{(k-1)/n})^2 \to \infty$, so (1) fails, implying that B^H cannot be a semimartingale.
- If $H > \frac{1}{2}$, Corollary 2.1 yields that $\sum_{k=1}^{n} (B_{k/n} - B_{(k-1)/n})^2 \to 0$. Let p be such that $1 < p < \frac{1}{H}$. We then have, still by Corollary 2.1, that $\sum_{k=1}^{n} \left| B^H_{k/n} - B^H_{(k-1)/n} \right|^p \to \infty$. Moreover, because of the (uniform) continuity of $t \mapsto B^H_t(\omega)$ on [0, 1], we have

  $$\sup_{1 \leqslant k \leqslant n} \left| B^H_{k/n} - B^H_{(k-1)/n} \right|^{p-1} \overset{\text{a.s.}}{\to} 0.$$

  Hence, using the inequality

  $$\sum_{k=1}^{n} \left| B^H_{k/n} - B^H_{(k-1)/n} \right|^p \leqslant \sup_{1 \leqslant k \leqslant n} \left| B^H_{k/n} - B^H_{(k-1)/n} \right|^{p-1} \times \sum_{k=1}^{n} \left| B^H_{k/n} - B^H_{(k-1)/n} \right|,$$

we deduce that $\sum_{k=1}^{n} \left| B^H_{k/n} - B^H_{(k-1)/n} \right| \to \infty$. These two facts being in contradiction with (2), B^H cannot be a semimartingale. □

2.5 Markov Property

Let X be a real-valued process. Recall that X is called a Markov process if it satisfies, for all Borel set $A \subset \mathbb{R}$ and all real numbers $t > s > 0$,

$$P(X_t \in A \,|\, X_u,\, u \leqslant s) = P(X_t \in A \,|\, X_s).$$

For fractional Brownian motion, we have the following result.

Theorem 2.3. *Let B^H be a fractional Brownian motion of Hurst index $H \in (0,1/2) \cup (1/2,1)$. Then B^H is not a Markov process.*

Proof. We proceed by contradiction. Assume that B^H is a Markov process. Since it is a Gaussian process as well, we must have (see, e.g., [25, Proposition 11.7] or [52, Chapter III, Exercise 1.13]), for all $0 \leqslant s \leqslant t \leqslant u$, that

$$E[B^H_s B^H_u] E[(B^H_t)^2] = E[B^H_s B^H_t] E[B^H_t B^H_u]. \tag{2.14}$$

Choose $u = 1$ in (2.14), and set, for $0 < s \leqslant 1$:

$$\phi_H(s) = E[B^H_s B^H_1] = \frac{1}{2}\left(1 + s^{2H} - (1-s)^{2H}\right) > 0. \tag{2.15}$$

Observe that

$$\frac{E[B^H_s B^H_t]}{E[(B^H_t)^2]} = \phi_H(s/t), \quad 0 < s \leqslant t \leqslant 1.$$

We deduce from (2.14) that

$$\phi_H(s) = \phi_H(s/t)\phi_H(t), \quad 0 < s \leqslant t \leqslant 1. \tag{2.16}$$

Set $\varphi_H(x) = \log \phi_H(e^{-x})$ for $x \geqslant 0$ and observe that

$$\varphi_H(0) = 0 \quad \text{and} \quad \lim_{x \to \infty} \varphi_H(x) = -\infty. \tag{2.17}$$

Moreover, the functional identity (2.16) implies that

$$\varphi_H(x+y) = \varphi_H(x) + \varphi_H(y), \quad x, y \geqslant 0,$$

so that, by differentiating (e.g. with respect to x), the function φ'_H is constant over $\mathbb{R}_+$. Using (2.17), we deduce the existence of $c > 0$ such that $\varphi_H(x) = -c\,x$ for all $x \geqslant 0$. Equivalently

$$\phi_H(s) = s^c, \quad 0 \leqslant s \leqslant 1. \tag{2.18}$$

By differentiating in (2.15), we get that $\phi''_H(s) = H(2H-1)\left(s^{2H-2} - (1-s)^{2H-2}\right)$. Since $H(2H-1) \neq 0$ and $2H-2 < 0$, we deduce that $\lim_{s \to 1} |\phi''_H(s)| = \infty$. But we also have $\phi''_H(s) = c(c-1)s^{c-2}$ by differentiating in (2.18), hence $\lim_{s \to 1} |\phi''_H(s)| = c|c-1| \neq \infty$, leading to a contradiction.

The proof of the theorem is done. □

2.6 Hurst Phenomenon

Fractional Brownian motion has been used successfully to model a variety of natural phenomenons. Following [56], let us see how it was introduced historically, and why its selfsimilarity index is called 'Hurst parameter'.

In the fifties, Hurst [23] studied the flow of water in the Nile river, and empirically highlighted a somewhat curious phenomenon. Let us denote by $X_1, X_2, \ldots$ the set of data observed by Hurst. The statistics he looked at was the so-called R/S-statistic (for 'rescaled range of the observations'), defined as

$$\frac{R}{S}(X_1,\ldots,X_n) := \frac{\max_{1\leqslant i\leqslant n}\left(S_i-\frac{i}{n}S_n\right)-\min_{1\leqslant i\leqslant n}\left(S_i-\frac{i}{n}S_n\right)}{\sqrt{\frac{1}{n}\sum_{i=1}^n\left(X_i-\frac{1}{n}S_n\right)^2}},$$

where $S_n = X_1 + \ldots + X_n$. It measures the ratio between the highest and lowest positions of the partial sums with respect to the straight line of uniform growth and the sample standard deviation.

As a crude approximation, let us first assume that the X_i's are i.i.d. with common mean $\mu \in \mathbb{R}$ and common variance $\sigma^2 > 0$. Because $t \mapsto \frac{1}{\sqrt{n}}(S_{[nt]} - [nt]\mu)$ is constant on each $(i/n, (i+1)/n)$ whereas $t \mapsto \frac{t}{\sqrt{n}}(S_n - n\mu)$ is monotonous, the maximum of $t \mapsto \frac{1}{\sqrt{n}}\big(S_{[nt]} - [nt]\mu - t(S_n - n\mu)\big)$ on $[0,1]$ is necessarily attained at a point of the form $t = \frac{i}{n}$, $i = 0,\ldots,n$. Therefore,

$$\sup_{t\in[0,1]}\frac{1}{\sqrt{n}}\big(S_{[nt]}-[nt]\mu-t(S_n-n\mu)\big) = \frac{1}{\sqrt{n}}\max_{0\leqslant i\leqslant n}\left(S_i-\frac{i}{n}S_n\right).$$

Similarly,

$$\inf_{t\in[0,1]}\frac{1}{\sqrt{n}}\big(S_{[nt]}-[nt]\mu-t(S_n-n\mu)\big) = \frac{1}{\sqrt{n}}\min_{0\leqslant i\leqslant n}\left(S_i-\frac{i}{n}S_n\right).$$

Hence,

$$\frac{1}{\sigma\sqrt{n}}\left\{\max_{1\leqslant i\leqslant n}\left(S_i-\frac{i}{n}S_n\right)-\min_{1\leqslant i\leqslant n}\left(S_i-\frac{i}{n}S_n\right)\right\}$$
$$= \phi\left(\frac{1}{\sigma\sqrt{n}}\big(S_{[n\cdot]}-[n\cdot]\mu\big)\right),$$

where

$$\phi(f) = \sup_{0\leqslant t\leqslant 1}\{f(t)-tf(1)\} - \inf_{0\leqslant t\leqslant 1}\{f(t)-tf(1)\}.$$

Thus, by applying the celebrated Donsker's theorem, we get that

$$\frac{1}{\sigma\sqrt{n}}\left\{\max_{1\leqslant i\leqslant n}\left(S_i-\frac{i}{n}S_n\right)-\min_{1\leqslant i\leqslant n}\left(S_i-\frac{i}{n}S_n\right)\right\}$$

converges in law to

$$\phi(W) = \sup_{0\leqslant t\leqslant 1}\{W_t - tW_1\} - \inf_{0\leqslant t\leqslant 1}\{W_t - tW_1\},$$

where W stands for a classical Brownian motion on [0, 1]. Finally, because

$$\sqrt{\frac{1}{n}\sum_{i=1}^{n}\left(X_i - \frac{1}{n}S_n\right)^2} \to \sigma \quad \text{a.s.}$$

(by the strong law of large numbers), we get that

$$\frac{1}{\sqrt{n}} \times \frac{R}{S}(X_1,\ldots,X_n) \overset{\text{law}}{\to} \sup_{0\leqslant t\leqslant 1}\{W_t - tW_1\} - \inf_{0\leqslant t\leqslant 1}\{W_t - tW_1\}$$

as $n \to \infty$. That is, in the case of i.i.d.observations the R/S-statistic grows as $\sqrt{n}$, where n denotes the sample size. But this is not what Hurst observed when he calculated the R/S-statistic on the Nile river data (between 622 and 1469). Instead, he found a growth of order $n^{0,74}$.

Is it possible to find a stochastic model explaining this fact? As we will see, fractional Brownian motion allows one to do so. Indeed, let $X_1, X_2, \ldots$ have now the form

$$X_i = \mu + \sigma(B_i^H - B_{i-1}^H), \tag{2.19}$$

with B^H a fractional Brownian motion of index $H = 0.74$. That is, the X_i's have again μ for mean and σ^2 for variance, but they are no longer independent. Due to the specific form of (2.19), it is readily checked, by a telescoping sum argument, that

$$\frac{R}{S}(X_1,\ldots,X_n) = \frac{\max_{1\leqslant i\leqslant n}\left(B_i^H - \frac{i}{n}B_n^H\right) - \min_{1\leqslant i\leqslant n}\left(B_i^H - \frac{i}{n}B_n^H\right)}{\sqrt{\frac{1}{n}\sum_{i=1}^{n}\left(B_i^H - B_{i-1}^H - \frac{1}{n}B_n^H\right)^2}}.$$

Using the selfsimilarity property of B^H (Proposition 2.2(1)), we get that

$$\begin{aligned}
&\frac{1}{n^H}\left\{\max_{1\leqslant i\leqslant n}\left(B_i^H - \frac{i}{n}B_n^H\right) - \min_{1\leqslant i\leqslant n}\left(B_i^H - \frac{i}{n}B_n^H\right)\right\}\\
&\overset{\text{law}}{=} \max_{1\leqslant i\leqslant n}\left(B_{i/n}^H - \frac{i}{n}B_1^H\right) - \min_{1\leqslant i\leqslant n}\left(B_{i/n}^H - \frac{i}{n}B_1^H\right)\\
&\overset{\text{a.s.}}{\to} \sup_{0\leqslant t\leqslant 1}\{B_t^H - tB_1^H\} - \inf_{0\leqslant t\leqslant 1}\{B_t^H - tB_1^H\} \quad \text{as } n\to\infty.
\end{aligned}$$

On the other hand, as $n \to \infty$ we have that $n^{-1}B_n^H \overset{\text{law}}{=} n^{H-1}B_1^H \overset{\text{a.s.}}{\to} 0$ whereas, by Theorem 2.1, $\frac{1}{n}\sum_{i=1}^{n}(B_i^H - B_{i-1}^H)^2 \overset{L^2}{\to} 1$. Putting all these facts together yields

$$\frac{1}{n^H} \times \frac{R}{S}(X_1,\ldots,X_n) \overset{\text{law}}{\to} \sup_{0\leqslant t\leqslant 1}\{B_t^H - tB_1^H\} - \inf_{0\leqslant t\leqslant 1}\{B_t^H - tB_1^H\}$$

as $n \to \infty$; hence the model (2.19) represents a plausible explanation to the phenomenon observed by Hurst in [23].

Chapter 3
Integration with Respect to Fractional Brownian Motion

We have just seen in Chapter 1 (Theorem 2.2) that, except when it is a standard Brownian motion, fractional Brownian motion B^H is not a semimartingale. As a result, the usual Itô calculus is not available for use, and alternate methods are required in order to define and solve differential equations of the type

$$X_t = X_0 + \int_0^t \sigma(X_s)dB_s^H + \int_0^t b(X_s)ds, \quad t \in [0, T]. \tag{3.1}$$

This chapter contains only a few examples of how one can obtain a stochastic calculus with a fractional Brownian motion as integrator. (For instance, we will not speak about the possibility to use Wick product and Malliavin calculus. For this, we refer the reader to [3, 45] and the references therein.)

When $H > 1/2$, it happens that the regularity of the sample paths of B^H is enough and allows for the solution to be defined pathwise using Young integral, see Section 3.1. Under such a regime, existence and uniqueness for (3.1) are shown in Section 3.2 under reasonable assumptions on the coefficients σ and b.

In the case that $H < 1/2$, a powerful approach (known as *rough path theory*) may be used to make sense of (3.1), at least if H is not too small. See [18, 28]. In this book, we only focus on a *very specific* situation when $H < 1/2$, precisely the case where the underlying dimension is one and when one seeks for a change of variable formula of the type

$$f(B_T^H) = f(0) + \int_0^T f'(B_s^H)dB_s^H + \text{a correction term}. \tag{3.2}$$

This analysis is done in Section 3.3.

3.1 Young Integral

Fix $T > 0$ as being the horizon time. (That is, in the sequel all the considered functions are going to be defined on the time interval $[0, T]$.) For any integer $l \geqslant 1$, we

Nourdin I.: *Selected Aspects of Fractional Brownian Motion*, B&SS,
DOI 10.1007/978-88-470-2823-4_3, © Springer-Verlag Italia 2012

denote by $\mathcal{C}^l$ the set of functions $g : [0, T] \to \mathbb{R}$ that are l times differentiable and whose lth derivative is continuous. We use the common convention that $\mathcal{C}^0$ denotes the set of continuous functions $g : [0, T] \to \mathbb{R}$. For any $\alpha \in [0, 1]$, we denote by C^α the set of Hölder continuous functions of index α, that is, the set of functions $f : [0, T] \to \mathbb{R}$ satisfying

$$|f|_\alpha := \sup_{0 \leqslant s < t \leqslant T} \frac{|f(t) - f(s)|}{(t-s)^\alpha} < \infty.$$

(Observe the calligraphic difference between $\mathcal{C}^l$ and C^α.) We also set $|f|_\infty = \sup_{t \in [0,T]} |f(t)|$, and we equip C^α with the norm

$$\|f\|_\alpha := |f|_\alpha + |f|_\infty.$$

We have $|f|_\infty \leqslant |f(t)| + T^\alpha |f|_\alpha$ for all $t \in [0, T]$.

Fix $f \in C^\alpha$, and consider the operator $T_f : \mathcal{C}^1 \to \mathcal{C}^1$ defined as

$$T_f(g)(t) = \int_0^t f(u) dg(u) = \int_0^t f(u) g'(u) du, \quad t \in [0, T].$$

For any $s, t \in [0, T]$, $s < t$, we have

$$T_f(g)(t) - T_f(g)(s) = \int_s^t f(u) g'(u) du = \lim_{n \to \infty} J_n(f, g, s, t),$$

with

$$J_n(f, g, s, t) = \sum_{k=1}^{2^n} f(s + k2^{-n}(t-s)) \big\{ g(s + k2^{-n}(t-s)) - g(s + (k-1)2^{-n}(t-s)) \big\}.$$

We can decompose

$$\begin{aligned} J_n(f, g, s, t) &= \sum_{k=1}^{2^{n-1}} f(s + 2k2^{-n}(t-s)) \\ &\quad \times \big(g(s + 2k2^{-n}(t-s)) - g(s + (2k-1)2^{-n}(t-s)) \big) \\ &\quad + \sum_{k=1}^{2^{n-1}} f(s + (2k-1)2^{-n}(t-s)) \\ &\quad \times \big(g(s + (2k-1)2^{-n}(t-s)) - g(s + (2k-2)2^{-n}(t-s)) \big). \end{aligned}$$

Thus

$$\begin{aligned} &J_n(f, g, s, t) - J_{n-1}(f, g, s, t) \\ &\quad = -\sum_{k=1}^{2^{n-1}} \big(f(s + (2k)2^{-n}(t-s)) - f(s + (2k-1)2^{-n}(t-s)) \big) \\ &\qquad \times \big(g(s + (2k-1)2^{-n}(t-s)) - g(s + (2k-2)2^{-n}(t-s)) \big), \end{aligned}$$

so that, for any $\beta \in [0,1]$,

$$\left| J_n(f,g,s,t) - J_{n-1}(f,g,s,t) \right| \leqslant \frac{1}{2}(t-s)^{\alpha+\beta} |f|_\alpha |g|_\beta 2^{-n(\alpha+\beta-1)}.$$

Now, fix $\beta > 1-\alpha$. By summing the previous inequality over $n \geqslant 1$, we get

$$\left| \int_s^t f(u)g'(u)du - f(t)(g(t)-g(s)) \right| \leqslant C_{\alpha,\beta} |f|_\alpha |g|_\beta (t-s)^{\alpha+\beta}, \tag{3.3}$$

with $C_{\alpha,\beta} = \frac{1}{2}\sum_{n=1}^\infty 2^{-n(\alpha+\beta-1)} < \infty$. We deduce that

$$\begin{aligned} \left| \int_s^t f(u)g'(u)ds \right| &\leqslant |f|_\infty |g|_\beta (t-s)^\beta + C_{\alpha,\beta} |f|_\alpha |g|_\beta (t-s)^{\alpha+\beta} \\ &\leqslant \left(1 + C_{\alpha,\beta} T^\alpha\right) \|f\|_\alpha |g|_\beta (t-s)^\beta. \end{aligned} \tag{3.4}$$

Consequently:

$$\left| \int_0^\cdot f(u)g'(u)du \right|_\beta \leqslant \left(1 + C_{\alpha,\beta} T^\alpha\right) \|f\|_\alpha |g|_\beta, \tag{3.5}$$

and

$$\left| \int_0^\cdot f(u)g'(u)du \right|_\infty \leqslant \left(1 + C_{\alpha,\beta} T^\alpha\right) T^\beta \|f\|_\alpha |g|_\beta. \tag{3.6}$$

Finally, by combining (3.5) with (3.6) and by using the crude bound $|g|_\beta \leqslant \|g\|_\beta$ as well, we get

$$\left\| \int_0^\cdot f(u)g'(u)du \right\|_\beta \leqslant \left(1 + C_{\alpha,\beta} T^\alpha\right)\left(1 + T^\beta\right) \|f\|_\alpha \|g\|_\beta. \tag{3.7}$$

The following result, which is central in the theory (as it will lead to the definition of the Young integral), is an immediate consequence of (3.7).

Theorem 3.1 (Young [67]). *Let $f \in C^\alpha$ with $\alpha \in (0,1)$, and let $\beta \in (0,1)$ be such that $\alpha + \beta > 1$. The linear operator $T_f : \mathscr{C}^1 \subset C^\beta \to C^\beta$ defined as $T_f(g) = \int_0^\cdot f(u)g'(u)du$ is continuous with respect to the norm $\|\cdot\|_\beta$. By density, it extends (in an unique way) to an operator $T_f : C^\beta \to C^\beta$.*

Definition 3.1. *Let $f \in C^\alpha$ and $g \in C^\beta$ with $\alpha + \beta > 1$. The Young integral $\int_0^\cdot f(u)dg(u)$ is (well-)defined as being $T_f(g)$.*

The Young integral obeys the following chain rule.

Theorem 3.2. *Let $\phi : \mathbb{R}^2 \to \mathbb{R}$ be a $\mathscr{C}^2$ function, and let $f, g \in C^\alpha$ with $\alpha \in (\frac{1}{2}, 1]$. Then $\int_0^\cdot \frac{\partial \phi}{\partial f}(f(u), g(u))df(u)$ and $\int_0^\cdot \frac{\partial \phi}{\partial g}(f(u), g(u))dg(u)$ are well-defined as Young integrals. Moreover, for all $t \in [0,T]$, we have*

$$\begin{aligned} \phi(f(t), g(t)) &= \phi(f(0), g(0)) + \int_0^t \frac{\partial \phi}{\partial f}(f(u), g(u))df(u) \\ &\quad + \int_0^t \frac{\partial \phi}{\partial g}\phi(f(u), g(u))dg(u). \end{aligned} \tag{3.8}$$

Proof. Because ϕ is $\mathcal{C}^2$ and f, g are C^α, the functions $u \mapsto \frac{\partial\phi}{\partial f}(f(u), g(u))$ and $u \mapsto \frac{\partial\phi}{\partial g}(f(u), g(u))$ are C^α too by the Mean Value Theorem. Consequently, since $2\alpha > 1$, the integrals $\int_0^t \frac{\partial\phi}{\partial f}(f(u), g(u))df(u)$ and $\int_0^t \frac{\partial\phi}{\partial g}(f(u), g(u))dg(u)$ are well-defined as Young integrals for any $t \in [0, T]$. When f and g belong to $\mathcal{C}^1$, the identity (3.8) is a consequence of the fundamental theorem of calculus. Finally, we may prove (3.8) in full generality by using a density argument. □

3.2 Solving Integral Equations

The material developed in the previous section allows one to solve integral equations driven by Hölder continuous functions of index bigger than $\frac{1}{2}$. In particular, it applies to sample paths of fractional Brownian motion of Hurst parameter $H > \frac{1}{2}$ (see indeed Proposition 1.6). We start with the multidimensional case and, then, we strengthen the statement in the one-dimensional case.

3.2.1 Multidimensional Case

Theorem 3.3. *Fix two integers $d, m \geqslant 1$, and let $g : [0, T] \to \mathbb{R}^m$ and $\sigma : \mathbb{R}^d \to \mathcal{M}_{d,m}(\mathbb{R})$. Write $g = (g_j)_{1\leqslant j\leqslant m}$ and $\sigma = (\sigma_{ij})_{1\leqslant i\leqslant d,\,1\leqslant j\leqslant m}$. Fix $\beta \in (\frac{1}{2}, 1)$, and assume that each g_j is β-Hölder continuous. Suppose moreover that each σ_{ij} is of class $\mathcal{C}^2$ and is bounded together with its two derivatives. Finally, consider an initial condition $a = (a_1, \ldots, a_d) \in \mathbb{R}^d$. Then, for all $\alpha \in (\frac{1}{2}, \beta)$, the integral equation*

$$x_i(t) = a_i + \sum_{j=1}^{m} \int_0^t \sigma_{ij}(x(u))dg_j(u), \quad i = 1, \ldots, d, \tag{3.9}$$

admits a unique solution $x = \{x_i\}_{1\leqslant i\leqslant d}$ on $[0, T]$ satisfying $|x_i|_\alpha < \infty$ for any $i = 1, \ldots, d$. In (3.9), the integrals with respect to g_j are understood in the Young sense (Definition 3.1).

Proof. To simplify the exposition, we 'only' do the proof for $m = d = 1$ (that is, in the one-dimensional case). The general case may be obtained *mutatis mutandis* with cumbersome notation, and wet let the details as a useful exercise.

Let $\sigma : \mathbb{R} \to \mathbb{R}$ be of class $\mathcal{C}^2$ and be bounded together with its derivatives. Let also $g : [0, T] \to \mathbb{R}$ be β-Hölder continuous, with $\beta \in (\frac{1}{2}, 1)$. Finally, let $a \in \mathbb{R}$ be a given initial condition. We have to show that, for all $\alpha \in (\frac{1}{2}, \beta)$, the integral equation

$$x(t) = a + \int_0^t \sigma(x(s))dg(s), \quad t \in [0, T], \tag{3.10}$$

admits a unique solution $x : [0, T] \to \mathbb{R}$ belonging to C^α.

Let $M > 0$ be such that $\max\{|\sigma|_\infty, |\sigma'|_\infty, |\sigma''|_\infty\} \leqslant M$. Fix $\alpha \in (\frac{1}{2}, \beta)$, and let $\tau > 0$ be small enough so that

$$C_{\alpha,\beta}\tau^\alpha \leqslant 1;\ 4M|g|_\beta \tau^{\beta-\alpha} \leqslant 1 \ \text{ and } \ \tau^\alpha M|g|_\beta\big(3 + C_{\alpha,\beta}\big) \leqslant \frac{1}{2}. \tag{3.11}$$

Here, $C_{\alpha,\beta} = \frac{1}{2}\sum_{n=1}^\infty 2^{-n(\alpha+\beta-1)} < \infty$. Consider the Banach space

$$B_a^\alpha := \{x : [0,\tau] \to \mathbb{R} : x(0) = a \ \text{ and } \ |x|_\alpha \leqslant 1\}$$

equipped with the norm $|\cdot|_\alpha$. Let $\mathcal{T}$ be the operator defined, for $x \in B_a^\alpha$, as

$$\mathcal{T}(x)(t) = a + \int_0^t \sigma(x(u))dg(u), \quad t \in [0,\tau].$$

Combining (3.11) with (3.4), for $x \in B_a^\alpha$ we have

$$|\mathcal{T}(x)|_\alpha \leqslant 2\|\sigma(x)\|_\alpha |g|_\beta \tau^{\beta-\alpha} \leqslant 2\big(|\sigma|_\infty + |\sigma'|_\infty |x|_\alpha\big)|g|_\beta \tau^{\beta-\alpha} \leqslant 1;$$

hence $\mathcal{T}(x) \in B_a^\alpha$. In other words, the ball B_a^α is invariant by the operator $\mathcal{T}$.

On the other hand, for any $x, y \in B_a^\alpha$ and $s, t \in [0,\tau]$, we have

$$\begin{aligned}
&|\sigma(x(t)) - \sigma(y(t)) - \sigma(x(s)) + \sigma(y(s))| \\
&\quad = \Big|\big(x(t) - y(t)\big)\int_0^1 \sigma'(\nu x(t) + (1-\nu)y(t))d\nu \\
&\qquad\qquad - \big(x(s) - y(s)\big)\int_0^1 \sigma'(\nu x(s) + (1-\nu)y(s))d\nu\Big| \\
&\quad = \Big|\big(x(t) - y(t) - x(s) + y(s)\big)\int_0^1 \sigma'(\nu x(t) + (1-\nu)y(t))d\nu \\
&\qquad\qquad + \big(x(s) - y(s)\big)\int_0^1 \big(\sigma'(\nu x(t) + (1-\nu)y(t))d\nu - \sigma'(\nu x(s) \\
&\qquad\qquad + (1-\nu)y(s))\big)d\nu\Big| \\
&\quad \leqslant |\sigma'|_\infty |x-y|_\alpha |t-s|^\alpha + |x-y|_\alpha \tau^\alpha |\sigma''|_\infty \big(|x|_\alpha + |y|_\alpha\big)|t-s|^\alpha \\
&\quad \leqslant |\sigma'|_\infty |x-y|_\alpha |t-s|^\alpha + 2|x-y|_\alpha \tau^\alpha |\sigma''|_\infty |t-s|^\alpha,
\end{aligned}$$

so that

$$|\sigma(x) - \sigma(y)|_\alpha \leqslant |x-y|_\alpha\big(|\sigma'|_\infty + 2\tau^\alpha|\sigma''|_\infty\big) \leqslant |x-y|_\alpha M(1 + 2\tau^\alpha).$$

Thus, for any $x, y \in B_a^\alpha$ and $s, t \in [0,\tau]$, we have, using moreover (3.3), that

$$\begin{aligned}
&|\mathcal{T}(x)(t) - \mathcal{T}(y)(t) - \mathcal{T}(x)(s) + \mathcal{T}(y)(s)| \\
&\quad = \left|\int_s^t \big(\sigma(x(u)) - \sigma(y(u))\big)dg(u)\right| \\
&\quad \leqslant \big|\sigma(x(t)) - \sigma(y(t))\big||g(t) - g(s)| + C_{\alpha,\beta}|\sigma(x) - \sigma(y)|_\alpha |g|_\beta |t-s|^{\alpha+\beta} \\
&\quad \leqslant |\sigma'|_\infty |x(t) - y(t)||g|_\beta |t-s|^\beta + C_{\alpha,\beta}|x-y|_\alpha M(1+2\tau^\alpha)|g|_\beta |t-s|^{\alpha+\beta} \\
&\quad \leqslant M|x-y|_\alpha \tau^\alpha |g|_\beta |t-s|^\beta + C_{\alpha,\beta}|x-y|_\alpha M(1+2\tau^\alpha)|g|_\beta |t-s|^{\alpha+\beta},
\end{aligned}$$

so that, because of (3.11),

$$
\begin{aligned}
|\mathcal{T}(x)-\mathcal{T}(y)|_\alpha &\leqslant |x-y|_\alpha \tau^\alpha |g|_\beta M\big(1+C_{\alpha,\beta}+2C_{\alpha,\beta}\tau^\alpha\big)\\
&\leqslant |x-y|_\alpha \tau^\alpha M|g|_\beta\big(3+C_{\alpha,\beta}\big) \leqslant \frac{1}{2}|x-y|_\alpha .
\end{aligned}
$$

Applying the fixed point argument to $\mathcal{T} : B_a^\alpha \to B_a^\alpha$ leads to a unique solution to (3.10) on $[0,\tau]$. One is then able to obtain the unique solution on an arbitrary interval $[0,k\tau]$, with $k \geqslant 1$, by patching solutions on $[j\tau,(j+1)\tau]$. Notice here that a crucial point, which allows one to use a constant step τ, is the fact that the conditions (3.11) do not depend on the initial condition a (indeed, it only relies on α, β, $|\sigma|_\infty$, $|\sigma'|_\infty$, $|\sigma''|_\infty$ and $|g|_\beta$). □

3.2.2 One-Dimensional Case

In this section, we strengthen Theorem 3.3 in the one-dimensional case, by being more precise with the shape of the solution. Let us introduce our new context. Let $g \in C^\beta$ with $\beta \in (\frac{1}{2},1)$, and let $\sigma, b : \mathbb{R} \to \mathbb{R}$ be two given functions. We aim to solve integral equations of the form

$$
x(t) = x(0) + \int_0^t \sigma(x(u))dg(u) + \int_0^t b(x(u))du, \quad t \in [0,T], \tag{3.12}
$$

where the unknown function $x : [0,T] \to \mathbb{R}$ is assumed to belong to C^β, and where the integral with respect to g is understood in the Young sense (Definition 3.1).

Theorem 3.4 (Doss [15] – Sussmann [62]). *Suppose that $\sigma : \mathbb{R} \to \mathbb{R}$ is bounded and of class $\mathcal{C}^2$, with bounded first and second derivatives. Suppose moreover that $b : \mathbb{R} \to \mathbb{R}$ is Lipschitz. Then the one-dimensional equation (3.12) admits a unique solution in C^β. This unique solution is given by*

$$
x(t) = \phi(g(t), y(t)), \quad t \in [0,T], \tag{3.13}
$$

for a suitable continuous function $\phi : \mathbb{R}^2 \to \mathbb{R}$ and a function $y : [0,T] \to \mathbb{R}$ which solves an ordinary differential equation.

Proof. Let $\phi : \mathbb{R}^2 \to \mathbb{R}$ be the solution of the ordinary differential equation

$$
\frac{\partial \phi}{\partial x} = \sigma \circ \phi, \quad \phi(0,y) = y.
$$

Notice that such a solution exists globally, thanks to our assumption. It is readily checked that

$$
\frac{\partial^2 \phi}{\partial x \partial y} = \sigma'(\phi)\frac{\partial \phi}{\partial y}, \quad \frac{\partial \phi}{\partial y}(0,y) = 1,
$$

so that

$$
\frac{\partial \phi}{\partial y}(x,y) = e^{\int_0^x \sigma'(\phi(u,y))du}.
$$

If A stands for a uniform bound of σ, σ' and σ'', we deduce

$$\frac{\partial \phi}{\partial y}(x, y) \leqslant e^{A|x|}.$$

This implies that

$$\left|\phi(x, y_1) - \phi(x, y_2)\right| \leqslant e^{A|x|}|y_1 - y_2|;$$

hence, if L is a Lipschitz constant for b, then

$$\left|b(\phi(x, y_1)) - b(\phi(x, y_2))\right| \leqslant Le^{A|x|}|y_1 - y_2|. \tag{3.14}$$

Moreover, by using the inequality $|e^{u_1} - e^{u_2}| \leqslant e^{|u_1|+|u_2|}|u_1 - u_2|$, we can write

$$\begin{aligned}
&\left|e^{-\int_0^x \sigma'(\phi(u,y_1))du} - e^{-\int_0^x \sigma'(\phi(u,y_2))du}\right| \\
&\quad \leqslant e^{\left|\int_0^x [\sigma'(\phi(u,y_1))du\right| + \left|\int_0^x \sigma'(\phi(u,y_2))]du\right|} \int_0^{|x|} \left|\sigma'(\phi(u, y_1)) - \sigma'(\phi(u, y_2))\right| du \\
&\quad \leqslant Ae^{2A|x|} \int_0^{|x|} \left|\phi(u, y_1) - \phi(u, y_2)\right| du \\
&\quad \leqslant A|x|e^{3A|x|}|y_1 - y_2|.
\end{aligned} \tag{3.15}$$

It follows from (3.14) and (3.15) that $\psi(x, y) = b(\phi(x, y))e^{-\int_0^x \sigma'(\phi(x,y))du}$ satisfies Lipschitz and growth conditions of the form

$$|\psi(x, y_1) - \psi(x, y_2)| \leqslant L_k|y_1 - y_2|; \quad -k \leqslant x, y_1, y_2 \leqslant k \tag{3.16}$$

$$|\psi(x, y)| \leqslant K_1 + K_k|y|; \quad |x| \leqslant k, \ y \in \mathbb{R}, \tag{3.17}$$

where the constants L_k and K_k depend on k.

Let $y : [0, T] \to \mathbb{R}$ be the solution to the ordinary differential equation

$$\dot{y}(t) = \psi(g(t), y(t)), \quad t \in [0, T], \quad y(0) = x(0).$$

Such a solution exists and is unique because of (3.16)-(3.17). Let $x : [0, T] \to \mathbb{R}$ be the function defined by (3.13). Thanks to the assumptions made on σ as well as the bounds shown above, we immediately prove that x belongs to $\mathcal{C}^\beta$. Moreover, a straightforward application of (3.8) shows that x satisfies (3.12).

We only sketch the proof of uniqueness. Let x be a solution to (3.13) belonging to $\mathcal{C}^\beta$. Let z be the function defined as $z(t) = \phi(-g(t), x(t))$. Using (3.8), we may show that $\dot{z}(t) = \psi(g(t), z(t))$ (with $z(0) = x(0)$). By a uniqueness argument in the ordinary differential equation defining y, we deduce that $z(t) = y(t)$ for all t. Hence, $y(t) = \phi(-g(t), x(t))$, which is equivalent to $x(t) = \phi(g(t), y(t))$. This finishes the proof of the uniqueness in (3.12). □

3.3 Dimension One and Hurst Index Less than 1/2

Let B^H be a fractional Brownian motion of index $H \in (0, 1/2)$. Using the infiniteness of the quadratic variation of B^H (which is an immediate consequence of Corollary 2.1 with $p = 2$), it is straightforward to check that the sample paths of B^H do not belong to $\cup_{\alpha \in (1/2,1]} C^\alpha$, implying in turn that one cannot apply Theorem 3.2 when $H < 1/2$. As a result, it happens that the Young integral is not a good candidate when wanting to integrate against a fBm of index less than $1/2$.

In this section, we will study what is undoubtedly the most simple situation that one can consider. It is nevertheless deep enough to make interesting phenomenons arise. Consider a function $f : \mathbb{R} \to \mathbb{R}$ (whose regularity will be made explicit later on). As far as a change of variable formula for $f(B_T^H)$ is concerned, it is reasonable to expect something like

$$f(B_T^H) = f(0) + \int_0^T f'(B_s^H) dB_s^H + \text{a correction term}, \tag{3.18}$$

where the exact meaning of the stochastic integral with respect to B^H remains to be given and the form of the correction term has to be discovered.

Let us first investigate what happens when one deals with a *forward* Riemann sum approach.

Definition 3.2. *Let $X = (X_t)_{t \in [0,T]}$ and $Y = (Y_t)_{t \in [0,T]}$ be two continuous stochastic processes. Provided the limit exists in probability, we define the forward integral of Y with respect to X as being*

$$\int_0^T Y_s d^- X_s = \lim_{n \to \infty} \sum_{k=0}^{n-1} Y_{kT/n} \big(X_{(k+1)T/n} - X_{kT/n} \big).$$

In order to have an idea whether this definition is or not relevant in our context, let us first study the very simple case $X = Y = B^H$. (We also fix $T = 1$ for the sake of simplicity.) We have

$$\begin{aligned} \big(B_1^H\big)^2 &= \sum_{k=0}^{n-1} \Big(\big(B_{(k+1)/n}^H\big)^2 - \big(B_{k/n}^H\big)^2 \Big) \\ &= 2 \sum_{k=0}^{n-1} B_{k/n}^H \big(B_{(k+1)/n}^H - B_{k/n}^H\big) + \sum_{k=0}^{n-1} \big(B_{(k+1)/n}^H - B_{k/n}^H\big)^2. \end{aligned}$$

Assume for the time being that $\int_0^1 B_s^H d^- B_s^H$ exists. We deduce that, as $n \to \infty$,

$$\sum_{k=0}^{n-1} \big(B_{(k+1)/n}^H - B_{k/n}^H\big)^2 \overset{\text{proba}}{\to} \big(B_1^H\big)^2 - 2 \int_0^1 B_s^H d^- B_s^H.$$

But this latter fact is clearly in contradiction with Corollary 2.1 ($p = 2$). We conclude that $\int_0^T B_s^H d^- B_s^H$ does not exist whenever $H < 1/2$, something which is inconceivable for a reasonable theory of integration with respect to B^H. So, we must change our Definition 3.2. Let us try the following one.

