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CURTIS WILSON

The Hill–Brown Theory of the Moon's Motion

Its Coming-to-be and Short-lived Ascendancy (1877–1984)

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We must find a theory that will work; *and that means something extremely difficult; for our theory must mediate between all previous truths and certain new experiences. It must derange common sense and previous belief as little as possible, and it must lead to some sensible terminus or other that can be verified exactly.*

– William James, *Pragmatism*, 1907 edition, p. 216

Preface

The Hill–Brown theory of the Moon's motion was constructed in the years from 1877 to 1908, and adopted as the basis for the lunar ephemerides in the nautical almanacs of the US, UK, Germany, France, and Spain beginning in 1923. At that time and for some decades afterward, it was the most accurate lunar theory ever constructed. Its accuracy was due, first, to a novel choice of "intermediary orbit" or first approximation, more nearly closing in on the Moon's actual motion than any elliptical orbit ever could, and secondly to the care and discernment and stick-to-it-ive-ness with which the further approximations ("perturbations" to this initial orbit) had been computed and assembled so as yield a final theory approximating the Moon's path in real space with an accuracy of a hundredth of an arc-second or better. The method by which the Hill–Brown lunar theory was developed held the potentiality for still greater accuracy.

The intermediary orbit of the Hill–Brown theory may be described as a periodic solution of a simplified three-body problem, with numerical parameters carried to 15 decimal places. George William Hill, a young American mathematician working for the U.S. Nautical Almanac Office, had proposed it, and computed the numerical parameters to their 15 places. A self-effacing loner, he had in his privately pursued studies come to see that the contemporary attempts at predicting the Moon's motion were guaranteed to fail in achieving a lunar ephemeris of the accuracy desired.

Of the two lunar theories vying for preeminence in the 1870s, one was the work of Peter Andreas Hansen. Hansen's theory had been adopted as the basis for the lunar ephemerides in the national almanacs beginning with the year 1862, and it would continue in that role through 1922. It was numerical rather than algebraic. This meant that numerical constants were introduced at an early stage of the computation. A consequence was that, beyond this stage, the course of the calculation was not traceable; the algebraic structure of the theory was lost from sight. The only way to make responsible corrections to the theory was to start over again from the beginning – a daunting prospect, given that Hansen's construction of the theory had occupied 20 years. Already in the 1870s Hansen's theory was known to be seriously in need of correction. Further corrections would be required for the theory to keep pace with ongoing improvements in the precision of celestial observations.

The second theory, that of Charles Delaunay, which had also required about 20 years for its construction, was entirely algebraic; its calculative paths were therefore clearly traceable. Its method, deriving ultimately from Lagrange, was elegant, and Hill was initially charmed with it. But then came a disillusioning discovery. In the higher-order approximations, the convergence slowed to a snail's pace, and the complexity of the computations increased staggeringly. For perturbations of higher order than the 7th, Delaunay resorted to "complements," guesses as to what the $(n+1)$th-order perturbation would be by extrapolation from already computed perturbations of the nth and $(n-1)$th order. The complements were later found to be quite unreliable. Delaunay's resort to "complements," Hill concluded, was an admission that his method had failed.

In Part I of the following study, I tell of the new method that Hill now envisaged for developing the lunar theory, a method suggested by Euler's lunar theory of 1772. In the form in which E.W. Brown carried it to completion, it was semi-numerical: the initial orbit (Hill called it the "variation curve") was given by the dynamics of a simplified three-body problem. The numerical input for this three-body problem was a single number, the ratio of the mean motion of the Sun to the synodic motion of the Moon. This number was as exactly known as any of the constants of astronomy, and therefore unlikely to require revision. The remainder of the theory, consisting of the thousands of terms necessary to "correct" the simplified model taken as starting-point, was to be literal or algebraic throughout, and therefore straightforwardly correctable. Part II tells how Brown, recruited by George Howard Darwin of Christ's College Cambridge as Hill's continuator, skillfully organized the long series of computations required for the completion of the Hill–Brown theory.

Can our story appropriately be ended here? I say No. In the 1930s, J. Leslie Comrie of the British Nautical Almanac Office hazarded the opinion that the Hill–Brown theory would remain the basis of the lunar ephemerides to the year 2000. In fact, it would be replaced after some 50 years, and in the meantime the lunar problem would be transformed out of recognition. Brown lived long enough (he died in July, 1938) to have a role in early phases of the new development. I devote Part III of my study to describing this transformation, really three revolutions wrapped into one.

To begin with, even before Hill had conceived of the Hill–Brown theory, two anomalies had been discovered in the Moon's motion – variations in its motion which gravitational theory could not account for; they would still be unresolved when Brown completed his Tables in 1919. In 1853 John Couch Adams had shown that Laplace's theory of the Moon's secular acceleration (published in 1787) could account for only about half the observed secular acceleration, leaving the other half unexplained. Secondly, Simon Newcomb in the 1860s discovered that, besides its steady acceleration, the Moon's motion was subject to additional variations, involving accelerations both positive and negative, lasting sometimes for decades and sometimes for shorter times. In 1939 it was at last shown conclusively that the first of these anomalies was due to a deceleration in the Earth's rotation, and that the second was due to erratic variations in that same rotation. The assembling of the data leading to this conclusion was the result of a cooperative effort on the part of many astronomers, including Brown. The final proof was worked out by H. Spencer Jones, H.M. Astronomer at the Cape of Good Hope, and published in 1939.

Jones's proof meant that astronomy was in need of a new clock. Since Antiquity astronomers had depended on the diurnal motion of the stars to measure time. They now knew that this motion, a reflection of the Earth's rotation, was not strictly uniform, but was slowing gradually and also varying erratically. A new method of measuring time was necessary if astronomy was to be a self-consistent enterprise.

In an initial effort to restore logical consistency to their science, astronomers invented the notion of Ephemeris Time. This was intended to be the time presupposed in the ephemerides of the Moon, Sun, and planets, which time was in turn supposed to be the time presupposed in dynamical theory – still, in the 1950s, largely Newtonian. Unfortunately, the ephemerides were only approximately in accord with dynamical theory, and were subject to repeated revision to bring them more exactly in accord with the underlying dynamical theory. Moreover, intervals of Ephemeris Time could be measured only for the past – a considerable inconvenience. Observations made in the present had to be made in Universal Time, the varying time given by the apparent diurnal motion of the stars. Time intervals in Universal Time were then corrected later through comparisons with the ephemerides.

A more convenient option became available in 1955, with the invention of the atomic clock. Its possibility had been suggested in 1945 by Isidore Rabi, the inventor of the magnetic resonance method for studying the structure of atoms and molecules. Quartz clocks could be calibrated against an atomic frequency, and thus brought to new levels of precision and accuracy as timepieces. By 1970 atomic clocks had been so improved as to be accurate to about 5 ns per day. An experiment carried out in 1971 proved that these clocks obeyed the rules of relativity theory: their rate of running was dependent on the gravitational fields and accelerated frames of reference in which they were placed. Here were new complexities and newly available levels of precision which practical astronomy needed to take into account.

The second revolution came about through the development of the electronic computer and its application in the calculations of astronomy. During the 1920s, J. Leslie Comrie of the British Nautical Almanac Office initiated the application of available punched card technology to the computation of ephemerides. He demonstrated these processes to E.W. Brown and his graduate student, Walter J. Eckert, and Eckert took up with enthusiasm the project of adapting computer programs to the needs of astronomy. By the late 1930s Eckert had succeeded in computerizing the processes whereby Brown had originally computed the 3000 or so terms of the Hill–Brown theory; the computerized computations showed that, with but few exceptions, Brown's results were extremely accurate. In 1948, with the cooperative help of Thomas J. Watson of IBM, Eckert completed the design and construction of the Selective Sequence Electronic Calculator. One of the intended uses of this instrument was to compute an ephemeris of the Moon directly from Brown's trigonometric series, thus obviating use of Brown's *Tables*, which had been found to introduce systematic error.

Later, with further increments in computer speed and reliability, efforts were made to re-do the development of the lunar theory by Delaunay's method. The old difficulty of slow convergence re-appeared, and it was found better to start from Hill's "Variation Curve," computed numerically; the theory as a whole, like Brown's, would thus be semi-numerical.

The third revolution concerned new types of data, above all, data giving the *distances* of celestial bodies. These types of data were introduced by radar-ranging, space-craft ranging, and after 1969 in the case of the Moon, laser-ranging. Earlier, the more accurate data had been angular, measuring the positions of celestial bodies laterally with respect to the line of sight. The new astronomical data, measuring the distances of celestial bodies, was more accurate by about four orders of magnitude. These types of data were the work of Jet Propulsion Laboratory (JPL), which had the task of sending spacecraft aloft and then astronauts to the Moon. The newer data types required the development of numerical integration techniques and more comprehensive (and relativistic) physical models. Laser light, and spacecraft sent aloft, achieved new wonders in determining the Moon's position, increasing the precision of the measurement by four orders of magnitude. The transformation completed itself in 1984, when responsibility for producing lunar ephemerides, and planetary ephemerides as well, passed from the Nautical Almanac Office in Washington, DC to Jet Propulsion Laboratory in Pasadena.

Without doubt, it was the end of an era.

But the mathematical and philosophical interest of an analytic solution to the lunar problem, in the Hill–Brown-Eckert manner, remains high. Such a solution reveals something of the nature and limitations of our knowledge of similar problems.

Annapolis, MD
January, 2010

Curtis Wilson

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Contents

Part I

Hill Lays the Foundation (1877–1878)

1

George William Hill, Mathematician

George William Hill (1838–1914), a mathematician with the U.S. Nautical Almanac Office from 1861 to 1892, in two papers of 1877 and 1878 laid the foundations of a new lunar theory, departing from a basic pattern that had characterized earlier algebraic theories of the Moon's motions with one exception, to be mentioned below. The first of Hill's papers was printed privately, but very quickly a copy (probably sent by Hill) reached the lunar theorist John Couch Adams of Cambridge University, and Adams called attention to its seminal importance in the Royal Astronomical Society's *Monthly Notices* for November of that year.[1] Wider recognition of its innovative character came during the course of the next decade. In 1887 Hill was awarded the Gold Medal of the Royal Astronomical Society. His sponsors for the award included Adams, George Howard Darwin, and the new president of the society, J.W.L. Glaisher, who devoted his presidential address to a précis and evaluation of Hill's two papers.

The starting-point for Hill's theory was a particular solution of two second-order differential equations expressing what Henri Poincaré would later call 'the restricted problem of three bodies' (*le problème restreint de trois corps*). These equations idealized the lunar problem, treating the Moon as of infinitesimal mass and as moving in the ecliptic plane, the Sun as having zero parallax, and the Earth as moving uniformly in a circle about the Sun. Hence, before this theory could yield the Moon's *actual* motions, it would need to be modified so as to allow for the inclusion of further "inequalities." In his paper of 1878 Hill proposed to treat the inequalities that are proportional to the sine of the lunar inclination, the solar eccentricity, and the solar parallax; but the memoir as published contains no further mention of these inequalities. In the 1880s and 1890s Hill published a number of papers on lunar inequalities; but by the 1890s, we learn, he had bequeathed the project of systematically developing his lunar theory to a younger man.

[1] J.C. Adams, "On the Motion of the Moon's Node in the Case When the Orbits of the Sun and Moon are Supposed to Have No Eccentricities, and When Their Mutual Inclination is Supposed Indefinitely Small," *Monthly Notices of the Royal Astronomical Society*, (hereinafter *MNRAS*) *38* (Nov., 1877), 43–49.

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This transfer was brought about by George Howard Darwin (1845–1912), son of Charles Darwin and an applied mathematician of Christ's College, Cambridge. Hill, writing on 10 December 1889 to Darwin in reply to Darwin's note of 22 November (no longer extant, apparently), explained what had kept him from further developing his lunar theory:

> My energies at present are devoted to the evolving a theory and tables of Jupiter and Saturn, and other projects have to be laid aside for this time. Thus it has happened that I have done scarcely anything beyond what you have seen in print, in reference to the Lunar Theory. It is very problematical whether I ever have an opportunity of continuing these researches. I should be glad to see Mr. Brown or any one else enter upon that field of labor....[2]

The Mr. Brown here mentioned was Ernest W. Brown (1866–1938), a student and protégé of Darwin's at Christ's College during the 1880s. In 1892 he was to migrate to the United States, take a position at Haverford College, and set himself to work on the elaboration of Hill's theory.

Meanwhile, at the urging of Simon Newcomb, Hill had committed himself to constructing a new theory of Jupiter and Saturn. In 1877, when Newcomb became director of the Nautical Almanac Office, he had envisaged two ambitious projects for his staff: the development of a set of planetary tables consistent in their assignment of masses to the planets (the planetary tables recently published by Le Verrier in Paris lacked such consistency), and the development of lunar tables more accurate than those currently available. The theory of Jupiter and Saturn was the most difficult of the planetary problems, and Newcomb asked Hill – whom he would later characterize as "easily ... the greatest master of mathematical astronomy during the last quarter of the nineteenth century"[3] – to take it on. This theory absorbed most of Hill's efforts from 1882 to 1892. He insisted on carrying out all the calculations himself, relying on an assistant only for verifications.

F.R. Moulton on Hill's death in 1914 wrote an appreciation of the man and his achievement. Hill, he says, was "retiring and modest to the verge of timidity.... He was absorbed in his own work but never inflicted it on others. In fact, he would hardly discuss it when others desired him to do so."[4] Moulton reports a conversation he had with Hill "after one of the meetings of the National Academy in Washington a few years ago" (Moulton does not specify the year, which was presumably in the twentieth century). It was a fine spring day; Hill had asked Moulton to join him in a walk, and was unusually forthcoming about his own earlier work:

> Hill told me that he thought the greatest piece of astronomical calculation ever carried out by one man was Delaunay's lunar theory, and that his work on Jupiter and Saturn came second. Now the greater part of this work was straight computation by methods which were largely due to Hansen, and

[2] Hill to G.H. Darwin, 10 Dec. 1889, University of Cambridge Library manuscript collection (hereinafter UCL.MS), DAR.251:3533; quoted with permission.

[3] S. Newcomb, *The Reminiscenses of an Astronomer* (Houghton: Mifflin, 1903), 218.

[4] F.R. Moulton, "George William Hill," *Popular Astronomy, 22* (1914), 391–400, 391.

> which could have been carried out under Hill's direction by men who did not have his great ability for original work. It seems probable that science lost much because Newcomb caused Hill to spend about eight years of the prime of life on this work. At any rate, this was the direct cause of his laying aside, as he thought for a time only, his researches on the lunar theory.[5]

Hill recognized that the working out of his lunar theory would involve much tedious calculation; he estimated it would require about 10 years, assuming a number of assistants to do the routine calculations. From a letter of Hill to Darwin of July 1886, we gain some sense of the strain that Hill felt when engaged in "that field of labor." Darwin had written to invite Hill to contribute a paper to a certain journal (unspecified in Hill's letter); but Hill is begging off:

> ...I have made arrangements for going off in a few days to the wilds of Canada to pass the vacation. The relaxation I get during the summer vacation is a matter of great importance to me, as by it I gain sufficient strength to keep in working trim for the following nine or ten months; and it is all the more effective, if, during the time, I can be absolutely free from the worry of scientific investigations.[6]

In 1892, at age 54, Hill retired from the Nautical Almanac Office, and returned to the family farm in West Nyack, New York, where he had always preferred to be. He was an amateur botanist, with considerable expertise in identifying wild plants, and he loved taking solitary walks and botanizing. From Washington he brought with him the still unfinished tables for Jupiter and Saturn, and completed them in West Nyack.

