S. Bellucci
Editor

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The Attractor Mechanism

Proceedings of the INFN-Laboratori Nazionali di Frascati School 2007

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With 52 Figures

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Preface

This book is based upon the lectures delivered from 18 to 22 June 2007 at the INFN-Laboratori Nazionali di Frascati School on Attractor Mechanism, directed by Stefano Bellucci, with the participation of prestigious lecturers, including S. Ferrara, M. Gnaydin, P. Levay, T. Mohaupt, and A. Zichichi. All lectures were given at a pedagogical, introductory level, a feature which is reflected in the specific "flavor" of this volume, which has also benefited much from the extensive discussions and related reworking of the various contributions.

This is the fourth volume in a series of books on the general topics of supersymmetry, supergravity, black holes, and the attractor mechanism. Indeed, based on previous meetings, three volumes have already been published:

BELLUCCI S. (2006). Supersymmetric Mechanics – Vol. 1: Supersymmetry, Noncommutativity and Matrix Models. (vol. 698, pp. 1–229). ISBN: 3-540-33313-4. Berlin, Heidelberg: Springer Verlag (Germany). Springer Lecture Notes in Physics Vol. 698.

BELLUCCI S., S. FERRARA, A. MARRANI. (2006). Supersymmetric Mechanics – Vol. 2: The Attractor Mechanism and Space Time Singularities. (vol. 701, pp. 1–242). ISBN-13: 9783540341567. Berlin, Heidelberg: Springer Verlag (Germany). Springer Lecture Notes in Physics Vol. 701.

BELLUCCI S. (2008). Supersymmetric Mechanics – Vol. 3: Attractors and Black Holes in Supersymmetric Gravity. (vol. 755, pp. 1–373). ISBN-13: 9783540795223. Berlin, Heidelberg: Springer Verlag (Germany). Springer Lecture Notes in Physics 755.

In this volume, we have included two contributions originating from short presentations of recent original results given by participants, i.e., Wei Li and Filipe Moura.

I thank all the lecturers and participants for contributing to the success of the School, which prompted the publication of this volume. I also thank Mrs. Silvia Colasanti for her generous efforts in the secretarial work and in various organizational aspects. My gratitude goes to INFN and in particular to Mario Calvetti for supporting the School. At this special time for me and my family, with the birth

of our longed for daughter Maristella recently and happily occurred, my thoughts go to my wife Gloria and our beloved Costanza, Eleonora, Annalisa, and Erica for supporting and encouraging me every day: their love gave me strength to complete this volume.

April 2010 *Stefano Bellucci*

Contents

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Chapter 1
SAM Lectures on Extremal Black Holes in $d = 4$ Extended Supergravity

Stefano Bellucci, Sergio Ferrara, Murat Günaydin, and Alessio Marrani

Abstract We report on recent results in the study of extremal black hole attractors in $N = 2$, $d = 4$ ungauged Maxwell–Einstein supergravities.

For homogeneous symmetric scalar manifolds, the three general classes of attractor solutions with non-vanishing Bekenstein–Hawking entropy are discussed. They correspond to three (inequivalent) classes of orbits of the charge vector, which sits in the relevant symplectic representation R_V of the U-duality group. Other than the $\frac{1}{2}$-BPS one, there are two other distinct non-BPS classes of charge orbits, one of which has vanishing central charge.

The complete classification of the U-duality orbits, as well as of the moduli spaces of non-BPS attractors (spanned by the scalars which are not stabilized at the black hole event horizon), is also reviewed.

Finally, we consider the analogous classification for $N \geqslant 3$-extended, $d = 4$ ungauged supergravities, in which also the $\frac{1}{N}$-BPS attractors yield a related moduli space.

1.1 Introduction

In the framework of *ungauged* Einstein supergravity theories in $d = 4$ space–time dimensions, the fluxes of the two-form electric-magnetic field strengths determine the charge configurations of stationary, spherically symmetric, asymptotically flat extremal black holes (BHs). Such fluxes sit in a representation R_V of the U-duality[1] group G_4 of the underlying $d = 4$ supergravity, defining the embedding of G into the larger symplectic group $Sp(2n, \mathbb{R})$. Moreover, after the study of [2], for

[1]Here U-duality is referred to as the "continuous" version, valid for large values of the charges, of the U-duality groups introduced by Hull and Townsend [1].

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symmetric scalar manifolds $\frac{G_4}{H_4}$ (see (1.1) below) the fluxes belong to distinct classes of orbits of the representation R_V, i.e., the R_V-representation space of G_4 is actually "stratified" into disjoint classes of orbits. Such orbits are defined and classified by suitable constraints on the (lowest order, actually unique) G-invariant $\mathcal{I}$ built out of the symplectic representation R_V.

For all $N \geqslant 3$, $d = 4$ supergravities the scalar manifold of the theory is an homogeneous *symmetric* space $\frac{G_4}{H_4}$. Thus, for such theories some relations between the coset expressions of the aforementioned orbits and different real (non-compact) forms of the stabilizer H_4 can be established [3]. It is here worth remarking that the "large" charge orbits (having $\mathcal{I} \neq 0$) support the *Attractor Mechanism* [4–8], whereas the "small" ones (having $\mathcal{I} = 0$) do not.

Recently, a number of papers have been devoted to the investigation of *extremal* BH *attractors* (see, e.g., [9–90]; for further developments and refs., see also, e.g., [91–95]), essentially because new classes of solutions to the so-called *Attractor Equations* were (re)discovered. Such new solutions have been found to determine non-BPS (*Bogomol'ny–Prasad–Sommerfeld*) BH horizon geometries, breaking *all* supersymmetries (*if any*).

The present report, originated from lectures given at the *School on Attractor Mechanism* (*SAM 2007*), held on June 18–22, 2007 at INFN National Laboratories in Frascati (LNF), Italy, is devoted to an introduction to the foundations of the theory of U-duality orbits in the theory of extremal BH attractors in $N = 2$, $d = 4$ MESGT's based on symmetric manifolds. Also $N \geqslant 3$-extended, $d = 4$ supergravities, as well as the issue of moduli spaces of attractor solutions, will be briefly considered. Our review incorporates some of the more recent developments that have taken place since *SAM2007*.

The plan of the report is as follows.

Section 1.2 is devoted to the treatment of $N = 2$, $d = 4$ MESGT's based on symmetric scalar manifolds.

In Sect. 1.2.1 we review some basic facts about such theories, concerning their "large" charge orbits and the relations with the extremal BH solutions to the corresponding Attractor Eqs.

Thence, Sect. 1.2.2 reports the general analysis, performed in [3], of the three classes of extremal BH attractors of $N = 2$, $d = 4$ symmetric *magic*[2] MESGT's, and of the corresponding classes of "large" charge orbits in the symplectic representation space of the relevant $d = 4$ U-duality group. In particular, the $\frac{1}{2}$-BPS solutions are treated in Sect. 1.2.2.1, while the two general species of non-BPS $Z \neq 0$ and non-BPS $Z = 0$ attractors are considered in Sect. 1.2.2.2.

The splittings of the mass spectra of $N = 2$, $d = 4$ symmetric *magic* MESGT's along their three classes of "large" charge orbits [3], and the related issues of

[2] These theories were called "magical" MESGT's in the original papers. In some of the recent literature they are referred to as "magic" MESGT's which we shall adopt in this review.

massless Hessian modes and moduli spaces of attractor solutions, are considered in Sect. 1.2.3.[3]

Section 1.2.4 deals with the crucial result that the massless Hessian modes of the effective BH potential $V_{\rm BH}$ of $N = 2$, $d = 4$ MESGT's based on symmetric scalar manifolds at its critical points actually correspond to "flat" directions. Such "flat" directions are nothing but the scalar degrees which are not stabilized at the event horizon of the considered $d = 4$ extremal BH, thus spanning a *moduli space* associated to the considered attractor solution. Nevertheless, the BH entropy is still well defined, because, due to the existence of such "flat" directions, it is actually *independent* on the unstabilized scalar degrees of freedom.

Actually, moduli spaces of attractors solutions exist *at least* for all *ungauged* supergravities based on homogeneous scalar manifolds. The classification of such moduli spaces (and of the corresponding supporting "large" orbits of U-duality for $N \geqslant 3$, $d = 4$ supergravities is reported in Sect. 1.3.

Section 1.4 concludes the present report, with some final comments and remarks.

1.2 $N = 2$, $d = 4$ Symmetric MESGT's

1.2.1 U-Duality "Large" Orbits

The critical points of the BH effective potential $V_{\rm BH}$ for all $N = 2$ symmetric special geometries in $d = 4$ are generally referred to as attractors. These extrema describe the "large" configurations (BPS as well as non-BPS) of $N = 2, 6, 8$ supergravities, corresponding to a finite, non-vanishing quartic invariant $\mathcal{I}_4$ and thus to extremal BHs with classical non-vanishing entropy $S_{\rm BH} \neq 0$. The related orbits in the $R_{\rm V}$ of the $d = 4$ U-duality group G_4 will correspondingly be referred to as "large" orbits. The attractor equations for BPS configurations were first studied in [4–7], and flow eqs. for the general case were given in [8].

Attractor solutions and their "large" charge orbits in $d = 5$ have been recently classified for the case of all rank-2 symmetric spaces in [24].

In [3] the results holding for $N = 8$, $d = 4$ supergravity were obtained also for the particular class of $N = 2$, $d = 4$ symmetric Maxwell–Einstein supergravity theories (MESGT's) [96–98], which we will now review. Such a class consists of $N = 2$, $d = 4$ supergravities sharing the following properties:

1. Beside the supergravity multiplet, the matter content is given only by a certain number $n_{\rm V}$ of Abelian vector multiplets.

[3] In the present report we do not consider the other $N = 2, d = 4$ MESGT's with symmetric scalar manifolds, given by the two infinite sequences $\frac{SU(1,1+n)}{U(1)\times SU(1+n)}$ and $\frac{SU(1,1)}{U(1)} \times \frac{SO(2,2+n)}{SO(2)\times SO(2+n)}$. These theories are treated in detail in the two Appendices of [3].

2. The space of the vector multiplets' scalars is an homogeneous symmetric special Kähler manifold, i.e., a special Kähler manifold with coset structure

$$\frac{G_4}{H_4} \equiv \frac{G}{H_0 \times U(1)}, \tag{1.1}$$

where $G \equiv G_4$ is a semisimple non-compact Lie group and $H_4 \equiv H_0 \times U(1)$ is its maximal compact subgroup (*mcs*) (with symmetric embedding, as understood throughout).
3. The charge vector in a generic (dyonic) configuration with $n_V + 1$ electric and $n_V + 1$ magnetic charges sits in a real (symplectic) representation R_V of G of $\dim(R_V) = 2(n_V + 1)$.

By exploiting such special features and relying on group theoretical considerations, in [3] the coset expressions of the various distinct classes of "large" orbits (of dimension $2n_V + 1$) in the R_V-representation space of G were related to different real (non-compact) forms of the compact group H_0. Correspondingly, the $N = 2$, $d = 4$ Attractor Eqs. were solved for all such classes, also studying the scalar mass spectrum of the theory corresponding to the obtained solutions.

The symmetric special Kähler manifolds of $N = 2$, $d = 4$ MESGT's have been classified in the literature (see, e.g., [99, 100] and refs. therein). All such theories can be obtained by dimensional reduction of the $N = 2$, $d = 5$ MESGT's that were constructed in [96–98]. The MESGT's with symmetric manifolds that originate from $d = 5$ all have cubic prepotentials determined by the norm form of the Jordan algebra of degree three that defines them [96–98].

The unique exception is provided by the infinite sequence ($n \in \mathbb{N} \cup \{0\}$, $n_V = n + r = n + 1$) [101]

$$\mathrm{I}_n : \frac{SU(1,1+n)}{U(1) \times SU(1+n)}, \quad r = 1, \tag{1.2}$$

where r stands for the *rank* of the coset throughout. This is usually referred to as *minimal coupling* sequence, and it is endowed with quadratic prepotential. It should be remarked that the $N = 2$ *minimally coupled* supergravity is the only (symmetric) $N = 2$, $d = 4$ MESGT which yields the *pure* $N = 2$ supergravity simply by setting $n = -1$ (see, e.g., [76] and refs. therein).

Only another infinite symmetric sequence exists, namely ($n \in \mathbb{N} \cup \{0, -1\}$, $n_V = n + r = n + 3$)

$$\mathrm{II}_n : \frac{SU(1,1)}{U(1)} \times \frac{SO(2,2+n)}{SO(2) \times SO(2+n)}, \quad r = 3. \tag{1.3}$$

This one has a $d = 5$ origin and its associated Jordan algebras are not simple. It is referred to as the "generic Jordan family" since it exists $\forall n \in \mathbb{N} \cup \{0, -1\}$. The first elements of such sequences (1.2) and (1.3) correspond to the following manifolds and holomorphic prepotential functions in special coordinates:

$$\mathrm{I}_0 : \frac{SU(1,1)}{U(1)}, \quad F(t) = -\frac{i}{2}\left(1 - t^2\right); \tag{1.4}$$

$$\mathrm{II}_{-1} : \frac{SU(1,1) \times SO(2,1)}{U(1) \times SO(2)} = \left(\frac{SU(1,1)}{U(1)}\right)^2, \quad F(s,t) = st^2; \tag{1.5}$$

$$\mathrm{II}_0 : \frac{SU(1,1) \times SO(2,2)}{U(1) \times SO(2) \times SO(2)} = \left(\frac{SU(1,1)}{U(1)}\right)^3, \quad F(s,t,u) = stu. \tag{1.6}$$

It is here worth remarking that the so-called t^3 model, corresponding to the following manifold and holomorphic prepotential function in special coordinates:

$$\frac{SU(1,1)}{U(1)}, \quad F(t) = t^3, \tag{1.7}$$

is an *isolated case* in the classification of symmetric SK manifolds (see, e.g., [102]; see also [103] and refs. therein), but it can be thought also as the "t^3 degeneration" of the stu model (see, e.g., [50]; see also Sect. 1.2.4 for a treatment of models I_0 and t^3).

As mentioned, all manifolds of type I correspond to quadratic prepotentials ($C_{ijk} = 0$), and all manifolds of type II correspond to cubic prepotentials (in special coordinates $F = \frac{1}{3!} d_{ijk} t^i t^j t^k$ and therefore $C_{ijk} = \mathrm{e}^K d_{ijk}$, where K denotes the Kähler potential and d_{ijk} is a completely symmetric rank-3 constant tensor). The 3-moduli case II_0 is the well-known stu model [104, 105] (see also, e.g., [77] and refs. therein), whose noteworthy *triality symmetry* has been recently related to *quantum information theory* [106–120].

Beside the infinite sequence II, there exist four other MESGT's defined by simple Euclidean Jordan algebras of degree three with the following rank-3 symmetric manifolds:

$$\mathrm{III} : \frac{E_{7(-25)}}{E_6 \times U(1)}; \tag{1.8}$$

$$\mathrm{IV} : \frac{SO^*(12)}{U(6)}; \tag{1.9}$$

$$\mathrm{V} : \frac{SU(3,3)}{S(U(3) \times U(3))} = \frac{SU(3,3)}{SU(3) \times SU(3) \times U(1)}; \tag{1.10}$$

$$\mathrm{VI} : \frac{Sp(6,\mathbb{R})}{U(3)}. \tag{1.11}$$

The $N = 2$, $d = 4$ MESGT's whose geometry of scalar fields is given by the manifolds III–VI are called "magic", since their symmetry groups are the groups of the famous Magic Square of Freudenthal, Rozenfeld and Tits associated with some remarkable geometries [121–123]. The four $N = 2$, $d = 4$ *magic* MESGT's III–VI, as their $d = 5$ versions, are defined by four simple Euclidean Jordan algebras $J_3^{\mathbb{O}}$, $J_3^{\mathbb{H}}$, $J_3^{\mathbb{C}}$ and $J_3^{\mathbb{R}}$ of degree 3 with irreducible norm forms, namely by the Jordan algebras of Hermitian 3×3 matrices over the four division algebras, i.e., respectively over the octonions $\mathbb{O}$, quaternions $\mathbb{H}$, complex numbers $\mathbb{C}$ and real numbers $\mathbb{R}$ [96–98, 124–127].

By denoting with n_V the number of vector multiplets coupled to the supergravity one, the total number of Abelian vector fields in the considered $N = 2$, $d = 4$ MESGT is $n_V + 1$; correspondingly, the real dimension of the corresponding scalar manifold is $2n_V = \dim(G) - \dim(H_0) - 1$. Since the $2(n_V + 1)$-dim. vector of extremal BH charge configuration is given by the fluxes of the electric and magnetic field-strength two-forms, it is clear that $\dim_{\mathbb{R}}(R_V) = 2(n_V + 1)$.

Since H_0 is a proper compact subgroup of the duality semisimple group G, one can decompose the $2(n_V + 1)$-dim. real symplectic representation R_V of G in terms of complex representations of H_0, obtaining in general the following decomposition scheme:

$$R_V \longrightarrow R_{H_0} + \overline{R_{H_0}} + \mathbf{1}_{\mathbb{C}} + \bar{\mathbf{1}}_{\mathbb{C}} = R_{H_0} + \mathbf{1}_{\mathbb{C}} + c.c., \tag{1.12}$$

where "$c.c.$" stands for the complex conjugation of representations throughout, and R_{H_0} is a certain complex representation of H_0.

The basic data of the cases I–VI listed above are summarized in Tables 1.1 and 1.2.

It was shown in [2] that $\frac{1}{2}$-BPS orbits of $N = 2$, $d = 4$ symmetric MESGT's are coset spaces of the form

$$\begin{aligned} \mathcal{O}_{\frac{1}{2}-\mathrm{BPS}} &= \frac{G}{H_0}, \\ \dim_{\mathbb{R}}\left(\mathcal{O}_{\frac{1}{2}-\mathrm{BPS}}\right) &= \dim(G) - \dim(H_0) = 2n_V + 1 = \dim_{\mathbb{R}}(R_V) - 1. \end{aligned} \tag{1.13}$$

Table 1.1 Data of the two sequences of symmetric $N = 2$, $d = 4$ MESGT's

	I	II
G	$SU(1, 1+n)$	$SU(1,1) \times SO(2, 2+n)$
H_0	$SU(1+n)$	$SO(2) \times SO(2+n)$
r	1	3
$\dim_{\mathbb{R}}\left(\frac{G}{H_0 \times U(1)}\right)$	$2(n+1)$	$2(n+3)$
n_V	$n + r = n + 1$	$n + r = n + 3$
R_V	$(\mathbf{2}(\mathbf{n}+\mathbf{2}))_{\mathbb{R}}$	$(\mathbf{2}(\mathbf{n}+\mathbf{4}))_{\mathbb{R}}$
R_{H_0}	$(\mathbf{n}+\mathbf{1})_{\mathbb{C}}$	$(\mathbf{n}+\mathbf{2}+\mathbf{1})_{\mathbb{C}}$
$\dim_{\mathbb{R}}(R_V)$	$2(n+2)$	$2(n+4)$
$\dim_{\mathbb{R}}(R_{H_0})$	$2(n+1)$	$2(n+3)$
R_V $\downarrow$ $R_{H_0} + \mathbf{1}_{\mathbb{C}} +$ $+c.c.$	$(\mathbf{2}(\mathbf{n}+\mathbf{2}))_{\mathbb{R}}$ $\downarrow$ $(\mathbf{n}+\mathbf{1})_{\mathbb{C}} + \mathbf{1}_{\mathbb{C}} +$ $+c.c.$	$(\mathbf{2}(\mathbf{n}+\mathbf{4}))_{\mathbb{R}}$ $\downarrow$ $(\mathbf{n}+\mathbf{2}+\mathbf{1})_{\mathbb{C}} + \mathbf{1}_{\mathbb{C}} +$ $+c.c.$

Table 1.2 Data of the four *magic* symmetric $N = 2, d = 4$ MESGT's. $\mathbf{14}'_{\mathbb{R}}$ is the rank-3 antisymmetric tensor representation of $Sp(6,\mathbb{R})$. In $(\mathbf{3},\mathbf{3}')_{\mathbb{C}}$ the prime distinguishes the representations of the two distinct $SU(3)$ groups

	III : $J_3^{\mathbb{O}}$	IV : $J_3^{\mathbb{H}}$	V : $J_3^{\mathbb{C}}$	VI : $J_3^{\mathbb{R}}$
G	$E_{7(-25)}$	$SO^*(12)$	$SU(3,3)$	$Sp(6,\mathbb{R})$
H_0	E_6	$SU(6)$	$SU(3)\times SU(3)$	$SU(3)$
r	3	3	3	3
$\dim_{\mathbb{R}}\left(\frac{G}{H_0\times U(1)}\right)$	54	30	18	12
n_{V}	27	15	9	6
R_{V}	$\mathbf{56}_{\mathbb{R}}$	$\mathbf{32}_{\mathbb{R}}$	$\mathbf{10}_{\mathbb{R}}$	$\mathbf{14}'_{\mathbb{R}}$
R_{H_0}	$\mathbf{27}_{\mathbb{C}}$	$\mathbf{15}_{\mathbb{C}}$	$(\mathbf{3},\mathbf{3}')_{\mathbb{C}}$	$\mathbf{6}_{\mathbb{C}}$
$\dim_{\mathbb{R}}(R_{\mathrm{V}})$	56	32	20	14
$\dim_{\mathbb{R}}(R_{H_0})$	54	30	18	12
R_{V} $\downarrow$ $R_{H_0}+\mathbf{1}_{\mathbb{C}}+$ $+c.c.$	$\mathbf{56}_{\mathbb{R}}$ $\downarrow$ $\mathbf{27}_{\mathbb{C}}+\mathbf{1}_{\mathbb{C}}+$ $+c.c.$	$\mathbf{32}_{\mathbb{R}}$ $\downarrow$ $\mathbf{15}_{\mathbb{C}}+\mathbf{1}_{\mathbb{C}}+$ $+c.c.$	$\mathbf{10}_{\mathbb{R}}$ $\downarrow$ $(\mathbf{3},\mathbf{3}')_{\mathbb{C}}+\mathbf{1}_{\mathbb{C}}+$ $+c.c.$	$\mathbf{14}'_{\mathbb{R}}$ $\downarrow$ $\mathbf{6}_{\mathbb{C}}+\mathbf{1}_{\mathbb{C}}+$ $+c.c.$

We need to consider the $N = 2$ Attractor Eqs.; these are nothing but the criticality conditions for the $N = 2$ BH effective potential [6, 128]

$$V_{\mathrm{BH}} \equiv |Z|^2 + G^{i\bar{i}} D_i Z \overline{D}_{\bar{i}} \overline{Z} \tag{1.14}$$

in the corresponding special Kähler geometry [8]:

$$\partial_i V_{\mathrm{BH}} = 0 \Longleftrightarrow 2\overline{Z} D_i Z + i C_{ijk} G^{j\bar{j}} G^{k\bar{k}} \overline{D}_{\bar{j}} \overline{Z} \overline{D}_{\bar{k}} \overline{Z} = 0, \forall i = 1, \ldots, n_{\mathrm{V}}. \tag{1.15}$$

C_{ijk} is the rank-3, completely symmetric, covariantly holomorphic tensor of special Kähler geometry, satisfying (see, e.g., [129])

$$\overline{D}_{\bar{l}} C_{ijk} = 0, \quad D_{[l} C_{i]jk} = 0, \tag{1.16}$$

where the square brackets denote antisymmetrization with respect to the enclosed indices.

For symmetric special Kähler manifolds the tensor C_{ijk} is covariantly constant:

$$D_i C_{jkl} = 0, \tag{1.17}$$

which further implies [97, 99]

$$G^{k\bar{k}} G^{r\bar{j}} C_{r(pq} C_{ij)k} \overline{C}_{\bar{k}\bar{i}\bar{j}} = \frac{4}{3} G_{(q|\bar{i}} C_{|ijp)}. \tag{1.18}$$

This equation is simply the $d = 4$ version of the "adjoint identity" satisfied by all (Euclidean) Jordan algebras of degree three that define the corresponding MESGT's in $d = 5$ [97, 99]:

$$d_{r(pq}d_{ij)k}d^{rkl} = \frac{4}{3}\delta^{l}_{(q}d_{ijp)}. \tag{1.19}$$

Z is the $N = 2$ "central charge" function, whereas $\{D_i Z\}_{i=1,\dots,n_V}$ is the set of its Kähler-covariant holomorphic derivatives, which are nothing but the "matter charge" functions of the system. Indeed, the sets[4] $\{q_0, q_i, p^0, p^i\} \in \mathbb{R}^{2n_V+2}$ and $\{Z, D_i Z\} \in \mathbb{C}^{n_V+1}$ (when evaluated at purely (q, p)-dependent critical values of the moduli) are two equivalent basis for the charges of the system, and they are related by a particular set of identities of special Kähler geometry [21, 22, 128]. The decomposition (1.12) corresponds to nothing but the splitting of the sets $\{q_0, q_i, p^0, p^i\}$ ($\{Z, D_i Z\}$) of $2n_V + 2$ ($n_V + 1$) real (complex) charges ("charge" functions) in q_0, p^0 (Z) (related to the graviphoton, and corresponding to $\mathbf{1}_{\mathbb{C}} + c.c.$) and in $\{q_i, p^i\}$ ($\{D_i Z\}$) (related to the n_V vector multiplets, and corresponding to $R_{H_0} + c.c.$).

In order to perform the subsequent analysis of orbits, it is convenient to use "flat" I-indices by using the (inverse) n_V-bein e^i_I of $\frac{G}{H_0 \times U(1)}$:

$$D_I Z = e^i_I D_i Z. \tag{1.20}$$

By switching to "flat" local I-indices, the special Kähler metric $G_{i\bar{j}}$ (assumed to be *regular*, i.e., strictly positive definite everywhere) will become nothing but the Euclidean n_V-dim. metric $\delta_{I\overline{J}}$. Thus, the attractor eqs. (1.15) can be "flattened" as follows:

$$\partial_I V_{\mathrm{BH}} = 0 \Longleftrightarrow 2\overline{Z}D_I Z + iC_{IJK}\delta^{J\overline{J}}\delta^{K\overline{K}}\overline{D_{\overline{J}}Z D_{\overline{K}}Z} = 0, \quad \forall I = 1, \dots, n_V. \tag{1.21}$$

Note that C_{IJK} becomes an H_0-invariant tensor [130]. This is possible because C_{ijk} in special coordinates is proportional to the invariant tensor d_{IJK} of the $d = 5$ U-duality group G_5. G_5 and H_0 correspond to two different real forms of the same Lie algebra [97].

As it is well known, $\frac{1}{2}$-BPS attractors are given by the following solution [8] of attractor eqs. (1.15) and (1.21):

$$Z \neq 0, \quad D_i Z = 0 \Leftrightarrow D_I Z = 0, \quad \forall i, I = 1, \dots, n_V. \tag{1.22}$$

Since the "flattened matter charges" $D_I Z$ are a vector of R_{H_0}, (2.90) directly yields that $\frac{1}{2}$-BPS solutions are manifestly H_0-invariant. In other words, since the $N = 2$, $\frac{1}{2}$-BPS orbits are of the form $\frac{G}{H_0}$, the condition for the $(n_V + 1)$-dim. complex vector $(Z, D_i Z)$ to be H_0-invariant is precisely given by (2.90), defining $N = 2$, $\frac{1}{2}$-BPS attractor solutions.

[4] We always consider the "classical" framework, disregarding the actual quantization of the ranges of the electric and magnetic charges q_0, q_i, p^0 and p^i. That is why we consider $\mathbb{R}^{2n_V+2}$ rather than the $(2n_V + 2)$-dim. charge lattice $\hat{\Gamma}_{(p,q)}$.

Thus, as for the $N = 8$, $d = 4$ attractor solutions (see, e.g., [3] and refs. therein), also for the $N = 2$, $d = 4$ $\frac{1}{2}$-BPS case *the invariance properties of the solutions at the critical point(s) are given by the maximal compact subgroup (mcs) of the stabilizer of the corresponding charge orbit,* which in the present case is the compact stabilizer itself. Thus, at $N = 2$ $\frac{1}{2}$-BPS critical points the following enhancement of symmetry holds:

$$\mathcal{S} \longrightarrow H_0, \tag{1.23}$$

where here and below $\mathcal{S}$ denotes the compact symmetry of a generic orbit of the real symplectic representation R_V of the $d = 4$ duality group G.

However, all the scalar manifolds of $N = 2$, $d = 4$ symmetric MESGT's have other species of *regular* critical points V_{BH} (and correspondingly other classes of "large" charge orbits).

Concerning the $N = 2$, $d = 4$ symmetric MESGT's, the rank-1 sequence I has one more, non-BPS class of orbits (with vanishing central charge), while all rank-3 aforementioned cases II–VI have two more distinct non-BPS classes of orbits, one of which with vanishing central charge.

The results about the classes of "large" charge orbits of $N = 2$, $d = 4$ symmetric MESGT's are summarized in Table 1.3.[5]

Table 1.3 "Large" orbits of $N = 2$, $d = 4$ symmetric MESGT's

	$\frac{1}{2}$-BPS orbits $\mathcal{O}_{\frac{1}{2}-\text{BPS}} = \frac{G}{H_0}$	Non-BPS, $Z \neq 0$ orbits $\mathcal{O}_{\text{non-BPS}, Z\neq 0} = \frac{G}{\hat{H}}$	Non-BPS, $Z = 0$ orbits $\mathcal{O}_{\text{non-BPS}, Z=0} = \frac{G}{\tilde{H}}$
I	$\frac{SU(1,n+1)}{SU(n+1)}$	–	$\frac{SU(1,n+1)}{SU(1,n)}$
II	$SU(1,1) \times \frac{SO(2,2+n)}{SO(2) \times SO(2+n)}$	$SU(1,1) \times \frac{SO(2,2+n)}{SO(1,1) \times SO(1,1+n)}$	$SU(1,1) \times \frac{SO(2,2+n)}{SO(2) \times SO(2,n)}$
III	$\frac{E_{7(-25)}}{E_6}$	$\frac{E_{7(-25)}}{E_{6(-26)}}$	$\frac{E_{7(-25)}}{E_{6(-14)}}$
IV	$\frac{SO^*(12)}{SU(6)}$	$\frac{SO^*(12)}{SU^*(6)}$	$\frac{SO^*(12)}{SU(4,2)}$
V	$\frac{SU(3,3)}{SU(3) \times SU(3)}$	$\frac{SU(3,3)}{SL(3,\mathbb{C})}$	$\frac{SU(3,3)}{SU(2,1) \times SU(1,2)}$
VI	$\frac{Sp(6,\mathbb{R})}{SU(3)}$	$\frac{Sp(6,\mathbb{R})}{SL(3,\mathbb{R})}$	$\frac{Sp(6,\mathbb{R})}{SU(2,1)}$

[5] We should note that the column on the right of Table 2 of [2] is not fully correct.

Indeed, that column coincides with the central column of Table 1.3 of the present paper (by disregarding case I and shifting $n \to n-2$ in case II), listing the non-BPS, $Z \neq 0$ orbits of $N = 2$, $d = 4$ symmetric MESGT's, which are all characterized by a strictly negative quartic E_7-invariant $\mathcal{I}_4$. This does not match what is claimed in [2], where such a column is stated to list the particular class of orbits with $I_4 > 0$ and eigenvalues of opposite sign in pair.

Actually, the statement of [2] holds true only for the case I (which, by shifting $n \to n-1$, coincides with the last entry of the column on the right of Table 2 of [2]). On the other hand, such a case is the only one which cannot be obtained from $d = 5$ by dimensional reduction. Moreover,

1.2.2 Classification of Attractors

The three classes of orbits in Table 1.3 correspond to the three distinct classes of solutions of the $N = 2$, $d = 4$ Attractor Eqs. (1.15) and (1.21).

1.2.2.1 $\frac{1}{2}$-BPS

As already mentioned, the class of $\frac{1}{2}$-BPS orbits corresponds to the solution (2.90) determining $N = 2$, $\frac{1}{2}$-BPS critical points of V_{BH}. Such a solution yields the following value of the BH scalar potential at the considered attractor point(s) [8]:

$$V_{\mathrm{BH},\frac{1}{2}-\mathrm{BPS}} = |Z|^2_{\frac{1}{2}-\mathrm{BPS}} + \left[G^{i\bar{i}} D_i Z \overline{D}_{\bar{i}} \overline{Z}\right]_{\frac{1}{2}-\mathrm{BPS}} = |Z|^2_{\frac{1}{2}-\mathrm{BPS}} . \tag{1.24}$$

The overall symmetry group at $N = 2$ $\frac{1}{2}$-BPS critical point(s) is H_0, stabilizer of $\mathcal{O}_{\frac{1}{2}-\mathrm{BPS}} = \frac{G}{H_0}$. The *symmetry enhancement* is given by (1.23). For such a class of orbits

$$I_{4,\frac{1}{2}-\mathrm{BPS}} = |Z|^4_{\frac{1}{2}-\mathrm{BPS}} > 0. \tag{1.25}$$

1.2.2.2 Non-BPS

The two classes of $N = 2$ non-BPS "large" charge orbits respectively correspond to the following solutions of $N = 2$ attractor eqs. (1.15):

$$\text{non-BPS, } Z \neq 0: \begin{cases} Z \neq 0, \\ D_i Z \neq 0 \quad \text{for some } i \in \{1, \ldots, n_V\}, \\ I_{4,\mathrm{non-BPS},Z\neq 0} = -\Big[|Z|^2_{\mathrm{non-BPS},Z\neq 0} \\ \qquad + \left(G^{i\bar{j}} D_i Z \overline{D}_{\bar{j}} \overline{Z}\right)_{\mathrm{non-BPS},Z\neq 0} \Big]^2 \\ \qquad = -16\, |Z|^4_{\mathrm{non-BPS},Z\neq 0} < 0; \end{cases} \tag{1.26}$$

---

it is the only one not having non-BPS, $Z \neq 0$ orbits, rather it is characterized only by a class of non-BPS orbits with $Z = 0$ and $\mathcal{I}_4 > 0$.

$$\text{non-BPS, } Z = 0\text{:} \begin{cases} Z = 0, \\ \\ D_i Z \neq 0 \quad \text{for some } i \in \{1, \ldots, n_V\}, \\ \\ I_{4,\text{non-BPS},Z=0} = \left(G^{i\bar{j}} D_i Z \overline{D}_{\bar{j}} \overline{Z} \right)^2_{\text{non-BPS},Z=0} > 0. \end{cases} \tag{1.27}$$

In the treatment given below, we will show how the general solutions of (1.15), respectively determining the two aforementioned classes of $N = 2$ non-BPS extremal BH attractors, can be easily given by using "flat" local I-coordinates in the scalar manifold.

In other words, we will consider the "flattened" attractor eqs. (1.21), which can be specialized in the "large" non-BPS cases as follows:

$$\text{non-BPS, } Z \neq 0\text{:} \quad 2\overline{Z} D_I Z = -i C_{IJK} \delta^{J\overline{J}} \delta^{K\overline{K}} \overline{D}_{\overline{J}} \overline{Z} \overline{D}_{\overline{K}} \overline{Z}; \tag{1.28}$$

$$\text{non-BPS, } Z = 0\text{:} \quad C_{IJK} \delta^{J\overline{J}} \delta^{K\overline{K}} \overline{D}_{\overline{J}} \overline{Z} \overline{D}_{\overline{K}} \overline{Z} = 0. \tag{1.29}$$

Thus, by respectively denoting with $\hat{H}$ ($\tilde{H}$) the stabilizer of the $N = 2$, non-BPS, $Z \neq 0$ ($Z = 0$) classes of orbits listed in Table 1.3, our claim is the following: *the general solution of* (1.28)–(1.29) *is obtained by retaining a complex charge vector* $(Z, D_I Z)$ *which is invariant under* $\hat{h}$ ($\frac{\tilde{h}}{U(1)}$)*, where* $\hat{h}$ ($\tilde{h}$) *is the mcs of* $\hat{H}$ ($\tilde{H}$)*.*[6]

As a consequence, *the overall symmetry group of the* $N = 2$*, non-BPS,* $Z \neq 0$ ($Z = 0$) *critical point(s) is* $\hat{h}$ ($\frac{\tilde{h}}{U(1)}$). Thus, at $N = 2$, non-BPS, $Z \neq 0$ ($Z = 0$) critical point(s) the following *enhancement of symmetry* holds

$$\begin{aligned} &N = 2, \text{non-BPS, } Z \neq 0 : \mathcal{S} \longrightarrow \hat{h} = mcs\left(\hat{H}\right); \\ &N = 2, \text{non-BPS, } Z = 0 : \mathcal{S} \longrightarrow \frac{\tilde{h}}{U(1)} = \frac{mcs(\tilde{H})}{U(1)}. \end{aligned} \tag{1.30}$$

It is worth remarking that the non-compact group $\hat{H}$ stabilizing the non-BPS, $Z \neq 0$ class of orbits of $N = 2$, $d = 4$ symmetric MESGT's, beside being a real

[6] Indeed, while H_0 is a proper compact subgroup of G, the groups $\hat{H}$, $\tilde{H}$ are real (non-compact) forms of H_0, as it can be seen from Table 1.3 (see also [131, 132]). Therefore in general they admit a *mcs* $\hat{h}$, $\tilde{h}$, which in turn is a (non-maximal) compact subgroup of G and a proper compact subgroup of H_0.

It is interesting to notice that in all cases (listed in Table 1.3) G always admits only two real (non-compact) forms $\hat{H}$, $\tilde{H}$ of H_0 as proper subgroups (consistent with the required dimension of orbits). The inclusion of $\hat{H}$, $\tilde{H}$ in G is such that in all cases $\hat{H} \times SO(1,1)$ and $\tilde{H} \times U(1)$ are different maximal non-compact subgroups of G.

Table 1.4 Stabilizers and corresponding maximal compact subgroups of the "large" classes of orbits of $N = 2$, $d = 4$ symmetric MESGT's. $\hat{H}$ and $\tilde{H}$ are real (non-compact) forms of H_0, the stabilizer of $\frac{1}{2}$-BPS orbits

	H_0	$\hat{H}$	$\tilde{H}$	$\hat{h} \equiv mcs\left(\hat{H}\right)$	$\tilde{h}' \equiv \frac{mcs(\tilde{H})}{U(1)}$
I	$SU(n+1)$	$-$	$SU(1,n)$	$-$	$SU(n)$
II	$SO(2) \times SO(2+n)$	$SO(1,1) \times SO(1,1+n)$	$SO(2) \times SO(2,n)$	$SO(1+n)$	$SO(2) \times SO(n)$
III	$E_6 \equiv E_{6(-78)}$	$E_{6(-26)}$	$E_{6(-14)}$	$F_4 \equiv F_{4(-52)}$	$SO(10)$
IV	$SU(6)$	$SU^*(6)$	$SU(4,2)$	$USp(6)$	$SU(4) \times SU(2)$
V	$SU(3) \times SU(3)$	$SL(3,\mathbb{C})$	$SU(2,1) \times SU(1,2)$	$SU(3)$	$SU(2) \times SU(2) \times U(1)$
VI	$SU(3)$	$SL(3,\mathbb{R})$	$SU(2,1)$	$SO(3)$	$SU(2)$

(non-compact) form of H_0, is isomorphic to the duality group G_5 of $N = 2, d = 5$ symmetric MESGT's.[7]

Since the scalar manifolds of $N = 2$, $d = 5$ symmetric MESGT's are endowed with a real special geometry [96–98], the complex representation R_{H_0} of H_0 decomposes in a pair of irreducible real representations $(R_{\hat{h}} + \mathbf{1})_{\mathbb{R}}$'s of $\hat{h} = mcs(\hat{H}) \subsetneq H_0$ (see Sect. 1.2.2.2, and in particular (1.31)). As we will see below, such a fact crucially distinguishes the non-BPS, $Z \neq 0$ and $Z = 0$ cases.

The stabilizers (and the corresponding *mcs*'s) of the non-BPS, $Z \neq 0$ and $Z = 0$ classes of orbits of $N = 2$, $d = 4$ symmetric MESGT's are given in Table 1.4.

Non-BPS, $Z \neq 0$

Let us start by considering the class of non-BPS, $Z \neq 0$ orbits of $N = 2$, $d = 4$ symmetric MESGT's.

As mentioned, the "flattened matter charges" $D_I Z$ are a vector of R_{H_0}. In general, R_{H_0} decomposes under the *mcs* $\hat{h} \subset \hat{H}$ as follows:

$$R_{H_0} \longrightarrow \left(R_{\hat{h}} + \mathbf{1}\right)_{\mathbb{C}}, \tag{1.31}$$

[7] Such a feature is missing in the $N = 2$, $d = 4$ symmetric MESGT's whose scalar manifolds belong to the sequence I given by (1.2), simply because such theories do not have a class of non-BPS, $Z \neq 0$ orbits.

where the r.h.s. is made of the complex singlet representation of $\hat{h}$ and by another non-singlet real representation of $\hat{h}$, denoted above with $R_{\hat{h}}$. As previously mentioned, despite being complex, $(R_{\hat{h}} + \mathbf{1})_{\mathbb{C}}$ is not charged with respect to $U(1)$ symmetry because, due to the five-dimensional origin of the non-compact stabilizer $\hat{H}$ whose *mcs* is $\hat{h}$, actually $(R_{\hat{h}} + \mathbf{1})_{\mathbb{C}}$ is nothing but the complexification of its real counterpart $(R_{\hat{h}} + \mathbf{1})_{\mathbb{R}}$. The decomposition (1.31) yields the following splitting of "flattened matter charges":

$$D_I Z \longrightarrow \left(D_{\hat{I}} Z, D_{\hat{I}_0} Z\right), \tag{1.32}$$

where $\hat{I}$ are the indices along the representation $R_{\hat{h}}$, and $\hat{I}_0$ is the $\hat{h}$-singlet index.

By considering the related attractor eqs., it should be noticed that the rank-3 symmetric tensor C_{IJK} in (1.28) corresponds to a cubic H_0-invariant coupling $(R_{H_0})^3$. By decomposing $(R_{H_0})^3$ in terms of representations of $\hat{h}$, one finds

$$\left(R_{H_0}\right)^3 \longrightarrow \left(R_{\hat{h}}\right)^3 + \left(R_{\hat{h}}\right)^2 \mathbf{1}_{\mathbb{C}} + \left(\mathbf{1}_{\mathbb{C}}\right)^3. \tag{1.33}$$

Notice that a term $R_{\hat{h}} \left(\mathbf{1}_{\mathbb{C}}\right)^2$ cannot be in such a representation decomposition, since it is not $\hat{h}$-invariant, and thus not H_0-invariant. This implies that components of the form $C_{\hat{I}\hat{I}_0\hat{I}_0}$ cannot exist. Also, a term like $(\mathbf{1}_{\mathbb{C}})^3$ can appear in the r.h.s. of the decomposition (1.28) since as we said the $\hat{h}$-singlet $\mathbf{1}_{\mathbb{C}}$, despite being complex, is *not* $U(1)$-charged.

It is then immediate to conclude that the solution of $N = 2$, $d = 4$ non-BPS, $Z \neq 0$ extremal BH attractor eqs. in "flat" indices (1.28) corresponds to keeping the "flattened matter charges" $D_I Z$ $\hat{h}$-invariant. By virtue of decomposition (1.33), this is obtained by putting

$$D_{\hat{I}} Z = 0, \quad D_{\hat{I}_0} Z \neq 0, \tag{1.34}$$

i.e., by putting all "flattened matter charges" to zero, except the one along the $\hat{h}$-singlet (and thus $\hat{h}$-invariant) direction in scalar manifold. By substituting the solution (1.34) in (1.28), one obtains

$$2\overline{Z} D_{\hat{I}_0} Z = -i C_{\hat{I}_0\hat{I}_0\hat{I}_0} \left(\overline{D}_{\overline{\hat{I}}_0} \overline{Z}\right)^2 \overset{Z\neq 0}{\Leftrightarrow} D_{\hat{I}_0} Z = -\frac{i}{2} \frac{C_{\hat{I}_0\hat{I}_0\hat{I}_0}}{\overline{Z}} \left(\overline{D}_{\overline{\hat{I}}_0} \overline{Z}\right)^2 \tag{1.35}$$

$$\Downarrow$$

$$\left|D_{\hat{I}_0} Z\right|^2 \left(1 - \frac{1}{4} \frac{\left|C_{\hat{I}_0\hat{I}_0\hat{I}_0}\right|^2}{|Z|^2} \left|D_{\hat{I}_0} Z\right|^2\right) = 0$$

$$\Updownarrow$$

$$\left|D_{\hat{I}_0} Z\right|^2 = 4\frac{|Z|^2}{\left|C_{\hat{I}_0\hat{I}_0\hat{I}_0}\right|^2}; \tag{1.36}$$

this is nothing but the general criticality condition of V_{BH} for the 1-modulus case in the locally "flat" coordinate $\hat{I}_0$, which in this case corresponds to the $\hat{h}$-singlet direction in the scalar manifold. Such a case has been thoroughly studied in non-flat i-coordinates in [21].

All $N = 2$, $d = 4$ symmetric MESGT's (disregarding the sequence I having $C_{ijk} = 0$) have a cubic prepotential ($F = \frac{1}{3!} d_{ijk} t^i t^j t^k$ in special coordinates), and thus in special coordinates it holds that $C_{ijk} = \mathrm{e}^K d_{ijk}$, with K and d_{ijk} respectively denoting the Kähler potential and the completely symmetric rank-3 constant tensor that is determined by the norm form of the underlying Jordan algebra of degree three [97]. In the cubic $n_{\mathrm{V}} = 1$-modulus case, by using (1.18) it follows that

$$\left(G^{1_{\mathrm{s}}\overline{1}_{\mathrm{s}}}\right)^3 |C_{1_{\mathrm{s}}1_{\mathrm{s}}1_{\mathrm{s}}}|^2 = |C_{1_{\mathrm{f}}1_{\mathrm{f}}1_{\mathrm{f}}}|^2 = \frac{4}{3}, \tag{1.37}$$

where the subscripts "s" and "f" respectively stand for "special" and "flat", denoting the kind of coordinate system being considered. By substituting (1.37) in (1.36) one obtains the result

$$\left|D_{\hat{I}_0} Z\right|^2 = 3\,|Z|^2\,. \tag{1.38}$$

Another way of proving (1.38) is by computing the quartic invariant along the $\hat{h}$-singlet direction, then yielding

$$I_{4,\mathrm{non-BPS,Z}\neq 0} = -16\,|Z|^2_{\mathrm{non-BPS,Z}\neq 0}\,. \tag{1.39}$$

The considered solution (1.34)–(1.36), (1.38) is the $N = 2$ analogue of the $N = 8$, $d = 4$ non-BPS "large" solution discussed in [23], and it yields the following value of the BH scalar potential at the considered attractor point(s) [21, 23]:

$$V_{\mathrm{BH,non-BPS,Z}\neq 0} = 4\,|Z|^2_{\mathrm{non-BPS,Z}\neq 0}\,. \tag{1.40}$$

Once again, as for the non-BPS $N = 8$ "large" solutions, we find the extra factor 4.

From the above considerations, *the overall symmetry group at $N = 2$ non-BPS, $Z \neq 0$ critical point(s) is $\hat{h}$, mcs of the non-compact stabilizer $\hat{H}$ of $\mathcal{O}_{\mathrm{non-BPS,Z}\neq 0}$.*

Non-BPS, $Z = 0$

Let us now move to consider the other class of non-BPS orbits of $N = 2$, $d = 4$ symmetric MESGT's.

It has $Z = 0$ and it was not considered in [2] (see also Footnote 4). We will show that the solution of the $N = 2$, $d = 4$, non-BPS, $Z = 0$ extremal BH attractor eqs.

(1.29) are the "flattened matter charges" $D_I Z$ which are invariant under $\frac{\tilde{h}}{U(1)}$, where $\tilde{h}$ is the *mcs* of $\tilde{H}$, the stabilizer of the class $\mathcal{O}_{\text{non-BPS,Z=0}} = \frac{G}{\tilde{H}}$.

Differently from the non-BPS, $Z \neq 0$ case, in the considered non-BPS, $Z = 0$ case there is always a $U(1)$ symmetry acting, since the scalar manifolds of $N = 2$, $d = 4$ symmetric MESGT's *all* have the group $\tilde{h}$ of the form

$$\tilde{h} = \tilde{h}' \times U(1), \quad \tilde{h}' \equiv \frac{\tilde{h}}{U(1)}. \tag{1.41}$$

The compact subgroups $\tilde{h}'$ for all $N = 2$, $d = 4$ symmetric MESGT's are listed in Table 1.4. In the case at hand, we thence have to consider the decomposition of the previously introduced complex representation R_{H_0} under the compact subgroup $\tilde{h}' \subsetneq H_0$. In general, R_{H_0} decomposes under $\tilde{h}' \subsetneq \tilde{H}$ as follows:

$$R_{H_0} \longrightarrow \left(\mathcal{W}_{\tilde{h}'} + \mathcal{Y}_{\tilde{h}'} + \mathbf{1}\right)_{\mathbb{C}}, \tag{1.42}$$

where in the r.h.s. the complex singlet representation of $\tilde{h}'$ and two complex non-singlet representations $\mathcal{W}_{\tilde{h}'}$ and $\mathcal{Y}_{\tilde{h}'}$ of $\tilde{h}'$ appear. In general, $\mathcal{W}_{\tilde{h}'}$, $\mathcal{Y}_{\tilde{h}'}$ and $\mathbf{1}_{\mathbb{C}}$ are charged (and thus not invariant) with respect to the $U(1)$ explicit factor appearing in (1.41). The decomposition (1.42) yields the following splitting of "flattened matter charges":

$$D_I Z \longrightarrow \left(D_{\tilde{I}'_{\mathcal{W}}} Z, D_{\tilde{I}'_{\mathcal{Y}}} Z, D_{\tilde{I}'_0} Z\right), \tag{1.43}$$

where $\tilde{I}'_{\mathcal{W}}$ and $\tilde{I}'_{\mathcal{Y}}$ respectively denote the indices along the complex representations $\mathcal{W}_{\tilde{h}'}$ and $\mathcal{Y}_{\tilde{h}'}$, and $\tilde{I}'_0$ is the $\tilde{h}'$-singlet index.

Once again, the related $N = 2$, $d = 4$ non-BPS, $Z = 0$ extremal BH attractor eqs. (1.29) contain the rank-3 symmetric tensor C_{IJK}, corresponding to a cubic H_0-invariant coupling $(R_{H_0})^3$. The decomposition of $(R_{H_0})^3$ in terms of representations of $\tilde{h}'$ yields

$$\left(R_{H_0}\right)^3 \longrightarrow \left(\mathcal{W}_{\tilde{h}'}\right)^2 \mathcal{Y}_{\tilde{h}'} + \left(\mathcal{Y}_{\tilde{h}'}\right)^2 \mathbf{1}_{\mathbb{C}}. \tag{1.44}$$

When decomposed under $\tilde{h}'$, $(R_{H_0})^3$ must be nevertheless $\tilde{h}$-invariant, and therefore, beside the $\tilde{h}'$-invariance, one has to consider the invariance under the $U(1)$ factor, too. Thus, terms of the form $(\mathcal{W}_{\tilde{h}'})^3$, $(\mathcal{Y}_{\tilde{h}'})^3$, $\mathcal{W}_{\tilde{h}'}(\mathbf{1}_{\mathbb{C}})^2$, $\mathcal{Y}_{\tilde{h}'}(\mathbf{1}_{\mathbb{C}})^2$ and $(\mathbf{1}_{\mathbb{C}})^3$ cannot exist in the $\tilde{h}$-invariant r.h.s. of decomposition (1.44).

Notice also that the structure of the decomposition (1.44) implies that components of the cubic coupling of the form $C_{\tilde{I}'_{\mathcal{W}}\tilde{I}'_0\tilde{I}'_0}$, $C_{\tilde{I}'_{\mathcal{Y}}\tilde{I}'_0\tilde{I}'_0}$ and $C_{\tilde{I}'_0\tilde{I}'_0\tilde{I}'_0}$ cannot exist. For such a reason, it is immediate to conclude that the solution of $N = 2$, $d = 4$ non-BPS, $Z = 0$ extremal BH attractor eqs. in "flat" indices (1.29) corresponds to keep the "flattened matter charges" $D_I Z$ $\tilde{h}'$-invariant. By virtue of decomposition (1.44), this is obtained by putting

$$D_{\tilde{I}'_{\mathcal{W}}} Z = 0 = D_{\tilde{I}'_{\mathcal{Y}}} Z, \quad D_{\tilde{I}'_0} Z \neq 0, \tag{1.45}$$

i.e., by putting all "flattened matter charges" to zero, except the one along the $\tilde{h}'$-singlet (and thus $\tilde{h}'$-invariant, but not $U(1)$-invariant and therefore not $\tilde{h}$-invariant) direction in the scalar manifold.

The considered solution (1.45) does not have any analogue in $N = 8$, $d = 4$ supergravity, and it yields the following value of the BH scalar potential at the considered attractor point(s):

$$\begin{aligned} V_{\text{BH,non-BPS,Z=0}} &= |Z|^2_{\text{non-BPS,Z=0}} + \left[G^{i\bar{i}} D_i Z \overline{D}_{\bar{i}} \overline{Z} \right]_{\text{non-BPS,Z=0}} \\ &= \left| D_{\tilde{I}'_0} Z \right|^2_{\text{non-BPS,Z=0}}. \end{aligned} \tag{1.46}$$

It is here worth remarking that in the stu model it can be explicitly computed that [50, 77]

$$V_{\text{BH,non-BPS,Z=0}} = \left| D_{\tilde{I}'_0} Z \right|^2_{\text{non-BPS,Z=0}} = |Z|^2_{\frac{1}{2}\text{-BPS}} = V_{\text{BH},\frac{1}{2}\text{-BPS}}. \tag{1.47}$$

From above considerations, *the overall symmetry group at $N = 2$ non-BPS, $Z = 0$ critical point(s) is $\tilde{h}' = \frac{\tilde{h}}{U(1)}$, $\tilde{h}$ being the mcs of the non-compact stabilizer $\tilde{H}$ of $\mathcal{O}_{\text{non-BPS,Z=0}}$.*

The general analysis carried out above holds for all $N = 2$, $d = 4$ symmetric *magic* MESGT's, namely for the irreducible cases III–VI listed in Tables 1.2 and 1.3. The cases of irreducible sequence I and of generic Jordan family II deserve suitable, slightly different treatments, respectively given in Appendices I and II of [3].

1.2.3 Critical Spectra and Massless Hessian Modes of V_{BH}

The effective BH potential V_{BH} gives different masses to the different BPS-phases of the considered symmetric $N = 2$, $d = 4$ MESGT's. The fundamental object to be considered in such a framework is the moduli-dependent $2n_V \times 2n_V$ Hessian matrix of V_{BH}, which in complex basis reads[8] [21]

$$\mathbf{H}^{V_{\text{BH}}} \equiv \begin{pmatrix} D_i D_j V_{\text{BH}} & D_i \overline{D}_{\bar{j}} V_{\text{BH}} \\ D_j \overline{D}_{\bar{i}} V_{\text{BH}} & \overline{D}_{\bar{i}} \overline{D}_{\bar{j}} V_{\text{BH}} \end{pmatrix} \equiv \begin{pmatrix} \mathcal{M}_{ij} & \mathcal{N}_{i\bar{j}} \\ \overline{\mathcal{N}}_{j\bar{i}} & \overline{\mathcal{M}}_{\bar{i}\bar{j}} \end{pmatrix}; \tag{1.48}$$

[8] The reported formulæ for $\mathcal{M}_{ij}$ and $\mathcal{N}_{i\bar{j}}$ hold for any special Kähler manifold. In the symmetric case formula (1.49) gets simplified using (1.17).

$$
\begin{aligned}
\mathcal{M}_{ij} &\equiv D_i D_j V_{\mathrm{BH}} = D_j D_i V_{\mathrm{BH}} \\
&= 4i\overline{Z} C_{ijk} G^{k\bar{k}} \overline{D}_{\bar{k}} \overline{Z} + i G^{k\bar{k}} G^{l\bar{l}} \left(D_j C_{ikl} \right) \overline{D}_{\bar{k}} \overline{Z} \overline{D}_{\bar{l}} \overline{Z};
\end{aligned}
\tag{1.49}
$$

$$
\begin{aligned}
\mathcal{N}_{i\bar{j}} &\equiv D_i \overline{D}_{\bar{j}} V_{\mathrm{BH}} = \overline{D}_{\bar{j}} D_i V_{\mathrm{BH}} \\
&= 2\left[G_{i\bar{j}} |Z|^2 + D_i Z \overline{D}_{\bar{j}} \overline{Z} + G^{l\bar{n}} G^{k\bar{k}} G^{m\bar{m}} C_{ikl} \overline{C}_{\bar{j}\bar{m}\bar{n}} \overline{D}_{\bar{k}} \overline{Z} D_m Z \right];
\end{aligned}
\tag{1.50}
$$

$$
\mathcal{M}^T = \mathcal{M}, \mathcal{N}^{\dagger} = \mathcal{N}. \tag{1.51}
$$

By analyzing $\mathbf{H}^{V_{\mathrm{BH}}}$ at critical points of V_{BH}, it is possible to formulate general conclusions about the mass spectrum of the corresponding extremal BH solutions with finite, non-vanishing entropy, i.e., about the mass spectrum along the related classes of "large" charge orbits of the symplectic real representation R_{V} of the $d = 4$ duality group G.

Let us start by remarking that, due to its very definition (2.84), the $N = 2$ effective BH potential V_{BH} is positive for any (not necessarily strictly) positive definite metric $G_{i\bar{i}}$ of the scalar manifold. Consequently, the *stable* critical points (i.e., the *attractors* in a strict sense) will necessarily be minima of such a potential. As already pointed out above and as done also in [21,22], the geometry of the scalar manifold is usually assumed to be *regular*, i.e., endowed with a metric tensor $G_{i\bar{j}}$ being strictly positive definite everywhere.

1.2.3.1 $\frac{1}{2}$-BPS

It is now well known that *regular* special Kähler geometry implies that *all* $N = 2$ $\frac{1}{2}$-BPS critical points of *all* $N = 2$, $d = 4$ MESGT's are stable, and therefore they are attractors in a strict sense. Indeed, the Hessian matrix $\mathbf{H}^{V_{\mathrm{BH}}}_{\frac{1}{2}-\mathrm{BPS}}$ evaluated at such points is strictly positive definite [8]:

$$
\begin{aligned}
&\mathcal{M}_{ij,\frac{1}{2}-\mathrm{BPS}} = 0, \\
&\mathcal{N}_{i\bar{j},\frac{1}{2}-\mathrm{BPS}} = 2\, G_{i\bar{j}}\Big|_{\frac{1}{2}-\mathrm{BPS}} |Z|^2_{\frac{1}{2}-\mathrm{BPS}} > 0,
\end{aligned}
\tag{1.52}
$$

where the notation ">0" is clearly understood as strict positive definiteness of the quadratic form related to the square matrix being considered. Notice that the Hermiticity and strict positive definiteness of $\mathbf{H}^{V_{\mathrm{BH}}}_{\frac{1}{2}-\mathrm{BPS}}$ are respectively due to the Hermiticity and strict positive definiteness of the Kähler metric $G_{i\bar{j}}$ of the scalar manifold.

By switching from the non-flat i-coordinates to the "flat" local I-coordinates by using the (inverse) Vielbein e_I^i of the scalar manifold, (1.52) can be rewritten as

$$\begin{aligned} \mathcal{M}_{IJ,\frac{1}{2}-\mathrm{BPS}} &= 0, \\ \mathcal{N}_{I\overline{J},\frac{1}{2}-\mathrm{BPS}} &= 2\delta_{I\overline{J}}\,|Z|^2_{\frac{1}{2}-\mathrm{BPS}} > 0. \end{aligned} \tag{1.53}$$

Thus, one obtains that in *all* $N = 2$, $d = 4$ MESGT's the $\frac{1}{2}$-BPS mass spectrum in "flat" coordinates is *monochromatic*, i.e., that all "particles" (i.e., the "modes" related to the degrees of freedom described by the "flat" local I-coordinates) acquire *the same* mass at $\frac{1}{2}$-BPS critical points of V_{BH}.

1.2.3.2 Non-BPS, $Z \neq 0$

In this case the result of [13] should apply, namely the Hessian matrix $\mathbf{H}^{V_{\mathrm{BH}}}_{\mathrm{non-BPS},Z\neq 0}$ should have $n_{\mathrm{V}} + 1$ strictly positive and $n_{\mathrm{V}} - 1$ vanishing real eigenvalues.

By recalling the analysis performed in Sect. 1.2.2, it is thence clear that such massive and massless non-BPS, $Z \neq 0$ "modes" fit distinct real representations of $\hat{h} = mcs(\hat{H})$, where $\hat{H}$ is the non-compact stabilizer of the class $\mathcal{O}_{\mathrm{non-BPS},Z\neq 0} = \frac{G}{\hat{H}}$ of non-BPS, $Z \neq 0$ "large" charge orbits.

This is perfectly consistent with the decomposition (1.31) of the complex representation R_{H_0} ($\dim_{\mathbb{R}} R_{H_0} = 2n_{\mathrm{V}}$) of H_0 in terms of representations of $\hat{h}$:

$$R_{H_0} \longrightarrow \left(R_{\hat{h}} + \mathbf{1}\right)_{\mathbb{C}} = \left(R_{\hat{h}} + \mathbf{1} + R_{\hat{h}} + \mathbf{1}\right)_{\mathbb{R}}, \quad \dim_{\mathbb{R}} \left(R_{\hat{h}}\right)_{\mathbb{R}} = n_{\mathrm{V}} - 1. \tag{1.54}$$

As yielded by the treatment given in Sect. 1.2.2.2, the notation "$(R_{\hat{h}} + \mathbf{1})_{\mathbb{C}} = (R_{\hat{h}} + \mathbf{1} + R_{\hat{h}} + \mathbf{1})_{\mathbb{R}}$" denotes nothing but the decomplexification of $(R_{\hat{h}} + \mathbf{1})_{\mathbb{C}}$, which is actually composed by a pair of real irreducible representations $(R_{\hat{h}} + \mathbf{1})_{\mathbb{R}}$ of $\hat{h}$.

Therefore, the result of [13] can be understood in terms of real representations of the *mcs* of the non-compact stabilizer of $\mathcal{O}_{\mathrm{non-BPS},Z\neq 0}$: the $n_{\mathrm{V}} - 1$ massless non-BPS, $Z \neq 0$ "modes" are in one of the two real $R_{\hat{h}}$'s of $\hat{h}$ in the r.h.s. of (1.54), say the first one, whereas the $n_{\mathrm{V}} + 1$ massive non-BPS, $Z \neq 0$ "modes" are split in the remaining real $R_{\hat{h}}$ of $\hat{h}$ and in the two real $\hat{h}$-singlets. The resulting interpretation of the decomposition (1.54) is

$$R_{H_0} \longrightarrow \begin{pmatrix} \left(R_{\hat{h}}\right)_{\mathbb{R}} \\ \\ n_{\mathrm{V}} - 1 \ \ \text{massless} \end{pmatrix} + \begin{pmatrix} \left(R_{\hat{h}}\right)_{\mathbb{R}} + \mathbf{1}_{\mathbb{R}} + \mathbf{1}_{\mathbb{R}} \\ \\ n_{\mathrm{V}} + 1 \ \ \text{massive} \end{pmatrix}. \tag{1.55}$$

It is interesting to notice once again that there is no $U(1)$ symmetry relating the two real $R_{\hat{h}}$'s (and thus potentially relating the splitting of "modes" along $\mathcal{O}_{\mathrm{non-BPS},Z\neq 0}$), since in *all* symmetric $N = 2$, $d = 4$ MESGT's $\hat{h}$ *never* contains an explicit factor $U(1)$ (as instead it *always* happens for $\tilde{h}$!); this can be related to the fact that the non-compact stabilizer is $\hat{H}$ whose *mcs* is $\hat{h}$.

1.2.3.3 Non-BPS, $Z = 0$

For the class $\mathcal{O}_{\text{non-BPS,Z=0}}$ of "large" non-BPS, $Z = 0$ orbits the situation changes, and the result of [13] no longer holds true, due to the local vanishing of Z.

In all *magic* $N = 2$, $d = 4$ MESGT's the complex representation R_{H_0} of H_0 decomposes under $\tilde{h}' = \frac{mcs(\tilde{H})}{U(1)}$ in the following way (see (1.42)):

$$R_{H_0} \longrightarrow \mathcal{W}_{\tilde{h}'} + \mathcal{Y}_{\tilde{h}'} + \mathbf{1}_{\mathbb{C}}, \tag{1.56}$$

where in the r.h.s. the complex $\tilde{h}'$-singlet and the complex non-singlet representations $\mathcal{W}_{\tilde{h}'}$ and $\mathcal{Y}_{\tilde{h}'}$ of $\tilde{h}'$ appear. Correspondingly, the decomposition of the H_0-invariant representation $(R_{H_0})^3$ in terms of representations of $\tilde{h}'$ reads (see (1.44))

$$\left(R_{H_0}\right)^3 \longrightarrow \left(\mathcal{W}_{\tilde{h}'}\right)^2 \mathcal{Y}_{\tilde{h}'} + \left(\mathcal{Y}_{\tilde{h}'}\right)^2 \mathbf{1}_{\mathbb{C}}. \tag{1.57}$$

Let us now recall that $\dim_{\mathbb{R}} R_{H_0} = 2n_{\text{V}}$ and $\dim_{\mathbb{R}} \mathbf{1}_{\mathbb{C}} = 2$, and let us define

$$\left.\begin{array}{r} \dim_{\mathbb{R}} \mathcal{W}_{\tilde{h}'} \equiv \mathbf{W}_{\tilde{h}'}; \\ \\ \dim_{\mathbb{R}} \mathcal{Y}_{\tilde{h}'} \equiv \mathbf{Y}_{\tilde{h}'}; \end{array}\right\} : \mathbf{W}_{\tilde{h}'} + \mathbf{Y}_{\tilde{h}'} + 2 = 2n_{\text{V}}. \tag{1.58}$$

Thus, it can generally be stated that the mass spectrum along $\mathcal{O}_{\text{non-BPS,Z=0}}$ of all *magic* $N = 2$, $d = 4$ symmetric MESGT's splits under $\tilde{h}' = \frac{mcs(\tilde{H})}{U(1)}$ as follows:

- The mass "modes" fitting the $\mathbf{W}_{\tilde{h}'}$ real degrees of freedom corresponding to the complex ($U(1)$-charged) non-$\tilde{h}'$-singlet representation $\mathcal{W}_{\tilde{h}'}$ (which does *not* couple to the complex $\tilde{h}'$-singlet in the H_0-invariant decomposition (1.57)) remain *massless*.
- The mass "modes" fitting the $\mathbf{Y}_{\tilde{h}'} + 2$ real degrees of freedom corresponding to the complex ($U(1)$-charged) non-$\tilde{h}'$-singlet representation $\mathcal{Y}_{\tilde{h}'}$ and to the ($U(1)$-charged) $\tilde{h}'$-singlet $\mathbf{1}_{\mathbb{C}}$ *all* become *massive*.

The resulting interpretation of the decomposition (1.56) is

$$R_{H_0} \longrightarrow \left(\begin{array}{c} \mathcal{W}_{\tilde{h}'} \\ \\ \mathbf{W}_{\tilde{h}'} \ \text{massless} \end{array}\right) + \left(\begin{array}{c} \mathcal{Y}_{\tilde{h}'} + \mathbf{1}_{\mathbb{C}} \\ \\ \mathbf{Y}_{\tilde{h}'} + 2 \ \text{massive} \end{array}\right). \tag{1.59}$$

The interpretations (1.56) and (1.59) show that, even though the complex representations $\mathcal{W}_{\tilde{h}'}$, $\mathcal{Y}_{\tilde{h}'}$ and $\mathbf{1}_{\mathbb{C}}$ of $\tilde{h}'$ are charged with respect to the explicit factor $U(1)$ *always* appearing in $\tilde{h}$, this fact does *not* affect in any way the splitting of the non-BPS, $Z = 0$ mass "modes".

The critical mass spectra of the irreducible sequence $\frac{SU(1,1+n)}{U(1)\times SU(1+n)}$ and of the reducible sequence $\frac{SU(1,1)}{U(1)} \times \frac{SO(2,2+n)}{SO(2)\times SO(2+n)}$ are treated in Appendices I and II of [3], respectively.

Generally, the Hessian $\mathbf{H}^{V_{\rm BH}}$ at *regular* $N = 2$, non-BPS critical points of $V_{\rm BH}$ exhibits the following features: it does not have "repeller" directions (i.e., strictly *negative* real eigenvalues), it has a certain number of "attractor" directions (related to strictly *positive* real eigenvalues), but it is also characterized by some *vanishing* eigenvalues, corresponding to massless non-BPS "modes".

A priori, in order to establish whether the considered $N = 2$, non-BPS critical points of $V_{\rm BH}$ are actually *attractors* in a strict sense, i.e., whether they actually are *stable minima* of $V_{\rm BH}$ in the scalar manifold, one should proceed further with covariant differentiation of $V_{\rm BH}$, dealing (at least) with third and higher-order derivatives.

The detailed analysis of the issue of stability of both classes of *regular* non-BPS critical points ($Z \neq 0$ and $Z = 0$) of $V_{\rm BH}$ in $N = 2$, $d = 4$ (symmetric) MESGT's was performed in [42]. In that paper it was found that, for *all* supergravities with homogeneous (not necessarily symmetric) scalar manifolds the massless Hessian modes are actually "flat" directions of $V_{\rm BH}$, i.e., that the Hessian massless modes persist, at the critical points of $V_{\rm BH}$ itself, at all order in covariant differentiation of $V_{\rm BH}$. This is reported in the next section.

1.2.4 From Massless Hessian Modes of $V_{\rm BH}$ to Moduli Spaces of Attractors

In $N = 2$ homogeneous (not necessarily symmetric) and $N > 2$-extended (all symmetric), $d = 4$ supergravities the Hessian matrix of $V_{\rm BH}$ at its critical points is in general *semi-positive definite*, eventually with some vanishing eigenvalues (*massless Hessian modes*), which actually are *flat* directions of $V_{\rm BH}$ itself [39, 42]. Thus, it can be stated that for all supergravities based on homogeneous scalar manifolds the critical points of $V_{\rm BH}$ which correspond to "large" black holes (i.e., for which one finds that $V_{\rm BH} \neq 0$) all are *stable*, up to some eventual *flat* directions.

As pointed out above, the Attractor Equations of $N = 2$, $d = 4$ MESGT with $n_{\rm V}$ Abelian vector multiplets may have *flat* directions in the non-BPS cases [39, 42], but *not* in the $\frac{1}{2}$-BPS one [8] (see (1.52) and (1.53) above).

Tables 1.5 and 1.6 respectively list the moduli spaces of non-BPS $Z \neq 0$ and non-BPS $Z = 0$ attractors for symmetric $N = 2$, $d = 4$ special Kähler geometries, for which a complete classification is available [42] (the attractor moduli spaces should exist also in homogeneous non-symmetric $N = 2$, $d = 4$ special Kähler geometries, but their classification is currently unknown). The general "rule

Table 1.5 Moduli spaces of non-BPS $Z \neq 0$ critical points of $V_{BH,N=2}$ in $N = 2, d = 4$ symmetric supergravities ($\hat{h}$ is the maximal compact subgroup of $\hat{H}$). They are the $N = 2, d = 5$ symmetric real special manifolds [42]

	$\frac{\hat{H}}{\hat{h}}$	r	$\dim_{\mathbb{R}}$
$\text{II}: \mathbb{R} \oplus \Gamma_{n+2}$ ($n = n_V - 3 \in \mathbb{N} \cup \{0, -1\}$)	$SO(1,1) \times \frac{SO(1,n+1)}{SO(n+1)}$	$1 (n = -1)$ $2 (n \geqslant 0)$	$n + 2$
$\text{III}: J_3^{\mathbb{O}}$	$\frac{E_{6(-26)}}{F_{4(-52)}}$	2	6
$\text{IV}: J_3^{\mathbb{H}}$	$\frac{SU^*(6)}{USp(6)}$	2	14
$\text{V}: J_3^{\mathbb{C}}$	$\frac{SL(3,C)}{SU(3)}$	2	8
$\text{VI}: J_3^{\mathbb{R}}$	$\frac{SL(3,\mathbb{R})}{SO(3)}$	2	5

Table 1.6 Moduli spaces of non-BPS $Z = 0$ critical points of $V_{BH,N=2}$ in $N = 2, d = 4$ symmetric supergravities ($\tilde{h}$ is the maximal compact subgroup of $\tilde{H}$). They are (non-special) symmetric Kähler manifolds [42]

	$\frac{\tilde{H}}{\tilde{h}} = \frac{\tilde{H}}{\tilde{h}' \times U(1)}$	r	$\dim_{\mathbb{C}}$
I : Quadratic sequence ($n = n_V - 1 \in \mathbb{N} \cup \{0\}$)	$\frac{SU(1,n)}{U(1) \times SU(n)}$	1	n
$\text{II}: \mathbb{R} \oplus \Gamma_{n+2}$ ($n = n_V - 3 \in \mathbb{N} \cup \{0, -1\}$)	$\frac{SO(2,n)}{SO(2) \times SO(n)}, n \geqslant 1$	$1 (n = 1)$ $2 (n \geqslant 2)$	n
$\text{III}: J_3^{\mathbb{O}}$	$\frac{E_{6(-14)}}{SO(10) \times U(1)}$	2	16
$\text{IV}: J_3^{\mathbb{H}}$	$\frac{SU(4,2)}{SU(4) \times SU(2) \times U(1)}$	2	8
$\text{V}: J_3^{\mathbb{C}}$	$\frac{SU(2,1)}{SU(2) \times U(1)} \times \frac{SU(1,2)}{SU(2) \times U(1)}$	2	4
$\text{VI}: J_3^{\mathbb{R}}$	$\frac{SU(2,1)}{SU(2) \times U(1)}$	1	2

of thumb" to construct the moduli space of a given attractor solution in the considered symmetric framework is to coset the *stabilizer* of the corresponding charge orbit by its *mcs*. By such a rule, the $\frac{1}{2}$-BPS attractors do *not* have an associated moduli space simply because the stabilizer of their supporting BH charge orbit is *compact*. On the other hand, *all* attractors supported by BH charge orbits whose stabilizer is *non-compact* exhibit a non-vanishing moduli space. furthermore, it should be noticed that the non-BPS $Z \neq 0$ moduli spaces are nothing but the symmetric real special scalar manifolds of the corresponding $N = 2$, $d = 5$ supergravity.

Nevertheless, it is worth remarking that some symmetric $N = 2$, $d = 4$ supergravities have no non-BPS *flat* directions at all.

The unique $n_V = 1$ symmetric models are the so-called t^2 and t^3 models; they are based on the rank-1 scalar manifold $\frac{SU(1,1)}{U(1)}$, but with different holomorphic prepotential functions.

The t^2 model is the first element ($n = 0$) of the sequence of irreducible symmetric special Kähler manifolds $\frac{SU(1,n+1)}{U(1)\times SU(n+1)}$ ($n_V = n + 1$, $n \in \mathbb{N}\cup\{0\}$) (see, e.g., [3] and refs. therein), endowed with *quadratic* prepotential. Its bosonic sector is given by the $(U(1))^6 \to (U(1))^2$ truncation of Maxwell–Einstein-axion-dilaton (super)gravity, i.e., of *pure* $N = 4$, $d = 4$ supergravity (see, e.g., [71] and [90] for recent treatments).

On the other hand, the t^3 model has *cubic* prepotential; as pointed out above, it is an *isolated case* in the classification of symmetric SK manifolds (see, e.g., [102]; see also [103] and refs. therein), but it can be thought also as the $s = t = u$ *degeneration* of the stu model. It is worth pointing out that the t^2 and t^3 models are based on the same rank-1 SK manifold, with different constant *scalar curvature*, which respectively can be computed to be (see, e.g., [35] and refs. therein)

$$\begin{aligned} &\tfrac{SU(1,1)}{U(1)},\ t^2 \text{ model}: R = -2; \\ &\tfrac{SU(1,1)}{U(1)},\ t^3 \text{ model}: R = -\tfrac{2}{3}. \end{aligned} \tag{1.60}$$

Beside the $\frac{1}{2}$-BPS attractors, the t^2 model admits only non-BPS $Z = 0$ critical points of V_{BH} with no *flat* directions. Analogously, the t^3 model admits only non-BPS $Z \neq 0$ critical points of V_{BH} with no *flat* directions.

For $n_V > 1$, the non-BPS $Z \neq 0$ critical points of V_{BH}, if any, all have *flat* directions, and thus a related moduli space (see Table 1.5). However, models with no non-BPS $Z = 0$ *flat* directions at all and $n_V > 1$ exist, namely they are the first and second element ($n = -1, 0$) of the sequence of reducible symmetric special Kähler manifolds $\frac{SU(1,1)}{U(1)} \times \frac{SO(2,n+2)}{SO(2)\times SO(n+2)}$ ($n_V = n + 3$, $n \in \mathbb{N}\cup\{0, -1\}$) (see, e.g., [3] and refs. therein), i.e., the so-called st^2 and stu models, respectively. The stu model [104, 105], see also, e.g., [77] and refs. therein) has two non-BPS $Z \neq 0$ *flat* directions, spanning the moduli space $SO(1,1) \times SO(1,1)$ (i.e., the scalar manifold of the stu model in $d = 5$), but *no* non-BPS $Z = 0$ *massless Hessian modes* at all. On the other hand, the st^2 model (which can be thought as the $t = u$ *degeneration* of the stu model) has one non-BPS $Z \neq 0$ *flat* direction, spanning the moduli space $SO(1,1)$ (i.e., the scalar manifold of the st^2 model in $d = 5$), but *no* non-BPS $Z = 0$ *flat* direction at all. The st^2 is the "smallest" symmetric model exhibiting a non-BPS $Z \neq 0$ *flat* direction.

Concerning the "smallest" symmetric models exhibiting a non-BPS $Z = 0$ *flat* direction they are the second ($n = 1$) element of the sequence $\frac{SU(1,n+1)}{U(1)\times SU(n+1)}$ and the third ($n = 1$) element of the sequence $\frac{SU(1,1)}{U(1)} \times \frac{SO(2,n+2)}{SO(2)\times SO(n+2)}$. In both cases, the unique non-BPS $Z = 0$ *flat* direction spans the non-BPS $Z = 0$ moduli space $\frac{SU(1,1)}{U(1)} \sim \frac{SO(2,1)}{SO(2)}$ (see Table 1.6), whose local geometrical properties however differ in the two cases (for the same reasons holding for the t^2 and t^3 models treated above).

We conclude by recalling that in [133–135] it was shown that the $N = 2$, $d = 5$ *magic* MESGT's defined by $J_3^{\mathbb{C}}$, $J_3^{\mathbb{H}}$ and $J_3^{\mathbb{O}}$ are simply the "lowest" members of three infinite families of unified $N = 2$, $d = 5$ MESGT's defined by Lorentzian Jordan algebras of degree > 3. The scalar manifolds of such theories are not homogeneous except for the "lowest" members. It would be interesting to extend the analysis of [24] and [3] to these theories in five dimensions and to their descendants in $d = 5$, respectively.

1.3 U-Duality "Large" Orbits and Moduli Spaces of Attractors in $N \geqslant 3$-Extended, $d = 4$ Supergravities

In $N \geqslant 3$-extended, $d = 4$ supergravities, whose scalar manifold is always symmetric, there are *flat* directions of V_{BH} at both its BPS and non-BPS critical points. As mentioned above, from a group-theoretical point of view this is due to the fact that the corresponding supporting BH charge orbits always have a *non-compact* stabilizer [42, 61]. The BPS *flat* directions can be interpreted in terms of left-over hypermultiplets' scalar degrees of freedom in the truncation down to the $N = 2$, $d = 4$ theories [39, 136]. In Tables 1.7 and 1.8 all (classes of) "large" charge orbits and the corresponding moduli spaces of attractor solution in $N \geqslant 3$-extended, $d = 4$ supergravities are reported [61].

Table 1.7 "Large" charge orbits of the real, symplectic R_{V} representation of the U-duality group G supporting BH attractors with non-vanishing entropy in $N \geqslant 3$-extended, $d = 4$ supergravities (n is the number of matter multiplets) [61]

	$\frac{1}{N}$-BPS orbits $\frac{G}{\mathcal{H}}$	Non-BPS, $Z_{AB} \neq 0$ orbits $\frac{G}{\hat{\mathcal{H}}}$	Non-BPS, $Z_{AB} = 0$ orbits $\frac{G}{\tilde{\mathcal{H}}}$
$N = 3$	$\frac{SU(3,n)}{SU(2,n)}$	–	$\frac{SU(3,n)}{SU(3,n-1)}$
$N = 4$	$SU(1,1) \times \frac{SO(6,n)}{SO(2)\times SO(4,n)}$	$SU(1,1) \times \frac{SO(6,n)}{SO(1,1)\times SO(5,n-1)}$	$SU(1,1) \times \frac{SO(6,n)}{SO(2)\times SO(6,n-2)}$
$N = 5$	$\frac{SU(1,5)}{SU(3)\times SU(2,1)}$	–	–
$N = 6$	$\frac{SO^*(12)}{SU(4,2)}$	$\frac{SO^*(12)}{SU^*(6)}$	$\frac{SO^*(12)}{SU(6)}$
$N = 8$	$\frac{E_{7(7)}}{E_{6(2)}}$	$\frac{E_{7(7)}}{E_{6(6)}}$	–

Table 1.8 Moduli spaces of BH attractors with non-vanishing entropy in $N \geqslant 3$-extended, $d = 4$ supergravities ($\mathfrak{h}$, $\hat{\mathfrak{h}}$ and $\tilde{\mathfrak{h}}$ are maximal compact subgroups of $\mathcal{H}$, $\hat{\mathcal{H}}$ and $\tilde{\mathcal{H}}$, respectively, and n is the number of matter multiplets) [61]

	$\frac{1}{N}$-BPS moduli space $\frac{\mathcal{H}}{\mathfrak{h}}$	Non-BPS, $Z_{AB} \neq 0$ moduli space $\frac{\hat{\mathcal{H}}}{\hat{\mathfrak{h}}}$	Non-BPS, $Z_{AB} = 0$ moduli space $\frac{\tilde{\mathcal{H}}}{\tilde{\mathfrak{h}}}$
$N = 3$	$\frac{SU(2,n)}{SU(2) \times SU(n) \times U(1)}$	–	$\frac{SU(3,n-1)}{SU(3) \times SU(n-1) \times U(1)}$
$N = 4$	$\frac{SO(4,n)}{SO(4) \times SO(n)}$	$SO(1,1) \times \frac{SO(5,n-1)}{SO(5) \times SO(n-1)}$	$\frac{SO(6,n-2)}{SO(6) \times SO(n-2)}$
$N = 5$	$\frac{SU(2,1)}{SU(2) \times U(1)}$	–	–
$N = 6$	$\frac{SU(4,2)}{SU(4) \times SU(2) \times U(1)}$	$\frac{SU^*(6)}{USp(6)}$	–
$N = 8$	$\frac{E_{6(2)}}{SU(6) \times SU(2)}$	$\frac{E_{6(6)}}{USp(8)}$	–

1.4 Conclusions

In the present report we dealt with results holding at the classical, Einstein supergravity level. It is conceivable that the *flat* directions of classical extremal BH attractors will be removed (i.e., lifted) by *quantum* (*perturbative* and *non-perturbative*) corrections (such as those coming from higher-order derivative contributions to the gravity and/or gauge sector) to the *classical* effective BH potential V_{BH}. Consequently, *at the quantum level, moduli spaces for attractor solutions may not exist at all* (and therefore also *the actual attractive nature of the critical points of* V_{BH} *might be destroyed*). *However, this may not be the case for* $N = 8$.

In the presence of *quantum* lifts of *classically flat* directions of the Hessian matrix of V_{BH} at its critical points, in order to answer the key question: "Do extremal BH attractors (in a strict sense) survive at the quantum level?", it is thus crucial to determine whether such lifts originate from Hessian modes with *positive* squared mass (corresponding to *attractive* directions) or with *negative* squared mass (i.e., *tachyonic*, *repeller* directions).

The fate of the unique non-BPS $Z \neq 0$ flat direction of the st^2 model in presence of the most general class of quantum perturbative corrections consistent with the axionic-shift symmetry has been studied in [80], showing that, as intuitively expected, the *classical solutions get lifted at the quantum level*. Interestingly, in [80] it is found that the *quantum* lift occurs more often towards *repeller* directions (thus destabilizing the whole critical solution, and *destroying the attractor in a strict sense*), than towards *attractive* directions. The same behavior may be expected for the unique non-BPS $Z = 0$ flat direction of the $n = 2$ element of the quadratic irreducible sequence and the $n = 3$ element of the cubic reducible sequence (see above).

Generalizing it to the presence of more than one *flat* direction, this would mean that *only a (very) few classical attractors do remain attractors in a strict sense at the quantum level*; consequently, *at the quantum* (*perturbative* and *non-perturbative*) *level the "landscape" of extremal BH attractors should be strongly constrained and reduced*.

Despite the considerable number of papers written on the *Attractor Mechanism* in the extremal BHs of the supersymmetric theories of gravitation in past years, still much remains to be discovered along the way leading to a deep understanding of the inner dynamics of (eventually extended) space–time singularities in supergravities, and hopefully of their fundamental high-energy counterparts, such as $d = 10$ superstrings and $d = 11$ M-theory.

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Chapter 2
Lectures on Spectrum Generating Symmetries and U-Duality in Supergravity, Extremal Black Holes, Quantum Attractors and Harmonic Superspace

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Abstract We review the underlying algebraic structures of supergravity theories with symmetric scalar manifolds in five and four dimensions, orbits of their extremal black hole solutions and the spectrum generating extensions of their U-duality groups. For 5D, $N = 2$ Maxwell–Einstein supergravity theories (MESGT) defined by Euclidean Jordan algebras, J, the spectrum generating symmetry groups are the conformal groups Conf(J) of J which are isomorphic to their U-duality groups in four dimensions. Similarly, the spectrum generating symmetry groups of 4D, $N = 2$ MESGTs are the quasiconformal groups QConf(J) associated with J that are isomorphic to their U-duality groups in three dimensions. We then review the work on spectrum generating symmetries of spherically symmetric stationary 4D BPS black holes, based on the equivalence of their attractor equations and the equations for geodesic motion of a fiducial particle on the target spaces of corresponding 3D supergravity theories obtained by timelike reduction. We also discuss the connection between harmonic superspace formulation of 4D, $N = 2$ sigma models coupled to supergravity and the minimal unitary representations of their isometry groups obtained by quantizing their quasiconformal realizations. We discuss the relevance of this connection to spectrum generating symmetries and conclude with a brief summary of more recent results.

2.1 Introduction

This review on spectrum generating symmetries in supergravity, extremal black holes, U-duality orbits, quantum attractor flows and harmonic superspace is based on four lectures given at the School on Attractor Mechanism (SAM 2007) in Frascati, Italy. The lectures were titled:

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1. Very special real geometry, Jordan algebras and attractors
2. Very special complex geometry, Freudenthal triple systems and attractors
3. Conformal and quasiconformal extensions of U-duality groups as spectrum generating symmetry groups
4. Harmonic superspace, quasiconformal groups and their minimal unitary representations

I will follow closely the material covered in the lectures and discuss briefly the results obtained since SAM 2007 at the end.

More specifically, Sect. 2.2 is a review of the U-duality symmetries of maximal supergravity in various dimensions. In Sects. 2.3 and 2.4 we review the $N = 2$ Maxwell–Einstein supergravity theories (MESGT) in five dimensions and the connection between Jordan algebras of degree 3 and the MESGTs with symmetric scalar manifolds. This is followed by a review of the symmetry groups of Jordan algebras using the language of space–time symmetry groups and a complete list of simple finite dimensional Jordan algebras and their automorphism (rotation), reduced structure (Lorentz) and Möbius (conformal) groups. In Sect. 2.6 we review the U-duality orbits of extremal black holes of 5D supergravity theories with symmetric target manifolds and discuss how these results lead to the proposal that 4D U-duality groups act as spectrum generating symmetry groups of the corresponding 5D theories. Section 2.7 is an overview of the 4D MESGTs with symmetric target spaces and their connection with Freudenthal triple systems. This is followed by a classification of the U-duality orbits of extremal black holes of $N = 2$ MESGTs with symmetric target manifolds and of $N = 8$ supergravity and the proposal that the three-dimensional U-duality groups act as spectrum generating quasiconformal groups of the corresponding 4D theories. In Sect. 2.9 we summarize the novel quasiconformal realizations of non-compact groups and their relation to Freudenthal triple systems. A precise and concrete implementation of the proposal that 3D U-duality groups act as spectrum generating symmetry groups of the corresponding 4D theories within the framework of spherically symmetric stationary BPS black holes is discussed in Sect. 2.10. We then review the connection between the harmonic superspace formulation of 4D, $N = 2$ sigma models coupled to supergravity and the minimal unitary representations of their isometry groups. For sigma models with symmetric target spaces we show that there is a remarkable map between the Killing potentials that generate their isometry groups in harmonic superspace and the minimal unitary representations of these groups obtained by quantizing their quasiconformal realizations. Implications of this result are also discussed. Section 2.12 is devoted to the M/superstring theoretic origins of $N = 2$ MESGTs with symmetric target spaces, in particular, the magical supergravity theories. We conclude with a brief discussion of the related developments that took place since SAM 2007 and some open problems.

2.2 U-Duality Symmetries of Maximal Supergravity in Various Dimensions

The maximal possible dimension for Poincare supergravity is 11 dimensions [1]. Eleven-dimensional supergravity involves a Majorana gravitino field, the elf-bein E_M^A and an anti-symmetric tensor of rank three A_{MNP} and was constructed in [2]. Lagrangian of its bosonic sector has a very simple form

$$\mathcal{L}_{11} = \frac{1}{\kappa_{11}^2}\Big[-\frac{1}{2}ER - \frac{1}{48}E(F_{MNPQ})^2 - \frac{\sqrt{2}}{3456}\,\varepsilon^{MNPQRSTUVWX}\, F_{MNPQ}\, F_{RSTU}\, A_{VWX} + \cdots \Big], \tag{2.1}$$

where the ellipses denote terms that involve fermions, R is the scalar curvature, E is the determinant of 11-dimensional elf-bein E_M^A and F_{MNPQ} is the field strength of the antisymmetric three-form field A_{MNP} with $M, N, P, \ldots = 0, 1, \ldots, 10$.

Under dimensional reduction to d dimensions, 11-dimensional supergravity results in the maximal supergravity with a global symmetry group $E_{((11-d)(11-d))}$. We shall refer to these global symmetry groups as the U-duality groups even though the term was originally used for discrete subgroups of these continuous groups which are believed to be non-perturbative symmetries of M-theory toroidally compactified to d dimensions [3].

In six dimensions maximal supergravity has the global symmetry group $E_{5(5)} = SO(5,5)$ and the scalar fields of this theory parametrize the symmetric space

$$\mathcal{M}_6 = \frac{SO(5,5)}{SO(5) \times SO(5)}. \tag{2.2}$$

In five dimensions the global symmetry group of maximal supergravity is $E_{6(6)}$ under which all the vector fields transform in the irreducible 27-dimensional representation of $E_{6(6)}$. The scalar fields of the theory parametrize the symmetric space

$$\mathcal{M}_5 = \frac{E_{6(6)}}{USp(8)}. \tag{2.3}$$

U-duality group $E_{6(6)}$ is a symmetry of the Lagrangian of ungauged maximal supergravity in five dimensions.

In four dimensions the maximal ungauged supergravity has the U-duality group $E_{7(7)}$ as an *on-shell symmetry* under which field strengths of the 28 vector fields of the theory and their magnetic duals transform in the 56-dimensional representation, which is the smallest non-trivial representation of $E_{7(7)}$. The 70 scalar fields of the theory parametrize the symmetric space

$$\mathcal{M}_4 = \frac{E_{7(7)}}{SU(8)}. \tag{2.4}$$

Symmetry of the Lagrangian of ungauged $N = 8$ supergravity theory in four dimensions depends on the real symplectic section chosen. Dimensional reduction from five dimensions leads to a symplectic section with $E_{6(6)} \times SO(1,1)$ symmetry of the Lagrangian such that under $E_{6(6)}$ electric field strengths transform in the reducible representation $(27+1)$. There exist also symplectic sections leading to Lagrangians with $SL(8,\mathbb{R})$ and $SU^*(8)$ symmetry groups. In the latter two cases the electric field strengths transform irreducibly in the 28-dimensional real representation of $SL(8,\mathbb{R})$ or $SU^*(8)$, respectively.

In three dimensions all the dynamical bosonic degrees of freedom of maximal supergravity can be dualized to scalar fields parametrizing the symmetric space [4]

$$\mathcal{M}_3 = \frac{E_{8(8)}}{SO(16)}. \tag{2.5}$$

2.3 5D, $N = 2$ Maxwell–Einstein Supergravity Theories

Certain matter coupled supergravity theories do admit global symmetry groups which we shall also refer to as U-duality groups. In this section we shall study the U-duality groups that arise in five-dimensional $N = 2$ Maxwell–Einstein supergravity theories (MESGT). Five-dimensional MESGTs that describe the coupling of an arbitrary number of $N = 2$ (Abelian) vector multiplets to $N = 2$ supergravity were constructed long ago in [5–8]. The fields of the graviton supermultiplet are the fünfbein e_μ^m, two gravitini ψ_μ^i $(i = 1,2)$, and a vector field A_μ (the "bare" graviphoton). A vector multiplet consists of a vector field A_μ, two "gaugini" λ^i and one real scalar ϕ. The bosonic part of five-dimensional $N = 2$ MESGT Lagrangian describing the coupling of $(n_V - 1)$ vector multiplets has a very simple form[1]

$$\begin{aligned} e^{-1}\mathcal{L}_{\text{bosonic}} = &-\frac{1}{2}R - \frac{1}{4}\overset{\circ}{a}_{IJ}F_{\mu\nu}^I F^{J\mu\nu} - \frac{1}{2}g_{xy}(\partial_\mu\varphi^x)(\partial^\mu\varphi^y) \\ &+ \frac{e^{-1}}{6\sqrt{6}}C_{IJK}\varepsilon^{\mu\nu\rho\sigma\lambda}F_{\mu\nu}^I F_{\rho\sigma}^J A_\lambda^K, \end{aligned} \tag{2.6}$$

where

$$\begin{aligned} I &= 1,\ldots,n_V \\ x &= 1,\ldots,(n_V - 1) \\ \mu,\nu,\ldots &= 0,1,2,3,4. \end{aligned}$$

Note that we combined the "bare graviphoton" with the other vector fields and labelled them with a single index I which runs from 1 to n_V. e and R denote the

[1] We use the conventions of [6].

fünfbein determinant and scalar curvature of spacetime, respectively. $F^I_{\mu\nu}$ are field strengths of the vector fields A^I_μ. The metric, g_{xy}, of the scalar manifold $\mathcal{M}_5$ and the "metric" $\overset{\circ}{a}_{IJ}$ of the kinetic energy term of vector fields both depend on the scalar fields φ^x. On the other hand, the completely symmetric tensor C_{IJK} is constant as required by local Abelian gauge symmetries of vector fields.

Remarkably, the entire 5D, $N = 2$ MESGT is uniquely determined by the constant tensor C_{IJK} [6]. In particular, geometry of the scalar manifold $\mathcal{M}_5$ is determined by C_{IJK} as follows. One defines a cubic polynomial, $\mathcal{V}(h)$, in n_V real variables h^I $(I = 1, \ldots, n_V)$ using the C-tensor,

$$\mathcal{V}(h) := C_{IJK} h^I h^J h^K \tag{2.7}$$

and a metric, a_{IJ}, of a n_V dimensional ambient space $\mathcal{C}_{n_V}$ coordinatized by h^I:

$$a_{IJ}(h) := -\frac{1}{3} \frac{\partial}{\partial h^I} \frac{\partial}{\partial h^J} \ln \mathcal{V}(h) \,. \tag{2.8}$$

It can then be proven that the $(n_V - 1)$-dimensional manifold, $\mathcal{M}_5$, of scalar fields φ^x can be represented as an hypersurface defined by the condition [6]

$$\mathcal{V}(h) = C_{IJK} h^I h^J h^K = 1, \tag{2.9}$$

in this ambient space $\mathcal{C}_{n_V}$.[2] The metric g_{xy} is simply the pull-back of (2.8) to $\mathcal{M}_5$:

$$g_{xy}(\varphi) = \left(h^I_x h^J_y a_{IJ} \right) |_{\mathcal{V}=1}, \tag{2.10}$$

where

$$h^I_x = \sqrt{\frac{3}{2}} \frac{\partial h^I}{\partial \phi^x} |_{\mathcal{V}=1}$$

and the "metric" $\overset{\circ}{a}_{IJ}(\varphi)$ of kinetic energy term of the vector fields is given by the componentwise restriction of the metric a_{IJ} of the ambient space $\mathcal{C}_{n_V}$ to $\mathcal{M}_5$:

$$\overset{\circ}{a}_{IJ}(\varphi) = a_{IJ}|_{\mathcal{V}=1}.$$

Riemann curvature tensor of the scalar manifold takes on a very simple form

$$K_{xyzu} = \frac{4}{3} \left(g_{x[u} g_{z]y} + T_{x[u}{}^{w} T_{z]yw} \right), \tag{2.11}$$

[2] The ambient space $\mathcal{C}_{n_V}$ is the five-dimensional counterpart of the hyper Kähler cone of the twistor space of the corresponding three-dimensional quaternionic geometry of the scalar manifold $\mathcal{M}_3$.

where T_{xyz} is a symmetric tensor that is the pull-back of the C-tensor to the hypersurface

$$T_{xyz} = h^I_x h^J_y h^K_z C_{IJK}. \tag{2.12}$$

Since the Riemann curvature tensor K_{xyzu} depends only on the metric g_{xy} and the tensor T_{xyz} it follows that the covariant constancy of T_{xyz} implies the covariant constancy of K_{xyzu}:

$$T_{xyz;w} = 0 \longrightarrow K_{xyzu;w} = 0.$$

Therefore scalar manifolds $\mathcal{M}_5$ with covariantly constant T tensors are locally symmetric spaces. If the scalar manifold $\mathcal{M}_5$ is homogeneous then the covariant constancy of T_{xyz} is equivalent to the "adjoint identity" for the C-tensor [6]:

$$C^{IJK} C_{J(MN} C_{PQ)K} = \delta^I_{(M} C_{NPQ)}, \tag{2.13}$$

where the indices are raised by the inverse $\overset{\circ}{a}{}^{IJ}$ of $\overset{\circ}{a}_{IJ}$. Furthermore, cubic forms defined by C_{IJK} of $N = 2$ MESGT's that satisfy the adjoint identity and lead to positive definite metrics g_{xy} and $\overset{\circ}{a}_{IJ}(\varphi)$ are in one-to-one correspondence with norm forms of Euclidean (formally real) Jordan algebras J of degree 3 [6]. The scalar manifolds of the corresponding MESGT's are of the form

$$\mathcal{M}_5 = \frac{\mathrm{Str}_0(J)}{\mathrm{Aut}(J)}, \tag{2.14}$$

where $\mathrm{Str}_0(J)$ is the invariance group of the norm $\mathbf{N}$ of J and $\mathrm{Aut}(J)$ is its automorphism group. These theories exhaust the list of 5D MESGTs with symmetric target spaces G/H such that G is a symmetry of the Lagrangian [9]. Remarkably, the list of cubic forms that satisfy the adjoint identity coincides also with the list of Legendre invariant cubic forms that were classified more recently by mathematicians [10]. Before we discuss the geometries of $N = 2$ MESGT's defined by Jordan algebras we shall take a detour and review some of the basic facts regarding Jordan algebras of degree 3 in the next subsection. For details and further references on Jordan algebras we refer to the monograph [11].

2.4 MESGT's with Symmetric Target Spaces and Euclidean Jordan Algebras of Degree 3

A Jordan algebra J over a field $\mathbb{F}$ is a commutative and non-associative algebra with a product $\circ$ that satisfies

$$X \circ Y = Y \circ X \in J, \quad \forall\ X, Y \in J, \tag{2.15}$$

and

$$X \circ (Y \circ X^2) = (X \circ Y) \circ X^2, \tag{2.16}$$

where $X^2 \equiv (X \circ X)$. Given a Jordan algebra J, one can define a norm form

$$\mathbf{N} : J \to \mathbb{R}$$

that satisfies the composition property [12]

$$\mathbf{N}[2X \circ (Y \circ X) - (X \circ X) \circ Y] = \mathbf{N}^2(X)\mathbf{N}(Y). \tag{2.17}$$

A Jordan algebra is said to be of degree, p, if its norm form satisfies $\mathbf{N}(\lambda X) = \lambda^p \mathbf{N}(X)$, where $\lambda \in \mathbb{R}$. A *Euclidean* Jordan algebra is a Jordan algebra for which the condition $X \circ X + Y \circ Y = 0$ implies that $X = Y = 0$ for all $X, Y \in J$. Euclidean Jordan algebras are sometimes called compact Jordan algebras since their automorphism groups are compact.

As explained above, given a Euclidean Jordan algebra of degree 3 one can identify its norm form $\mathbf{N}$ with the cubic polynomial $\mathcal{V}$ defined by the C-tensor of a 5D, $N = 2$ MESGT with a symmetric scalar manifold [6]. Euclidean Jordan algebras of degree 3 fall into an infinite family of non-simple Jordan algebras which are direct sums of the form

$$J = \mathbb{R} \oplus \Gamma_{(1,n-1)}, \tag{2.18}$$

where $\Gamma_{(1,n-1)}$ is an n-dimensional Jordan algebra of degree 2 associated with a quadratic norm form in n dimensions that has a "Minkowskian signature" $(+,-,\ldots,-)$ and $\mathbb{R}$ is the one-dimensional Jordan algebra. This infinite family of reducible Jordan algebras of degree 3 exists for any n and is referred to as the generic Jordan family. The scalar manifolds of corresponding 5D, $N = 2$ MESGT's are

$$\mathcal{M}_5(\mathbb{R} \oplus \Gamma_{(1,n-1)}) = \frac{SO(n-1,1)}{SO(n-1)} \times SO(1,1). \tag{2.19}$$

An irreducible realization of $\Gamma_{(1,n-1)}$ is provided by $(n-1)$ $(2^{[n/2]} \times 2^{[n/2]})$ Dirac gamma matrices γ^i $(i, j, \ldots = 1, \ldots, (n-1))$ of an $(n-1)$ dimensional Euclidean space together with the identity matrix $\gamma^0 = \mathbf{1}$ and the Jordan product $\circ$ being defined as one half the anticommutator:

$$\begin{aligned}
\gamma^i \circ \gamma^j &= \frac{1}{2}\{\gamma^i, \gamma^j\} = \delta^{ij}\gamma^0, \\
\gamma^0 \circ \gamma^0 &= \frac{1}{2}\{\gamma^0, \gamma^0\} = \gamma^0, \\
\gamma^i \circ \gamma^0 &= \frac{1}{2}\{\gamma^i, \gamma^0\} = \gamma^i \ .
\end{aligned} \tag{2.20}$$

The quadratic norm of a general element $\mathbb{X} = X_0\gamma^0 + X_i\gamma^i$ of $\Gamma_{(1,n-1)}$ is defined as

$$\mathbf{Q}(\mathbb{X}) = \frac{1}{2^{[n/2]}} Tr\, \mathbb{X}\overline{\mathbb{X}} = X_0 X_0 - X_i X_i \ ,$$

where

$$\overline{\mathbb{X}} \equiv X_0\gamma^0 - X_i\gamma^i.$$

The norm of a general element $y \oplus \mathbb{X}$ of the non-simple Jordan algebra $J = \mathbb{R} \oplus \Gamma_{(1,n-1)}$ is then simply

$$\mathbf{N}(y \oplus \mathbb{X}) = y\mathbf{Q}(\mathbb{X}), \tag{2.21}$$

where $y \in \mathbb{R}$.

In addition to this generic reducible infinite family, there exist four simple Euclidean Jordan algebras of degree 3. They are realized by Hermitian (3×3)-matrices over the four division algebras $\mathbb{A} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ (reals $\mathbb{R}$, complex numbers $\mathbb{C}$, quaternions $\mathbb{H}$ and octonions $\mathbb{O}$)

$$J = \begin{pmatrix} \alpha & Z & \overline{Y} \\ \overline{Z} & \beta & X \\ Y & \overline{X} & \gamma \end{pmatrix}, \tag{2.22}$$

where $\alpha, \beta, \gamma \in \mathbb{R}$ and $X, Y, Z \in \mathbb{A}$ with the Jordan product being one half the anticommutator. They are denoted as $J_3^{\mathbb{R}}$, $J_3^{\mathbb{C}}$, $J_3^{\mathbb{H}}$, $J_3^{\mathbb{O}}$, respectively. The corresponding $N = 2$ MESGTs are called "magical supergravity theories"[5]. The scalar manifolds of the magical supergravity theories in five dimensions are the irreducible symmetric spaces

$$\begin{aligned} J_3^{\mathbb{R}}: &\quad \mathcal{M} = SL(3,\mathbb{R})/SO(3), \\ J_3^{\mathbb{C}}: &\quad \mathcal{M} = SL(3,\mathbb{C})/SU(3), \\ J_3^{\mathbb{H}}: &\quad \mathcal{M} = SU^*(6)/USp(6), \\ J_3^{\mathbb{O}}: &\quad \mathcal{M} = E_{6(-26)}/F_4. \end{aligned} \tag{2.23}$$

The cubic norm form, $\mathbf{N}$, of the simple Jordan algebras is given by the determinant of the corresponding Hermitian (3×3)-matrices.

$$\mathbf{N}(J) = \alpha\beta\gamma - \alpha X\overline{X} - \beta Y\overline{Y} - \gamma Z\overline{Z} + 2Re(XYZ). \tag{2.24}$$

$Re(XYZ)$ denotes the real part of XYZ

$$Re(XYZ) = Re(X(YZ)) = Re((XY)Z) = \frac{1}{2}(XYZ + \overline{XYZ}), \tag{2.25}$$

where bar denotes conjugation in the underlying division algebra.

A real quaternion $X \in \mathbb{H}$ can be expanded as

$$\begin{aligned} X &= X_0 + X_1 j_1 + X_2 j_2 + X_3 j_3, \\ \overline{X} &= X_0 - X_1 j_1 - X_2 j_2 - X_3 j_3, \\ X\overline{X} &= X_0^2 + X_1^2 + X_2^2 + X_3^2, \end{aligned} \tag{2.26}$$

where the imaginary units j_i $(i = 1, 2, 3)$ satisfy

$$j_i j_j = -\delta_{ij} + \epsilon_{ijk} j_k. \tag{2.27}$$

A real octonion $X \in \mathbb{O}$ has an expansion

$$\begin{aligned} X &= X_0 + X_1 j_1 + X_2 j_2 + X_3 j_3 + X_4 j_4 + X_5 j_5 + X_6 j_6 + X_7 j_7, \\ \overline{X} &= X_0 - X_1 j_1 - X_2 j_2 - X_3 j_3 - X_4 j_4 - X_5 j_5 - X_6 j_6 - X_7 j_7, \\ X\overline{X} &= X_0^2 + \sum_{A=1}^{7} (X_A)^2, \end{aligned} \tag{2.28}$$

where the seven imaginary units j_A $(A = 1, 2, \ldots, 7)$ satisfy

$$j_A j_B = -\delta_{AB} + \eta_{ABC} j_C. \tag{2.29}$$

The G_2 invariant tensor η_{ABC} is completely antisymmetric and in the conventions of [13] its nonvanishing components take on the values

$$\eta_{ABC} = 1 \Leftrightarrow (ABC) = (123), (471), (572), (673), (624), (435), (516). \tag{2.30}$$

The bosonic content and scalar manifold of $N = 6$ supergravity is the same as that of the $N = 2$ MESGT defined by the simple Euclidean Jordan algebra $J_3^{\mathbb{H}}$[5], namely

$$\mathcal{M}_5 = \frac{SU^*(6)}{USp(6)}. \tag{2.31}$$

Therefore its invariant C-tensor is simply the one given by the cubic norm of $J_3^{\mathbb{H}}$.

The C-tensor C_{IJK} of $N = 8$ supergravity in five dimensions can be identified with the symmetric tensor given by the cubic norm of the split exceptional Jordan algebra $J_3^{\mathbb{O}_s}$ [14, 15] defined over split octonions $\mathbb{O}_s$. The automorphism group of the split exceptional Jordan algebra defined by 3×3 Hermitian matrices over the split octonions $\mathbb{O}_s$

$$J^s = \begin{pmatrix} \alpha & Z^s & \overline{Y}^s \\ \overline{Z}^s & \beta & X^s \\ Y^s & \overline{X}^s & \gamma \end{pmatrix}, \tag{2.32}$$

where X^s, Y^s, Z^S are split octonions, is the noncompact group $F_{4(4)}$ with the maximal compact subgroup $USp(6) \times USp(2)$. Its reduced structure group is $E_{6(6)}$ under which the C-tensor is invariant. $E_{6(6)}$ is the U-duality group of maximal supergravity in five dimensions whose scalar manifold is

$$E_{6(6)}/USp(8).$$

2.5 Rotation (Automorphism), Lorentz (Reduced Structure) and Conformal (Möbius) Groups of Jordan Algebras

Above we reviewed briefly the connections between Jordan algebras of degree 3 and supergravity theories. Jordan algebras were used in the very early days of spacetime supersymmetry to define generalized spacetimes that naturally extend the description of four-dimensional Minkowski spacetime and its symmetry groups in terms of 2×2 complex Hermitian matrices. This was mainly motivated by attempts to find the super analogs of the exceptional Lie algebras [16] before a complete classification of finite dimensional simple Lie superalgebras was given by Kac [17].

As is well-known the twistor formalism in four-dimensional space–time ($d = 4$) leads naturally to the representation of spacetime coordinates x_μ in terms of 2×2 Hermitian matrices over the field of complex numbers $\mathbb{C}$:

$$x = x_\mu \sigma^\mu . \tag{2.33}$$

Hermitian matrices over the field of complex numbers close under the symmetric anti-commutator product and form a simple Jordan algebra denoted as $J_2^{\mathbb{C}}$. Therefore one can regard the four-dimensional Minkowski coordinate vectors as elements of the Jordan algebra $J_2^{\mathbb{C}}$ [16, 18]. Then the rotation, Lorentz and conformal groups in four dimensions correspond simply to the automorphism, reduced structure and Möbius (linear fractional) groups of the Jordan algebra $J_2^{\mathbb{C}}$ [16, 18]. The reduced structure group $\mathrm{Str}_0(J)$ of a Jordan algebra J is simply the invariance group of its norm form $\mathbf{N}(J)$, while the structure group $\mathrm{Str}(J) = \mathrm{Str}_0(J) \times SO(1,1)$ is defined as the invariance group of $\mathbf{N}(J)$ modulo an overall nonzero global scale factor. This correspondence was then used to define generalized space–times coordinatized by elements of general Jordan algebras whose rotation $\mathrm{Rot}(J)$, Lorentz $\mathrm{Lor}(J)$ and conformal $\mathrm{Conf}(J)$ groups are identified with the automorphism $\mathrm{Aut}(J)$, reduced structure $\mathrm{Str}_0(J)$ and Möbius $\mathrm{M\ddot{o}b}(J)$ groups of J [16, 18–20]. Denoting as $J_n^{\mathbb{A}}$ the Jordan algebra of $n \times n$ Hermitian matrices over the *division* algebra $\mathbb{A}$ and the Jordan algebra of Dirac gamma matrices in d (Euclidean) dimensions as $\Gamma_{(1,d)}$ we list the symmetry groups of generalized space–times defined by simple Euclidean (formally real) Jordan algebras in Table 2.1.

Note that for Euclidean Jordan algebras $\Gamma_{(1,d)}$ the automorphism, reduced structure and Möbius groups are simply the rotation, Lorentz and conformal groups of $(d+1)$-dimensional Minkowski spacetime. There exist the following special isomorphisms between the Jordan algebras of 2×2 Hermitian matrices over the four division algebras and the Jordan algebras of gamma matrices:

$$J_2^{\mathbb{R}} \simeq \Gamma_{(1,2)}; \quad J_2^{\mathbb{C}} \simeq \Gamma_{(1,3)}; \quad J_2^{\mathbb{H}} \simeq \Gamma_{(1,5)}; \quad J_2^{\mathbb{O}} \simeq \Gamma_{(1,9)}. \tag{2.34}$$

The Minkowski spacetimes they correspond to are precisely the critical dimensions for the existence of super Yang–Mills theories as well as of the classical Green–Schwarz superstrings. These Jordan algebras are all quadratic and their norm forms

Table 2.1 Complete list of simple Euclidean Jordan algebras and their rotation (automorphism), Lorentz (reduced structure) and Conformal (linear fractional) groups

J	Rot(J)	Lor(J)	Conf(J)
$J_n^{\mathbb{R}}$	$SO(n)$	$SL(n,\mathbb{R})$	$Sp(2n,\mathbb{R})$
$J_n^{\mathbb{C}}$	$SU(n)$	$SL(n,\mathbb{C})$	$SU(n,n)$
$J_n^{\mathbb{H}}$	$USp(2n)$	$SU^*(2n)$	$SO^*(4n)$
$J_3^{\mathbb{O}}$	F_4	$E_{6(-26)}$	$E_{7(-25)}$
$\Gamma_{(1,d)}$	$SO(d)$	$SO(d,1)$	$SO(d,2)$

The symbols $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$ represent the four division algebras. For the Jordan algebras $J_n^{\mathbb{A}}$ of $n \times n$ hermitian matrices over $\mathbb{A}$ the norm form is the determinantal form (or its generalization to the quaternionic and octonionic matrices)

are precisely the quadratic invariants constructed using the Minkowski metric. The spacetimes defined by simple Jordan algebras of degree 3 can be interpreted as extensions of Minkowskian spacetimes in critical dimensions by bosonic spinorial coordinates plus a dilaton and the adjoint identity implies the Fierz identities for the existence of the corresponding supersymmetric theories [21, 22].

We should note two important facts about Table 2.1. First, the conformal groups of generalized space–times defined by Euclidean (formally real) Jordan algebras all admit positive energy unitary representations. Hence they can be given a causal structure with a unitary time evolution as in four-dimensional Minkowski space–time [23, 24]. Second is the fact that the maximal compact subgroups of the generalized conformal groups of formally real Jordan algebras are simply the compact real forms of their structure groups (which are the products of their generalized Lorentz groups with dilatations).