Definition 3.3. *Let $X = (X_t)_{t\in[0,T]}$ and $Y = (Y_t)_{t\in[0,T]}$ be two continuous stochastic processes. Provided the limit exists in probability, we define the symmetric integral of Y with respect to X as being*

$$\int_0^T Y_s d^\circ X_s = \lim_{n\to\infty} \sum_{k=0}^{n-1} \frac{1}{2}\big(Y_{(k+1)T/n} + Y_{kT/n}\big)\big(X_{(k+1)T/n} - X_{kT/n}\big). \quad (3.19)$$

If B^H is a fBm of index $H \geqslant 1/2$ and if $f : \mathbb{R} \to \mathbb{R}$ of class C^2 then, by relying to the fact that the quadratic variation of B^H is finite, it is not very difficult to prove that $\int_0^T f'(B_s^H)d^\circ B_s^H$ exists and that we have

$$f(B_T^H) = f(0) + \int_0^T f'(B_s^H)d^\circ B_s^H. \quad (3.20)$$

Setting $f(x) = x^2$, (3.20) says that

$$(B_T^H)^2 = 2\int_0^T B_s^H d^\circ B_s^H. \quad (3.21)$$

If $H < 1/2$, we have just seen that the forward integral $\int_0^T B_s^H d^- B_s^H$ does not exist. But (3.21) is still valid; in fact, using the identity

$$\big(B_1^H\big)^2 = 2\sum_{k=0}^{n-1} \frac{B_{(k+1)/n}^H + B_{k/n}^H}{2}\big(B_{(k+1)/n}^H - B_{k/n}^H\big), \quad (3.22)$$

we can immediately see that (3.21) holds for any $0 < H < 1$. The natural question which arises is the following: is (3.20) valid for any $0 < H < 1$? The answer is no. In reality, taking $f(x) = x^3$, similarly to (3.22) we can expand as follows

$$\begin{aligned}\big(B_1^H\big)^3 &= \sum_{k=0}^{n-1}\Big(\big(B_{(k+1)/n}^H\big)^3 - \big(B_{k/n}^H\big)^3\Big) \\ &= 3\sum_{k=0}^{n-1}\frac{1}{2}\big((B_{(k+1)/n}^H)^2 + (B_{k/n}^H)^2\big)\big(B_{(k+1)/n}^H - B_{k/n}^H\big) \\ &\quad -\frac{1}{2}\sum_{k=0}^{n-1}\big(B_{(k+1)/n}^H - B_{k/n}^H\big)^3.\end{aligned}$$

For $\int_0^1 (B_s^H)^2 d^\circ B_s^H$ to exist, we deduce that it is necessary and sufficient that the cubic variation of B^H, defined as

$$\lim_{n\to\infty}\sum_{k=0}^{n-1}\big(B_{(k+1)/n}^H - B_{k/n}^H\big)^3, \quad (3.23)$$

exists in probability. Set $X_k = B^H_k - B^H_{k-1}$, $k \geqslant 1$, and observe that

$$E[X_k X_l] = \rho(k-l) = \frac{1}{2}\big(|k-l+1|^{2H} + |k-l-1|^{2H} - 2|k-l|^{2H}\big).$$

We have

$$\rho(k) = \rho(-k) \sim H(2H-1)k^{2H-2} \quad \text{as } |k| \to \infty,$$

so $\sum_{k\in\mathbb{Z}} |\rho(k)| < \infty$ (recall that $H < 1/2$). By applying the (forthcoming) Breuer–Major Theorem 7.2 with $\phi(x) = x^3$ (we have $\phi(x) = 3H_1(x) + H_3(x)$, hence the Hermite rank of ϕ is one), we deduce that

$$\frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} (B^H_{k+1} - B^H_k)^3 \overset{\text{law}}{\to} \mathcal{N}(0, \sigma^2_H) \quad \text{as } n \to \infty, \tag{3.24}$$

where (since $\sum_{k\in\mathbb{Z}} \rho(k) = 0$ by a telescoping sum argument)

$$\sigma^2_H = \frac{3}{4} \sum_{k\in\mathbb{Z}} \big(|k+1|^{2H} + |k-1|^{2H} - 2|k|^{2H}\big)^3 > 0. \tag{3.25}$$

By the selfsimilarity property of B^H, (3.24) is clearly equivalent to

$$n^{3H-\frac{1}{2}} \sum_{k=0}^{n-1} (B^H_{(k+1)/n} - B^H_{k/n})^3 \overset{\text{law}}{\to} \mathcal{N}(0, \sigma^2_H) \quad \text{as } n \to \infty,$$

from which we deduce that the limit in (3.23) exists in probability if and only if $H > \frac{1}{6}$. That is, the integral $\int_0^1 (B^H_s)^2 d^\circ B^H_s$ exists if and only if $H > 1/6$. As a result the formula (3.20), which holds true when $H \geqslant 1/2$, cannot be extended to the case $H \leqslant 1/6$. On the other hand, this observation asks the following important question: is (3.20) correct for all $H > 1/6$? The next result provides a positive answer.

Theorem 3.5 (Cheridito–Nualart [10] – Gradinaru *et al.* [19]). *Let B^H be a fractional Brownian motion with Hurst index $H > \frac{1}{6}$, fix $T > 0$ and let $f : \mathbb{R} \to \mathbb{R}$ be a $\mathscr{C}^\infty$ function with polynomial growth together with all its derivatives. Then $\int_0^T f'(B^H_s) d^\circ B^H_s$ exists (in the sense of (3.19)) and we have*

$$f(B^H_T) = f(0) + \int_0^T f'(B^H_s) d^\circ B^H_s. \tag{3.26}$$

Our proof of Theorem 3.5 follows [37], and will rely on a fine estimate of the weighted power variations associated to fractional Brownian motion. Before giving it, let us observe that (3.26) shows no correction term. But if we allow the limit in (3.19) to be only *in law* and if we consider the critical case $H = 1/6$, it is worth noting that, in this case, we get a chain rule formula involving a correction term, which takes the surprising form of a classical Itô integral with respect to an *independent* Brownian motion W. Namely, for f as in Theorem 3.5 and with $\kappa = \sigma_{1/6} > 0$ given by (3.25), we have

$$f(B_T^{1/6}) = f(0) + \int_0^T f'(B_s^{1/6})d^\circ B_s^{1/6} - \frac{\kappa}{12}\int_0^T f'''(B_s^{1/6})dW_s. \tag{3.27}$$

The proof of (3.27) (which is out of the scope of this book) can be found in Nourdin, Réveillac and Swanson [43].

Let us now do the proof of Theorem 3.5.

Proof. Without loss of generality and in order to simplify the exposition, we assume that $T = 1$. Also, we set

$$\Delta B_{k/n}^H = B_{(k+1)/n}^H - B_{k/n}^H, \quad k = 0, \dots, n-1,$$

and we recall that $H \in (\frac{1}{6}, \frac{1}{2})$. We can write, for any $a, b \in \mathbb{R}$,

$$\begin{aligned}
f(b) &= f(a) + f'(a)(b-a) + \frac{1}{2}f''(a)(b-a)^2 + \frac{1}{6}f^{(3)}(a)(b-a)^3 \\
&\quad + \frac{1}{24}f^{(4)}(a)(b-a)^4 + \frac{1}{120}f^{(5)}(a)(b-a)^5 + \frac{1}{120}\int_a^b f^{(6)}(t)(b-t)^5 dt \\
f(b) &= f(a) + f'(b)(b-a) - \frac{1}{2}f''(b)(b-a)^2 + \frac{1}{6}f^{(3)}(b)(b-a)^3 \\
&\quad - \frac{1}{24}f^{(4)}(b)(b-a)^4 + \frac{1}{120}f^{(5)}(b)(b-a)^5 - \frac{1}{120}\int_a^b f^{(6)}(t)(t-a)^5 dt \\
f''(b) &= f''(a) + f^{(3)}(a)(b-a) + \frac{1}{2}f^{(4)}(a)(b-a)^2 + \frac{1}{6}f^{(5)}(a)(b-a)^3 \\
&\quad + \frac{1}{6}\int_a^b f^{(6)}(t)(b-t)^3 dt \\
f^{(3)}(b) &= f^{(3)}(a) + f^{(4)}(a)(b-a) + \frac{1}{2}f^{(5)}(a)(b-a)^2 + \frac{1}{2}\int_a^b f^{(6)}(t)(b-t)^2 dt
\end{aligned}$$

$$\begin{aligned}
f^{(4)}(b) &= f^{(4)}(a) + f^{(5)}(a)(b-a) + \int_a^b f^{(6)}(t)(b-t)dt \\
f^{(5)}(b) &= f^{(5)}(a) + \int_a^b f^{(6)}(t)dt,
\end{aligned}$$

so that

$$\begin{aligned}
f(b) &= f(a) + \frac{1}{2}\big(f'(b) + f'(a)\big)(b-a) - \frac{1}{12}f^{(3)}(a)(b-a)^3 \\
&\quad - \frac{1}{24}f^{(4)}(a)(b-a)^4 - \frac{1}{80}f^{(5)}(a)(b-a)^5 + R(a,b),
\end{aligned}$$

with

$$R(a,b) = \frac{1}{240}\int_a^b f^{(6)}(t)\Big((b-t)^5 - (t-a)^5 - 10(b-t)^3(b-a)^2 + 10(b-t)^2(b-a)^3 - 5(b-t)(b-a)^4 + (b-a)^5\Big)dt.$$

Observe that

$$|R(a,b)| \leqslant \frac{1}{8}\sup_{t\in[a,b]}|f^{(6)}(t)|(b-a)^6, \quad a,b\in\mathbb{R}. \tag{3.28}$$

Setting $a = B^H_{k/n}$ and $b = B^H_{(k+1)/n}$ and summing over $k = 0, \ldots, n-1$ yields

$$\begin{aligned} f(B^H_1) = {} & f(0) + \frac{1}{2}\sum_{k=0}^{n-1}\big(f'(B^H_{(k+1)/n}) + f'(B^H_{k/n})\big)\,\Delta B^H_{k/n} \\ & -\frac{1}{12}\sum_{k=0}^{n-1} f^{(3)}(B^H_{k/n})\big(\Delta B^H_{k/n}\big)^3 - \frac{1}{24}\sum_{k=0}^{n-1} f^{(4)}(B^H_{k/n})\big(\Delta B^H_{k/n}\big)^4 \\ & -\frac{1}{80}\sum_{k=0}^{n-1} f^{(5)}(B^H_{k/n})\big(\Delta B^H_{k/n}\big)^5 + \sum_{k=0}^{n-1} R(B^H_{k/n}, B^H_{(k+1)/n}). \end{aligned} \tag{3.29}$$

Using (3.28), we have

$$\begin{aligned} \left|\sum_{k=0}^{n-1} R(B^H_{k/n}, B^H_{(k+1)/n})\right| & \leqslant \frac{1}{8}\sum_{k=0}^{n-1}\sup_{\frac{k}{n}\leqslant u\leqslant\frac{k+1}{n}}|f^{(6)}(B^H_u)|(\Delta B^H_{k/n})^6 \\ & \leqslant \frac{1}{8}\sup_{u\in[0,1]}|f^{(6)}(B^H_u)|\sum_{k=0}^{n-1}(\Delta B^H_{k/n})^6, \end{aligned}$$

and this term tends to zero in probability as $n\to\infty$ since, by (2.12),

$$n^{6H-1}\sum_{k=0}^{n-1}(\Delta B^H_{k/n})^6 \xrightarrow{L^2} 15,$$

implying in turn (because $H > \frac{1}{6}$) that $\sum_{k=0}^{n-1}(\Delta B^H_{k/n})^6 \overset{\text{proba}}{\to} 0$ as $n\to\infty$. Let us expand the monomials x^m, $m = 1,2,3,4,5$, in terms of the Hermite polynomials:

$$\begin{aligned} x &= H_1(x) \\ x^2 &= H_2(x) + 1 \\ x^3 &= H_3(x) + 3\,H_1(x) \\ x^4 &= H_4(x) + 6\,H_2(x) + 3 \\ x^5 &= H_5(x) + 10\,H_3(x) + 15\,H_1(x). \end{aligned}$$

Also, for any integer $q \geqslant 1$ and any continuous function $g : \mathbb{R} \to \mathbb{R}$, let us introduce the following quantity:

$$V_n^{(q)}(g) = \sum_{k=0}^{n-1} g(B_{k/n}^H) H_q(n^H \Delta B_{k/n}^H), \quad n \geqslant 1.$$

We then have

$$\sum_{k=0}^{n-1} f^{(3)}(B_{k/n}^H)(\Delta B_{k/n}^H)^3 = n^{-3H} V_n^{(3)}(f^{(3)}) + 3n^{-3H} V_n^{(1)}(f^{(3)}), \tag{3.30}$$

$$\begin{aligned}\sum_{k=0}^{n-1} f^{(4)}(B_{k/n}^H)(\Delta B_{k/n}^H)^4 & \\ &= n^{-4H} V_n^{(4)}(f^{(4)}) + 6n^{-4H} V_n^{(2)}(f^{(4)}) + 3n^{-4H} \sum_{k=0}^{n-1} f^{(4)}(B_{k/n}^H) \\ &= n^{-4H} V_n^{(4)}(f^{(4)}) + 3n^{-4H} V_n^{(2)}(f^{(4)}) + 3n^{-2H} \sum_{k=0}^{n-1} f^{(4)}(B_{k/n}^H)(\Delta B_{k/n}^H)^2,\end{aligned} \tag{3.31}$$

$$\begin{aligned}\sum_{k=0}^{n-1} f^{(5)}(B_{k/n}^H)(\Delta B_{k/n}^H)^5 & \\ &= n^{-5H} V_n^{(5)}(f^{(5)}) + 10n^{-5H} V_n^{(3)}(f^{(5)}) + 15n^{-5H} V_n^{(1)}(f^{(5)}).\end{aligned} \tag{3.32}$$

The following result, which is of independent interest, provides the asymptotic behavior of $V_n^{(q)}(g)$ for any value of $q \geqslant 2$. (It is worth noticing that the picture when $H \geqslant \frac{1}{2}$ is also known, but it will not be used here. See [37].)

Theorem 3.6 (Nourdin–Nualart–Réveillac–Tudor). *Let B^H be a fractional Brownian motion of Hurst index $H \in (0, \frac{1}{2})$, let $q \geqslant 2$ be an integer, and let $g : \mathbb{R} \to \mathbb{R}$ be a $\mathscr{C}^\infty$ function with polynomial growth together with all its derivatives. Let W denote a classical Brownian motion independent of B^H, and set*

$$\sigma_{q,H} = \sqrt{\frac{1}{2^q} \sum_{k \in \mathbb{Z}} \left(|k+1|^{2H} + |k-1|^{2H} - 2|k|^{2H}\right)^q}.$$

Then, the following convergences hold true as $n \to \infty$:

1. *If $0 < H < \frac{1}{2q}$, then*

$$n^{qH-1} V_n^{(q)}(g) \xrightarrow{L^2} \frac{(-1)^q}{2^q} \int_0^1 g^{(q)}(B_s^H) ds.$$

2. *If $H = \frac{1}{2q}$, then*

$$\frac{1}{\sqrt{n}} V_n^{(q)}(g) \xrightarrow{\text{law}} \frac{(-1)^q}{2^q} \int_0^1 g^{(q)}(B_s^H) ds + \sigma_{q,1/(2q)} \int_0^1 g(B_s^H) dW_s.$$

3. *If* $\frac{1}{2q} < H < \frac{1}{2}$, *then*

$$\frac{1}{\sqrt{n}} V_n^{(q)}(g) \overset{\text{law}}{\to} \sigma_{q,H} \int_0^1 g(B_s^H) dW_s.$$

Proof. See [35, 37] for $H < \frac{1}{2q}$, [36, 42] for $H = \frac{1}{2q}$ and [37] for $H > \frac{1}{2q}$. □

We will also need the asymptotic behavior of $V_n^{(1)}(g)$, which is not given in the previous theorem but can be deduced from it.

Proposition 3.1. *Let B^H be a fractional Brownian motion of Hurst index $H \in (0, \frac{1}{2})$ and let $g : \mathbb{R} \to \mathbb{R}$ be a $\mathscr{C}^\infty$ function with polynomial growth together with all its derivatives. Then, as $n \to \infty$,*

$$n^{H-1} V_n^{(1)}(g) \overset{\text{proba}}{\to} -\frac{1}{2} \int_0^1 g'(B_s^H) ds. \tag{3.33}$$

If $\frac{1}{6} < H < \frac{1}{2}$, *we also have*

$$n^{-3H} V_n^{(1)}(g) + \frac{1}{2} n^{-2H} \sum_{k=0}^{n-1} g'(B_{k/n}^H) \big(\Delta B_{k/n}^H\big)^2 \overset{\text{proba}}{\to} 0. \tag{3.34}$$

Proof. Let us first prove (3.33). We have

$$\begin{aligned}
\int_{B_{k/n}^H}^{B_{(k+1)/n}^H} g(u) du &= g(B_{k/n}^H) \Delta B_{k/n}^H + \frac{1}{2} g'(B_{k/n}^H) (\Delta B_{k/n}^H)^2 \\
&\quad + \int_{B_{k/n}^H}^{B_{(k+1)/n}^H} \big(g'(t) - g'(B_{k/n}^H)\big)(B_{(k+1)/n}^H - t) dt \\
&= g(B_{k/n}^H) \Delta B_{k/n}^H + \frac{1}{2n^{2H}} g'(B_{k/n}^H)(H_2(n^H \Delta B_{k/n}^H) + 1) \\
&\quad + \int_{B_{k/n}^H}^{B_{(k+1)/n}^H} \big(g'(t) - g'(B_{k/n}^H)\big)(B_{(k+1)/n}^H - t) dt,
\end{aligned}$$

so that, by summing over $k = 0, \ldots, n-1$ and by multiplying by n^{2H-1},

$$\begin{aligned}
n^{2H-1} \int_0^{B_1^H} g(u) du &= n^{H-1} V_n^{(1)}(g) + \frac{1}{2n} V_n^{(2)}(g') + \frac{1}{2n} \sum_{k=0}^{n-1} g'(B_{k/n}^H) \\
&+ n^{2H-1} \sum_{k=0}^{n-1} \int_{B_{k/n}^H}^{B_{(k+1)/n}^H} \big(g'(t) - g'(B_{k/n}^H)\big)(B_{(k+1)/n}^H - t) dt.
\end{aligned} \tag{3.35}$$

In (3.35), we have

$$
\begin{aligned}
& n^{2H-1}\left|\sum_{k=0}^{n-1}\int_{B^H_{k/n}}^{B^H_{(k+1)/n}}\big(g'(t)-g'(B^H_{k/n})\big)(B^H_{(k+1)/n}-t)dt\right| \\
&\leqslant \max_{k=0,\dots,n-1}\sup_{\frac{k}{n}\leqslant u\leqslant\frac{k+1}{n}}|g'(B^H_u)-g'(B^H_{k/n})|\times\frac{1}{2}n^{2H-1}\sum_{k=0}^{n-1}(\Delta B^H_{k/n})^2 \\
&\leqslant \sup_{\substack{u,v\in[0,1]\\|u-v|\leqslant 1/n}}|g'(B^H_u)-g'(B^H_v)|\times\frac{1}{2}n^{2H-1}\sum_{k=0}^{n-1}(\Delta B^H_{k/n})^2
\end{aligned}
$$

and this last term tends to zero in probability as $n\to\infty$ because, on one hand,

$$
n^{2H-1}\sum_{k=0}^{n-1}(\Delta B^H_{k/n})^2 \overset{L^2}{\to} 1
$$

by (2.12) and, on the other hand,

$$
\sup_{\substack{u,v\in[0,1]\\|u-v|\leqslant 1/n}}|g'(B^H_u)-g'(B^H_v)| \overset{\text{a.s.}}{\to} 0
$$

by the uniform continuity of the sample paths of $g'\circ B^H$ on the compact interval [0, 1]. Since $H<\frac{1}{2}$, we have that $n^{2H-1}\int_0^{B^H_1}g(u)du \overset{\text{a.s.}}{\to} 0$ as $n\to\infty$. By Theorem 3.6, we have that $\frac{1}{n}V_n^{(2)}(g')\to 0$ in law, hence in probability. Finally, the Riemann sum $\frac{1}{n}\sum_{k=0}^{n-1}g'(B^H_{k/n})$ converges almost surely to $\int_0^1 g'(B^H_s)ds$ as $n\to\infty$. Putting all these limits in (3.35) yields (3.33).

Let us now prove (3.34). We have

$$
\begin{aligned}
& \int_{B^H_{k/n}}^{B^H_{(k+1)/n}} g(u)du \\
&= g(B^H_{k/n})\Delta B^H_{k/n}+\frac{1}{2}g'(B^H_{k/n})(\Delta B^H_{k/n})^2+\frac{1}{6}g''(B^H_{k/n})(\Delta B^H_{k/n})^3 \\
&\quad +\frac{1}{6}\int_{B^H_{k/n}}^{B^H_{(k+1)/n}} g'''(t)(B^H_{(k+1)/n}-t)^3dt \\
&= g(B^H_{k/n})\Delta B^H_{k/n}+\frac{1}{2}g'(B^H_{k/n})(\Delta B^H_{k/n})^2 \\
&\quad +\frac{1}{6}n^{-3H}g''(B^H_{k/n})H_3(n^H\Delta B^H_{k/n})+\frac{1}{2}n^{-3H}g''(B^H_{k/n})(n^H\Delta B^H_{k/n}) \\
&\quad +\frac{1}{6}\int_{B^H_{k/n}}^{B^H_{(k+1)/n}} g'''(t)(B^H_{(k+1)/n}-t)^3dt,
\end{aligned}
$$

so that, by summing over $k = 0, \dots, n-1$ and by multiplying by n^{-2H},

$$\begin{aligned} n^{-2H} \int_0^{B_1^H} g(u)du \\ &= n^{-3H} V_n^{(1)}(g) + \frac{1}{2} n^{-2H} \sum_{k=0}^{n-1} g'(B_{k/n}^H)\big(\Delta B_{k/n}^H\big)^2 \\ &+ \frac{1}{6} n^{-5H} V_n^{(3)}(g'') + \frac{1}{2} n^{-5H} V_n^{(1)}(g'') \\ &+ \frac{1}{6} n^{-2H} \sum_{k=0}^{n-1} \int_{B_{k/n}^H}^{B_{(k+1)/n}^H} g'''(t)(B_{(k+1)/n}^H - t)^3 dt. \end{aligned} \tag{3.36}$$

It is obvious that $n^{-2H} \int_0^{B_1^H} g(u)du \overset{\text{a.s.}}{\to} 0$. Moreover, from Theorem 3.6 and since $H > \frac{1}{6} > \frac{1}{10}$, we have that $n^{-5H} V_n^{(3)}(g'') \overset{\text{proba}}{\to} 0$. Also, from (3.33) and since $H > \frac{1}{6}$, we deduce that $n^{-5H} V_n^{(1)}(g'') \overset{\text{proba}}{\to} 0$. Finally, we have

$$\begin{aligned} n^{-2H} &\left| \sum_{k=0}^{n-1} \int_{B_{k/n}^H}^{B_{(k+1)/n}^H} g'''(t)(B_{(k+1)/n}^H - t)^3 dt \right| \\ &\leqslant \frac{n^{-2H}}{4} \sum_{k=0}^{n-1} \sup_{\frac{k}{n} \leqslant u \leqslant \frac{k+1}{n}} |g'''(B_u^H)| (\Delta B_{k/n}^H)^4 \\ &\leqslant \frac{n^{-2H}}{4} \sup_{u \in [0,1]} |g'''(B_u^H)| \sum_{k=0}^{n-1} (\Delta B_{k/n}^H)^4, \end{aligned}$$

and this term tends to zero in probability as $n \to \infty$ since, by (2.12),

$$n^{4H-1} \sum_{k=0}^{n-1} (\Delta B_{k/n}^H)^4 \overset{L^2}{\to} 3,$$

implying in turn (because $H > \frac{1}{6}$) that $n^{-2H} \sum_{k=0}^{n-1} (\Delta B_{k/n}^H)^4 \overset{\text{proba}}{\to} 0$ as $n \to \infty$. Putting all these facts together in (3.36) concludes the proof of (3.34). □

We can now finish the proof of Theorem 3.5. From Theorem 3.6 and since $H > \frac{1}{6}$, we immediately deduce that $n^{-4H} V_n^{(2)}(f^{(4)}) \overset{\text{proba}}{\to} 0$, $n^{-3H} V_n^{(3)}(f^{(3)}) \overset{\text{proba}}{\to} 0$, $n^{-5H} V_n^{(3)}(f^{(5)}) \overset{\text{proba}}{\to} 0$, $n^{-4H} V_n^{(4)}(f^{(4)}) \overset{\text{proba}}{\to} 0$ and $n^{-5H} V_n^{(5)}(f^{(5)}) \overset{\text{proba}}{\to} 0$. From Proposition 3.1, $n^{-5H} V_n^{(1)}(f^{(5)}) \overset{\text{proba}}{\to} 0$ and

$$n^{-3H} V_n^{(1)}(f^{(3)}) + \frac{1}{2} n^{-2H} \sum_{k=0}^{n-1} f^{(4)}(B_{k/n}^H)(\Delta B_{k/n}^H)^2 \overset{\text{proba}}{\to} 0.$$

Putting all these limits in the decompositions (3.30)-(3.31)-(3.32) and then in (3.29) yield the desired conclusion (3.26). □

Chapter 4
Supremum of the Fractional Brownian Motion

For all $x > 0$, one has

$$\begin{aligned}\left(\frac{1}{x} - \frac{1}{x^3}\right) e^{-x^2/2} &= \int_x^\infty e^{-y^2/2} \left(1 - \frac{3}{y^4}\right) dy \\ &\leqslant \int_x^\infty e^{-y^2/2} dy \leqslant \frac{1}{x} \int_x^\infty y e^{-y^2/2} dy = \frac{1}{x} e^{-x^2/2},\end{aligned}$$

from which we deduce that

$$\lim_{x\to\infty} x^{-2} \log \int_x^\infty e^{-y^2/2} dy = -1/2. \tag{4.1}$$

Let $(B_t^H)_{t\geqslant 0}$ be a fractional Brownian motion of Hurst index $H \in (0, 1)$ and consider its running maximum

$$M_t^H = \sup_{u\in[0,t]} B_u^H, \quad t \geqslant 0.$$

When $H = 1/2$, it is well-known that $M_1^{1/2} \overset{\text{law}}{=} |\mathcal{N}(0, 1)|$ (see, e.g., [52, Proposition 3.7]). Relying to this property as well as (4.1), it is straightforward to check that

$$\lim_{x\to\infty} x^{-2} \log P(M_1^{1/2} \geqslant x) = -1/2, \tag{4.2}$$

$$\lim_{x\to 0+} (\log x)^{-1} \log P(M_1^{1/2} \leqslant x) = 1. \tag{4.3}$$

The exact distribution of M_1^H is still an open problem for $H \neq 1/2$. In this chapter, we shall extend (4.2)–(4.3) to any value of H. It is our opinion that the extension of (4.3) (due to Molchan [31]) is one of the most beautiful result dealing with fractional Brownian motion.

Nourdin I.: *Selected Aspects of Fractional Brownian Motion*, B&SS,
DOI 10.1007/978-88-470-2823-4_4, © Springer-Verlag Italia 2012

4.1 Asymptotic Behavior at Infinity

The following result is the extension of (4.2) to any value of H.

Theorem 4.1. *For any* $H \in (0,1)$*, we have*

$$\lim_{x\to\infty} x^{-2}\log P(M_1^H \geqslant x) = -1/2.$$

Actually, Theorem 4.1 is an easy corollary of the following result, which is of the upmost importance in the modern theory of Gaussian processes. It asserts that the supremum of a Gaussian process roughly behaves like a single Gaussian variable with variance equal to the largest variance achieved by the entire process.

Theorem 4.2. *Let* $X = (X_t)_{t\in[0,1]}$ *be a centered and continuous Gaussian process. Set* $\sigma^2 = \sup_{t\in[0,1]} \mathrm{Var}(X_t)$*. Then* $m := E[\sup_{u\in[0,1]} X_u]$ *is finite and we have, for all* $x > m$*,*

$$P\left(\sup_{u\in[0,1]} X_u \geqslant x\right) \leqslant e^{-\frac{(x-m)^2}{2\sigma^2}}.$$

As we just said, Theorem 4.2 implies Theorem 4.1.

Proof of Theorem 4.1. We notice first that

$$\frac{1}{\sqrt{2\pi}}\left(\frac{1}{x}-\frac{1}{x^3}\right)e^{-\frac{x^2}{2}} \leqslant \frac{1}{\sqrt{2\pi}}\int_x^\infty e^{-\frac{y^2}{2}}dy = P(B_1^H \geqslant x) \leqslant P(M_1^H \geqslant x),$$

implying in turn that

$$\liminf_{x\to\infty} x^{-2}\log P(M_1^H \geqslant x) \geqslant -1/2. \tag{4.4}$$

On the other hand, Theorem 4.2 implies, for any x large enough,

$$\log P\left(M_1^H \geqslant x\right) \leqslant -\frac{(x-E[M_1^H])^2}{2\sup_{t\in[0,1]}\mathrm{Var}(B_t^H)} = -\frac{(x-E[M_1^H])^2}{2}, \tag{4.5}$$

implying in turn that

$$\limsup_{x\to\infty} x^{-2}\log P(M_1^H \geqslant x) \leqslant -1/2. \tag{4.6}$$

By combining (4.6) and (4.4), we conclude the proof of Theorem 4.1. □

Let us now prove Theorem 4.2. In preparation to this, we recall that any positive definite symmetric matrix $C \in \mathcal{M}_d(\mathbb{R})$ admits a unique square root (that is, there is a unique positive definite symmetric matrix $\sqrt{C} \in \mathcal{M}_d(\mathbb{R})$ satisfying $(\sqrt{C})^2 = C$), and we prove the following result.

Theorem 4.3. *Let $C \in \mathcal{M}_d(\mathbb{R})$ be a symmetric positive definite matrix, let $G \sim \mathcal{N}_d(0, C)$, and let $\phi : \mathbb{R}^d \to \mathbb{R}$ be a Lipschitz and $\mathcal{C}^1$ function. Then, for all $x > 0$ and with*

$$M = \sup_{z \in \mathbb{R}^d} \|\sqrt{C}\, \nabla\phi(z)\|_{\mathbb{R}^d} \in [0, \infty),$$

we have

$$P\big(\phi(G) \geqslant E[\phi(G)] + x\big) \leqslant \exp\left(-\frac{x^2}{2M^2}\right)$$
$$P\big(\phi(G) \leqslant E[\phi(G)] - x\big) \leqslant \exp\left(-\frac{x^2}{2M^2}\right).$$

The proof of Theorem 4.3 (shown to us by Christian Houdré [21]) relies on the following lemma.

Lemma 4.1. *Let $C \in \mathcal{M}_d(\mathbb{R})$ be a symmetric positive definite matrix, let $G \sim \mathcal{N}_d(0, C)$, and let $\phi, \psi : \mathbb{R}^d \to \mathbb{R}$ be two Lipschitz and $\mathcal{C}^1$ functions. Then*

$$\mathrm{Cov}\big(\phi(G), \psi(G)\big) = \int_0^1 E\langle\sqrt{C}\, \nabla\phi(G_\alpha), \sqrt{C}\, \nabla\psi(H_\alpha)\rangle_{\mathbb{R}^d}\, d\alpha, \tag{4.7}$$

where

$$(G_\alpha, H_\alpha) \sim \mathcal{N}_{2d}\left(0, \begin{pmatrix} C & \alpha C \\ \alpha C & C \end{pmatrix}\right), \quad 0 \leqslant \alpha \leqslant 1.$$

Proof. By bilinearity and approximation, it is enough to show (4.7) for $\phi(x) = e^{i\langle t,x\rangle_{\mathbb{R}^d}}$ and $\psi(x) = e^{i\langle s,x\rangle_{\mathbb{R}^d}}$ when $s, t \in \mathbb{R}^d$ are given (and fixed once for all). Set

$$\varphi_\alpha(t, s) = E[e^{i\langle \binom{t}{s}, \binom{G_\alpha}{H_\alpha}\rangle_{\mathbb{R}^{2d}}}].$$

We have

$$\int_0^1 E\langle\sqrt{C}\nabla\phi(G_\alpha), \sqrt{C}\nabla\psi(H_\alpha)\rangle_{\mathbb{R}^d}\, d\alpha = -\langle\sqrt{C}t, \sqrt{C}s\rangle_{\mathbb{R}^d} \int_0^1 \varphi_\alpha(t, s) d\alpha. \tag{4.8}$$

Observe that $G_\alpha \overset{\text{law}}{=} H_\alpha \overset{\text{law}}{=} G \sim \mathcal{N}_d(0, C)$, that H_0 and G_0 are independent and that $H_1 = G_1$ a.s. Hence,

$$\mathrm{Cov}(\phi(G), \psi(G)) = \varphi_1(t, s) - \varphi_0(t, s) = \int_0^1 \frac{\partial}{\partial\alpha}\varphi_\alpha(t, s) d\alpha.$$

Since

$$\begin{pmatrix} C & \alpha C \\ \alpha C & C \end{pmatrix} = \alpha \begin{pmatrix} C & C \\ C & C \end{pmatrix} + (1-\alpha)\begin{pmatrix} C & 0 \\ 0 & C \end{pmatrix},$$

we have, with $\varphi_G(t) = E[e^{i\langle t,G\rangle_{\mathbb{R}^d}}]$,

$$\varphi_\alpha(t, s) = \varphi_1(t, s)^\alpha \varphi_0(t, s)^{1-\alpha} = \varphi_G(t+s)^\alpha \varphi_G(t)^{1-\alpha} \varphi_G(s)^{1-\alpha}.$$

Consequently,

$$\frac{\partial}{\partial\alpha}\varphi_\alpha(t,s) = \big(\log\varphi_G(t+s) - \log\varphi_G(t) - \log\varphi_G(s)\big)\varphi_\alpha(t,s),$$

implying in turn, since $\varphi_G(t) = e^{-\frac{1}{2}\|\sqrt{C}t\|^2_{\mathbb{R}^d}}$ (see (1.1),

$$\varphi_1(t,s) - \varphi_0(t,s) = -\langle\sqrt{C}t, \sqrt{C}s\rangle_{\mathbb{R}^d}\int_0^1 \varphi_\alpha(t,s)d\alpha. \tag{4.9}$$

The two right-hand sides in (4.8) and (4.9) being the same, the proof of Lemma 4.1 is complete. □

We are now able to prove Theorem 4.3.

Proof of Theorem 4.3. Replacing ϕ by $\phi - E[\phi(G)]$ if necessary, we may assume that $E[\phi(G)] = 0$ without loss of generality. By Lemma 4.1, we can write

$$\begin{aligned} E[\phi(G)e^{t\phi(G)}] &= \mathrm{Cov}\big(\phi(G), e^{t\phi(G)}\big) \\ &= t\int_0^1 E\big[\langle\sqrt{C}\,\nabla\phi(G_\alpha), \sqrt{C}\,\nabla\phi(H_\alpha)\rangle_{\mathbb{R}^d} e^{t\phi(H_\alpha)}\big]d\alpha \\ &\leqslant t\int_0^1 E\big[\|\sqrt{C}\,\nabla\phi(G_\alpha)\|_{\mathbb{R}^d}\|\sqrt{C}\,\nabla\phi(H_\alpha)\|_{\mathbb{R}^d} e^{t\phi(H_\alpha)}\big]d\alpha \\ &\leqslant tM^2\int_0^1 E\big[e^{t\phi(H_\alpha)}\big]d\alpha = tM^2\,E\big[e^{t\phi(G)}\big], \end{aligned}$$

where, in the last equality, we used that $H_\alpha \overset{\text{law}}{=} G$ for all $\alpha \in [0,1]$. Thus,

$$\frac{\partial}{\partial t}E[e^{t\phi(G)}] = E[\phi(G)e^{t\phi(G)}] \leqslant tM^2\,E[e^{t\phi(G)}],$$

so that, after integration,

$$E[e^{t\phi(G)}] \leqslant e^{\frac{t^2M^2}{2}}, \quad t > 0.$$

Using Markov inequality and then setting $t = x/M^2$ (which is the optimal choice), we get, for any $x > 0$,

$$P(\phi(G) \geqslant x) \leqslant e^{-tx}E[e^{t\phi(G)}] \leqslant e^{-tx+\frac{t^2M^2}{2}} \leqslant \exp\left(-\frac{x^2}{2M^2}\right).$$

By replacing ϕ by $-\phi$, we deduce

$$P(\phi(G) \leqslant -x) = P(-\phi(G) \geqslant x) \leqslant \exp\left(-\frac{x^2}{2M^2}\right)$$

as well, which concludes the proof of the theorem. □

As a corollary of Theorem 4.3, we get the following result.

Corollary 4.1. *Fix $d \geqslant 1$ and let $G = (G_1, \dots, G_d)$ be a centered Gaussian vector of $\mathbb{R}^d$. Then, for all $x > 0$,*

$$P\left(\max_{1\leqslant i\leqslant d} G_i \geqslant E\left[\max_{1\leqslant i\leqslant d} G_i\right] + x\right) \leqslant \exp\left(-\frac{x^2}{2\max\limits_{1\leqslant i\leqslant d} \operatorname{Var}(G_i)}\right) \tag{4.10}$$

$$P\left(\max_{1\leqslant i\leqslant d} G_i \leqslant E\left[\max_{1\leqslant i\leqslant d} G_i\right] - x\right) \leqslant \exp\left(-\frac{x^2}{2\max\limits_{1\leqslant i\leqslant d} \operatorname{Var}(G_i)}\right). \tag{4.11}$$

Proof. Using $\max\limits_{1\leqslant i\leqslant d} |x_i| \leqslant \sqrt{\sum_{i=1}^d x_i^2}$, notice first that $E\left[\left|\max_{1\leqslant i\leqslant d} G_i\right|\right] < \infty$. Let $C = (C_{i,j})_{1\leqslant i,j\leqslant d}$ denote the covariance matrix of G and, for any $\beta > 0$, let

$$\phi_\beta(z) = \frac{1}{\beta}\log\left(\sum_{i=1}^d e^{\beta z_i}\right), \quad z \in \mathbb{R}^d.$$

For any $u, v \in \mathbb{R}^d$ and $\beta > 0$, we have

$$\begin{aligned}\left|\phi_\beta(u) - \phi_\beta(v)\right| &= \left|\int_0^1 \langle \nabla\phi_\beta(tu + (1-t)v), u - v\rangle_{\mathbb{R}^d}\, dt\right| \\ &\leqslant \max_{z\in\mathbb{R}^d} \|\nabla\phi_\beta(z)\|_{\mathbb{R}^d} \|u - v\|_{\mathbb{R}^d} \leqslant \|u - v\|_{\mathbb{R}^d},\end{aligned}$$

where in the last inequality we used that

$$\|\nabla\phi_\beta(z)\|^2_{\mathbb{R}^d} = \sum_{i=1}^d \left(\frac{e^{\beta z_i}}{\sum_{j=1}^d e^{\beta z_j}}\right)^2 \leqslant \sum_{i=1}^d \frac{e^{\beta z_i}}{\sum_{j=1}^d e^{\beta z_j}} = 1.$$

That is, ϕ_β is 1-Lipschitz continuous on $\mathbb{R}^d$. On the other hand,

$$\begin{aligned}\|\sqrt{C}\,\nabla\phi_\beta(z)\|^2_{\mathbb{R}^d} &= \sum_{i,j=1}^d C_{i,j} \frac{e^{\beta(z_i+z_j)}}{\left(\sum_{k=1}^d e^{\beta z_k}\right)^2} \\ &\leqslant \max_{i,j=1,\dots,d} |C_{i,j}| \sum_{i,j=1}^d \frac{e^{\beta(z_i+z_j)}}{\left(\sum_{k=1}^d e^{\beta z_k}\right)^2} \\ &= \max_{i,j=1,\dots,d} |C_{i,j}| = \max_{i=1,\dots,d} \operatorname{Var}(G_i),\end{aligned} \tag{4.12}$$

where the last equality comes from the Cauchy-Schwarz inequality. By applying Theorem 4.3, we deduce that, for all $x > 0$,

$$P\left(\phi_\beta(G) \geqslant E[\phi_\beta(G)] + x\right) \leqslant \exp\left(-\frac{x^2}{2\max_{i=1,\dots,d}\operatorname{Var}(G_i)}\right),$$
$$P\left(\phi_\beta(G) \leqslant E[\phi_\beta(G)] - x\right) \leqslant \exp\left(-\frac{x^2}{2\max_{i=1,\dots,d}\operatorname{Var}(G_i)}\right).$$

Finally, observe that $\beta \to \infty$ implies (4.10)-(4.11). Indeed,

$$\begin{aligned}\max_{1\leqslant i\leqslant d} z_i &= \frac{1}{\beta}\log\left(e^{\beta\max_{1\leqslant i\leqslant d} z_i}\right) \leqslant \phi_\beta(z)\\ &\leqslant \frac{1}{\beta}\log\left(d e^{\beta\max_{1\leqslant i\leqslant d} z_i}\right) = \frac{1}{\beta}\log d + \max_{1\leqslant i\leqslant d} z_i,\end{aligned}$$

implying in turn that $E[\phi_\beta(G)] \to E[\max_{1\leqslant i\leqslant d} G_i]$ and $\phi_\beta(z) \to \max_{1\leqslant i\leqslant d} z_i$ for all $z \in \mathbb{R}^d$. □

Corollary 4.1 implies Theorem 4.2.