In tackling the problem of Jupiter and Saturn, Hill considered the possibility of using Delaunay's method – the method Delaunay had applied to the Moon[7]; it had not previously been applied to planetary perturbations. He abandoned this idea, however, and adopted instead a modification of the method of Hansen's *Auseinandersetzung*.[8] Hansen had already applied an early version of his method to Jupiter and Saturn, thus providing a model.[9] Hill apparently judged that Hansen's processes would lead more swiftly to the result aimed at than the extensive transformations required by Delaunay's method.

In 1895 Hill was chosen president of the American Mathematical Society for the 1895–1896 term. His presidential address, delivered on 27 December 1895, concerned "the Progress of Celestial Mechanics since the Middle of the Century."[10]

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5 Ibid., 398.

6 CUL. MS. DAR.251: 2614, Hill to Darwin, 12 July 1886.

7 See Hill's article, "Notes on the Theories of Jupiter and Saturn," *The Analyst*, VIII (1881), 33–40, 89–93; *The Collected Mathematical Works of George William Hill*, I, 351–363.

8 P.A. Hansen, *Auseinandersetzung einer zweckmässigen Methode zur Berechnung der absoluten Störungen der kleinen Planeten*, in *Abhandlungen der Königlich Sächsischen Gesellschaft der Wissenschaften, 5* (1859): 43–218; 6(1859), 3–147.

9 *Untersuchung über die gegenseitigen Störungen des Jupiters und Saturns*, Berlin, 1831.

10 *Bulletin of the American Mathematical Society*, second series, *II* (1896), 125–136; *The Collected Mathematical Works of George William Hill*, IV, 99–110.

Nowhere in it does he mention his own lunar theory; he deals solely with the work of Delaunay, Gyldén, and Poincaré. Brown, having gone to New York to hear it, reported to Darwin that "it wasn't particularly interesting."[11] Hill had mastered an enormous amount of the detail of celestial mechanics, including the crucial details that had led him to his new lunar theory. But he was not particularly successful at transmitting to others a larger view. Frank Schlesinger's account of Hill's lecturing on his specialty at Columbia University for a semester in 1899 tells us that the lecturer was tense and that the three graduate students who constituted his audience were awed and uncomprehending.[12] As Newcomb will remark later, Hill lacked the teaching faculty.[13]

The archives of the Naval Observatory Library contain an undated, typed memorandum of three pages, giving Hill's assessment of the status of the lunar problem and his estimate as to what the development of the new lunar theory he had laid the foundations of would require (for the text, see the Appendix). A reference there to a memoir by Radau – it dealt with the planetary perturbations of the Moon and had appeared in the *Bulletin astronomique* in April and May, 1892 – is consonant with the memorandum's having been drawn up around the time of Hill's retirement. The addressee of the memorandum is not specified, but in it Hill refers three times to "Prof. Newcomb," in particular mentioning Newcomb's corrections of Hansen's lunar tables. Hansen's lunar tables had been adopted as the basis for the lunar ephemerides in the British *Nautical Almanac* and the French *Connaissance des Temps* beginning in 1862; with corrections introduced by Newcomb they were adopted for the American lunar ephemerides beginning in 1883. Hill, while respectful of Newcomb's endeavors, is in effect criticizing Newcomb's attempt to "make do" with Hansen's theory.

Hill allows that, from a purely practical point of view, Hansen's tables, with minor corrections, might be used for an indefinite time without serious error. But the comparison Newcomb has made (in *Astronomical Papers prepared for the use of the American Ephemeris and Nautical Almanac*, I, 1882, 57–107) between the terms in Hansen's theory and those in Delaunay's shows discrepancies in the values of the coefficients amounting in some cases to $0''.5$; some of these were probably due to numerical mistakes made by Hansen. "It is not creditable to the advanced science of the present day," Hill remarks, "that we should be in any uncertainty in this respect." He goes on to urge that, "in treating this subject, we should start from a foundation reasonably certain in its details, all known forces being taken correctly into account." Hansen's theory, in his opinion, could not furnish such a foundation. "To pass from Hansen to a theory absolutely unencumbered with empiricism is a matter of difficulty. It is not even certain that the figures in [Hansen's *Tables de la lune*, 1857] are actually founded on the formulas of the introduction [to those tables]."

[11] Brown to Darwin, 12 January 1896, CUL. MS. DAR.251: 477.
[12] F. Schlesinger, "Recollections of George William Hill", *Publications of the Astronomical Society of the Pacific*, 49, 5–12.
[13] S. Newcomb, *The Reminiscenses of an Astronomer*, 218.

Hill is endorsing the further development of his own lunar theory, urging that Ernest W. Brown be encouraged in the computations he has commenced.

> Aid should be given in order that we may have the results sooner.... I estimate that on this plan new tables could be prepared and ready for use in ten years. Of course, sufficient computing force must be given to the undertaker of this project, perhaps three persons might suffice.

Hill's confidence in his theory was not misplaced. Brown in the course of his work demonstrated the superior accuracy of the new theory compared to earlier theories, including Hansen's and Delaunay's.

Hill does not imagine that the new tables will resolve all difficulties. Unknown causes are acting, producing unsolved puzzles that are unlikely to be cleared up in a mere decade.

> The comparison of [the new] theory with observation will give residuals which are the combined effects of the necessary changes in the values of the arbitrary constants and the action of the unknown causes. The latter undoubtedly exist, and I am afraid the period of observation is too short to show their real law.

Here Hill may have in mind Newcomb's earlier discovery that Hansen's tables were well fitted to lunar observations from 1750 to 1850, but deviated from observations made before and after that period. As Newcomb discovered, Hansen had altered numerically the theoretical value of the perturbations of the Moon due to Venus, attempting in this way to accommodate these earlier and later observations, while claiming that this was the sole piece of sheer empiricism in his tables. Newcomb at the date Hill writes is still tinkering with this term – a mistaken effort in Hill's view. Hill's own guess is that the discrepancies are due to the attractions of meteors, a guess that will prove equally illusory.

The Moon's motion, it was found, departed from Hansen's tables in two ways that Newtonian theory could not account for. First, the Moon was accelerating over the centuries. Part of this acceleration was derivable from planetary perturbation of the Earth, but the rest was not. Delaunay suggested that the excess acceleration might be due to a deceleration in the Earth's rotation caused by tidal friction. The second effect was a fluctuation in the Moon's motion; its speed, besides accelerating, was altering in seemingly random ways. Like the excess acceleration, the fluctuations might be attributable to alterations in the Earth's rotation. But demonstrating these conjectures would take some doing, and would not be accomplished till 1939. The introduction of the atomic clock in 1955 will put the assignment of these effects to changes in the Earth's rotation beyond possible doubt. Both the tidal deceleration and the fluctuations remain subjects of ongoing research today. In Part III of our study we shall enter into more detail concerning this topic, insofar as it is relevant to lunar astronomy.

Hill's innovations in the lunar theory led to two later developments in mathematics that we shall touch on in passing. In computing the motion of the Moon's perigee he found himself confronted with an infinite determinant, which he succeeded in solving. This feat sparked the interest and admiration of Henri Poincaré, and Poincaré's ensuing investigation of infinite determinants then led to a considerable mathematical development in later decades.[14] Secondly, Hill's detailed working out of a periodic solution of the three-body problem brought such periodic solutions to the attention of mathematicians, including, again, Poincaré. Such periodic solutions became for Poincaré the point of departure for explorations of the phase space of the three-body problem – researches which cast new light on the theory of differential equations as well as on the nature of classical mechanics.[15] In the present study we focus on the lunar theory itself: Hill's promising beginnings, and Brown's elaboration of them into a complete lunar theory.

[14] See M. Bernkopf, "A History of Infinite Matrices," *Archive for History of Exact Sciences, 4* (1967–1968), 308–358, especially 313ff.

[15] See J. Barrow-Green, *Poincaré and the Three Body Problem* (Providence, RI: American Mathematical Society; London: London Mathematical Society, 1997).

2

Lunar Theory from the 1740s to the 1870s – A Sketch

The attempt to cope with Newton's three-body problem not geometrically as Newton had done but algebraically, using the calculus in the form elaborated by Leibniz, got under way in the 1740s. That this attempt had not been made earlier appears to have been due to lack of an appreciation, among Continental mathematicians, of the importance of trigonometric functions for the solution of certain differential equations; they failed to develop systematically the differential and integral calculus of these functions. Newton had used derivatives and anti-derivatives of sines and cosines, but had not explained these operations to his readers. Roger Cotes, in his posthumous *Harmonia mensurarum* of 1722, articulated some of the rules of this application of the calculus. But Euler, in 1739, was the first to provide a systematic account of it. In the process he introduced the modern notation for the trigonometric functions, and made evident their role qua functions. Thus sines and cosines having as argument a linear function of the time, t, could now be differentiated and integrated by means of the chain rule. Differential equations giving the gravitational forces acting on a body could be formulated and solved – though only by approximation.

Euler was the first to exploit these possibilities in computing the perturbations of the Moon. The tables resulting from his calculation were published in 1746, without explanation of the procedures whereby they had been derived.

In March of 1746 the prize commission of the Paris Academy of Sciences, meeting to select a prize problem for the Academy's contest of 1748, chose the mutual perturbations of Jupiter and Saturn. Since Kepler's time, Jupiter had been accelerating and Saturn slowing down, and in other ways deviating from the Keplerian rules. Newton assumed the deviations to be due to the mutual attraction of the two planets, and proposed coping with the deviations in Saturn by referring Saturn's motion to the center of gravity of Jupiter and the Sun, and assuming an oscillation in Saturn's apsidal line. These proposals do not appear to have led to helpful results. The contest of 1748 was the first academic contest of the eighteenth century in which a case of the three-body problem was posed for solution.

The winning essay was Euler's; it was published in 1749. It was not successful in accounting for the anomalies in the motions of Saturn and Jupiter, but its

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technical innovations proved to be crucially important in later celestial mechanics. One of them was the invention of trigonometric series – a series in which the arguments of the successive sinusoidal terms are successive integral multiples of an angular variable. Euler's angle in the case of Jupiter and Saturn was the difference in mean heliocentric longitude between the two planets, which runs through 360° in the course of about 20 years. As it does this, the distance between the two planets varies by a factor of about 3.4, and hence the forces they exert on each other vary by a factor of about $(3.4)^2 = 11.6$. The expression of the perturbing force by means of a trigonometric series enabled Euler to solve the differential equations of motion to a first-order approximation. Trigonometric series later found other applications in celestial mechanics, for instance in expressing the coordinates of the Moon in terms of the mean anomaly, and the relations between mean anomaly, eccentric anomaly, and true anomaly.

A second seminal innovation in Euler's essay was his use of multiple observations in refining the values of certain coefficients. It was the first explicit appeal in mathematical astronomy to a statistical procedure. The method of least squares had not yet been invented. Euler's procedure involved forming the differential corrections for the coefficients in question, then selecting observations in which a given coefficient could be expected to be large, and solving the resulting equations approximately by neglecting terms that were relatively small. Tobias Mayer soon put this procedure to use in the lunar theory.

The lunar problem differs significantly from the planetary problem. The distance from the Moon of the chief perturbing body, the Sun, changes by only about 1/390th of its value during the course of a month, and the resulting perturbation is so minimal that it can be ignored in the first approximation. What primarily causes the lunar perturbations is the *difference* between the forces that the Sun exerts on the Moon and on the Earth. Were the Moon entirely unperturbed by the Sun, it would move about the Earth in an ellipse, one focus of which would be occupied by the Earth's center of mass; a limiting case being a circle concentric to the Earth. But as Newton showed in Corollaries 2–5 of Proposition I.66 of his *Principia*, if the Moon's pristine orbit about the Earth were a concentric circle, the effect of the Sun's extra force, over and above the force it exerts on the Earth, would be to flatten the circle in the direction of the line connecting the Earth with the Sun (the line of syzygies), decreasing its curvature there, while increasing it in the quadratures (where the angle between the Sun and Moon is 90°). Also, the Moon's angular speed about the Earth would be greater in the line of syzygies than in the quadratures. The variation in angular speed had been discovered by Tycho in the 1590s, and was named by him the "Variation." Newton derived a quantitative measure of the Variation in Propositions III.26–29 of the *Principia*, showing (on the assumption again of the Moon's having pristinely a circular orbit) that the Moon's displacement from its mean place would reach a maximum of $35'10''$ in the octants of the syzygies, and the oval into which the circle is stretched would have its major axis about one-seventieth longer than its minor axis.

Astronomers had found the eccentricity of the Moon's orbit to be, on average, about one-twentieth of the semi-major axis; were the Sun not perturbing the Moon, such an eccentricity would imply an elliptical orbit with the major axis exceeding the minor by only about 1/800th. Thus eccentricity by itself distorts the shape of the Moon's orbit less than solar perturbation. On the other hand, it causes a greater departure of the Moon from its mean motion, rising to a maximum displacement of nearly 6° approximately midway between perigee and apogee. (This departure from the mean motion is what led astronomers to assume an eccentric lunar orbit in the first place.) The true orbit of the Moon, Newton implies, is a kind of blend of the Variation oval and the eccentric ellipse – "an oval of another kind."[16]

When Newton undertook to derive a quantitative measure of the Moon's apsidal motion, probably in 1686, he attempted to meld the effects of these two orbits; his procedure was bold but unjustifiable. From this leap in the dark he later retreated, apparently recognizing its illegitimacy.[17]

The first published lunar theory giving explicit derivation of the inequalities by means of the Leibnizian calculus was Alexis-Claude Clairaut's *Théorie de la lune* (1752). Clairaut and Jean le Rond d'Alembert, both members of the prize commission for the Paris Academy's contest of 1748, had been occupied with the lunar theory since the commission met in the spring of 1746. Both of them discovered, early on, that their calculations yielded in the first approximation only about half the motion of the Moon's apse. With respect to the other known inequalities of the Moon, their calculations had yielded reasonably good approximations. Neither Clairaut nor d'Alembert supposed that the second-order approximation would be able to remove the large discrepancy in the apsidal motion. In September 1747 Clairaut learned that Euler in his lunar calculations had found the same discrepancy. The three mathematicians were calculating along rather different routes; hence the apsidal discrepancy did not appear to be an artifact of a particular procedure. Clairaut presented this discovery to the Paris Academy in November 1747, proposing that a term be added to Newton's inverse-square gravitational law, with the additional force varying inversely as the fourth power of the distance; the coefficient of this second term was to be adjusted so as to yield the missing apsidal motion. The proposal met with vigorous protest from Buffon, who regarded a two-term law as metaphysically repugnant.

Clairaut's proposal to modify the gravitational law was in accord with an idea suggested earlier by John Keill – that the inverse-square law holding for interplanetary distances might take on a modified form at smaller distances, so as to account for the forces involved in, for instance, capillary and chemical actions. Euler, by contrast, thought the gravitational law would fail at very large distances, for he attributed all forces to the impact of bodies, and gravitational force to the pressure of an aether; but the aether responsible for the "attraction" toward a particular celestial body would presumably extend only a finite distance from the body. D'Alembert, differing from both Clairaut and Euler, regarded the inverse-square law of gravitation

[16] See D.T. Whiteside, *The Mathematical Papers of Isaac Newton*, VI, (Cambridge: Cambridge University Press, 1974) 519.