Conformal group Conf(J) of a Jordan algebra J is generated by translations $T_{\mathbf{a}}$, special conformal generators $K_{\mathbf{a}}$, dilatations and Lorentz transformations $M_{\mathbf{ab}}$ $(\mathbf{a}, \mathbf{b} \in J)$. Lorentz transformations and dilatations generate the structure algebra $\mathfrak{str}(J)$ of J [16, 19, 20]. Lie algebra $\mathfrak{conf}(J)$ of the conformal group Conf(J) has a 3-grading with respect to the generator D of dilatations:

$$\mathfrak{conf}(J) = K_{\mathbf{a}} \oplus M_{\mathbf{ab}} \oplus T_{\mathbf{b}}. \tag{2.35}$$

Action of $\mathfrak{conf}(J)$ on the elements $\mathbf{x}$ of a Jordan algebra J are as follows [20]:

$$\begin{aligned} T_{\mathbf{a}}\mathbf{x} &= \mathbf{a}, \\ M_{\mathbf{ab}}\mathbf{x} &= \{\mathbf{abx}\}, \\ K_{\mathbf{a}}\mathbf{x} &= -\frac{1}{2}\{\mathbf{xax}\}, \end{aligned} \tag{2.36}$$

where $\{\mathbf{abx}\}$ is the Jordan triple product

$$\{\mathbf{abx}\} := \mathbf{a} \circ (\mathbf{b} \circ \mathbf{x}) - \mathbf{b} \circ (\mathbf{a} \cdot \mathbf{x}) + (\mathbf{a} \circ \mathbf{b}) \circ \mathbf{x}$$
$$\mathbf{a}, \mathbf{b}, \mathbf{x} \in J$$

with $\circ$ denoting the Jordan product. They satisfy the commutation relations

$$
\begin{aligned}
[T_{\mathbf{a}}, K_{\mathbf{b}}] &= M_{\mathbf{ab}}, \\
[M_{\mathbf{ab}}, T_{\mathbf{c}}] &= T_{\{\mathbf{abc}\}}, \\
[M_{\mathbf{ab}}, K_{\mathbf{c}}] &= K_{\{\mathbf{bac}\}}, \\
[M_{\mathbf{ab}}, M_{\mathbf{cd}}] &= M_{\{\mathbf{abc}\}\mathbf{d}} - M_{\{\mathbf{bad}\}\mathbf{c}},
\end{aligned}
$$

corresponding to the well-known Tits–Kantor–Koecher construction of Lie algebras from Jordan triple systems [25–27]. The generators $M_{\mathbf{ab}}$ can be decomposed as

$$
M_{\mathbf{ab}} = D_{\mathbf{a},\mathbf{b}} + L_{\mathbf{a}\cdot\mathbf{b}}, \tag{2.37}
$$

where $D_{\mathbf{a},\mathbf{b}}$ are the derivations that generate the automorphism (rotation) group of J

$$
D_{\mathbf{a},\mathbf{b}}\mathbf{x} = \mathbf{a} \circ (\mathbf{b} \circ \mathbf{x}) - \mathbf{b} \circ (\mathbf{a} \circ \mathbf{x})
$$

and L_c denotes multiplication by the element $c \in J$. The dilatation generator D is proportional to the multiplication operator by the identity element of J.

Choosing a basis e_I and a conjugate basis $\tilde{e}^I$ of a Jordan algebra J transforming covariantly and contravariantly, respectively, under the action of the Lorentz (reduced structure) group of J one can expand an element $\mathbf{x} \in J$ as

$$
\mathbf{x} = e_I q^I = \tilde{e}^I q_I .
$$

In this basis one can write the generators of $\mathfrak{conf}(J)$ as differential operators acting on the "coordinates" q^I [20]. These generators can be twisted by a unitary character λ and take on a simple and elegant form

$$
\begin{aligned}
T_I &= \frac{\partial}{\partial q^I}, \\
R^I_J &= -\Lambda^{IK}_{JL} q^L \frac{\partial}{\partial q^K} - \lambda \delta^I_J, && (2.38) \\
K^I &= \frac{1}{2} \Lambda^{IK}_{JL} q^J q^L \frac{\partial}{\partial q^K} + \lambda q^I, && (2.39)
\end{aligned}
$$

where

$$
\Lambda^{IJ}_{KL} := \delta^I_K \delta^J_L + \delta^I_L \delta^J_K - \frac{4}{3} C^{IJM} C_{KLM}.
$$

They satisfy the commutation relations

$$
\begin{aligned}
[T_I, K^J] &= -R^J_I, && (2.40) \\
[R^J_I, T_K] &= \Lambda^{JL}_{IK} T_L, && (2.41) \\
[R^J_I, K^K] &= -\Lambda^{JL}_{IK} K^L. && (2.42)
\end{aligned}
$$

The generator of the rotation (automorphism) subgroup are simply

$$A_{IJ} = R^J_I - R^I_J. \tag{2.43}$$

2.6 U-Duality Orbits of Extremal Black Hole Solutions of 5D Supergravity Theories with Symmetric Target Manifolds and Their Spectrum Generating Conformal Extensions

Orbits of the spherically symmetric stationary BPS black holes (BH) with non-vanishing entropy under the action of U-duality groups of $N = 2$ MESGT's with symmetric target spaces were given in [14]. In the same work the orbits with non-vanishing cubic invariants that are non-BPS were also classified. These latter orbits describe extremal non-BPS black holes and corresponding solutions to the attractor equations were given in [28]. In this section we shall review the solutions to the attractor equations in 5D MESGTs for extremal black holes, BPS as well as non-BPS, following [28].

Let us denote the $(n+1)$ dimensional charge vector in an extremal BH background as q_I. It is given by

$$q_I = \int_{S^3} H_I = \int_{S^3} \overset{\circ}{a}_{IJ} * F^J \quad (I = 0, 1, \ldots, n). \tag{2.44}$$

The black hole potential [29, 30] that determines the attractor flow takes on the following form for $N = 2$ MESGTs:

$$V(\phi, q) = q_I \overset{\circ}{a}{}^{IJ} q_J, \tag{2.45}$$

where $\overset{\circ}{a}{}^{IJ}$ is the inverse of the metric $\overset{\circ}{a}_{IJ}$ of the kinetic energy term of the vector fields. In terms of the central charge function

$$Z = q_I h^I$$

the potential can be written as

$$V(q, \phi) = Z^2 + \frac{3}{2} g^{xy} \partial_x Z \partial_y Z, \tag{2.46}$$

where

$$\partial_x Z = q_I h^I_{,x} = \sqrt{\frac{2}{3}} h^I_x.$$

The critical points of the potential are determined by the equation

$$\partial_x V = 2(2Z\partial_x Z - \sqrt{3/2}T_{xyz}g^{yy'}g^{zz'}\partial_{y'}Z\partial_{z'}Z) = 0. \tag{2.47}$$

The BPS attractors are given by the solutions satisfying [31, 32]

$$\partial_x Z = 0 \tag{2.48}$$

at the critical points. The non-BPS attractors are given by non-trivial solutions [28]

$$2Z\partial_x Z = \sqrt{\frac{3}{2}}T_{xyz}\partial^y Z\partial^z Z \tag{2.49}$$

such that

$$\partial^x Z \equiv g^{xx'}\partial_{x'}Z \neq 0$$

at the critical points. Equation (2.49) can be inverted using the relation

$$q_I = h_I Z - \frac{3}{2}h_{I,x}\partial^x Z. \tag{2.50}$$

For BPS attractors satisfying $\partial_x Z = 0$ this gives

$$q_I = h_I Z \tag{2.51}$$

and for non-BPS attractors satisfying $\partial_x Z \neq 0$ we get

$$q_I = h_I Z - (3/2)^{3/2}\frac{1}{2Z}h_{I,x}T^{xyz}\partial_y Z\partial_z Z. \tag{2.52}$$

Since the BPS attractor solution with non-vanishing entropy [31, 32] is given by $\partial_x Z = 0$, which is invariant under the automorphism group $\mathrm{Aut}(J)$ of the underlying Jordan algebra J, the orbits of BPS black hole solutions are of the form

$$\mathcal{O}_{\mathrm{BPS}} = \mathrm{Str}_0(J)/\mathrm{Aut}(J) \tag{2.53}$$

and were listed in column 1 of Table 1 of [14] which we reproduce in Table 2.2.

The orbits for extremal non-BPS black holes with non-vanishing entropy are of the form

$$\mathcal{O}_{\mathrm{non-BPS}} = \mathrm{Str}_0(J)/\mathrm{Aut}(J_{(1,2)}), \tag{2.54}$$

where $\mathrm{Aut}(J_{(1,2)})$ is a noncompact real form of the automorphism group of J and were listed in column 2 of Table 1 of [14], which we reproduce in Table 2.3.

The entropy S of an extremal black hole solution of $N = 2$ MESGT with charges q_I is determined by the value of the black hole potential V at the attractor points

$$S = (V_{\mathrm{critical}})^{3/4}. \tag{2.55}$$

Table 2.2 Orbits of spherically symmetric stationary BPS black hole solutions in 5D MESGTs defined by Euclidean Jordan algebras J of degree 3. U-duality and stability groups are given by the Lorentz (reduced structure) and rotation (automorphism) groups of J

J	$\mathcal{O}_{\mathrm{BPS}} = \mathrm{Str}_0(J)/\mathrm{Aut}(J)$
$J_3^{\mathbb{R}}$	$SL(3,\mathbb{R})/SO(3)$
$J_3^{\mathbb{C}}$	$SL(3,\mathbb{C})/SU(3)$
$J_3^{\mathbb{H}}$	$SU^*(6)/USp(6)$
$J_3^{\mathbb{O}}$	$E_{6(-26)}/F_4$
$\mathbb{R} \oplus \Gamma_{(1,n-1)}$	$SO(n-1,1) \times SO(1,1)/SO(n-1)$

Table 2.3 Orbits of non-BPS extremal black holes of $N = 2$ MESGT's with non-vanishing entropy in $d = 5$. The first column lists the Jordan algebras of degree 3 that define these theories. The third column lists the maximal compact subgroups K of the stability group $\mathrm{Aut}(J_{(1,2)})$

J	$\mathcal{O}_{\mathrm{non-BPS}} = \mathrm{Str}_0(J)/\mathrm{Aut}(J_{(1,2)})$	$K \subset \mathrm{Aut}(J_{(1,2)})$
$J_3^{\mathbb{R}}$	$SL(3,\mathbb{R})/SO(2,1)$	SO(2)
$J_3^{\mathbb{C}}$	$SL(3,\mathbb{C})/SU(2,1)$	$SU(2) \times U(1)$
$J_3^{\mathbb{H}}$	$SU^*(6)/USp(4,2)$	$USp(4) \times USp(2)$
$J_3^{\mathbb{O}}$	$E_{6(-26)}/F_{4(-20)}$	$SO(9)$
$\mathbb{R} \oplus \Gamma_{(1,n-1)}$	$SO(n-1,1) \times SO(1,1)/SO(n-2,1)$	$SO(n-2)$

For $N = 2$ MESGTs defined by Jordan algebras of degree 3, the tensor C_{IJK} is an invariant tensor of the U-duality group $\mathrm{Str}_0(J)$. Similarly the tensor T_{abc} with "flat" indices

$$T_{abc} = e_a^x e_b^y e_c^z T_{xyz},$$

where e_a^x is the n-bein on the n-dimensional scalar manifold with metric g_{xy} is an invariant tensor of the maximal compact subgroup $\mathrm{Aut}(J)$ of $\mathrm{Str}_0(J)$. In terms of flat indices the attractor equation becomes

$$2Z\partial_a Z = \sqrt{3/2}T_{abc}\partial^b Z \partial^c Z. \tag{2.56}$$

Thus for BPS attractor solution $\partial_a Z = 0$ one finds

$$S_{\mathrm{BPS}} = (V_{\mathrm{BPS}})^{3/4} = Z_{\mathrm{BPS}}^{3/2}. \tag{2.57}$$

For extremal non-BPS attractors $\partial_a Z \neq 0$, squaring the criticality condition one finds

$$4Z^2\partial_a Z \partial_a Z = \frac{3}{2} T_{abc} T_{ab'c'} \partial_b Z \partial_c Z \partial'_b Z \partial'_c Z. \tag{2.58}$$

Then using the identity

$$T_{a(bc} T^a_{b'c')} = \frac{1}{2} g_{(bc} g_{b'c')}$$

valid only for MESGTs defined by Jordan algebras of degree 3 one obtains

$$\partial_a Z \partial_a Z = \frac{16}{3} Z^2. \tag{2.59}$$

Hence the entropy of extremal non-BPS black holes are given by[3]

$$S_{\text{non-BPS}} = V_{\text{non-BPS}}^{4/3} = \left(Z^2 + \frac{3}{2} \partial_a Z \partial_a Z \right)^{3/4} = (3 Z_{\text{non-BPS}})^{3/2}. \tag{2.60}$$

By differentiating (2.47) one finds a general expression for the Hessian of the black hole potential around the critical points

$$\begin{aligned} \frac{1}{4} D_x \partial_y V &= \frac{2}{3} g_{xy} Z^2 + \partial_x Z \partial_y Z - 2\sqrt{\frac{2}{3}} T_{xyz} g^{zw} \partial_w Z Z \\ &\quad + T_{xpq} T_{yzs} g^{pz} g^{qq'} g^{ss'} \partial_{q'} Z \partial_{s'} Z \\ &= \frac{2}{3} \left(g_{xz} Z - \sqrt{\frac{3}{2}} T_{xzp} \partial^p Z \right) \left(g_{yz} Z - \sqrt{\frac{3}{2}} T_{yzq} \partial^q Z \right) \\ &\quad + \partial_x Z \partial_y Z. \end{aligned} \tag{2.61}$$

Thus for BPS critical points for which we have $\partial_x Z = 0$ the Hessian is given simply as

$$\partial_x \partial_y V = \frac{8}{3} g_{xy} Z^2, \tag{2.62}$$

which is the same result as in $d = 4$ [30]. Since the metric of the scalar manifold is positive definite the above formula implies that the scalar fluctuations have positive square mass reflecting the attractor nature of the BPS critical points.

For non-BPS extremal critical points of the black hole potential the Hessian has flat directions and is positive semi-definite [28].

The orbits of BPS black hole solutions of $N = 8$ supergravity theory in five dimensions were also given in [14]. The $1/8$ BPS black holes with non-vanishing entropy has the orbit

$$\mathcal{O}_{1/8-\text{BPS}} = \frac{E_{6(6)}}{F_{4(4)}} = \frac{\text{Str}_0(J_3^{\mathbb{O}_S})}{\text{Aut}(J_3^{\mathbb{O}_S})}, \tag{2.63}$$

where $\mathbb{O}_S$ stands for the split octonions and $J_3^{\mathbb{O}_S}$ is the split exceptional Jordan algebra. Note that in contrast to the exceptional $N = 2$ MESGT theory defined by the real exceptional Jordan algebra that has two different orbits with nonvanishing

[3] We should stress that both for BPS as well as extremal non-BPS black holes the quantities appearing in the above formulas are evaluated at the corresponding attractor points.

entropy the maximal supergravity has only one such orbit. On the other hand maximal supergravity theory admits 1/4 and 1/2 BPS black hole solutions with vanishing entropy [33]. Their orbits under U-duality are [14]

$$\mathcal{O}_{1/4-\mathrm{BPS}} = \frac{E_{6(6)}}{O(5,4)\circledS T_{16}}, \tag{2.64}$$

$$\mathcal{O}_{1/2-\mathrm{BPS}} = \frac{E_{6(6)}}{O(5,5)\circledS T_{16}}, \tag{2.65}$$

where $\circledS$ denotes the semi-direct product and T_{16} is the group of translations transforming in the spinor (16) of $SO(5,5)$. Vanishing entropy means vanishing cubic norm. Thus the black hole solutions corresponding to vanishing entropy has additional symmetries beyond the five-dimensional U-duality group, namely they are invariant under the generalized special conformal transformations of the underlying Jordan algebras. This is complete parallel to the invariance of light-like vectors under special conformal transformations in four dimensional Minkowski spacetime which can be coordinatized by the elements of the Jordan algebra $J_2^{\mathbb{C}}$. Acting on a black hole solution with non-vanishing entropy these special conformal transformations change their norms and hence the entropy. Hence the conformal groups of Jordan algebras were proposed as spectrum generating symmetry groups of black hole solutions of MESGTs defined by them [14, 15, 34, 35]. Since the conformal groups of Jordan algebras of degree 3 are isomorphic to the U-duality groups of the corresponding four dimensional supergravity theories obtained by dimensional reduction this implies that the four-dimensional U-duality groups must act as spectrum generating symmetry groups of the corresponding five-dimensional theories. Since it was first made, there have been several works relating black hole solutions in four and five dimensions (4D/5D lift) [36–39] that lend support to the proposal that four-dimensional U-duality groups act as spectrum generating conformal symmetry groups of five-dimensional supergravity theories from which they descend.

2.7 4D, $N = 2$ Maxwell–Einstein Supergravity Theories with Symmetric Target Spaces and Freudenthal Triple Systems

Under dimensional reduction on a torus 5D, $N = 2$ MESGTs with $(n_V - 1)$ vector multiplets lead to 4D, $N = 2$ MESGTs with n_V vector multiplets, with the extra vector multiplet coming from the 5D graviton supermultiplet. The metric of the target space of the four-dimensional scalar fields of dimensionally reduced theories were first given in [6] in the so-called unbounded realization of their geometries. More precisely, the resulting four-dimensional target spaces are generalized upper half-spaces (tube domains) over the convex cones defined by the cubic norm. They are parameterized by complex coordinates [6],

$$z^I := \frac{1}{\sqrt{3}} A^I + \frac{i}{\sqrt{2}} \tilde{h}^I, \tag{2.66}$$

where A^I denote the 4D scalars descending from the 5D vectors. Imaginary components of z^I are given by

$$\tilde{h}^I := e^{\sigma} h^I , \tag{2.67}$$

where σ is the scalar field (dilaton) coming from 5D graviton and h^I were defined above. They satisfy the positivity condition

$$\mathcal{V}(\tilde{h}^I) = C_{IJK}\tilde{h}^I\tilde{h}^J\tilde{h}^K = e^{3\sigma} > 0.$$

Geometry of four-dimensional $N = 2$ MESGTs obtained by dimensional reduction from five dimensions (R-map) was later referred to as "very special geometry" and studied extensively.[4] The full bosonic sector of 4D theories obtained by dimensional reduction from gauged 5D, $N = 2$ Yang–Mills Einstein supergravity coupled to tensor multiplets and their reformulation in the standard language of special Kähler geometry was given in [41], which we follow in our summary here, restricting ourselves to the ungauged MESGT theory without tensors.

As is well-known one can interpret the n_V complex coordinates z^I of dimensionally reduced MESGTs as inhomogeneous coordinates of a $(n_V + 1)$-dimensional complex vector with coordinates X^A

$$X^A = \begin{pmatrix} X^0 \\ X^I \end{pmatrix} = \begin{pmatrix} 1 \\ z^I \end{pmatrix}, \tag{2.68}$$

where the capital Latin indices $A, B, C, \ldots$ run from 0 to n_V. Taking as "prepotential" the cubic form defined by the C-tensor coming from five dimensions

$$F(X^A) = -\frac{\sqrt{2}}{3} C_{IJK} \frac{X^I X^J X^K}{X^0} \tag{2.69}$$

and using the symplectic section

$$\begin{pmatrix} X^A \\ F_A \end{pmatrix} = \begin{pmatrix} X^A \\ \partial_A F \end{pmatrix} \equiv \begin{pmatrix} X^A \\ \frac{\partial F}{\partial X^A} \end{pmatrix} \tag{2.70}$$

one gets the Kähler potential

$$\begin{aligned} \mathcal{K}(X, \overline{X}) &:= -\ln [i\overline{X}^A F_A - iX^A \overline{F}_A] \\ &= -\ln \left[i\frac{\sqrt{2}}{3} C_{IJK}(z^I - \bar{z}^I)(z^J - \bar{z}^J)(z^K - \bar{z}^K) \right], \end{aligned} \tag{2.71}$$

which agrees precisely with the Kähler potential obtained in [6]. The "period matrix" that determines the kinetic terms of the vector fields in four dimensions

[4] See for example [40] and the references therein.

is given by

$$\mathcal{N}_{AB} := \overline{F}_{AB} + 2i\,\frac{\mathrm{Im}(F_{AC})\mathrm{Im}(F_{BD})X^C X^D}{\mathrm{Im}(F_{CD})X^C X^D}, \tag{2.72}$$

where $F_{AB} \equiv \partial_A \partial_B F$, etc. Components of the resulting period matrix $\mathcal{N}_{AB}$ under dimensional reduction are

$$\mathcal{N}_{00} = -\frac{2\sqrt{2}}{9\sqrt{3}} C_{IJK} A^I A^J A^K - \frac{i}{3}\left(e^{\sigma} \overset{\circ}{a}_{IJ} A^I A^J + \frac{1}{2} e^{3\sigma}\right), \tag{2.73}$$

$$\mathcal{N}_{0I} = \frac{\sqrt{2}}{3} C_{IJK} A^J A^K + \frac{i}{\sqrt{3}} e^{\sigma} \overset{\circ}{a}_{IJ} A^J, \tag{2.74}$$

$$\mathcal{N}_{IJ} = -\frac{2\sqrt{2}}{\sqrt{3}} C_{IJK} A^K - i e^{\sigma} \overset{\circ}{a}_{IJ}. \tag{2.75}$$

The prepotential (2.69) leads to the Kähler metric

$$g_{I\overline{J}} \equiv \partial_I \partial_{\overline{J}} \mathcal{K} = \frac{3}{2} e^{-2\sigma} \overset{\circ}{a}_{IJ} \tag{2.76}$$

for the scalar manifold $\mathcal{M}_4$ of four-dimensional theory, where $\overset{\circ}{a}_{IJ}$ is the "metric" of the kinetic energy term of the vector fields of the 5D theory. Above we denoted the field strength of the vector field that comes from the graviton in five dimensions as $F^0_{\mu\nu}$. The bosonic sector of dimensionally reduced Lagrangian can then be written as

$$e^{-1}\mathcal{L}^{(4)} = -\frac{1}{2}R - g_{I\overline{J}}(\partial_\mu z^I)(\partial^\mu \bar{z}^J) + \frac{1}{4}\mathrm{Im}(\mathcal{N}_{AB})F^A_{\mu\nu}F^{\mu\nu B} - \frac{1}{8}\mathrm{Re}(\mathcal{N}_{AB})\epsilon^{\mu\nu\rho\sigma}F^A_{\mu\nu}F^B_{\rho\sigma}. \tag{2.77}$$

Since the complex scalar fields z^I of the four-dimensional theory are restricted to the domain $\mathcal{V}(Im(z)) > 0$, the scalar manifolds of 4D, $N = 2$ MESGT's defined by Euclidean Jordan algebras J of degree 3 are simply the Köcher "upper half spaces" of the corresponding Jordan algebras, which belong to the family of Siegel domains of the first kind [42]. The "upper half spaces" of Jordan algebras can be mapped into bounded symmetric domains, which can be realized as hermitian symmetric spaces of the form

$$\mathcal{M}_4 = \frac{\mathrm{Conf}(J)}{\widetilde{\mathrm{Str}}J}, \tag{2.78}$$

where $\mathrm{Conf}(J)$ is the conformal group of the Jordan algebra J and its maximal compact subgroup $\widetilde{\mathrm{Str}}J$ is the compact real form of its structure group $\mathrm{Str}(J)$. The Kähler potential (2.71) that one obtains directly under dimensional reduction from five dimensions is given by the "cubic light-cone"

$$\mathcal{V}(z-\bar{z}) = C_{IJK}(z^I - \bar{z}^I)(z^J - \bar{z}^J)(z^K - \bar{z}^K), \tag{2.79}$$

which is manifestly invariant under the five-dimensional U-duality group $\mathrm{Str}_0(J)$ and real translations

$$Re(z^I) \Rightarrow Re(z^I) + a^I,$$
$$a^I \in \mathbb{R},$$

which follows from Abelian gauge invariances of vector fields of the five-dimensional theory. Under dilatations it gets simply rescaled. Infinitesimal action of special conformal generators K^I of $\mathrm{Conf}(J)$ on the "cubic light-cone" yields [15]

$$K^I \mathcal{V}(z-\bar{z}) = (z^I + \bar{z}^I)\mathcal{V}(z-\bar{z}), \tag{2.80}$$

which can be integrated to give the global transformation of the form

$$\mathcal{V}(z-\bar{z}) \Longrightarrow f(z^I)\bar{f}(\bar{z}^I)\mathcal{V}(z-\bar{z}). \tag{2.81}$$

This shows that the cubic light-cone defined by $\mathcal{V}(z-\bar{z}) = 0$ is invariant under the full conformal group $\mathrm{Conf}(J)$. Furthermore, the above global conformal group action leaves the metric $g_{I\bar{J}}$ invariant since it simply induces a Kähler transformation of the Kähler potential $\ln \mathcal{V}(z-\bar{z})$.

In $N = 2$ MESGTs defined by Euclidean Jordan algebras J of degree 3, one-to-one correspondence between vector fields of five-dimensional theories (and hence their charges) and elements of J gets extended, in four dimensions, to a one-to-one correspondence between field strengths of vector fields *plus* their magnetic duals and Freudenthal triple systems defined over J [6, 14, 15, 22, 34]. An element X of Freudenthal triple system (FTS) $\mathcal{F}(J)$ [43, 44] over J can be represented formally as a 2×2 "matrix":

$$X = \begin{pmatrix} \alpha & \mathbf{x} \\ \mathbf{y} & \beta \end{pmatrix} \in \mathcal{F}(J), \tag{2.82}$$

where $\alpha, \beta \in \mathbb{R}$ and $\mathbf{x}, \mathbf{y} \in J$.

Denoting the "bare" four-dimensional graviphoton field strength and its magnetic dual as $F^0_{\mu\nu}$ and $\tilde{F}^{\mu\nu}_0$, respectively, we have the correspondence

$$\begin{pmatrix} F^0_{\mu\nu} & F^I_{\mu\nu} \\ \tilde{F}^{\mu\nu}_I & \tilde{F}^{\mu\nu}_0 \end{pmatrix} \Longleftrightarrow \begin{pmatrix} e_0 & e_I \\ \tilde{e}^I & \tilde{e}^0 \end{pmatrix} \in \mathcal{F}(J),$$

where $e_I(\tilde{e}^I)$ are the basis elements of J (its conjugate $\tilde{J}$). Consequently, one can associate with a black hole solution with electric and magnetic charges (fluxes) $\left(q_0, q_I, p^0, p^I\right)$ of the 4D MESGT defined by J an element of the FTS $\mathcal{F}(J)$

$$\begin{pmatrix} p^0 e_0 & p^I e_I \\ q_I \tilde{e}^I & q_0 \tilde{e}^0 \end{pmatrix} \in \mathcal{F}(J). \tag{2.83}$$

Table 2.4 Scalar manifolds $\mathcal{M}_d$ of $N = 2$ MESGT's defined by Euclidean Jordan algebras J of degree 3 in $d = 3, 4, 5$ dimensions. $J_3^{\mathbb{A}}$ denotes the Jordan algebra of 3×3 Hermitian matrices over the division algebra $\mathbb{A} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$. The last row $\mathbb{R} \oplus \Gamma_{(1,n-1)}$ are the reducible Jordan algebras which are direct sums of Jordan algebras $\Gamma_{(1,n-1)}$ defined by a quadratic form $\mathbb{Q}$ of Minkowskian signature and one-dimensional Jordan algebra $\mathbb{R}$. $\widetilde{\mathrm{Str}}(J)$ and $\widetilde{\mathrm{Conf}}(J)$ denote the compact real forms of the structure group $\mathrm{Str}(J)$ and conformal group $\mathrm{Conf}(J)$ of a Jordan algebra J. QConf $(\mathcal{F}(J))$ denotes the quasiconformal group defined by the FTS $\mathcal{F}(J)$ defined over J

J	$\mathcal{M}_5 =$ $\mathrm{Str}_0(J)/\mathrm{Aut}(J)$	$\mathcal{M}_4 =$ $\mathrm{Conf}(J)/\widetilde{\mathrm{Str}}(J)$	$\mathcal{M}_3 =$ $\mathrm{QConf}(\mathcal{F}(J))/\widetilde{\mathrm{Conf}}(J) \times \mathrm{SU}(2)$
$J_3^{\mathbb{R}}$	$\mathrm{SL}(3,\mathbb{R})/\mathrm{SO}(3)$	$\mathrm{Sp}(6,\mathbb{R})/\mathrm{U}(3)$	$\mathrm{F}_{4(4)}/\mathrm{USp}(6) \times \mathrm{SU}(2)$
$J_3^{\mathbb{C}}$	$\mathrm{SL}(3,\mathbb{C})/\mathrm{SU}(3)$	$\mathrm{SU}(3,3)/\mathrm{S}\,(\mathrm{U}(3) \times \mathrm{U}(3))$	$\mathrm{E}_{6(2)}/\mathrm{SU}(6) \times \mathrm{SU}(2)$
$J_3^{\mathbb{H}}$	$\mathrm{SU}^*(6)/\mathrm{USp}(6)$	$\mathrm{SO}^*(12)/\mathrm{U}(6)$	$\mathrm{E}_{7(-5)}/\mathrm{SO}(12) \times \mathrm{SU}(2)$
$J_3^{\mathbb{O}}$	$\mathrm{E}_{6(-26)}/\mathrm{F}_4$	$\mathrm{E}_{7(-25)}/\mathrm{E}_6 \times \mathrm{U}(1)$	$\mathrm{E}_{8(-24)}/\mathrm{E}_7 \times \mathrm{SU}(2)$
$\mathbb{R} \oplus \Gamma_{(1,n-1)}$	$\frac{\mathrm{SO}(n-1,1) \times \mathrm{SO}(1,1)}{\mathrm{SO}(n-1)}$	$\frac{\mathrm{SO}(n,2) \times \mathrm{SU}(1,1)}{\mathrm{SO}(n) \times \mathrm{SO}(2) \times \mathrm{U}(1)}$	$\frac{\mathrm{SO}(n+2,4)}{\mathrm{SO}(n+2) \times \mathrm{SO}(4)}$

U-duality group G_4 of such a four-dimensional MESGT acts as automorphism group of the FTS $\mathcal{F}(J)$, which is endowed with an invariant symmetric quartic form and a skew-symmetric bilinear form. The entropy of an extremal black with charges (p^0, p^I, q_0, q_I) is determined by the quartic invariant $\mathcal{Q}_4(q, p)$ of $\mathcal{F}(J)$. With this identification the orbits of extremal black holes of 4D, $N = 2$ MESGT's with symmetric scalar manifolds were classified in [14, 45].

Upon further dimensional reduction to three dimensions (C-map) $N = 2$ MESGTs lead to $N = 4$, $d = 3$ quaternionic Kähler σ models coupled to supergravity [6, 46]. In Table 2.4 we give the symmetry groups of $N = 2$ MESGTs defined by Euclidean Jordan algebras in $d = 5, 4$ and 3 dimensions and their scalar manifolds. We should note that five and three-dimensional U-duality symmetry groups $\mathrm{Str}_0(J)$ and $\mathrm{QConf}(J)$, respectively, act as symmetries of supergravity Lagrangians, while four-dimensional U-duality groups $\mathrm{Conf}(J)$ are on-shell symmetries.

2.8 U-Duality Orbits of Extremal Black Holes of 4D, $N = 2$ MESGTs with Symmetric Scalar Manifolds and of $N = 8$ Supergravity and Their Spectrum Generating Quasiconformal Extensions

The discussion of the orbits extremal black holes of extended supergravity theories with symmetric scalar manifolds were covered in Sergio Ferrara's lectures [47]. Referring to Ferrara's lectures for details including the related recent developments I will briefly summarize the results for $N = 2$ MESGTs and $N = 8$ supergravity in this section following [45]. As in the five-dimensional case, the 4D extremal black hole attractor equations are simply the criticality conditions for the black hole scalar potential which can be written as [29, 48]

$$V_{BH} \equiv |Z|^2 + G^{I\overline{J}}(D_I Z)(\overline{D}_{\overline{J}}\overline{Z}), \tag{2.84}$$

where Z is the central charge function. The criticality condition is [30]:

$$\partial_I V_{BH} = 0 \tag{2.85}$$

implies

$$2\overline{Z}D_I Z + iC_{IJK}G^{J\overline{J}}G^{K\overline{K}}\overline{D}_{\overline{J}}\overline{Z}\,\overline{D}_{\overline{K}}\overline{Z} = 0. \tag{2.86}$$

C_{IJK} is the completely symmetric, covariantly holomorphic tensor of special Kähler geometry that satisfies

$$\overline{D}_{\overline{L}}C_{IJK} = 0, \quad D_{[L}C_{I]JK} = 0, \tag{2.87}$$

where square brackets denote antisymmetrization. For symmetric special Kähler manifolds the tensor C_{IJK} is covariantly constant:

$$D_I C_{JKL} = 0, \tag{2.88}$$

which implies the four-dimensional counterpart of the adjoint identity [6, 49]

$$G^{K\overline{K}}G^{M\overline{J}}C_{M(PQ}C_{IJ)K}\overline{C}_{\overline{KIJ}} = \frac{4}{3}C_{(IJP}G_{Q)\overline{I}}. \tag{2.89}$$

The $\frac{1}{2}$-BPS attractors are given by the following solution of attractor equations [30]

$$Z \neq 0, \quad D_I Z = 0 \quad \forall \ I = 1, \ldots, n_V. \tag{2.90}$$

The orbits of the 1/2-BPS black hole solutions with positive quartic invariants of $N = 2$ MESGTs with symmetric target spaces were given in [14] and are listed in column 1 of Table 2.5. In [14] a second family of orbits with non-vanishing quartic invariants were also given. They correspond to non-BPS extremal black holes and the respective solutions to the attractor equations were given in [45]. In addition there exist another family of non-BPS extremal black holes with non-vanishing quartic invariant and vanishing central charge [45]. The complete list of orbits of BPS and extremal non-BPS black holes is given in Table 2.5.

The orbits of black hole solutions of 4D $N = 8$ supergravity under the action of U-duality group $E_{7(7)}$ were given in [14]. There exist two classes of non-degenerate charge orbits of black hole solutions with non-vanishing quartic invariant I_4 constructed from the electric and magnetic charges transforming in **56** of $E_{7(7)}$ [14]. Depending on the sign of I_4, one finds

Table 2.5 Non-degenerate orbits of $N = 2$, $D = 4$ MESGTs with symmetric scalar manifolds. Except for the first row all such theories originate from five dimensions and are defined by Jordan algebras that are indicated in the first column

J	$\frac{1}{2}$-BPS orbits $\mathcal{O}_{\frac{1}{2}-\mathrm{BPS}}$	Non-BPS, $Z \neq 0$ orbits $\mathcal{O}_{\mathrm{non-BPS},Z\neq 0}$	Non-BPS, $Z = 0$ orbits $\mathcal{O}_{\mathrm{non-BPS},Z=0}$
–	$\frac{SU(1,n+1)}{SU(n+1)}$	–	$\frac{SU(1,n+1)}{SU(1,n)}$
$\mathbb{R} \oplus \Gamma_{(1,n-1)}$	$\frac{SU(1,1)\otimes SO(2,2+n)}{SO(2)\otimes SO(2+n)}$	$\frac{SU(1,1)\otimes SO(2,2+n)}{SO(1,1)\otimes SO(1,1+n)}$	$\frac{SU(1,1)\otimes SO(2,2+n)}{SO(2)\otimes SO(2,n)}$
$J_3^{\mathbb{O}}$	$\frac{E_{7(-25)}}{E_6}$	$\frac{E_{7(-25)}}{E_{6(-26)}}$	$\frac{E_{7(-25)}}{E_{6(-14)}}$
$J_3^{\mathbb{H}}$	$\frac{SO^*(12)}{SU(6)}$	$\frac{SO^*(12)}{SU^*(6)}$	$\frac{SO^*(12)}{SU(4,2)}$
$J_3^{\mathbb{C}}$	$\frac{SU(3,3)}{SU(3)\otimes SU(3)}$	$\frac{SU(3,3)}{SL(3,\mathbb{C})}$	$\frac{SU(3,3)}{SU(2,1)\otimes SU(1,2)}$
$J_3^{\mathbb{R}}$	$\frac{Sp(6,\mathbb{R})}{SU(3)}$	$\frac{Sp(6,\mathbb{R})}{SL(3,\mathbb{R})}$	$\frac{Sp(6,\mathbb{R})}{SU(2,1)}$

$$I_4 > 0 : \mathcal{O}_{\frac{1}{8}-\mathrm{BPS}} = \frac{E_{7(7)}}{E_{6(2)}} \Longleftrightarrow \frac{1}{8}\text{-BPS}; \tag{2.91}$$

$$I_4 < 0 : \mathcal{O}_{\mathrm{non-BPS}} = \frac{E_{7(7)}}{E_{6(6)}} \Longleftrightarrow \text{non-BPS}. \tag{2.92}$$

This is to be contrasted with the non-degenerate orbits of the exceptional $N = 2$ supergravity with U-duality group $E_{7(-25)}$, which has three non-degenerate orbits, one BPS and two non-BPS one of which has vanishing central charge. On the other hand, in $N = 8$ supergravity one has $1/4$ and $1/2$ BPS black holes with vanishing entropy [33]. The "light-like" orbits of these BPS black hole solutions with vanishing quartic invariant were given in [14]. There are three distinct cases depending on the number of vanishing "eigenvalues" that lead to vanishing $\mathcal{Q}_4$. The generic light-like orbit for which a single eigenvalue vanishes is

$$\frac{E_{7(7)}}{F_{4(4)} \circledS T_{26}}, \tag{2.93}$$

where T_{26} is a 26-dimensional Abelian subgroup of $E_{7(7)}$ and $\circledS$ denotes semi-direct product. The critical light-like orbit has two vanishing eigenvalues and correspond to the 45-dimensional orbit

$$\frac{E_{7(7)}}{O(6,5) \circledS (T_{32} \oplus T_1)}. \tag{2.94}$$

The doubly critical light-like orbit with three vanishing eigenvalues is given by the 28-dimensional quotient space

$$\frac{E_{7(7)}}{E_{6(6)} \circledS T_{27}}. \tag{2.95}$$

As discussed above, four-dimensional U-duality groups G_4 were proposed as spectrum generating conformal symmetry groups in five dimensions that leave a cubic light-cone invariant. This raises the question, first investigated in [15], whether the three-dimensional U-duality groups G_3 could act as spectrum generating "conformal" groups of corresponding four-dimensional supergravity theories. It is easy to show that there exist three-dimensional U-duality groups that do not have any conformal realizations in general. Some other three-dimensional U-duality groups do not admit conformal realizations on the $2n_V + 2$ dimensional space of the FTS that defines the four dimensional theory. However as was shown in [15] the three-dimensional U-duality groups G_3 all have novel geometric realizations as quasi-conformal groups on the vector spaces of FTS's extended by an extra singlet coordinate that leave invariant a generalized light-cone with respect to a quartic distance function. The quasiconformal actions of three-dimensional U-duality groups G_3 were then proposed as spectrum generating symmetry groups of corresponding four-dimensional supergravity theories [15, 34, 35, 50–52]. We shall denote the quasiconformal groups defined over FTS's $\mathcal{F}$ extended by a singlet coordinate as QConf($\mathcal{F}$). If the FTS is defined over a Jordan algebra J of degree 3 we shall denote the corresponding quasiconformal groups either as QConf($\mathcal{F}(J)$) or simply as QConf(J). The construction given in [15] is covariant with respect to the automorphism group of the FTS, which is isomorphic to the 4D U-duality group of the corresponding supergravity. For $N = 2$ MESGTs defined by Jordan algebras of degree 3, quasiconformal group actions of their three-dimensional U-duality groups G_3 were given explicitly in [22], in a basis covariant with respect to U-duality groups G_6 of corresponding six-dimensional supergravity theories.

2.9 Quasiconformal Realizations of Lie Groups and Freudenthal Triple Systems

In this section we shall review the general theory of quasiconformal realizations of noncompact groups over Freudenthal triple systems that was given in [15].

Every simple Lie algebra $\mathfrak{g}$ of dimension greater than three can be given a 5-graded decomposition,[5] determined by one of its generators Δ, such that grade ± 2 subspaces are one-dimensional:

$$\mathfrak{g} = \mathfrak{g}^{-2} \oplus \mathfrak{g}^{-1} \oplus \mathfrak{g}^{0} \oplus \mathfrak{g}^{+1} \oplus \mathfrak{g}^{+2}, \tag{2.96}$$

[5] This is to be contrasted with the three grading of generalized conformal groups. No real forms of exceptional Lie algebras G_2, F_4 and E_8 admit such a three grading.

where

$$\mathfrak{g}^0 = \mathfrak{h} \oplus \Delta \tag{2.97}$$

and

$$[\Delta, \mathfrak{t}] = m\mathfrak{t} \quad \forall \mathfrak{t} \in \mathfrak{g}^m,\ m = 0, \pm 1, \pm 2. \tag{2.98}$$

Given such a 5-graded Lie algebra it can be constructed over a Freudenthal triple system $\mathcal{F}$ which we shall denote as $\mathfrak{g}(\mathcal{F})$ [44,53]. A Freudenthal triple system (FTS) is defined as a vector space $\mathcal{F}$ equipped with a triple product (X, Y, Z)

$$(X, Y, Z) \in \mathcal{F} \qquad \forall \ X, Y, Z \in \mathcal{F} \tag{2.99}$$

that satisfies the identities

$$\begin{aligned}
(X, Y, Z) &= (Y, X, Z) + 2\,\langle X, Y\rangle Z\,,\\
(X, Y, Z) &= (Z, Y, X) - 2\,\langle X, Z\rangle Y\,,\\
\langle (X, Y, Z), W\rangle &= \langle (X, W, Z), Y\rangle - 2\,\langle X, Z\rangle\langle Y, W\rangle\,,\\
(X, Y, (V, W, Z)) &= (V, W, (X, Y, Z) + ((X, Y, V), W, Z)\\
&\quad + (V, (Y, X, W), Z)\,.
\end{aligned} \tag{2.100}$$

and admits a skew symmetric bilinear form

$$\langle X, Y\rangle = -\langle Y, X\rangle \in \mathbb{R}, \quad \forall \ X, Y \in \mathcal{F}.$$

In the corresponding construction of $\mathfrak{g}(\mathcal{F})$ one labels the generators belonging to subspace $\mathfrak{g}^{+1}$ by the elements of $\mathcal{F}$

$$U_A \in \mathfrak{g}^{+1} \leftrightarrow A \in \mathcal{F} \tag{2.101}$$

and through the involution, that reverses the grading, elements of $\mathfrak{g}^{-1}$ can also be labeled by elements of $\mathcal{F}$

$$\tilde{U}_A \in \mathfrak{g}^{-1} \leftrightarrow A \in \mathcal{F}. \tag{2.102}$$

Elements of $\mathfrak{g}^{\pm 1}$ generate the full Lie algebra $\mathfrak{g}(\mathcal{F})$ by commutation. The generators belonging to grade zero and grade ± 2 subspaces are labelled by a pair of elements of $\mathcal{F}$

$$\begin{aligned}
[U_A, \tilde{U}_B] &\equiv S_{AB} \in \mathfrak{g}^0, \qquad (2.103)\\
[U_A, U_B] &\equiv -K_{AB} \in \mathfrak{g}^2,\\
[\tilde{U}_A, \tilde{U}_B] &\equiv -\tilde{K}_{AB} \in \mathfrak{g}^{-2}.
\end{aligned}$$

Commutation relations of the generators of the Lie algebra $\mathfrak{g}$ can all be expressed in terms of the Freudenthal triple product (A, B, C)

$$
\begin{aligned}
[S_{AB}, U_C] &= -U_{(A,B,C)}, \\
[S_{AB}, \tilde{U}_C] &= -\tilde{U}_{(B,A,C)}, \\
[K_{AB}, \tilde{U}_C] &= U_{(A,C,B)} - U_{(B,C,A)}, \\
[\tilde{K}_{AB}, U_C] &= \tilde{U}_{(B,C,A)} - \tilde{U}_{(A,C,B)}, \\
[S_{AB}, S_{CD}] &= -S_{(A,B,C)D} - S_{C(B,A,D)}, \\
[S_{AB}, K_{CD}] &= K_{A(C,B,D)} - K_{A(D,B,C)}, \\
[S_{AB}, \tilde{K}_{CD}] &= \tilde{K}_{(D,A,C)B} - \tilde{K}_{(C,A,D)B}, \\
[K_{AB}, \tilde{K}_{CD}] &= S_{(B,C,A)D} - S_{(A,C,B)D} - S_{(B,D,A)C} + S_{(A,D,B)C}.
\end{aligned} \tag{2.104}
$$

Since the grade ± 2 subspaces are one-dimensional their generators can be written as

$$K_{AB} := K_{\langle A,B \rangle} := \langle A, B \rangle K, \tag{2.105}$$

$$\tilde{K}_{AB} := \tilde{K}_{\langle A,B \rangle} := \langle A, B \rangle \tilde{K}. \tag{2.106}$$

Now the defining identities of a FTS imply that

$$S_{AB} - S_{BA} = -2\langle A, B \rangle \Delta, \tag{2.107}$$

where Δ is the generator that determines the 5-grading

$$
\begin{aligned}
[\Delta, U_A] &= U_A, \\
[\Delta, \tilde{U}_A] &= -\tilde{U}_A, \\
[\Delta, K] &= 2K, \\
[\Delta, \tilde{K}] &= -2\tilde{K},
\end{aligned} \tag{2.108}
$$

and generates a distinguished $sl(2)$ subalgebra together with K, $\tilde{K}$

$$[K, \tilde{K}] = -2\Delta. \tag{2.109}$$

The 5-grading of $\mathfrak{g}$ can then be recast as

$$\mathfrak{g} = \tilde{K} \oplus \tilde{U}_A \oplus [S_{(AB)} + \Delta] \oplus U_A \oplus K,$$

where

$$S_{(AB)} := \frac{1}{2}(S_{AB} + S_{BA})$$

are the generators of the automorphism group Aut($\mathcal{F}$) of $\mathcal{F}$ that commute with Δ

$$[\Delta, S_{(AB)}] = 0. \tag{2.110}$$

The remaining non-zero commutators are

$$\begin{aligned}
[U_A, \tilde{U}_B] &= S_{(AB)} - \langle A, B\rangle \Delta, \\
[K, \tilde{U}_A] &= -2\tilde{U}_A, \\
[\tilde{K}, U_A] &= 2\tilde{U}_A, \\
[S_{(AB)}, K] &= 0.
\end{aligned} \tag{2.111}$$

Every FTS $\mathcal{F}$ admits a completely symmetric quadrilinear form which induces a quartic norm $\mathcal{Q}_4$. For an element $X \in \mathcal{F}$ the quartic norm is

$$\mathcal{Q}_4(X) := \frac{1}{48}\langle (X, X, X), X\rangle, \tag{2.112}$$

which is invariant under the automorphism group Aut($\mathcal{F}$) of $\mathcal{F}$ generated by $S_{(AB)}$.

As was shown in [15] one can realize the 5-graded Lie algebra $\mathfrak{g}$ non-linearly as a quasiconformal Lie algebra over a vector space $\mathcal{T}$ coordinatized by the elements X of the FTS $\mathcal{F}$ plus an extra singlet variable x [15, 22]:

$$\begin{aligned}
&K(X) = 0, \quad U_A(X) = A, \quad S_{AB}(X) = (A, B, X), \\
&K(x) = 2, \quad U_A(x) = \langle A, X\rangle, \quad S_{AB}(x) = 2\langle A, B\rangle x, \\
&\tilde{U}_A(X) = \frac{1}{2}(X, A, X) - Ax, \\
&\tilde{U}_A(x) = -\frac{1}{6}\langle (X, X, X), A\rangle + \langle X, A\rangle x, \\
&\tilde{K}(X) = -\frac{1}{6}(X, X, X) + Xx, \\
&\tilde{K}(x) = \frac{1}{6}\langle (X, X, X), X\rangle + 2x^2.
\end{aligned} \tag{2.113}$$

The quasiconformal action of the Lie algebra $\mathfrak{g}(\mathcal{F})$ on the space $\mathcal{T}$ has a beautiful geometric interpretation. To see this one defines the quartic norm of a vector $\mathcal{X} = (X, x)$ in the space $\mathcal{T}$ as

$$\mathcal{N}_4(\mathcal{X}) := \mathcal{Q}_4(X) - x^2, \tag{2.114}$$

where $\mathcal{Q}_4(X)$ is the quartic norm of $X \in \mathcal{F}$ and then a "distance" function between any two points $\mathcal{X} = (X, x)$ and $\mathcal{Y} = (Y, y)$ in $\mathcal{T}$ as

$$d(\mathcal{X}, \mathcal{Y}) := \mathcal{N}_4(\delta(\mathcal{X}, \mathcal{Y}), \tag{2.115}$$

where $\delta(\mathcal{X}, \mathcal{Y})$ is the "symplectic" difference of two vectors $\mathcal{X}$ and $\mathcal{Y}$:

$$\delta(\mathcal{X}, \mathcal{Y}) := (X - Y, x - y + \langle X, Y\rangle) = -\delta(\mathcal{Y}, \mathcal{X}). \tag{2.116}$$

One can then show that the light-like separations with respect to this quartic distance function

$$d(\mathcal{X}, \mathcal{Y}) = 0 \tag{2.117}$$

is left invariant under quasiconformal group action [15]. In other words quasiconformal groups are the invariance groups of "light-cones" defined by a quartic distance function.

2.10 3D U-Duality Groups as Spectrum Generating Quasiconformal Groups of 4D Supergravity Theories and Quantum Attractor Flows

As explained above the vector field strengths plus their magnetic duals of a 4D supergravity defined by a Jordan algebra J of degree 3 are in one-to-one correspondence with the elements of the Freudenthal triple $\mathcal{F}(J)$ defined over J. The automorphism group of $\mathcal{F}(J)$ is the U-duality group G_4 of the supergravity defined by J and is isomorphic to the conformal group Conf(J) of J. Furthermore, U-duality symmetry groups G_3 of the 3D supergravity theories they reduce to under dimensional reduction are the quasiconformal groups QConf(J) of $\mathcal{F}(J)$. The U-duality groups of $N = 2$ MESGTs defined by Jordan algebras of degree 3 in five, four and three dimensions are also the isometry groups of their scalar manifolds in the respective dimensions. In five dimensions scalar manifolds are

$$\mathcal{M}_5 = \frac{\mathrm{Str}_0(J)}{\mathrm{Aut}(J)},$$

where $\mathrm{Str}_0(J)$ and $\mathrm{Aut}(J)$ are the reduced structure and automorphism groups of J, respectively. The scalar manifolds of these theories in four dimensions are

$$\mathcal{M}_4 = \frac{\mathrm{Conf}(J)}{\widetilde{\mathrm{Str}}_0(J) \times U(1)},$$

where $\mathrm{Conf}(J)$ is the conformal group of the Jordan algebra J and $\widetilde{\mathrm{Str}}_0(J)$ is the compact form of the reduced structure group. Upon further dimensional reduction to three dimensions they lead to scalar manifolds of the form

$$\mathcal{M}_3 = \frac{\mathrm{QConf}(J)}{\widetilde{\mathrm{Conf}}(J) \times SU(2)},$$

where $\mathrm{QConf}(J)$ is the quasiconformal group associated with the Jordan algebra J and $\widetilde{\mathrm{Conf}}(J)$ is the compact real form of the conformal group of J. The complete list of the symmetric scalar manifolds in five, four and three dimensions are given in Table 2.4.

In the original proposal of [15] that the three-dimensional U-duality groups act as spectrum generating quasiconformal groups of the corresponding four-dimensional supergravity theories the extra singlet coordinate that extends the 56-dimensional charge space (p^A, q_A) of black hole solutions of $N = 8$ supergravity was interpreted as the entropy s of the black hole. The light cone condition on the 57 dimensional charge-entropy vector (p^A, q_A, s) on which G_3 acts as a quasiconformal group then gives the well-established relation between the entropy s to the quartic invariant $\mathcal{Q}_4$ constructed out of the charges

$$s^2 = \mathcal{Q}_4(p^A, q_A).$$

A concrete and precise implementation of the proposal that three-dimensional U-duality groups must act as spectrum generating quasiconformal groups of spherically symmetric stationary BPS black holes of four-dimensional supergravity theories, was given in [51, 52, 54] which we will summarize in this section.[6] The basic starting point of these works is the fact that the attractor equations [29, 56] for a spherically symmetric stationary black hole of four-dimensional supergravity theories are equivalent to the equations for geodesic motion of a fiducial particle on the moduli space $\mathcal{M}_3^*$ of the three-dimensional supergravity obtained by reduction on a time-like circle. The connection between the stationary black holes of 4D gravity coupled to matter and geodesic motion of a fiducial particle on the pseudo-Riemannian manifold $\mathcal{M}_3$ coupled to gravity in three dimensions on a timelike circle was first observed in [57].[7] More specifically, a 4D supergravity theory with symmetric scalar manifold $\mathcal{M}_4 = G_4/K_4$ reduces on a space-like circle to a 3D supergravity with scalar manifold

$$\mathcal{M}_3 = \frac{G_3}{K_3},$$

where K_3 is the maximal compact subgroup of G_3. The same theory dimensionally reduced on a time-like circle leads to a theory with scalar manifold of the form [57]

$$\mathcal{M}^* = \frac{G_3}{H_3},$$

where H_3 is a certain noncompact real form of K_3. Then the stationary, spherically symmetric solutions of the four-dimensional equations of motion are equivalent to geodesic trajectories on the three-dimensional scalar manifold $\mathcal{M}_3^* = G_3/H_3$ [57]. For $N = 2$ MESGTs defined by Euclidean Jordan algebras of degree 3 the resulting spaces $\mathcal{M}_3^*$ are para-quaternionic symmetric spaces of the form

[6] See also [55].

[7] This was used in [58, 59] to construct static and rotating black holes in heterotic string theory.

Table 2.6 Number of supercharges n_Q, 4D vector fields n_V, scalar manifolds of supergravity theories before and after reduction along a timelike Killing vector from $D=4$ to $D=3$, and associated Jordan algebras J. Isometry groups of 4D and 3D supergravity theories are given by the conformal, Conf(J), and quasiconformal groups, QConf(J), of J, respectively

n_Q	n_V	M_4	$\mathcal{M}_3^*$	J
8	1	$\emptyset$	$\frac{U(2,1)}{U(1,1)\times U(1)}$	$\mathbb{R}$
8	2	$\frac{SL(2,\mathbb{R})}{U(1)}$	$\frac{G_{2,2}}{SO(2,2)}$	$\mathbb{R}$
8	7	$\frac{Sp(6,\mathbb{R})}{SU(3)\times U(1)}$	$\frac{F_{4(4)}}{Sp(6,\mathbb{R})\times SL(2,\mathbb{R})}$	$J_3^{\mathbb{R}}$
8	10	$\frac{SU(3,3)}{SU(3)\times SU(3)\times U(1)}$	$\frac{E_{6(2)}}{SU(3,3)\times SL(2,\mathbb{R})}$	$J_3^{\mathbb{C}}$
8	16	$\frac{SO^*(12)}{SU(6)\times U(1)}$	$\frac{E_{7(-5)}}{SO^*(12)\times SL(2,\mathbb{R})}$	$J_3^{\mathbb{H}}$
8	28	$\frac{E_{7(-25)}}{E_6\times U(1)}$	$\frac{E_{8(-24)}}{E_{7(-25)}\times SL(2,\mathbb{R})}$	$J_3^{\mathbb{O}}$
8	$n+2$	$\frac{SL(2,\mathbb{R})}{U(1)}\times\frac{SO(n,2)}{SO(n)\times SO(2)}$	$\frac{SO(n+2,4)}{SO(n,2)\times SO(2,2)}$	$\mathbb{R}\oplus\Gamma_{(1,n-1)}$
16	$n+2$	$\frac{SL(2,\mathbb{R})}{U(1)}\times\frac{SO(n-4,6)}{SO(n-4)\times SO(6)}$	$\frac{SO(n-2,8)}{SO(n-4,2)\times SO(2,6)}$	$\mathbb{R}\oplus\Gamma_{(5,n-5)}$
24	16	$\frac{SO^*(12)}{SU(6)\times U(1)}$	$\frac{E_{7(-5)}}{SO^*(12)\times SL(2,\mathbb{R}}$	$J_3^{\mathbb{H}}$
32	28	$\frac{E_{7(7)}}{SU(8)}$	$\frac{E_{8(8)}}{SO^*(16)}$	$J_3^{\mathbb{O}_s}$

$$\mathcal{M}_3^* = \frac{\text{QConf}(J)}{\text{Conf}(J)\times SU(1,1)}, \tag{2.118}$$

where Conf(J) is the isometry group of the scalar manifold $\mathcal{M}_4$ of the four-dimensional theory. In Table 2.6 we reproduce a table from [51] giving a complete list of supergravity theories whose four-dimensional isometry groups are conformal groups Conf(J) of a Jordan algebra of degree 3 and the resulting scalar manifolds $\mathcal{M}_3^*$, which include all $N \geq 4$ supergravity theories as well as $N=2$ MESGTs defined by Euclidean Jordan algebras.

For dimensionally reducing the $D=4$ theory along a timelike direction one makes the standard Kaluza–Klein-type ansatz [57]

$$ds_4^2 = -e^{2U}(dt+\omega)^2 + e^{-2U}ds_3^2, \tag{2.119}$$

which results in 3D Euclidean gravity coupled to scalars, vectors and fermions. Vector fields in three dimensions can be dualized to scalars and the bosonic sector is described simply by the three-dimensional metric ds_3^2 and scalar fields ϕ^a. The three-dimensional scalar fields ϕ^a consist of the scalars coming from the 4D theory, plus electric and magnetic potentials from the reduction of 4D vector fields A_t^A and their duals, plus the scale factor U and the twist potential dual to the shift ω defined in (2.119). The resulting manifold $\mathcal{M}_3^*$ of scalar fields can be thought of as analytic continuation of the Riemannian manifold M_3 obtained from reduction on a space-like circle.

For spherically symmetric configurations, the metric on three-dimensional slices can be written as

$$\mathrm{d}s_3^2 = N^2(\rho)\,\mathrm{d}\rho^2 + r^2(\rho)\left[\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right], \tag{2.120}$$

where ρ is the radial coordinate that plays the role of time in radial quantization. The bosonic part of the action then becomes

$$S = \int \mathrm{d}\rho \left[\frac{N}{2} + \frac{1}{2N}\left(\dot{r}^2 - r^2 G_{ab}\dot{\phi}^a\dot{\phi}^b\right)\right], \tag{2.121}$$

where the dot denotes derivative with respect to ρ and G_{ab} is the metric on $\mathcal{M}_3^*$. Thus four-dimensional equations of motion are equivalent to geodesic motion of a fiducial particle on a real cone $\mathbb{R} \times \mathcal{M}_3^*$ over $\mathcal{M}_3^*$. The equation of motion for the lapse function N, which is an auxiliary field, imposes the Hamiltonian constraint

$$H = p_r^2 - \frac{1}{r^2}G^{ab}\,p_a p_b - 1 \equiv 0, \tag{2.122}$$

where p_r and p_a are the canonical conjugates to r and ϕ^a, respectively. This constraint fixes the mass of the fiducial particle on the cone to be 1. Note that for BPS black holes, one may choose $N = 1, \rho = r, p_r = 1$. With this choice the problem reduces to *light-like* geodesic motion on $\mathcal{M}_3^*$, with affine parameter $\tau = 1/r$. The magnetic and electric charges of the black hole are simply Noether charges P^A, Q_A associated with the generators of 4D gauge transformations in the isometry group G_3 acting on $\mathcal{M}_3^*$. These charges generate an Heisenberg subalgebra under Poisson brackets

$$[P^A, Q_B]_{\mathrm{PB}} = 2\delta_B^A\, K\,, \tag{2.123}$$

where the "central charge" K is the NUT charge of the black hole [39, 51, 52, 60].[8] The conserved charge of the isometry that corresponds to rescalings of the time-time component g_{tt} of the metric is the ADM mass M that satisfies

$$[M, P^A]_{\mathrm{PB}} = P^A, \quad [M, Q_A]_{\mathrm{PB}} = Q_A, \quad [M, K]_{\mathrm{PB}} = 2K\,. \tag{2.124}$$

[8] The solutions with $K \neq 0$ have closed timelike curves when lifted back to four dimensions, as a consequence of the off-diagonal term $\omega = K\cos\theta\mathrm{d}\phi$ in the metric (2.119). Therefore real four-dimensional black holes require taking the "central charge" $K \to 0$ limit.

For supergravity theories whose scalar manifolds $\mathcal{M}_3^* = G_3/H_3$ are homogeneous or symmetric spaces there exist additional conserved charges associated with the additional isometries. For $N = 2$ MESGTs defined by Euclidean Jordan algebras J of degree 3 the full isometry group of $\mathcal{M}_3^*$ is the quasiconformal group QConf(J) that has a five grading with respect to the generator M

$$\mathfrak{qconf}(J) = \tilde{K} \oplus (\tilde{P}^A, \tilde{Q}_A) \oplus (\mathfrak{conf}(J) + M) \oplus (P^A, Q_A) \oplus K, \qquad (2.125)$$

where $\mathfrak{conf}(J)$ is the Lie algebra of the 4D U-duality group Conf(J), which commutes with ADM mass generator M.

For spherically symmetric stationary solutions, the supersymmetry variation of the fermionic fields λ^α are of the general form [61]

$$\delta\lambda^\alpha = V_i^\alpha \epsilon^i, \qquad (2.126)$$

where ϵ^i is the supersymmetry parameter and V_i^α is a matrix linear in the velocities $\dot{\phi}^a \equiv \frac{\partial \phi}{\partial \tau}$ on $\mathcal{M}_3^*$. For general $\mathcal{N} = 2$ MESGTs reduced to $d = 3$, the indices $i = 1, 2$ and $\alpha = 1, \ldots, 2n_{\mathrm{V}} + 2$ transform as fundamental representations of the restricted holonomy group $Sp(2, \mathbb{R}) \times Sp(2n_{\mathrm{V}} + 2, \mathbb{R})$ of para-quaternionic geometry. For supersymmetric backgrounds this variation vanishes for some non-zero ϵ^i. One can show that this is equivalent to the system of equations [39, 51, 52, 60]:

$$\frac{\mathrm{d}z^I}{\mathrm{d}\tau} = -\mathrm{e}^{U+\mathrm{i}\alpha} g^{I\bar{J}} \partial_{\bar{J}} |Z|, \qquad (2.127)$$

$$\frac{\mathrm{d}U}{\mathrm{d}\tau} + \frac{i}{2} K = -2\mathrm{e}^{U+\mathrm{i}\alpha} |Z|, \qquad (2.128)$$

where

$$Z(P, Q, K) = \mathrm{e}^{\mathcal{K}/2} \Big[(Q_A - 2K\tilde{\zeta}_A) X^A - (P^A + 2K\zeta^A) F_A \Big] \qquad (2.129)$$

is the central charge function.[9]

For vanishing NUT charge K, the above equations take the form of the standard attractor flow equations describing the radial evolution of the scalars towards the black hole horizon [29, 30, 56, 62, 63]

$$\frac{\mathrm{d}U}{\mathrm{d}\tau} = -2\mathrm{e}^U |Z| \qquad (2.130)$$

$$\frac{\mathrm{d}z^I}{\mathrm{d}\tau} = -\, \mathrm{e}^U \, g^{I\bar{J}} \partial_{\bar{J}} |Z| \qquad (2.131)$$

with the central charge function

[9] The phase α is to be chosen such that $\mathrm{d}U/\mathrm{d}\tau$ is real.

$$Z(P, Q, K=0) = \mathrm{e}^{\mathcal{K}/2}\left[Q_A X^A - P^A F_A\right].$$

The equivalence of attractor flow of $N=2$ supergravity in $d=4$ and supersymmetric geodesic motion on M_3^* was pointed out in [64].

The scalar fields $\tilde{\zeta}_A, \zeta^A$ conjugate to the charges P^A and Q_A evolve according to

$$\begin{aligned}\frac{\mathrm{d}\zeta^A}{\mathrm{d}\tau} &= -\frac{1}{2}\mathrm{e}^{2U}\left[(Im\mathcal{N})^{-1}\right]^{AB} \\ &\quad \times\left[Q_A - 2K\tilde{\zeta}_A - [Re\mathcal{N}]_{BC}(P^C + 2K\zeta^C)\right] \\ \frac{\mathrm{d}\tilde{\zeta}_A}{\mathrm{d}\tau} &= -\frac{1}{2}\mathrm{e}^{2U}[Im\mathcal{N}]_{AB}\left(P^B + 2K\zeta^A\right) - [Re\mathcal{N}]_{AB}\frac{\mathrm{d}\zeta^J}{\mathrm{d}\tau}, \end{aligned} \quad (2.132)$$

where $\mathcal{N}_{AB}$ is the period matrix of special geometry [48].

For $\mathcal{N}=2$ MESGTs defined by Euclidean Jordan algebras J of degree 3 the holonomy group of $\mathcal{M}_3^*$ is $\mathrm{Conf}(J)\times Sp(2,\mathbb{R}) \subset Sp(2n_{\mathrm{V}}+4,\mathbb{R})$. The full phase space is $8n_{\mathrm{V}}+8$-dimensional and for BPS black holes supersymmetry leads to $2n_{\mathrm{V}}+1$ first class constraints which reduce the dimension of the phase space to $4n_{\mathrm{V}}+6 = (8n_{\mathrm{V}}+8) - 2(2n_{\mathrm{V}}+1)$. This reduced phase of BPS black holes can be identified with the twistor space of $\mathcal{M}_3$ of complex dimension $(2n_{\mathrm{V}}+3)$ [51, 52, 54, 65].

The twistor space of the scalar manifold

$$\mathcal{M}_3 = \frac{\mathrm{QConf}(J)}{\widetilde{\mathrm{Conf}}(J)\times SU(2)}$$

of dimensionally reduced $N=2$ MESGT defined by a Jordan algebra J is

$$\mathcal{Z}_3 = \frac{\mathrm{QConf}(J)}{\widetilde{\mathrm{Conf}}(J)\times SU(2)} \times \frac{SU(2)}{U(1)} = \frac{\mathrm{QConf}(J)}{\widetilde{\mathrm{Conf}}(J)\times U(1)}. \quad (2.133)$$

The quasiconformal group action of a group G extends to its complexification [15]. Consequently, quasiconformal actions of three-dimensional U-duality groups $\mathrm{QConf}(J)$ on the space with real coordinates $\mathcal{X}=(X,x)$ extend naturally to the complex coordinates $\mathcal{Z}=(Z,z)$ of corresponding twistor spaces $\mathcal{Z}_3$ [54]. The Kähler potential of the Kähler–Einstein metric of the twistor space is given precisely by the "quartic light-cone" of quasiconformal geometry in these coordinates

$$\mathcal{K}(\mathcal{Z},\overline{\mathcal{Z}}) = \ln d(\mathcal{Z},\overline{\mathcal{Z}}) = \ln\left[\mathcal{Q}_4(\mathbf{Z}-\overline{\mathbf{Z}}) + (\mathbf{z}-\bar{\mathbf{z}}+\langle\mathbf{Z},\overline{\mathbf{Z}}\rangle)^2\right]. \quad (2.134)$$

The Kähler potential is manifestly invariant under the Heisenberg symmetry group corresponding to "symplectic translations" generated by Q_A, P^A and K. Under the global action of "symplectic special conformal generators" $\tilde{Q}_A$, $\tilde{P}^A$ and $\tilde{K}$ the quartic light-cone transforms as [15, 54]

$$d(\mathcal{Z},\overline{\mathcal{Z}}) \Longrightarrow f(\mathbf{Z},\mathbf{z})\bar{f}(\overline{\mathbf{Z}},\bar{\mathbf{z}})d(\mathcal{Z},\overline{\mathcal{Z}}), \tag{2.135}$$

which correspond to Kähler transformations of the Kähler potential (2.134) of the twistor space and hence leaves the Kähler metric invariant. These results were first established for quaternionic symmetric spaces [54] and there exist analogous Kähler potentials for more general quaternionic manifolds invariant only under the Heisenberg symmetries generated by Q_A, P^A and K that are in the C-map [65]. As will be discussed in the next section, the correspondence established between harmonic superspace formulation of 4D, $N = 2$ sigma models coupled to $N = 2$ supergravity and quasiconformal realizations of their isometry groups [66] implies that Kähler potentials of quartic light-cone type must exist for all quaternionic target manifolds and not only those that are in the C-map.

The quantization of the motion of fiducial particle on $\mathcal{M}_3^*$ leads to quantum mechanical wave functions that provide the basis of a unitary representation of the isometry group G_3 of $\mathcal{M}_3^*$. BPS black holes correspond to a special class of geodesics which lift holomorphically to the twistor space $\mathcal{Z}_3$ of $\mathcal{M}_3^*$. Spherically symmetric stationary BPS black holes of $N = 2$ MESGT's are described by holomorphic curves in $\mathcal{Z}_3$ [51, 52, 54, 65]. Therefore for theories defined by Jordan algebras J of degree 3, the relevant unitary representations of the isometry groups QConf(J) for BPS black holes are those induced by their holomorphic actions on the corresponding twistor spaces $\mathcal{Z}_3$, which belong in general to quaternionic discrete series representations [54]. For rank two quaternionic groups $SU(2,1)$ and $G_{2(2)}$ unitary representations induced by the geometric quasiconformal actions were studied in great detail in [54].

2.11 Harmonic Superspace, Minimal Unitary Representations and Quasiconformal Group Actions

In this section we shall review the connection between the harmonic superspace (HSS) formulation of 4D, $N = 2$ supersymmetric quaternionic Kähler sigma models that couple to $N = 2$ supergravity and the minimal unitary representations of their isometry groups [66]. We shall then discuss the relevance of these results to the proposal that quasiconformal extensions of U-duality groups of four-dimensional $N = 2$ MESGTs must act as spectrum generating symmetry groups [15, 34, 51, 52, 54], which extends the proposal that the conformal extensions of U-duality groups of $N = 2, d = 5$ MESGTs act as spectrum generating symmetry groups [14, 15, 34].

2.11.1 4D, $N = 2$ σ-Models Coupled to Supergravity in Harmonic Superspace

The target spaces of $N = 2$ supersymmetric σ-models coupled to $N = 2$ supergravity in four dimensions are quaternionic Kähler manifolds [61]. They can be formulated in a manifestly supersymmetric form in harmonic superspace [67–70] which we shall review briefly following [70]. In harmonic superspace approach the metric on a quaternionic target space of $N = 2$ sigma model is given by a quaternionic potential $\mathcal{L}^{(+4)}$, which is the analog of Kähler potentials of complex Kähler manifolds.

The $N = 2$ harmonic superspace action for the general $4n$-dimensional quaternionic σ-model has the simple form [70][10]

$$S = \int \mathrm{d}\zeta^{(-4)}\mathrm{d}u\{Q^+_\alpha D^{++} Q^{+\alpha} - q^+_i D^{++} q^{+i} + \mathcal{L}^{(+4)}(Q^+, q^+, u^-)\}, \quad (2.136)$$

where the integration is over the *analytic* superspace coordinates $\zeta, u^\pm_i$. The hypermultiplet superfields $Q^+_\alpha(\zeta, u),\ \alpha = 1, \ldots, 2n$ and the supergravity hypermultiplet compensators $q^+_i(\zeta, u),\ (i = 1, 2)$ are analytic $N = 2$ superfields. The $u^\pm_i, (i = 1, 2)$ are the $S^2 = \frac{SU_A(2)}{U_A(1)}$ isospinor harmonics that satisfy

$$u^{+i} u^-_i = 1$$

and D^{++} is a supercovariant derivative with respect to harmonics with the property

$$D^{++} u^-_i = u^+_i .$$

The analytic subspace of the full $N = 2$ harmonic superspace involves only half the Grassmann variables with coordinates ζ^M and $u^\pm_i$

$$\zeta^M := \{x^\mu_A,\ \theta^{a+},\ \bar{\theta}^{\dot{a}+}\}, \quad (2.137)$$

where

$$x^\mu_A := x^\mu - 2i\theta^{(i}\sigma^\mu\bar{\theta}^{j)} u^+_i u^-_j ,$$
$$\theta^{a+} := \theta^{ai} u^+_i ,$$
$$\bar{\theta}^{\dot{a}+} := \bar{\theta}^{\dot{a}i} u^+_i ,$$
$$\theta^{(i}\sigma^\mu\bar{\theta}^{j)} u^+_i u^-_j := \theta^{(ai}(\sigma^\mu)_{a\dot{a}}\bar{\theta}^{\dot{a}j)} u^+_i u^-_j ,$$
$$\mu = 0, 1, 2, 3; \quad a = 1, 2; \quad \dot{a} = 1, 2.$$

---

[10] The conventions for indices in this section are independent of the conventions of previous sections and follow closely the conventions used in [66]. The number of plus (+) or minus (−) signs in a superscript or subscript denote the $U_A(1)$ charges. If the $U_A(1)$ charge $n > 2$, it is indicated as $(+n)$.

Furthermore the analytic subspace does not involve $U(1)$ charge -1 projections of the Grassmann variables and still closes under $N = 2$ supersymmetry transformations. It satisfies a "reality condition" with respect to the conjugation$\widetilde{\ }$

$$\begin{aligned} \tilde{x}^{\mu} &= x^{\mu}, \\ \tilde{\theta}^{+} &= \bar{\theta}^{+}, \\ \tilde{\bar{\theta}}^{+} &= -\theta^{+}, \\ \tilde{u}^{i\pm} &= -u_i^{\pm}, \\ \tilde{u}_i^{\pm} &= u^{i\pm}, \end{aligned} \tag{2.138}$$

which is a product of complex conjugation and anti-podal map on the sphere S^2 [71].

The quaternionic potential $\mathcal{L}^{(+4)}$ is a homogeneous function in Q_{α}^{+} and q_i^{+} of degree 2 and has $U(1)$-charge $+4$. It does not depend on u^{+} and is, otherwise, an arbitrary "real function" with respect to the involution $\sim$. We shall suppress the dependence of all the fields on the harmonic superspace coordinates ζ^{M} and $u_i^{\pm}$ in what follows.

The action (7.81) is of the form of Hamiltonian mechanics with the harmonic derivative D^{++} playing the role of time derivative [70–72] and with the superfields Q^{+} and q^{+} corresponding to phase space coordinates under the Poisson brackets

$$\{f, g\}^{--} = \frac{1}{2}\Omega^{\alpha\beta}\frac{\partial f}{\partial Q^{+\alpha}}\frac{\partial g}{\partial Q^{+\beta}} - \frac{1}{2}\epsilon^{ij}\frac{\partial f}{\partial q^{+i}}\frac{\partial g}{\partial q^{+j}}, \tag{2.139}$$

$\Omega^{\alpha\beta}$ and ϵ^{ij} are the invariant antisymmetric tensors of $Sp(2n)$ and $Sp(2)$, respectively, which are used to raise and lower indices

$$\begin{aligned} Q^{+\alpha} &= \Omega^{\alpha\beta} Q_{\beta}^{+}, \\ q^{+i} &= \epsilon^{ij} q_j^{+}, \end{aligned} \tag{2.140}$$

and satisfy[11]

$$\Omega^{\alpha\beta}\Omega_{\beta\gamma} = \delta_{\gamma}^{\alpha}, \tag{2.141}$$

$$\epsilon^{ij}\epsilon_{jk} = \delta_k^i. \tag{2.142}$$

The quaternionic potential $\mathcal{L}^{(+4)}$ plays the role of the Hamiltonian and we shall refer to it as such following [70].

Isometries of the σ-model (7.81) are generated by Killing potentials $K_A^{++}(Q^{+}, q^{+}, u^{-})$ that obey the conservation law

$$\partial^{++} K_A^{++} + \{K_A^{++}, \mathcal{L}^{(+4)}\}^{--} = 0\,, \tag{2.143}$$

[11] Note that the conventions of [70] for the symplectic metric, which we follow in this section, differ from those of [73].

where ∂_{++} is defined as

$$\partial_{++} = u^{+i} \frac{\partial}{\partial u^{-i}}.$$

They generate the Lie algebra of the isometry group

$$\{K_A^{++}, K_B^{++}\}^{--} = f_{AB}{}^C K_C^{++} \tag{2.144}$$

under Poisson brackets (2.139).

The "Hamiltonians" $\mathcal{L}^{(+4)}$ of all $N = 2$ σ-models coupled to N = 2 supergravity with irreducible symmetric target manifolds were given in [70]. The quaternionic symmetric spaces, sometimes known as Wolf spaces [74], that arise in $N = 2$ supergravity are of the non-compact type. For each simple Lie group there is a unique non-compact quaternionic symmetric space. Below we give a complete list of these spaces:

$$\begin{gathered} \frac{SU(n,2)}{U(n)\times Sp(2)}, \\ \frac{SO(n,4)}{SO(n)\times SU(2)\times Sp(2)}, \\ \frac{USp(2n,2)}{Sp(2n)\times Sp(2)}, \\ \frac{G_{2(+2)}}{SU(2)\times Sp(2)}, \\ \frac{F_{4(+4)}}{Sp(6)\times Sp(2)}, \\ \frac{E_{6(+2)}}{SU(6)\times Sp(2)}, \\ \frac{E_{7(-5)}}{SO(12)\times Sp(2)}, \\ \frac{E_{8(-24)}}{E_7\times Sp(2)}. \end{gathered} \tag{2.145}$$

Given a target space $\frac{G}{H\times Sp(2)}$ of $N = 2$ σ model coordinatized by Q_α^+ and q_i^+, every generator Γ_A of G maps to a function $K_A^{++}(Q^+, q^+, u^-)$ such that the action of K_A^{++} is given via the Poisson brackets (2.139). Furthermore, it can be shown that the Hamiltonian $\mathcal{L}^{(+4)}$ depends only on Q_α^+ and the combination $(q^+u^-) \equiv q^{+i}u_i^-$ [70]

$$\mathcal{L}^{(+4)} = \mathcal{L}^{(+4)}(Q^+, (q^+u^-)) \tag{2.146}$$

and can be written as

$$\mathcal{L}^{(+4)} = \frac{P^{(+4)}(Q^+)}{(q^+u^-)^2}. \tag{2.147}$$

The fourth order polynomial $P^{(+4)}$ is given by

$$P^{(+4)}(Q^+) = \frac{1}{12} S_{\alpha\beta\gamma\delta}\ Q^{+\alpha} Q^{+\beta} Q^{+\gamma} Q^{+\delta}, \tag{2.148}$$

where $S_{\alpha\beta\gamma\delta}$ is a completely symmetric invariant tensor of H. In terms of matrices $t^a_{\alpha\beta} = t^a_{\beta\alpha}$, $a = 1, 2, \ldots, \dim(H)$ given by the action of the generators K_a^{++} of H on the coset space generators $K_{i\alpha}^{++}$

$$\{K_a^{++}, K_{i\alpha}^{++}\} = t^{\beta}_{a\alpha} K_{i\beta}^{++} \tag{2.149}$$

the invariant tensor reads as [70]

$$S_{\alpha\beta\gamma\delta} = h_{ab} t^a_{\ \alpha\beta} t^b_{\ \gamma\delta} + \Omega_{\alpha\gamma}\Omega_{\beta\delta} + \Omega_{\alpha\delta}\Omega_{\beta\gamma}\ , \tag{2.150}$$

where h_{ab} is the Killing metric of H.

The Killing potentials that generate the isometry group G are given by [70]

$$\mathbf{Sp(2)}: \quad K_{ij}^{++} = 2(q_i^+ q_j^+ - u_i^- u_j^- \mathcal{L}^{(+4)}), \tag{2.151}$$

$$\mathbf{H}: \quad K_a^{++} = t_{a\alpha\beta} Q^{+\alpha} Q^{+\beta}, \tag{2.152}$$

$$\mathbf{G/H \times Sp(2)}: \quad K_{i\alpha}^{++} = 2q_i^+ Q_\alpha^+ - u_i^-(q^+u^-)\partial_\alpha^- \mathcal{L}^{(+4)}\ , \tag{2.153}$$

where

$$\partial_\alpha^- := \frac{\partial}{\partial Q^{+\alpha}},$$
$$t_{a\alpha\beta} = \Omega_{\beta\gamma} t_{a\alpha}^{\ \ \gamma}.$$

The Killing potentials K_{ij}^{++} that generate $Sp(2)$ are conserved for any arbitrary polynomial $P^{(+4)}(Q^+)$. Since t^a are also the representation matrices of the generators of H acting on $Q^{+\alpha}$ one finds that the fourth order polynomial $P^{(+4)}$ is proportional to the quadratic "Casimir function" $h^{ab} K_a^{++} K_b^{++}$ of H. Furthermore, $P^{(+4)}$ can also be expressed in terms of Killing potentials of the coset space generators, or in terms of the $Sp(2)$ Killing potentials as follows [70]:

$$P^{(+4)} = -\frac{1}{16}\epsilon^{ij}\Omega^{\alpha\beta} K_{i\alpha}^{++} K_{j\beta}^{++} = -\frac{1}{8} K^{++ij} K_{ij}^{++}. \tag{2.154}$$

2.11.2 Minimal Unitary Representations of Non-compact Groups from Their Quasiconformal Realizations

Before discussing its relationship to the HSS formulation of $N = 2$ sigma models given in the previous section, we shall review the unified construction of minimal unitary representations of noncompact groups obtained by quantization of their geometric realizations as quasiconformal groups following [73] which generalizes the results of [22, 75, 76].

Consider the 5-graded decomposition of the Lie algebra $\mathfrak{g}$ of a noncompact group G of quaternionic type and label the generators such that

$$\begin{aligned} \mathfrak{g} &= \mathfrak{g}^{-2} \oplus \mathfrak{g}^{-1} \oplus (\mathfrak{h} \oplus \Delta) \oplus \mathfrak{g}^{+1} \oplus \mathfrak{g}^{+2}, \\ \mathfrak{g} &= E \oplus E^{\alpha} \oplus (J^{a} + \Delta) \oplus F^{\alpha} \oplus F, \end{aligned} \tag{2.155}$$

where Δ is the generator that determines the 5-grading. Generators J^a of $\mathfrak{h}$ satisfy

$$[J^a, J^b] = f^{ab}{}_c J^c, \tag{2.156a}$$

where $a, b, \ldots = 1, \ldots, D = \dim(H)$. We shall denote the symplectic representation by which $\mathfrak{h}$ acts on the subspaces $\mathfrak{g}^{\pm 1}$ as ρ

$$[J^a\,, E^{\alpha}] = (\lambda^a)^{\alpha}{}_{\beta} E^{\beta}, \qquad [J^a\,, F^{\alpha}] = (\lambda^a)^{\alpha}{}_{\beta} F^{\beta}, \tag{2.156b}$$

where E^{α}, $\alpha, \beta, \ldots = 1, \ldots, N = \dim(\rho)$ are generators that span the subspace $\mathfrak{g}^{-1}$

$$[E^{\alpha}\,, E^{\beta}] = 2\Omega^{\alpha\beta} E \tag{2.156c}$$

and F^{α} are generators that span $\mathfrak{g}^{+1}$

$$[F^{\alpha}\,, F^{\beta}] = 2\Omega^{\alpha\beta} F. \tag{2.156d}$$

$\Omega^{\alpha\beta}$ is the symplectic invariant "metric" of the representation ρ. The remaining nonvanishing commutation relations of $\mathfrak{g}$ are given by

$$\begin{aligned} F^{\alpha} &= [E^{\alpha}\,, F], & [\Delta, E^{\alpha}] &= -E^{\alpha}, \\ E^{\alpha} &= [E\,, F^{\alpha}], & [\Delta, F^{\alpha}] &= F^{\alpha}, \\ [E^{\alpha}, F^{\beta}] &= -\Omega^{\alpha\beta}\Delta + \epsilon \lambda^{\alpha\beta}_{a} J^{a}, & [\Delta, E] &= -2E, \\ & & [\Delta, F] &= 2F, \end{aligned} \tag{2.156e}$$

where ϵ is a constant parameter whose value depends on the Lie algebra $\mathfrak{g}$. Note that the positive (negative) grade generators form a Heisenberg subalgebra since

$$[E^{\alpha}, E] = 0 \tag{2.157}$$

with the grade $+2$ (-2) generator $F(E)$ acting as its central charge.

In the minimal unitary realization of noncompact groups [73], negative grade generators are expressed as bilinears of symplectic bosonic oscillators ξ^{α} satisfying the canonical commutation relations[12]

$$[\xi^\alpha, \xi^\beta] = \Omega^{\alpha\beta} \tag{2.158}$$

and an extra coordinate y.[13]

$$E = \frac{1}{2}y^2, \qquad E^\alpha = y\,\xi^\alpha, \qquad J^a = -\frac{1}{2}\lambda^a{}_{\alpha\beta}\xi^\alpha\xi^\beta. \tag{2.159}$$

The quadratic Casimir operator of the Lie algebra $\mathfrak{h}$ is

$$\mathcal{C}_2(\mathfrak{h}) = \eta_{ab} J^a J^b, \tag{2.160}$$

where η_{ab} is the Killing metric of the subgroup H, which is isomorphic to the automorphism group of the underlying FTS $\mathcal{F}$. The quadratic Casimir $C_2(\mathfrak{h})$ is equal to the quartic invariant of H in the representation ρ modulo an additive constant that depends on the normal ordering chosen, namely

$$I_4(\xi^\alpha) = S_{\alpha\beta\gamma\delta}\xi^\alpha\xi^\beta\xi^\gamma\xi^\delta = C_2(\mathfrak{h}) + \mathfrak{c}, \tag{2.161}$$

where $\mathfrak{c}$ is a constant and

$$S_{\alpha\beta\gamma\delta} := \lambda_{a(\alpha\beta}\lambda^a_{\gamma\delta)}.$$

The grade $+2$ generator F has the general form

$$F = \frac{1}{2}p^2 + \frac{\kappa(\mathcal{C}_2(\mathfrak{h}) + \mathfrak{C})}{y^2}, \tag{2.162}$$

where p is the momentum conjugate to the singlet coordinate y

$$[y, p] = i \tag{2.163}$$

and κ and $\mathfrak{C}$ are some constants depending on the Lie algebra $\mathfrak{g}$. The grade $+1$ generators are then given by commutators

$$F^\alpha = [E^\alpha, F] = i p\,\xi^\alpha + \kappa y^{-1}[\xi^\alpha, \mathcal{C}_2]. \tag{2.164}$$

If H is simple or Abelian they take the general form [73]

$$F^\alpha = i p\,\xi^\alpha - \kappa y^{-1}[2\,(\lambda^a)^\alpha{}_\beta\xi^\beta J_a + C_\rho\,\xi^\alpha], \tag{2.165}$$

[12] In this section we follow the conventions of [73]. The indices $\alpha, \beta, \ldots$ are raised and lowered with the antisymmetric symplectic metric $\Omega^{\alpha\beta} = -\Omega^{\beta\alpha}$ that satisfies $\Omega^{\alpha\beta}\Omega_{\gamma\beta} = \delta^\alpha_\gamma$ and $V^\alpha = \Omega^{\alpha\beta}V_\beta$, and $V_\alpha = V^\beta\Omega_{\beta\alpha}$.

[13] In the corresponding geometric quasiconformal realization of the group G over an $(N+1)$-dimensional space this coordinate corresponds to the singlet of H.

where C_ρ is the eigenvalue of the second order Casimir of H in the representation ρ. Furthermore, one finds

$$[E^\alpha, F^\beta] = -\Delta \Omega^{\alpha\beta} - 6\kappa (\lambda^a)^{\alpha\beta} J_a, \tag{2.166}$$

where $\Delta = -\frac{i}{2}(yp + py)$.[14]

For simple H the quadratic Casimir operator of the Lie algebra $\mathfrak{g}$ can be calculated simply [73]

$$\mathcal{C}_2(\mathfrak{g}) = J^a J_a + \frac{2\,C_\rho}{N+1}\left(\frac{1}{2}\Delta^2 + EF + FE\right) - \frac{C_\rho}{N+1}\Omega_{\alpha\beta}(E^\alpha F^\beta + F^\beta E^\alpha). \tag{2.167}$$

The quadratic Casimir of $sl(2)$ generated by E, F and Δ can be expressed in terms of the quadratic Casimir $J^a J_a$ of H:

$$\frac{1}{2}\Delta^2 + EF + FE = \kappa(J^a J_a + \mathfrak{C}) - \frac{3}{8} \tag{2.168}$$

as well as the contribution of the coset generators F^α and E^β to $\mathcal{C}_2(\mathfrak{g})$ to the Casimir of $\mathfrak{g}$

$$\Omega_{\alpha\beta}(E^\alpha F^\beta + F^\beta E^\alpha) = 8\,\kappa J^a J_a + \frac{N}{2} + \kappa C_\rho N. \tag{2.169}$$

Thus the quadratic Casimir of $\mathfrak{g}$ reduces to a c-number [73]

$$\mathcal{C}_2(\mathfrak{g}) = \mathfrak{C}\left(\frac{8\kappa C_\rho}{N+1} - 1\right) - \frac{3}{4}\frac{C_\rho}{N+1} - \frac{N}{2}\frac{C_\rho}{N+1} - \frac{\kappa C_\rho^2 N}{N+1} \tag{2.170}$$

as required by irreducibility of the minimal representation and is a general feature for minimal unitary realizations of all simple groups G obtained from their quasiconformal realizations [22, 73, 75, 76].

2.11.3 Harmonic Superspace Formulation of N = 2 Sigma Models and Minimal Unitary Representations of Their Isometry Groups

Let us now show that there is a precise mapping between the Killing potentials of the isometry group G of the sigma model in harmonic superspace and the generators of its minimal unitary realization [66]. This is best achieved by rewriting the Killing potentials in terms of $SU(2)_A$ invariant canonical variables, which are defined as

[14] In the most general case, where H is not necessarily simple or Abelian, one finds that the commutator of E^α and F^β has the same form as above.

follows

$$\sqrt{2}q^{+i}u_i^- := w, \tag{2.171}$$
$$\sqrt{2}q^{+i}u_i^+ := p^{++}. \tag{2.172}$$

The poisson brackets of q^{+i}

$$\{q^{+i}, q^{+j}\} = -\frac{1}{2}\epsilon^{ij} \tag{2.173}$$

imply that

$$\{w, p^{++}\} = -1. \tag{2.174}$$

Under the conjugation $\widetilde{\ }$ we have

$$\widetilde{\widetilde{q^{+i}}} = -q^{+i},$$
$$\widetilde{\widetilde{u_i^{\pm}}} = -u_i^{\pm},$$

which imply

$$\widetilde{\widetilde{w}} = w, \tag{2.175}$$
$$\widetilde{\widetilde{p^{++}}} = p^{++}. \tag{2.176}$$

The Hamiltonian can then be written as

$$\mathcal{L}^{(+4)} = \frac{2P^{(+4)}(Q^+)}{w^2}. \tag{2.177}$$

The $SU_A(2)$ invariant Killing potentials that generate the isometry group G are then

$$\begin{array}{lll}
\mathbf{Sp(2)}: & M^{(+4)} & := M^{++++} = K_{ij}^{++}u^{+i}u^{+j} = (p^{++})^2 - \frac{2P^{(+4)}(Q^+)}{w^2}, \\
& M^{++} & = K_{ij}^{++}(u^{+i}u^{-j} + u^{+j}u^{-i}) = wp^{++} + p^{++}w, \\
& M^0 & = K_{ij}^{++}u^{-i}u^{-j} = w^2, \\
& & \\
\mathbf{H}: & K_a^{++} & = t_{a\alpha\beta}Q^{+\alpha}Q^{+\beta}, \\
& & \\
\mathbf{G/H}\times\mathbf{Sp(2)}: & T_\alpha^{(+3)} & := T_\alpha^{+++} = K_{i\alpha}^{++}u^{+i} = -\sqrt{2}\{p^{++}Q_\alpha^+ - \frac{\partial_\alpha^- P^{(+4)}(Q^+)}{w}\}, \\
& T_\alpha^+ & = K_{i\alpha}^{++}u^{-i} = -\sqrt{2}wQ_\alpha^+.
\end{array} \tag{2.178}$$

Comparing the above Killing potentials of the isometry group G with the generators of the minimal unitary realization of G given above we have one-to-one

Table 2.7 Correspondence between the quantities of the harmonic space formulation of $N = 2$ sigma models coupled to supergravity and the operators that enter in the minimal unitary realizations of their isometry groups

σ-Model with isometry group G in HSS	Minimal unitary representation of G
w	y
p^{++}	p
$\{\,,\,\}$	$i[\,,\,]$
$Q^{+\alpha}$	ξ^α
$P^{(+4)}(Q^+)$	$I_4(\xi)$
$K^{a++} = t^a_{\alpha\beta} Q^{+\alpha} Q^{+\beta}$	$J^a = \lambda^a_{\alpha\beta}\xi^\alpha\xi^\beta$
$T^{+++}_\alpha = K^{++}_{i\alpha} u^{+i}$	F^α
$T^+_\alpha = K^{++}_{i\alpha} u^{-i}$	E^α
M^{++++}	F
M^0	E
M^{++}	Δ

correspondence between the elements of harmonic superspace (HSS) and those of minimal unitary realizations (MINREP) given in Table 2.7.

The Poisson brackets (PB) $\{,\}$ in HSS formulation go over to i times the commutator $[,]$ in the minimal unitary realization and the harmonic superfields w, p^{++} corresponding to supergravity hypermultiplet compensators, that are canonically conjugate under PB map to the canonically conjugate coordinate and momentum operators y, p. Similarly, the harmonic superfields $Q^{+\alpha}$ that form $N/2$ conjugate pairs under Poisson brackets map into the symplectic bosonic oscillators ξ^α on the MINREP side. One finds a normal ordering ambiguity in the quantum versions of the quartic invariants. The classical expression relating the quartic invariant polynomial $P^{(+4)}$ to the quadratic Casimir function in HSS formulation differs from the expression relating the quartic quantum invariant I_4 to the quadratic Casimir of H by an additive c-number depending on the ordering chosen. The consistent choices for the quadratic Casimirs for all noncompact groups of quaternionic type were given in [22, 73, 76].

Furthermore, the mapping between HSS formulation of $N = 2$ sigma models and minimal unitary realizations of their isometry groups G extends also to the equations relating the quadratic Casimir of the subgroup H to the quadratic Casimir of $Sp(2)$ subgroup and to the contribution of the coset generators $G/H \times Sp(2)$ to the quadratic Casimir of G modulo some additive constants due to normal ordering.

On the MINREP side we are working with a realization in terms of quantum mechanical coordinates and momenta, while in HSS side the corresponding quantities are classical harmonic analytic superfields. The above correspondence can be extended to the full quantum correspondence on both sides by reducing the 4D $N = 2$ σ model to one dimension and quantizing it to get a supersymmetric quantum mechanics (with eight supercharges). The bosonic spectrum of the corresponding quantum mechanics must furnish a minimal unitary representation of the isometry group, which extends to a fully supersymmetric spectrum.

2.11.4 Implications

The mapping between the formulation of $N = 2$, $d = 4$ quaternionic Kähler σ models in HSS and the minimal unitary realizations of their isometry groups reviewed above is quite remarkable. It implies that the *fundamental spectra* of the quantum $N = 2$, quaternionic Kähler σ models in $d = 4$ and their lower dimensional counterparts must fit into the minimal unitary representations of their isometry groups. The fundamental spectra consist of states created by the action of harmonic analytic superfields at a given point in analytic superspace with coordinates ζ^M on the vacuum of the theory. The above analysis shows that the states created by purely bosonic components of analytic superfields will fit into the minimal unitary representation of the corresponding isometry group. Since the analytic superfields are unconstrained, the bosonic spectrum must extend to full $N = 2$ supersymmetric spectrum (eight supercharges) by the action of the fermionic components of the superfields.

The minimal unitary representations for general noncompact groups are the analogs of the singleton representations of symplectic groups $Sp(2n, \mathbb{R})$ [73]. The singleton realizations of $Sp(2n, \mathbb{R})$ correspond to free field realizations, i.e. their generators can be written as bilinears of bosonic oscillators.[15] As a consequence the tensoring procedure becomes simple and straightforward for the symplectic groups [77]. However, in general, the minimal unitary realization of a noncompact group is "interacting" and the corresponding generators are nonlinear in terms of the coordinates and momenta, which makes the tensoring problem highly nontrivial. For the quantum $N = 2$ quaternionic Kähler σ models one then has to tensor the fundamental supersymmetric spectra with each other repeatedly to obtain the full "perturbative" spectra. The full "nonperturbative" spectra in quantum HSS will, in general, contain states that do not form full $N = 2$ supermultiplets, such as $1/2$ BPS black holes, etc.

We should also stress that the fundamental spectrum is generated by the action of analytic harmonic superfields involving an infinite number of auxiliary fields. Once the auxiliary fields are eliminated the dynamical components of the superfields become complicated nonlinear functions of the physical bosonic and fermionic fields. Therefore the "fundamental spectrum" in HSS is in general not the simple Fock space of free bosons and fermions.

The $N = 2$, $d = 4$ MESGT's under dimensional reduction lead to $N = 4$, $d = 3$ supersymmetric σ models with quaternionic Kähler manifolds $\mathcal{M}_3$ (C-map). After T-dualizing the three-dimensional theory one can lift it back to four dimension, thereby obtaining an $N = 2$ sigma model coupled to supergravity that is in the mirror image of the original $N = 2$ MESGT. In previous sections we discussed how quantizing spherically symmetric stationary BPS black hole solutions of 4D, $N = 2$ MESGTs defined by Jordan algebras naturally lead to quaternionic discrete

[15] For symplectic groups the quartic invariant vanishes and hence the minimal unitary realization becomes a free field construction.

series representations of their three dimensional U-duality groups QConf(J). The results summarized in this section suggest even a stronger result, namely, quantized solutions (spectra) of a 4D, $N = 2$ MESGT, that allow dimensional reduction to three dimensions, must fit into the minimal unitary representation of its three-dimensional U-duality group and those representations obtained by tensoring of minimal representations.

2.12 M/Superstring Theoretic Origins of $N = 2$ MESGTs with Symmetric Scalar Manifolds

Both the maximal supergravity and the exceptional $N = 2$ MESGT defined by the exceptional Jordan algebra $J_3^{\mathbb{O}}$ have exceptional groups of the E series as their U-duality symmetry groups in five, four and three dimensions. The numbers of vector fields of these two theories are the same in five and four dimensions. However, the real forms of their U-duality groups are different. In the Table 2.8 we list the U-duality groups of the $N = 2$ exceptional MESGT and those of maximal $N = 8$ supergravity. Just like the $N = 8$ supergravity the fields of the exceptional $N = 2$ MESGT could not be identified with quarks and leptons so as to obtain a unified theory of all interactions without invoking a composite scenario [5,78]. However, it was observed that unlike the maximal $N = 8$ supergravity theory one might be able to couple matter multiplets to the exceptional supergravity theory which could be identified with quarks and leptons [5,78]. These two theories have a common sector which is the $N = 2$ MESGT defined by the quaternionic Jordan algebra $J_3^{\mathbb{H}}$ [5]. The existence of the exceptional MESGT begs the question whether there exists a larger theory that can be "truncated" to both the exceptional MESGT and the maximal supergravity theory [78]. After the discovery of Green–Schwarz anomaly cancellation and the development of Calabi–Yau technology this question evolved into the question whether one could obtain the exceptional 4D and 5D MESGTs as low energy effective theories of type II superstring theory or M-theory on some exceptional Calabi–Yau manifold [8]. In fact more generally one would like to know if $N = 2$ MESGTs with symmetric target spaces can be obtained from M/superstring theory on some Calabi–Yau manifold with or without hypermultiplet couplings. The compactifications of M/Superstring theory over generic Calabi–Yau manifolds do not, in general, lead to symmetric or homogeneous scalar manifolds in the corresponding five or four-dimensional theories [79, 80]. In the mathematics literature this was posed as the question whether Hermitian symmetric spaces could arise as moduli spaces of deformations of Hodge structures of Calabi–Yau manifolds [81]. In particular, the difficulty of obtaining the exceptional moduli space $\frac{E_{7(-25)}}{E_6 \times U(1)}$ was highlighted by Gross [81].

Using methods developed earlier in [82] Sen and Vafa constructed dual pairs of compactifications of type II superstring with $N = 2, 4$ and $N = 6$ supersymmetries in $d = 4$ by orbifolding $T^4 \times S^1 \times S^1$ [83]. Remarkably, the low energy effective theory of one of the compactifications they study is precisely the magical $N = 2$

Table 2.8 U-duality groups of the exceptional $N = 2$ MESGT defined by the exceptional Jordan algebra and of the maximal $N = 8$ supergravity in five, four and three spacetime dimensions. In the last column we list the U-duality group of the $N = 2$ MESGT defined by $J_3^{\mathbb{H}}$ that is a common sector of these two theories

d	$J_3^{\mathbb{O}}$ MESGT	$N = 8$ Supergravity	$J_3^{\mathbb{H}}$ MESGT
5	$E_{6(-26)}$	$E_{6(6)}$	$SU^*(6)$
4	$E_{7(-25)}$	$E_{7(7)}$	$SO^*(12)$
3	$E_{8(-24)}$	$E_{8(8)}$	$E_{7(-5)}$

MESGT defined by the quaternionic Jordan algebra $J_3^{\mathbb{H}}$ without any hypermultiplets [84, 85]. This follows from the fact that the resulting $N = 2$ theory has the same bosonic field content as the $N = 6$ supergravity which they also obtain via orbifolding. As was shown in [5], the bosonic sector of $N = 6$ supergravity and the $N = 2$ MESGT defined by $J_3^{\mathbb{H}}$ are identical and their scalar manifold in $d = 4$ is $SO^*(12)/U(6)$.[16] This theory is self-dual with the dilaton belonging to a vector multiplet. In addition to the magical $N = 2$ MESGT defined by $J_3^{\mathbb{H}}$, Sen and Vafa gave several other examples of compactifications with $N = 2$ supersymmetry and symmetric target manifolds in $d = 4$. One is the STU model coupled to four hypermultiplets with scalar manifold

$$\mathcal{M}_V \times \mathcal{M}_H = \left[\frac{SU(1,1)}{U(1)}\right]^3 \times \frac{SO(4,4)}{SO(4) \times SO(4)}, \tag{2.179}$$

which is also self-dual. Another theory they construct leads to $N = 2$ supergravity belonging to the generic Jordan family defined by the Jordan algebra $J = \mathbb{R} \oplus \Gamma_{(1,6)}$ with the target space

$$\mathcal{M}_V = \frac{SO(6,2) \times SU(1,1)}{SO(6) \times U(1) \times U(1)}$$

which is not self-dual.

A well-known theory with a symmetric target space that descends from type II string theory is the FSHV model [86]. It is obtained by compactification on a self-mirror Calabi–Yau manifold with Hodge numbers $h^{(1,1)} = h^{(2,1)} = 11$ and has the 4D scalar manifold

$$\mathcal{M}_V \times \mathcal{M}_H = \frac{SO(10,2) \times SU(1,1)}{SO(10) \times U(1) \times U(1)} \times \frac{SO(12,4)}{SO(12) \times SO(4)} \tag{2.180}$$

corresponding to the MESGT defined by the Jordan algebra $(\mathbb{R} \oplus \Gamma_{(1,9)})$ coupled to 12 hypermultiplets.

[16] The authors of [83] appear to be unaware of this fact. It is easy to show that the resulting $N = 2$ supergravity with 15 vector multiplets can not belong to the generic Jordan family with the scalar manifold $\frac{SO(14,2)\times SU(1,1)}{SO(14)\times U(1)\times U(1)}$.

It was known for sometime (M. Günaydin and E. Sezgin, unpublished) that there exists a six-dimensional $(1,0)$ supergravity theory, which is free from gravitational anomalies, with 16 vector multiplets, 9 tensor multiplets and 28 hypermultiplets, that parametrize the exceptional quaternionic symmetric space $E_{8(-24)}/E_7 \times SU(2)$.[17] This theory reduces to the exceptional supergravity theory defined by $J_3^{\mathbb{O}}$ coupled to the hypermultiplets in lower dimensions. It has the 4D scalar manifold

$$\mathcal{M}_V \times \mathcal{M}_H = \frac{E_{7(-25)}}{E_6 \times U(1)} \times \frac{E_{8(-24)}}{E_7 \times SU(2)}$$

and in three dimensions the moduli space becomes

$$\frac{E_{8(-24)}}{E_7 \times SU(2)} \times \frac{E_{8(-24)}}{E_7 \times SU(2)}. \tag{2.181}$$

The moduli space of the FSHV model is a subspace of this doubly exceptional moduli space. The existence of an anomaly free theory in $d = 6$ that reduces to this doubly exceptional theory whose moduli space includes that of the FSHV model suggests strongly that there must exist an exceptional Calabi–Yau manifold such that M/superstring theory compactified over it yields this doubly exceptional theory as was argued in [84, 85].

Some important developments about the stringy origins of magical supergravity theories that took place after SAM 2007 is summarized in the next section.

2.13 Recent Developments and Some Open Problems

Since they were delivered, a great deal of progress has been made on the main topics covered in these lecture, which I will try to summarize briefly in this section.

We discussed at some length the proposals that four and three dimensional U-duality groups act as spectrum generating conformal and quasiconformal groups of five and four dimensional supergravity theories with symmetric scalar manifolds respectively. Applying these proposals in succession implies that three-dimensional U-duality groups should act as spectrum generating symmetry groups of the five-dimensional theories from which they descend. Several results that have been obtained recently lend further support to these proposals. The authors of [87, 88] used solution generating techniques to relate the known black hole solutions of simple five-dimensional ungauged supergravity and those of the 5D STU model to each other and generate new solutions by the action of corresponding 3D U-duality groups $G_{2(2)}$ and $SO(4,4)$, respectively. For simple ungauged 5D supergravity similar results were also obtained in [89] and for gauged 5D supergravity in [90].

[17] This theory remains anomaly free if one replaces the hypermultiplet sector with any 112 (real) dimensional quaternionic sigma model.

Attractor flows for non-BPS extremal black holes that are described by certain pseudo-Riemannian symmetric sigma models coupled to 3D supergravity was carried out in [91].[18] Extremal BPS and non-BPS black holes of supergravity theories with symmetric target spaces were also studied in [93, 94]. For a more complete up-to-date list of references on extremal BPS and non-BPS attractors and their orbit structures I refer to [47].

As we discussed above the quantization of the attractor flows of stationary spherically symmetric 4D $N = 2$ BPS black holes leads to wave-functions that form the basis of quaternionic discrete series representations of the corresponding 3D U-duality groups QConf(J). Unitary representations of two quaternionic groups of rank two, namely $SU(2,1)$ and $G_{2(2)}$, induced by their geometric quasiconformal actions were studied in [54]. The starting point of the constructions of unitary representations given in [54] are the spherical vectors with respect to the maximal compact subgroups $SU(2)\times U(1)$ and $SU(2)\times SU(2)$, respectively, under their quasiconformal actions. In a recent paper with Pavlyk [95] we gave a unified realization of three-dimensional U-duality groups of all $N = 2$ MESGTs defined by Euclidean Jordan algebras of degree 3 as spectrum generating quasiconformal groups covariant with respect to their 5D U-duality groups. The spherical vectors of quasiconformal realizations of all these groups twisted by a unitary character with respect to their maximal compact subgroups as well as their quadratic Casimir operators and their values were also given in [95]. In a subsequent paper [96] we extended these results to a unified realization of split exceptional groups $F_{4(4)}, E_{6(6)}, E_{7(7)}, E_{8(8)}$ and of $SO(n + 3, m + 3)$ as quasiconformal groups that is covariant with respect to their subgroups $SL(3, R), SL(3, R)\times SL(3, R), SL(6, R), E_{6(6)}$ and $SO(n, m)\times SO(1, 1)$, respectively, and determined their spherical vectors. We also gave their quadratic Casimir operators and determined their values in terms of ν and the dimension n_V of the underlying Jordan algebras. For $\nu = -(n_V + 2) + i\rho$ the quasiconformal action induces unitary representations on the space of square integrable functions in $(2n_V + 3)$ variables, that belong to the principle series. For special discrete values of ν the quasiconformal action leads to unitary representations belonging to the discrete series and their continuations. For the quaternionic real forms of 3D U-duality groups these discrete series representations belong to the quaternionic discrete series.

As I discussed in Sect. 2.11, the minimal unitary representations of non-compact groups are the analogs of the singleton representations of the symplectic groups $Sp(2n, \mathbb{R})$. Now $Sp(4, \mathbb{R})$ is the covering group 4D anti-de Sitter or 3D conformal group $SO(3, 2)$. For higher AdS or conformal groups the analogous representations were referred to as doubletons [97, 98]. Singleton or doubleton representations of AdS or conformal groups $SO(d, 2)$ are singular positive energy (lowest weight) unitary representations, which are in the continuation of holomorphic discrete representations. Even though they do not belong to the holomorphioc discrete series,

[18] Preliminary results of this work was reported by Li [92] in this school after these lectures were delivered.

by tensoring singletons or doubletons one can obtain the entire holomorphic discrete series representations of AdS or conformal groups. This fact lies at the heart of AdS/CFT dualities [99]. In fact, the entire K–K spectrum of IIB supergravity over $AdS_5 \times S^5$ was obtained by tensoring the CPT invariant doubleton supermultiplet repeatedly with itself in [98]. Again it was pointed out in [98] that the doubleton supermultiplet decouples from the K–K spectrum as gauge modes and its field theory is the conformally invariant $N = 4$ super Yang–Mills theory that lives on the boundary of AdS_5 which is the 4D Minkowski space. Similar results were obtained for 11-dimensional supergravity on $AdS_4 \times S^7$ [100] and on $AdS_7 \times S^4$ [97].

Similarly, the minimal unitary representations of noncompact groups of quaternionic type do not belong to the quaternionic discrete series, but are in their singular continuations [101]. I argued in Sect. 2.11 that the "fundamental spectrum" of $N = 2$ sigma models coupled to supergravity with symmetric target manifolds $G/K \times SU(2)$ must belong to the minimal unitary representation of G obtained by quantization of the geometric quasiconformal realization of G. On the other hand quantization of the geodesic motion describing the dynamics of spherically symmetric BPS black holes lead to wave functions belonging to the quaternionic discrete series representations of G [54]. This implies that by tensoring minimal unitary representations one should be able to obtain the quaternionic discrete series representations. Decomposition of tensor products of minimal unitary representations into its irreducible pieces is much harder for general noncompact groups since their minimal representations are, in general, not of the lowest weight type and their realizations are non-linear (interacting)! This problem is currently under investigation [102].

Several novel results were obtained regarding the stringy origins of magical supergravity theories since SAM 2007. In [103] the hyper-free $N = 2$ string models based on asymmetric orbifolds with $N = (4, 1)$ world-sheet superconformal symmetry using 2D fermionic construction were given. Among these models two of them correspond to two of the magical supergravity theories, namely the $J_3^{\mathbb{C}}$ MESGT with the moduli space

$$\mathcal{M}_4 = \frac{SU(3,3)}{SU(3) \times SU(3) \times U(1)}$$

and the $J_3^{\mathbb{H}}$ theory with the moduli space

$$\mathcal{M}_4 = \frac{SO^*(12)}{U(6)}.$$

In [104], Bianchi and Ferrara reconsidered the string derivation of FSHV model over the Enriques Calabi–Yau and argued that the exceptional supergravity theory defined by octonionic $J_3^{\mathbb{O}}$ admits a string interpretation closely related to the Enriques model. They show that the uplift of the exceptional MESGT to $d = 6$ has 16 Abelian vectors which is related to the rank of Type I and heterotic strings.

In mathematics literature, Todorov gave a construction of Calabi–Yau n-folds whose moduli spaces are locally symmetric spaces [105]. This family of CY manifolds of complex dimension n are double covers of the projective n dimensional spaces ramified over $2n+2$ hyperplanes. They have the Hodge numbers

$$h^{(n-p,p)} = \binom{n}{p}^2$$

and have the second Betti numbers

$$b_2 = \binom{(2n+2)}{2} + 1.$$

Type IIB superstring theory compactified over such a CY threefold constructed by Todorov leads to the magical $N = 2$ MESGT defined by the complex Jordan algebra $J_3^{\mathbb{C}}$ with vector moduli space

$$\mathcal{M}_4 = \frac{SU(3,3)}{SU(3)\times SU(3)\times U(1)}$$

coupled to $(h^{(1,1)}+1) = 30$ hypermultiplets. Todorov promises a sequel to his paper in which he would show that there are no instanton corrections to the moduli space of this Calabi–Yau manifold.

The fact that some $N = 2$ MESGTs with symmetric target spaces, in particular the magical supergravity theories, can be obtained from M/Superstring theory is of utmost importance. It implies that the corresponding supergravity theories have quantum completions and discrete arithmetic subgroups of their U-duality groups will survive at the non-perturbative regime of M/Superstring theory [3]. The relevant unitary representations of the spectrum generating symmetry groups will then be some "automorphic representations". If and how some of the results obtained for continuous U-duality groups extend to their discrete subgroups is a wide open problem.

Another open problem is the quantization of 4D, $N = 2$ sigma models that couple to supergravity in harmonic superspace and their lower dimensional descendents.

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Chapter 3
Attractors, Black Holes and Multiqubit Entanglement

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Abstract Recently a striking correspondence has been established between quantum information theory and black hole solutions in string theory. For the intriguing mathematical coincidences underlying this correspondence the term "Black Hole Analogy" has been coined. The basic correspondence of the analogy is the one between the entropy formula of certain stringy black hole solutions on one hand and entanglement measures for qubit and qutrit systems on the other. In these lecture notes we develop the basic concepts of multiqubit entanglement needed for a clear exposition of the Black Hole Analogy. We show that using this analogy we can rephrase some of the well-known results and awkward looking expressions of supergravity in a nice form by employing some multiqubit entangled states depending on the quantized charges and the moduli. It is shown that the attractor mechanism in this picture corresponds to a distillation procedure of highly entangled graph states at the black hole horizon. As a further insight we also find a very interesting connection between error correcting codes, designs and the classification of extremal BPS and non-BPS black hole solutions.

3.1 Introduction

In a recent series of papers [1–5] some interesting multiple relations have been established between quantum information theory and the physics of stringy black hole solutions. The activity in this field has started with the observation of Duff [1] that the macroscopic black hole entropy for the BPS STU model can be expressed as an entanglement invariant characterizing three-qubit entanglement. Later Kallosh and Linde [2] have shown that for this model the different classes of black hole solutions correspond to the so called stochastic local operations and classical

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communication (SLOCC) classes of entanglement types characterizing three-qubit entanglement. As a next step in [3] it has been shown that the well-known process of moduli stabilization based on the supersymmetric attractor mechanism [6–9] in the entanglement picture corresponds to a distillation procedure of a GHZ-like state with maximum tripartite entanglement. This nice correspondence is based on the similar symmetry properties of the corresponding physical systems. For the STU model the symmetry group in question is $SL(2,\mathbf{R})^{\otimes 3}$ coming from the structure of the moduli space $SL(2,\mathbf{R})/U(1) \times SL(2,\mathbf{R})/U(1) \times SL(2,\mathbf{R})/U(1)$, and for the quantum information theoretic scenario it is the group $SL(2,\mathbf{C})^{\otimes 3}$ related to the SLOCC group $GL(2,\mathbf{C})^{\otimes 3}$. Due to the very special structure of the STU model at first sight it seems that this black hole analogy should run out of steam for black hole solutions corresponding to moduli spaces not exhibiting a product structure. However, later work [4, 5, 10] originating from the insight of Kallosh and Linde [2] revealed that the black hole entropy as a function of the 56 charges (28 electric and 28 magnetic) expressed in terms of Cartan's quartic invariant occurring in the context of $N = 8$, $d = 4$ supergravity [11] with moduli space $E_{7(7)}/SU(8)$ can be understood as a special type of entangled system of seven qubits based on the discrete geometry of the Fano plane consisting of seven points and seven lines. Moreover, different types of consistent truncations of this $N = 8$, $d = 4$ model in the entanglement picture can be understood via restriction to entangled subsets of the Fano plane. For example the STU model arising as a consistent truncation with eight charges (four electric and four magnetic) is obtained by keeping merely one point of the Fano plane.