Proof of Theorem 4.2. It is divided into two steps.

Step 1. Assume for the time being that $m = \infty$. Fix $x > 0$. By monotone convergence and continuity, we have

$$E\left[\sup_{k=1,\dots,2^n} X_{k2^{-n}}\right] \to m = \infty.$$

Hence, let n be large enough so that $E\left[\sup_{k=1,\dots,2^n} X_{k2^{-n}}\right] \geqslant 2x$. If $\sup_{u\in[0,1]} X_u \leqslant x$, then

$$\begin{aligned}\sup_{k=1,\dots,2^n} X_{k2^{-n}} - E\left[\sup_{k=1,\dots,2^n} X_{k2^{-n}}\right] &\leqslant \sup_{u\in[0,1]} X_u - E\left[\sup_{k=1,\dots,2^n} X_{k2^{-n}}\right]\\ &\leqslant -x.\end{aligned}$$

Thus, using (4.11) as well,

$$\begin{aligned}P\left(\sup_{u\in[0,1]} X_u \leqslant x\right) &\leqslant P\left(\sup_{k=1,\dots,2^n} X_{k2^{-n}} - E\left[\sup_{k=1,\dots,2^n} X_{k2^{-n}}\right] \leqslant -x\right)\\ &\leqslant e^{-\frac{x^2}{2\sigma^2}}.\end{aligned}$$

By letting $x \to \infty$ and by noticing that $\sup_{u\in[0,1]} X_u < \infty$ a.s. (since X has continuous paths), we get a contradiction. Hence $m < \infty$.

Step 2. By applying Corollary 4.1, we get

$$P\left(\sup_{k=1,\dots,2^n} X_{k2^{-n}} \geqslant E\left[\sup_{k=1,\dots,2^n} X_{k2^{-n}}\right] + x\right) \leqslant \exp\left(-\frac{x^2}{2\sigma^2}\right).$$

Fatou's lemma together with monotone convergence (see Step 1) then implies

$$P\left(\sup_{u\in[0,1]} X_u \geqslant m + x\right) \leqslant \exp\left(-\frac{x^2}{2\sigma^2}\right),$$

and the proof of Theorem 4.2 is concluded. □

4.2 Asymptotic Behavior at Zero

The following result is the extension of (4.3) to any value of H. We follow Molchan [31] and Aurzada [2].

Theorem 4.4 (Molchan). *For any $H \in (0,1)$, we have*

$$\lim_{x\to 0^+} (\log x)^{-1} \log P(M_1^H \leqslant x) = \frac{1-H}{H}.$$

The proof of Theorem 4.4 is divided into two parts: Lemma 4.3 and Lemma 4.4. Also, we shall make use of the following auxiliary result.

Lemma 4.2. *For any $H \in (0,1)$, we have*

$$E\left[\left(\int_0^t e^{B_u^H}\,du\right)^{-1}\right] \sim H\,E[M_1^H]\,t^{H-1} \quad \text{as } t \to \infty.$$

Proof. For any $x > 0$, we have (using in particular that $B^H \overset{\text{law}}{=} -B^H$)

$$\begin{aligned} P\left(\sup_{u\in[0,1]} |B_u^H| \geqslant x\right) &\leqslant P\left(\sup_{u\in[0,1]} B_u^H \geqslant x\right) + P\left(\sup_{u\in[0,1]} -B_u^H \geqslant x\right) \\ &= 2P(M_1^H \geqslant x). \end{aligned}$$

Together with (4.5), this implies that $\sup_{u\in[0,1]} |B_u^H|$ has all exponential moments (that is, for all $\theta \in \mathbb{R}$,

$$E\left[e^{\theta \sup_{u\in[0,1]} |B_u^H|}\right] < \infty). \tag{4.13}$$

By selfsimilarity, $\sup_{u\in[0,t]} |B_u^H|$ has all exponential moments for all $t > 0$ as well. In particular, we can freely interchange expectation and differentiation in the forthcoming calculations.

Now, fix $t > 0$ and observe that $(B_u^H)_{u\in[0,t]} \overset{\text{law}}{=} (B_{t-u}^H - B_t^H)_{u\in[0,t]}$; indeed, these two processes are centered, Gaussian and have the same covariance. We deduce that

$$\begin{aligned}
&E\left[\left(\int_0^t e^{B_u^H}du\right)^{-1}\right] = E\left[e^{B_t^H}\left(\int_0^t e^{B_{t-u}^H}du\right)^{-1}\right] \\
&= E\left[e^{B_t^H}\left(\int_0^t e^{B_u^H}du\right)^{-1}\right] = E\left[\frac{\partial}{\partial t}\left(\log\int_0^t e^{B_u^H}du\right)\right] \\
&= \frac{\partial}{\partial t}E\left[\log\int_0^t e^{B_u^H}du\right] = \frac{\partial}{\partial t}\left(E\left[\log\int_0^1 e^{t^H B_u^H}du\right] + \log t\right) \text{ by selfsimilarity} \\
&= E\left[\frac{\partial}{\partial t}\left(\log\int_0^1 e^{t^H B_u^H}du\right)\right] + \frac{1}{t} = Ht^{H-1}E\left[\frac{\int_0^1 B_u^H e^{t^H B_u^H}du}{\int_0^1 e^{t^H B_u^H}du}\right] + \frac{1}{t}.
\end{aligned} \tag{4.14}$$

Since B^H has continuous paths, we have

$$\frac{\int_0^1 B_u^H e^{t^H B_u^H}du}{\int_0^1 e^{t^H B_u^H}du} \to M_1^H \quad \text{as } t \to \infty.$$

Moreover,

$$\left|\frac{\int_0^1 B_u^H e^{t^H B_u^H}du}{\int_0^1 e^{t^H B_u^H}du}\right| \leqslant \sup_{u\in[0,1]}|B_u^H| \text{ a.s.}$$

By dominated convergence, the desired conclusion follows by letting $t \to \infty$ in (4.14). □

Lemma 4.3. *For any* $H \in (0,1)$*, we have*

$$\liminf_{x\to 0+}\frac{\log P(M_1^H \leqslant x)}{\log x} \geqslant \frac{1-H}{H}. \tag{4.15}$$

Proof. Set $x(t) = 3t^{-H}\big(\sqrt{\log t} + \log t\big)$, $t > 0$. Since $x'(t) \sim -3H\frac{\log t}{t^{1+H}}$ as $t \to \infty$, one can choose $M > 0$ large enough so that $x : [M,\infty) \to (0, x(M)]$ is a strictly decreasing bijection. For $x \geqslant M$, let $t(x)$ be uniquely defined by

$$x = 3\,t(x)^{-H}\big(\sqrt{\log t(x)} + \log t(x)\big). \tag{4.16}$$

Since $t(x) \to \infty$ as $x \to 0$, we deduce from (4.16) that, as $x \to 0$,

$$\frac{\log x}{\log t(x)} \to -H \quad \text{or, equivalently,} \quad \frac{\log t(x)}{\log x} \to -\frac{1}{H}. \tag{4.17}$$

Assume for the time being that, for some $c > 0$ and for all t large enough,

$$P\left(M_1^H \leqslant 3t^{-H}\big(\sqrt{\log t} + \log t\big)\right)e^{3\sqrt{\log t}}\,t^{1-H} \geqslant c. \tag{4.18}$$

Taking the logarithm and setting $t = t(x)$ yield, for all $x > 0$ small enough,

$$\log P(M_1^H \leqslant x) + 3\sqrt{\log t(x)} + (1-H)\log t(x) \geqslant \log c,$$

implying in turn (4.15), see also (4.17).

So, let us show that (4.18) holds true. To this aim, assume that t is an integer for the sake of simplicity and consider

$$\Omega_t^1 = \left\{ \inf_{u\in[0,1]} B_u^H \geqslant -3\sqrt{\log t} \right\},$$
$$\Omega_t^2 = \bigcap_{i=0}^{t-1} \left\{ \sup_{u,v\in[i,i+1]} (B_u^H - B_v^H) \leqslant 3\sqrt{\log t} \right\}.$$

For any $t \geqslant 1$, we have

$$E\left[\left(\int_0^t e^{B_u^H} du \right)^{-1} \right] \leqslant A_1 + A_2 + A_3 + A_4,$$

where

$$A_1 = E\left[\left(\int_0^1 e^{B_u^H} du \right)^{-1} 1_{\{M_t^H \leqslant 3\sqrt{\log t} + 3\log t\}\cap\Omega_t^1} \right]$$
$$A_2 = E\left[\left(\int_0^1 e^{B_u^H} du \right)^{-1} 1_{(\Omega_t^1)^c} \right]$$
$$A_3 = E\left[\left(\int_0^t e^{B_u^H} du \right)^{-1} 1_{\{M_t^H > 3\sqrt{\log t} + 3\log t\}\cap\Omega_t^2} \right]$$
$$A_4 = E\left[\left(\int_0^1 e^{B_u^H} du \right)^{-1} 1_{(\Omega_t^2)^c} \right].$$

Study of A_1. We have

$$A_1 \leqslant e^{3\sqrt{\log t}} P\big(M_t^H \leqslant 3\sqrt{\log t} + 3\log t\big)$$
$$= e^{3\sqrt{\log t}} P\big(M_1^H \leqslant 3t^{-H}(\sqrt{\log t} + \log t)\big).$$

Study of A_2. We have

$$E\left[\left(\int_0^1 e^{B_u^H} du \right)^{-2} \right] \leqslant E\left[e^{-2\inf_{u\in[0,1]} B_u^H} \right] = E\left[e^{2\sup_{u\in[0,1]}(-B_u^H)} \right]$$
$$= E\left[e^{2M_1^H} \right] < \infty \text{ by Theorem 4.2.} \tag{4.19}$$

Also, as $t \to \infty$,

$$\begin{aligned} P((\Omega_t^1)^c) &= P\left(\inf_{u\in[0,1]} B_u^H < -3\sqrt{\log t}\right) = P\left(\inf_{u\in[0,1]}(-B_u^H) < -3\sqrt{\log t}\right) \\ &= P\left(M_1^H > 3\sqrt{\log t}\right) \\ &= O(t^{\varepsilon-\frac{9}{2}}) \quad \text{by Theorem 4.2, for all } \varepsilon > 0 \\ &= O(t^{-2}). \end{aligned}$$

Using Cauchy-Schwarz, we deduce that $A_2 = O(t^{-1})$ as $t \to \infty$.

Study of A_3. We have

$$\Omega_t^2 \cap \left\{M_t^H > 3\sqrt{\log t} + 3\log t\right\} \subset \bigcup_{i=0}^{t-1} \left\{\inf_{u\in[i,i+1]} B_u^H \geqslant 3\log t\right\}.$$

Indeed, let $i = 0, \ldots, t-1$ and $s \in [i, i+1]$ be such that $M_t^H = B_s^H$; if $M_t^H > 3\sqrt{\log t} + 3\log t$ and $\sup_{u,v\in[i,i+1]}(B_u^H - B_v^H) \leqslant 3\sqrt{\log t}$ then, for all $u \in [i, i+1]$,

$$\begin{aligned} 0 &\leqslant B_s^H - 3\sqrt{\log t} - 3\log t = B_s^H - B_u^H + B_u^H - 3\sqrt{\log t} - 3\log t \\ &\leqslant B_u^H - 3\log t, \end{aligned}$$

so that $\inf_{u\in[i,i+1]} B_u^H \geqslant 3\log t$. Thus,

$$A_3 \leqslant \sum_{i=0}^{t-1} E\left[\left(\int_0^t e^{B_u^H}\, du\right)^{-1} 1_{\{\inf_{u\in[i,i+1]} B_u^H \geqslant 3\log t\}}\right] \leqslant t^{-3}\sum_{i=0}^{t-1} 1 = t^{-2}.$$

Study of A_4. We have, as $t \to \infty$,

$$\begin{aligned} P\left((\Omega_t^2)^c\right) &\leqslant \sum_{i=0}^{t-1} P\left(\sup_{u,v\in[i,i+1]} B_u^H - B_v^H > 3\sqrt{\log t}\right) \\ &= tP\left(\sup_{u,v\in[0,1]} B_u^H - B_v^H > 3\sqrt{\log t}\right) \\ &= O(t^{\varepsilon-\frac{7}{2}}) \text{ by Theorem 4.2, for all } \varepsilon > 0 \\ &= O(t^{-2}). \end{aligned}$$

Recall also from (4.19) that $E\left[\left(\int_0^1 e^{B_u^H}\, du\right)^{-2}\right] < \infty$. Using Cauchy-Schwarz, we deduce that $A_4 = O(t^{-1})$ as $t \to \infty$.

Conclusion. By putting all these estimates together, we get that

$$E\left[\left(\int_0^t e^{B_u^H}\, du\right)^{-1}\right] \leqslant e^{3\sqrt{\log t}}\, P\left(M_1^H \leqslant 3t^{-H}(\sqrt{\log t} + \log t)\right) + O(t^{-1}),$$

as $t \to \infty$. But $E[(\int_0^t e^{B_u^H} du)^{-1}] \sim E[M_1^H]t^{H-1}$ as $t \to \infty$ by Lemma 4.2, so (4.18) holds true for t large enough. □

Lemma 4.4. *For any $H \in (0, 1)$, we have*

$$\limsup_{x \to 0^+} (\log x)^{-1} \log P(M_1^H \leqslant x) \leqslant \frac{1 - H}{H}. \tag{4.20}$$

Proof. To limit the size of the book, we only do the proof for $H \in (\frac{1}{2}, 1)$. We follow the approach developed in [2]. (The case $H \in (0, \frac{1}{2})$ is slightly more difficult to handle, and we refer to [2] for the details.) During the proof of (4.20), we shall make use of the following classical and very important result.

Lemma 4.5 (Slepian [58]). *Fix some integers $d \geqslant 2$ and $k_d, \ldots, k_1 \geqslant 1$, and consider a centered Gaussian vector*

$$(G_1^1, \ldots, G_{k_1}^1, \ldots, G_1^d, \ldots, G_{k_d}^d).$$

Assume that $E[G_a^i G_b^j] \geqslant 0$ for all $i \neq j$, $a = 1, \ldots, k_i$ and $b = 1, \ldots, k_j$. Then, for any $x_1, \ldots, x_d \in \mathbb{R}$,

$$\begin{aligned} &P\big(\max\{G_1^1, \ldots, G_{k_1}^1\} \leqslant x_1\big) \ldots P\big(\max\{G_1^d, \ldots, G_{k_d}^d\} \leqslant x_d\big) \\ &\leqslant P\Big(\max\{G_1^1, \ldots, G_{k_1}^1\} \leqslant x_1, \ldots, \max\{G_1^d, \ldots, G_{k_d}^d\} \leqslant x_d\Big). \end{aligned}$$

Proof. By reasoning by induction, we suppose without loss of generality that $d = 2$. Let us first change the notation to simplify the exposition. We want to prove that, if $G = (G_1, \ldots, G_{p+q})$ is a given centered Gaussian vector such that

$$E[G_i G_j] \geqslant 0, \quad i = 1, \ldots, p, \quad j = p + 1, \ldots, p + q,$$

and if $\widehat{G} = (\widehat{G}_1, \ldots, \widehat{G}_{p+q})$ is another centered Gaussian vector such that

$$E[\widehat{G}_i \widehat{G}_j] = \begin{cases} E[G_i G_j] & \text{if } (i, j) \in \{1, \ldots, p\}^2 \cup \{p + 1, \ldots, p + q\}^2 \\ 0 & \text{otherwise} \end{cases},$$

then, for all $x, y \in \mathbb{R}$,

$$\begin{aligned} &P\big(\max\{G_1, \ldots, G_p\} \leqslant x\big) P\big(\max\{G_{p+1}, \ldots, G_{p+q}\} \leqslant y\big) \\ &= P\Big(\max\{\widehat{G}_1, \ldots, \widehat{G}_p\} \leqslant x, \max\{\widehat{G}_{p+1}, \ldots, \widehat{G}_{p+q}\} \leqslant y\Big) \\ &\leqslant P\big(\max\{G_1, \ldots, G_p\} \leqslant x, \max\{G_{p+1}, \ldots, G_{p+q}\} \leqslant y\big). \end{aligned}$$

Without loss of generality, let us assume that G and $\widehat{G}$ are independent. Let $\phi(u) = f_1(u_1) \ldots f_{p+q}(u_{p+q})$, $u \in \mathbb{R}^{p+q}$, where each f_i is a positive, decreasing and smooth enough function. For $t \in [0,1]$, set $\varphi(t) = E\big[\phi(\sqrt{t}G + \sqrt{1-t}\widehat{G})\big]$. We have

$$\begin{aligned}
E[\phi(G)] - E[\phi(\widehat{G})] &= \int_0^1 \varphi'(t)dt \\
&= \sum_{i=1}^{p+q} E\left[\frac{\partial\phi}{\partial u_i}(\sqrt{t}G + \sqrt{1-t}\widehat{G})\left(\frac{1}{2\sqrt{t}}G_i - \frac{1}{2\sqrt{1-t}}\widehat{G}_i\right)\right] \\
&= \frac{1}{2}\sum_{i,j=1}^{p+q} E\left[\frac{\partial^2\phi}{\partial u_i \partial u_j}(\sqrt{t}G + \sqrt{1-t}\widehat{G})\right]\Big(E[G_iG_j] - E[\widehat{G}_i\widehat{G}_j]\Big) \\
&= \sum_{i=1}^{p}\sum_{j=p+1}^{p+q} E\left[\frac{\partial^2\phi}{\partial u_i \partial u_j}(\sqrt{t}G + \sqrt{1-t}\widehat{G})\right] E[G_iG_j].
\end{aligned}$$

But when $i \neq j$, we have

$$\frac{\partial^2\phi}{\partial u_i \partial u_j}(u) = f_i'(u_i)f_j'(u_j)\prod_{k\neq i,j} f_k(u_k) \geqslant 0.$$

We deduce that

$$E[\phi(G)] = E\left[\prod_{i=1}^{p+q} f_i(G_i)\right] \geqslant E\left[\prod_{i=1}^{p+q} f_i(\widehat{G}_i)\right] = E[\phi(\widehat{G})],$$

and the conclusion is attained by taking, for each i, a sequence $f_i^{(n)}$ of positive, decreasing, $\mathcal{C}^2$ approximations to the indicator $1_{(-\infty,x]}$ ($i = 1,\dots,p$) or $1_{(-\infty,y]}$ ($i = p+1,\dots,p+q$). □

Corollary 4.2. *Let B^H be a fractional Brownian motion of Hurst index $H \in (\frac{1}{2},1)$. Then, for all $t \geqslant 1$, all $x_1, x_2 \geqslant 0$ and all $x_3 \in \mathbb{R}$,*

$$\begin{aligned}
&P\left(\sup_{0\leqslant s\leqslant 1} B_s^H \leqslant x_1\right) P\left(\sup_{1\leqslant s\leqslant t}(B_s^H - B_1^H) \leqslant x_2\right) P\big(B_1^H \leqslant x_3\big) \\
&\leqslant P\left(\sup_{0\leqslant s\leqslant 1} B_s^H \leqslant x_1,\ \sup_{1\leqslant s\leqslant t}(B_s^H - B_1^H) \leqslant x_2,\ B_1^H \leqslant x_3\right).
\end{aligned}$$

Proof. For any $s \in [0,1]$ and $u \in [1,+\infty)$, we have, using (2.2) since $H > \frac{1}{2}$,

$$E[B_s^H(B_u^H - B_1^H)] = H(2H-1)\int_0^s dx \int_1^u dy(y-x)^{2H-2} \geqslant 0$$

$$E[B_s^H B_1^H] = H(2H-1)\int_0^s dx \int_0^1 dy|y-x|^{2H-2} \geqslant 0.$$

Slepian's Lemma 4.5 then implies that

$$\begin{aligned}
P\left(\sup_{k=1,\dots,2^n} B_{k2^{-n}}^H \leqslant x_1\right) P\left(\sup_{k=1,\dots,2^n} (B_{1+(t-1)k2^{-n}}^H - B_1^H) \leqslant x_2\right) P\left(B_1^H \leqslant x_3\right) \\
\leqslant P\left(\sup_{k=1,\dots,2^n} B_{k2^{-n}}^H \leqslant x_1, \sup_{k=1,\dots,2^n} (B_{1+(t-1)k2^{-n}}^H - B_1^H) \leqslant x_2, B_1^H \leqslant x_3\right).
\end{aligned}$$

A monotone convergence argument as $n \to \infty$ allows one to conclude the proof. □

We are now in a position to prove Lemma 4.4 for $H \in (\frac{1}{2}, 1)$. Fix $t > 1$ and let

$$\phi(s) = 1_{[0,(2\log t)^{1/H})}(s) - 2\log t \, 1_{[(2\log t)^{1/H},t]}(s), \quad s \geqslant 0.$$

We have

$$\begin{aligned}
E\left[\left(\int_0^t e^{B_s^H} ds\right)^{-1}\right] &\geqslant E\left[1_{\{\forall s \leqslant t:\, B_s^H \leqslant \phi(s)\}} \left(\int_0^t e^{B_s^H} ds\right)^{-1}\right] \\
&\geqslant P\left(\forall s \leqslant t : B_s^H \leqslant \phi(s)\right) \left(\int_0^t e^{\phi(s)} ds\right)^{-1}.
\end{aligned}$$

For t large enough, observe that

$$\int_0^t e^{\phi(s)} ds = e(2\log t)^{1/H} + \frac{1}{t} - \frac{(2\log t)^{1/H}}{t^2} \sim_{t\to\infty} e(2\log t)^{1/H},$$

implying in turn that $\int_0^t e^{\phi(s)} ds \leqslant 2e(2\log t)^{1/H}$ for t large enough. Therefore, for t large enough,

$$P\left(\forall s \leqslant t : B_s^H \leqslant \phi(s)\right) \leqslant 2e(2\log t)^{1/H} E\left[\left(\int_0^t e^{B_s^H} ds\right)^{-1}\right]. \tag{4.21}$$

On the other hand, by (4.15) we have that

$$(\log t)^{\frac{1}{H}-1} P\left(M_1^H \leqslant \frac{1}{2\log t}\right) \to \infty \quad \text{as } t \to \infty.$$

Therefore, for t large enough, $(\log t)^{\frac{1}{H}} P\left(M_1^H \leqslant \frac{1}{2\log t}\right) \geqslant \frac{1}{P(B_1^H \leqslant -2)}$, implying in turn:

$$
\begin{aligned}
&(\log t)^{1-\frac{1}{H}} P(M_t^H \leqslant 1) \\
&\quad \leqslant P\left(M_1^H \leqslant \frac{1}{2\log t}\right) P(M_t^H \leqslant 1) P(B_1^H \leqslant -2) \\
&\quad = P\left(\sup_{0\leqslant s\leqslant 1} B_s^H \leqslant \frac{1}{2\log t}\right) P\left(\sup_{1\leqslant s\leqslant t+1} B_{s-1}^H \leqslant 1\right) P(B_1^H \leqslant -2) \\
&\quad = P\left(\sup_{0\leqslant s\leqslant 1} B_s^H \leqslant \frac{1}{2\log t}\right) P\left(\sup_{1\leqslant s\leqslant t+1} (B_s^H - B_1^H) \leqslant 1\right) P(B_1^H \leqslant -2) \\
&\qquad\qquad \text{by stationarity} \\
&\quad \leqslant P\left(\sup_{0\leqslant s\leqslant 1} B_s^H \leqslant \frac{1}{2\log t}, \sup_{1\leqslant s\leqslant t+1} (B_s^H - B_1^H) \leqslant 1,\ B_1^H \leqslant -2\right) \\
&\qquad\qquad \text{by Corollary 4.2} \\
&\quad \leqslant P\left(\sup_{0\leqslant s\leqslant 1} B_s^H \leqslant \frac{1}{2\log t}, \sup_{1\leqslant s\leqslant t} B_s^H \leqslant -1\right) \\
&\quad = P\left(\sup_{0\leqslant s\leqslant (2\log t)^{\frac{1}{H}}} B_s^H \leqslant 1, \sup_{(2\log t)^{\frac{1}{H}}\leqslant s\leqslant t(2\log t)^{\frac{1}{H}}} B_s^H \leqslant -2\log t\right) \\
&\qquad\qquad \text{by selfsimilarity} \\
&\quad \leqslant P\left(\sup_{0\leqslant s\leqslant (2\log t)^{\frac{1}{H}}} B_s^H \leqslant 1, \sup_{(2\log t)^{\frac{1}{H}}\leqslant s\leqslant t} B_s^H \leqslant -2\log t\right) \\
&\quad = P\left(\forall s \leqslant t :\ B_s^H \leqslant \phi(s)\right) \\
&\quad \leqslant 2e(2\log t)^{\frac{1}{H}} E\left[\left(\int_0^t e^{B_s^H} ds\right)^{-1}\right] \quad \text{by (4.21)} \\
&\quad \leqslant c\,(\log t)^{\frac{1}{H}} t^{H-1} \quad \text{by Lemma 4.2, for some } c > 0.
\end{aligned} \tag{4.22}
$$

Using first the selfsimilarity property of B^H and then (4.22), we deduce that

$$P(M_1^H \leqslant x) = P(M_{x^{-H}}^H \leqslant 1) \leqslant c(-\log x)^{\frac{2-H}{H}} x^{\frac{1-H}{H}},$$

for any $x > 0$ small enough, frow which it is immediate that (4.20) holds. □

Chapter 5
Malliavin Calculus in a Nutshell

In this chapter, we introduce the reader to the basic operators of Malliavin calculus. This is because, in the next chapter, we shall use this framework to study the convergence in law of some functionals involving fractional Brownian motion. For the sake of simplicity and to avoid useless technicalities, we only consider the case where the underlying Gaussian process, fixed once for all, is a two-sided classical Brownian motion $W = (W_t)_{t\in\mathbb{R}}$ (see (1.8)) defined on some probability space $(\Omega, \mathcal{F}, P)$; we further assume that the σ-field $\mathcal{F}$ is generated by W.

For a detailed exposition of Malliavin calculus and for missing proofs, we refer the reader to the textbooks [39, 45].

5.1 Itô Stochastic Calculus

In this section, we survey some of the basic properties of the stochastic integral of adapted processes with respect to W, as introduced by Itô. (For a detailed exposition of Itô stochastic calculus and for missing proofs, we refer the reader to the classical textbook [52].)

For each $t \in \mathbb{R}$, let $\mathcal{F}_t$ be the σ-field generated by the random variables $\{W_s,\ s \leqslant t\}$ together with the null sets of $\mathcal{F}$.

Definition 5.1. *A stochastic process $u = (u_t)_{t\in\mathbb{R}}$ is called adapted*[1] *if u_t is $\mathcal{F}_t$-measurable for any $t \in \mathbb{R}$.*

We denote by $L^2(\mathbb{R}\times\Omega) = L^2(\mathbb{R}\times\Omega, \mathcal{B}(\mathbb{R})\otimes\mathcal{F}, \lambda\times P)$ (where λ stands for the Lebesgue measure) the set of square integrable processes, and by $L^2_a(\mathbb{R}\times\Omega)$ the subspace of adapted processes. Let $\mathcal{E}$ be the class of elementary adapted processes,

[1] Any adapted process u that is either càdlàg or càglàd admits a progressively measurable version. We will always assume that we are dealing with it, meaning that the restriction of u to the product $(-\infty, t]\times\Omega$ is $\mathcal{B}((-\infty, t])\otimes\mathcal{F}_t$-measurable for all t.

Nourdin I.: *Selected Aspects of Fractional Brownian Motion*, B&SS,
DOI 10.1007/978-88-470-2823-4_5, © Springer-Verlag Italia 2012

that is, a process u belongs to $\mathcal{E}$ if it has the form

$$u_t = \sum_{i=1}^{n} F_i 1_{[t_i, t_{i+1})}(t) \tag{5.1}$$

where $t_1 < \ldots < t_{n+1}$, and every F_i is an $\mathcal{F}_{t_i}$-measurable and square integrable random variable.

Definition 5.2. *For an adapted process u of the form (5.1), the random variable*

$$\int_{-\infty}^{\infty} u_s dW_s = \sum_{i=1}^{n} F_i \left(W_{t_i+1} - W_{t_i}\right) \tag{5.2}$$

is called the Itô integral of u with respect to W.

The Itô integral of elementary processes is a linear functional that takes values on $L^2(\Omega)$ and has the following basic features, coming mainly from the independence property of the increments of W:

$$E\left[\int_{\mathbb{R}} u_s dW_s\right] = 0 \tag{5.3}$$

$$E\left[\int_{\mathbb{R}} u_s dW_s \times \int_{\mathbb{R}} v_s dW_s\right] = E\left[\int_{\mathbb{R}} u_s v_s ds\right]. \tag{5.4}$$

Thanks to the isometry (5.4), the definition of $\int_{\mathbb{R}} u_s dW_s$ is extended to all adapted process u of $L_a^2(\mathbb{R}\times\Omega)$, and (5.3)–(5.4) continue to hold in this more general setting.

When $u \in L^2(\mathbb{R})$ is deterministic, it is straightforward to show (using (5.2) as well as a characteristic function argument) that

$$\int_{\mathbb{R}} u_s dW_s \sim \mathcal{N}\left(0, \int_{\mathbb{R}} u_s^2 ds\right). \tag{5.5}$$

One of the most important tool in the Itô stochastic calculus is its associated change of variable formula.

Theorem 5.1. *Fix $d \geqslant 1$, let $x_1, \ldots, x_d \in \mathbb{R}$ and let $F : \mathbb{R}^d \to \mathbb{R}$ be a $\mathcal{C}^2$-function. Also, for $i = 1, \ldots, d$, suppose that u_i and v_i are measurable and adapted processes verifying $\int_{-\infty}^{t} u_i(s)^2 ds < \infty$ a.s. and $\int_{-\infty}^{t} |v_i(s)| ds < \infty$ a.s. for every $t \in \mathbb{R}$. Set $X_t^i = x_i + \int_{-\infty}^{t} u_i(s) dW_s + \int_{-\infty}^{t} v_i(s) ds$ for any $i = 1, \ldots, d$. Then*

$$\begin{aligned} F(X_t^1, \ldots, X_t^d) = {} & F(x_1, \ldots, x_d) + \sum_{i=1}^{d} \int_{-\infty}^{t} \frac{\partial F}{\partial x_i}(X_s^1, \ldots, X_s^d) u_i(s) dW_s \\ & + \sum_{i=1}^{d} \int_{-\infty}^{t} \frac{\partial F}{\partial x_i}(X_s^1, \ldots, X_s^d) v_i(s) ds \\ & + \frac{1}{2} \sum_{i,j=1}^{d} \int_{-\infty}^{t} \frac{\partial^2 F}{\partial x_i \partial x_j}(X_s^1, \ldots, X_s^d) u_i(s) u_j(s) ds. \end{aligned} \tag{5.6}$$

5.2 Multiple Wiener–Itô Integrals and Wiener Chaoses

Let $f \in L^2(\mathbb{R}^q)$. Let us see how one could give a 'natural' meaning to the q-fold multiple integral

$$I_q^W(f) = \int_{\mathbb{R}^q}' f(s_1, \ldots, s_q) dW_{s_1} \ldots dW_{s_q},$$

where the prime indicates that one does not integrate over the hyperdiagonals $t_i = t_j$, $i \neq j$. To achieve this goal, we shall use an iterated Itô integral; the following heuristic 'calculations' are thus natural within this framework:

$$\begin{aligned}
&\int_{\mathbb{R}^q}' f(s_1, \ldots, s_q) dW_{s_1} \ldots dW_{s_q} \\
&= \sum_{\sigma \in \mathfrak{S}_q} \int_{\mathbb{R}^q}' f(s_1, \ldots, s_q) 1_{\{s_{\sigma(1)} > \ldots > s_{\sigma(q)}\}} dW_{s_1} \ldots dW_{s_q} \\
&= \sum_{\sigma \in \mathfrak{S}_q} \int_{-\infty}^{\infty} dW_{s_{\sigma(1)}} \int_{-\infty}^{s_{\sigma(1)}} dW_{s_{\sigma(2)}} \ldots \int_{-\infty}^{s_{\sigma(q-1)}} dW_{s_{\sigma(q)}} f(s_1, \ldots, s_q) \\
&= \sum_{\sigma \in \mathfrak{S}_q} \int_{-\infty}^{\infty} dW_{t_1} \int_{-\infty}^{t_1} dW_{t_2} \ldots \int_{-\infty}^{t_{q-1}} dW_{t_q} f(t_{\sigma^{-1}(1)}, \ldots, t_{\sigma^{-1}(q)}) \\
&= \sum_{\sigma \in \mathfrak{S}_q} \int_{-\infty}^{\infty} dW_{t_1} \int_{-\infty}^{t_1} dW_{t_2} \ldots \int_{-\infty}^{t_{q-1}} dW_{t_q} f(t_{\sigma(1)}, \ldots, t_{\sigma(q)}). \qquad (5.7)
\end{aligned}$$

Now, we can use (5.7) as a natural candidate for being $I_q^W(f)$.

Definition 5.3. *Let $q \geqslant 1$ be an integer.*
1. *When $f \in L^2(\mathbb{R}^q)$, we set*

$$I_q^W(f) = \sum_{\sigma \in \mathfrak{S}_q} \int_{-\infty}^{\infty} dW_{t_1} \int_{-\infty}^{t_1} dW_{t_2} \ldots \int_{-\infty}^{t_{q-1}} dW_{t_q} f(t_{\sigma(1)}, \ldots, t_{\sigma(q)}). \qquad (5.8)$$

The random variable $I_q^W(f)$ is called the qth multiple Wiener-Itô integral of f.
2. *The set $\mathcal{H}_q^W$ of random variables of the form $I_q^W(f)$, $f \in L^2(\mathbb{R}^q)$, is called the qth Wiener chaos of W. We also use the convention $\mathcal{H}_0^W = \mathbb{R}$.*

The following properties are readily checked.

Proposition 5.1. *Let $q \geqslant 1$ be an integer and let $f \in L^2(\mathbb{R}^q)$.*
1. *If f is symmetric, then*

$$I_q^W(f) = q! \int_{-\infty}^{\infty} dW_{t_1} \int_{-\infty}^{t_1} dW_{t_2} \ldots \int_{-\infty}^{t_{q-1}} dW_{t_q} \ f(t_1, \ldots, t_q). \qquad (5.9)$$

2. *We have*

$$I_q^W(f) = I_q^W(\widetilde{f}), \qquad (5.10)$$

where $\widetilde{f}$ stands for the symmetrization of f given by

$$\widetilde{f}(t_1,\ldots,t_q) = \frac{1}{q!}\sum_{\sigma\in\mathfrak{S}_q} f(t_{\sigma(1)},\ldots,t_{\sigma(q)}). \tag{5.11}$$

3. *Let $D_q \subset \mathbb{R}^q$ be the collection of the hyperdiagonals of $\mathbb{R}^q$, i.e.*

$$D_q = \{(t_1,\ldots,t_q)\in\mathbb{R}^q : t_i = t_j \text{ for some } i\neq j\}.$$

When $[a_1,b_1[\times\ldots\times[a_q,b_q[\cap D_q = \emptyset$ (that is, when $[a_1,b_1[,\ldots,[a_q,b_q[$ are disjoint intervals of $\mathbb{R}$), we have

$$I_q^W(1_{]a_1,b_1]}\otimes\ldots\otimes 1_{]a_q,b_q]}) = \big(W_{b_1}-W_{a_1}\big)\ldots\big(W_{b_q}-W_{a_q}\big). \tag{5.12}$$

(In (5.12) and throughout the book, we write $f_1\otimes\ldots\otimes f_q$ to indicate the tensor product of $f_1,\ldots,f_q$, defined by $(f_1\otimes\ldots\otimes f_q)(t_1,\ldots,t_q) = f_1(t_1)\ldots f_q(t_q)$.)
4. *For any $p,q\geqslant 1$, $f\in L^2(\mathbb{R}^p)$ and $g\in L^2(\mathbb{R}^q)$,*

$$E[I_q^W(f)] = 0 \tag{5.13}$$
$$E[I_p^W(f)I_q^W(g)] = p!\langle\widetilde{f},\widetilde{g}\rangle_{L^2(\mathbb{R}^p)} \quad \text{if } p=q \tag{5.14}$$
$$E[I_p^W(f)I_q^W(g)] = 0 \quad \text{if } p\neq q. \tag{5.15}$$

Multiple Wiener-Itô integrals and Hermite polynomials are intimately connected.

Proposition 5.2. *Let $e\in L^2(\mathbb{R})$ have norm 1, let $q\geqslant 1$ be an integer, and recall the Definition 1.4 of Hermite polynomials. Then*

$$H_q\left(\int_{\mathbb{R}} e(s)dW_s\right) = I_q^W(e^{\otimes q}). \tag{5.16}$$

Proof. Let $t\in\mathbb{R}$ and, for any $x\in\mathbb{R}$ and $a\geqslant 0$, set $h_q(x,a) = a^{q/2}H_q(x/\sqrt{a})$ if $a\neq 0$ and $h_q(x,0) = x^q$. Using Proposition 1.3(1), it is readily checked that $\left(\frac{1}{2}\frac{\partial^2}{\partial x^2}+\frac{\partial}{\partial a}\right)h_q = 0$ and $\frac{\partial}{\partial x}h_q = qh_{q-1}$. Itô's formula (5.6) implies then that

$$\begin{aligned}
&h_q\left(\int_{-\infty}^t e(u)dW_u, \int_{-\infty}^t e^2(u)du\right)\\
&= q\int_{-\infty}^t dW_{t_1}\, e(t_1)\, h_{q-1}\left(\int_{-\infty}^{t_1} e(u)dW_u, \int_{-\infty}^{t_1} e^2(u)du\right)\\
&= \ldots\\
&= q!\int_{-\infty}^t dW_{t_1}\, e(t_1)\int_{-\infty}^{t_1} dW_{t_2}\, e(t_2)\ldots\int_{-\infty}^{t_{q-2}} dW_{t_{q-1}} e(t_{q-1})\\
&\qquad\qquad \times h_1\left(\int_{-\infty}^{t_{q-1}} e(u)dW_u, \int_{-\infty}^{t_{q-1}} e^2(u)du\right)\\
&= q!\int_{-\infty}^t dW_{t_1}\, e(t_1)\int_{-\infty}^{t_1} dW_{t_2}\, e(t_2)\ldots\int_{-\infty}^{t_{q-1}} dW_{t_q} e(t_q).
\end{aligned}$$

By choosing $t = \infty$ and because $\int_{\mathbb{R}} e^2(u)du = 1$, we get

$$H_q\left(\int_{\mathbb{R}} e(u)dW_u\right) = I_q^W(e^{\otimes q})$$

and the proof of (5.16) is complete. □

The space $L^2(\Omega)$ can be decomposed into the infinite orthogonal sum of the spaces $\mathcal{H}_q^W$. (It is precisely here that we need to assume that the σ-field $\mathcal{F}$ is generated by W.) It follows that any square-integrable random variable $F \in L^2(\Omega)$ admits the following chaotic expansion:

$$F = E[F] + \sum_{q=1}^{\infty} I_q^W(f_q), \tag{5.17}$$

where the functions $f_q \in L^2(\mathbb{R}^q)$ are symmetric and uniquely determined by F.