[17] See my "Newton on the Moon's Variation and Apsidal Motion," in *Isaac Newton's Natural Philosophy* (eds. Jed Z. Buchwald and I. Bernard Cohen: Cambridge, MA: The MIT Press, 2001), 155–168.

as sufficiently confirmed by the empirical evidence Newton had supplied; the cause of the discrepancy in apsidal motion, he advised, should be sought in the action of a separate force, such as magnetism, reaching from the Earth to the Moon.

The issue was resolved in the spring of 1749, when Clairaut proceeded to a second-order approximation. In the new calculation, certain terms deriving from the transverse component of the perturbing force proved after integration to have very small divisors; the re-calculated coefficients were thus extremely large. These revisions led in turn to a value for the apsidal motion nearly equal to the observed value. The inverse-square law, it appeared, required no alteration.[18] On the other hand, the slow convergence revealed in the initial analytic assault on the lunar theory was to prove a persistent difficulty.

Euler published a detailed lunar theory in 1753. Its primary purpose was to confirm or disconfirm Clairaut's new result by an entirely different route. Euler eliminated the radius vector from his calculations, since it did not admit of precise measurement by the means then available (namely, micrometer measurements of the Moon's diameter). He took his value for the apsidal motion from observation, but in his equations assumed that the inverse-square law required modification by the addition of a term which he symbolized by μ. The end-result of his calculation was that μ was negligible and could be set equal to zero.

D'Alembert had registered his early writings on the lunar theory with the Paris Academy's secretary, but learning of Clairaut's new result, stipulated that they should not be published. In 1754 he published a lunar theory re-worked from the earlier versions, but now incorporating a multi-stage derivation of the apsidal motion. He gave four successive approximations, with algebraic formulas for the first two. Whether further approximations would continue to converge toward the observational value, he pointed out, remained a question. Neither he nor Clairaut searched for the deeper cause of the slow convergence they had encountered.

The predictive accuracy achieved in the lunar theories of our three mathematicians was between 3 and 5 arc-minutes – not particularly better than the accuracy of a Newtonian-style lunar theory, such as Le Monnier published in his *Institutions astronomiques* of 1746.

The first lunar tables accurate enough to give the position of the Moon to two arc-minutes, and hence to give navigators the geographical longitude to 1°, were those of Tobias Mayer (1723–1762)), published initially in 1753. They were later refined and submitted to the British Admiralty. In 1760 James Bradley, the Astronomer Royal, compared them with 1100 observations made at Greenwich, and found $1'.25$ as the upper bound of the errors. The Admiralty Board at length adopted Mayer's tables as the basis for the lunar ephemerides in the *Nautical Almanac*, which appeared annually beginning in 1767. Whence the superior accuracy of Mayer's tables?

We are unable at the present time to answer this question definitively, but it appears that empirical comparisons had much to do with the accuracy achieved.

18 A somewhat fuller account is given in "Newton on the Moon's Variation and Apsidal Motion," as cited in the preceding note, 173ff.

Mayer began with a Newtonian-style theory.[19] At some date he carried out an analytical development of the lunar theory, following, with some variations, the pattern laid out in Euler's theory of Jupiter and Saturn of 1749; he carried the analysis so far as to exhaust, as he said, "nearly all my patience." Many of the inequalities, he found, could not be deduced theoretically with the desired accuracy unless the calculation were carried still farther. From Euler's prize essay on Saturn's inequalities he had learned how the constants of a theory could be differentially corrected by comparison with large numbers of equations of condition based on observations; and he had applied such a process in determining the Moon's librations (slight variations in the face that the Moon presents to an Earth-bound observer, due primarily to variations in the Moon's orbital speed combined with the Moon's almost exactly uniform axial rotation). But of the processes he used in determining the Moon's motions in longitude, he gives us no description. We know that he assembled a large store of lunar observations, many of them his own, including extremely accurate ones based on the Moon's occultations of stars. Presumably he once more constructed Eulerian-style equations of condition, solved them approximately, and thus refined the coefficients of his theoretically derived terms to achieve a superior predictive accuracy.

Mayer's tables, being semi-empirical, did not answer the theoretical question as to whether the Newtonian law could account for all lunar inequalities. But they met the navigator's practical need, supplying a method for determining longitude at sea – at first the only method generally available. In later years, as marine chronometers became more affordable and reliable, the chronometric method was understandably preferred. The chronometer gave the time at Greenwich, and this, subtracted from local time as determined from the Sun, gave the difference in longitude from Greenwich. The method of lunar distances, by contrast, required a much more extended calculation. The latter method was long retained, however, as supplying both an economical substitute for the chronometrical method and an important check on it.

In 1778 Charles Mason revised Mayer's tables, relying on 1137 observations due to Bradley, and using, we assume, a similar deployment of equations of condition. It was in the same way, apparently, that Tobias Bürg revised Mason's tables early in the 1800s; he used 3000 of the Greenwich lunar observations made by Maskelyne between 1760 and 1793. From Mayer's theoretical derivation (published by the Admiralty in 1767), Mason deduced eight new terms, and Bürg added six more, to be included in the tables. But the accuracy of the tables depended crucially on the empirical refining of constants.

When Laplace undertook to deduce the lunar motions from the gravitational law, he saw these semi-empirical tables as setting a standard of accuracy difficult to surpass (*Mécanique Céleste*, Book VII, Introduction). Laplace's theory was considerably more accurate than the earlier analytical theories of Clairaut, Euler, and d'Alembert. This was principally because of Laplace's discovery of new inequalities by deduction from the gravitational law. Among these new inequalities were

[19] Private communication from Steven Wepster of the Mathematics Department, University of Utrecht.

two arising from the Earth's oblateness (the decreasing curvature of its surface from equator to poles). Moreover, Laplace for the first time supplied a gravitational explanation for the Moon's secular acceleration, as arising indirectly from the secular diminution of the eccentricity of the Earth's orbit; his deduced value for it was in good agreement with observations. (In the 1850s it would be found to be theoretically in error, so that a drastic reinterpretation was required – a topic that we shall return to in Part III.) The greatest difference between the predictions of Laplace's theory and Bürg's tables was 8.3 arc-seconds; thus the theoretical deduction fell little short of the accuracy attainable by comparisons with observations. The day was coming, Laplace confidently predicted, when lunar tables could be based on universal gravitation alone, borrowing from observation solely the data required to determine the arbitrary constants of integration.

Bürg's tables were published by the French Bureau des Longitudes in 1806. In 1811 J.K. Burckhardt presented new lunar tables to the Bureau; they were based on 4000 observations as well as on the terms newly discovered by Laplace. A commission compared Bürg's and Burckhardt's tables with observations of the Moon's longitudes and latitudes from around the orbit, using the method of least squares to assess the goodness of fit (this appears to have been the first published use of MLS). In 167 observations of the Moon's longitude, the root mean square error of Bürg's tables was $6''.5$, compared with $5''.2$ for Burckhardt's tables; in 137 observations of the Moon's latitudes, the corresponding numbers were $6''.0$ and $5''.5$. Consequently Burckhardt's tables were adopted as the basis of the lunar ephemerides in the French *Connaissance des Temps* and in the British *Nautical Almanac*. They would continue in that role, with some later corrections, through 1861.

For its prize contest of 1820, the Paris Academy of Sciences, at Laplace's urging, proposed the problem of forming tables of the Moon's motion as accurate as the best current tables [i.e., Burckhardt's] on the basis of universal gravitation alone. Two memoirs were submitted, one by the Baron de Damoiseau (1768–1846), director of the observatory of the École Militaire in Paris, the other by Giovanni Plana (1781–1864) and Francesco Carlini (1783–1862), directors, respectively, of the observatories in Turin and Milan. Both memoirs were Laplacian in method. Damoiseau proceeded more systematically than had Laplace. From the start he put the reciprocal radius vector (u) equal to $u_0 + \delta u$, and the tangent of the latitude (s) equal to $s_0 + \delta s$, where u_0 and s_0 are the elliptic values of u and s, and δu and δs are the modifications produced by perturbation. He developed the expressions for u and s to the sixth order inclusive in the lunar and solar eccentricities and inclination of the lunar orbit, whereas Laplace had stopped at the fourth order. He put δu, and also δs, equal to a set of sinusoidal terms, with the coefficient of each such term containing an undetermined factor; there were 85 such factors in the expression for δu and 37 in the expression for δs. Substituting the expressions for u and s into the differential equations, replacing the arbitrary constants by their empirical values, and setting the coefficient of each sine and cosine term equal to zero, Damoiseau obtained 207 equations of condition, which he solved by successive approximations for the undetermined factors. Because he substituted numerical values of the arbitrary constants from the start, his theory is called a *numerical* theory; it is to be contrasted with a *literal* theory in which the coefficients are expressed as algebraic functions of the arbitrary constants. Comparing Damoiseau's tables with 120 observations, and finding them to be of the same order of accuracy as Burckhardt's tables, the prize commission deemed them worthy of the prize.

Plana and Carlini in their memoir undertook to achieve a strictly literal solution of the differential equations. The coefficients of the sinusoidal terms of the theory are functions of certain constants of the theory – the orbital eccentricities of the Moon and the Sun, the tangent of the Moon's orbital inclination to the ecliptic, the ratio of the Sun's and Moon's mean motions, the ratio of the mean Moon-Earth and Sun-Earth distances. But these functions are far too complicated to be represented analytically, except in the form of infinite series in the powers and products of the constants involved. Our authors accordingly introduced such series into the representation of the theory – an important innovation, revealing the causal provenance of each term, and permitting the effect of any revision of a constant to be immediately calculated. The numerical factor that multiplies any term in such a series can be determined not merely approximately but exactly, as a numerical fraction, and the approximate character of the coefficient is due only to the series having to be broken off after a finite number of terms rather than being summed as a whole.[20] Unfortunately, for some of the series the rate of convergence was excruciatingly slow. Where denominators were produced by the integrations, Plana and Carlini developed their reciprocals as series and multiplied them into the numerators, often with a decrease in rate of convergence. At the time of the contest deadline they had not yet constructed tables, but they showed that their coefficients for the inequalities in longitude were in close agreement with Burckhardt's. In view of the immense labor that their memoir embodied, and the value of the resulting analytic expressions, the Academy decreed that they, like Damoiseau, should receive the full value of the prize as originally announced.

Plana went on to achieve a more complete development of the Plana-Carlini theory in three large volumes published in 1832. Here the dependent variables u and s emerge in successive approximations. Volume II gives the results accurate to the fifth order of small quantities, while Volume III gives the developments required to proceed to still higher orders.

The lunar theories of Clairaut, d'Alembert, Laplace, Damoiseau, and Plana all took as independent variable the true anomaly v, expressing the true longitude of the Moon from the lunar apse. Hence the variables u and s were obtained as functions of v, and so also was the mean anomaly ($[nt + \varepsilon]$ in Laplace's notation, where n is the mean rate of motion, t is the time, and ε the mean longitude at epoch). The resulting series, Laplace stated, converged more rapidly than the series obtained when the independent variable was the mean anomaly. The choice of v as independent variable meant that, to obtain u, s, and v as functions of t, it was necessary to obtain v as a function of the mean anomaly by reversion of the series for $nt + \varepsilon$ in terms

[20] A number of the points made here are due to J.C. Adams, "Address on presenting the Gold Medal of the Royal Astronomical Society to M. Charles Delaunay," *The Scientific Papers of John Couch Adams*, *I*, 328–340.

of ν. This operation becomes increasingly laborious as higher-order approximations are undertaken, and in 1833 Siméon-Denis Poisson (1781–1840) proposed that it be avoided by taking t as independent variable from the start. His former student Count Philippe G.D. de Pontécoulant was the first to carry through a complete development of the lunar theory on this plan. It was published in 1846 as Volume IV of Pontécoulant's *Théorie du système du monde*.

After completing the analytic development, Pontécoulant substituted empirical values for the constants in his formulas, and compared the resulting coefficients of terms in the longitude with those given by Damoiseau, Plana, and Burckhardt. His and Plana's coefficients agreed closely, despite the difference in their methods. Of Pontécoulant's 95 longitudinal terms, Plana gave 92. In eleven cases of discrepancy Pontécoulant traced the difference to errors in Plana's derivations – errors later verified and acknowledged by Plana. The differences between Pontécoulant's and Burckhardt's coefficients were generally small; in two cases they exceeded $2''$, and in 16 they exceeded $1''$. Pontécoulant believed the fault lay with the observations on which Burckhardt's tables were based.

In 1848 G.B. Airy published a reduction of the Greenwich lunar observations for the period 1750–1830. To compare the sequence of resulting positions of the Moon with theory, he turned to Damoiseau's tables of 1824, but with the coefficients modified to agree with Plana's theory, including all corrections so far found necessary. From Plana's theory and the observations, Airy then obtained corrected orbital elements for the Moon. Airy's lunar elements were the basis on which Benjamin Peirce of Harvard founded his *Tables of the Moon* (1853, 1865), from which were derived the lunar ephemerides published in the *American Ephemeris and Nautical Almanac* from its inception in 1855 through 1882.

For accuracy, however, lunar theories and tables from Damoiseau's to Pontécoulant's were outdistanced by the *Tables de la lune* of Peter Andreas Hansen (1795–1874), published in 1857. Deriving perturbations from gravitation alone, Hansen achieved an accuracy superior to Burckhardt's. His tables were adopted for the British and French national ephemerides beginning with the year 1862, and for the American *Nautical Almanac* beginning with the year 1883; they would remain in that role till 1922.

Hansen's method differed from that of any earlier theory. He had devised his way of computing perturbations in the course of preparing a memoir for submission in the Berlin Academy's contest of 1830. The problem posed by the Academy concerned Laplace's and Plana's conflicting results for second-order perturbations of Saturn due to Jupiter. Contestants were asked to clarify the issues involved.

The difficulty in deriving analytically the motion of the Moon's apse in the 1740s had led to the recognition that perturbations must necessarily be computed by successive approximations. Often the first approximation would prove sufficiently precise, but if greater precision were needed, the approximations could be arranged in a series with respect to powers of the perturbing force. For instance, to compute Saturn's perturbations of the first order with respect to Jupiter's perturbing force, you started from assumed approximate motions for the two planets (motions, say, following Kepler's "laws"), and on this basis calculated the attractions whereby Jupiter perturbs Saturn. To obtain the second-order perturbations of Saturn, the first-order perturbations of Jupiter due to Saturn, as well as the first-order perturbations of Saturn due to Jupiter, had to be taken into account. Thus the approximations initially assumed were to be progressively refined. When the corrections became smaller than the currently attainable observational precision, the result could be accepted as sufficiently precise.

Laplace gave no systematic procedure for perturbations beyond those of first-order. Second-order perturbations, he believed, would need to be calculated only in special cases – where, for instance, the first-order perturbations were large. He failed to recognize the need for a systematic way of obtaining higher-order perturbations. It would later become evident that he had omitted second-order perturbations as large as those he calculated. Nor did Plana, though questioning Laplace's second-order results, supply a systematic procedure.

A systematic and rigorous procedure for first- and higher-order perturbations, however, was already at hand. It utilized formulas in the second edition of Lagrange's *Mécanique analytique* (1814). These formulas expressed the time-rates of change of the orbital elements as functions of these same elements and of the partial derivatives of the disturbing function with respect to them. (The disturbing function, a Lagrangian innovation, is a potential function from which the force in any direction can be derived by partial differentiation.) These formulas were rigorous, and remarkable in their independence of the time. Lagrange was imagining the planet or satellite as moving at each instant in an ellipse characterized by its six orbital elements, with the elements changing from instant to instant due to perturbation. Second- and higher-order perturbations were derivable by applying the well known "Taylor's theorem".