Based on recent results in the mathematics literature [12, 13] it was conjectured [5] that discrete geometric structures associated with the exceptional groups occurring in the magic square of Freudenthal and Tits might show up in other entangled systems which in turn can provide interesting connections to magic supergravities. In this context see the interesting paper of Duff and Ferrara [14] stretching the validity of the black hole analogy to the realm of black hole solutions in $d = 5$ based on the group $E_{6(6)}$ connected to the bipartite entanglement of three qutrits.

Interestingly these striking mathematical coincidences can be related to classical error correcting codes. For example the possibility for extending the black hole analogy further to the $N = 8$, $d = 4$ context is intimately connected to the classical $(7, 3, 1)$ Hamming code. This is a code consisting of 16 codewords of 7 bits, encoding 4 message bits. The code, capable of correcting 1 bit flip errors, is connected to the incidence structure of the Fano plane. One of the aims of these lecture notes is to show that all the relevant information concerning the structure of the $E_{7(7)}$ symmetric black hole entropy formula and the corresponding U-duality transformations can nicely be derived from the properties of the Hamming code [5].

Classical error correcting codes have their corresponding quantum counterparts. In the rapidly evolving field of quantum information theory the theory of quantum error correcting codes based on some well-known classical ones is under intense scrutiny these days. Hence it is an interesting idea to check whether the elements of quantum error correction can be found in the black hole context. In our recent paper [15] on the structure of BPS and non-BPS extremal black hole solutions in the

STU model we have shown that a formalism based on the idea of error correction can really give some additional insights. We have shown that the black hole potential can be expressed as one-half the norm of a suitably chosen three-qubit entangled state containing the quantized charges and the moduli. The extremization of the black hole potential in terms of this entangled state amounts to either suppressing bit flip errors (BPS-case) or allowing very special types of flips transforming the states between different classes of non-BPS solutions. We have illustrated our results for the example of a D2–D6 system. In this case the bit flip errors were corresponding to sign flip ones of the charges originating from the number of D2 branes. It turned out that after moduli stabilization the states depending entirely on the charges are maximally entangled graph-states (of the triangle graph) well-known from quantum information theory.

The aim of these lecture notes is twofold. First we would like to present a detailed derivation of these results. Second we would like to introduce the reader having expertise in the field of stringy black hole solutions also to the existing theory of multipartite quantum entanglement. In order to achieve this task we start by carefully developing the basic concepts of multiqubit entanglement needed for a clear exposition of the black hole analogy. We presented slightly more results from quantum information theory than needed for understanding the applications within the realm of stringy black holes. These extra results might serve as a good starting point for the reader to get some inspiration for developing the analogy further. We hope that these entanglement based observations provide some additional insight into the development of this interesting field.

3.2 Bipartite Systems

3.2.1 Pure States

In this section we start reviewing the basic properties of multipartite entangled systems. The standard references for the material reviewed here are of Nielsen and Chuang [16] and Bengtsson and Zyczkowski [17].

Let us consider a quantum system consisting of two subsystems A and B. Moreover, let us suppose that the systems have been in contact for some time and the interactions resulted in a joint system AB characterized by a complete set of properties. Let us now look at the subsystems A and B *inside* AB. If it is impossible to assign our complete set of properties also to both constituents individually then we say the system AB is *entangled*. Alternatively, we can define a composite system to be *not entangled* if and only if it is possible for both constituents to posses the complete set of properties. After giving a precise mathematical meaning to this physical situation it can be shown that this definition for a composite system of *distinguishable particles* being *not entangled* is equivalent to the separability of the wave function representing the composite system in Hilbert space. More precisely, let us consider the situation where the Hilbert spaces representing systems A and B are $\mathcal{H}_\mathrm{A}$ and $\mathcal{H}_\mathrm{B}$ with dimensions M and N respectively. Then the physical state of the combined system AB is generally represented as

$$|\Psi\rangle = \sum_{a=1}^{M}\sum_{b=1}^{N} \Psi_{ab}|a\rangle_\mathrm{A} \otimes |b\rangle_\mathrm{B} \in \mathcal{H}_\mathrm{A} \otimes \mathcal{H}_\mathrm{B}, \tag{3.1}$$

where the states $|a\rangle_\mathrm{A}$ and $|b\rangle_\mathrm{B}$ form an orthonormal base for $\mathcal{H}_\mathrm{A}$ and $\mathcal{H}_\mathrm{B}$ respectively. If it is possible to find two states $|\psi\rangle \in \mathcal{H}_\mathrm{A}$ and $|\varphi\rangle \in \mathcal{H}_\mathrm{B}$ such that

$$|\Psi\rangle = |\psi\rangle \otimes |\varphi\rangle \tag{3.2}$$

then we say that the system AB is *not entangled* or *separable*.

How to find a criteria for checking whether a bipartite state such as (3.1) is entangled or not? A standard way of checking is based on the *Schmidt decomposition*. The main idea is that using the singular value decomposition of the $M \times N$ matrix Ψ_{ab} it is possible to find a new set of basis vectors for $\mathcal{H}_\mathrm{A}$ and $\mathcal{H}_\mathrm{B}$ such that

$$|\Psi\rangle = \sum_{i=1}^{\min\{M,N\}} r_i|i\rangle_\mathrm{A} \otimes |i\rangle_\mathrm{B}. \tag{3.3}$$

The *positive real numbers* r_i, $i = 1, 2, \ldots \min\{M, N\}$ are the Schmidt coefficients of the state. If the number of nonzero Schmidt coefficients is strictly greater than 1 then $|\Psi\rangle$ is entangled otherwise it is separable.

It is instructive to recast the mathematical process of finding the Schmidt form of a particular bipartite state in the language of *reduced density matrices*. The density matrix corresponding to the pure state of (3.1) is $|\Psi\rangle\langle\Psi|$. One can then define the reduced density matrices ϱ_A and ϱ_B as follows

$$\varrho_\mathrm{A} = \mathrm{Tr}_\mathrm{B}|\Psi\rangle\langle\Psi\rangle, \qquad \varrho_\mathrm{B} = \mathrm{Tr}_\mathrm{A}|\Psi\rangle\langle\Psi\rangle, \tag{3.4}$$

where $\mathrm{Tr}_{\mathrm{A,B}}$ refers to taking the partial trace with respect to the corresponding subsystem. Using these definitions one can check that the matrix representation of ϱ_A and ϱ_B in the basis $|a\rangle_\mathrm{A}$ and $|b\rangle_\mathrm{B}$ is

$$\varrho_{\mathrm{A}aa'} = (\Psi\Psi^\dagger)_{aa'}, \qquad \bar{\varrho}_{\mathrm{B}bb'} = (\Psi^\dagger\Psi)_{bb'}, \tag{3.5}$$

where the overline refers to complex conjugation. From this form it is clear that the nonzero eigenvalues of ϱ_A and ϱ_B are the same moreover, the square roots of these nonnegative eigenvalues (their number is precisely $\min\{M, N\}$) gives the Schmidt coefficients r_i. The new basis vectors (the Schmidt basis) are obtained by finding the eigenvectors of ϱ_A and ϱ_B equivalent to finding the *local unitary* transformations $U \in U(M)$ and $V \in U(N)$ transforming the corresponding reduced density matrices to the diagonal form. Notice that the basis vectors $|i\rangle_\mathrm{A}$ and $|i\rangle_\mathrm{B}$ appearing in (3.3) are merely the ones from this new set of basis vectors which correspond to the common nonzero eigenvalues of $\varrho_{\rm A}$ and $\varrho_{\rm B}$.

From this two important results follow. First, a pure bipartite state $|\Psi\rangle$ is entangled if and only if the corresponding reduced density matrices are mixed (i.e. their ranks are greater than 1). Alternatively $|\Psi\rangle$ is separable if and only if the reduced density matrices representing the states of subsystems A and B inside AB are pure, i.e. are of rank 1. As a second result, since the ranks of $\varrho_{\rm A}$ and $\varrho_{\rm B}$ and the particular values of r_i are invariant under local unitary transformations we have learnt that the entanglement properties of a bipartite system are invariant under local unitary transformations of the form

$$|\Psi\rangle \mapsto (U \otimes V)|\Psi\rangle, \qquad U \otimes V \in U(M) \times U(N). \tag{3.6}$$

Hence the classification of different entanglement types of bipartite systems is effected by calculating the *Schmidt rank*, i.e. the rank of either of the reduced density matrices. States $|\Psi\rangle$ and $|\Phi\rangle$ with different Schmidt ranks belong to inequivalent entanglement classes under the group of local unitary transformations $U(M)\times U(N)$. Moreover, states with a fixed rank but different values of r_i indicate further refinement of different entanglement types.

The next question is: given the numbers r_i how to quantify bipartite entanbreak glement? It turns out that a useful measure of bipartite entanglement is the von-Neumann entropy of $|\Psi\rangle$ defined as follows:

$$S(\Psi) = -{\rm Tr}\varrho_{\rm A}\log_2\varrho_{\rm A} = -{\rm Tr}\varrho_{\rm B}\log_2\varrho_{\rm B} = -\sum_{i=1}^{\min\{M,N\}} r_i^2\log_2 r_i^2. \tag{3.7}$$

Alternatively one can define a whole class of entanglement measures the Rényi entropies as (since the nonzero eigenvalues of $\varrho_{\rm A}$ and $\varrho_{\rm B}$ are the same in the following we leave the subscripts from ϱ)

$$S_\alpha(\Psi) = \frac{1}{1-\alpha}\log_2{\rm Tr}\varrho^\alpha, \qquad \alpha = 1, 2, \ldots. \tag{3.8}$$

Notice that $\lim_{\alpha\to 1} S_\alpha(\Psi) = S(\Psi)$, i.e. the limit $\alpha \to 1$ corresponds to the von-Neumann entropy. Note, that sometimes the quantities ${\rm Tr}\varrho^\alpha$, $\alpha = 2, 3\ldots$ are also used to quantify bipartite entanglement. For an $N\times N$ density matrix particularly important is the quantity $\frac{N}{N-1}(1-{\rm Tr}\varrho^2)$ which is called the *concurrence* squared. Since for a pure state $\varrho^2 = \varrho$ then regarding ϱ as the reduced density matrix coming from a bipartite state $|\Psi\rangle$ the concurrence can be used as a measure of entanglement. Obviously for separable states the concurrence is zero. For unnormalized states the concurrence is

$$\mathcal{C}^2 \equiv \frac{N}{N-1}\left[({\rm Tr}\varrho)^2 - {\rm Tr}\varrho^2\right]. \tag{3.9}$$

The simplest of all bipartite systems is the one of two qubits. In this case $\mathcal{H}_{\rm A} \simeq \mathcal{H}_{\rm B} = \mathbf{C}^2$, and the matrix Ψ_{ab} of (3.1) is a 2×2 one. In this case

$$|\Psi\rangle = \sum_{a,b=0,1} \Psi_{ab}|ab\rangle, \quad \text{where} \quad |ab\rangle = |a\rangle \otimes |b\rangle \in \mathbf{C}^2 \otimes \mathbf{C}^2. \tag{3.10}$$

Since for 2×2 matrices we have $(\mathrm{Tr}M)^2 - \mathrm{Tr}M^2 = 2\mathrm{Det}M$ then using (3.5), and (3.9) with $N = 2$ we have

$$\mathcal{C}^2 = 4\mathrm{Det}\varrho_{\mathrm{A}} = 4\mathrm{Det}\varrho_{\mathrm{B}} = |2\mathrm{Det}\Psi|^2. \tag{3.11}$$

For normalized states ($\mathrm{Tr}\varrho_{\mathrm{A}} = \mathrm{Tr}\varrho_{\mathrm{B}} = 1$) it is easy to show that

$$0 \le \mathcal{C}(\Psi) = 2|\mathrm{Det}\Psi| \le 1. \tag{3.12}$$

For separable states like $\frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle) = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, $\mathcal{C} = 0$, and for *maximally entangled states*, like the Bell-state $\frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$ we have $\mathcal{C} = 1$. Notice that in the first case $\varrho_{\mathrm{A}} = \varrho_{\mathrm{B}} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ hence $\varrho^2 = \varrho$, i.e. they are pure states, and in the second $\varrho_{\mathrm{A}} = \varrho_{\mathrm{B}} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ hence they are maximally mixed ones.

By diagonalizing the reduced density matrices it is straightforward to show that the two Schmidt coefficients are

$$r_{\pm}^2 = \frac{1}{2}(1 \pm \sqrt{1 - \mathcal{C}(\Psi)^2}), \tag{3.13}$$

and the von-Neumann entropy is

$$0 \le S(\Psi) = -r_+^2 \log_2 r_+^2 - r_-^2 \log_2 r_-^2 \le 1. \tag{3.14}$$

Since under the local unitary group $U(2) \times U(2)$ the matrix Ψ transforms as $\Psi \mapsto U\Psi V^{\mathrm{T}}$ it is obvious that the concurrence $\mathcal{C} = 2|\mathrm{Det}\Psi|$ is left invariant as it has to be. Intuitively it is clear that entanglement should not be changed under *local* manipulations. Hence being invariant under local unitaries the concurrence, the von-Neumann entropy (and similarly the Rényi entropies) are good entanglement measures for the two-qubit system.

From the previous discussion we have learnt that the Schmidt rank is invariant under the local unitary group $U(M) \times U(N)$, hence using such transformations separable states cannot be converted to entangled ones and vice versa. However, this property is also preserved by a more general set of local transformations. Such transformations are of the form $GL(M, \mathbf{C}) \times GL(N, \mathbf{C})$, i.e. they are the complex linear invertible transformations. Although these transformations are not preserving the norm of entangled states, but in spite of this they still have found their important place in quantum information theory. The transformations of the form $GL(N, \mathbf{C}) \times$

$GL(M, \mathbf{C})$ are called transformations of *stochastic local operations and classical communication* or SLOCC transformations in short.

The motivation for their occurrence in quantum information theory is as follows. Let us imagine the subsystems A and B of the combined system AB characterized by the entangled state $|\Psi\rangle$ are spatially separated but the entanglement between them is still preserved. Now A and B can exploit their entanglement by using local manipulations, moreover they can inform each other on the results of their local actions via using classical communication channels (e.g. ordinary phones). The quantum protocols arising in this way can be used for the processing of quantum information. The most spectacular example of this kind is quantum teleportation of an unknown state from A to B. In a particular type of protocols one is interested in local manipulations effected by A and B that can convert an arbitrary state $|\Psi\rangle$ to another prescribed one $|\Phi\rangle$ not with certainty but with some probability of success in either direction. Such transformations are not necessarily unitary but they are certainly invertible. These transformations supplemented by the usual classical channels are the ones of SLOCC transformations. Of course measurements being projectors are excluded from the set of SLOCC transformations. In order to implement such transformations one imagines A and B having their corresponding *local* environments E_{A} and E_{B}. Under the unitary evolutions of E_{A} and E_{B} corresponding to some local manipulations the embedded systems A and B are not necessarily evolve unitarily and they can give rise to local $GL(M, \mathbf{C}) \times GL(N, \mathbf{C})$ transformations.

The entanglement classes under the SLOCC group for two qubits are very simple. We have merely *two* classes: the separable class of rank 1 and the entangled one of rank 2. These classes are characterized by the conditions $\mathcal{C} = 0$ and $\mathcal{C} \neq 0$ respectively since the relevant SLOCC group in this case is $GL(2, \mathbf{C}) \times GL(2, \mathbf{C})$ acting in the form

$$|\Psi\rangle \to (\mathcal{A}\otimes\mathcal{B})|\Psi\rangle, \qquad \Psi \mapsto \mathcal{A}\Psi\mathcal{B}^{\mathrm{T}}, \qquad \mathcal{A}\otimes\mathcal{B} \in GL(2, \mathbf{C})\times GL(2, \mathbf{C}), \quad (3.15)$$

can change the *nonzero* values of $\mathcal{C}$, but they cannot change the rank. By comparison under the action of the subgroup of transformations of local unitaries $U(2)\times U(2)$ we have infinitely many classes labelled by different values of $\mathcal{C}$ that are left invariant. A very important subgroup of SLOCC transformations is the one of determinant one transformations, i.e. $SL(2, \mathbf{C}) \times SL(2, \mathbf{C})$. They are obviously leaving invariant the concurrence $\mathcal{C}$.

3.2.2 Mixed States

Having discussed the entanglement properties of bipartite pure states now we turn to the problem of mixed state entanglement. Mixed states for a bipartite system AB are characterized by a density matrix $\varrho : \mathcal{H}_{\mathrm{A}} \otimes \mathcal{H}_{\mathrm{B}} \to \mathcal{H}_{\mathrm{A}} \otimes \mathcal{H}_{\mathrm{B}}$ which is Hermitian, positive semidefinite and having trace one. There are many possible ways of expressing a particular ϱ as a mixture of pure states. A general decomposition is of the form

$$\varrho = \sum_{j=1}^{K} s_j |\chi_j\rangle\langle\chi_j|, \qquad \text{where} \qquad |\chi_j\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B, \qquad \sum_{j=1}^{K} s_j = 1, \quad (3.16)$$

with nonnegative real numbers s_j, and the states $|\chi_j\rangle$, $j = 1, 2, \ldots K$ are normalized but not necessarily orthogonal. Moreover, K can even be greater than NM the dimension of $\mathcal{H}_A \otimes \mathcal{H}_B$. Of course being a Hermitian matrix, among decompositions of the (3.16) form we have the canonical one

$$\varrho = \sum_{i=1}^{MN} \lambda_i |\psi_i\rangle\langle\psi_i|, \qquad \text{where} \qquad |\psi_i\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B, \quad \sum_{i=1}^{MN} \lambda_i = 1, \quad (3.17)$$

in terms of the nonnegative eigenvalues λ_i and eigenvectors $|\psi_i\rangle$ now spanning an orthonormal basis. This non uniqueness of the density matrix ϱ can be expressed in a precise way using the mixture theorem [18,19] stating that a density matrix having the canonical form of (3.17) can also be written in the form of (3.16) *if and only if* there exists a $K \times K$ unitary matrix $\mathcal{U}$ such that

$$|\chi_j\rangle = \frac{1}{\sqrt{s_j}} \sum_{i=1}^{MN} \mathcal{U}_{ji} \sqrt{\lambda_i} |\psi_i\rangle. \quad (3.18)$$

Given ϱ this theorem provides all the possible ways in which our mixed state density matrix can be expressed as an ensemble of pure states. Observe that the matrix $\mathcal{U}$ is not acting on the Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$ but on vectors whose components are state vectors. Notice also that in (3.18) only the first *MN* columns of $\mathcal{U}$ appear, the remaining $K - NM$ columns are merely added to be able to refer to the matrix as an unitary one. This theorem tells us that the pure states that can make up the ensemble for ϱ are linearly dependent on the *MN* vectors that make up the eigenensemble.

For a bipartite system AB it is clear that the vectors $|\chi_j\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ that make up the ensemble for ϱ can both be entangled or separable pure states. We adopt the definition that ϱ is entangled if it cannot be represented as a mixture of separable pure states. A measure of mixed state entanglement which will be of some interest for us is the entanglement of formation $E_f(\varrho)$ defined as

$$E_f(\varrho) = \inf \sum_{j=1}^{K} s_j S(\chi_j), \quad (3.19)$$

where S is the von-Neumann entropy as defined in (3.7) and the infimum is taken over all pure-state ensembles for ϱ.

Let us now specialize again to the important case of two qubits, and define the Wootters spin flip $\tilde{\varrho}$ of our 4×4 density matrix $\varrho : \mathbf{C}^2 \otimes \mathbf{C}^2 \to \mathbf{C}^2 \otimes \mathbf{C}^2$ as

$$\tilde{\varrho} \equiv \varepsilon \otimes \varepsilon \varrho^{\mathrm{T}} \varepsilon \otimes \varepsilon, \qquad \varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{3.20}$$

Define the generalization of the concurrence $\mathcal{C}(\varrho)$ for mixed states as [20]

$$0 \leq \mathcal{C}(\varrho) \equiv \max\{0, \Lambda_1 - \Lambda_2 - \Lambda_3 - \Lambda_4\} \leq 1, \tag{3.21}$$

where $\Lambda_1 \geq \Lambda_2 \geq \Lambda_3 \geq \Lambda_4$ are the square roots of the eigenvalues of the matrix $\tilde{\varrho}\varrho$. (It turns out that the eigenvalues of $\varrho\tilde{\varrho}$ are nonnegative real numbers.) Now one can prove that

$$E_f(\varrho) = -R_+^2 \log_2 R_+^2 - R_-^2 \log_2 R_-^2, \qquad R_\pm^2 = \frac{1}{2}\left(1 \pm \sqrt{1 - \mathcal{C}(\varrho)}\right). \tag{3.22}$$

Hence states with $\mathcal{C}(\varrho) = 0$ ($E_f(\varrho) = 0$) are separable and states with $\mathcal{C}(\varrho) \neq 0$ ($E_f(\varrho) \neq 0$) are entangled. (Compare (3.22) with the pure state result of (3.13)–(3.14).) One can show that specifying to a pure state $\varrho = |\Psi\rangle\langle\Psi|$ (3.21) gives back the pure state concurrence $\mathcal{C}(\Psi)$, and the entanglement of formation in this case is $E_f(\Psi) = S(\Psi)$.

As an example let us consider the two-qubit mixed state density matrix

$$\varrho = \frac{1}{4}(1 - x)I + x|\Phi\rangle\langle\Phi| = \frac{1}{4}\begin{pmatrix} 1+x & 0 & 0 & 2x \\ 0 & 1-x & 0 & 0 \\ 0 & 0 & 1-x & 0 \\ 2x & 0 & 0 & 1+x \end{pmatrix},$$
$$|\Phi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle), \tag{3.23}$$

where $x \in [0, 1]$. Then one calculates $\Lambda_1 = (1 + 3x)/4$ and $\Lambda_2 = \Lambda_3 = \Lambda_4 = (1 - x)/4$ hence this density matrix is separable for $x \leq 1/3$ and for $x > 1/3$ it is entangled. Moreover, for the entangled states the value of their concurrence is $\mathcal{C}(\varrho) = (3x - 1)/2$. The maximum value is achieved for $x = 1$ which is a pure state $|\Phi\rangle\langle\Phi|$. Notice also that the maximally mixed state ϱ with $x = 0$ is separable. Hence in this case mixedness and entanglement are complementary notions.

3.2.3 Real Bipartite Pure States

We will see that in the black hole analogy the states we are interested in are real ones, meaning that they are either having real (integer) amplitudes, or they are complex states local unitary equivalent to ones with real amplitudes.

First we discuss real two-qubit states as the ones

$$|\Psi\rangle = \sum_{A,B=0}^{1} \Psi_{AB}|AB\rangle, \qquad \text{with} \qquad \begin{pmatrix} \Psi_{00} & \Psi_{01} \\ \Psi_{10} & \Psi_{11} \end{pmatrix} \in M(2,\mathbf{R}). \tag{3.24}$$

These are called "rebits" in the literature [21]. Notice that we have switched to the comfortable labelling scheme for the amplitudes as Ψ_{AB} where the first index refers to the first subsystem (Alice) and the second index to the second one (Bob). Let us choose a basis in $M(2,\mathbf{R})$ regarded as a vector space over $\mathbf{R}$ as follows

$$E_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad E_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3.25}$$

Then we can write

$$\begin{pmatrix} \Psi_{00} & \Psi_{01} \\ \Psi_{10} & \Psi_{11} \end{pmatrix} = \sum_{m=0}^{3} \Psi_m \frac{1}{\sqrt{2}} E_m, \tag{3.26}$$

where

$$\begin{aligned} \Psi_0 &= \frac{1}{\sqrt{2}}(\Psi_{00} + \Psi_{11}), \quad \Psi_1 = \frac{1}{\sqrt{2}}(\Psi_{01} - \Psi_{10}), \\ \Psi_3 &= \frac{1}{\sqrt{2}}(\Psi_{00} - \Psi_{11}), \quad \Psi_2 = \frac{1}{\sqrt{2}}(\Psi_{01} + \Psi_{10}). \end{aligned} \tag{3.27}$$

Alternatively one can write

$$|\Psi\rangle = \sum_{m=0}^{3} \Psi_m |m\rangle, \tag{3.28}$$

where $|m\rangle, m = 0,1,2,3$ defines the so called Bell basis, e.g. $|2\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$ e.t.c. Notice that (3.24) and (3.28) give rise to an expansion with respect to a separable and entangled basis respectively.

Motivated by (3.20) let us now define for an arbitrary element M of $M(2,\mathbf{R})$ the Wootters spin flip operation as

$$\tilde{M} \equiv -\varepsilon M^{\mathrm{T}} \varepsilon = (\mathrm{Det} M) M^{-1}. \tag{3.29}$$

Then we can write the formula for the concurrence of (3.11) as

$$\mathcal{C} = 2|\mathrm{Det}\Psi| = |\mathrm{Tr}(\tilde{\Psi}\Psi)| = |\mathrm{Tr}(\Psi\varepsilon\Psi^{\mathrm{T}}\varepsilon)|, \tag{3.30}$$

where

$$\mathrm{Tr}(\tilde{\Psi}\Psi) = \varepsilon^{A_1A_2}\varepsilon^{B_1B_2}\Psi_{A_1B_1}\Psi_{A_2B_2}$$
$$= \begin{pmatrix}\Psi_{00} & \Psi_{01} & \Psi_{10} & \Psi_{11}\end{pmatrix}\begin{pmatrix}0 & 0 & 0 & 1\\ 0 & 0 & -1 & 0\\ 0 & -1 & 0 & 0\\ 1 & 0 & 0 & 0\end{pmatrix}\begin{pmatrix}\Psi_{00}\\ \Psi_{01}\\ \Psi_{10}\\ \Psi_{11}\end{pmatrix}. \tag{3.31}$$

For two rebits with amplitudes a_{AB} and b_{AB} this motivates the definition of the bilinear form

$$\cdot\ :\ M(2,\mathbf{R})\times M(2,\mathbf{R}) \to \mathbf{R} \tag{3.32}$$
$$(a,b) \mapsto a\cdot b = \sum_{\mu\nu=0}^{3} g_{\mu\nu}a^{\mu}b^{\nu} = \varepsilon^{A_1A_2}\varepsilon^{B_1B_2}a_{A_1B_1}b_{A_2B_2},$$

where $a^{\mu} \equiv (a^0, a^1, a^2, a^3)^{\mathrm{T}} = (a_{00}, a_{01}, a_{10}, a_{11})^{\mathrm{T}}$, and $g_{\mu\nu}$ corresponds to the matrix $\varepsilon\otimes\varepsilon$ with the explicit form as appearing in (3.31).

The analogue of the SLOCC subgroup $SL(2,\mathbf{C})\times SL(2,\mathbf{C})$ for the real case is $SL(2,\mathbf{R})\times SL(2,\mathbf{R})$ acting on the amplitudes as

$$\Psi \mapsto \mathcal{A}\Psi\mathcal{B}^{\mathrm{T}}, \qquad \mathcal{A}\otimes\mathcal{B} \in SL(2,\mathbf{R})\times SL(2,\mathbf{R}). \tag{3.33}$$

Since $\mathcal{A}^{\mathrm{T}}\varepsilon\mathcal{A} = \varepsilon$ the bilinear form of (3.32) is obviously left invariant by the $SL(2,\mathbf{R})\times SL(2,\mathbf{R})$ action.

Notice also that

$$2\mathrm{Det}\Psi = 2(\Psi_{00}\Psi_{11} - \Psi_{01}\Psi_{10}) = \Psi_0^2 + \Psi_1^2 - \Psi_2^2 - \Psi_3^2. \tag{3.34}$$

Hence in the entangled Bell basis the SLOCC subgroup is $SO(2,2) \simeq SL(2,\mathbf{R})\times SL(2,\mathbf{R})$. Hence working in the Bell basis $|m\rangle, m = 0,1,2,3$ rather than the computational base $|AB\rangle$, $A, B = 0,1$ the bilinear form is given by the $SO(2,2)$ invariant form.

In closing this subsection we remark that the basis vectors E_{m}, $m = 0,1,2,3$ of (3.25) can be regarded as matrix representatives of the basis vectors e_0, e_1, e_2, e_3 of the algebra of split-quaternions $\mathbf{H}_s$. Then an arbitrary element $x \in \mathbf{H}_s$ corresponding to an element $X \in M(2,\mathbf{R})$ can be expressed as $\sum_{m=0}^{3} x_m e_m$. In this picture the Wootters spin flip corresponds to the conjugation in $\mathbf{H}_s$ $\bar{x} \mapsto \tilde{X}$, and the norm $N(x) = x\bar{x} \mapsto \mathrm{Det}X$ is related to the entanglement measure the concurrence. Moreover defining the bilinear form

$$\langle x, y\rangle \equiv N(x+y) - N(x) - N(y), \tag{3.35}$$

we see that it corresponds to the SLOCC invariant bilinear form of (3.32), i.e. $\langle x, y\rangle \mapsto \mathrm{Tr}(X\tilde{Y})$. This interesting correspondence between the algebra of split-quaternions and the entanglement properties of two rebits will be useful for establishing further results in the stringy black hole context.

3.2.4 Real Bipartite Mixed States

In the case of quantum mechanics with states represented by Hermitian operators acting on a finite dimensional vector space over the complex numbers we know that an arbitrary operator represented by a $K \times K$ matrix can be split into its antisymmetric and symmetric parts of dimension $K(K-1)/2$ and $K(K+1)/2$ respectively. The real vector space of Hermitian operators (representing the physical states) takes advantage of both the antisymmetric and symmetric parts since a Hermitian operator H can be written as $H = S + iA$. However, when considering operators in quantum mechanics over the real numbers physical states are represented by merely the symmetric part, i.e. $H = S$.

The consequence of this is that when combining two subsystems based on the spaces $\mathcal{H}_{\mathrm{A}}$ of dimension M and $\mathcal{H}_{\mathrm{B}}$ of dimension N, the composite space of symmetric matrices of dimension $MN(MN+1)/2$ based on the MN dimensional tensor product space has the structure

$$\mathcal{S}_{\mathrm{AB}} = (\mathcal{S}_{\mathrm{A}} \otimes \mathcal{S}_{\mathrm{B}}) \oplus (\mathcal{A}_{\mathrm{A}} \otimes \mathcal{A}_{\mathrm{B}}), \tag{3.36}$$

where $\mathcal{S}_{\mathrm{A}}$ and $\mathcal{A}_{\mathrm{A}}$, etc., refers to the symmetric (antisymmetric) subspaces respectively. Hence joint states with components in $\mathcal{A}_{\mathrm{A}} \otimes \mathcal{A}_{\mathrm{B}}$ are automatically entangled since product states cannot have a component of this form.

For the special case of a rebit, density matrices are symmetric hence they are spanned merely by the three symmetric matrices E_0, E_2 and E_3 of (3.25) corresponding to the split quaternionic basis vectors $e_{0,2,3}$. For two rebits the density matrix is a 4×4 symmetric matrix with 10 independent components which can be spanned by the 9 combinations of the form $E_\alpha \otimes E_\beta$ where $\alpha, \beta = 0, 2, 3$ belonging to $\mathcal{S}_{\mathrm{A}} \otimes \mathcal{S}_{\mathrm{B}}$ *and* the combination $E_1 \otimes E_1 = \varepsilon \otimes \varepsilon$ belonging to $\mathcal{A}_{\mathrm{A}} \otimes \mathcal{A}_{\mathrm{B}}$. Any state containing a component proportional to $\varepsilon \otimes \varepsilon$ is necessarily entangled relative to the real vector space because on the individual spaces $\mathcal{H}_{\mathrm{A}}$ and $\mathcal{H}_{\mathrm{B}}$ the states have no components proportional to ε.

These observations motivate the following definition for the mixed state concurrence for rebits

$$0 \leq \mathcal{C}(\varrho) = |\mathrm{Tr}(\varrho\varepsilon \otimes \varepsilon)| \leq 1. \tag{3.37}$$

Notice, however that this definition is *not* merely the restriction of (3.21) to real density operators. In order to show this just take the density matrix

$$\varrho = \frac{1}{4}(I \otimes I - \varepsilon \otimes \varepsilon), \tag{3.38}$$

to calculate the complex concurrence (3.20) giving zero, but the real concurrence (3.37) is giving the value one. Hence this state is separable in complex quantum mechanics but maximally entangled in its real version. The complex separability is obvious from the decomposition

$$\varrho = \frac{1}{2}\left(\frac{1}{2}(I + i\varepsilon) \otimes \frac{1}{2}(I + i\varepsilon) + \frac{1}{2}(I - i\varepsilon) \otimes \frac{1}{2}(I - i\varepsilon)\right). \tag{3.39}$$

which is of course not a legitimate mixture in the real case.

Notice also that using the notation of (3.32) the real concurrence of (3.37) can also be written in the alternative form

$$\mathcal{C}(\varrho) = |g_{\mu\nu}\varrho^{\mu\nu}| = |\varrho^{\mu}_{\mu}|. \tag{3.40}$$

It can also be shown that the real entanglement of formation, defined via a similar process as in the complex case, is also given by (3.22) with the important difference that now we have to substitute the real mixed state concurrence in the relevant formula.

3.3 Multipartite Systems

3.3.1 Pure States, Three Qubits

For multipartite systems we have n subsystems the states of which are represented by finite dimensional Hilbert spaces V_A, V_B, $V_C \ldots$, etc., with dimensions d_A, d_B, $d_\mathrm{C}, \ldots$. A vector with d components is called a *qudit*. Frequently investigated examples involve the cases of $d_\mathrm{A} = d_\mathrm{B} = d_\mathrm{C} = \cdots = 2$ (multiqubit systems) and $d_\mathrm{A} = d_\mathrm{B} = d_\mathrm{C} = \cdots = 3$ multi*qutrit* ones, or their combinations. In this paper we restrict our attention to multiqubit systems. In particular we will be interested in 3, 4 and 7 qubit systems which have some relevance to the physics of stringy black hole solutions.

Let us first discuss three-qubit systems. A pure state $|\Psi\rangle$ of three-qubits can be written in the form

$$|\Psi\rangle = \sum_{A,B,C=0}^{1} \Psi_{ABC}|ABC\rangle, \quad |ABC\rangle = |A\rangle \otimes |B\rangle \otimes |C\rangle \in V_A \otimes V_B \otimes V_C, \tag{3.41}$$

where $V_A \otimes V_B \otimes V_C = (\mathbf{C}^2)^{\otimes 3} \simeq \mathbf{C}^8$. We are interested in classifying states of different entanglement types [22]. There are two classification schemes of importance to quantum information theory. The first is based on the notion of *local unitary equivalence* and the second one on equivalence under stochastic local operations and classical communication (SLOCC). According to the first scheme

$$|\Psi\rangle \sim |\Phi\rangle \quad \text{iff} \quad |\Psi\rangle = (U \otimes V \otimes W)|\Phi\rangle, \quad U \otimes V \otimes W \in U(2)^{\otimes 3}. \tag{3.42}$$

In words: two states are equivalent iff there exist local unitary transformations transforming one state to the other. The set of equivalence classes provides one possible set of different entanglement types. However, for practical purposes a coarse graining of these entanglement classes was needed by enlarging the group from $U(2)$ to $GL(2) \equiv GL(2, \mathbf{C})$. The SLOCC equivalence classes are defined as

$$|\Psi\rangle \simeq |\Phi\rangle \quad \text{iff} \quad |\Psi\rangle = (\mathcal{A} \otimes \mathcal{B} \otimes \mathcal{C})|\Phi\rangle, \quad \mathcal{A} \otimes \mathcal{B} \otimes \mathcal{C} \in GL(2)^{\otimes 3}. \tag{3.43}$$

Clearly local unitary equivalence implies SLOCC equivalence but not in the other way.

It turns out that there are only two SLOCC entanglement types of genuine three-qubit entanglement or in other words three qubits can be entangled in two different ways. The two different SLOCC equivalence classes are represented by the states

$$|GHZ\rangle = \frac{1}{2}(|000\rangle + |111\rangle), \qquad |W\rangle = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle). \tag{3.44}$$

The first is called the Greenberger–Horne–Zeilinger (GHZ) state. This state represents states of maximal three-partite entanglement. Such states will play an important role in establishing a three-qubit interpretation of black hole solutions in the STU model.

The remaining entanglement classes describe separable states. They are either of type $A(BC)$, $B(AC)$ and $C(AB)$, i.e. they are the biseparable states, or they are completely separable states of the form $(A)(B)(C)$. For example the state

$$|A(BC)\rangle = \frac{1}{\sqrt{2}}|0\rangle(|01\rangle + |10\rangle), \tag{3.45}$$

is biseparable, and the one $|000\rangle$ is separable.

How can we understand these classes in terms of polynomial invariants that measure different types of tripartite entanglement? Based on reduced density matrices it is easy to define local unitary invariants that differentiate between different types of biseparable cases.

For this purpose let us define the reduced density matrices

$$\varrho_1 = \text{Tr}_{23}|\Psi\rangle\langle\Psi|, \qquad \varrho_2 = \text{Tr}_{13}|\Psi\rangle\langle\Psi|, \qquad \varrho_3 = \text{Tr}_{12}|\Psi\rangle\langle\Psi|, \tag{3.46}$$

where from now on we refer to subsystems A, B, C as $1, 2, 3$. These states are one-qubit mixed states represented by 2×2 matrices. However, we can also form two-qubit density matrices

$$\varrho_{23} = \text{Tr}_1|\Psi\rangle\langle\Psi|, \qquad \varrho_{13} = \text{Tr}_2|\Psi\rangle\langle\Psi|, \qquad \varrho_{12} = \text{Tr}_3|\Psi\rangle\langle\Psi|. \tag{3.47}$$

These are 4×4 matrices. As we see all these density matrices are coming from the pure state $|\Psi\rangle\langle\Psi|$. Based on the results of (3.5), (3.9) and (3.11) it is clear that the quantities $0 \leq 4\mathrm{Det}\varrho_{1,2,3} \leq 1$ are entanglement measures of biseparability. For example a three-qubit state $|\Psi\rangle$ is $1(23)$ separable if and only if $4\mathrm{Det}\varrho_1 = 0$. Hence it is useful to define the quantities

$$\tau_{1(23)} = 4\mathrm{Det}\varrho_1, \quad \tau_{2(13)} = 4\mathrm{Det}\varrho_2, \quad \tau_{3(12)} = 4\mathrm{Det}\varrho_3. \tag{3.48}$$

These measures of biseparability are obviously $U(2) \times U(4)$ invariants where $U(2)$ is acting on the qubit whose separability properties we are interested in. Notice that all three measures taken together (forming, e.g. their arithmetic mean) measure $(1)(2)(3)$ separability and having the invariance group $U(2) \times U(2) \times U(2)$.

Apart from biseparability we can also ask the question: how much 1 and 2 are entangled *within* the tripartite system 123? In order to answer this question we note that this property is described by the entanglement of formation of (3.22) or the mixed state concurrence of (3.21) of the density matrix ϱ_{12}. Since this quantity is again coming from a bipartite split $1(23)$ of the form $\mathbf{C}^2 \otimes \mathbf{C}^4$ the 4×4 density matrix ϱ_{12} can have at most two nonzero eigenvalues. (See the discussion after (3.5).) These two nonzero eigenvalues of ϱ_{12} are the same as the ones of ϱ_3. As a result when calculating the mixed state concurrence (3.21) we obtain two eigenvalues Λ_1 and Λ_2. Then one can show that the mixed state concurrence squared in this special case is

$$C_{12}^2 \equiv C^2(\varrho_{12}) = \mathrm{Tr}(\varrho_{12}\tilde{\varrho}_{12}) - 2\Lambda_1\Lambda_2. \tag{3.49}$$

It is interesting to observe at this point that according to (3.31)–(3.32) and (3.49) for the *special* mixed state coming from a three-qubit state we can symbolically write

$$C^2(\varrho) = \varrho^{\mu\nu}\varrho_{\mu\nu} - 2\Lambda_1\Lambda_2, \tag{3.50}$$

which is to be compared with the square of (3.40), i.e. $C^2(\varrho) = (\varrho_\mu^\mu)^2$ found for a *general* mixed state of two rebits.

Similarly to C_{12} one can form the quantities $C_{23} = C(\varrho_{23})$ and $C_{13} = C(\varrho_{13})$. Let us now define the *two-tangle*s as

$$\tau_{12} = C_{12}^2, \qquad \tau_{13} = C_{13}^2, \qquad \tau_{23} = C_{23}^2. \tag{3.51}$$

An important question to be asked is the following: how the quantities $\tau_{1(23)}$, τ_{12} and τ_{13} are related to each other? Can it happen for example that 1 is entangled with 23 taken together but at the same time 1 is *not* entangled with 2 and 3 individually? To a mind trained in classical relationships this question seems absurd. For example we cannot imagine a mother loving both of her kids taken together but not loving them individually. A classical tripartite state of the mother and her two kids of that kind is impossible. However, the GHZ state is precisely of this kind. Indeed a calculation shows that

$$\tau_{1(23)} = 1, \qquad \tau_{12} = \tau_{13} = 0, \tag{3.52}$$

exemplifying precisely this absurd situation. On the other hand the $|W\rangle$ state of (3.44) shows a trade-off between different types of tangles, since for this state one finds

$$\tau_{1(23)} = \tau_{12} + \tau_{13}. \tag{3.53}$$

Let us now define the *residual tangle* or three-tangle τ_{123} as the quantity

$$\tau_{123} \equiv \tau_{1(23)} - \tau_{12} - \tau_{13} = \tau_{2(13)} - \tau_{12} - \tau_{23} = \tau_{3(12)} - \tau_{13} - \tau_{23}. \tag{3.54}$$

Indeed, this definition makes sense since according to a calculation τ_{123} is permutation invariant as clearly expressed by the second and the third of the equalities of (3.54). An explicit expression for the three-tangle is given by the formula [23]

$$0 \leq \tau_{123} = 4|D(\Psi)| = 4\Lambda_1\Lambda_2 \leq 1, \tag{3.55}$$

where

$$\begin{aligned} D(\Psi) \equiv\ & \Psi_{000}^2\Psi_{111}^2 + \Psi_{001}^2\Psi_{110}^2 + \Psi_{010}^2\Psi_{101}^2 + \Psi_{011}^2\Psi_{100}^2 \\ & -2(\Psi_{000}\Psi_{001}\Psi_{110}\Psi_{111} + \Psi_{000}\Psi_{010}\Psi_{101}\Psi_{111} \\ & + \Psi_{000}\Psi_{011}\Psi_{100}\Psi_{111} + \Psi_{001}\Psi_{010}\Psi_{101}\Psi_{110} \\ & + \Psi_{001}\Psi_{011}\Psi_{110}\Psi_{100} + \Psi_{010}\Psi_{011}\Psi_{101}\Psi_{100}) \\ & \times 4(\Psi_{000}\Psi_{011}\Psi_{101}\Psi_{110} + \Psi_{001}\Psi_{010}\Psi_{100}\Psi_{111}) \end{aligned} \tag{3.56}$$

is called Cayley's hyperdeterminant [24]. From the definition of τ_{123} or from (3.52)–(3.53) one can see that

$$\tau_{123}(GHZ) = 1, \qquad \tau_{123}(W) = 0, \tag{3.57}$$

hence for the GHZ state the residual tangle is of the maximum value. From (3.54) and (3.55) for a general three-qubit state we get the Coffmann–Kundu–Wootters inequalities [23]

$$\tau_{12} + \tau_{13} \leq \tau_{1(23)}, \qquad \tau_{12} + \tau_{23} \leq \tau_{2(13)}, \qquad \tau_{13} + \tau_{23} \leq \tau_{3(12)}. \tag{3.58}$$

These inequalities describe how entanglement as a resource can be distributed between three parties. Sometimes these relations are called entanglement monogamy relations reflecting the fact that entanglement cannot be shared for free.

Now we would like to discuss the invariance properties of the three-tangle τ_{123} of (3.55). First we observe that Cayley's hyperdeterminant $D(\Psi)$ of (3.56) can be expressed in the following form

$$\begin{aligned} -2D(\Psi) = \ & \varepsilon^{A_1A_3}\varepsilon^{B_1B_2}\varepsilon^{C_1C_2}\varepsilon^{A_2A_4}\varepsilon^{B_3B_4}\varepsilon^{C_3C_4} \\ & \times \Psi_{A_1B_1C_1}\Psi_{A_2B_2C_2}\Psi_{A_3B_3C_3}\Psi_{A_4B_4C_4}. \end{aligned} \tag{3.59}$$

This form is not reflecting the permutation invariance of $D(\Psi)$ since it is of the form

$$D(\Psi) = -\frac{1}{2}\varepsilon^{A_1A_3}\varepsilon^{A_2A_4}\left(\Psi_{A_1}\cdot\Psi_{A_2}\right)\left(\Psi_{A_3}\cdot\Psi_{A_4}\right), \tag{3.60}$$

(here we have used the definition of (3.32)) where the first qubit is playing a special role. Hence a final formula for τ_{123} where qubit A is singled out is

$$\tau_{123} = 4|\left((\Psi_0\cdot\Psi_1)^2 - (\Psi_0\cdot\Psi_0)(\Psi_1\cdot\Psi_1)\right)|, \tag{3.61}$$

here $\Psi_0 \equiv \Psi_{0BC}$ and $\Psi_1 \equiv \Psi_{1BC}$. Now it is straightforward to check that the basic expression of (3.56) is permutation invariant hence our expression (3.61) is unchanged with qubit B or qubit C playing a special role (we could have used the alternative definitions $\Psi_0 \equiv \Psi_{A0C}$ and $\Psi_1 \equiv \Psi_{A1C}$ or $\Psi_0 \equiv \Psi_{AB0}$ and $\Psi_1 \equiv \Psi_{AB1}$ in (3.61)).

Our expression (3.61) clearly shows that τ_{123} is invariant under SLOCC transformations (see (3.43)) with $\mathcal{A}\otimes\mathcal{B}\otimes\mathcal{C} \in SL(2,\mathbf{C})\times SL(2,\mathbf{C})\times SL(2,\mathbf{C})$. Indeed, under transformations of the form $I\otimes\mathcal{B}\otimes\mathcal{C}$ τ_{123} is invariant due to the definition of the $\cdot$ bilinear form (see (3.32)). Moreover, (3.60) as the determinant of the 2×2 matrix $\Psi_{\mathrm{A}}\cdot\Psi_{A'}$ is clearly invariant under transformations of the form $\mathcal{A}\otimes I\otimes I$.

It is important to realize that for the full SLOCC group $GL(2,\mathbf{C})^{\otimes 3}$ the quantity τ_{123} is not invariant but picks up a factor corresponding to the determinants of the matrices $\mathcal{A},\mathcal{B},\mathcal{C}$. However, the two genuine three-qubit entanglement classes labelled by $\tau_{123}\neq 0$ (GHZ-class) and $\tau_{123} = 0$ (W-class) are obviously invariant under the full SLOCC group. (Here we assumed that all types of separable states are excluded, i.e. we are looking merely at the states having $\tau_{1(23)}\neq 0$, $\tau_{2(13)}\neq 0$ and $\tau_{3(12)}\neq 0$.)

Now for our later discussion of Cartan's quartic invariant as an entanglement measure within the context of $N=8$, $d=4$ extremal black hole solutions we introduce some more notation. For two 3-qubit states with amplitudes Ψ_{ABC} and Φ_{ABC} *where qubit A plays a special role* let us introduce the quartic form

$$\begin{aligned}\mathcal{Q}(\Psi,\Phi) \equiv{}& \varepsilon^{A_1A_3}\varepsilon^{B_1B_2}\varepsilon^{C_1C_2}\varepsilon^{A_2A_4}\varepsilon^{B_3B_4}\varepsilon^{C_3C_4}\\ &\times\Psi_{A_1B_1C_1}\Psi_{A_2B_2C_2}\Phi_{A_3B_3C_3}\Phi_{A_4B_4C_4}.\end{aligned} \tag{3.62}$$

It is clear that

$$\mathcal{Q}(\Psi,\Psi) = -2D(\Psi). \tag{3.63}$$

Let us now consider two 3-qubit states with amplitudes Ψ_{ABC} and Σ_{ADE} made of five different qubits A, B, C, D, E with qubit A as the common one. For this situation we employ a new notation

$$\begin{aligned}\Psi^2\Sigma^2 \equiv \mathcal{Q}(\Psi,\Sigma) ={}& \varepsilon^{A_1A_3}\varepsilon^{B_1B_2}\varepsilon^{C_1C_2}\varepsilon^{A_2A_4}\varepsilon^{D_3D_4}\varepsilon^{E_3E_4}\\ &\times\Psi_{A_1B_1C_1}\Psi_{A_2B_2C_2}\Sigma_{A_3D_3E_3}\Sigma_{A_4D_4E_4}.\end{aligned} \tag{3.64}$$

We can easily remember the structure of this quantity by writing it into the form

$$\Psi^2\Sigma^2 \equiv (\Psi_0\cdot\Psi_0)(\Sigma_1\cdot\Sigma_1)-2(\Psi_0\cdot\Psi_1)(\Sigma_0\cdot\Sigma_1)+(\Psi_1\cdot\Psi_1)(\Sigma_0\cdot\Sigma_0), \quad (3.65)$$

where we have again used (3.32). As a shorthand notation we sometimes use the alternative expression

$$\Psi^4 \equiv \mathcal{Q}(\Psi,\Psi) = -2D(\Psi). \quad (3.66)$$

For three 3-qubit states with *integer* amplitudes Ψ_{ABC}, Σ_{ADE} and Ξ_{AFG} formed out of seven qubits A, B, C, D, E, F, G it turns out that the quantity

$$\Psi^4+\Sigma^4+\Xi^4+2\left(\Psi^2\Sigma^2+\Sigma^2\Xi^2+\Sigma^2\Xi^2\right), \quad (3.67)$$

will play an important role within the context of $N=4$, $d=4$ extremal black holes with U-duality group $SL(2,\mathbf{Z})\times SO(6,6,\mathbf{Z})$.

In closing this subsection we comment on the geometry of three-qubit entanglement [25]. Let us single out as usual qubit A, then define two four-vectors Z^μ and W^μ $\mu=0,1,2,3$ as

$$\begin{pmatrix}Z^0\\Z^1\\Z^2\\Z^3\end{pmatrix}\equiv\begin{pmatrix}\Psi_{000}\\\Psi_{001}\\\Psi_{010}\\\Psi_{011}\end{pmatrix}, \qquad \begin{pmatrix}W^0\\W^1\\W^2\\W^3\end{pmatrix}\equiv\begin{pmatrix}\Psi_{100}\\\Psi_{101}\\\Psi_{110}\\\Psi_{111}\end{pmatrix}. \quad (3.68)$$

These two four vectors define a plane in $\mathbf{C}^4$. The geometry of three-qubit entanglement can be described as the geometry of the manifold of two planes in $\mathbf{C}^4$ which is just the Grassmannian $Gr(4,2)$. Since the four vectors Z and W are defined up to a nonzero complex number we can alternatively regard the pair (Z,W) as a *line* in $\mathbf{CP}^3$. Hence for each three-qubit state we can associate a line in $\mathbf{CP}^3$. The coordinates describing such lines are the Plücker coordinates defined as

$$P^{\mu\nu} \equiv Z^\mu W^\nu - Z^\nu W^\mu. \quad (3.69)$$

Then the three-tangle τ_{123} can be written in the nice form

$$\tau_{123} = 2|P^{\mu\nu}P_{\mu\nu}|. \quad (3.70)$$

On $\mathbf{C}^4$ we have the bilinear form of (3.32) at our disposal. A vector $N^\mu \in \mathbf{C}^4$ satisfying the quadratic constraint $N\cdot N = 0$ is called null. In the $\mathbf{CP}^3$ picture such vectors give rise to a quadric surface in $\mathbf{CP}^3$. The geometry of three-qubit entanglement can be analysed by clarifying the relationship between this quadric and the lines corresponding to our three-qubit states. It turns out that lines intersecting the fixed quadric at two points are belonging to the GHZ class, and lines touching the quadric at one point are the ones of the W-class. The lines which are lying inside the quadric are of two types corresponding to the biseparable classes $B(AC)$ and $C(AB)$. These cases precisely correspond to $P^{\mu\nu}$ being either self-dual or anti-self-dual. The biseparable case $A(BC)$ is represented by a point in $\mathbf{CP}^3$ *off* the quadric. The totally separable class corresponds to the degenerate situation of a point lying on the quadric.

3.3.2 Pure States, Four Qubits

A four qubit state can be written in the form

$$|\Psi\rangle = \sum_{A,B,C,D=0}^{1} \Psi_{ABCD}|ABCD\rangle,$$
$$|ABCD\rangle \equiv |A\rangle \otimes |B\rangle \otimes |C\rangle \otimes |D\rangle \in V_A \otimes V_B \otimes V_C \otimes V_D, \tag{3.71}$$

where $V_{A,B,C,D} \equiv \mathbf{C}^2$. Let the subgroup of stochastic local operations and classical communication representing admissible fourpartite protocols be $SL(2,\mathbf{C})^{\otimes 4}$ acting on $|\Psi\rangle$ as

$$|\Psi\rangle \mapsto (\mathcal{A} \otimes \mathcal{B} \otimes \mathcal{C} \otimes \mathcal{D})|\Psi\rangle, \quad \mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D} \in SL(2,\mathbf{C}). \tag{3.72}$$

Our aim in this subsection is to give a unified description of four-qubit states taken together with their SLOCC transformations and their associated invariants. As we will see states and transformations taken together can be described in a unified manner using the group $SO(4,4,\mathbf{C})$. This point of view is based on the idea of a dual characterization of four-qubits as states and at the same time as transformations. Entangled states representing configurations of quantum entanglement in this picture are also regarded as *operators*. An entangled state is a pattern of entanglement, however this pattern of entanglement can also be regarded as a one acting on other patterns of entanglement to produce new kind of entanglement. Such considerations will play an important role in the description of the fundamental representation of E_7 in terms of seven three-qubit systems, and the 112 generators of E_7 not belonging to the subgroup $SL(2)^{\otimes 7}$ in terms of seven four-qubit ones.

Let us introduce the 2×2 matrices

$$E_{00} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad E_{01} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad E_{10} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad E_{11} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{3.73}$$

Then we can arrange the 16 complex amplitudes appearing in Ψ_{ABCD} in a 4×4 matrix in three different ways

$$\sum_{ABCD=0}^{1} \Psi_{ABCD} E_{AC} \otimes E_{BD} = \begin{pmatrix} \Psi_{0000} & \Psi_{0001} & \Psi_{0010} & \Psi_{0011} \\ \Psi_{0100} & \Psi_{0101} & \Psi_{0110} & \Psi_{0111} \\ \Psi_{1000} & \Psi_{1001} & \Psi_{1010} & \Psi_{1011} \\ \Psi_{1100} & \Psi_{1101} & \Psi_{1110} & \Psi_{1111} \end{pmatrix} \tag{3.74}$$

$$\sum_{ABCD=0}^{1} \Psi_{ABCD} E_{AB} \otimes E_{CD} = \begin{pmatrix} \Psi_{0000} & \Psi_{0001} & \Psi_{0100} & \Psi_{0101} \\ \Psi_{0010} & \Psi_{0011} & \Psi_{0110} & \Psi_{0111} \\ \Psi_{1000} & \Psi_{1001} & \Psi_{1100} & \Psi_{1101} \\ \Psi_{1010} & \Psi_{1011} & \Psi_{1110} & \Psi_{1111} \end{pmatrix} = \begin{pmatrix} X & Y \\ W & Z \end{pmatrix}$$

$$\sum_{ABCD=0}^{1} \Psi_{ABCD} E_{AB} \otimes E_{DC} = \begin{pmatrix} \Psi_{0000} & \Psi_{0010} & \Psi_{0100} & \Psi_{0110} \\ \Psi_{0001} & \Psi_{0011} & \Psi_{0101} & \Psi_{0111} \\ \Psi_{1000} & \Psi_{1010} & \Psi_{1100} & \Psi_{1110} \\ \Psi_{1001} & \Psi_{1011} & \Psi_{1101} & \Psi_{1111} \end{pmatrix} = \begin{pmatrix} X^{\mathrm{T}} & Y^{\mathrm{T}} \\ W^{\mathrm{T}} & Z^{\mathrm{T}} \end{pmatrix}$$

where the 2×2 matrices X, Y, W, Z are introduced merely to illustrate the block structure of the relevant matrices. Notice also that the first matrix is obtained by arranging the components of X, Y, W, Z as the first, second, third and fourth rows.

It is useful to define a new set of 4×4 matrices by multiplying these by $\varepsilon \otimes \varepsilon$ from the right, i.e.

$$D_1(\Psi) = \sum_{A,B,C,D=0}^{1} \Psi_{ABCD} E_{AC}\varepsilon \otimes E_{BD}\varepsilon, \tag{3.75}$$

$$D_2(\Psi) = \sum_{A,B,C,D=0}^{1} \Psi_{ABCD} E_{AB}\varepsilon \otimes E_{CD}\varepsilon, \tag{3.76}$$

$$D_3(\Psi) = \sum_{A,B,C,D=0}^{1} \Psi_{ABCD} E_{AB}\varepsilon \otimes E_{DC}\varepsilon. \tag{3.77}$$

These matrices can be regarded as matrix representatives of maps acting on pairs of qubits associated to a four-qubit state $|\Psi\rangle$, i.e.

$$D_1(|\Psi\rangle) : V_C \otimes V_D \to V_A \otimes V_B, \tag{3.78}$$

$$D_2(|\Psi\rangle) : V_B \otimes V_D \to V_A \otimes V_C, \tag{3.79}$$

$$D_3(|\Psi\rangle) : V_B \otimes V_C \to V_A \otimes V_D. \tag{3.80}$$

Then using the definition of the Wootters spin-flip operation (3.20) we can define three 8×8 matrices representing a four-qubit state as follows

$$R_I(\Psi) = \begin{pmatrix} 0 & D_I(\Psi) \\ -\tilde{D}_I(\Psi) & 0 \end{pmatrix}, \quad I = 1, 2, 3. \tag{3.81}$$

These are matrix representations of $|\Psi\rangle$ regarded as an operator intertwining different pairs of qubits. The carrier space of R_1 is $(V_A \otimes V_B) \oplus (V_C \otimes V_D)$, of R_2 is $(V_A \otimes V_C) \oplus (V_B \otimes V_D)$ and of R_3 is $(V_A \otimes V_D) \oplus (V_B \otimes V_C)$. Obviously this construction is related to $8_v - 8_s - 8_c$ triality of $SO(4, 4, \mathbf{C})$.

How to also include the SLOCC subgroup $SL(2, \mathbf{C})^{\otimes 4}$ into this picture? An infinitesimal transformation of this kind is of the form

$$\mathcal{A} \otimes \mathcal{B} \otimes \mathcal{C} \otimes \mathcal{D} = I \otimes I \otimes I \otimes I + \alpha\varepsilon \otimes I \otimes I \otimes I + I \otimes \beta\varepsilon \otimes I \otimes I \\ + I \otimes I \otimes \gamma\varepsilon \otimes I + I \otimes I \otimes I \otimes \delta\varepsilon + \cdots, \tag{3.82}$$

where the 2×2 matrices $\alpha, \beta, \gamma, \delta$ are *symmetric*, hence having three independent components. The dots are indicating omission of terms of higher order. In this way we have identified the Lie-algebra $sl(2)$ with the space of symmetric complex matrices over the two-dimensional complex vector space V, hence we have $sl(2) = S^2 V$. The necessity of the special form $\alpha\varepsilon, \beta\varepsilon, \gamma\varepsilon, \delta\varepsilon$ used in (3.82) is verified by noticing that the infinitesimal form of $\mathcal{A}\varepsilon\mathcal{A}^{\mathrm{T}} = \varepsilon$ valid for $SL(2)$ matrices implies that $(\alpha\varepsilon)\varepsilon + \varepsilon(\alpha\varepsilon)^{\mathrm{T}} = 0$, i.e. $\alpha = \alpha^{\mathrm{T}}$. The set of infinitesimal $SL(2)^{\otimes 4}$ transformations is characterized by 12 complex parameters and a four-qubit state $|\Psi\rangle$ by 16 complex ones. Hence altogether we have 28 complex parameters that should give rise to three 8-dimensional representations of the Lie-algebra $so(4, 4, \mathbf{C})$. This conjecture is easily verified by filling in the block diagonal entries of (3.81) in the following way:

$$R_1(\alpha, \beta, \gamma, \delta; \Psi) = \begin{pmatrix} \alpha\varepsilon \otimes I + I \otimes \beta\varepsilon & D_1(\Psi) \\ -\tilde{D}_1(\Psi) & \gamma\varepsilon \otimes I + I \otimes \delta\varepsilon \end{pmatrix}, \quad V_{AB} \oplus V_{CD}, \tag{3.83}$$

$$R_2(\alpha, \beta, \gamma, \delta; \Psi) = \begin{pmatrix} \alpha\varepsilon \otimes I + I \otimes \gamma\varepsilon & D_2(\Psi) \\ -\tilde{D}_2(\Psi) & \beta\varepsilon \otimes I + I \otimes \delta\varepsilon \end{pmatrix}, \quad V_{AC} \oplus V_{BD}, \tag{3.84}$$

$$R_3(\alpha, \beta, \gamma, \delta; \Psi) = \begin{pmatrix} \alpha\varepsilon \otimes I + I \otimes \delta\varepsilon & D_3(\Psi) \\ -\tilde{D}_3(\Psi) & \beta\varepsilon \otimes I + I \otimes \gamma\varepsilon \end{pmatrix}, \quad V_{AD} \oplus V_{BC}, \tag{3.85}$$

where we also indicated the structure of the representation space on which these matrices act. (We used the shorthand notation $V_{AB} \equiv V_A \otimes V_B$, etc.) These matrices are indeed belonging to the Lie-algebra $so(4, 4, \mathbf{C})$ since

$$R_I G + G R_I^{\mathrm{T}} = 0, \quad I = 1, 2, 3, \qquad \text{where} \qquad G = \begin{pmatrix} \varepsilon \otimes \varepsilon & 0 \\ 0 & \varepsilon \otimes \varepsilon \end{pmatrix}, \tag{3.86}$$

and after using the Bell-base in the relevant two-qubit spaces, G can be converted to a matrix with four $+1$ and four -1 in the diagonal (see (3.27) and (3.34)).

Let us now discuss briefly the structure of invariants [26, 27] under the SLOCC subgroup $SL(2)^{\otimes 4}$. Let us define the quantities

$$\mathcal{L} \equiv \mathrm{Det}D_1(\Psi), \qquad \mathcal{M} \equiv \mathrm{Det}D_2(\Psi), \qquad \mathcal{N} \equiv \mathrm{Det}D_3(\Psi). \tag{3.87}$$

They are of fourth order in the amplitudes of $|\Psi\rangle$. Then it can be shown that they are $SL(2)^{\otimes 4}$ invariants. However, since $8_v - 8_s - 8_c$ triality links the representations D_1, D_2 and D_3 they are not independent. One can show that

$$\mathcal{M} = \mathcal{L} + \mathcal{N}. \tag{3.88}$$

Let us chose $\mathcal{L}$ and $\mathcal{M}$ as the independent invariants. In fact they are also algebraically independent. It turns out that we have four algebraically independent $SL(2)^{\otimes 4}$ invariants. We can easily find a third one by experimenting with a structure similar to the one of Cayley's hyperdeterminant

$$\begin{aligned}\varepsilon^{A_1A_3}\varepsilon^{B_1B_2}\varepsilon^{C_1C_2}\varepsilon^{D_1D_2}\varepsilon^{A_2A_4}\varepsilon^{B_3B_4}\varepsilon^{C_3C_4}\varepsilon^{D_3D_4}&\\ \times\Psi_{A_1B_1C_1D_1}\Psi_{A_2B_2C_2D_2}\Psi_{A_3B_3C_3D_3}\Psi_{A_4B_4C_4D_4}.&\end{aligned} \tag{3.89}$$

In this formula qubit A is again playing a special role, hence similar to (3.60)–(3.61) it can be written in the form

$$2\left((\Psi_0 \circ \Psi_0)(\Psi_1 \circ \Psi_1) - (\Psi_0 \circ \Psi_1)^2\right), \tag{3.90}$$

where the $\circ$ bilinear form is now with respect to the 8×8 matrix $\varepsilon \otimes \varepsilon \otimes \varepsilon$ since the vectors $\Psi_0 \equiv \Psi_{0BCD}$ and $\Psi_1 \equiv \Psi_{1BCD}$ are having now eight components. However, since $\varepsilon \otimes \varepsilon \otimes \varepsilon$ is antisymmetric the terms $\Psi_0 \cdot \Psi_0$ and $\Psi_1 \cdot \Psi_1$ are zero. Hence we are left with an invariant of *second order* in the amplitudes of the form $\Psi_0 \cdot \Psi_1$. Using the explicit form of this quantity let us define the invariant

$$I_1 = \frac{1}{2}(\Psi_0\Psi_{15} - \Psi_1\Psi_{14} - \Psi_2\Psi_{13} + \Psi_3\Psi_{12} - \Psi_4\Psi_{11} + \Psi_5\Psi_{10} + \Psi_6\Psi_9 - \Psi_7\Psi_8), \tag{3.91}$$

where in order to see the structure of this invariant more clearly we switched to decimal labelling of the 16 amplitudes. It is important to realize (by converting back to binary labelling) that I_1 is also permutation invariant. Let us define as our second basic invariant the fourth order combination

$$I_2 = \frac{1}{6}(4I_1^2 + 2\mathcal{L} - 4\mathcal{M}). \tag{3.92}$$

The choice for this strange looking combination will be motivated later. For the third basic invariant we just take $\mathcal{L}$, i.e.

$$I_3 \equiv \mathcal{L}. \tag{3.93}$$

There is a fourth basic algebraically independent invariant I_4 which is of order 6. In order to construct it let us denote the *rows* of the matrix of (3.74) by x, y, w, z regarded as four-vectors. Then using our bilinear form $\cdot$ of (3.32) we can write I_1 in the alternative form

$$I_1 = \frac{1}{2}(x \cdot z - y \cdot w). \tag{3.94}$$

Then defining the *duals* of the four-vectors x, y, w, z as

$$\begin{aligned} x^*_\mu &= -\epsilon_{\mu\nu\rho\sigma} y^\nu w^\rho z^\sigma, \qquad & y^*_\nu &= \epsilon_{\mu\nu\rho\sigma} x^\mu w^\rho z^\sigma, \\ w^*_\rho &= -\epsilon_{\mu\nu\rho\sigma} x^\mu y^\nu z^\sigma, \qquad & z^*_\sigma &= \epsilon_{\mu\nu\rho\sigma} x^\mu y^\nu w^\rho, \end{aligned} \tag{3.95}$$

we can define an $SL(2)^{\otimes 4}$ invariant as

$$I_4 = \frac{1}{2}(x^* \cdot z^* - y^* \cdot w^*). \tag{3.96}$$

However, since the definition of the dual is based on the special role for qubit A, I_4 is not invariant under permutations of the four qubits.

For our later discussion of E_7 and Cartan's quartic invariant it is important to realize that our basic invariants I_1, I_2, I_3, I_4 can easily be obtained in the nice unified $SO(4, 4, \mathbf{C})$ representation of four-qubits and their SLOCC subgroup $SL(2)^{\otimes 4}$. For this purpose just take the 8×8 matrix $R \equiv R_1(\Psi)$ of (3.81) and calculate the invariants of R. Since on the block off-diagonal matrix R the SLOCC subgroup $SL(2)^{\otimes 4}$ as the space of block diagonal matrices acts naturally one expects to get all of our invariants in a very simple way. For example one observes that

$$\mathrm{Tr}R^2 = -2\mathrm{Tr}(D_1\tilde{D}_1) = -8I_1, \qquad \mathrm{Det}(R) = \mathcal{L}^2 = I_3^2, \qquad \mathrm{Pf}(RG) = \mathcal{L} = I_3. \tag{3.97}$$

Notice that the last equality relates the Pfaffian of the 8×8 antisymmetric matrix RG to the invariant I_3, where G is defined by (3.86). Similarly calculating $\mathrm{Tr}(R^4)$, $\mathrm{Tr}(R^6)$ and combining these invariants one can recover all of our algebraically independent invariants.

We will not discuss here the full SLOCC classification of four qubits. We just remark that the basic result states [28] that four qubits can be entangled in nine different ways. It is to be contrasted with the two entanglement classes obtained for three qubits. It can be shown that a generic state of four qubits can always be transformed to the form

$$\begin{aligned} |G_{abcd}\rangle = {} & \frac{a+d}{2}(|0000\rangle + |1111\rangle) + \frac{a-d}{2}(|0011\rangle + |1100\rangle) \\ & + \frac{b+c}{2}(|0101\rangle + |1010\rangle) + \frac{b-c}{2}(|0110\rangle + |1001\rangle), \end{aligned} \tag{3.98}$$

where a, b, c, d are complex numbers. This class corresponds to the GHZ class found in the three-qubit case. For this state the reduced density matrices obtained by tracing out all but one party are proportional to the identity. This is the state with maximal four-partite entanglement. Another interesting property of this state is that it does not contain true three-partite entanglement. A straightforward calculation shows that the values of our invariants (I_1, I_2, I_3, I_4) occurring for the state $|G_{abcd}\rangle$ representing the generic SLOCC class are

$$I_1 = \frac{1}{4}[a^2 + b^2 + c^2 + d^2], \quad I_2 = \frac{1}{6}[(ab)^2 + (ac)^2 + (ad)^2 + (bc)^2 + (bd)^2 + (cd)^2], \tag{3.99}$$

$$I_4 = \frac{1}{4}[(abc)^2 + (abd)^2 + (acd)^2 + (bcd)^2], \quad I_3 = abcd, \tag{3.100}$$

hence the values of the invariants $(4I_1, 6I_2, I_3^2, 4I_4)$ are given in terms of the elementary symmetric polynomials in the variables $(x_1, x_2, x_3, x_4) = (a^2, b^2, c^2, d^2)$.

Let us finally comment on the structure of the hyperdeterminant D_4 for the four-qubit system. As we already know for two-qubit systems the determinant $D_2 = \Psi_{00}\Psi_{11} - \Psi_{01}\Psi_{10}$ is related to the *concurrence* of (3.12) as $\mathcal{C} = 2|D_2|$ characterizing two-qubit entanglement. Similarly for three-qubits we have seen that the basic quantity is the three-tangle $\tau_{123} = 4|D_3|$ which is related to the hyperdeterminant D_3 of (3.56) of a $2 \times 2 \times 2$ tensor formed from the eight complex amplitudes Ψ_{ABC}. D_3 is an irreducible polynomial in the eight amplitudes which is the sum of 12 terms of degree four. It is known that the next item in the line namely the hyperdeterminant D_4 of format $2 \times 2 \times 2 \times 2$ is a polynomial of degree 24 in the 16 amplitudes Ψ_{ABCD} which has 2,894,276 terms. It can be shown that D_4 can be expressed in terms of our fundamental invariants in the form

$$256 D_4 = S^3 - 27T^2 \tag{3.101}$$

where

$$S = (I_4^2 - I_2^2) + 4(I_2^2 - I_1 I_3), \quad T = (I_4^2 - I_2^2)(I_1^2 - I_2) + (I_3 - I_1 I_2)^2. \tag{3.102}$$

In order to relate D_4 to our unifying $SO(4, 4, \mathbf{C})$ structure for four qubits one can prove that D_4 is just the discriminant of the polynomial

$$p[I_1, I_2, I_3, I_4; \lambda] \equiv \lambda^4 - (4I_1)\lambda^3 + (6I_2)\lambda^2 - (4I_4)\lambda + I_3^2, \tag{3.103}$$

where λ are the doubly degenerate eigenvalues obtained from the characteristic polynomial $\mathrm{Det}(R - \lambda\mathbf{1})$ of the matrix $R = R_1(\Psi)$ of (3.81). Moreover, on the generic class $|G_{abcd}\rangle$ the value of D_4 can be expressed as

$$D_4 = \frac{1}{256}\Pi_{i<j}(x_i - x_j)^2 = \frac{1}{256}V(a^2, b^2, c^2, d^2)^2, \tag{3.104}$$

where $(x_1, x_2, x_3, x_4) \equiv (a^2, b^2, c^2, d^2)$ and V is the Vandermonde determinant.

3.4 Error Correction, Hadamard Matrices and Graph States

3.4.1 Errors

In this subsection we briefly summarize some more background material from quantum information theory needed later to establish a three-qubit interpretation of BPS and non BPS black hole solutions in the STU model. As we know a qubit is an element of a two-dimensional complex vector space $\mathbf{C}^2$ with basis vectors (computational base) denoted by $|0\rangle$ and $|1\rangle$. These correspond to the usual basis vectors that are eigenvectors of the Pauli matrix σ_3. This operator is conventionally denoted by $\mathcal{Z}$ and is called the phase flip operator. Hence we have

$$\mathcal{Z}|0\rangle = |0\rangle, \qquad \mathcal{Z}|1\rangle = -|1\rangle. \tag{3.105}$$

The Pauli matrix σ_1 (conventionally denoted by X) is used to represent bit flips

$$X|0\rangle = |1\rangle, \qquad X|1\rangle = |0\rangle. \tag{3.106}$$

The orthogonal projectors $P_\pm$ are defined as

$$P_\pm = \frac{1}{2}(I \pm \mathcal{Z}), \tag{3.107}$$

where I is the 2×2 unit matrix.

In quantum information theory, especially in quantum error correction the discrete Fourier or Hadamard transformed base is often used. The Hadamard transformed basis vectors are denoted by $|\bar{0}\rangle$ and $|\bar{1}\rangle$ and defined as

$$|\bar{0}\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \qquad |\bar{1}\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle). \tag{3.108}$$

They are sometimes alternatively denoted by $|+\rangle$ and $|-\rangle$ since they are eigenvectors of the bit flip operator X with eigenvalues ± 1. These basis vectors can also be defined by introducing the unitary operator of Hadamard transformation

$$|\bar{0}\rangle = H|0\rangle, \qquad |\bar{1}\rangle = H|1\rangle, \qquad \text{i.e.} \quad H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \tag{3.109}$$

Since $HXH = \mathcal{Z}$ and $H\mathcal{Z}H = X$ the operator X is acting on the Hadamard transformed base as a phase flip operator and vice versa. The important corollary of this observation is that in the theory of quantum error correction once we have found a means for correcting bit flip errors using a discrete Fourier transform the same technique can be used for correcting phase flip ones.

For three-qubit systems we use the Hadamard transformation $H^{\otimes 3} = H \otimes H \otimes H$ represented by the 8×8 matrix

$$H^{\otimes 3} = \frac{1}{\sqrt{8}} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{pmatrix}. \quad (3.110)$$

Labelling the rows of this matrix in the binary form one can verify that we have for example

$$|\overline{110}\rangle = \frac{1}{\sqrt{8}}(|000\rangle + |001\rangle - |010\rangle - |011\rangle - |100\rangle - |101\rangle + |110\rangle + |111\rangle) \quad (3.111)$$

coming from the sign combinations of the seventh row. Adding and subtracting the first and last rows of the matrix $H^{\otimes 3}$ reveals that

$$\frac{1}{\sqrt{2}}(|000\rangle + |111\rangle) = \frac{1}{2}(|\overline{000}\rangle + |\overline{011}\rangle + |\overline{101}\rangle + |\overline{110}\rangle), \quad (3.112)$$

$$\frac{1}{\sqrt{2}}(|000\rangle - |111\rangle) = \frac{1}{2}(|\overline{111}\rangle + |\overline{100}\rangle + |\overline{010}\rangle + |\overline{001}\rangle). \quad (3.113)$$

This shows that the relative phase of the states $|000\rangle$ and $|111\rangle$ in a multipartite superposition can be detected in the Hadamard transformed base via a parity check (in (3.112) the number of 1's is *even* and in (3.113) it is *odd*). This is a crucial observation for developing quantum error correcting codes [16, 29, 30].

3.4.2 The (7, 4, 3) Hamming Code, Designs and Steiner Triple Systems

Quantum error correcting codes [16] are quantum versions of the well-known classical error correcting codes developed in the middle of the twentieth century. Concerning quantum errors we will only need the elementary observations having already been discussed in the previous paragraph. However, for an entanglement based understanding of the $E_{7(7)}$ symmetric macroscopic black hole entropy formula within the context of $N = 8$, $d = 4$ supergravity we have to learn something more about classical linear codes, Steiner triple systems and designs [31].

For this purpose let us delete the first column of the Hadamard matrix of (3.110), and let us replace the -1s with 0s in the remaining 8×7 matrix. Alternatively we can replace the $+1$s with 0s and the -1s with 1s. Then we obtain the following matrices which are complements of each other

$$\begin{pmatrix} 1\,1\,1\,1\,1\,1\,1 \\ 0\,1\,0\,1\,0\,1\,0 \\ 1\,0\,0\,1\,1\,0\,0 \\ 0\,0\,1\,1\,0\,0\,1 \\ 1\,1\,1\,0\,0\,0\,0 \\ 0\,1\,0\,0\,1\,0\,1 \\ 1\,0\,0\,0\,0\,1\,1 \\ 0\,0\,1\,0\,1\,1\,0 \end{pmatrix}, \qquad \begin{pmatrix} 0\,0\,0\,0\,0\,0\,0 \\ 1\,0\,1\,0\,1\,0\,1 \\ 0\,1\,1\,0\,0\,1\,1 \\ 1\,1\,0\,0\,1\,1\,0 \\ 0\,0\,0\,1\,1\,1\,1 \\ 1\,0\,1\,1\,0\,1\,0 \\ 0\,1\,1\,1\,1\,0\,0 \\ 1\,1\,0\,1\,0\,0\,1 \end{pmatrix}. \tag{3.114}$$

Our aim is to regard the rows of these matrices as seven binary digit codewords encoding messages of *four* digits. For this purpose let us now regard the *first*, *second* and *fourth* digits as *check digits*. The remaining ones are the *message digits*. Hence for example the codeword $(\mathbf{0}, \mathbf{1}, 0, \mathbf{1}, 0, 1, 0)$ encodes the message 0010 and the check digits are **011**.

Now we would like to send four message bits through a noisy channel. For this purpose we encode our 16 possible 4 digit message bits into our 16 seven digit long codewords as discussed above. The encoding procedure explained above can also be described formally as the one using the generator matrix **G**

$$\mathbf{G} \equiv \begin{pmatrix} 1\,1\,0\,1 \\ 1\,0\,1\,1 \\ 1\,0\,0\,0 \\ 0\,1\,1\,1 \\ 0\,1\,0\,0 \\ 0\,0\,1\,0 \\ 0\,0\,0\,1 \end{pmatrix} \tag{3.115}$$

by acting with it on the 4 digit **m** message vector regarded as a column vector as **Gm**. For example 1101 is encoded into the codeword 1010101). Let us suppose that the noisy channel has the effect of flipping merely one of the seven bits. The receiver would like to know whether the seven bit sequence received by here is corrupted or not. Moreover, if it is corrupted she would like to correct it unambiguously. In order to see that she can perform this task just notice that all of our codewords are differing from each other *at least in three digits*. If we define the Hamming distance between two codewords as the number of places in which the codewords differ we see that all pairs of our codewords have distance at least three. Now if one error is made in the transmission then the received binary sequence will still be closer to the original one than to any other. As a result the received sequence can be unambiguously corrected by choosing the codeword from the list which is the closest to it.

A nice way of describing the decoding process is effected by employing the *parity check matrix*

$$\mathbf{H} = \begin{pmatrix} 0\,0\,0\,1\,1\,1\,1 \\ 0\,1\,1\,0\,0\,1\,1 \\ 1\,0\,1\,0\,1\,0\,1 \end{pmatrix}, \tag{3.116}$$

which is composed of the *fifth, third* and *second* lines of the second matrix of (3.114). As we see the columns of $\mathbf{H}$ contain the binary representation of the integers $1, 2, 3, 4, 5, 6, 7$ labelling the digits of the codewords from the left to the right. It is straightforward to check that all of our 16 codewords are annihilated by $\mathbf{H}$, i.e. for a word $\mathbf{w}$ we have $\mathbf{H}\mathbf{w}^{\mathrm{T}} = (000)$ mod2. However, for a corrupted codeword $\mathbf{w} + \mathbf{e}_k$ with the kth digit of $\mathbf{w}$ is flipped the parity check $\mathbf{H}(\mathbf{w}+\mathbf{e})^{\mathrm{T}} = \mathbf{H}\mathbf{e}^{\mathrm{T}}$ gives the binary representation of k, hence the error can be corrected unambiguously. The code we have just described is the $(7, 4, 3)$ Hamming code. The notation refers to the number of digits of the codeword, the number of message digits and the Hamming distance.

The Hamming code has an important and intimate connection to designs and Steiner triple systems. A (v, k, λ) design is a collection of k element subsets (called *blocks*) of a v-element set $\mathbf{S}$, where $k < \lambda$, such that each pair of elements of $\mathbf{S}$ occur together in exactly λ blocks. Such a design is also known as a *balanced incomplete block design* (BIBD). The adjective "balanced" refers to the existence of λ, and "incomplete" refers to the requirement that $k < \lambda$ (so that no blocks contain all the elements). As an example of a BIBD let us consider seven numbers $\mathbf{S} \equiv \{1, 2, 3, 4, 5, 6, 7\}$. Let us form groups from these blocks that are triples as

$$(246) \qquad (145) \qquad (347) \qquad (123) \qquad (257) \qquad (167) \qquad (356). \tag{3.117}$$

Then we see that $k = 3 < v = 7$, and each pair of numbers occur together in exactly $\lambda = 1$ blocks. Hence the arrangement as given by (3.117) is a (7,3,1) BIBD. A $(v, 3, 1)$ design is called a Steiner triple system (STS). Hence our example in (3.117) is also a STS. It is an important theorem that a STS exists if and only if $v \equiv 1$ or 3 mod 6.

One can also show that if a (v, k, λ) design has b blocks then each element occurs in precisely r blocks, where

$$\lambda(v - 1) = r(k - 1), \qquad \text{and} \qquad bk = vr. \tag{3.118}$$

The incidence matrix of a (v, k, λ) design is the $b \times v$ matrix $\mathcal{I}_{ij}$ which is 1 if the ith block contains the jth element, and 0 otherwise. In our example $b = 7$, hence the incidence matrix is a 7×7 matrix which is just the first matrix of (3.114) after omitting its first row. Hence 7 of the nontrivial codewords of the $(7, 4, 3)$ Hamming code define a $(7, 3, 1)$ STS.

It can be shown that generally the number of blocks is greater than or equal the number of points of $\mathbf{S}$, i.e. $b \geq v$. BIBDs for which $b = v$ are called symmetric BIBDs. Note that for a symmetric design we have $r = k$ hence every block contains k elements, and at the same time every element is in k blocks. Moreover every pair of elements is in λ blocks and every pair of blocks intersect in λ elements. Obviously our $(7,3,1)$ design is symmetric, since the number of blocks and the number of elements of $\mathbf{S}$ are both 7. Moreover, the number of elements in a block is 3, and every element is in three blocks. One can also see that every pair of blocks intersect in just one element.

Designs of the form $(n^2+n+1, n+1, 1)$ where $n \geq 2$ are called *finite projective planes* of order n. It is easy to check that these designs are automatically symmetric. Moreover, the elements and blocks of such designs mimic the properties of points and lines in projective geometry, namely any two lines intersect in one point, and any two points lie on a unique line. The $(7,3,1)$ design of (3.117) is a projective plane of order 2. This is called the Fano plane. Instead of the representation as given by (3.117) it is instructive to have the pictorial representation as given by Fig. 3.1.

We see that we have seven points and seven lines, and each line is containing three points, and three lines intersect in a unique point. The complements of the lines are called quadrangles. We have seven quadrangles, and every pair of points is contained in exactly *two* quadrangles. It is easy to see that the seven points and the seven quadrangles form a $(7,4,2)$ design. Generally if we have a design (v,k,λ) $\mathbf{D}$ with blocks $\mathbf{B}_1, \ldots \mathbf{B}_b$, then the sets $\overline{\mathbf{B}}_i = \mathbf{S} - \mathbf{B}_i$ form a $(v, v-k, \lambda')$ design $\overline{\mathbf{D}}$ which is called the *complementary design*, provided $\lambda' = b - 2r + \lambda > 0$. Our design of quadrangles (the $(7,4,2)$ one)

$$(1357) \quad (2367) \quad (1256) \quad (4567) \quad (1346) \quad (2345) \quad (1247) \qquad (3.119)$$

forms the complementary design to the one of lines (the $(7,3,1)$ one). Clearly the incidence matrix of the $(7,4,2)$ design of quadrangles is given by the 7×7 matrix obtained from the second matrix of (3.114) after omitting the first line. We will show later that the $(7,3,1)$ design will play an important role in the construction of the 56-dimensional fundamental representation of E_7 in terms of seven tripartite states formed from seven qubits, and the complementary $(7,4,2)$ design will be relevant for the construction of the 133-dimensional adjoint representation. It will turn out

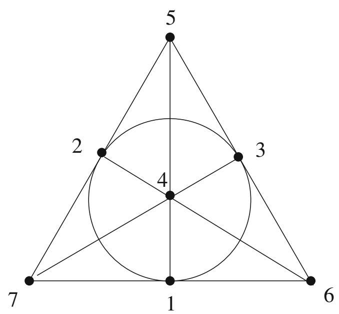

Fig. 3.1 The Fano plane with seven points and seven lines. The set of triples labelling the lines gives rise to the $(7,3,1)$ design. The complements of triples (quadrangles) define the complementary $(7,4,2)$ design

that the seven quadrangles of the $(7, 4, 2)$ design will give rise to seven 4-qubit states formed from seven qubits describing the $7 \times 16 = 112$ E_7 generators not belonging to the $7 \times 3 = 21$ dimensional SLOCC subalgebra $sl(2)^{\oplus 7}$.

The last topic we would like to discuss here is the connection found between Hadamard matrices (that formed the basis of our construction of the $(7, 4, 3)$ Hamming code) and $BIBD$s. A Hadamard matrix of order n is a matrix with entries ± 1 satisfying $HH^{\mathrm{T}} = H^T H = nI$. It is known that if there exist a Hadamard matrix H of order $n > 2$, then n must be a multiple of 4. Moreover, an important theorem states that a Hadamard matrix of order $4m$ exists if and only if a $(4m - 1, 2m - 1, m - 1)$ design exists. For $m = 2$ we get our design $(7, 3, 1)$ which is according to this theorem clearly connected to the Hadamard matrix of order 8 of (3.110) which formed the basis of our constructions.

The tensor product construction yielding (3.110) can obviously continued to produce Hadamard matrices of order 2^m. Another simple method for constructing Hadamard matrices is Paley's method. Take a prime number of the form $p = 4m - 1$. Form the set of squares of the numbers $1, 2, 3, \ldots (p - 1)/2$ modulo p. The $(p - 1)/2$ *different* numbers obtained in this way are called quadratic residues modulo p. The remaining $(p - 1)/2$ numbers are the quadratic non-residues modulo p. For example take $p = 7$ which is of the desired form. Then the quadratic residues are $1, 2, 4$ and the quadratic nonresidues are $3, 5, 6$. One can define the Dirichlet character $\chi(n)$ modulo p as the function which takes the value $+1$ for quadratic residues, -1 for quadratic nonresidues, and 0 for integers being multiples of p. Then let us form the matrix

$$\mathcal{Q}_{\alpha\beta} = \chi(\beta - \alpha), \qquad \alpha, \beta = 1, 2, \ldots p = 4m - 1. \tag{3.120}$$

For example for $p = 7$ we get the matrix

$$\mathcal{Q} = \begin{pmatrix} 0 & 1 & 1 & -1 & 1 & -1 & -1 \\ -1 & 0 & 1 & 1 & -1 & 1 & -1 \\ -1 & -1 & 0 & 1 & 1 & -1 & 1 \\ 1 & -1 & -1 & 0 & 1 & 1 & -1 \\ -1 & 1 & -1 & -1 & 0 & 1 & 1 \\ 1 & -1 & 1 & -1 & -1 & 0 & 1 \\ 1 & 1 & -1 & 1 & -1 & -1 & 0 \end{pmatrix}. \tag{3.121}$$

Now according to Paley the matrix

$$\Omega \equiv \begin{pmatrix} 1 & E \\ E^{\mathrm{T}} & \mathcal{Q} - I \end{pmatrix}, \tag{3.122}$$

where E is the row vector containing p 1s and I is the $p \times p$ identity matrix is a Hadamard matrix. Hence for our case with $p = 7$ we have

$$\Omega = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & -1 & 1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 & -1 & -1 & -1 & 1 \\ 1 & 1 & 1 & -1 & 1 & -1 & -1 & -1 \end{pmatrix}. \tag{3.123}$$

Notice that this Hadamard matrix is clearly cyclic unlike our one of (3.110). Replacing the -1s with 0s then omitting the first row and column (moreover, for later convenience putting the last row to the first place), we get

$$\begin{pmatrix} 1&1&0&1&0&0&0 \\ 0&1&1&0&1&0&0 \\ 0&0&1&1&0&1&0 \\ 0&0&0&1&1&0&1 \\ 1&0&0&0&1&1&0 \\ 0&1&0&0&0&1&1 \\ 1&0&1&0&0&0&1 \end{pmatrix} \tag{3.124}$$

the incidence matrix of a $(7, 3, 1)$ symmetric BIBD which is again just the Fano plane with a different labelling used for its points. The permutation which takes the points of Fig. 3.1. to this new Fano plane is (2)(4)(16)(375). It is well-known that the symmetry group of the Fano plane is of order 168, hence the number of different Fano planes is $7!/168 = 30$. It is also known that the space of Fano planes is consisting of two different orbits with respect to the alternating group A_7, consisting of 15–15 Fano planes. Since the permutation taking the Fano plane of Fig. 3.1. to the new one is an odd one these two descriptions are inequivalent with respect to the group A_7.

3.4.3 Graph States

Let us finally introduce the notion of a *graph state* [32] which will play a dominant role in the classification of extremal BPS and non-BPS black hole solutions in the STU model. Consider a simple graph G which contains neither loops nor multiple edges. Let us denote its vertices by V its edges by E. The main idea is to prepare n-qubits (their number is the same as the number of vertices V) in some initial vector $|\psi\rangle \in \mathbf{C}^2 \otimes \cdots \otimes \mathbf{C}^2$ and then couple them according to some interaction pattern represented by G. It turns out that the interaction pattern can be completely specified by G if it is of the form

$$U^I_{xy} = \mathrm{e}^{-ig_{xy}\mathcal{H}^I_{xy}}, \qquad \mathcal{H}^I_{xy} = I \otimes \cdots I \otimes \mathcal{Z}_x \otimes I \cdots I \otimes \mathcal{Z}_y \otimes I \cdots \otimes I. \tag{3.125}$$

Here g_{xy} are coupling constants which are the same for every pair of vertices $x, y \in V$. Notice that $\mathcal{H}^I_{xy}$ is the Ising Hamiltonian operating only between the vertices $x, y \in V$ that are linked by an edge E. The statement is that the interaction pattern assigned to the graph G in which the qubits interact according to some two-particle unitaries chosen from a commuting set of interactions is up to phase factors and local $\mathcal{Z}$ rotations is the one of (3.125). In the theory of graph states the initial state $|\psi\rangle$ associated to the graph G is a separable one which is usually of the form $|\overline{00\cdots 0}\rangle$, but we can choose any one of the 2^n Hadamard transformed basis vectors. For convenience the unitary operators appearing in the definition of a graph state are not the ones of (3.125) but rather the combination

$$U_{xy}(g_{xy}) = \mathrm{e}^{-ig_{xy}/4}\mathrm{e}^{ig_{xy}\mathcal{Z}_x/4}\mathrm{e}^{ig_{xy}\mathcal{Z}_y/4}U^I_{xy}(g_{xy}/4). \tag{3.126}$$

Here the operators having a particular label are merely acting on the corresponding qubit, on the remaining ones the unit matrix is operating. Notice that U_{xy} is the same as the Ising one up to a phase and two $\mathcal{Z}$ rotations. The unitary U_{xy} with $g_{xy} \equiv \pi$ is just the controlled phase gate

$$\mathcal{U}_{xy} \equiv U_{xy}(\pi) = (P_+)_x \otimes I_y + (P_-)_x \otimes \mathcal{Z}_y = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{3.127}$$

where only the relevant 4×4 part of the $2^n \times 2^n$ matrix was displayed. Now we can define a graph state as

$$|G\rangle \equiv \prod_{\{x,y\}\in E} \mathcal{U}_{x,y}|\overline{00\cdots 0}\rangle = \prod_{\{x,y\}\in E} \mathcal{U}_{x,y}|++\cdots+\rangle, \tag{3.128}$$

Such states for $n = 3$ based on the triangle graph will be appearing in connection with non-BPS solutions with the central charge $Z \neq 0$.

3.5 STU Black Holes

Having discussed all the basic results we need from the theory of multiqubit entanglement now we start applying these for obtaining additional insight into the structure of extremal stringy black hole solutions. As a first example to show how such techniques can be used we embark in a detailed analysis of the well-known black hole solutions in the STU model. We consider ungauged $N = 2$ supergravity in $d = 4$ coupled to n vector multiplets. The $n = 3$ case corresponds to the *STU* model. The bosonic part of the action (without hypermultiplets) is

$$\mathcal{S} = \frac{1}{16\pi}\int d^4x\sqrt{|g|}\{-\frac{R}{2} + G_{a\bar{b}}\partial_\mu z^a \partial_\nu \bar{z}^{\bar{b}} g^{\mu\nu} + (\mathrm{Im}\mathcal{N}_{IJ}\mathcal{F}^I\mathcal{F}^J + \mathrm{Re}\mathcal{N}_{IJ}\mathcal{F}^I {}^*\mathcal{F}^J)\} \tag{3.129}$$

Here $\mathcal{F}^I$, and ${}^*\mathcal{F}^I$, $I = 1, 2, \ldots, n+1$ are two-forms associated to the field strengths $\mathcal{F}^I_{\mu\nu}$ of $n+1$ $U(1)$ gauge-fields and their duals. The z^a $a = 1, \ldots, n$ are complex scalar (moduli) fields that can be regarded as local coordinates on a projective special Kähler manifold $\mathcal{M}$. This manifold for the STU model is $SL(2,\mathbf{R})/U(1) \times SL(2,\mathbf{R})/U(1) \times SL(2,\mathbf{R})/U(1)$. In the following we will denote the three complex scalar fields as

$$z^1 \equiv S = S_1 + iS_2, \qquad z^2 \equiv T = T_1 + iT_2, \qquad z^3 \equiv U = U_1 + iU_2. \tag{3.130}$$

With these notations we have

$$G_{1\bar{1}} = G_{S\overline{S}} = \frac{1}{4S_2^2}, \quad G_{2\bar{2}} = G_{T\overline{T}} = \frac{1}{4T_2^2}, \quad G_{3\bar{3}} = G_{U\overline{U}} = \frac{1}{4U_2^2}, \tag{3.131}$$

with the other components like G_{11}, $G_{\bar{1}\bar{1}}$ and $G_{1\bar{2}}$, etc., are zero. The metric above can be derived from the Kähler potential

$$K = -\ln(-8U_2T_2S_2) \tag{3.132}$$

as $G_{a\bar{b}} = \partial_a\partial_{\bar{b}}K$. In order to ensure the positivity of e^K (needed later) we demand that S, T and U should have *negative* imaginary parts.

For the STU model the scalar dependent vector couplings $\mathrm{Re}\mathcal{N}_{IJ}$ and $\mathrm{Im}\mathcal{N}_{IJ}$ take the following form

$$\mathrm{Re}\mathcal{N}_{IJ} = \begin{pmatrix} 2U_1T_1S_1 & -U_1T_1 & -U_1S_1 & -T_1S_1 \\ -U_1T_1 & 0 & U_1 & T_1 \\ -U_1S_1 & U_1 & 0 & S_1 \\ -T_1S_1 & T_1 & S_1 & 0 \end{pmatrix}, \tag{3.133}$$

$$\mathrm{Im}\mathcal{N}_{IJ} = U_2T_2S_2 \begin{pmatrix} 1 + \left(\frac{S_1}{S_2}\right)^2 + \left(\frac{T_1}{T_2}\right)^2 + \left(\frac{U_1}{U_2}\right)^2 & -\frac{S_1}{S_2^2} & -\frac{T_1}{T_2^2} & -\frac{U_1}{U_2^2} \\ -\frac{S_1}{S_2^2} & \frac{1}{S_2^2} & 0 & 0 \\ -\frac{T_1}{T_2^2} & 0 & \frac{1}{T_2^2} & 0 \\ -\frac{U_1}{U_2^2} & 0 & 0 & \frac{1}{U_2^2} \end{pmatrix}. \tag{3.134}$$

We note that these scalar dependent vector couplings can be derived from the holomorphic prepotential

$$F(X) = \frac{X^1X^2X^3}{X^0}, \qquad X^I = (X^0, X^0z^a), \tag{3.135}$$

via the standard procedure characterizing special Kähler geometry [33]. For the explicit expressions for $\mathcal{N}_{IJ}$ for general cubic holomorphic potentials see the recent paper of Ceresole et al. [34].

For the physical motivation of (3.129) we note that when type IIA string theory is compactified on a T^6 one recovers $N = 8$ supergravity in $d = 4$ with 28 vectors and 70 scalars taking values in the symmetric space $E_{7(7)}/SU(8)$. This $N = 8$ model with an on shell U-duality symmetry $E_{7(7)}$ has a consistent $N = 2$ truncation with four vectors and three complex scalars which is just the STU model [35].

Now we briefly recall the basic facts concerning static, spherically symmetric, extremal black hole solutions associated to the (3.129) action. Let us consider the static spherically symmetric ansatz for the metric

$$\mathrm{d}s^2 = \mathrm{e}^{2\mathcal{U}}\mathrm{d}t^2 - \mathrm{e}^{-2\mathcal{U}}\left[\frac{c^4}{\sinh^4 c\tau}\mathrm{d}\tau^2 + \frac{c^2}{\sinh^2 c\tau}\mathrm{d}\Omega^2\right], \tag{3.136}$$

here $\mathcal{U} \equiv \mathcal{U}(\tau)$, $c^2 = 2\mathcal{S}\mathcal{T}$, where $\mathcal{S}$ is the entropy and $\mathcal{T}$ is the temperature of the black hole. The coordinate τ is a "radial" one, at infinity ($\tau \to 0$) we will be interested in solutions reproducing the Minkowski metric. $\mathrm{d}\Omega^2$ is the usual metric of the unit two-sphere in terms of polar coordinates θ and φ. Our extremal black holes will correspond to the limit $c \to 0$, i.e. having vanishing Hawking temperature. Putting this ansatz into (3.129) we obtain a one-dimensional effective Lagrangian for the radial evolution of the quantities $\mathcal{U}(\tau)$, $z^a(\tau)$, as well as the electric $\xi^I(\tau)$, and magnetic $\chi_I(\tau)$ potentials defined as [36]

$$\mathcal{F}^I_{t\tau} = \partial_\tau \xi^I, \qquad \mathcal{G}_{It\tau} \equiv -i\,\mathrm{Im}\mathcal{N}_{IJ}\,(*\mathcal{F})^J_{t\tau} - \mathrm{Re}\mathcal{N}_{IJ}\mathcal{F}^J_{t\tau} = \partial_\tau \chi_I, \tag{3.137}$$

$$\mathcal{L}(\mathcal{U}(\tau), z^a(\tau), \bar{z}^{\bar{a}}(\tau)) = \left(\frac{\mathrm{d}\mathcal{U}}{\mathrm{d}\tau}\right)^2 + G_{a\bar{a}}\frac{dz^a}{\mathrm{d}\tau}\frac{\mathrm{d}\bar{z}^{\bar{a}}}{\mathrm{d}\tau} + \mathrm{e}^{2\mathcal{U}}V_{\mathrm{BH}}(z, \bar{z}, p, q), \tag{3.138}$$

and the constraint

$$\left(\frac{\mathrm{d}\mathcal{U}}{\mathrm{d}\tau}\right)^2 + G_{a\bar{a}}\frac{dz^a}{\mathrm{d}\tau}\frac{\mathrm{d}\bar{z}^{\bar{a}}}{\mathrm{d}\tau} - \mathrm{e}^{2\mathcal{U}}V_{\mathrm{BH}}(z, \bar{z}, p, q) = c^2. \tag{3.139}$$

Here our quantity of central importance is the Black Hole potential V_{BH} is depending on the moduli as well on the quantized charges defined by

$$p^I = \frac{1}{4\pi}\int_{S^2}\mathcal{F}^I, \qquad q_I = \frac{1}{4\pi}\int_{S^2}\mathcal{G}_I. \tag{3.140}$$

Its explicit form is given by

$$V_{\mathrm{BH}} = \frac{1}{2}\begin{pmatrix} p^I & q_I \end{pmatrix}\begin{pmatrix} (\mu + \nu\mu^{-1}\nu)_{IJ} & -(\nu\mu^{-1})_I^J \\ -(\mu^{-1}\nu)^I_J & (\mu^{-1})^{IJ} \end{pmatrix}\begin{pmatrix} p^J \\ q_J \end{pmatrix}, \tag{3.141}$$

where the matrices $\nu = \mathrm{Re}\mathcal{N}$ and $\mu = \mathrm{Im}\mathcal{N}$ are the ones of (3.133) and (3.134). The explicit form of μ^{-1} is

$$\mu^{-1} = \frac{1}{U_2 T_2 S_2} \begin{pmatrix} 1 & S_1 & T_1 & U_1 \\ S_1 & |S|^2 & S_1 T_1 & S_1 U_1 \\ T_1 & S_1 T_1 & |T|^2 & T_1 U_1 \\ U_1 & S_1 U_1 & T_1 U_1 & |U|^2 \end{pmatrix}. \tag{3.142}$$

An alternative expression for V_{BH} can be given in terms of the central charge of $N = 2$ supergravity, i.e. the charge of the graviphoton.

$$V_{\mathrm{BH}} = Z\overline{Z} + G^{a\bar{b}}(D_a Z)(\overline{D}_{\bar{b}}\overline{Z}), \tag{3.143}$$

where for the STU model

$$Z = \mathrm{e}^{K/2} W = \mathrm{e}^{K/2}(q_0 + Sq_1 + Tq_2 + Uq_3 + UTSp^0 - UTp^1 - USp^2 - TSp^3), \tag{3.144}$$

and D_a is the Kähler covariant derivative

$$D_a Z = (\partial_a + \frac{1}{2}\partial_a K)Z. \tag{3.145}$$

Here $W(U, T, S) \equiv W(U, T, S; p, q)$ is the superpotential.

For extremal black hole solutions ($c = 0$) the geometry is given by the line element

$$ds^2 = \mathrm{e}^{2\mathcal{U}} dt^2 - \mathrm{e}^{-2\mathcal{U}} \left[\frac{\mathrm{d}\tau^2}{\tau^4} + \frac{1}{\tau^2}(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\varphi^2) \right]. \tag{3.146}$$

The requirement for the solution to have finite horizon area is

$$\mathrm{e}^{-2\mathcal{U}} \to \left(\frac{A}{4\pi}\right)\tau^2, \qquad \text{as} \qquad \tau \to -\infty, \tag{3.147}$$

which using the new variable $r = -1/\tau$ is yielding for the near horizon geometry the $AdS_2 \times S^2$ form

$$\mathrm{d}s^2 = \left(\frac{4\pi}{A}\right) r^2 \mathrm{d}t^2 - \left(\frac{A}{4\pi}\right) \left[\frac{\mathrm{d}r^2}{r^2} + (\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\varphi^2) \right]. \tag{3.148}$$

A particularly important subclass of solutions are the double-extremal solutions [37–39]. These solutions have everywhere-constant moduli. These black holes pick up the frozen values of the moduli that extremize the black hole mass at infinity. The frozen values of the scalar fields are the ones at the horizon. These solutions are of Reissner–Nordström type with constant scalars defined by the critical point of the black hole potential V_{BH}

$$\partial_a V_{\mathrm{BH}} = 0, \qquad z_{\mathrm{fix}}(p,q) = z_\infty = z_{\mathrm{horizon}}. \tag{3.149}$$

For such double-extremal black hole solutions the value of A in (3.148) the area of the horizon is defined by the value of the black hole potential at the horizon [36]

$$\frac{A}{4\pi} = V_{\mathrm{BH}}(z_{\mathrm{horizon}}, \bar{z}_{\mathrm{horizon}}, p, q). \tag{3.150}$$

Although our considerations in the following sections can obviously generalized for solutions of more general type [40–43], in order to simplify presentation in the following we restrict our attention to this particular subclass of double-extremal solutions.

3.5.1 The Black Hole Potential as the Norm of a Three-Qubit State

In order to exhibit the interesting structure of the black hole potential (3.141) first we make some preliminary definitions. As was observed by Duff [1] it is useful to reorganize the charges of the STU model into the eight amplitudes of a three-qubit state

$$|\psi\rangle = \sum_{l,k,j=0}^{1} \psi_{lkj}|lkj\rangle \qquad |lkj\rangle \equiv |l\rangle_U \otimes |k\rangle_T \otimes |j\rangle_S, \tag{3.151}$$

where

$$\begin{pmatrix} p^0 & p^1 & p^2 & p^3 \\ q_0 & q_1 & q_2 & q_3 \end{pmatrix} = \begin{pmatrix} \psi_{000} & \psi_{001} & \psi_{010} & \psi_{100} \\ -\psi_{111} & \psi_{110} & \psi_{101} & \psi_{011} \end{pmatrix}. \tag{3.152}$$

Notice, however that our identification of the amplitudes of the three-qubit state and the charges is slightly different from the one used by Duff [1]. Moreover, we have introduced the convention of labelling the qubits from the right to the left. Also we will regard the first, second and third qubits as the ones associated to some fictitious subsystems S (Sarah), T (Tom), and U (Ursula). The state $|\psi\rangle$ is a three-qubit state of a very special kind. First of all unlike the one of (3.41) this state defined by the charges need not have to be normalized. Moreover, the amplitudes of this state are not complex numbers but *integers*. In the following we will refer to this state as the *reference state*. Now we are going to define a new unnormalized three-qubit state $|\Psi\rangle$ which is depending on the charges *and also the moduli* [3]. This new state will be a three-qubit state with eight complex amplitudes. However, as we will see it is really a *real three-qubit state*, since it is $SU(2)^{\otimes 3}$ equivalent to a one with eight real amplitudes [3]. So this state is equivalent to an entangled unnormalized one composed of three rebits (see Sect. 3.2.3).

In order to motivate our definition of the new state $|\Psi\rangle$ we notice that

$$V_{\mathrm{BH}} = -\frac{1}{2}\frac{1}{U_2 T_2 S_2}\langle\psi| \begin{pmatrix} |U|^2 & -U_1 \\ -U_1 & 1 \end{pmatrix} \otimes \begin{pmatrix} |T|^2 & -T_1 \\ -T_1 & 1 \end{pmatrix} \otimes \begin{pmatrix} |S|^2 & -S_1 \\ -S_1 & 1 \end{pmatrix} |\psi\rangle. \quad (3.153)$$

In order to prove this calculate the 8×8 matrix in the middle with rows and columns labelled in the binary form $000, 001, 010, 011, 100, 101, 110, 111$, and regard $|\psi\rangle$ as the column vector $(\psi_{000}, \psi_{001}, \ldots, \psi_{111})^{\mathrm{T}}$ and $\langle\psi|$ the corresponding row vector. It is straightforward to see that the resulting expression is the same as the one that can be obtained using (3.133), (3.134), (3.141) and (3.142). For establishing this result note, however the different labelling of rows and columns of matrices in (3.141) (which is based on the symplectic structure) and (3.153) (based on the binary labelling).

Now we define the state $|\Psi\rangle$ as

$$|\Psi(U,T,S;p,q)\rangle = \mathrm{e}^{\mathrm{i}\Phi}\mathrm{e}^{K/2} \begin{pmatrix} \overline{U} & -1 \\ -U & 1 \end{pmatrix} \otimes \begin{pmatrix} \overline{T} & -1 \\ -T & 1 \end{pmatrix} \otimes \begin{pmatrix} \overline{S} & -1 \\ -S & 1 \end{pmatrix} |\psi\rangle. \quad (3.154)$$

With the choice for the phase factor $\mathrm{e}^{\mathrm{i}\Phi} = \mathrm{e}^{-3\mathrm{i}\pi/4}$ the resulting matrices in the threefold tensor product are all $SL(2,\mathbf{C})$ ones. They are explicitly given by

$$\mathcal{A}_{\mathrm{S}} \equiv \frac{\mathrm{e}^{-\mathrm{i}\pi/4}}{\sqrt{-2S_2}} \begin{pmatrix} \overline{S} & -1 \\ -S & 1 \end{pmatrix} = \mathrm{e}^{\mathrm{i}\pi/4}\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} \frac{1}{\sqrt{-S_2}} \begin{pmatrix} -S_2 & 0 \\ -S_1 & 1 \end{pmatrix}, \quad (3.155)$$

$$\mathcal{B}_{\mathrm{T}} \equiv \frac{\mathrm{e}^{-\mathrm{i}\pi/4}}{\sqrt{-2T_2}} \begin{pmatrix} \overline{T} & -1 \\ -T & 1 \end{pmatrix} = \mathrm{e}^{\mathrm{i}\pi/4}\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} \frac{1}{\sqrt{-T_2}} \begin{pmatrix} -T_2 & 0 \\ -T_1 & 1 \end{pmatrix}, \quad (3.156)$$

$$\mathcal{C}_{\mathrm{U}} \equiv \frac{\mathrm{e}^{-\mathrm{i}\pi/4}}{\sqrt{-2U_2}} \begin{pmatrix} \overline{U} & -1 \\ -U & 1 \end{pmatrix} = \mathrm{e}^{\mathrm{i}\pi/4}\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} \frac{1}{\sqrt{-U_2}} \begin{pmatrix} -U_2 & 0 \\ -U_1 & 1 \end{pmatrix}. \quad (3.157)$$

With this notation we have $|\Psi\rangle = \mathcal{C}_{\mathrm{U}} \otimes \mathcal{B}_{\mathrm{T}} \otimes \mathcal{A}_{\mathrm{S}}|\psi\rangle$. This means that the states $|\Psi\rangle$ for all values of the moduli are in the $SL(2,\mathbf{C})^{\otimes 3}$ orbit of the reference state $|\psi\rangle$ of (3.151) defined by the charges. This means that the value of the three-tangle of τ_{123} (3.55) is the same for both $|\psi\rangle$ and $|\Psi\rangle$. Obviously the state $|\Psi\rangle$ is an unnormalized three-qubit one with eight complex amplitudes. However, according to (3.155)–(3.157) it is *not* a genuine complex three-qubit state but rather a one which is $SU(2)^{\otimes 3}$ equivalent to a real one. This should not come as a surprise since the symmetry group associated with the STU model is not $SL(2,\mathbf{C})^{\otimes 3}$ but rather $SL(2,\mathbf{R})^{\otimes 3}$.

Using (3.153) now we are ready to write the black hole potential in the following nice form

$$V_{\mathrm{BH}} = \frac{1}{2}||\Psi||^2. \quad (3.158)$$

Here the norm is defined using the usual scalar product in $\mathbf{C}^8 \simeq \mathbf{C}^2 \otimes \mathbf{C}^2 \otimes \mathbf{C}^2$ with complex conjugation in the first factor. Since the norm is invariant under $U(2)^{\otimes 3}$ our choice of the phase factor $\mathrm{e}^{\mathrm{i}\Phi}$ is not relevant in the structure of V_{BH}. In the following for the sake of calculational simplicity we set in (3.154) $\Phi \equiv 0$. However, in this convenient "gauge" the three-tangle τ_{123} for the charge-dependent $|\psi\rangle$ and the charge and moduli-dependent $|\Psi\rangle$ will no longer be the same. Hence the charge and moduli-dependent $|\Psi\rangle$ in the "gauge" $\Phi \equiv 0$ will be in the same SLOCC (i.e. $GL(2,\mathbf{C})^{\otimes 3}$) but not in the same $SL(2,\mathbf{C})^{\otimes 3}$ orbit as the charge-dependent reference state $|\psi\rangle$. Moreover, we could have defined a new moduli dependent real state instead of the complex one $|\Psi\rangle$ by using merely the $SL(2,\mathbf{R})$ matrices of (3.155)–(3.157) for their definition. However, we prefer the complex form of (3.154) since it will be useful later.

It is instructive to write out explicitly the amplitudes of our complex three-qubit state $|\Psi\rangle$. After recalling the definition of the superpotential $W(U,T,S)$ of (3.144) they are

$$\Psi_{000} = \mathrm{e}^{K/2} W(\overline{U},\overline{T},\overline{S}), \qquad \Psi_{111} = -\mathrm{e}^{K/2} W(U,T,S), \tag{3.159}$$
$$\Psi_{110} = \mathrm{e}^{K/2} W(U,T,\overline{S}), \qquad \Psi_{001} = -\mathrm{e}^{K/2} W(\overline{U},\overline{T},S), \tag{3.160}$$
$$\Psi_{101} = \mathrm{e}^{K/2} W(U,\overline{T},S), \qquad \Psi_{010} = -\mathrm{e}^{K/2} W(\overline{U},T,\overline{S}), \tag{3.161}$$
$$\Psi_{011} = \mathrm{e}^{K/2} W(\overline{U},T,S), \qquad \Psi_{100} = -\mathrm{e}^{K/2} W(U,\overline{T},\overline{S}). \tag{3.162}$$

We can summarize this as

$$\Psi_{lkj} = (-1)^{l+k+j} \mathrm{e}^{K/2} W_{lkj}, \quad \text{where} \quad W_{101} \equiv W(U,\overline{T},S), \quad \text{etc.} \tag{3.163}$$

Notice also that we have the property

$$\Psi_{000} = -\overline{\Psi_{111}}, \quad \Psi_{110} = -\overline{\Psi_{001}}, \quad \Psi_{101} = -\overline{\Psi_{010}}, \quad \Psi_{011} = -\overline{\Psi_{100}}. \tag{3.164}$$

Using this in (3.158) we can write V_{BH} in the alternative form

$$V_{\mathrm{BH}} = \mathrm{e}^K \left(|W(U,T,S)|^2 + |W(U,T,\overline{S})|^2 + |W(U,\overline{T},S)|^2 + |W(\overline{U},T,S)|^2 \right), \tag{3.165}$$

in agreement with the result found in (A.39) of the Appendix of Kallosh et al. [44].