The following result contains a very useful property of Wiener multiple integrals (the so-called *hypercontractivity* property) which states that all the $L^r(\Omega)$-norms are equivalent for multiple Wiener-Itô integrals of a given order.

Theorem 5.2 (Nelson [32]). *Let $f \in L^2(\mathbb{R}^q)$ with $q \geqslant 1$. Then, for all $r \in [2, +\infty)$,*

$$E\big[|I_q^W(f)|^r\big] \leqslant [(r-1)^q q!]^{r/2} \|f\|_{L^2(\mathbb{R}^q)}^r < \infty. \tag{5.18}$$

Proof. See, e.g., [39, Corollary 2.8.14]. □

As an application, let us deduce from Theorem 5.2 that the law of any random vector composed of either single or double Wiener-Itô integrals is determined by its joint moments. It is an interesting result (that we shall use in the proof of Theorem 7.3) because the laws of multiple integrals of order strictly greater than two are, in general, not determined by their moments. See Slud [59] or Janson [24, Chapter VI] for a complete picture.

Proposition 5.3. *Let $e_1, \ldots, e_r$ be functions in $L^2(\mathbb{R})$ and $f_1, \ldots, f_d$ be symmetric functions in $L^2(\mathbb{R}^2)$. Then, the law of the random vector*

$$(Z_1, \ldots, Z_{r+d}) := (I_1^W(e_1), \ldots, I_W^1(e_r), I_2^W(f_1), \ldots, I_2^W(f_d))$$

is determined by its joint moments. That is, if $(Y_1, \ldots, Y_{r+d})$ is a random vector satisfying

$$E[Y_1^{n_1} \ldots Y_{r+d}^{n_{r+d}}] = E[Z_1^{n_1} \ldots Z_{r+d}^{n_{r+d}}]$$

for each choice of $n_1, \ldots, n_{r+d} \in \mathbb{N}$, then

$$(Y_1, \ldots, Y_{r+d}) \overset{\text{law}}{=} (Z_1, \ldots, Z_{r+d}).$$

Proof. Since

$$E[e^{t|I_1^W(e)|}] = \frac{1}{\sqrt{2\pi}\|e\|}\int_{\mathbb{R}} e^{t|x|}e^{-\frac{x^2}{2\|e\|^2}}dx < \infty \tag{5.19}$$

for any $e \in L^2(\mathbb{R}) \setminus \{0\}$ and $t > 0$, it is sufficient to show that, for any symmetric function $f \in L^2(\mathbb{R}^2)$, there exists $t > 0$ such that

$$E[e^{t|I_2^W(f)|}] < \infty. \tag{5.20}$$

(It is indeed a routine exercise to deduce the desired conclusion from (5.19)-(5.20).) To do so, without loss of generality we may and will assume that $2\|f\|_{L^2(\mathbb{R}^2)} = E[I_2^W(f)^2] = 1$, so that (5.18) implies that, for every $r \geqslant 2$,

$$E[|I_2^W(f)|^r]^{1/r} \leqslant r - 1,$$

implying in turn that $P(|I_2^W(f)| > u) \leqslant u^{-r}(r-1)^r$ for every $u > 0$. Choosing $r = r(u) = 1 + u/e$, the previous relation shows that $P(|I_2^W(f)| > u) \leqslant e^{-u/e}$, for every $u > e$. By a Fubini argument,

$$E[e^{t|I_2^W(f)|}] = 1 + t\int_0^\infty e^{tu}P\big(|I_2^W(f)| > u\big)du,$$

hence $E[e^{t|I_2^W(f)|}] < \infty$ for every $t \in [0, 1/e)$. The proof is concluded. □

Another interesting application of Theorem 5.2 is the following result.

Proposition 5.4. *Let $(f_n)_{n\geqslant 1}$ be a sequence of non-zero symmetric elements of $L^2(\mathbb{R}^q)$. If the sequence $(I_q^W(f_n))_{n\geqslant 1}$ converges in law, then*

$$\sup_{n\geqslant 1} E\big[|I_q^W(f_n)|^r\big] < \infty \quad \textit{for every } r > 0. \tag{5.21}$$

Proof. Recall the Paley's inequality: if F is a positive random variable with $E[F] = 1$ and if $\theta \in (0, 1)$, then

$$E[F^2]\,P(F > \theta) \geqslant (1-\theta)^2. \tag{5.22}$$

(The proof of (5.22) is easy: consider the decomposition $F = F1_{\{F>\theta\}} + F1_{\{F\leqslant\theta\}}$, take the expectation and use Cauchy-Schwarz to deduce that $1 \leqslant \sqrt{E[F^2]}\sqrt{P(F > \theta)} + \theta$.) Combining (5.18) for $r = 4$ with (5.22), we get for every $\theta \in (0, 1)$ and with $F = I_q^W(f_n)^2/E[I_q^W(f_n)^2]$,

$$P\big(I_q^W(f_n)^2 > \theta E[I_q^W(f_n)^2]\big) \geqslant (1-\theta)^2\frac{\big(E[I_q^W(f_n)^2]\big)^2}{E[I_q^W(f_n)^4]} \geqslant (1-\theta)^2 9^{-q}. \tag{5.23}$$

The sequence $(I_q^W(f_n))_{n\geqslant 1}$ converging in law, it is tight and we can choose $M > 0$ large enough such that $P(I_q^W(f_n)^2 > M) < 9^{-q-1}$ for all $n \geqslant 1$. On the other hand, by (5.23) with $\theta = 2/3$ and all n, we have

$$P\big(I_q^W(f_n)^2 > M\big) < 9^{-q-1} \leqslant P\big(I_q^W(f_n)^2 > \frac{2}{3}E[I_q^W(f_n)^2]\big).$$

As a consequence, $\sup_{n\geqslant 1} E[I_q^W(f_n)^2] \leqslant \frac{3M}{2}$. Applying (5.18) we conclude (5.21). □

Multiple Wiener-Itô integrals are linear by construction. Let us see how they behave with respect to multiplication. To this aim, we need to introduce the concept of *contractions*.

Definition 5.4. *When $r \in \{1, \dots, p \wedge q\}$, $f \in L^2(\mathbb{R}^p)$ and $g \in L^2(\mathbb{R}^q)$, we write $f \otimes_r g$ to indicate the rth contraction of f and g, defined as being the element of $L^2(\mathbb{R}^{p+q-2r})$ given by*

$$(f \otimes_r g)(t_1, \dots, t_{p+q-2r}) \tag{5.24}$$
$$= \int_{\mathbb{R}^r} f(t_1, \dots, t_{p-r}, x_1, \dots, x_r) g(t_{p-r+1}, \dots, t_{p+q-2r}, x_1, \dots, x_r) \times dx_1 \dots dx_r.$$

By convention, we set $f \otimes_0 g = f \otimes g$ as being the tensor product of f and g, that is, $(f \otimes g)(t_1, \dots, t_{p+q}) = f(t_1, \dots, t_p) g(t_{p+1}, \dots, t_{p+q})$.

Observe that

$$\|f \otimes_r g\|_{L^2(\mathbb{R}^{p+q-2r})} \leqslant \|f\|_{L^2(\mathbb{R}^p)} \|g\|_{L^2(\mathbb{R}^q)}, \quad r = 0, \dots, p \wedge q \tag{5.25}$$

by Cauchy-Schwarz, and that $f \otimes_p g = \langle f, g \rangle_{L^2(\mathbb{R}^p)}$ when $p = q$. The next result is the *product formula* between multiple Wiener-Itô integrals.

Theorem 5.3. *Let $p, q \geqslant 1$ and let $f \in L^2(\mathbb{R}^p)$ and $g \in L^2(\mathbb{R}^q)$ be two symmetric functions. Then*

$$I_p^W(f) I_q^W(g) = \sum_{r=0}^{p \wedge q} r! \binom{p}{r} \binom{q}{r} I_{p+q-2r}^W(f \widetilde{\otimes}_r g), \tag{5.26}$$

where $f \widetilde{\otimes}_r g$ stands for the symmetrization of $f \otimes_r g$ (see (5.11)).

Proof. See, e.g., [45, page 12]. □

5.3 Malliavin Derivatives

Let $F \in L^2(\Omega)$ and consider its chaotic expansion (5.17).

Definition 5.5. 1. *When $m \geqslant 1$ is an integer, we say that F belongs to the Sobolev-Watanabe space $\mathbb{D}^{m,2}$ if*

$$\sum_{q=1}^{\infty} q^m q! \|f_q\|_{L^2(\mathbb{R}^q)}^2 < \infty. \tag{5.27}$$

2. *When (5.27) holds with $m = 1$, the Malliavin derivative $DF = (D_t F)_{t\in\mathbb{R}}$ of F is the element of $L^2(\Omega \times \mathbb{R})$ given by*

$$D_t F = \sum_{q=1}^{\infty} q I_{q-1}^W \left(f_q(\cdot, t)\right). \tag{5.28}$$

It is clear by construction that D is a linear operator. Using the orthogonality and the isometry properties of multiple Wiener-Itô integrals, it is easy to compute the L^2-norm of DF in terms of the kernels f_q appearing in the chaotic expansion (5.17) of F:

Proposition 5.5. *Let $F \in \mathbb{D}^{1,2}$. We have*

$$E\left[\|DF\|^2_{L^2(\mathbb{R})}\right] = \sum_{q=1}^{\infty} q q! \|f_q\|^2_{L^2(\mathbb{R}^q)}.$$

Proof. By (5.28), we can write

$$\begin{aligned} E\left[\|DF\|^2_{L^2(\mathbb{R})}\right] &= \int_{\mathbb{R}} E\left[\left(\sum_{q=1}^{\infty} q I_{q-1}^W \left(f_q(\cdot, t)\right)\right)^2\right] dt \\ &= \sum_{p,q=1}^{\infty} pq \int_{\mathbb{R}} E\left[I_p^W \left(f_p(\cdot, t)\right) I_{q-m}^W \left(f_q(\cdot, t)\right)\right] dt. \end{aligned}$$

Using (5.15), we deduce that

$$E\left[\|DF\|^2_{L^2(\mathbb{R})}\right] = \sum_{q=1}^{\infty} q^2 \int_{\mathbb{R}} E\left[I_{q-1}^W \left(f_q(\cdot, t)\right)^2\right] dt.$$

Finally, using (5.14), we get that

$$E\left[\|DF\|^2_{L^2(\mathbb{R})}\right] = \sum_{q=1}^{\infty} q^2 (q-1)! \int_{\mathbb{R}} \|f_q(\cdot, t)\|^2_{L^2(\mathbb{R}^{q-1})} dt = \sum_{q=1}^{\infty} q q! \|f_q\|^2_{L^2(\mathbb{R}^q)}.$$

□

Let H_q denote the qth Hermite polynomial (for some $q \geqslant 1$) and let $e \in L^2(\mathbb{R})$ have norm 1. Recall (5.16) and Proposition 1.3(1). We deduce that, for any $t \geqslant 0$,

$$\begin{aligned} D_t\left(H_q\left(\int_{\mathbb{R}} e(s) dW_s\right)\right) &= D_t(I_q^W(e^{\otimes q})) = q I_{q-1}^W (e^{\otimes q-1}) e(t) \\ &= q H_{q-1}\left(\int_{\mathbb{R}} e(s) dW_s\right) e(t) = H_q'\left(\int_{\mathbb{R}} e(s) dW_s\right) D_t\left(\int_{\mathbb{R}} e(s) dW_s\right). \end{aligned}$$

More generally, the Malliavin derivative D verifies the *chain rule*:

Theorem 5.4. *Let $\varphi : \mathbb{R} \to \mathbb{R}$ be both of class $\mathcal{C}^1$ and Lipschitz, and let $F \in \mathbb{D}^{1,2}$. Then, $\varphi(F) \in \mathbb{D}^{1,2}$ and*

$$D_t \varphi(F) = \varphi'(F) D_t F, \quad t \in \mathbb{R}. \tag{5.29}$$

Proof. See, e.g., [45, Proposition 1.2.3]. □

5.4 Generator of the Ornstein–Uhlenbeck Semigroup

Recall the definition (5.27) of the Sobolev–Watanabe spaces $\mathbb{D}^{m,2}$, $m \geqslant 1$, and that the symmetric kernels $f_q \in L^2(\mathbb{R}^q)$ are uniquely defined through (5.17).

Definition 5.6. 1. *The generator of the Ornstein–Uhlenbeck semigroup is the linear operator L defined on $\mathbb{D}^{2,2}$ by*

$$LF = -\sum_{q=0}^{\infty} q I_q^W(f_q).$$

2. *The pseudo-inverse of L is the linear operator L^{-1} defined on $L^2(\Omega)$ by*

$$L^{-1}F = -\sum_{q=1}^{\infty} \frac{1}{q} I_q^W(f_q).$$

It is obvious that, for any $F \in L^2(\Omega)$, we have that $L^{-1}F \in \mathbb{D}^{2,2}$ and

$$LL^{-1}F = F - E[F]. \tag{5.30}$$

Our terminology for L^{-1} is explained by the identity (5.30). Another crucial property of L is contained in the following result.

Proposition 5.6. *Let $F \in \mathbb{D}^{2,2}$ and $G \in \mathbb{D}^{1,2}$. Then*

$$E[LF \times G] = -E[\langle DF, DG\rangle_{L^2(\mathbb{R})}]. \tag{5.31}$$

Proof. By bilinearity and approximation, it is enough to show (5.31) for $F = I_p^W(f)$ and $G = I_q^W(g)$ with $p, q \geqslant 1$ and $f \in L^2(\mathbb{R}_+^p)$, $g \in L^2(\mathbb{R}_+^q)$ symmetric. When $p \neq q$, we have

$$E[LF \times G] = -pE[I_p^W(f) I_q^W(g)] = 0$$

and

$$E[\langle DF, DG\rangle_{L^2(\mathbb{R})}] = pq \int_0^\infty E[I_{p-1}^W(f(\cdot,t)) I_{q-1}^W(g(\cdot,t))]dt = 0$$

by (5.15), so the desired conclusion holds true in this case. When $p = q$, we have

$$E[LF \times G] = -qE[I_q^W(f) I_q^W(g)] = 0 = -qq!\langle f, g\rangle_{L^2(\mathbb{R}^q)}$$

and

$$\begin{aligned} &E[\langle DF, DG\rangle_{L^2(\mathbb{R})}] \\ &= q^2 \int_0^\infty E[I_{q-1}^W(f(\cdot,t)) I_{q-1}^W(g(\cdot,t))]dt \\ &= q^2(q-1)! \int_{\mathbb{R}} \langle f(\cdot,t), g(\cdot,t)\rangle_{L^2(\mathbb{R}^{p-1})} dt = qq!\langle f, g\rangle_{L^2(\mathbb{R}^q)} \end{aligned}$$

by (5.14), so the desired conclusion holds true also in this case. □

We are now in a position to state and prove an integration by parts formula which will play a crucial role in the sequel.

Theorem 5.5. *Let $\varphi : \mathbb{R} \to \mathbb{R}$ be both of class $\mathcal{C}^1$ and Lipschitz, and let $F \in \mathbb{D}^{1,2}$ be such that $E[F] = 0$. Then*

$$E\big[F\varphi(F)\big] = E\big[\varphi'(F)\langle DF, -DL^{-1}F\rangle_{L^2(\mathbb{R})}\big]. \tag{5.32}$$

Proof. Using the assumptions made on F and φ, we can write:

$$\begin{aligned} E\big[F\varphi(F)\big] &= E\big[L(L^{-1}F)\varphi(F)\big] \quad \text{(by (5.30))} \\ &= E\left[\langle D\varphi(F), -DL^{-1}F\rangle_{L^2(\mathbb{R})}\right] \quad \text{(by (5.31))} \\ &= E\left[\varphi'(F)\langle D\varphi(F), -DL^{-1}F\rangle_{L^2(\mathbb{R})}\right] \quad \text{(by (5.29))}, \end{aligned}$$

which is the announced formula. □

In (5.32), the random variable $\langle DF, -DL^{-1}F\rangle_{L^2(\mathbb{R})}$ belongs to $L^1(\Omega)$ since $F \in \mathbb{D}^{1,2}$. Indeed,

$$\begin{aligned} E\big[|\langle DF, -DL^{-1}F\rangle_{L^2(\mathbb{R})}|\big] &\leqslant \sqrt{E[\|DF\|^2_{L^2(\mathbb{R})}]}\,\sqrt{E[\|DL^{-1}F\|^2_{L^2(\mathbb{R})}]} \\ &= \sqrt{\sum_{q=1}^{\infty} qq!\|f_q\|^2_{L^2(\mathbb{R}^q)}}\,\sqrt{\sum_{q=1}^{\infty}(q-1)!\|f_q\|^2_{L^2(\mathbb{R}^q)}} \\ &\leqslant \sum_{q=1}^{\infty} qq!\|f_q\|^2_{L^2(\mathbb{R}^q)} = E[\|DF\|^2_{L^2(\mathbb{R})}] < \infty. \end{aligned} \tag{5.33}$$

Theorem 5.5 admits a useful extension to indicator functions. Before stating and proving it, we recall the following classical result from measure theory.

Proposition 5.7. *Let B be a Borel set in $\mathbb{R}$, assume that $B \subset [-A, A]$ for some $A > 0$, and let μ be a finite measure on $[-A, A]$. Then, there exists a sequence (h_n) of continuous functions with support included in $[-A, A]$ and such that $h_n(x) \in [0, 1]$ and $1_B(x) = \lim_{n\to\infty} h_n(x)$ μ-a.e.*

Proof. This is an immediate corollary of Lusin's theorem, see e.g. [54, page 56]. □

Corollary 5.1. *Let B be a Borel set in $\mathbb{R}$, assume that $B \subset [-A, A]$ for some $A > 0$, and let $F \in \mathbb{D}^{1,2}$ be such that $E[F] = 0$. Then*

$$E\left[F\int_{-\infty}^{F} 1_B(x)dx\right] = E\big[1_B(F)\langle DF, -DL^{-1}F\rangle_{L^2(\mathbb{R})}\big].$$

Proof. Let λ denote the Lebesgue measure and let P_F denote the law of F. By Proposition 5.7 with $\mu = (\lambda + P_F)|_{[-A,A]}$ (that is, μ is the restriction of $\lambda + P_F$ to $[-A, A]$), there is a sequence (h_n) of continuous functions with support included in

$[-A, A]$ and such that $h_n(x) \in [0,1]$ and $1_B(x) = \lim_{n\to\infty} h_n(x)$ μ-a.e. In particular, $1_B(x) = \lim_{n\to\infty} h_n(x)$ λ-a.e. and P_F-a.e. By Theorem 5.5, we have moreover that

$$E\left[F\int_{-\infty}^{F} h_n(x)dx\right] = E\big[h_n(F)\langle DF, -DL^{-1}F\rangle_{L^2(\mathbb{R})}\big].$$

The dominated convergence applies and yields the desired conclusion. □

As a corollary of both Theorem 5.5 and Corollary 5.1, we shall prove that the law of any multiple Wiener-Itô integral is always absolutely continuous with respect to the Lebesgue measure except, of course, when its kernel is identically zero. This result was obtained by Shigekawa in [57].

Corollary 5.2 (Shigekawa). *Let $q \geqslant 1$ be an integer and let f be a non zero element of $L^2(\mathbb{R}^q)$. Then the law of $F = I_q^W(f)$ is absolutely continuous with respect to the Lebesgue measure.*

Proof. Without loss of generality, we further assume that f is symmetric. The proof is by induction on q. When $q = 1$, the desired property is readily checked because $I_1^W(f) \sim \mathcal{N}(0, \|f\|^2_{L^2(\mathbb{R})})$. Now, let $q \geqslant 2$ and assume that the statement of Corollary 5.2 holds true for $q-1$, that is, assume that the law of $I_{q-1}^W(g)$ is absolutely continuous for any symmetric element g of $L^2(\mathbb{R}^{q-1})$ such that $\|g\|_{L^2(\mathbb{R}^{q-1})} > 0$. Let f be a symmetric element of $L^2(\mathbb{R}^q)$ with $\|f\|_{L^2(\mathbb{R}^q)} > 0$. Let $h \in L^2(\mathbb{R})$ be such that $\left\|\int_0^\infty f(\cdot,s)h(s)ds\right\|_{L^2(\mathbb{R}^{q-1})} \neq 0$. (Such an h necessarily exists because, otherwise, we would have that $f(\cdot,s) = 0$ for almost all $s \geqslant 0$ which, by symmetry, would imply that $f \equiv 0$; this would be in contradiction with our assumption.) Using the induction assumption, we have that the law of

$$\langle DF, h\rangle_{L^2(\mathbb{R})} = \int_{\mathbb{R}} D_sF\, h(s)ds = qI_{q-1}^W\left(\int_{\mathbb{R}} f(\cdot,s)h(s)ds\right)$$

is absolutely continuous with respect to the Lebesgue measure. In particular,

$$P(\langle DF, h\rangle_{L^2(\mathbb{R})} = 0) = 0,$$

implying in turn, because $\{\|DF\|_{L^2(\mathbb{R})} = 0\} \subset \{\langle DF, h\rangle_{L^2(\mathbb{R})} = 0\}$, that

$$P(\|DF\|_{L^2(\mathbb{R})} > 0) = 1. \tag{5.34}$$

Now, let B be a Borel set in $\mathbb{R}$. Using Corollary 5.1, we can write, for every $n \geqslant 1$,

$$\begin{aligned} E\left[1_{B\cap[-n,n]}(F)\frac{1}{q}\|DF\|^2_{L^2(\mathbb{R})}\right] &= E\left[1_{B\cap[-n,n]}(F)\langle DF, -DL^{-1}F\rangle_{L^2(\mathbb{R})}\right] \\ &= E\left[F\int_{-\infty}^{F} 1_{B\cap[-n,n]}(y)dy\right]. \end{aligned}$$

Assume that the Lebesgue measure of B is zero. The previous equality implies that

$$E\left[1_{B\cap[-n,n]}(F)\frac{1}{q}\|DF\|^2_{L^2(\mathbb{R})}\right] = 0, \quad n \geqslant 1.$$

But (5.34) holds as well, so $P(F \in B \cap [-n, n]) = 0$ for all $n \geqslant 1$. By monotone convergence, we actually get $P(F \in B) = 0$. This shows that the law of F is absolutely continuous with respect to the Lebesgue measure. The proof of Corollary 5.2 is concluded. □

Chapter 6
Central Limit Theorem on the Wiener Space

A widely acclaimed achievement of probability has been its success in approximating the distributions of arbitrarily complicated random variables in terms of a rather small number of 'universal' distributions. Central limit theorem, which proves convergence to the Gaussian law, is the best known among this type of results. However, for practical purposes, it is much more important to know how accurate such an approximation is, and this is of course a more difficult question to answer. For instance, central limit theorem was known already around 1715 (and in full generality by 1900), whereas the corresponding approximation theorem of Berry and Esseen was only proved in 1941. Stein's method, introduced in 1972 in [60], offers a general means of solving such problems.

The goal of this chapter (which, in a sense, is a summary of the content of the book [39]) is to explain how to combine Stein's method with Malliavin calculus in order to assess the distance between the laws of regular Brownian functionals and a one-dimensional Gaussian distribution. Notably, we deduce a complete characterization of Gaussian approximations inside a fixed Wiener chaos, which is systematically stronger than the popular method of moments.

In Section 6.4, one uses the approach developed in this chapter to study the asymptotic behavior of the quadratic variation of fBm.

6.1 Stein's Lemma for Gaussian Approximations

We start by introducing the distance we shall use to measure the closeness of the laws of random variables.

Definition 6.1. *The total variation distance between the laws of two real-valued random variables Y and Z is defined by*

$$d_{TV}(Y,Z) = \sup_{B\in\mathscr{B}(\mathbb{R})} \big|P(Y\in B) - P(Z\in B)\big|, \tag{6.1}$$

where $\mathscr{B}(\mathbb{R})$ stands for the set of Borel sets in $\mathbb{R}$.

Nourdin I.: *Selected Aspects of Fractional Brownian Motion*, B&SS,
DOI 10.1007/978-88-470-2823-4_6,

When $B \in \mathscr{B}(\mathbb{R})$, we have that $P(Y \in B \cap [-n,n]) \to P(Y \in B)$ and $P(Z \in B \cap [-n,n]) \to P(Z \in B)$ as $n \to \infty$ by the monotone convergence theorem. So, without loss we may restrict the supremum in (6.1) to be taken over *bounded* Borel sets, that is,

$$d_{TV}(Y,Z) = \sup_{\substack{B \in \mathscr{B}(\mathbb{R}) \\ B \text{ bounded}}} \big|P(Y \in B) - P(Z \in B)\big|. \tag{6.2}$$

The following statement contains all the elements of Stein's method that are needed for our discussion. For more details and the heuristic behind the method, one can consult the recent books [8, 39] and the references therein.

Lemma 6.1 (Stein). *Let $N \sim \mathscr{N}(0,1)$ be a standard Gaussian random variable. Let $h : \mathbb{R} \to [0,1]$ be any continuous function. Define $f_h : \mathbb{R} \to \mathbb{R}$ by*

$$f_h(x) = e^{\frac{x^2}{2}} \int_{-\infty}^{x} \big(h(a) - E[h(N)]\big) e^{-\frac{a^2}{2}} da \tag{6.3}$$

$$= -e^{\frac{x^2}{2}} \int_{x}^{\infty} \big(h(a) - E[h(N)]\big) e^{-\frac{a^2}{2}} da. \tag{6.4}$$

Then f_h is of class $\mathscr{C}^1$, and satisfies $|xf_h(x)| \leqslant 1$, $|f_h'(x)| \leqslant 2$ and

$$f_h'(x) = xf_h(x) + h(x) - E[h(N)] \tag{6.5}$$

for all $x \in \mathbb{R}$.

Proof. The equality between (6.3) and (6.4) comes from

$$0 = E\big[h(N) - E[h(N)]\big]$$
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \big(h(a) - E[h(N)]\big) e^{-\frac{a^2}{2}} da.$$

Using (6.4) we have, for $x \geqslant 0$:

$$\big|xf_h(x)\big| = \left| xe^{\frac{x^2}{2}} \int_{x}^{+\infty} \big(h(a) - E[h(N)]\big) e^{-\frac{a^2}{2}} da \right|$$
$$\leqslant xe^{\frac{x^2}{2}} \int_{x}^{+\infty} e^{-\frac{a^2}{2}} da \leqslant e^{\frac{x^2}{2}} \int_{x}^{+\infty} ae^{-\frac{a^2}{2}} da = 1.$$

Using (6.3) we have, for $x \leqslant 0$:

$$\big|xf_h(x)\big| = \left| xe^{\frac{x^2}{2}} \int_{-\infty}^{x} \big(h(a) - E[h(N)]\big) e^{-\frac{a^2}{2}} da \right|$$
$$\leqslant |x|e^{\frac{x^2}{2}} \int_{|x|}^{+\infty} e^{-\frac{a^2}{2}} da \leqslant e^{\frac{x^2}{2}} \int_{|x|}^{+\infty} ae^{-\frac{a^2}{2}} da = 1.$$

The identity (6.5) is readily checked to be true. In particular, we deduce that

$$|f_h'(x)| \leqslant |xf_h(x)| + |h(x) - E[h(N)]| \leqslant 2$$

for all $x \in \mathbb{R}$. The proof of the lemma is complete. □

6.2 Combining Stein's Method with Malliavin Calculus

We now derive a bound for the Gaussian approximation of a centered Malliavin-differentiable random variable, following the approach initiated in [38].

Theorem 6.1 (Nourdin–Peccati). *Let W be a two-sided classical Brownian motion defined on some probability space $(\Omega, \mathcal{F}, P)$, and assume further that the σ-field $\mathcal{F}$ is generated by W. Let the notation of Chapter 5 prevail. Consider $F \in \mathbb{D}^{1,2}$ with $E[F] = 0$. Then, with $N \sim \mathcal{N}(0,1)$,*

$$d_{TV}(F, N) \leqslant 2\, E\left[\left|1 - \langle DF, -DL^{-1}F\rangle_{L^2(\mathbb{R})}\right|\right]. \tag{6.6}$$

Proof. Let B be a bounded Borel set in $\mathbb{R}$. Let $A > 0$ be such that $B \subset [-A, A]$. Let λ denote the Lebesgue measure and let P_F denote the law of F. By Proposition 5.7 with $\mu = (\lambda + P_F)|_{[-A-,A]}$ (that is, μ is the restriction of $\lambda + P_F$ to $[-A, A]$), there is a sequence (h_n) of continuous functions such that $h_n(x) \in [0,1]$ and $1_B(x) = \lim_{n\to\infty} h_n(x)$ μ-a.e. By the dominated convergence theorem, $E[h_n(F)] \to P(F \in B)$ and $E[h_n(N)] \to P(N \in B)$ as $n \to \infty$. On the other hand, using Lemma 6.1 as well as (5.32) we can write, for each n,

$$\begin{aligned}\big|E[h_n(F)] - E[h_n(N)]\big| &= \big|E[f'_{h_n}(F)] - E[F f_{h_n}(F)]\big| \\ &= \big|E[f'_{h_n}(F)(1 - \langle DF, -DL^{-1}F\rangle_{L^2(\mathbb{R})}]\\ &\leqslant 2\,E\big[|1 - \langle DF, -DL^{-1}F\rangle_{L^2(\mathbb{R})}|\big].\end{aligned}$$

Letting n goes to infinity yields

$$\big|P(F \in B) - P(N \in B)\big| \leqslant 2\,E\big[|1 - \langle DF, -DL^{-1}F\rangle_{L^2(\mathbb{R})}|\big],$$

which, together with (6.2), implies the desired conclusion. □

6.3 Wiener Chaos and the Fourth Moment Theorem

In this section, we apply Theorem 6.1 to chaotic random variables, that is, to random variables having the specific form of a multiple Wiener-Itô integral. We begin with a technical lemma.

Lemma 6.2. *Let $q \geqslant 1$ be an integer and consider a symmetric function $f \in L^2(\mathbb{R}^q)$. Set $F = I_q^W(f)$ and $\sigma^2 = E[F^2] = q!\|f\|^2_{L^2(\mathbb{R}^q)}$. The following two identities hold:*

$$E\left[\left(\sigma^2 - \frac{1}{q}\|DF\|^2_{L^2(\mathbb{R})}\right)^2\right] = \sum_{r=1}^{q-1} \frac{r^2}{q^2}\, r!^2 \binom{q}{r}^4 (2q-2r)!\|f \widetilde{\otimes}_r f\|^2_{L^2(\mathbb{R}^{2q-2r})} \tag{6.7}$$

and

$$E[F^4]-3\sigma^4 = \frac{3}{q}\sum_{r=1}^{q-1} r r!^2 \binom{q}{r}^4 (2q-2r)!\|f\widetilde{\otimes}_r f\|^2_{L^2(\mathbb{R}^{2q-2r})} \tag{6.8}$$

$$= \sum_{r=1}^{q-1} q!^2\binom{q}{r}^2\left\{\|f\otimes_r f\|^2_{L^2(\mathbb{R}^{2q-2r})} + \binom{2q-2r}{q-r}\|f\widetilde{\otimes}_r f\|^2_{L^2(\mathbb{R}^{2q-2r})}\right\}. \tag{6.9}$$

In particular,

$$E\left[\left(\sigma^2-\frac{1}{q}\|DF\|^2_{L^2(\mathbb{R})}\right)^2\right] \leqslant \frac{q-1}{3q}\big(E[F^4]-3\sigma^4\big). \tag{6.10}$$

Proof. For any $t\in\mathbb{R}$, we have $D_tF = qI^W_{q-1}\big(f(\cdot,t)\big)$ so that, using the product formula (5.26),

$$\begin{aligned}
\frac{1}{q}\|DF\|^2_{L^2(\mathbb{R})} &= q\int_{\mathbb{R}} I^W_{q-1}\big(f(\cdot,t)\big)^2 dt \\
&= q\int_{\mathbb{R}}\sum_{r=0}^{q-1} r!\binom{q-1}{r}^2 I^W_{2q-2-2r}\big(f(\cdot,t)\widetilde{\otimes}_r f(\cdot,t)\big)dt \\
&= q\int_{\mathbb{R}}\sum_{r=0}^{q-1} r!\binom{q-1}{r}^2 I^W_{2q-2-2r}\big(f(\cdot,t)\otimes_r f(\cdot,t)\big)dt \\
&= q\sum_{r=0}^{q-1} r!\binom{q-1}{r}^2 I^W_{2q-2-2r}\left(\int_{\mathbb{R}} f(\cdot,t)\otimes_r f(\cdot,t)dt\right) \\
&= q\sum_{r=0}^{q-1} r!\binom{q-1}{r}^2 I^W_{2q-2-2r}(f\otimes_{r+1} f) \\
&= q\sum_{r=1}^{q} (r-1)!\binom{q-1}{r-1}^2 I^W_{2q-2r}(f\otimes_r f) \\
&= q!\|f\|^2_{L^2(\mathbb{R}^q)} + q\sum_{r=1}^{q-1}(r-1)!\binom{q-1}{r-1}^2 I^W_{2q-2r}(f\otimes_r f).
\end{aligned} \tag{6.11}$$

Since $E[F^2] = q!\|f\|^2_{L^2(\mathbb{R}^q)} = \sigma^2$, the identity (6.7) follows now from (6.11) and the orthogonality/isometry properties of multiple Wiener-Itô integrals.

Recall the hypercontractivity property (5.18) of multiple Wiener-Itô integrals, and observe the relationships $-L^{-1}F = \frac{1}{q}F$ and $D(F^3) = 3F^2DF$. Hence, by combining formula (5.32) with an approximation argument (the derivative of $\varphi(x) = x^3$ being not bounded), we infer that

$$E[F^4] = E\big[F\times F^3\big] = \frac{3}{q}E\big[F^2\|DF\|^2_{L^2(\mathbb{R})}\big]. \tag{6.12}$$

Moreover, the product formula (5.26) yields

$$F^2 = I_q^W(f)^2 = \sum_{s=0}^{q} s! \binom{q}{s}^2 I_{2q-2s}^W(f \widetilde{\otimes}_s f). \tag{6.13}$$

By combining this last identity with (6.11) and (6.12), we obtain (6.8) and finally (6.10).

It remains to prove (6.9). Let σ be a permutation of $\{1,\ldots,2q\}$ (this fact is written in symbols as $\sigma \in \mathfrak{S}_{2q}$). If $r \in \{0,\ldots,q\}$ denotes the cardinality of $\{\sigma(1),\ldots,\sigma(q)\} \cap \{1,\ldots,q\}$ then it is readily checked that r is also the cardinality of $\{\sigma(q+1),\ldots,\sigma(2q)\} \cap \{q+1,\ldots,2q\}$ and that

$$\begin{aligned}
&\int_{\mathbb{R}^{2q}} f(t_1,\ldots,t_q) f(t_{\sigma(1)},\ldots,t_{\sigma(q)}) f(t_{q+1},\ldots,t_{2q}) \\
&\qquad\qquad\qquad \times f(t_{\sigma(q+1)},\ldots,t_{\sigma(2q)}) dt_1 \ldots dt_{2q} \\
&= \int_{\mathbb{R}^{2q-2r}} (f \otimes_r f)(x_1,\ldots,x_{2q-2r})^2 dx_1 \ldots dx_{2q-2r} \\
&= \|f \otimes_r f\|^2_{L^2(\mathbb{R}^{2q-2r})}.
\end{aligned} \tag{6.14}$$

Moreover, for any fixed $r \in \{0,\ldots,q\}$, there are $\binom{q}{r}^2 (q!)^2$ permutations $\sigma \in \mathfrak{S}_{2q}$ such that $\#\{\sigma(1),\ldots,\sigma(q)\} \cap \{1,\ldots,q\} = r$. (Indeed, such a permutation is completely determined by the choice of: (1) r distinct elements $y_1,\ldots,y_r$ of $\{1,\ldots,q\}$; (2) $q-r$ distinct elements $y_{r+1},\ldots,y_q$ of $\{q+1,\ldots,2q\}$; (3) a bijection between $\{1,\ldots,q\}$ and $\{y_1,\ldots,y_q\}$; (4) a bijection between $\{q+1,\ldots,2q\}$ and $\{1,\ldots,2q\} \backslash \{y_1,\ldots,y_q\}$.) Now, observe that the symmetrization of $f \otimes f$ is given by

$$f \widetilde{\otimes} f(t_1,\ldots,t_{2q}) = \frac{1}{(2q)!} \sum_{\sigma \in \mathfrak{S}_{2q}} f(t_{\sigma(1)},\ldots,t_{\sigma(q)}) f(t_{\sigma(q+1)},\ldots,t_{\sigma(2q)}).$$

Therefore,

$$\begin{aligned}
&\|f \widetilde{\otimes} f\|^2_{L^2(\mathbb{R}^{2q})} \\
&= \frac{1}{(2q)!^2} \sum_{\sigma,\sigma' \in \mathfrak{S}_{2q}} \int_{\mathbb{R}^{2q}} f(t_{\sigma(1)},\ldots,t_{\sigma(q)}) f(t_{\sigma(q+1)},\ldots,t_{\sigma(2q)}) \\
&\qquad \times f(t_{\sigma'(1)},\ldots,t_{\sigma'(q)}) f(t_{\sigma'(q+1)},\ldots,t_{\sigma'(2q)}) dt_1 \ldots dt_{2q} \\
&= \frac{1}{(2q)!} \sum_{\sigma \in \mathfrak{S}_{2q}} \int_{\mathbb{R}^{2q}} f(t_1,\ldots,t_q) f(t_{q+1},\ldots,t_{2q}) \\
&\qquad \times f(t_{\sigma(1)},\ldots,t_{\sigma(q)}) f(t_{\sigma(q+1)},\ldots,t_{\sigma(2q)}) dt_1 \ldots dt_{2q} \\
&= \frac{1}{(2q)!} \sum_{r=0}^{q} \sum_{\substack{\sigma \in \mathfrak{S}_{2q} \\ \{\sigma(1),\ldots,\sigma(q)\} \cap \{1,\ldots,q\} = r}} \int_{\mathbb{R}^{2q}} f(t_1,\ldots,t_q) f(t_{q+1},\ldots,t_{2q}) \\
&\qquad \times f(t_{\sigma(1)},\ldots,t_{\sigma(q)}) f(t_{\sigma(q+1)},\ldots,t_{\sigma(2q)}) dt_1 \ldots dt_{2q}.
\end{aligned}$$

Using (6.14), we deduce that

$$(2q)!\|f\widetilde{\otimes}f\|^2_{L^2(\mathbb{R}^{2q})} = 2(q!)^2\|f\|^4_{L^2(\mathbb{R}^q)} + (q!)^2\sum_{r=1}^{q-1}\binom{q}{r}^2\|f\otimes_r f\|^2_{L^2(\mathbb{R}^{2q-2r})}. \tag{6.15}$$

Using the orthogonality/isometry properties of multiple Wiener-Itô integrals, the identity (6.13) yields

$$\begin{aligned} E[F^4] &= \sum_{r=0}^{q}(r!)^2\binom{q}{r}^4(2q-2r)!\|f\widetilde{\otimes}_r f\|^2_{L^2(\mathbb{R}^{2q-2r})} \\ &= (2q)!\|f\widetilde{\otimes}f\|^2_{L^2(\mathbb{R}^{2q})} + (q!)^2\|f\|^4_{L^2(\mathbb{R}^q)} \\ &\quad + \sum_{r=1}^{q-1}(r!)^2\binom{q}{r}^4(2q-2r)!\|f\widetilde{\otimes}_r f\|^2_{L^2(\mathbb{R}^{2q-2r})}. \end{aligned}$$

By inserting (6.15) in the previous identity (and because $(q!)^2\|f\|^4_{L^2(\mathbb{R}^q)} = E[F^2]^2 = \sigma^4$), we get (6.9). □

As a consequence of Lemma 6.2, we deduce the following bound on the total variation distance for the Gaussian approximation of a normalized multiple Wiener-Itô integral.