This procedure, however, was time-consuming. The perturbations of all six orbital elements had to be computed, whereas it was only the perturbations of the coordinates, three in number, that were required practically. The perturbations of the elements were often larger than those of the coordinates, so that a smaller quantity would have to be determined from the difference of two larger ones, giving a result of uncertain precision. Hansen therefore set out to transform Lagrange's formulas, so as to obtain a more direct route from disturbing function to the perturbations of the coordinates.

Two simultaneous processes had to be taken into account: the continuous change in shape and orientation of the instantaneous elliptical orbit in which the perturbed body was conceived to be traveling, and the body's motion along this protean orbit. The first of these processes was expressible through the Lagrangian formulas giving the rates of change of the orbital elements. The second process was governed by well-known elliptical formulas: the true anomaly of the body (its longitude from perihelion) was given, through an auxiliary variable, in terms of the mean anomaly; and the radius vector was given in terms of the true anomaly.

The main focus of Hansen's method was on the perturbations affecting the orbital motion in the instantaneous plane (he treated the perturbations in the position of the instantaneous plane separately). Here two processes needed to be kept distinct: change in shape and size of the ellipse and motion of the body along it. For this purpose Hansen introduced two variables for the time: t for the time in which changes in orbital elements are registered, τ for the time in which the motion along the orbit occurs. Eventually the two times would be identified as one, the single time of the ongoing, twofold process.

To have a single variable that would incorporate both aspects of this double process, Hansen introduced ζ as a function of both t and τ. To define it quantitatively, he stipulated that the true anomaly λ should be a function of ζ, and through ζ of t and τ. Hence

$$\frac{\partial\lambda}{\partial t} = \frac{\partial\lambda}{\partial\zeta} \times \frac{\partial\zeta}{\partial t},$$
$$\frac{\partial\lambda}{\partial\tau} = \frac{\partial\lambda}{\partial\zeta} \times \frac{\partial\zeta}{\partial\tau}. \qquad \text{(Ha.1)}$$

The quotient of the first of these equations by the second is

$$\frac{\partial\zeta/\partial t}{\partial\zeta/\partial\tau} = \frac{\partial\lambda/\partial t}{\partial\lambda/\partial\tau}. \qquad \text{(Ha.2)}$$

Now $\partial\lambda/\partial t$ is given in terms of the Lagrangian formulas for rates of change of the orbital elements; and $\partial\lambda/\partial\tau$ in terms of known elliptical formulas. Hence the quotient on the right side of (Ha.2) is expressible in terms of explicitly defined quantities.

To obtain an expression for ζ, Hansen proceeded by successive approximations. In the first approximation, he set $\partial\zeta/\partial\tau$ equal to 1, so that $\zeta = \tau$. Equation (Ha.2) then simplifies to an expression for $\partial\zeta/\partial t$ which can be integrated with respect to t, yielding a first-order expression for ζ. Differentiating this expression with respect to τ, Hansen obtained an improved value of $\partial\zeta/\partial\tau$, which he substituted back into (Ha.2). The resulting expression when integrated with respect to t gave the second-order approximation to ζ. Higher-order approximations were obtained by repeating this process. At the end of each stage of approximation, Hansen replaced τ by t, and ζ by z. Thus in descriptions of Hansen's method the variable z is sometimes referred to as "the perturbed time", and nz as "the perturbed mean anomaly."

The foregoing sketch omits crucial detail, such as the steps required to determine the arbitrary constants introduced by the integrations, the processes for determining the radius vector as a function of ζ, and the procedure for finding the instantaneous plane in which the instantaneous ellipse is located. Among features distinguishing Hansen's development of the theory were his use of harmonic analysis (or "special values"), as advocated by Gauss, in determining the disturbing function, and his application of Bessel functions in the expansions. Like Damoiseau before him, he insisted on a *numerical* rather than a *literal* form for his theory, and introduced approximate numerical values for the orbital elements at an early stage, so as to avoid the problems of slow convergence of series encountered by Plana, and to make sure that all terms greater than an agreed-upon minimum would be included.

After completing his memoir on the mutual perturbations of Jupiter and Saturn (*Untersuchung über die gegenseitigen Störungen des Jupiters und Saturns*, Berlin, 1831), Hansen set out to apply his new method to the lunar problem. He described this application in his *Fundamenta nova investigationis orbitae verae quam luna perlustrat* (Gotha, 1838). Is the method really suitable to the lunar problem? Brouwer and Clemence in their *Methods of Celestial Mechanics* suggest that it is not. They give high marks to Hansen's method in its application to planetary perturbations, but they describe his adaptation of it to the lunar problem as a *tour de force*.[21] The method as set forth in the *Fundamenta* presents new complications, not easily susceptible of schematic description. We mention here only certain major new features. A full account is given by Ernest W. Brown in his *Introductory Treatise on the Lunar Theory*, Chapter X.

Hansen's earlier treatment of the latitudes had lacked rigor, while the lunar latitudes require an especially careful development. In the *Fundamenta* Hansen succeeded in deriving them as accurately as could be wished, taking account of the motions of the ecliptic as well as those of the instantaneous plane of the lunar orbit with respect to a fixed plane. Comparing the different derivations of the perturbations in latitude put forward by the celestial mechanicians of his day, the mathematician Richard Cayley found Hansen's alone to be strictly rigorous.[22]

A special difficulty in the lunar theory comes from the relatively large motions of the Moon's perigee and node in each lunar month, much larger proportionately than the motions of the perihelion and node of any planet during its sidereal period. In his theory of Jupiter and Saturn, Hansen had permitted terms proportional to the time (t) and its square (t^2) to be present, but in the lunar case such terms would quickly become embarrassingly large. To avoid them Hansen introduced a factor y, such that the mean rate of the perigee's advance is ny, where n is the mean rate of advance in longitude, and y is constant so long as only the perturbations due to the Sun are considered. He likewise used y in defining the mean rate of recession of the lunar node.

Another new feature in the *Fundamenta* was the introduction of a function W which, integrated twice, gave the perturbations in the instantaneous plane of the orbit. Initial values for the mean anomaly and radius vector were taken from an auxiliary ellipse of fixed eccentricity and unvarying transverse axis, the mean motion on it having a fixed rate n_0, and the perigee progressing at the steady rate $n_0 y$. The perturbed mean anomaly, nz, was obtained by the integration of W, and then substituted into the standard elliptical formulas to yield the true anomaly. To find the perturbed radius vector r, Hansen stipulated that $r = r_0(1 + \nu)$, where r_0 is the radius vector in the auxiliary ellipse, and ν is a small fraction which represents the perturbations and is obtained from the integration of W.

Hansen's lunar theory, Brown tells us, was "much the most difficult to understand of any of those given up to the present time [1896]." Presumably Hill, at an early stage in his studies, became acquainted with it, but there are no references to it in his writings of the 1870s. To Hansen's work on Jupiter and Saturn, on the contrary, Hill

21 D. Brouwer and G.M. Clemence, *Methods of Celestial Mechanics* (New York: Academic, 1961), 335, 416.

22 See R. Cayley, "A Memoir on the Problem of Disturbed Elliptic Motion," *Memoirs of the Royal Astronomical Society, 27* (1859), 1.

refers explicitly in an article of 1873 concerning a long-term inequality of Saturn; and a publication of 1874 shows his intensive study of Hansen's *Auseinandersetzung einer zweckmässigen Methode zur Berechnung der absoluten Störungen der kleinen Planeten*.[23] When in the decade 1882–1892 he developed the theory of Jupiter and Saturn, he chose to apply the method of the *Auseinandersetzung*, with the modification of taking the mean anomaly as independent variable, whereas Hansen had chosen the eccentric anomaly for this role.

Hill in an article of 1883 takes issue with Hansen's assertion that the long-period inequalities of the Moon due to planetary action are difficult to compute, and proposes an elegant method deriving from Cauchy.[24] Hill's memorandum regarding new tables of the Moon, which we have reproduced in an Appendix, makes evident Hill's strongly negative assessment of the future of Hansen's lunar theory. He saw no way in which, by various adjustments, it could be brought up to the standards of exactness and clarity he regarded as obligatory for the celestial mechanics of his day. He envisaged a theory transparent in the sense that each derived effect was clearly traceable back to the assumptions and numerical constants on which it depended. Hansen's theory could not be so described. When E.W. Brown's *An Introductory Treatise on the Lunar Theory* (Cambridge University Press, 1896) appeared, Hill wrote Brown to compliment him on the book, but, as Brown reports it, with one criticism:

> He thinks it would have been better to leave out Hansen – because he says 'it will probably never be used again'! Otherwise he is complimentary – but I don't think he appreciates what a student beginning the subject wants.[25]

Delaunay's lunar theory initially aroused Hill's enthusiastic allegiance. It had been published in two huge volumes in 1860 and 1867, and Hill had begun studying it early in the 1870s. This study influenced his interests and thinking pervasively, as articles published in *The Analyst* in 1874 and 1875 testify.[26] Delaunay had not given a derivation of the Hamiltonian-style canonical equations on which he based his theory, referring instead to a memoir by Binet published in 1841.[27] (Binet was the first to develop canonical equations in which the variables are the elliptical elements

[23] The reference is given in note 8.

[24] G.W. Hill, "On certain possible abbreviations in the computation of the long-period inequalities of the Moon's motion due to the direct action of the planets," *American Journal of Mathematics, 6* (1883), 115–130.

[25] E.W. Brown to G.H. Darwin, 21 March 1896, CUL. MS. DAR.251:479.

[26] "Remarks on the Stability of Planetary Systems," *The Analyst*, *I* (1874), 53–60; "The Differential Equations of Dynamics," ibid., 200–203; "On the Development of the Perturbative Function in Periodic Series," *The Analyst*, *II* (1875), 161–180.

[27] M.J. Binet, "Mémoire sur la variation des constants arbitraires dans les formulas générales de la dynamique," *Journal de l'École Polytechnique*, Vingt-Huitième Cahier, *T.XVII* (1841), 1–94. Binet's work derives, not from Hamilton or Jacobi, but from Poisson (personal communication from Michiyo Nakane; see M. Nakane and C.G. Fraser, "The Early History of Hamilton-Jacobi Dynamics 1834–1837," *Centaurus, 44* (2002), 161–227.)

of motion of a planet or satellite; Delaunay used them with one change, indicated below.) In an article published in 1876, Hill derived Delaunay's equations, relying not on Lagrange's and Poisson's brackets, which, as he acknowledged, permitted the equations to be established in a very elegant manner, but "on more direct and elementary considerations."[28] He evidently saw his role here as that of presenting to American mathematicians a sophisticated development with which they were presumably unfamiliar. His first sentence conveys what he saw in it:

> The method of treating the lunar theory adopted by Delaunay is so elegant that it cannot fail to become in the future the classic method of treating all the problems of celestial mechanics.

The rudiments of Delaunay's method may be described as follows.[29] Let R be the disturbing function, and let the elements selected as canonical be: ℓ, the mean anomaly; g, the angle between the node on a fixed plane and the perigee; h, the angle between the node and a fixed line in the fixed plane; $L = \sqrt{(a\mu)}$, where a is the semi-major axis and μ is the sum of the masses of the Earth and the Moon; $G = L\sqrt{(1 - e^2)}$, where e is the eccentricity; and $H = G \cos i$, where i is the orbital inclination. The mean anomaly $\ell = nt + \varepsilon$ is an unexpected choice for an element since it is not a constant in the unperturbed elliptical orbit; Delaunay introduced it to replace one of Binet's elements (*viz.*, a factor entering into n), to avoid the emergence of terms proportional to t in the partial derivatives of R. Of Delaunay's set of elements, Hill remarks that "it does not appear that a better can be selected." For the disturbed ellipse Delaunay then obtained the canonical equations

$$\begin{aligned} \frac{dL}{dt} &= \frac{\partial R}{\partial \ell}, \quad \frac{dG}{dt} = \frac{\partial R}{\partial g}, \quad \frac{dH}{dt} = \frac{\partial R}{\partial h}, \\ \frac{d\ell}{dt} &= -\frac{\partial R}{\partial L}, \quad \frac{dg}{dt} = -\frac{\partial R}{\partial G}, \quad \frac{dh}{dt} = -\frac{\partial R}{\partial H}. \end{aligned} \tag{D.1}$$

Delaunay developed R as a series of cosines of multiples of the angles ℓ, g, h, and ℓ', where ℓ' is the mean anomaly of the Sun. If the unperturbed values of these variables are identified by the subscript "0," the resulting series may be written

$$R = F + \sum A \cos[i_1(nt + \ell_0) + i_2 g_0 + i_3 h_0 + i_4(n't + \ell')],$$

where i_1, i_2, i_3, i_4 are integers, and the summation extends to all sets of integers leading to detectable inequalities.

Delaunay's strategy in solving the equations was to separate R into two parts, R_1 and $R - R_1$, where R_1 is a single term in R, and solve the equations with R_1

[28] G.W. Hill, "Demonstration of the Differential Equations Employed by Delaunay in the Lunar Theory," *The Analyst, III* (1876), 655–670.

[29] Our account is based on Delaunay's earliest description: "Mémoire sur une nouvelle méthode pour la determination du movement de la lune," *Comptes rendus hebdomadaires des séances de l'Académie des Sciences, 22* (1846), 32–37.

substituted for R. The solution gave him new values of L, ℓ, G, g, H, h, which he substituted into $R - R_1$. Then he repeated the process, separating $R - R_1$ into two parts, R_2 and $R - R_1 - R_2$, and solving the equations with R_2 substituted for R. The process was to be repeated over and over, at each stage removing the largest remaining term in R, until all significant terms were removed. At each stage the literal expressions of L, ℓ, G, g, H, h approached more nearly to their final form.

According to Hill, Delaunay's procedure was based on the most advanced and elegant formulation of dynamics available, and it provided complete transparency in the relations between causes and derived effects. A distinct advantage was its enabling the calculator to focus on one term of the disturbing function at a time. Later, Hill would come to regard the large number of tedious transformations entailed by the method as a serious drawback.

At some point in the mid-1870s, Hill became aware of a more serious difficulty in the application of Delaunay's method to the Moon – a difficulty which, from the human standpoint, looked fatal. The series determining the coefficients of some perturbation terms converged so slowly that to obtain a result of the desired precision required a quite unreasonable expenditure of time and effort. It was the same difficulty that Plana and Pontécoulant had encountered, and that had led Hansen to choose a numerical form for his theory. Delaunay carried the development of his series to the eighth and sometimes to the ninth order of small quantities, still without attaining a final result of sufficient precision to match the precision of contemporary observations. Seeing the daunting number of further terms that would have to be calculated if he were to proceed to the next higher order, he introduced "probable complements", based on the rate of decrease of the last two or three terms calculated. Newcomb later found these complements, though in some cases roughly correct, quite illusory in others. Delaunay's method, after seeming to promise exact science, was here showing itself irremediably inexact.

For Hill, the recognition of this difficulty was a trumpet call. A new beginning was necessary. Hill opted for a radical departure from the tradition of past lunar theory.