As a next step we would like to clarify the meaning of the complex amplitudes Ψ_{lkj}. For this we have to look at the structure of covariant derivatives. Using (3.145) we have $D_a W = \partial_a W + (\partial_a K) W$ so for example

$$D_S W(U,T,S) = \frac{W(U,T,\overline{S})}{\overline{S} - S}. \tag{3.166}$$

Since the nonzero components of the Christoffel symbols are

$$\Gamma^{S}_{SS} = \frac{2}{\overline{S} - S}, \quad \Gamma^{T}_{TT} = \frac{2}{\overline{T} - T}, \quad \Gamma^{U}_{UU} = \frac{2}{\overline{U} - U}, \tag{3.167}$$

we have [44]

$$D_S D_T W(U,T,S) = \frac{W(U,\overline{T},\overline{S})}{(\overline{S} - S)(\overline{T} - T)}, \quad D_S D_S W(U,T,S) = 0, \quad \text{etc.} \tag{3.168}$$

It is convenient to introduce flat covariant derivatives. Let $\delta_{\hat{a}\hat{\bar{b}}}$ be the flat Euclidean metric. Then we define the vielbein $e^{\hat{a}}_{a}$ via the expression $G_{a\bar{b}} = e^{\hat{a}}_{a} e^{\hat{\bar{b}}}_{\bar{b}} \delta_{\hat{a}\hat{\bar{b}}}$. Using (3.131) we get for the nonzero components of the *inverse* vielbein

$$\begin{aligned} e^{S}_{\hat{S}} &= i(S - \overline{S}) = -2S_2, \\ e^{T}_{\hat{T}} &= i(T - \overline{T}) = -2T_2, \\ e^{U}_{\hat{U}} &= i(U - \overline{U}) = -2U_2. \end{aligned} \tag{3.169}$$

The flat covariant derivatives are defined by $D_{\hat{a}} = e^{a}_{\hat{a}} D_a$. Using (3.166) and (3.169) we see that

$$D_{\hat{S}}\Psi_{111} = i\Psi_{110}, \quad D_{\hat{T}}\Psi_{111} = i\Psi_{101}, \quad D_{\hat{U}}\Psi_{111} = i\Psi_{011}, \tag{3.170}$$

$$D_{\hat{\overline{S}}}\Psi_{111} = D_{\hat{\overline{T}}}\Psi_{111} = D_{\hat{\overline{U}}}\Psi_{111} = 0. \tag{3.171}$$

It is straightforward to verify that the action of the operators $D_{\hat{a}}$ and $D_{\hat{\bar{a}}}$ on the remaining amplitudes follows the same pattern. We can neatly summarize their action after defining the raising and lowering operators $S_{\pm}$

$$S_+|0\rangle = |1\rangle, \qquad S_+|1\rangle = 0, \qquad S_-|0\rangle = 0, \qquad S_-|1\rangle = |0\rangle. \tag{3.172}$$

Hence the flat covariant derivatives are transforming between the eight amplitudes Ψ_{lkj} and the combinations like $I \otimes I \otimes S_{\pm}$ are transforming between the eight basis vectors $|lkj\rangle$ of the three qubit state $|\Psi\rangle$. In fact one can verify that

$$\frac{1}{i} D_{\hat{S}}|\Psi\rangle = (I \otimes I \otimes S_+)|\Psi\rangle, \qquad -\frac{1}{i} D_{\hat{\overline{S}}}|\Psi\rangle = (I \otimes I \otimes S_-)|\Psi\rangle, \tag{3.173}$$

$$\frac{1}{i} D_{\hat{T}}|\Psi\rangle = (I \otimes S_+ \otimes I)|\Psi\rangle, \qquad -\frac{1}{i} D_{\hat{\overline{T}}}|\Psi\rangle = (I \otimes S_- \otimes I)|\Psi\rangle, \tag{3.174}$$

$$\frac{1}{i} D_{\hat{U}}|\Psi\rangle = (S_+ \otimes I \otimes I)|\Psi\rangle, \qquad -\frac{1}{i} D_{\hat{\overline{U}}}|\Psi\rangle = (S_- \otimes I \otimes I)|\Psi\rangle. \tag{3.175}$$

Hence the flat covariant derivatives are acting on our three-qubit state $|\Psi\rangle$ as the operators of *projective errors* known from the theory of quantum error correction (see Sect. 3.4.1). Alternatively one can look at the action of the combination $(D_{\hat{a}} - D_{\hat{\bar{a}}})/i$

$$\frac{1}{i}(D_{\hat{S}} - D_{\hat{\bar{S}}})|\Psi\rangle = (I \otimes I \otimes X)|\Psi\rangle, \quad \text{etc.,} \tag{3.176}$$

where $I \otimes I \otimes X$ is the operator of bit-flip error acting on the *first qubit*.

Having clarified the meaning of the entangled three-qubit state $|\Psi\rangle$ and the flat covariant derivatives as error operations acting on it, in light of these result in the next section we would like to obtain some additional insight on the structure of BPS and non-BPS black hole solutions.

3.5.2 BPS and Non-BPS Solutions

As it is well-known [36,44,45] the extremization of the black-hole potential (3.143) with respect to the moduli yields the following set of equations

$$\partial_a V_{\mathrm{BH}} = \mathrm{e}^K \left(G^{b\bar{c}}(D_a D_b W)\overline{D}_{\bar{c}}\overline{W} + 2(D_a W)\overline{W} \right) = 0, \tag{3.177}$$

$$\partial_{\bar{a}} V_{\mathrm{BH}} = \mathrm{e}^K \left(G^{\bar{b}c}(D_{\bar{a}} D_{\bar{b}} \overline{W})D_c W + 2(\overline{D}_{\bar{a}}\overline{W})W \right) = 0. \tag{3.178}$$

Assuming $W \neq 0$ expressing $\overline{D}_{\bar{a}}\overline{W}$ from (3.178), and substituting the resulting expression back to (3.177) yields an equation [45] of the form

$$M_a^b (D_b W) = 0. \tag{3.179}$$

For the STU-model for the matrix M_a^b we get the following expression

$$\begin{pmatrix} 4\Psi_7\Psi_0 - \Psi_4\Psi_3 - \Psi_5\Psi_2 & \frac{T_2}{S_2}\Psi_6\Psi_2 & \frac{U_2}{S_2}\Psi_6\Psi_4 \\ \frac{S_2}{T_2}\Psi_5\Psi_1 & 4\Psi_7\Psi_0 - \Psi_6\Psi_1 - \Psi_4\Psi_3 & \frac{U_2}{T_2}\Psi_5\Psi_4 \\ \frac{S_2}{U_2}\Psi_3\Psi_1 & \frac{T_2}{U_2}\Psi_3\Psi_2 & 4\Psi_7\Psi_0 - \Psi_6\Psi_1 - \Psi_5\Psi_2 \end{pmatrix},$$

where we used the decimal notation $(\Psi_{000}, \ldots, \Psi_{111}) = (\Psi_0, \ldots, \Psi_7)$. Expressing the covariant derivatives $D_a W$ in terms of the corresponding amplitudes using (3.170) and (3.171), we obtain the explicit expression for (3.179)

$$(2\Psi_7\Psi_0 - \Psi_5\Psi_2 - \Psi_4\Psi_3)\Psi_6 = 0, \tag{3.180}$$

$$(2\Psi_7\Psi_0 - \Psi_4\Psi_3 - \Psi_6\Psi_1)\Psi_5 = 0, \tag{3.181}$$

$$(2\Psi_7\Psi_0 - \Psi_6\Psi_1 - \Psi_5\Psi_2)\Psi_3 = 0. \tag{3.182}$$

Recall also that $\Psi_{7-\alpha} = -\bar{\Psi}_\alpha$ where $\alpha = 0, 1, \ldots, 7$ which is just the decimal form of (3.164). The determinant of M_a^b is

$$\frac{1}{4}\mathrm{Det}M = |\Psi_0|^2(4|\Psi_0|^2 - |\Psi_1|^2 - |\Psi_2|^2 - |\Psi_4|^2) - |\Psi_1\Psi_2\Psi_4|^2. \tag{3.183}$$

Using these results we can conclude that there are two different types of solutions for $Z \neq 0$.

I. BPS solutions

$$\Psi_1 = \Psi_2 = \Psi_4 = 0, \qquad \mathrm{Det}M \neq 0. \tag{3.184}$$

II. Non-BPS solutions

$$|\Psi_0|^2 = |\Psi_1|^2 = |\Psi_2|^2 = |\Psi_4|^2, \qquad \mathrm{Det}M = 0. \tag{3.185}$$

Notice that the amplitudes $\Psi_0 = \Psi_{000}$ and $\Psi_7 = \Psi_{111}$ are playing a special role in the STU model. Indeed they are related to the central charge and its complex conjugate as

$$Z = -\Psi_7, \qquad \overline{Z} = \Psi_0. \tag{3.186}$$

For the type of solutions considered here $Z \neq 0$, hence the corresponding amplitudes are never zero. We should remark, however at this point that there are solutions belonging to a third class [46,47]: the ones with $Z \equiv 0$. The structure of these solutions has recently been studied in the context of the STU-model [48]. In the next sections we are focusing merely on classes I and II where an interpretation of known results in the language of quantum information theory is straightforward. It is easy to extend our considerations also to the third class however, we postpone the investigation of these solutions for the special case of the D2–D6 system until Sect. 3.5.6. Until then let us try to find a quantum information theoretic interpretation for the two types of solutions found above.

3.5.3 Entanglement and BPS Solutions

We know that for BPS black holes at the horizon ($r = 0$) we have $D_a Z \equiv 0$. From the amplitudes of (3.184) and their complex conjugates we see that the only non-vanishing amplitudes of $|\Psi\rangle$ at the horizon are Ψ_{000} and Ψ_{111}, hence for the BPS case

$$|\Psi(0)\rangle = \overline{Z}|000\rangle - Z|111\rangle. \tag{3.187}$$

This state is of the generalized GHZ form of maximal tripartite entanglement (see (3.44)). The form of the black hole potential at the horizon is

$$V_{\mathrm{BH}} = \frac{1}{2}(|\Psi_{000}|^2 + |\Psi_{111}|^2) = |Z|^2 = M^2_{\mathrm{BPS}}. \tag{3.188}$$

Notice, that for double-extremal black holes (3.187) and (3.188) are valid even away from the horizon. However, for BPS solutions of more general type $|\Psi\rangle$ as a function of τ (or r) is of the general form of (3.154) with the moduli $S(\tau)$, $T(\tau)$ and $U(\tau)$ being solutions for the equations of motion for the moduli [40–43]. Of course these solutions at the horizon ($r = 0$) will again be attracted to the very special form of $|\Psi\rangle$ as dictated by (3.187)–(3.188). Hence the first interpretation of the attractor mechanism for the BPS case is that of a quantum information theoretic distillation of a GHZ-like state (3.187) at the horizon from a one of the general form (3.151). As we reach the horizon the conditions

$$D_S Z = D_T Z = D_U Z = 0, \qquad Z \neq 0. \tag{3.189}$$

guarantee that

$$\Psi_{110} = \Psi_{101} = \Psi_{011} = \Psi_{001} = \Psi_{010} = \Psi_{100} = 0, \tag{3.190}$$

hence we are left merely with the GHZ components Ψ_{000} and Ψ_{111}.

Equations (3.190) can be used to express the stabilized values of the moduli in terms of the charges [38]. For obtaining also some geometric insight we proceed as follows [3]. First we define the following set of four-vectors

$$n_S = \begin{pmatrix} 1 \\ T \\ U \\ TU \end{pmatrix}, \quad n_T = \begin{pmatrix} 1 \\ S \\ U \\ US \end{pmatrix}, \quad n_U = \begin{pmatrix} 1 \\ S \\ T \\ ST \end{pmatrix}. \tag{3.191}$$

Notice that these are null with respect to our metric (3.32), i.e. $n \cdot n = 0$ due to their tensor product structure (e.g. $n_S = (1, U)^t \otimes (1, T)^t$). Then we obtain for the BPS constraints the following form

$$(\overline{S}\xi_S - \eta_S) \cdot n_S = 0, \tag{3.192}$$
$$(\overline{T}\xi_T - \eta_T) \cdot n_T = 0, \tag{3.193}$$
$$(\overline{U}\xi_U - \eta_U) \cdot n_U = 0 \tag{3.194}$$

and their complex conjugates. Here the charge four-vectors are defined as

$$\xi_S = \begin{pmatrix} p^0 \\ p^2 \\ p^3 \\ q_1 \end{pmatrix}, \quad \eta_S = \begin{pmatrix} p^1 \\ q_3 \\ q_2 \\ -q_0 \end{pmatrix}, \tag{3.195}$$

$$\xi_T = \begin{pmatrix} p^0 \\ p^1 \\ p^3 \\ q_2 \end{pmatrix}, \quad \eta_T = \begin{pmatrix} p^2 \\ q_3 \\ q_1 \\ -q_0 \end{pmatrix}, \tag{3.196}$$

$$\xi_U = \begin{pmatrix} p^0 \\ p^1 \\ p^2 \\ q_3 \end{pmatrix}, \quad \eta_S = \begin{pmatrix} p^3 \\ q_2 \\ q_1 \\ -q_0 \end{pmatrix}. \tag{3.197}$$

Notice that due to our labelling convention of the reference state (3.151) with amplitudes ψ_{lkj} we have $\xi_{\mathrm{S}} \equiv \psi_{lk0}$, $\eta_{\mathrm{S}} \equiv \psi_{lk1}$, $\xi_{\mathrm{T}} \equiv \psi_{l0j}$, etc. After algebraic manipulations it can be shown that (3.192)–(3.194) can be satisfied provided

$$(\overline{S}\xi_S - \eta_S)^2 = 0, \quad (\overline{T}\xi_T - \eta_T)^2 = 0, \quad (\overline{U}\xi_U - \eta_U)^2 = 0, \tag{3.198}$$

meaning that these vectors are also null, i.e. they are lying on the corresponding quadric surface of $\mathbf{CP^3}$ (see the discussion following (3.69)). From these quadratic equations the stabilized values of the moduli are

$$\begin{aligned} S(0) &= \frac{(\xi_S \cdot \eta_S) + i\sqrt{-D}}{(\xi_S \cdot \xi_S)}, \\ T(0) &= \frac{(\xi_T \cdot \eta_T) + i\sqrt{-D}}{(\xi_T \cdot \xi_T)}, \\ U(0) &= \frac{(\xi_U \cdot \eta_U) + i\sqrt{-D}}{(\xi_U \cdot \xi_U)}. \end{aligned} \tag{3.199}$$

Notice that in these equations we have chosen the positive sign since according to a calculation

$$|\overline{S}\xi_1 - \eta_1|^2 = -\frac{1}{2}\frac{(\xi^{(T)} \cdot \xi^{(T)})(\xi^{(U)} \cdot \xi^{(U)})}{(\xi^{(S)} \cdot \xi^{(S)})}, \tag{3.200}$$

hence the quantities $(\xi_S \cdot \xi_S)$ can be chosen negative, hence the imaginary parts of the frozen values for the moduli are indeed negative. Notice also that the quantity $\overline{S}\xi_S - \eta_{\mathrm{S}}$ defines a line in $\mathbf{CP^3}$, and the solution $S(0)$ of (3.199) gives the intersection points of this line with the fixed quadric surface of null vectors defined by the bilinear form of (3.32).

Having calculated the attractor values for the moduli we can obtain the explicit form of the three-qubit state of (3.187). The result is

$$|\Psi(0)\rangle = (-D)^{1/4}[\mathrm{e}^{\mathrm{i}\delta}|000\rangle - \mathrm{e}^{-\mathrm{i}\delta}|111\rangle]. \tag{3.201}$$

Here

$$\begin{aligned} D = {} & (p \circ q)^2 - 4[(p^1 q_1)(p^2 q_2) + (p^1 q_1)(p^3 q_3) + (p^2 q_2)(p^3 q_3)] \\ & + 4p^0 q_1 q_2 q_3 - 4q_0 p^1 p^2 p^3, \end{aligned} \tag{3.202}$$

where $p \circ q = p^0 q_0 + p^1 q_1 + p^2 q_2 + p^3 q_3$, is Cayley's hyperdeterminant of (3.56) and

$$\tan\delta = \sqrt{-D}\frac{p^0}{2p^1p^2p^3 - p^0(p\circ q)}. \tag{3.203}$$

For the BPS solution to be consistent we have to require $-D > 0$ otherwise the scalar fields are real and the Kähler potential is not defined. Using (3.150) and (3.188) and the Bekenstein–Hawking entropy formula $\mathcal{S} = A/4$ we get the well-known result

$$\mathcal{S} = \pi\sqrt{-D}. \tag{3.204}$$

Notice however, that apart from reproducing the result of Behrndt et al. [38] we have also calculated a useful quantity namely our entangled three-qubit state at the horizon. As we will see in the following this quantity will give us extra information on the nature of both BPS and non-BPS solutions.

As an important special case (to be also discussed later in the non-BPS context) let us consider the D2–D6 system [44]. In this case $q_0 = p^1 = p^2 = p^3 = 0$ and the superpotential is of the form $W = UTSp^0 + Sq_1 + Tq_2 + Uq_3$. Using (3.201)–(3.203) the three-qubit entangled state at the horizon is

$$|\Psi(0)\rangle = i\sqrt{2}(-p^0q_1q_2q_3)^{1/4}(|000\rangle + |111\rangle), \tag{3.205}$$

where $-p^0q_1q_2q_3 > 0$. Equation (3.205) is just the (unnormalized) canonical GHZ-state. Notice that for the charge dependent reference state $|\psi\rangle$ of (3.151) $D(\psi) = 4p^0q_1q_2q_3 < 0$, but $D(\Psi(0)) = (\Psi_{000}\Psi_{111})^2 = -4p^0q_1q_2q_3 > 0$. This change of sign is due to our choice of "gauge" $\Phi \equiv 0$ in (3.154). From (3.187) we see that for this D2–D6 system the value of the central charge at the horizon is [34] $Z = -i\sqrt{2}(-p^0q_1q_2q_3)^{1/4}$.

Until this point we have discussed a quantum information theoretic reinterpretation of the attractor mechanism for BPS black-hole solutions. In this picture we are looking at the dynamical system as a one starting from the asymptotically Minkowski geometry where $|\Psi(r)\rangle$ is of the general form (3.154), and when reaching the horizon with $AdS_2 \times S^2$ geometry one is left with $|\Psi(0)\rangle$, a GHZ-like state.

However we have an alternative way of interpretation. In this picture one is starting from the *horizon* with the state $|\Psi(0)\rangle$. We know that this state is of the GHZ (i.e. maximally entangled) form of (3.187). According to (3.189) and the interpretation of the action of the flat covariant derivatives as error operators (see (3.173)–(3.175)) we see that in the BPS case our GHZ-state $|\Psi(0)\rangle$ is protected from bit flip errors. The BPS conditions of (3.189) are precisely the ones of suppressing the bit flip errors for the three-qubit state $|\Psi(0)\rangle$ characterizing the extremal BPS black-hole solution. Notice also that bit flips in the computational base correspond to phase flips in the Hadamard transformed base (see Sect. 3.4.1). Using the definition of (3.110) we can write (3.205) in the form

$$|\Psi(0)\rangle = i(-p^0q_1q_2q_3)^{1/4}[|\overline{000}\rangle + |\overline{011}\rangle + |\overline{101}\rangle + |\overline{110}\rangle]. \tag{3.206}$$

Hence the observation that for the state $|\Psi(0)\rangle$ bit flip errors in the computational base are suppressed also means that errors of the form

$$(I \otimes I \otimes X)|\Psi(0)\rangle = i(-p^0 q_1 q_2 q_3)^{1/4}[\overline{000}\rangle - |\overline{011}\rangle - |\overline{101}\rangle + |\overline{110}\rangle] \quad (3.207)$$

changing the relative phase of the states in the Hadamard transformed base are not allowed. Moreover it is instructive to consider the state (3.206) together with the "reference" state $|\psi\rangle$ which is also depending merely on the charges

$$|\psi\rangle = p^0|000\rangle + q_3|011\rangle + q_2|101\rangle + q_1|110\rangle. \quad (3.208)$$

Hence for the D2–D6 system the charge dependent state resulting from moduli stabilization (3.206) is arising from the reference state (3.208) via discrete Fourier (Hadamard) transformation and uniformization of the amplitudes. Moreover, a comparison of (3.207) with (3.208) suggests that these bit flip errors are somehow connected to sign flip errors of the charges corresponding to $D2$ branes. This conjecture will be verified in the next subsection.

3.5.4 Entanglement and Non-BPS Solutions

In order to gain some insight into the structure of non-BPS solutions provided by quantum information theory we consider the specific example of the D2–D6 system. By minimizing the effective potential the solutions to the moduli are [44]

$$S = \pm i\sqrt{\frac{q_2 q_3}{p^0 q_1}}, \qquad T = \pm i\sqrt{\frac{q_1 q_3}{p^0 q_2}}, \qquad U = \pm i\sqrt{\frac{q_1 q_2}{p^0 q_3}}, \quad p^0 q_1 q_2 q_3 > 0, \quad (3.209)$$

where the sign combinations not violating the positivity of e^K are

$$\{(-,-,-),(-,+,+),(+,-,+),(+,+,-)\}. \quad (3.210)$$

In the work of Kallosh et al. [44] it was also checked that these solutions are forming stable attractors, meaning that the extremum of the black hole potential is also a minimum. In the following we would like to use these solutions to calculate $|\Psi(0)\rangle$ and study its behavior with respect to bit flip errors.

For the $(-,-,-)$ class straightforward calculation gives the result

$$\Psi_{000} = \Psi_{111} = -\frac{i}{\sqrt{8}}(p^0 q_1 q_2 q_3)^{1/4}\{\mathrm{sgn}(p^0) - \mathrm{sgn}(q_3) - \mathrm{sgn}(q_2) - \mathrm{sgn}(q_1)\}, \quad (3.211)$$

$$\Psi_{011} = \Psi_{100} = -\frac{i}{\sqrt{8}}(p^0 q_1 q_2 q_3)^{1/4}\{\mathrm{sgn}(p^0) - \mathrm{sgn}(q_3) + \mathrm{sgn}(q_2) + \mathrm{sgn}(q_1)\}, \quad (3.212)$$

$$\Psi_{101} = \Psi_{010} = -\frac{i}{\sqrt{8}}(p^0 q_1 q_2 q_3)^{1/4}\{\text{sgn}(p^0) + \text{sgn}(q_3) - \text{sgn}(q_2) + \text{sgn}(q_1)\}, \tag{3.213}$$

$$\Psi_{110} = \Psi_{001} = -\frac{i}{\sqrt{8}}(p^0 q_1 q_2 q_3)^{1/4}\{\text{sgn}(p^0) + \text{sgn}(q_3) + \text{sgn}(q_2) - \text{sgn}(q_1)\}. \tag{3.214}$$

For definiteness we consider the case $p^0 > 0, q_1 > 0, q_2 > 0, q_3 > 0$ which is compatible with the constraint $p^0 q_1 q_2 q_3 > 0$. In this case we obtain the state

$$|\Psi(0)\rangle_{---} = \omega\{|000\rangle - |001\rangle - |010\rangle - |011\rangle - |100\rangle - |101\rangle - |110\rangle + |111\rangle\}, \tag{3.215}$$

where

$$\omega = \frac{i}{\sqrt{2}}(p^0 q_1 q_2 q_3)^{1/4}. \tag{3.216}$$

From this state we see that $Z = -\Psi^{111} = -\omega$ in agreement with (4.16) of Ceresole et al. [34]. In the Hadamard transformed basis this state takes the form

$$|\Psi(0)\rangle_{---} = -i(p^0 q_1 q_2 q_3)^{1/4}\{|\overline{000}\rangle - |\overline{011}\rangle - |\overline{101}\rangle - |\overline{110}\rangle\}. \tag{3.217}$$

Comparing (3.206) and (3.217) we see that the basic difference between the BPS and non-BPS case is the change of sign in the combination $p^0 q_1 q_2 q_3$ and also *the appearance of a nontrivial relative phase between the Hadamard transformed basis vectors*.

Let us now consider the class $(-,+,+)$. Since for the $(-,-,-)$ class we had $q_1 S_2 = -\text{sgn}(q_1)\sqrt{q_1 q_2 q_3/p^0}$, $q_2 T_2 = -\text{sgn}(q_2)\sqrt{q_1 q_2 q_3/p^0}$, and $q_3 U_2 = -\text{sgn}(q_3)\sqrt{q_1 q_2 q_3/p^0}$ then going from the class $(-,-,-)$ to the one of $(-,+,+)$ amounts to changing the signs of q_1 and q_2. (Remember our convention of labelling everything from the right to the left.) As a result according to (3.212) the amplitudes Ψ_{011} and Ψ_{100} will be positive and the remaining ones are negative. The resulting state in this case is of the form

$$|\Psi(0)\rangle = \omega\{-|000\rangle - |001\rangle - |010\rangle + |011\rangle + |100\rangle - |101\rangle - |110\rangle - |111\rangle\} \tag{3.218}$$

or in the Hadamard transformed base

$$|\Psi(0)\rangle_{-++} = -i(p^0 q_1 q_2 q_3)^{1/4}\{|\overline{000}\rangle - |\overline{011}\rangle + |\overline{101}\rangle + |\overline{110}\rangle\}. \tag{3.219}$$

We can summarize these observations for all classes of non-BPS attractors with $Z \neq 0$ for the D2–D6 system as

$$|\Psi(0)\rangle_{\gamma\beta\alpha} = -i(p^0 q_1 q_2 q_3)^{1/4}\{|\overline{000}\rangle + \gamma|\overline{011}\rangle + \beta|\overline{101}\rangle + \alpha|\overline{110}\rangle\}, \tag{3.220}$$

where $(\gamma, \beta, \alpha) = \{(-,-,-), (-,+,+), (+,-,+), (+,+,-)\}$. Notice also that for example

$$(I \otimes I \otimes X)|\Psi(0)\rangle_{---} = |\Psi(0)\rangle_{++-},$$
$$(I \otimes I \otimes X)|\Psi(0)\rangle_{-++} = |\Psi(0)\rangle_{+-+}, \qquad \text{e.t.c.} \tag{3.221}$$

This means that the bit flip operators $I \otimes I \otimes X$, $I \otimes X \otimes I$ and $X \otimes I \otimes I$ are transforming in between the admissible classes of (3.210). The rule of transformation is: those class labels that are in the same slot as the bit flip operator X *are not changed*, while the remaining ones are flipped.

What about physics? The non-BPS black holes corresponding to attractors of a D2–D6 system can be characterized by the four three-qubit entangled states of (3.220) depending merely on the charges. This equation should be taken together with the other charge-dependent state of (3.206). It is clear from (3.221) that the error operation on the *first* qubit which is changing the signs of β and γ (the entries of the *second and third* slots) is corresponding to a sign change of q_2 and q_3 (the *second* and *third*) of the charges in the reference state (3.208). Generally a bit flip error on the jth qubit corresponds to a sign flip of the kth an lth charge q_k and q_l where $j \neq k \neq l$, and $j, k, l = 1, 2, 3$.

At this point we can obtain an additional insight into the BPS case as well. Looking back at (3.207) which again corresponds to sign flips of charges q_2 and q_3, we understand that in the BPS case *sign flips of these kind are suppressed*. Although these sign flips are not changing the sign of the combination $p^0 q_1 q_2 q_3$ they are not allowed due to supersymmetry. On the other hand for non-BPS black holes flipping the signs of a pair of charges corresponds to changing the sign of the number of $D2$ branes. (Negative number of branes correspond to positive number of antibranes of the same kind.) These transformations can be regarded as bit flip errors transforming one non-BPS solution to the other. Moreover, according to (3.173)–(3.175) we also see that these bit flip errors have their origin in the action of the flat covariant derivatives on our moduli dependent entangled state of (3.154).

In closing this section we make an additional interesting observation. As we have already realized for the BPS case, at the horizon the form of the three-qubit entangled state will be of very special form. For the D2–D6 system it is proportional to the canonical *GHZ* state. What about the non-BPS case? Comparing (3.205) for the BPS and (3.215) for the non-BPS $(-,-,-)$-class we see that unlike the GHZ state (3.215) does not seem to be related to any three-qubit state of special importance in quantum information theory. However, the state of (3.215) is a particularly nice example of a *graph-state* [32]. Graph states are under intense scrutiny these days due to the special role they are playing in quantum error correction, and in the study of correlations in wave functions of many body systems. Here we would like to show that the non-BPS states associated to the classes of (3.210) are (unnormalized) graph states based on the simple triangle graph.

In order to see this let us recall the results of Sect. 3.4.3 and take an equilateral triangle. Now associate to its vertices the two-dimensional complex Hilbert spaces

$\mathcal{H}_{\rm S}$, $\mathcal{H}_{\rm T}$ and $\mathcal{H}_{\rm U}$ of Sarah, Tom and Ursula. Let us now chose a particular two-qubit state from each of these spaces. First let us define

$$\begin{aligned}|\pm\rangle_{\rm S} &= \frac{1}{\sqrt{2}}(|0\rangle_{\rm S} \pm |1\rangle_{\rm S}),\\ |\pm\rangle_{\rm T} &= \frac{1}{\sqrt{2}}(|0\rangle_{\rm T} \pm |1\rangle_{\rm T}),\\ |\pm\rangle_{\rm U} &= \frac{1}{\sqrt{2}}(|0\rangle_{\rm U} \pm |1\rangle_{\rm U}),\end{aligned} \tag{3.222}$$

which are just the Hadamard transformed states $|\bar{0}\rangle$ and $|\bar{1}\rangle$ of the ones $|0\rangle$ and $|1\rangle$, and associate to the triangle graph the three-qubit state

$$|---\rangle \equiv |-\rangle_{\rm U} \otimes |-\rangle_{\rm T} \otimes |-\rangle_{\rm S}. \tag{3.223}$$

As we know a graph state is arising by specifying the interactions between the states of the vertices along the three edges of the triangle. For graph states the interactions are of the following form

$$\begin{aligned}V_{TS} &= I \otimes I \otimes P_+ + I \otimes \mathcal{Z} \otimes P_-,\\ V_{UT} &= I \otimes P_+ \otimes I + \mathcal{Z} \otimes P_- \otimes I.\\ V_{US} &= I \otimes I \otimes P_+ + \mathcal{Z} \otimes I \otimes P_-,\end{aligned} \tag{3.224}$$

where for the definitions of the 2×2 matrices $P_\pm$ and $\mathcal{Z}$ see (7.84) and (3.105). Now it is straightforward to check that the graph state

$$|G\rangle_{---} = V_{TS} V_{UT} V_{US} |---\rangle, \tag{3.225}$$

is up to the factor $\sqrt{8}\omega$ is precisely the state of (3.215). Moreover, had we chosen the state

$$|-++\rangle = |-\rangle_{\rm U} \otimes |+\rangle_{\rm T} \otimes |+\rangle_{\rm S} \tag{3.226}$$

as the starting state attached to the corresponding vertices of the triangle graph we would have obtained the other graph state

$$|G\rangle_{-++} = V_{TS} V_{UT} V_{US} |-++\rangle, \tag{3.227}$$

which is up to $-\sqrt{8}\omega$ is just the state of (3.218) corresponding to the non-BPS class $(-++)$. The remaining cases are obtained by permutation of the signs $(-++)$. Hence we managed to demonstrate that the entangled states corresponding to non-BPS black hole solutions for the D2–D6 system characterized by the condition $Z \neq 0$ are just graph states associated to the simple triangle graph.

3.5.5 $\mathcal{N} = 8$ Reinterpretation of the STU-Model: Density Matrices

In this subsection using some more results from quantum information theory we would like to comment on the embedding of the solutions of the $d = 4$, $\mathcal{N} = 2$ STU-model in $d = 4$, $\mathcal{N} = 8$ supergravity [35]. As we have seen the $Z \neq 0$ extremal black-hole solutions of the STU-model can be given a nice interpretation in terms of a moduli and charge-dependent *pure* three-qubit entangled state. How to describe the embedding of these solutions in the ones of $\mathcal{N} = 8$ supergravity? In the next section we will see that one way to do this is to consider the *pure* state tripartite entanglement of seven qubits [4, 5, 10]. However, here we would like to describe the solutions in the $\mathcal{N} = 8$ context using *mixed* three-qubit states, characterized by a density matrix with special properties.

The main idea is to associate the matrix of the central charge Z_{AB}, $A, B = 0, 1, \ldots, 7$ to a *bipartite* system consisting of two indistinguishable fermionic subsystems with $2M = \mathcal{N} = 8$ single-particle states. This system is characterized by the pure state

$$|\chi\rangle = \sum_{A,B=0}^{2M-1} Z_{AB}\hat{c}^{\dagger}_{\mathrm{A}}\hat{c}^{\dagger}_{\mathrm{B}}|\Omega\rangle \in \mathcal{A}(\mathbf{C}^{2M} \otimes \mathbf{C}^{2M}), \tag{3.228}$$

where

$$\{\hat{c}_{\mathrm{A}}, \hat{c}^{\dagger}_{\mathrm{B}}\} = \delta_{AB}, \quad \{\hat{c}_{\mathrm{A}}, \hat{c}_{\mathrm{B}}\} = 0, \quad \{\hat{c}^{\dagger}_{\mathrm{A}}, \hat{c}^{\dagger}_{\mathrm{B}}\} = 0, \quad A, B = 0, \ldots 2M - 1. \tag{3.229}$$

Here Z is a $2M \times 2M$ complex antisymmetric matrix, $\hat{c}_{\mathrm{A}}$ and $\hat{c}^{\dagger}_{\mathrm{A}}$ are fermionic annihilation and creation operators, $|\Omega\rangle$ is the fermionic vacuum and the symbol $\mathcal{A}$ refers to antisymmetrisation [49, 50]. It can be shown [49] that the normalization condition $\langle\chi|\chi\rangle = 1$ implies that $2\mathrm{Tr}ZZ^{\dagger} = 1$. However, since our states in the black hole analogy are unnormalized we do not need this condition.

As was demonstrated in the literature [49] local unitary transformations $U \otimes U$ with $U \in U(2M)$ acting on $\mathbf{C}^{2M} \otimes \mathbf{C}^{2M}$ do not change the fermionic correlations and under such transformations Z transforms as

$$Z \mapsto UZU^{\mathrm{T}}. \tag{3.230}$$

In the black hole context for $2M = \mathcal{N} = 8$ the group $U(8)$ is the automorphism group of the $\mathcal{N} = 8$, $d = 4$ supersymmetry algebra.

Since the fermions are indistinguishable, the reduced one-particle density matrices are equal and have the form [51]

$$\rho = ZZ^{\dagger}. \tag{3.231}$$

(For normalized states we have $\rho = 2ZZ^{\dagger}$ in order to have $\mathrm{Tr}\rho = 1$. However, for unnormalized states, our concern here, we prefer to swallow the factor of 2 in (3.231).) However now we cannot pretend that any of the one-particle density matrices describes the properties of precisely the first or the second subsystem. ρ describes the properties of a randomly chosen subsystem that cannot be better identified [52]. A useful measure describing fermionic entanglement for $M = 2$ (which corresponds to $N = 4$ supergravity) is [49, 50]

$$\eta \equiv 8|Z_{01}Z_{23} - Z_{02}Z_{13} + Z_{03}Z_{12}| = 8|\mathrm{Pf}(Z)|. \tag{3.232}$$

For normalized states $0 \leq \eta \leq 1$. For $M > 2$ similar measures related to the Pfaffian in higher dimensions have also been considered [53]. A fermionic analogue of the usual Schmidt decomposition of (3.3) can also be introduced. According to this result [49] (which is just a reinterpretation of an old result of Zumino [54]) there exists an unitary matrix $\mathcal{U} \in U(2M)$ such that

$$\Lambda = \mathcal{U}Z\mathcal{U}^{\mathrm{T}}, \qquad \Lambda = \bigoplus_{j=0}^{M-1} \zeta_j \varepsilon, \quad \varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{3.233}$$

The number of *nonzero* complex numbers ζ_j, $j = 0, 1, \ldots, M-1$ is called the *Slater rank* of the fermionic state. A fermionic state is called *entangled* if its Slater rank is greater than 1. For $M = 2$ a sufficient and necessary condition for having Slater rank 1 states is the vanishing of η, i.e. the Pfaffian of Z (see (3.232)). (For $M > 2$ similar conditions can be found in the literature [53, 55].) Such states can always be written in terms of one Slater determinant, i.e. in this case Z_{AB} is a *separable bivector*. Note, that to the process of obtaining the block diagonal form (3.233) in the black hole picture corresponds the one of finding the canonical form of the central charge matrix Z_{AB}.

One can alternatively characterize bipartite entanglement by the entropies of von Neumann and Rényi of (3.7) and (3.8). Similar to our treatment of Sect. 3.2.1 for fermionic states one calculates the eigenvalues $|\zeta_j|^2$ of the reduced density matrix ρ of (3.231). Then the entropies have the form (compare with (3.7)–(3.8))

$$S_1 = 1 - \sum_{j=0}^{M-1} |\zeta_j|^2 \log_2 |\zeta_j|^2, \quad S_\alpha = 1 + \frac{1}{1-\alpha} \log_2 \sum_{j=0}^{M-1} |\zeta_j|^{2\alpha}. \tag{3.234}$$

The fact that for fermionic systems these entropies satisfy the bound $1 \leq S_\alpha$ can be traced back to the fact that for fermionic density matrices the so-called generalized Pauli principle holds [50]. This is to be contrasted with the lower bound $0 \leq S_\alpha$ coming from (3.14) which holds for all bipartite systems with distinguishable subsystems. Some special cases of S_α are often encountered, for example the quantity

$$\mathrm{Tr}\rho^2 = \sum_{j=0}^{M-1} |\zeta_j|^4, \tag{3.235}$$

is called *the purity* of the mixed state ρ. Obviously one has $S_2 = -\log_2[\mathrm{Tr}\rho^2]$.

Let us now consider the central charge matrix in the $N = 8$ theory

$$Z_{AB} = f^{\Lambda\Sigma}_{AB} Q_{\Lambda\Sigma} - h_{\Lambda\Sigma,AB} P^{\Lambda\Sigma}, \quad A, \dots \Sigma = 1, \dots 8, \quad A < B, \Lambda < \Sigma, \tag{3.236}$$

where the charge vector $(Q^{\Lambda\Sigma}, P_{\Lambda\Sigma})$ is forming the fundamental representation of $E_{7(7)}$. The beins $f^{\Lambda\Sigma}_{AB}(\phi)$ and $h_{AB,\Lambda\Sigma}(\phi)$ are depending on the 70 scalar fields of the coset $E_{7(7)}/SU(8)$. The black hole potential for $N = 8, d = 4$ supergravity has the following form [35]

$$\mathcal{V}_{\text{BH}}(\phi; Q, P) = Z_{AB}\overline{Z}^{AB} = \text{Tr}ZZ^{\dagger} = \frac{1}{2}\text{Tr}\rho, \tag{3.237}$$

where subscripts A, B label an $\mathbf{8}$ and superscripts label an $\bar{\mathbf{8}}$ of $SU(8)$. Hence $\overline{Z}^{AB}$ refers to the complex conjugate of the central charge. (Summation is understood only for $A < B$.) Notice that using (3.231) we have also introduced the (unnormalized) reduced density matrix. It is hermitian $\rho = \rho^{\dagger}$, positive $\rho \geq 0$, however now it is *not* satisfying the additional normalization condition $\text{Tr}\rho = 1$. Equation (3.237) has to be compared with our previous result of (3.158). Both of these equations express the black hole potential as half of the "norm" of a moduli and charge dependent state. However, for the $N = 8$ case it is a *mixed state*. Since the $N = 2$ STU model can be regarded as a consistent truncation of the $N = 8$ case, one might suspect that the mixed state ρ is somehow related to the pure one Ψ of (3.154). Using the result of Ferrara and Kallosh [35] we can easily establish the desired relationship. Indeed it has been shown that the algebraic attractor equations of the $N = 8$ theory can be identified with the corresponding $N = 2$ attractor equations, under the correspondence

$$\zeta_0 = iZ, \qquad \zeta_1 = \overline{D_{\hat{S}}Z}, \qquad \zeta_2 = \overline{D_{\hat{T}}Z}, \qquad \zeta_3 = \overline{D_{\hat{U}}Z}. \tag{3.238}$$

Using (3.144), (3.159), (3.164) and (3.170)–(3.171) we can identify these with the components of $|\Psi\rangle$ of (3.154) as

$$i\zeta_0 = \Psi_{111}, \qquad i\zeta_1 = \Psi_{001}, \qquad i\zeta_2 = \Psi_{010}, \qquad i\zeta_3 = \Psi_{100}. \tag{3.239}$$

We emphasize that these components are not all independent but related to each other via (3.164).

Now we use instead of the labeling $A, B = 0, 1, \dots 7$ the binary one of 000, $001, \dots, 111$ to write the density matrix in the form

$$\rho = \sum_{lkj=0}^{1} |\Psi_{lkj}|^2 |lkj\rangle\langle lkj| = |Z|^2[|000\rangle\langle 000| + |111\rangle\langle 111|] + \cdots. \tag{3.240}$$

Here the vectors $|lkj\rangle$ are the *eigenvectors* of the matrix $ZZ^{\dagger}$ depending on the remaining charges and moduli. In this way we managed to represent ρ as a mixed state, where the eight weights appearing in the mixture are determined by the eight moduli-dependent amplitudes of the pure state of the *STU* model. They are multiplying the three-qubit pure states $|lkj\rangle\langle lkj|$ the mixture is composed of.

However, as we already know from (3.16) a density matrix as a convex linear combination of different types of pure states can be written in many different ways [18, 19]. The one based on the eigenvectors of ρ is just one of them. Of course the "quantum" ensembles to be considered here has to be chosen from the subclass compatible with the U-duality group $E_{7(7)}$. It would be nice to establish an explicit correspondence between consistent truncations [4,5] of the $N = 8$ model other than the $N = 2$ STU one and these alternative decompositions of ρ.

The possibility of interpreting $\rho = ZZ^\dagger$ as a mixed three-qubit state depending on the 56 charges and 70 moduli fields has further illuminating aspects. It is well-known that the entropy formula for regular $N = 8$ black holes in four dimensions can be given in terms of the square root of the magnitude of the unique Cartan–Cremmer–Julia quartic invariant J_4 [35, 56, 57] constructed from the fundamental **56** of the group $E_{7(7)}$. Using the definition of ρ J_4 can be expressed as

$$J_4 = \text{Tr}\rho^2 - \frac{1}{4}(\text{Tr}\rho)^2 + 8\text{RePf}(Z), \qquad \rho = ZZ^\dagger. \tag{3.241}$$

Notice that the terms contributing to J_4 are the *purity* (3.235) (which is related to Renyi's entropy S_2), one-fourth of the norm squared and eight times the real part of a quantity similar to the fermionic entanglement measure η of (3.232). All these terms are invariant under the subgroup $SU(8)$ of local unitary transformations. However, their particular combination is invariant under the larger group $E_{7(7)}$ as well. It is tempting to interpret J_4 as an entanglement measure for a *special* subclass of three-qubit mixed states. Apart from the fact that ρ is an 8×8 matrix the three-qubit reinterpretation will also be justified in the next section where we show that the **56** of E_7 can be described in terms of seven 3-qubit states. We note in this context that finding a suitable measure of entanglement for mixed states is a difficult problem. We remark that the only explicit formula known is the celebrated one of Wootters for two-qubit mixed states [20] the one we have already discussed in (3.22). J_4 might possibly serve as an entanglement measure for three-qubit mixed states having doubly degenerate eigenvalues which is related to the fact that the purification of ρ is the fermionic entangled state of (3.228).

Using this density matrix picture let us now look at the BPS and non-BPS solutions as embedded in the corresponding $N = 8$ ones. According to (3.238)–(3.239) for the BPS case we have

$$\zeta_0 \neq 0, \quad \zeta_1 = \zeta_2 = \zeta_3 = 0, \qquad \mathcal{S} = \pi|Z|^2_{\text{BPS}}, \tag{3.242}$$

where the central charge is calculated at the attractor point. The corresponding density matrix has the form

$$\rho_{\text{BPS}} = |Z|^2_{\text{BPS}}\{|000\rangle\langle 000| + |111\rangle\langle 111|\}, \tag{3.243}$$

which is a state of *Slater rank 1*. Hence for BPS states the corresponding fermionic purification (3.228) can be expressed using Z_{AB} as a separable bivector. This state is consisting of merely one Slater determinant expressed in terms of two states with eight single particle states.

For the non-BPS case we have

$$|\zeta_0| = |\zeta_1| = |\zeta_2| = |\zeta_3| \neq 0, \qquad \mathcal{S} = 4\pi |Z|^2_{\text{nonBPS}}, \tag{3.244}$$

with the corresponding mixed state

$$\rho_{\text{NBPS}} = |Z|^2 \sum_{lkj} |lkj\rangle\langle lkj|, \tag{3.245}$$

which is a state of Slater rank 4. According to Schliemann et al. [49] a fermionic state is called entangled if and only if its Slater rank is strictly greater than 1. Hence class I solutions correspond to non-entangled, and class II solutions correspond to entangled fermionic purifications. We have to be careful however, not to conclude that BPS solutions are represented by non-entangled fermionic purifications and non-BPS solutions with entangled ones. This is because we have not analysed solutions of class III namely the ones with $Z \equiv 0$. For these solutions we have [48] for example

$$\zeta_0 = 0, \qquad \zeta_1 \neq 0, \qquad \zeta_2 = 0, \qquad \zeta_3 = 0, \tag{3.246}$$

hence these solutions also give rise to Slater rank 1 (i.e. non-entangled) states. This is because from the $N = 8$ perspective $N = 2$ non-BPS $Z = 0$ solutions are originated from the $N = 2$ BPS ones by simply exchanging the eigenvalues $|\zeta_0|^2$ and $|\zeta_1|^2$ of ρ.

3.5.6 A Unified Picture for the D2–D6 System

In order to make the picture complete, let us also include for the D2–D6 system the non-BPS solutions of type III. For such solutions we have $Z \equiv 0$. Let us chose the signs for the charges as follows [34]

$$p^0 < 0, \qquad q_3 < 0, \qquad q_2 < 0, \qquad q_1 > 0. \tag{3.247}$$

For this combination the solutions are [48]

$$S = -\frac{i}{q_1}\lambda, \qquad T = \frac{i}{q_2}\lambda, \qquad U = \frac{i}{q_3}\lambda, \qquad \lambda = \sqrt{-\frac{q_1 q_2 q_3}{p^0}}. \tag{3.248}$$

A calculation of the three-qubit entangled state (3.154) shows that

$$|\Psi\rangle_{q_1>0} = i\sqrt{2}(-p^0 q_1 q_2 q_3)^{1/4}(|001\rangle + |110\rangle). \tag{3.249}$$

It is a GHZ-like state obtained from the canonical GHZ state corresponding to the BPS solutions by applying the bit flip error operation $I \otimes I \otimes X$. This is consistent with our interpretation that non-vanishing covariant derivatives of Z at the attractor point (in this case $D_S Z \neq 0$) are represented by bit flip errors. By permutation symmetry the remaining two cases with the sign of q_2 and then the sign of q_3 is chosen to be positive will result in the states

$$|\Psi\rangle_{q_2>0} = i\sqrt{2}(-p^0 q_1 q_2 q_3)^{1/4}(|010\rangle + |101\rangle), \tag{3.250}$$

and

$$|\Psi\rangle_{q_3>0} = i\sqrt{2}(-p^0 q_1 q_2 q_3)^{1/4}(|100\rangle + |011\rangle), \tag{3.251}$$

corresponding to bit flip errors $I \otimes X \otimes I$ and $X \otimes I \otimes I$ ($D_T Z \neq 0$ and $D_U Z \neq 0$).

In the Hadamard transformed basis the connection between sign flip errors of charges and phase flip errors is displayed explicitly. In this case we have

$$|\Psi\rangle_{q_1>0} = i(-p^0 q_1 q_2 q_3)^{1/4}\{|\overline{000}\rangle - |\overline{011}\rangle - |\overline{101}\rangle + |\overline{110}\rangle\}. \tag{3.252}$$

Comparing this with the corresponding state for the BPS solution ($p^0 < 0, q_3 > 0, q_2 > 0, q_1 > 0$)

$$|\Psi\rangle_{q_3>0,q_2>0,q_1>0} = i(-p^0 q_1 q_2 q_3)^{1/4}\{|\overline{000}\rangle + |\overline{011}\rangle + |\overline{101}\rangle + |\overline{110}\rangle\}. \tag{3.253}$$

and the reference state

$$|\psi\rangle = p^0|000\rangle + q_3|011\rangle + q_2|101\rangle + q_1|110\rangle, \tag{3.254}$$

clearly shows that at the attractor point the phase flip error $I \otimes I \otimes \mathcal{Z}$ in the Hadamard transformed base transforming the BPS solution to the non-BPS $Z = 0$ one corresponds to a simultaneous sign flip in the charges q_2 and q_3.

Now we realize that there is a possibility to present a unified formalism for the characterization of all extremal black hole solutions found for the D2–D6 system. In order to do this let us call the charge configuration related to the BPS case the *standard* one. Hence for the standard configuration we have

$$p^0 < 0, \qquad q_3 > 0, \qquad q_2 > 0, \qquad q_1 > 0. \tag{3.255}$$

Our aim is to describe all the remaining classes of solutions as deviations from this one. This viewpoint is justified by the fact that for the BPS solutions bit flip errors corresponding to sign changes of charges are suppressed, but for the remaining non-BPS cases they are not. Let us define a map

$$(\text{sgn}(p^0), \text{sgn}(q_3), \text{sgn}(q_2), \text{sgn}(q_1)) \mapsto (d, c, b, a), \qquad d, c, b, a = 0, 1 \tag{3.256}$$

in the following way. For the standard configuration we define $(dcba) \equiv (0000)$. The occurrence of 1s in some of the slots is indicating a sign flip of the corresponding charge with respect to the standard configuration. Hence for example the class label (0110) refers to the state of (3.252) with the signs of the charges q_3 and q_2 have been changed. (Compare (3.252) and (3.253).) Then recalling our result in (3.220) for the remaining non-BPS classes of solutions we define

$$|\Psi(0)\rangle_{dcba} = i[-(-1)^d p^0 q_1 q_2 q_3]^{1/4}\{e^{id\pi}|\overline{000}\rangle + e^{ic\pi}|\overline{011}\rangle + e^{ib\pi}|\overline{101}\rangle + e^{ia\pi}|\overline{110}\rangle\}. \tag{3.257}$$

BPS (class I) solutions have the label (0000), no charge flips. For class II solutions a quick check shows that the class (1000) corresponds to our state of (3.217), and the one with label (1011) to the one of (3.219). This class can be characterized with and *odd* number of sign flips. In the first case only one charge has been flipped (p^0), in the second three (p^0, q_2 and q_1). For class III (non-BPS, $Z = 0$) solutions correspond to states like (3.252) with class label (0110). They have an even number of sign flips.

Notice also that classes I and III have the same charge orbit structure [47] (that correspond to two separated branches of a disconnected manifold)and both of them have an even number of charge flip errors. Class II solutions have two subclasses. Its is also useful to recall that the configuration with one charge error is known to be upliftable to a $d = 5$ BPS solution, and the other ones with three errors are upliftable to $d = 5$ non-BPS ones [34].

There is however an important distinction to be made between charge flip errors and bit flip errors. We have a nice correspondence between bit flip errors and sign flip ones only for the $D2$-brane charges q_1, q_2 and q_3. The sign flip error of the $D6$-brane charge p^0 cannot be understand in terms of quantum information theory within the STU model. However, by embedding this model into the $N = 8$ one we have seen that the class $p^0 < 0$ corresponds to mixed states with fermionic purification having Slater rank 1, and $p^0 > 0$ with fermionic purifications having Slater rank 4. Notice also that from (3.158) and (3.257) we immediately obtain the result (see (4.20) of the paper of Ceresole et al. [34]).

$$V_{\rm BH}(0) = 2|p^0 q_1 q_2 q_3|^{1/2} = \sqrt{|D|}. \tag{3.258}$$

We remark that the dual situation for extremal black hole solutions (i.e. the D0–D4) system is showing similar features. In this case states very similar to the ones of (3.257) can be introduced. This class of states will contain the basis states $|\overline{111}\rangle$, $|\overline{100}\rangle$, $|\overline{010}\rangle$ and $|\overline{001}\rangle$, i.e. states with opposite parity than the ones of (3.257). Of course our interpretation in terms of charge and bit flip errors still survives.

3.5.7 Real States in the STU Model

As we have seen the states occurring in the STU model are real states meaning that they are either real or local unitary equivalent to real ones. Let us consider now a pure real realization of our state of (3.154)

$$|\hat{\Psi}(U,T,S;p,q)\rangle \equiv C_U \otimes B_T \otimes A_S|\psi\rangle, \tag{3.259}$$

where now we omit the phase factors and the unitary matrices from the expressions of (3.155)–(3.157). So for example we have

$$A_S = \frac{1}{\sqrt{-S_2}}\begin{pmatrix} -S_2 & 0 \\ -S_1 & 1 \end{pmatrix}. \tag{3.260}$$

Then we can write [3]

$$M_{\text{BPS}}^2 = \frac{1}{8}(||\hat{\Psi}||^2 - \text{Tr}(\hat{\varrho}_{TU}\varepsilon \otimes \varepsilon) - \text{Tr}(\hat{\varrho}_{SU}\varepsilon \otimes \varepsilon) - \text{Tr}(\hat{\varrho}_{ST}\varepsilon \otimes \varepsilon)). \tag{3.261}$$

The first term is just $\frac{1}{8}$ times the norm of our entangled state, i.e. $\frac{1}{4}$ times the black hole potential (see (3.158)). The remaining terms can be related to the real concurrence calculated for the reduced density matrices after the single qubits S, T and U are traced out respectively (see (3.37)).

Since $|\hat{\Psi}\rangle$ and $|\Psi'\rangle$ are unitarily related we have $||\hat{\Psi}||^2 = ||\Psi'||^2$. Moreover, the extremal BPS mass squared can also be written in the form [3] $M_{\text{BPS}}^2(0) = \frac{1}{2}\overline{\mathcal{C}}(0)$ where $\overline{\mathcal{C}}(0) = \frac{1}{3}(\mathcal{C}_{ST}(0) + \mathcal{C}_{TU}(0) + \mathcal{C}_{SU}(0))$ is the average *real* concurrence. Hence the entropy for the BPS STU black hole can be written in the alternative forms [3]

$$S_{\text{BH}} = \frac{\pi}{2}\sqrt{\hat{\tau}_{STU}(0)} = \frac{\pi}{2}\overline{\mathcal{C}}(0) = \frac{\pi}{2}||\hat{\Psi}(0)||^2. \tag{3.262}$$

Notice that in these expressions all quantities are expressed in terms of the real moduli dependent three-qubit state $|\hat{\Psi}\rangle$ calculated with the frozen values at the horizon. Of course due to the $SL(2,\mathbf{R})$ invariance of the three-tangle we have $\hat{\tau}_{STU} = \tau_{STU}$ so it has the same value, no matter we use the reference state $|\psi\rangle$ with integer or the one $|\hat{\Psi}\rangle$ with moduli dependent real amplitudes. However, the norm and the average real concurrence depends on the values of the moduli in a nontrivial way. Indeed the combination in (3.261) of these quantities gives M_{BPS}^2 to be extremized. However, quite remarkably according to (3.262) all three quantities are frozen to the same value at the horizon.

Let us try to understand these results using our example of the D2–D6 system. After tracing out one of the qubits the universal formula of (3.257) for the reduced density matrices gives the form

$$\varrho_{ST} = (-\delta p^0 q_1 q_2 q_3)^{1/4} \begin{pmatrix} 1 & 0 & 0 & \gamma\delta \\ 0 & 1 & \alpha\beta & 0 \\ 0 & \alpha\beta & 1 & 0 \\ \gamma\delta & 0 & 0 & 1 \end{pmatrix}, \tag{3.263}$$

$$\varrho_{SU} = (-\delta p^0 q_1 q_2 q_3)^{1/4} \begin{pmatrix} 1 & 0 & 0 & \beta\delta \\ 0 & 1 & \alpha\gamma & 0 \\ 0 & \alpha\gamma & 1 & 0 \\ \beta\delta & 0 & 0 & 1 \end{pmatrix}, \tag{3.264}$$

$$\varrho_{TU} = (-\delta p^0 q_1 q_2 q_3)^{1/4} \begin{pmatrix} 1 & 0 & 0 & \alpha\delta \\ 0 & 1 & \gamma\beta & 0 \\ 0 & \gamma\beta & 1 & 0 \\ \alpha\delta & 0 & 0 & 1 \end{pmatrix}, \tag{3.265}$$

where

$$\alpha = \mathrm{e}^{\mathrm{i}\pi a}, \quad \beta = \mathrm{e}^{\mathrm{i}\pi b}, \quad \gamma = \mathrm{e}^{\mathrm{i}\pi c}, \quad \delta = \mathrm{e}^{\mathrm{i}\pi d}, \tag{3.266}$$

with $(dcba)$ being the binary labels of the corresponding state of (3.257). For the non-BPS class II solutions ($Z \neq 0$) these density matrices are of the form (see for example the choices (1000) and (1011))

$$\varrho = \sqrt{p^0 q_1 q_2 q_3}(I \otimes I \pm \varepsilon \otimes \varepsilon). \tag{3.267}$$

On the other hand for BPS solutions, and non-BPS ones with $Z = 0$ we have

$$\varrho = \sqrt{-p^0 q_1 q_2 q_3}(I \otimes I \pm X \otimes X). \tag{3.268}$$

We would like to transform these states to the purely real basis obtained by acting with the matrices $C_U \otimes B_T \otimes A_s$ of (3.260) on the reference state of (3.151). First notice that the two-qubit density matrices of (3.267)–(3.268) are in the Hadamard transformed base. In order to transform back to the computational base note that $HXH = \mathcal{Z}$, $H\mathcal{Z}H = X$ and $H\varepsilon H = -\varepsilon$. Hence (3.267) will be the same but (3.268) is changed to $\varrho' \simeq (I \otimes I \pm \mathcal{Z} \otimes \mathcal{Z})$. To obtain the form of our density matrices in the real base we have to further transform these matrices with $\mathcal{U} \otimes \mathcal{U}$ where

$$\mathcal{U} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} \tag{3.269}$$

see the unitary matrices occurring in (3.155)–(3.157). Since $\mathcal{U}^\dagger \varepsilon \mathcal{U} = iX$ and $\mathcal{U}^\dagger \mathcal{Z} \mathcal{U} = i\varepsilon$ after the transformation $\hat{\varrho} = \mathcal{U}^\dagger \otimes \mathcal{U}^\dagger \varrho' \mathcal{U} \otimes \mathcal{U}$ we get the final result

$$\hat{\varrho} = \sqrt{p^0 q_1 q_2 q_3}(I \otimes I \pm X \otimes X) \tag{3.270}$$

for all the one-qubit reduced density matrices of the non-BPS ($Z \neq 0$) case and

$$\hat{\varrho} = \sqrt{-p^0 q_1 q_2 q_3}(I \otimes I \pm \varepsilon \otimes \varepsilon) \tag{3.271}$$

for the BPS and non-BPS $Z = 0$ ones.

According to the results of Sect. 3.2.4 these mixed two-rebit states are *separable* for the non-BPS $Z \neq 0$ cases and *entangled* for the BPS and the non-BPS $Z = 0$ ones. Now using (3.37) for the BPS case it is easy to show that at the attractor point we have

$$||\hat{\Psi}(0)||^2 = \overline{\mathcal{C}}(0) = \sqrt{\hat{\tau}_{STU}(0)} = 4\sqrt{-p^0 q_1 q_2 q_3} \tag{3.272}$$

giving rise to (3.262) as claimed. Hence the three quantities namely the norm squared, the average real concurrence and the square-root of the three-tangle gives rise to the same value at the attractor point. Notice however, that for non-BPS $Z \neq 0$ solutions we have $\mathcal{C}(0) = 0$, but $||\hat{\Psi}(0)||^2 = \sqrt{\hat{\tau}_{STU}(0)} \neq 0$. Hence the tripartite entanglement present in the graph states corresponding to these types of non-BPS solutions is sensitive to the loss of one of the qubits (i.e. for tracing them out). Notice that these states are the ones that can be obtained from the BPS ones by an *odd number of sign flips*.

3.5.8 Summary

Let us summarize the main results we have found from the entanglement interpretation of the STU model. We have introduced a three-qubit entangled *pure* state which is depending on the charges and the moduli (3.154). The different components of this pure state are obtained by replacing in the superpotential W some of the moduli with their complex conjugates (see (3.159)–(3.162)). In terms of this unnormalized pure state the black hole potential can be written as one-half the norm of this state (3.158). The flat covariant derivatives with respect to the moduli are acting on this pure state as bit flip errors (3.173)–(3.175). In other words: the representatives of the flat moduli dependent covariant derivatives at the attractor point are the bit flip errors. Using our entangled state BPS and non-BPS ($Z = 0$) solutions can be characterized as the ones containing only *GHZ* components or their bit-flipped versions (3.190), (3.201), (3.205) and (3.249)–(3.251).

For non-BPS ($Z \neq 0$) solutions the corresponding states have amplitudes with equal magnitudes (3.185), meaning that these states are linear combinations of all states with suitable phase factors as expansion coefficients. For the D2–D6 (and its dual D2–D6) systems the expansion coefficients are just positive or negative signs, and the corresponding states are graph states. For the D2–D6 (D0–D4) systems in the Hadamard (discrete Fourier) transformed base the states at the attractor point show a universal behaviour (see (3.257)). The bit flip errors in this base correspond to phase flip ones, which correspond to the sign flips of the charges q_1, q_2, q_3, (p^1, p^2, p^3). For BPS solutions bit flips are suppressed, but for non-BPS solutions they are not. We managed to embed the $N = 2$ STU model characterized by a three-qubit *pure* state to the $N = 8$ one characterized by a three-qubit *mixed* one. These mixed states have fermionic purifications. Fermionic purifications with Slater rank 1 describe BPS, and the ones with Slater rank 4 non-BPS solutions. In the STU truncation for the D2–D6 system these classes correspond to the charge configurations with $p^0 < 0$ and $p^0 > 0$ respectively. Finally we remark that the nice universal behavior we have found (see (3.257) and its D0–D4 analogue) are also physically relevant cases that correspond to *stable* minima of V_{BH} [44, 47, 58], i.e. they are all attractors in a strict sense.

3.6 $N = 8$ Supergravity and the Tripartite Entanglement of Seven Qubits

3.6.1 The Representation Space for the **56** *of* E_7

In Sect. 3.5.5, we have discussed the embedding of the $N = 2$ STU model in $N = 8$ supergravity. This model can be obtained from 11-dimensional M-theory or type IIA string theory via compactification on a seven- or six-dimensional torus. We have learnt that the $N = 2$ STU truncation can be described as a one characterized by a pure three-qubit state coming from a mixed three-qubit one which is related to the $N = 8$ model. In this picture the $N = 8$ model which has 56 charges (28 electric and 28 magnetic) and 70 scalar (moduli) fields truncates to the $N = 2$ STU one which has eight charges (four electric and four magnetic) with six scalar fields (three complex ones). The STU truncation in the entanglement representation amounts to regarding the density matrix related to the $N = 8$ model as a mixture of pure states corresponding to its canonical form (see (3.17)). However, there are other consistent truncations. For example we also have the possibility to truncate to $N = 4$ supergravity with $SL(2) \times SO(6,6)$ symmetry which is a maximal subgroup of the $E_{7(7)}$ on shell symmetry group. Though we have the freedom to consider other decompositions of the density matrix related to the canonical one via a suitable unitary transformation (see (3.18)), however, it is not at all obvious how to do this consistently to get other truncations. Hence in this section we chose a different route and try to describe the black hole solutions of $N = 8$ supergravity with a *pure* entangled state with more than three qubits.

Our success with the three-qubit interpretation of the STU model is clearly related to the underlying $SL(2,\mathbf{R})^{\otimes 3}$ symmetry group of the corresponding $N = 2$ supergravity which can be related to *real states* or *rebits* which are also transforming according to the $(2,2,2)$ of the complex SLOCC subgroup $SL(2,\mathbf{C})^{\otimes 3}$ of a three-qubit system. However, in the $N = 8$ context the symmetry group in question is $E_{7(7)}$ which is not of the product form hence a qubit interpretation seems to be impossible. However, we know that the 56 charges of the $N = 8$ model are transforming according to the fundamental 56-dimensional representation of $E_{7(7)}$. We can try to arrange these 56 charges as the integer-valued amplitudes of a reference state. This is the same starting point as the one we adopted in (3.151). However, 56 is not a power of 2 so the entanglement of this reference state if it exists at all should be of unusual kind. A trivial observation is that $56 = 7 \times 8$ hence the direct sum of seven copies of three-qubit state spaces produces the right count. Moreover, a multiqubit description is possible if the complexification of $E_{7(7)}$, i.e. $E_7(\mathbf{C})$ contains the product of some number of copies of the SLOCC subgroup $SL(2,\mathbf{C})$. Since the rank of E_7 is seven we expect that it should contain seven copies of $SL(2,\mathbf{C})$ groups. Hence this 56-dimensional representation space might be constructed as some combination of tripartite states of seven qubits. From the work of Duff and Ferrara we know that this construction is indeed possible. The relevant decomposition of the 56 of E_7 with respect to the $SL(2)^{\otimes 7}$ subgroup is [4]

$$\begin{aligned}\mathbf{56} \to\ & (\mathbf{2},\mathbf{2},\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{1}) + (\mathbf{1},\mathbf{2},\mathbf{2},\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1}) + (\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{2},\mathbf{1},\mathbf{2},\mathbf{1}) \\ & + (\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{2},\mathbf{1},\mathbf{2}) + (\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{2},\mathbf{1}) \\ & + (\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{2}) + (\mathbf{2},\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2}). \end{aligned} \tag{3.273}$$

Let us now replace formally the 2s with 1s, and the 1s with 0s, and form a 7×7 matrix by regarding the seven vectors obtained in this way as its rows. The result we get is the incidence matrix of the Fano plane in the cyclic (Paley) realization (see (3.124)). Hence the multiqubit state we are searching for is some sort of generalized graph state associated with the graph of the Fano plane. More precisely here we are having a multiqubit state associated with not a graph but a design (BIBD) a creature we are already familiar with from Sect. 3.4.2.

Though the realization with a cyclic incidence matrix is particularly appealing however, we would rather use the other realization based on (3.110) since it is directly related to Hamming's error correcting code (3.114). From the physical point of view our choice is justified by our success of characterizing the BPS and non-BPS black hole solutions of the STU truncation within an error correcting framework.

Let us then use the first of the two matrices of (3.114) as the incidence matrix of the Fano plane, i.e. our (7, 3, 1) design. Let us reproduce here this incidence matrix with the following labelling for the rows (r) and columns (c)

$$\begin{pmatrix} r/c & A & B & C & D & E & F & G \\ a & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ b & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ c & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ d & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ e & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ f & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ g & 0 & 0 & 1 & 0 & 1 & 1 & 0 \end{pmatrix} \mapsto \begin{pmatrix} a_{BDF} \\ b_{ADE} \\ c_{CDG} \\ d_{ABC} \\ e_{BEG} \\ f_{AFG} \\ g_{CEF} \end{pmatrix}, \tag{3.274}$$

where we also displayed the important fact that this labelling automatically defines the index structure for the amplitudes of seven 3-qubit states formed out of seven qubits A, B, C, D, E, F, G (Alice, Bob, Charlie, Daisy, Emma, Fred and George). This convention also fixes the labelling of lines and points of the Fano plane, see Fig. 3.2.

Moreover, according to Fig. 3.3. to the points we can associate qubits, i.e. two-dimensional complex vector spaces $V_A, V_B, \ldots, V_G$ and to the lines three-qubit systems, i.e. vector spaces $V_{BDF}, V_{ADE}, \ldots, V_{CEF}$ where for example $V_{BDF} \equiv V_B \otimes V_D \otimes V_F$.

Let us now make a list of the three-qubit Hilbert spaces $\mathcal{H}_\sigma$, $\sigma \in \mathbf{Z}_2^3 - (000)$

$$\begin{aligned}
\mathcal{H}_{100} &= V_2 \otimes V_4 \otimes V_6 = V_B \otimes V_D \otimes V_F \qquad a_{BDF} \qquad (246),\\
\mathcal{H}_{010} &= V_1 \otimes V_4 \otimes V_5 = V_A \otimes V_D \otimes V_E \qquad b_{ADE} \qquad (145),\\
\mathcal{H}_{110} &= V_3 \otimes V_4 \otimes V_7 = V_C \otimes V_D \otimes V_G \qquad c_{CDG} \qquad (347),\\
\mathcal{H}_{001} &= V_1 \otimes V_2 \otimes V_3 = V_A \otimes V_B \otimes V_C \qquad d_{ABC} \qquad (123),\\
\mathcal{H}_{101} &= V_2 \otimes V_5 \otimes V_7 = V_B \otimes V_E \otimes V_G \qquad e_{BEG} \qquad (257),\\
\mathcal{H}_{011} &= V_1 \otimes V_6 \otimes V_7 = V_A \otimes V_F \otimes V_G \qquad f_{AFG} \qquad (167),\\
\mathcal{H}_{111} &= V_3 \otimes V_5 \otimes V_6 = V_C \otimes V_E \otimes V_F \qquad g_{CEF} \qquad (356).
\end{aligned} \tag{3.275}$$

Here the correspondence between the three-qubit amplitudes and the Steiner triples (3.117) is also indicated. Notice also that the ordering of the spaces $\mathcal{H}_\sigma$ is related to

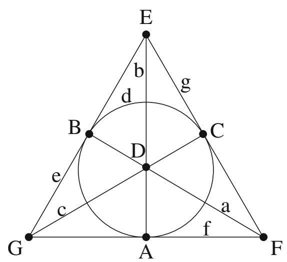

Fig. 3.2 Our labelling convention for the points and lines of the Fano plane

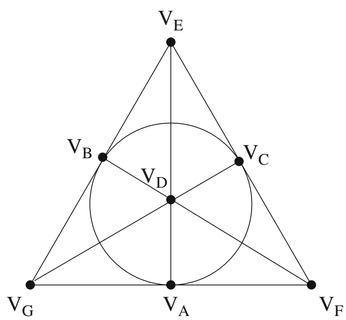

Fig. 3.3 The Fano plane with two-dimensional complex vector spaces (qubits) attached to its points

the ordering of the nontrivial codewords of the first matrix of (3.114) according to the rule

$$1, 2, 3, 4, 5, 6, 7 \to (100), (010), (110), (001), (101), (011), (111),$$

which is the reverse binary labelling.

Now we fix our convention for the representation space of the 56 of E_7 in terms of the spaces $\mathcal{H}_\sigma$ as

$$\begin{aligned} \mathcal{H} \equiv\ & \mathcal{H}_{001} \oplus \mathcal{H}_{010} \oplus \mathcal{H}_{011} \oplus \mathcal{H}_{100} \oplus \mathcal{H}_{101} \oplus \mathcal{H}_{110} \oplus \mathcal{H}_{111} \\ & V_{ABC} \oplus V_{ADE} \oplus V_{AFG} \oplus V_{BDF} \oplus V_{BEG} \oplus V_{CDG} \oplus V_{CEF}, \end{aligned} \tag{3.276}$$

i.e. we switch back to the usual binary labelling.

3.6.2 The Generators of E_7

The Lie-algebra of $E_7(\mathbf{C})$ has 133 complex dimensions. According to the decomposition of (3.276) we have $sl(2,\mathbf{C})^{\oplus 7}$ as a subalgebra of complex dimension $7 \times 3 = 21$. These 21 generators are acting on $\mathcal{H}$ according to the pattern of (3.276) via the well known action of the SLOCC subgroup.

How to define the remaining 112 generators, and how do they act on $\mathcal{H}$? Let us consider the complements of the lines of the Fano plane of Fig. 3.3. These seven sets of four points form seven *quadrangles*. Moreover, we know from (3.119) that these quadrangles form the complementary $(7, 4, 2)$ design to the $(7, 3, 1)$ Steiner triple system (3.117) on which our construction of the representation space $\mathcal{H}$ was based. Since we have already attached to the points of the Fano plane qubits, and this assignment automatically defined our three-qubit states corresponding to the lines, it then follows that the quadrangles define seven 4-qubit states. They are forming the 112-dimensional complex vector space

$$W \equiv V_{DEFG} \oplus V_{BCFG} \oplus V_{BCDE} \oplus V_{ACEG} \oplus V_{ACDF} \oplus V_{ABEF} \oplus V_{ABDG}, \tag{3.277}$$

which we can hopefully use as the space of E_7 generators not belonging to the SLOCC subalgebra. Let us denote the basis vectors of the corresponding four-qubit spaces in the computational base as $(T_{ACEG}, \dots, T_{ABDG})$. This means that for example T_{1011} for one of the subspaces with a fixed index structure corresponds to the amplitude of the four-qubit state $|1011\rangle$ having zeros everywhere except at the entry 1011. The indication that we are on the right track for defining the e_7 algebra via four-qubit states is coming from the possibility of defining the Lie-bracket on W according to the pattern

$$[T_{ACEG}, T_{BC'FG'}] = \Phi(ACEG, BC'FG')\varepsilon_{CC'}\varepsilon_{GG'}T_{ABEF}, \tag{3.278}$$

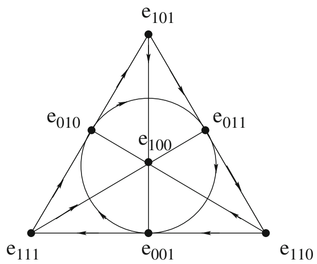

Fig. 3.4 The octonionic multiplication table represented by the Fano plane. The seven octonionic units $e_{001}, \ldots, e_{111}$ are multiplied according to the orientation given to the lines. For example $e_{010}e_{110} = e_{100}$

where in this example the pair CG is the one occurring in both of the quadrangles $ACEG$ and $BCFG$. Indeed, the map underlying this example is the unique SLOCC equivariant map up to scalar and it must be nonzero due to the properties of a semisimple Lie-algebra. The map Φ is restricted by the antisymmetry and the Jacobi identity, however the crucial restriction on Φ is coming from the observation that the Fano plane also serves as a nice mnemonic for the octonionic multiplication table (Fig. 3.4).

Hence if the seven quadrangles are corresponding to the seven imaginary octonionic units it can be shown that the map Φ can take the values ± 1 on the set of pairs of quadrangles.

Let us label the four-qubit states associated with quadrangles according to the rule

$$W = \mathcal{W}_{001} \oplus \mathcal{W}_{010} \oplus \mathcal{W}_{011} \oplus \mathcal{W}_{100} \oplus \mathcal{W}_{101} \oplus \mathcal{W}_{110} \oplus \mathcal{W}_{111}, \qquad (3.279)$$

i.e. the four-qubit states T_σ belonging to the subspace $\mathcal{W}_\sigma$ has the same label as their complementary three-qubit states belonging to the subspace $\mathcal{H}_\sigma$. For example T^{101} is belonging to the subspace $\mathcal{W}_{101}$ having the index structure T^{101}_{ACDF}. Since the upper index defines the structure of the lower ones uniquely we can safely omit it, hence it can be shown that the final result for the commutator for our 112 generators has the form

$$[T_\sigma, T_\tau] = \Phi(\sigma, \tau) T_{\sigma+\tau}, \qquad (3.280)$$

where the explicit form for $\Phi(\sigma, \tau)$ is arising from the octonionic multiplication rule $e_\sigma e_\tau = \Phi(\sigma, \tau) e_{\sigma+\tau}$ as defined again by the Fano plane. Note, that in (3.280) it is understood that pairs of common implicit indices are contracted by two εs as in (3.278).

Now let us discuss the commutators involving also the remaining 21 generators. Define the $sl(2)$ generators as maps acting on the corresponding qubits. For example the ones acting on qubit A have the form

$$s_{AA'} \equiv \frac{1}{2}(E_{AA'}\varepsilon + E_{AA'}\varepsilon), \qquad (3.281)$$

where for the definitions see (3.73). (The labels AA' correspond to the three possible generators s_{00}, s_{11} and $s_{01} = s_{10}$.) Then the commutator of the elements of $sl(2)^{\oplus 7}$ with the ones belonging to W of (3.279) is defined by the action of the SLOCC group on the relevant indices of the four-qubit states. Hence for example T_{ACEG} transforms as a $(2,1,2,1,2,1,2)$ under $sl(2)^{\oplus 7}$.

Since the commutators for the $sl(2)^{\oplus 7}$ subalgebra are the usual ones based on the realization of (3.281), the only type of commutator we have not discussed yet is the one of the form $[T_{DEFG}, T_{D'E'F'G'}]$, i.e. the ones where the index structure of the four-qubit states coincides. Since according to Sect. 3.3.2 four-qubit states taken together with their SLOCC groups acting on them can be described by the $so(4,4,\mathbf{C})$ algebra this commutator can be easily found. It is

$$[T_{DEFG}, T_{D'E'F'G'}] = \varepsilon_{DD'}\varepsilon_{EE'}\varepsilon_{FF'}S_{GG'} + \varepsilon_{DD'}\varepsilon_{EE'}\varepsilon_{GG'}S_{FF'} + \varepsilon_{DD'}\varepsilon_{FF'}\varepsilon_{GG'}S_{EE'} + \varepsilon_{EE'}\varepsilon_{FF'}\varepsilon_{GG'}S_{DD'}, \quad (3.282)$$

where $S_{GG'} \equiv (0,0,0,0,0,0,s_{GG'})$, etc.

Finally let us summarize the structure we have found. Let

$$\mathcal{W}_{000} \equiv sl(2)^{\oplus 7}. \quad (3.283)$$

Define the 133-dimensional complex vector space as

$$\mathcal{W} \equiv \bigoplus_{\sigma \in \mathbf{Z}_2^3} \mathcal{W}_\sigma = \mathcal{W}_{000} \oplus W \quad (3.284)$$

with the Lie-bracket $[\quad]$ as defined above. Then [5, 12, 13]

$$e_7 = (\mathcal{W}, [\quad]). \quad (3.285)$$

As we see $\mathcal{W}$ has a deep connection with the division algebra of octonions, in technical terms e_7 as a vector space has an octonionic grading.

3.6.3 The Generators of e_7 as a Set of Tripartite Transformations

In this subsection we would like to describe the 56 of e_7 in terms of tripartite transformations. In other words we would like to construct the action of the e_7 generators on the representation space $\mathcal{H}$ given by the direct sum form of (3.276). Obviously the action of the $sl(2)^{\oplus 7}$ generators is defined by a decomposition similar to the one of (3.273). The only thing we have to modify is the positions for the dublets. Indeed (3.273) is based on the Paley form of the Hadamard matrix, but our decomposition of $\mathcal{H}$ is based on the form which is related to the classical Hamming code. So we have to put the dublets into the slots where in the Hamming code we have 1s. (See the first code of (3.114).)

So we are left with a definition for the action of the 7×16 generators belonging to the subspace W. Let us consider one of the quadrangles, e.g. $DEFG$ forming one of the blocks of the complementary design. To this quadrangle we associate the 16 generators T^{001}_{DEFG}. In order to see how these generators act let us write $\mathcal{H}$ in the following form

$$\mathcal{H} = V_{ABC} \oplus V_A \otimes (V_{DE} \oplus V_{FG}) \oplus V_B \otimes (V_{DF} \oplus V_{EG}) \oplus V_C \otimes (V_{DG} \oplus V_{EF}). \quad (3.286)$$

Now according to the triality construction of (3.78)–(3.81) to the abstract generators T^{001}_{DEFG} we can associate three different 8×8 representations

$$R_I(T^{001}_{DEFG}) = \begin{pmatrix} 0 & D_I(T^{001}_{DEFG}) \\ -\tilde{D}_I(T^{001}_{DEFG}) & 0 \end{pmatrix}, \quad (3.287)$$

where the 4×4 blocks have the form

$$\begin{aligned} D_1(T^{001}_{DEFG}) &= E_{DF}\varepsilon \otimes E_{EG}\varepsilon : \quad V_{FG} \to V_{DE}, \\ D_2(T^{001}_{DEFG}) &= E_{DE}\varepsilon \otimes E_{FG}\varepsilon : \quad V_{EG} \to V_{DF}, \\ D_3(T^{001}_{DEFG}) &= E_{DE}\varepsilon \otimes E_{GF}\varepsilon : \quad V_{EF} \to V_{DG}. \end{aligned} \quad (3.288)$$

From this it is clear that the 56×56 matrix $\mathcal{R}$ representing T^{001}_{DEFG} is of the form

$$\mathcal{R}(T^{001}) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I \otimes D_1 & 0 & 0 & 0 & 0 \\ 0 & -I \otimes \tilde{D}_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I \otimes D_2 & 0 & 0 \\ 0 & 0 & 0 & -I \otimes \tilde{D}_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I \otimes D_3 \\ 0 & 0 & 0 & 0 & 0 & -I \otimes \tilde{D}_3 & 0 \end{pmatrix}, \quad (3.289)$$

where for simplicity we left the indices $DEFG$ corresponding to the 16 different matrices implicit. They can be recovered from (3.288). From (3.288) we see that the entanglement transformation can be written as a 7×7 diagonal matrix with entries $(0, I \otimes D_1, -I \otimes \tilde{D}_1, I \otimes D_2, -I \otimes \tilde{D}_2, I \otimes D_3, -I \otimes \tilde{D}_3)$ times a 7×7 permutation matrix corresponding to the permutation (1)(23)(45)(67).

The following set of 16 matrices we try to find is $\mathcal{R}(T^{010}_{BCFG})$. This choice corresponds to the set of generators belonging to the subspace $\mathcal{W}_{010}$. The complement of $BCFG$ is the line ADE, hence we write down another decomposition starting with V_{ADE}

$$V_{ADE} \oplus V_A \otimes (V_{BC} \oplus V_{FG}) \oplus V_D \otimes (V_{BF} \oplus V_{CG}) \oplus V_E \otimes (V_{BG} \oplus V_{CF}). \quad (3.290)$$

This space is related to our representation space by some permutation changing the blocks $\mathcal{H}_\sigma$ and at the same time exchanging some of the qubits. Clearly apart from these permutations and the change of labels $DEFG \to BCFG$ the basic structure of the representation is the same. Proceeding as before we get for $\mathcal{R}(T^{010})$

$$\begin{pmatrix} 0 & 0 & I\otimes D_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -I\otimes\tilde{D}_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & (I\otimes D_2)_{12} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & (I\otimes D_3)_{12} \\ 0 & 0 & 0 & -(I\otimes\tilde{D}_2)_{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -(I\otimes\tilde{D}_3)_{12} & 0 & 0 \end{pmatrix}. \tag{3.291}$$

Here for example

$$(I\otimes D_2(T^{010}_{BCFG}))_{12} = (I\otimes E_{BC}\varepsilon\otimes E_{FG}\varepsilon)_{12} = E_{BC}\varepsilon\otimes I\otimes E_{FG}\varepsilon. \tag{3.292}$$

This matrix can again be written as the product of a diagonal matrix times the matrix representing the permutation (2)(13)(46)(57).

Let us now find the representation matrix $\mathcal{R}(T^{011}_{BCDE})$. Since the complement of the quadrangle $BCDE$ in this case is AFG we have a decomposition starting with V_{AFG} of the form

$$V_{AFG}\oplus V_A\otimes(V_{BC}\oplus V_{DE})\oplus V_F\otimes(V_{BD}\oplus V_{CE})\oplus V_G\otimes(V_{BE}\oplus V_{CD}). \tag{3.293}$$

Now the permutation exchanging the different three-qubit state spaces is (3)(12)(47)(56) and the matrix $\mathcal{R}(T^{011})$ is

$$\begin{pmatrix} 0 & I\otimes D_1 & 0 & 0 & 0 & 0 & 0 \\ -I\otimes\tilde{D}_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & (I\otimes D_2)_{13} \\ 0 & 0 & 0 & 0 & 0 & (I\otimes D_3)_{13} & 0 \\ 0 & 0 & 0 & 0 & -(I\otimes\tilde{D}_3)_{13} & 0 & 0 \\ 0 & 0 & 0 & -(I\otimes\tilde{D}_2)_{13} & 0 & 0 & 0 \end{pmatrix}. \tag{3.294}$$

Here for example

$$(I\otimes D_3(T^{011}_{BCDE}))_{13} = (I\otimes E_{BC}\varepsilon\otimes E_{ED}\varepsilon)_{13} = E_{ED}\varepsilon\otimes E_{BC}\varepsilon\otimes I. \tag{3.295}$$

Proceeding similarly the remaining generators can be constructed. (For some subtleties however, see [5]). The basic observation is that the generators in question are always of the form consisting of a block-diagonal matrix containing tripartite transformations times a permutation. The structure of the permutation is uniquely determined by the action of the Z_2^3 label σ of T^σ on the Z_2^3 label $\tau(\text{mod } 2)$ of the component $\mathcal{H}_\tau$ corresponding to the tripartite state in the direct sum of $\mathcal{H}$ in (3.276). Notice, that the generators we have constructed are the ones having a block diagonal structure with respect to our choice $\mathcal{H}$ for the representation space. By working out the corresponding permutations one can see that the representation matrices of the remaining generators T^{100}, T^{101}, T^{110} and T^{111} are not block diagonal ones. Clearly for the representatives of T^{001}, T^{010} and T^{011} the block-diagonal structure consisting of a 24-dimensional and a 32-dimensional block corresponds to the decomposition

$$\mathbf{56} \to (\mathbf{2}, \mathbf{12}) \oplus (\mathbf{1}, \mathbf{32}), \tag{3.296}$$

with respect to the maximal subgroup $SL(2, \mathbf{C}) \times SO(6, 6, \mathbf{C})$.

Notice that the $(\mathbf{2}, \mathbf{12})$ part of the representation space $\mathcal{H}$ consists of the amplitudes of the form

$$\begin{pmatrix} d_{ABC} \\ b_{ADE} \\ f_{AFG} \end{pmatrix} \in \mathcal{H}_{(\mathbf{2},\mathbf{12})} \equiv \mathcal{H}_{001} \oplus \mathcal{H}_{010} \oplus \mathcal{H}_{011}, \tag{3.297}$$

and the $(\mathbf{1}, \mathbf{32})$ part of the ones

$$\begin{pmatrix} a_{BDF} \\ e_{BEG} \\ c_{CDG} \\ g_{CEF} \end{pmatrix} \in \mathcal{H}_{(\mathbf{1},\mathbf{32})} \equiv \mathcal{H}_{100} \oplus \mathcal{H}_{101} \oplus \mathcal{H}_{110} \oplus \mathcal{H}_{111}. \tag{3.298}$$

We see that the $(\mathbf{2}, \mathbf{12})$ space consists of all the amplitudes sharing qubit A in common, and the $(\mathbf{1}, \mathbf{32})$ one of all the ones *excluding* qubit A. Moreover, for the $(\mathbf{2}, \mathbf{12})$ part the $3 \times 16 = 48$ generators corresponding to the quadrangles *DEFG*, *BCFG* and *BCDE* are all built up from quantities of the form $I \otimes D_r$, $r = 1, 2, 3$, with the 2×2 identity matrix acting on qubit A is the dummy label for the action of $SL(2)_{\text{A}}$ (3 generators). The remaining 6×3 generators are coming from the diagonal blocks not displayed in (3.289), (3.291), (3.294). On the other hand the $(\mathbf{1}, \mathbf{32})$ part is built up from quantities like

$$I \otimes E_{DE}\varepsilon \otimes E_{FG}\varepsilon, \quad E_{DE}\varepsilon \otimes I \otimes E_{FG}\varepsilon, \quad E_{FG}\varepsilon \otimes E_{DE}\varepsilon \otimes I, \tag{3.299}$$

with the corresponding permutations of the letters D, E, F, G related to triality and similar expressions involving the remaining quadrangles *BCFG* and *BCDE*.

It is also clear that by looking back to the explicit form of the operators $\mathcal{R}(T^{001})$ ((3.276) and (3.289)) we can understand the further decomposition

$$(\mathbf{2}, \mathbf{12}) \oplus (\mathbf{1}, \mathbf{32}) \to (\mathbf{2}, \mathbf{2}, \mathbf{2}, \mathbf{1}) \oplus (\mathbf{2}, \mathbf{1}, \mathbf{1}, \mathbf{8_v}) \oplus (\mathbf{1}, \mathbf{2}, \mathbf{1}, \mathbf{8_s}) \oplus (\mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{8_s}), \tag{3.300}$$

with respect to the inclusion $SL(2) \times SL(2) \times SL(2) \times SO(4,4) \subset SL(2) \times SO(6,6)$.

The main conclusion of this subsection is that the generators of $e_7(\mathbf{C})$ can be written as combinations of tripartite entanglement transformations. Some of them are of SLOCC form (the ones operating in the diagonal blocks) and the others are establishing correlations between the different tripartite states. Notice also that the representation theoretic details are entirely encoded in the $(7,3,1)$ design and its complementary $(7,4,2)$ one, which are related to lines and quadrangles of the smallest discrete projective plane: the Fano plane. Moreover, all these designs are described in a unified form in the nontrivial codewords of the $(7,4,1)$ Hamming code (3.114). The Hamming code in turn is clearly related to the Hadamard matrix (3.110) which is the discrete Fourier transform on three-qubits. Since Hadamard transformations of this kind played a crucial role in our obtaining a nice characterization of BPS and non-BPS solutions to the STU truncation, this gives a hint that black hole solutions of more general type might be understood in a framework related to error correcting codes. In order to gain more insight into such issues now we turn to the structure of the $E_{7(7)}$ symmetric black hole entropy formula related to Cartan's quartic invariant of (3.241).

3.6.4 Cartan's Quartic Invariant as an Entanglement Measure

Let us now consider the problem of finding an appropriate measure of entanglement for the tripartite entanglement of our seven qubits. We have seen that there are seven tripartite systems associated to the seven lines of the Fano plane. Moreover, we know that the unique $SL(2,\mathbf{C})^{\otimes 3}$ and triality invariant for three-qubit systems is τ_{123} related to Cayley's hyperdeterminant (3.56). Since we have seven tripartite systems we are searching for an $E_7(\mathbf{C})$ invariant which is quartic in the amplitudes and when it is restricted to any of the subsystems corresponding to the lines of the Fano plane gives rise to Cayley's hyperdeterminant. Based on a result of algebraic geometry [13] which states that there is an invariant quartic form on $\mathcal{H}$ which is also the unique $W(E_7)$ (the Weyl group of $E(7)$) invariant quartic form, whose restriction to the lines of the Fano plane is proportional to Cayley's hyperdeterminant. From this result it follows that this quartic invariant we are searching for should contain the sum of *seven* copies of the expression for Cayley's hyperdeterminant.

The invariant in question is of course Cartan's quartic invariant J_4 well-known from studies concerning $SO(8)$ supergravity [57, 59–61], the one we have already referred to in (3.241). J_4 is the singlet in the tensor product representation $56 \times 56 \times 56 \times 56$. Its explicit form in connection with stringy black holes with their $E_{7(7)}$ symmetric area form [60] is given either in the Cremmer-Julia form [59] in terms of the complex 8×8 central charge matrix Z (rewritten in (3.241)) or in the Cartan form [57] in terms of two real 8×8 ones $\mathcal{P}$ and $\mathcal{Q}$ containing the quantized electric and magnetic charges of the black hole. Its new form in terms of the 56 *complex* amplitudes of our seven qubits has been calculated by Duff and Ferrara [4]

$$\begin{aligned}J_4 = {} & \frac{1}{2}(a^4 + b^4 + c^4 + d^4 + e^4 + f^4 + g^4) \\ & + 2\big[a^2b^2 + b^2c^2 + c^2d^2 + d^2e^2 + e^2f^2 + f^2g^2 + g^2a^2 \\ & + a^2c^2 + b^2d^2 + c^2e^2 + d^2f^2 + e^2g^2 + f^2a^2 + g^2b^2 \\ & + a^2d^2 + b^2e^2 + c^2f^2 + d^2g^2 + e^2a^2 + f^2b^2 + g^2c^2\big] \\ & + 8[aceg + bcfg + abef + defg + acdf + bcde + abdg], \quad (3.301)\end{aligned}$$

where we have used the definitions from (3.65)–(3.66) and we have for example

$$bcde = \varepsilon^{A_1A_3}\varepsilon^{B_3B_4}\varepsilon^{C_2C_3}\varepsilon^{D_1D_2}\varepsilon^{E_1E_4}\varepsilon^{G_2G_4}b_{A_1D_1E_1}c_{C_2D_2G_2}d_{A_3B_3C_3}e_{B_4E_4G_4}. \quad (3.302)$$

Notice that according to our labelling convention as given by (3.264)–(3.270) the terms containing *four* tripartite systems can be written symbolically as

$$\begin{aligned} aceg &= \psi^{100}\psi^{110}\psi^{101}\psi^{111}, \\ bcfg &= \psi^{010}\psi^{110}\psi^{011}\psi^{111}, \\ abef &= \psi^{100}\psi^{010}\psi^{101}\psi^{011}, \\ defg &= \psi^{001}\psi^{101}\psi^{011}\psi^{111}, \\ acdf &= \psi^{100}\psi^{110}\psi^{001}\psi^{011}, \\ bcde &= \psi^{010}\psi^{110}\psi^{001}\psi^{101}, \\ abdg &= \psi^{100}\psi^{010}\psi^{001}\psi^{111}. \end{aligned} \quad (3.303)$$

Notice that the sum of the $\mathbf{Z}_2^3$ labels always gives (000) mod 2 corresponding to the fact that the resulting combination has no $\mathbf{Z}_2^3$ "charge", i.e. it is belonging to the singlet of E_7 as it has to be. The remaining terms of J_4 containing *two* and *one* tripartite states obviously share the same property. Do not confuse however, the upper indices, e.g. in ψ^{001} with the lower ones occurring in (3.41), e.g. ψ_{ABC}. Upper indices label the superselection sectors, i.e. the different types of tripartite systems and lower indices label the components with respect to the basis vectors. So for example ψ^{001}_{ABC} is just another notation for d_{ABC} according to the labelling scheme of (3.275). However, it is interesting to realize that the sum of the lower indices (regarded as elements of $\mathbf{Z}_2^3$) occurring in the terms of the expression for Cayley's hyperdeterminant (3.56) gives again (000). Moreover, some of the combinations in (3.303) are having the same form as the ones in (3.56). This coincidence might be an indication that using the 56 amplitudes in the purely $\mathbf{Z}_2^3$ labelled form ψ^{σ}_{ijk}, the quartic invariant J_4 can be expressed in a very compact form reflecting additional symmetry properties.

Another important observation is that the terms occurring in the (3.301) expression for J_4 can be understood using the *dual* Fano plane. To see this note, that the Fano plane is a projective plane hence we can use projective duality to exchange the role of lines and planes. Originally we attached qubits to the points, and tripartite systems to the lines of the Fano plane. Now we take the dual perspective, and attach the tripartite states to the points and qubits to the lines of the dual Fano plane see

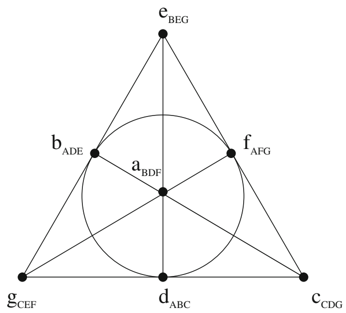

Fig. 3.5 The dual Fano plane. To its points now we attached three-qubit states with the representative amplitudes indicated. To the lines we associate the common qubits these tripartite states share

Fig. 3.5. In the ordinary Fano plane the fact that three lines are intersecting in a unique point corresponded to the fact that any three entangled tripartite systems share a unique qubit. In the dual perspective this entanglement property corresponds to the geometric one that three points are always lying on a unique line. For example let us consider the three points corresponding to the tripartite states belonging to the subspaces $\mathcal{H}_\sigma$, with $\sigma = (001), (010), (011)$. According to (3.275) to these subspaces correspond the amplitudes d, b, and f. Looking at Fig. 3.5 these amplitudes define the corresponding points lying on the line dbf. This line is defined by the common qubit these tripartite states share, i.e. qubit A.

In the dual Fano plane we have seven points, with seven tripartite states attached to them. The corresponding entanglement measures are proportional to seven copies of Cayley's hyperdeterminant, then in J_4 we have the terms $a^4, b^4, c^4, d^4, e^4, f^4$ and g^4. We also have seven lines with three tripartite states on each of them. We can group the 21 terms of the form a^2b^2, etc., into seven groups associated to such lines. According to the explicit formula (3.67) they are describing the pairwise entanglement between the three different tripartite systems (sharing a common qubit). For example for the line dbf we have the terms b^2d^2, d^2f^2 and b^2f^2 describing such pairwise entanglements. Finally we have seven quadrangles (as complements to the lines) with four entangled tripartite systems. They are precisely the ones as listed in (3.303) giving rise to the last seven terms in J_4. Hence the terms in J_4 are of three type

$$\begin{aligned} \text{POINT} &\leftrightarrow 1 \quad \text{TRIPARTITE} \quad \text{STATE} \leftrightarrow a^4, \dots, \\ \text{LINE} &\leftrightarrow 3 \quad \text{TRIPARTITE} \quad \text{STATES} \leftrightarrow (b^2d^2, d^2f^2, b^2f^2), \dots, \\ \text{QUADRANGLE} &\leftrightarrow 4 \quad \text{TRIPARTITE} \quad \text{STATES} \leftrightarrow aceg, \dots. \end{aligned} \tag{3.304}$$

It is useful to remember that the tripartite states forming lines are *sharing* a qubit, and the tripartite ones forming the quadrangles which are complements to these lines are *excluding* the corresponding ones.

We have already seen that the truncation of our system with seven tripartite states to a single tripartite one yields the three-tangle $\tau_3 = 4|D(\psi)|$ as the natural measure of entanglement. Here ψ can denote any of the amplitudes from the set $\psi^\sigma, \sigma \in \mathbf{Z_2}^3 - (000)$. In the black hole analogy where instead of the complex amplitudes of ψ we use integer ones corresponding to the quantized charges the scenario we get is the one of the STU model which we have already discussed in Sect. 3.5. In this case the black hole entropy is given by the formula

$$S = \pi\sqrt{|D(\psi)|} = \frac{\pi}{2}\sqrt{\tau_3^{(1)}} \tag{3.305}$$

i.e. it is related to the three-tangle $\tau_3^{(1)} \equiv \tau_{123}$. The upper index indicates that we have merely one tripartite system. The geometric picture suggested by our use of the dual Fano plane is that of a truncation of the entangled design to a single point.

Consider now a truncation of the seven qubit system to one of the lines of the dual Fano plane Fig. 3.5. Let us take for example the line dbf. As the measure of entanglement for this case we define

$$\tau_3^{(3)} = 2|b^4 + d^4 + f^4 + 2(b^2d^2 + d^2f^2 + b^2f^2)|, \tag{3.306}$$

where the notation $\tau_3^{(3)}$ indicates that now we have three tripartite states. Now we write the state corresponding to the line dbf in the form

$$|\psi\rangle = \sum_{ABCDEFG=0,1} |A\rangle \otimes (d_{ABC}|BC\rangle + b_{ADE}|DE\rangle + f_{AFG}|FG\rangle). \tag{3.307}$$

This notation clearly displays that this state is an entangled one of qubit A with the remaining ones $(BC)(DE)(FG)$. Recalling that on this state the $(2, 12)$ of $SL(2) \times SO(6, 6)$ acts we can write this as

$$|\psi\rangle = \sum_{A\mu} \psi_{A\mu}|A\rangle \otimes |\mu\rangle, \quad A = 0, 1, \quad \mu = 1, 2, \dots, 12. \tag{3.308}$$

Let us discuss the role the group $SL(2) \times SO(6, 6)$ plays in the quantum information theoretic context. $SL(2)$ corresponds to the usual SLOCC protocols. The second one $SO(6, 6)$ contains two different types of transformations. One set corresponds to the remaining part of the SLOCC group, i.e. $SL(2)^{\otimes 6}$ (18 generators). The other set defines transformations transforming states between the different tripartite sectors with different $\mathbf{Z}_2^3$ charge. As we know these transformations are generated by three sets of four-qubit states (3×16 generators).

Denote by ψ the 2×12 matrix of (3.308). For its components $\psi_{A\mu}$ we introduce the notation

$$p^{\mu} \equiv \psi_{0\mu} = \begin{pmatrix} d_{0BC} \\ b_{0DE} \\ f_{0FG} \end{pmatrix}, \quad q^{\mu} \equiv \psi_{1\mu} = \begin{pmatrix} d_{1BC} \\ b_{1DE} \\ f_{1FG} \end{pmatrix}. \tag{3.309}$$

Now we employ the notation

$$\mathbf{pq} \equiv \mathcal{G}_{\mu\nu} p^{\mu} q^{\nu} = p^{\mu} q_{\mu} \equiv d_0 \cdot d_1 + b_0 \cdot b_1 + f_0 \cdot f_1, \quad \mu, \nu = 1, 2, \ldots, 12, \tag{3.310}$$

where the 12×12 matrix $\mathcal{G}$ with 4×4 blocks as elements has the form

$$\mathcal{G} = \begin{pmatrix} \varepsilon \otimes \varepsilon & 0 & 0 \\ 0 & \varepsilon \otimes \varepsilon & 0 \\ 0 & 0 & \varepsilon \otimes \varepsilon \end{pmatrix}, \tag{3.311}$$

and the Plücker coordinates

$$P^{\mu\nu} \equiv p^{\mu} q^{\nu} - p^{\nu} q^{\mu}, \tag{3.312}$$

to get for the invariant $\tau_3^{(3)}$ the following expression

$$\tau_3^{(3)} = 2|P_{\mu\nu} P^{\mu\nu}| = 4|(\mathbf{pp})(\mathbf{qq}) - (\mathbf{pq})^2|. \tag{3.313}$$

In the black hole analogy using for p^{μ} and q^{μ} instead of complex numbers integers corresponding to quantized charges of electric and magnetic type the measure of entanglement in (3.306) can be related to the black hole entropy

$$S = \frac{\pi}{2} \sqrt{\tau_3^{(3)}}, \tag{3.314}$$

coming from the truncation of the $N = 8$ case with $E_{7(7)}$ symmetry to the $N = 4$ one [4, 44, 62] with symmetry group $SL(2) \times SO(6, 6)$.

From the 2×12 matrix ψ of (3.308) we can form the one $\varrho \equiv \psi\psi^{\dagger}$ which is just the reduced density matrix of qubit A the one all of our tripartite systems share. (See (3.4)–(3.5).) From (3.11) we know that for normalized states $\langle\psi|\psi\rangle = 1$ the measure

$$0 \le \tau_{1(234567)} = 4\overline{P_{\alpha\beta}} P^{\alpha\beta} = 4|\mathrm{Det}\varrho| \le 1, \quad \alpha, \beta = 1, 2, \ldots, 12, \tag{3.315}$$

i.e. the concurrence squared gives information on the degree of separability of qubit A from the rest of the system. Here unlike in (3.313) summation is understood with respect to the 12×12 unit matrix.

Now we can prove that for normalized states

$$0 \le \tau_3^{(3)} \le 1. \tag{3.316}$$

Indeed after noticing that

$$\varepsilon \otimes \varepsilon = UU^T, \qquad U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & i & i & 0 \\ 0 & -1 & 1 & 0 \\ i & 0 & 0 & -i \end{pmatrix} \in SU(4) \tag{3.317}$$

with the help of U we can transform the four components of the amplitudes $d_{0BC}, \ldots, f_{1FG}$ to the so-called magic base [20] consisting of the four famous Bell-states with suitable phase factors included. (Algebraically this is the base related to biquaternions, the complex generalization of the split-quaternions discussed in Sect. 3.2.3.) Then we have $\tau_3^{(3)} = 2|P_{\alpha\beta}P^{\alpha\beta}|$ where $P^{\alpha\beta}$ refers to the components of the Plücker matrix in the magic base and summation is now with respect to $\delta_{\alpha\beta}$. Since $\varrho = \psi\psi^\dagger$ is invariant with respect to this transformation $\psi \to \psi\mathcal{U}$ where $\mathcal{U} = U \oplus U \oplus U$ the expression in (3.315) is not changed. Using (3.315) and the triangle inequality

$$0 \leq 4|P_{\alpha\beta}P^{\alpha\beta}| \leq 4\overline{P_{\alpha\beta}}P^{\alpha\beta} \leq 1, \tag{3.318}$$

hence we get (3.316).

An immediate consequence of this is that $\tau_3^{(3)}$ vanishes for systems where qubit A is separable from the rest. Similar conclusions can be drawn from the vanishing of the *six* quantities (based on the remaining six qubits) defined accordingly. The six new quantities $\tau_3^{(3)}$ are vanishing when any qubit located at the vertices of the Fano plane is separable from the tripartite systems associated with the three lines the qubit is lying on. It is important to realize however, that one can also get $\tau_3^{(3)} = 0$ by choosing $b_{100} = b_{010} = b_{001} = d_{100} = d_{010} = d_{001} = f_{100} = f_{010} = f_{001} = 1/3$. This state corresponds to the situation of choosing three different tripartite states belonging to the class of the $|W\rangle$ state of (3.44). These tripartite states are genuine entangled three-qubit ones which retain maximal bipartite entanglement when any one of the three qubits is traced out.

Having discussed the truncation to a line of the dual Fano plane, now we consider the complementary situation, i.e. truncation to a quadrangle. By a quadrangle as usual we mean the complement of a line. We have seen that there is a complementary relationship between the entanglement properties as well. Three tripartite systems associated to a line share a common qubit, and four tripartite systems associated to the complement of this line exclude precisely this qubit. Hence we are expecting this relationship to be manifest in the special form of an entanglement measure characterizing this situation.

As an example let us consider again the line dbf and its complement the quadrangle $aceg$. We define the quantity

$$\tau_3^{(4)} = 2|a^4+c^4+e^4+g^4+2(a^2c^2+a^2e^2+a^2g^2+c^2e^2+c^2g^2+e^2g^2)+8aceg|. \tag{3.319}$$

Here the notation $\tau_3^{(4)}$ refers to the situation of entangling *four* tripartite systems. In the following we prove that $\tau_4^{(4)}$ is the entanglement measure characterizing the configuration complementary to the one of the previous subsection.

The first observation is a group theoretic one. The amplitudes b, d and f are transforming according to the $(\mathbf{2}, \mathbf{12})$ and the complementary ones a, c, e and g according to the $(\mathbf{1}, \mathbf{32})$ of $SL(2) \times SO(6, 6,)$, i.e. they are spinors under $SO(6, 6)$. This fact is clearly displayed in our explicit matrix representation equations (3.289), (3.291) and (3.294). Hence our invariant $\tau_3^{(4)}$ should also be regarded as the singlet in the symmetric tensor product of 4 spinor representations of $SO(6, 6)$.

Our second observation is based on the black hole analogy. Let us relate our (unnormalized) amplitudes $a, b, \ldots, g$ to the quantized charges of the $E_{7(7)}$ symmetric area form [60] of the black hole. In this case we have $7 \times 8 = 56$ integers regarded as amplitudes of a seven qubit system associated to the entangled design defined by the Fano plane. These amplitudes correspond to the two 8×8 antisymmetric matrices of charges $\mathcal{P}$ and $\mathcal{Q}$. Then the Cartan form of our quartic invariant $J_4(\mathcal{P}, \mathcal{Q})$ is [57]

$$J_4(\mathcal{P}, \mathcal{Q}) = -\mathrm{Tr}(\mathcal{Q}\mathcal{P}\mathcal{Q}\mathcal{P}) + \frac{1}{4}(\mathrm{Tr}\mathcal{Q}\mathcal{P})^2 - 4\left(\mathrm{Pf}(\mathcal{P}) + \mathrm{Pf}(\mathcal{Q})\right). \tag{3.320}$$

(Compare this form with the one of (3.244) given in terms of the central charge matrix.) In the context of toroidal compactifications of M-theory or type II string theory the antisymmetric matrices $\mathcal{P}$ and $\mathcal{Q}$ may be identified as [63]

$$\mathcal{Q} = \begin{pmatrix} [D2]^{mn} & [F1]^m & [kkm]^m \\ -[F1]^m & 0 & [D6] \\ -[kkm]^m & -[D6] & 0 \end{pmatrix},$$
$$\mathcal{P} = \begin{pmatrix} [D4]_{mn} & [NS5]_m & [kk]_m \\ -[NS5]_m & 0 & [D0] \\ -[kk]_m & -[D0] & 0 \end{pmatrix}, \quad m, n = 1, \ldots 6. \tag{3.321}$$

Here, $[D2]^{mn}$ denotes a $D2$ brane wrapped along the directions mn of a six-dimensional torus T^6. $[D4]_{mn}$ corresponds to $D4$-branes wrapped on all directions *but* mn, $[kk]_m$ denotes a momentum state along direction m, $[kkm]^m$ a Kaluza–Klein five-monopole localized along the direction m, $[F1]^m$ a fundamental string winding along direction m, and $[NS5]_m$ a $NS5$-brane wrapped on all directions but m. Then the $N = 4$ truncation where

$$\mathcal{Q} = \begin{pmatrix} 0 & [F1]^m & [kkm]^m \\ -[F1]^m & 0 & 0 \\ -[kkm]^m & 0 & 0 \end{pmatrix}, \quad \mathcal{P} = \begin{pmatrix} 0 & [NS5]_m & [kk]_m \\ -[NS5]_m & 0 & 0 \\ -[kk]_m & 0 & 0 \end{pmatrix}, \tag{3.322}$$

should corresponds to the case of our truncation to a line (e.g. the one dbf). In this case our $\tau_3^{(3)}$ is just the quartic invariant with respect to $SL(2) \times SO(6, 6)$. The complementary case

$$\mathcal{Q} = \begin{pmatrix} [D2]^{mn} & 0 & 0 \\ 0 & 0 & [D6] \\ 0 & -[D6] & 0 \end{pmatrix}, \quad \mathcal{P} = \begin{pmatrix} [D4]_{mn} & 0 & 0 \\ 0 & 0 & [D0] \\ 0 & -[D0] & 0 \end{pmatrix}, \qquad (3.323)$$

of the $N = 2$ truncation should correspond to our restriction to quadrangles (e.g. the one $aceg$). The resulting quartic invariant, also based on the Jordan algebra [63] $J_3^{\mathbf{H}}$ should be related to our $\tau_3^{(4)}$. For an explicit proof of our claim what we need is a correspondence between the amplitudes $a, b, \ldots g$ and the components of $\mathcal{P}$ and $\mathcal{Q}$. This would also establish an explicit connection between our 56 of E_7 in terms of seven qubits and the one of Cartan [57] in terms of the antisymmetric matrices $\mathcal{P}$ and $\mathcal{Q}$.

In order to prove our claim by establishing this correspondence in these special cases we proceed as follows. We already know that our expression for the entanglement measure associated with J_4 should give the three-tangle τ_{123} when restricting to a point of the dual Fano plane. Let us consider this point to be g, i.e. the amplitude g_{CEF}. We arrange the 2×4 complex amplitudes of g_{CEF} in $\mathcal{Q}$ and $\mathcal{P}$ as follows

$$\mathcal{P} = \begin{pmatrix} g_{001} & 0 & 0 & 0 \\ 0 & g_{010} & 0 & 0 \\ 0 & 0 & g_{100} & 0 \\ 0 & 0 & 0 & g_{111} \end{pmatrix} \otimes \varepsilon, \qquad \mathcal{Q} = \begin{pmatrix} g_{110} & 0 & 0 & 0 \\ 0 & g_{101} & 0 & 0 \\ 0 & 0 & g_{011} & 0 \\ 0 & 0 & 0 & g_{000} \end{pmatrix} \otimes \varepsilon. \qquad (3.324)$$

Then from (3.66) we get

$$J_4 = -D(g) = \frac{1}{2} g^4, \qquad (3.325)$$

Hence it is natural to define a normalized measure of entanglement for our seven qubit system as

$$\tau_7 \equiv 4|J_4|. \qquad (3.326)$$

Indeed, for normalized states truncation to a single tripartite system gives rise to the three-tangle τ_{123} satisfying the constraint $0 \leq \tau_{123} \leq 1$. Moreover, for the important special case of putting GHZ states to the seven vertices of the dual Fano plane ($a_{000} = a_{111} = b_{000} = \cdots = g_{111} = 1/\sqrt{14}$) we get $\tau_7 = 1$.

In the black hole analogy however, the amplitudes are integers and no normalization condition is used. The special case having only $g \neq 0$ is the case of the STU model. Notice that the amplitudes g_{CEF} are occurring as the entries in the canonical form of the antisymmetric matrices $\mathcal{P}$ and $\mathcal{Q}$. Hence we expect that this special choice will be reflected in our choice for filling in the missing entries of the matrices $\mathcal{P}$ and $\mathcal{Q}$ in the more general cases. For normalized states truncation to the tripartite systems bdf lying on a line of the dual Fano plane we choose

$$\mathcal{P} = \begin{pmatrix} 0 & 0 & 0 & f_0^{\mathrm{T}} \\ 0 & 0 & 0 & b_0^{\mathrm{T}} \\ 0 & 0 & 0 & d_0^{\mathrm{T}} \\ -f_0 & -b_0 & -d_0 & 0 \end{pmatrix}, \qquad \mathcal{Q} = \begin{pmatrix} 0 & 0 & 0 & -\tilde{f}_1 \\ 0 & 0 & 0 & -\tilde{b}_1 \\ 0 & 0 & 0 & -\tilde{d}_1 \\ \tilde{f}_1^{\mathrm{T}} & \tilde{b}_1^{\mathrm{T}} & \tilde{d}_1^{\mathrm{T}} & 0 \end{pmatrix}. \tag{3.327}$$

Here the elements of these matrices are 2×2 matrices constructed as follows. As we have stressed in our chosen arrangement the role of qubits C, E, and F are special. These qubits are contained in the corresponding three-qubit amplitudes d_{ABC}, b_{ADE} and f_{AFG}. We split the eight components of these amplitudes into two 2×2 matrices based on the positions of the special qubits (CEF) they contain

$$\begin{aligned} d_0 &= d_{AB0}, \quad d_1 = d_{AB1}, \quad b_0 = b_{AD0}, \\ b_1 &= b_{AD1}, \quad f_0 = f_{A0G}, \quad f_1 = f_{A1G}. \end{aligned} \tag{3.328}$$

In order to check that $4|J_4(\mathcal{P}, \mathcal{Q})|$ restricted to the line dbf indeed gives back our expression for $\tau_3^{(3)}$ of (3.306) we note that in this case we can write J_4 in the form

$$J_4 = 4\mathrm{Det}(X^T Y) - (\mathrm{Tr}(X^T Y))^2, \qquad X = \begin{pmatrix} d_0^{\mathrm{T}} \\ b_0^{\mathrm{T}} \\ f_0^{\mathrm{T}} \end{pmatrix}, \quad Y = \begin{pmatrix} \tilde{d}_1 \\ \tilde{b}_1 \\ \tilde{f}_1 \end{pmatrix}. \tag{3.329}$$

Using the identity valid for 2×2 matrices

$$\mathrm{Det}(A + B) = \mathrm{Det}A + \mathrm{Det}B + \mathrm{Tr}(A\tilde{B}) \tag{3.330}$$

and grouping the terms we get $4|J_4(\mathcal{P}, \mathcal{Q})| = \tau_3^{(3)}$.