Theorem 6.2 (Nourdin–Peccati). *Let $q \geqslant 1$ be an integer and consider a symmetric function $f \in L^2(\mathbb{R}^q)$. Set $F = I_q^W(f)$, assume that $E[F^2] = 1$, and let $N \sim \mathcal{N}(0,1)$. Then*

$$d_{TV}(F,N) \leqslant 2\sqrt{\frac{q-1}{3q}\big|E[F^4]-3\big|}. \tag{6.16}$$

Proof. Since $L^{-1}F = -\frac{1}{q}F$, we have $\langle DF, -DL^{-1}F\rangle_{L^2(\mathbb{R})} = \frac{1}{q}\|DF\|^2_{L^2(\mathbb{R})}$. So, we only need to apply Theorem 6.1 and then formula (6.10) to conclude. □

The estimate (6.16) allows one to deduce the following characterization of CLTs on Wiener chaos.

Corollary 6.1 (Nualart–Peccati [46]). *Let $q \geqslant 1$ be an integer and consider a sequence (f_n) of symmetric functions of $L^2(\mathbb{R}^q)$. Set $F_n = I_q^W(f_n)$ and assume that $E[F_n^2] \to \sigma^2 > 0$ as $n \to \infty$. Then, as $n \to \infty$, the following four assertions are equivalent:*

1. $F_n \xrightarrow{\text{law}} N \sim \mathcal{N}(0,\sigma^2)$;
2. $E[F_n^4] \to E[N^4] = 3\sigma^4$;
3. $\|f_n\widetilde{\otimes}_r f_n\|_{L^2(\mathbb{R}^{2q-2r})} \to 0$ *for all* $r = 1,\dots,q-1$.
4. $\|f_n \otimes_r f_n\|_{L^2(\mathbb{R}^{2q-2r})} \to 0$ *for all* $r = 1,\dots,q-1$.

Proof. Without loss of generality, we may and do assume that $\sigma^2 = 1$ and $E[F_n^2] = 1$ for all n. The implication (2) → (1) is a direct application of Theorem 6.2. The implication (1) → (2) comes from the Continuous Mapping Theorem together with an approximation argument (observe that $\sup_{n\geqslant 1} E[F_n^4] < \infty$ by the hypercontractivity relation (5.18)). The equivalence between (2) and (3) is an immediate consequence of (6.8). The implication (4) → (3) is obvious since $\| f_n \widetilde{\otimes}_r f_n \| \leqslant \| f_n \otimes_r f_n \|$, whereas the implication (2) → (4) follows from (6.9). □

6.4 Quadratic Variation of the Fractional Brownian Motion

In this section, we use Theorem 6.1 in order to derive an explicit bound for the second-order approximation of the quadratic variation of a fractional Brownian motion on [0, 1]. We mainly follow [38] and [5].

Let $B^H = (B_t^H)_{t\geqslant 0}$ be a fractional Brownian motion with Hurst index $H \in (0,1)$. A natural question is the identification of the Hurst parameter from real data. To do so, it is popular and classical to use the quadratic variation (on, say, [0, 1]), which is observable and given by

$$S_n = \sum_{k=0}^{n-1} (B^H_{(k+1)/n} - B^H_{k/n})^2, \quad n \geqslant 1.$$

Recall from (2.12) that

$$n^{2H-1} S_n \overset{\text{proba}}{\to} 1 \quad \text{as } n \to \infty. \tag{6.17}$$

We deduce that the estimator $\widehat{H}_n$, defined as

$$\widehat{H}_n = \frac{1}{2} - \frac{\log S_n}{2 \log n},$$

satisfies $\widehat{H}_n \overset{\text{proba}}{\to} 1$ as $n \to \infty$. To study its asymptotic normality, consider

$$F_n = \frac{n^{2H}}{\sigma_n} \sum_{k=0}^{n-1} \big[(B^H_{(k+1)/n} - B^H_{k/n})^2 - n^{-2H} \big] \overset{\text{(law)}}{=} \frac{1}{\sigma_n} \sum_{k=0}^{n-1} \big[(B^H_{k+1} - B^H_k)^2 - 1 \big],$$

where $\sigma_n > 0$ is so that $E[F_n^2] = 1$. We then have the following result.

Theorem 6.3. *Let $N \sim \mathcal{N}(0,1)$ and assume that $H \leqslant 3/4$. Then,* $\lim_{n\to\infty} \sigma_n^2/n = 2\sum_{r\in\mathbb{Z}} \rho^2(r)$ *if $H \in (0, \frac{3}{4})$, with $\rho : \mathbb{Z} \to \mathbb{R}$ given by*

$$\rho(r) = \frac{1}{2}\big(|r+1|^{2H} + |r-1|^{2H} - 2|r|^{2H} \big), \tag{6.18}$$

and $\lim_{n\to\infty} \sigma_n^2/(n \log n) = \frac{9}{16}$ *if $H = \frac{3}{4}$. Moreover, there exists a constant $c_H > 0$ (depending only on H) such that, for every $n \geqslant 1$,*

$$d_{TV}(F_n, N) \leqslant c_H \times \begin{cases} \frac{1}{\sqrt{n}} & \text{if } H \in (0, \frac{5}{8}) \\ \frac{(\log n)^{3/2}}{\sqrt{n}} & \text{if } H = \frac{5}{8} \\ n^{4H-3} & \text{if } H \in (\frac{5}{8}, \frac{3}{4}) \\ \frac{1}{\log n} & \text{if } H = \frac{3}{4} \end{cases}. \tag{6.19}$$

As an immediate consequence of Theorem 6.3, provided $H < 3/4$ we obtain that

$$\sqrt{n}\big(n^{2H-1}S_n - 1\big) \overset{\text{law}}{\to} \mathcal{N}\big(0, 2\sum_{r\in\mathbb{Z}} \rho^2(r)\big) \quad \text{as } n \to \infty, \tag{6.20}$$

implying in turn

$$\sqrt{n}\log n\big(\widehat{H}_n - H\big) \overset{\text{law}}{\to} \mathcal{N}\big(0, \frac{1}{2}\sum_{r\in\mathbb{Z}} \rho^2(r)\big) \quad \text{as } n \to \infty. \tag{6.21}$$

Indeed, we can write

$$\log x = x - 1 - \int_1^x du \int_1^u \frac{dv}{v^2} \quad \text{for all } x > 0,$$

so that (by considering $x \geqslant 1$ and $0 < x < 1$)

$$\big|\log x + 1 - x\big| \leqslant \frac{(x-1)^2}{2}\left\{1 + \frac{1}{x^2}\right\} \quad \text{for all } x > 0.$$

As a result,

$$\sqrt{n}\log n\big(\widehat{H}_n - H\big) = -\frac{\sqrt{n}}{2}\log(n^{2H-1}S_n) = -\frac{\sqrt{n}}{2}(n^{2H-1}S_n - 1) + R_n$$

with

$$|R_n| \leqslant \frac{\big(\sqrt{n}(n^{2H-1}S_n - 1)\big)^2}{4\sqrt{n}}\left\{1 + \frac{1}{(n^{2H-1}S_n)^2}\right\}.$$

Using (6.17) and (6.20), it is clear that $R_n \overset{\text{proba}}{\to} 0$ as $n \to \infty$ and then that (6.21) holds true.

Now we have motivated it, let us go back to the proof of Theorem 6.3. We will need the following ancillary result.

Lemma 6.3. 1. *For any $r \in \mathbb{Z}$, let $\rho(r)$ be defined by (6.18). If $H \neq \frac{1}{2}$, one has $\rho(r) \sim H(2H-1)|r|^{2H-2}$ as $|r| \to \infty$. If $H = \frac{1}{2}$ and $|r| \geqslant 1$, one has $\rho(r) = 0$. Consequently, $\sum_{r\in\mathbb{Z}} \rho^2(r) < \infty$ if and only if $H < \frac{3}{4}$.*

2. *For all $\alpha > -1$, we have $\sum_{r=1}^{n-1} r^\alpha \sim \frac{n^{\alpha+1}}{\alpha+1}$ as $n \to \infty$.*

Proof. 1. The sequence ρ is symmetric, that is, one has $\rho(n) = \rho(-n)$. When $r \to \infty$,

$$\rho(r) = H(2H-1)r^{2H-2} + o(r^{2H-2}).$$

Using the usual criterion for convergence of Riemann sums, we deduce that $\sum_{r\in\mathbb{Z}} \rho^2(r)$ is finite if and only if $4H - 4 < -1$ if and only if $H < \frac{3}{4}$.

2. For $\alpha > -1$, we have:

$$\frac{1}{n}\sum_{r=1}^{n}\left(\frac{r}{n}\right)^\alpha \longrightarrow \int_0^1 x^\alpha dx = \frac{1}{\alpha+1} \quad \text{as } n \to \infty.$$

We deduce that $\sum_{r=1}^{n} r^\alpha \sim \frac{n^{\alpha+1}}{\alpha+1}$ as $n \to \infty$. □

We are now in a position to prove Theorem 6.3.

Proof of Theorem 6.3. Thanks to the selfsimilarity property of B^H (Proposition 2.2(1)), we may replace F_n in the proof by

$$F_n = \frac{1}{\sigma_n} \sum_{k=0}^{n-1} \big[(B^H_{k+1} - B^H_k)^2 - 1\big].$$

Using Proposition 2.3, we have moreover that there exists a sequence $\{e_k\}_{k\in\mathbb{N}}$ (the explicit expression of the e_k's is not needed here) such that

$$\{B^H_{k+1} - B^H_k : k \in \mathbb{N}\} \overset{\text{law}}{=} \left\{\int_{\mathbb{R}} e_k(s)dW_s : k \in \mathbb{N}\right\} = \left\{I_1^W(e_k) : k \in \mathbb{N}\right\},$$

where W stands for a two-sided Brownian motion and $I_p^W(\cdot)$, $p \geqslant 1$, denotes the pth multiple Wiener-Itô integral associated to W. Observe in particular that, for all $k, l \in \mathbb{N}$,

$$\int_{\mathbb{R}} e_k(s)e_l(s)ds = E[(B^H_{k+1} - B^H_k)(B^H_{l+1} - B^H_l)] = \rho(k-l) \tag{6.22}$$

with ρ given by (6.18). As a consequence, we may replace F_n by

$$F_n = \frac{1}{\sigma_n} \sum_{k=0}^{n-1} \left[\left(I_1^W(e_k)\right)^2 - 1\right]$$

in this proof without loss of generality. Now, using the product formula (5.26) (or, equivalently, Itô's formula), we deduce that

$$F_n = I_2^W(f_n), \quad \text{with } f_n = \frac{1}{\sigma_n} \sum_{k=0}^{n-1} e_k \otimes e_k.$$

Let us compute the exact value of σ_n. By the isometry formula (5.14) we can write

$$1 = E[F_n^2] = 2\|f_n\|^2_{L^2(\mathbb{R}^2)} = \frac{2}{\sigma_n^2} \sum_{k,l=0}^{n-1} \langle e_k, e_l\rangle^2_{L^2(\mathbb{R})} = \frac{2}{\sigma_n^2} \sum_{k,l=0}^{n-1} \rho^2(k-l).$$

That is,

$$\sigma_n^2 = 2 \sum_{k,l=0}^{n-1} \rho^2(k-l) = 2 \sum_{|r|<n} (n-|r|)\rho^2(r).$$

Assume that $H < \frac{3}{4}$ and write

$$\frac{\sigma_n^2}{n} = 2 \sum_{r\in\mathbb{Z}} \rho^2(r) \left(1 - \frac{|r|}{n}\right) 1_{\{|r|<n\}}.$$

Since $\sum_{r\in\mathbb{Z}}\rho^2(r) < \infty$ by Lemma 6.3, we obtain by dominated convergence that, when $H < \frac{3}{4}$,

$$\lim_{n\to\infty}\frac{\sigma_n^2}{n} = 2\sum_{r\in\mathbb{Z}}\rho^2(r). \tag{6.23}$$

Assume now that $H = \frac{3}{4}$. We then have $\rho^2(r) \sim \frac{9}{64|r|}$ as $|r| \to \infty$, implying in turn

$$n\sum_{|r|<n}\rho^2(r) \sim \frac{9n}{64}\sum_{0<|r|<n}\frac{1}{|r|} \sim \frac{9n\log n}{32}$$

and

$$\sum_{|r|<n}|r|\rho^2(r) \sim \frac{9}{64}\sum_{|r|<n}1 \sim \frac{9n}{32}$$

as $n \to \infty$. Hence, when $H = \frac{3}{4}$,

$$\lim_{n\to\infty}\frac{\sigma_n^2}{n\log n} = \frac{9}{16}. \tag{6.24}$$

On the other hand, recall that the convolution of two sequences $\{u(n)\}_{n\in\mathbb{Z}}$ and $\{v(n)\}_{n\in\mathbb{Z}}$ is the sequence $u * v$ defined as $(u * v)(j) = \sum_{n\in\mathbb{Z}} u(n)v(j-n)$, and observe that $(u * v)(l-i) = \sum_{k\in\mathbb{Z}} u(k-l)v(k-i)$ whenever $u(n) = u(-n)$ and $v(n) = v(-n)$ for all $n \in \mathbb{Z}$. Set

$$\rho_n(k) = |\rho(k)|1_{\{|k|\leqslant n-1\}}, \quad k\in\mathbb{Z},\, n\geqslant 1.$$

We then have (using (6.7) for the first equality, and noticing that $f_n\otimes_1 f_n = f_n\widetilde{\otimes}_1 f_n$),

$$\begin{aligned}
&E\left[\left(1-\frac{1}{2}\|D[I_2(f_n)]\|^2_{L^2(\mathbb{R})}\right)^2\right]\\
&= 8\,\|f_n\otimes_1 f_n\|^2_{L^2(\mathbb{R}^2)} = \frac{8}{\sigma_n^4}\sum_{i,j,k,l=0}^{n-1}\rho(k-l)\rho(i-j)\rho(k-i)\rho(l-j)\\
&\leqslant \frac{8}{\sigma_n^4}\sum_{i,l=0}^{n-1}\sum_{j,k\in\mathbb{Z}}\rho_n(k-l)\rho_n(i-j)\rho_n(k-i)\rho_n(l-j)\\
&= \frac{8}{\sigma_n^4}\sum_{i,l=0}^{n-1}(\rho_n*\rho_n)(l-i)^2\\
&\leqslant \frac{8n}{\sigma_n^4}\sum_{k\in\mathbb{Z}}(\rho_n*\rho_n)(k)^2 = \frac{8n}{\sigma_n^4}\|\rho_n*\rho_n\|^2_{\ell^2(\mathbb{Z})}.
\end{aligned}$$

Recall Young's inequality: if $s, p, q \geqslant 1$ are such that $\frac{1}{p}+\frac{1}{q} = 1+\frac{1}{s}$, then

$$\|u*v\|_{\ell^s(\mathbb{Z})} \leqslant \|u\|_{\ell^p(\mathbb{Z})}\|v\|_{\ell^q(\mathbb{Z})}. \tag{6.25}$$

Let us apply (6.25) with $u = v = \rho_n$, $s = 2$ and $p = \frac{4}{3}$. We get

$$\|\rho_n * \rho_n\|^2_{\ell^2(\mathbb{Z})} \leqslant \|\rho_n\|^4_{\ell^{\frac{4}{3}}(\mathbb{Z})},$$

so that

$$E\left[\left(1 - \frac{1}{2}\|D[I_2(f_n)]\|^2_{L^2(\mathbb{R})}\right)^2\right] \leqslant \frac{8n}{\sigma_n^4}\left(\sum_{|k|<n} |\rho(k)|^{\frac{4}{3}}\right)^3. \tag{6.26}$$

Recall the asymptotic behavior of $\rho(k)$ as $|k| \to \infty$ from Lemma 6.3(1). Hence

$$\sum_{|k|<n} |\rho(k)|^{\frac{4}{3}} = \begin{cases} O(1) & \text{if } H \in (0, \frac{5}{8}) \\ O(\log n) & \text{if } H = \frac{5}{8} \\ O(n^{(8H-5)/3}) & \text{if } H \in (\frac{5}{8}, 1). \end{cases} \tag{6.27}$$

Assume first that $H < \frac{3}{4}$ and recall (6.23). This, together with (6.26) and (6.27), imply that

$$\begin{aligned} E\left[\left|1 - \frac{1}{2}\|D[I_2(f_n)]\|^2_{L^2(\mathbb{R})}\right|\right] &\leqslant \sqrt{E\left[\left(1 - \frac{1}{2}\|D[I_2(f_n)]\|^2_{L^2(\mathbb{R})}\right)^2\right]} \\ &\leqslant c_H \times \begin{cases} \frac{1}{\sqrt{n}} & \text{if } H \in (0, \frac{5}{8}) \\ \frac{(\log n)^{3/2}}{\sqrt{n}} & \text{if } H = \frac{5}{8} \\ n^{4H-3} & \text{if } H \in (\frac{5}{8}, \frac{3}{4}) \end{cases}. \end{aligned}$$

Therefore, the desired conclusion holds for $H \in (0, \frac{3}{4})$ by applying Theorem 6.1. Assume now that $H = \frac{3}{4}$ and recall (6.24). This, together with (6.26) and (6.27), imply that

$$\begin{aligned} E\left[\left|1 - \frac{1}{2}\|D[I_2(f_n)]\|^2_{L^2(\mathbb{R})}\right|\right] &\leqslant \sqrt{E\left[\left(1 - \frac{1}{2}\|D[I_2(f_n)]\|^2_{L^2(\mathbb{R})}\right)^2\right]} \\ &= O(1/\log n), \end{aligned}$$

and leads to the desired conclusion for $H = \frac{3}{4}$ as well. □

6.5 Multivariate Gaussian Approximation

We conclude this chapter with the proof of Theorem 6.6, which is going to be crucial in our (modern) proof of the Breuer–Major theorem in the next chapter. We start by a kind of multivariate counterpart of Theorem 6.1.

Theorem 6.4 (Nourdin–Peccati–Réveillac [40]). *Fix $d \geqslant 2$, and let $F = (F_1, \ldots, F_d)$ be a random vector such that $F_i \in \mathbb{D}^{1,2}$ with $E[F_i] = 0$ for any i. Let $C \in \mathcal{M}_d(\mathbb{R})$ be a symmetric and positive matrix, and let $N \sim \mathcal{N}_d(0, C)$. Then, for any $h : \mathbb{R}^d \to \mathbb{R}$ belonging to $\mathcal{C}^2$ and such that*

$$\|h''\|_\infty := \max_{i,j=1,\ldots,d} \sup_{x\in\mathbb{R}^d} \left| \frac{\partial^2 h}{\partial x_i \partial x_j}(x) \right| < \infty,$$

we have

$$\big|E[h(F)] - E[h(N)]\big| \leqslant \frac{1}{2}\|h''\|_\infty \sum_{i,j=1}^{d} E\Big[\big|C_{i,j} - \langle DF_j, -DL^{-1}F_i\rangle_{L^2(\mathbb{R})}\big|\Big]. \quad (6.28)$$

Proof. Without loss of generality, we assume that N is independent of the underlying Brownian motion W. Let h be as in the statement of the theorem. For any $t \in [0, 1]$, set $\Psi(t) = E\big[h\big(\sqrt{1-t}\,F + \sqrt{t}\,N\big)\big]$, so that

$$E[h(N)] - E[h(F)] = \Psi(1) - \Psi(0) = \int_0^1 \Psi'(t)dt.$$

We easily see that Ψ is differentiable on $(0, 1)$ with

$$\Psi'(t) = \sum_{i=1}^{d} E\left[\frac{\partial h}{\partial x_i}\big(\sqrt{1-t}\,F + \sqrt{t}\,N\big)\left(\frac{1}{2\sqrt{t}}N_i - \frac{1}{2\sqrt{1-t}}F_i\right)\right].$$

By integrating by parts, we can write

$$\begin{aligned}
&E\left[\frac{\partial h}{\partial x_i}\big(\sqrt{1-t}\,F + \sqrt{t}\,N\big)N_i\right] \\
&\quad = E\left\{E\left[\frac{\partial h}{\partial x_i}\big(\sqrt{1-t}\,x + \sqrt{t}\,N\big)N_i\right]_{|x=F}\right\} \\
&\quad = \sqrt{t}\sum_{j=1}^{d} C_{i,j}\, E\left\{E\left[\frac{\partial^2 h}{\partial x_i \partial x_j}\big(\sqrt{1-t}\,x + \sqrt{t}\,N\big)\right]_{|x=F}\right\} \\
&\quad = \sqrt{t}\sum_{j=1}^{d} C_{i,j}\, E\left[\frac{\partial^2 h}{\partial x_i \partial x_j}\big(\sqrt{1-t}\,F + \sqrt{t}\,N\big)\right].
\end{aligned}$$

By using (5.32) in order to perform the integration by parts, we can also write

$$\begin{aligned}
&E\left[\frac{\partial h}{\partial x_i}(\sqrt{1-t}F+\sqrt{t}N)F_i\right]\\
&= E\left\{E\left[\frac{\partial h}{\partial x_i}(\sqrt{1-t}F+\sqrt{t}x)F_i\right]_{|x=N}\right\}\\
&= \sqrt{1-t}\sum_{j=1}^{d} E\left\{E\left[\frac{\partial^2 h}{\partial x_i\partial x_j}(\sqrt{1-t}F+\sqrt{t}x)\langle DF_j,-DL^{-1}F_i\rangle_{L^2(\mathbb{R})}\right]_{|x=N}\right\}\\
&= \sqrt{1-t}\sum_{j=1}^{d} E\left[\frac{\partial^2 h}{\partial x_i\partial x_j}(\sqrt{1-t}F+\sqrt{t}N)\langle DF_j,-DL^{-1}F_i\rangle_{L^2(\mathbb{R})}\right].
\end{aligned}$$

Hence

$$\Psi'(t)=\frac{1}{2}\sum_{i,j=1}^{d} E\left[\frac{\partial^2 h}{\partial x_i\partial x_j}(\sqrt{1-t}F+\sqrt{t}N)\left(C_{i,j}-\langle DF_j,-DL^{-1}F_i\rangle_{L^2(\mathbb{R})}\right)\right],$$

and the desired conclusion follows. □

A consequence of Theorem 6.4 is the following result, which asserts roughly speaking that, for a sequence of vectors of multiple Wiener-Itô integrals, componentwise convergence to Gaussian always implies joint convergence. This result, which is nothing but the multivariate counterpart to Theorem 6.1, will allow to effectively study the Gaussian approximation of general functionals by using their chaotic expansion, see Theorem 6.6.

Theorem 6.5 (Peccati–Tudor [48]). *Let $d \geqslant 2$ and $q_d,\ldots,q_1 \geqslant 1$ be some fixed integers. Consider vectors*

$$F_n=(F_{1,n},\ldots,F_{d,n})=(I_{q_1}^W(f_{1,n}),\ldots,I_{q_d}^W(f_{d,n})),\quad n\geqslant 1,$$

with $f_{i,n}\in L^2(\mathbb{R}^{q_i})$ symmetric. Let $C\in\mathcal{M}_d(\mathbb{R})$ be a symmetric and positive matrix. Assume that

$$\lim_{n\to\infty} E[F_{i,n}F_{j,n}]=C_{i,j},\quad 1\leqslant i,j\leqslant d. \tag{6.29}$$

Then, as $n\to\infty$, the following two conditions are equivalent:

1. *F_n converges in law to $\mathcal{N}_d(0,C)$;*
2. *$F_{i,n}$ converges in law to $\mathcal{N}(0,C_{i,i})$ for every $i=1,\ldots,d$.*

Proof. The implication (1) → (2) being trivial, we only concentrate on (2) → (1). So, assume (2) and let us show that (1) holds true. Thanks to (6.28), we are left to show that, for each $i,j=1,\ldots,d$,

$$\langle DF_{j,n},-DL^{-1}F_{i,n}\rangle_{L^2(\mathbb{R})}=\frac{1}{q_i}\langle DF_{j,n},DF_{i,n}\rangle_{L^2(\mathbb{R})}\overset{L^2(\Omega)}{\longrightarrow} C(i,j)\quad\text{as } n\to\infty. \tag{6.30}$$

We consider all the possible cases for q_i and q_j.

First case: $q_i = q_j = 1$. We have $\langle DF_{j,n}, DF_{i,n}\rangle_{L^2(\mathbb{R})} = \langle f_{i,n}, f_{j,n}\rangle_{L^2(\mathbb{R})} = E[F_{i,n}F_{j,n}]$. But it is our assumption that $E[F_{i,n}F_{j,n}] \to C(i,j)$ so (6.30) holds true in this case.

Second case: $q_i = 1$ and $q_j \geqslant 2$ (a similar analysis might be done whenever $q_j = 1$ and $q_i \geqslant 2$). We have $\langle DF_{j,n}, DF_{i,n}\rangle_{L^2(\mathbb{R})} = \langle f_{i,n}, DF_{j,n}\rangle_{L^2(\mathbb{R})} = I_{q_j-1}(f_{i,n} \otimes_1 f_{j,n})$. We deduce that

$$\begin{aligned}
E[\langle DF_{j,n}, DF_{i,n}\rangle^2_{L^2(\mathbb{R})}] &= (q_j-1)!\|f_{i,n}\widetilde{\otimes}_1 f_{j,n}\|^2_{L^2(\mathbb{R}^{q_j-1})} \\
&\leqslant (q_j-1)!\|f_{i,n}\otimes_1 f_{j,n}\|^2_{L^2(\mathbb{R}^{q_j-1})} \\
&= (q_j-1)!\langle f_{i,n}\otimes f_{i,n}, f_{j,n}\otimes_{q_j-1} f_{j,n}\rangle_{L^2(\mathbb{R}^2)} \\
&\leqslant (q_j-1)!\|f_{i,n}\|^2_{L^2(\mathbb{R})}\|f_{j,n}\otimes_{q_j-1} f_{j,n}\|_{L^2(\mathbb{R}^2)} \\
&= (q_j-1)!E[F^2_{i,n}]\|f_{j,n}\otimes_{q_j-1} f_{j,n}\|_{L^2(\mathbb{R}^2)}.
\end{aligned}$$

At this stage, observe the following two facts. First, because $q_i \neq q_j$, we have $C(i,j) = 0$ necessarily. Second, since $E[F^2_{j,n}] \to C(j,j)$ and $F_{j,n} \stackrel{\text{Law}}{\to} \mathcal{N}(0, C(j,j))$, we have by Theorem 6.1 that $\|f_{j,n}\otimes_{q_j-1} f_{j,n}\|_{L^2(\mathbb{R}^2)} \to 0$. Hence, (6.30) holds true in this case as well.

Third case: $q_i = q_j \geqslant 2$. We have, using the product formula (5.26),

$$\begin{aligned}
&\frac{1}{q_i}\langle DF_{j,n}, DF_{i,n}\rangle_{L^2(\mathbb{R})} \\
&= q_i\int_0^\infty I_{q_i}(f_{i,n}(\cdot,t))I_{q_i}(f_{j,n}(\cdot,t))dt \\
&= q_i\sum_{r=0}^{q_i-1} r!\binom{q_i-1}{r}^2 I_{2q_i-2-2r}\left(\int_0^\infty f_{i,n}(\cdot,t)\otimes_r f_{j,n}(\cdot,t)dt\right) \\
&= q_i\sum_{r=0}^{q_i-1} r!\binom{q_i-1}{r}^2 I_{2q_i-2-2r}\left(f_{i,n}\otimes_{r+1} f_{j,n}\right) \\
&= q_i!\langle f_{i,n}, f_{j,n}\rangle_{L^2(\mathbb{R}^{q_i})} + q_i\sum_{r=1}^{q_i-1}(r-1)!\binom{q_i-1}{r-1}^2 I_{2q_i-2r}(f_{i,n}\otimes_r f_{j,n}) \\
&= E[F_{i,n}F_{j,n}] + q_i\sum_{r=1}^{q_i-1}(r-1)!\binom{q_i-1}{r-1}^2 I_{2q_i-2r}(f_{i,n}\otimes_r f_{j,n}).
\end{aligned}$$

We deduce that

$$
\begin{aligned}
&E\left[\left(\frac{1}{q_i}\langle DF_{j,n}, DF_{i,n}\rangle_{L^2(\mathbb{R})} - C(i,j)\right)^2\right] \\
&\quad = \big(E[F_{i,n}F_{j,n}] - C(i,j)\big)^2 \\
&\qquad + q_i^2 \sum_{r=1}^{q_i-1} (r-1)!^2 \binom{q_i-1}{r-1}^4 (2q_i-2r)! \| f_{i,n} \widetilde{\otimes}_r f_{j,n}\|^2_{L^2(\mathbb{R}^{2q_i-2r})}.
\end{aligned}
$$

The first term of the right-hand side tends to zero by assumption. For the second term, we can write, whenever $r \in \{1, \dots, q_i - 1\}$,

$$
\begin{aligned}
\| f_{i,n} \widetilde{\otimes}_r f_{j,n}\|^2_{L^2(\mathbb{R}^{2q_i-2r})} &\leqslant \| f_{i,n} \otimes_r f_{j,n}\|^2_{L^2(\mathbb{R}^{2q_i-2r})} \\
&= \langle f_{i,n} \otimes_{q_i-r} f_{i,n}, f_{j,n} \otimes_{q_i-r} f_{j,n}\rangle_{L^2(\mathbb{R}^{2r})} \\
&\leqslant \| f_{i,n} \otimes_{q_i-r} f_{i,n}\|_{L^2(\mathbb{R}^{2r})} \| f_{j,n} \otimes_{q_i-r} f_{j,n}\|_{L^2(\mathbb{R}^{2r})}.
\end{aligned}
$$

Since $F_{i,n} \overset{\text{Law}}{\to} \mathcal{N}(0, C(i,i))$ and $F_{j,n} \overset{\text{Law}}{\to} \mathcal{N}(0, C(j,j))$, by Theorem 6.1 we have that $\| f_{i,n} \otimes_{q_i-r} f_{i,n}\|_{L^2(\mathbb{R}^{2r})} \| f_{j,n} \otimes_{q_i-r} f_{j,n}\|_{L^2(\mathbb{R}^{2r})} \to 0$, thereby showing that (6.30) holds true in our third case.

Fourth case: $q_j > q_i \geqslant 2$ (a similar analysis might be done whenever $q_i > q_j \geqslant 2$). We have, using the product formula (5.26),

$$
\begin{aligned}
&\frac{1}{q_i}\langle DF_{j,n}, DF_{i,n}\rangle_{L^2(\mathbb{R})} \\
&\quad = q_j \int_0^\infty I_{q_i}(f_{i,n}(\cdot,t)) I_{q_j}(f_{j,n}(\cdot,t)) dt \\
&\quad = q_j \sum_{r=0}^{q_i-1} r! \binom{q_i-1}{r}\binom{q_j-1}{r} I_{q_i+q_j-2-2r}\left(\int_0^\infty f_{i,n}(\cdot,t) \otimes_r f_{j,n}(\cdot,t) dt\right) \\
&\quad = q_j \sum_{r=0}^{q_i-1} r! \binom{q_i-1}{r}\binom{q_j-1}{r} I_{q_i+q_j-2-2r}\left(f_{i,n} \otimes_{r+1} f_{j,n}\right) \\
&\quad = q_j \sum_{r=1}^{q_i} (r-1)! \binom{q_i-1}{r-1}\binom{q_j-1}{r-1} I_{q_i+q_j-2r}(f_{i,n} \otimes_r f_{j,n}).
\end{aligned}
$$

We deduce that

$$
\begin{aligned}
&E\left[\frac{1}{q_i}\langle DF_{j,n}, DF_{i,n}\rangle^2_{L^2(\mathbb{R})}\right] \\
&= q_j^2 \sum_{r=1}^{q_i} (r-1)!^2 \binom{q_i-1}{r-1}^2 \binom{q_j-1}{r-1}^2 (q_i+q_j-2r)! \| f_{i,n} \widetilde{\otimes}_r f_{j,n}\|^2_{L^2(\mathbb{R}^{q_i+q_j-2r})}.
\end{aligned}
$$

For any $r \in \{1, \ldots, q_i\}$, we have

$$
\begin{aligned}
\| f_{i,n} \widetilde{\otimes}_r f_{j,n} \|^2_{L^2(\mathbb{R}^{q_i+q_j-2r})} &\leqslant \| f_{i,n} \otimes_r f_{j,n} \|^2_{L^2(\mathbb{R}^{q_i+q_j-2r})} \\
&= \langle f_{i,n} \otimes_{q_i-r} f_{i,n}, f_{j,n} \otimes_{q_j-r} f_{j,n} \rangle_{L^2(\mathbb{R}^{2r})} \\
&\leqslant \| f_{i,n} \otimes_{q_i-r} f_{i,n} \|_{L^2(\mathbb{R}^{2r})} \| f_{j,n} \otimes_{q_j-r} f_{j,n} \|_{L^2(\mathbb{R}^{2r})} \\
&\leqslant \| f_{i,n} \|^2_{L^2(\mathbb{R}^{q_i})} \| f_{j,n} \otimes_{q_j-r} f_{j,n} \|_{L^2(\mathbb{R}^{2r})}.
\end{aligned}
$$

Since $F_{j,n} \stackrel{\text{Law}}{\to} \mathcal{N}(0, C(j,j))$ and $q_j - r \in \{1, \ldots, q_j - 1\}$, by Theorem 6.1 we have that $\| f_{j,n} \otimes_{q_j-r} f_{j,n} \|_{L^2(\mathbb{R}^{2r})} \to 0$. We deduce that (6.30) holds true in our fourth case.

Summarizing, we have that (6.30) is true for any i and j, and the proof of the theorem is done. □

Remark 6.1. If the integers $q_d, \ldots, q_1$ are pairwise disjoint in Theorem 6.5, then (6.29) is automatically verified with $C_{i,j} = 0$ for all $i \neq j$ (see (5.15)).

We shall now prove a criterion of asymptotic normality for centered random vectors of $L^2(\Omega)$.

Theorem 6.6. *Fix an integer $d \geqslant 1$, and let $F_n = (F_n^1, \ldots, F_n^d)$, $n \geqslant 1$, be a sequence of random vectors in $L^2(\Omega)$ such that $E[F_n^i] = 0$ for all i and n. For any i and n, consider the chaotic expansion (5.17) of F_n^i, that is,*

$$
F_n^i = \sum_{l=1}^{\infty} I_l^W (f_{n,l}^i),
$$

with $f_{n,l}^i \in L^2(\mathbb{R}^l)$ symmetric for all i, l and n. Suppose in addition that:

1. *for fixed $i, j = 1, \ldots, d$ and $l \geqslant 1$, $C_{i,j,l} = \lim_{n\to\infty} l! \langle f_{n,l}^i, f_{n,l}^j \rangle_{L^2(\mathbb{R}^l)}$ exists;*
2. *for $i, j = 1, \ldots, d$, $\sum_{l=1}^{\infty} |C_{i,j,l}| < \infty$;*
3. *for $l \geqslant 2$, $i = 1, \ldots, d$ and $r = 1, \ldots, l-1$, $\| f_{n,l}^i \otimes_r f_{n,l}^i \|_{L^2(\mathbb{R}^{2l-2r})} \to 0$ as $n \to \infty$;*
4. *for $i = 1, \ldots, d$, $\lim_{N\to\infty} \sup_{n \geqslant 1} \sum_{l=N+1}^{\infty} l! \| f_{n,l}^i \|^2_{L^2(\mathbb{R}^l)} = 0$.*

Then $F_n \stackrel{\text{Law}}{\to} \mathcal{N}_d(0, C)$ as $n \to \infty$, where $C = (C_{i,j})_{1 \leqslant i,j \leqslant d}$ is given by

$$
C_{i,j} = \sum_{l=1}^{\infty} C_{i,j,l}.
$$

Remark 6.2. Of course, condition (3) can be replaced by any of the equivalent assertions (1)–(4) of Corollary 6.1.