All lunar theorists from Kepler to Delaunay, Euler alone excepted, had taken a solution of the *two-body* problem in Newtonian theory – a circular or elliptical orbit of the Moon about the Earth – as their starting-point, then superimposed on it periodic variations as required by theory or observation. John Couch Adams, in opening his lectures on lunar theory in the 1860s, called this procedure "the method of the Lunar Theory":

> The Earth and Moon describe orbits round the Sun which are approximately ellipses, and the Moon might be regarded as one of the planets; but this point of view would not be a simple one; the disturbing action of the Earth would be too great, though it is never so great as the direct attraction of the Sun, that is to say, never great enough to make the Moon's path convex to the Sun. The more convenient method is to refer the motion of the Moon to the Earth, and counting only the difference of the attractions of the Sun

> upon the Earth and upon the Moon, to find how this distorts the otherwise elliptical relative orbit. This is the method of the Lunar Theory.[30]

In contrast, Hill will take as starting-point an oval orbit of the Moon about the Earth – a circle flattened toward the Sun by the difference between the Sun-induced, Sun-ward accelerations of the Moon and of the Earth; it is a periodic solution of a simplified version of the *three-body* problem. It was in fact the same curve that Newton had found as an effect of the Sun's action on the Moon. As previously noted, and unbeknownst to Hill, Newton in the 1680s had computed an ellipse which approximated this "Variation curve" with considerable accuracy. And J.A. Euler, son of Leonhard Euler, had calculated in 1766 the first two terms giving the Moon's motion on the Variation curve (Euler's coefficient for the second term is mistaken, owing to a simple numerical error).[31] Young Euler's article contains the statement, "I dare assert that if anyone succeeded in finding a perfect solution [to the problem of the Variation], he would scarcely find any further difficulty in determining the true motion of the real Moon." Leonhard Euler, the father, was likely the source of this claim, but its decisive substantiation would have to await the elaboration of the Hill–Brown lunar theory. We shall find Hill proceeding just as if he had read and accepted Euler's pronouncement (we have no evidence that he in fact had seen it).

In his final lunar theory, published in 1772, Leonhard Euler chose rotating rectangular coordinates, the x and y coordinates rotating in the plane of the ecliptic about the z axis with the mean angular speed of the Moon. His objective was to obtain series that converged rapidly. He separated the periodic developments of the lunar coordinates into classes according to the parameters on which they depended: K, the eccentricity of the lunar orbit; i, the inclination of the lunar orbit to the ecliptic; κ, the eccentricity of the solar orbit; a, the ratio of the Sun's parallax to the Moon's parallax; p, the difference between the mean motion of the Moon and the mean motion of the Sun, which Euler calls the "mean elongation;" q, the mean anomaly of the Moon; r, the mean argument of latitude; and t, the mean anomaly of the Sun. The stress on inequalities dependent on p does not appear to be present; p is simply one of eight parameters on which the Moon's motion depends. Euler was thus proposing to develop his mathematical theory systematically in terms of the successive powers and products – of one, two, three, and higher dimensions – of these small parameters. It was a new way of proceeding, which could guarantee the correctness of the theory to any pre-chosen level of precision.

Why did Euler (father and/or son) in the paper of 1766 claim that, given a perfect solution of the problem of the Variation, the further development of the lunar theory would be without difficulty? Euler does not say, but the following considerations were probably part of his thinking.

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[30] J.C. Adams, "Lectures on the Lunar Theory," in *The Scientific Papers of John Couch Adams*, II (Cambridge: Cambridge University Press, 1900), 6. The lectures were given with successive refinements from 1860 to 1889.

[31] J.A. Euler, "Réflexions sur la variation de la lune," *Histoire de l'Académie Royale des Sciences et Belles-Lettres*, Berlin, 1766, 334–353.

For earlier investigators, the Variation was a single term, a sinusoidal term discovered empirically by Tycho, with argument equal to twice the difference between the mean longitudes of the Moon and the Sun, i.e., $2(n - n')$. Euler's paper of 1766 derives not only this term but a second term, with double the argument of the first term, i.e., $4(n - n')$. Euler knew his solution to be an approximation at best; with more investment of labor, further terms could be derived. This discovery can have been a stepping-stone to Euler's project in the theory of 1772: to develop the entire lunar theory in terms of the powers and products of small parameters, by successive approximations.

But secondly, Euler may have come to see the Variation as more intrinsic to the lunar problem than any of the other inequalities found in the Moon's motion – inequalities dependent on eccentricity, inclination, or parallax. Let us imagine the eccentricities K and κ, the inclination i, and the ratio a of solar parallax to lunar parallax diminishing so as to become negligible or zero; a "Variation" would still be present in the Moon's motion, provided only that the Moon's mean motion n and the Sun's mean motion n' differed. Deriving the resulting motion of the Moon would be solving an essentially three-body problem. To cope with it, the analyst would no doubt proceed by successive approximations. The parameter in terms of which to develop these approximations could be $m = n'/n$ or $\mathbf{m} = n'/(n - n')$. While Newton was able to show by qualitative geometrical arguments that the Variation curve is some kind of oval, flattened along the line of syzygies, it is important to note that, except for successive approximations in terms of m or $\mathbf{m}$, no other avenue to learning the precise nature of the Variation orbit and motion was – or yet today is – known.

It will be worth our while to review certain general qualitative features of the Variation; see the figure below. The Moon moves about the Earth **E** in an orbit **abcd**, while the Earth-Moon system moves about the Sun **S**; we have exaggerated both the size of the orbit **abcd** relative to the distance **SE**, and the flattening of this orbit. The period of the Earth about the Sun, reckoned with respect to the stars, is 365.256 days. The period of the Moon about the Earth, reckoned again with respect to the stars, is 27.321 days. These two numbers, with their ratio, have been – at least until the introduction of atomic clocks in 1955 – as accurately known as any constants in all of astronomy.

Two further numbers are needed to determine the ratio of the forces of the Sun and Earth on the Moon. These two numbers are the Earth-Sun distance and the Moon-Earth distance. The mean ratio of these distances was already known in the

1760s to be about 380 or 390 to 1. J.A. Euler in his paper of 1766 assumed a solar parallax of 9″; this with the known lunar parallax of close to 57′.0 implies a ratio of 380:1. The senior Euler in his lunar theory of 1772 used the value 390:1 for this ratio. The accepted value today is about 389:1. These data, along with Proposition 4 of Book I of Newton's *Principia*, yield a value for the ratio of the Earth's force on the Moon to the Sun's force on the Moon. With Newton's value for the solar parallax, 10.5 arcseconds (corresponding to an Earth-Sun distance of 19,644 Earth radii), the Sun's force on the Moon comes out to be 1.8 times the Earth's force on the Moon. With Euler's value of 1/390 for the ratio of parallaxes, the Sun's force on the Moon is found to be 2.18 times the Earth's force.

Since the two forces act constantly, the Moon's path must at each instant be curved concavely toward *both* the Sun and the Earth. To understand how this can be, consider the Moon moving from **a**, where it is a new Moon, to **b**, where it is at the first quarter. Its path **ab** is shown in the preceding figure as convex toward the Sun, but this is an illusion due to the diagram's failing to incorporate time and motion. The Moon requires 7.4 days to move from **a** to **b**, an arc which at the Sun subtends an angle of 8.8 arcminutes, or less than one-sixth of a degree. But in 7.4 days the whole Earth-Moon system moves through 7°.293 about the Sun. The relatively tiny motion that takes the Moon around the Earth is dwarfed with respect to the larger sweep that takes the Earth-Moon system about the Sun. This larger sweep moves the Moon in an arc always concave toward the Sun, while the Moon creeps round the arc **ab** which, reckoned in the moving space with Earth at its origin, is always concave toward the Earth. Since the curvatures are inversely as the radii, the Moon's orbit about the Earth has a curvature 389 times the curvature of the Moon's path about the Sun. The curvatures are directly as the accelerative forces, but inversely as the 3/2 powers of the linear velocities. Given that the accelerative force of the Sun on the Moon is 2.18 times the accelerative force of the Earth on the Moon, the much larger curvature of the Moon's path about the Earth compared to the curvature of its path about the Sun is due to the much smaller linear velocity of the Moon's motion about the Earth – only about 1/90th of its velocity about the Sun.

The Variation, more than the other parametric dependencies of the Moon's motion considered by Euler, must have led him to ponder more deeply the dynamic complexities presented by our Moon's motion. The curve the Moon follows in space is fully determinate, yet its essence, its mathematical formula, its exact individuality, is unknown, except the parameters governing it be extracted by successive approximations, step by step. Newton approximated the Variation curve with an ellipse, but it is not an ellipse or any other curve with a finitely expressible formula. In this respect the Variation resembles the lunar theory as a whole; the exact character of the motion is hidden in the dynamics. These realizations must have led Euler to propose that, of all the problems in the Moon's motion, the problem of the Variation should be tackled first, and independently of the other lunar inequalities.

Hill's acquaintance with Euler's theory came about in his undergraduate study at Rutgers in 1855–1859 under Theodore Strong, professor of mathematics. Strong, Hill later recalled, was old-fashioned, and liked to go back to Euler for all his theorems, asserting that "Euler is our Great Master."[32] Hill, in the introduction to his paper of 1878, explicitly cites Euler's lunar theory of 1772 as providing the model for his own partition of the inequalities into classes. Hill's papers and Euler's lunar theory also agreed in using rotating rectangular coordinates, but for Hill the coordinates rotated with the mean speed of the Sun, not the Moon.

The Eulerian roots of Hill's new theory are important. Without Hill's having previously become acquainted with Euler's theory of 1772, he might never have thought of developing the lunar theory along Eulerian lines.

Also important, however, were the respects in which Hill went beyond Euler. First, he had studied the methods of Hansen and Delaunay. Hansen's *Untersuchung* showed how all terms with coefficients greater than a pre-specified lower bound could be obtained – a kind of result that no earlier mathematical astronomer had achieved. Delaunay's completely literal lunar theory permitted each perturbational term to be traced back to the assumptions on which it was based. Hill undoubtedly saw the exactitude and transparency thus illustrated as standards that a new theory ought to meet.

Crucial to Hill's new solution of the lunar problem was the Jacobian integral, an integral of the equations of motion for a restricted form of the three-body problem. Nothing similar was available to Euler, who had long struggled to integrate the equations of the general three-body problem, and had at last given up the attempt. In the lunar case he made no use of general integrals, such as those for *vis viva* and angular momentum. Knowing in advance that the Moon's position depended on certain parameters, he formed differential equations each of which contained trigonometric terms deriving from just one of these parameters or the product of two or more, and solved the equations one after another by the method of undetermined coefficients. He did not attempt to calculate the motions of the apsidal and nodal lines, but used the values for these constants that Mayer had derived from observation. Other constants besides those introduced by integration, he suggested, might have to be evaluated observationally. His primary aim was to achieve a precision of one minute of arc, matching the precision of the available observations.

In contrast, Hill's solution will be controlled by the *vis viva* integral due to C.G.J. Jacobi and first published in the *Comptes rendus* of the Paris Academy in 1836.[33] According to Jacobi in his *Vorlesungen über Dynamik*, Euler had regarded the *vis viva* integral as valid only about a *fixed* center of attraction, whereas the Jacobian integral was here applied to a *moving* center; Jacobi credits Lagrange with the extension to moving centers.[34] For Hill, the Jacobian integral did yeoman service in determining the properties of the motion. It enabled him, for instance, to obtain

[32] See E. Hogan, "Theodore Strong and Ante-bellum American Mathematics," *Historia Mathematica*, 8 (1981), 435–455.

[33] C.G.J. Jacobi, *Comptes rendus de l'Académie des Sciences de Paris*, III, 5961; reprinted in C.G.J. Jacobi's *Gesammelte Werke*, IV (ed. K. Weierstrasse: Berlin: Reimer 1886), 35–38.

[34] C.G.J. Jacobi, Vorlesungen über Dynamik, in *Gesammelte Werke, Supplementband* (ed. A. Clebsch Berlin: Reimer, 1884), 10. For a detailed account of Jacobi's likely path in deriving his integral, see pp. 195–201 of M. Nakane and C.G. Fraser, "The Early History of Hamilton-Jacobi Dynamics 1834–1837," *Centaurus, 44* (2002), 161–227. The

the constants of the Variation orbit in *literal* form, as series in the constant **m**. By its means he obtained the terms of the Variation in longitude and radius vector with a precision far greater than ever before achieved.

Another important feature of Hill's treatment of the lunar problem was his use of the imaginary exponential as it relates to the cosine and sine:

$$e^{\pm(\sqrt{-1})\theta} = \cos\theta \pm \sqrt{-1}\sin\theta.$$

This relation had been used by d'Alembert in his lunar theory of 1754, but had not been employed by later celestial mechanicians until Cauchy started promoting it in the 1840s. The expression of cosines and sines of angles by the imaginary exponential is particularly useful when infinite series are to be multiplied. Hill's theory relied heavily on such multiplications. The expression of trigonometrical series by imaginary exponentials reduced the multiplications to a simple addition of exponents.

Hill's first use of this device was in his paper "On the Development of the Perturbative Function in Periodic Series," published in *The Analyst* in 1875.[35] This paper makes reference to a memoir of 1860 by Puiseux, also dealing with the development of the perturbing function.[36] Puiseux advocated use of the imaginary exponential with the mean anomaly or its multiples as argument:

> The consideration of this new variable allows us not only to assign the limits within which the coordinates remain convergent, but, as M. Cauchy has remarked, to calculate without difficulty the general terms of these developments. Moreover, the same method applied to the perturbing function furnishes the general term of this function developed according to the sines and cosines of multiple arcs of the mean anomalies of the two planets. The coefficients of the sine and cosine of a given argument are thus obtained directly in the form of series proceeding according to the integral powers of the two eccentricities, of the sine of the mutual half-inclination of the orbits, and of the ratio of the major axes – that is, under the most appropriate form for use in celestial mechanics.[37]

Puiseux is here following in the footsteps of A.-L. Cauchy, who in the Paris Academy *Comptes rendus* of the 1840s wrote frequently on ways to make rigorous and to streamline celestial mechanics. Puiseux refers in particular to Cauchy's report, in the *Comptes rendus* for 1845,[38] for a commission reviewing a memoir by Le Verrier on an inequality in the mean motion of the minor planet Pallas. The minor

reconstructed derivation involves a time-dependent potential and thus a non-conservative dynamical system.

[35] *The Analyst*, *II*, 161–180; *Collected Mathematical Works of G.W. Hill*, *I*, 206–226.

[36] Puiseux, "Mémoire sur le développement en séries des coordonnées des planets et de la fonction perturbatrice," *Journal de mathématiques pures et appliqués*, Deuxième Série, *V* (1860), 65–102, 105–120.

[37] Ibid., 65.

[38] XX, 767–786.

planets so far discovered – there were just four of them – had all proved troublesome: orbital elements calculated from 1 year's observations disagreed with the next year's observations, and so it was unclear how to proceed in determining perturbations. Le Verrier had found that 7 times the mean motion of Pallas minus 18 times the mean motion of Jupiter was a very small angle (*viz.*, $27'11''$); an inequality with a period of 83 years would result, but being of the eleventh order in the eccentricities and inclinations, the question was whether it was in fact detectable. Only a detailed computation could decide the matter. The available methods for computing it stemmed essentially from Laplace, and were exceedingly laborious. Le Verrier carried out this computation, and found the maximum value of the inequality to be $14'55''$, and the phase difference from the mean anomaly, $-29°7'$. The commission desired to check Le Verrier's result without having to repeat his long calculation.