Finally we consider the complementary situation, i.e. restriction to the quadrangle $aceg$. Let us consider the matrices

$$\mathcal{P} = \begin{pmatrix} g_{001}\varepsilon & c_1^{\mathrm{T}} & e_1^{\mathrm{T}} & 0 \\ -c_1 & g_{010}\varepsilon & a_1^{\mathrm{T}} & 0 \\ -e_1 & -a_1 & g_{100}\varepsilon & 0 \\ 0 & 0 & 0 & g_{111}\varepsilon \end{pmatrix}, \qquad \mathcal{Q} = \begin{pmatrix} g_{110}\varepsilon & -\tilde{c}_0 & -\tilde{e}_0 & 0 \\ \tilde{c}_0^{\mathrm{T}} & g_{101}\varepsilon & -\tilde{a}_0 & 0 \\ \tilde{e}_0^{\mathrm{T}} & \tilde{a}_0^{\mathrm{T}} & g_{011}\varepsilon & 0 \\ 0 & 0 & 0 & g_{000}\varepsilon \end{pmatrix}. \tag{3.331}$$

Here the elements of these matrices are again 2×2 matrices. The tripartite systems with amplitudes c_{CDG}, e_{BEG} and a_{BDF} are again containing our special qubits C, E and F. The matrices occurring in the entries of $\mathcal{P}$ and $\mathcal{Q}$ are

$$\begin{aligned} c_0 &= c_{0DG}, \quad c_1 = c_{1DG}, \quad e_0 = e_{B0G}, \\ e_1 &= e_{B1G}, \quad a_0 = a_{BD0}, \quad a_1 = a_{BD1}. \end{aligned} \tag{3.332}$$

Then a straightforward but tedious calculation shows that using $\mathcal{P}$ and $\mathcal{Q}$ of (3.331) we get

$$4|J_4(\mathcal{P},\mathcal{Q})| = \tau_3^{(4)}. \tag{3.333}$$

Notice that the structure of the matrices $\mathcal{P}$ and $\mathcal{Q}$ follows the pattern

$$\begin{pmatrix} * & GD & GB & GA \\ DG & * & DB & DA \\ BG & BD & * & BA \\ AG & AD & AB & * \end{pmatrix}, \qquad \begin{pmatrix} 7 & 6 & 5 & 3 \\ 6 & 7 & 4 & 2 \\ 5 & 4 & 7 & 1 \\ 3 & 2 & 1 & 7 \end{pmatrix}, \tag{3.334}$$

where the symbol $*$ refers to the positions for the amplitudes g_{CEF} composed from the special qubits *CEF* (see also (3.324)). For the meaning of the decimal labels just convert the binary ones of (3.275) and use the associated amplitudes. Notice also that the split of the relevant amplitudes to two four component ones, e.g. a_{BD0} and a_{BD1} corresponds to the structure of split-octonions. To get some hints for this connection see the similar construction of $SO(2,2)$ and the split quaternions in (3.34)–(3.35).

Using these results it is clear now that in the black hole analogy truncation to a line of our entangled system corresponds to the one of truncating the $N = 8$ case with moduli space $E_{7(7)}/SU(8)$ to the $N = 4$ one with moduli space $(SL(2)/U(1)) \times (SO(6,6)/SO(6) \times SO(6))$. Moreover, the truncation to a quadrangle complementary to this line gives rise to the $N = 2$ truncation [64] with the moduli space being $SO^*(12)/U(6)$. It is also known [47] that the manifold $SO^*(12)/U(6)$ is the largest one which can be obtained as a consistent truncation of the $N = 8$, $d = 4$ supergravity based on $E_{7(7)}/SU(8)$.

3.7 Conclusions

Recently there has been much progress in two seemingly unrelated fields of theoretical physics. One of them is quantum information theory which concerns the study of quantum entanglement and its possible applications such as quantum teleportation, cryptography and computing. The other is the physics of stringy black holes which has provided spectacular results such as the black hole attractor mechanism and the microscopic calculation of the black-hole entropy related to the nonperturbative symmetries found between different string theories. In these lecture notes we have shown that there are some interesting mathematical coincidences between these different strains of knowledge. The results we have established here are intriguing mathematical connections arising from the similar symmetry properties of entangled systems and the web of dualities in string theory. In order to refer to such coincidences Duff and Ferrara coined the term *the black hole analogy*. We hope that we have convinced the reader that this analogy can be quite useful in repackaging some of the well-known results and awkward looking expressions of supergravity into a nice form by employing some multiqubit entangled states depending on the charges and the moduli. Moreover, this repackaging has given hints for understanding extremal BPS and non-BPS solutions within an error correction based picture. At the moment we are not aware whether this nice picture will survive or not in more general scenarios. In order to settle this issue these generalizations should be explored further.

In this respect note the very interesting connection we have found between error correcting codes (classical and quantum) and the classification of black-hole solutions. It is a well-known mathematical fact that error correcting codes are related to designs. Here we also established connections between designs (the $(7, 3, 1)$ design and its complement) and the representation theory of the exceptional group E_7. However, for duality symmetries in string theory (due to quantization of charge) the interesting objects are merely suitable discrete subgroups like the one in our case: $E_7(\mathbf{Z})$. Other discrete symmetry groups are also occurring in this context as automorphism groups of designs, for example in our case the 168 element symmetry group of the Fano plane (i.e. the simple group $L_2(7) \simeq L_3(2)$). To cap all this at the attractor point we can have multiqubit (or qudit) states of very special type (graph states) attached to the vertices of the corresponding designs. (See (3.257) and our example with the STU truncations based on the dual Fano plane.) The *charge codes* of these states might be connected to further discrete symmetries. Since design theory is also linked to the structure of finite simple groups, structures which in turn can be related to string theory [65] and black hole solutions [66] this ideas are definitely worth exploring.

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Chapter 4
From Special Geometry to Black Hole Partition Functions

Thomas Mohaupt

Abstract These notes are based on lectures given at the Erwin-Schrödinger Institute in Vienna in 2006/2007 and at the 2007 School on Attractor Mechanism in Frascati. Lecture I reviews special geometry from the superconformal point of view. Lecture II discusses the black hole attractor mechanism, the underlying variational principle and black hole partition functions. Lecture III applies the formalism introduced in the previous lectures to large and small BPS black holes in $N = 4$ supergravity. Lecture IV is devoted to the microscopic description of these black holes in $N = 4$ string compactifications. The lecture notes include problems which allow the readers to develop some of the key ideas by themselves. Appendix A reviews special geometry from the mathematical point of view. Appendix B provides the necessary background in modular forms needed for understanding S-duality and string state counting.

4.1 Introduction

Recent years have witnessed a renewed interest in the detailed study of supersymmetric black holes in string theory. This has been triggered by the work of H. Ooguri, A. Strominger and C. Vafa [1], who introduced the so-called mixed partition function for supersymmetric black holes, and who formulated an intriguing conjecture about its relation to the partition function of the topological string. The ability to test these ideas in a highly non-trivial way relies on two previous developments, which have been unfolding over the last decade. The first is that string theory provides models of black holes at the fundamental or 'microscopic' level, where microstates can be identified and counted with high precision, at least for supersymmetric black holes [2–4]. The second development is that one can handle subleading

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contributions to the thermodynamical or 'macroscopic' black hole entropy. The macroscopic description of black holes is provided by solutions to the equations of motion of effective, four-dimensional supergravity theories, which approximate the underlying string theory at length scales which are large compared to the string, Planck and compactification scale. In this framework subleading contributions manifest themselves as higher derivative terms in the effective action. For a particular class of higher derivative terms in $N = 2$ supergravity, which are usually referred to as 'R^2-terms', it is possible to construct exact near-horizon asymptotic solutions and to compute the black hole entropy to high precision [5, 6]. The subleading corrections to the macroscopic entropy agree with the subleading contributions to the microscopic entropy, provided that the area law for the entropy is replaced by Wald's generalized formula, which applies to any diffeomorphism invariant Lagrangian [7].

The main tools which make it possible to handle the R^2-terms are the superconformal calculus, which allows the off-shell construction of $N = 2$ supergravity coupled to vector multiplets, and the so-called special geometry, which highly constrains the vector multiplet couplings. The reason for this simplification is that scalars and gauge fields sit in the same supermultiplet, so that the electric–magnetic duality of the gauge fields imprints itself on the whole multiplet. As a result the complicated structure of the theory, including an infinite class of higher derivative terms, becomes manageable and transparent, once all quantities are organised such that they transform as functions or vectors under the symplectic transformations which implement electric–magnetic duality. This is particularly important if the $N = 2$ supergravity theory is the effective field theory of a string compactification, because string dualities form a subset of these symplectic transformations.

In these lectures we give a detailed account of the whole story, starting from the construction of $N = 2$ supergravity, proceeding to the definition of black hole partition functions, and ending with microscopic state counting. In more detail, the first lecture is devoted to special geometry, the superconformal calculus and the construction of $N = 2$ supergravity with vector multiplets, including the R^2-terms. The essential concept of gauge equivalence is explained using non-supersymmetric toy examples. When reviewing the construction of $N = 2$ supergravity we focus on the emergence of special geometry and stress the central role of symplectic covariance. Appendix A, which gives an account of special geometry from the mathematical point of view, provides an additional perspective on the subject. Lecture II starts by reviewing the concept of BPS or supersymmetric states and solitons. Its main point is the black hole variational principle, which underlies the black hole attractor equations. Based on this, conjectures about the relation between the macroscopically defined black hole free energy and the microscopically defined black hole partition functions are formulated. We do not only discuss how R^2-terms enter into this, but also give a detailed discussion of the crucial role played by the so-called non-holomorphic corrections, which are essential for making physical quantities, such as the black hole entropy, duality invariant.

The second half of the lectures is devoted to tests of the conjectures formulated in Lecture II. For concreteness and simplicity, I only discuss the simplest string compactification with $N = 4$ supersymmetry, namely the compactification of the heterotic string on T^6. After explaining how the $N = 2$ formalism can be used to analyse $N = 4$ theories, we will see that $N = 4$ black holes are governed by a simplified, reduced variational principle for the dilaton. There are two different types of supersymmetric black holes in $N = 4$ compactifications, called 'large' and 'small' black holes, and we summarize the results on the entropy for both of them.

With Lecture IV we turn to the microscopic side of the story. While the counting of $\frac{1}{2}$-BPS states, corresponding to small black holes, is explained in full detail, we also give an outline of how this generalises to $\frac{1}{4}$-BPS states, corresponding to large black holes. With the state degeneracy at hand, the corresponding black hole partition functions can be computed and confronted with the predictions made on the basis of the macroscopically defined free energy. We give a critical discussion of the results and point out which open problems need to be addressed in the future. While Appendix A reviews Kähler and special Kähler geometry from the mathematical point of view, Appendix B collects some background material on modular forms.

The selection of the material and the presentation are based on two principles. The first is to give a pedagogical account, which should be accessible to students, postdocs, and researchers working in other fields. The second is to present this field from the perspective which I found useful in my own work. For this reason various topics which are relevant or related to the subject are not covered in detail, in particular the topological string, precision state counting for other $N = 4$ compactifications and for $N = 2$ compactifications, and the whole field of non-supersymmetric extremal black holes. But this should not be a problem, given that these topics are already covered by other excellent recent reviews and lectures notes. See in particular [40] for an extensive review of the entropy function formalism and non-supersymmetric black holes, and [9] for a review emphasizing the role of the topological string. The selection of references follows the same principles. I have not tried to give a complete account, but to select those references which I believe are most useful for the reader. The references are usually given in paragraphs entitled 'Further reading and references' at the end of sections or subsections.

At the ends of Lectures I and IV, I formulate exercises which should be instructive for beginners. The solutions of these exercises are available upon request. In addition, some further exercises are suggested within the lectures.

4.2 Lecture I: Special Geometry

Our first topic is the so-called special geometry which governs the couplings of $N = 2$ supergravity with vector multiplets. We start with a review of the Stückelberg mechanism for gravity, explain how this can be generalized to the gauge equivalence between gravity and a gauge theory of the conformal group, and then sketch how this can be used to construct $N = 2$ supergravity in the framework of the superconformal tensor calculus.

4.2.1 Gauge Equivalence and the Stückelberg Mechanism for Gravity

The Einstein–Hilbert action

$$S[g] = -\frac{1}{2\kappa^2} \int \mathrm{d}^n x \sqrt{-g} R \tag{4.1}$$

is not invariant under local dilatations

$$\delta g_{\mu\nu} = -2\Lambda(x) g_{\mu\nu}. \tag{4.2}$$

However, we can enforce local dilatation invariance at the expense of introducing a 'compensator'. Let $\phi(x)$ be a scalar field, which transforms as

$$\delta\phi = \frac{1}{2}(n-2)\Lambda\phi. \tag{4.3}$$

Then the action

$$\tilde{S}[g,\phi] = -\int \mathrm{d}^n x \sqrt{-g} \left(\phi^2 R - 4\frac{n-1}{n-2} \partial_\mu \phi \partial^\mu \phi \right) \tag{4.4}$$

is invariant under local dilatations. If we impose the 'dilatational gauge'

$$\phi(x) = a = \text{const.}, \tag{4.5}$$

we obtain the gauge fixed action

$$\tilde{S}_{\text{g.f.}} = -a^2 \int \mathrm{d}^n x \sqrt{-g} R. \tag{4.6}$$

This is proportional to the Einstein–Hilbert action (4.1), and becomes equal to it if we choose the constant a to satisfy $a^2 = \frac{1}{2\kappa^2}$.

The actions $S[g]$ and $\tilde{S}[g,\phi]$ are said to be 'gauge equivalent'. We can go from $S[g]$ to $\tilde{S}[g,\phi]$ by adding the compensator ϕ, while we get from $\tilde{S}[g,\phi]$ to $S[g]$ by gauge fixing the additional local scale symmetry. Both theories are equivalent, because the extra degree of freedom ϕ is balanced by the additional symmetry.

There is an alternative view of the relation between $S[g]$ and $\tilde{S}[g,\phi]$. If we perform the field redefinition

$$g_{\mu\nu} = \phi^{(n-2)/4} \tilde{g}_{\mu\nu}, \tag{4.7}$$

then

$$S[g] = \tilde{S}[\tilde{g},\phi]. \tag{4.8}$$

Conversely, starting from $\tilde{S}[\tilde{g}, \phi]$, we can remove ϕ by a field-dependent gauge transformation with parameter $\exp(\Lambda) = \frac{b}{\phi}$, where $b = \text{const}$. The field redefinition (4.7) decomposes the metric into its trace (a scalar) and its traceless part (associated with the graviton). This is analogous to the Stückelberg mechanism for a massive vector field, which decomposes the vector field into a massless vector (the transverse part) and a scalar (the longitudinal part), and which makes the action invariant under $U(1)$ gauge transformations.

We conclude with some further remarks:

1. The same procedure can be applied in the presence of matter. The compensator field has to be added in such a way that it compensates for the transformation of matter fields under dilatations. Derivatives need to be covariantized with respect to dilatations. (We will see how this works in Sect. 4.2.2.)
2. It is possible to write down a dilatation invariant action for gravity, which only involves the metric and its derivatives, but this action is quadratic rather than linear in the curvature:[1]

$$S[g] = \int \mathrm{d}^4x \sqrt{-g} \left(R_{\mu\nu} R^{\mu\nu} - \frac{1}{3} R^2 \right). \tag{4.9}$$

   This actions contains terms with up to four derivatives. These and other higher derivative terms typically occur when quantum or stringy corrections to the Einstein–Hilbert action are taken into account.
3. When looking at $\tilde{S}[g, \phi]$, one sees that the kinetic term for the scalar ϕ has the 'wrong' sign, meaning that the kinetic energy is not positive definite. This signals that ϕ is not a matter field, but a compensator.

4.2.2 Gravity as a Constrained Gauge Theory of the Conformal Group

Let us recall some standard concepts of gauge theory. Given a reductive[2] Lie algebra with generators X_A and relations $[X_A, X_B] = f_{AB}^C X_C$, we define a Lie algebra valued gauge field (connection)

$$h_\mu = h_\mu^A X_A. \tag{4.10}$$

The corresponding covariant derivative (frequently also called the connection) is

$$D_\mu = \partial_\mu - i h_\mu, \tag{4.11}$$

[1] In contrast to other formulae in this subsection, the following formula refers specifically to $n = 4$ dimensions.

[2] A direct sum of simple and abelian Lie algebras.

where it is understood that h_μ operates on the representation of the field on which D_μ operates. The field strength (curvature) is

$$R^A_{\mu\nu} = 2\partial_{[\mu} h^A_{\nu]} + 2h^B_{[\mu} h^C_{\nu]} f_{BC}{}^A. \tag{4.12}$$

We now specialize to the conformal group, which is generated by translations P^a, Lorentz transformations M^{ab}, dilatations D and special conformal transformations K^a. Here $a, b = 0, 1, 2, 3$ are internal indices. We denote the corresponding gauge fields (with hindsight) by $e^a_\mu, \omega^{ab}_\mu, b_\mu, f^a_\mu$, where μ is a space–time index. The corresponding field strength are denoted $R(P)^a_{\mu\nu}$, $R(M)^{ab}_{\mu\nu}$, $R(D)_{\mu\nu}$, $R(K)^a_{\mu\nu}$.

So far the conformal transformations have been treated as internal symmetries, acting as gauge transformations at each point of space–time, but not acting on space–time. The set-up is precisely as in any standard gauge theory, except that our gauge group is not compact and wouldn't lead to a unitary Yang–Mills-type theory.

But now the so-called conventional constraints are imposed, which enforce that the local translations are identified with diffeomorphisms of space–time, while the local Lorentz transformations become Lorentz transformations of local frames:

1. The first constraint is

$$R(P)^a_{\mu\nu} = 0. \tag{4.13}$$

It can be shown that this implies that local translations act as space–time diffeomorphisms, modulo gauge transformations. As a result, the M-connection ω^{ab}_μ becomes a dependent field, and can be expressed in terms of the P-connection e^a_μ and the D-connection b_μ:

$$\omega^{ab}_\mu = \omega(e)^{ab}_\mu - 2e^{[a}_\mu e^{b]\nu} b_\nu, \tag{4.14}$$

$$\omega(e)_{\mu b}{}^c = \frac{1}{2} e_\mu{}^a (-\Omega_{ab}{}^c + \Omega_b{}^c{}_a + \Omega^c{}_{ab}), \tag{4.15}$$

$$\Omega^c_{ab} = e^\mu_a e^\nu_b \left(\partial_\mu e^c_\nu - \partial_\nu e^c_\mu\right), \tag{4.16}$$

$$\text{where} \quad e^a_\mu e^\nu_a = \delta^\nu_\mu.$$

2. The second constraint imposes 'Ricci-flatness' on the M-curvature:

$$e^\nu_b R(M)^{ab}_{\mu\nu} = 0. \tag{4.17}$$

This constraint allows to solve for the K-connection:

$$f^a_\mu = \frac{1}{2} e^{\nu a} \left(R_{\mu\nu} - \frac{1}{6} R g_{\mu\nu} \right), \tag{4.18}$$

where

$$R^{ab}_{\mu\nu} := R(\omega)^{ab}_{\mu\nu} := 2\partial_{[\mu}\omega^{ab}_{\nu]} - 2\omega^{ac}_{[\mu}\omega^{db}_{\nu]}\eta_{cd} \tag{4.19}$$

is the part of the M-curvature which does not involve the K-connection:

$$R(M)^{ab}_{\mu\nu} = R(\omega)^{ab}_{\mu\nu} - 4f^{[a}_{[\mu} e^{b]}_{\nu]}. \tag{4.20}$$

By inspection of (4.15) and (4.19) we can identify $\omega(e)_\mu^{ab}$ with the spin connections, $R_{\mu\nu}^{ab}$ with the space–time curvature, e_μ^a with the vielbein and $\Omega_{ab}^{\ c}$ with the anholonomity coefficients.[3] While ω_μ^{ab} and f_μ^a are now dependent quantities, the D-connection b_μ is still an independent field. However, it can be shown that b_μ can be gauged away using K-transformations, and the vielbein e_μ^a remains as the only independent physical field. Thus we have matched the field content of gravity. To obtain the Einstein–Hilbert action, we start from the conformally invariant action for a scalar field ϕ:

$$S = -\int \mathrm{d}^4 x e \phi (D_c)^2 \phi, \tag{4.21}$$

where $(D_c)^2 = D_\mu D^\mu$ is the conformal D'Alambert operator. In the K-gauge $b_\mu = 0$ this becomes

$$S = \int \mathrm{d}^4 x e \left(\partial_\mu \phi \partial^\mu \phi - \frac{1}{6} R \phi^2 \right). \tag{4.22}$$

As in our discussion of the Stückelberg mechanism, we can now impose the D-gauge $\phi = \phi_0 =$ const. to obtain the Einstein–Hilbert action. Observe that the kinetic term for ϕ has again the 'wrong' sign, indicating that this field is a compensator. Note that the Einstein–Hilbert action is obtained from a conformal matter action, and not from a Yang–Mills-type action with Lagrangian $\sim (R(M)_{\mu\nu}^{ab})^2$. As we have seen already in the discussion of the Stückelberg mechanism, such actions are higher order in derivatives, and become interesting once we want to include higher order corrections to the Einstein–Hilbert action.

4.2.3 Rigid $N = 2$ Vector Multiplets

Before we can adapt the method of the previous section to the case of $N = 2$ supergravity, we need to review rigidly supersymmetric $N = 2$ vector multiplets. An $N = 2$ off-shell vector multiplet has the following components:

$$\left(X, \lambda_i, A_\mu | Y_{ij} \right). \tag{4.23}$$

X is a complex scalar and λ_i is a doublet of Weyl spinors. The $N = 2$ supersymmetry algebra has the R-symmetry group $SU(2) \times U(1)$, and the index $i = 1, 2$ belongs to the fundamental representation of $SU(2)$. A_μ is a gauge field, and Y_{ij} is an $SU(2)$-triplet ($Y_{ij} = Y_{ji}$) of scalars, which is subject to the reality constraint $\overline{Y}^{ij} = Y_{ij}$.[4] All together there are eight bosonic and eight fermionic degrees of freedom.

[3] The anholonomity coefficients measure the deviation of a given frame (choice of basis of tangent space at each point) from a coordinate frame (choice of basis corresponding to the tangent vector fields of a coordinate system).

[4] $SU(2)$ indices are raised and lowered with the invariant tenor $\varepsilon_{ij} = -\varepsilon_{ji}$.

If we build an action with abelian gauge symmetry, then the gauge field A_μ will only enter through its field strength $F_{\mu\nu} = 2\partial_{[\mu}A_{\nu]}$, which is part of a so-called restricted[5] chiral $\mathcal{N} = 2$ multiplet

$$\underline{X} = \left(X, \lambda_i, F^-_{\mu\nu}, \ldots | Y_{ij}, \ldots\right), \tag{4.24}$$

where the omitted fields are dependent. $F^-_{\mu\nu}$ is the anti-selfdual part of the field strength $F_{\mu\nu}$. The selfdual part $F^+_{\mu\nu}$ resides in the complex conjugate of the above multiplet, together with the complex conjugate scalar $\overline{X}$ and fermions of the opposite chirality.

We take an arbitrary number $n + 1$ of such multiplets and label them by $I = 0, 1, \ldots, n$. The general Lagrangian is given by a chiral integral over $N = 2$ superspace,

$$\mathcal{L}^{\text{rigid}} = \int \mathrm{d}^4\theta F(\underline{X}^I) + \text{c.c.}, \tag{4.25}$$

where $F(\underline{X}^I)$ is a function which depends arbitrarily on the restricted chiral superfields $\underline{X}^I$ but not on their complex conjugates. Restricting the superfield $F(\underline{X}^I)$ to its lowest component, we obtain a holomorphic function $F(X^I)$ of the scalar fields, called the prepotential. The bosonic part of the resulting component Lagrangian is given by the highest component of the same superfield and reads

$$\mathcal{L}^{\text{rigid}} = i(\partial_\mu F_I \partial^\mu \overline{X}^I - \partial_\mu \overline{F}_I \partial^\mu X^I) + \frac{i}{4} F_{IJ} F^{-I}_{\mu\nu} F^{-J|\mu\nu} - \frac{i}{4} \overline{F}_{IJ} F^{+I}_{\mu\nu} F^{+J|\mu\nu}. \tag{4.26}$$

Here $\overline{X}^I$ is the complex conjugate of X^I, etc., and

$$F_I = \frac{\partial F}{\partial X^I}, \quad F_{IJ} = \frac{\partial^2 F}{\partial X^I \partial X^J}, \quad \text{etc.} \tag{4.27}$$

The equations of motion for the gauge fields are[6]

$$\partial_\mu \left(G_I^{-|\mu\nu} - G_I^{+|\mu\nu}\right) = 0, \tag{4.28}$$

$$\partial_\mu \left(F_I^{-|\mu\nu} - F_I^{+|\mu\nu}\right) = 0. \tag{4.29}$$

Equations (4.28) are the Euler–Lagrange equations resulting from variations of the gauge fields A^I_μ. We formulated them using the dual gauge fields

[5] While a general chiral $N = 2$ chiral multiplet has $16 + 16$ components, a restricted chiral multiplet is obtained by imposing additional conditions and has only $8 + 8$ (independent) components. Moreover, the anti-selfdual tensor field $F^-_{\mu\nu}$ of a restricted chiral multiplet is subject to a Bianchi identity, which allows to interpret it as a field strength.

[6] As an additional exercise, convince yourself that you get the Maxwell equations if the gauge couplings are constant.

$$G_I^{\pm|\mu\nu} := 2i \frac{\partial \mathcal{L}}{\partial F_{\mu\nu}^{I\pm}}. \tag{4.30}$$

Equations (4.29) are the corresponding Bianchi identities. The combined set of field equations is invariant under linear transformations of the $2n+2$ field strength $(F^{I\pm}, G_I^{\pm})^T$. Since the dual field strength are dependent quantities, we would like to interpret the rotated set of field equations as the Euler–Lagrange equations and Bianchi identities of a 'dual' Lagrangian. Up to rescalings of the field strength, this restricts the linear transformations to the symplectic group $Sp(2n+2, \mathbb{R})$. These symplectic rotations generalize the electric–magnetic duality transformations of Maxwell theory.[7]

Since $G_{\mu\nu}^{I-} \propto F_{IJ} F_{\mu\nu}^{J-}$, the gauge couplings F_{IJ} must transform fractionally linearly:

$$\mathbb{F} \to (W + V\mathbb{F})(U + Z\mathbb{F})^{-1}, \tag{4.31}$$

where $\mathbb{F} = (F_{IJ})$ and

$$\begin{pmatrix} U & Z \\ W & V \end{pmatrix} \in Sp(2n+2, \mathbb{R}). \tag{4.32}$$

This transformation must be induced by a symplectic rotation of the scalars. This is the case if $(X^I, F_I)^T$ transforms linearly, with the same matrix as the field strength.

Quantities which transform linearly, such as the field strength $(F_{\mu\nu}^{I\pm}, G_{I|\mu\nu}^{\pm})^T$ and the scalars $(X^I, F_I)^T$ are called symplectic vectors. A function $f(X)$ is called a symplectic function if

$$f(X) = \tilde{f}(\tilde{X}). \tag{4.33}$$

Note that the prepotential $F(X)$ is *not* a symplectic function, but transforms in a rather complicated way. However, we can easily construct examples of symplectic functions, by contracting symplectic vectors. The following symplectic functions will occur in the following:

$$K = i\left(X^I \overline{F}_I - F_I \overline{X}^I\right), \tag{4.34}$$

$$\mathcal{F}_{\mu\nu}^- = X^I G_{I|\mu\nu}^- - F_I F_{\mu\nu}^{I-}. \tag{4.35}$$

The scalar part of the action (4.26) can be rewritten as follows:

$$\mathcal{L}_{\text{scalar}}^{\text{rigid}} = -N_{IJ} \partial_\mu X^I \partial^\mu X^J, \tag{4.36}$$

where

$$N_{IJ} = -i\left(F_{IJ} - \overline{F}_{IJ}\right) = \frac{\partial^2 K}{\partial X^I \partial \overline{X}^J}. \tag{4.37}$$

[7] To see this more clearly, take F_{IJ} to be constant and restrict yourself to one single gauge field. The resulting $Sp(2, \mathbb{R}) \simeq SL(2, \mathbb{R})$ mixes the field strength with its Hodge dual.

N_{IJ} can be interpreted as a Riemannian metric on the target manifold of the scalars X^I, which we denote M. In fact, N_{IJ} is a Kähler metric with Kähler potential (4.34). Thus the scalar manifold M is a Kähler manifold. Moreover, M is a non-generic Kähler manifold, because its Kähler potential can be expressed in terms of the holomorphic prepotential $F(X^I)$. Such manifolds are called 'affine special Kähler manifolds.'

An intrinsic definition of affine special Kähler manifolds can be given in terms of the so-called special connection ∇ (which is different from the Levi–Civita connection of the metric N_{IJ}). This is explained in Appendix A. Equivalently, an affine special Kähler manifold can be characterised (locally) by the existence of a so-called Kählerian Lagrangian immersion

$$\Phi : M \to T^*\mathbb{C}^{n+1} \simeq \mathbb{C}^{2n+2}. \tag{4.38}$$

In this construction the special Kähler metric of M is obtained by pulling back a flat Kähler metric from $T^*\mathbb{C}^{n+1}$. In other words, all specific properties of M are encoded in the immersion Φ. Since the immersion is Lagrangian, it has a generating function, which is nothing but the prepotential: $\Phi = \mathrm{d}F$. The immersed manifold M is (generically) the graph of a map $X^I \to W_I = F_I(X)$, where (X^I, W_I) are symplectic coordinates on $T^*\mathbb{C}^{n+1}$. Along the immersed manifold, half of the coordinates of $T^*\mathbb{C}^{n+1}$ become functions of the other half: the X^I are coordinates on M while the W_I can be expressed in terms of the X^I using the prepotential as $W_I = \frac{\partial F}{\partial X^I}$. We refer the interested reader to Appendix A for more details on the mathematical aspects of this construction.

4.2.4 Rigid Superconformal Vector Multiplets

The superconformal calculus provides a systematic way to obtain the Lagrangian of $N = 2$ Poincaré supergravity by exploiting its gauge equivalence with $N = 2$ conformal supergravity. This proceeds in the following steps:

1. Construct the general Lagrangian for rigid superconformal vector multiplets
2. Gauge the superconformal group to obtain conformal supergravity
3. Gauge fix the additional transformations to obtain Poincaré supergravity

One can use the gauge equivalence to study Poincaré supergravity in terms of conformal supergravity, which is useful because one can maintain manifest symplectic covariance. In practice one might gauge fix some transformations, while keeping others intact, or use gauge invariant quantities.

As a first step, we need to discuss the additional constraints resulting from rigid $N = 2$ superconformal invariance. Besides the conformal generators P^a, M^{ab}, D, K^a, the $N = 2$ superconformal algebra contains the generators A and V^Λ of the $U(1) \times SU(2)$ R-symmetry, the supersymmetry generators Q and the special supersymmetry generators S. Note that the superconformal algebra has a second set of supersymmetry transformations which balances the additional bosonic symmetry transformations.

The dilatations and chiral $U(1)$ transformations naturally combine into complex scale transformations. The scalars have scaling weight $w = 1$ and $U(1)$ charge $c = -1$:

$$X^I \to \lambda X^I, \quad \lambda = |\lambda| \mathrm{e}^{-\mathrm{i}\phi} \in \mathbb{C}^*. \tag{4.39}$$

Scale invariance of the action requires that the prepotential is homogenous of degree 2:

$$F(\lambda X^I) = \lambda^2 F(X^I). \tag{4.40}$$

Geometrically, this implies that the scalar manifold M of rigid superconformal vector multiplets is a complex cone. Such manifolds are called 'conical affine special Kähler manifolds'.

4.2.5 $N = 2$ *Conformal Supergravity*

The construction of $N = 2$ supergravity now proceeds along the lines of the $N = 0$ example given in Sect. 4.2.2. Starting from (4.25), one needs to covariantize all derivatives with respect to superconformal transformations. The corresponding gauge fields are: e_μ^a (Translations), ω_μ^{ab} (Lorentz transformations), b_μ (Dilatations), f_μ^a (special conformal transformations), A_μ (chiral $U(1)$ transformations), $\mathcal{V}_{\mu i}^{\,j}$ ($SU(2)$ transformations), ψ_μ^i (supersymmetry transformations) and ϕ_μ^i (special supersymmetry transformations).

As in Sect. 4.2.2 one needs to impose constraints, which then allow to solve for some of the gauge fields. The remaining, independent gauge fields belong to the Weyl multiplet,

$$\left(e_\mu^a, \psi_\mu^i, b_\mu, A_\mu, \mathcal{V}_{\mu i}^{\,j} | T_{ab}^-, \chi^i, D \right), \tag{4.41}$$

together with the auxiliary fields T_{ab}^- (anti-selfdual tensor), χ^i (spinor doublet) and D (scalar). The only physical degrees of freedom contributed to Poincaré supergravity from this multiplet are the graviton e_μ^a and the two gravitini ψ_μ^i. The other connections can be gauged away or become dependent fields upon gauge fixing.

While covariantization of (4.25) with respect to superconformal transformations leads to a conformal supergravity Lagrangian with up to two derivatives in each term, it is also possible to include a certain class of higher derivative terms. This elaborates on the previous observation that one can also construct a Yang–Mills like action quadratic in the field strength. The field strength associated with the Weyl multiplet form a reduced chiral tensor multiplet $\underline{W}_{ab}$, whose lowest component is the auxiliary tensor field T_{ab}^-. The highest component contains, among other terms, the Lorentz curvature, which after superconformal gauge fixing becomes the anti-selfdual Weyl tensor ${}^-C_{\mu\nu\rho\sigma}^-$. By contraction of indices one can form the (unreduced) chiral multiplet $\underline{W}^2 = \underline{W}_{ab}\underline{W}^{ab}$, which is also referred to as 'the'

Weyl multiplet. While its lowest component is $\hat{A} = (T^{-}_{ab})^2$, the highest component contains, among other terms, the square of the anti-selfdual Weyl tensor. Higher curvature terms can now be incorporated by allowing the prepotential to depend *explicitly* on the Weyl multiplet: $F(X^I) \to F(X^I, \hat{A})$. Dilatation invariance requires that this (holomorphic) function must be (graded) homogenous of degree 2:

$$F(\lambda X^I, \lambda^2 \hat{A}) = \lambda^2 F(X^I, \hat{A}). \tag{4.42}$$

We refrain from writing down the full bosonic Lagrangian. However it is instructive to note that the scalar part, which is the analogue of (4.21) reads

$$8\pi \mathrm{e}^{-1} \mathcal{L}_{\mathrm{scalar}} = i \left(\overline{F}_I D^a D_a X^I - F_I D^a D_a \overline{X}^I \right). \tag{4.43}$$

Here D_a is the covariant derivative with respect to all superconformal transformations.

4.2.6 $N = 2$ Poincaré Supergravity

Our goal is to construct the coupling of n vector multiplets to $N = 2$ Poincaré supergravity. The gauge equivalent superconformal theory involves the Weyl multiplet and $n + 1$ vector multiplets, one of which acts a compensator. Moreover, one needs to add a second compensating multiplet, which one can take to be a hypermultiplet. The second compensator does not contribute any physical degrees of freedom to the vector multiplet sector. This is different for the compensating vector multiplet. The physical fields in the $N = 2$ supergravity multiplet are the graviton e^a_μ, the gravitini ψ^i_μ and the graviphoton $\mathcal{F}_{\mu\nu}$. While the first two fields come from the Weyl multiplet, the graviphoton is a linear combination of the field strength of all the $n + 1$ superconformal vector multiplets:

$$\mathcal{F}^{-}_{\mu\nu} = X^I G^{-}_{I|\mu\nu} - F_I F^{I-}_{\mu\nu}. \tag{4.44}$$

At the two-derivative level, one obtains $T^{-}_{\mu\nu} = \mathcal{F}^{-}_{\mu\nu}$ when eliminating the auxiliary tensor by its equation of motion. Note, however, that once higher derivative terms have been added, this relation becomes more complicated, and can only be solved iteratively in derivatives.

While all $n + 1$ gauge fields of the superconformal theory correspond to physical fields of the Poincaré supergravity theory, one of the superconformal scalars acts as a compensator for the complex dilatations. Gauge fixing imposes one complex condition on $n + 1$ complex scalars, which leaves n physical complex scalars. Geometrically, the scalar manifold of the Poincaré supergravity theory arises by taking the quotient of the 'superconformal' scalar manifold by the action of the complex dilatations.

To see what happens with the scalars, we split the superconformal covariant derivative D_μ into the covariant derivative $\mathcal{D}_\mu$, which contains the connections for $M, D, U(1), SU(2)$, and the remaining connections. Then the scalar term (4.43) becomes

$$8\pi \mathrm{e}^{-1}\mathcal{L}_{\text{scalar}} = i\left(\overline{F}_I \mathcal{D}^a\mathcal{D}_a X^I - F_I \mathcal{D}^a\mathcal{D}_a \overline{X}^I\right) - i\left(F_I\overline{X}^I - \overline{F}_I X^I\right)\left(\frac{1}{6}R - D\right). \tag{4.45}$$

In absence of higher derivative terms, the only other term containing the auxiliary field D is

$$8\pi \mathrm{e}^{-1}\mathcal{L}_{\text{comp}} = \chi\left(\frac{1}{6}R + \frac{1}{2}D\right), \tag{4.46}$$

where χ depends on the compensating hypermultiplet. The equation of motion for D is solved by[8]

$$\frac{1}{2}\chi = i\left(F_I\overline{X}^I - \overline{F}_I X^I\right). \tag{4.47}$$

When substituting this back, D cancels out, and we obtain

$$8\pi \mathrm{e}^{-1}(\mathcal{L}_{\text{scalar}} + \mathcal{L}_{\text{comp}}) = i\left(\overline{F}_I \mathcal{D}^a\mathcal{D}_a X^I - F_I \mathcal{D}^a\mathcal{D}_a \overline{X}^I\right) + \left(i\left(F_I\overline{X}^I - \overline{F}_I X^I\right)\right)\left(-\frac{1}{2}R\right). \tag{4.48}$$

The second line gives the standard Einstein–Hilbert term, in Planckian units $G_N = 1$,

$$8\pi \mathrm{e}^{-1}\mathcal{L} = -\frac{1}{2}R + \cdots, \tag{4.49}$$

once we impose the D-gauge

$$i\left(F_I\overline{X}^I - X^I\overline{F}_I\right) = 1. \tag{4.50}$$

Geometrically, imposing the D-gauge amounts to taking the quotient of the scalar manifold M with respect to the (real) dilatations $X^I \to |\lambda| X^I$. The chiral $U(1)$ transformations act isometrically on the quotient, and therefore we can take a further quotient by imposing a $U(1)$ gauge. The resulting manifold $\overline{M} = M/\mathbb{C}^*$ is the scalar manifold of the Poincaré supergravity theory. It is a Kähler manifold, whose Kähler potential can be expressed in terms of the prepotential $F(X^I)$. The target manifolds of vector multiplets of in $N = 2$ Poincaré supergravity are called '(projective) special Kähler manifolds.'

[8] Thus, at the two-derivative level, D just acts as a Lagrange multiplier. This changes once higher-derivative terms are added, but we won't discuss the implications here.

To see how the geometry of $\overline{M}$ arises, consider the scalar sigma model given by the first line of (4.48)

$$8\pi e^{-1}\mathcal{L}_{\text{sigma}} = i\left(\mathcal{D}_\mu F_I \mathcal{D}^\mu \overline{X}^I - \mathcal{D}_\mu X^I \mathcal{D}^\mu \overline{F}_I\right) \tag{4.51}$$

$$= -N_{IJ}\mathcal{D}_\mu X^I \mathcal{D}^\mu \overline{X}^J, \tag{4.52}$$

where

$$N_{IJ} = 2\mathrm{Im}F_{IJ} = -i(F_{IJ} - \overline{F}_{IJ}), \tag{4.53}$$

and

$$\mathcal{D}_\mu X^I = (\partial_\mu + iA_\mu)X^I, \quad \mathcal{D}_\mu \overline{X}^I = (\partial_\mu - iA_\mu)X^I, \tag{4.54}$$

$$\mathcal{D}_\mu F_I = (\partial_\mu + iA_\mu)F_I, \quad \mathcal{D}_\mu \overline{F}_I = (\partial_\mu - iA_\mu)\overline{F}_I. \tag{4.55}$$

We imposed the K-gauge $b_\mu = 0$, so that only the $U(1)$ gauge field A_μ appears in the covariant derivative. This gauged non-linear sigma model is the only place where A_μ occurs in the Lagrangian. A_μ can be eliminated by solving its equation of motion

$$A_\mu = \frac{1}{2}\left(\overline{F}_I \overset{\leftrightarrow}{\partial}_\mu X^I - \overline{X}^I \overset{\leftrightarrow}{\partial}_\mu F_I\right). \tag{4.56}$$

Substituting this back, we obtain the non-linear sigma model

$$\begin{aligned} 8\pi e^{-1}\mathcal{L}_{\text{sigma}} &= -\left(N_{IJ} + e^{\mathcal{K}}(N\overline{X})_I(NX)_J\right)\partial_\mu X^I \partial^\mu \overline{X}^J \\ &=: -M_{IJ}\partial_\mu X^I \partial^\mu \overline{X}^J. \end{aligned} \tag{4.57}$$

Here we suppress indices which are summed over:

$$(NX)_I := N_{IJ}X^J, \quad \text{etc.}$$

The scalar metric M_{IJ} has two null directions

$$X^I M_{IJ} = 0 = M_{IJ}\overline{X}^J. \tag{4.58}$$

This does not imply that the kinetic term for the physical scalars is degenerate, because M_{IJ} operates on the 'conformal scalars' X^I, which are subject to dilatations and $U(1)$-transformations. We have already gauge-fixed the dilatations by imposing the D-gauge. We could similarly impose a gauge condition for the $U(1)$ transformations, but it is more convenient to introduce the gauge invariant scalars

$$Z^I = \frac{X^I}{X^0}. \tag{4.59}$$

One of these scalars is trivial, $Z^0 = 1$, while the others $z^i = Z^i$, $i = 1, \ldots, n$ are the physical scalars of the Poincaré supergravity theory. Using the transversality relations (4.58) and the homogeneity of the prepotential, we can rewrite the Lagrangian in terms of the gauge-invariant scalars Z^I:

$$8\pi e^{-1}\mathcal{L}_{\text{sigma}} = -g_{IJ}\partial_\mu Z^I \partial^\mu \overline{Z}^J,$$

where

$$g_{IJ} = -\frac{N_{IJ}}{(ZN\overline{Z})} + \frac{(N\overline{Z})_I(NZ)_J}{(ZN\overline{Z})^2}. \tag{4.60}$$

Note that we have used the homogeneity of the prepotential to rewrite it and its derivatives in terms of the Z^I:

$$F(X) = (X^0)^2 F(Z), \quad F_I(X) = X^0 F_I(Z), \quad F_{IJ}(X) = F_{IJ}(Z), \quad \text{etc.}$$

One can show that g_{IJ} has the following properties:

1. g_{IJ} is degenerate along the complex direction Z^I, or, in other words, along the orbits of the $\mathbb{C}^*$-action. We will call this direction the vertical direction. As we will see below the vertical directions correspond to unphysical excitations.
2. g_{IJ} is non-degenerate along the horizontal directions, which form the orthogonal complement of the horizontal direction with respect to the non-degenerate metric N_{IJ}. As we will see below, this implies a non-degenerate kinetic term for the physical scalars.
3. g_{IJ} is positive definite along the horizontal directions if and only if N_{IJ} has signature $(2, 2n)$ or $(2n, 2)$. This corresponds to the case where N_{IJ} has opposite signature along the vertical and horizontal directions. We need to impose this to have standard kinetic terms for the physical scalars.
4. g_{IJ} can be obtained from a Kähler potential which in turn can be expressed by the prepotential of the underlying superconformal theory:

$$g_{IJ} = \frac{\partial^2 K}{\partial Z^I \partial \overline{Z^J}}, \quad K = -\log\left(i\left(F_I\overline{Z}^I - Z^I\overline{F}_I\right)\right).$$

Here it is understood that we only set $Z^0 = 1$ at the end.

Since $Z^0 = 1$, and, hence, $\partial_\mu Z^0 = 1$, the Lagrangian only depends on in the physical scalars $z^i = Z^i$, $i = 1, \ldots, n$. Following conventions in the literature, we distinguish holomorphic indices i and anti-holomorphic indices $\overline{i}$ when using the physical scalars z^i, despite that we do not make such a distinction for X^I, Z^I, etc. Thus the complex conjugate of $z^i = Z^i$ is denoted $\overline{z}^{\overline{i}} = \overline{Z}^i$.

To express the Lagrangian in terms of the physical scalars, we define

$$\mathcal{F}(z^1, \ldots, z^n) := F(Z^0, Z^1, \ldots, Z^n).$$

The Lagrangian only depends on the horizontal part of g_{IJ}, which is denoted $g_{i\overline{j}}$, and which is given by

$$g_{i\overline{j}} = \frac{\partial^2 K}{\partial z^i \partial \overline{z}^{\overline{j}}} \tag{4.61}$$

with Kähler potential

$$K = -\log\left(2i(\mathcal{F} - \overline{\mathcal{F}}) - i(z^i - \overline{z}^{\overline{i}})(\mathcal{F}_i + \overline{\mathcal{F}}_{\overline{i}})\right), \tag{4.62}$$

where $\mathcal{F}_i = \frac{\partial \mathcal{F}}{\partial z^i}$. The Lagrangian takes the form

$$8\pi \mathrm{e}^{-1} \mathcal{L}_{\text{sigma}} = -g_{i\overline{j}} \partial_\mu z^i \partial^\mu \overline{z}^{\overline{j}}.$$

Geometrically, we have performed a quotient of the rigid superconformal scalar manifold M by the $\mathbb{C}^*$-action and obtained the metric $g_{i\overline{j}}$ of the scalar manifold $\overline{M}$ of the Poincaré supergravity theory in terms of special coordinates z^i. Metrics and manifolds obtained in this way are called 'projective special Kähler metrics' and 'projective special Kähler manifolds,' respectively. One can reformulate the theory in terms of general holomorphic coordinates, but we will not pursue this here. The special coordinates are physically distinguished, because they are the lowest components of Poincaré vector multiplets. They are also natural from the geometrical point of view, because they can be defined in terms of intrinsic properties of M, as explained in more details in Appendix A.

Since the z^i are not part of a symplectic vector, the action of the symplectic transformations in the scalar sector is complicated. Therefore it is often more convenient to work on the rigid scalar manifold M using the 'conformal scalars' X^I and the symplectic vector $(X^I, F_I)^T$. As we have seen, the superconformal and the super Poincaré theory are gauge-equivalent, and we know how to go back and forth between the two. The advantage of the superconformal picture is that there is an equal number of gauge fields and scalars, which all sit in vector multiplets. Therefore symplectic transformations act in a simple way on the scalars.

Let us finally have a brief look at the higher derivative terms. We expand the function $F(X^I, \hat{A})$ in $\hat{A}$:

$$F(X^I, \hat{A}) = \sum_{g=0}^{\infty} F^{(g)}(X^I) \hat{A}^g. \tag{4.63}$$

While $F^{(0)}(X^I) = F(X^I)$ is the prepotential, the functions $F^{(g)}(X^I)$ with $g > 0$ are coupling functions multiplying various higher derivative terms. The most prominent class of such terms are

$$F^{(g)}(X^I)(^- C^-_{\mu\nu\rho\sigma})^2 (T^-_{\mu\nu})^{2g-2} + \text{c.c.}, \tag{4.64}$$

where ${}^{-}C^{-}_{\mu\nu\rho\sigma}$ is the anti-selfdual Weyl tensor and $T^{-}_{\mu\nu}$ is the anti-selfdual auxiliary field in the Weyl multiplet. To lowest order in derivatives, this field equals the anti-selfdual graviphoton field strength $\mathcal{F}^{-}_{\mu\nu}$. Therefore such terms are related to effective couplings between two gravitons and $2g-2$ graviphotons.

$N=2$ supergravity coupled to vector multiplets (and hypermultiplets) arises by dimensional reduction of type-II string theory on Calabi–Yau threefolds. Terms of the above form arise from loop diagrams where the external states are two gravitons and $2g-2$ graviphotons, while an infinite number of massive strings states runs in the loop. It turns out that in the corresponding string amplitudes only genus-g diagrams contribute, and that only BPS states make a net contribution. Moreover these amplitudes are 'topological': upon topological twisting of the world sheet theory the couplings $F^{(g)}(X^I)$ turn into the genus-g free energies (logarithms of the partition functions) of the topological type-II string. This means that the couplings $F^{(g)}(X^I)$ can be computed, at least in principle.

Further Reading and References

Besides original papers, my main sources for this lecture are the 1984 Trieste lecture notes of de Wit [10], and an (unpublished) Utrecht PhD thesis [11]. Roughly the same material was covered in Chap. 3 of my review [12]. Readers who would like to study special geometry and $N=2$ supergravity in the superconformal approach in detail should definitely look into the original papers, starting with [13, 14]. Electric–magnetic duality in the presence of R^2-corrections was investigated in [15, 16], and is reviewed in [12]. Special geometry has been reformulated in terms of general (rather than special) holomorphic coordinates [17–19]. We will not discuss this approach in these lectures and refer the reader to [20] for a review of $N=2$ supergravity within this framework. The intrinsic definition of special Kähler geometry in terms of the special connection ∇ was proposed in [21]. The equivalent characterisation by a Kählerian Lagrangian immersion into a complex symplectic vector space is described in [22]. The resulting modern formulation of special geometry was used systematically in [23–25] to explore the special geometry of Euclidean supersymmetric theories. Key references about the topological string and its role in computing couplings in the effective action are [26] and [27]. See also [9, 28] for a review of the role of the topological string for black holes.

4.2.7 Problems

Problem 4.1. The Stückelberg mechanism for gravity.

Compute the variation of the Einstein–Hilbert action

$$S[g] = -\frac{1}{2\kappa^2}\int d^n x \sqrt{-g}R \tag{4.65}$$

and the variation of the action

$$\tilde{S}[g,\phi] = -\int \mathrm{d}^n x \sqrt{-g}\left(\phi^2 R - 4\frac{n-1}{n-2}\partial_\mu\phi\partial^\mu\phi\right) \tag{4.66}$$

under local dilatations

$$\delta g_{\mu\nu} = -2\Lambda(x) g_{\mu\nu}, \quad \delta\phi = \frac{1}{2}(n-2)\Lambda\phi. \tag{4.67}$$

You can use that

$$\begin{aligned} \delta\sqrt{-g} &= -n\Lambda\sqrt{-g}, \\ g^{\mu\nu}R_{\mu\nu} &= -2(n-1)\nabla^2\Lambda. \end{aligned} \tag{4.68}$$

You should find that (4.66) is invariant while (4.65) is not, as explained in Lecture I. Convince yourself that you can obtain (4.65) from (4.66) by gauge fixing.

If you are not familiar with the Stückelberg mechanism, use what you have learned to make the action of a free massive vector field invariant with respect to local $U(1)$ transformations.

Problem 4.2. Einstein–Hilbert action from conformal matter.

Show that the Einstein–Hilbert action (4.65) can be obtained from the conformally invariant matter action

$$S = -\int \mathrm{d}^4 x e\phi D_c^2\phi, \tag{4.69}$$

where $D_c^2 = D_\mu D^\mu$ is the conformal D'Alambert operator, by gauge fixing the K- and D-transformations.

Instruction: the scalar field ϕ is neutral under K-transformations and transforms with weight $w = 1$ under D_μ. Its first and second conformally covariant derivatives are

$$D_\mu\phi = \partial_\mu\phi - b_\mu\phi, \tag{4.70}$$

$$D_\mu D^a\phi = (\partial_\mu - 2b_\mu)D^a\phi - \omega_\mu^{ab} D_b\phi + f_\mu^a\phi. \tag{4.71}$$

The K-connection f_μ^a appears in the second line because the D-connection b_μ transforms non-trivially under K. Note that b_μ is the only field in the problem which transforms non-trivially under K, and that $D^2\phi$ is invariant under K. The K-transformations can be gauged fixed by setting $b_\mu = 0$. (In fact, it is clear that b_μ will cancel out of (4.69). Why?) Use this together with the result of Problem 4.1 to obtain the Einstein–Hilbert action (4.65) by gauge fixing (4.69).

4.3 Lecture II: Attractor Mechanism, Variational Principle, and Black Hole Partition Functions

We are now ready to look at BPS black holes in $N = 2$ supergravity with vector multiplets. First we review the concept of a BPS state.

4.3.1 BPS States

The N-extended four-dimensional supersymmetry algebra has the following form:

$$\{Q^A_\alpha, Q^{+B}_{\dot\beta}\} = 2\sigma^\mu_{\alpha\dot\beta}\delta^{AB}P_\mu,$$
$$\{Q^A_\alpha, Q^B_\beta\} = \epsilon_{\alpha\beta}Z^{AB}.$$

$A, B, \ldots = 1, \ldots, N$ label the supercharges, which we have taken to be Weyl spinors. The generators $Z^{AB} = -Z^{BA}$ are central, i.e. they commute with all generators of the Poincaré Lie superalgebra. On irreducible representations they are complex multiples of the unit operators. One can then skew-diagonalise the antisymmetric constant matrix Z^{AB}, and the skew eigenvalues $Z_1, Z_2, \ldots$ are known as the central charges carried by the representation. The eigenvalue of the Casimir operator $P^\mu P_\mu$ is $-M^2$, where M is the mass. Using the algebra one can derive the BPS inequality

$$M^2 \geq |Z_1|^2 \geq |Z_2|^2 \geq \cdots \geq 0,$$

where we have labeled the central charges according to the size of their absolute values. Thus the mass is bounded from below by the central charges. Whenever a bound on the mass is saturated, some of the supercharges operate trivially on the representation, and therefore the representation is smaller than a generic massive representation. Such multiplets are called shortened multiplets or BPS multiplets. The extreme case is reached when all bounds are saturated, $M = |Z_1| = |Z_2| = \cdots$. In these representations half of the supercharges operate trivially, and the representation has as many states as a massless one. These multiplets are called short multiplets or $\frac{1}{2}$-BPS multiplets.

Here are some examples of $N = 2$ multiplets:

1. $M > |Z|$: these are generic massive multiplets. One example is the 'long' vector multiplet, which has $8 + 8$ on-shell degrees of freedom.
2. $M = |Z|$: these are short or $\frac{1}{2}$-BPS multiplet. Examples are hypermultiplets and 'short' vector multiplets, which both have $4 + 4$ on-shell degrees of freedom. The short vector multiplet is the 'Higgsed' version of the massless vector multiplet discussed earlier in these lectures.[9] The long vector multiplet combines the

[9] This has $8 + 8$ off-shell degrees of freedom and $4 + 4$ on-shell degrees of freedom.

degrees of freedom of a hypermultiplet and a short vector multiplet. This shows that one cannot expect that the number of BPS multiplets is conserved when deforming the theory (by moving through its moduli space of vacua), because BPS multiplets can combine into non-BPS multiplets. However the difference between the number of hypermultiplets and short vector multiplets is preserved under multiplet recombination and has the chance of being an ‘index’.

Let us give some examples of $N = 4$ multiplets:

1. $M > |Z_1| > |Z_2|$: these are generic massive multiplets. The number of states is 2^8.[10]
2. $M = |Z_1| > |Z_2|$: these are called intermediate or $\frac{1}{4}$-BPS multiplets. One quarter of the supercharges operate trivially, and they have (a multiple of) 2^6 states.
3. $M = |Z_1| = |Z_2|$: these are short or $\frac{1}{2}$-BPS multiplets, with (a multiple of) 2^4 states. One example are short $N = 4$ vector multiplets which have $8 + 8$ states, as many as a massless $N = 4$ vector multiplet. Short or massless multiplets have the same field content as an large $N = 2$ vector multiplet, or, equivalently, as a short or massless $N = 2$ vector multiplet plus a hypermultiplet.

Finally, there can of course also be singlets under the supersymmetry algebra, states which are completely invariant. Such states are maximally supersymmetric and can therefore be interpreted as supersymmetric ground states.

Further Reading and References

This section summarises basic facts about the representation theory of Poincaré Lie superalgebras, which can be found in textbooks on supersymmetry, i.p. in Chap. II of [29] and Chap. 8 of [30].

4.3.2 BPS Solitons and BPS Black Holes

One class of BPS states are states in the Hilbert space which sit in BPS representations. They correspond to fundamental fields in the Lagrangian, which transform in BPS representations of the supersymmetry algebra. Another class of BPS states is provided by non-trivial static solutions of the field equations, which have finite mass and are non-singular. Such objects are called solitons and interpreted as extended particle-like collective excitations of the theory.

[10] We are referring here to representations of the algebra generated by the supercharges. Irreducible representations of the full Poincaré Lie superalgebra are obtained by replacing the lowest weight state by any irreducible representation of the little group. Their dimension is therefore a multiple of 2^8.

Because of the finite mass condition they have to approach Minkowski space at infinity[11] and can be classified according to their transformation under the asymptotic Poincaré Lie superalgebra generated by the Noether charges. If this representation is BPS, the soliton is called a BPS soliton. The corresponding field configuration admits Killing spinors, i.e. there are choices of the supersymmetry transformation parameters $\epsilon(x)$ such that the field configuration is invariant:

$$\delta_{\epsilon(x)} \Phi(x)\big|_{\Phi_0(x)} = 0.$$

Here Φ is a collective notation for all fundamental fields, and Φ_0 is the invariant field configuration. The maximal number of linearly independent Killing spinors equals the number N of supercharges. Solutions with N Killing spinors are completely invariant under supersymmetry and qualify as supersymmetric ground states.[12] Generic solitonic solutions of the field equations do not have Killing spinors and correspond to generic massive representations. Solitonic solutions with $\frac{N}{n}$ Killing spinors are invariant under $\frac{1}{n}$ of the asymptotic symmetry algebra and correspond to $\frac{1}{n}$-BPS representations.[13]

The particular type of solitons we are interested in are black hole solutions of $N = 2$ supergravity. Black holes are asymptotically flat, have a finite mass, and are 'regular' in the sense that they do not have naked singularities. For static four-dimensional black holes in Einstein–Maxwell type theories with matter, the BPS bound coincides with the extremality bound. Therefore BPS black holes are extremal black holes, with vanishing Hawking temperature. Since this makes them stable against decay through Hawking radiation, the interpretation as a particle-like solitonic excitation appears to be reasonable.

We will restrict ourselves in the following to static, spherically symmetric $\frac{1}{2}$-BPS solutions of $N = 2$ supergravity with n vector multiplets. Such solutions describe single black holes.[14] As a first step, let us ignore higher derivative terms and work with a prepotential of the form $F(X)$.

In an asymptotically flat space–time, we can define electric and magnetic charges by integrating the flux of the gauge fields over an asymptotic two-sphere at infinity:

$$\begin{pmatrix} p^I \\ q_I \end{pmatrix} = \begin{pmatrix} \oint F^I_{\mu\nu} \mathrm{d}^2 \Sigma^{\mu\nu} \\ \oint G_{I|\mu\nu} \mathrm{d}^2 \Sigma^{\mu\nu} \end{pmatrix}. \tag{4.72}$$

By construction, the charges form a symplectic vector $(p^I, q_I)^T$. The central charge under the asymptotic Poincaré Lie superalgebra is given by the charge associated

[11] We only consider theories where Minkowski space is a supersymmetric ground state.

[12] Minkowski space is a trivial example. Here all Killing spinors are constant (in linear coordinates).

[13] More precisely, the collective modes generated by the broken supersymmetries fall into such representations.

[14] There are also static multi-black hole solutions, which we will not discuss here.

with the graviphoton:

$$Z = \oint \mathcal{F}^{-}_{\mu\nu} \mathrm{d}^2\Sigma^{\mu\nu} = \oint \left(F^{I-}_{\mu\nu} F_I - G^{-}_{I|\mu\nu} X^I \right) \mathrm{d}^2\Sigma^{\mu\nu} = p^I F_I(\infty) - q_I X^I(\infty). \tag{4.73}$$

This is manifestly invariant under symplectic transformations. By common abuse of terminology, the symplectic function

$$Z = p^I F_I - q_I X^I$$

is also called the central charge, despite that it is actually a function of the scalars which are in turn functions on space–time.

A static, spherically symmetric metric can be brought to the following form:[15]

$$\mathrm{d}s^2 = -\mathrm{e}^{2g(r)}\mathrm{d}t^2 + \mathrm{e}^{2f(r)}(\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2), \tag{4.74}$$

with two arbitrary functions $f(r)$, $g(r)$ of the radial variable r. We also impose that the solution has four Killing spinors. In this case one can show that $g(r) = -f(r)$. For the gauge fields and scalars we impose the same symmetry requirements as for the metric. Therefore each gauge field has only two independent components, one electric and one magnetic, which are functions of r:

$$F^I_{\underline{tr}} = F^I_E(r), \quad F^I_{\underline{\theta\phi}} = F^I_M(r).$$

Here $\underline{t}, \underline{r}, \underline{\theta}, \underline{\phi}$ are tangent space indices.[16]

The physical scalar fields z^i can be functions of the radial variable r, $z^i = z^i(r)$. In order to maintain symplectic covariance, we work in the gauge-equivalent superconformal theory and use the conformal scalars X^I. It turns out to be convenient to rescale the scalars and to define

$$Y^I(r) = \overline{Z}(r) X^I(r),$$

where $Z(r)$ is the 'central charge'. Note that

$$|Z|^2 = \overline{Z}Z = \overline{Z}\left(p^I F_I(X) - q_I X^I\right) = p^I F_I(Y) - q_I Y^I,$$

where we used that F_I is homogenous of degree one.

In the following we will focus on the near-horizon limit. In the isotropic coordinates used in (4.74), the horizon is located at $r = 0$. The scalar fields show a

[15] The solution can be constructed without fixing the coordinate system, but we present it in this way for pedagogical reasons.

[16] If we use world indices, F^I_M depends on the angular variables. This dependence is trivial in the sense that it disappears when the tensor components are evaluated in an orthonormal frame.

very particular behaviour in this limit: irrespective of their 'initial values' $z^i(\infty)$ at spatial infinity, they approach fixed point values $z^i_* = z^i(p^I, q_I)$ at the horizon. This behaviour was discovered by Ferrara, Kallosh and Strominger and is called the black hole attractor mechanism. The fixed point values are determined by the attractor equations, which can be brought to the following, manifestly symplectic form:

$$\begin{pmatrix} Y^I - \overline{Y}^I \\ F_I - \overline{F}_I \end{pmatrix}_* = i \begin{pmatrix} p^I \\ q_I \end{pmatrix}.$$

Here and in the following '$*$' indicates the evaluation of a quantity on the horizon. Depending on the explicit form of the prepotential it may or may not be possible to solve this set of algebraic equations to obtain explicit formulae for the scalars as functions of the charges. The remaining data of the near-horizon solution are the metric and the gauge fields. The near-horizon metric takes the form

$$\mathrm{d}s^2 = -\frac{r^2}{|Z_*|^2}\mathrm{d}t^2 + \frac{|Z_*|^2}{r^2}\mathrm{d}r^2 + |Z_*|^2 \mathrm{d}\Omega^2_{(2)},$$

where Z_* is the horizon value of the central charge,

$$|Z_*|^2 = \left(p^I F_I(Y) - q_I Y^I\right)_* .$$

The near horizon geometry is therefore $AdS^2 \times S^2$, with curvature radius $R = |Z_*|^2$. This is a maximally symmetric space, or more precisely the product of two maximally symmetric spaces. The gauge fields become covariantly constant in the near horizon limits, i.e., they become fluxes whose strength is characterized by the charges (p^I, q_I). In suitable coordinates[17] one simply has

$$F^I_E = q_I, \quad F^I_M = p^I.$$

$AdS^2 \times S^2$, supported by fluxes and constant scalars is a generalisation of the Bertotti–Robinson solution of Einstein–Maxwell theory.

This generalised Bertotti–Robinson solution is not only the near horizon solution of BPS black holes, but also an interesting solution in its own right. It can be shown that it is the most general static fully supersymmetric solution (eight Killing spinors) of $N = 2$ supergravity with vector multiplets. Note that the attractor equations follow from imposing full supersymmetry, or, equivalently, the field equations. Thus in a Bertotti–Robinson background the scalars cannot take arbitrary values. This is easily understood by interpreting the solution as a flux compactification of four-dimensional supergravity on S^2. Since S^2 is not Ricci flat, flux must be switched on to solve the field equations. The dimensionally reduced theory is a

[17] Essentially, $r \to \frac{1}{r}$ combined with a rescaling of t. In these coordinates it becomes manifest that the metric is conformally flat.

gauged supergravity theory with a non-trivial scalar potential with a non-degenerate AdS^2 ground state and fixed moduli.

The BPS black hole solution, which has only four Killing spinor, interpolates between two supersymmetric ground states with eight Killing spinors. At infinity it approaches Minkowski space, and in this limit the values of the scalars are arbitrary, because the four-dimensional supergravity theory has no scalar potential and a moduli space of vacua, parameterised by the scalars. At the horizon we approach another supersymmetric ground state, but here the scalars have to flow to the fixed point values dictated by the attractor equations. The black hole solution can be viewed as a dynamical system for the radial evolution of the scalars[18] from arbitrary initial values at $r = \infty$ to fixed point values at $r = 0$.

For completeness we mention that not all flows correspond to regular black holes. For non-generic choices of the charges (typically when switching off sufficiently many charges) the scalar fields can run off to the boundary of moduli space. In these cases $|Z_*|^2$ becomes zero or infinity, so that there is no black hole horizon. The original derivation of the attractor equations was in fact motivated by this observation: if one imposes that the scalars do not run off to infinity at the horizon, this implies that the solution must approach a supersymmetric ground state, which in turn implies that the geometry is Bertotti–Robinson and that the scalars take fixed point values. In this context the attractor equations were called stabilisation equations, because they forbid that the moduli run off.

There can also be more complicated phenomena if the flow crosses, at finite r, a line of marginal stability, where the BPS spectrum changes, or if it runs into a boundary point or other special point in the moduli space. We will concentrate on regular black hole solutions here, and make some comments on so-called small black holes later.

The attractor behaviour of the scalars is important for the consistency of black hole thermodynamics. The laws of black hole mechanics, combined with the Hawking effect, suggest that a black hole has a macroscopic (thermodynamical) entropy proportional to its area A:

$$S_{\text{macro}} = \frac{A}{4}.$$

The corresponding microscopic (statistical) entropy is given by the state degeneracy[19]

$$S_{\text{micro}} = \log \#\{\text{Microstates corresponding to given macrostate}\}.$$

Both entropies should be equal, at least asymptotically in the semi-classical limit (which, for non-rotating black holes, is the limit of large mass and charges). Therefore it should not be possible to change the area continuously. This is precisely what the attractor mechanism guarantees.

[18] The other non-trivial data, namely metric and gauge fields can be expressed in terms of the scalars.

[19] The macrostate of a black hole is given by its mass, angular momentum and conserved charges.

From the near horizon geometry we can read off that the area of the black hole is $A = 4\pi|Z_*|^2$. The entropy is given by the following symplectic function of the charges:

$$S_{\text{macro}} = \frac{A}{4} = \pi|Z|_*^2 = \pi|p^I F_I(X) - q_I X^I|_*^2 = \pi\left(p^I F_I(Y) - q_I Y^I\right)_* .$$

Further Reading and References

For a general introduction to solitons (and instantons), see for example the book by Rajaraman [31]. The idea to interpret extremal black holes as supersymmetric solitons is due to Gibbons [32] (see also [33]). There are many good reviews on BPS solitons in string theory, in particular [34] and [35]. The black hole attractor mechanism was discovered by Ferrrara, Kallosh and Strominger [36]. This section is heavily based on a paper written jointly with Cardoso, de Wit and Käppeli [6], where we proved that the attractor mechanism is not only sufficient, but also necessary for $\frac{1}{2}$-BPS solution, and that the Bertotti–Robinson solution is the only static solution preserving full supersymmetry.

We mentioned that not all attractor flows correspond to regular black holes solutions. One phenomenon which can occur is that the solution becomes singular before the horizon is reached (i.e. the solution becomes singular at finite values of r.) In string theory such singularities can usually be explained by a breakdown of the effective field theory. In particular, for domain walls and black holes in five-dimensional string compactifications it has been shown that one always reaches an internal boundary of moduli space before the singularity forms [37, 38]. When the properties of the internal boundary are taken into account, the solutions becomes regular.[20] In four dimensions the variety of phenomena appears to be more complex. There are so-called split attractor flows, which correspond to situations where the flow crosses a line of marginal stability [39]. This has the effect that solutions which look like single-centered black hole solutions when viewed form infinity, turn out to be complicated composite objects when viewed from nearby. The role of lines of marginal stability has been studied recently in great detail in [40].

While we only consider BPS black holes in these lectures, many features also hold for non-BPS extremal black holes. This was already observed in [41], and has become a major field of activity starting from [42, 43]. Black holes which are not BPS but still extremal can be described in terms of first order flow equations [44–47]. Alternatively, they can be described in terms of harmonic maps, which provides an interesting link to Hessian and para-complex geometry and allows to construct multi-centered extremal non-BPS solutions systematically [48]. The structure of non-BPS attractors in has been studied extensively in recent years [40, 49, 50].

[20] At internal boundaries one typically encounters additional massless states, and this changes the flow corresponding to the solution.

4.3.3 The Black Hole Variational Principle

Almost immediately after the black hole attractor mechanism was discovered, it was observed that the attractor equations follow from a variational principle. More recently it has been realized that this variational principle plays an important role in black hole thermodynamics and can be used to relate macrophysics (black hole solutions of effective supergravity) to microphysics (string theory, and in particular BPS partition functions and the topological string) in an unexpectedly direct way.

To explain the variational principle we start by defining the 'entropy function'

$$\Sigma(Y,\overline{Y},p,q) := \mathcal{F}(Y,\overline{Y}) - q_I(Y^I + \overline{Y}^I) + p^I(F_I + \overline{F}_I),$$

where $F(Y,\overline{Y})$ is the 'free energy'

$$\mathcal{F}(Y,\overline{Y}) = -i(F_I\overline{Y}^I - Y^I\overline{F}_I).$$

The terminology will become clear later. If we extremize the entropy function with respect to the scalars, the equations characterising critical points of Σ are precisely the attractor equations:

$$\frac{\partial\Sigma}{\partial Y^I} = 0 = \frac{\partial\Sigma}{\partial\overline{Y}^I} \Longleftrightarrow \begin{pmatrix} Y^I - \overline{Y}^I \\ F_I - \overline{F}_I \end{pmatrix}_* = i\begin{pmatrix} p^I \\ q_I \end{pmatrix}.$$

And if we evaluate the entropy function at its critical point, we obtain the entropy, up to a conventional factor:

$$\pi\Sigma_* = S_{\text{macro}}(p,q).$$

The geometrical meaning of the entropy function becomes clear if we use the special affine coordinates

$$\begin{aligned} x^I &= \mathrm{Re}Y^I, \\ y_I &= \mathrm{Re}F_I(Y), \end{aligned} \tag{4.75}$$

instead of the special coordinates $Y^I = x^I + \mathrm{i}u^I$. The special affine coordinates $(q^a) = (x^I, y_I)^T$ have the advantage that they form a symplectic vector. In special affine coordinates, the special Kähler metric can be expressed in terms of a real Kähler potential $H(x^I, y_I)$, called the Hesse potential. The Hesse potential is related to the prepotential by a Legendre transform, which replaces $u^I = \mathrm{Im}Y^I$ by $y_I = \mathrm{Re}(Y^I)$ as an independent field:

$$H(x^I, y_I) = 2\left(\mathrm{Im}F(x^I + iu^I(x,y)) - y^I u_I(x,y)\right),$$

where

$$y_I = \frac{\partial \mathrm{Im} F}{\partial u^I}.$$

If we express the entropy function in terms of special affine coordinates, we find:

$$\Sigma(x, y, q, p) = 2H(x, y) - 2q_I x^I + 2p^I y_I,$$

where

$$2H(x, y) = \mathcal{F}(Y, \overline{Y}) = -i(F_I \overline{Y}^I - Y^I \overline{F}_I).$$

Thus, up to a factor, the Hesse potential is the free energy. The critical points of the entropy function satisfy the black hole attractor equations, which in special affine coordinates take the following form:

$$\frac{\partial H}{\partial x^I} = q_I, \quad \frac{\partial H}{\partial y_I} = -p^I.$$

The black hole entropy is obtained by substituting the critical values into the entropy function:

$$S_{\text{macro}}(p, q) = 2\pi \left(H - x^I \frac{\partial H}{\partial x^I} - y_I \frac{\partial H}{\partial y_I} \right)_* .$$

This shows that, up to a factor, the macroscopic black hole entropy is Legendre transform of the Hesse potential. Note that at the horizon the scalar fields are determined by the charges, so that the charges provide coordinates on the scalar manifold. More precisely, the charges are not quite coordinates, because they can only take discrete values, but by the attractor equations they are proportional to continuous quantities which provide coordinates. The attractor equations can be rewritten in the form

$$\begin{pmatrix} 2u^I \\ 2v_I \end{pmatrix} = \begin{pmatrix} p^I \\ q_I \end{pmatrix}, \tag{4.76}$$

where $u^I = \mathrm{Im} Y^I$ and $v_I = \mathrm{Im} F_I$. It can be shown that (u^I, v_I) is another system of special affine coordinates. Thus the attractor equations specify a point on the scalar manifold in terms of the coordinates (u^I, v_I). The extremisation of the entropy function can be viewed as a Legendre transform from one set of special affine coordinates to another.

The special affine coordinates (x^I, y_I) also have a direct relation to the gauge fields, which even holds away from the horizon. By the gauge field equations of motion in a static (or stationary) background the scalars (x^I, y_I) are proportional to the electrostatic and magnetostatic potentials (ϕ^I, χ_I):

$$\begin{pmatrix} 2x^I \\ 2y_I \end{pmatrix} = \begin{pmatrix} \phi^I \\ \chi_I \end{pmatrix}.$$

Thermodynamically, the electrostatic and magnetostatic potentials are the chemical potentials associated with the electric and magnetic charges in a grand canonical ensemble.

Further Reading and References

The black hole variational principle described in this section was formulated by Behrndt et al. in [51]. The reformulation in terms of real coordinates is relatively recent [52]. The relation of the black hole variational principle to the work of Ooguri, Strominger and Vafa [1] will be explained in the following sections. Sen's entropy function (see [40] for a review and references), which can be used to establish the attractor mechanism for general extremal black holes, irrespective of supersymmetry and details of the Lagrangian, can be viewed as a generalisation of the entropy function discussed here, in the sense that the two entropy functions differ by terms which vanish in BPS backgrounds [53].[21]

Another variational approach to extremal black holes is based on the black hole effective potential [41]. The idea is to use the symmetries of static, spherically symmetric black holes to reduce the dynamics to the one of particle moving in an effective potential. This does not rely on supersymmetry and has become, besides Sen's entropy function, the second approach for studying the attractor mechanism for non-BPS black holes [42]. The two approaches are related because they both rely on using symmetry properties to obtain an effective lower description, see [53] for details.

4.3.4 Canonical, Microcanonical and Mixed Ensemble

For a grand canonical ensemble, the first law of thermodynamics takes the following form:

$$\delta E = T\delta S - p\delta V + \mu_i \delta N_i .$$

Here E is the energy, T the temperature, S the entropy, p the pressure, V the volume, μ_i the chemical potential and N_i the particle number of the i-th species of particles. In relativistic systems the particle number is replaced by the conserved charge under a gauge symmetry. For a general stationary black hole, the first law of black hole mechanics has the same structure:

$$\delta M = \frac{\kappa_S}{2\pi}\delta A + \omega\delta J + \phi^I \delta q_I + \chi_I \delta p^I .$$

Here M is the mass, κ_S the surface gravity, A the area, ω the rotation velocity, J the angular momentum, and ϕ^I, χ_I, p^I, q_I are the electric and magnetic potentials and

[21] To be precise, Sen's formalism is based on an entropy function which is based on the 'mixed' rather than the 'canonical' ensemble. This is explained in the next section.

charges. The Hawking effect and the generalized second law of thermodynamics suggest to take the formal analogy between thermodynamics and black hole physics seriously. In particular, the Hawking temperature of a black hole is $T = \frac{\kappa_S}{2\pi}$, which fixes the relation between area and entropy to be $S = \frac{A}{4}$.

In thermodynamics we consider other ensembles as well. The canonical ensemble is obtained by freezing the particle number while the microcanonical ensemble is obtained by freezing the energy as well. In general, the result for a thermodynamical quantity will depend on the ensemble one uses. However, all ensembles give the same result in the thermodynamical limit.

We will only discuss non-rotating black holes, $\omega = 0$. The analogous ensemble in thermodynamics does not seem to have a particular name, but, by common abuse of terminology, we will call this the canonical ensemble. Moreover, we only consider extremal black holes, with zero temperature. For $\kappa_S = 0$ the first law does not give directly a relation between mass and entropy, but we can interpret extremal black holes as limits of non-extremal ones. The independent variables in the canonical ensemble are the potentials $(\phi^I, \chi_I) \propto (x^I, y_I)$. This ensemble corresponds to a situation where the electric and magnetic charge is allowed to fluctuate, while the corresponding chemical potentials are prescribed. The ensemble obtained by fixing the electric and magnetic charges is called the microcanonical ensemble. Here the independent variables are $(p^I, q_I) \propto (u^I, v_I)$.

At the microscopic ('statistical mechanics') level, all three ensembles are characterised by a corresponding partition function. The microcanonical partition function is simply given by the microscopic state degeneracy:

$$Z_{\text{micro}}(p,q) = d(p,q),$$

where $d(p,q)$ is the number of microstates of a BPS black hole with charges p^I, q_I. The microscopic (statistical) entropy of the black hole is

$$S_{\text{micro}}(p,q) = \log d(p,q).$$

The partition function of the canonical ensemble is obtained by a formal discrete Laplace transform:

$$Z_{\text{can}}(\phi,\chi) = \sum_{p,q} d(p,q) \mathrm{e}^{\pi(q\phi - p\chi)}. \tag{4.77}$$

This relation can be inverted (formally):

$$d(p,q) = \oint \mathrm{d}\phi \mathrm{d}\chi Z_{\text{can}}(\phi,\chi) \mathrm{e}^{-\pi(q\phi - p\phi)}.$$

These partition functions are supposed to provide the microscopic description of BPS black holes. The macroscopic description is provided by black hole solutions of the effective supergravity theory, through the attractor equations, the macroscopic entropy and the entropy function. The variational principle suggests that the Hesse potential should be interpreted as the BPS black hole free energy with respect to the microscopic ensemble. This leads to the conjecture

$$e^{2\pi H(\phi,\chi)} \approx Z_{\rm can} = \sum_{p,q} d(p,q) e^{\pi[q_I\phi^I - p^I\chi_I]}, \tag{4.78}$$

or, using special coordinates instead of special affine coordinates:

$$e^{\pi\mathcal{F}(Y,\overline{Y})} \approx Z_{\rm can} = \sum_{p,q} d(p,q) e^{\pi[q_I(Y^I+\overline{Y}^I)-p^I(F_I+\overline{F}_I)]}. \tag{4.79}$$

Here '$\approx$' means asymptotic equality in the limit of large charges, which is the semi-classical and thermodynamic limit. Ideally, one would hope to find an exact relation between macroscopic and microscopic quantities, but so far there is only good evidence for a weaker, asymptotic relation. We can formally invert (4.78), (4.79) to obtain a prediction for the state degeneracy in terms of the macroscopically defined free energy:

$$d(p,q) \approx \int dx dy e^{\pi\Sigma(x,y)} \approx \int dY d\overline{Y} |\det[\mathrm{Im} F_{KL}]| e^{\pi\Sigma(Y,\overline{Y})}$$

Observe that this formula is manifestly invariant under symplectic transformations, because

$$dx dy := \prod_{I,J} dx^I dy_J = (dx^I \wedge dy_I)^{\rm top}$$

is the natural volume form on the scalar manifold (the top exterior power of the symplectic form $dx^I \wedge dy_I$), and $\Sigma(x,y)$ is a symplectic function.[22] Note that there is a non-trivial Jacobian if we go to special coordinates.

By the variational principle, the saddle point value of $\pi\Sigma$ is the macroscopic entropy. Therefore it is obvious that microscopic and macroscopic entropy agree to leading order in a saddle point evaluation of the integral:

$$e^{S_{\rm micro}(p,q)} = d(p,q) \approx e^{S_{\rm macro}(p,q)(1+\cdots)}.$$

However, in general the microscopic entropy (defined through state counting) and the macroscopic entropy (defined geometrically through the area law) will be different. The reason is that the macroscopic entropy is the Legendre transform of the canonical free energy, while the microcanonical and canonical partition functions are related by the Laplace transform (4.77). The Legendre transform between canonical free energy and macroscopic entropy provides the leading order approximation of this Laplace transform. In other words, the macroscopic entropy is not

[22] Observe that the relevant scalar manifold is M rather than $\overline{M}$.

computed in the microcanonical ensemble, and we can only expect it to agree with the microscopic entropy in the thermodynamical limit.

The Mixed Ensemble

We will now consider the so-called mixed ensemble, where the independent variables are p^I and ϕ^I. This corresponds to a situation where the magnetic charge is fixed while the electric charge fluctuates and the electrical potential is prescribed. This ensemble has the disadvantage that the independent variables do not form a symplectic vector, which obscures symplectic covariance. However, the mixed ensemble is natural in the functional integral framework, and one obtains a direct relation between black hole thermodynamics and the topological string.

The partition function of the mixed ensemble is obtained from the microcanonical partition function through a Laplace transform with respect to half of the variables:

$$Z_{\text{mix}}(p,\phi) = \sum_q d(p,q)\mathrm{e}^{\pi q\phi},$$

$$d(p,q) = \oint Z_{\text{mix}}(p,\phi)\mathrm{e}^{-\pi q\phi}.$$

Let us discuss this ensemble from the macroscopic point of view. In our previous treatment of the variational principle, we extremized the entropy function with respect to all scalar fields/potentials at once. This extremisation process can be broken up into several steps. The 'magnetic' attractor equations

$$Y^I - \overline{Y}^I = \mathrm{i}p^I$$

fix the imaginary parts of the Y^I:

$$Y^I = \frac{1}{2}(\phi^I + \mathrm{i}p^I).$$

If we substitute this into Σ we obtain a reduced entropy function:

$$\Sigma(\phi,p,q)_{\text{mix}} = \mathcal{F}_{\text{mix}}(p,\phi) - q_I\phi^I,$$

where

$$\mathcal{F}_{\text{mix}}(p,\phi) = 4\,\mathrm{Im}F(Y,\overline{Y})$$

is interpreted as the free energy in the mixed ensemble. Σ_{mix} can be interpreted as the entropy function in the mixed ensemble, because there is a new, reduced variational principle in the following sense: if we extremize Σ_{mix} with respect to the remaining scalars $\phi^I = \frac{1}{2}\mathrm{Re}Y^I$, then we obtain the remaining 'electric' attractor equations:

$$F_I - \overline{F}_I = q_I.$$

If this is substituted back into the mixed entropy function, we obtain the macroscopic entropy:

$$S_{\text{macro}}(p,q) = \pi \Sigma_{\text{mix},*}.$$

The extremisation of the mixed entropy function defines a Legendre transform between the mixed free energy and the entropy. Note that the mixed free energy is the imaginary part of the prepotential.

The mixed free energy should be related to the mixed partition function. One conceivable relation is the original 'OSV-conjecture'

$$\mathrm{e}^{\pi \mathcal{F}_{\text{mix}}(p,\phi)} \approx Z_{\text{mix}}(p,\phi). \tag{4.80}$$

To leading order in a saddle point approximation the variational principle guarantees that macroscopic and microscopic entropy agree. But one disadvantage of the mixed ensemble is that the independent variables p^I, ϕ^I do not form a symplectic vector. Therefore symplectic covariance is obscure.

Let us then compare (4.80) to the symplectically covariant conjecture (4.79) involving the canonical ensemble. Since the variational principle can be broken up into two steps, we can perform a partial saddle point approximation of (4.79) with respect to the imaginary parts of the scalars and obtain

$$d(p,q) \approx \int \mathrm{d}\phi \sqrt{|\det \mathrm{Im} F_{IJ}|}\,\mathrm{e}^{\pi[\mathcal{F}_{\text{mix}}(p,\phi) - q\phi]}.$$

This can be formally inverted with the result:

$$\sqrt{\Delta^-}\,\mathrm{e}^{\pi \mathcal{F}_{\text{mix}}(p,\phi)} \approx Z_{\text{mix}} = \sum_q d(p,q)\mathrm{e}^{\pi q_I \phi^I}. \tag{4.81}$$

Thus by imposing symplectic covariance we predict the presence of a non-trivial 'measure factor' in the mixed ensemble.

Further Reading and References

The idea to interpret the (partial) Legendre transform of the black hole entropy as a free energy (in the mixed ensemble) is due to Ooguri, Strominger and Vafa [1] and has triggered an immense number of publications which elaborate on their observation. Our presentation, which is based on [53], uses the variational principle of [51] to reformulate the 'OSV-conjecture' in a manifestly symplectically covariant way.

4.3.5 R^2-Corrections

Non-trivial tests of conjectures about state counting and partition functions depend on the ability to compute subleading corrections to the macroscopic entropy. Such corrections are due to quantum and stringy corrections to the effective action, which manifest themselves as higher derivative terms. Within the superconformal calculus one class of such terms can be handled by giving the prepotential an explicit dependence on the lowest component of the Weyl multiplet. Incidentally, in type-II Calabi–Yau compactifications the same class of terms is controlled by the topologically twisted world sheet theory. Therefore these higher derivative couplings can be computed, at least in principle.

It is possible to find the most general stationary $\frac{1}{2}$-BPS solution for a general prepotential of the form $F(X^I, \hat{A})$, at least iteratively. Here we restrict ourselves to the near-horizon limit of static, spherically symmetric single black hole solutions. It is convenient to introduce rescaled variables $Y^I = \overline{Z} X^I$ and $\Upsilon = \overline{Z}^2 \hat{A}$, and by homogeneity we get a rescaled prepotential $F(Y^I, \Upsilon) = \overline{Z}^2 F(X^I, \hat{A})$. The near horizon solution is completely determined by the generalized attractor equations

$$\begin{pmatrix} Y^I - \overline{Y}^I \\ F_I(Y, \Upsilon) - \overline{F}_I(\overline{Y}, \overline{\Upsilon}) \end{pmatrix}_* = i \begin{pmatrix} p^I \\ q_I \end{pmatrix}, \quad \Upsilon_* = -64. \tag{4.82}$$

This is symplectically covariant, because $(Y^I, F_I(Y, \Upsilon))^T$ is a symplectic vector. The variable Υ is invariant and takes a particular numerical value at the horizon. The geometry is still $AdS^2 \times S^2$, but the radius and therefore the area is modified by the higher derivative corrections:

$$A = 4\pi |p^I F_I(X, \hat{A}) - q_I X^I|^2_* = 4\pi \left(p^I F_I(Y, \Upsilon) - q_I Y^I \right)_* .$$

But this is not the only modification of the entropy, because in theories with higher curvature terms the entropy is not determined by the area law. Wald has shown by a careful derivation of the first law of black hole mechanics for generally covariant Lagrangians (admitting higher curvature terms) that the definition of the entropy must be modified, if the first law is still to be valid. Entropy, mass, angular momentum and charges can be defined as surface charges, which are the Noether charges related to the Killing vectors of the space–time. The entropy is given by the integral of a Noether two-form over the event horizon:

$$S = \oint Q.$$

The symmetry associated with this Noether charge is the one generated by the so-called horizontal Killing vector field. For static black holes this is the timelike Killing vector field associated with the time-independence of the background, while for rotating black holes it is a linear combination of the timelike and the axial Killing vector field. In practice the Noether charge can be expressed in terms of variational derivatives of the Lagrangian with respect to the Riemann tensor:

$$S = \oint \frac{\delta \mathcal{L}}{\delta R_{\mu\nu\rho\sigma}} \varepsilon_{\mu\nu} \varepsilon_{\rho\sigma} \sqrt{h} \mathrm{d}^2 \Omega_2 .$$

Here $\varepsilon_{\mu\nu}$ is the normal bivector, normalized to $\varepsilon_{\mu\nu}\varepsilon^{\mu\nu} = -2$ and $\sqrt{h}\mathrm{d}^2\Omega$ is the induced volume element of the horizon. If one evaluates this formula for $N = 2$ supergravity with prepotential $F(Y^I, \Upsilon)$, the result is

$$S_{\mathrm{macro}} = \pi \left((p^I F_I(Y, \Upsilon) - q_I Y^I) - 256 \mathrm{Im} \left(\frac{\partial F}{\partial \Upsilon} \right) \right)_* . \tag{4.83}$$

This is the sum of two symplectic functions. The first term corresponds to the area law while the second is an explicit modification. This modification is crucial for the matching of subleading contributions to the macroscopic and microscopic entropy in string theory.

R^2-corrections can be incorporated into the variational principle in a straightforward way. One defines a generalized Hesse potential as the Legendre transform of (two times the imaginary part of) the prepotential $F(Y^I, \Upsilon)$:

$$H(x, y, \Upsilon, \overline{\Upsilon}) = 2 \left(\mathrm{Im} F(Y^I, \Upsilon) - y_I u^I \right),$$

where

$$y_I = \mathrm{Re} F_I(Y^I, \Upsilon) = \frac{\partial \mathrm{Im} F(Y^I, \Upsilon)}{\partial u^I}.$$

The canonical free energy is

$$\mathcal{F}(Y, \overline{Y}) = 2H(x, y) = -i(\overline{Y}^I F_I - Y^I \overline{F}_I) - 2i(\Upsilon F_\Upsilon - \overline{\Upsilon} \overline{F}_{\overline{\Upsilon}}).$$

Here and in the following we adopt a notation where we usually suppress the dependence on Υ, unless where we want to emphasize that R^2-corrections have been taken into account. The entropy function takes the form

$$\Sigma(x, y, p, q) = 2(H - qx + py),$$

where H is now the generalized Hesse potential. It is straightforward to show that the extremization of this entropy function gives the attractor equations (4.82), and that its critical values gives the entropy (4.83): $S_{\mathrm{macro}} = \pi \Sigma_*$.

Further Reading and References

R^2-corrections to BPS solutions of $N = 2$ supergravity with vector (and hyper) multiplets were first obtained in [5] in the near horizon limit. The comparison with subleading corrections to state counting in $N = 2$ string compactifications [3, 4] showed that is crucial to use Wald's modified definition of the black hole entropy [7]. This approach assumes a Lagrangian which is covariant under diffeomorphisms, and identifies the correct definition of the entropy by imposing the validity of the first law of black hole mechanics. The entropy is found to be a Noether surface charge, which can be expressed in terms of variational derivatives of the Lagrangian [54]. The full derivation is quite intricate, and while no concise complete review is available, some elements of it have been reformulated in [55] from a more conventional gauge theory perspective. Otherwise, see [12] for a more detailed account on Wald's entropy formula and its merits in string theory. Sen's entropy function formalism [40] is based on Wald's definition of black hole entropy.

The general class of stationary $\frac{1}{2}$-BPS solutions in $N = 2$ supergravity with R^2-terms was described in [6]. The generalisation of the black hole variational principle to include R^2 terms was found in [52].

4.3.6 Non-holomorphic Corrections

There is a further type of corrections which need to be taken into account, the so-called non-holomorphic corrections. One way of deducing that such corrections must be present is to investigate the transformation properties of the entropy under string dualities, specifically under S-duality and T-duality. We will discuss an instructive example in Sect. 4.4.3. The consequence is that the entropy and the attractor equations can only be duality invariant, if there are additional contributions to the entropy and to the symplectic vector $(Y^I, F_I(Y, \Upsilon))$, which cannot be derived from a holomorphic prepotential $F(Y, \Upsilon)$. This is related to a generic feature of string-effective actions and their couplings. One has to distinguish between two types of effective actions. The Wilsonian action is always local and the corresponding Wilsonian couplings are holomorphic functions of the moduli (in supersymmetric theories). The other type of effective action is the generating functional of the scattering amplitudes. If massless modes are present this is in general non-local, and the associated physical couplings have a more complicated, non-holomorphic dependence on the moduli. Both types of actions differ by threshold corrections associated with the massless modes, which can be computed by field theoretic methods. The supergravity actions which we have constructed and discussed so far are based on a holomorphic prepotential and have to be interpreted as Wilsonian actions. Their couplings are holomorphic, and they are different from the physical couplings, which can be extracted from string scattering amplitudes. The Wilsonian couplings are not necessarily invariant under symmetries, such as string dualities, whereas the physical couplings are. The same distinction between holomorphic, but non-covariant quantities and non-holomorphic, but covariant quantities occurs for the topological string, which is the tool used to compute the couplings. Here the non-holomorphicity arises from the integration over the world-sheet moduli space, and it is encoded in the holomorphic anomaly equations.

In the following we will describe a general formalism for incorporating non-holomorphic corrections to the attractor equations and the entropy. This formalism is model-independent (as such), but we should stress that it is inspired by the example which we are going to discuss in Sect. 4.4.3. While it has been shown to work in $N = 4$ compactifications, it is not clear a priori whether the non-holomorphic modifications that are introduced are general enough to cover generic $N = 2$

compactifications. Moreover, it should be interesting to investigate the relation between this formalism and the holomorphic anomaly equation of the topological string in more detail.

The basic assumption underlying the formalism is that all non-holomorphic modifications are captured by a single real-valued function $\Omega(Y, \overline{Y}, \Upsilon, \overline{\Upsilon})$, which is required to be (graded) homogenous of degree 2:

$$\Omega(\lambda Y^I, \overline{\lambda Y}^I, \lambda^2 \Upsilon, \overline{\lambda}^2 \overline{\Upsilon}) = |\lambda|^2 \Omega(Y, \overline{Y}, \Upsilon, \overline{\Upsilon}).$$

We then define a generalized Hesse potential by taking the Legendre transform of $\mathrm{Im}F + \Omega$:

$$\hat{H}(x, y) = 2\left(\mathrm{Im}F(x + \mathrm{i}u, \Upsilon) + \Omega(x, y, \Upsilon, \overline{\Upsilon}) - qx + p\hat{y}\right), \tag{4.84}$$

where

$$\hat{y}_I = y_I + \mathrm{i}(\Omega_I - \Omega_{\overline{I}}). \tag{4.85}$$

Clearly, this modification is only non-trivial if Ω is not a harmonic function, because otherwise it could be absorbed by redefining the holomorphic function F.

We now take the generalized Hesse potential as our canonical free energy and define the entropy function

$$\Sigma = 2(\hat{H} - qx + p\hat{y}). \tag{4.86}$$

By variation of the entropy function with respect to $x, \hat{y}$ we obtain the attractor equations

$$\frac{\partial \hat{H}}{\partial x} = q, \quad \frac{\partial \hat{H}}{\partial \hat{y}} = -p, \tag{4.87}$$

and by substituting the critical values back into the entropy function we obtain the macroscopic black hole entropy

$$S_{\mathrm{macro}} = \pi \Sigma_* = 2\pi \left(\hat{H} - x\frac{\partial \hat{H}}{\partial x} - \hat{y}\frac{\partial \hat{H}}{\partial \hat{y}} \right)_* . \tag{4.88}$$

In practice, one works with special coordinates rather than special affine coordinates, because explicit expressions for subleading contributions to the couplings are only known in terms of complex coordinates. In special coordinates the entropy function has the following form:

$$\Sigma(Y, \overline{Y}, p, q) = \mathcal{F}(Y, \overline{Y}, \Upsilon, \overline{\Upsilon}) - q_I(Y^I + \overline{Y}^I) + p^I(F_I + \overline{F}_I + 2\mathrm{i}(\Omega_I - \Omega_{\overline{I}})),$$

with canonical free energy

$$\mathcal{F}(Y, \overline{Y}, \Upsilon, \overline{\Upsilon}) = -\mathrm{i}(\overline{Y}^I F_I - Y^I \overline{F}_I) - 2\mathrm{i}(\Upsilon F_\Upsilon - \overline{\Upsilon}\overline{F}_{\overline{\Upsilon}}) + 4\Omega - 2(Y^I - \overline{Y}^I)(\Omega_I - \Omega_{\overline{I}}).$$

The attractor equations are

$$\begin{pmatrix} Y^I - \overline{Y}^I \\ F_I - \overline{F}_I + 2\mathrm{i}(\Omega_I + \Omega_{\overline{I}}) \end{pmatrix} = \begin{pmatrix} p^I \\ q_I \end{pmatrix},$$

and the entropy is

$$S_{\text{macro}} = \pi \left(|Z|^2 - 256 \mathrm{Im}(F_\Upsilon + \mathrm{i}\Omega_\Upsilon) \right)_* . \tag{4.89}$$

By inspection, the net effect of the non-holomorphic corrections is to replace $F \to F + 2\mathrm{i}\Omega$ in the entropy function and in the attractor equations, but $F \to F + \mathrm{i}\Omega$ in the definition of the Hesse potential and in the entropy.[23]

As before we can impose half of the attractor equations and go from the canonical to the mixed ensemble. The modified mixed free energy is found to be

$$\mathcal{F}_{\text{mix}} = 4(\mathrm{Im} F + \Omega).$$

Since the non-holomorphic modifications are enforced by duality invariance, they are relevant for the conjectures about the relation between macroscopic quantities (free energy and macroscopic entropy) and microscopic quantities (partition functions and microscopic entropy).

Our basic conjecture is that the canonical free energy, including non-holomorphic modifications, is related to the canonical partition function by

$$\mathrm{e}^{2\pi H(x,y)} \approx Z_{\text{can}} = \sum_{p,q} d(p,q) \mathrm{e}^{2\pi [q_I x^I - p^I \hat{y}_I]}. \tag{4.90}$$

In special coordinates, this reads

$$\mathrm{e}^{\pi \mathcal{F}(Y,\overline{Y})} \approx Z_{\text{can}} = \sum_{p,q} d(p,q) \mathrm{e}^{\pi [q_I (Y^I + \overline{Y}^I) - p^I (\hat{F}_I + \overline{\hat{F}}_I)]}. \tag{4.91}$$

We can formally invert these formulae to get a prediction of the state degeneracy in terms of macroscopic quantities:

$$d(p,q) \approx \int \mathrm{d}x \mathrm{d}\hat{y} \mathrm{e}^{\pi \Sigma(x,\hat{y})} \approx \int \mathrm{d}Y \mathrm{d}\overline{Y} \Delta^-(Y,\overline{Y}) \mathrm{e}^{\pi \Sigma(Y,\overline{Y})}, \tag{4.92}$$

where we defined

$$\Delta^\pm(Y,\overline{Y}) = \left| \det \left[\mathrm{Im} F_{KL} + 2\mathrm{Re}(\Omega_{KL} \pm \Omega_{K\overline{L}}) \right] \right|. \tag{4.93}$$

[23] As an exercise, the curious reader is encouraged to verify this statement by himself, starting from the definition of the generalized Hesse potential and re-deriving all the formulae step by step.

In saddle point approximation, we predict the following relation between the microscopic and the macroscopic entropy:

$$e^{S_{\rm micro}(p,q)} = d(p,q) \approx e^{S_{\rm macro}(p,q)} \sqrt{\frac{\Delta^-}{\Delta^+}} \approx e^{S_{\rm macro}(p,q)(1+\cdots)}.$$

Here we used that both the measure factor Δ^- and the fluctuation determinant Δ^+ are subleading in the limit of large charges.

We can also perform a partial saddle point approximation

$$d(p,q) \approx \int d\phi \sqrt{\Delta^-(p,\phi)} e^{\pi[\mathcal{F}_{\rm mix}(\phi,p) - q_I \phi^I]}$$

and get a conjecture for the relation between the mixed free energy and the mixed partition function:

$$\sqrt{\Delta^-} e^{\pi \mathcal{F}_{\rm mix}(p,\phi)} \approx Z_{\rm mix} = \sum_q d(p,q) e^{\pi q_I \phi^I}. \tag{4.94}$$

The conjecture put forward by Ooguri, Strominger and Vafa is

$$e^{\pi \mathcal{F}^{\rm hol}_{\rm mix}(p,\phi)} \approx Z^{(\rm mix)}_{\rm BH} = \sum_q d(p,q) e^{\pi q_I \phi^I}. \tag{4.95}$$

This differs from (4.94) in two ways: (1) the measure factor Δ^- is absent, and (2) the mixed free energy does not include contributions from non-holomorphic terms. Since these modifications are subleading, the black hole variational principle guarantees that both formulae agree to leading order for large charges. As indicated by our presentation, we expect that the measure factor and the non-holomorphic contributions to the free energy are present, because they are needed for symplectic covariance and duality invariance. In fact, the presence of subleading modifications in (4.94) has been verified, and we will review this later.

The Relation to the Topological String

One nice feature of (4.95) is that provides a direct link between the mixed black hole partition function and the partition function of the topological string. The coupling functions $F^{(g)}(X)$ in the effective action of type-II strings compactified on a Calabi–Yau threefold are related to particular set of 'topological' amplitudes. If one performs a topological twist of the world-sheet conformal field theory, the function $F^{(g)}(X)$ becomes the free-energies of the twisted theory on a world-sheet of genus g. The generalized prepotential $F(X, \hat{A})$ is therefore proportional to the all-genus free energy, i.e., to the logarithm of the all-genus partition function $Z_{\rm top}$ of the topological string. As we have seen, the mixed free energy $\mathcal{F}^{\rm hol}_{\rm mix}$ is proportional to the imaginary part of $F(X, \hat{A})$. Taking into account conventional normalization factors, (4.95) can be rewritten in the following, suggestive form:

$$Z_{\mathrm{BH}}^{\mathrm{mix}} \approx |Z_{\mathrm{top}}|^2. \tag{4.96}$$

However, general experience with holomorphic quantities in supersymmetric theories suggests that such a relation should not be expected to be exact, but should be modified by a non-holomorphic factor.[24] And indeed, work done over the last years on state counting and partition functions in $N = 2$ compactifications, has established that the holomorphic factorisation of the black hole partition function holds to leading order, but is spoiled by subleading corrections. The underlying microscopic picture is that the black hole corresponds, modulo string dualities, to a system of branes and antibranes. To leading order, when interactions can be neglected, this leads to the holomorphic factorisation.

Currently, the detailed microscopic interpretation of the modified conjecture (4.91), (4.94) and its relation to the topological string is still an open question. In the following two lectures, we will discuss how the general ideas explained in this lecture can be tested in concrete examples.

Further Reading and References

This section is mostly based on [52], where we used the results of [56] to formulate a modified version of the 'OSV conjecture' [1]. The relation between Wilsonian and physical couplings in string effective actions was worked out in [57] and is reviewed in [58]. Concrete examples for the failure of physical quantities of supersymmetric theories to show holomorphic factorisation are provided by mass formulae (see, e.g. [59]) and by the path integral measure of the non-critical string (see, e.g. [60] for a discussion). The topological string can be used to derive the physical couplings of $N = 2$ compactifications [26, 27]. In this case the non-holomorphic corrections are captured by the holomorphic anomaly equations. The relation between these and symplectic covariance in supergravity have been discussed in [61], while the relevance of non-holomorphic corrections for black hole entropy was explained in [56]. The role of non-holomorphic corrections for the microscopic aspects of the OSV conjecture has been addressed in [62]. The ramifications of the OSV conjecture for 'topological M-theory', and the role of non-holomorphic corrections in this context have been discussed in [60, 63]. More recent work on the relation between holomorphicity and modularity includes [64, 65].

References for tests of the OSV conjecture will be given in Lecture IV.

[24] One example is the mass formula $M^2 = \mathrm{e}^{-K}|\mathcal{M}|^2$ for orbifold models, where K is the Kähler potential and $\mathcal{M}$ is the chiral mass which depends holomorphically on the moduli. In this case the presence of the non-holomorphic factor e^{-K} can be inferred from T-duality. Another example, which has been pointed out to me by S. Shatashvili, is the path integral measure for strings. While it shows holomorphic factorisation for critical strings, this is spoiled by a correction factor, namely the exponential of the Liouville action, for the generic, non-critical case.

4.4 Lecture III: Black Holes in $N = 4$ Supergravity

4.4.1 $N = 4$ Compactifications

The dynamics of string compactifications with $N = 4$ supersymmetry is considerably more restricted than the dynamics of $N = 2$ compactifications. In particular, the classical S- and T-duality symmetries are exact, and there are fewer higher derivative terms. Therefore $N = 4$ compactifications can be used to test conjectures by precision calculations. We consider the simplest example, the compactification of the heterotic string on a six-torus. This is equivalent to the compactification of the type-II string on $K3 \times T^2$, but we will mostly use the heterotic language.

The massless spectrum consists of the $N = 4$ supergravity multiplet (graviton, four gravitini, six graviphotons, four fermions, one complex scalar, which is, in heterotic $N = 4$ compactifications, the dilaton) together with 22 $N = 4$ vector multiplets (one gauge boson, four gaugini, six scalars). Since the gravity multiplet contains six graviphotons, the resulting gauge group is $U(1)^{28}$ (at generic points of the moduli space). The corresponding electric and magnetic charges each live on a copy of the Narain lattice $\Gamma = \Gamma_{22;6}$, which is an even self-dual lattice of signature $(22, 6)$:

$$(p, q) \in \Gamma \oplus \Gamma.$$

Locally, the moduli space is

$$\mathcal{M} \simeq \frac{SL(2,\mathbb{R})}{SO(2)} \otimes \frac{SO(22,6)}{SO(22) \otimes SO(6)},$$

where the first factor is parameterised by the (four-dimensional, heterotic) dilaton S,

$$S = \mathrm{e}^{-2\phi} + \mathrm{i}a.$$

The vacuum expectation value of ϕ is related to the four-dimensional heterotic string coupling g_S by $\mathrm{e}^{\langle\phi\rangle} = g_S$, and a is the universal axion (the dual of the universal antisymmetric tensor field). The global moduli space is obtained by modding out by the action of the duality group

$$SL(2,\mathbb{Z})_S \otimes SO(22,6,\mathbb{Z})_T.$$

The T-duality group $SO(22, 6, \mathbb{Z})_T$ is a perturbative symmetry under which the dilaton S is inert, and which acts linearly on the Narain lattice Γ. The S-duality group $SL(2, \mathbb{Z})_S$ is a non-perturbative symmetry, which acts on the dilaton by fractional linear transformations,

$$S \to \frac{aS + \mathrm{i}b}{-\mathrm{i}cS + d}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}), \tag{4.97}$$

while it acts linearly on the charge lattice $\Gamma \oplus \Gamma$ by

$$\begin{pmatrix} p \\ q \end{pmatrix} \longrightarrow \begin{pmatrix} a\ \mathbb{I}_{28} & b\ \mathbb{I}_{28} \\ c\ \mathbb{I}_{28} & d\ \mathbb{I}_{28} \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}. \tag{4.98}$$

Using the Narain scalar product, we can form three quadratic T-duality invariants out of the charges: $p^2, q^2, p \cdot q$. Under S-duality these quantities form a 'vector', i.e., they transform in the **3**-representation, which is the fundamental representation of $SO(2,1) \simeq SL(2)$. The scalar product of two such S-duality vectors is an S-duality singlet. One particularly important example is the S- and T-duality invariant combination of charges

$$p^2 q^2 - (p \cdot q)^2,$$

which discriminates between different types of BPS multiplets. Recall that the $N = 4$ algebra has two complex central charges. Short ($\frac{1}{2}$-BPS) multiplets satisfy

$$M = |Z_1| = |Z_2| \Leftrightarrow p^2 q^2 - (p \cdot q)^2 = 0,$$

whereas intermediate ($\frac{1}{4}$-BPS) multiplets satisfy

$$M = |Z_1| > |Z_2| \Leftrightarrow p^2 q^2 - (p \cdot q)^2 \neq 0.$$

4.4.2 $N = 4$ Supergravity in the $N = 2$ Formalism

In constructing BPS black hole solutions, we can make use of the $N = 2$ formalism. The $N = 4$ gravity multiplet decomposes into the $N = 2$ gravity multiplet, one vector multiplet (which contains the dilaton), and two gravitino multiplets (each consisting of a gravitino, two graviphotons, and one fermion). Each $N = 4$ vector multiplet decomposes into an $N = 2$ vector multiplet plus a hypermultiplet. We will truncate out the gravitino and hypermultiplets and work with the resulting $N = 2$ vector multiplets. This means that we 'loose' four electric and four magnetic charges, corresponding the four gauge fields in the gravitino multiplets. But as we will see we can use T-duality to obtain the entropy formula for the full $N = 4$ theory.

At the two-derivative level, the effective action is an $N = 2$ vector multiplet action with prepotential

$$F(Y) = -\frac{Y^1 Y^a \eta_{ab} Y^b}{Y^0}, \tag{4.99}$$

where

$$Y^a \eta_{ab} Y^b = Y^2 Y^3 - (Y^4)^2 - (Y^5)^2 - \cdots.$$

The dilaton is given by

$$S = -\mathrm{i}\frac{Y^1}{Y^0}.$$

The corresponding scalar manifold is (locally)

$$\overline{M} \simeq \frac{SL(2,\mathbb{R})}{SO(2)} \otimes \frac{SO(22,2)}{SO(22)\otimes SO(2)},$$

with duality group $SL(2,\mathbb{Z})_S \otimes SO(22,2,\mathbb{Z})_T$.

The prepotential (4.99) corresponds to a choice of the symplectic frame where the symplectic vector of the scalars is $(Y^I, F_I(Y))^T$. The magnetic and electric charges corresponding to this frame are denoted (p^I, q_I). This symplectic frame is called the supergravity frame in the following. Heterotic string perturbation theory distinguishes a different symplectic frame, called the heterotic frame, which is defined by imposing that all gauge coupling go to zero in the limit of weak string coupling $g_S \to 0$ (equivalent to $S \to \infty$). In this frame p^1 is an electric charges while q_1 is a magnetic charge. An alternative way of defining the heterotic frame is to impose that the electric charges are those which are carried by heterotic strings, while magnetic and dyonic charges are carried by solitons (wrapped five-branes). The heterotic frame has the particular property that 'there is no prepotential' (see also Appendix A). The symplectic transformation relating the heterotic frame and the supergravity frame is $p^1 \to q_1$, $q_1 \to -p^0$. If one applies this transformation to $(Y^I, F_I)^T$, then the transformed Y^I are dependent and do not form a coordinate system on M (the complex cone over $\overline{M}$), while the transformed F_I do not form the components of a gradient.

Since one frame is not adapted to string perturbation theory while the other is inconvenient, one uses a hybrid formalism, where calculations are performed in the supergravity frame but interpreted in the heterotic frame. The vectors of physical electric and magnetic charges are

$$\begin{aligned} q &= (q_0, p^1, q_a) \in \Gamma, \\ p &= (p_0, -q_1, p^a) \in \Gamma, \end{aligned} \tag{4.100}$$

where $a, b = 2, \ldots$. In this parametrisation, the explicit expressions for the T-duality invariant scalar products are

$$\begin{aligned} q^2 &= 2(q_0 p^1 - \frac{1}{4} q_a \eta^{ab} q_b), \\ p^2 &= 2(-p^0 q_1 - p^a \eta_{ab} p^b), \\ p \cdot q &= q_0 p^0 - q_1 p^1 + q_2 p^2 + q_3 p^3 + \cdots, \end{aligned} \tag{4.101}$$

where

$$\begin{aligned} p^a \eta_{ab} p^b &= p^2 p^3 - (p^4)^2 - (p^5)^2 - \cdots, \\ q_a \eta^{ab} q_b &= 4 q_2 q_3 - (q_4)^2 - (q_5)^2 - \cdots. \end{aligned} \tag{4.102}$$

In the heterotic frame, S-duality acts according to (4.98), and the three quadratic T-duality invariants transform in the vector representation of $SO(2,1) \simeq SL(2)$, where $SO(2,1)$ is realised as the invariance group of the indefinite bilinear form $a_1a_2 - a_3^2$. The scalar product of two S-duality vectors is a scalar, and the quartic S- and T-duality invariant of the charges is

$$q^2p^2 - (p\cdot q)^2 = \left(q^2, p^2, p\cdot q\right)\begin{pmatrix} 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} q^2 \\ p^2 \\ p\cdot q \end{pmatrix}.$$

For a prepotential of the form (4.99) the attractor equations can be solved in closed form, and the resulting formula for the entropy is

$$S_{\text{macro}} = \pi\sqrt{p^2q^2 - (p\cdot q)^2}. \tag{4.103}$$

This formula is manifestly invariant under $SL(2,\mathbb{Z})_S \otimes SO(22,2,\mathbb{Z})_T$, and we can reconstruct the eight missing charges by passing to the corresponding invariant of the full duality group $SL(2,\mathbb{Z})_S \otimes SO(22,6,\mathbb{Z})_T$. This result agrees with the direct derivation of the solution within $N = 4$ supergravity.

When using the prepotential (4.99) we neglect higher derivative corrections to the effective action. Therefore the solution is only valid if both the string coupling and the curvature are small at the event horizon. This is the case if the charges are uniformly large in the following sense:

$$q^2p^2 \gg (p\cdot q)^2 \gg 1.$$

Note that if the scalars take values inside the moduli space[25] then $q^2 < 0$ and $p^2 < 0$ in our parametrisation.

From the entropy formula (4.103) it is obvious that there are two different types of BPS black holes in $N = 4$ theories:

- If $p^2q^2 - (p\cdot q)^2 \neq 0$ the black hole is $\frac{1}{4}$-BPS and has a finite horizon. These are called large black holes.
- If $p^2q^2 - (p\cdot q)^2 = 0$ the black hole is $\frac{1}{2}$-BPS and has a vanishing horizon. These are called small black holes. They are null singular, which means that the event horizon coincides with the singularity.

Further Reading and References

The conventions used in this section are those of [56]. See there for more information and references about the relation between $N = 4$ and $N = 2$ compactifications.

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[25] The moduli space is realised as an open domain in $\mathbb{R}^n$, which is given by a set of inequalities. In our parametrisation one of these inequalities is $\text{Re}S = e^{-2\phi} > 0$, which implies that the dilaton lives in a half plane (the right half plane). Solutions where $\text{Re}S < 0$ at the horizon are therefore unphysical. Similar remarks apply to the other moduli.

The entropy for large black holes in $N=4$ compactifications was computed in [66, 67] and rederived using the $N=2$ formalism in [56].

4.4.3 R^2-Corrections for $N=4$ Black Holes

Let us now incorporate higher derivative corrections. Since no treatment within $N=4$ supergravity is available, it is essential that we can fall back onto the $N=2$ formalism. One simplifying feature of $N=4$ compactifications is that all higher coupling functions $F^{(g)}(Y)$ with $g>1$ vanish. The only higher derivative coupling is $F^{(1)}(Y)$, which, moreover, only depends on the dilaton S. The generalized prepotential takes the following form:

$$F(Y,\Upsilon) = -\frac{Y^1 Y^a \eta_{ab} Y^b}{Y^0} + F^{(1)}(S)\Upsilon.$$

In order to find duality covariant attractor equations and a duality invariant entropy, we must incorporate the non-holomorphic corrections to the Wilsonian coupling $F^{(1)}(Y)$, which are encoded in a homogenous, real valued, non-harmonic function $\Omega(Y,\overline{Y},\Upsilon,\overline{\Upsilon})$.