Proof of Theorem 6.6. Set $F_{n,N}^i = \sum_{l=1}^N I_l(f_{n,l}^i)$ as well as

$$C_N = \left(\sum_{l=1}^N C_{i,j,l}\right)_{1\leqslant i,j\leqslant d} \in \mathcal{M}_d(\mathbb{R}). \tag{6.31}$$

It is readily checked that the symmetric matrix C_N is positive: indeed, for any $x \in \mathbb{R}^d$, we have

$$\begin{aligned}\sum_{i,j=1}^d x_i x_j \sum_{l=1}^N C_{i,j,l} &= \lim_{n\to\infty} \sum_{i,j=1}^d x_i x_j \sum_{l=1}^N l!\langle f_{n,l}^i, f_{n,l}^j\rangle_{L^2(\mathbb{R}^l)} \\ &= \lim_{n\to\infty} \sum_{l=1}^N l! \left\| \sum_{i=1}^d x_i f_{n,l}^i \right\|_{L^2(\mathbb{R}^l)}^2 \geqslant 0.\end{aligned}$$

Similarly, C is positive. Let $G_N \sim \mathcal{N}_d(0, C_N)$ and $G \sim \mathcal{N}_d(0, C)$. For any $t \in \mathbb{R}^d$, we have

$$\begin{aligned}&\left|E[e^{i\langle t,F_n\rangle_{\mathbb{R}^d}}] - E[e^{i\langle t,G\rangle_{\mathbb{R}^d}}]\right| \\ &\leqslant \left|E[e^{i\langle t,F_n\rangle_{\mathbb{R}^d}}] - E[e^{i\langle t,F_{n,N}\rangle_{\mathbb{R}^d}}]\right| + \left|E[e^{i\langle t,F_{n,N}\rangle_{\mathbb{R}^d}}] - E[e^{i\langle t,G_N\rangle_{\mathbb{R}^d}}]\right| \\ &\quad + \left|E[e^{i\langle t,G_N\rangle_{\mathbb{R}^d}}] - E[e^{i\langle t,G\rangle_{\mathbb{R}^d}}]\right| \\ &= a_{n,N} + b_{n,N} + c_N.\end{aligned}$$

Thanks to (2), observe that

$$c_N = \left| e^{-\frac{1}{2}\langle C_N t,t\rangle_{\mathbb{R}^d}} - e^{-\frac{1}{2}\langle Ct,t\rangle_{\mathbb{R}^d}} \right| \leqslant \frac{1}{2}\|t\|_{\mathbb{R}^d}^2 \|C - C_N\|_{\mathcal{M}_d(\mathbb{R})} \to 0$$

as $N \to \infty$. On the other hand, due to (4),

$$\begin{aligned}\sup_{n\geqslant 1} a_{n,N} &\leqslant \|t\|_{\mathbb{R}^d} \sup_{n\geqslant 1} \sqrt{E[\|F_n - F_{n,N}\|_{\mathbb{R}^d}^2]} \\ &= \|t\|_{\mathbb{R}^d} \sup_{n\geqslant 1} \sqrt{\sum_{i=1}^d \sum_{l=N+1}^\infty l!\|f_{n,l}^i\|_{L^2(\mathbb{R}^l)}^2} \to 0 \text{ as } N \to \infty.\end{aligned}$$

Fix $\varepsilon > 0$. We can choose and fix N large enough so that $\sup_{n\geqslant 1} a_{n,N} \leqslant \varepsilon/3$ and $c_N \leqslant \varepsilon/3$. Due to (1), (3) and (5.15), we deduce from Corollary 6.1 and Theorem 6.5 that, as $n \to \infty$,

$$\left(I_1(f_{n,1}^1), \ldots, I_1(f_{n,1}^d), \ldots, I_N(f_{n,N}^1), \ldots, I_N(f_{n,N}^d)\right) \overset{\text{Law}}{\to} \mathcal{N}_{Nd}(0, M),$$

where $M = (M_{a,b})_{1\leqslant a,b\leqslant Nd} \in \mathcal{M}_{Nd}(\mathbb{R})$ is given by

$$M_{a,b} = \begin{cases} 0 & \text{if } a = (k-1)d + i \text{ and } b = (l-1)d + j \text{ with } 1 \leqslant k \neq l \leqslant N \\ & \text{and } 1 \leqslant i, j \leqslant d \\ C_{i,j,l} & \text{if } a = (l-1)d + i \text{ and } b = (l-1)d + j \text{ with } 1 \leqslant l \leqslant N \\ & \text{and } 1 \leqslant i, j \leqslant d. \end{cases}$$

In particular, $F_{n,N} \xrightarrow{\text{Law}} G_N \sim \mathcal{N}_d(0, C_N)$ as $n \to \infty$ (with C_N given by (6.31)), so that $b_{n,N} \leqslant \varepsilon/3$ if n is large enough. Summarizing, we have shown that $\left|E[e^{i\langle t,F_n\rangle_{\mathbb{R}^d}}] - E[e^{i\langle t,G\rangle_{\mathbb{R}^d}}]\right| \leqslant \varepsilon$ if n is large enough, which is the desired conclusion. □

Chapter 7
Weak Convergence of Partial Sums of Stationary Sequences

Normalized sums of i.i.d. random variables satisfy the usual central limit theorem. But this is not necessarily the case if the i.i.d. random variables are replaced by a stationary sequence with long-range dependence, that is, with a correlation which decays slowly as the lag tends to infinity.

In this chapter, we study in details three situations. The first one corresponds to the celebrated Breuer–Major theorem, where we have weak convergence to classical Brownian motion. In the second one, we will obtain weak convergence to the fractional Brownian motion. Finally, the so-called Rosenblatt process (which may be seen as a suitable non-Gaussian generalization of fBm) will appear in the limit of the third one.

7.1 General Framework

Let $X = (X_k)_{k\in\mathbb{N}}$ be a stationary Gaussian sequence with $E[X_k] = 0$ and $E[X_k^2] = 1$, and let $\rho(k-l) = E[X_k X_l]$ be its correlation kernel. (Observe that ρ is symmetric, that is, $\rho(n) = \rho(-n)$ for all $n \geqslant 1$.) Consider the linear span $\mathscr{H}$ of X, that is, $\mathscr{H}$ is the closed linear subspace of $L^2(\Omega)$ generated by $(X_k)_{k\in\mathbb{N}}$. It is a real separable Hilbert space and, consequently, there exists an isometry $\Phi : \mathscr{H} \to L^2(\mathbb{R})$. For any $k \in \mathbb{N}$, set $e_k = \Phi(X_k)$; we then have, for all $k, l \in \mathbb{N}$,

$$\int_{\mathbb{R}} e_k(s)e_l(s)ds = E[X_k X_l] = \rho(k-l), \tag{7.1}$$

so that

$$\{X_k : k \in \mathbb{N}\} \overset{\text{law}}{=} \left\{\int_{\mathbb{R}} e_k(s)dW_s : k \in \mathbb{N}\right\} = \left\{I_1^W(e_k) : k \in \mathbb{N}\right\},$$

where W is a two-sided Brownian motion. Since the forthcoming limits only involve the distribution of the X_k's, we assume from now on without loss of generality that $X_k = I_1^W(e_k)$.

Nourdin I.: *Selected Aspects of Fractional Brownian Motion*, B&SS,
DOI 10.1007/978-88-470-2823-4_7, © Springer-Verlag Italia 2012

Let $\phi \in L^2\big(\mathbb{R}, \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx\big)$ be such that

$$\int_{\mathbb{R}} \phi(x) e^{-x^2/2} dx = 0, \tag{7.2}$$

and let $H_0(x) = 1$, $H_1(x) = x$, $H_2(x) = x^2 - 1, \ldots$, denote the sequence of Hermite polynomials (Definition 1.4). According to Proposition 1.3(2) and because it satisfies (7.2), one can decompose ϕ in terms of Hermite polynomials as

$$\phi(x) = \sum_{l \geqslant q} a_l H_l(x), \tag{7.3}$$

where $q \geqslant 1$ and $a_q \neq 0$. The integer q is called the *Hermite rank* of ϕ and we have $\mathrm{Var}(\phi(N)) = \sum_{l \geqslant q} a_l^2 l! < \infty$ if $N \sim \mathcal{N}(0,1)$. Set

$$V_n(\phi, t) = \sum_{k=1}^{[nt]} \phi(X_k) = \sum_{l \geqslant q} a_l \sum_{k=1}^{[nt]} H_l(X_k), \quad t \geqslant 0. \tag{7.4}$$

When $\sum_{k \in \mathbb{Z}} |\rho(k)|^q = \infty$ (we then say that X has *long-range dependence*), we will specialize our study to the case where ρ has the form

$$\rho(k) = k^{-D} L(k), \quad k \geqslant 1, \tag{7.5}$$

with $0 < D < \frac{1}{q}$ and $L : (0, \infty) \to (0, \infty)$ slowly varying at infinity and bounded away from 0 and infinity on every compact subset of $[0, \infty)$.

A slowly varying function at infinity L is such that

$$\lim_{x \to \infty} L(cx)/L(x) = 1 \quad \text{for all } c > 0. \tag{7.6}$$

Constants and logarithm satisfy (7.6). A useful property of slowly varying functions is the *Potter's bound* (see [6, Theorem 1.5.6(2)]): for every $\delta > 0$, there is $C = C(\delta) > 1$ such that, for all $x, y > 0$,

$$\frac{L(x)}{L(y)} \leqslant C \max\left\{ \left(\frac{x}{y}\right)^{\delta}, \left(\frac{x}{y}\right)^{-\delta} \right\}. \tag{7.7}$$

We also have the following crucial result.

Theorem 7.1 (Karamata). *Let $L : (0, \infty) \to (0, \infty)$ be slowly varying at infinity. Assume further that L is bounded away from 0 and infinity on every compact subset of* $[0, \infty)$. *Then, for any $\alpha \in (0, 1)$,*

$$\sum_{j=1}^{n} j^{-\alpha} L(j) \sim \frac{n^{1-\alpha} L(n)}{1 - \alpha} \quad \text{as } n \to \infty. \tag{7.8}$$

Proof. We have

$$\frac{1}{n^{1-\alpha}L(n)}\sum_{j=1}^{n} j^{-\alpha}L(j) = \int_0^1 l_n(x)dx,$$

with

$$l_n(x) = \sum_{j=1}^{n} \frac{L(j)}{L(n)}\left(\frac{j}{n}\right)^{-\alpha} 1_{[\frac{j-1}{n},\frac{j}{n})}(x).$$

Since $L(j)/L(n) = L(n(j/n))/L(n) \to 1$ for fixed j/n as $n \to \infty$, one has $l_n(x) \to l_\infty(x)$ for $x \in (0,1)$, where $l_\infty(x) = x^{-\alpha}$. By choosing a small enough $\delta > 0$ so that $\alpha + \delta < 1$ and $L(j)/L(n) \leqslant C(j/n)^{-\delta}$ (this is possible thanks to (7.7)), we get that

$$|l_n(x)| \leqslant C\sum_{j=1}^{[nt]}\left(\frac{j}{n}\right)^{-(\alpha+\delta)} 1_{[\frac{j-1}{n},\frac{j}{n})}(x) \leqslant Cx^{-(\alpha+\delta)}$$

for all $x \in (0,1)$. The function in the bound is integrable on $(0,1)$. Hence, the dominated convergence theorem yields

$$\frac{1}{n^{1-\alpha}L(n)}\sum_{j=1}^{n} j^{-\alpha}L(j) \to \int_0^1 l_\infty(x)dx = \int_0^1 x^{-\alpha}dx = \frac{1}{1-\alpha},$$

which is equivalent to (7.8). □

For more about slow varying functions, we refer the reader to the book [6], which is the classical reference on the subject.

7.2 Central Limit Theorem

As shown by the following important result, partial sums associated to 'any' stationary times series with *short-range dependence* (that is, such that $\sum_{k\in\mathbb{Z}} |\rho(k)|^q < \infty$) always converge to the classical Brownian motion (see Remark 7.1(3) for an example built from a fractional Brownian motion). Our proof does not rely on cumulants and diagrams as in the seminal paper [7] by Breuer and Major, but on the material developed in Chapter 6.

Theorem 7.2 (Breuer–Major). *Let $\phi : \mathbb{R} \to \mathbb{R}$ be the function given by* (7.3) *(with Hermite rank $q \geqslant 1$) and recall the definition (7.4) of $V_n(\phi,\cdot)$. If $\sum_{k\in\mathbb{Z}} |\rho(k)|^q$ is finite, then, as $n \to \infty$,*

$$\frac{V_n(\phi,\cdot)}{\sqrt{n}} \overset{\text{f.d.d.}}{\to} \sqrt{\sum_{l\geqslant q} a_l^2 l! \sum_{k\in\mathbb{Z}} \rho(k)^l} \times B^{1/2}, \tag{7.9}$$

with $B^{1/2}$ a classical Brownian motion. (The fact that $\sum_{l\geq q} a_l^2 l! \sum_{k\in\mathbb{Z}} \rho(k)^l$ is well-defined and positive is part of the conclusion.)

Remark 7.1. 1. With extra efforts (we omit the details here), it is possible to prove that the convergence (7.9) actually holds in the Skorohod space of càdlàg functions.

2. It is worthwhile noting that the classical Donsker's theorem is a particular case of Theorem 7.2. Indeed, let $Y_1, Y_2, \dots$ be any sequence of square integrable i.i.d. random variables. It is a well-known result that any random variable Y has the same law than $F_Y^{-1}(U)$, where U is uniformly distributed on $(0,1)$ and

$$F_Y^{-1}(u) = \inf\{s \in \mathbb{R} : u \leqslant F(s)\}, \quad u \in (0,1),$$

is the pseudo-inverse of $F_Y(y) = P(Y \leqslant y)$, $y \in \mathbb{R}$. In particular, if $X \sim \mathcal{N}(0,1)$, then $U := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{W} e^{-u^2/2} du$ is uniformly distributed on $(0,1)$. Thus, if $X_1, X_2, \dots \sim \mathcal{N}(0,1)$ are i.i.d. and if Y is a given random variable, then the sequence $\{Y_k\}_{k \geqslant 1}$ defined by $Y_k = \phi(X_k)$, where

$$\phi(x) = F_Y^{-1}\left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2} du \right), \quad x \in \mathbb{R},$$

is composed of i.i.d. random variables with the same law than that of Y, so that we recover Donsker's theorem by applying Theorem 7.2. (Observe that $\rho(k) = 0$ if $k \neq 0$ and that $\sum_{l \geqslant q} a_l^2 l! = \mathrm{Var}(Y_1)$.)

3. Let B^H be a fractional Brownian motion with Hurst index $H \in (0,1)$. An explicit example of centered stationary Gaussian sequence satisfying $\sum_{k \in \mathbb{Z}} |\rho(k)|^q < \infty$ is given by the increments

$$X_k = B_{k+1}^H - B_k^H$$

whenever $H < 1 - 1/(2q)$. (See also Lemma 6.3.)

Now, let us proceed with the proof of Theorem 7.2.

Proof of Theorem 7.2. Our main tool is Theorem 6.6. According to Section 7.1, we may and will assume that $X_k = I_1^W(e_k)$ for all k. Since $E[X_k^2] = \|e_k\|^2_{L^2(\mathbb{R})} = 1$ we have, by (7.3) and (5.16), that

$$\frac{1}{\sqrt{n}} V_n(\phi, t) = \frac{1}{\sqrt{n}} \sum_{k=1}^{[nt]} \phi(X_k) = \frac{1}{\sqrt{n}} \sum_{k=1}^{[nt]} \sum_{l \geqslant q} a_l H_l(X_k) = \sum_{l \geqslant q} I_l^W(f_{n,l}(t,\cdot)),$$

where the symmetric function $f_{n,l}(t,\cdot) \in L^2(\mathbb{R}^l)$ is given by

$$f_{n,l}(t, x_1, \dots, x_l) = \frac{a_l}{\sqrt{n}} \sum_{k=1}^{[nt]} e_k(x_1) \dots e_k(x_l).$$

Fix $d \geqslant 1$ and $t_1, \dots, t_d \geqslant 0$, and set $F_n^i = \sum_{l \geqslant q} I_l^W(f_{n,l}(t_i,\cdot))$, $i = 1, \dots, d$. To conclude the proof, we need to check that $F_n = (F_n^1, \dots, F_n^d) \overset{\text{Law}}{\to} \mathcal{N}_d(0, C)$ as

$n \to \infty$, where $C = (C_{i,j})_{1 \leqslant i,j \leqslant d} \in \mathcal{M}_d(\mathbb{R})$ is given by

$$C_{i,j} = t_i \wedge t_j \times \sum_{l \geqslant q} l! a_l^2 \sum_{v \in \mathbb{Z}} \rho(v)^l.$$

To do so, we will check the four conditions (1)–(4) of Theorem 6.6.

Condition 1. Fix an integer $l \geqslant q$ and let $t > s \geqslant 0$ be two given real numbers. We have

$$\begin{aligned}
&2\langle f_{n,l}(s,\cdot,), f_{n,l}(t,\cdot)\rangle_{L^2(\mathbb{R}^l)} \\
&= \| f_{n,l}(t,\cdot,)\|^2_{L^2(\mathbb{R}^l)} + \| f_{n,l}(s,\cdot,)\|^2_{L^2(\mathbb{R}^l)} - \| f_{n,l}(t,\cdot,) - f_{n,l}(s,\cdot)\|^2_{L^2(\mathbb{R}^l)} \\
&= \frac{a_l^2}{n} \sum_{i,j=1}^{[nt]} \rho(i-j)^l + \frac{a_l^2}{n} \sum_{i,j=1}^{[ns]} \rho(i-j)^l - \frac{a_l^2}{n} \sum_{i,j=1}^{[nt]-[ns]} \rho(i-j)^l \\
&= a_l^2 \sum_{v \in \mathbb{Z}} \rho(v)^l \frac{[nt]-|v|}{n} 1_{\{|v|<[nt]\}} + a_l^2 \sum_{v \in \mathbb{Z}} \rho(v)^l \frac{[ns]-|v|}{n} 1_{\{|v|<[ns]\}} \\
&\quad - a_l^2 \sum_{v \in \mathbb{Z}} \rho(v)^l \frac{[nt]-[ns]-|v|}{n} 1_{\{|v|<[nt]-[ns]\}}.
\end{aligned}$$

It follows from the dominated convergence theorem that, as $n \to \infty$,

$$l!\langle f_{n,l}(t_i,\cdot,), f_{n,l}(t_j,\cdot)\rangle_{L^2(\mathbb{R}^l)} \longrightarrow C_{i,j,l} := t_i \wedge t_j \times l! a_l^2 \sum_{v \in \mathbb{Z}} \rho(v)^l. \tag{7.10}$$

Condition 2. Since $E[X_v^2] = 1$ for all v, we have by Cauchy-Schwarz that $|\rho(v)| \leqslant 1$ for all v. We can thus write

$$\sum_{l \geqslant q} l! a_l^2 \sum_{v \in \mathbb{Z}} \rho(v)^l \leqslant \sum_{l \geqslant q}^{\infty} l! a_l^2 \times \sum_{v \in \mathbb{Z}} |\rho(v)|^q = E[\phi^2(X_1)] \sum_{v \in \mathbb{Z}} |\rho(v)|^q < \infty.$$

Condition 3. Fix $l \geqslant q$, $l \neq 1$. For all $n \geqslant 1$, $r = 1, \ldots, l-1$ and $t \in (0, \infty)$, we have

$$\begin{aligned}
&\| f_{n,l}(t,\cdot) \otimes_r f_{n,l}(t,\cdot)\|^2_{L^2(\mathbb{R}^{2l-2r})} \\
&\quad = \frac{a_l^4}{n^2} \sum_{i,j,u,v=1}^{[nt]} \rho(u-v)^r \rho(i-j)^r \rho(u-i)^{l-r} \rho(v-j)^{l-r}.
\end{aligned}$$

Consequently, using $\left|\rho(u-v)^r \rho(u-i)^{l-r}\right| \leqslant |\rho(u-v)|^l + |\rho(u-i)|^l$, we obtain that

$$\begin{aligned}
&\| f_{n,l}(t,\cdot) \otimes_r f_{n,l}(t,\cdot)\|^2_{L^2(\mathbb{R}^{2l-2r})} \\
&\leqslant \frac{a_l^4}{n^2} \sum_{i,j,u,v=1}^{[nt]} |\rho(u-v)|^l \left(|\rho(i-j)|^r |\rho(v-j)|^{l-r} + |\rho(i-j)|^{l-r} |\rho(v-j)|^r\right) \\
&\leqslant \frac{a_l^4}{n^2} \sum_{u\in\mathbb{Z}} |\rho(u)|^l \sum_{i,j,v=1}^{[nt]} \left(|\rho(i-j)|^r |\rho(v-j)|^{l-r} + |\rho(i-j)|^{l-r} |\rho(v-j)|^r\right) \\
&\leqslant \frac{2a_l^4}{n} \sum_{u\in\mathbb{Z}} |\rho(u)|^l \sum_{|i|\leqslant[nt]} |\rho(i)|^r \sum_{|j|\leqslant[nt]} |\rho(j)|^{l-r} \\
&= 2a_l^4 \sum_{u\in\mathbb{Z}} |\rho(u)|^l \times n^{-1+\frac{r}{l}} \sum_{|i|\leqslant[nt]} |\rho(i)|^r \times n^{-1+\frac{l-r}{l}} \sum_{|j|\leqslant[nt]} |\rho(j)|^{l-r}.
\end{aligned}$$

Therefore, to conclude that $\| f_{n,l}(t,\cdot)\otimes_r f_{n,l}(t,\cdot)\|_{L^2(\mathbb{R}^{2l-2r})} \to 0$, it remains to prove that, for any $r = 1, \ldots, l-1$,

$$n^{-1+\frac{r}{l}} \sum_{|j|\leqslant[nt]} |\rho(j)|^r \to 0. \tag{7.11}$$

For that, fix $\delta \in (0,t)$, and decompose the sum in (7.11) as

$$\sum_{|j|\leqslant[nt]} = \sum_{|j|\leqslant[n\delta]} + \sum_{[n\delta]<|j|\leqslant[nt]}.$$

By the Hölder inequality we obtain (recall that $\sum_{j\in\mathbb{Z}} |\rho(j)|^l \leqslant \sum_{j\in\mathbb{Z}} |\rho(j)|^q < \infty$)

$$n^{-1+l/q} \sum_{|j|\leqslant[n\delta]} |\rho(j)|^r \leqslant n^{-1+r/l} \left(2[n\delta]+1\right)^{1-r/l} \left(\sum_{j\in\mathbb{Z}} |\rho(j)|^l\right)^{r/l} \leqslant c\delta^{1-r/l},$$

where c is some constant, as well as

$$n^{-1+r/l} \sum_{[n\delta]<|j|\leqslant n} |\rho(j)|^r \leqslant \left(\sum_{[n\delta]<|j|\leqslant n} |\rho(j)|^l\right)^{r/l}.$$

The first term converges to 0 as δ goes to zero (because $1 \leqslant r \leqslant l-1$), and the second also converges to 0 for fixed δ and $n \to \infty$. This proves that (7.11) holds true.

Condition 4. For $t \in (0,\infty)$ and $N \geqslant q$, we have

$$\sum_{l=N+1}^{\infty} l!\|f_{n,l}(t,\cdot)\|^2_{L^2(\mathbb{R}^l)} = \frac{1}{n}\sum_{l=N+1}^{\infty} a_l^2 l! \sum_{i,j=1}^{[nt]} \rho(i-j)^l$$
$$\leqslant \sum_{l=N+1}^{\infty} a_l^2 l! \sum_{v\in\mathbb{Z}} |\rho(v)|^l$$
$$\leqslant \sum_{v\in\mathbb{Z}} |\rho(v)|^q \times \sum_{l=N+1}^{\infty} a_l^2 l!,$$

so that $\lim_{N\to\infty}\sup_{n\geqslant 1}\sum_{l=N+1}^{\infty} l!\|f_{n,l}(t,\cdot)\|^2_{L^2(\mathbb{R}^l)} = 0$. The proof of Theorem 7.2 is concluded. □

7.3 Non-Central Limit Theorem

In this section, we investigate what happens when, contrary to what is assumed in Section 7.2, we have $\sum_{k\in\mathbb{Z}} |\rho(k)|^q = \infty$. Actually, to go further we need to precise the behavior of ρ at infinity. We will specialize our study to the case where ρ has the form (7.5) with $0 < D < \frac{1}{q}$ and $L : (0,\infty) \to (0,\infty)$ a function which is slowly varying at infinity (which is further bounded away from 0 and infinity on every compact subset of $[0,\infty)$). We then have the following result, that goes back to Taqqu [63] (see also Davydov [12]) and Dobrushin-Major [14].

Theorem 7.3 (Dobrushin–Major–Taqqu). *Let $\phi : \mathbb{R} \to \mathbb{R}$ be the function given by* (7.3) *and recall the definition* (7.4) *of $V_n(\phi,\cdot)$.*

1. *Assume the Hermite rank of ϕ is $q = 1$, and that ρ satisfies (7.5) with* $0 < D < 1$. *Then, as $n \to \infty$,*

$$\frac{V_n(\phi,\cdot)}{n^{1-D/2}\sqrt{L(n)}} \overset{\text{f.d.d.}}{\to} \frac{a_1}{\sqrt{(1-D/2)(1-D)}} \times B^{1-D/2}, \tag{7.12}$$

   with $B^{1-D/2}$ a fractional Brownian motion of parameter $H = 1 - D/2$.

2. *Assume the Hermite rank of ϕ is $q = 2$, and that ρ satisfies* (7.5) *with* $0 < D < 1/2$. *Then, as $n \to \infty$,*

$$\frac{V_n(\phi,\cdot)}{n^{1-D}L(n)} \overset{\text{f.d.d.}}{\to} \frac{a_2}{\sqrt{(1-D)(1-2D)}} \times R^{1-D}, \tag{7.13}$$

   with R^{1-D} a Rosenblatt process of parameter $H = 1 - D$ (see Definition 7.1).

The Rosenblatt process appearing in (7.13) is defined as follows.

Definition 7.1. *Let $H \in (1/2, 1)$. The Rosenblatt process of parameter H is the stochastic process $(R_t^H)_{t\geq 0}$ defined by the double Wiener-Itô integral*

$$R_t^H = I_2^W\big(f_H(t,\cdot)\big), \quad t \geqslant 0, \tag{7.14}$$

where

$$f_H(t,x,y) = \frac{\sqrt{\frac{H}{2}(2H-1)}}{\beta(\frac{H}{2}, 1-H)} \int_0^t (s-x)_+^{\frac{H}{2}-1}(s-y)_+^{\frac{H}{2}-1} ds, \tag{7.15}$$

with β the usual Beta function.

For any $s, t \geqslant 0$, we have the crucial relationship

$$\int_{\mathbb{R}} (t-x)_+^{\frac{H}{2}-1}(s-x)_+^{\frac{H}{2}-1} dx = \beta(H/2, 1-H)|t-s|^{H-1}. \tag{7.16}$$

Indeed, for $t \geqslant s$,

$$\begin{aligned}
\int_{\mathbb{R}} (t-x)_+^{\frac{H}{2}-1}(s-x)_+^{\frac{H}{2}-1} dx &= \int_{-\infty}^{s} (t-x)^{\frac{H}{2}-1}(s-x)^{\frac{H}{2}-1} dx \\
&= \int_0^\infty (t-s+u)^{\frac{H}{2}-1} u^{\frac{H}{2}-1} du = (t-s)^{H-1} \int_0^\infty (1+v)^{\frac{H}{2}-1} v^{\frac{H}{2}-1} dv \\
&= (t-s)^{H-1} \int_0^1 w^{-H}(1-w)^{\frac{H}{2}-1} dw = \beta(H/2, 1-H)(t-s)^{H-1}.
\end{aligned}$$

Using (7.16) and (2.2), it is straightforward to check that, for any $t \geqslant 0$,

$$\begin{aligned}
E[R_t^H R_s^H] &= 2\int_{\mathbb{R}^2} f_H(t,x,y) f_H(s,x,y) dx dy \\
&= H(2H-1) \int_{[0,t]\times[0,s]} |u-v|^{2H-2} du dv \\
&= \frac{1}{2}\big(t^{2H} + s^{2H} - |t-s|^{2H}\big).
\end{aligned}$$

7.3.1 Computation of Cumulants

The Rosenblatt process is a double Wiener-Itô integral. As such, it enjoys useful properties, that we derive now in full generality. Let $f \in L^2(\mathbb{R}^2)$ be a given symmetric kernel. One of the most effective ways of dealing with $I_2^W(f)$ is to associate to f the following Hilbert-Schmidt operator:

$$A_f : L^2(\mathbb{R}) \to L^2(\mathbb{R});\ g \mapsto f \otimes_1 g = \int_{\mathbb{R}} f(\cdot, y) g(y) dy. \tag{7.17}$$

We write $\{\lambda_{f,j} : j \geqslant 1\}$ and $\{e_{f,j} : j \geqslant 1\}$, respectively, to indicate the eigenvalues of A_f and the corresponding eigenvectors (forming an orthonormal system in $L^2(\mathbb{R})$).

Some useful relations between all these objects are given in the next proposition. The proof, which is omitted here, relies on elementary functional analysis (see e.g. Section 6.2 in [20]).

Proposition 7.1. *Let f be a symmetric element of $L^2(\mathbb{R}^2)$, and let the above notation prevail.*

1. *The series $\sum_{j=1}^{\infty} \lambda_{f,j}^p$ converges for every $p \geqslant 2$, and f admits the expansion*

$$f = \sum_{j=1}^{\infty} \lambda_{f,j} \; e_{f,j} \otimes e_{f,j}, \tag{7.18}$$

where the convergence takes place in $L^2(\mathbb{R}^2)$.

2. *For every $p \geqslant 2$, one has the relations*

$$\mathrm{Tr}(A_f^p) = \int_{\mathbb{R}^p} dx_1 \dots dx_p \prod_{i=1}^{p} f(x_i, x_{i+1}) = \sum_{j=1}^{\infty} \lambda_{f,j}^p, \tag{7.19}$$

with the convention $x_{p+1} = x_1$, and where $\mathrm{Tr}(A_f^p)$ stands for the trace of the pth power of A_f.

In the following statement we collect some facts concerning the law of a random variable of the type $I_2^W(f)$.

We recall that, given a random variable F with moments of all order, the cumulant of order $p \geqslant 1$ of F is defined by

$$\kappa_p(F) = (-i)^p \frac{\partial^p}{\partial t^p}|_{t=0} \log \phi_F(t),$$

where $\phi_F(t) = E[e^{itF}]$ stand for the characteristic function of F. It can be shown that the moments of a given random variable are completely determined by its cumulants, and vice versa. For more background on cumulants, see the recent book by Peccati and Taqqu [47].

Proposition 7.2. *Let f be a symmetric element of $L^2(\mathbb{R}^2)$.*

1. *The following equality holds:*

$$I_2^W(f) \stackrel{\text{Law}}{=} \sum_{j=1}^{\infty} \lambda_{f,j} \left(N_j^2 - 1\right), \tag{7.20}$$

where $(N_j)_{j \geqslant 1}$ is a sequence of independent $\mathcal{N}(0,1)$ random variables, and the series converges in $L^2(\Omega)$.

2. *For every $p \geqslant 2$,*

$$\kappa_p\left(I_2^W(f)\right) = 2^{p-1}(p-1)! \int_{\mathbb{R}^p} dx_1 \dots dx_p \prod_{i=1}^{p} f(x_i, x_{i+1}), \tag{7.21}$$

with the convention $x_{p+1} = x_1$.

Proof. Relation (7.20) is an immediate consequence of (7.18), of the identity

$$I_2^W\left(e_{f,j}\otimes e_{f,j}\right) = I_1^W\left(e_{f,j}\right)^2 - 1,$$

as well as of the fact that the family $\left\{e_{f,j}\right\}$ is orthonormal (implying that the sequence $\{I_1^W\left(e_{f,j}\right) : j \geqslant 1\}$ is composed of independent $\mathcal{N}(0,1)$ random variables).

To prove (7.21), first observe that (7.20) implies that

$$E\left[e^{itI_2^W(f)}\right] = \prod_{j=1}^{\infty} \frac{e^{-it\lambda_{f,j}}}{\sqrt{1-2it\lambda_{f,j}}}.$$

Thus, standard computations give

$$\begin{aligned}\log E\left[e^{itI_2^W(f)}\right] &= -it\sum_{j=1}^{\infty}\lambda_{f,j} - \frac{1}{2}\sum_{j=1}^{\infty}\log\left(1-2it\lambda_{f,j}\right)\\ &= \frac{1}{2}\sum_{p=2}^{\infty}\frac{(2it)^p}{p}\sum_{j=1}^{\infty}\lambda_{f,j}^p = \sum_{p=2}^{\infty}2^{p-1}\frac{(it)^p}{p}\sum_{j=1}^{\infty}\lambda_{f,j}^p.\end{aligned} \tag{7.22}$$

We can now identify the coefficients in the series (7.22), so to deduce that

$$\kappa_1\left(I_2^W(f)\right) = E\left[I_2^W(f)\right] = 0,$$

and

$$\frac{2^{p-1}}{p}\sum_{j=1}^{\infty}\lambda_{f,j}^p = \frac{\kappa_p\left(I_2^W(f)\right)}{p!},$$

thus obtaining the desired conclusion, see also (7.19). □

The following lemma gives a further expression of (7.21) for functions of interest.

Lemma 7.1. *Let $T \subset \mathbb{R}$, let $e : T\times\mathbb{R} \to \mathbb{R}$ be a measurable function, fix two integers $d \geqslant 1$ and $p \geqslant 2$, and let $\nu_1,\ldots,\nu_d$ be given signed measures on $\mathbb{R}$. Assume further that*

$$\sum_{j=1}^{d}\int_{\mathbb{R}} dx\sqrt{\int_T e(t,x)^2|\nu_j|(dt)} < \infty.$$

Finally, for any $j = 1,\ldots,d$, define $f_j \in L^2(\mathbb{R}^2)$ to be

$$f_j(x,y) = \int_T e(s,x)e(s,y)\nu_j(ds), \quad x,y\in\mathbb{R}.$$

We then have, for all $\theta_1,\ldots,\theta_d \in \mathbb{R}$,

$$\begin{aligned}\kappa_p\left(\sum_{j=1}^{d}\theta_j I_2^W(f_j)\right) &= 2^{p-1}(p-1)!\sum_{j_1,\ldots,j_p=1}^{d}\theta_{j_1}\ldots\theta_{j_p}\\ &\quad\times\int_{T^p}\nu_{j_1}(ds_1)\ldots\nu_{j_p}(ds_p)\prod_{i=1}^{p}\int_{\mathbb{R}}e(s_i,x)e(s_{i+1},x)dx,\end{aligned} \tag{7.23}$$

with the convention $s_{p+1} = s_1$.

Proof. Using (7.21), we can write

$$\kappa_p\Big(\sum_{j=1}^d \theta_j I_2^W(f_j)\Big) = \kappa_p\Big(I_2^W\Big(\sum_{j=1}^d \theta_j f_j\Big)\Big)$$

$$= 2^{p-1}(p-1)! \int_{\mathbb{R}^p} dx_1 \dots dx_p \prod_{i=1}^p \Big(\sum_{j=1}^d \theta_j f_j(x_i, x_{i+1})\Big)$$

$$= 2^{p-1}(p-1)! \sum_{j_1,\dots,j_p=1}^d \theta_{j_1} \dots \theta_{j_p} \int_{\mathbb{R}^p} dx_1 \dots dx_p \prod_{i=1}^p f_{j_i}(x_i, x_{i+1})$$

$$= 2^{p-1}(p-1)! \sum_{j_1,\dots,j_p=1}^d \theta_{j_1} \dots \theta_{j_p} \int_{\mathbb{R}^p} dx_1 \dots dx_p \prod_{i=1}^p \int_T e(s, x_i) e(s, x_{i+1}) \nu_{j_i}(ds)$$

$$= 2^{p-1}(p-1)! \sum_{j_1,\dots,j_p=1}^d \theta_{j_1} \dots \theta_{j_p} \int_{T^p} \nu_{j_1}(ds_1) \dots \nu_{j_p}(ds_p) \int_{\mathbb{R}} e(s_i, x) e(s_{i+1}, x) dx.$$

□

Let us come back to the Rosenblatt process R^H. In the next result, we give a formula for the cumulants of any linear combination built from R^H at given times $t_1, \dots, t_d$, $d \geqslant 2$.

Proposition 7.3. *Fix* $d \geqslant 1$, *let* $t_1, \dots, t_d$ *be positive real numbers, and let* R^H *be a Rosenblatt process of parameter* $H \in (1/2, 1)$. *Then, for all* $p \geqslant 2$ *and all* $\theta_1, \dots, \theta_d \in \mathbb{R}$,

$$\kappa_p\Big(\sum_{j=1}^d \theta_j R^H(t_j)\Big) = 2^{p/2-1}(p-1)! H^{p/2}(2H-1)^{p/2} \tag{7.24}$$

$$\times \sum_{j_1,\dots,j_p=1}^d \theta_{j_1} \dots \theta_{j_p} \int_0^{t_{j_1}} \dots \int_0^{t_{j_p}} ds_1 \dots ds_p \prod_{i=1}^p |s_{i+1} - s_i|^{H-1},$$

with the convention $s_{p+1} = s_1$.

Proof. This is an immediate consequence of (7.15) and Lemma 7.1 with $T = \mathbb{R}_+$, measures

$$\nu_i(ds) = \frac{\sqrt{\frac{H}{2}(2H-1)}}{\beta(H/2, 1-H)} 1_{[0,t_i]}(s) ds$$

and the function $e(s, x) = (s-x)_+^{\frac{H}{2}-1}$. □

Corollary 7.1. *For any* $H \in (\frac{1}{2}, 1)$, *the Rosenblatt process* R^H *has stationary increments and is selfsimilar with parameter H.*

Proof. Since the law of R^H is determined by its moments (see Proposition 5.3) or equivalently by its cumulants, it suffices to use expression (7.24). Let $h > 0$. Replacing $R^H(t_i)$ by $R^H(t_i + h) - R^H(h)$ in the left-hand side of (7.24) changes the integrals $\int_0^{t_i}$ in the right-hand side by integrals $\int_h^{t_i+h}$. Since this does not modify the right-hand side, the process R^H has stationary increments. To prove selfsimilarity, let $a > 0$, replace each $t_1, \ldots, t_p$ by $at_1, \ldots, at_p$ in (7.24) and note that the right-hand side is then multiplied by a factor a^{pH}. □

7.3.2 Proof of Theorem 7.3

The starting point of the proofs of (7.12) and (7.13) is the same, and rely on the following lemma.