Cauchy had already shown how to do this: derive a *general* term of the perturbing function algebraically, then substitute into it the numbers appropriate to the inequality in question. No one earlier had carried out such a procedure. Applying it to Le Verrier's inequality, Cauchy first obtained a maximum of $15'6.6''$ and a phase difference of $-29°3'55''$, and then by a slightly different calculative route a maximum of $15'6.3''$ and a phase difference of $-29°3'25''$. The results agreed closely with each other and differed but slightly from Le Verrier's result; the difference, according to Cauchy, was of the order of the error arising from Le Verrier's use of 7-place logarithms.

Since Hill gives us no specific references, we do not know which of Cauchy's writings he read. He was clearly aware of Cauchy's insistence on quantifying the error committed in breaking off an infinite series at any particular point. Hill in his paper of 1878 stated:

> I regret that, on account of the difficulty of the subject and the length of the investigation it seems to require, I have been obliged to pass over the important questions of the limits between which the series are convergent, and of the determination of superior limits to the errors committed in stopping short at definite points. There cannot be a reasonable doubt that, in all cases, where we are compelled to employ infinite series in the solution of a problem, analysis is capable of being perfected to the point of showing us within what limits our solution is legitimate, and also of giving us a limit which its error cannot surpass. When the coordinates are developed in ascending powers of the time, or in ascending powers of a parameter attached as a multiplier to the disturbing forces, certain investigations of Cauchy afford us the means of replying to these questions. But when, for powers of the time, are substituted circular functions of it, and the coefficients of these are expanded in powers and products of certain parameters produced from the combination of the masses with certain of the arbitrary constants introduced by integration, it does not appear that anything in the writings of Cauchy will help us to the conditions of convergence.[39]

[39] *The Collected Mathematical Works of George William Hill*, *I*, 287.

Thus Hill recognized the legitimacy of Cauchy's demand for tests of convergence, and though he was unable to give error-terms for the series he used, he demonstrated, as we shall see, that the apparent convergence of these series was exceedingly rapid. Both in his use of the imaginary exponential and in his concern with convergence, we must recognize the influence of Cauchy.

When Hill was first appointed to the Nautical Almanac Office in 1861, he spent a year or two in Cambridge, Massachusetts[40]; the office was located there from its inception in 1849 till 1866, primarily in order to benefit from the guidance of Benjamin Peirce, professor of mathematics at Harvard. Peirce had introduced Cauchy's work, including the *Cours d'analyse* of 1821, into the Harvard curriculum.[41] It can have been during Hill's time in Cambridge that he gained some acquaintance with Cauchy's writings. But he soon obtained permission to do his work at the family farm in West Nyack, and we do not know what works he had in his library there.[42]

In his paper of 1878, Hill solved his differential equations while leaving out of account the lunar orbit's eccentricity and its inclination to the ecliptic; he thus obtained a periodic orbit. His paper of 1877, proceeding from that same periodic orbit, introduced eccentricity into the problem, and set out to solve the differential equations that thus resulted. In this way he arrived at an infinite determinant, a kind of problem he was the first to confront. In the course of solving it he made crucial use of a summation which may be written as

$$\sum_{i=-\infty}^{+\infty} \frac{1}{\theta \pm i} = \pi \cot \pi\theta,$$

where θ is a constant. This formula, according to Hill, was "well known". It had first been derived by Euler, with the daring manipulative virtuosity for which he is famous, in a paper published in 1743[43]; and it is also given in Euler's *Introductio in analysin infinitorum*, I.[44] It can be derived more soberly in accordance with Cauchy's theory of residues, and is so derived in *Théorie des fonctions doublement périodiques* by Briot and Bouquet, published in 1859[45]; this book was a standard text for complex

40 R.C. Archibald, *A Semicentennial History of the American Mathematical Society, 1888–1938* (New York: American Mathematical Society, 1938), 117.

41 K.H. Parshall and D.E. Rowe, *The Emergence of the American Mathematical Research Community, 1876–1900: J.J. Sylvester, Felix Klein, and E.H. Moore* (Providence, RI American Mathematical Society, 1994), 18.

42 Hill bequeathed his library to Columbia University, according to his will, dated 15 April 1897, and published in the *Columbiana* at that time. But a list of the books thus donated to Columbia does not appear to have survived.

43 *Leonhardi Euleri Opera Omnia*, *I.17*, 15.

44 Ibid., I.8, 191. See also J.A. Euler, *Introduction to Analysis of the Infinite*, I (tr. John D. Blanton: New York, Springer-Verlag, 1988), 149.

45 See C. Briot and C. Bouquet, *Théorie des fonctions doublement périodiques* (Paris: Mallet-Bachelier, 1859), 126.

function theory in the late nineteenth century, and Hill may have met with his "well known" formula there.

In his founding of a new and more exact lunar theory, Hill was powerfully assisted by what he had learned from his study of Euler's writings during his college days and by his later solitary study of the mathematical literature of his own day. His construction of the new lunar theory was also solitary. Among mathematical astronomers in America, his preparation was altogether unique. Without that preparation, it is hard to see how his two seminal papers of 1877 and 1878 could have come to be.

3

Hill on the Motion of the Lunar Perigee

Of Hill's two innovative papers on the lunar theory, the first, "On the part of the motion of the lunar perigee which is a function of the mean motions of the sun and moon" (Cambridge, MA: John Wilson, 1877, 28pp; reprinted in *Acta Mathematica 8* (1886), pp. 1–36) was by far the most esoteric in its subject matter and hyper-refined in the methods it employed. The second paper introduces the new lunar theory in a more pedestrian and reader-friendly way, as the reader will discover in our later section on "Hill's Variation Curve." The first paper must have made a stunning impression on those readers who were prepared to appreciate it; it is a blockbuster of a paper, astonishing in what and how it achieves. We shall attempt to make its essential steps understandable for readers with a moderate amount of training in algebra and the calculus.

The first paper was initially published privately at the author's expense. The second paper, "Researches in the Lunar Theory," was published in the first three issues of the first volume of the *American Journal of Mathematics* in 1878. Most of Hill's earlier papers, and a few later ones (up to 1881), were published in *The Analyst*, a recreationally oriented American journal of pure and applied mathematics published from 1874 to 1883;[46] for his lunar papers, however, Hill apparently did not consider *The Analyst* a suitable vehicle. The first volume of the *American Journal of Mathematics*, in which Hill published his second paper, did not exist when Hill completed his first paper. He surely knew that he had achieved something important, and must have wanted to see it quickly in print. He opted for a private printing of 200 copies; this had the advantage of giving him control of the distribution. As noted earlier, John Couch Adams received a copy shortly after the article appeared – no doubt sent by Hill.

In opening his essay of 1877, Hill remarks that lunar theorists since the publication of Newton's *Principia* have been puzzled to account for the lunar perigee's motion, simply because they could not conceive that terms of the second and higher

[46] Parshall and Rowe, *The Emergence of the American Mathematical Community 1876–1900*, 51, 85.

C. Wilson, *The Hill–Brown Theory of the Moon's Motion*, Sources and Studies
in the History of Mathematics and Physical Sciences, DOI 10.1007/978-1-4419-5937-9_3,

orders with respect to the disturbing force produced more than half of it. Nor, he asserts, has the problem yet been satisfactorily solved:

> The rate of motion of the lunar perigee is capable of being determined from observation with about a thirteenth of the precision of the rate of mean motion in longitude. Hence if we suppose that the mean motion of the moon, in the century and a quarter which has elapsed since Bradley began to observe, is known within $3''$, it follows that the motion of the perigee can be got to within about 500,000th of the whole. None of the values hitherto computed from theory agrees as closely as this with the value derived from observation.

The perigee moves about $40°40'$ per year; hence in the 125 years since Bradley it has moved about $5085°$. Hill is asserting that this total motion can be determined by observation with a precision of about $3 \times 13 = 39$ arc-seconds $= 0°.010833$, which is approximately the 500,000th part of the whole. Lunar theorists had not yet come near to achieving so precise a determination.

> Hence I propose, in this memoir, to compute the value of this quantity, so far as it depends on the mean motions of the sun and moon, with a degree of accuracy that shall leave nothing further to be desired.

It is only part of the motion of the lunar perigee that Hill is here aiming to calculate, for the complete motion of the perigee depends in some measure on the eccentricities of the Moon's and Earth's orbits, and on the inclination of the Moon's orbit to the ecliptic. But the part Hill will be calculating – the part dependent on the constant **m** – will prove to be the main part. Indeed, the value Hill will obtain from his calculation will differ from the observational value by no more than 1/70th of the latter. Think of it! – a discrepancy of 1/70th instead of the one-half that Euler, Clairaut, and d'Alembert were initially confronted with. With the result of Hill's calculation in hand, it will no longer be a wild surmise that the Moon's path is more nearly approximated by the Variation curve than by any Earth-focused ellipse. And this curve is totally definable in terms of the small parameter **m**, the cause of all the problems of slow convergence that had stymied the earlier investigators.

The mathematical development in Hill's paper of 1877 assumes that the lunar inequalities depending solely on the parameter **m** – except for the motion of the apse – have already been obtained. In other respects, Hill no doubt intends his paper of 1877 to be self-contained, but his explanations here are remarkable for their concision. We can promise the reader that certain concepts presented here with the briefest characterization – the Jacobian integral, for instance, and the potential function Ω – will in our resume of the second paper be more fully explained.

Hill begins by presenting the differential equations in the form[47]

$$\frac{d^2x}{dt^2} = \frac{\partial\Omega}{\partial x}, \quad \frac{d^2y}{dt^2} = \frac{\partial\Omega}{\partial y}. \tag{I.1}$$

[47] In our numbering of equations, "I" stands for the paper of 1877, and "II" for the paper of 1878.

These are the equations Jacobi started from, in the paper of 1836 introducing the Jacobian integral. The variables x and y are the Moon's rectangular coordinates with respect to the Earth's center. Ω is the potential function, so that $\partial\Omega/\partial x$ and $\partial\Omega/\partial y$ express the net forces exerted on the Moon in the x- and y-directions. Hill leaves unspecified the terms of which Ω consists, and proceeds at once to the integral. As integrating factors ("Eulerian multipliers," he calls them) he proposes

$$F = \frac{dx}{dt} + n'y, \quad G = \frac{dy}{dt} - n'x,$$

where n' is the angular motion of the Sun about the Earth or of the Earth about the Sun, here taken to be uniform and circular. The first equation of (I.1) is to be multiplied by F, and the second by G; the resulting equations are then added together. The result (which Hill does not write out) is

$$\frac{d^2x}{dt^2}\frac{dx}{dt} + \frac{d^2y}{dt^2}\frac{dy}{dt} - n'\left(\frac{d^2y}{dt^2} - y\frac{d^2x}{dt^2}\right) = \frac{\partial\Omega}{\partial x}\frac{dx}{dt} + \frac{\partial\Omega}{\partial y}\frac{dy}{dt} - n'\left(x\frac{\partial\Omega}{\partial y} - y\frac{\partial\Omega}{\partial x}\right). \tag{I.2}$$

Note that the third term on the right is identically equal to the third term on the left, by (I.1). The time-integral of (I.2), Hill then claims, is

$$\frac{dx^2 + dy^2}{2dt^2} - n'\left(\frac{xdy - ydx}{dt}\right) = \Omega + C, \tag{I.3}$$

where C is the constant of integration.

That the left-hand side of (I.3) is the integral of the left-hand side of (I.2) is easily verified. On the right-hand side of (I.2) the first two terms give the indirect dependence of Ω on t through the variables x and y. Assuming that Ω depends in addition on t directly, we should have

$$\frac{d\Omega}{dt} = \frac{\partial\Omega}{\partial x}\frac{dx}{dt} + \frac{\partial\Omega}{\partial y}\frac{dy}{dt} + \frac{\partial\Omega}{\partial t}.$$

Then, for the right-hand side of (I.3) to be the integral of the right-hand side of (I.2), we must have

$$\frac{\partial\Omega}{\partial t} = n'y\frac{\partial\Omega}{\partial x} - n'x\frac{\partial\Omega}{\partial y}.$$

The latter equation can be verified if the terms of which Ω is composed are known. We find them, not in Hill, but in Jacobi:

$$\frac{1}{2}\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right] - n'\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right) = \frac{M}{(x^2+y^2)^{1/2}}$$
$$+ m'\left[\frac{1}{[(x - a'\cos n't)^2 + (y - a'\sin n't)^2]^{1/2}} - \frac{x\cos n't + y\sin n't}{a'^2}\right] + \text{const.} \tag{I.4}$$

(We have omitted the terms involving the variable z, since Hill confines the orbit to the x-y plane.) In (I.4) a' is the Earth-Sun distance, M is the Earth's mass, and m' the Sun's mass. The Moon is assumed to be without mass. Taking the right-hand side of (I.4) as an expression of $\Omega +$ *const.*, we find by a straightforward calculation that

$$\frac{\partial \Omega}{\partial t} = n' \left(y \frac{\partial \Omega}{\partial x} - x \frac{\partial \Omega}{\partial y} \right),$$

as required.

Assuming that (I.1) and (I.3) have together been solved for the Variation orbit, Hill now proposes to investigate the effect of small departures from that solution. The Variation curve, as we shall see in our resume of Hill's second paper, is an oval symmetrical with respect to the rotating x- and y-coordinate axes, with origin at the Earth's center. Let x_0 and y_0 be the variables for the Variation orbit, and let the (I.1) and (I.2) be written with x and y thus distinguished by subscript 0.[48] Hill is asking what happens to the orbit when increments ∂x and ∂y are added, respectively, to x_0 and y_0 in the differential equations.

The increments ∂x and ∂y will destroy the symmetry, making the Moon's path eccentric with respect to the Earth's center, so as to have perigee(s) and apogee(s). For in the absence of perturbation, the Moon would move in a circle or else in an ellipse with a center eccentric to the Earth's center. The eccentricity can be expected to remain when solar perturbation supervenes. Newton, as mentioned earlier, had thought in terms of somehow melding the properties of the Variation and those of the ellipse, but he lacked a legitimate mathematical technique for doing this. Hill, with Euler's guidance, is setting out to combine the effects as determined by their defining parameters, **e** and **m**. This he can easily do, using the exponential expression of sines and cosines. It is a matter of adding exponents.

"Let us suppose," Hill writes, "...that it is desired to get [the inequalities] which are multiplied by the simple power of [the eccentricity]." Given this statement, the reader may be surprised to find that the eccentricity **e** does not figure as a quantity in the calculations of the paper we are examining. But the increments δx, δy *do* produce eccentricity. Hill's remark means that the increments are small enough so that their squares and their product can be neglected. Given eccentricity, there will be a perigee and an apogee, and solar perturbation will cause these points of the orbit to move forward. Hill aims in this paper to determine that motion, insofar as it depends on **m**. Such a determination is prerequisite for determining the mean anomaly in the resulting orbit, and hence for determining the inequalities proportional to **e**.