One way to find this function is to compute the physical coupling of the curvature-squared term in string theory. Since this coupling depends on the dilaton (but not on the other moduli), it can receive non-perturbative corrections (though no perturbative ones). At this point one has to invoke the duality between the heterotic string on T^6 and the type-IIA string on $K3\times T^2$. Since the heterotic dilaton corresponds to a geometric type-IIA modulus, the exact result can be found by a perturbative calculation in the IIA theory. This calculation is one-loop, and can be done exactly in α', because there is no dependence on the K3-moduli.

Alternatively, one can start with the perturbative heterotic coupling and infer the necessary modifications of the attractor equations and of the entropy by imposing S-duality invariance. It turns out that there is a minimal S-duality invariant completion, which in principle could differ from the full result by further subleading S-duality invariant terms. But for the case at hand the minimal S-duality completion turns out to give complete result.

At tree level, the coupling function $F^{(1)}$) is given by

$$F^{(1)}_{\text{tree}}(S) = c_1 \mathrm{i} S, \quad \text{where} \quad c_1 = -\frac{1}{64}.$$

We know a priori that there can be instanton corrections $\mathcal{O}(\mathrm{e}^{-S})$. The function $F^{(1)}(S)$ determines the 'R^2-couplings'

$$\mathcal{L}_{R^2} \simeq \frac{1}{g^2} C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} + \Theta C_{\mu\nu\rho\sigma} \tilde{C}^{\mu\nu\rho\sigma}, \tag{4.104}$$

where $C_{\mu\nu\rho\sigma}$ is the Weyl tensor, through $g^{-2} \simeq \mathrm{Im}F^{(1)}$ and $\theta \simeq \mathrm{Re}F^{(1)}$. Therefore $\mathrm{Im}F^{(1)}$ must be an S-duality invariant function, whereas $\mathrm{Re}F^{(1)}$ must only be invariant up to discrete shifts. According to (4.97), the linear tree-level piece is not invariant. Restrictions on the functional dependence of $F^{(1)}$ on S result from the requirement that the S-duality transformation (4.97) of the dilaton induces the symplectic transformation (4.98) of the symplectic vector $(Y^I, F_I)^T$. This implies that

$$f(S) := -\mathrm{i}\frac{\partial F^{(1)}}{\partial S}$$

must transform with weight 2:

$$f\left(\frac{aS - \mathrm{i}b}{\mathrm{i}cS + d}\right) = (\mathrm{i}cS + d)^2 f(S).$$

A classical result in the theory of modular forms[26] implies that $f(S)$, (and, hence, $F^{(1)}$) cannot be holomorphic. The holomorphic object which comes closest to transforming with weight 2 is the holomorphic second Eisenstein series

$$G_2(S) = -4\pi\partial_S \eta(S),$$

where $\eta(S)$ is the Dedekind η-function.[27] To obtain a function which transforms with weight 2 one needs to add a non-holomorphic term and obtains the non-holomorphic second Eisenstein series:

$$G_2(S, \overline{S}) = G_2(S) - \frac{2\pi}{S + \overline{S}}.$$

This is the only candidate for $f(S)$. We will write $f(S, \overline{S})$ in the following, to emphasize that this function is non-holomorphic. We need to check that we get the correct asymptotics in the weak coupling limit $S \to \infty$. Since $F^{(1)} \to c_1 \mathrm{i}S$, we know that $f(S, \overline{S})$ must go to a constant. This is indeed true for the second Eisenstein series (the non-holomorphic term is subleading):

$$G_2(S, \overline{S}) \to \frac{\pi^2}{3},$$

and therefore the minimal choice for $f(S, \overline{S})$ is

$$f(S, \overline{S}) = c_1 \frac{3}{\pi^2} G_2(S, \overline{S}).$$

[26] We refer the reader to Appendix B for a brief review of modular forms and references.

[27] Here $G_2(S)$ is short for $G_2(\mathrm{i}S)$, etc.

This can be integrated, and we obtain the non-holomorphic function

$$F^{(1)}(S,\overline{S}) = -\mathrm{i}c_1 \frac{6}{\pi}\left(\log \eta^2(S) + \log(S+\overline{S})\right). \tag{4.105}$$

This function generates a symplectic vector $(Y^I, F_I(Y,\overline{Y}))^T$ with the correct behaviour under S-duality. Moreover, the function $p^I F_I(Y,\overline{Y}) - q_I Y^I$, which is proportional to the area, is S-duality invariant. However $F^{(1)}(S,\overline{S})$ is not S-duality invariant, but transforms as follows:

$$F^{(1)}(S,\overline{S}) \to F^{(1)}(S,\overline{S}) + \mathrm{i}c_1 \frac{6}{\pi} \log\left(-\mathrm{i}c\overline{S} + d\right).$$

This was to be expected, because derivatives (and, hence, integrals) of modular forms are not modular forms but transform with additional terms. The function $F^{(1)}(S,\overline{S})$ was constructed by requiring that its derivative is a modular form of weight 2. Therefore it does not quite transform as a modular form of weight zero (modular function). In order to get an S-duality invariant function, we need to add a further non-holomorphic piece:

$$F^{(1)}_{\mathrm{phys}}(S,\overline{S}) = F^{(1)}(S,\overline{S}) + \mathrm{i}c_1\frac{3}{\pi}\log(S+\overline{S}) = F^{(1)}(S)_{\mathrm{hol}} + \mathrm{i}c_1\frac{6}{\pi}\log(S+\overline{S}),$$

where

$$F^{(1)}_{\mathrm{hol}}(S) = -\mathrm{i}c_1\frac{6}{\pi}\log\eta^2(S).$$

The invariant function $F^{(1)}_{\mathrm{phys}}$ is the minimal S-duality completion of the R^2-coupling (4.104). An explicit calculation of this coupling in string theory shows that this is in fact the full R^2-coupling.

Since the entropy must be S-duality invariant, it is also clear that the correct way of generalizing the holomorphic function $F^{(1)}(S)$ in the entropy formula is[28]

$$\mathcal{S}_{\mathrm{macro}} = \pi\left[(p^I F_I(Y,\overline{Y}) - q_I Y^I) + 4\mathrm{Im}\left(\Upsilon F^{(1)}_{\mathrm{phys}}(S,\overline{S})\right)\right]_* .$$

Note that the non-holomorphic modifications are purely imaginary. Therefore they only modify the R^2-coupling $g^{-2} \simeq \mathrm{Im} F^{(1)}$ and reside in a real-valued, non-harmonic function Ω. In the following we find it convenient to absorb the holomorphic function $\Upsilon F^{(1)}(S)$ into Ω:

$$\begin{aligned}\Omega(S,\overline{S},\Upsilon,\overline{\Upsilon}) &= \mathrm{Im}\left(\Upsilon F^{(1)}(S,\overline{S}) + \Upsilon i c_1\frac{3}{\pi}\log(S+\overline{S})\right)\\ &= \mathrm{Im}\left(\Upsilon F^{(1)}(S) - \Upsilon i c_1 \frac{3}{\pi}\log(S+\overline{S})\right).\end{aligned} \tag{4.106}$$

[28] Remember $\Upsilon_* = -64$.

This function encodes all higher derivative corrections to the tree-level prepotential.

We already mentioned that the holomorphic R^2-corrections correspond to instantons. To make this explicit we expand $F^{(1)}_{\rm hol}(S)$ for large S:

$$F^{(1)}_{\rm hol}(S) \simeq \log \eta^{24}(S) = -2\pi S - 24 e^{-2\pi S} + \mathcal{O}(e^{-4\pi S}).$$

This shows that the R^2-coupling has a classical piece proportional to S, followed by an infinite series of instanton corrections, which correspond to wrapped five-branes.

Further Reading and References

This section is based on [56]. The treatment of the non-holomorphic corrections illustrates the general formalism introduced in [52]. In fact, the formalism is modelled on this example, and it is not excluded that generic $N = 2$ compactifications need more general modifications. The R^2-term in the effective action for $N = 4$ compactifications was computed in [68].

4.4.4 The Reduced Variational Principle for N = 4 Theories

It is possible and in fact instructive to analyse the attractor equations and entropy without using the explicit form of Ω. Using that Ω depends on the dilaton S, but not on the other moduli $T^a \simeq (Y^a/Y^0)$, one can solve all but two of the attractor equations explicitly. The remaining two 'dilaton attractor equations' are the only ones which involve Ω, and they determine the dilaton as a function of the charges. Substituting the solved attractor equations into the entropy function, we obtain the following, reduced entropy function:

$$\Sigma(S, \overline{S}, p, q) = -\frac{q^2 - i p \cdot q (S - \overline{S}) + p^2 |S|^2}{S + \overline{S}} + 4\Omega(S, \overline{S}, \Upsilon, \overline{\Upsilon}). \tag{4.107}$$

Extremisation of this function yields the remaining dilatonic attractor equations

$$\partial_S \Sigma = 0 = \partial_{\overline{S}} \Sigma \quad \Leftrightarrow \quad \text{Dilaton attractor equations,}$$

and its critical value gives the entropy:

$$\begin{aligned} S_{\rm macro}(p, q) &= \pi \Sigma_*(p, q) \\ &= \left(-\frac{q^2 - i p \cdot q (S - \overline{S}) + p^2 |S|^2}{S + \overline{S}} + 4\Omega(S, \overline{S}, \Upsilon, \overline{\Upsilon}) \right)_{|\partial_S \Sigma = 0} . \end{aligned} \tag{4.108}$$

The entropy function is manifestly S-duality and T-duality invariant, provided that Ω is an S-duality invariant function.[29]

Further Reading and References

The observation that all but two of the $N = 4$ attractor equations can be solved, even in presence of R^2-terms, was already made in [56] and exploited in [69] and [52].

4.4.5 Small N = 4 Black Holes

Let us now have a second look at small black holes. For convenience we take them to be electric black holes, $p = 0$. By this explicit choice, S-duality is no longer manifest, but T-duality remains manifest. As we saw above, as long as $\Omega = 0$ the area of a $\frac{1}{2}$-BPS black hole vanishes, $A = 0$, and therefore the Bekenstein–Hawking entropy is zero, too. In fact, the moduli also show singular behaviour, and, in particular, the dilaton runs of to infinity at the horizon $S_* = \infty$. Thus small black holes live on the boundary of moduli space.

The lowest order approximation to the R^2-coupling is to take its classical part,

$$F^{(1)} \simeq \log \eta^{24}(S) = -2\pi S + \mathcal{O}(\mathrm{e}^{-2\pi S}),$$

and to neglect all instanton and non-holomorphic corrections. In this approximation one can solve the dilatonic attractor equations explicitly. This results in the following, non-vanishing and T-duality invariant area:

$$A = 8\pi \sqrt{\frac{1}{2}|q^2|} \neq 0.$$

Thus the R^2-corrections smooth out the null-singularity and create a finite horizon. We need to impose that $|q^2| \gg 1$ in order that the dilaton S is large,[30] which we need to impose because we neglect subleading corrections to the R^2-coupling. Note that in contrast to the two-derivative approximation the dilaton is now finite at the horizon. Thus not only the metric but also the moduli are smoothed by the higher derivative corrections. The horizon area is small in string units, even though it is large in Planck units. This motivates the terminology ‘small black holes.’

The resulting Bekenstein–Hawking entropy is

$$S_{\text{Bekenstein–Hawking}} = \frac{A}{4} = 2\pi \sqrt{\frac{1}{2}|q^2|}.$$

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[29] $\frac{1}{S+\overline{S}}(1, |S|^2, -i(S - \overline{S}))$ transforms as an $SO(2,1)$ vector under S-duality, and therefore the contraction with the vector $(q^2, p^2, p \cdot q)$ gives an invariant.

[30] In our parametrisation $q^2 < 0$, if the horizon values of the scalars are inside the moduli space.

However, since the area law does not apply to theories with higher curvature terms, the correct way to compute the macroscopic black hole entropy is (4.83). Evaluating this for the case at hand gives

$$S_{\text{macro}} = \frac{A}{4} + \text{correction} = \frac{A}{4} + \frac{A}{4} = \frac{A}{2} = 4\pi\sqrt{\frac{1}{2}|q^2|}.$$

In this particular case the correction is as large as the area term itself. Later we will have the opportunity to confront both formulae with string microstate counting.

In the limit of large S the next subleading correction comes from the non-holomorphic corrections $\propto \log(S + \overline{S})$. We can still find an explicit formula for the entropy:

$$S_{\text{macro}} = 4\pi\sqrt{\frac{1}{2}|q^2|} - 6\log|q^2|,$$

which we will compare to microstate counting later.

If we include further corrections, ultimately the full series of instanton corrections encoded in $\log\eta^{24}(S)$, we cannot find an explicit formula for the entropy anymore. However, we know that the exact macroscopic entropy is given as the solution of the extremisation problem for the dilatonic entropy function (4.107). This can be used for a comparison with state counting.

Further Reading and References

The observation that R^2-corrections smooth or 'cloak' the null singularity of small black holes was made in [70]. This result follows immediately from [56].

4.5 Lecture IV: $N = 4$ State Counting and Black Hole Partition Functions

The BPS spectrum of the heterotic string on T^6 consists of the excited modes of the heterotic string itself, and solitons. Heterotic string states are labeled by 28 quantum numbers: 6 winding numbers, 6 discrete momenta around T^6 and 16 charges of the unbroken $U(1)^{16} \subset E_8 \otimes E_8$ gauge group. They combine into 22 left- and 6 right-moving momenta, which take values in the Narain lattice:

$$(p_L; p_R) \in \Gamma.$$

Modular invariance of the world sheet conformal field theory implies that the lattice Γ must be even and selfdual with respect to the bilinear form $p_L^2 - p_R^2$, which has signature $(+)^{22}(-)^6$. From the four-dimensional point of view, the 28 left- and right-moving momenta are the 28 electric charges with respect to the generic gauge group $U(1)^{16+6+6}$: $q = (p_L; p_R) \in \Gamma$.

Similarly, the winding states of heterotic five-branes carry magnetic charges $p \in \Gamma^* = \Gamma$. If purely electric or purely magnetic states satisfy a BPS bound, they must be $\frac{1}{2}$-BPS states, because $p^2q^2 - p \cdot q = 0$ if either $p = 0$ or $q = 0$. However, there are also dyonic solitonic states with $q^2p^2 - p \cdot q \neq 0$, which are $\frac{1}{4}$-BPS. By string–black hole complementarity, the BPS states with charges $(p, q) \in \Gamma \oplus \Gamma$ should be the microstates of $N = 4$ black holes with the same charges. We will now discuss how these states are counted and compare our results to the macroscopic black hole entropy and free energy.

4.5.1 Counting $\frac{1}{2}$-BPS States

Without loss of generality, we take the $\frac{1}{2}$-BPS states to be electric, $p = 0$. Such states correspond to excitations of the heterotic string, and are called Dabholkar–Harvey states. Recall that the world-sheet theory of the heterotic string has two different sectors. The left-moving sector consists of 24 world sheet bosons (using the light cone gauge), namely the left-moving projections of the eight coordinates transverse to the world sheet, and 16 bosons with values in the maximal torus of $E_8 \otimes E_8$. The right-moving sector consists of the right-moving projections of the eight transverse coordinates, together with eight right-moving fermions. This sector is supersymmetric in the two-dimensional, world-sheet sense. World-sheet supersymmetry combined with a condition on the spectrum of charges implies the existence of an extended chiral algebra on the world-sheet, which is equivalent to $N = 4$ supersymmetry in the ten-dimensional, space–time sense. The generators of the space–time supersymmetry algebra are build exclusively out of right-moving degrees of freedom. To obtain BPS states one needs to put the right-moving sector into its ground state, but still has the freedom to excite the left-moving sector. A basis of such states is

$$\alpha^{i_1}_{-m_1} \alpha^{i_2}_{-m_2} \cdots |q\rangle \otimes \mathbf{16}, \tag{4.109}$$

where $\alpha^{i_k}_{-m_l}$ are creation operators for the oscillation modes of the string. The indices $i_k = 1, \ldots, 24$ label the directions transverse to the world-sheet of the string, while $m_k = 1, 2, 3, \ldots$ label the oscillation modes. $q = (p_L; p_R) = \Gamma$ are the electric charges, which correspond to the winding modes, momentum modes and $U(1)^{16}$ charges. **16** denotes the ground state of the right-moving sector, which carries the degrees of freedom of an $N = 4$ vector multiplet (with 16 on-shell degrees of freedom). States of this form are invariant under as many supercharges as the right-moving ground state, and therefore they belong to $\frac{1}{2}$-BPS multiplets. To be physical, the state must satisfy the level matching condition,

$$N - 1 + \frac{1}{2}p_L^2 = \tilde{N} + \frac{1}{2}p_R^2, \tag{4.110}$$

where $N, \tilde{N}$ are the total left-moving and right-moving excitation numbers. BPS states have $\tilde{N} = 0$, and therefore the excitation level is fixed by the charges:

$$N - 1 + \frac{1}{2}p_L^2 = \tilde{N} + \frac{1}{2}p_R^2 \Rightarrow N = \frac{1}{2}p_R^2 - \frac{1}{2}p_L^2 - 1 = -q^2 - 1 = |q^2| - 1. \quad (4.111)$$

This is equivalent to the statement that the mass saturates the BPS bound. Note that $q^2 < 0$ for physical BPS states. For large charges we can use $N \approx |q^2|$.

The problem of counting $\frac{1}{2}$-BPS states amounts to counting in how many ways a given total excitation number $N \approx |q^2|$ can be distributed among the creation operators α^i_{-m}. If we ignore the additional space–time index $i = 1, \ldots, 24$, this becomes the classical problem of counting the partitions of an integer N, which was studied by Hardy and Ramanujan. The space–time index i adds an additional 24-fold degeneracy, and we might say that we have to count partitions of N into integers with 24 different 'colours'. Incidentally exactly the same problems arises (up to the overall factor 16 from the degeneracy of the right-moving ground state) when counting the physical states of the open bosonic string. From the world-sheet perspective, both problems amount to finding the partition function of 24 free bosons, which is a standard problem in quantum statistics and conformal field theory.

The reader is encouraged to solve Problem 3, which is to derive the following formula for the state degeneracy:

$$d(q) = d(q^2) = 16 \oint d\tau \frac{e^{i\pi\tau q^2}}{\eta^{24}(\tau)}, \quad (4.112)$$

where $\tau = \tau_1 + i\tau_2 \in \mathcal{H}$, where $\mathcal{H} = \{\tau \in \mathbb{C} | \tau_2 > 0\}$ is the upper half plane and where $\eta(\tau)$ is the Dedekind η-function. The integration contour runs through a strip of width one in the upper half plane, i.e., it connects two points $\tau^{(1)}$ and $\tau^{(2)} = \tau^{(1)} + 1$. Since the integrand is periodic under $\tau \to \tau + 1$ (which is a general property of modular forms), this integration contour is effectively closed. (It becomes a closed contour when going to the new variable $e^{2\pi i\tau}$, which takes values in the interior of the unit disc.)

In its present form this expression is not very useful, because we want to know $d(q)$ explicitly, at least asymptotically for large values of $|q^2|$. This type of problem was studied already by Hardy and Ramanujan, and a method for solving it exactly was found by Rademacher. For our specific problem with 24 'colours' the Rademacher expansion takes the following form:

$$d(q^2) = 16 \sum_{c=1}^{\infty} c^{-14} \mathrm{Kl}\left(\frac{1}{2}|q^2|, -1; c\right) \hat{I}_{13}\left(\frac{4\pi}{c}\sqrt{\frac{1}{2}|q^2|}\right), \quad (4.113)$$

where $\hat{I}_{13} =$ is the modified Bessel function of index 13, and $Kl(l, m; c)$ are the so-called Kloosterman sums.

Modified Bessel functions have the following integral representation:

$$\hat{I}_\nu(z) = -\mathrm{i}(2\pi)^\nu \int_{\epsilon-\mathrm{i}\infty}^{\epsilon+\mathrm{i}\infty} \frac{\mathrm{d}t}{t^{\nu+1}} \mathrm{e}^{t+\frac{z^2}{4t}},$$

and their asymptotics for $\mathrm{Re}(z) \to \infty$ is

$$\hat{I}_\nu(z) \approx \frac{\mathrm{e}^z}{\sqrt{2}} \left(\frac{z}{4\pi}\right)^{-\nu-\frac{1}{2}} \left(1 - \frac{2\nu^2-1}{8z} + \mathcal{O}(z^{-2})\right).$$

We will not need the values of the Kloostermann sums, except that $Kl(l,m;1) = 1$.

In the limit of large $|q^2|$ the term with $c = 1$ is leading, while the terms with $c > 1$ are exponentially suppressed

$$d(q^2) = 16\, \hat{I}_{13}\left(4\pi\sqrt{\frac{1}{2}|q^2|}\right) + \mathcal{O}\left(\mathrm{e}^{-|q^2|}\right).$$

Using the asymptotics of Bessel functions, this can be expanded in inverse powers of $|q^2|$:

$$S_{\mathrm{micro}}(q^2) = \log d(q^2) \approx 4\pi\sqrt{\frac{1}{2}|q^2|} - \frac{27}{4}\log|q^2| + \frac{15}{2}\log(2) - \frac{675}{32\pi|q^2|} + \cdots.$$

The first two terms correspond to a saddle point evaluation of the integral representation (4.112): The first term is the value of integrand at its saddle point, while the second term is the 'fluctuation determinant'. The derivation of the first two terms using a saddle point approximation of (4.112) is left to the reader as Problem 4. A derivation of the full Rademacher expansion (4.113) can be found in the literature.

Further Reading and References

An excellent and accessible account on the Rademacher expansion can be found in [71]. See in particular the appendix of this paper for two versions of the proof of the Rademacher expansion. We have also borrowed some formulae from [72, 73], who have studied the state counting for $\frac{1}{2}$-BPS states in great detail, including various $N = 4$ and $N = 2$ orbifolds of the toroidal $N = 4$ compactification considered in this lecture.

4.5.2 State Counting for $\frac{1}{4}$-BPS States

For the problem of counting $\frac{1}{4}$-BPS states the dual type-II picture of the $N = 4$ compactification is useful. Here all the heterotic $\frac{1}{2}$- and $\frac{1}{4}$-BPS states arise as winding states of the NS-five-brane. It is believed that the dynamics of an NS-five-brane is described by a string field theory whose target space is the world volume of the five-brane. If one assumes that the counting of BPS states is not modified by interactions, the problem of state counting reduces to counting states in a multi-string Fock space. For $\frac{1}{2}$-BPS states the resulting counting problem is found to be equivalent to the one described in the last section, as required by consistency. For $\frac{1}{4}$-BPS states the counting problem is more complicated, but one can derive the following integral representation:

$$d(p,q) = \oint \mathrm{d}\rho \mathrm{d}\sigma \mathrm{d}v \frac{\mathrm{e}^{\mathrm{i}\pi\left[\rho p^2+\sigma q^2+(2v-1)pq\right]}}{\Phi_{10}(\rho,\sigma,v)}. \tag{4.114}$$

This formula requires some explanation. Essentially it is a generalisation of (4.112), where the single complex variable τ has been replaced by three complex variables ρ,σ,τ, which live in the so called rank-2 Siegel upper half space $\mathcal{S}_2$. In general the rank-n Siegel upper half space is the space of all symmetric $(n\times n)$-matrices with positive definite imaginary part. This is a symmetric space,

$$\mathcal{S}_n \simeq \frac{Sp(2n)}{U(n)},$$

which can be viewed as a generalisation of the upper half plane

$$\mathcal{H} = \frac{Sp(2)}{U(1)} = \mathcal{S}_1.$$

The group $Sp(2n,\mathbb{Z})$ acts by fractional linear transformations on the $(n\times n)$ matrices $\Omega\in\mathcal{S}_n$,

$$\Omega \to (A\Omega+B)(C\Omega+D)^{-1}, \quad \text{where} \quad \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2n).$$

The discrete subgroup $Sp(2n,\mathbb{Z})$ is a generalisation of the modular group $Sp(2,\mathbb{Z}) \simeq SL(2,\mathbb{Z})$, and there is a corresponding theory of Siegel modular forms. A Siegel modular form is said to have weight $2k$, if it transforms as

$$\Phi(\Omega) \to \Phi\left((A\Omega+B)(C\Omega+D)^{-1}\right) = (\det(C\Omega+D))^k\Phi(\Omega).$$

In the rank-2 case, we can parameterise the matrix Ω as

$$\Omega = \begin{pmatrix} \rho & v \\ v & \sigma \end{pmatrix},$$

and positive definiteness of the imaginary part implies that

$$\rho_2 > 0, \quad \sigma_2 > 0, \quad \rho_2\sigma_2 - v^2 > 0,$$

where $\rho = \rho_1 + \mathrm{i}\rho_2$, etc.

In the theory of rank-2 Siegel modular forms, the analogon of the weight-12 cusp form $\eta^{24}(\tau)$ is the weight-10 Siegel cusp form $\Phi_{10}(\rho,\sigma,v)$, which enters into the state counting formula (4.114). Like modular forms, Siegel modular forms are periodic in the real parts of the variables ρ,σ,v. The integration contour in the Siegel half space is along a path of the form $\rho \to \rho+1$, $\sigma \to \sigma+1$, $v \to v+1$, which is effectively a closed contour since the integrand is periodical.[31] The state counting formula (4.114) is manifestly T-duality invariant. It is also formally S-duality invariant, in the sense that S-duality transformations can be compensated by $Sp(4,\mathbb{Z})$ transformations of the integration variables.

As in the $\frac{1}{2}$-BPS case one would like to evaluate (4.114) asymptotically, in the limit of large charges $q^2p^2-(p\cdot q)^2 \gg 1$. One important difference between Φ_{10} and η^{24} is that the Siegel cusp form has zeros in the interior of the Siegel half space $\mathcal{S}_2$, namely at $v=0$ and its images under $Sp(4,\mathbb{Z})$. The v-integral therefore evaluates the residues of the integrand. At $v=0$, the asymptotics of Φ_{10} is

$$\Phi \simeq_{v=0} v^2\eta^{24}(\rho)\eta^{24}(\sigma).$$

The asymptotics at the other zeros can be found by applying $Sp(4,\mathbb{Z})$ transformations.

If one sets the magnetic charges to zero, the residue at $v=0$ is the only one which contributes to (4.114). This can be used to derive the $\frac{1}{2}$-BPS formula (4.112) as a special case of (4.114).[32]

For $\frac{1}{4}$-BPS states it can be shown that for large charges the dominant contribution comes from the residue at

$$D = v + \rho\sigma - v^2 = 0,$$

while all other residues are exponentially suppressed. Neglecting the subleading residues, one can perform the v-integral. The remaining integral has the following structure:

$$d(p,q) = \oint \mathrm{d}\rho\mathrm{d}\sigma \mathrm{e}^{\mathrm{i}\pi(X_0+X_1)(\rho,\sigma)}\Delta(\rho,\sigma). \tag{4.115}$$

The parametrisation has been chosen such that X_1 and Δ are subleading for large charges.

This integral can be evaluated in a saddle point approximation, analogous to (4.112). The leading term for large charges is given by the approximate saddle point value of the integrand,

$$d(p,q) \approx \mathrm{e}^{\mathrm{i}\pi\, X_0|_*} = \mathrm{e}^{\pi\sqrt{p^2q^2-(pq)^2}}. \tag{4.116}$$

This result is manifestly T- and S-duality invariant.

[31] In the numerator one has to use that the Narain lattice is even selfdual.

[32] Incidentally, the problem is equivalent to the factorisation of a genus-2 string partition function into two genus-1 string partition functions.

A refined approximation can be obtained as follows: one identifies the exact critical point of $e^{i\pi X} = e^{i\pi(X_0+X_1)}$, expands the integrand $e^{i\pi X}\Delta$ to second order and performs a Gaussian integral. This is different from a standard saddle point approximation, where one would expand around the critical point of the full integrand $e^{i\pi X}\Delta$. This modification is motivated by the observation that the critical point of $i\pi X$ agrees exactly with the critical point of the reduced dilatonic entropy function (4.107), which gives the exact macroscopic entropy:

$$i\pi X_* = \pi\Sigma_* = S_{\text{macro}}(p,q).$$

At the critical point one has the following relation between the critical values of ρ, σ and the fixed point value of the dilaton:

$$\rho_* = \frac{i|S_*|^2}{S_* + \overline{S}_*}, \quad \sigma_* = \frac{i}{S_* + \overline{S}_*}.$$

One might think that the subleading contributions from Δ spoil the resulting equality between microscopic and macroscopic entropy. However, these cancel against the contributions from the Gaussian integration (the 'fluctuation determinant'), at least to leading order in an expansion in inverse powers of the charges:

$$e^{S_{\text{micro}}(p,q)} = d(p,q) \approx e^{\pi\Sigma_* + \cdots} = e^{S_{\text{macro}}(p,q)+\cdots}. \quad (4.117)$$

This shows that the modified saddle point approximation is compatible with a systematic expansion in large charges. Moreover, there is an intriguing direct relation between the saddle point approximation of the exact microscopic state degeneracy (4.114) and the black hole variational principle.

Further Reading and References

The state counting formula for $\frac{1}{4}$-BPS states in $N = 4$ compactifications was proposed in [74]. There several ways of deriving it were discussed, which provide very strong evidence for the formula. Further evidence was obtained more recently in [75], by using the relation between four-dimensional and five-dimensional black holes [76]. While the leading order matching between state counting and black hole entropy was already observed in [74], the subleading corrections were obtained in [69] by using the modified saddle point evaluation explained above.

Another line of development is the generalisation from toroidal compactifications to a class of $N = 4$ orbifolds, the so-called CHL-models [77, 78]. The issue of choosing integration countours is actually more subtle than apparent from our review, see [79, 80] for a detailed account. For a comprehensive account of Siegel modular forms, see [81].

4.5.3 Partition Functions for Large Black Holes

The strength of this result becomes even more obvious when we use it to compare the (microscopically defined) black hole partition function to the (macroscopically defined) free energy.

One way of doing this is to the evaluate mixed partition function $Z_{\text{mix}}(p,\phi) = \sum_q d(p,q)\mathrm{e}^{\pi q_I \phi^I}$ using integral representation (4.114) of $d(p,q)$. The result can be brought to the following form

$$Z_{\text{mix}}(p,\phi) = \sum_{\text{shifts}} \sqrt{\tilde{\Delta}(p,\phi)}\mathrm{e}^{\pi \mathcal{F}_{\text{mix}}(p,\phi)}. \tag{4.118}$$

$\mathcal{F}_{\text{mix}}$ is the black hole free energy, including all, both the holomorphic and the non-holomorphic corrections:

$$\mathcal{F}_{\text{mix}}(p,\phi) = \frac{1}{2}(S+\overline{S})\left(p^a \eta_{ab} p^b - \phi^a \eta_{ab}\phi^b\right) - \mathrm{i}(S-\overline{S})p^a \eta_{ab}\phi^b$$
$$+ 4\Omega(S,\overline{S},\Upsilon,\overline{\Upsilon}).$$

By imposing the magnetic attractor equations in the transition to the mixed ensemble, the dilaton has become a function of the electric potentials and the magnetic charges:

$$S = \frac{-\mathrm{i}\phi^1 + p^1}{\phi^0 + \mathrm{i}p^0}.$$

The mixed partition functions is by construction invariant under shifts $\phi \to \phi + 2\mathrm{i}$. The mixed free energy is found to have a different periodicity, and this manifests itself by the appearance of a finite sum over additional shifts of ϕ in (4.118). As predicted on the basis of symplectic covariance, the relation between the partition function and the free energy is modified by a 'measure factor' $\tilde{\Delta}^-$, which we do not need to display explicitly. This factor agrees with the measure factor Δ^- in (4.94), which we found by imposing symplectic covariance in the limit of large charges:

$$\tilde{\Delta}^- \approx \Delta^-.$$

Since we already made a partial saddle point approximation when going from the canonical to the mixed ensemble, we could not expect an exact agreement. It is highly non-trivial that we can match the full mixed free energy, including the infinite series of instanton corrections. Moreover, we have established that there is a non-trivial measure factor, which agrees to leading order with the one constructed by symmetry considerations.

Further Reading, References, and Some Comments

The idea to evaluate the mixed partition function using microscopic state counting in order to check the OSV conjecture for $N = 4$ compactifiactions was first used in [82]. This confirmed the expectation that the OSV conjecture needs to be modified by a measure factor once subleading corrections are taken into account. This result was generalized in [52], where we showed that the measure factor agrees asymptotically with our conjecture which is based on imposing symplectic covariance. Above, we pointed out that in (4.118) we obtain the full mixed $\mathcal{F}_{\text{mix}}$, including the non-holomorphic corrections. Of course, this way of organising the result is motivated by our approach to non-holomorphic corrections, and it is consistent to regard these contributions as part of the measure factor, as other authors appear to do. Further work is needed, in particular on the role played by the non-holomorphic corrections in the microscopic description, before we can decide which way interpreting the partition function is more adequate. Let us also mention that while we specifically considered toroidal $N = 4$ compactifications in this section, all results generalise to CHL models.

There has also been much activity on $N = 2$ compactifications over the last years. Much of this work has focussed on establishing and explaining the asymptotic factorisation

$$Z_{\text{mix}} \simeq |Z_{\text{top}}|^2$$

predicted by the OSV conjecture [83–86]. The strategy pursued in these papers is to use string-dualities, in particular the AdS_3/CFT_2-correspondence, to reformulate the problem in terms of two-dimensional conformal field theory. In comparison to the simpler $N = 4$ models, the relevant microscopic partition functions are related to the so-called elliptic genus of the underlying CFT. Roughly, the elliptic genus is a 'BPS partition function', i.e. a partition function which has been modified by operator insertions such that it only counts BPS states. The main problem is to find a suitable generalisation of the Rademacher expansion which allows to evaluate these BPS partition functions asymptotically for large charges. The picture emerging from this treatment is that the black hole can be described microscopically (modulo string dualities) as a non-interacting state of branes and anti-branes. This explains the asymptotic factorisation.

But as we have stressed throughout, non-holomorphic corrections are expected to manifest themselves at the subleading level, which microscopically correspond to interactions between branes and antibranes. And indeed, a more recent refined analysis [40] has revealed the presence of a measure factor, which agrees with the one found in [82] and [52] in the limit of large charges.

There is one further point which we need to comment on. During this lecture we have tentatively assumed that 'state counting' means literally to count all the BPS states. But, as we have mentioned previously, the BPS spectrum changes when crossing a line of marginal stability. This is a possible cause for discrepancies between state counting and thermodynamical entropy, because they are computed in different regions of the parameter space. In their original work [1] therefore conjectured that the microscopical entropy entering into the OSV conjecture is an 'index', i.e. a weighted sum over states which remains invariant when crossing lines of marginal stability. The detailed study of [72, 73] showed that it is very hard in practice to discriminate between absolute versus weighted state counting. While one example appeared to support absolute state counting, it was pointed out later that there are several candidates for the weighted counting [40]. One intriguing proposal is that the correct absolute state counting is in fact captured by an index, once it is taken into account that states which are stable in the free limit become unstable once interactions are taken into account [40].

In conclusion, the OSV conjecture appears to work well in the semi-classical approximation, if supplemented by a measure factor. The concrete proposal discussed in these lectures works correctly in this limit. It is less clear what is the status of the original, more ambitious goal of finding an exact relation [1], which would have various ramifications, such as helping to find a non-perturbative definition of the topological string [1], formulating a mini-superspace approximation of stringy quantum cosmology [87], studying $N = 1$ compactifications via 'topological M-theory' [60, 63], and approaching the vacuum selection problem of string theory by invoking an 'entropic principle' [88–90].

4.5.4 Partition Functions for Small Black Holes

The counting of $\frac{1}{2}$-BPS states gave rise to the following microscopic entropy:

$$S_{\text{micro}} \approx \log \hat{I}_{13} \left(4\pi \sqrt{\frac{1}{2}|q^2|} \right) \approx 4\pi \sqrt{\frac{1}{2}|q^2|} - \frac{27}{4} \log |q^2| + \cdots . \quad (4.119)$$

This is to be compared with the macroscopic entropy. Including the classical part of the R^2-coupling and the non-holomorphic corrections, but neglecting instantons, this is

$$S_{\text{macro}} = 4\pi \sqrt{\frac{1}{2}|q^2|} - 6 \log |q^2| + \cdots . \quad (4.120)$$

While the leading terms agree, the first subleading term comes with a slightly different coefficient. However, we have seen that both entropies belong to different ensembles, so that we can only expect that they agree in the thermodynamical limit. Since we have a conjecture about the exact (or at least asymptotically exact) relation between both entropies, we can check whether the shift in the coefficient of the subleading term is predicted correctly. Our conjecture about the relation between the canonical free energy and the canonical partition function predicts the following relation (see Sect. 4.3.6):

$$S_{\text{micro}} = S_{\text{macro}} + \log \sqrt{\frac{\Delta^-}{\Delta^+}}.$$

This shows that both entropy are indeed different if the measure factor Δ^- and the fluctuation determinant Δ^+ are different. For dyonic black holes we found that both were equal, up to subleading contribution. Unfortunately our relation is not useful for small black holes, because

$$\begin{aligned} \Delta^- &= 0, \quad \text{up to non-holomorphic terms and instantons,} \\ \Delta^+ &= 0, \quad \text{up to instantons.} \end{aligned}$$

Since the measure factor and the fluctuation determinant are degenerate (up to subleading contributions) the saddle point approximation is not well defined. This reflect that small black hole live on the boundary of moduli space.

We can still test our conjecture about the relation between the mixed partition and the mixed free energy, in particular the presence of a measure factor and the role of non-holomorphic contributions. This requires to evaluate

$$\exp(S_{\text{micro}}) = d(p^1, q) \approx \int \mathrm{d}\phi \sqrt{\Delta^-(p^1, \phi)} \mathrm{e}^{\pi[\mathcal{F}_{\text{mix}}(p^1,\phi) - q_I \phi^I]},$$

where a non-vanishing Δ^- is obtained by including the non-holomorphic corrections.[33] We still neglect the contributions of the instantons.

The integral can be evaluated, with the result:

$$d(p^1, q) \approx \int \frac{\mathrm{d}S \mathrm{d}\overline{S}}{(S + \overline{S})^{14}} \sqrt{S + \overline{S} - \frac{12}{2\pi}} \exp\left[-\frac{\pi q^2}{S + \overline{S}} + 2\pi(S + \overline{S})\right]. \tag{4.121}$$

Here the integrals over $\phi^a = \phi^2, \phi^3, \ldots, \phi^{27}$ have been performed and the remaining integrals over ϕ^0 and ϕ^1 have been expressed in terms of the dilaton S. If we approximate

$$\sqrt{S + \overline{S} - \frac{12}{2\pi}} \approx \sqrt{S + \overline{S}},$$

this becomes the integral representation of a modified Bessel function.

Then our conjecture predicts

$$S_{\text{micro}}^{(\text{predicted})} \approx \log \hat{I}_{13-\frac{1}{2}} \left(4\pi \sqrt{\frac{1}{2}|q^2|}\right) \approx 4\pi \sqrt{\frac{1}{2}|q^2|} - \frac{13}{2} \log|q^2| + \cdots, \tag{4.122}$$

while the entropy obtained from state counting is

$$S_{\text{micro}} \approx \log \hat{I}_{13} \left(4\pi \sqrt{\frac{1}{2}|q^2|}\right) \approx 4\pi \sqrt{\frac{1}{2}|q^2|} - \frac{27}{4} \log|q^2| + \cdots. \tag{4.123}$$

[33] Remember that p^1 is an electric charge for the heterotic string. We take $q_1 = 0$, because this is a magnetic charge.

Thus there is a systematic mismatch in the index of the Bessel function, and while the leading terms agree, the coefficients of the log-terms and all the following inverse-power terms mismatch.

This result can be compared with the original OSV-conjecture, where one does not have a measure factor, and where only holomorphic contributions to the free energy are taken into account:

$$d(p^1,q) \approx \int \mathrm{d}\phi \mathrm{e}^{\pi[\mathcal{F}_{\mathrm{OSV}}(p^1,\phi)-q_I\phi^I]} \approx (p^1)^2 \hat{I}_{15}\left(4\pi\sqrt{\frac{1}{2}|q^2|}\right),$$

$$S_{\mathrm{micro}}^{(\mathrm{predicted})} = 4\pi\sqrt{\frac{1}{2}|q^2|} - \frac{31}{4}\log|q^2| + \log(p^1)^2 + \cdots . \quad (4.124)$$

In this case the index of the Bessel function deviates even more, and in addition there is an explicit factor $(p^1)^2$ which spoils T-duality. This clearly shows that the OSV conjecture needs to be modified by a measure factor.

When deriving (4.124), we have integrated over 28 potentials ϕ^I, as we have done in our discussion of large black holes, and in (4.121). There is one subtlety to be discussed here. The full $N=4$ theory has 28 gauge fields, but we have used the $N=2$ formalism. Since we disregard the gravitini multiplets (and the hypermultiplets), we work with a truncation to a subsector consisting of the $N=2$ gravity multiplet and 23 vector multiplets. This theory only has 24 gauge fields, and therefore it only has 24 electrostatic potentials ϕ^I. However, at the end we should reconstruct the missing four gauge potentials, and as we have seen when recovering the $N=4$ entropy formula using the $N=2$ formalism, this extension is uniquely determined by T-duality. As we have seen this prescription works for large black holes, but for small black holes we do not quite obtain the right index for the Bessel function.

However, the correct index for the Bessel function is obtained when using the unmodified OSV conjecture, but integrating only over 24 instead of 28 electrostatic potentials:

$$d(p^1,q) \approx \int \mathrm{d}\phi \mathrm{e}^{\pi[\mathcal{F}_{\mathrm{OSV}}(p^1,\phi)-q_I\phi^I]} \approx (p^1)^2 \hat{I}_{13}\left(4\pi\sqrt{\frac{1}{2}|q^2|}\right), \quad (4.125)$$

$$S_{\mathrm{micro}}^{(\mathrm{predicted})} = 4\pi\sqrt{\frac{1}{2}|q^2|} - \frac{27}{4}\log|q^2| + \log(p^1)^2 + \cdots . \quad (4.126)$$

Note that this does not cure the problem with the prefactor $(p^1)^2$, which is incompatible with T-duality. It is intriguing, but at the same time puzzling that the correct value for the index is obtained by reducing the number of integrations. However, it is not clear how to interpret this restriction. Moreover, it is unavoidable to include a measure factor to implement T-duality, and this is very likely to have an effect on the index.

Further Reading and References

In this section, we followed [52], and compared the result with the calculation based on the original OSV conjecture [72, 73]. Both approaches find agreement for the leading term, but disagreement for the subleading terms. Moreover, when sticking to the original OSV conjecture, the result is not compatible with T-duality. Further problems and subtleties with the OSV conjecture for $\frac{1}{2}$-BPS black holes have been discussed in detail in [72, 73]. One obvious explanation for these difficulties is that in the 'would-be leading' order approximation small black holes are singular: they have a vanishing horizon area and the moduli take values at the boundary of the moduli space. While the higher curvature smooth the null singularity, leading to agreement between macroscopic and microscopic entropy to leading order in the charges, the semi-classical expansion is still ill defined, since one attempts to expand around a singular configuration. Apparently one needs to find a different way of organising the expansion, if some version of the OSV conjecture is to hold at the semi-classical level. A more drastic alternative is that the OSV conjecture simply does not apply to small black holes. But since the mismatch of the subleading corrections appears to follow some systematics, there is room for hope. The situation is less encouraging for the non-perturbative corrections coming form instantons. As observed both in [72, 73] and in [52] the analytical structure of the terms observed in microscopic state counting is different from the one expected on the basis of the OSV conjecture.

4.5.5 Problems

Problem 4.3. Counting states of the open bosonic string.

In the light cone gauge, a basis for the Hilbert space of the open bosonic string (neglecting the center of mass momentum) is given by

$$\alpha^{i_1}_{-m_1}\alpha^{i_2}_{-m_2}\cdots|0\rangle, \tag{4.127}$$

where $i_k = 1, \ldots, 24$ and $m_k = 1, 2, 3, \ldots$. States with the same (total) excitation number $n = m_1 + m_2 + \ldots$ have the same mass. Incidentally, the problem of counting states of the open bosonic string with given mass, is the same as counting the number of $\frac{1}{2}$-BPS states for the heterotic string, compactified on T^6, with given charges $q \in \Gamma_{\text{Narain}}$.

The number of states with given excitation number n is encoded in the partition function

$$Z(q) = \mathrm{Tr}\, q^N, \tag{4.128}$$

where the trace is over the Hilbert space of physical string states (light cone gauge), $q \in \mathbb{C}$, and N is the number operator with eigenvalues $n = 0, 1, 2, 3, \ldots$. Evaluation of the trace gives

$$Z(q) = \left(\prod_{l=1}^{\infty} (1 - q^l) \right)^{-c}, \quad |q| < 1, \tag{4.129}$$

where $c = D - 2 = 24$ is the number of space–time dimensions transverse to the string world sheet (the physical excitations). The number d_n of string states at level n is encoded in the Taylor expansion

$$Z(q) = \sum_{n=0}^{\infty} d_n q^n. \tag{4.130}$$

Verify that d_n counts string states, for small $n = 0, 1, 2, \ldots$ Do this either for the critical open bosonic string, $c = 24$, or for just one string coordinate, $c = 1$. The latter is the classical problem of counting partitions of an integer. It is instructive to evaluate d_n both directly, by reorganising the product representation (4.129) into a Taylor series, and by the integral representation of d_n obtained by inverting (4.130).

Hints: Note that

$$Z(q) = q\Delta^{-1}(q), \tag{4.131}$$

where $\Delta(q) = \eta^{24}(q)$ is the cusp form (η is the Dedekind eta-function). $\Delta(q)$ is a modular form of weight 12 and has the following expansion around the cusp $q = 0$:

$$\Delta(q) = q - 24q^2 + 252q^3 - 1472q^4 + 4830q^5 + \mathcal{O}(q^6). \tag{4.132}$$

$\Delta(q)$ has no zeros for $0 < |q| < 1$.
d_n can be computed by a contour integral in the unit disc $|q| < 1$.

Problem 4.4. The asymptotic state density of the open bosonic string.

Given the information provided in Problem 4.3, compute the asymptotic number of open bosonic string states d_n for $n \to \infty$. (You may restrict yourself to the case $c = 24$, which corresponds to the critical dimension.)

Instructions:

1. The unit disc $|q| < 1$ can be mapped to the semi-infinite strip $-\frac{1}{2} < \tau_1 < \frac{1}{2}$, $\tau_2 > 0$ in the complex τ-plane, $\tau = \tau_1 + i\tau_2$ by

$$q = e^{2\pi i \tau}. \tag{4.133}$$

   (Like other modular forms, Δ extends to a holomorphic function on the whole upper half plane by periodicity in τ_1.)
   Rewrite the contour integral for d_n as a contour integral over τ.
2. Use the modular properties of $\Delta(\tau)$ to find the behaviour of the integrand close to $\tau = 0$ from the known behaviour of $\Delta(\tau)$ at $\tau = i\infty \Leftrightarrow q = 0$. Show that for $n \to \infty$ the integrand has a sharp saddle point. Use this to evaluate the contour

integral in saddle point approximation. (Expand the integrand to second order around the saddle point and perform the resulting Gaussian integral.)

3. The correct result is

$$d_n \approx \text{Const. } e^{4\pi\sqrt{n}} n^{-\frac{27}{4}}. \tag{4.134}$$

A Kähler Manifolds and Special Kähler Manifolds

In this appendix we review Kähler manifolds and special Kähler manifolds from the mathematical perspective. The first part is devoted to the basic definitions and properties of complex, hermitian and Kähler manifolds. For a more extensive review we recommend the book by Nakahara [91], and, for readers with a stronger mathematical inclination, the concise lecture notes by Ballmann [92]. The characterisation of complex and Kähler manifolds in terms of holonomy groups can be found in [93]. The second part reviews special Kähler manifolds and is mostly based on [21, 22] with supplements from [23–25]. A review of special geometry from a modern perspective can also be found in [94].

A para-complex variant of special geometry, which applies to the target manifolds of Euclidean $N = 2$ theories has been developed in [23, 24]. The framework of ϵ-Kähler manifolds, which has been employed in [25], is particularly suitable for treating Euclidean supersymmetry and standard (Lorentzian) supersymmetry in parallel.

A.1 Complex and Almost Complex Manifolds

Let M be a differentiable manifold of dimension $2n$.

Definition 4.1. An *almost complex structure* I on M is tensor field of type $(1,1)$ with the property that (pointwise)

$$I^2 = -\text{Id}.$$

In components, using real coordinates $\{x^m | m = 1, \dots, 2n\}$, this condition reads

$$I^m_{\ p} I^p_{\ n} = -\delta^m_n. \tag{4.135}$$

Definition 4.2. An almost complex structure is called *integrable* if the associated Nijenhuis tensor N_I vanishes for all vector fields X, Y on M:

$$N_I(X,Y) := [IX, IY] + [X,Y] - I[X, IY] - I[IX, Y] = 0.$$

The expression for N_I in terms of local coordinates $\{x^m\}$ can be found by substituting the coordinate expressions $X = X^m \partial_m$, $Y = Y^m \partial_m$ for the vector fields.[34] We will not need this explicitly.

Remark. The integrability of an almost complex structure is equivalent to the existence of local complex coordinates $\{z^i | i = 1, \ldots, n\}$. An integrable almost complex structure is therefore also simply called a *complex structure*.

Definition 4.3. A manifold which is equipped with an (almost) complex structure is called an (almost) complex manifold.

Remark. The existence of an (almost) complex structure can be rephrased in terms of holonomy. An almost complex structure is a $GL(n, \mathbb{C})$ structure, and a complex structure is a torsion-free $GL(n, \mathbb{C})$ structure.

A.2 Hermitian Manifolds

Let (M, I) be a complex manifold and let g be a (pseudo-)Riemannian metric on M.

Definition 4.4. (M, g, I) is called a *hermitian* manifold, if I generates isometries of g:

$$I^* g = g. \tag{4.136}$$

Remark. Condition (4.136) is equivalent to saying that

$$g(IX, IY) = g(X, Y),$$

for all vector fields X, Y on M. In local coordinates the condition reads

$$g_{pq} I^p_{\ m} I^q_{\ n} = g_{mn}. \tag{4.137}$$

Remark. If the metric is indefinite, (M, g, I) is called pseudo-hermitian, but we will usually drop the prefix 'pseudo-'.

On a hermitian manifold one can define the so-called *fundamental two-form*:

$$\omega(X, Y) := g(IX, Y),$$

or, in coordinates,

$$\omega_{mn} = -g_{mp} I^p_{\ n}. \tag{4.138}$$

[34] In fact, it is sufficient to substitute a basis of coordinate vector fields $\{\partial_m\}$ to obtain the components $N_{mn} = N(\partial_m, \partial_n)$.

Note that $\omega_{mn} = -\omega_{nm}$, because g_{mn} is symmetric, while I satisfies (4.135) and (4.137). Moreover the two-form ω is non-degenerate, because g is.

Equation (4.138) can be solved for the metric g or for the complex structure I:

$$g_{mn} = \omega_{mk} I^k_{\ n}, \\ I^m_{\ n} = -g^{mk}\omega_{kn}. \tag{4.139}$$

Thus, if any two of the three data g (metric), I (complex structure) or ω (fundamental two-form) are given on a hermitian manifold, the third is already determined.

When we use complex coordinates $\{z^i\}$, the complex structure only has 'pure' components:

$$I^i_{\ j} = i\delta^i_{\ j}, \quad I^{\bar{i}}_{\ \bar{j}} = -i\delta^{\bar{i}}_{\ \bar{j}}.$$

For a hermitian metric the pure components vanish, $g_{ij} = 0$ and $g_{\bar{i}\bar{j}} = 0$. Only the 'mixed' components $g_{i\bar{j}}$ and $g_{\bar{i}j} = \overline{g_{i\bar{j}}}$ remain. Note that the matrix $g_{i\bar{j}}$ is hermitian. The fundamental two-form also only has mixed components, and $\omega_{i\bar{j}} = i g_{i\bar{j}}$. Thus in complex coordinates the matrices representing the metric and the fundamental two-form are hermitian and anti-hermitian, respectively, while in real coordinates they are symmetric and antisymmetric, respectively.

On a hermitian manifold the metric

$$g = g_{i\bar{j}} \left(\mathrm{d}z^i \otimes \bar{z}^{\bar{j}} + \mathrm{d}\bar{z}^{\bar{j}} \otimes \mathrm{d}z^i \right)$$

and the fundamental two-form

$$\omega = i g_{i\bar{j}} \mathrm{d}z^i \wedge \mathrm{d}\bar{z}^{\bar{j}} = i g_{i\bar{j}} \left(\mathrm{d}z^i \otimes \mathrm{d}\bar{z}^{\bar{j}} - \mathrm{d}\bar{z}^{\bar{j}} \otimes \mathrm{d}z^i \right)$$

can be combined into the hermitian form

$$\tau = g_{i\bar{j}} \mathrm{d}z^i \otimes \mathrm{d}\bar{z}^{\bar{j}} = \frac{1}{2}(g - i\omega).$$

The hermitian form defines a hermitian metric on the complexified tangent bundle $TM_{\mathbb{C}}$ of M. All statements and formulae in this section apply irrespective of g being positive definite or indefinite (but non-degenerate).

A.3 Kähler Manifolds

Definition 4.5. A *Kähler manifold* (M, g, I) is a hermitian manifold where the fundamental form is closed:

$$\mathrm{d}\omega = 0.$$

Remark. Equivalently, one can impose that the complex structure is parallel with respect to the Levi–Civita connection,

$$\nabla^{(g)} I = 0.$$

Comment: hermitian manifolds are characterised by 'pointwise' compatibility conditions between metric, complex structure and fundamental form. For Kähler manifolds one imposes a stronger compatibility condition: the complex structure I must be parallel (covariantly constant) with respect to the Levi–Civita connection $\nabla^{(g)}$. Since the metric g itself is parallel by definition of $\nabla^{(g)}$, parallelity of I is equivalent to the parallelity of the fundamental form ω. Moreover, it can be shown that if ω is closed, it is automatically parallel with respect to $\nabla^{(g)}$.

The fundamental form of a Kähler manifold is called its *Kähler form.* It can be shown that a Kähler metric can be expressed in terms of a real-analytic function, the Kähler potential, by[35]

$$g_{i\bar{j}} = \frac{\partial^2 K(z,\bar{z})}{\partial z^i \partial \bar{z}^{\bar{j}}}.$$

The Kähler form can also be expressed as the second derivative of the Kähler potential:

$$\omega = i\partial\bar{\partial}K = i g_{i\bar{j}} \mathrm{d}z^i \wedge \mathrm{d}\bar{z}^{\bar{j}}, \quad \text{where} \quad \partial = \mathrm{d}z^i \partial_i, \quad \bar{\partial} = \mathrm{d}\bar{z}^{\bar{j}} \partial_{\bar{j}}$$

are the Dolbeault operators (holomorphic exterior derivatives).

Remark. If the metric g is positive definite, a Kähler manifold can be defined equivalently as a $2n$-dimensional manifold with a torsion-free $U(n)$ structure. Note that $U(n) \simeq GL(n,\mathbb{C}) \cap SO(2n) \subset GL(2n,\mathbb{R})$, which shows that $U(n)$ holonomy implies that there is a connection such that both the metric and the complex structure are parallel.

Remark. If the metric is not positive definite, $U(n)$ is replaced by a suitable non-compact form. Pseudo-hermitian manifolds with closed fundamental form are called pseudo-Kähler manifolds. We have seen in the main text that the (conical affine special) Kähler manifolds occurring in the construction of supergravity theories within the superconformal calculus always have indefinite signature, because the compensator of complex dilatations has a kinetic term with an inverted sign. We usually omit the prefix 'pseudo-' in the following and in the main text.

A.4 Affine Special Kähler Manifolds

Special Kähler manifolds are distinguished by the fact that the Kähler potential $K(z,\bar{z})$ can itself be expressed in terms of a holomorphic prepotential $F(z)$. The intrinsic definition of such manifolds is as follows [21].

[35] In fact, this might serve as yet another equivalent definition of a Kähler manifold.

Definition 4.6. An *affine special Kähler manifold* (M, g, I, ∇) is a Kähler manifold (M, g, I) equipped with a flat, torsion-free connection ∇, which has the following properties:

1. The connection is symplectic, i.e., the Kähler form is parallel

$$\nabla \omega = 0.$$

2. The complex structure satisfies

$$\mathrm{d}^{\nabla} I = 0,$$

which means, in local coordinates, that

$$\nabla_{[m} I^{p}_{\ n]} = 0.$$

Remark. The complex structure is not parallel with respect to the special connection ∇, but only 'closed' (regarding I as a vector-valued one-form). This, together with the fact that ∇ is flat shows that the connections ∇ and $\nabla^{(g)}$ are different, except for the trivial case of a flat Levi–Civita connection.

It can be shown that the existence of a special connection ∇ is equivalent to the existence of a Kählerian Lagrangian immersion of M into a model vector space, namely the standard complex vector space of doubled dimension [22]. Let us review this construction in some detail.

The standard complex symplectic vector space of complex dimension $2n$ is $V = T^*\mathbb{C}^n$. As a vector space, this is isomorphic to $\mathbb{C}^{2n}$. Let z^i be linear coordinates on $\mathbb{C}^n$ and w_i be coordinates on $T_z\mathbb{C}^n$. Then we can take (z^i, w_i) as coordinates on $T^*\mathbb{C}^n$, and the symplectic form is

$$\Omega_V = \mathrm{d}z^i \wedge \mathrm{d}w_i .$$

If we interpret V as a phase space, then the z^i are the coordinates and the w_i are the associated momenta. Symplectic rotations of (z^i, w_i) give rise to different 'polarisations' (choices of coordinates vs. momenta) of V.

The vector space V can be made a Kähler manifold in the following way: starting form the antisymmetric complex bilinear form Ω_V one can define an hermitian sesquilinear form γ_V by applying complex conjugation in the second argument of Ω, plus multiplication by i:

$$\gamma_V = \mathrm{i}\left(\mathrm{d}z^i \otimes \mathrm{d}\overline{w}_i - \mathrm{d}w_i \otimes \mathrm{d}\overline{z}^i\right) .$$

The real part of γ_V is a flat Kähler metric of signature $(2n, 2n)$:

$$g_V = \mathrm{Re}(\gamma_V) = \mathrm{i}\left(\mathrm{d}z^i \otimes_{\mathrm{sym}} \mathrm{d}\overline{w}_i - \mathrm{d}w_i \otimes_{\mathrm{sym}} \mathrm{d}\overline{z}^i\right) ,$$

while the imaginary part is the associated Kähler form:

$$\omega_V = \mathrm{Im}(\gamma_V) = \mathrm{d}z^i \wedge \mathrm{d}\overline{w}_i - \mathrm{d}w_i \wedge \mathrm{d}\overline{z}^i .$$

Now consider the immersion of a manifold M into V. An immersion is a map with invertible differential. An immersion need not be an invertible map, but it can be made invertible by restriction. Invertible immersion are called embeddings. (Intuitively, the difference between immersions and embeddings is that embeddings are not allowed to have self-intersections, or points where two image points come arbitrarily close.)

Definition 4.7. An immersion Φ of a complex manifold M into a Kähler manifold is called *Kählerian*, if it is holomorphic and if the pullback $g = \Phi^* g_V$ of the Kähler metric is nondegenerate.

Remark. Equivalently, one can require that the pullback of the hermitian form or of the Kähler form is non-degenerate.

Definition 4.8. An immersion Φ of a complex manifold M into a complex symplectic manifold is called *Lagrangian* if the pullback of the complex symplectic form vanishes, $\Phi^* \Omega_V = 0$.

Remark. For generic choices of coordinates, a Lagrangian immersion Φ is generated by a holomorphic function F on M, i.e. $\Phi = dF$.

It has been shown that for any affine special Kähler manifold of complex dimensions n there exists[36] a Kählerian Lagrangian immersion into $V = T^*\mathbb{C}^n$. Moreover every Kählerian Lagrangian immersion of an n-dimensional complex manifold M into V induces on it the structure of an affine special Kähler manifold.

By the immersion Φ, the special Kähler manifold M is mapped into V as the graph[37] of a map $z^i \to w_i = \frac{\partial F}{\partial z^i}$, where F is the prepotential of the special Kähler metric, which is the generating function of the immersion: $\Phi = \mathrm{d}F$. Using the immersion, one obtains 'special' coordinates on M by picking half of the coordinates (z^i, w_i) of V (say, the z^i). Along the graph, the other half of the coordinates of V are dependent quantities, and can be expressed through the prepotential: $w_i = w_i(z) = \frac{\partial F}{\partial z^i}$. The special Kähler metric g, the Kähler form ω and the hermitian form γ on M are the pullbacks of the corresponding data g_V, ω_V, τ_V of V under the immersion.

Remark. For non-generic choices of Φ the immersed M may be not a graph. Then the z^i do not provide local coordinates, the w_i are not the components of a gradient, and Φ does not have a generating function, i.e., 'there is no prepotential'.[38] This

[36] Locally, and if the manifold is simply connected even globally.

[37] More precisely, the image is *generically* the graph of map. We comment on non-generic immersions below.

[38] In the physics literature, this phenomenon and its consequences have been discussed in detail in [95, 96].

is not a problem, since one can work perfectly well by using only the symplectic vector (z^i, w_i). Moreover, by a symplectic transformation one can always make the situation generic and go to a symplectic basis (polarisation of V) which admits a prepotential.

Remark. In the main text we denoted the component expression for the affine special Kähler metric on M by N_{IJ} instead of $g_{i\bar{j}}$. The scalar fields X^I correspond to the special coordinates z^i. More precisely, the scalar fields can be interpreted as compositions of maps from space–time into M with coordinate maps $M \supset U \to \mathbb{C}^n$. The key formulae which express the Kähler potential and the metric in terms of the prepotential are (4.34) and (4.37).

A.4.1 Special Affine Coordinates and the Hesse Potential

Kähler manifolds are in particular symplectic manifolds, because the fundamental form is both non-degenerate and closed. The additional structure on affine special Kähler manifolds is the special connection ∇, which is both flat and symplectic (i.e. the symplectic form ω is parallel with respect to ∇.)[39] As a consequence, there exist ∇-affine (real) coordinates $x^i, y_i, i = 1, \ldots, n$ on M,

$$\nabla \mathrm{d}x^i = 0, \quad \nabla \mathrm{d}y_i = 0,$$

which are adapted to the symplectic structure,

$$\omega = 2\mathrm{d}x^i \wedge \mathrm{d}y_i .$$

The relation between these special affine coordinates and the special coordinates z^i can be elucidated by using the immersion of M into V. We can decompose z^i, w_i into their real and imaginary parts:

$$z^i = x^i + \mathrm{i}u^i, \quad w_i = y_i + \mathrm{i}v_i .$$

Then the Kähler form ω_V takes the form

$$\omega_V = \mathrm{d}x^i \wedge \mathrm{d}y_i + \mathrm{d}u^i \wedge \mathrm{d}v_i .$$

Using that the pullback of the complex symplectic form Ω_V vanishes, one finds that the pullback of ω_V is[40]

$$\omega = \Phi^* \omega_V = 2\mathrm{d}x^i \wedge \mathrm{d}y_i .$$

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[39] It is of course also parallel with respect to the Levi–Civita connection $\nabla^{(g)}$, but the Levi–Civita connection is not flat (except in trivial cases).

[40] For notational simplicity, we denote the pulled back coordinates $\Phi^* x^i, \Phi^* y^i$ by x^i, y_i.

Thus the special real coordinates form the real part of the symplectic vector (z^i, w_i). The real and imaginary parts of $z^i = x^i + u^i$ also form a system of real coordinates on M, which is induced by the complex coordinate system z^i, but not adapted to the symplectic structure (since x^i, u^i do not form a symplectic vector). The change of coordinates

$$(x^i, u^i) \to (x^i, y_i)$$

can be viewed as a Legendre transform, because

$$y_i = \mathrm{Re}\left(\frac{\partial F}{\partial z^i}\right) = \frac{\partial \mathrm{Im} F}{\partial \mathrm{Im} z^i} = \frac{\partial \mathrm{Im} F}{\partial u_i}. \tag{4.140}$$

The Legendre transform maps the imaginary part of the prepotential to the Hesse potential

$$H(x, y) = 2\left(\mathrm{Im} F(x + \mathrm{i}u(x, y)) - u_i y^i\right).$$

A Hesse potential is a real Kähler potential, i.e., a potential for the metric, but based on real rather than complex coordinates. Denoting the affine special coordinates by $\{q^a | a = 1, \ldots, 2n\} = \{x^i, y_i | i = 1, \ldots, n\}$, the special Kähler metric on M is given by

$$g = \frac{\partial^2 H}{\partial q^a \partial q^b} \mathrm{d}q^a \otimes_{\mathrm{sym}} \mathrm{d}q^b.$$

The special connection present on an affine special Kähler manifold is not unique. The $U(1)$ action generated by the complex structure generates a one-parameter family of such connections. Each of these comes with its corresponding special affine coordinates. The imaginary part (u^i, v_i) of the symplectic vector (z^i, w_i) provides one of these special affine coordinate systems. The coordinate systems (x^i, y_i) and (u^i, v_i) both occur naturally in the construction of BPS black hole solutions.

A.5 Conical Affine Special Kähler Manifolds and Projective Special Kähler Manifolds

Definition 4.9. A *conical* affine special Kähler manifold (M, g, I, ∇, ξ) is an affine special Kähler manifold endowed with a vector field ξ such that

$$\nabla^{(g)}\xi = \nabla\xi = \mathrm{Id}. \tag{4.141}$$

The condition $\nabla^{(g)}\xi = \mathrm{Id}$ implies that ξ is a homothetic Killing vector field, and that it is hypersurface orthogonal. Then one can introduce adapted coordinates $\{r, v^a\}$ such that

$$\xi = r\frac{\partial}{\partial r}$$

and

$$g = \mathrm{d}r^2 + r^2 g_{ab}(v)\mathrm{d}v^a \mathrm{d}v^b.$$

Thus M is a real cone. However, in our case M carries additional structures, and ξ satisfies the additional condition $\nabla\xi = \mathrm{Id}$. It can be shown that this implies that M has a freely acting $U(1)$ isometry, with Killing vector field $I\xi$. The surfaces $r = \text{const.}$ are the level surfaces of the moment map of this isometry. Therefore the isometry preserves the level surfaces, and $M \subset T^*\mathbb{C}^{n+1}$ has the structure of a complex cone, with $\mathbb{C}^*$-action generated by $\{\xi, I\xi\}$.

One can choose special affine coordinates such that ξ has the form[41]

$$\xi = q^a \frac{\partial}{\partial q^a} = x^i \frac{\partial}{\partial x^i} + y_i \frac{\partial}{\partial y_i}. \tag{4.142}$$

Moreover, it can be shown that the existence of a vector field ξ which satisfies (4.141) is equivalent to the condition that the prepotential is homogenous of degree 2:

$$F(\lambda z^i) = \lambda^2 F(z^i),$$

where $z^i \to \lambda z^i$ is the action of $\mathbb{C}^*$ on the (conical) special coordinates $\{z^i\}$ associated with the (conical) special affine coordinates $\{x^i, y_i\}$. In special coordinates, ξ takes the form[42]

$$\xi = z^i \frac{\partial}{\partial z^i}.$$

The quotient $\overline{M} = M/\mathbb{C}^*$ is a Kähler manifold which inherits its metric from M. Manifolds which are obtained from conical affine special Kähler manifolds in this way are called *projective special Kähler manifolds*. These are the scalar manifolds of vector multiplets in $N = 2$ Poincaré supergravity. The corresponding conical affine special Kähler manifold is the target space of a gauge equivalent theory of superconformal vector multiplets. As we have seen from the physical perspective one can go back and forth between M and $\overline{M}$. Geometrically, M can be regarded as a $\mathbb{C}^*$-bundle over $\overline{M}$. In turn M itself is embedded into $V = T^*\mathbb{C}^{n+1}$, where $n+1$ is the complex dimensions of M. In the main text the D-gauge is fixed by imposing

$$-i(X^I \overline{F}_I - F_I \overline{X}^I) = 1$$

on the symplectic vector (X^I, F_I). Geometrically, this means that (X^I, F_I) is required to be a unitary section of the so-called universal line bundle over $\overline{M}$. Instead of using unitary sections, one can also reformulate the theory in terms of holomorphic sections of the universal bundle. This is frequently done when working with general (in contrast to special) coordinates, see [20]. For a more detailed account on the universal bundle, see [25].

[41] These are called conical special affine coordinates, but we will usually drop 'conical'.

[42] Note that this is equivalent to (4.142) if and only if the prepotential is homogenous of degree 2.

In the main text we gave explicit formulae for various quantities defined on projective special Kähler manifolds in the notation used in the supergravity literature. In particular, (4.61) and (4.62) are the expressions for the metric and Kähler potential in terms of special coordinates on $\overline{M}$. There we also discussed the relation between the signatures of the special Kähler metrics on M and $\overline{M}$ The 'horizontal' metric g_{IJ} (4.60) vanishes along the vertical directions (the directions orthogonal to $\overline{M}$ under the natural projection with respect to the special Kähler metric of M), but it is non-degenerate along the horizontal directions (the directions which project orthogonally onto $\overline{M}$). If the metric of M is complex Lorentzian $(\mp, \mp, \pm, \ldots, \pm)$, then the metric defined on $\overline{M}$ by projection is even positive definite. This defines a projective special Kähler metric on $\overline{M}$, for which an explicit formula in terms of special coordinates is given by (4.61), (4.62).

B Modular Forms

Here we summarize some standard results on modular forms. See [97] for a more detailed account. As we mentioned in the main text, the theory of Siegel modular forms is a generalisation of the theory of 'standard' modular forms reviewed here. Some facts are stated in the main text. For a detailed account on Siegel modular forms see for example [81].

The action of the modular group $PSL(2,\mathbb{Z}) \simeq SL(2,\mathbb{Z})/\mathbb{Z}_2$ on the upper half plane $\mathcal{H} = \{\tau \in \mathbb{C} | \text{Im}\tau > 0\}$ is

$$\tau \to \tau' = \frac{a\tau + b}{c\tau + d}, \quad \text{where} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}).$$

The modular group is generated by the two transformations[43]

$$T : \tau \to \tau + 1, \quad S : \tau \to -\frac{1}{\tau}.$$

The interior of the standard fundamental domain for this group action is

$$\mathcal{F} = \left\{ \tau \in \mathcal{H} | -\frac{1}{2} < \text{Re}\tau < \frac{1}{2}, |\tau| > 1 \right\}.$$

The full domain is obtained by adding a point at infinity, denoted $i\infty$, and identifying points on the boundary which are related by the group action. The point $i\infty$ is called the cusp point.

[43] The notation T and S is standard in the mathematical literature, and does not refer to T- or S-duality. However, there are several examples where either T-duality or S-duality acts by $PSL(2,\mathbb{Z})$ transformations on complex fields.

A function on $\mathcal{H}$ is said to transform with (modular) weight k:

$$\phi(\tau') = (c\tau + d)^k \phi(\tau).$$

A function on $\mathcal{H}$ is called a modular function, a modular form, a cusp form, if it is meromorphic, holomorphic, vanishing at the cusp, respectively.

The ring of modular forms is generated by the Eisenstein series G_4, G_6, which have weights 4 and 6 respectively. The (normalized[44]) Eisenstein series of weight k is defined by

$$G_k(\tau) = \frac{(k-1)!}{2(2\pi i)^k} \sum_{m,n}{}' \frac{1}{(m\tau + n)^k},$$

where the sum is over all pairs of integers (m, n) except $(0, 0)$. The sum converges absolutely for $k > 2$ and vanishes identically for odd k. For $k = 2$ the sum is only conditionally convergent, and one can define two functions with interesting properties. The holomorphic second Eisenstein series is defined by

$$G_k(\tau) = \frac{(k-1)!}{(2\pi i)^k} \sum_{n=1}^{\infty} \frac{1}{n^k} + \sum_{m=1}^{\infty} \left(\frac{(k-1)!}{(2\pi i)^k} \sum_{n \in \mathbb{Z}} \frac{1}{(m\tau + n)^k} \right),$$

with $k = 2$ (the same organisation of the sum can be used for $k > 2$). The non-holomorphic second Eisenstein series is defined by

$$\overline{G}_2(\tau, \overline{\tau}) = -\frac{1}{8\pi^2} \lim_{\epsilon \to 0+} \left(\sum_{m,n}{}' \frac{1}{(m\tau + n)|m\tau + n|^{\epsilon}} \right).$$

Both are related by

$$\overline{G}_2(\tau, \overline{\tau}) = G_2(\tau) + \frac{1}{8\pi\tau_2}.$$

While the non-holomorphic $\overline{G}_2(\tau, \overline{\tau})$ transforms with weight two, the holomorphic function $G_2(\tau)$ transforms with an extra term:

$$G_2\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^2 G_2(\tau) - \frac{c(c\tau + d)}{4\pi i}.$$

There is no modular form of weight two: $G_2(\tau)$ is holomorphic but does not strictly transform with weight two, while $G_2(\tau, \overline{\tau})$ transforms with weight two but is not holomorphic.

There is a unique cusp form Δ_{12} of weight 12, which can be expressed in terms of the Dedekind η-function by

[44] With these prefactors, the coefficients of an expansion in $q = e^{2\pi i \tau}$ are rational numbers. In fact, they are related to the Bernoulli numbers.

$$\Delta(\tau) = \eta^{24}(\tau),$$

where

$$\Delta(\tau) = \eta^{24}(\tau) = q \prod_{l=1}^{\infty} (1-q^l)^{-24},$$
$$\eta(\tau) = q^{\frac{1}{24}} \prod_{l=1}^{\infty} (1-q^l)^{-1}. \tag{4.143}$$

The Dedekind η-function is a modular form of weight $\frac{1}{2}$ with multiplier system, i.e. a 'modular form up to phase':

$$\eta(\tau+1) = \mathrm{e}^{\frac{2\pi i}{24}} \eta(\tau), \quad \eta\left(-\frac{1}{\tau}\right) = \sqrt{-i\tau}\,\eta(\tau).$$

Modular forms are periodic under $\tau \to \tau + 1$ and therefore they have a Fourier expansion in $\tau_1 = \mathrm{Re}\tau$. It is convenient to introduce the variable

$$q = \mathrm{e}^{2\mathrm{i}\pi\tau}.$$

In the main text we avoid using the variable q, because it might be confused with the electric charge vector $q \in \Gamma$. The transformation $\tau \to q$ maps the semi-infinite strip $\{\tau \in \mathbb{C} | \, |\tau_1| \leq 1, \tau_2 > 0\} \subset \mathcal{H}$ onto the unit disc $\{q \in \mathbb{C} | \, |q| < 1\} \subset \mathbb{C}$. In particular, the cusp $\tau = i\infty$ is mapped to the origin $q = 0$. The Fourier expansion in τ_1 maps to a Laurent expansion in q, known as the q-expansion.

The q-expansion of the cusp form $\Delta_{12} = \eta^{24}$ is

$$\eta^{24}(q) = q - 24q^2 + 252q^3 + \cdots .$$

In the main text we express modular forms in terms of variables which live in right half plane rather than in the upper half plane, e.g., the heterotic dilaton S, where $\tau = iS$. For notational simplicity we then write $\eta(S)$ instead of $\eta(iS)$.

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Chapter 5
Complexity at the Fundamental Level

Antonino Zichichi

PURPOSE

Purpose of this lecture is to show that Complexity in the real world exists, no matter the Mass–Energy and Space–Time scales considered, including the fundamental one.

To prove this it is necessary:

1. To identify the experimentally observable effects which call for the existence of Complexity;
2. To analyse how we have discovered the most advanced frontier of Science: the SM&B (Standard Model and Beyond);
3. To construct the platonic version of this frontier: i.e., what would be the ideal platonic Simplicity.

It is often stated that Science is able to make predictions and that these predictions are the source of the greatest achievements in human knowledge.

As we prove that Complexity exists at the fundamental level of scientific knowledge, i.e., physics, it is necessary to establish the correct relation between Complexity and Predictions.

5.1 The Basic Points on Complexity and Predictions

What are the experimental evidences for *Complexity* to *exist*, and for *Predictions* to *exist*?

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5.1.1 Complexity

The experimental evidences for the *existence* of *Complexity are two:*

1. The Anderson-Feynman-Beethoven-type phenomena (AFB), i.e., phenomena whose laws and regularities ignore the existence of the Fundamental Laws of Nature from which they originate;
2. The Sarajevo-type effects, i.e., Unexpected Events of quasi irrelevant magnitude which produce Enormous Consequences (UEEC).

5.1.2 Predictions

The experimental evidences for the *existence* of *Predictions* are the very many results of scientific reproducible experiments.

Quantum Electro-Dynamics (QED) is the best example. The anomalous magnetic moments, in symbols ($g - 2$), of the electron (e) and of the muon (μ):

$$(g - 2)_{e,\mu}$$

are theoretically computed at an extraordinary level of precision (few parts in ten billion parts for the electron) and are experimentally verified to be correct.

Can

$$(g - 2)_{e,\mu}$$

be theoretically predicted before the discovery of the Maxwell equations and the existence of QED? The answer is obviously no.

5.1.3 Complexity and Predictions

Predictions at the *fundamental level of scientific knowledge* depend on *UEEC events.*

For example: it is the discovery of the laws governing electric, magnetic, and optical phenomena (all totally unpredicted) which produced the mathematical structure called QED.

The mathematical structure was not discovered before the innumerable series of *UEEC* events was found in electricity, magnetism, and optics. This series of UEEC events allowed Maxwell to express 200 years of experimental discoveries in a set of four equations.

The mathematical formalism comes *after* a totally unexpected discovery: an *UEEC event* which no one was able *to predict.*

To our knowledge rigorous predictions exist only in Science. These predictions are based on the mathematical description of the UEEC events. This description can either be the result of new mathematics (example the Dirac δ-function) or the use of existing mathematical formalism (example: the Einstein use Ricci tensor calculus).

The UEEC event at the origin of the Dirac equation is the fact that the electron was not a "scalar" particle but a spin $^1/_2$ object.

The UEEC event at the origin of Einstein mathematical formulation of the gravitational forces are the discoveries of

$$\text{Galilei} \ \ (F = mg)$$

and of

$$\text{Newton} \left(F = G \frac{m_1 \cdot m_2}{R_{12}^2} \right)$$

These are just two examples of the fact that the greatest steps in the *progress of Science* come from totally unpredicted discoveries.

This is the reason why we need to perform experiments, as Galileo Galilei realized 400 years ago. Even when we have a mathematical formalism coming from a series of UEEC events, if this formalism opens a new frontier, as it is the case for the Superworld, the experimental proof is needed to verify the validity of the new theoretical frontier.

Today we have a reasonable mathematical formalism to describe the *Superworld*, but in order to know if the Superworld exists we need the experimentally reproducible proof for its existence.

5.2 AFB Phenomena from Beethoven to the Superworld

Let me now mention a few other examples of AFB phenomena in Science.

5.2.1 Beethoven and the Laws of Acoustics

Beethoven could compose superb masterpieces of music without any knowledge of the laws governing acoustic phenomena. But these masterpieces could not exist if the laws of acoustics were not there.

5.2.2 The Living Cell and QED

To study the mechanisms governing a living cell, we do not need to know the laws of electromagnetic phenomena whose advanced formulation is QED. All mechanisms needed for life are examples of purely electromagnetic processes. If QED was not there, Life could not exist.

5.2.3 Nuclear Physics and the UEEC Events

This year is the centenary of the birth of Hideki Yukawa, the father of theoretical nuclear physics. In 1935 the existence of a particle, with mass intermediate (this is the origin of "mesotron" now "meson") between the light electron, m_e, and the heavy nucleon (proton or neutron), m_N, was proposed by Yukawa [1].

This intermediate mass value was deduced by Yukawa from the range of the nuclear forces. Contrary to the general wisdom of the time, Yukawa was convinced that the particles known (electrons, protons, neutrons, and photons), could not explain how protons and neutrons are bound into the extremely small dimensions of a nucleus.

In order to make this "prediction," Yukawa needed the Heisenberg uncertainty principle: a totally unexpected theoretical discovery. The origin of it was the totally unexpected discovery of the dual nature of the electron (wave and particle) and of the photon (wave and particle). Heisenberg himself tried to explain the binding forces between the proton and the neutron, via the exchange of electrons, in order not to postulate the existence of a new particle.

The very light electron, m_e, could not stay in the very small dimension of the nucleus. The author of the uncertainty principle and father, with Dirac and Pauli, of Quantum Mechanics, did not realise this contradiction. The need for a new particle was the reason.

What no-one was able to predict is the "gold-mine" hidden in the production, the decay and the intrinsic structure of this "particle." This "gold-mine" is still being explored nowadays and its present frontier is the Quark-Gluon-Coloured-World (QGCW) [2]. I have recently described [3] the unexpected conceptual developments coming from the study of the production, the decay and the intrinsic structure of the Yukawa particle.

Let me just quote the most relevant ones: chirality-invariance, spontaneous symmetry breaking (SSB), symmetry breaking of fundamental invariance laws (P, C, T), anomalies, and "anomaly-free condition," existence of a third family of fundamental fermions, gauge principle for non-Abelian forces, instantons and existence of a pseudoscalar particle made of the quanta of a new fundamental force of Nature acting between the constituents of the Yukawa particle.

It is considered standard wisdom the fact that nuclear physics is based on perfectly sound theoretical predictions. People forget the impressive series of UEEC events discovered in the Yukawa gold mine.

Let me quote just three of them:

1. The first experimental evidence for a cosmic ray particle believed to be the Yukawa meson was a lepton: the muon.
2. The decay-chain: $\pi \rightarrow \mu \rightarrow$ e was found to break the symmetry laws of Parity and Charge Conjugation.
3. The intrinsic structure of the Yukawa particle was found to be governed by a new fundamental force of Nature, Quantum ChromoDynamics: QCD.

5.2.4 *Nuclear Physics and QCD*

Proton and neutron interactions appear as if a fundamental force of nature is at work: the nuclear force, with its rules and its regularities. These interactions ignore that protons and neutrons are made with quarks and gluons.

Nuclear physics does not appear to care about the existence of QCD, although all phenomena occurring in nuclear physics have their roots in the interactions of quarks and gluons. In other words, protons and neutrons behave like Beethoven: they interact and build up nuclear physics without "knowing" the laws governing QCD.

The most recent example of Anderson-Feynman-Beethoven-type phenomenon: *the World could not care less about the existence of the Superworld.*

5.3 UEEC Events, from Galilei up to Present Days

In Fig. 5.1 there is a sequence of *UEEC* events from Galilei to Fermi–Dirac and the "strange particles." In Figs. 5.2–5.4, from Fermi–Dirac to the construction of the Standard Model and in Fig. 5.5 a synthesis of the UEEC events in what we now call the Standard Model and Beyond (SM&B).

'UEEC'
TOTALLY UNEXPECTED DISCOVERIES
FROM GALILEI TO FERMI-DIRAC AND THE 'STRANGE' PARTICLES

I	Galileo Galilei discovery of $F = mg$.
II	Newton discovery of $F = G\,\dfrac{m_1 \cdot m_2}{R_{12}^2}$
III	Maxwell discovers the unification of electricity, magnetism and optical phenomena, which allows him to conclude that light is a vibration of the EM field.
IV	Planck discovery of $h \neq 0$.
V	Lorentz discovers that space and time cannot be both real.
VI	Einstein discovers the existence of time-like and space-like worlds. Only in the time-like world, simultaneity does not change with changing observer.
VII	Rutherford discovers the nucleus.
VIII	Hess discovers the cosmic rays.
IX	Dirac discovers his equation, which opens new horizons, including the existence of the antiworld.
X	Fermi discovers the weak forces.
XI	Fermi and Dirac discover the Fermi–Dirac statistics.
XII	The 'strange particles' are discovered in the Blackett Lab.

Fig. 5.1 "UEEC" totally unexpected discoveries. From Galilei to Fermi–Dirac and the "Strange" Particles

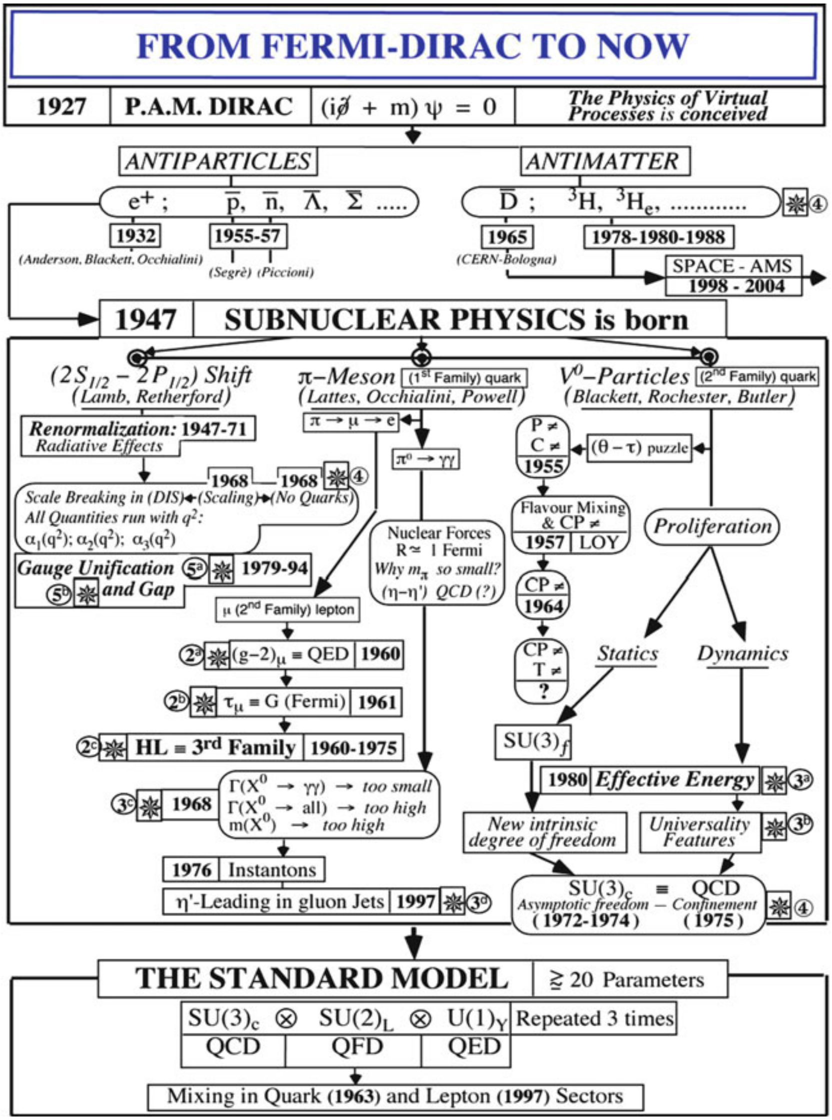


Fig. 5.2 From Fermi–Dirac to now

A few cases (seven) where I have been directly involved are summarised in Fig. 5.6.

Each *UEEC* event is coupled with a *despite*, to emphasize the reason why the event is unexpected.

The SM&B is the greatest synthesis of all times in the study of the fundamental phenomena governing the Universe in all its structures.

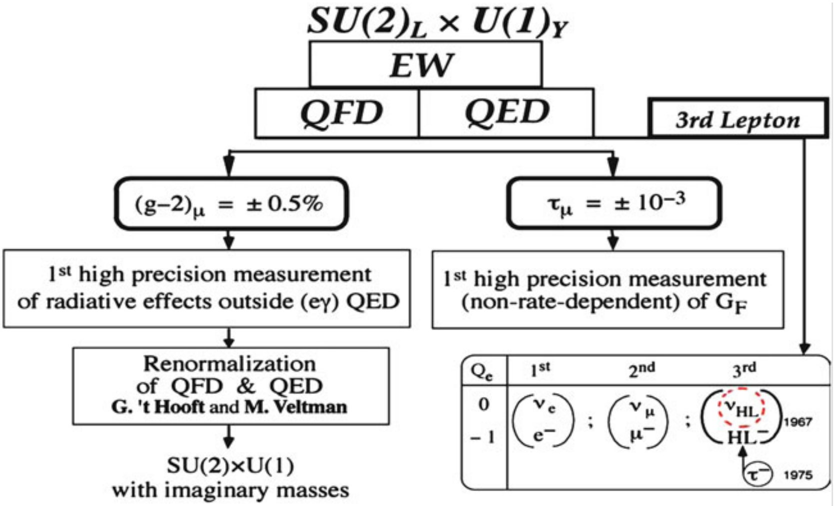


Fig. 5.3 Details from Fig. 5.2, concerning $SU(2)_L$ and $U(1)_Y$.

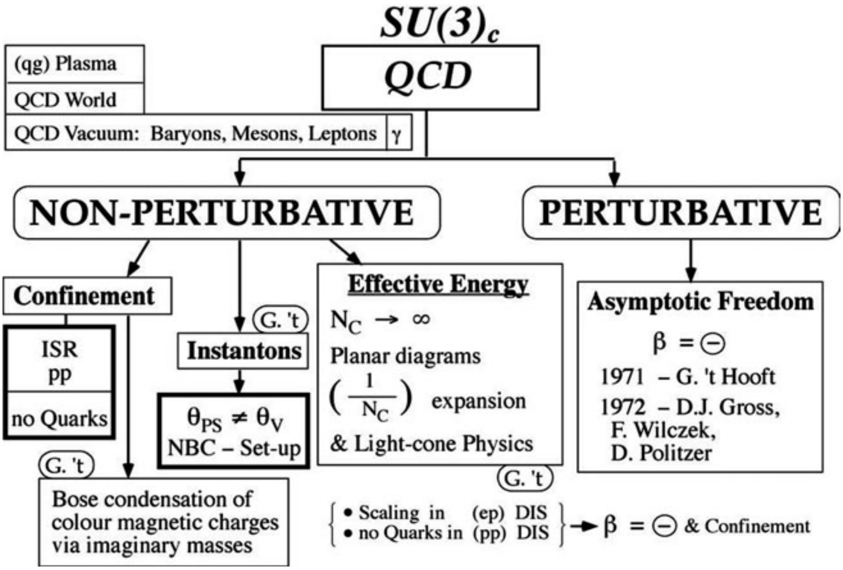


Fig. 5.4 Details from Fig. 5.2, concerning $SU(3)_c$.

The basic achievements of the SM&B have been obtained via UEEC events; moreover, the SM&B could not care less about the existence of Platonic Simplicity. An example is shown in Fig. 5.7 where the straight line (small dots) would be the Platonic simple solution toward the Unification of all Fundamental Forces. But the effective unification is expected to be along the sequence of points (the big ones) calculated using the Renormalization Group Equations (RGEs) [4].

SM&B

THE STANDARD MODEL AND BEYOND

① RGEs ($\alpha_i (i \equiv 1, 2, 3)$; m_j ($j \equiv q, l, G, H$)) : $f(k^2)$.
- GUT ($\alpha_{GUT} \cong 1/24$) & GAP ($10^{16} - 10^{18}$) GeV.
- SUSY (to stabilize $m_F/m_P \cong 10^{-17}$).
- RQST (to quantize Gravity).

② Gauge Principle (hidden and expanded dimensions).
— How a Fundamental Force is generated: SU(3); SU(2); U(1) and Gravity.

③ The Physics of Imaginary Masses: SSB.
— The Imaginary Mass in SU(2)×U(1) produces masses ($m_{W^\pm}$; m_{Z^0}; m_q; m_l), including $m_\gamma = 0$.
— The Imaginary Mass in SU(5) ⇒ SU(3)×SU(2)×U(1) or in any higher (not containing U(1) Symmetry Group ⇒ SU(3)×SU(2)×U(1) produces Monopoles.
— The Imaginary Mass in $SU(3)_c$ generates Confinement.

④ Flavour Mixings & CP ≠ , T ≠ .
— No need for it but it is there.

⑤ Anomalies & Instantons.
— Basic Features of all Non-Abelian Forces.

Note: q ≡	quark and squark;	m_F ≡	Fermi mass scale;
l ≡	lepton and slepton;	m_P ≡	Planck mass scale;
G ≡	Gauge boson and Gaugino;	k ≡	quadrimomentum;
H ≡	Higgs and Shiggs;	C ≡	Charge Conjugation;
RGEs ≡	Renormalization Group Equations;	P ≡	Parity;
GUT ≡	Grand Unified Theory;	T ≡	Time Reversal;
SUSY ≡	Supersymmetry;	≠ ≡	Breakdown of Symmetry Operators.
RQST ≡	Relativistic Quantum String Theory;		
SSB ≡	Spontaneous Symmetry Breaking.		

The five basic steps in our understanding of nature. ① The renormalization group equations (RGEs) imply that the gauge couplings (α_i) and the masses (m_j) all run with k^2. It is this running which allows GUT, suggests SUSY and produces the need for a non point-like description (RQST) of physics processes, thus opening the way to quantize gravity. ② All forces originate in the same way: the gauge principle. ③ Imaginary masses play a central role in describing nature. ④ The mass-eigenstates are mixed when the Fermi forces come in. ⑤ The Abelian force QED has lost its role of being the guide for all fundamental forces. The non-Abelian gauge forces dominate and have features which are not present in QED.

Fig. 5.5 SM&B the standard model and beyond

Platonic Simplicity for the unification of all forces (see Addendum 1) and Platonic Supersymmetry for the existence of the Superworld (see Addendum 2) are violated at every corner in the process of construction of the SM&B [5], as reported in Addendum 3. These violations are the proof that Complexity exists at the fundamental level of scientific knowledge where we have proved that AFB phenomena and UEEC events are present. The conclusion is that Complexity exists at the elementary level. In fact, starting from Platonic Simplicity, the SM&B needs a series of “ad hoc” inputs [5].

UEEC EVENTS
IN THE CONSTRUCTION OF THE
SM&B ≡ MY PERSONAL EXPERIENCE

① ***The 3rd lepton***, **HL** (now called τ) with its own neutrino, ν_{HL} (now called ν_τ),
despite the abundance of neutrinos: ν_e and ν_μ.

② ***Antimatter***
despite S-matrix and C, P, CP, T breakings.

③ ***Nucleon Time-like*** *EM structure*
despite S-matrix.

④ ***No quarks*** *in violent (pp) collisions*
despite scaling.

⑤ ***Meson mixings***
$\theta_V \neq \theta_{PS}$ **:** (51°) ≠ (10°) ≠ 0 **despite** $SU(3)_{uds}$.

⑥ ***Effective energy*****:** the QCD-light
despite QCD.

⑦ ***The running*** *of* $\alpha_1\ \alpha_2\ \alpha_3$ *versus* ***energy*** *at* **a point E_{GU}.**
(1979) (1991) despite straight line convergence.

EGM

Fig. 5.6 UEEC events in the construction of the SM&B ≡ my personal experience

5.4 Seven Definitions of Complexity

People speak of "Complexity" as a source of new insights in physics, biology, geology, cosmology, social sciences and in all intellectual activities which look at the world through the lens of a standard analysis in terms of either Simplicity or Complexity. But "Complexity" is ill-defined, as shown by the existence of at least seven definitions of Complexity.

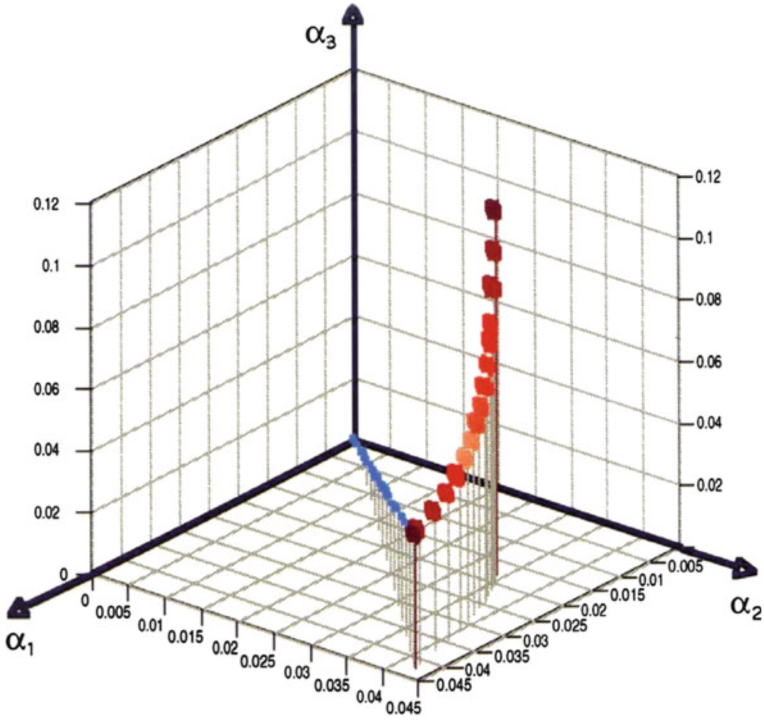
DEFINITION NUMBER 1

Complexity is a property of systems that are somewhere in between a completely random and a completely regular state, often described by a highly nonlinear set of equations but sometimes not describable by equations at all.

DEFINITION NUMBER 2

Bad ones:

1. Chaos.
2. The need for lengthy calculations.
3. The need for many distinct variables.



The points have a sequence of 100 GeV in energy. The last point where the 'ideal' platonic straight line intercepts the theoretical prediction is at the energy of the Grand Unification. This corresponds to $E_{GU} = 10^{16.2}$ GeV. Other detailed information on the theoretical inputs: the number of fermionic families, N_F, is 3; the number of Higgs particles, N_H, is 2. The input values of the gauge couplings at the Z^0-mass is $\alpha_3 (M_Z) = 0.118 \pm 0.008$; the other input is the ratio of weak and electromagnetic couplings also measured at the Z^0-mass value: $\sin^2 \theta_W(M_Z) = 0.2334 \pm 0.0008$.

Fig. 5.7 'Ideal' platonic straight line intercepting the theoretical prediction at the energy of the Grand Unification

Better ones:

4. Unexpected difficulty when attempting to describe something in a precisely formulated theory.
5. What is left over after all systematic approaches failed.
6. But it could also be that: Complexity is an excuse for sloppy thinking.

DEFINITION NUMBER 3

The Complexity of a theory (problem) is the minimum amount of computer time and storage required to simulate (solve) it to a specified level of precision.

DEFINITION NUMBER 4

If we admit that biological or linguistic evolution, or financial dynamics are complex phenomena, then their typical dynamics is somehow between strong chaos (i.e., positive Lyapunov exponents) and simple orbits (i.e., negative Lyapunov exponents). In other words, Complexity (or at least some form of it) is deeply related to the edge of chaos (i.e., vanishing maximal Lyapunov exponent). As the edge of chaos appears to be related paradigmatically to an entropy index "q" different from unity, there must be some deep connection between Complexity and generalized entropies such as "S_q."

DEFINITION NUMBER 5

From the mathematical point of view:

- A problem can be polynomial, which means that it is not too hard to predict surprises.
- A problem can be NP or NP-complete, which represent different degrees of difficulty in predicting surprises.
  - Surprises means: UEEC event.
  - That degree of difficulty can be associated with the level of Complexity.

DEFINITION NUMBER 6

A system is "complex" when it is no longer useful to describe it in terms of its fundamental constituents.

DEFINITION NUMBER 7

The simplest definition of Complexity: **"***Complexity is the opposite of Simplicity.*" This is why we have studied the platonic Standard Model (Addendum 1) and its extension to the platonic Superworld (Addendum 2).

These seven definitions of Complexity must be compared with the whole of our knowledge (see later) in order to focus our attention on the key features needed to study our real world.

5.5 Complexity Exists at all Scales

The Logic of Nature allows the existence of a large variety of structures with their regularities and laws which appear to be independent from the basic constituents and fundamental laws of Nature which govern their interactions.

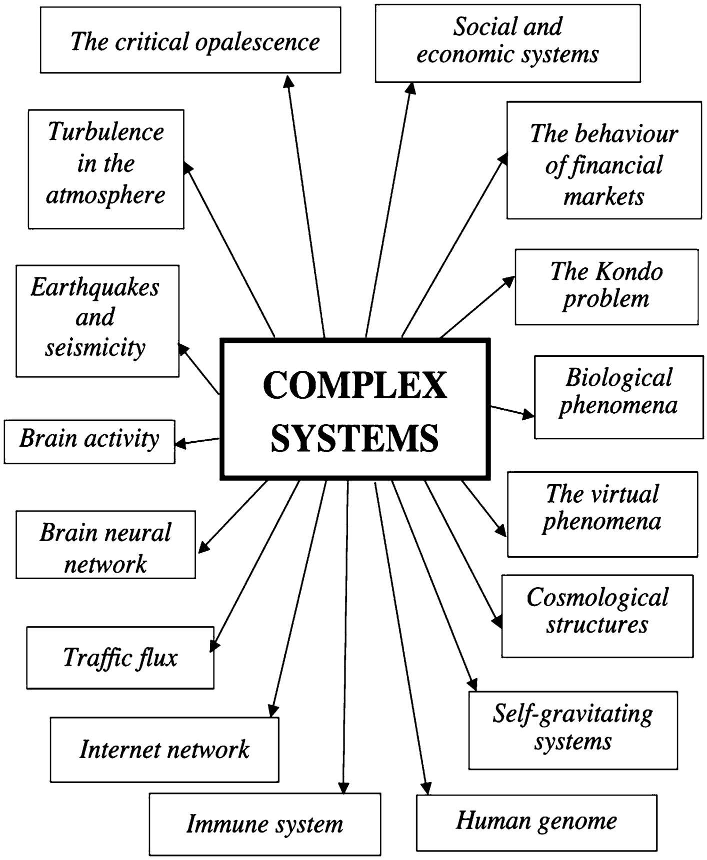


Fig. 5.8 Complex systems

But, without these laws it would be impossible to have the real world which is in front of us and of which we are part of. A series of complex systems is shown in Fig. 5.8.

As you can see, we go from traffic flux, to the internet network, to earthquakes and seismicity, to social and economic systems, to the behavior of financial markets, to the study of cosmological structures, and so on.

There is no question that nature shows structures which are considered complex on the basis of AFB and UEEC events (as shown in Fig. 5.9).

The only certainty about Complexity is the existence of the experimentally observable effects: UEEC & AFB. These effects exist at all scales, and therefore Complexity exists at all scales, as illustrated in Fig. 5.9.

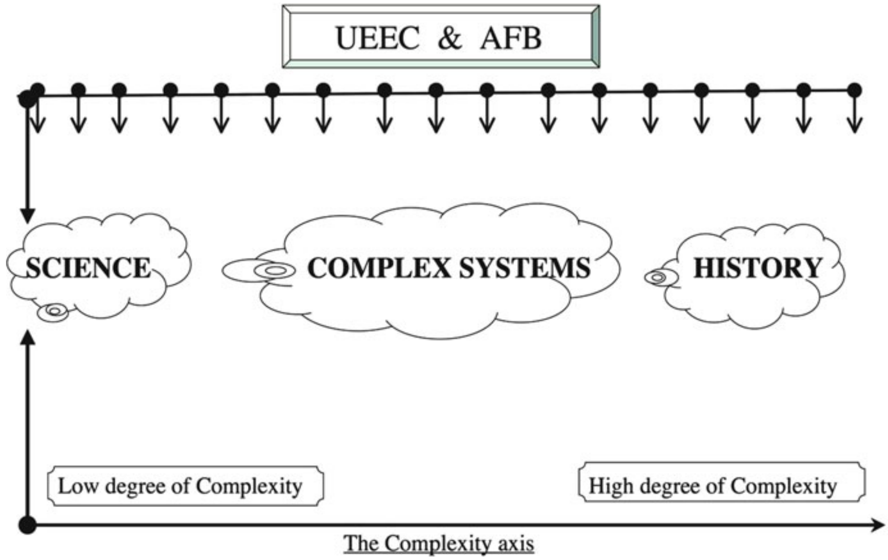


Fig. 5.9 The complexity axis

5.6 Science, from Planck to Complexity

Four centuries of Galilean research work based on Reductionism, i.e., on the identification of the simplest elements in the study of Nature, has allowed us to get the greatest achievement of Science, i.e., the so called *Standard Model* and its extension (SM&B), illustrated before in Fig. 5.5.

This extension predicts the Grand Unification Theory, the existence of the Superworld and the resolution of the quantum-gravity problem via the powerful theoretical structure of Relativistic Quantum String Theory (RQST). All these developments started 30 years ago when a great scientific novelty came; all experimental discoveries obtained with our powerful accelerators were to be considered only matters of extremely low energy.

The scale of energy on which to direct the attention to understand the Logic that rules the world, from the tiniest structures to the galactic ones, had to be shifted at a much higher level: to the mass–energy named after Planck, E_{Planck}, something like 17 powers of ten above the Fermi scale, E_{Fermi}, that already seemed to be an extremely high level of energy.

Now, after 30 years, it comes about the novelty of our time, illustrated in Fig. 5.10: Complexity exists at the fundamental level [5]. In fact, AFB and UEEC events exist at all scales, as reported in Chap. 5.

This result is corroborated from the mathematical structure (the only one) to be in a position of describing all that happens at the Planck scale: the RQST.

This mathematical structure produces innumerable minima of energy, named *Landscape.*

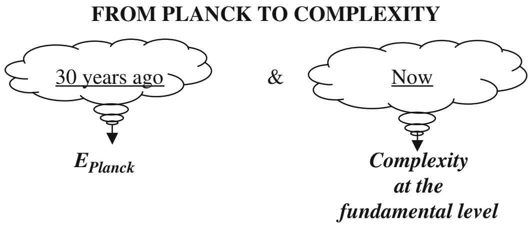


Fig. 5.10 From Planck to complexity

The theoretical discovery of the *Landscape* (Leonard Susskind) [6], has been followed by another formidable discovery in mathematical physics: the most rigorous model of RQST (Raphael Bousso and Joseph Polchinski) is *NP-complete* (Michael R. Douglas and Frederik Denef) [7].

This discovery corroborates all that we have put in evidence during the last 5 years [8–11]: *Complexity exists at the fundamental level* [5].

We do not know what will be the final outcome of String Theory.

What we know is that: "The world appears to be complex at every scale. Therefore, we must expect a continued series of surprises that we cannot easily predict."

5.7 The Two Asymptotic Limits: History and Science

The real world seems characterized by two basic features, which are one on the opposite side of the other: *Simplicity* and *Complexity*.

It is generally accepted that *Simplicity* is the outcome of *Reductionism*, while *Complexity* is the result of *Holism*.

The most celebrated example of *Simplicity* is *Science* while the most celebrated example of *Complexity* is *History*.

Talking about asymptotic limits, the general trend is to consider *History* as the asymptotic limit of *Holism* and of *Complexity*; *Science* as the asymptotic limit of *Reductionism* and of *Simplicity*, as illustrated in Fig. 5.11.

The Logic of Nature allows the existence of Science (the asymptotic limit of Simplicity) and of History (the asymptotic limit of Complexity), which share a property, common to both of them.

It is interesting to define Science and History in terms of this property, probably the only one, which they share; i.e., Evolution.

- Science is the Evolution of our Basic Understanding of the laws governing the world in its Structure ≡ *EBUS*.
- History is the Evolution of the World in its Real Life ≡ *EWRL*.

In Table 5.1 we compare these two supposedly asymptotic limits – History and Science – on the basis of "What if ?"; a condition elaborated by the specialists in what is now known as "virtual history" [12].

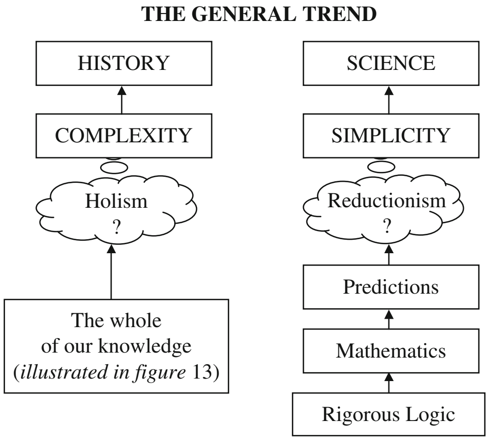


Fig. 5.11 The general trend

On the basis of "What if?" these specialists conclude that the world would not be as it is, if one, or few, or any number of "What if?" had not been as History tells us. This is not the case for Science. The world would have exactly the same laws and regularities, whether Galileo Galilei or somebody else had discovered

$$F = mg\,?,$$

and so on for all the other scientific discoveries.

It is in the consequences of "What if?" that the two asymptotic limits of Simplicity and Complexity seem to diverge, despite the fact that the sequence of "What if?" in Science belongs to the "totally unexpected events" (UEEC) exactly like the others listed in the column of History.

5.8 Conclusions

We have proved that AFB and UEEC – which are at the origin of Complexity, with its consequences permeating all our existence, from molecular biology to life in all its innumerable forms up to our own, including History – do exist at the fundamental level [8–11] and [5].

It turns out that Complexity in the real world exists, no matter the mass–energy and space–time scales considered.

Therefore, the only possible prediction is that:

- *Totally Unexpected Effects* should *show up*.
- ***Effects***, which are impossible to be predicted on the basis of *present knowledge.*

Table 5.1 "What if?"

	In history ≡ EWRL		In science ≡ EBUS
I	What if Julius Caesar had been assassinated many years before?	*I*	What if Galileo Galilei had not discovered that $F = mg$?
II	What if Napoleon had not been born?	*II*	What if Newton had not discovered that $F = G\frac{m_1 \cdot m_2}{R_{12}^2}$
III	What if America had been discovered few centuries later?	*III*	What if Maxwell had not discovered the unification of electricity, magnetism and optical phenomena, which allowed him to conclude that light is a vibration of the EM field?
IV	What if Louis XVI had been able to win against the "Storming of the Bastille"?	*IV*	What if Planck had not discovered that $h \neq 0$
V	What if the 1908 Tunguska Comet had fallen somewhere in Europe instead of Tunguska in Siberia?	*V*	What if Lorentz had not discovered that space and time cannot be both real?
VI	What if the killer of the Austrian Archduke Francisco Ferdinand had been arrested the day before the Sarajevo event?	*VI*	What if Einstein had not discovered the existence of time-like and space-like real worlds? Only in the time-like world, simultaneity does not change, with changing observer.
VII	What if Lenin had been killed during his travelling through Germany?	*VII*	What if Rutherford had not discovered the nucleus?
VIII	What if Hitler had not been appointed Chancellor by the President of the Republic of Weimar Paul von Hindenburg?	*VIII*	What if Hess had not discovered the cosmic rays?
IX	What if the first nuclear weapon had been built either by Japan before Pearl Arbour (1941) or by Hitler in 1942 or by Stalin in 1943?	*IX*	What if Dirac had not discovered his equation, which opens new horizons, including the existence of the antiworld?
X	What if Nazi Germany had defeated the Soviet Union?	X	What if Fermi had not discovered the weak forces?
XI	What if Karol Wojtyla had not been elected Pope, thus becoming John Paul II?	*XI*	What if Fermi and Dirac had not discovered the Fermi–Dirac statistics?
XII	What if the USSR had not collapsed?	XII	What if the "strange particles" had not been discovered in the Blackett Lab?

We should be prepared with *powerful experimental instruments, technologically at the frontier of our knowledge*, to discover Totally Unexpected Events in all laboratories, the world over (including CERN in Europe, Gran Sasso in Italy, and other facilities in Japan, USA, China, and Russia). All the pieces of the Yukawa gold mine could not have been discovered if the experimental technology was not at the frontier of our knowledge.

Example: the cloud-chambers (Anderson, Neddermeyer), the photographic emulsions (Lattes, Occhialini, Powell), the high power magnetic fields (Conversi, Pancini, Piccioni), and the powerful particle accelerators and associated detectors for the discovery – the world over – of the intrinsic structure of the Yukawa particle (quarks and gluons). This means that we must be prepared with the most advanced technology for the discovery of totally unexpected events like the ones found in the Yukawa gold mine.

The mathematical descriptions, and therefore the predictions come after an UEEC event, never before.

Recall:

- The *discoveries in Electricity, Magnetism,* and *Optics* (UEEC).
- *Radioactivity* (UEEC).
- The *Cosmic Rays* (UEEC).
- The *Weak Forces* (UEEC).
- The *Nuclear Physics* (UEEC).
- The *Strange Particles* (UEEC).
- The *three Columns* (UEEC).
- The *origin of the Fundamental Forces* (UEEC).

The present status of Science is reported in Fig. 5.12.

It could be that Science will be mathematically proved to be "NP-complete." This is the big question for the immediate future [13].

It is, therefore, instructive to see how Science fits in the whole of our knowledge as reported in Fig. 5.13.

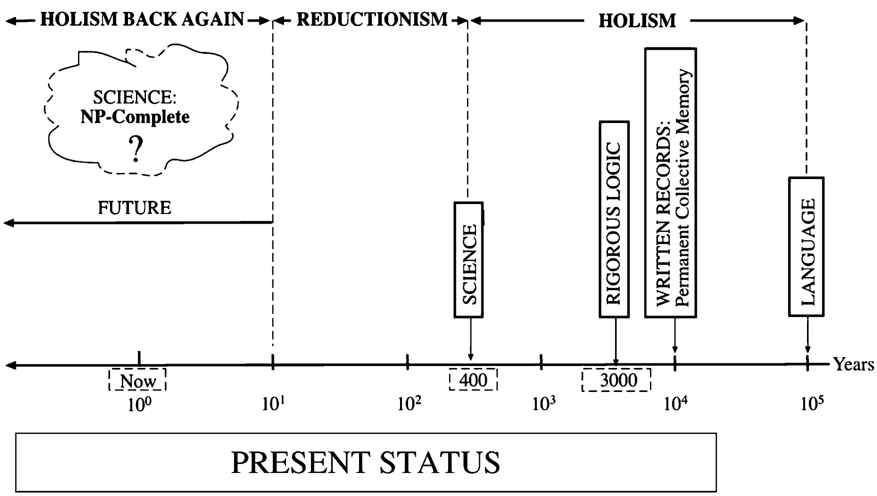


Fig. 5.12 The present status of Science

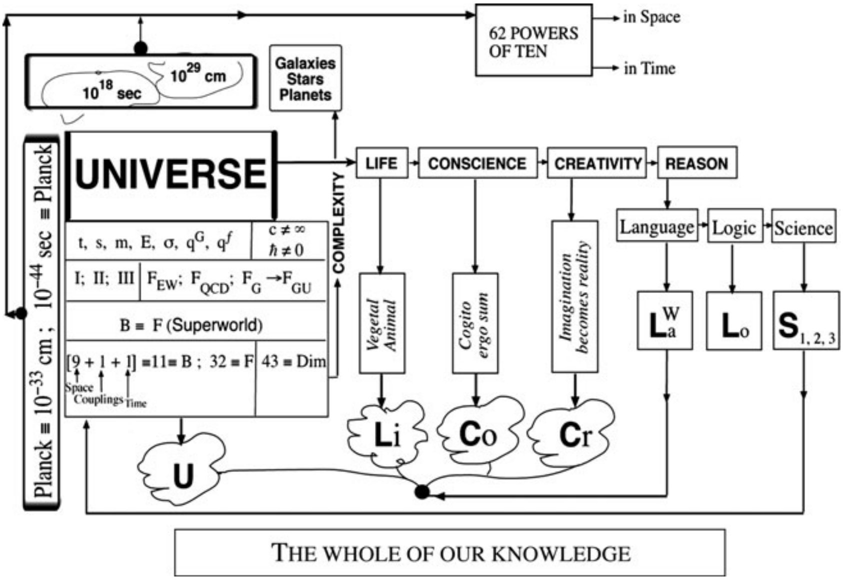


Fig. 5.13 Science fits in the whole of our knowledge

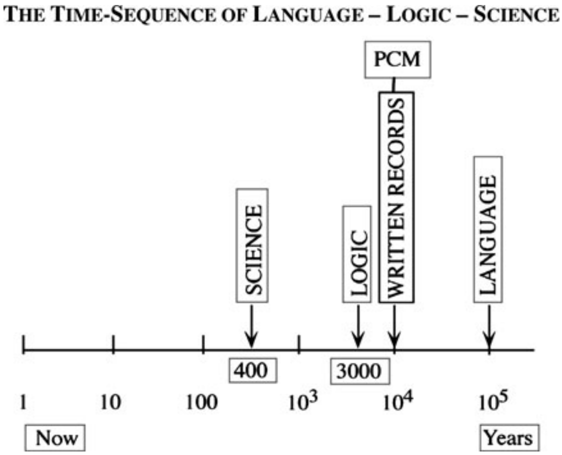


Fig. 5.14 The time-sequence of Language – Logic – Science

Let me point out that Science is the consequence of us being the only form of leaving matter endowed with Reason, from where the sequence of Language–Logic–Science has been originated [14]. The time-sequence of Language–Logic–Science is shown in Fig. 5.14.