Lemma 7.2 (Reduction). *Let $\phi : \mathbb{R} \to \mathbb{R}$ be the function given by* (7.3) *(with Hermite rank $q \geqslant 1$). Let X be a stationary Gaussian sequence as in Section 7.1 and assume that its covariance kernel ρ satisfies* (7.5) *with $0 < D < 1/q$. Decompose ϕ as $\phi = a_q H_q + \widehat{\phi}$, and recall the definition* (7.4) *of $V_n(\phi, \cdot)$. Then, for any fixed $t \geqslant 0$, $V_n(\widehat{\phi}, t) \to 0$ in $L^2(\Omega)$ as $n \to \infty$.*

Proof. To simplify the exposition, we assume without loss of generality that $t = 1$. By Cauchy-Schwarz, we have $|\rho(j)| \leqslant 1$ for all $j \geqslant 1$. For any integer l larger than or equal to $q + 1$, we can thus write

$$E\left[\left(\frac{1}{n^{1-qD/2}L(n)^{q/2}}\sum_{k=1}^{n} H_l(X_k)\right)^2\right] = \frac{l!}{n^{2-qD}L(n)^q}\sum_{k,k'=1}^{n}\rho(k-k')^l$$

$$\leqslant \frac{l!}{n^{2-qD}L(n)^q}\left(n + 2\sum_{1\leqslant k'<k\leqslant n}|\rho(k-k')|^q\right)$$

$$\leqslant \frac{l!}{n^{1-qD}L(n)^q}\left(1 + 2\sum_{j=1}^{n} j^{-qD}L(j)^q\right).$$

We deduce using Proposition 1.3(3) that, for any $r \geqslant q$,

$$E\left[\left(\frac{1}{n^{1-qD/2}L(n)^{q/2}}\sum_{l=r}^{\infty} a_l \sum_{k=1}^{n} H_l(X_k)\right)^2\right]$$
$$\leqslant \frac{1}{n^{1-qD}L(n)^q}\left(1 + 2\sum_{j=1}^{n} j^{-qD}L(j)^q\right)\sum_{l=r}^{\infty} a_l^2 l!. \tag{7.25}$$

Now, observe the following three facts:

1. by Karamata's Theorem 7.1, we have

$$\sup_{n\geqslant 1}\frac{1}{n^{1-qD}L(n)^q}\sum_{j=1}^{n} j^{-qD}L(j)^q < \infty;$$

2. since $qD < 1$, we have that $n^{qD-1}L(n)^{-q} \to 0$ (to see this, use (7.7) with $1/L$ instead of L, the function $1/L$ being slowly varying as well);
3. because $\mathrm{Var}(\phi(N)) = \sum_{l=q}^{\infty} a_l^2 l! < \infty$ (with $N \sim \mathcal{N}(0,1)$), we have that $\sum_{l=r}^{\infty} a_l^2 l! \to 0$ as $r \to \infty$.

We deduce from (7.25) and (1)–(2)–(3) that

$$\lim_{r\to\infty} \sup_{n\geqslant 1} E\left[\left(\frac{1}{n^{1-qD/2}L(n)^{q/2}} \sum_{l=r}^{\infty} a_l \sum_{k=1}^{n} H_l(X_k)\right)^2\right] = 0. \tag{7.26}$$

On the other hand, fix $\varepsilon \in (0,1]$. There exists an integer $M > 0$ large enough so that, for all $j > M$,

$$|\rho(j)| = j^{-D}L(j) \leqslant \varepsilon \leqslant 1.$$

For any integer l larger than or equal to $q+1$, we can write

$$\begin{aligned} E\left[\left(\frac{1}{n^{1-qD/2}L(n)^{q/2}} \sum_{k=1}^{n} H_l(X_k)\right)^2\right] &= \frac{l!}{n^{2-qD}L(n)^q} \sum_{k,k'=1}^{n} \rho(k-k')^l \\ &\leqslant \frac{l!}{n^{2-qD}L(n)^q}\left(\sum_{\substack{k,k'=1\\|k-k'|\leqslant M}}^{n} 1 + 2\varepsilon \sum_{\substack{k,k'=1\\k>k'+M}}^{n} |\rho(k-k')|^q\right) \\ &\leqslant \frac{l!}{n^{1-qD}L(n)^q}\left((2M+1) + 2\varepsilon \sum_{j=1}^{n} j^{-qD}L(j)^q\right). \end{aligned}$$

Karamata's Theorem 7.1 leads to

$$\limsup_{n\to\infty} E\left[\left(\frac{1}{n^{1-qD/2}L(n)^{q/2}} \sum_{k=1}^{n} H_l(X_k)\right)^2\right] \leqslant \frac{2l!\varepsilon}{1-qD}.$$

Since $\varepsilon > 0$ is arbitrary, this implies that

$$\frac{1}{n^{1-qD/2}L(n)^{q/2}} \sum_{k=1}^{n} H_l(X_k) \overset{L^2}{\to} 0 \quad \text{as } n \to \infty \tag{7.27}$$

for all fixed $l \geqslant q+1$.

Finally, the combination of (7.26) and (7.27) implies the desired conclusion. □

Proof of (7.12). Thanks to Lemma 7.2 we can assume that $\phi(x) = a_1 H_1(x) = a_1 x$. We are thus left to show that

$$Z_n \overset{\text{f.d.d.}}{\to} \frac{1}{\sqrt{(1-D/2)(1-D)}} \times B^{1-D/2}, \tag{7.28}$$

with

$$Z_n(t) = \frac{1}{n^{1-D/2}\sqrt{L(n)}} \sum_{k=1}^{[nt]} X_k, \quad t \geqslant 0, \tag{7.29}$$

and $B^{1-D/2}$ a fractional Brownian motion of parameter $H = 1 - D/2$.

As a first step, we claim that, for any $t \geqslant 0$ and as $n \to \infty$,

$$\frac{1}{n^{2-D}L(n)} \sum_{k,k'=1}^{[nt]} \rho(k-k') \to \frac{t^{2-D}}{(1-D/2)(1-D)}. \tag{7.30}$$

Indeed, decompose the left-hand side as

$$\frac{1}{n^{2-D}L(n)} \sum_{\substack{k,k'=1,\dots,[nt] \\ |k-k'|\leqslant 2}} \rho(k-k') + \frac{1}{n^{2-D}L(n)} \sum_{\substack{k,k'=1,\dots,[nt] \\ |k-k'|\geqslant 3}} \rho(k-k').$$

Using $|\rho(k)| \leqslant 1$, the first term is bounded by $5tn^{D-1}/L(n)$ and therefore tends to zero as $n \to \infty$ (recall that $D < 1$). Concerning the second term, one has, using (7.5),

$$\frac{1}{n^{2-D}L(n)} \sum_{\substack{k,k'=1,\dots,[nt] \\ |k-k'|\geqslant 3}} \rho(k-k') = \int_{\mathbb{R}_+^2} l_n(x,x')dxdx',$$

with

$$l_n(x,x') = \sum_{\substack{k,k'=1,\dots,[nt] \\ |k-k'|\geqslant 3}} \left|\frac{k-k'}{n}\right|^{-D} \frac{L(|k-k'|)}{L(n)} 1_{[\frac{k-1}{n},\frac{k}{n})}(x) 1_{[\frac{k'-1}{n},\frac{k'}{n})}(x').$$

For fixed $|k-k'|/n$ and as $n \to \infty$, one has

$$L(|k-k'|)/L(n) = L(n \times |k-k'|/n)/L(n) \to 1,$$

so that $l_n(x,x') \to l_\infty(x,x')$ for any $x, x' \in \mathbb{R}_+$, where

$$l_\infty(x,x') = \left|x-x'\right|^{-D} 1_{[0,t]^2}(x,x').$$

Let us show that l_n is dominated by an integrable function. If $k - k' \geqslant 3$ (the case where $k' - k \geqslant 3$ is similar by symmetry), $x \in [\frac{k-1}{n}, \frac{k}{n})$ and $x' \in [\frac{k'-1}{n}, \frac{k'}{n})$, then

$$\frac{3}{n} \leqslant \frac{k-k'}{n} \leqslant x + \frac{1}{n} - x',$$

so that $x - x' \geqslant \frac{2}{n}$, implying in turn

$$\frac{k-k'}{n} \geqslant x - x' - \frac{1}{n} \geqslant \frac{x-x'}{2}.$$

Since $D < 1$, one can choose a small enough δ so that $D + \delta < 1$ and $L(k - k')/L(n) \leqslant C((k - k')/n)^{-\delta}$ (this is possible thanks to (7.7)). We get that

$$|l_n(x,x')| \leqslant C \sum_{\substack{k,k'=1,\ldots,[nt] \\ |k-k'|\geqslant 3}} \left|\frac{k-k'}{n}\right|^{-D-\delta} 1_{[\frac{k-1}{n},\frac{k}{n})}(x) 1_{[\frac{k'-1}{n},\frac{k'}{n})}(x')$$
$$\leqslant C\, 2^{D+\delta} 1_{[0,t]^2}(x,x') \left|x - x'\right|^{-D-\delta}.$$

The function in the bound is integrable on $\mathbb{R}_+^2$. Hence, the dominated convergence theorem applies and yields, as $n \to \infty$,

$$\frac{1}{n^{2-D}L(n)} \sum_{\substack{k,k'=1,\ldots,[nt] \\ |k-k'|\geqslant 3}} \rho(k-k')$$
$$\to \int_{\mathbb{R}_+^2} l_\infty(x,x') dx dx' = \int_{[0,t]^2} \left|x - x'\right|^{-D} dx dx' = \frac{t^{2-D}}{(1 - D/2)(2 - D)}.$$

This concludes the proof of (7.30).

Now, consider $t_d > \ldots > t_1 \geqslant 0$, and recall the definition (7.29) of $Z_n(t)$. For any $j \geqslant i$, one has, as $n \to \infty$,

$$E[Z_n(t_j)Z_n(t_i)]$$
$$= \frac{1}{2}E[Z_n(t_i)^2] + \frac{1}{2}E[Z_n(t_j)^2] - \frac{1}{2}E[(Z_n(t_j) - Z_n(t_i))^2]$$
$$= \frac{1}{2n^{2-D}L(n)}\Bigg(\sum_{k,k'=1}^{[nt_i]} \rho(k-k') + \sum_{k,k'=1}^{[nt_j]} \rho(k-k') - \sum_{k,k'=1}^{[nt_j]-[nt_i]} \rho(k-k')\Bigg).$$

Using (7.30), we deduce that

$$E[Z_n(t_j)Z_n(t_i)] \to \frac{1}{(2-D)(1-D)}\big(t_i^{2-D} + t_j^{2-D} - (t_j - t_i)^{2-D}\big). \tag{7.31}$$

Since $\{X_k\}_{k\geqslant 1}$ is a centered Gaussian family, so is $\{Z_n(t_i)\}_{1\leqslant i\leqslant d}$, and (7.31) implies that (7.12) holds. □

Proof of (7.13). Thanks to Lemma 7.2 we can assume that $\phi(x) = a_2 H_2(x) = a_2(x^2 - 1)$. We are thus left to show that

$$F_n \overset{\text{f.d.d.}}{\to} \frac{1}{\sqrt{(1-D)(1-2D)}} \times R^{1-D}, \tag{7.32}$$

with

$$F_n(t) = \frac{1}{n^{1-D}L(n)} \sum_{k=1}^{[nt]} (X_k^2 - 1), \quad t \geqslant 0, \tag{7.33}$$

and R^{1-D} a Rosenblatt process of parameter $H = 1 - D/2$.

Recall from Section 7.1 that one can assume without loss of generality that $X_k = I_1^W(e_k)$, with $\{e_k\} \subset L^2(\mathbb{R})$ such that (7.1) is satisfied. Using the product formula (5.26), we deduce that $F_n(t) = I_2^W(f_n(t,\cdot))$ for all $t \geqslant 0$, with

$$f_n(t,x,y) = \frac{1}{n^{1-D}L(n)} \sum_{k=1}^{[nt]} e_k(x)e_k(y), \quad x, y \in \mathbb{R}_+^2.$$

Now, consider $t_d > \ldots > t_1 \geqslant 0$. Using (7.1) and Lemma 7.1 with $T = \mathbb{R}_+$, measures

$$\nu_i(ds) = \frac{1}{n^{1-D}L(n)} \sum_{k=1}^{[nt_i]} \delta_k(ds) \quad \text{(with } \delta_k \text{ the Dirac mass at } k\text{)}$$

and the function $e(k,x) = e_k(x)$, we get, for any $p \geqslant 2$ and $\theta_1, \ldots, \theta_d \in \mathbb{R}$,

$$\begin{aligned} &\kappa_p\Big(\sum_{j=1}^{d} \theta_j F_n(t_j)\Big) \\ &= \frac{2^{p-1}(p-1)!}{n^{p-pD}L(n)^p} \sum_{j_1,\ldots,j_p=1}^{d} \theta_{j_1}\ldots\theta_{j_d} \sum_{k_1=1}^{[nt_{j_1}]} \cdots \sum_{k_p=1}^{[nt_{j_p}]} \prod_{i=1}^{p} \rho(k_{i+1}-k_i), \end{aligned} \tag{7.34}$$

with the convention $k_{p+1} = k_1$. To obtain the limit of (7.34) as $n \to \infty$ and thus to conclude the proof of (7.12), we proceed in five steps.

Step 1 (Determination of the main term). We split the sum $\sum_{k_1=1}^{[nt_{j_1}]} \ldots \sum_{k_p=1}^{[nt_{j_p}]}$ in the right-hand side of (7.34) into

$$\sum_{\substack{k_1=1,\ldots,[nt_{j_1}] \\ \cdots \\ k_p=1,\ldots,[nt_{j_p}] \\ \forall i:\,|k_{i+1}-k_i|\geqslant 3}} + \sum_{\substack{k_1=1,\ldots,[nt_{j_1}] \\ \cdots \\ k_p=1,\ldots,[nt_{j_p}] \\ \exists i:\,|k_{i+1}-k_i|\leqslant 2}}, \tag{7.35}$$

and we show that the second sum in (7.35) is asymptotically negligible as $n \to \infty$. Up to reordering, it is enough to show that

$$R_n := \frac{1}{n^{p-pD}L(n)^p} \sum_{\substack{k_1=1,\ldots,[nt_{j_1}] \\ \cdots \\ k_p=1,\ldots,[nt_{j_p}] \\ |k_1-k_p|\leqslant 2}} \prod_{i=1}^{p} \rho(k_{i+1}-k_i) \tag{7.36}$$

tends to zero as $n \to \infty$. In (7.36), let us bound $\rho(k_p - k_{p-1})\rho(k_1 - k_p)$ by 1. We get that

$$R_n \leqslant \frac{5}{n^{p-pD}L(n)^p} \sum_{k_1=1}^{[nt_{j_1}]} \cdots \sum_{k_{p-1}=1}^{[nt_{p-1}]} \prod_{i=1}^{p-2} \rho(k_{i+1} - k_i).$$

Set $g_n(t,x) = n^{D/2-1}L(n)^{-1/2}\sum_{k=1}^{[nt]} e_k(x)$, and observe that

$$\begin{aligned}
&\frac{1}{n^{(p-1)-(p-2)D}L(n)^{p-2}} \sum_{k_1=1}^{[nt_{j_1}]} \cdots \sum_{k_{p-1}=1}^{[nt_{p-1}]} \prod_{i=1}^{p-2} \rho(k_{i+1} - k_i) \\
&= \Big\langle \Big(\ldots \Big(\big(g_n(t_{j_1}, \cdot) \otimes_1 f_n(t_2, \cdot) \big) \otimes_1 f_n(t_3, \cdot) \Big) \ldots \Big) \otimes_1 f_n(t_{p-2}, \cdot), g_n(t_{p-1}, \cdot) \Big\rangle_{L^2(\mathbb{R})} \\
&\leqslant \|g_n(t_{j_1}, \cdot)\|_{L^2(\mathbb{R})} \|f_n(t_2, \cdot)\|_{L^2(\mathbb{R}^2)} \ldots \|f_n(t_{p-2}, \cdot)\|_{L^2(\mathbb{R}^2)} \|g_n(t_{p-1}, \cdot)\|_{L^2(\mathbb{R})}.
\end{aligned}$$

But

$$\begin{aligned}
\sup_{n\geqslant 1} \|f_n(t,\cdot)\|^2_{L^2(\mathbb{R}^2)} &= \sup_{n\geqslant 1} \frac{1}{n^{2-2D}L(n)^2} \sum_{k,k'=1}^{[nt]} \rho(k-k')^2 \\
&= \sup_{n\geqslant 1} \frac{1}{n^{2-2D}L(n)^2} \sum_{|k|<[nt]} \rho(k)^2 \big([nt] - |k|\big) \\
&\leqslant \sup_{n\geqslant 1} \frac{[nt]}{n^{2-2D}L(n)^2} \left(1 + 2\sum_{k=1}^{[nt]-1} k^{-2D}L(k)^2\right) < \infty,
\end{aligned}$$

where the finiteness holds because of Theorem 7.1. Similarly, we have

$$\begin{aligned}
\sup_{n\geqslant 1} \|g_n(t,\cdot)\|^2_{L^2(\mathbb{R})} &= \sup_{n\geqslant 1} \frac{1}{n^{2-D}L(n)} \sum_{k,k'=1}^{[nt]} \rho(k-k') \\
&= \sup_{n\geqslant 1} \frac{1}{n^{2-D}L(n)} \sum_{|k|<[nt]} \rho(k) \big([nt] - |k|\big) \\
&\leqslant \sup_{n\geqslant 1} \frac{[nt]}{n^{2-D}L(n)^2} \left(1 + 2\sum_{k=1}^{[nt]-1} k^{-D}L(k)\right) < \infty.
\end{aligned}$$

We deduce that $R_n = O(n^{2D-1}L(n)^{-2})$, so that $R_n \to 0$ as $n \to \infty$ since $2D < 1$.

Step 2 (Expressing sums as integrals). We now consider the first term in (7.35) and express it as an integral, so to apply the dominated convergence theorem. We have, using the specific form (7.5) of ρ,

$$\frac{1}{n^{p-pD}L(n)^p} \sum_{\substack{k_1=1,\dots,[nt_{j_1}] \\ \dots \\ k_p=1,\dots,[nt_{j_p}] \\ \forall i: |k_{i+1}-k_i|\geqslant 3}} \prod_{i=1}^{p} \rho(k_{i+1}-k_i)$$

$$= \frac{1}{n^p} \sum_{\substack{k_1=1,\dots,[nt_{j_1}] \\ \dots \\ k_p=1,\dots,[nt_{j_p}] \\ \forall i: |k_{i+1}-k_i|\geqslant 3}} \prod_{i=1}^{p} \left| \frac{k_{i+1}-k_i}{n} \right|^{-D} \frac{L(|k_{i+1}-k_i|)}{L(n)}$$

$$= \int_0^\infty \dots \int_0^\infty l_n(x_1,\dots,x_p)dx_1\dots dx_p,$$

where

$$l_n(x_1,\dots,x_p) = \sum_{\substack{k_1=1,\dots,[nt_{j_1}] \\ \dots \\ k_p=1,\dots,[nt_{j_p}] \\ \forall i: |k_{i+1}-k_i|\geqslant 3}} \prod_{i=1}^{p} \left| \frac{k_{i+1}-k_i}{n} \right|^{-D} \frac{L(|k_{i+1}-k_i|)}{L(n)}$$
$$\times 1_{[\frac{k_1-1}{n},\frac{k_1}{n})}(x_1)\dots 1_{[\frac{k_p-1}{n},\frac{k_p}{n})}(x_p).$$

Step 3 (Pointwise convergence). We show the pointwise convergence of l_n. Since, for fixed $|k_{i+1}-k_i|/n$ and as $n \to \infty$, one has

$$L(|k_{i+1}-k_i|)/L(n) = L(n \times |k_{i+1}-k_i|/n)/L(n) \to 1,$$

one deduces that $l_n(x_1,\dots,x_p) \to l_\infty(x_1,\dots,x_p)$ for any $x_1,\dots,x_p \in \mathbb{R}_+$, where

$$l_\infty(x_1,\dots,x_p) = 1_{[0,t_{j_1}]}(x_1)\dots 1_{[0,t_{j_p}]}(x_p) \prod_{i=1}^{p} |x_{i+1}-x_i|^{-D},$$

with the convention $x_{p+1} = x_1$.

Step 4 (Domination). We show that l_n is dominated by an integrable function. If $k_{i+1}-k_i \geqslant 3$ (the case where $k_{i+1}-k_i \geqslant 3$ is similar by symmetry), $x_i \in [\frac{k_i-1}{n}, \frac{k_i}{n})$ and $x_{i+1} \in [\frac{k_{i+1}-1}{n}, \frac{k_{i+1}}{n})$, then

$$\frac{3}{n} \leqslant \frac{k_{i+1}-k_i}{n} \leqslant x_{i+1} + \frac{1}{n} - x_i,$$

so that $x_{i+1} - x_i \geqslant \frac{2}{n}$, implying in turn

$$\frac{k_{i+1}-k_i}{n} \geqslant x_{i+1} - x_i - \frac{1}{n} \geqslant \frac{x_{i+1}-x_i}{2}.$$

Since $2D < 1$, choose a small enough δ so that $2D + 2\delta < 1$ and

$$\frac{L(|k_{i+1} - k_i|)}{L(n)} \leqslant C \left(\frac{|k_{i+1} - k_i|}{n}\right)^{-\delta}$$

(this is possible thanks to (7.7)). We get that

$$\begin{aligned} &|l_n(x_1, \ldots, x_p)| \\ &\leqslant C \sum_{\substack{k_1=1,\ldots,[nt_{j_1}] \\ \ldots \\ k_p=1,\ldots,[nt_{j_p}] \\ \forall i: |k_{i+1}-k_i| \geqslant 3}} \prod_{i=1}^{p} \left| \frac{k_{i+1} - k_i}{n} \right|^{-D-\delta} 1_{[\frac{k_1-1}{n}, \frac{k_1}{n})}(x_1) \ldots 1_{[\frac{k_p-1}{n}, \frac{k_p}{n})}(x_p) \\ &\leqslant C\, 2^{(D+\delta)p} 1_{[0,t_{j_1}]}(x_1) \ldots 1_{[0,t_{j_p}]}(x_p) \prod_{i=1}^{p} |x_{i+1} - x_i|^{-D}. \end{aligned}$$

The function in the bound is integrable on $\mathbb{R}_+^p$, see indeed Proposition 7.3 with $H = 1 - D > \frac{1}{2}$.

Step 5 (Dominated convergence). By combining the results of Steps 2 to 4, we get that the dominated convergence theorem applies and yields

$$\begin{aligned} &\kappa_p\left(\sum_{j=1}^{d} \theta_j F_n(t_j)\right) \\ &\to 2^{p-1}(p-1)! \sum_{j_1,\ldots,j_p=1}^{d} \theta_{j_1} \ldots \theta_{j_p} \int_0^{t_{j_1}} dx_1 \ldots \int_0^{t_{j_p}} dx_p \prod_{i=1}^{p} |x_{i+1} - x_i|^{-D}. \end{aligned}$$

We recognize that, up to a multiplicative constant, this is the quantity in (7.24) with $H = 1 - D$. More precisely,

$$\kappa_p\left(\sum_{j=1}^{d} \theta_j F_n(t_j)\right) \to \frac{1}{\sqrt{(1-D)^p(1-2D)^p}} \kappa_p\left(\sum_{j=1}^{d} \theta_j R^{1-D}(t_j)\right),$$

which concludes the proof of (7.12) thanks to Proposition 5.3 and the fact that the knowledge of moments is equivalent to the knowledge of cumulants. □

Chapter 8
Non-Commutative Fractional Brownian Motion

In the previous chapter, we showed that normalized sums associated to a stationary sequence with long-range dependence may yield a non-central limit theorem (Theorem 7.3). Here, motivated by the fact that there is often a close correspondence between classical probability and free probability, we want to investigate whether similar non-central results hold in the free probability setting. This leads to the definition of the non-commutative fractional Brownian motion. In passing, we will also prove the free counterpart of Breuer–Major theorem 7.2.

Our main reference for free probability is the book [33] by Nica and Speicher. The needed material on Wigner multiple integrals is taken from the seminal paper [4] by Biane and Speicher.

8.1 Free Probability in a Nutshell

In the same way as calculus provides a nice setting for studying limits of sums and classical Brownian motion provides a nice setting for studying limits of random walks, free probability provides a convenient framework for investigating limits of random matrices. (See chapter 5 of the excellent book [1] by Anderson, Guionnet and Zeitouni for more details about this.) In this section, we survey the main concepts of free probability theory that are useful in the sequel.

8.1.1 Non-Commutative Probability Space

Definition 8.1. *A non-commutative probability space is a von Neumann algebra $\mathscr{A}$ (that is, an algebra of bounded operators on a complex separable Hilbert space, closed under adjoint and convergence in the weak operator topology) equipped with a trace φ, that is, a unital linear functional (meaning preserving the identity) which is weakly continuous, positive (meaning $\varphi(X) \geq 0$ whenever X is a non-negative element of $\mathscr{A}$; i.e. whenever $X = YY^*$ for some $Y \in \mathscr{A}$), faithful (meaning that*

Nourdin I.: *Selected Aspects of Fractional Brownian Motion*, B&SS,
DOI 10.1007/978-88-470-2823-4_8, © Springer-Verlag Italia 2012

if $\varphi(YY^*) = 0$ *then* $Y = 0$*), and tracial (meaning that* $\varphi(XY) = \varphi(YX)$ *for all* $X, Y \in \mathcal{A}$*, even though in general* $XY \neq YX$*).*

We will not need to use the full force of this definition, only some of its consequences. The reader is referred to [33] for a systematic presentation.

8.1.2 Random variables

Definition 8.2. *In a non-commutative probability space, we refer to the self-adjoint elements of the algebra as random variables.*

Any random variable X has a law μ_X, which is defined as follows.

Proposition 8.1. *Let* $(\mathcal{A}, \varphi)$ *be a non-commutative probability space and let* X *be a random variable. There exists a unique probability measure* μ_X *on* $\mathbb{R}$ *with compact support and the same moments as* X*. More precisely, the support* μ_X *is a subset of* $[-\|X\|, \|X\|]$ *(with* $\|\cdot\|$ *the operator norm) and* μ_X *is such that*

$$\int_{\mathbb{R}} Q(x) d\mu_X(x) = \varphi(Q(X)), \tag{8.1}$$

for all real polynomial Q*.*

Proof. See [33, Proposition 3.13]. □

8.1.3 Convergence in law

Definition 8.3. *Let* $(\mathcal{A}, \varphi)$ *be a non-commutative probability space.*
1. *We say that a sequence* $(X_{1,n}, \dots, X_{k,n})$*,* $n \geqslant 1$*, of random vectors (meaning that each* $X_{i,n}$ *is a self-adjoint operator in* $(\mathcal{A}, \varphi)$*) converges in law to a random vector* $(X_{1,\infty}, \dots, X_{k,\infty})$*, and we write*

$$(X_{1,n}, \dots, X_{k,n}) \overset{\text{law}}{\to} (X_{1,\infty}, \dots, X_{k,\infty}),$$

to indicate the convergence in the sense of (joint) moments, that is,

$$\lim_{n\to\infty} \varphi\left(Q(X_{1,n}, \dots, X_{k,n})\right) = \varphi\left(Q(X_{1,\infty}, \dots, X_{k,\infty})\right), \tag{8.2}$$

for any polynomial Q *in* k *non-commuting variables.*
2. *We say that a sequence* (F_n) *of non-commutative stochastic processes (meaning that each* F_n *is a one-parameter family of self-adjoint operators* $F_n(t)$ *in* $(\mathcal{A}, \varphi)$*) converges in the sense of finite-dimensional distributions to a non-commutative stochastic process* F_∞*, and we write*

$$F_n \overset{\text{f.d.d.}}{\to} F_\infty,$$

to indicate that, for any integer $k \geqslant 1$ *and any* $t_1, \dots, t_k \geqslant 0$*,*

$$(F_n(t_1), \dots, F_n(t_k)) \overset{\text{law}}{\to} (F_\infty(t_1), \dots, F_\infty(t_k)).$$

8.1.4 Free Independence

In the free probability setting, the notion of *independence* (introduced by Voiculescu in [64]) goes as follows.

Definition 8.4. *Let $\mathcal{A}_1, \ldots, \mathcal{A}_p$ be unital subalgebras of $\mathcal{A}$. Let $X_1, \ldots, X_m$ be random variables chosen from among the $\mathcal{A}_i$'s such that, for $1 \leq j < m$, two consecutive elements X_j and X_{j+1} do not come from the same $\mathcal{A}_i$, and such that $\varphi(X_j) = 0$ for each j. The subalgebras $\mathcal{A}_1, \ldots, \mathcal{A}_p$ are said to be free (or freely independent) if, in this circumstance,*

$$\varphi(X_1 X_2 \cdots X_p) = 0. \tag{8.3}$$

Random variables are called freely independent if the unital algebras they generate are freely independent.

Freeness is in general much more complicated than classical independence. For example, if X, Y are free and $m, n \geqslant 1$, then by (8.3),

$$\varphi\big((X^m - \varphi(X^m)1)(Y^n - \varphi(Y^n)1)\big) = 0.$$

By expanding (and using the linear property of φ), we get

$$\varphi(X^m Y^n) = \varphi(X^m)\varphi(Y^n), \tag{8.4}$$

which is what we would expect under classical independence. But, by (8.3), we also have

$$\varphi\big((X - \varphi(X)1)(Y - \varphi(Y)1)(X - \varphi(X)1)(Y - \varphi(Y)1)\big) = 0.$$

By expanding and by using (8.4) and the tracial property of φ (for instance $\varphi(XYX) = \varphi(X^2Y)$) we get

$$\varphi(XYXY) = \varphi(Y)^2\varphi(X^2) + \varphi(X)^2\varphi(Y^2) - \varphi(X)^2\varphi(Y)^2,$$

which is different from $\varphi(X^2)\varphi(Y^2)$, which is what one would have obtained if X and Y were classical independent random variables. Nevertheless, if $X_1, \ldots, X_d$ are freely independent, then their joint moments are determined by the moments of X_1, ..., X_d separately, as in the classical case.

8.1.5 Semicircular Distribution

Definition 8.5. *The semicircular distribution $\mathcal{S}(m, \sigma^2)$, with mean $m \in \mathbb{R}$ and variance $\sigma^2 > 0$, is the probability distribution*

$$\mathcal{S}(m, \sigma^2)(dx) = \frac{1}{2\pi\sigma^2}\sqrt{4\sigma^2 - (x-m)^2}\, 1_{\{|x-m| \leq 2\sigma\}}\, dx. \tag{8.5}$$

By convention, $\mathcal{S}(m, 0)$ is the Dirac mass δ_m.

A simple calculation shows that the odd centered moments of $\mathcal{S}(m,\sigma^2)$ are all zero, whereas its even centered moments are given by (scaled) *Catalan numbers*: for non-negative integers k,

$$\int_{m-2\sigma}^{m+2\sigma} (x-m)^{2k}\,\mathcal{S}(m,\sigma^2)(dx) = C_k\sigma^{2k},$$

where $C_k = \frac{1}{k+1}\binom{2k}{k}$ (see, e.g., [33, Lecture 2]). In particular, the variance is σ^2 while the centered fourth moment is $2\sigma^4$. The semicircular distribution plays here the role of the Gaussian distribution. It has the following similar properties.

Proposition 8.2. 1. *If* $S \sim \mathcal{S}(m,\sigma^2)$ *and* $a,b \in \mathbb{R}$, *then* $aS+b \sim \mathcal{S}(am+b, a^2\sigma^2)$.
2. *If* $S_1 \sim \mathcal{S}(m_1,\sigma_1^2)$ *and* $S_2 \sim \mathcal{S}(m_2,\sigma_2^2)$ *are freely independent, then* $S_1 + S_2 \sim \mathcal{S}(m_1+m_2, \sigma_1^2+\sigma_2^2)$.

Proof. The first property is immediately shown. The second property can be verified using the R-transform (which characterizes the law and linearizes free independence, meaning that, if X and Y are freely independent, then $R_{X+Y}(z) = R_X(z) + R_Y(z)$, $z \in \mathbb{C}$). Since the R-transform of the semicircular law is $R_{\mathcal{S}(m,\sigma^2)}(z) = m + \sigma^2 z$ (see [33, Formula (11.13)]), the desired conclusion follows easily. Details are left to the reader. □

8.1.6 Free Brownian Motion

Definition 8.6. 1. *A one-sided free Brownian motion* $S = \{S_t\}_{t\geq 0}$ *is a non-commutative stochastic process with the following defining characteristics:*

(i) $S_0 = 0$.

(ii) *For* $t_2 > t_1 \geq 0$, *the law of* $S_{t_2} - S_{t_1}$ *is the semicircular distribution of mean 0 and variance* $t_2 - t_1$.

(iii) *For all* n *and* $t_n > \cdots > t_2 > t_1 > 0$, *the increments* S_{t_1}, $S_{t_2} - S_{t_1}$, …, $S_{t_n} - S_{t_{n-1}}$ *are freely independent.*

2. *A two-sided free Brownian motion* $S = \{S_t\}_{t\in\mathbb{R}}$ *is defined to be*

$$S_t = \begin{cases} S_t^1 & \text{if } t \geq 0 \\ S_{-t}^2 & \text{if } t < 0 \end{cases},$$

where S^1 *and* S^2 *are two freely independent one-sided free Brownian motions.*

8.1.7 Wigner Integral

From now on, we suppose that $L^2(\mathbb{R}^q)$ stands for the set of all real-valued square-integrable functions on $\mathbb{R}^q$. When $q = 1$, we only write $L^2(\mathbb{R})$ to simplify the notation.

Let $S = \{S_t\}_{t\in\mathbb{R}}$ be a two-sided free Brownian motion. Let us quickly sketch out the construction of the *Wigner integral* of f with respect to S. For an indicator function $f = 1_{[u,v]}$, the Wigner integral of f is defined by

$$\int_{\mathbb{R}} 1_{[u,v]}(x)dS_x = S_v - S_u.$$

We then extend this definition by linearity to simple functions of the form $f = \sum_{i=1}^{k} \alpha_i 1_{[u_i,v_i]}$, where $[u_i, v_i]$ are disjoint intervals of $\mathbb{R}$. Simple computations show that

$$\int_{\mathbb{R}} f(x)dS_x \sim \mathcal{S}\left(0, \int_{\mathbb{R}} f^2(x)dx\right) \quad (8.6)$$

$$\varphi\left(\int_{\mathbb{R}} f(x)dS_x \times \int_{\mathbb{R}} g(x)dS_x\right) = \langle f, g\rangle_{L^2(\mathbb{R})}. \quad (8.7)$$

By approximation, the definition of $\int_{\mathbb{R}} f(x)dS_x$ is extended to all $f \in L^2(\mathbb{R})$, and (8.6)-(8.7) continue to hold in this more general setting.

8.1.8 Semicircular Sequence and Semicircular Process

The following definition is the free counterpart of Definition 1.2.

Definition 8.7. 1. *Let $k \geqslant 1$. A random vector $(X_1, \ldots, X_k)$ is said to have a k-dimensional semicircular distribution if, for every $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$, the random variable $\lambda_1 X_1 + \ldots + \lambda_k X_k$ has a semicircular distribution. In this case, one says that the random variables $X_1, \ldots, X_k$ are jointly semicircular or, alternatively, that $(X_1, \ldots, X_k)$ is a semicircular vector.*

2. *Let I be an arbitrary set. A semicircular family indexed by I is a collection of random variables $\{X_i : i \in I\}$ such that, for every $k \geqslant 1$ and every $(i_1, \ldots, i_k) \in I^k$, the vector $(X_{i_1}, \ldots, X_{i_k})$ has a k-dimensional semicircular distribution.*
3. *When $X = \{X_i : i \in I\}$ is a semicircular family for which I is denumerable (resp. for which $I = \mathbb{R}_+$), we say that X is a semicircular sequence (resp. semicircular process).*

The distribution of any centered semicircular family $\{X_i : i \in I\}$ turns out to be uniquely determined by its covariance function $\Gamma : I^2 \to \mathbb{R}$ given by $\Gamma(i, j) = \varphi(X_i X_j)$. (This is an easy consequence of [33, Corollary 9.20].) When $I = \mathbb{N}$, the family is said to be *stationary* if $\Gamma(i, j) = \Gamma(|i - j|)$ for all $i, j \in \mathbb{N}$.

Let $X = \{X_k : k \in \mathbb{N}\}$ be a centered semicircular sequence and consider the linear span $\mathcal{H}$ of X, that is, $\mathcal{H}$ is the closed linear subspace of $L^2(\varphi)$ generated by X. It is a real separable Hilbert space and, consequently, there exists an isometry $\Phi : \mathcal{H} \to L^2(\mathbb{R})$. For any $k \in \mathbb{N}$, set $e_k = \Phi(X_k)$; we have, for all $k, l \in \mathbb{N}$,

$$\int_{\mathbb{R}} e_k(x)e_l(x)dx = \varphi(X_k X_l) = \Gamma(k, l). \quad (8.8)$$

Thus, since the covariance function Γ of X characterizes its distribution, we have

$$\{X_k : k \in \mathbb{N}\} \stackrel{\text{law}}{=} \left\{ \int_{\mathbb{R}} e_k(x) dS_x : k \in \mathbb{N} \right\},$$

with the notation of Section 8.1.7.

8.1.9 Multiple Wigner Integral

Let $S = \{S_t\}_{t \in \mathbb{R}}$ be a two-sided free Brownian motion, and let $q \geqslant 1$ be an integer. When f belongs to $L^2(\mathbb{R}^q)$ (recall from Section 8.1.7 that it means, in particular, that f is real-valued), we write f^* to indicate the function of $L^2(\mathbb{R}^q)$ given by $f^*(t_1, \dots, t_q) = f(t_q, \dots, t_1)$.

Following [4], let us quickly sketch out the construction of the *multiple Wigner integral* of f with respect to S. Let $D_q \subset \mathbb{R}^q$ be the collection of all diagonals, i.e.

$$D_q = \{(t_1, \dots, t_q) \in \mathbb{R}^q : t_i = t_j \text{ for some } i \neq j\}.$$

For an indicator function $f = 1_A$, where $A \subset \mathbb{R}^q$ has the form $A = [a_1, b_1[\times \dots \times [a_q, b_q[$ with $A \cap D_q = \emptyset$, the qth multiple Wigner integral of f is defined by

$$I_q^S(f) = (S_{b_1} - S_{a_1}) \dots (S_{b_q} - S_{a_q}).$$

We then extend this definition by linearity to simple functions of the form $f = \sum_{i=1}^k \alpha_i 1_{A_i}$, where $A_i = [a_1^i, b_1^i[\times \dots \times [a_q^i, b_q^i[$ are disjoint q-dimensional rectangles as above which do not meet the diagonals. Simple computations show that

$$\varphi(I_q^S(f)) = 0 \tag{8.9}$$

$$\varphi(I_q^S(f) I_q^S(g)) = \langle f, g^* \rangle_{L^2(\mathbb{R}^q)}. \tag{8.10}$$

By an isometry argument, the definition of $I_q^S(f)$ is extended to all $f \in L^2(\mathbb{R}^q)$, and (8.9)–(8.10) continue to hold in this more general setting. If one wants $I_q^S(f)$ to be a random variable in the sense of Section 8.1.2, it is necessary that f be *mirror symmetric*, that is, $f = f^*$, in order to ensure that $I_q^S(f)$ is self-adjoint, namely $(I_q^S(f))^* = I_q^S(f)$. Observe that $I_1^S(f) = \int_{\mathbb{R}} f(x) dS_x$ (see Section 8.1.7) when $q = 1$. We have moreover

$$\varphi(I_p^S(f) I_q^S(g)) = 0 \text{ when } p \neq q, \ f \in L^2(\mathbb{R}^p) \text{ and } g \in L^2(\mathbb{R}^q). \tag{8.11}$$

Definition 8.8. *When $r \in \{1, \dots, p \wedge q\}$, $f \in L^2(\mathbb{R}^p)$ and $g \in L^2(\mathbb{R}^q)$, we write $f \overset{r}{\frown} g$ to indicate the rth contraction of f and g, defined as being the element of $L^2(\mathbb{R}^{p+q-2r})$ given by*

$$\begin{aligned} & f \overset{r}{\frown} g(t_1, \dots, t_{p+q-2r}) \\ &= \int_{\mathbb{R}^r} f(t_1, \dots, t_{p-r}, x_1, \dots, x_r) g(x_r, \dots, x_1, t_{p-r+1}, \dots, t_{p+q-2r}) \\ &\qquad dx_1 \dots dx_r. \end{aligned} \tag{8.12}$$

By convention, set $f \overset{0}{\frown} g = f \otimes g$ as being the tensor product of f and g.