To arrive at differential equations for δx and δy, Hill first substitutes x_0 and y_0, then $x_0 + \delta x$ and $y_0 + \delta y$, for x and y in the two equations of (I.1), then takes the difference of the corresponding equations so as to eliminate x_0 and y_0. The result is

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[48] This notation is due to Brouwer and Clemence, *Methods of Celestial Mechanics*, 350ff.

$$\frac{d^2\delta x}{dt^2} = \left(\frac{\partial^2\Omega}{\partial x^2}\right)_0 \delta x + \left(\frac{\partial^2\Omega}{\partial x \partial y}\right)_0 \delta y,$$

$$\frac{d^2\delta y}{dt^2} = \left(\frac{\partial^2\Omega}{\partial y^2}\right)_0 \delta y + \left(\frac{\partial^2\Omega}{\partial x \partial y}\right)_0 \delta x.$$

The zero subscripts indicate that the partial derivatives are to be evaluated using the variables of the Variation orbit. If with Hill we put

$$H = \left(\frac{\partial^2\Omega}{\partial x^2}\right)_0, \quad J = \left(\frac{\partial^2\Omega}{\partial x \partial y}\right)_0, \quad K = \left(\frac{\partial^2\Omega}{\partial y^2}\right)_0,$$

the equations take the form

$$\frac{d^2\delta x}{dt^2} = H\delta x + J\delta y, \frac{d^2\delta y}{dt^2} = K\delta y + J\delta x. \tag{I.5}$$

Next, Hill carries out the analogous operation on (I.3), discarding terms in which δx and δy are squared or multiply each other; the result is

$$\frac{dx_0}{dt}\frac{d(\delta x)}{dt} + \frac{dy_0}{dt}\frac{d(\delta y)}{dt} - n'\left(x_0\frac{d(\delta y)}{dt} - y_0\frac{d(\delta x)}{dt} + \frac{dy_0}{dt}\delta x - \frac{dx_0}{dt}\delta y\right)$$
$$= \left(\frac{\partial\Omega}{\partial x}\right)_0 \delta x + \left(\frac{\partial\Omega}{\partial y}\right)_0 \delta y + \delta C.$$

According to Hill, δC if developed in ascending powers of the eccentricity is found to contain only even powers of **e**; therefore in the approximation we are here exploring, we shall have $\delta C = 0$. Also, in accordance with (I.1), the first-order partial derivatives of Ω are

$$F\frac{d(\delta x)}{dt} + G\frac{d(\delta y)}{dt} - \frac{dF}{dt}\delta x - \frac{dG}{dt}\delta y = 0. \tag{I.6}$$

This equation, Hill observes, is identically satisfied by the solution $\delta x = F$ and $\delta y = G$. The same solution satisfies equations (I.5), giving

$$\frac{d^2F}{dt^2} = HF + JG, \quad \frac{d^2G}{dt^2} = KG + JF. \tag{I.5a}$$

This solution, being composed of terms having the same argument as the Variation, tells us nothing about an orbit incorporating the increments $\delta x, \delta y$. To obtain the latter orbit, Hill proposes a solution of the form $\delta x = F\rho, \delta y = G\sigma$, where ρ and σ are new variables. The use of F and G in this manner – a well-known technique – will enable Hill to reduce the order of his final differential equation. Introducing these new variables into (I.5) and (I.6), and making use of (I.5a), he finds

$$F\frac{d^2\rho}{dt^2} + 2\frac{dF}{dt}\frac{d\rho}{dt} + JG(\rho - \sigma) = 0,$$

$$G\frac{d^2\sigma}{dt^2} + 2\frac{dG}{dt}\frac{d\sigma}{dt} + JF(\sigma - \rho) = 0,$$

$$F^2\frac{d\rho}{dt} + G^2\frac{d\sigma}{dt} = 0.$$

Deriving from the first of these equations an expression for σ, he substitutes it into the third equation, and so obtains

$$\frac{d^3\rho}{dt^3} + \frac{d}{dt}\left[\ln\frac{F^3}{JG}\right]\frac{d^2\rho}{dt^2} + \left[\frac{J(F^2+G^2)}{FG} + \frac{JG}{F}\frac{d}{dt}\left(\frac{2}{JG}\frac{dF}{dt}\right)\right]\frac{d\rho}{dt} = 0. \quad \text{(I.7)}$$

Hill's final move is to introduce the substitution

$$\frac{d\rho}{dt} = \sqrt{\frac{JG}{F}}w. \quad \text{(I.8)}$$

This yields, after algebraic reductions, the differential equation

$$\frac{d^2w}{dt^2} + \theta w = 0, \quad \text{(I.9)}$$

where θ can be expressed by

$$\theta = \frac{J(F^2+G^2)}{FG} + \frac{d^2\cdot\ln(JFG)}{2dt^2} - \left[\frac{d\cdot\ln(JFG)}{2dt}\right]^2. \quad \text{(I.10)}$$

(I.10) shows that θ depends solely on the variables x_0, y_0, their time-derivatives, and the derivative J of Ω with respect to them. Interchanging F and G leaves θ unchanged; thus if ρ had been eliminated instead of σ, a formally identical equation would have resulted. According to Hill,

> ...we arrive always at the same value of θ, no matter what variables have been used to express the original differential equations. From this we may conclude that θ depends only on the relative position of the Moon with reference to the Sun, and that it can be developed in a periodic series of the form
>
> $$\theta = \theta_0 + \theta_1\cos 2\tau + \theta_2\cos 4\tau + \cdots,$$
>
> in which τ denotes the mean angular distance of the two bodies.

Here θ_0, θ_1, etc., are constants, and $\tau = v(t - t_0)$, in which v is the frequency of the moon's synodic motion, and t_0 the time of the moon's conjunction with the Sun.

In this passage Hill does not explain why θ should be an infinite series of cosines, nor why the arguments of the cosines should be the even multiples of τ. Could not the function $\cos\tau$, for instance, be included in θ, since it gives the same value whenever the Moon is at the same angular distance from the Sun? The reason θ must be an infinite series of cosines with arguments that are *even* multiples of τ can be elucidated as follows. F and G are linear functions of x_0, y_0, $dx_0/d\tau$, $dy_0/d\tau$; the latter variables are given by infinite series of sinusoidal functions, with arguments that are *odd* multiples of τ. These representations of x_0 and y_0 were chosen initially (as explained in Hill's paper of 1878) in order to obtain a periodic orbit. Hence, in (I.10), F^2, G^2, and FG are infinite series of sinusoidal functions with arguments that are *even* multiples of τ (exponential expression of the sines and cosines makes this obvious). Moreover, J, the mixed partial derivative of Ω which multiplies F^2, G^2, and FG in (I.10), can also be expressed by an infinite series in which the arguments of the sinusoidal functions are even multiples of τ. Thus θ is represented by an infinite series of sinusoidal terms, in which the arguments are necessarily even multiples of τ.

Hill does not say a word about what the new variable w represents.

George Howard Darwin, in his lectures on Hill's lunar theory,[49] points out that it is a common procedure in dynamics to consider "free oscillations" about a steady state (free oscillations are contrasted to forced oscillations, produced by an external force). If the Variation orbit is taken as the steady state, then the obvious oscillations to consider are those normal and tangential to the Variation curve. Let δp and δs represent these oscillations. If φ is the inclination of the outward normal of the Variation curve to the x-axis, then

$$\delta x = \delta p \cos\varphi - \delta s \sin\varphi.$$
$$\delta y = \delta p \sin\varphi + \delta s \cos\varphi.$$

The sine and cosine of the angle φ are furnished by the relations $dx_0/d\tau = -V \sin\varphi$, $dy_0/d\tau = V \cos\varphi$, where V is the orbital speed:

$$V = \left[\left(\frac{dx_0}{d\tau}\right)^2 + \left(\frac{dy_0}{d\tau}\right)^2\right]^{1/2}.$$

By the foregoing equivalences Darwin eliminates δx and δy from (I.5) and (I.6) in favor of δp and δs. He also eliminates x_0, y_0, and their derivatives, using initial terms from Hill's infinite series for x_0 and y_0 (he stops at terms in which the multiple of τ is 3). He thus obtains the following approximate differential equation for δp:

$$\frac{d^2\delta p}{d\tau^2} + \delta p\left[1 + 2\underline{m} - \frac{1}{2}\underline{m}^2 - 15\underline{m}^2 \cos 2\tau\right] = 0.$$

This resembles Hill's equation for w, except that θ here consists of a constant and only one sinusoidal term, rather than an infinite series of such terms. A similar equation can be obtained for δs.

If θ were a constant, (I.9) would describe simple harmonic motion, with a solution of the form $w = A\cos ft$ where f is a frequency equal to $\sqrt{\theta}$. But θ consists of an initial constant (θ_0), plus an infinity of terms that vary, $\sum_{i=1}^{\infty} \theta_i \cos 2i\tau$. We shall

[49] G. H. Darwin, "Hill's Lunar Theory," *Scientific Papers of George Howard Darwin, 5* (Cambridge: Cambridge University Press, 1916), 27ff.

find that $\theta_0 > \theta_1 > \theta_2 > \cdots$, with θ_0 a good deal larger than its successors (it is more than ten times θ_1). Can the simple harmonic solution based on setting $\theta = \theta_0$ serve as a first approximation in a sequence of successive approximations leading to the final solution? This idea, Darwin shows, leads into in a cul-de-sac.

Darwin puts Hill's equation in the form

$$\frac{d^2w}{dt^2} + (\theta_0 + 2\theta_1 \cos 2\tau + 2\theta_2 \cos 4\tau + \cdots)w = 0,$$

and takes

$$w = A\cos[t\sqrt{\theta_0} + \varepsilon]$$

as the first approximation. Substituting this expression in the term multiplied by θ_1, and neglecting θ_2, θ_3, etc., he obtains the equation

$$\frac{d^2w}{dt^2} + \theta_0 w + A\theta_1\{\cos[t(\sqrt{\theta_0} + 2) + \varepsilon] + \cos[t(\sqrt{\theta_0} - 2) + \varepsilon]\} = 0.$$

Solving this by the usual rules he obtains the second approximation:

$$w = A\left\{\cos[t\sqrt{\theta_0} + \varepsilon] + \frac{\theta_1\cos[t(\sqrt{\theta_0} + 2) + \varepsilon]}{4(\sqrt{\theta_0} + 1)} - \frac{\theta_1\cos[t(\sqrt{\theta_0} - 2) + \varepsilon]}{4(\sqrt{\theta_0} - 1)}\right\}.$$

If this value of w is substituted into the terms of the differential equation having the coefficients θ_1 and θ_2, terms in $\cos[t(\sqrt{\theta_0} + 4) + \varepsilon)]$ and $\cos[t(\sqrt{\theta_0} - 4) + \varepsilon]$ are produced; and so are terms in $\cos[t\sqrt{\theta_0} + \varepsilon]$—a term of exactly the same kind as that assumed for the first approximation. As a consequence, in the next stage of the approximation a secular term having the form $Ct\sin[t\sqrt{\theta_0} + \varepsilon]$ arises. Such a term would come to dominate the solution and there falsify it.

A remedy would seem to be to start over again, using a first approximation of the form $w = A\cos[ct + \varepsilon]$, where c differs slightly from $\sqrt{\theta_0}$. But the process of successive approximations still circles back on itself, generating terms that modify the values of terms ostensibly determined earlier in the process. Evidently we are in need of a procedure that is holistic in the sense of taking account from the start of all the terms that can significantly influence the solution.

Since the reduction of Θ, in the form previously given, namely $(\theta_0 + \theta_1 \cos 2\tau + \theta_2 \cos 4\tau + \cdots)$, presents difficulties, Hill proposes to derive another form from differential equations in terms of coordinates expressing the relative position of the moon to the sun. He introduces rectangular coordinates x and y, rotating in the plane of the ecliptic with constant angular speed, in such a way that the axis of x passes constantly through the center of the sun. He adopts the imaginary variables

$$u = x + y\sqrt{-1}, \quad s = x - y\sqrt{-1},$$

and puts $\varepsilon^{\tau\sqrt{-1}} = \zeta$, where ε is the basis of natural logarithms. In addition he introduces the operator $D = -\frac{d}{d\tau}\sqrt{-1}$, so that

$$D(a\zeta^v) = va\zeta^v.$$

He makes the parameter **m** to be, as in our earlier discussion, the ratio of the synodic month to the sidereal year, or $\mathbf{m} = n'/(n - n')$. With μ as the sum of the masses of the earth and the moon, he puts $\kappa = \mu/(n - n')^2$. Finally, he defines the potential function by

$$\Omega = \frac{\kappa}{\sqrt{us}} + \frac{3}{8}\mathrm{m}^2(u + s)^2.$$

With these preliminaries, he can now derive differential equations of the moon's motion. A step-by-step derivation of the differential equations will be given in our resume of Hill's second paper, using the Lagrangian algorithm for extracting equations from the expressions for the potential function and the kinetic energy.

In the exposition of his first paper, Hill merely gives the result:

$$D^2u + 2\mathrm{m}Du + 2\frac{\partial\Omega}{\partial s} = 0,$$

$$D^2s - 2\mathrm{m}Ds + 2\frac{\partial\Omega}{\partial u} = 0. \tag{I.11}$$

Multiplying the first of these by Ds, the second by Du, adding the products and integrating the resulting equation, he obtains the Jacobian integral:

$$DuDs + 2\Omega = 2C. \tag{I.11a}$$

Subjecting (I.11) and (I.lla) to the operation δ yields the three equations

$$D^2\delta u + 2\mathrm{m}D\delta u + 2\left(\frac{\partial^2\Omega}{\partial u\partial s}\right)_0 \delta u + 2\left(\frac{\partial^2\Omega}{\partial s^2}\right)_0 \delta s = 0,$$

$$D^2\delta s - 2\mathrm{m}D\delta s + 2\left(\frac{\partial^2\Omega}{\partial u\partial s}\right)_0 \delta s + 2\left(\frac{\partial^2\Omega}{\partial u^2}\right)_0 \delta u = 0,$$

$$DuD\delta s + DsD\delta u + 2\left(\frac{\partial\Omega}{\partial u}\right)_0 \delta u + 2\left(\frac{\partial\Omega}{\partial s}\right)_0 \delta s = 0. \tag{I.12}$$

These equations still hold if δ is changed into D, since they then become the derivatives of (I.11) and (I.lla). Hence $\delta u = Du_0$, $\delta s = Ds$ constitute a particular solution of (I.12). This solution reveals nothing about the effect of the free oscillations δu, δs on the Variation orbit. As before, Hill uses the particular solution to reduce the order of the final solution. He adopts new variables v and w such that $\delta u = Du \cdot v$, $\delta s = Ds \cdot w$. When these are substituted into (I.12), and the second and third derivatives of u and s are eliminated by means of (I.11) and (I.lla), the result is

$$Du_0 \cdot D^2v - 2\left[2\left(\frac{\partial\Omega}{\partial s}\right)_0 + \mathrm{m}Du_0\right]Dv - 2\left(\frac{\partial^2\Omega}{\partial s^2}\right)_0 Ds_0 \cdot (v - w) = 0,$$

$$Ds_0 \cdot D^2 w - 2[2\left(\frac{\partial \Omega}{\partial u}\right)_0 + \mathrm{m} Ds_0] Dw - 2\left(\frac{\partial^2 \Omega}{\partial u^2}\right)_0 Du_0 \cdot (w - v) = 0,$$

$$Du_0 Ds_0 \cdot D(v + w) - 2\left[\left(\frac{\partial \Omega}{\partial s}\right)_0 Ds_0 - \left(\frac{\partial \Omega}{\partial u}\right)_0 Du_0 + \mathrm{m} Du_0 Ds_0\right](v - w) = 0. \tag{I.13}$$