The experimental evidence is that UEEC events dominate our life as the evolution of the world in its real life (EWRL ≡ History) and the evolution of our basic understanding of the laws governing the world (EBUS ≡ Science) do show.

The present status of physics is confronted with Ten Challenges, see Addendum 4). The next UEEC event must be outside these Ten Challenges. We should be aware of the fact that it would be great if, for the first time in the 400 years of Galilean Science, the sequence of UEEC events could enjoy a formidable stop.

The final question is: why the greatest achievements of Science have always been originated by totally unexpected events?

Addendum 1: The Platonic Grand Unification

The simplest way to have a Platonic Grand Unification is to have one and only one basic fundamental particle, B. This particle must obey the very simple symmetry law which puts fermions and bosons on the same basis. This basic fundamental particle B can therefore exist either as being a boson B_B or as being a fermion B_F. To this Symmetry Law we add the *Gauge Principle* and the SSB, which represent the conceptual structure of the Standard Model.

The Gauge Principle corresponds to a special property of the energy density. The Lagrangian has to contain quantities which obey the following invariance property: in ordinary four-dimensional space–time and in intrinsic spaces with 1, 2, and 3 complex dimensions we can perform changes in the Lagrangian. These changes do not affect the energy density, provided that these changes follow the group properties of Poincare (for the ordinary four dimensional Space–Time) and U(1), SU(2), SU(3) for the intrinsic spaces, with 1, 2, 3 complex dimensions. The Lagrangian must contain scalar fields with imaginary masses, in order to produce the SSB mechanism.

A synthesis of the Platonic Grand Unification and the deviations needed is reported in Fig. 5.15.

Let us consider first B_B. The fundamental forces exist because a basic fundamental boson B_B exists. Figure 5.16 illustrates the simple sequence which generates all known forces of nature.

At the bottom of Fig. 5.16 there is the QFD force, illustrated in Fig. 5.17. The "platonic" Simplicity suffers a further deviation.

In fact, we need to introduce many complications. The quarks and the leptons are "mixed."

This mixing is indicated by the index m, while the indices "u" and "d" refer to the two types of flavors (up-type) and (down-type) which are present in each of the three families: 1, 2, 3. There is a further complication.

The two mixings for the "up" and the "down" flavors must be different. In the case of the quark, this mixing is experimentally measured. In the case of the leptons, the experimental results are with nearly half a century of delay, compared with the quark case.

Mixing and violation of symmetry laws (for charge conjugation, C, parity, P, and the product of the two, CP) are well-established in the quark case. In the leptonic sector, only future experiments will tell us if the same symmetry laws are violated.

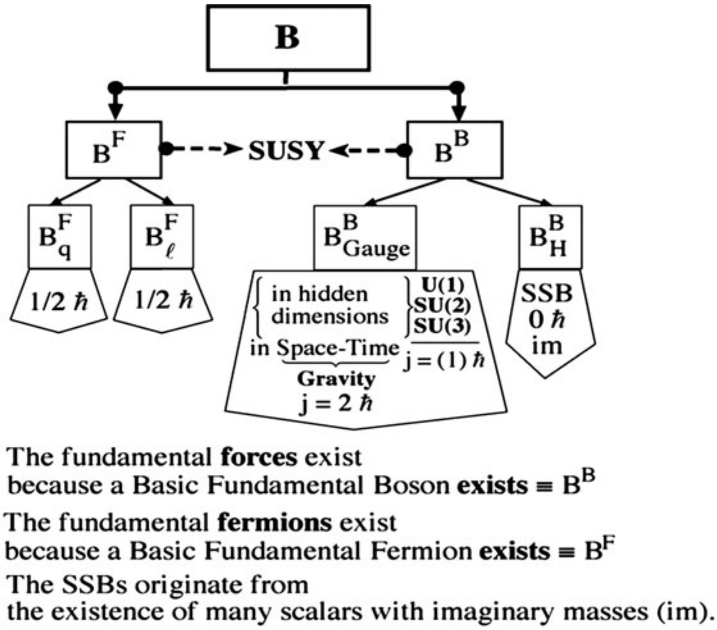


Fig. 5.15 Platonic Grand Unification and the needed deviations

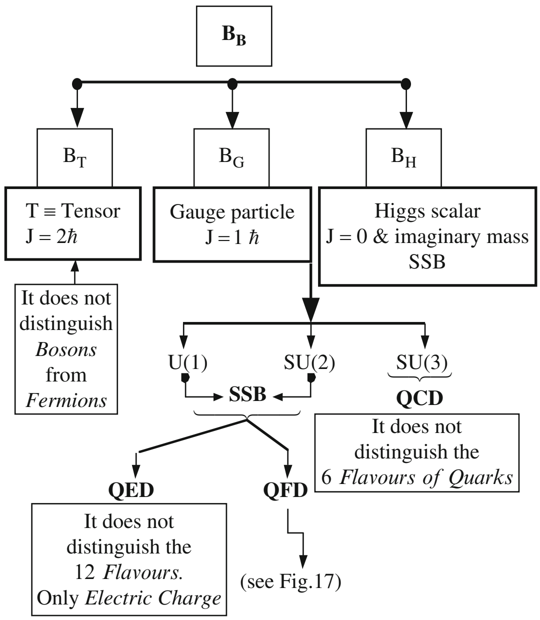


Fig. 5.16 The simple sequence generating all known forces of nature

QFD	
$(q^u_m)_{1,2,3}$ $(q^d_m)_{1,2,3}$	$(\ell^u_m)_{1,2,3}$ $(\ell^d_m)_{1,2,3}$
Experimentally proved	*Experimentally proved*
mixing & C ≠ ; P ≠ ; CP≠	?

Fig. 5.17 The QFD force

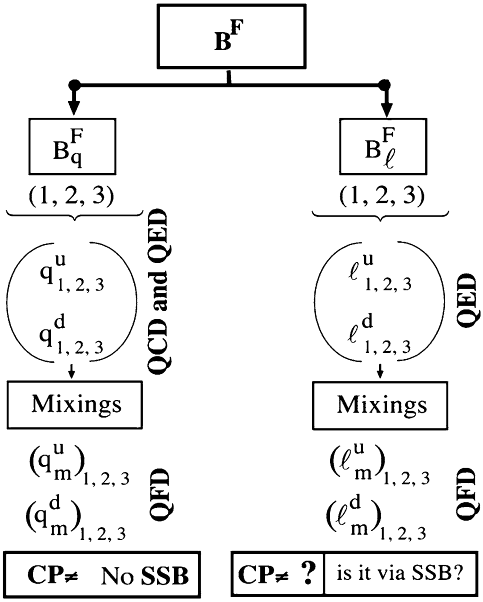


Fig. 5.18 Deviations needed from the Platonic Simplicity

There is no known reason why all these details – mixing of states and symmetry law violations – are needed. They have been experimentally discovered and show how many deviations from the simple "platonic" structure are needed. So far we have developed the sequence of Platonic Deviations from *Simplicity,* starting from the basic fundamental boson B_B.

We now consider the basic fundamental fermion B_F and show in Fig. 5.18 the deviations needed from the Platonic Simplicity. The BF must have "quark" and "lepton" flavors, repeated three times, with two flavors each time. There are three *families* of quarks and leptons. The total number of flavors is *12: 6 for quarks, 6 for leptons.*

We postulate the same number of quarks and leptons for Simplicity. It will then be found that this is a necessary and sufficient condition in order to have "anomaly free" theories. This is an important ingredient in theoretical model building. The "anomaly-free condition" explains why the number of fundamental quark-fermions must be equal to the number of fundamental lepton-fermions. This allows a theoretical prediction to be made for the existence of the *heaviest quark*, in addition to the b-quark in the 3rd family of elementary fermions, *the top-quark.*

Why so many quarks and leptons? The answer will probably come from the superspace with 43 dimensions compactified into $(3 + 1)$.

The quark sector interacts with two forces, QCD and QED, while the lepton sector interacts using only QED. The QFD force comes into play only after all the mixings come in. No one knows why all these deviations from the Platonic Simplicity are needed.

The bold symbols, QCD, QED in the column

$$B_{\mathrm{q}}^{\mathrm{F}}$$

indicate that the six quark flavors interact via these two forces. In the lower part of the same column, the "mixing" indicates that the quark states are no longer "pure" states. They are "mixed"; only these mixed states

$$\left(q_{\mathrm{m}}^{\mathrm{u}}\right)_{1,2,3} \quad \text{and} \quad \left(q_{\mathrm{m}}^{\mathrm{d}}\right)_{1,2,3}$$

interact via the QFD forces.

The column later

$$B_{\ell}^{\mathrm{F}}$$

has the same structure, but the "mixings" are not the same as in the "quark" column.

Furthermore, no one knows at present if the symmetry CP is violated as it is in the quark case. This is why in the box CP $\neq$ there is a question mark. Another detail needs to be specified.

In the quark case, the CP symmetry breaking, CP $\neq$, has been experimentally established not to be via the basic Standard Model mechanism, SSB.

A further deviation from simplicity.

In the leptonic case, we do not know if the CP symmetry is violated. It could be it is. In this case it will be interesting to know if it follows the SSB mechanism.

All these question marks are evidence of further deviations from the simple Platonic descriptions of natural phenomena.

The synthesis of the Platonic Grand Unification and the deviations needed is reported again in Fig. 5.15.

Addendum 2: The Platonic Supersymmetry

The Platonic concept of Supersymmetry is schematically reported in Fig. 5.19, where the basic point for a Platonic concept of Supersymmetry is given; i.e., the only fermions with spin *(1/2 h)* allowed to exist would be the "gauginos."

THE PLATONIC CONCEPT OF SUPERSYMMETRY

The **Gauge Principle** should generate a
Gauge Force ⟶ **Gauge Bosons**
If *NATURE* was platonically *SUPERSYMMETRIC*
Supersymmetry Transformation should generate **Gauginos**

1st **DEVIATION FROM PLATONIC SIMPLICITY**
OUR FERMIONS ARE NOT THE GAUGINOS

2nd **DEVIATION FROM PLATONIC SIMPLICITY**
THE FUNDAMENTAL FERMIONS ARE OF TWO DIFFERENT CLASSES: LEPTONS AND QUARKS

3rd **DEVIATION FROM PLATONIC SIMPLICITY**
THERE IS NOT ONLY ONE BUT THREE FAMILIES OF FUNDAMENTAL FERMIONS

4th **DEVIATION FROM PLATONIC SIMPLICITY**
THE FUNDAMENTAL FERMIONS BECAME MIXED WHEN THE WEAK FORCES ARE SWITCHED ON: MIXINGS EXIST

5th **DEVIATION FROM PLATONIC SIMPLICITY**
THERE ARE DIFFERENT MIXINGS

Fig. 5.19 The Platonic concept of Supersymmetry

If the only allowed fermions would be the "gauginos," there would be no way to have quarks and leptons.

Our fermions are not the gauginos. A deviation is needed. And this is the first one. Our fermions are in fact of two classes: quarks and leptons. Another deviation is needed to introduce quarks and leptons. And this is not enough: one family would not suffice. We need another deviation, the third one, in order to produce three families. Once again this is not enough.

We need a further deviation: the fundamental fermions became mixed when the weak forces are switched on. This fourth deviation is followed by another one, the fifth: the mixing of states in the quark sector and in the leptonic sector is different.

Having proved that Platonic Simplicity is not at work in the Grand Unification and in the law of supersymmetry we go on illustrating a few examples of other deviations from Platonic Simplicity in the detailed construction of the Standard Model (SM). These deviations are coupled to UEEC events.

Addendum 3: Examples of UEEC Events in the Construction of the Standard Model and Beyond

The Standard Model (SM) is the greatest synthesis of all times in the study of fundamental phenomena governing the Universe in its microscopic structure. We will see that the basic achievements of the SM have been obtained via UEEC events; moreover, the SM could not care less about the existence of Platonic Simplicity. Platonic Simplicity is violated at every corner in the process of construction of the SM.

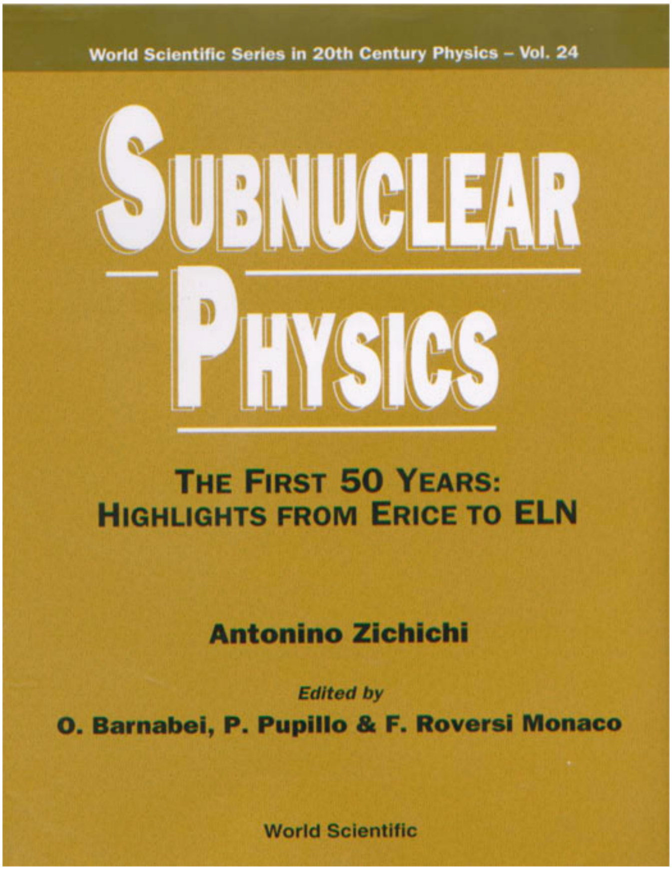


Fig. 5.20 The front-page of the volume "Subnuclear Physics - the first 50 years"

The conclusion is that Complexity exists at the elementary level. In fact, starting from Platonic Simplicity, the Standard Model needs a series of "ad hoc" inputs. These inputs are the proof that at the fundamental level of scientific knowledge there is experimental evidence for the existence of the AFB phenomena and the UEEC events.

On page 209 of my book (whose front-page is reported in Fig. 5.20) "*Subnuclear Physics – the first 50 years*" [15] the following sentence is quoted: "*Copying is easy, logical reasoning is difficult.*" The author being Gerardus 't Hooft, Erice 1997.

With his statement, my friend Gerardus 't Hooft, worded his view on the progress made, and progress still to be made, in theoretical physics.

On the occasion of his 60th Anniversary Celebrations I gave my own testimony on the validity of 't Hooft's statement in experimental physics [2]. In fact, in experimental physics as well, it is not enough to have an original idea. My great teacher, Patrick Blackett, used to tell us, young fellows of his group: "We *experimentalists are not like theorists: the originality of an idea is not for being printed in a paper, but for being shown in the implementation of an original experiment."* To reach the level of performing an original experiment corresponds to bring "logical reasoning" to its asymptotic limit of reality.

Thus, both in *theory* and in *experiment*, the progress of physics is due to those who have the perseverance of not only having an original idea, but of investigating its logical structure in terms of its consequences. At the end of this series of creative steps what is always found is an UEEC event.

A few cases where I have been directly involved are summarized in Fig. 5.6. Each UEEC event is coupled with a *despite*, in order to emphasize the reason why the event is unexpected.

- **The third lepton.** *UEEC event no. 1*

In the late fifties, I realized that the (π–μ) case was unique. This is why the muon was so obviously present everywhere. If a new lepton of 1 GeV mass (or heavier) would have been there, no one would have seen it; if the n was not there, the only way to have muons was via electromagnetic production processes. If a third lepton, heavier than the muon existed, its production could be via electromagnetic processes.

In fact, even if another meson existed in the heavy mass region, and was strongly produced in all proton accelerators, this meson would strongly decay into many pions. A third lepton could not easily be produced as decay-product of heavy mesons.

The absence of a third lepton in the so many final states produced in high energy interactions at CERN and other proton accelerators was not to be considered a fundamental absence, but a consequence of the fact that a third lepton could only be produced via electromagnetic processes, as for example "time-like" photons in ($p\bar{p}$) or (e^+e^-) annihilation.

The uniquencesses of the (π–μ) case sparked the idea of searching for a third lepton in the appropriate production processes. This is how the study for the correct production and decay processes in order to search for a third lepton started [16].

I did not limit myself to discussing this topic with a few colleagues; I followed Blackett's teaching. And this is how I realized that the best "signature" for a heavy lepton would have been "eμ" acoplanar pairs; this is how I invented the "preshower" to improve electron identification by many orders of magnitude; this is why I studied how to improve muon identification; this is how I experimentally established that the best production mechanism could not be ($p\bar{p}$), but (e^+e^-) annihilation.

If the ADONE energy would have been increased, as firmly requested by me, we would have discovered here in Frascati, first of all the

$$J/\psi$$

and than the third lepton.

- **Matter–Antimatter Symmetry.** *UEEC event no. 2*

In the sixties, the need to check the symmetry between nuclear matter and antimatter came to the limelight.

The reason being the apparent triumph of the S-matrix theory to describe strong interactions and the violation of the "well-established" symmetry operators (C, P, CP, T) in weak interactions and in the K-meson decay physics.

When the discovery of scaling in Deep Inelastic Scattering (DIS) and the non-breaking of the protons in high-energy collisions come in the late sixties, the basic structure of all RQFT were put in serious difficulties, and therefore the validity of the celebrated CPT theorem. On the other hand, the basic reason why nuclear antimatter had to exist was CPT.

In the early sixties the first example of nuclear antimatter, the antideuteron, had been searched for and found not to be there at the level of one antideuteron per 10^7 pions produced.

I did not limit myself to the saying that it would have been important to build a beam of negatively charged "partially separated" particles in order to have a very high intensity.

I did not limit myself to suggesting a very advanced electronic device in order to increase, by an order of magnitude, the accuracy for time-of-flight (TOF) measurements.

I did bring all my ideas to the point of full implementation in a detailed experiment, where the antideuteron was found, thus proving nuclear matter–antimatter symmetry. Therefore, credence could be given to CPT and to RQFT.

The matter–antimatter symmetry is related to the basic distinction between matter and mass. This is illustrated in Fig. 5.21 in a very synthetic form.

From the Greeks who associated "stability" of matter with "heaviness" to our present understanding, the number of Sarajevo-type events is really impressive.

There are in fact seven decades of developments which started *from* the antielectron and C-invariance and brought us to the discovery of nuclear antimatter and to the unification of all gauge forces with all deviations from simplicity.

These steps are reported in Fig. 5.22, which looks as complex and full of deviations from simplicity as a page of History (EWRL), despite being a page of Science (EBUS).

- **The nucleon time-like electromagnetic structure (form factors).**

UEEC event no. 3

For a long time QED was taken as the ideal model to describe a fundamental force of nature, such as the nuclear forces (proposed by Yukawa) and the weak forces (proposed by Fermi). The mathematical description of these forces had to be like QED, i.e., a RQFT. Many unexpected experimental discoveries started to create difficulties; these discoveries included the violation of the symmetry operators (parity **P,** charge conjugation **C,** and time reversal **T),** mentioned in UEEC no. 2.

Mass ≢ Matter

$$|m_i\rangle \equiv \text{Mass} \equiv \text{Antimass} \equiv |\overline{m}_i\rangle$$

$\mathbf{i \equiv 1}$ (Intrinsic); $\mathbf{i \equiv 2}$ (Confinement); $\mathbf{i \equiv 3}$ (Binding)

$$C\,|m_i\rangle = |m_i\rangle \qquad \boxed{***}$$

$$\mathbf{i \equiv 1, 2, 3}$$

$$|m_i Q_j\rangle \equiv \text{Matter} \not\equiv \text{Antimatter} \equiv |m_i \overline{Q}_j\rangle$$

$Q_j \equiv$ Flavour Charges

$$\mathbf{j \equiv (u \;\; d \;\; c \;\; s \;\; t \;\; b) \equiv (1, 2, 3, 4, 5, 6)}$$

$$\mathbf{(\nu_e \; e^- \; \nu_\mu \; \mu^- \; \nu_{HL} \; HL^-) \equiv (7, 8, 9, 10, 11, 12)}$$

$$\nu_{HL} \downarrow \tau^-$$

$$C\,|m_i Q_j\rangle = |m_i \overline{Q}_j\rangle \qquad \boxed{***}$$

$$\mathbf{i \equiv 1, 2, 3;\;\; J \equiv 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12.}$$

Fig. 5.21 The matter–antimatter symmetry related to the basic distinction between matter and mass

This is how the scattering matrix, **S**, become the way out. But the **S**-matrix was the negation of RQFT.

In fact, the field concept involves a larger set of functions than those derived by the analytic continuation of the S-matrix. But no one knew (and even now knows) how to construct fields purely in terms of analytic scattering amplitudes. Scattering amplitudes are "on the mass shell" while fields imply extension to "off the mass shell."

Form factors are not scattering amplitudes, nevertheless they do exist and they are due to the virtual phenomena produced by all possible interactions (strong, electromagnetic, and weak). The conjectured analyticity properties of the nuclear scattering matrix are a very restricted concept, if compared with the concept of a "field."

S-matrix theory is not designed to describe experiments in which interactions between particle states do take place while momentum measurements are being performed. In other words all the physics due to "virtual processes" fell outside the physics described by the S-matrix theory, which had a period of large success in the description of strong interactions [43].

This apparent triumph of the S-matrix [43] put in serious troubles the foundations of RQFT. On the other hand these theories were restricted to be of Abelian

THE INCREDIBLE STORY
TO DISENTANGLE THE ORIGIN OF THE STABILITY OF MATTER
SEVEN DECADES FROM THE ANTIELECTRON TO ANTIMATTER
AND THE UNIFICATION OF ALL GAUGE FORCES

- **The validity of C invariance from 1927 to 1957.**

After the discovery by Thomson in 1897 of the first example of an elementary particle, the Electron, it took the genius of Dirac to theoretically discover the Antielectron thirty years after Thomson.

1927 → Dirac equation [17]; the existence of the antielectron is, soon after, theoretically predicted. Only a few years were needed, after Dirac's theoretical discovery, to experimentally confirm (Anderson, Blackett and Occhialini [18]) the existence of the Dirac antielectron.

1930-1957 → **Discovery of the C operator** [(charge conjugation) H. Weyl and P.A.M. Dirac [19]]; discovery of the P Symmetry Operator [E.P. Wigner, G.C. Wick and A.S. Wightman [20, 21]]; discovery of the T operator (time reversal) [E.P. Wigner, J. Schwinger and J.S. Bell [22, 23, 24, 25]]; discovery of the CPT Symmetry Operator from RQFT (1955-57) [26].

1927-1957 → Validity of C invariance: e^+ [18]; $\bar{p}$ [27]; $\bar{n}$ [28]; $K_2^0 \to 3\pi$ [29] but see LOY [30].

- **The new era starts: C ≠ ; P ≠ ; CP ≠** (*) .

1956 → Lee & Yang P ≠ ; C ≠ [31].

1957 → Before the experimental discovery of P ≠ & C ≠, Lee, Oehme, Yang (LOY) [30] point out that the existence of the second neutral K-meson, $K_2^0 \to 3\pi$, is proof neither of C invariance nor of CP invariance. Flavour antiflavour mixing does not imply CP invariance.

1957 → C.S. Wu et al. P ≠ ; C ≠ [32]; CP ok [33].

1964 → $K_2^0 \to 2\pi \equiv K_L$: CP ≠ [34].

1947-1967 → QED divergences & Landau poles.

1950-1970 → The crisis of RQFT & the triumph of S-matrix theory (i.e. the negation of RQFT).

1965 → Nuclear antimatter is (experimentally) discovered [35]. See also [36].

1968 → The discovery [37] at SLAC of Scaling (free quarks inside a nucleon at very high q^2) but in violent (pp) collisions no free quarks at the ISR are experimentally found [38]. Theorists consider Scaling as being evidence for RQFT not to be able to describe the Physics of Strong Interactions. The only exception is G. 't Hooft who discovers in 1971 that the β-function has negative sign for non-Abelian theories [15].

1971-1973 → β = – ; 't Hooft; Politzer; Gross & Wilczek. The discovery of **non-Abelian** gauge theories. Asymptotic freedom in the interaction between quarks and gluons [15].

1974 → All gauge couplings α_1 α_2 α_3 run with q^2 but they do not converge towards a unique point.

1979 → A.P. & A.Z. point out that the new degree of freedom due to SUSY allows the three couplings $\boldsymbol{\alpha_1}$ $\boldsymbol{\alpha_2}$ $\boldsymbol{\alpha_3}$ **, to converge towards a unique point** [39].

1980 → QCD has a 'hidden' side: the multitude of final states for each pair of interacting particles: (e^+e^- ; $p\bar{p}$; πp; Kp; νp; pp; etc.)
The introduction of the Effective Energy allows to discover the Universality properties [40] in the multihadronic final states.

1992 → All gauge couplings converge towards a unique point at the gauge unification energy: $E_{GU} \cong 10^{16}$ GeV with $\alpha_{GU} \cong 1/24$ [41, 42] .

1994 → The Gap [4] between E_{GU} & the String Unification Energy: $E_{SU} \cong E_{Planck}$.

1995 → **CPT loses its foundations at the Planck scale (T.D. Lee)** [43].

1995-1999 → **No CPT theorem from M-theory (B. Greene)** [44].

1995-2000 → A.Z. points out the need for new experiments to establish if matter-antimatter symmetry or asymmetry are at work.

(*) The symbol ≠ stands for 'Symmetry Breakdown'.

Fig. 5.22 Seven decades of developments leading to the discovery of nuclear antimatter and the unification of all gauge forces

nature, since the non-Abelian ones were shrouded by even more mystifying problems. The "prediction" was that the "time-like" electromagnetic structure of the nucleon had not been there. A totally unexpected result [44, 45] came with the experiment performed at CERN to study the annihilation process between a proton and an antiproton ($\bar{p}p$) giving rise to a "virtual photon" (γ) transforming into an electron–antielectron pair (e^+e^-). The corresponding reaction is $\bar{p}p \rightarrow \gamma \rightarrow e^+e^-$.

The experimental results [44,45] proved that the nucleon had a very large "time-like" electromagnetic form factor: totally unexpected.

- **The proton does not break into three quarks despite 1968 Panofsky.**

UEEC event no. 4

When in 1968 I heard Pif (W.K.H.) Panofsky reporting in Vienna on (ep) deep-inelastic-scattering, whose immediate consequence was that "partons" inside a proton behaved as "free" particles, I did not limit myself to the saying that it would have been interesting to check if, in violent (pp) collisions, "free" partons were produced.

As the "partons" were suspected to be the quarks earlier suggested by M. Gell-Mann and G. Zweig (we now know that partons can also be gluons), the experiment needed was a search for fractionally charged particles in the final states of violent (pp) interactions at the CERN ISR.

To perform the experiment, a new type of plastic scintillator was needed, with very long attenuation length because the counters had to be put inside a very big magnet. These scintillators did not exist on the market. We studied the problem and built the most powerful and sensitive scintillators. The result was that free quarks were not produced, despite the violent (pp) collisions.

- **The mesonic mixings:** $\theta_{PS} \neq \theta_V \neq 0$. *UEEC event no. 5*

The problem of concern in the physics of strong interactions was the "mixing" in meson physics. It was necessary to know why this mixing was there and why the vector mesons (ρ, ω, ϕ) did not show the same behavior as the pseudoscalar mesons (π, η, η').

At the end of the "logical reasoning" in terms of experimental searches never conducted before (Fig. 5.23 is the cover-page of a volume dedicated to this topic), the result was that the mesonic mixing was there and the two mixing angles were drastically different: $\theta_{PS} \neq \theta_V$.

This is what Dick Dalitz defined the most significant results from all mesonic physics [46].

Let me show Fig. 5.24 which illustrates the difference existing between the two mesonic mixing angles, pseudoscalar and vector: $\theta_{PS} \neq \theta_V$. They should both be zero if SU $(3)_{uds}$ was a good Symmetry.

The existence of instantons was not known. They came after the discovery that $\theta_{PS} \neq \theta_V$. A strong supporter of my experiment was Richard Dalitz, to whom I would like to dedicate the results reported in Fig. 5.24.

Let me go back to "logical reasoning" in experimental physics. I did not limit myself to the saying that the most appropriate way to study this problem [(e^+e^-) colliders did not yet exist], was to measure with the best possible accuracy the electromagnetic decay rates of the vector mesons

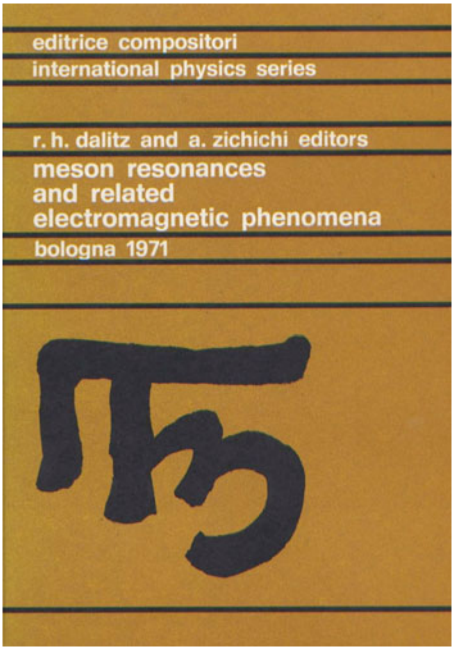


Fig. 5.23 The cover-page of a volume dedicated to "logical reasoning" in terms of experimental searches

$$(\rho \to e^+e^-),$$
$$(\omega \to e^+e^-),$$
$$(\phi \to e^+e^-),$$

and to see if the heaviest meson (known at that time with the symbol X^0) was decaying into two γ's ($X^0 \to \gamma\gamma$).

These were times when experimental physics was dominated by bubble chambers. I designed and built a nonbubble-chamber detector, NBC; it consisted of an original neutron missing mass spectrometer coupled with a powerful electromagnetic detector which allowed to clearly identify all final states of the decaying mesons into (e^+e^-) or ($\gamma\gamma$) pairs. The mass of the meson (be it pseudoscalar or vector) was measured by the neutron missing mass spectrometer. The two "mixing angles," the pseudoscalar θ_{PS} and the vector θ_V, where directly measured (without using the masses) to be, not as expected by $SU(3)_{uds}$, i.e., $\theta_{PS} = \theta_V = 0$, but, $\theta_{PS} \neq 0$, $\theta_V \neq 0$ and totally different $\theta_{PS} \neq \theta_V$. Many years were needed for Gerardus 't Hooft instantons to explain why $\theta_{PS} \simeq 10°$ and $\theta_V \simeq 51°$.

- **The Gribov QCD light.** *UEEC event no. 6*

When the physics of strong interactions finally became the physics of quarks and gluons, QCD had a problem, defined by Gribov as being its "hidden side": i.e., the

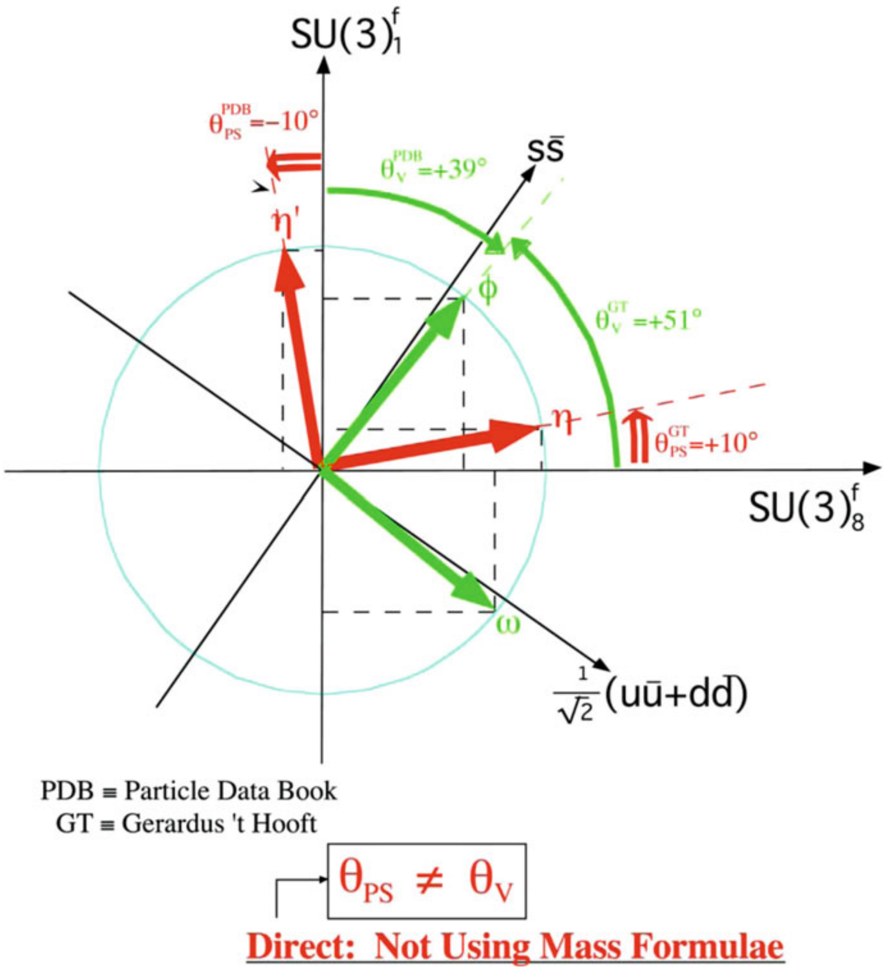


Fig. 5.24 The difference existing between the two mesonic mixing angles, pseudoscalar and vector

large number of different final states produced by different pairs of interacting particles, such as (πp, pp, $\bar{\text{p}}$p, Kp, e^+e^-, νp, μp, ep, etc.). I did not limit myself to suggesting that a totally different approach was needed to put all these final states on the same basis. I found what this basis could be and this is how the "Effective Energy" became the correct quantity to be measured in each interaction. The "Effective Energy" was not predicted by QCD.

To perform this study, it was necessary to analyze tens of thousands of (pp) interactions at the ISR. This was done despite all the difficulties to be overcome. And this is how what Vladimir Gribov defined the QCD light was discovered (Figs. 5.25 and 5.26).

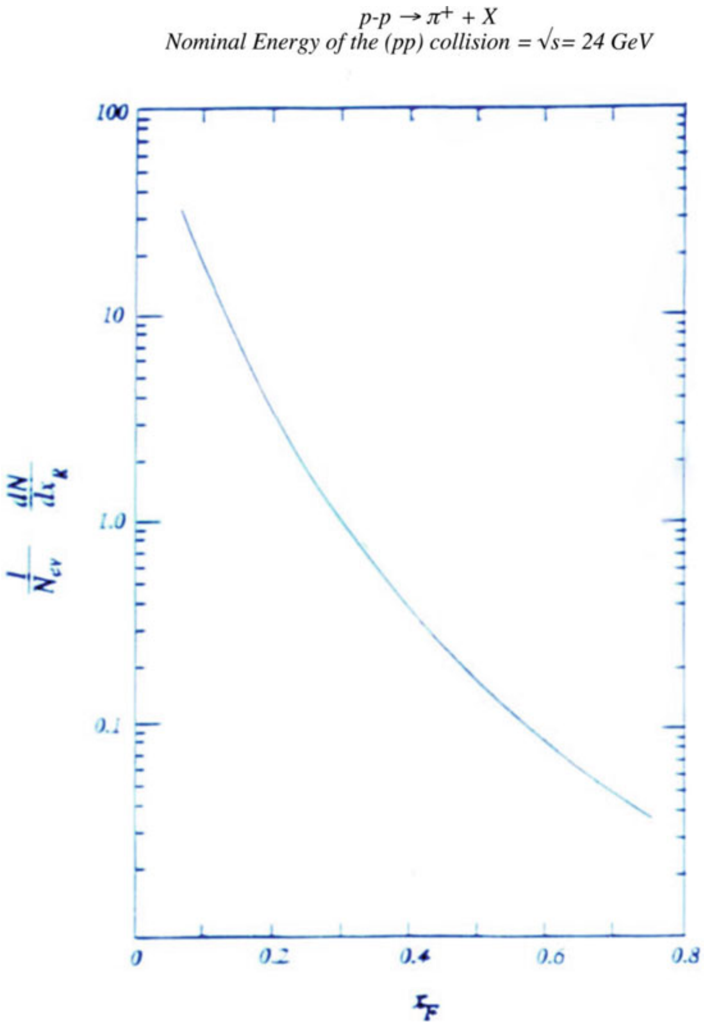


Fig. 5.25 The spectrum of the (pp) reaction. The horizontal axis is for the fractional energy of the pion (also called Feynman x), while the vertical axis is for the number of pions having fractional energy x_F

Gribov pointed out what follows. Newton discovered that QED light is the sum of different colors. In QCD we have quarks and gluons interacting and producing Jets made of many pions, as for example in the (pp) reaction

$$\mathrm{pp} \to \pi + X$$

whose spectrum is shown in Fig. 5.25. The horizontal axis is for the fractional energy of the pion (also called Feynman x), while the vertical axis is for the number of pions having fractional energy x_F.

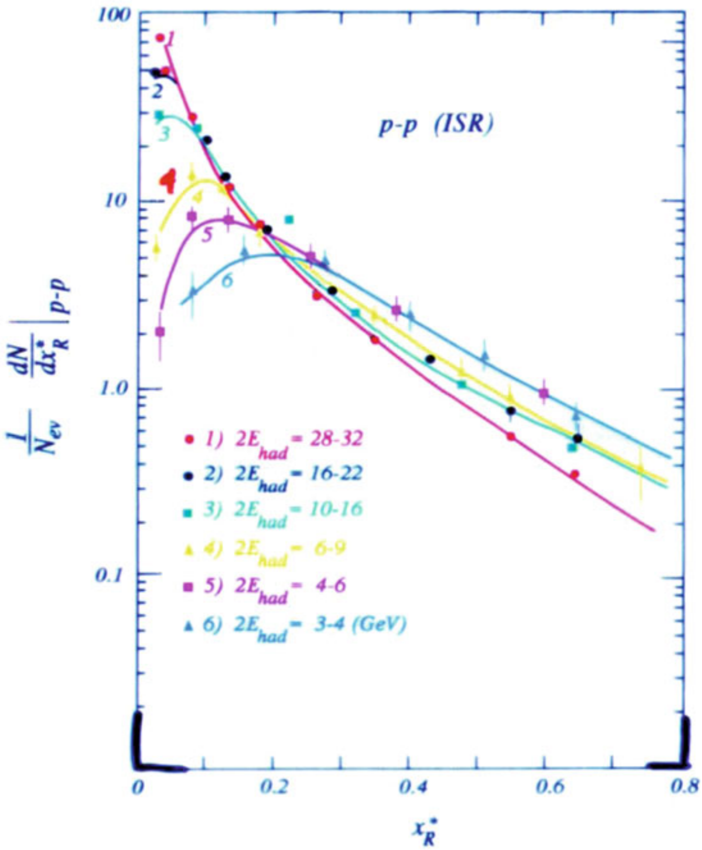


Fig. 5.26 The "effective energy" at the origin. The same initial state (pp) produces many final states with different "effective energies"

$$p - p \rightarrow \pi^{+} + X$$

Nominal Energy of the (pp) collision $= \sqrt{s} = 24\,GeV$

The spectrum reported in Fig. 5.25 is an example of *QCD light.*

The "effective energy" shown in Fig. 5.26 is at its origin, despite being totally unexpected. In fact the same initial state (pp) produces many final states each one having different "effective energy," called $E_{\rm had}$ in Fig. 5.26.

Each $E_{\rm had}$ has a given π-spectrum. The sum of all these spectra gives the total spectrum of Fig. 5.25. If, instead of (pp) we study other initial states, purely hadronic (kp), ($\bar{p}$p) or mixed (lepton-hadron) (ep) (νp) (μp) and even (γp), no matter what the initial states is, the only relevant quantity is the "effective energy." Each "effective energy" produces the same π-spectrum in analogy with the QED light of given color.

So, when a new problem appears, the only way out is to bring the logical reasoning – be it of experimental, theoretical or technical nature – to the deepest level of consequences. At the very end of this "logical reasoning" what is found is a UEEC event: i.e., a result which was totally unexpected.

This is how progress is made in the most advanced frontier of reductionism: physics.

- **The Grand Unification in the Real World.** *UEEC event no. 7*

We now move toward the unification of all fundamental forces. This is really a set of UEEC events, as we will see in this chapter.

The grand unification in the real world depends on how the gauge couplings ($\alpha_1\alpha_2\alpha_3$) change with energy. It is this change which allows the fundamental forces to converge toward a unique origin.

The mathematical structure describing this "change" is a system of coupled differential non linear equations, called the "renormalization group equations," RGEs, reported later.

- **The Renormalization Group Equations.**

The lines in Fig. 5.27 are the result of calculations executed with a supercomputer using the following system of equations:

$$\mu \frac{d\alpha_i}{d\mu} = \frac{b_i}{2\pi}\alpha_i^2 + \sum_j \frac{bij}{8\pi^2}\alpha_j\alpha_i^2. \tag{5.1}$$

This is a system of coupled nonlinear differential equations (RGEs) that describes the Superworld, from the maximum level of energy (Planck scale) to our world at the minimum of energy.

The results reported in Fig. 5.27 are the most exact use of the renormalization group equations for the running of the three gauge couplings $\alpha_1\alpha_2\alpha_3$. The unification of all forces and the threshold, where to find the first particle of the Superworld, with its problems are reported in Figs. 5.27 and 5.28, respectively.

During more than 10 years (from 1979 to 1991), no one had realized [47] that the energy threshold for the existence of the Superworld was strongly dependent on the "running" of the masses. This is now called: the EGM effect (from the initials of Evolution of Gaugino Masses).

To compute the energy threshold using only the "running" of the gauge couplings ($\alpha_1, \alpha_2, \alpha_3$) corresponds to neglecting nearly three orders of magnitude in the energy threshold for the discovery of the first particle (the lightest) of the Superworld [48], as illustrated in Fig. 5.28.

This is just a further example of comparison between the "Platonic" Simplicity and the "real world," when we deal with the Grand Unification.

Talking about supersymmetry, there is another important step: how we go from pure theoretical speculations to phenomenology. This is not an easy task.

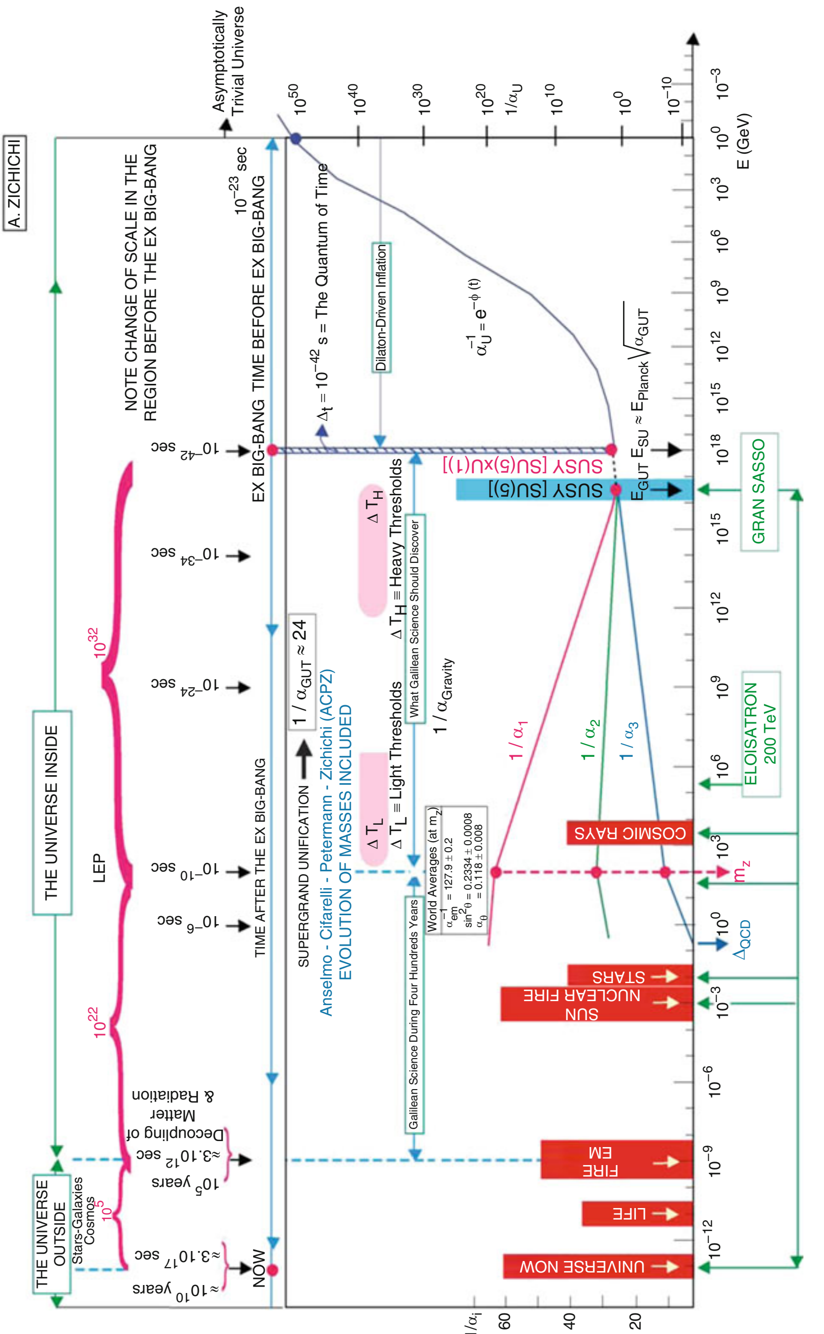


Fig. 5.27 Calculations carried out with a supercomputer showing the most exact use of the renormalization group equations for the running of the three gauge couplings

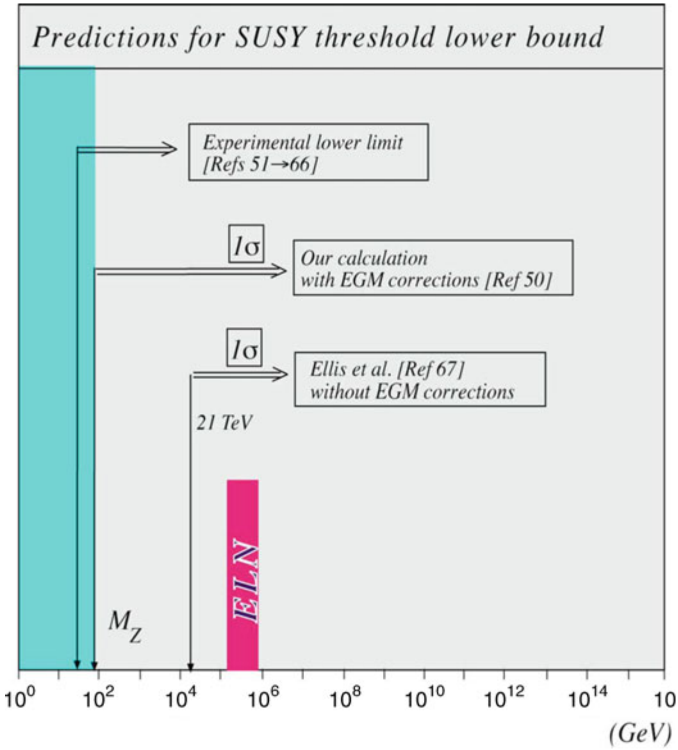


Fig. 5.28 Predictions for the SUSY threshold lower bound

The proof is given in Fig. 5.29 where it is shown how many important properties of the physics to be described have been neglected by some authors (AdBF) [47] whose claim was to "predict" the energy scale at which supersymmetry is broken.

In order to attempt to give such a prediction, there are at least five "details" to be taken into account, as reported in the last five columns (6–10) of Fig. 5.29.

It is interesting to study the point where the three gauge couplings meet (the GUT point). This is neither at the "String Unification Point," E_{SU}, nor at the Planck scale, as reported in Fig. 5.30.

There is in fact a "gap" of few orders of magnitudes between these points. A detailed study of this gap has been performed by ACZ [4].

In Fig. 5.7, there is a different way of reporting the results obtained using the same (5.1) mathematical structure (RGEs). The three axis are the gauge couplings $\alpha_1\alpha_2\alpha_3$ and the other details are given in the figure caption.

After we have published these results [37], the $(\alpha_1\alpha_2\alpha_3)$ graph has been given the name of "action space." In this space we have emphasized the "straight" line as being the one which would naively be considered the "platonic" way of imagining the changes of $\alpha_1\alpha_2\alpha_3$ in order to meet at the same point E_{GUT}.

●	①	②	③	④	⑤	⑥	⑦	⑧	⑨	⑩
Authors	*Input data*	*Errors*	*EC*	M_{SUSY}	*CC*	*UC*	ΔT_L	M_X	ΔT_H	*EGM*
ACPZ [4, 41, 50, 68→71]	WA	± 2 σ	all possible solutions (24)	Yes	physical	Yes	Yes	Yes	Yes	Yes

Authors	*Input data*	*Errors*	*EC*	M_{SUSY}	*CC*	*UC*	ΔT_L	M_X	ΔT_H	*EGM*
AdBF [49]	only one experiment	± 1 σ	only one solution	Yes	Geometrical	No	No	No	No	No

① *WA* ≡ *World Average*
② *Errors* ≡ *Uncertainty taken from all data (World Average) or from a single experiment*
③ *EC* ≡ *Evolution of Couplings*
④ M_{SUSY} ≡ *Mass Scale assumed to represent the Supersymmetry Scale breaking*
⑤ *CC* ≡ *Convergence of Couplings*
⑥ *UC* ≡ *Unification of Couplings above* E_{GUT}
⑦ ΔT_L ≡ *Low Energy threshold*
⑧ M_X ≡ *Mass Scale at the breaking of the Grand Unified Theory to the SU(3)xSU(2)xU(1)*
⑨ ΔT_H ≡ *High Energy threshold*
⑩ *EGM* ≡ *Evolution of Gaugino Masses*

Fig. 5.29 Difficulties in supersymmetry, when going from pure theoretical speculations to phenomenology

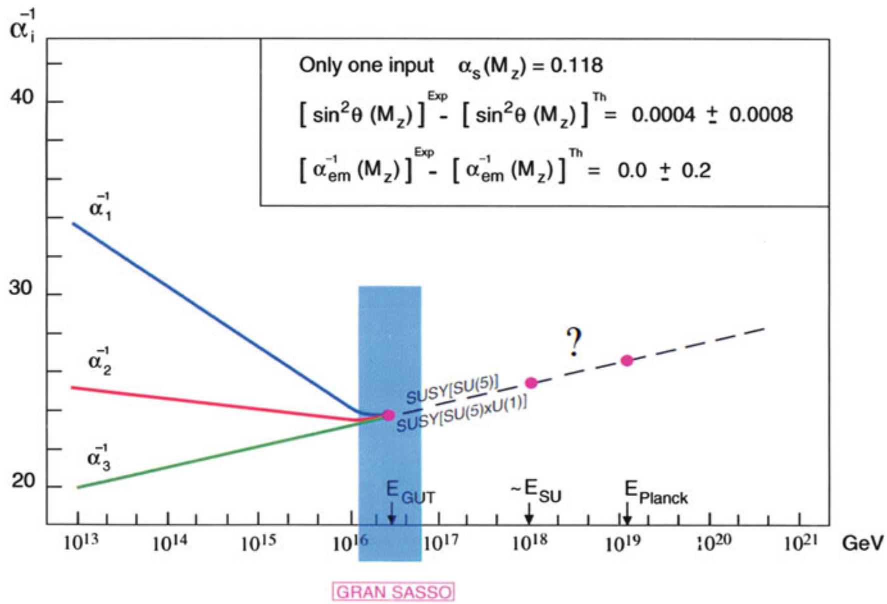


Fig. 5.30 The point where the three gauge couplings meet (the GUT point)

The "Platonic" Simplicity would indicate the series of points making up the straight line as the platonic ideally simple solution. The real solution is the sequence of points which totally deviate from the straight line.

The points have a sequence of 100 GeV in energy. The last point where the "ideal" platonic straight line intercepts the theoretical prediction is at the energy of the Grand Unification. This corresponds to $E_{GU} = 10^{16.2}$ GeV. Other detailed information on the theoretical inputs: the number of fermionic families, N_F, is 3; the number of Higgs particles, N_H, is 2. The input values of the gauge couplings at the Z^0-mass is $\alpha_3\ (M_Z) = 0.118 \pm 0.008$; the other input is the ratio of weak and electromagnetic couplings also measured at the Z^0-mass value: $\sin^2 \theta_W(M_Z) = 0.2334 \pm 0.0008$.

Finally, in Fig. 5.31 we show how the Planck energy could go down to the Fermi energy scale if one extra dimension could be compactified, as suggested by Ignatios Antoniadis.

The "origin of space–time" is indicated immediately above E_{GUT} since it is there where all theoretical speculations stop to be coupled with even a very small amount of experimental finding.

In fact, even E_{GUT} is the result of extrapolation (using the most accurate mathematical description) from 10^2 GeV up to 10^{16} GeV, i.e., over 14 orders of magnitude in energy.

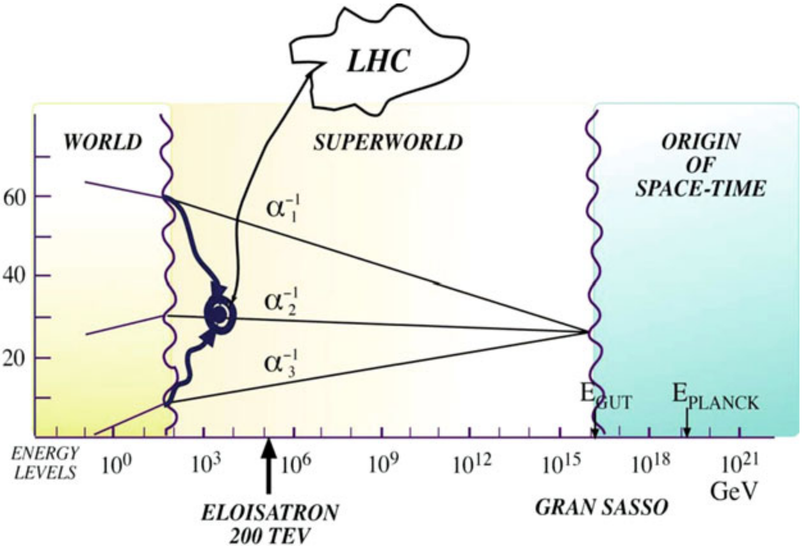


Fig. 5.31 The Planck energy could go down to the Fermi energy scale if one extra dimension could be compactified

Addendum 4: The Ten Challenges of Subnuclear Physics (Figs. 5.32 and 5.33)

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THE TEN CHALLENGES OF SUBNUCLEAR PHYSICS

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ABSTRACT

The vitality of Subnuclear Physics is proved by the existence of its challenges, which are presented and discussed with reference to their implementation in the near future.

Contents

International School of Subnuclear Physics
39th Course : New Fields And Strings In Subnuclear Physics
Erice, 29 August - 7 September 2001

Fig. 5.32 The Ten Challenges of Subnuclear Physics, front page

THE TEN CHALLENGES
OF SUBNUCLEAR PHYSICS

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A — **Introduction.**

The next decade will be a decisive one for Subnuclear Physics. It is necessary to think in a critical and constructive way in order to be sure that its future will be granted.

The last decade has been devoted to the search for the first example of superparticles (be it the neutralino or the gravitino) and for the lightest Higgs. None of them has been found. On the other hand, no one is able to "predict" the Energy-threshold for the two challenges of our field: the Spontaneous Symmetry Breaking (SSB) and the Supersymmetry (SUSY). These two thresholds, on a qualitative ground, should be nearly degenerate with the Fermi scale and therefore be potentially "round the corner".

Nevertheless, the point I want to emphasize is that it would be a serious mistake to focus all our attention on SSB and SUSY. The healthy future of Subnuclear Physics is granted by the existence of ten challenges, each one being of extraordinary scientific value; ten, not only the two mentioned above (4 and 5 in our list below).

B — **The ten challenges.**

Here is the list.

1 Non-perturbative QCD.
2 Anomalies and Instantons.
3 The Physics of NSSB (non-Spontaneous Symmetry Breaking: CP ≠, T ≠, CPT ≠ (*) Matter-Antimatter Symmetry).
4 The Physics of Imaginary Masses: SSB (part of this is the Higgs particle/particles).
5 The Physics of 43 dimensions (part of this is Supersymmetry).
6 Flavour mixing in the quark sector.
7 Flavour mixing in the leptonic sector.
8 The problem of the missing mass in the Universe.
9 The problem of the Hierarchy.
10 The Physics at the Planck scale and the number of expanded dimensions.

And now the comments with reference to their implementation in the near future.

(*) The symbol ≠ means that a Symmetry law is non spontaneously broken as it happens with C, P, CP and T). [C (charge conjugation, i.e. interchange of charges with anti-charges); P (parity, i.e. interchange of left and right); T (inversion of the arrow of Time)]. The products CP and CPT mean the simultaneous Symmetry laws for all operations CP and CPT, respectively. The existence of Matter-Antimatter Asymmetry would be a proof of CPT ≠ .

Fig. 5.33 The Ten Challenges of Subnuclear Physics, the list

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- 31st, 32nd and 33th Course of the International School of Solid State Physics, Erice (Italy), July 2004, "*Complexity at the Elementary Level.*"
- 42nd International School of Subnuclear Physics, Erice (Italy), August–September 2004, "*Complexity at the Elementary Level.*"
- Trinity College, Dublin (Ireland), February 2005, "*Complexity at the Elementary Level.*"
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Chapter 6
Non-supersymmetric Attractors in Symmetric Coset Spaces

Wei Li

Abstract We develop a method of constructing generic black hole attractor solutions, both BPS and non-BPS, single-centered as well as multi-centered, in a large class of 4D $\mathcal{N} = 2$ supergravities coupled to vector-multiplets with cubic prepotentials. The method is applicable to models for which the 3D moduli spaces obtained via c^*-map are symmetric coset spaces. All attractor solutions in such a 3D moduli space can be constructed algebraically in a unified way. Then the 3D attractor solutions are mapped back into four dimensions to give 4D extremal black holes.

6.1 Introduction

The attractor mechanism for supersymmetric (BPS) black holes was discovered in 1995 [1]: at the horizon of a supersymmetric black hole, the moduli are completely determined by the charges of the black hole, independent of their asymptotic values. In 2005, Sen showed that all extremal black holes, both supersymmetric and non-supersymmetric (non-BPS), exhibit attractor behavior [2]: it is a result of the near-horizon geometry of extremal black holes, rather than supersymmetry. Since then, non-BPS attractors have been a very active field of research (see for instance [3–8, 10–15, 41]). In particular, a microstate counting for certain non-BPS black holes was proposed in [16]. Moreover, a new extension of topological string theory was suggested to generalize the Ooguri–Strominger–Vafa (OSV) formula so that it also applies to non-supersymmetric black holes [17].

Both BPS and non-BPS attractor points are simply determined as the critical points of the black hole potential V_{BH} [7, 18]. However, it is much easier to solve the full BPS attractor flow equations than to solve the non-BPS ones: the supersymmetry condition reduces the second-order equations of motion to first-order ones. Once the BPS attractor moduli are known in terms of D-brane charges, the full BPS

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attractor flow can be generated via a harmonic function procedure, i.e., by replacing the charges in the attractor moduli with corresponding harmonic functions:

$$t_{\mathrm{BPS}}(x) = t^{*}_{\mathrm{BPS}}\left(p^I \to H^I(\boldsymbol{x}), q_I \to H_I(\boldsymbol{x})\right). \tag{6.1}$$

In particular, when the harmonic functions $(H^I(\boldsymbol{x}), H_I(\boldsymbol{x}))$ are multi-centered, this procedure generates multi-centered BPS solutions [19].

The existence of multi-centered BPS bound states is crucial in understanding the microscopic entropy counting of BPS black holes and the exact formulation of OSV formula [20]. One can imagine that a similarly important role could be played by multi-centered non-BPS solutions in understanding non-BPS black holes microscopically. However, the multi-centered non-BPS attractor solutions have not been constructed until [21], on which this talk is based. In fact, even their existence has been in question.

In the BPS case, the construction of multi-centered attractor solutions is a simple generalization of the full attractor flows of single-centered black holes: one needs simply to replace the single-centered harmonic functions in a single-centered BPS flow with multi-centered harmonic functions. However, the full attractor flow of a generic single-centered non-BPS black hole has not been solved analytically, due to the difficulty of solving second-order equations of motion. Ceresole et al. obtained an equivalent first-order equation for non-BPS attractors in terms of a "fake superpotential," but the fake superpotential can only be explicitly constructed for special charges and asymptotic moduli [22, 23]. Similarly, the harmonic function procedure was only shown to apply to a special subclass of non-BPS black holes, but has not been proven for generic cases [11].

In this talk, we will develop a method of constructing generic black hole attractor solutions, both BPS and non-BPS, single-centered as well as multi-centered, in a large class of 4D $\mathcal{N} = 2$ supergravities coupled to vector-multiplets with cubic prepotentials. The method is applicable to models for which the 3D moduli spaces obtained via c^*-map are symmetric coset spaces. All attractor solutions in such a 3D moduli space can be constructed algebraically in a unified way. Then the 3D attractor solutions are mapped back into four dimensions to give 4D extremal black holes.

The outline of the talk is as follows. Section 6.2 lays out the framework and presents our solution generating procedures; Sect. 6.3 focuses on the theory of 4D $\mathcal{N} = 2$ supergravity coupled to one vector-multiplet, and shows in detail how to determine the attractor flow generators; Sect. 6.4 then uses these generators to construct single-centered attractors, both BPS and non-BPS, and proves that generic non-BPS solutions cannot be generated via the harmonic function procedure; Sect. 6.5 constructs multi-centered solutions, and shows the great contrasts between BPS and non-BPS ones. We end with a discussion on various future directions.

6.2 Framework

6.2.1 3D Moduli Space $\mathcal{M}_{3D}$

The technique of studying stationary configurations of 4D supergravities by dimensionally reducing the 4D theories to 3D non-linear σ-models coupled to gravity was described in the pioneering work [24]. The 3D moduli space for 4D $\mathcal{N} = 2$ supergravity coupled to n_V vector-multiplets is well-studied, for example in [25–28]. Here we briefly review the essential points.

The bosonic part of the 4D action is

$$S = -\frac{1}{16\pi}\int \mathrm{d}^4x\sqrt{-g^{(4)}}\Big[R - 2G_{i\bar{j}}\mathrm{d}t^i \wedge *_4\mathrm{d}\bar{t}^{\bar{j}} - F^I \wedge G_I\Big], \tag{6.2}$$

where $I = 0, 1, \ldots, n_V$, and $G_I = (\mathrm{Re}\mathcal{N})_{IJ}F^J + (\mathrm{Im}\mathcal{N})_{IJ} * F^J$. For a theory endowed with a prepotential $F(X)$, $\mathcal{N}_{IJ} = F_{IJ} + 2i\frac{(\mathrm{Im}F\cdot X)_I(\mathrm{Im}F\cdot X)_J}{X\cdot\mathrm{Im}F\cdot X}$ where $F_{IJ} = \partial_I\partial_J F(X)$ [28]. We will consider generic stationary solutions, allowing non-zero angular momentum. The ansatz for the metric and gauge fields are

$$\mathrm{d}s^2 = -\mathrm{e}^{2U}(\mathrm{d}t + \boldsymbol{\omega})^2 + \mathrm{e}^{-2U}\mathbf{g}_{ab}\mathrm{d}x^a\mathrm{d}x^b, \tag{6.3}$$
$$A^I = A_0^I(\mathrm{d}t + \boldsymbol{\omega}) + \mathbf{A}^I, \tag{6.4}$$

where $\mathbf{g}_{ab}$ is the 3D space metric and bold fonts denote three-dimensional fields and operators. The variables are $3n_V + 2$ scalars $\{U, t^i, \bar{t}^{\bar{i}}, A_0^I\}$, and $n_V + 2$ vectors $\{\boldsymbol{\omega}, \mathbf{A}^I\}$.

The existence of a time-like isometry allows us to reduce the 4D theory to a 3D non-linear σ-model on this isometry. Dualizing the vectors $\{\boldsymbol{\omega}, \mathbf{A}^I\}$ to the scalars $\{\sigma, B_I\}$, and renaming A_0^I as A^I, we arrive at the 3D Lagrangian, which is a non-linear σ-model minimally coupled to 3D gravity:[1]

$$\mathcal{L} = \frac{1}{2}\sqrt{\mathbf{g}}\left(-\frac{1}{\kappa}\mathbf{R} + \partial_a\phi^m\partial^a\phi^n g_{mn}\right), \tag{6.5}$$

where ϕ^n are the $4(n_V+1)$ moduli fields $\{U, t^i, \bar{t}^{\bar{i}}, \sigma, A^I, B_I\}$, and g_{mn} is the metric of the 3D moduli space $\mathcal{M}_{3D}$, whose line element is

$$\mathrm{d}s^2 = \mathbf{d}U^2 + \frac{1}{4}\mathrm{e}^{-4U}\left(\mathbf{d}\sigma + A^I\mathbf{d}B_I - B_I\mathbf{d}A^I\right)^2 + g_{i\bar{j}}(t,\bar{t})\mathbf{d}t^i\cdot\mathbf{d}\bar{t}^{\bar{j}}$$
$$+\frac{1}{2}\mathrm{e}^{-2U}\Big[\left(\mathrm{Im}\mathcal{N}^{-1}\right)^{IJ}\left(\mathbf{d}B_I + \mathcal{N}_{IK}\mathbf{d}A^K\right)\cdot\left(\mathbf{d}B_J + \overline{\mathcal{N}}_{JL}\mathbf{d}A^L\right)\Big]. \tag{6.6}$$

[1] Note that the black hole potential term in 4D breaks down into kinetic terms of the 3D moduli, thus there is no potential term for the 3D moduli.

The resulting $\mathcal{M}_{3D}$ is a para-quaternionic-Kähler manifold, with special holonomy $Sp(2,\mathbb{R}) \times Sp(2n_V + 2,\mathbb{R})$ [29]. It is the analytical continuation of the quaternionic-Kähler manifold with special holonomy $USp(2,\mathbb{R}) \times USp(2n_V + 2,\mathbb{R})$ studied in [26]. Thus the vielbein has two indices (α, A), transforming under $Sp(2,\mathbb{R})$ and $Sp(2n_V + 2,\mathbb{R})$, respectively. The para-quaternionic vielbein is the analytical continuation of the quaternionic vielbein computed in [26]. This procedure is called the c^*-map [29], as it is the analytical continuation of the c-map in [25, 26].

The isometries of the $\mathcal{M}_{3D}$ descends from the symmetry of the 4D system. In particular, the gauge symmetries in 4D give the shift isometries of $\mathcal{M}_{3D}$, whose associated conserved charges are

$$q_I \mathrm{d}\tau = J_{A^I} = P_{A^I} - B_I P_\sigma, \quad p^I \mathrm{d}\tau = J_{B_I} = P_{B_I} + A^I P_\sigma, \quad a\mathrm{d}\tau = J_\sigma = P_\sigma, \tag{6.7}$$

where the $\{P_\sigma, P_{A^I}, P_{B_I}\}$ are the momenta. Here τ is the affine parameter defined as $\mathrm{d}\tau \equiv -*_3 \sin\theta \mathrm{d}\theta \mathrm{d}\phi$. (p^I, q_I) are the D-brane charges, and a the NUT charge. A non-zero a gives rise to closed time-like curves, so we will set $a = 0$ from now on.

6.2.2 Attractor Flow Equations

The E.O.M. of 3D gravity is Einstein's equation:

$$\mathbf{R}_{ab} - \frac{1}{2}\mathbf{g}_{ab}\mathbf{R} = \kappa T_{ab} = \kappa \left(\partial_a \phi^m \partial_b \phi^n g_{mn} - \frac{1}{2}\mathbf{g}_{ab} \partial_c \phi^m \partial^c \phi^n g_{mn} \right) \tag{6.8}$$

and the E.O.M. of the 3D moduli are the geodesic equations in $\mathcal{M}_{3D}$:

$$\nabla_a \nabla^a \phi^n + \Gamma^n_{mp} \partial_a \phi^m \partial^a \phi^p = 0. \tag{6.9}$$

It is not easy to solve a non-linear σ-model that couples to gravity. However, the theory greatly simplifies when the 3D spatial slice is flat: the dynamics of the moduli are decoupled from that of 3D gravity:

$$T_{ab} = 0 = \partial_a \phi^m \partial_b \phi^n g_{mn} \quad \text{and} \quad \partial_a \partial^a \phi^n + \Gamma^n_{mp} \partial_a \phi^m \partial^a \phi^p = 0. \tag{6.10}$$

In particular, a single-centered attractor flow then corresponds to a null geodesics in $\mathcal{M}_{3D}$: $\mathrm{d}s^2 = \mathrm{d}\phi^m \mathrm{d}\phi^n g_{mn} = 0$.

The condition of the 3D spatial slice being flat is guaranteed for BPS attractors, both single-centered and multi-centered, by supersymmetry. Furthermore, for single-centered attractors, both BPS and non-BPS, extremality condition ensures the flatness of the 3D spatial slice. In this paper, we will impose this flat 3D spatial slice condition on all multi-centered non-BPS attractors we are looking for. They correspond to the multi-centered solutions that are directly "assembled" by single-centered attractors, and have properties similar to their single-centered constituents: they live in certain null totally geodesic sub-manifolds of $\mathcal{M}_{3D}$. We will discuss the relaxation of this condition at the end of the paper.

To summarize, the problem of finding 4D single-centered black hole attractors can be translated into finding appropriate null geodesics in $\mathcal{M}_{3D}$, and that of finding 4D multi-centered black hole bound states into finding corresponding 3D multi-centered solutions living in certain null totally geodesic sub-manifold of $\mathcal{M}_{3D}$.

The null geodesic that corresponds to a 4D black hole attractor is one that terminates at a point on the $U \to -\infty$ boundary and in the interior region with respect to all other coordinates of the moduli space $\mathcal{M}_{3D}$. However, it is difficult to find such geodesics since a generic null geodesic flows to the boundary of $\mathcal{M}_{3D}$. For BPS attractors, the termination of the null geodesic at its attractor point is guaranteed by the constraints imposed by supersymmetry. For non-BPS attractor, one need to find the constraints without the aid of supersymmetry. We will show that this can be done for models with $\mathcal{M}_{3D}$ that are symmetric coset spaces. Moreover, the method can be easily generalized to find the multi-centered non-BPS attractor solutions.

6.2.3 Models with $\mathcal{M}_{3D}$ Being Symmetric Coset Spaces

A homogeneous space $\mathcal{M}$ is a manifold on which its isometry group $\mathbf{G}$ acts transitively. It is isomorphic to the coset space $\mathbf{G}/\mathbf{H}$, with $\mathbf{G}$ being the isometry group and $\mathbf{H}$ the isotropy group. For $\mathcal{M}_{3D} = \mathbf{G}/\mathbf{H}$, $\mathbf{H}$ is the maximal compact subgroup of $\mathbf{G}$ when one compactifies on a spatial isometry down to $(1, 2)$ space, or the analytical continuation of the maximal compact subgroup when one compactifies on the time isometry down to $(0, 3)$ space.

The Lie algebra $\mathbf{g}$ has Cartan decomposition: $\mathbf{g} = \mathbf{h} \oplus \mathbf{k}$ where

$$[\mathbf{h}, \mathbf{h}] = \mathbf{h}, \qquad [\mathbf{h}, \mathbf{k}] = \mathbf{k}. \tag{6.11}$$

When $\mathbf{G}$ is semi-simple, the coset space $\mathbf{G}/\mathbf{H}$ is symmetric, meaning:

$$[\mathbf{k}, \mathbf{k}] = \mathbf{h}. \tag{6.12}$$

The building block of the non-linear σ-model with symmetric coset space $\mathcal{M}_{3D}$ as target space is the coset representative M, from which the left-invariant current is constructed

$$J = M^{-1}\mathrm{d}M = J_{\mathbf{k}} + J_{\mathbf{h}}, \tag{6.13}$$

where $J_{\mathbf{k}}$ is the projection of J onto the coset algebra $\mathbf{k}$. The lagrangian density of the σ-model with target space $\mathbf{G}/\mathbf{H}$ is then given by $J_{\mathbf{k}}$ as

$$\mathcal{L} = \mathrm{Tr}(J_{\mathbf{k}} \wedge *_3 J_{\mathbf{k}}). \tag{6.14}$$

The symmetric coset space has the nice property that its geodesics $M(\tau)$ are simply generated by exponentiation of the coset algebra **k**:

$$M(\tau) = M_0 e^{k\tau/2} \quad \text{with} \quad k \in \mathbf{k}, \tag{6.15}$$

where M_0 parameterizes the initial point of the geodesic, and the factor $\frac{1}{2}$ in the exponent is for later convenience. A null geodesic corresponds to $|k|^2 = 0$. Therefore, in the symmetric coset space $\mathcal{M}_{3D}$, the problem of finding the null geodesics that terminate at attractor points is translated into finding the appropriate constraints on the null elements of the coset algebra **k**.

The theories with 3D moduli spaces $\mathcal{M}_{3D}$ that are symmetric coset spaces include: D-dimensional gravity toroidally compactified to four dimensions, all 4D $\mathcal{N} > 2$ extended supergravities, and certain 4D $\mathcal{N} = 2$ supergravities coupled to vector-multiplets with cubic prepotentials. The entropies in the last two classes are U-duality invariant. In this talk, we will focus on the last class. The discussion on the first class can be found in [21].

6.2.3.1 Parametrization of $\mathcal{M}_{3D}$

The symmetric coset space $M_{3D} = \mathbf{G}/\mathbf{H}$ can be parameterized by exponentiation of the solvable subalgebra *solv* of **g**:

$$\mathcal{M}_{3D} = \mathbf{G}/\mathbf{H} = e^{solv} \qquad \text{with} \qquad \mathbf{g} = \mathbf{h} \oplus solv \tag{6.16}$$

The solvable subalgebra *solv* is determined via Iwasawa decomposition of **g**. Being semi-simple, **g** has Iwasawa decomposition: $\mathbf{g} = \mathbf{h} \oplus \mathbf{a} \oplus \mathbf{n}$, where **a** is the maximal abelian subspace of **k**, and **n** the nilpotent subspace of the positive root space Σ^+ of **a**. The solvable subalgebra $solv = \mathbf{a} \oplus \mathbf{n}$. Each point ϕ^n in $\mathcal{M}_{3D}$ corresponds to a solvable element $\Sigma(\phi) = e^{solv}$, thus the solvable elements can serve as coset representatives.

We briefly explain how to extract the values of moduli from the coset representative M. Since M is defined up to the action of the isotropy group **H**, we need to construct from M an entity that encodes the values of moduli in an **H**-independent way. The symmetric matrix S defined as

$$S \equiv M S_0 M^{\mathrm{T}} \tag{6.17}$$

has such a property, where S_0 is the signature matrix.[2] Moreover, as the isometry group **G** acts transitively on the space of matrices with signature S_0, the space of

[2] In all systems considered in the present work, the isotropy group **H** is the maximal orthogonal subgroup of **G**: $H S_0 H^{\mathrm{T}} = S_0$, for any $H \in \mathbf{H}$. Therefore, S is invariant under the **H**-action $M \to MH$.

possible S is the same as the symmetric coset space $\mathcal{M}_{3D} = \mathbf{G}/\mathbf{H}$. Therefore, we can read off the values of moduli from S in an $\mathbf{H}$-independent way.

The non-linear σ-model with target space $\mathcal{M}_{3D}$ can also be described in terms of S instead of M. First, the left-invariant current of S is $J_S = S^{-1}\mathrm{d}S$, which is related to $J_{\mathbf{k}}$ by

$$J_S = S^{-1}\mathrm{d}S = 2\left(S_0 M^{\mathrm{T}}\right)^{-1} J_{\mathbf{k}}\left(S_0 M^{\mathrm{T}}\right) \tag{6.18}$$

The lagrangian density in terms of S is thus $\mathcal{L} = \frac{1}{4}\mathrm{Tr}(J_S \wedge *_3 J_S)$. The equation of motion is the conservation of current:

$$\nabla \cdot J = \nabla \cdot (S^{-1}\nabla S) = 0, \tag{6.19}$$

where we have dropped the subscript S in J_S, since we will only be dealing with this current from now on.

6.2.4 *Example:* $n_{\mathrm{V}} = 1$

In this talk, we will perform the explicit computation only for the simplest case: 4D $\mathcal{N} = 2$ supergravity coupled to one vector-multiplet. The generalization to generic n_{V} is straightforward. The 3D moduli space $\mathcal{M}_{3D}$ for $n_{\mathrm{V}} = 1$ is an eight-dimensional quaternionic kähler manifold, with special holonomy $Sp(2,\mathbb{R}) \times Sp(4,\mathbb{R})$. Computing the killing symmetries of the metric (6.6) with $n_{\mathrm{V}} = 1$ shows that it is a coset space $G_{2(2)}/(SL(2,\mathbb{R}) \times SL(2,\mathbb{R}))$.[3] Figure 6.1 shows the root diagram of $G_{2(2)}$ in its Cartan decomposition.

The six roots on the horizontal and vertical axes $\{L_{\mathrm{h}}^{\pm}, L_{\mathrm{h}}^{3}, L_{\mathrm{v}}^{\pm}, L_{\mathrm{v}}^{3}\}$ generate the isotropy subgroup $\mathbf{H} = SL(2,\mathbb{R})_{\mathrm{h}} \times SL(2,\mathbb{R})_{\mathrm{v}}$. The two vertical columns of eight roots $a_{\alpha A}$ generate the coset algebra $\mathbf{k}$, with index α labeling a spin-1/2 representation of $SL(2,\mathbb{R})_{\mathrm{h}}$ and index A a spin-3/2 representation of $SL(2,\mathbb{R})_{\mathrm{v}}$.

The Iwasawa decomposition, $\mathbf{g} = \mathbf{h} \oplus solv$ with $solv = \mathbf{a} \oplus \mathbf{n}$, is shown in Fig. 6.2. The two Cartan generators $\{\mathbf{u}, \mathbf{y}\}$ form $\mathbf{a}$, while $\mathbf{n}$ is spanned by $\{\mathbf{x}, \boldsymbol{\sigma}, \mathbf{A}^0, \mathbf{A}^1, \mathbf{B}_1, \mathbf{B}_0\}$. $\{\mathbf{u}, \mathbf{y}\}$ generates the rescaling of $\{u, y\}$, where $u \equiv \mathrm{e}^{2U}$, and $\{\mathbf{x}, \boldsymbol{\sigma}, \mathbf{A}^0, \mathbf{A}^1, \mathbf{B}_1, \mathbf{B}_0\}$ generates the translation of $\{x, \sigma, A^0, A^1, B_1, B_0\}$ [27].

The moduli space $\mathcal{M}_{3D}$ can be parameterized by solvable elements:

$$\Sigma(\phi) = \mathrm{e}^{(\ln u)\mathbf{u}/2 + (\ln y)\mathbf{y}}\mathrm{e}^{x\mathbf{x} + A^I \mathbf{A}^I + B_I \mathbf{B}_I + \sigma\boldsymbol{\sigma}}. \tag{6.20}$$

The symmetric matrix S can then be expressed in terms of the eight moduli ϕ^n:

$$S(\phi) = \Sigma(\phi) S_0 \Sigma(\phi)^{\mathrm{T}}, \tag{6.21}$$

[3] Other work on this coset space has appeared recently, including [30–32].

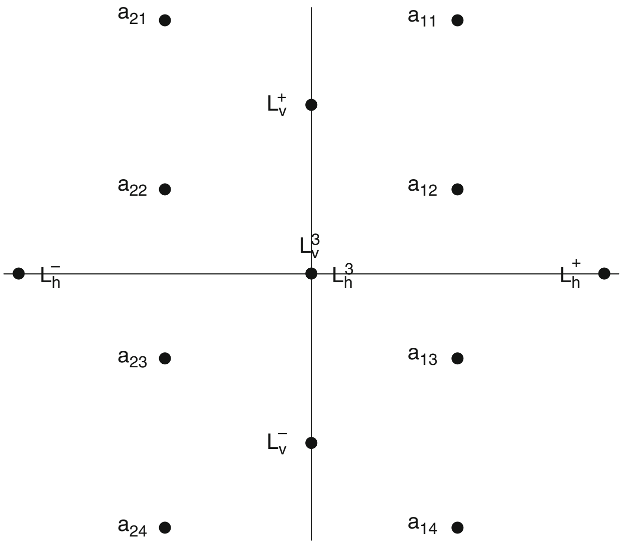

Fig. 6.1 Root diagram of $G_{2(2)}$ in Cartan decomposition

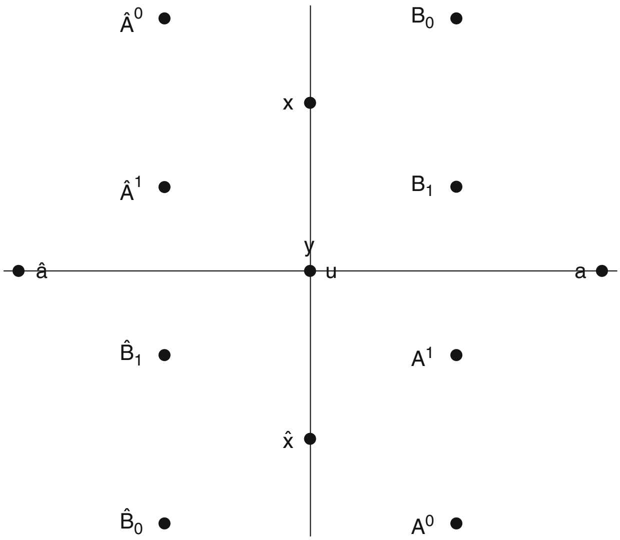

Fig. 6.2 Root diagram of $G_{2(2)}$ in Iwasawa decomposition. $\{\mathbf{u}, \mathbf{y}, \mathbf{x}, \boldsymbol{\sigma}, \mathbf{A}^0, \mathbf{A}^1, \mathbf{B}_1, \mathbf{B}_0\}$ generates the solvable subgroup *Solv*

which shows how to extract the values of moduli from S even when S is not constructed from the solvable elements, since it is invariant under **H**-action.

6.3 Generators of Attractor Flows

In this section, we will solve 3D attractor flow generators k as in (6.15). We will prove that extremality condition ensures that they are nilpotent elements of the coset algebra $\mathbf{k}$. In particular, for $n_V = 1$, both BPS and non-BPS generators are third-degree nilpotent. However, despite this common feature, k_{BPS} and k_{NB} differ in many aspects.

6.3.1 Construction of Attractor Flow Generators

6.3.1.1 Construction of k_{BPS}

Since the 4D BPS attractor solutions are already known, one can easily obtain the BPS flow generator k_{BPS} in the 3D moduli space $\mathcal{M}_{3D}$.

The generator k_{BPS} can be expanded by coset elements $a_{\alpha A}$ as $k_{BPS} = a_{\alpha A} C^{\alpha A}$, where $C^{\alpha A}$ are conserved along the flow. On the other hand, since the conserved currents in the homogeneous space are constructed by projecting the one-form valued Lie algebra $g^{-1} \cdot \mathrm{d}g$ onto $\mathbf{k}$, a procedure that also gives the vielbein: $J_{\mathbf{k}} = g^{-1}\mathrm{d}g|_{\mathbf{k}} = a_{\alpha A} V^{\alpha A}$, the vielbein $V^{\alpha A}$ are also conserved along the flow: $\frac{\mathrm{d}}{\mathrm{d}\tau}\left(V_n^{\alpha A}\dot{\phi}^n\right) = 0$. Since both the expansion coefficients $C^{\alpha A}$ and the vielbein $V^{\alpha A}$ transform as $(\mathbf{2}, \mathbf{4})$ of $SL(2,\mathbb{R})_h \times SL(2,\mathbb{R})_v$ and are conserved along the flow, they are related by

$$C^{\alpha A} = V_n^{\alpha A}\dot{\phi}^n \tag{6.22}$$

up to an overall scaling factor.

In terms of the vielbein $V^{\alpha A}$, the supersymmetry condition that gives the BPS attractors is $V^{\alpha A} = z^\alpha V^A$ [29, 32, 33]. Using (6.22), we conclude that the 3D BPS flow generator k_{BPS} has the expansion

$$k_{BPS} = a_{\alpha A} z^\alpha C^A. \tag{6.23}$$

A 4D supersymmetric black hole is labeled by four D-brane charges (p^0, p^1, q_1, q_0). A 3D attractor flow generator k_{BPS} has five parameters $\{C^A, z\}$. As will be shown later, z drops off in the final solutions of BPS attractor flows, under the zero NUT charge condition. Thus the geodesics generated by k_{BPS} are indeed in a four-parameter family.

k_{BPS} can be obtained by a twisting procedure as follows. First, define a k^0_{BPS} which is spanned by the four coset generators with positive charges under $SL(2,\mathbb{R})_h$:

$$k^0_{BPS} \equiv a_{1A} C^A \tag{6.24}$$

then, conjugate k^0_{BPS} with lowering operator L^-_{h}:

$$k_{\mathrm{BPS}} = \mathrm{e}^{-zL^-_{\mathrm{h}}} k^0_{\mathrm{BPS}} \mathrm{e}^{zL^-_{\mathrm{h}}}. \tag{6.25}$$

Using properties of k^0_{BPS}, it is easy to check that k_{BPS} is null:

$$|k_{\mathrm{BPS}}|^2 = 0. \tag{6.26}$$

More importantly, k_{BPS} is found to be third-degree nilpotent:

$$k^3_{\mathrm{BPS}} = 0. \tag{6.27}$$

A natural question then arises: Is the nilpotency condition of k_{BPS} a result of supersymmetry or extremality? If latter, we can use the nilpotency condition as a constraint to solve for the non-BPS attractor generators k_{NB}. We will prove that this is indeed the case.

6.3.1.2 Extremality Implies Nilpotency of Flow Generators

We will now prove that all attractor flow generators, both BPS and Non-BPS, are nilpotent elements in the coset algebra **k**. It is a result of the near-horizon geometry of extremal black holes.

The near-horizon geometry of a 4D attractor is $AdS_2 \times S^2$, i.e.

$$\mathrm{e}^{-U} \to \sqrt{V_{\mathrm{BH}}}|_* \tau \quad \text{as} \quad \tau \to \infty. \tag{6.28}$$

As the flow goes to the near-horizon, i.e., as $u = \mathrm{e}^{2U} \to 0$, the solvable element $M = \mathrm{e}^{(\ln u)\mathbf{u}/2+\cdots}$ is a polynomial function of τ:

$$M(\tau) \sim u^{-\ell/2} \sim \tau^{\ell}, \tag{6.29}$$

where $-\ell$ is the lowest eigenvalue of **u**.

On the other hand, since the geodesic flow is generated by $k \in \mathbf{k}$ via $M(\tau) = M_0 \mathrm{e}^{k\tau/2}$, $M(\tau)$ is an exponential function of τ. To reconcile the two statements, the attractor flow generator k must be nilpotent:

$$k^{\ell+1} = 0, \tag{6.30}$$

where the value of ℓ depends on the particular moduli space under consideration. In $G_{2(2)}/SL(2,\mathbb{R})^2$, by looking at the weights of the fundamental representation, we see that $\ell = 2$, thus

$$k^3 = 0. \tag{6.31}$$

The nilpotency condition of the flow generators also automatically guarantees that they are null:

$$k^3 = 0 \quad \Longrightarrow \quad (k^2)^2 = 0 \quad \Longrightarrow \quad \mathrm{Tr}(k^2) = 0. \tag{6.32}$$

6.3.1.3 Construction of k_{NB}

To construct non-BPS attractor flows, one needs to find third-degree nilpotent elements in the coset algebra **k** that are distinct from the BPS ones. In the real $G_{2(2)}/SL(2,\mathbb{R})^2$, there are two third-degree nilpotent orbits in total [34]. We have shown that $k_{\mathrm{BPS}} = \mathrm{e}^{-zL_{\mathrm{h}}^-} k^0_{\mathrm{BPS}} \mathrm{e}^{zL_{\mathrm{h}}^-}$, with k^0_{BPS} spanned by the four generators with positive charge under $SL(2,\mathbb{R})_{\mathrm{h}}$.

Since there are only two $SL(2,\mathbb{R})$'s inside **H**, a natural guess for k_{NB} is that it can be constructed by the same twisting procedure with $SL(2,\mathbb{R})_{\mathrm{h}}$ replaced by $SL(2,\mathbb{R})_{\mathrm{v}}$:

$$k_{\mathrm{NB}} = e^{-zL_{\mathrm{v}}^-} k^0_{\mathrm{NB}} e^{zL_{\mathrm{v}}^-} \qquad \text{with} \qquad k^0_{\mathrm{NB}} \equiv a_{\alpha a} C^{\alpha a}, \qquad \alpha, a = 1, 2, \tag{6.33}$$

where k^0_{NB} is spanned by the four generators with positive charge under $SL(2,\mathbb{R})_{\mathrm{v}}$.

Using properties of k^0_{NB}, one can easily show that k_{NB} defined above is indeed third-degree nilpotent:

$$k^3_{\mathrm{NB}} = 0 \tag{6.34}$$

That is, k_{NB} defined in (6.33) generates non-BPS attractor flows in $\mathcal{M}_{3D}$.

A 4D non-BPS extremal black hole is labeled by four D-brane charges (p^0, p^1, q_1, q_0). Similar to the BPS case, the 3D attractor flow generator k_{NB} has five parameters $\{C^{\alpha a}, z\}$. As will be shown later, z can be determined in terms of $\{C^{\alpha a}\}$ using the zero NUT charge condition, thus the geodesics generated by k_{NB} are also in a four-parameter family.

6.3.2 Properties of Attractor Flow Generators

We choose the representation of $G_{2(2)}$ group to be the symmetric 7×7 matrices that preserve a non-degenerate three-form w_{ijk} such that $\eta_{is} \equiv w_{ijk} w_{stu} w_{mno} \epsilon^{jktumno}$ is a metric with signature $(4, 3)$ and normalized as $\eta^2 = 1$. We decompose **7** as $\mathbf{3}\oplus\bar{\mathbf{3}}\oplus\mathbf{1}$ of $SL(3,\mathbb{R})$ and choose the non-zero components of w, $\mathbf{3}\wedge\mathbf{3}\wedge\mathbf{3}$, $\bar{\mathbf{3}}\wedge\bar{\mathbf{3}}\wedge\bar{\mathbf{3}}$ and $\mathbf{3}\otimes\bar{\mathbf{3}}\otimes\mathbf{1}$, as

$$w = \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3 + \mathrm{d}y^1 \wedge \mathrm{d}y^2 \wedge \mathrm{d}y^3 - \frac{1}{\sqrt{2}} \mathrm{d}x_a \wedge \mathrm{d}y^a \wedge \mathrm{d}z, \tag{6.35}$$

which gives $\eta = \mathrm{d}x_a \mathrm{d}y^a - \mathrm{d}z^2$. Written explicitly, an element of $G_{2(2)}$ Lie algebra is

$$g = \begin{pmatrix} A_{i_1}^{j_1} & \epsilon_{i_1 j_2 k} v^k & \sqrt{2} w_{i_1} \\ \epsilon^{i_2 j_1 k} w_k & -A_{j_2}^{i_2} & -\sqrt{2} v^{i_2} \\ -\sqrt{2} v^{j_1} & \sqrt{2} w_{j_2} & 0 \end{pmatrix}. \tag{6.36}$$

Here A is a traceless 3×3 matrix. The signature matrix S_0 is thus Diag $[1, -1, -1, 1, -1, -1, 1]$.

The real $G_{2(2)}$ group has two third-degree nilpotent orbits. In both orbits, k^2 is of rank 2 and has Jordan form with two blocks of size 3. Thus k^2 can be written as

$$k^2 = \sum_{a,b=1,2} v_a v_b^{\mathrm{T}} c_{ab} S_0 \tag{6.37}$$

with v_a null and orthogonal to each other: $v_a \cdot v_b \equiv v_a^{\mathrm{T}} S_0 v_b = 0$, and c_{ab} depends on the particular choice of k. Therefore, k can be expressed as

$$k = \sum_{a=1,2} \left(v_a w_a^{\mathrm{T}} + w_a v_a^{\mathrm{T}} \right) S_0, \tag{6.38}$$

where each w_a is orthogonal to both v_a: $w_a \cdot v_b = 0$, and w_a satisfies $w_a \cdot w_b = c_{ab}$. Next we solve for v_a and w_a for k_{BPS} and k_{NB} and compare their properties.

6.3.2.1 Properties of k_{BPS}

The null space of k^2 is five-dimensional, with v_a spanning its two-dimensional complement. For k_{BPS}, v_a^{BPS} and w_a^{BPS} in (6.38) are solved in terms of C^A and z.

In basis (6.36), from inspection of k_{BPS}^2, we find that v_a^{BPS} can always be chosen to have the form:[4]

$$v_1^{\mathrm{BPS}} = (V_1, -\eta_1 V_1, 0) \qquad v_2^{\mathrm{BPS}} = \left(-V_2, \eta_1 V_2, \sqrt{2} \right), \tag{6.39}$$

where η_1 is a 3D signature matrix $\eta = \mathrm{Diag}[1, -1, -1]$, and V_a are two three-vectors satisfying

$$V_1 \cdot V_1 = 0, \quad V_1 \cdot V_2 = 0, \quad V_2 \cdot V_2 = -1. \tag{6.40}$$

We drop the superscript "BPS" for V_a here since, as will be shown later, v_a^{NB} can also be written in terms of V_a, though in a slightly different form. Note that for k_{BPS}, V_2 is defined up to a shift of V_1: $V_2 \to V_2 - c V_1$, since any linear combination of v_a^{BPS} forms a new set of v_a^{BPS}.

[4] There are some freedom on the choice of (v_a, w_a): a rotational freedom: $(v_a, w_a) \to (R_{ab} v_b, R_{ab} w_b)$ with R orthogonal; and a rescaling freedom: $(v_a, w_a) \to (v_a r, w_a / r)$.

Written in twistor representation,[5] V_a are given by the twistors z and u as

$$V_1^{\alpha\beta} = 2z^\alpha z^\beta, \quad V_2^{\alpha\beta} = z^\alpha u^\beta + z^\beta u^\alpha, \tag{6.41}$$

where we have used the rescaling freedom to set $z^1 u^2 - z^2 u^1 = 1$. Note that for k_{BPS}, the twistor u is arbitrary, due to the shift freedom of V_2.

The condition $w_a^{\text{BPS}} \cdot v_b^{\text{BPS}} = 0$ dictates that w_a^{BPS} has the form:

$$w_1^{\text{BPS}} = \left(W_1^{\text{BPS}}, \eta_1 W_1^{\text{BPS}}, 0\right), \quad w_2^{\text{BPS}} = \left(W_2^{\text{BPS}}, \eta_1 W_2^{\text{BPS}}, 0\right) \tag{6.42}$$

with W_a^{BPS} solved as

$$\left(W_1^{\text{BPS}}, W_2^{\text{BPS}}\right)^{\alpha\beta} = \left(P^{\alpha\beta\gamma} u_\gamma, P^{\alpha\beta\gamma} z_\gamma\right), \tag{6.43}$$

where the totally symmetric $P^{\alpha\beta\gamma}$ is defined in terms of C^A as

$$P^{111} = C^1, \quad P^{112} = C^2, \quad P^{122} = C^3, \quad P^{222} = C^4. \tag{6.44}$$

In summary, v_a^{BPS} span a one-dimensional space (since u is arbitrary) and w_a^{BPS} span a four-dimensional space.

6.3.3 *Properties of* k_{NB}

$(v_a^{\text{NB}}, w_a^{\text{NB}})$ are solved in terms of $\{C^{\alpha a}, z\}$. The forms of v_a^{NB} are only slightly different from those of v_a^{BPS}:

$$v_1^{\text{NB}} = (V_1, \eta_1 V_1, 0), \quad v_2^{\text{NB}} = \left(V_2, -\eta_1 V_2, \sqrt{2}\right), \tag{6.45}$$

where V_a are the same three-vectors given in (6.41), with one major difference: the twistor u is no longer arbitrary, but is determined by $C^{\alpha a}$ as

$$u = \frac{u^2}{u^1} = \frac{C^{22}}{C^{12}} \tag{6.46}$$

[5] With the inner product of two three-vectors defined as $V_a \cdot V_b \equiv V_a^{\text{T}} \eta_1 V_b$, the twistor representation of a three-vector $V = (x, y, z)$ can be chosen as

$$\sigma_V = x\sigma_0 + y\sigma_3 + z\sigma_1 = \begin{pmatrix} x+y & z \\ z & x-y \end{pmatrix}.$$

since the V_2 in v_a^{NB} no longer has the shift freedom.

The forms of w_a^{NB} are also only slightly different from the BPS ones (6.42):

$$w_1^{\mathrm{NB}} = \left(W_1^{\mathrm{NB}}, -\eta_1 W_1^{\mathrm{NB}}, 0\right), \quad w_2^{\mathrm{NB}} = \left(W_2^{\mathrm{NB}}, \eta_1 W_2^{\mathrm{NB}}, 0\right) \tag{6.47}$$

with W_a^{NB} solved in terms of $\{C^{\alpha a}, z, u\}$ as

$$\left(W_1^{\mathrm{NB}}\right)^{\alpha\beta} = u^\alpha u^\beta + \left(C^{11}u^2 - C^{12}u^1\right) z^\alpha z^\beta, \tag{6.48}$$

$$\left(W_2^{\mathrm{NB}}\right)^{\alpha\beta} = \left(z^\alpha u^\beta + u^\alpha z^\beta\right) + \left(C^{21} - C^{11}z - 3u^1\right) z^\alpha z^\beta. \tag{6.49}$$

Since the value of u imposes an extra constraint on the vectors w_a^{NB} via (6.46), w_a^{NB} span a three-dimensional space instead of a four-dimensional one as in the BPS case (6.43). In summary, in contrast to the BPS case, v_a^{NB} span a two-dimensional space and w_a^{NB} span a three-dimensional one.

6.4 Single-Centered Attractor Flows

Having solved the attractor flow generators for both BPS and non-BPS case, we are ready to construct single-centered attractor flows. A geodesic starting from arbitrary asymptotic moduli is given by $M(\tau) = M_0 \mathrm{e}^{k\tau/2}$, which gives the flow of S as $S(\tau) = M_0 \mathrm{e}^{k\tau} S_0 M_0^{\mathrm{T}}$, which in turn can be written as $S(\tau) = \mathrm{e}^{K(\tau)} S_0$, where $K(\tau)$ is a matrix function. From now on, we use capital K to denote the matrix function which we exponentiate to generate attractor solutions.

The current of S is

$$J = S^{-1}\nabla S = S_0 \left(\nabla K + [\nabla K, K] + \frac{1}{2}[[\nabla K, K], K] + \cdots\right) S_0. \tag{6.50}$$

The equation of motion is the conservation of currents: $\nabla \cdot (S^{-1}\nabla S) = 0$, which is solved by $K(\tau)$ being harmonic:

$$\nabla^2 K(\tau) = 0, \quad \Longrightarrow \quad K(\tau) = k\tau + g. \tag{6.51}$$

g parameterizes the asymptotic moduli. Using the **H**-action, we can adjust g such that $g \in \mathbf{k}$, and g has the same properties as the flow generator k, namely, $g^3 = 0$ and g^2 is of rank 2. Therefore, for single-centered flow given by $S(\tau) = e^{K(\tau)} S_0$, the harmonic matrix function $K(\tau)$ has the same properties as the flow generator k:

$$K^3(\tau) = 0 \quad \text{and} \quad K^2(\tau) \text{ rank 2}. \tag{6.52}$$

To find the harmonic $K(\tau)$ that satisfies the constraints (6.52), recall that the constraints dictate $K(\tau)$ to have the form:

$$K(\tau) = \sum_{a=1,2} \left(v_a(\tau) w_a(\tau)^T + w_a(\tau) v_a(\tau)^T \right) S_0 \tag{6.53}$$

with $v_a(\tau)$ being null and $w_a(\tau)$ orthogonal to $v_b(\tau)$ for all τ. Then the constraints (6.52) can simply be solved by choosing $v_a(\tau)$ to be the constant null vectors $v_a(\tau) = v_a$ and $w_a(\tau)$ to be harmonic vectors which are everywhere orthogonal to v_b:

$$w_a(\tau) = w_a \tau + m_a \quad \text{with} \quad w_a \cdot v_b = m_a \cdot v_b = 0. \tag{6.54}$$

The two 7-vectors w_a's contain the information of the black hole charges, and the two 7-vectors m_a's contain that of asymptotic moduli.

To summarize, the single-centered attractor flow starting from an arbitrary asymptotic moduli is generated by $S(\tau) = e^{K(\tau)} S_0$, with harmonic matrix function $K(\tau) = k\tau + g$ where

$$k = \sum_{a=1,2} \left[v_a w_a^T + w_a v_a^T \right] S_0 \quad \text{and} \quad g = \sum_{a=1,2} \left[v_a m_a^T + m_a v_a^T \right] S_0. \tag{6.55}$$

Since k and g share the same set of null vectors v_a and both w_a and m_a are orthogonal to v_b, g has the same form as that of flow generator k, namely:

$$g_{\mathrm{BPS}} = a_{\alpha A} z^\alpha G^A, \quad g_{\mathrm{NB}} = \mathrm{e}^{-zL_{\mathrm{v}}^-} \left(a_{\alpha a} G^{\alpha a} \right) \mathrm{e}^{zL_{\mathrm{v}}^-}, \tag{6.56}$$

which guarantees that g is also third-degree nilpotent. Moreover, that g and k have the same form implies $[[k, g], g] = 0$, thus the current is reduced to

$$J = \frac{S_0(k + \frac{1}{2}[k, g]) S_0}{r^2} \hat{\boldsymbol{r}} \tag{6.57}$$

from which we can solve v_a and w_a in terms of charges and asymptotic moduli.

Now that we are able to construct arbitrary attractor flows in the 3D moduli space, we can lift them to the 4D black hole attractor solution. First, in representation given by (6.36), the 4D moduli $t = x + \mathrm{i}y$ can be extracted from the symmetric matrix S via

$$x(\tau) = -\frac{S_{35}(\tau)}{S_{33}(\tau)}, \quad y = \sqrt{\frac{S_{33}(\tau) S_{55}(\tau) - S_{35}(\tau)^2}{S_{33}(\tau)^2}} \tag{6.58}$$

and $u = \mathrm{e}^{2U}$ via

$$u = \frac{1}{\sqrt{S_{33}(\tau) S_{55}(\tau) - S_{35}(\tau)^2}}. \tag{6.59}$$

Since both k and g are third-degree nilpotent, $S(\tau)$ is a quadratic function of τ. Moreover, since g has the same form as k, $S(\tau)$ is composed of harmonic functions of τ: $H^A(\tau) \equiv C^A \tau + G^A$ for BPS attractors and $H^{\alpha a}(\tau) \equiv C^{\alpha a} \tau + G^{\alpha a}$ for non-BPS attractors.[6] Generic single-centered attractor flows with arbitrary charges and asymptotic moduli can thus be generated. The attractor moduli are read off from $S(\tau)$ with $\tau \to \infty$, and asymptotic moduli with $\tau \to 0$.

The D-brane charges can be read off from the charge matrix defined as $\mathbf{Q} \equiv \frac{1}{4\pi}\int \nabla \cdot J$. The 4D gauge currents sit in the current $J = S^{-1}\nabla S$ as

$$(J_{31}, J_{51}, J_{72}, J_{12}, J_{32}) = \left(\sqrt{2}J_{B_0}, -\sqrt{2}J_{B_1}, \frac{2}{3}J_{A^1}, \sqrt{2}J_{A^0}, -2J_\sigma\right). \quad (6.60)$$

Therefore $\mathbf{Q}$ relates to the D-brane charge (p^0, p^1, q_1, q_0) and the vanishing NUT charge a by

$$(\mathbf{Q}_{31}, \mathbf{Q}_{51}, \mathbf{Q}_{72}, \mathbf{Q}_{12}) = \left(\sqrt{2}p^0, -\sqrt{2}p^1, \frac{2}{3}q_1, \sqrt{2}q_0\right), \quad \mathbf{Q}_{32} = -2a = 0. \quad (6.61)$$

6.4.1 Single-Centered BPS Attractor Flows

As an example, a single-centered BPS black hole constructed by lifting the attractor solution in $\mathcal{M}_{3D}$ is shown in Fig. 6.3. It has D-brane charges $(p^0, p^1, q_1, q_0) = (5, 2, 7, -3)$. The four flows, starting from different asymptotic moduli, terminate at the attractor point $(x^*_{\text{BPS}}, y^*_{\text{BPS}})$ with different tangent directions. The reason is that the mass matrix of the black hole potential V_{BH} at the BPS critical point has two identical eigenvalues, thus there is no preferred direction for the geodesics to flow to the attractor point.

We now discuss in detail how to determine k_{BPS} and g_{BPS} for given charges and asymptotic moduli. There are nine parameters in k_{BPS} and g_{BPS}: $\{C^A, G^A, z\}$, since the twistor u is arbitrary. On the other hand, there are eight constraints in a given attractor flow: four D-brane charges (p^I, q_I), the vanishing NUT charge a, and the asymptotic moduli (x_0, y_0, u_0).[7] We will use these eight constraints to fix C^A and G^A in k_{BPS} and g_{BPS}, leaving the twistor z unfixed.

Integrating the current (6.57) for BPS case produces five coupled equations:

$$\mathbf{Q}_{\text{BPS}} = S_0\left(k_{\text{BPS}} + \frac{1}{2}[k_{\text{BPS}}, g_{\text{BPS}}]\right)S_0, \quad (6.62)$$

[6] Space prohibits listing the rather lengthy result of $S(\tau)$, readers can consult (6.2) and (6.3) of [21] for BPS attractors, and (6.28) for non-BPS ones.

[7] The asymptotic value of u can be fixed to an arbitrary value by a rescaling of time and the radial distance. We will set $u_0 = 1$.

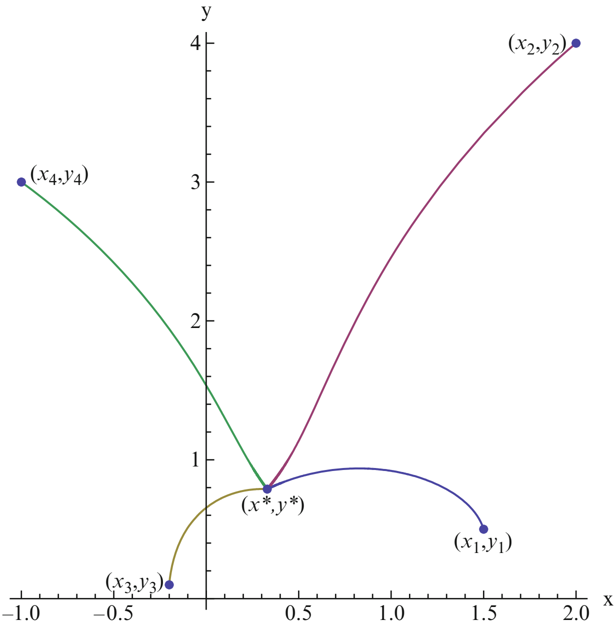

Fig. 6.3 BPS flow with charge $(p^0, p^1, q_1, q_0) = (5, 2, 7, -3)$ and attractor point $(x^*, y^*) = (0.329787, 0.788503)$. The initial points of each flow are given by $(x_1 = 1.5, y_1 = 0.5)$, $(x_2 = 2, y_2 = 4)$, $(x_3 = -0.2, y_3 = 0.1)$, $(x_4 = -1, y_4 = 3)$

where $[k_{\rm BPS}, g_{\rm BPS}] = \langle C, G\rangle \Theta$, with $\langle C, G\rangle \equiv C^1G^4 - 3C^2G^3 + 3C^3G^2 - C^4G^1$, and $\Theta \equiv -\frac{4}{1+z^2} \mathrm{e}^{-zL_{\rm h}^-} L_{\rm h}^+ \mathrm{e}^{zL_{\rm h}^-}$.

In order to show that the BPS flow can be expressed in terms of harmonic functions: $H(\tau) = Q\tau + h$, with $Q \equiv (p^I, q_I)$ and $h \equiv (h^I, h_I)$, we will solve $g_{\rm BPS}$ in terms of h instead of (x_0, y_0, u_0). h relates to the asymptotic moduli by

$$(x_0, y_0, u_0)_{\rm BPS} = (x, y, u)^*_{\rm BPS}(Q \to h) \tag{6.63}$$

and there is one extra degree of freedom to be fixed later.

First, for later convenience, we separate from $g_{\rm BPS}$ a piece that has the same dependence on (h, z) as $k_{\rm BPS}$ on (Q, z):

$$g_{\rm BPS} = g_{{\rm BPS},h} + \Lambda \quad \text{with} \quad g_{{\rm BPS},h} \equiv k_{\rm BPS}(Q \to h, z), \tag{6.64}$$

that is, $g_{{\rm BPS},h} = a_{\alpha A} z^\alpha G_{\rm h}^A$ with $G_{\rm h}^A \equiv C^A(Q \to h)$. We can use the unfixed degree of freedom in h to set $\langle C, G_h\rangle = 0$, so that (6.62) simplifies into

$$\mathbf{Q}_{\rm BPS} = S_0 \left(k_{\rm BPS} + \frac{1}{2} [k_{\rm BPS}, \Lambda] \right) S_0. \tag{6.65}$$

Λ can then be determined using the three constraints from (6.63) and the zero NUT charge condition in (6.65): $\Lambda = a_{\alpha A} z^{\alpha} E^A$ with $E^1 = -E^3 = -\frac{1}{1+z^2}$ and $E^2 = -E^4 = \frac{z}{1+z^2}$. The form of Λ will ensure that the twistor z drops off in the final attractor flow solution written in terms of Q and h.

The remaining four conditions in the coupled equations (6.65) determine C^A as functions of D-brane charges and the twistor z: $C^A = C^A(Q,z)$.[8] Then G_h^A are given by $G_h^A = C^A(Q \to h, z)$. The product $\langle C^A, G_h^A \rangle$ is proportional to the symplectic product of (p^I, q_I) and (h^I, h_I):

$$\langle C^A, G_h^A \rangle = \frac{2}{1+z^2} < Q, h >, \text{ where } < Q, h > \equiv p^0 h_0 + p^1 h_1 - q_1 h^1 - q_0 h^0. \tag{6.66}$$

The condition $\langle C^A, G_h^A \rangle = 0$ is then the integrability condition on h: $< Q, h >= 0$.

BPS attractor flows in terms of (p^I, q_I) and (h^I, h_I) are obtained by substituting solutions of $C^A(Q,z)$ and $G_h^A(h,z)$ into the flow of $S(\tau)$. The attractor moduli are determined by the charges as

$$x^*_{\text{BPS}} = -\frac{p^0 q_0 + p^1 \frac{q_1}{3}}{2\left[(p^1)^2 + p^0 \frac{q_1}{3}\right]}, \quad y^*_{\text{BPS}} = \frac{\sqrt{J_4(p^0, p^1, \frac{q_1}{3}, q_0)}}{2\left[(p^1)^2 + p^0 \frac{q_1}{3}\right]}, \tag{6.67}$$

where $J_4(p^0, p^1, q_1, q_0)$ is the quartic $E_{7(7)}$ invariant:

$$J_4(p^0, p^1, q_1, q_0) = 3(p^1 q_1)^2 - 6(p^0 q_0)(p^1 q_1) - (p^0 q_0)^2 - 4(p^1)^3 q_0 + 4p^0 (q_1)^3 \tag{6.68}$$

thus $J_4(p^0, p^1, \frac{q_1}{3}, q_0)$ is the discriminant of charges. Charges with positive (negative) $J_4(p^0, p^1, \frac{q_1}{3}, q_0)$ form a BPS (non-BPS) black hole. The attractor value of u is $u^*_{\text{BPS}} = 1/\sqrt{J_4(p^0, p^1, \frac{q_1}{3}, q_0)}$. The constraint on h from $u_0 = 1$ is then $J_4(h^0, h^1, \frac{h_1}{3}, h_0) = 1$. The attractor moduli (6.67) match those from Type II string compactified on diagonal T^6, with $q_1 \to \frac{q_1}{3}$.