Since f and g are not necessarily symmetric functions, the position of the identified variables $x_1, \ldots, x_r$ in (8.12) is important, in contrast to what happens in classical probability, see (5.24). Observe moreover that

$$\|f \overset{r}{\frown} g\|_{L^2(\mathbb{R}^{p+q-2r})} \leqslant \|f\|_{L^2(\mathbb{R}^p)} \|g\|_{L^2(\mathbb{R}^q)} \tag{8.13}$$

by Cauchy-Schwarz, and also that $f \overset{p}{\frown} g = \langle f, g^* \rangle_{L^2(\mathbb{R}^p)}$ when $p = q$.

We have the following *product formula*, see [4, Proposition 5.3.3] and compare with (5.26).

Theorem 8.1 (Biane–Speicher). *Let $f \in L^2(\mathbb{R}^p)$ and $g \in L^2(\mathbb{R}^q)$. Then*

$$I_p^S(f) I_q^S(g) = \sum_{r=0}^{p\wedge q} I_{p+q-2r}^S(f \overset{r}{\frown} g). \tag{8.14}$$

We deduce the following useful result, which is the free counterpart of (5.16):

Corollary 8.1. *Let $e \in L^2(\mathbb{R})$ and $q \geqslant 1$. Then*

$$U_q\left(\int_{\mathbb{R}} e(x) dS_x\right) = I_q^S(e^{\otimes q}). \tag{8.15}$$

Here, $U_0(x) = 1$, $U_1(x) = x$, $U_2(x) = x^2 - 1$, $U_3(x) = x^3 - 2x$, ..., is the sequence of Tchebycheff polynomials of second kind (determined by the recursion $U_{q+1} = xU_q - U_{q-1}$) and $\int_{\mathbb{R}} e(x) dS_x$ is understood as a Wigner integral (as defined previously).

Proof. The proof is by induction on $q \geqslant 1$. The case $q = 1$ is obvious, as $U_1 = X$ and $I_1^S(e) = \int_{\mathbb{R}} e(x) dS_x$. Assume now that (8.15) is shown for $1, \ldots, q$, and let us prove it for $q + 1$. We have, using respectively our induction property, (8.14) and the calculation $e \overset{1}{\frown} (e^{\otimes q}) = \|e\|^2_{L^2(\mathbb{R})} e^{\otimes q-1} = e^{\otimes q-1}$,

$$\begin{aligned} &U_{q+1}\left(\int_{\mathbb{R}} e(x) dS_x\right) \\ &= \int_{\mathbb{R}} e(x) dS_x \times U_q\left(\int_{\mathbb{R}} e(x) dS_x\right) - U_{q-1}\left(\int_{\mathbb{R}} e(x) dS_x\right) \\ &= I_1^S(e) I_q^S(e^{\otimes q}) - I_{q-1}^S(e^{\otimes q-1}) = I_{q+1}^S(e^{\otimes q+1}). \end{aligned}$$

Hence, the desired conclusion is proved by induction. □

8.1.10 Wiener–Wigner Transfer Principle

We state a Wiener–Wigner transfer principle for translating results between the classical and free chaoses. This transfer principle will allow us to easily prove the free version of the Breuer–Major Theorem 7.2. It is important to note that Theorem 8.2 requires the strong assumption that the kernels are symmetric in both the classical and free cases. While this is no loss of generality in the Wiener chaos, it applies to only a small subspace of the Wigner chaos of orders 3 or higher.

Theorem 8.2 (Kemp–Nourdin–Peccati–Speicher [26, 41]). *Let $d, q_1, \ldots, q_d \geqslant 1$ be some fixed integers, consider a positive definite symmetric matrix $C \in \mathcal{M}_d(\mathbb{R})$ and let W (resp. S) be a classical (resp. free) Brownian motion. Let $(G_1, \ldots, G_d)$ be a Gaussian vector and $(S_1, \ldots, S_d)$ be a semicircular vector, both centered with covariance C. For each $i = 1, \ldots, d$, we consider a sequence $\{f_{i,n}\}_{n\geqslant 1}$ of symmetric functions in $L^2(\mathbb{R}^{q_i})$. Then, as $n \to \infty$,*

$$\big(I^S_{q_1}(f_{1,n}), \ldots, I^S_{q_d}(f_{d,n})\big) \overset{\text{law}}{\to} (S_1, \ldots, S_d)$$

if and only if

$$\big(I^W_{q_1}(f_{1,n}), \ldots, I^W_{q_d}(f_{d,n})\big) \overset{\text{law}}{\to} \big(\sqrt{q_1!}G_1, \ldots, \sqrt{q_d!}G_d\big).$$

Proof. In order to keep the size of the book within bounds, we only consider the one-dimensional case, that is, we do the proof by assuming that $d = 1$. The complete proof can be found in [41].

More precisely, we shall prove the following result. Let $q \geqslant 2$ be a given integer (the case $q = 1$ being trivial), let $\sigma > 0$ be a non-negative real number, let W (resp. S) be a classical (resp. free) Brownian motion, and let $\{f_n\}_{n\geqslant 1}$ be a sequence of symmetric functions in $L^2(\mathbb{R}^q)$. Then, as $n \to \infty$, we have equivalence between

$$I^S_q(f_n) \overset{\text{law}}{\to} \mathcal{S}(0, \sigma^2) \tag{8.16}$$

and

$$I^W_q(f_n) \overset{\text{law}}{\to} \mathcal{N}(0, q!\sigma^2). \tag{8.17}$$

The proof of (8.16)↔(8.17) is divided into several steps.

Step 1 (Expressing the moments of $I^S_q(f)$). Let f be a symmetric function in $L^2(\mathbb{R}^q)$ and assume that $\|f\|_{L^2(\mathbb{R}^q)} = 1$. Fix $k \geqslant 3$. By iterating the product formula (8.14), we can write

$$I^S_q(f)^k = \sum_{\mathrm{r}\in A_{k,q}} I^S_{kq-2|\mathrm{r}|}\Big((\ldots((f \overset{r_1}{\frown} f) \overset{r_2}{\frown} f)\ldots) \overset{r_{k-1}}{\frown} f\Big),$$

where

$$\begin{aligned} A_{k,q} = \big\{&\mathrm{r} = (r_1, \ldots, r_{k-1}) \in \{0, 1, \ldots, q\}^{k-1} : r_2 \leqslant 2q - 2r_1, \\ &r_3 \leqslant 3q - 2r_1 - 2r_2, \ldots, r_{k-1} \leqslant (k-1)q - 2r_1 - \ldots - 2r_{k-2}\big\}, \end{aligned}$$

and $|\mathrm{r}| = r_1 + \ldots + r_{k-1}$. By taking the φ-trace in the previous expression and taking into account that (8.9) holds, we deduce that

$$\varphi(I^S_q(f)^k) = \sum_{\mathrm{r}\in B_{k,q}} (\ldots((f \overset{r_1}{\frown} f) \overset{r_2}{\frown} f)\ldots) \overset{r_{k-1}}{\frown} f, \tag{8.18}$$

with

$$B_{k,q} = \{\mathrm{r} = (r_1, \ldots, r_{k-1}) \in A_{k,q} : 2|\mathrm{r}| = kq\}.$$

If $\mathrm{r} \in B_{k,q}$ then $2r_1 + \ldots + 2r_{k-1} = kq$ and $r_{k-1} \leqslant (k-1)q - 2r_1 - \ldots - 2r_{k-2}$, implying in turn that

$$2r_{k-1} = q + (k-1)q - 2r_1 - \ldots - 2r_{k-2} \geqslant q + r_{k-1},$$

that is, $r_{k-1} = q$. As a result, (8.18) becomes

$$\varphi(I_q^S(f)^k) = \sum_{\mathrm{r} \in C_{k,q}} \left\langle (\ldots((f \overset{r_1}{\frown} f) \overset{r_2}{\frown} f) \ldots) \overset{r_{k-2}}{\frown} f, f \right\rangle_{L^2(\mathbb{R}^q)}, \tag{8.19}$$

with

$$\begin{aligned} C_{k,q} = \{ & \mathrm{r} = (r_1, \ldots, r_{k-2}) \in \{0, 1, \ldots, q\}^{k-2} : r_2 \leqslant 2q - 2r_1, \\ & r_3 \leqslant 3q - 2r_1 - 2r_2, \ldots, r_{k-2} \leqslant (k-2)q - 2r_1 - \ldots - 2r_{k-3}, \\ & 2r_1 + \ldots + 2r_{k-2} = (k-2)q \}. \end{aligned}$$

Let us decompose $C_{k,q}$ into $D_{k,q} \cup E_{k,q}$, with $D_{k,q} = C_{k,q} \cap \{0, q\}^{k-2}$ and $E_{k,q} = C_{k,q} \setminus D_{k,q}$. We then have

$$\begin{aligned} \varphi(I_q^S(f)^k) = & \sum_{\mathrm{r} \in D_{k,q}} \left\langle (\ldots((f \overset{r_1}{\frown} f) \overset{r_2}{\frown} f) \ldots) \overset{r_{k-2}}{\frown} f, f \right\rangle_{L^2(\mathbb{R}^q)} \\ & + \sum_{\mathrm{r} \in E_{k,q}} \left\langle (\ldots((f \overset{r_1}{\frown} f) \overset{r_2}{\frown} f) \ldots) \overset{r_{k-2}}{\frown} f, f \right\rangle_{L^2(\mathbb{R}^q)}. \end{aligned}$$

Using the two relationships $f \overset{0}{\frown} f = f \otimes f$ and $f \overset{q}{\frown} f = \|f\|^2_{L^2(\mathbb{R}^q)} = 1$, it is evident that, for all $\mathrm{r} \in D_{k,q}$,

$$\left\langle (\ldots((f \overset{r_1}{\frown} f) \overset{r_2}{\frown} f) \ldots) \overset{r_{k-2}}{\frown} f, f \right\rangle_{L^2(\mathbb{R}^q)} = 1.$$

We deduce that

$$\varphi(I_q^S(f)^k) = \#D_{k,q} + \sum_{\mathrm{r} \in E_{k,q}} \left\langle (\ldots((f \overset{r_1}{\frown} f) \overset{r_2}{\frown} f) \ldots) \overset{r_{k-2}}{\frown} f, f \right\rangle_{L^2(\mathbb{R}^q)}.$$

On the other hand, by applying (8.19) with $q = 1$ and $f = 1_{[0,1]}$, we get that

$$\begin{aligned} \varphi(S_1^k) &= \varphi(I_1^S(1_{[0,1]})^k) \\ &= \sum_{\mathrm{r} \in C_{k,1}} \left\langle (\ldots((1_{[0,1]} \overset{r_1}{\frown} 1_{[0,1]}) \overset{r_2}{\frown} 1_{[0,1]}) \ldots) \overset{r_{k-2}}{\frown} 1_{[0,1]}, 1_{[0,1]} \right\rangle_{L^2(\mathbb{R})} \\ &= \sum_{\mathrm{r} \in C_{k,1}} 1 = \#C_{k,1}. \end{aligned}$$

Since it is easily checked that $D_{k,q}$ is in bijection with $C_{k,1}$ (indeed, divide the r_i's in $D_{k,q}$ by q), we deduce that

$$\varphi(I_q^S(f)^k) = \varphi(S_1^k) + \sum_{\mathrm{r}\in E_{k,q}} \langle (\ldots((f \overset{r_1}{\frown} f) \overset{r_2}{\frown} f)\ldots) \overset{r_{k-2}}{\frown} f, f\rangle_{L^2(\mathbb{R}^q)}. \quad (8.20)$$

Step 2 (Fourth moment). Let us specialize (8.20) to the case $k = 4$. We then have

$$\begin{aligned}\varphi(I_q^S(f)^4) &= \varphi(S_1^4) + \sum_{r=1}^{q-1} \langle (f \overset{r}{\frown} f) \overset{q-r}{\frown} f, f\rangle_{L^2(\mathbb{R}^q)} \\ &= 2 + \sum_{r=1}^{q-1} \| f \overset{r}{\frown} f\|^2_{L^2(\mathbb{R}^{2q-2r})}. \qquad (8.21)\end{aligned}$$

Step 3 (Proof of (8.16)→(8.17)). Assume that (8.16) holds true. Without loss of generality, one can assume that $\sigma = 1$ and that $\varphi(I_q^S(f_n)^2) = \|f_n\|^2_{L^2(\mathbb{R}^q)} = 1$ for any n (instead of $\lim_{n\to\infty} \varphi(I_q^S(f_n)^2) = 1$). Due to (8.16), we have $\varphi(I_q^S(f_n)^4) \to \varphi(S_1^4) = 2$ as $n \to \infty$. We deduce from (8.21) that, for any $r = 1, \ldots, q-1$ and as $n \to \infty$,

$$\|f_n \otimes_r f_n\|^2_{L^2(\mathbb{R}^{2q-2r})} = \|f_n \overset{r}{\frown} f_n\|^2_{L^2(\mathbb{R}^{2q-2r})} \to 0. \qquad (8.22)$$

This, together with $E[I_q^W(f_n)^2] = q!\|f_n\|^2_{L^2(\mathbb{R}^q)} = q!$ for any n, implies by Corollary 6.1 that (8.17) holds true.

Step 4 (Proof of (8.17)→(8.16)). Assume that (8.17) holds true. First, using Proposition 5.4, we deduce that $E[I_q^W(f_n)^2] = q!\|f_n\|^2_{L^2(\mathbb{R}^q)} \to q!\sigma^2$ as $n \to \infty$. Without loss of generality, one can assume that $\sigma = 1$ and that $\|f_n\|^2_{L^2(\mathbb{R}^q)} = 1$ for any n. We deduce from Corollary 6.1 that (8.22) holds true. Fix $k \geqslant 3$, let $(r_1, \ldots, r_{k-2}) \in E_{k,q}$ and let $j \in \{1, \ldots, k-2\}$ be the smallest integer such that $r_j \in \{1, \ldots, q-1\}$. Then:

$$\begin{aligned}&\left|\langle (\ldots((f_n \overset{r_1}{\frown} f_n) \overset{r_2}{\frown} f_n)\ldots) \overset{r_{k-2}}{\frown} f_n, f_n\rangle_{L^2(\mathbb{R}^q)}\right| \\ &= \left|\langle (\ldots((f_n \overset{r_1}{\frown} f_n) \overset{r_2}{\frown} f_n)\ldots \overset{r_{j-1}}{\frown} f_n) \overset{r_j}{\frown} f_n) \overset{r_{j+1}}{\frown} f_n)\ldots) \overset{r_{k-2}}{\frown} f_n, f_n\rangle_{L^2(\mathbb{R}^q)}\right| \\ &= \left|\langle (\ldots((f_n \otimes \ldots \otimes f_n) \overset{r_j}{\frown} f_n) \overset{r_{j+1}}{\frown} f_n)\ldots) \overset{r_{k-2}}{\frown} f_n, f_n\rangle_{L^2(\mathbb{R}^q)}\right| \\ &\qquad\qquad\qquad\qquad \text{(since } f_n \overset{q}{\frown} f_n = 1\text{)} \\ &= \left|\langle (\ldots((f_n \otimes \ldots \otimes f_n) \otimes (f_n \overset{r_j}{\frown} f_n)) \overset{r_{j+1}}{\frown} f_n)\ldots) \overset{r_{k-2}}{\frown} f_n, f_n\rangle_{L^2(\mathbb{R}^q)}\right| \\ &\leqslant \|(f_n \otimes \ldots \otimes f_n) \otimes (f_n \overset{r_j}{\frown} f_n)\| \|f_n\|^{k-j-1} \quad \text{(Cauchy-Schwarz)} \\ &= \|f_n \overset{r_j}{\frown} f_n\| \quad \text{(since } \|f_n\|^2_{L^2(\mathbb{R}^q)} = 1\text{)} \\ &\to 0 \quad \text{as } n \to \infty \quad \text{by (8.22).}\end{aligned}$$

Therefore, we deduce from (8.20) that $\varphi(I_q^S(f_n)^k) \to \varphi(S_1^k)$ as $n \to \infty$, which is equivalent to (8.16). □

8.2 Non-Commutative Fractional Brownian Motion

We are now in a position to define the non-commutative fractional Brownian motion.

Definition 8.9. *Let $H \in (0,1)$. A non-commutative fractional Brownian motion (ncfBm in short) of Hurst parameter H is a centered semicircular process $S^H = \{S^H_t\}_{t\geqslant 0}$ (in the sense of Definition 8.7) with covariance function*

$$\varphi(S^H_t S^H_s) = \frac{1}{2}\big(t^{2H} + s^{2H} - |t-s|^{2H}\big). \tag{8.23}$$

It is readily checked that $S^{1/2}$ is nothing but a one-sided free Brownian motion. Immediate properties of S^H, obtained by reasoning as in the proof of Proposition 2.2, include the selfsimilarity property and the stationary property of the increments. Conversely, ncfBm of parameter H is the only standardized semicircular process to verify these two properties, since they determine the covariance (8.23), see again Proposition 2.2.

In the classical probability case, we derived in Section 2.3 three representations for the fractional Brownian motion. These representations continue to hold *mutatis mutandis* for ncfBm, by replacing the Wiener integral by its Wigner counterpart. For example, and as in (2.5), we have here

$$S^H \overset{\text{law}}{=} \frac{1}{c_H}\left(\int_0^\infty \big((t-u)^{H-\frac{1}{2}} - (-u)^{H-\frac{1}{2}}\big)dS_u + \int_0^t (t-u)^{H-\frac{1}{2}}dS_u\right),$$

with S a two-sided free Brownian motion.

As an illustration, let us show that normalized sums of semicircular sequences can converge to ncfBm.

Proposition 8.3. *Let $\{X_k : k \in \mathbb{N}\}$ be a stationary semicircular sequence with $\varphi(X_k) = 0$ and $\varphi(X_k^2) = 1$, and suppose that its correlation kernel $\rho(k-l) = \varphi(X_k X_l)$ verifies*

$$\sum_{k,l=1}^{n} \rho(k-l) \sim K n^{2H} L(n) \quad \text{as } n \to \infty, \tag{8.24}$$

with $L : (0,\infty) \to (0,\infty)$ slowly varying at infinity (see (7.6)), $0 < H < 1$ and $K > 0$. Consider the sequence (Z_n) of non-commutative stochastic processes given by

$$Z_n(t) = \frac{1}{n^H\sqrt{L(n)}}\sum_{k=1}^{[nt]} X_k, \quad t \geqslant 0.$$

Then $Z_n \overset{\text{f.d.d.}}{\to} \sqrt{K}\, S^H$ as $n \to \infty$, where S^H is a non-commutative fractional Brownian motion.

Proof. For any $t \geqslant s \geqslant 0$, we have, as $n \to \infty$,

$$\begin{aligned}
&\varphi\left[Z_n(t)Z_n(s)\right] \\
&= \frac{1}{2}\varphi\left[Z_n(t)^2\right] + \frac{1}{2}\varphi\left[Z_n(s)^2\right] - \frac{1}{2}\varphi\left[(Z_n(t)-Z_n(s))^2\right] \\
&= \frac{1}{2n^{2H}L(n)}\sum_{i,j=1}^{[nt]}\rho(i-j) + \frac{1}{2n^{2H}L(n)}\sum_{i,j=1}^{[ns]}\rho(i-j) \\
&\qquad - \frac{1}{2n^{2H}L(n)}\sum_{i,j=1}^{[nt]-[ns]}\rho(i-j) \\
&\to \frac{K}{2}\left(t^{2H} + s^{2H} - (t-s)^{2H}\right) = K\,\varphi(S^H(t)S^H(s)).
\end{aligned}$$

Since the X_k's are centered and jointly semicircular, the process Z_n is centered and semicircular as well, and the desired conclusion follows. □

8.3 Central and Non-Central Limit Theorems

8.3.1 General Framework

Let $Y = \{Y_k : k \in \mathbb{N}\}$ be a stationary semicircular sequence with $\varphi(Y_k) = 0$ and $\varphi(Y_k^2) = 1$, and let $\rho(k-l) = \varphi(Y_k Y_l)$ be its correlation kernel. (Observe that ρ is symmetric, that is, $\rho(n) = \rho(-n)$ for all $n \geqslant 1$.) Let $U_0(x) = 1$, $U_1(x) = x$, $U_2(x) = x^2 - 1$, $U_3(x) = x^3 - 2x, \ldots$, denote the sequence of Tchebycheff polynomials of second kind (determined by the recursion $xU_k = U_{k+1} + U_{k-1}$), and consider a polynomial $Q \in \mathbb{R}[X]$ of the form

$$Q(x) = \sum_{l \geqslant q} a_l U_l(x), \tag{8.25}$$

with $q \geqslant 1$ and $a_q \neq 0$, and where only a finite number of coefficients a_l are non zero. The integer q is called the *Tchebycheff rank* of Q. Finally, set

$$W_n(Q,t) = \sum_{k=1}^{[nt]} Q(Y_k) = \sum_{l \geqslant q} a_l \sum_{k=1}^{[nt]} U_l(Y_k), \quad t \geqslant 0. \tag{8.26}$$

The main result of this section goes as follows:

Theorem 8.3 (Nourdin–Taqqu [44]). *Let Q be the polynomial defined by* (8.25) *and let $W_n(Q,\cdot)$ be defined by* (8.26).

1. *If* $\sum_{k\in\mathbb{Z}} |\rho(k)|^q$ *is finite, then, as* $n \to \infty$,

$$\frac{W_n(Q,\cdot)}{\sqrt{n}} \overset{\text{f.d.d.}}{\to} \sqrt{\sum_{l\geqslant q} a_l^2 \sum_{k\in\mathbb{Z}} \rho(k)^l} \times S^{1/2}, \tag{8.27}$$

   with $S^{1/2}$ *a free Brownian motion.*
2. *Let* $L : (0,\infty) \to (0,\infty)$ *be a function which is slowly varying at infinity and bounded away from 0 and infinity on every compact subset of* $[0,\infty)$, *assume that* $q = 1$ *and that* ρ *has the form*

$$\rho(k) = k^{-D} L(k), \quad k \geqslant 1, \tag{8.28}$$

   with $0 < D < 1$. *Then, as* $n \to \infty$,

$$\frac{W_n(Q,\cdot)}{n^{1-D/2}\sqrt{L(n)}} \overset{\text{f.d.d.}}{\to} \frac{a_1}{\sqrt{(1-D/2)(1-D)}} \times S^{1-D/2}, \tag{8.29}$$

   with $S^{1-D/2}$ *a non-commutative fractional Brownian motion of parameter* $H = 1 - D/2$.

In [44], we derive more generally the limit in law of $W_n(Q,\cdot)$ (properly normalized) for $q \geqslant 2$ and when ρ satisfies (8.28) with $0 < D < \frac{1}{q}$.

8.3.2 Proof of the Central Limit Theorem (8.27)

Consider the Gaussian counterpart (in the usual probabilistic sense) of $\{Y_k\}_{k\in\mathbb{N}}$, namely $X = \{X_k\}_{k\in\mathbb{N}}$ where X is a stationary *Gaussian* sequence with mean 0 and same correlation ρ.

We assume in this proof that $\sum_{k\in\mathbb{Z}} |\rho(k)|^q < \infty$; this implies $\sum_{k\in\mathbb{Z}} |\rho(k)|^l < \infty$ for all $l \geqslant q$. Since Q given by (8.25) is a polynomial, we can choose N large enough so that $a_l = 0$ for all $l \geqslant N$. Set

$$V_n(H_l, t) = \sum_{k=1}^{[nt]} H_l(X_k), \quad t \geqslant 0, \quad l = q, \ldots, N.$$

Breuer–Major Theorem 7.2, together with Theorem 6.5, implies that

$$\left(\frac{V_n(H_q,\cdot)}{\sqrt{n}}, \ldots, \frac{V_n(H_N,\cdot)}{\sqrt{n}}\right) \tag{8.30}$$

converges as $n \to \infty$ in the sense of finite-dimensional distributions to

$$\left(\sigma_q \sqrt{q!}\, B_q, \ldots, \sigma_N \sqrt{N!}\, B_N\right),$$

where $\sigma_l^2 := \sum_{k\in\mathbb{Z}} \rho(k)^l$ $(l = q, \dots, N)$, and $B_q, \dots, B_N$ are independent classical Brownian motions. (The fact that $\sum_{k\in\mathbb{Z}} \rho(k)^l \geqslant 0$ is part of the conclusion.) On the other hand, using both (5.16) and (8.15), we get, for any $l = q, \dots, N$, that

$$V_n(H_l, t) = I_l^W\left(\sum_{k=1}^{[nt]} e_k^{\otimes l}\right) \quad \text{and} \quad W_n(U_l, t) = I_l^S\left(\sum_{k=1}^{[nt]} e_k^{\otimes l}\right),$$

where the sequence $\{e_k\}_{k\in\mathbb{N}}$ is as in (8.8), I_l^W stands for the multiple Wiener-Itô integral of order l and I_l^S stands for the multiple Wigner integral of order l. We observe that the kernel $\sum_{k=1}^{[nt]} e_k^{\otimes l}$ is a symmetric function of $L^2(\mathbb{R}^l)$. Therefore, according to Theorem 8.2, we deduce that the free counterpart of (8.30) holds as well, that is, we have that

$$\left(\frac{W_n(U_q,\cdot)}{\sqrt{n}}, \dots, \frac{W_n(U_N,\cdot)}{\sqrt{n}}\right),$$

converges as $n \to \infty$ in the sense of finite-dimensional distributions to

$$\left(\sigma_q\, S_q, \dots, \sigma_N\, S_N\right),$$

where $S_q, \dots, S_N$ denote freely independent free Brownian motions. The desired conclusion (8.27) follows then as a consequence of this latter convergence, together with the decomposition (8.25) of Q and the identity in law (see Section 8.1.5):

$$a_q\sigma_q S_q + \dots + a_N\sigma_N S_N \overset{\text{law}}{=} \sqrt{a_q^2\sigma_q^2 + \dots + a_N^2\sigma_N^2} \times S^{1/2}. \qquad \square$$

8.3.3 *Proof of the Non-Central Limit Theorem* (8.29)

Since the following result is a literal extension of Lemma 7.2, we let the details of its proof to the reader.

Lemma 8.1 (Reduction). *Let $Q \in \mathbb{R}[X]$ be the polynomial given by (8.25) (with Tchebicheff rank $q \geqslant 1$). Let Y be a stationary semicircular sequence as in Section 8.3.1 and assume that its covariance kernel ρ satisfies (8.28) with $0 < D < 1/q$. Decompose Q as $Q = a_qU_q + R$, and recall the definition (8.26) of $W_n(Q,\cdot)$. Then, for any fixed $t \geqslant 0$, $W_n(R,t) \to 0$ in $L^2(\varphi)$ as $n \to \infty$.*

We can now proceed with the proof of (8.29).

Proof of (8.29). Thanks to Lemma 8.1 we can assume that $Q = a_1U_1 = a_1X$. We are thus left to show that

$$\frac{1}{n^{1-D/2}\sqrt{L(n)}} \sum_{k=1}^{[n\cdot]} Y_k \overset{\text{f.d.d.}}{\to} \frac{1}{\sqrt{(1-D/2)(1-D)}} \times S^{1-D/2}, \tag{8.31}$$

where $S^{1-D/2}$ is a non-commutative fractional Brownian motion of parameter $H = 1 - D/2$. But (8.31) is in fact a direct consequence of Proposition 8.3 together with (7.30).

References

1. Anderson G., Guionnet A., Zeitouni O.: An Introduction to Random Matrices. Cambridge University Press, Cambridge (2010)
2. Aurzada F.: On the one-sided exit problem for fractional Brownian motion. Electr. Comm. Probab. **16**, 392–404 (2011)
3. Biagini F., Hu Y., Øksendal B., Zhang T.: Stochastic Calculus for Fractional Brownian Motion and Applications. Springer-Verlag London (2010)
4. Biane P., Speicher R.: Stochastic analysis with respect to free Brownian motion and analysis on Wigner space. Probab. Theory Rel. Fields **112**, 373–409 (1998)
5. Biermé H., Bonami A., Léon J.: Central Limit Theorems and Quadratic Variations in terms of Spectral Density. Electron. J. Probab. **16**, 362–395 (2011)
6. Bingham N.H., Goldie C.M., Teugels J.L.: Regular Variation, 2nd edn. Cambridge University Press, Cambridge (1989)
7. Breuer P., Major P.: Central limit theorems for non-linear functionals of Gaussian fields. J. Mult. Anal. **13**, 425–441 (1983)
8. Chen L.H.Y., Goldstein L., Shao Q.M.: Normal Approximation by Stein's Method. Probability and Its Applications. Springer-Verlag, Berlin Heidelberg (2010)
9. Cheridito P.: Mixed fractional Brownian motion. Bernoulli **7**, 913–934 (2001)
10. Cheridito P., Nualart D.: Stochastic integral of divergence type with respect to fractional Brownian motion with Hurst parameter $H \in (0, \frac{1}{2})$. Ann. Inst. H. Poincaré Probab. Statist. **41**(6), 1049–1081 (2005)
11. Cohen S., Istas J.: Fractional Fields and Applications. Springer-Verlag (2011)
12. Davydov Y.A.: The invariance principle for stationary processes. Theor. Probab. Appl. **15**, 487–498 (1970)
13. Decreusefond L., Üstünel A.S.: Stochastic Analysis of the Fractional Brownian Motion. Pot. Anal. **10**(2), 177–214 (1999)
14. Dobrushin R.L., Major P.: Non-central limit theorems for non-linear functions of Gaussian fields. Z. Wahrscheinlichkeitstheorie verw. Gebiete **50**, 27–52 (1979)
15. Doss H.: Liens entre équations différentielles stochastiques et ordinaires. Annales de l'I.H.P. sec. B **13**(2), 99–125 (1977)
16. Dudley R.M.: Real Analysis and Probability, 2nd edn. Cambridge University Press, Cambridge (2003)
17. Embrechts P., Maejima M.: Selfsimilar processes. Princeton University Press, Princeton (2002)
18. Friz P., Victoir N.: Multidimensional stochastic processes as rough paths. Cambridge Studies in Advanced Mathematics 120. Cambridge University Press, Cambridge (2010).
19. Gradinaru M., Nourdin I., Russo F., Vallois P.: m-order integrals and generalized Itô's formula: the case of a fractional Brownian motion with any Hurst index. Ann. Inst. H. Poincaré Probab. Statist. **41**(4) 781–806 (2005)

20. Hirsch F., Lacombe G.: Elements of Functional Analysis. Springer-Verlag, New York (1999)
21. Houdré C.: An Elementary Proof of the Gaussian Concentration Inequality. Private communication (2008)
22. Hu Y.: Integral transformations and anticipative calculus for fractional Brownian motions. Mem. Amer. Math. Soc. **175**(825) (2005)
23. Hurst H.E.: Long-term storage capacity in reservoirs. Trans. Amer. Soc. Civil Eng. **116**, 400–410 (1951)
24. Janson S.: Gaussian Hilbert Spaces. Cambridge University Press, Cambridge (1997)
25. Kallenberg O.: Foundations of modern probability. Springer-Verlag, New York (1997)
26. Kemp T., Nourdin I., Peccati G., Speicher R.: Wigner chaos and the fourth moment. Ann. Probab., in press (2011)
27. Kolmogorov A.N.: Wienersche Spiralen und einige andere interessante Kurven im Hilbertschen Raum. Comptes Rendus (Doklady) de l'Académie des Sciences de l'URSS (N.S.) **26**, 115–118 (1940)
28. Lyons T., Qian Z.: System control and rough paths. Oxford Mathematical Monographs. Oxford University Press, Oxford (2002)
29. Mandelbrot B.B., Van Ness J.W.: Fractional Brownian motions, fractional noises and applications. SIAM Review **10**, 422–437 (1968)
30. Mishura Y.S.: Stochastic Calculus for Fractional Brownian Motion and Related Processes. Springer-Verlag, Berlin Heidelberg (2008)
31. Molchan G.M.: Maximum of a fractional Brownian motion: probabilities of small values. Comm. Math. Phys. **205**(1), 97–111 (1999)
32. Nelson E.: The free Markov field. J. Funct. Analysis **12**, 211–227 (1973)
33. Nica A., Speicher R.: Lectures on the Combinatorics of Free Probability. Cambridge University Press, Cambridge (2006)
34. Norros I., Valkeila E., Virtamo J.: An elementary approach to a Girsanov formula and other analytical results on fractional Brownian motions. Bernoulli **5**(4), 571–587 (1999)
35. Nourdin I.: Asymptotic behavior of certain weighted quadratic and cubic variations of fractional Brownian motion. Ann. Probab. **36**(6), 2159–2175 (2008)
36. Nourdin I., Nualart D.: Central limit theorems for multiple Skorohod integrals. J. Theoret. Probab. **23**(1), 39–64 (2010)
37. Nourdin I., Nualart D., Tudor C.A.: Central and non-central limit theorems for weighted power variations of fractional Brownian motion. Ann. Inst. H. Poincaré Probab. Statist. **46**(4), 1055–1079 (2010)
38. Nourdin I., Peccati G.: Stein's method on Wiener chaos. Probab. Theory Rel. Fields **145**(1) 75–118 (2009)
39. Nourdin I., Peccati G.: Normal Approximations Using Malliavin Calculus: from Stein's Method to Universality. Cambridge Tracts in Mathematics. Cambridge University Press, Cambridge (2012)
40. Nourdin I., Peccati G., Réveillac A.: Multivariate normal approximation using Stein's method and Malliavin calculus. Ann. Inst. H. Poincaré (B) Probab. Statist. **46**(1), 45–58 (2010)
41. Nourdin I., Peccati G., Speicher R.: Multidimensional semicircular limits on the free Wigner chaos. Ascona 2011 Proceedings, to appear.
42. Nourdin I., Réveillac A.: Asymptotic behavior of weighted quadratic variations of fractional Brownian motion: the critical case $H = 1/4$. Ann. Probab. **37**(6) 2200–2230 (2009)
43. Nourdin I., Réveillac A., Swanson J.: The weak Stratonovich integral with respect to fractional Brownian motion with Hurst parameter $1/6$. Electron. J. Probab. **15**(70) 2117–2162 (2010)

44. Nourdin I., Taqqu M.S.: Central and non-central limit theorems in a free probability setting. J. Theoret. Probab, to appear (2012)
45. Nualart D.: The Malliavin calculus and related topics, 2nd edn. Springer-Verlag, Berlin (2006)
46. Nualart D., Peccati G.: Central limit theorems for sequences of multiple stochastic integrals. Ann. Probab. **33**(1), 177–193 (2005)
47. Peccati G., Taqqu M.S.: Wiener Chaos: Moments, Cumulants and Diagrams. Springer-Verlag, Milan (2011)
48. Peccati G., Tudor C.A.: Gaussian limits for vector-valued multiple stochastic integrals. Séminaire de Probabilités XXXVIII. LNM 1857. Springer-Verlag, Berlin Heidelberg, pp. 247–262 (2005)
49. Pipiras V., Taqqu M.S.: Long-Range Dependence and Self-Similarity. Forthcoming book (2012)
50. Prakasa Rao B.L.S.: Statistical Inference for Fractional Diffusion Processes. Wiley Series in Probability and Statistics. John Wiley & Sons, Ltd., Chichester (2010)
51. Protter P.E.: Stochastic Integration and Differential Equations, 2nd edn. Springer-Verlag, Berlin (2004)
52. Revuz D., Yor M.: Continuous martingales and Brownian motion. Springer-Verlag, Berlin Heidelberg New York (1999)
53. Rogers L.C.G.: Arbitrage with fractional Brownian motion. Math. Finance **7**, 95–105 (1997)
54. Rudin W.: Real and Complex Analysis, 3rd edn. McGraw-Hill, New York (1987)
55. Samorodnitsky G.: Long range dependence. Foundations and Trends in Stochastic Systems **1**(3), 163–257 (2006)
56. Samorodnitsky G., Taqqu M.S.: Stable non-Gaussian random processes. Chapman and Hall, New York (1994)
57. Shigekawa I.: Derivatives of Wiener functionals and absolute continuity of induced measures. J. Math. Kyoto Univ. **20**(2), 263–289 (1980)
58. Slepian D.: The one-side barrier problem for Gaussian noise. Bell System Tech. J. **41**, 463–01 (1962)
59. Slud E.V.: The moment problem for polynomial forms in normal random variables. Ann. Probab. **21**(4), 2200–2214 (1993)
60. Stein C.: A bound for the error in the normal approximation to the distribution of a sum of dependent random variables. In: Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability, vol. II: Probability theory, pp. 583–602. University of California Press, Berkeley, CA (1972)
61. Stein C.: Approximate computation of expectations. Institute of Mathematical Statistics Lecture Notes, Monograph Series 7. Institute of Mathematical Statistics, Hayward, CA (1986)
62. Sussmann H.J.: An interpretation of stochastic differential equations as ordinary differential equations which depend on a sample point. Bull. Amer. Math. Soc. **83**, 296–298 (1977)
63. Taqqu M.S.: Weak convergence to fractional Brownian motion and to the Rosenblatt process. Z. Wahrscheinlichkeitstheorie verw. Gebiete **31**, 287–302 (1975)
64. Voiculescu D.V.: Symmetries of some reduced free product C^*-algebras. In: Operator algebras and their connection with topology and ergodic theory. Lecture Notes in Mathematics 1132, pp. 556–588. Springer-Verlag, Berlin Heidelberg (1985)
65. Voiculescu D.V.: Limit laws for random matrices and free product. Invent. Math. **104**, 201–220 (1991)

66. Wigner E.P.: On the distribution of the roots of certain symmetric matrices. Ann. of Math. **67**(2), 325–327 (1958)
67. Young L.C.: An inequality of the Hölder type connected with Stieltjes integration. Acta Math. **67**, 251–282 (1936)

Index

B&SS – Bocconi & Springer Series

THE ONLINE VERSION OF THE BOOKS PUBLISHED IN THE SERIES IS AVAILABLE ON SpringerLink

1. G. Peccati, M.S. Taqqu
   Wiener Chaos: Moments, Cumulants and Diagrams – A survey with computer implementation
   2011, XIV+274 pp, ISBN 978-88-470-1678-1

2. A. Pascucci
   PDE and Martingale Methods in Option Pricing
   2011, XVIII+720 pp, ISBN 978-88-470-1780-1

3. F. Hirsch, C. Profeta, B. Roynette, M. Yor
Peacocks and Associated Martingales, with Explicit Constructions
2011, XXXII+384 pp, ISBN 978-88-470-1907-2

4. I. Nourdin
Selected Aspects of Fractional Brownian Motion
2012, X+122 pp, ISBN 978-88-470-2822-7

For further information, please visit the following link:
http://www.springer.com/series/8762