Now we will prove that the BPS attractor flows constructed above can indeed be generated by the "naive" harmonic function procedure, namely, by replacing charges Q in the attractor moduli with the corresponding harmonic functions $Q\tau + h$. First, using the properties of Λ, the flow of $t = x + iy$ can be generated from the attractor moduli by replacing k_{BPS} with the harmonic function $k_{\text{BPS}}\tau + g_{\text{BPS},h}$:

$$t_{\text{BPS}}(\tau) = t^*_{\text{BPS}}\left(k_{\text{BPS}} \to k\tau + g_{\text{BPS},h}\right). \tag{6.69}$$

Then, since k_{BPS} and $g_{\text{BPS},h}$ share the same twistor z, this is equivalent to replacing C^A with harmonic functions $H^A(\tau) = C^A \tau + G_h^A$ while keeping the twistor z fixed

[8] See (6.18) of [21] for the full solutions.

$$t_{\mathrm{BPS}}(\tau) = t^*_{\mathrm{BPS}}\left(C^A \to C^A\tau + G^A_h, z\right). \tag{6.70}$$

Finally, since C^A is linear in Q and G^A_h linear in h, and since z drops off after plugging in the solutions $C^A(Q,z)$ and $G^A_h(h,z)$, we conclude that the flow of $t_{\mathrm{BPS}}(\tau)$ is given by replacing the charges Q in the attractor moduli with the corresponding harmonic functions $Q\tau + h$:

$$t_{\mathrm{BPS}}(\tau) = t^*_{\mathrm{BPS}}(Q \to Q\tau + h). \tag{6.71}$$

6.4.2 Single-Centered Non-BPS Attractor Flows

A non-BPS attractor flow with generic charges and asymptotic moduli can be generated using the method detailed earlier. Figure 6.4 shows an example of non-BPS attractor flow with charges $(p^0, p^1, q_1, q_0) = (5, 2, 7, 3)$. Note that $J_4(5, 2, 7/3, 3) < 0$, so this is indeed a non-BPS black hole.

Unlike the BPS attractor flows, all non-BPS flows starting from different asymptotic moduli reach the attractor point with the same tangent direction. The reason is,

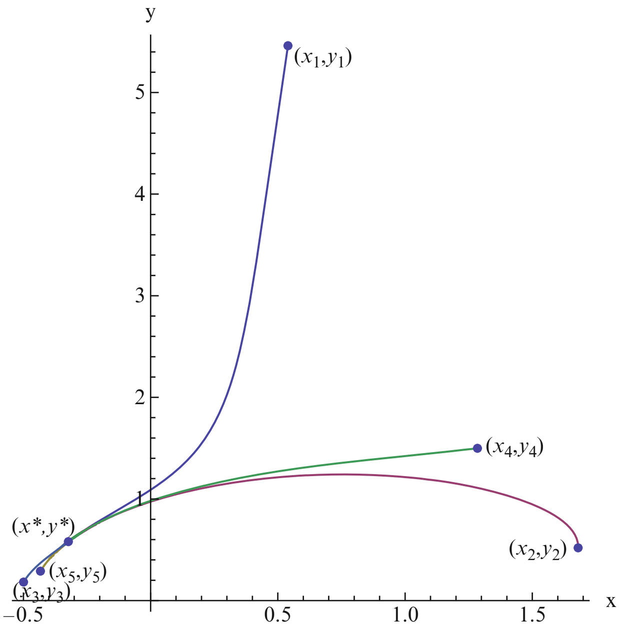

Fig. 6.4 Non-BPS flow with charges $(p^0, p^1, q_1, q_0) = (5, 2, 7, 3)$ and attractor point $(x^*, y^*) = (-0.323385, 0.580375)$. The initial points of each flow are given by: $(x_1 = 0.539624, y_1 = 5.461135)$, $(x_2 = 1.67984, y_2 = 0.518725)$, $(x_3 = -0.432811, y_3 = 0.289493)$, $(x_4 = 1.28447, y_4 = 1.49815)$, $(x_5 = -0.499491, y_5 = 0.181744)$

unlike the BPS case, the mass matrix of the black-hole potential V_{BH} at a non-BPS critical point has two different eigenvalues. The common tangent direction for the non-BPS flows corresponds to the eigenvector associated with the smaller mass.

Now we discuss how to determine k_{NB} and g_{NB} for given D-brane charges and asymptotic moduli. Unlike the BPS case, there are only eight parameters in k_{NB} and g_{NB}: the two twistors $\{z, u\}$ and $\{C^{\alpha a}, G^{\alpha a}\}$ under the constraints $u = \frac{C^{22}}{C^{12}} = \frac{G^{22}}{G^{12}}$. On the other hand, there are still eight constraints in a given non-BPS attractor flow as in the BPS case. Therefore, while k_{BPS} and g_{BPS} can parameterize black holes with arbitrary (p^I, q_I) and (x_0, y_0) while leaving $\{z, u\}$ free, all the parameters in k_{NB} and g_{NB}, including $\{z, u\}$, will be fixed.

Another major difference from the BPS case is that

$$[k_{\mathrm{NB}}, g_{\mathrm{NB}}] = 0 \tag{6.72}$$

guaranteed by the form of v_a^{NB} and w_a^{NB}. Thus the charge equation (6.62) becomes simply

$$\mathbf{Q}_{\mathrm{NB}} = S_0 (k_{\mathrm{NB}}) S_0. \tag{6.73}$$

Unlike the BPS case, g_{NB} does not enter the charge equations, thus cannot be used to eliminate the dependence on the twistor z. The three degrees of freedom in g_{NB} are simply fixed by the asymptotic moduli (x_0, y_0) and $u_0 = 1$, without invoking the zero NUT charge condition. The four D-brane charges equations in (6.73) determine $C^{\alpha a} = C^{\alpha a}(Q, z)$,[9] which then fixes u via $u = \frac{C^{22}}{C^{12}}$. Finally, the zero NUT charge condition imposes a degree-six equation on twistor z:

$$p^0 z^6 + 6p^1 z^5 - (3p^0 + 4q_1) z^4 - 4(3p^1 - 2q_0) z^3 + (3p^0 + 4q_1) z^2 + 6p^1 z - p^0 = 0. \tag{6.74}$$

Similar to the BPS case, the full non-BPS attractor flow can be generated from the attractor moduli by replacing $C^{\alpha a}$ with the harmonic function $H^{\alpha a}(\tau) = C^{\alpha a}\tau + G^{\alpha a}$, while keeping z fixed as in (6.70):

$$t_{\mathrm{NB}}(\tau) = t^*_{\mathrm{NB}}(C^{\alpha a} \to C^{\alpha a}\tau + G^{\alpha a}, z). \tag{6.75}$$

However, there are two important differences. First, the harmonic functions $H^{\alpha a}$ have to satisfy the constraint:[10]

$$\frac{H^{22}(\tau)}{H^{12}(\tau)} = u = \frac{C^{22}}{C^{12}} = \frac{G^{22}}{G^{12}}. \tag{6.76}$$

---

[9] See (6.35) of [21] for the full solution.

[10] This does not impose any constraint on the allowed asymptotic moduli since there are still three degrees of freedom in $G^{\alpha a}$ to account for (x_0, y_0, u_0). We will see later that its multi-centered counterpart helps impose a stringent constraint on the allowed D-brane charges in multi-centered non-BPS solutions.

Second, unlike the BPS flow, a generic non-BPS flow cannot be given by the "naive" harmonic function procedure:

$$t_{\rm NB}(\tau) \neq t^*_{\rm NB}(Q \to Q\tau + h). \tag{6.77}$$

The reason is that the twistor z in a non-BPS solution is no longer free as in the BPS case, but is determined in terms of D-brane charges via (6.74). Thus replacing Q with $Q\tau + h$, for generic Q and h, would not leave z invariant. That is, replacing $C^{\alpha a}$ in the attractor moduli with harmonic functions $H^{\alpha a}(\tau)$ is not equivalent to replacing the charges Q with $H = Q\tau + h$ as in the BPS case (6.71).

It is interesting to find the subset of non-BPS single-centered flows that *can* be constructed via the "naive" harmonic function procedure. The $n_{\rm V} = 1$ system can be considered as the STU model with the three moduli (S, T, U) identified. Since the STU model has an $SL(2,\mathbb{Z})^3$ duality symmetry at the level of E.O.M., the $n_{\rm V} = 1$ system has an $SL(2,\mathbb{Z})$ duality symmetry coming from identifying these three $SL(2,\mathbb{Z})$'s, namely, $\hat{\Gamma} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $ad - bc = 1$. The modulus $t = x + {\rm i}y$ transforms as $t \to \hat{\Gamma} t = \frac{at+b}{ct+d}$, and the transformation on the charges is given by [35].

Given an arbitrary charge Q, there exists a transformation $\hat{\Gamma}_Q$ such that $Q = \hat{\Gamma}_Q Q_{40}$ for some D4–D0 charge system $Q_{40} = (0, p^1, 0, q_0)$. The solution of (6.74) with charge $Q = \hat{\Gamma}_Q Q_{40}$ has a root $z = \frac{a \pm \sqrt{a^2+c^2}}{c}$, independent of Q_{40}. Thus for arbitrary $h_{40} = (0, h^1, 0, h_0)$, replacing Q with $Q\tau + \hat{\Gamma}_Q h_{40}$ would leave the twistor z invariant. We thus conclude that the non-BPS single-centered attractor flows that can be generated from their attractor moduli via the "naive" harmonic function procedure are only those with (Q, h) being the image of a single transformation $\hat{\Gamma}$ on a D4–D0 system (Q_{40}, h_{40}):

$$t_{\rm NB}(\tau) = t^*_{\rm NB}(\hat{\Gamma} Q_{40} \to \hat{\Gamma} Q_{40}\tau + \hat{\Gamma} h_{40}). \tag{6.78}$$

6.5 Multi-Centered Attractor Flows

Similar to the single-centered attractor solutions, the multi-centered ones are constructed by exponentiating harmonic matrix functions $K(\mathbf{x})$:

$$S(\mathbf{x}) = {\rm e}^{K(\mathbf{x})} S_0. \tag{6.79}$$

Recall that for single-centered attractors, using the $\mathbf{H}$-action on g, $K(\tau) = k\tau + g$ can be adjusted to have the same properties as the flow generator k as in (6.52). For BPS multi-centered solutions, supersymmetry guarantees that the matrix function $K(\mathbf{x})$ also has the same properties as the generator k:

$$K^3(\mathbf{x}) = 0, \quad \text{and} \quad K^2(\mathbf{x}) \text{ rank } 2. \tag{6.80}$$

We will impose these constraints on all non-BPS multi-centered solutions as well, since presently we are more interested in the multi-centered solutions that are "assembled" by individual single-centered attractors and thus have similar properties to their single-centered constituents. It is certainly interesting to see if there exist non-BPS multi-centered solutions with $K(\mathbf{x})$ not sharing the constraints (6.80) satisfied by the flow generator $k_{\rm NB}$.

The harmonic matrix function $K(\mathbf{x})$ satisfying all the above constraints is solved to be

$$K(\mathbf{x}) = \sum_i \frac{k_i}{|\boldsymbol{x} - \boldsymbol{x}_i|} + g, \tag{6.81}$$

where

$$k_i = \sum_{a=1,2} \left[v_a (w_a)_i^T + (w_a)_i v_a^T\right] S_0 \quad \text{and} \quad g = \sum_{a=1,2} \left[v_a m_a^T + m_a v_a^T\right] S_0 \tag{6.82}$$

with v_a being the same two constant null vectors in single-centered k, and the 7-vectors $(w_a)_i$ contain the information of the D-brane charges of center-i, and the two 7-vectors m_a's contain that of asymptotic moduli. Both $(w_a)_i$ and m_a are orthogonal to v_b. Since v_a only depends on the twistor $\{z, u\}$, and w_a are linear in C^A or $C^{\alpha a}$, the above generating procedure is equivalent to replacing C^A and $C^{\alpha a}$ with the multi-centered harmonic functions $H^A(\boldsymbol{x})$ and $H^{\alpha a}(\boldsymbol{x})$ while keeping the twistor $\{z, u\}$ fixed.

6.5.1 Multi-centered BPS Attractors

Using $\mathbf{Q}_i$ to denote the charge matrix of center-i, we have $5N$ coupled equations from $\mathbf{Q}_i = \frac{1}{4\pi}\int_i \nabla \cdot J$:

$$\mathbf{Q}_{\text{BPS},i} = S_0 \left(k_{\text{BPS},i} + \frac{1}{2}\left[k_{\text{BPS},i}, g_{\text{BPS}}\right] + \frac{1}{2}\sum_j \frac{\left[k_{\text{BPS},i}, k_{\text{BPS},j}\right]}{|\boldsymbol{x}_i - \boldsymbol{x}_j|} \right) S_0. \tag{6.83}$$

We now show in detail how to determine $k_{\text{BPS},i}$ and g_{BPS} for given charges and asymptotic moduli using (6.83). There are $4(N+1)+1$ parameters in $k_{\text{BPS},i}$ and g_{BPS}: $\{C_i^A, G^A, z\}$, since the twistor u is arbitrary. Different from the single-centered BPS case, there are also $3N-3$ degrees of freedom from the positions of centers on L.H.S. of (6.83). On the other hand, there are $5N+3$ constraints in a given BPS multi-centered attractor: $4N$ D-brane charges $(p_i^I, q_{I,i})$, N vanishing NUT charges, the asymptotic moduli (x_0, y_0) and $u_0 = 1$. We will use these $5N+3$ constraints to fix the $4(N+1)$ parameters $\{C_i^A, G^A\}$ in $k_{\text{BPS},i}$ and g_{BPS}, and impose $N-1$ constraints on the distances between the N centers, while leaving the twistor z free.

First, integrating $\nabla \cdot J$ over the sphere at the infinity gives the sum of the above N matrix equations: $\mathbf{Q}^{\text{tot}}_{\text{BPS}} = S_0(k^{\text{tot}}_{\text{BPS}} + \frac{1}{2}[k^{\text{tot}}_{\text{BPS}}, g_{\text{BPS}}])S_0$, which is the same as the charge equation for a single-center attractor with charge $Q_{\rm BPS}^{\rm tot}$. This determines g to be $g = g_h + \Lambda$, same as the single-centered case as in (6.64), using the three asymptotic moduli (x_0, y_0, u_0) and the constraint of zero total NUT charge. The h's are fixed by the asymptotic moduli and the integrability condition $< Q_{\rm BPS}^{\rm tot}, h >= 0$.

It is easy to see that the solutions of C_i^A are simply given by the single-centered solutions $C^A = C^A(Q, z)$ with Q replaced by Q_i. Thus the flow generator of each center $k_{{\rm BPS},i}$ (given by $k_{{\rm BPS},i} = a_{\alpha A} z^\alpha C_i^A$) satisfies

$$\mathbf{Q}_{{\rm BPS},i} = S_0(k_{{\rm BPS},i} + \frac{1}{2}[k_{{\rm BPS},i}, \Lambda])S_0, \tag{6.84}$$

which is the multi-centered generalization of the single-centered condition (6.65).

Using the solutions of $k_{{\rm BPS},i}$ and $g_{\rm BPS}$, the charge equations (6.83) become

$$\begin{aligned}\mathbf{Q}_{{\rm BPS},i} = S_0\Bigg(& k_{{\rm BPS},i} + \frac{1}{2}\,[k_{{\rm BPS},i}, \Lambda] \\ & + \left[< Q_{{\rm BPS},i}, h > + \sum_j \frac{< Q_{{\rm BPS},i}, Q_{{\rm BPS},j} >}{|\boldsymbol{x}_i - \boldsymbol{x}_j|}\right]\Theta\Bigg) S_0\end{aligned} \tag{6.85}$$

from which we subtract (6.84) to produce the integrability condition

$$< Q_{{\rm BPS},i}, h > + \sum_j \frac{< Q_{{\rm BPS},i}, Q_{{\rm BPS},j} >}{|\boldsymbol{x}_i - \boldsymbol{x}_j|} = 0. \tag{6.86}$$

The sum of the N equations in the integrability condition (6.86) reproduces the constraint on h: $< Q_{\rm BPS}^{\rm tot}, h > = 0$. Thus the remaining $N-1$ equations impose $N-1$ constraints on the relative positions between the N centers. The angular momentum $\mathbf{J}$, defined via $\omega_i = 2\epsilon_{ijk} J^j \frac{x^k}{r^3}$ as $r \to \infty$, is non-zero:

$$\mathbf{J} = \frac{1}{2}\sum_{i<j} \frac{\boldsymbol{x}_i - \boldsymbol{x}_j}{|\boldsymbol{x}_i - \boldsymbol{x}_j|}\langle Q_{{\rm BPS},i}, Q_{{\rm BPS},j}\rangle. \tag{6.87}$$

Thus we have shown that our multi-centered BPS attractor solutions reproduce those found in [19]. Same arguments as in the single-centered BPS case shows that multi-centered BPS attractors can be generated by replacing the charges in the attractor moduli with corresponding multi-centered harmonic functions:

$$t_{\rm BPS}(\mathbf{x}) = t^*_{\rm BPS}\left(Q_{\rm BPS} \to \sum_i \frac{Q_{{\rm BPS},i}}{|\boldsymbol{x} - \boldsymbol{x}_i|} + h\right). \tag{6.88}$$

6.5.2 Multi-centered Non-BPS Attractors

A multi-centered non-BPS attractor has $3(N+1)+2$ parameters inside its non-BPS generators $\{k_{NB,i}, g_{\rm NB}\}$: $\{C_i^{\alpha a}, G^{\alpha a}\}$ under the constraint (6.89) plus two twistors $\{z, u\}$. Given $\{k_{NB,i}, g_{\rm NB}\}$ in terms of $\{C_i^{\alpha a}, G^{\alpha a}, z, u\}$, the non-BPS multi-centered solution is the same as the single-centered one with $H^{\alpha a}(\tau)$ replaced by multi-centered harmonic functions $H^{\alpha a}(\boldsymbol{x}) = \sum_i \frac{C_i^{\alpha a}}{|\boldsymbol{x}-\boldsymbol{x}_i|} + G^{\alpha a}$ satisfying the constraint

$$u = \frac{H_i^{22}(\boldsymbol{x})}{H_i^{12}(\boldsymbol{x})} = \frac{C_i^{22}}{C_i^{12}} = \frac{G^{22}}{G^{12}}. \tag{6.89}$$

However, the process of determining $k_{NB,i}$ and $g_{\rm NB}$ in terms of charges and asymptotic moduli for a non-BPS multi-centered attractor is very different from its BPS counterpart.

The reason is that the charge equations for a non-BPS multi-centered solution simplifies a great deal since

$$[k_{NB,i}, k_{NB,j}] = 0 \quad \text{and} \quad [k_{NB,i}, g_{\rm NB}] = 0 \tag{6.90}$$

guaranteed by the forms of $(w_a^{\rm NB})_i$ and $m_a^{\rm NB}$. Therefore, the $5N$ equations (6.83) decouple into N sets of five coupled equations:

$$\mathbf{Q}_{NB,i} = S_0(k_{NB,i})S_0. \tag{6.91}$$

As in the single-centered non-BPS case, $g_{\rm NB}$ does not enter the charge equations (6.91), and its three degrees of freedom can be completely fixed by the given asymptotic moduli (x_0, y_0) and $u_0 = 1$ without using the zero NUT charge condition. More importantly, unlike BPS multi-centered solutions, the positions of centers $\mathbf{x}_i$ do not appear in the charge equations (6.91), thus receive no constraint: all centers are free. Finally, since we are using the remaining $3N + 2$ parameters $\{C^{\alpha a}, z, u\}$ to parameterize a N-centered attractor solution under $5N$ constraints coming from charge equations (6.91), there need to be $2N-2$ constraints imposed on the D-brane charges.

As in the BPS multi-centered attractors, solutions of $C_i^{\alpha a}$ are given by the single-centered non-BPS solutions $C^{\alpha a} = C^{\alpha a}(Q, z)$ with Q replaced by Q_i. The solutions of twistors z and u are the same as the single-centered ones with charges $Q_{\rm NB}$ replaces by $Q_{\rm NB}^{\rm tot}$. Among the aforementioned $2N-2$ constraints, $N-1$ come from demanding that all centers have the same twistor z, which follows from the zero NUT charge condition at each center, and the other $N-1$ come from demanding that they have the same twistor u as in (6.89). Solving these $2N-2$ constraints shows that all the charges $\{Q_{NB,i}\}$ are the image of a single duality transformation $\hat{\Gamma}$ on a multi-centered D4–D0 system $\{Q_{NB,40,i}\}$:

$$Q_{NB,i} = \hat{\Gamma}\, Q_{NB,40,i}. \tag{6.92}$$

The charges at different centers are all mutually local

$$\langle Q_{NB,i}, Q_{NB,j} \rangle = 0. \tag{6.93}$$

Like non-BPS single-centered attractors, the generic non-BPS multi-centered attractors cannot be generated via the "naive" harmonic function procedure, except for those with $\{Q_{NB,i}, h\}$ being the image of a single $\hat{\Gamma}$ on a pure D4–D0 system $\{Q_{NB,40,i}, h_{40}\}$:

$$t_{\mathrm{NB}}(\mathbf{x}) = t^*_{\mathrm{NB}} \left(\hat{\Gamma} Q_{NB,40} \to \sum_i \frac{\hat{\Gamma} Q_{NB,40,i}}{|\boldsymbol{x} - \boldsymbol{x}_i|} + \hat{\Gamma} h_{40} \right). \tag{6.94}$$

In summary, the non-BPS multi-centered attractors are drastically different from their BPS counterparts: there is no constraint imposed on the positions of the centers, but instead on the allowed charges $Q_{NB,i}$: they have to be mutually local. The result is that the centers can move freely, and there is no intrinsic angular momentum in the system.

6.6 Conclusion and Discussion

In this talk, we summarized the construction of generic single-centered and multi-centered extremal black hole solutions in theories whose 3D moduli spaces are symmetric coset spaces. In this construction, all attractors, both BPS and non-BPS, single-centered as well as multi-centered, are treated on an equal footing. The single-centered black hole attractors correspond to those null geodesics in $\mathcal{M}_{3D}$ that are generated by exponentiating appropriate nilpotent elements in the coset algebra. The multi-centered black hole attractors are given by 3D solutions that live in certain null totally geodesic sub-manifolds of $\mathcal{M}_{3D}$. The construction of multi-centered attractors, even that of non-BPS ones, is merely a straightforward generalization of the single-centered construction.

We presented a detailed computation in the theory of 4D $\mathcal{N}=2$ supergravity coupled to one vector-multiplet, whose 3D moduli space is the symmetric coset space $G_{2(2)}/SL(2,\mathbb{R})^2$. The attractor flow generators are third-degree nilpotent elements in the coset algebra. We explicitly constructed generic attractor solutions, both single-centered and multi-centered, and showed that while the BPS attractors can be generated from the attractor moduli via the "naive" harmonic function procedure, the generic non-BPS attractors cannot be generated this way.

In the $n_{\mathrm{V}}=1$ model, besides the BPS generator, there is only one extra third-degree nilpotent orbit to serve as non-BPS flow generators. Hence there is only one type of non-BPS single-centered attractor. In models with bigger symmetric moduli spaces, there should be more than one type of non-BPS generator. These would give rise to different types of non-BPS attractor flows, which might have different stability properties.

All multi-centered non-BPS attractors constructed in this work follow from the ansatz in which 3D gravity is assumed to decouple from the moduli. The multi-centered non-BPS black holes are found to be very different from their BPS counterparts: the charges of all centers are constrained to be mutually local, while the positions of centers are completely free. Thus the non-BPS multi-centered attractor is not a "bound state" and carries no intrinsic angular momentum.

We would like to construct true multi-centered non-BPS "bound states", i.e., solutions with constraints on the positions of centers but not on the charges. There are two possible ways to achieve this. First, one could adopt a more general ansatz in which 3D gravity is coupled to the moduli. For axisymmetric configurations, the inverse scattering method could be used to perform an exact analysis. One could also search in models with bigger moduli spaces. It is very likely that in bigger moduli spaces, there exist true multi-centered non-BPS "bound states" even within the ansatz with 3D gravity decoupled from moduli. We are also interested in the possibility of generating multi-centered non-BPS solutions with each center having different types of non-BPS generators k_{NB}.

Finally, with the hope of studying non-BPS extremal black holes in 4D $\mathcal{N}=2$ supergravity coupled to n_{V} vector-multiplets with more generic pre-potential, we would like to generalize our method to non-symmetric homogeneous spaces, and even to generic moduli spaces eventually.

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Chapter 7
Higher-Order String Effective Actions and Off-Shell $d = 4$ Supergravity

Filipe Moura

Abstract We study higher-derivative corrections to supergravity theories and their supersymmetrization, concentrating on theories in $d = 4$.

7.1 Introduction and Plan

Remarkable results have been achieved recently on black hole physics in string theory, among which the microscopic interpretation of the entropy and the attractor mechanism. Supersymmetry has played a crucial role in these results.

Black holes can appear already at the supergravity level, when string theories are compactified and (nonperturbative) p-branes are wrapped around nontrivial cycles of the compactification manifold. But black holes can also be formed from elementary perturbative string excitations; however, in this case the area of their horizons vanishes at the supergravity level (these are called small black holes). In order to prevent a naked singularity and get a finite horizon area, one needs to consider the effect of higher-order string corrections to supergravity. These terms appear in string theory effective actions as α' corrections, both at string tree level and higher string loops. They also affect classical black holes, since they introduce corrections to the supergravity equations of motion.

These are some reasons that motivate us to study higher-derivative corrections to supergravity theories and their supersymmetrization. This is what we do in the following, concentrating on theories in $d = 4$.

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We begin by reviewing four-dimensional superspace supergravity. We present curved superspace geometry, for arbitrary $\mathcal{N}$, including torsion, curvature and Bianchi identities. We motivate the choice of torsion constraints.

Next we move to the particular cases of $\mathcal{N} = 1, 2$. In both cases we show how Poincaré supergravity can be obtained from conformal supergravity by introducing a nonconformal constraint. We see how different choices of this nonconformal constraint lead to different versions of the Poincaré off-shell theory, with distinct compensating multiplets and sets of auxiliary fields. For those versions of $\mathcal{N} = 1, 2$ supergravities known as "old minimal", we present the solutions to the Bianchi identities, their field content and we show how to write superspace actions for these theories and their extensions using chiral densities and chiral projectors.

We then apply this formalism to the supersymmetrization of higher-derivative terms in $\mathcal{N} = 1, 2$ supergravities. As a concrete application, we study the supersymmetrization of $\mathcal{R}^4$ terms, which are required as string corrections to those theories. We write down the $\mathcal{R}^4$ terms which appear in the α'^3 type II and heterotic superstring effective actions. In $d = 4$ there are two of these terms. One of them is the square of the Bel-Robinson tensor. We work out its $\mathcal{N} = 1, 2$ supersymmetrizations, and we verify for both cases, with this term, that some auxiliary fields can be eliminated and some cannot. We identify these auxiliary fields and we interpret these results, which should be generalized to other supersymmetric higher-derivative terms, in terms of the breaking of conformal supergravity we discussed before.

The other $\mathcal{R}^4$ term cannot be directly supersymmetrized, as in $\mathcal{N} = 1$ it violates chirality. We show how to circumvent this problem in $\mathcal{N} = 1$ and we argue that it should not be possible in $\mathcal{N} = 8$.

We conclude by discussing possible applications of these results to open problems on black holes in string theory.

7.2 Superspace Geometry

7.2.1 Vielbein, Connection, Torsion and Curvature

Curved superspace is a manifold parameterized by the usual commuting x-space coordinates x^μ, plus a set of anticommuting spinorial coordinates, their number depending on the number of space-time dimensions in question and the number of supersymmetries $\mathcal{N}$. In four dimensions, we have

$$z^\Pi = \left(x^\mu, \theta^a_A, \theta^a_{\dot{A}}\right) \tag{7.1}$$

with $\mu = 0, \ldots, 3$, $A, \dot{A} = 1, 2$, $a = 1, \ldots, \mathcal{N}$.

Symmetries that are manifest in curved superspace are general supercoordinate transformations, with parameters ξ^Λ, and tangent space (structure group) transformations, with parameters Λ^{MN}. Curved superspace coordinates transform under general reparameterizations as

$$z^{\Pi} \to z'^{\Pi} = z^{\Pi} + \xi^{\Pi} \tag{7.2}$$

with $\xi^{\Pi} = \left(\xi^{\mu}, \xi^{a}_{A}, \xi^{a}_{\dot{A}}\right)$ defined as arbitrary functions of z^{Π}. ξ^{μ} corresponds to the usual x-space diffeomorphisms (Einstein transformations); $\xi^{a}_{A}, \xi^{a}_{\dot{A}}$ are their supersymmetric extension: the local supersymmetry transformations.

The main geometric objects of curved superspace are the supervielbein E^{M}_{Π} and the superconnection $\Omega^{P}_{\Lambda N}$. These objects transform under general supercoordinate transformations as

$$\delta E^{N}_{\Pi} = \xi^{\Lambda} \partial_{\Lambda} E^{N}_{\Pi} + \left(\partial_{\Pi} \xi^{\Lambda}\right) E^{N}_{\Lambda}, \tag{7.3}$$

$$\delta \Omega^{N}_{\Lambda M} = \xi^{\Pi} \partial_{\Pi} \Omega^{N}_{\Lambda M} + \left(\partial_{\Lambda} \xi^{\Pi}\right) \Omega^{N}_{\Pi M}. \tag{7.4}$$

The supervielbein relates the curved indices to the tangent space group ones, which we take to be $\mathrm{SO}(1,3) \times \mathrm{U}(\mathcal{N})$, with parameters $\Lambda_{MN} = (\Lambda_{mn}, \Lambda_{BbAa}, \Lambda_{\dot{B}b\dot{A}a})$. These parameters can still be decomposed in Lorentz and $\mathrm{U}(\mathcal{N})$ parts as

$$\Lambda_{BbAa} = \epsilon_{ba} \Lambda_{BA} + \epsilon_{BA} \widetilde{\Lambda}_{ba}, \; \Lambda_{\dot{B}b\dot{A}a} = \epsilon_{ba} \Lambda_{\dot{B}\dot{A}} + \epsilon_{\dot{B}\dot{A}} \widetilde{\Lambda}_{ba}, \tag{7.5}$$

satisfying

$$\Lambda_{BA} = \Lambda_{AB}, \; \Lambda_{\dot{B}\dot{A}} = \Lambda_{\dot{A}\dot{B}}, \; \Lambda_{A\dot{A}B\dot{B}} = 2\varepsilon_{\dot{A}\dot{B}} \Lambda_{AB} + 2\varepsilon_{AB} \Lambda_{\dot{A}\dot{B}} = -\Lambda_{B\dot{B}A\dot{A}}.$$

The $\mathrm{U}(\mathcal{N})$ parameters can still be decomposed into $\mathrm{SU}(\mathcal{N})$ and $\mathrm{U}(1)$ parts:

$$\widetilde{\Lambda}_{ba} = \Lambda_{ba} - \frac{1}{2} \epsilon_{ba} \Lambda, \; \Lambda^{a}_{a} = 0. \tag{7.6}$$

About our choice of structure group, two remarks must be made. Although the superconformal algebra is $\mathrm{SU}(2,2|\mathcal{N})$, the superspace we have introduced is perfectly adequate for the description of conformal supergravity. This is because from the additional parameters of $\mathrm{SU}(2,2|\mathcal{N})$, special conformal boosts get absorbed into general coordinate transformations, while Weyl (dilatations) and special supersymmetry transformations will appear as extra symmetries.

In principle we could have chosen some other structure group: if we wanted a superspace formulation that mimicked the x-space formulation of general relativity, the natural choice of structure group would rather contain the orthosymplectic group $\mathrm{OSp}(1,3|4)$ instead of the Lorentz group, but this would lead to problems. Indeed, any superspace formulation of supergravity requires the introduction of too many fields, through the supervielbeins and the superconnections. The gauge invariances of the theory allow one to eliminate some of the degrees of freedom, but that is still not enough. In order to have a plausible theory, in any superspace formulation one needs to put constraints on covariant objects, so that the excess of fields (some of them of spin exceeding two) can be eliminated. It can be shown (for instance, in [1]) that with such a choice of tangent group one would not be able to put an adequate set of constraints that could remove all the unwanted fields. The largest group that allows that set of constraints is precisely the one we took.

The supervielbein and superconnection transform under the structure group as

$$\delta E_{\Pi}^{N} = -E_{\Pi}^{M} \Lambda_{M}^{N}, \tag{7.7}$$

$$\delta \Omega_{\Lambda M}^{N} = -\partial_{\Lambda} \Lambda_{M}^{N} + \Omega_{\Lambda M}^{S} \Lambda_{S}^{N} + \Omega_{\Lambda R}^{N} \Lambda_{M}^{R} (-)^{(M+R)(N+R)}. \tag{7.8}$$

The superconnection is a structure algebra-valued (i.e., in the Lie algebra of the structure group) object, which can of course also be decomposed in its Lorentz and U($\mathcal{N}$) parts. Specifically, the Lorentz part $\Omega_{\Lambda M}^{Lor\,N}$ is written as

$$\Omega_{\Lambda M}^{Lor\,N} = \begin{pmatrix} \Omega_{\Lambda m}^{n} & 0 & 0 \\ 0 & -\frac{1}{4}\Omega_{\Lambda}^{mn} (\sigma_{mn})_{B}^{A} & 0 \\ 0 & 0 & \frac{1}{4}\Omega_{\Lambda}^{mn} (\sigma_{mn})_{\dot{B}}^{\dot{A}} \end{pmatrix}. \tag{7.9}$$

Having the superconnection, we define a supercovariant derivative:

$$D_{\Lambda} = \partial_{\Lambda} + \frac{1}{2}\Omega_{\Lambda}^{MN} J_{MN}, \ \nabla_{M} = E_{M}^{\Lambda} D_{\Lambda}. \tag{7.10}$$

J_{MN} are the generators of the structure group $((\sigma_{mn})_{B}^{A}, (\sigma_{mn})_{\dot{B}}^{\dot{A}}$ in the spinorial representation of the Lorentz group). We define the (super)torsions T_{MN}^{P} and (super)curvatures R_{MN}^{PQ} as

$$\begin{aligned} T_{MN}^{R} &= E_{M}^{\Lambda} \left(\partial_{\Lambda} E_{N}^{\Pi}\right) E_{\Pi}^{R} + \Omega_{MN}^{R} - (-)^{MN} (M \leftrightarrow N) \\ &= E_{M}^{\Lambda} \left(D_{\Lambda} E_{N}^{\Pi}\right) E_{\Pi}^{R} - (-)^{MN} (M \leftrightarrow N), \end{aligned} \tag{7.11}$$

$$\begin{aligned} R_{MN}^{RS} &= E_{M}^{\Lambda} E_{N}^{\Pi} \left\{\partial_{\Lambda} \Omega_{\Pi}^{RS} + \Omega_{\Lambda}^{RK} \Omega_{\Pi K}^{S} - (-)^{\Lambda\Pi} (\Lambda \leftrightarrow \Pi)\right\} \\ &= E_{M}^{\Lambda} E_{N}^{\Pi} \left\{D_{\Lambda} \Omega_{\Pi}^{RS} - (-)^{\Lambda\Pi} (\Lambda \leftrightarrow \Pi)\right\}. \end{aligned} \tag{7.12}$$

The curvatures are structure algebra-valued and, therefore, can also be decomposed in their Lorentz and U($\mathcal{N}$) parts. Because of (7.9), we have

$$\begin{aligned} R_{MNC\dot{C}D\dot{D}} &= 2\epsilon_{\dot{C}\dot{D}} R_{MNCD} + 2\epsilon_{CD} R_{MN\dot{C}\dot{D}}, \\ R_{MNmn} &= -\frac{1}{2}\sigma_{mn}^{CD} R_{MNCD} - \frac{1}{2}\sigma_{mn}^{\dot{C}\dot{D}} R_{MN\dot{C}\dot{D}}. \end{aligned} \tag{7.13}$$

From the definitions (7.10), (7.11) and (7.12) we have, for the supercommutator of covariant derivatives,

$$[\nabla_M, \nabla_N\} = T_{MN}^{R}\nabla_R + \frac{1}{2}R_{MN}^{RS}J_{RS}. \tag{7.14}$$

Torsions and curvatures satisfy Bianchi identities. One of the most important consequences of these identities is the fact that the curvatures can be expressed completely in terms of the torsions. This statement, known as Dragon's theorem [2], is also a consequence of the curvatures being Lie-algebra valued. This fact has no place in general relativity, where curvatures and torsions are independent, and one can constrain the torsion to vanish leaving a nonvanishing curvature. In superspace, the torsion is the main object determining the geometry. The curvature Bianchi identity is therefore redundant; all the information contained in it is also contained in the torsion Bianchi identity, which is written as

$$\begin{aligned}
&- (-)^{(M+N)R}\,\nabla_R T_{MN}^{F} + (-)^{(N+R)M}\,T_{NR}^{S}T_{SM}^{F} + (-)^{(N+R)M}\,R_{NRM}^{F}\\
&+ (-)^{MN}\,\nabla_N T_{MR}^{F} - (-)^{NR}\,T_{MR}^{S}T_{SN}^{F} - (-)^{NR}\,R_{MRN}^{F}\\
&- \nabla_M T_{NR}^{F} + T_{MN}^{S}T_{SR}^{F} + R_{MNR}^{F} = 0.
\end{aligned} \tag{7.15}$$

7.2.2 Variational Equations

Arbitrary variations of supervielbein and superconnection are given by [3]

$$H_M^{N} = E_M^{\Lambda}\delta E_{\Lambda}^{N},\ \Phi_{MN}^{P} = E_M^{\Lambda}\delta\Omega_{\Lambda N}^{P}. \tag{7.16}$$

From (7.11) and (7.16), we derive the arbitrary variation of the torsion:

$$\begin{aligned}
\delta T_{MN}^{R} = &-H_M^{S}T_{SN}^{R} + (-)^{MN}\,H_N^{S}T_{SM}^{R} + T_{MN}^{S}H_S^{R}\\
&-\nabla_M H_N^{R} + (-)^{MN}\,\nabla_N H_M^{R} + \Phi_{MN}^{R} - (-)^{MN}\,\Phi_{NM}^{R}.
\end{aligned} \tag{7.17}$$

By matching (7.16) to the variations under general coordinate and structure group transformations, one can solve for H_M^N and Φ_{MN}^P in terms of the transformation parameters, torsions and curvatures as

$$H_M^{N} = \xi^P T_{PM}^{N} + \nabla_M\xi^N + \Lambda_M^{N},\ \Phi_{MN}^{P} = \xi^Q R_{QMN}^{P} - \nabla_M\Lambda_N^{P}. \tag{7.18}$$

Until a gauge for the general coordinate and structure group transformations has not been fixed, any solution for H_M^N and Φ_{MN}^P is valid up to the transformations

$$\delta H_M^{N} = \nabla_M\tilde{\xi}^N - \xi^P T_{PM}^{N},\ \delta\Phi_{MN}^{P} = \tilde{\xi}^Q R_{QMN}^{P}, \tag{7.19}$$

$$\delta H_M^{N} = \widetilde{\Lambda}_M^{N},\ \delta\Phi_{MN}^{P} = \nabla_M\widetilde{\Lambda}_N^{P}. \tag{7.20}$$

Even fixing those gauges does not fix all the degrees of freedom of H_M^N [4–6]. Namely, $H = -\frac{1}{4}H_m^m$ remains an unconstrained superfield and parameterizes the super-Weyl transformations, which include the dilatations and the special supersymmetry transformations.

7.2.3 Choice of Constraints

As we previously mentioned, the superspace formulation of supergravity requires the introduction of too many fields, some of those having spins higher than 2. The only natural way to eliminate the undesired fields and get only those belonging to an irreducible representation of supersymmetry is to place constraints in the theory. Since those constraints should be valid in any frame of reference, they should be put only in covariant objects; and since, as we saw, we can express the curvatures in terms of the torsions, we choose to put the constraints in the torsions. Therefore, using the gauge freedom from (7.17), we analyze, from lower to upper dimensions, which torsions we can constrain.

At dimension 0, we have the torsion parts T_{AB}^{abm}, $T_{A\dot{B}}^{abm}$ and their complex conjugates. Considering the flat superspace limit for $T_{A\dot{B}}^{abm}$, we write

$$T_{A\dot{B}}^{abm} = -2i\varepsilon^{ab}\sigma_{A\dot{B}}^{m} + \tilde{T}_{A\dot{B}}^{abm}. \tag{7.21}$$

From (7.17), one finds [7] that the only parts of the torsion which cannot be absorbed by H_n^m, H_{aB}^{Ab}, $H_{a\dot{B}}^{Ab}$ and their complex conjugates are

$$\tilde{T}^{ab}_{A\dot{B}C\dot{C}} = \tilde{T}^{ab}_{\underline{A\dot{B}C\dot{C}}},\ T^{ab}_{ABC\dot{C}} = T^{\underline{ab}}_{\underline{ABC}\dot{C}}, \tag{7.22}$$

$\tilde{T}^{ab}_{\underline{A\dot{B}C\dot{C}}}$ being traceless in a, b. Since these fields have spin greater than two and therefore it would be impossible to describe any dynamics in their presence, we set them to zero:

$$\tilde{T}^{ab}_{\underline{A\dot{B}C\dot{C}}} = 0,\ T^{\underline{ab}}_{\underline{ABC}\dot{C}} = 0. \tag{7.23}$$

One must emphasize that these are the *only* constraints which have to be postulated (i.e., no other choice could be made to these specific parts of the torsion). All the other constraints are *conventional*, which means they must exist, but other choices could have been made. Conventional constraints correspond to redefinitions of the supervielbein and superconnection.

We are then left with

$$T_{A\dot{B}}^{abm} = -2i\varepsilon^{ab}\sigma_{A\dot{B}}^{m},\ T_{AB}^{abm} = 0. \tag{7.24}$$

As we will see, in $\mathcal{N} = 1, 2$ theories the constraint $T^{abm}_{AB} = 0$ has a geometrical meaning, and will be called "representation preserving". The constraint in $T^{abm}_{A\dot{B}}$ is just conventional.

At dimension $\frac{1}{2}$, it can be shown that, by adequate choices of the suitable parts of H^{M}_{N} and Φ^{P}_{MN} [7], we may set

$$T_{Aa\dot{B}b\dot{C}c} = 0,\ T_{AaBbCc} = 0,\ T^{amn}_{A} = 0. \tag{7.25}$$

At dimension 1, an appropriate redefinition of the Lorentz connection through an adequate choice of Φ^{p}_{mn} gives the usual constraint in Riemannian geometry

$$T^{p}_{mn} = 0. \tag{7.26}$$

Also, an adequate choice of Φ^{b}_{ma} allow us to constrain $R^{c\dot{C}}_{Ccab}$, and to have

$$T^{bCa}_{C\dot{C}B} = \beta T^{Cba}_{C\dot{C}\ \ B}. \tag{7.27}$$

This constraint establishes an identity between two a priori different superfields. The numerical parameter β depends on the choice that was made for $R^{c\dot{C}}_{Ccab}$, but it will have no impact on the theory, since this is a conventional constraint.

The Bianchi identities are valid, no matter which constraints we have. But once some of the torsions are constrained, the Bianchi identities become equations for the unconstrained torsions and curvatures. These equations are not independent, and need to be solved systematically. This has been achieved, in conformal supergravity, for arbitrary $\mathcal{N}$ [5]. One can conclude that off-shell conformal supergravity exists and is consistent for $\mathcal{N} \leq 4$. For $\mathcal{N} \geq 6$, an off-shell theory is not consistent [5, 6]. That does not rule out on-shell theories, but those have not been found. For $\mathcal{N} = 5$ nothing has been concluded. Thus for $\mathcal{N} > 4$ the situation is rather unclear. We will only review the $\mathcal{N} = 1, 2$ cases, because those are the ones we will need. For a more complete discussion the reader is referred to [6].

In $\mathcal{N} = 1, 2$ one can put chirality constraints in superfields. An antichiral superfield $\Phi_{...}$ satisfies

$$\nabla^{a}_{A}\Phi_{...} = 0 \tag{7.28}$$

(the hermitian conjugated equation defines a chiral superfield). This constraint on the superfield must be compatible with the solution to the Bianchi identities; an integrability condition must be verified (that is why general chiral superfields only exist for $\mathcal{N} = 1, 2$, as we will see; for other values of $\mathcal{N}$, a chirality condition may result only from the solution to the Bianchi identities, in the superfields introduced in this process).

$\mathcal{N} = 1, 2$ Poincaré supergravities can be obtained from the corresponding conformal theories by consistent couplings to compensating multiplets that break superconformal invariance and local U($\mathcal{N}$). There are different possible choices of compensating multiplets, leading to different formulations of the Poincaré theory. What is special about these theories is the existence of a completely off-shell formalism. This means that, for each of these theories, a complete set of auxiliary fields is known (actually, there exist three known choices for each theory). In superspace this means that, after imposing constraints on the torsions, we can completely solve the Bianchi identities without using the field equations [5,8], and there is a perfect identification between the superspace and x-space descriptions. We will review how is this achieved for the "old minimal" $\mathcal{N} = 1, 2$ cases.

7.3 $\mathcal{N} = 1$ Supergravity in Superspace

7.3.1 $\mathcal{N} = 1$ Superspace Geometry and Constraints

$\mathcal{N} = 1$ superspace geometry is a simpler particular case of the general $\mathcal{N}$ case we saw in the previous section. Namely, the internal group indices $a, b, \cdots$ do not exist. The structure group is at most $\mathrm{SO}(1,3) \times \mathrm{U}(1)$ (in the Poincaré theory we will consider, it is actually just the Lorentz group). To write any $\mathrm{U}(\mathcal{N})$ valued formula in the $\mathcal{N} = 1$ case, one simply has to decompose that formula under $\mathrm{U}(\mathcal{N})$ and take simply the group singlets.

Specific to $\mathcal{N} = 1, 2$ are the representation-preserving constraints, required by the above mentioned integrability condition for the existence of antichiral superfields, defined by (7.28). For $\mathcal{N} = 1$, these constraints are the following:

$$T_{AB}^{\dot{C}} = 0, \; T_{AB}^{m} = 0. \tag{7.29}$$

Conventional constraints allow us to express the superconnection in terms of the supervielbein. Namely, the constraint $T_{mnp} = 0$ allow us to solve for the bosonic connection Ω_{mn}^{p}, exactly as in general relativity. Constraints $T_{AB}^{C} = 0$ allow us to solve for Ω_{AB}^{C}, and $T_{A\underline{\dot{B}}}^{\underline{\dot{C}}} = 0$, for $\Omega_{A\dot{B}}^{\dot{C}}$. But in $\mathcal{N} = 1$ supergravity one can even go further, and solve for the supervielbein parts with bosonic tangent indices E_{n}^{Π} in terms of the other parts of the supervielbein. The conventional constraints that allow for that are $T_{A\dot{B}}^{m} = -2i\sigma_{A\dot{B}}^{m}$, $T_{A\dot{B}}^{\dot{C}} - \frac{1}{4} T_{A}^{mn} \left(\sigma_{mn}\right)_{\dot{B}}^{\dot{C}} = 0$.

In Sect. 7.2.3, we required a stronger constraint, which in $\mathcal{N} = 1$ language is written as $T_{A}^{mn} = 0$. We can still require that as a conventional constraint, if we take for structure group $\mathrm{SO}(1,3) \times \mathrm{U}(1)$. In the formulations in which $\mathrm{U}(1)$ is not gauged, only the constraints above are taken for the conformal theory, but an extra constraint will be necessary in order to obtain the Poincaré theory. We will analyze the possible cases next.

7.3.2 From Conformal to Poincaré Supergravity

To obtain $\mathcal{N} = 1$ Poincaré supergravity from conformal supergravity, we must adopt constraints which do not preserve the superconformal invariance. However, we must not break all superconformal invariance, since that would be equivalent to fixing all the superconformal gauges, and we would be left only with the fields which are inert under superconformal gauge choices, i.e., the fields of the Weyl multiplet e_μ^m, ψ_μ^A and A_μ. As we will see, this will be the case either with gauged or with ungauged U(1).

7.3.2.1 Ungauged U(1)

To determine the nonconformal constraints, we must first determine the transformation properties of the supervielbeins and superconnections.

In Lorentz superspace, the super Weyl parameter L is complex. We define

$$E'^{\Pi}_A = e^L E^{\Pi}_A, \; E'^{\Pi}_{\dot{A}} = e^{\overline{L}} E^{\Pi}_{\dot{A}}. \tag{7.30}$$

Since, with our choice of constraints, supervielbeins and superconnections can all be expressed in terms of the spinor vielbeins, we only need these transformation properties; conventional constraints are valid for any set of vielbeins and, therefore, they are automatically satisfied when one replaces E^{Π}_A, $E^{\Pi}_{\dot{A}}$ by their rescaled values. Then it can be proven [9] that the representation preserving constraints are invariant under (7.30). If these constraints were not invariant under Weyl transformations, then chiral multiplets could not exist in the background of conformal supergravity.

A complex scalar superfield can be decomposed in local superspace into chiral and linear parts. After breaking part of the super-Weyl group, the parameters $L, \overline{L}$ will be restricted such that a linear combination of them will be either chiral or linear. In the first case, one needs a dimension $\frac{1}{2}$ constraint; in this second, one of dimension 1. The only left unconstrained objects of dimension $\frac{1}{2}$ and 1 are, respectively, the torsion component T^m_{Am} and the superfield $R = R^{AB}_{AB}$. These superfields transform under the super-Weyl group as $\left(\nabla^2 = \nabla^A \nabla_A\right)$ [9, 10]

$$T'^m_{Am} = e^L \left(T^m_{Am} + 2\nabla_A \left(2L + \overline{L}\right)\right), \; R' = 3\left(\nabla^2 + \frac{1}{3}R\right) e^{2L}. \tag{7.31}$$

We can break the super Weyl invariance by imposing as a constraint

$$T^m_{Am} = 0. \tag{7.32}$$

For that to be consistent, we must impose that $2L + \overline{L}$ is antichiral:

$$\nabla_A \left(2L + \overline{L}\right) = 0. \tag{7.33}$$

What is left from the super-Weyl group is the so-called Howe–Tucker group [4]: the supervielbeins transforming as in (7.30), with $L, \overline{L}$ satisfying (7.33).

This constraint leads to the "old minimal" formulation of $\mathcal{N} = 1$ Poincaré supergravity [11, 12]. To the Weyl multiplet of conformal supergravity we are adding a compensating chiral multiplet with $8 + 8$ components.

Another possibility to break the super Weyl invariance is to set the constraint $R = 0$; the remaining super Weyl invariance contains a parameter L that now is an antilinear superfield: $\nabla^2 L = 0$. This constraint leads to the nonminimal formulation of $\mathcal{N} = 1$ Poincaré supergravity [13]. To the Weyl multiplet of conformal supergravity we are adding a compensating linear multiplet having $12 + 12$ components. This way, we have fermionic auxiliary fields.

Both constraints can be generalized. On dimensional grounds, the most general nonconformal constraint one may take is given by [9, 10]

$$C = -\frac{1}{3}R + \frac{n+1}{3n+1}\nabla^A T^m_{Am} - \left(\frac{n+1}{3n+1}\right)^2 T^{Am}_m T^n_{An} = 0. \tag{7.34}$$

n is a numerical parameter. This constraint transforms, for small L, as

$$\delta C = 2LC - 2\left(\nabla^2 - 2\frac{n+1}{3n+1}T^{Am}_m \nabla_A\right)\left(L - \frac{n+1}{3n+1}\left(2L + \overline{L}\right)\right). \tag{7.35}$$

For a generic choice of n, the constraint $R = 0$ is necessary and we have a nonminimal formulation. Taking $n = -\frac{1}{3}$ corresponds to the "old minimal" formulation we saw.

Another interesting case occurs by taking $n = 0$: only $L + \overline{L}$ appears in δC, such that the (axial) U(1) local gauge invariance, which we did not include in the structure group, is actually conserved, with parameter $L - \overline{L}$. This corresponds to the "new minimal" (also known as "axial") formulation of $\mathcal{N} = 1$ Poincaré supergravity [14], in which one introduces a compensating tensor multiplet having $8 + 8$ components.

Whichever constraint we choose, the irreducible parameter invariance of the resulting geometry corresponds to the compensating multiplet. This invariance allows for redefinition of torsions and, after gauge-fixing, for the fields of the compensating multiplet to appear in the final theory, with the original symmetry completely broken. These are very generical features, which we will also meet in the formulation of the $\mathcal{N} = 2$ theory.

7.3.2.2 Gauged U(1)

Let's now start from a $SO(1,3) \times U(1)$ superspace. From (7.8), the fermionic part of the U(1) connection transforms under U(1) as (A is a "flat" index):

$$\delta \Omega_A = -\nabla_A \Lambda - \Lambda \Omega_A \tag{7.36}$$

while, from (7.16), under a general transformation we have

$$\delta\Omega_A = \Phi_A - H_A^{\ M}\Omega_M. \tag{7.37}$$

In U(1) superspace, after fixing the constraints it can be shown [5] that one has $H_{AB} = \frac{1}{2}\varepsilon_{AB}H$, $\Phi_A = \frac{3}{2}\nabla_A H$, $H = -\frac{1}{4}H_m^{\ m}$ being an unconstrained superfield defined in Sect. 7.2.2 which parameterizes the super-Weyl transformations. Overall, Ω_A transforms as

$$\delta\Omega_A = \nabla_A\left(\frac{3}{2}H - \Lambda\right) + \left(\frac{1}{2}H - \Lambda\right)\Omega_A. \tag{7.38}$$

From this transformation law, by setting the constraint $\Omega_A = 0$, we see that we break the superconformal and local U(1) symmetries and restrict the combination $\frac{3}{2}H - \Lambda$ to a compensating chiral multiplet. This is the "old minimal" formulation of $\mathcal{N} = 1$ Poincaré supergravity [11, 12]. Other formulations have a treatment similar to the ungauged U(1) case. From now on, by $\mathcal{N} = 1$ Poincaré supergravity we always mean the "old minimal" formulation with $n = -1/3$.

7.3.3 The Chiral Compensator and the Chiral Measure

The superspace approach we have discussed has the inconvenience of involving a large number of fields and a large symmetry group. This way, one must put constraints and choose a particular gauge to establish the compatibility to the x-space theory (see Sect. 7.3.5). There is an approach which uses from the beginning fewer fields and a smaller symmetry group (holomorphic general coordinate transformations) [1, 9, 10, 15–17]. In this approach we take two chiral superspaces with complex coordinates (y^μ, θ), $(\overline{y^\mu}, \overline{\theta})$, which are related by complex conjugation. In four-component spinor notation, $\theta = \frac{1}{2}(1+\gamma_5)\Theta$, $\overline{\theta} = \frac{1}{2}(1-\gamma_5)\Theta$. One also has

$$\frac{1}{2}\left(y^\mu + \overline{y^\mu}\right) = x^\mu,\ \ y^\mu - \overline{y^\mu} = 2iH^\mu\left(x, \Theta\right). \tag{7.39}$$

This way, the imaginary part of the coordinates $y^\mu, \overline{y^\mu}$ is interpreted as an axial vector superfield, while the real part is identified with real spacetime. One has then in the combined $8 + 4$ dimensional space $(y^\mu, \overline{y^\mu}, \theta, \overline{\theta})$ a $4 + 4$ dimensional hypersurface defined by $y^\mu - \overline{y^\mu} = 2iH^\mu(y^\mu + \overline{y^\mu}, \theta, \overline{\theta})$. When one shifts points by a coordinate transformation, the hypersurface itself is deformed in such a way that the new points lie on the new hypersurface. These hypersurfaces, each characterized by the superfield $H^\mu(y^\mu + \overline{y^\mu}, \theta, \overline{\theta})$, represent each a real superspace like the one we have been working with.

The holomorphic coordinate transformations form a supergroup. If one puts no further restriction on their parameters, one is led to conformal supergravity. However, Poincaré supergravity is described by the very natural subgroup of unimodular holomorphic transformations, which satisfy

$$\text{sdet}\frac{\partial\left(y^{\mu\prime},\theta'\right)}{\partial\left(y^{\mu},\theta\right)}=1. \tag{7.40}$$

One can take Poincaré supergravity is a gauge theory with the gravitational superfield $H^{\mu}(x,\theta)$ as a dynamical object and the supergroup of holomorphic coordinate transformations being the gauge group [16]. But one can also remove the constraint (7.40) and handle arbitrary holomorphic transformations at the cost of the appearance of a compensating superfield. In the "old minimal" $n=-1/3$ theory, this superfield, which we define as $\varphi(y^{\mu},\theta)$, is holomorphic and is called the chiral compensator. It transforms as [9, 10]

$$\varphi\left(y^{\mu},\theta\right)=\left[\text{sdet}\frac{\partial\left(y^{\mu\prime},\theta'\right)}{\partial\left(y^{\mu},\theta\right)}\right]^{\frac{1}{3}}\varphi\left(y^{\mu\prime},\theta'\right). \tag{7.41}$$

One can then find a coordinate system in which $\varphi(y^{\mu},\theta)=1$. Clearly, all the holomorphic coordinate transformations preserving this gauge are unimodular; this way, we recover the gauge group of Poincaré supergravity. Poincaré supergravity is then a theory of two dynamical objects [15] – the gravitational superfield $H^{\mu}(x^{\mu},\theta,\overline{\theta})$ and the chiral compensator $\varphi(y^{\mu},\theta)$ – transforming under the supergroup of holomorphic coordinate transformations, and defined in real superspaces, given by the hypersurfaces above.

The chiral compensator allows us to define an invariant chiral measure in superspace. Since

$$d^4y\,d^2\theta=\text{sdet}\frac{\partial\left(y^{\mu},\theta\right)}{\partial\left(y^{\mu\prime},\theta'\right)}d^4y'd^2\theta', \tag{7.42}$$

we have

$$\varphi^3\left(y^{\mu},\theta\right)d^4y\,d^2\theta=\varphi'^3\left(y^{\mu\prime},\theta'\right)d^4y'd^2\theta'. \tag{7.43}$$

We define then the chiral density [3, 9, 10, 17] as

$$\epsilon=\varphi^3. \tag{7.44}$$

From the transformation law of φ, one can see that ϵ transforms under supercoordinate transformations with parameters ξ^{Λ} as

$$\delta\epsilon=-\partial_{\Lambda}\left(\epsilon\xi^{\Lambda}(-)^{\Lambda}\right). \tag{7.45}$$

Instead of choosing the gauge $\varphi\left(y^{\mu},\theta\right)=1$, it is more convenient to choose a Wess–Zumino gauge for H^{μ}, in which this superfield is expressed only in terms of the physical and auxiliary fields from the supergravity multiplet. After fixing the remaining gauge freedom, the same is valid for ϵ.

7.3.4 Solution to the Bianchi Identities in "old minimal" $\mathcal{N} = 1$ Poincaré Supergravity

The full off-shell solution to the Bianchi identities, given the representation-preserving and conventional constraints in Sect. 7.3.1 and the nonconformal constraint $T^m_{Am} = 0$, is standard textbook material which we do not include here [8, 18]. The results, in our conventions, may be seen in [19]. It can be shown that, as a result of $T^m_{Am} = 0$ and the conventional constraint $T^{\dot{C}}_{A\dot{B}} - \frac{1}{4} T^{mn}_{A} (\sigma_{mn})^{\dot{C}}_{\dot{B}} = 0$, one actually has simply $T^m_{Am} = 0$ and actually recovers the conventional constraint from the approach with gauged U(1).

The off-shell solutions are described in terms of the supergravity superfields $R = R^{AB}_{AB}$, G_m, W_{ABC}, their complex conjugates and their covariant derivatives. R and $W_{\dot{A}\dot{B}\dot{C}}$ are antichiral:

$$\nabla^A R = 0, \ \nabla_A W_{\dot{A}\dot{B}\dot{C}} = 0. \tag{7.46}$$

In $\mathcal{N} = 1$, chiral superfields may exist with any number of undotted indices (but no dotted indices). Chiral projectors exist; when acting with them on any superfield with only undotted indices, a chiral superfield always results. For scalar superfields the antichiral projector is given by $\left(\nabla^2 + \frac{1}{3}R\right)$.

The torsion constraints imply the following off-shell differential relations (not field equations) between the $\mathcal{N} = 1$ supergravity superfields:

$$\nabla^A G_{A\dot{B}} = \frac{1}{24} \nabla_{\dot{B}} R, \tag{7.47}$$

$$\nabla^A W_{ABC} = i \left(\nabla_{B\dot{A}} G^{\dot{A}}_{C} + \nabla_{C\dot{A}} G^{\dot{A}}_{B} \right), \tag{7.48}$$

which, together with the torsion conventional constraints, imply the relation

$$\nabla^2 \overline{R} - \overline{\nabla}^2 R = 96 i \nabla^n G_n. \tag{7.49}$$

7.3.5 From Superspace to x-Space

Another special feature of pure $\mathcal{N} = 1$ four-dimensional supergravity is that its action in superspace is known. It is written as the integral, over the whole superspace, of the superdeterminant of the supervielbein [1, 3]:

$$\mathcal{L}_{SG} = \frac{1}{2\kappa^2} \int E \, d^4\theta \,, E = \mathrm{sdet} E^M_{\Lambda}. \tag{7.50}$$

On dimensional grounds, this is the only possible action. The $\frac{1}{2\kappa^2}$ factor is necessary to reproduce the x-space results; in principle, one could multiply this action by any dimension zero unconstrained scalar, but that object does not exist. In this action, and in actions written as $d^4\theta$ integrals, the indices of the θ-variables are curved, i.e., they vary under Einstein transformations.

In order to determine the component expansion of this action, the best is certainly not to determine directly all the components of E, but rather to determine the component expansion of the supergravity superfields. For that, we use the method of gauge completion [18, 20]. The basic idea behind it is to relate in superspace some superfields and superparameters at $\theta = 0$ (which we symbolically denote with a vertical bar on the right) with some x space quantities, and then to require compatibility between the x space and superspace transformation rules [11, 12].

We make the following identification for the supervielbeins at $\theta = 0$ $E_{\Pi}^{N}\big|$:

$$E_{\Pi}^{N}\Big| = \begin{bmatrix} e_{\mu}^{m} & \frac{1}{2}\psi_{\mu}^{A} & \frac{1}{2}\psi_{\mu}^{\dot{A}} \\ 0 & \delta_{B}^{A} & 0 \\ 0 & 0 & \delta_{\dot{B}}^{\dot{A}} \end{bmatrix}. \tag{7.51}$$

In the same way, we gauge the fermionic superconnection at order $\theta = 0$ to zero and we can set its bosonic part equal to the usual spin connection:

$$\Omega_{\mu m}^{n}\Big| = \omega_{\mu m}^{n}(e, \psi), \; \Omega_{A m}^{n}\Big|, \; \Omega_{\dot{A} m}^{n}\Big| = 0. \tag{7.52}$$

The spin connection is given, in $\mathcal{N} = 1$ supergravity, by

$$\begin{aligned} \omega_{\mu m}^{n}(e, \psi) = \omega_{\mu m}^{n}(e) - \frac{i}{4}\kappa^2 \Big(& \psi_{\mu A}\sigma_{m}^{A\dot{A}}\psi_{\dot{A}}^{n} - \psi_{\mu A}\sigma^{nA\dot{A}}\psi_{m\dot{A}} + \psi_{mA}\sigma_{\mu}^{A\dot{A}}\psi_{\dot{A}}^{n} \\ & + \psi_{\mu\dot{A}}\sigma_{m}^{A\dot{A}}\psi_{A}^{n} - \psi_{\mu\dot{A}}\sigma^{nA\dot{A}}\psi_{mA} + \psi_{m\dot{A}}\sigma_{\mu}^{A\dot{A}}\psi_{A}^{n}\Big). \end{aligned} \tag{7.53}$$

$\omega_{\mu m}^{n}(e)$ is the connection from general relativity. We also identify, at the same order $\theta = 0$, the superspace vector covariant derivative (with an Einstein indice) with the curved space covariant derivative: $D_{\mu}\big| = \mathcal{D}_{\mu}$. These gauge choices are all preserved by supergravity transformations.

As a careful analysis using the solution to the Bianchi identities and the off-shell relations among the supergravity superfields $\overline{R}, G_n, W_{ABC}$ shows, the component field content of these superfields is all known once we know

$$\overline{R}\big|, \nabla_A\overline{R}\big|, \nabla^2\overline{R}\big|, G_{A\dot{A}}\big|, \nabla_{\underline{A}}G_{\underline{B}\dot{A}}\Big|, \nabla_{\underline{\dot{A}}}\nabla_{\underline{A}}G_{\underline{B}\underline{\dot{B}}}\Big|, W_{ABC}\big|, \nabla_{\underline{D}}W_{\underline{ABC}}\big|.$$

All the other components and higher derivatives of $\overline{R}, G_{A\dot{A}}, W_{ABC}$ can be written as functions of these previous ones. In order to determine the "basic" components, first we solve for superspace torsions and curvatures in terms of supervielbeins and superconnections using (7.51) and (7.52); then we identify them with the off-shell solution to the Bianchi identities [9, 18, 20]:[1]

$$\overline{R}\big| = 4\,(M + iN)\,,\ G_{A\dot{A}}\big| = \frac{1}{3}A_{A\dot{A}},\ W_{ABC}\big| = -\frac{1}{4}\psi_{\underline{A}}^{\ \dot{C}}{}_{\underline{BC}\underline{C}} - \frac{i}{4}A_{\underline{A}}^{\dot{C}}\,\psi_{\underline{B}\dot{C}\underline{C}}.$$

$R|$ and $G_m|$ are auxiliary fields. A_μ, a gauge field in conformal supergravity, is an auxiliary field in Poincaré supergravity. The (anti)chirality condition on $R, \overline{R}$ implies their $\theta = 0$ components (resp. the auxiliary fields $M - iN, M + iN$) lie in antichiral/chiral multiplets (the compensating multiplets); (7.47) shows the spin-1/2 parts of the gravitino lie on the same multiplets (because, as we will see in the next section, $\nabla_A G_{B\dot{B}}$, at $\theta = 0$, is the gravitino curl) and, according to (7.49), so does $\partial^\mu A_\mu$.

$\nabla_A \overline{R}\big|, \nabla_{\underline{A}} G_{\underline{B}\dot{A}}\big|$ also come straightforwardly from comparison to the solution to the Bianchi identities [9,19]. Finding $\nabla^2\overline{R}\big|, \nabla_{\underline{\dot{A}}}\nabla_{\underline{A}} G_{\underline{B}\dot{\underline{B}}}\big|, \nabla_{\underline{D}} W_{\underline{ABC}}\big|$ is a bit more involved: one must identify the (super)curvature $R_{\mu\nu}^{mn}$ with the x-space curvature $\mathcal{R}_{\mu\nu}^{mn}$, multiply by the inverse supervielbeins $E_M^\mu E_N^\nu$, identify with the solution to the Bianchi identities for R_{MN} and extract the field contents by convenient index manipulation. The field content of these components will include the Riemann tensor in one of its irreducible components, respectively the Ricci scalar, the Ricci tensor and the selfdual Weyl tensor ($\mathcal{W}_{ABCD} := -\frac{1}{8}\mathcal{W}^{+}_{\mu\nu\rho\sigma}\sigma^{\mu\nu}_{\underline{AB}}\sigma^{\rho\sigma}_{\underline{CD}}$, $\mathcal{W}^{\mp}_{\mu\nu\rho\sigma} := \frac{1}{2}(\mathcal{W}_{\mu\nu\rho\sigma} \pm \frac{i}{2}\varepsilon^{\lambda\tau}_{\mu\nu}\mathcal{W}_{\lambda\tau\rho\sigma})$). The full results are derived in [19]; at the linearized level,

$$\nabla^2\overline{R}\big| = -8\mathcal{R} + \ldots,\ \nabla_{\underline{A}}\nabla_{\underline{\dot{A}}} G_{\underline{B}\dot{\underline{B}}}\Big| = -\frac{1}{2}\sigma^{\mu}_{\underline{A\dot{A}}}\sigma^{\nu}_{\underline{B\dot{B}}}\left(\mathcal{R}_{\mu\nu} - \frac{1}{4}g_{\mu\nu}\mathcal{R}\right) + \ldots,$$

$$\nabla_{\underline{A}} W_{\underline{BCD}}\big| = -\frac{1}{8}\mathcal{W}^{+}_{\mu\nu\rho\sigma}\sigma^{\mu\nu}_{\underline{AB}}\sigma^{\rho\sigma}_{\underline{CD}} + \ldots,\ \nabla^2 W^2\big| = -2\mathcal{W}_{+}^2 + \ldots, \quad (7.54)$$

$$\nabla_{\underline{\dot{A}}} W_{\underline{\dot{B}\dot{C}\dot{D}}}\Big| = -\frac{1}{8}\mathcal{W}^{-}_{\mu\nu\rho\sigma}\sigma^{\mu\nu}_{\underline{\dot{A}\dot{B}}}\sigma^{\rho\sigma}_{\underline{\dot{C}\dot{D}}} + \ldots,\ \overline{\nabla}^2\overline{W}^2\Big| = -2\mathcal{W}_{-}^2 + \ldots \quad (7.55)$$

Knowing these components, we can compute, in x-space, any action which involves the supergravity multiplet. In order to do that, we need to know how to convert superspace actions to x-space actions.

Consider the coupling of a real scalar superfield to supergravity given by

$$\mathcal{L} = \frac{1}{2\kappa^2}\int E\Phi d^4\theta = \frac{3}{4\kappa^2}\int \left[\frac{E}{\overline{R}}\left(\overline{\nabla}^2 + \frac{1}{3}\overline{R}\right) + \frac{E}{R}\left(\nabla^2 + \frac{1}{3}R\right)\right]\Phi d^4\theta$$

$$= \frac{3}{4\kappa^2}\int\left(-\frac{1}{4}\frac{\overline{D}^2 E}{\overline{R}}\right)\left[\left(\overline{\nabla}^2 + \frac{1}{3}\overline{R}\right)\Phi\right] d^2\theta + \text{h.c.}. \quad (7.56)$$

[1] $\psi_{\mu\nu}^{B} = \mathcal{D}_\mu\psi_\nu^B - \mathcal{D}_\nu\psi_\mu^B$ is the gravitino curl.

$D_A = \left(E^{-1}\right)_A^{\ M} \nabla_M$ is the superspace covariant derivative with an Einstein index. In the previous equation, the operator $d^2\overline{\theta} = -\frac{1}{4}\overline{D}^2$ should apply to all the integrand, and not only to E. But, knowing that we can choose the gauge (7.51), we have $D_A| = \nabla_A|$ and therefore, to order $\theta = 0$, we have

$$D_{\dot{A}} \left[\frac{1}{\overline{R}} \left(\overline{\nabla}^2 + \frac{1}{3}\overline{R} \right) \right] \Big| = \nabla_{\dot{A}} \left[\frac{1}{\overline{R}} \left(\overline{\nabla}^2 + \frac{1}{3}\overline{R} \right) \right] \Big| = 0. \tag{7.57}$$

A "rigid" or "curved" superfield whose $\theta = 0$ component vanishes in any frame is identically zero (for a proof see [1]). Therefore, we conclude that we have $D_{\dot{A}} \left[\frac{1}{\overline{R}} \left(\overline{\nabla}^2 + \frac{1}{3}\overline{R} \right) \right] = 0$, and we may write (7.56).

In the particular gauge (7.51), we can write the chiral density (7.44) as

$$\epsilon = \frac{1}{4} \frac{\overline{D}^2 E}{\overline{R}}. \tag{7.58}$$

The proof of this fact requires the knowledge of the solution of the supergravity constraints in terms of unconstrained superpotentials [15]. Indeed, one of these prepotentials is identical to the chiral compensator. Equation (7.58) is obtained from expressing the supertorsions in terms of the prepotentials [9, 10].

The expansion in components of the chiral density is derived, in the same gauge, by requiring that $2\epsilon| = e$ and using its transformation law (7.45) [17]. In its expression, the θ-variables carry Lorentz indices. In these new θ-variables, the coefficients of the θ-expansion of chiral superfields are precisely their covariant derivatives [6, 18]. A chiral superfield has no $\overline{\theta}$'s in its expansion. This makes superspace integration much easier. For $\mathcal{N} = 1, 2$, when we write full superspace integrals the θ-variables carry Einstein indices, but when the integrals are in half superspace ($d^2\theta$ in $\mathcal{N} = 1$, $d^4\theta$ in $\mathcal{N} = 2$), they carry Lorentz indices. Therefore, one finally has for (7.56)

$$\mathcal{L} = -\frac{3}{4\kappa^2} \int \epsilon \left[\left(\overline{\nabla}^2 + \frac{1}{3}\overline{R} \right) \Phi \right] d^2\theta + \text{h.c.}. \tag{7.59}$$

By writing (7.56) on this form, one can identify the lagrangian of supergravity minimally coupled to a chiral field [18, 21]. The lagrangian of pure supergravity is simply obtained by taking $\Phi = 1$.

7.4 $\mathcal{N} = 2$ Supergravity in Superspace

7.4.1 $\mathcal{N} = 2$ Conformal Supergravity

The $\mathcal{N} = 2$ Weyl multiplet has 24 + 24 degrees of freedom. Its field content is given by the graviton $e_\mu^{\ m}$, the gravitinos $\psi_\mu^{\ Aa}$, the U(2) connection $\widetilde{\Phi}_\mu^{\ ab}$, an antisymmetric tensor W_{mn} which we decompose as $W_{A\dot{A}B\dot{B}} = 2\varepsilon_{\dot{A}\dot{B}} W_{AB} + 2\varepsilon_{AB} W_{\dot{A}\dot{B}}$, a spinor Λ^a_A and, as auxiliary field, a dimension 2 scalar I. In superspace, a gauge choice can be made (in the supercoordinate transformation) such that the graviton and the gravitinos are related to $\theta = 0$ components of the supervielbein (symbolically $E_\Pi^N|$):

$$E_\Pi^N\Big| = \begin{bmatrix} e_\mu^m & \frac{1}{2}\psi_\mu^{Aa} & \frac{1}{2}\psi_\mu^{\dot{A}a} \\ 0 & -\delta_B^A\delta_b^a & 0 \\ 0 & 0 & -\delta_{\dot{B}}^{\dot{A}}\delta_b^a \end{bmatrix}. \tag{7.60}$$

In the same way, we gauge the fermionic part of the Lorentz superconnection at order $\theta = 0$ to zero and we can set its bosonic part equal to the usual spin connection:

$$\Omega_{\mu m}^n\Big| = \omega_{\mu m}^n\left(e, \psi^a\right), \; \Omega_{Aam}^n\Big|, \; \Omega_{\dot{A}am}^n\Big| = 0. \tag{7.61}$$

The U(2) superconnection $\widetilde{\Phi}_\Pi^{ab}$ is such that $\widetilde{\Phi}_\mu^{ab}\Big| = \widetilde{\Phi}_\mu^{ab}$. The other fields are the $\theta = 0$ component of some superfield, which we write in the same way.

The chiral superfield W_{AB} is the basic object of $\mathcal{N} = 2$ conformal supergravity, in terms of which its action is written. Other theories with different $\mathcal{N}$ have an analogous superfield (e.g., W_{ABC} in $\mathcal{N} = 1$).

In U(2) $\mathcal{N} = 2$ superspace there is an off-shell solution to the Bianchi identities. The torsions and curvatures can be expressed in terms of superfields W_{AB}, Y_{AB}, $U_{A\dot{A}}^{ab}$, X_{ab}, their complex conjugates and their covariant derivatives. Of these four superfields, only W_{AB} transforms covariantly under super-Weyl transformations. The other three superfields transform non-covariantly; they describe all the non-Weyl covariant degrees of freedom in the transformation parameter H, and can be gauged away by a convenient (Wess–Zumino) gauge choice. Another nice feature of $\mathcal{N} = 2$ superspace is that there exists, analogously to the $\mathcal{N} = 1$ case, a chiral density ϵ which allows us to write chiral actions [22].

7.4.2 Degauging U(1)

The first step for obtaining the Poincaré theory is to couple to the conformal theory an abelian vector multiplet (with central charge), described by a vector A_μ, a complex scalar, a Lorentz-scalar SU(2) triplet and a spinorial SU(2) doublet. The vector A_μ is the gauge field of central charge transformations; it corresponds, in superspace, to a 1-form A_Π with a U(1) gauge invariance (the central charge transformation). This 1-form does not belong to the superspace geometry. Using the U(1) gauge invariance we can set the gauge $A_\Pi| = \left(A_\mu, 0\right)$. The field strength $F_{\Pi\Sigma}$ is a two-form defined as $F_{\Pi\Sigma} = 2D_{[\Pi} A_{\Sigma\}}$ or, with flat indices, $F_{MN} = 2\nabla_{[M} A_{N\}} + T_{MN}^P A_P$. It satisfies its own Bianchi identities $D_{[\Gamma} F_{\Pi\Sigma\}} = 0$ or, with flat indices,

$$\nabla_{[M} F_{NP\}} + T_{MN|}^Q F_{Q|P\}} = 0. \tag{7.62}$$

Here we split the U(2) superconnection $\widetilde{\Phi}_{\Pi}^{ab}$ into a SU(2) superconnection Φ_{Π}^{ab} and a U(1) superconnection φ_{Π}; only the later acts on A_{Π}: $\widetilde{\Phi}_{\Pi}^{ab} = \Phi_{\Pi}^{ab} - \frac{1}{2}\varepsilon^{ab}\varphi_{\Pi}$. One has to impose covariant constraints on its components (like in the torsions), in order to construct invariant actions:

$$F_{AB}^{ab} = 2\sqrt{2}\varepsilon_{AB}\varepsilon^{ab}F, \;\; F_{A\dot{B}}^{ab} = 0. \tag{7.63}$$

By solving the F_{MN} Bianchi identities with these constraints, we conclude that they define an off-shell $\mathcal{N} = 2$ vector multiplet, given by the $\theta = 0$ components of the superfields $A_\mu, F, F_A^a = \frac{i}{2}F_{A\dot{A}}^{\dot{A}a}, F_b^a = \frac{1}{2}\left(-\nabla_b^B F_B^a + F\overline{X}_b^a + \overline{F}X_b^a\right)$. $F_b^a\big|$ is an auxiliary field; $F_a^a = 0$ if the multiplet is abelian (as it has to be in this context). $\overline{F}$ is a Weyl covariant chiral superfield, with nonzero U(1) and Weyl weights. A superconformal chiral lagrangian for the vector multiplet is

$$\mathcal{L} = \int \overline{\epsilon}F^2 d^4\overline{\theta} + \text{h.c.}. \tag{7.64}$$

In order to get a Poincaré theory, we must break the superconformal and local abelian (from the U(1) subgroup of U(2) – not the gauge invariance of A_μ) invariances. For that, we set the Poincaré gauge $F = \overline{F} = 1$. As a consequence, from the Bianchi and Ricci identities we get

$$\varphi_A^a = 0, \;\; F_a^A = 0. \tag{7.65}$$

Furthermore, $U_{A\dot{A}}^{ab}$ is an SU(2) singlet, to be identified with the bosonic U(1) connection (now an auxiliary field):

$$U_{A\dot{A}}^{ab} = \varepsilon^{ab}U_{A\dot{A}} = \varepsilon^{ab}\varphi_{A\dot{A}}. \tag{7.66}$$

Other consequences are

$$F_{A\dot{A}B\dot{B}} = \sqrt{2}i\left[\varepsilon_{AB}\left(W_{\dot{A}\dot{B}} + Y_{\dot{A}\dot{B}}\right) + \varepsilon_{\dot{A}\dot{B}}\left(W_{AB} + Y_{AB}\right)\right], \tag{7.67}$$

$$F_b^a = X_b^a, \tag{7.68}$$

$$\overline{X_{ab}} = X^{ab}. \tag{7.69}$$

(7.67) shows that W_{mn} is now related to the vector field strength F_{mn}. Y_{mn} emerges as an auxiliary field, like X_{ab} (from (7.68)). We have, therefore, the minimal field representation of $\mathcal{N} = 2$ Poincaré supergravity, with a local SU(2) gauge symmetry and 32 + 32 off-shell degrees of freedom:

$$e_\mu^m, \psi_\mu^{Aa}, A_\mu, \Phi_\mu^{ab}, Y_{mn}, U_m, \Lambda_A^a, X_{ab}, I. \tag{7.70}$$

Although the algebra closes with this multiplet, it does not admit a consistent lagrangian because of the higher-dimensional scalar I [23].

7.4.3 Degauging SU(2)

The second step is to break the remaining local SU(2) invariance. This symmetry can be partially broken (at most, to local SO(2)) through coupling to a compensating so-called "improved tensor multiplet" [24, 25], or broken completely. We take the later possibility. There are still two different versions of off-shell $\mathcal{N} = 2$ supergravity without SO(2) symmetry, each with different physical degrees of freedom. In both cases we start by imposing a constraint on the SU(2) parameter L^{ab} which restricts it to a compensating nonlinear multiplet [26] (at the linearized level, $\nabla_A^{\underline{a}} L^{\underline{bc}} = 0$). From the transformation law of the SU(2) connection $\delta\Phi_M^{ab} = -\nabla_M L^{ab}$ we can get the required condition for L^{ab} by imposing the following constraint on the fermionic connection:

$$\Phi_A^{abc} = 2\varepsilon^{a\underline{b}}\rho_A^{\underline{c}}. \tag{7.71}$$

This constraint requires introducing a new fermionic superfield ρ_A^a. We also introduce its fermionic derivatives P and H_m. The previous SU(2) connection Φ_μ^{ab} is now an unconstrained auxiliary field. The divergence of H_m is constrained, though, at the linearized level by the condition $\nabla^m H_m = \frac{1}{3}R - \frac{1}{12}I$, which is equivalent to saying that I is no longer an independent field. This constraint implies that only the transverse part of H_m belongs to the nonlinear multiplet; its divergence lies in the original Weyl multiplet. From the structure equation (7.12) and the definition (7.71), we can derive off-shell relations for the (still SU(2) covariant) derivatives of ρ_A^a. Altogether, these component fields form then the "old minimal" $\mathcal{N} = 2$ $40 + 40$ multiplet [27]: $e_\mu^m, \psi_\mu^{Aa}, A_\mu, \Phi_\mu^{ab}, Y_{mn}, U_m, \Lambda_A^a, X_{ab}, H_m, P, \rho_A^a$. This is "old minimal" $\mathcal{N} = 2$ supergravity, the formulation we are working with. The final lagrangian can be found in [26, 28]. The other possibility (also with SU(2) completely broken) is to further restrict the compensating non-linear multiplet to an on-shell scalar multiplet [29]. This reduction generates a minimal $32 + 32$ multiplet (not to be confused with (7.70)) with new physical degrees of freedom. We will not further pursue this version of $\mathcal{N} = 2$ supergravity.

7.4.4 From $\mathcal{N} = 2$ SU(2) Superspace to x-Space

Our choices for torsion constraints in $\mathcal{N} = 2$ are very similar to the ones for generic $\mathcal{N}$ presented in Sect. 7.2.3, the only difference being that, like in $\mathcal{N} = 1$, we have the representation-preserving constraints $T_{AB}^{abm}, T_{AaBb\dot{C}c} = 0$. In conformal supergravity, all torsions and curvatures can be expressed in terms of the basic superfields $W_{AB}, Y_{AB}, U_{A\dot{A}}, X_{ab}$. After breaking of superconformal invariance and local U(2), the basic superfields in the Poincaré theory become the physical field W_{AB} and the auxiliary field ρ_A^a [30]. All torsions and curvatures can be expressed off-shell in terms of these superfields, their complex conjugates and derivatives [28]. $W_{AB}|$, at the linearized level, is related to the field strength of the physical vector field A_μ

(the graviphoton): from (7.67),

$$W_{AB}| = -\frac{i}{2\sqrt{2}}\sigma^{mn}_{AB}F_{mn} - Y_{AB} - \frac{i}{4}\sigma^{mn}_{AB}\left(\psi^{Cc}_{m}\psi_{nCc} + \psi^{\dot{C}c}_{m}\psi_{n\dot{C}c}\right). \quad (7.72)$$

$X^{ab} = \frac{1}{2}\left(\nabla^{\dot{A}\underline{a}} - 2\rho^{\dot{A}\underline{a}}\right)\rho^{\underline{b}}_{\dot{A}}$, $Y_{AB} = -\frac{i}{2}\left(\nabla^{a}_{\underline{A}} + 2\rho^{a}_{\underline{A}}\right)\rho_{\underline{B}a}$, $P = i\nabla^{\dot{A}a}\rho_{\dot{A}a}$, $U_{A\dot{A}} = \frac{1}{4}\left(\nabla^{a}_{A}\rho_{\dot{A}a} + \nabla^{a}_{\dot{A}}\rho_{Aa} + 4\rho^{a}_{A}\rho_{\dot{A}a}\right)$, $\Phi^{ab}_{A\dot{A}} = \frac{i}{2}\left(\nabla^{\underline{a}}_{A}\rho^{\underline{b}}_{\dot{A}} - \nabla^{\underline{a}}_{\dot{A}}\rho^{\underline{b}}_{A} - 4\rho^{\underline{a}}_{A}\rho^{\underline{b}}_{\dot{A}}\right)$, $H_{A\dot{A}} = -i\nabla^{a}_{A}\rho_{\dot{A}a} + i\nabla^{a}_{\dot{A}}\rho_{Aa}$, $\Lambda^{Aa} = -i\nabla^{A}_{b}X^{ab}$ are auxiliary fields at $\theta = 0$; $I = i\nabla^{\dot{A}a}\Lambda_{\dot{A}a} - i\nabla^{Aa}\Lambda_{Aa}$ is a dependent field. In the linearized approximation,

$$\begin{aligned}
W_{BCAa}| &= \frac{i}{2}\nabla_{Ba}W_{CA}\,| - \frac{i}{6}\left(\varepsilon_{BC}\Lambda_{Aa} + \varepsilon_{BA}\Lambda_{Ca}\right)\Big| = -\frac{1}{4}\psi_{ABCc} + \dots,\\
Y_{BC\dot{A}a}| &= -\frac{i}{2}\nabla_{\dot{A}a}Y_{BC}\Big| = -\frac{1}{8}\psi_{BC\dot{A}a} + \dots,\\
W_{ABCD}| &= \left(\frac{i}{4}\nabla^{b}_{\underline{A}}\nabla_{\underline{B}b} - 2Y_{\underline{AB}}\right)W_{\underline{CD}}\Big| = -\frac{1}{8}\mathcal{W}^{+}_{\mu\nu\rho\sigma}\sigma^{\mu\nu}_{\underline{AB}}\sigma^{\rho\sigma}_{\underline{CD}} + \dots, \quad (7.73)\\
P_{AB\dot{A}\dot{B}}| &= \left(\frac{i}{8}\nabla^{b}_{\underline{A}}\nabla_{\underline{B}b}Y_{\dot{A}\dot{B}} + \text{h.c.}\right)\Big|\dots = \frac{1}{2}\sigma^{\mu}_{\underline{A\dot{C}}}\sigma^{\nu}_{\underline{B\dot{D}}}\left(\mathcal{R}_{\mu\nu} - \frac{1}{4}g_{\mu\nu}\mathcal{R}\right)\dots,\\
R| &= \left(\frac{i}{4}\,\nabla^{\dot{A}a}\nabla^{\dot{B}}_{a}W_{\dot{A}\dot{B}} - \frac{1}{4}\nabla^{Aa}\nabla^{b}_{A}X_{ab} + \text{h.c.}\right)\Big| + \dots = -\mathcal{R} + \dots
\end{aligned}$$

7.4.5 The Chiral Density and the Chiral Projector

The action of $\mathcal{N} = 2, d = 4$ Poincaré supergravity is written in superspace as

$$\mathcal{L}_{SG} = -\frac{3}{4\kappa^2}\int \overline{\epsilon}d^4\overline{\theta} + \text{h.c.}. \quad (7.74)$$

The expansion of the chiral density ϵ in components, which allows us to write chiral actions, can be seen in [28]. From the solution to the Bianchi identities one can check that the following object is an antichiral projector [6]:

$$\nabla^{Aa}\nabla^{b}_{A}\left(\nabla^{B}_{a}\nabla_{Bb} + 16X_{ab}\right) - \nabla^{Aa}\nabla^{B}_{a}\left(\nabla^{b}_{A}\nabla_{Bb} - 16iY_{AB}\right). \quad (7.75)$$

When one acts with this projector on any scalar superfield, one gets an antichiral superfield (with the exception of W_{AB}, only scalar chiral superfields exist in curved $\mathcal{N} = 2$ superspace; other types of chiral superfields are incompatible with the solution to the Bianchi identities). Together with ϵ, this projector allows us to write more general actions in superspace.

7.5 Superstring α'^3 Effective Actions and $\mathcal{R}^4$ Terms in $d = 4$

In $d = 4$, there are only two independent real scalar polynomials made from four powers of the Weyl tensor [31], given by

$$\mathcal{W}_+^2 \mathcal{W}_-^2 = \mathcal{W}^{ABCD} \mathcal{W}_{ABCD} \mathcal{W}^{\dot{A}\dot{B}\dot{C}\dot{D}} \mathcal{W}_{\dot{A}\dot{B}\dot{C}\dot{D}}, \tag{7.76}$$

$$\mathcal{W}_+^4 + \mathcal{W}_-^4 = \left(\mathcal{W}^{ABCD} \mathcal{W}_{ABCD}\right)^2 + \left(\mathcal{W}^{\dot{A}\dot{B}\dot{C}\dot{D}} \mathcal{W}_{\dot{A}\dot{B}\dot{C}\dot{D}}\right)^2. \tag{7.77}$$

We now write the effective actions for type IIB, type IIA and heterotic superstrings in $d = 4$, after compactification from $d = 10$ in an arbitrary manifold, in the Einstein frame (considering only terms which are simply powers of the Weyl tensor, without any other fields except their couplings to the dilaton, and introducing the $d = 4$ gravitational coupling constant κ):

$$\frac{\kappa^2}{e}\mathcal{L}_{\mathrm{IIB}}\bigg|_{\mathcal{R}^4} = -\frac{\zeta(3)}{32} e^{-6\phi} \alpha'^3 \mathcal{W}_+^2 \mathcal{W}_-^2 - \frac{1}{2^{11}\pi^5} e^{-4\phi} \alpha'^3 \mathcal{W}_+^2 \mathcal{W}_-^2, \tag{7.78}$$

$$\frac{\kappa^2}{e}\mathcal{L}_{\mathrm{IIA}}\bigg|_{\mathcal{R}^4} = -\frac{\zeta(3)}{32} e^{-6\phi} \alpha'^3 \mathcal{W}_+^2 \mathcal{W}_-^2 - \frac{1}{2^{12}\pi^5} e^{-4\phi} \alpha'^3 \left[\left(\mathcal{W}_+^4 + \mathcal{W}_-^4\right) + 224 \mathcal{W}_+^2 \mathcal{W}_-^2\right], \tag{7.79}$$

$$\frac{\kappa^2}{e}\mathcal{L}_{\mathrm{het}}\bigg|_{\mathcal{R}^2+\mathcal{R}^4} = -\frac{1}{16} e^{-2\phi} \alpha' \left(\mathcal{W}_+^2 + \mathcal{W}_-^2\right) + \frac{1}{64}\left(1 - 2\zeta(3)\right) e^{-6\phi} \alpha'^3 \mathcal{W}_+^2 \mathcal{W}_-^2 - \frac{1}{3 \times 2^{12}\pi^5} e^{-4\phi} \alpha'^3 \left[\left(\mathcal{W}_+^4 + \mathcal{W}_-^4\right) + 20 \mathcal{W}_+^2 \mathcal{W}_-^2\right]. \tag{7.80}$$

These are only the moduli-independent $\mathcal{R}^4$ terms from these actions. Strictly speaking not even these terms are moduli-independent, since they are all multiplied by the volume of the compactification manifold, a factor we omitted for simplicity. But they are always present, no matter which compactification is taken. The complete action, for every different manifold, includes many other moduli-dependent terms which we do not consider here: we are mostly interested in a $\mathbb{T}^6$ compactification.

At string tree level, for all these theories in $d = 4$ only $\mathcal{W}_+^2 \mathcal{W}_-^2$ shows up. Because of its well known $d = 10$ SL$(2, \mathbb{Z})$ invariance, in type IIB theory only the combination $\mathcal{W}_+^2 \mathcal{W}_-^2$ is present in the $d = 4$ effective action (7.78). In the other theories, $\mathcal{W}_+^4 + \mathcal{W}_-^4$ shows up at string one loop level. For type IIA, the reason is the difference between the left and right movers in the relative GSO projection at one string loop, because of this theory being nonchiral. Heterotic string theories have $\mathcal{N} = 1$ supersymmetry in ten dimensions, which allows corrections to the sigma model already at order α', including $\mathcal{R}^2$ corrections (forbidden in type II theories in $d = 10$). Because of cancelation of gravitational anomalies, another $\mathcal{R}^4$

contribution is needed in heterotic theories, which when reduced to $d = 4$ gives rise to (7.76) and (7.77).

Next we consider the supersymmetrization of these $\mathcal{R}^4$ terms in $d = 4$.

7.5.1 $\mathcal{N} = 1, 2$ Supersymmetrization of $\mathcal{W}_+^2 \mathcal{W}_-^2$

The supersymmetrization of the square of the Bel-Robinson tensor $\mathcal{W}_+^2 \mathcal{W}_-^2$ has been known for a long time, in simple [19, 32] and extended [33, 34] four dimensional supergravity.

7.5.1.1 $\mathcal{N} = 1$

In $\mathcal{N} = 1$, the lagrangian to be considered is (α is a numerical constant)

$$\mathcal{L}_{SG} + \mathcal{L}_{\mathcal{R}^4} = \frac{1}{2\kappa^2} \int E \left(1 + \alpha\kappa^6 W^2 \overline{W}^2\right) d^4\theta. \tag{7.81}$$

From (7.54) and (7.55), the α term represents the supersymmetrization of $\mathcal{W}_+^2 \mathcal{W}_-^2$. To compute the variation of this action, we obviously need the constrained variation of W_{ABC}. The details of this calculation are presented in [19], and so is the final result for $\int \delta[E(1 + \alpha\kappa^6 W^2 \overline{W}^2)] d^4\theta$, which we do not reproduce here again. From this result, the $R, \overline{R}$ field equations are given by

$$R = 6\alpha\kappa^6 \frac{\overline{W}^2 \nabla^2 W^2}{1 - 2\alpha\kappa^6 W^2 \overline{W}^2} = 6\alpha\kappa^6 \overline{W}^2 \nabla^2 W^2 + 12\alpha^2\kappa^{12} \overline{W}^4 W^2 \nabla^2 W^2. \tag{7.82}$$

From (7.49), we can easily determine $\nabla^n G_n$. This way, auxiliary fields belonging to the compensating chiral multiplet can be eliminated on-shell. This is not the case for the auxiliary fields which come from the Weyl multiplet (A_m), as we obtained, also in [19], a complicated differential field equation for G_m.

7.5.1.2 $\mathcal{N} = 2$

Analogously to $\mathcal{N} = 1$, we write the $\mathcal{N} = 2$ supersymmetric $\mathcal{R}^4$ lagrangian in superspace, using the chiral projector and the chiral density, as a correction to the pure supergravity lagrangian [34] (α is again a numerical constant):

$$\begin{aligned} \mathcal{L}_{SG} + \mathcal{L}_{\mathcal{R}^4} = \int \overline{\epsilon} \Bigg[-\frac{3}{4\kappa^2} + \alpha\kappa^4 \Big(\nabla^{Aa} \nabla_A^b \left(\nabla_a^B \nabla_{Bb} + 16 X_{ab}\right) \\ - \nabla^{Aa} \nabla_a^B \left(\nabla_A^b \nabla_{Bb} - 16 i Y_{AB}\right)\Big) W^2 \overline{W}^2 \Bigg] d^4\overline{\theta} + \text{h.c.}. \end{aligned} \tag{7.83}$$

From the component expansion (7.73), the α term clearly contains $e\mathcal{W}_+^2 \mathcal{W}_-^2$.

At this point we proceed with the calculation of the components of (7.83) and analysis of its field content. For that, we use the differential constraints from the solution to the Bianchi identities and the commutation relations. The process is straightforward but lengthy [34]. The results can be summarized as follows: with the correction (7.83), auxiliary fields X_{ab}, $\Lambda_{\dot{C}c}$, $Y_{\dot{A}\dot{B}}$, U_m and Φ_m^{ab} get derivatives, and the same should be true for their field equations; therefore, these superfields cannot be eliminated on-shell. We also fully checked that superfields P and H_m do not get derivatives (with the important exception of $\nabla^m H_m$) and, therefore, have algebraic field equations which should allow for their elimination on shell. The only auxiliary field remaining is ρ_A^a. We did not analyze its derivatives because that would require computing a big number of terms and, for each term, a huge number of different contributions. Its derivatives should cancel, though: otherwise, we would have a field (ρ_A^a) with a dynamical field equation while having two fields obtained from its spinorial derivatives (P and the transverse part of H_m) without such an equation. ρ_A^a, like P and transverse H_m, are intrinsic to the "old minimal" version of $\mathcal{N} = 2$ supergravity; they all belong to the same nonlinear multiplet. The physical theory does not depend on these auxiliary fields and, therefore, it seems natural that they can be eliminated from the classical theory and its higher-derivative corrections.

7.5.2 $\mathcal{N} = 1$ Supersymmetrization of $\mathcal{W}_+^4 + \mathcal{W}_-^4$

For the term $\mathcal{W}_+^4 + \mathcal{W}_-^4$ there is a "no-go theorem", which goes as follows [35]: for a polynomial $I(\mathcal{W})$ of the Weyl tensor to be supersymmetrizable, each one of its terms must contain equal powers of $\mathcal{W}_{\mu\nu\rho\sigma}^+$ and $\mathcal{W}_{\mu\nu\rho\sigma}^-$. The whole polynomial must then vanish when either $\mathcal{W}_{\mu\nu\rho\sigma}^+$ or $\mathcal{W}_{\mu\nu\rho\sigma}^-$ do.

The derivation of this result is based on $\mathcal{N} = 1$ chirality arguments, which require equal powers of the different chiralities of the gravitino in each term of a superinvariant. The rest follows from the supersymmetric completion. That is why the only exception to this result is $\mathcal{W}^2 = \mathcal{W}_+^2 + \mathcal{W}_-^2$: in $d = 4$ this term is part of the Gauss–Bonnet topological invariant (it can be made equal to it with suitable field redefinitions). This term plays no role in the dynamics and it is automatically supersymmetric; its supersymmetric completion is 0 and therefore does not involve the gravitino.

The derivation of [35] has been obtained using $\mathcal{N} = 1$ supergravity, whose supersymmetry algebra is a subalgebra of $\mathcal{N} > 1$. Therefore, it should remain valid for extended supergravity too. But one must keep in mind the assumptions which were made, namely the preservation by the supersymmetry transformations of R-symmetry which, for $\mathcal{N} = 1$, corresponds to U(1) and is equivalent to chirality. In extended supergravity theories R-symmetry is a global internal U $(\mathcal{N})$ symmetry, which generalizes (and contains) U(1) from $\mathcal{N} = 1$.

Preservation of chirality is true for pure $\mathcal{N} = 1$ supergravity, but to this theory and to most of the extended supergravity theories one may add matter couplings and extra terms which violate U(1) R-symmetry and yet can be made supersymmetric, inducing corrections to the supersymmetry transformation laws which do not preserve U(1) R-symmetry.

Having this in mind [36], we consider a chiral multiplet, represented by a chiral superfield $\mathbf{\Phi}$ (we could take several chiral multiplets Φ_i, which show up after $d = 4$ compactifications of superstring and heterotic theories and truncation to $\mathcal{N} = 1$ supergravity, but we restrict ourselves to one for simplicity), and containing a scalar field $\Phi = \mathbf{\Phi}|$, a spin$-\frac{1}{2}$ field $\nabla_A \mathbf{\Phi}|$, and an auxiliary field $F = -\frac{1}{2}\nabla^2\mathbf{\Phi}\big|$. This superfield and its hermitian conjugate couple to $\mathcal{N} = 1$ supergravity in its simplest version through a superpotential

$$P(\mathbf{\Phi}) = d + a\mathbf{\Phi} + \frac{1}{2}m\mathbf{\Phi}^2 + \frac{1}{3}g\mathbf{\Phi}^3 \tag{7.84}$$

and a Kähler potential $K\left(\mathbf{\Phi}, \overline{\mathbf{\Phi}}\right) = -\frac{3}{\kappa^2}\ln\left(-\frac{\Omega(\mathbf{\Phi},\overline{\mathbf{\Phi}})}{3}\right)$, with

$$\Omega\left(\mathbf{\Phi}, \overline{\mathbf{\Phi}}\right) = -3 + \mathbf{\Phi}\overline{\mathbf{\Phi}} + c\mathbf{\Phi} + \bar{c}\overline{\mathbf{\Phi}}. \tag{7.85}$$

In order to include the term (7.77), we take the following effective action:

$$\begin{aligned}\mathcal{L} = &-\frac{1}{6\kappa^2}\int E\left[\Omega\left(\mathbf{\Phi}, \overline{\mathbf{\Phi}}\right) + \alpha'^3\left(b\mathbf{\Phi}\left(\nabla^2 W^2\right)^2 + \bar{b}\overline{\mathbf{\Phi}}\left(\overline{\nabla}^2\overline{W}^2\right)^2\right)\right]d^4\theta \\ &-\frac{2}{\kappa^2}\left(\int \epsilon P(\mathbf{\Phi})\, d^2\theta + \text{h.c.}\right).\end{aligned} \tag{7.86}$$

If one expands (7.86) in components, one does not directly get (7.77), but one should look at the auxiliary field sector. Because of the presence of the higher-derivative terms, the auxiliary field from the original conformal supermultiplet A_m also gets higher derivatives in its equation of motion, and therefore it cannot be simply eliminated [19, 34]. Because the auxiliary fields M, N belong to the chiral compensating multiplet, their field equation should be algebraic, despite the higher derivative corrections [19,34]. That calculation should still require some effort; plus, those M, N auxiliary fields should not generate by themselves terms which violate U(1) R-symmetry: these terms should only occur through the elimination of the chiral multiplet auxiliary fields $F, \bar{F}$. This is why we will only be concerned with these auxiliary fields, which therefore can be easily eliminated through their field equations [21]. The final result, taking into account only terms up to order α'^3, is

$$\begin{aligned}\kappa^2\mathcal{L}_{F,\overline{F}} = &-15e\frac{(3 + c\bar{c})}{(3 + 4c\bar{c})^2}\left(m\bar{a}\Phi + \overline{m}a\overline{\Phi}\right)\left(c\Phi + \bar{c}\overline{\Phi}\right) \\ &+ e\frac{2c^3\bar{c}^3 + 60c^2\bar{c}^2 + 117c\bar{c} - 135}{(3 + 4c\bar{c})^3}a\bar{a}\Phi\overline{\Phi} \\ &- 36\alpha'^3 e\left(b\bar{c}\left(\nabla^2 W^2\right)^2\Big| + \bar{b}c\left(\overline{\nabla}^2\overline{W}^2\right)^2\Big|\right)\end{aligned}$$

$$
\begin{aligned}
&\times \frac{a\bar{a} + m\bar{a}\Phi + \overline{m}a\overline{\Phi} + g\bar{a}\Phi^2 + \overline{g}a\overline{\Phi}^2 + m\overline{m}\Phi\overline{\Phi}}{(3+4c\bar{c})^2} \\
&- 3\alpha'^3 a\bar{a}\frac{74c^2\bar{c}^2 + 192c\bar{c} - 657}{(3+4c\bar{c})^4}\Phi\overline{\Phi}\left(b\bar{c}\left(\nabla^2 W^2\right)^2\right| \\
&+ \bar{b}c\left(\overline{\nabla}^2\overline{W}^2\right)^2\left|\right) + 15\alpha'^3 e\frac{a\bar{a} + m\bar{a}\Phi + \overline{m}a\overline{\Phi}}{(3+4c\bar{c})^3} \\
&\times\left[\left(c^2\left(21+4c\bar{c}\right)\Phi + \left(-9+6c\bar{c}\right)\overline{\Phi}\right)\bar{b}\left(\overline{\nabla}^2\overline{W}\right)^2\right| + \text{h.c.}\right] + \ldots
\end{aligned}
\tag{7.87}
$$

This way we are able to supersymmetrize $\mathcal{W}_+^4 + \mathcal{W}_-^4$, although we had to introduce a coupling to a chiral multiplet. Since from (7.54) and (7.55) the factor in front of $\mathcal{W}_+^4$ (resp. $\mathcal{W}_-^4$) in (7.87) is given by $\frac{-144b\bar{c}a\bar{a}}{(3+4c\bar{c})^2}$ (resp. $\frac{-144\bar{b}ca\bar{a}}{(3+4c\bar{c})^2}$), for this supersymmetrization to be effective, the factors a from $P(\Phi)$ in (7.84) and c from $\Omega\left(\Phi,\overline{\Phi}\right)$ in (7.85) (and of course b from (7.86)) must be nonzero.

7.5.2.1 $\mathcal{W}_+^4 + \mathcal{W}_-^4$ in Extended Supergravity

$\mathcal{W}_+^4 + \mathcal{W}_-^4$ must also arise in extended $d = 4$ supergravity theories, for the reasons we saw, but the "no-go" result of [35] should remain valid, since it was obtained for $\mathcal{N} = 1$ supergravity, which can always be obtained by truncating any extended theory. For extended supergravities, the chirality argument should be replaced by preservation by supergravity transformations of U(1), which is a part of R-symmetry.

$\mathcal{N} = 2$ supersymmetrization of $\mathcal{W}_+^4 + \mathcal{W}_-^4$ should work in a way similar to what we saw for $\mathcal{N} = 1$. $\mathcal{N} = 2$ chiral superfields must be Lorentz and SU(2) scalars but they can have an arbitrary U(1) weight, which allows supersymmetric U(1) breaking couplings.

A similar result should be more difficult to implement for $\mathcal{N} \geq 3$, because there are no generic chiral superfields. Still, there are other multiplets than the Weyl, which one can consider in order to couple to $\mathcal{W}_+^4 + \mathcal{W}_-^4$ and allow for its supersymmetrization. The only exception is $\mathcal{N} = 8$ supergravity, a much more restrictive theory because of its higher amount of supersymmetry. In this case one can only take its unique multiplet, which means there are no extra matter couplings one can consider. We have shown that the $\mathcal{N} = 8$ supersymmetrization of $\mathcal{W}_+^4 + \mathcal{W}_-^4$, coupled to scalar fields from the Weyl multiplet, is not allowed even at the linearized level [37]. In $\mathcal{N} = 8$ superspace one can only have SU(8) invariant terms, and we argued $\mathcal{W}_+^4 + \mathcal{W}_-^4$ should be only SU(4) $\otimes$ SU(4) invariant. If that is the case, in order to supersymmetrize this term besides the supergravity multiplet one must introduce U-duality multiplets, with massive string states and nonperturbative states. The fact that one cannot supersymmetrize in $\mathcal{N} = 8$ a term which string theory requires to be supersymmetric, together with the fact that one needs to consider nonperturbative states (from U-duality multiplets) in order to understand a perturbative contribution may be seen as indirect evidence that $\mathcal{N} = 8$ supergravity is indeed in the swampland [38]. We believe that topic deserves further study.

7.6 Applications to Black Holes in String Theory

String-corrected black holes have been a very active recent topic of research, for which one needs to know the string effective actions to a certain order in α'. Topics which have been studied include finding α'-corrected black hole solutions by themselves, but also studying their properties like the entropy. One of the biggest successes of string theory was the calculation of the microscopic entropy of a class of supersymmetric black holes and the verification that this result corresponds precisely to the macroscopic result of Bekenstein and Hawking. Clearly it is very important to find out if and how this correspondence extends to the full string effective action, without α' corrections.

Because of different α' corrections each quantity gets, typically the entropy does not equal one quarter of the horizon area for black holes with higher derivative terms. In order to compute the entropy for these black holes, a formula has been developed by Wald [39]. When this formula is applied to extremal (not necessarily supersymmetric) black holes, one arrives at the entropy functional formalism developed by Sen (for a complete review see [40]). This formalism can be summarized as follows: one considers a black hole solution from a lagrangian $\mathcal{L}$ with gravity plus some gauge fields and massless scalars in d dimensions. The near horizon limit of such black hole corresponds to $AdS_2 \times S^{d-2}$ geometry, with two parameters v_1, v_2. Also close to the horizon, the gauge fields are parameterized by sets of electric (e_i) and magnetic (p_a) charges, and the scalar fields by constants u_s. The parameters $(\mathbf{u}, \mathbf{v}, \mathbf{e}, \mathbf{p})$ are up to now arbitrary and, therefore, the solution is off-shell. Next we define the function (to be evaluated in the near horizon limit)

$$f\left(\mathbf{u}, \mathbf{v}, \mathbf{e}, \mathbf{p}\right) = \int_{S^{d-2}} \sqrt{-g}\mathcal{L}\, d\Omega_{d-2}.$$

The on-shell values of $\mathbf{u}, \mathbf{v}, \mathbf{e}$ for a given theory are found through the relations

$$\frac{\partial f}{\partial u_s} = 0,\ \frac{\partial f}{\partial v_j} = 0,\ \frac{\partial f}{\partial e_i} = q_i,$$

which also reproduce the equations of motion. Then, using Wald's formulation, Sen derived the black hole entropy, given by

$$S = 2\pi \left(e_i \frac{\partial f}{\partial e_i} - f \right).$$

This process has been verified for extremal (supersymmetric or not) black holes in generic d dimensions. In particular, it has been tested with off shell formulations of supergravity [41] (these formulations are known for $\mathcal{N} = 1$ in $d = 4$ or $\mathcal{N} = 2$ in $d = 4, 5, 6$). When one considers black holes in these theories, auxiliary fields must also be considered in f, necessarily as independent fields (since for this functional we take an a-priori off-shell solution). As we have seen, when considering theories with higher-derivative corrections, some of these auxiliary fields can still be eliminated, but others become dynamical. Clearly a precise knowledge of the behavior of the different auxiliary fields, like we have studied, is essential if one wishes to determine the higher-derivative corrections to black hole properties such as the entropy.

A particularly well studied case [42] (which has been reviewed in this volume [43]) is that of BPS black holes in $d = 4, \mathcal{N} = 2$ supergravity coupled to n vector multiplets, to which are associated n scalar fields X^I and n vector fields A^I_μ. The holomorphic higher-derivative corrections associated to these black holes are given as higher genus contributions to the prepotential, in the form of a function

$$F(X^I, \hat{A}) = \sum_{g=0}^{\infty} F^{(g)}(X^I)\hat{A}^g, \tag{7.88}$$

$\hat{A}$ being a scalar field which, in our conventions, is given by $\hat{A} = W^{AB} W_{AB}\big|$. From (7.72), one sees that $\hat{A}$ is related to the square of the selfdual part of the graviphoton field strength $F_{\mu\nu}$, but also to the square of the auxiliary field Y_{AB} (which, as we saw, may become dynamical in the presence of higher-derivative terms). From (7.73), one immediately sees that a lagrangian containing $F(X^I, \hat{A})$ as an F-term includes $\mathcal{W}^2$ terms, each multiplied by terms depending on moduli and on powers of either $F_{\mu\nu}$ or Y_{mn}. These Y_{mn} factors may generate terms with higher powers of the Weyl tensor $\mathcal{W}_{\mu\nu\rho\sigma}$.

After some rescaling (in order to have manifest symplectic covariance), $\hat{A}$ becomes the variable Υ, which at the horizon takes a particular numerical value ($\Upsilon = -64$ in the conventions of [43]). This value is universal, independent of the model taken (i.e., for any function $F(X^I, \hat{A})$ of the form (7.88)), as long as the black hole solution under consideration is supersymmetric. There may exist other near-horizon configurations (corresponding to nonsupersymmetric black holes) which extremize the entropy function but correspond to different attractor equations and different values for Υ. These values are not universal: each solution has its own (constant) Υ.

The generalized prepotential (7.88) does not represent the full set of higher derivative corrections one must consider in a supersymmetric theory in $d = 4$, even for a black hole solution. There are also the nonholomorphic corrections, which are necessary for the entropy to be invariant under string dualities, as discussed in [43]. At the time, the way to incorporate these corrections into the attractor mechanism is still under study. On general grounds, if Υ is coupled to the nonholomorphic corrections, then it should in principle get a different value. This (still unknown) different value for Υ should also in principle depend on the model which we are taking. Because of this nonuniversality, we cannot simply take a general expression for the nonholomorphic corrections: we really need each term, to the order we are working, in the effective action. For that, in the cases when auxiliary fields (namely Υ) exist and are part of the higher derivative correction terms (as studied in [44]), we must know exactly their behavior in the presence of such corrections, in the way we presented on the first part of these notes.

7.7 Summary and Discussion

We computed the $\mathcal{R}^4$ terms in the superstring effective actions in four dimensions. We showed that besides the usual square of the Bel-Robinson tensor $\mathcal{W}_+^2\mathcal{W}_-^2$, the other possible $\mathcal{R}^4$ term in $d = 4$, $\mathcal{W}_+^4 + \mathcal{W}_-^4$, was also part of two of those actions at one string loop. We then studied their supersymmetrization.

For $\mathcal{W}_+^2\mathcal{W}_-^2$ we wrote down its supersymmetrization directly in $\mathcal{N} = 1$ and $\mathcal{N} = 2$ superspace, taking advantage of the off-shell formulation of these theories. The terms we wrote down were off-shell; in both cases we tried to obtain the on-shell action by eliminating the auxiliary fields. We noticed that some auxiliary fields could be eliminated, while others couldn't.

A careful analysis shows that, in both cases we studied, the auxiliary fields that can be eliminated in the supersymmetrization of $\mathcal{W}_+^2\mathcal{W}_-^2$ come from multiplets which, on-shell, have no physical fields; while the auxiliary fields that get derivatives come from multiplets with physical fields on-shell (the graviton, the gravitino(s) and, in $\mathcal{N} = 2$, the vector). Our general conjecture for supergravity theories with higher derivative terms, which is fully confirmed in the "old minimal" $\mathcal{N} = 1, 2$ cases with $\mathcal{W}_+^2\mathcal{W}_-^2$, can now be stated: the auxiliary fields which come from multiplets with on-shell physical fields cannot be eliminated, but the ones that come from compensating multiplets that, on shell, have no physical fields, can. In order to get more evidence for it, the analysis we made should also be extended to the other different versions of these supergravity theories, and with other higher derivative terms.

We moved on to try to supersymmetrize $\mathcal{W}_+^4 + \mathcal{W}_-^4$, but we faced a previous result stating that supersymmetrization could not be achieved because in $\mathcal{N} = 1$ it would violate chirality, which is preserved in pure supergravity. The way we found to circumvent this problem was to couple $\mathcal{W}_+^4 + \mathcal{W}_-^4$ to a chiral multiplet and, after eliminating its auxiliary fields, obtain that same term on-shell. We worked this out in $\mathcal{N} = 1$ supergravity and the same should be possible in $\mathcal{N} = 2$. For $\mathcal{N} = 8$ that should not be possible any longer, because there are no other multiplets we could use to couple to $\mathcal{W}_+^4 + \mathcal{W}_-^4$ that could help us: the Weyl multiplet is the only one allowed in this theory. This is a sign that $\mathcal{N} = 8$ supergravity is indeed in the swampland.

We ended by discussing applications of these results to black holes in string theory, namely the attractor mechanism and the calculation of the black hole entropy in the presence of higher derivative terms. We considered extremal black holes in

d dimensions, through Sen's entropy functional formalism, and in particular BPS black holes in $d = 4, \mathcal{N} = 2$ supergravity. In all cases we concluded that, having those applications in mind, when auxiliary fields exist, one needs to know exactly their behavior in the presence of such higher derivative corrections.

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