Hill multiplies the first equation of (I.13) by Ds_0 and the second by Du_0 and takes their difference. The resulting equation, along with the third equation of (I.13), will be his basis for the solution of the problem. For brevity he writes

$$\Delta = \left(\frac{\partial \Omega}{\partial s}\right)_0 Ds_0 - \left(\frac{\partial \Omega}{\partial u}\right)_0 Du_0 + \mathrm{m} Du_0 Ds_0,$$

and puts

$$\rho = v + w, \sigma = v - w.$$

His two equations then take the form

$$Du_0 Ds_0 \cdot D\rho - 2\Delta \cdot \sigma = 0,$$

$$D[Du_0 Ds_0 \cdot D\sigma] - 2\Delta \cdot D\rho - 2\left[\left(\frac{\partial^2 \Omega}{\partial u^2}\right)_0 Du_0^2 + \left(\frac{\partial^2 \Omega}{\partial s^2}\right)_0 Ds_0^2\right]\sigma = 0. \tag{I.14}$$

Eliminating $D\rho$ between the two equations of (I.14), he obtains an equation in which the single variable representing the free oscillation is σ:

$$D[Du_0 Ds_0 \cdot D\sigma] - 2\left[\left(\frac{\partial^2 \Omega}{\partial u^2}\right)_0 Du_0^2 + \left(\frac{\partial^2 \Omega}{\partial s^2}\right)_0 Ds_0^2 + \frac{2\Delta^2}{Du_0 Ds_0}\right]\sigma = 0.$$

To remove the term involving $D\sigma$, he makes the substitution

$$\sigma = \frac{w}{\sqrt{Du_0 Ds_0}}.$$

The product $Du_0 Ds_0$, be it noted, is the negative of the square of the speed in the Variation orbit. With this substitution, he obtains a differential equation for w:

$$D^2 w = \theta w. \tag{I.15}$$

The coefficient θ can be put in the form

$$\frac{1}{Du_0 Ds_0}\left[\left(\frac{\partial^2 \Omega}{\partial u^2}\right)_0 Du_0^2 - 2\left(\frac{\partial^2 \Omega}{\partial u \partial s}\right)_0 Du_0 Ds_0 + \left(\frac{\partial^2 \Omega}{\partial s^2}\right)_0 Ds_0^2\right]$$

$$+ 3\left(\frac{\Delta}{Du_0 Ds_0}\right)^2 + \mathrm{m}^2.$$

The partial derivatives that appear here are determined from the formula:

$$\Omega = \frac{\kappa}{\sqrt{us}} + \frac{3}{8}\mathrm{m}^2(u+s)^2.$$

Also, $Du_0 Ds_0$ is replaced in accordance with the Jacobian integral by $2C - 2\Omega$. With these substitutions, θ becomes

$$\frac{\kappa}{(u_0 s_0)^{3/2}} + \frac{3}{8}\frac{\frac{\kappa}{(u_0 s_0)^{5/2}}[u_0 Ds_0 - s_0 Du_0]^2 + \mathrm{m}^2(Du_0 - Ds_0)^2}{C-\Omega} + \frac{3}{4}\left[\frac{\Delta}{C-\Omega}\right]^2 + \mathrm{m}^2.$$

This expression, Hill tells us, is suitable for development in infinite series, when the method of special values (harmonic analysis) is used. The quadrant from $\tau = 0^\circ$ to $\tau = 90^\circ$ is divided into a certain number of equal parts, say six, and from the values of u_0, s_0, Du_0, Ds_0 at the dividing points the corresponding values of θ are determined. From the latter, by a well-known process, the coefficients of the periodic terms of θ are determined. Hill thus obtains the following expression for θ:

$$\begin{aligned}\theta = 1.15884\ 39395\ 96583 & \\ - 0.11408\ 80374\ 93807 & \cos 2\tau \\ + 0.00076\ 64759\ 95109 & \cos 4\tau \\ - 0.00001\ 83465\ 77790 & \cos 6\tau \\ + 0.00000\ 01088\ 95009 & \cos 8\tau \\ - 0.00000\ 00020\ 98671 & \cos 10\tau \\ + 0.00000\ 00000\ 12103 & \cos 12\tau \\ - 0.00000\ 00000\ 00211 & \cos 14\tau.\end{aligned} \tag{I.16}$$

Hill also develops, with analytic ploys of considerable ingenuity, a literal formula for θ in terms of $\mathbf{m}$, accurate to the order of $\mathbf{m}^{10}$:

$$\begin{aligned}\theta = {} & 1 + 2\mathrm{m} - \frac{1}{2}\mathrm{m}^2 + \frac{3}{2}\mathrm{m}^2 a_1 + 54a_1^2 + (12-4\mathrm{m})a_1 a_{-1} + (6-4\mathrm{m})a_{-1}^2 \\ & + \left[(6+12\mathrm{m})a_1 + (6+8\mathrm{m})a_{-1} - \frac{3}{2}\mathrm{m}^2\right](\zeta^2 + \zeta^{-2}) \\ & + \left[\begin{array}{l} 20\mathrm{m}a_2 + (16+20\mathrm{m})a_{-2} - (9+40\mathrm{m})a_1^2 \\ +6a_1 a_{-1} + (7+4\mathrm{m})a_{-1}^2 - \frac{3}{2}\mathrm{m}^2(a_1 - a_{-1}) \end{array}\right](\zeta^4 + \zeta^{-4}).\end{aligned} \tag{I.17}$$

As this formula is not necessary to the central argument, we shall not examine its derivation.

We turn now to Hill's solution of (I.15). To begin with, he reformulates θ as a series of exponential terms. Thus, in place of the formula $\sum_{i=0}^{\infty} \theta_i \cos 2i\tau$ with the summation running from zero to infinity, he substitutes $\sum_{-\infty}^{+\infty} \theta_i \zeta^{2i}$, with the summation running from minus to plus infinity. In the latter formula we are to understand that $\theta_i = \theta_{-i}$. With the exception of θ_0, which retains its previous value, the new θ_i are the halves of the θ_i in the earlier formula. The symbol ζ stands as before for $\varepsilon^{\tau\sqrt{-1}}$, where ε is Hill's symbol for the base of natural logarithms. When the index i is negative, the exponent of ζ in Hill's new summation formula is negative. Thus Hill's new formula gives us the well-known exponential expression for the cosine:

$$2\cos 2i\tau = \varepsilon^{2i\tau\sqrt{-1}} + \varepsilon^{-2i\tau\sqrt{-1}}.$$

As the form of a possible solution of (I.15), Hill proposes

$$w = \sum_{-\infty}^{+\infty} b_i \zeta^{c+2i}. \tag{I.18}$$

Here c is the ratio of the synodic to the anomalistic month. Observation gives this constant as approximately 29.53/27.55, but the point now is to determine it from theory insofar as it depends on **m** alone. Under this restriction, c will give the rate at which w runs through its cycle, from perigee back to perigee. The b_i are also unknown constants, and a complete solution of (I.15) would require determining them; but Hill's aim in the present paper is solely to determine c.

The presence of $2i$ in the exponent of ζ in (I.17) is necessary because θ contains, besides the constant θ_0, terms of the form $\theta_i \cos 2i\tau$; such terms when multiplied by ζ^c will necessarily produce terms containing ζ^{c+2i}. Hence, for $D^2 w$ to be equal to θw as (I.15) requires, w must contain the factor ζ^{c+2i} from the start.

To solve (I.15), Hill uses the method of undetermined coefficients – his preferred method as it was Euler's. If we compute $D^2 w$ for a particular index j, we obtain $(c+2j)^2 b_j \zeta^{c+2j}$. The expression of θw on the right-hand side of the equation will contain all products of $\sum \theta_i \zeta^{2i}$ by $\sum b_i \zeta^{c+2(j-i)}$ such that the resulting exponent of ζ is $c+2j$. Using distinct indices in the two sums, we may write θw as follows:

$$\sum_{i=-\infty}^{+\infty} \theta_i \zeta^{2i} \times \sum_{k=-\infty}^{+\infty} b_k \zeta^{c+2k} = \sum_{i=-\infty}^{+\infty} \sum_{k=-\infty}^{+\infty} \theta_i b_k \zeta^{c+2(i+k)}.$$

The terms that we want from these products will be those in which $(i+k) = j$, in other words the sum $\sum_{i=-\infty}^{+\infty} \theta_i b_{j-i} \zeta^{c+2j}$. Although the number of terms in this sum is infinite, Hill's calculations have indicated that for large $|\pm i|$ the θ_i diminish sharply; hence the terms with i large should prove negligible. Hill moves the term $\theta_0 b_0 \zeta^{c+2j}$ from the right to the left side of the equation (changing its sign, of course), and replaces $(c+2j)^2 - \theta_0$ in each equation by the symbol $[j]$. Dividing ζ^{c+2j} out of each equation, he then gives explicitly

$$\begin{array}{llllll}
\cdots + [-2]b_{-2} & -\theta_1 b_{-1} & -\theta_2 b_0 & -\theta_3 b_1 & -\theta_4 b_2 & - \cdots = 0, \\
\cdots - \theta_1 b_{-2} & +[-1]b_{-1} & -\theta_1 b_0 & -\theta_2 b_1 & -\theta_3 b_2 & - \cdots = 0, \\
\cdots - \theta_2 b_{-2} & -\theta_1 b_{-1} & +[0]b_0 & -\theta_1 b_1 & -\theta_2 b_2 & - \cdots = 0, \\
\cdots - \theta_3 b_{-2} & -\theta_2 b_{-1} & -\theta_1 b_0 & +[1]b_1 & -\theta_1 b_2 & - \cdots = 0, \\
\cdots - \theta_4 b_{-2} & -\theta_3 b_{-1} & -\theta_2 b_0 & -\theta_1 b_1 & +[2]b_2 & - \cdots = 0.
\end{array}$$

For each $[j]$, where j is any positive or negative integer, there is an equation, and each equation contains an infinite number of terms. The equations are homogeneous, each term having one of the b's as linear factor. For a trivial solution, one could set all the b's equal to zero. Are non-trivial solutions possible?

If the equations were finite in number, containing a number of unknowns equal to the number of equations, it can be proved that a non-trivial solution would be possible if and only if the *determinant* of the equations were equal to zero. This determinant is composed of the coefficients. For the five terms of the five equations given above, it would be

$$\begin{vmatrix} [-2] & -\theta_1 & -\theta_2 & -\theta_3 & -\theta_4 \\ -\theta_1 & [-1] & -\theta_1 & -\theta_2 & -\theta_3 \\ -\theta_2 & -\theta_1 & [0] & -\theta_1 & -\theta_2 \\ -\theta_3 & -\theta_2 & -\theta_1 & [1] & -\theta_1 \\ -\theta_4 & -\theta_3 & -\theta_2 & -\theta_1 & [2] \end{vmatrix} \tag{I.19}$$

This determinant contains, in the bracketed quantities $[j]$, the unknown quantity c. Hence, were (I.19) the determinant in question, c might be determined in such a way as to make the determinant zero. As Hill puts it,

> ...we get a symmetrical determinant involving c, which, equated to zero, determines this quantity.

But, Hill's determinant is infinite. Do the rules for ordinary determinants apply? Hill believes they do, for he takes the infinite determinant as the limit of a sequence of finite determinants:

> The question of the convergence, so to speak, of a determinant, consisting of an infinite number of constituents, has nowhere, so far as I am aware, been discussed. All such determinants must be regarded as having a central constituent; when, in computing in succession the determinants formed from the $3^2, 5^2, 7^2$, & c., constituents symmetrically situated with respect to the central constituent, we approach, without limit, a determinate magnitude, the determinant may be called convergent, and the determinate magnitude is its value.

In a trial computation, Hill writes out the 3×3 determinant at the center of (I.19), and sets it equal to zero:

$$\begin{vmatrix} [-1] & -\theta_1 & -\theta_2 \\ -\theta_1 & [0] & -\theta_1 \\ -\theta_2 & -\theta_1 & [1] \end{vmatrix} = [-1][0][1] - \theta_1^2\{[1] + [-1]\} - 2\theta_1^2\theta_2 + \theta_2^2[0] = 0.$$

He proposes to neglect terms of the order of $\mathbf{m}^5 = 0.000003454$. One such term is $-2\theta_1^2\theta_2$, which proves to be equal to -0.000002494. The final term, $\theta_2^2[0]$, with a provisional value of [0] calculated from the observational value of c, proves to be

−0.000000001, hence also negligible. Hill then puts the equation, with these terms deleted and the symbols $[j]$ replaced by what they signify, in the form

$$[(c^2+4-\theta_0)^2-16c^2][c^2-\theta_0]-2\theta_1^2[c^2+4-\theta_0]=0.$$

(Achieving this result takes a little doing.)

A nearly exact solution, Hill shows, can be obtained by means of two further deletions. To show that they are reasonable, we write out the equation in an expanded form, and, substituting the observational value of c, namely 1.071713598, compute the numerical value of each term; these values, with the factor 10^6 omitted, are placed below each term:

$$(c^2-\theta_0)^3+8(c^2-\theta_0)^2+16(1-c^2)(c^2-\theta_0)-2\theta_1^2(c^2-\theta_0)-8\theta_1^2=0.$$

$$-1.084 \qquad 844.418 \qquad 24422.213 \qquad -66.862 \qquad -26032.152$$

The first term is much the smallest, and the fourth is but 8% of the next larger term; Hill neglects both. The remaining terms reduce to

$$c^4-2c^2-\theta_0^2+2\theta_0-\theta_1^2=0.$$

The solution of this is

$$c=\sqrt{1+\sqrt{(\theta_0^2-1)^2-\theta_1^2}}=1.0715632.$$

The observational value is larger by 0.014%.

If Hill is on the right track, the calculated value should indeed err in the direction of smallness. For c is the ratio of the synodic month to the anomalistic month. If dw/dt is the mean sidereal rate of motion of the lunar perigee, and n, n' are the mean sidereal rates of motion of the Moon and the Sun, then this ratio can be expressed as

$$c=\frac{n-\frac{dw}{dt}}{n-n'}.$$

Thus c will come out larger if dw/dt is smaller. But the calculation has neglected the lunar orbit's inclination with respect to the ecliptic, and this inclination has the effect of diminishing the Sun's action on the Moon. The calculation therefore makes the Sun's action too great, hence gives too great a ratio of dw/dt to n, and thus too small a value for c.

But now Hill commences his serious assault on the infinite determinant. He represents it by $D(c)$, and asks us to observe that $D(c)=D(-c)$: the determinant is an even function. Moreover, $D(c)=D(c\pm 2i)$, where i can be any integer: $D(c)$ is thus periodic. According to Hill:

> It will occur immediately to every one that the properties we have stated of the roots of $D(c)=0$ are precisely those of the transcendental equation
>
> $$\cos(\pi x)-a=0;$$

of which, if x_0 is one of the roots, the whole series of roots is represented by $\pm x_0 + 2i$. Hence we must necessarily have, identically,

$$D(c) = A[\cos(\pi c) - \cos(\pi c_0)],$$

A being some constant independent of c.

With a view to evaluating A, Hill introduces Euler's infinite product for $\cos(\pi c)$, namely, $\cos(\pi c) = \prod_{k=0}^{\infty}\left(1 - \frac{4c^2}{(2k+1)^2}\right)$. Hill may have read Euler's derivation of this formula, and of a parallel formula for $\sin(\pi c)$, in Euler's *Introductio in analysin infinitorum*, Lausanne, 1748.[50] When an approximation to $\cos(\pi c)$ is obtained from the first $(n+1)$ factors in the foregoing infinite product, the highest power of c (namely c^{2n}) will have the coefficient

$$\frac{-4}{1^2} \times \frac{-4}{3^2} \times \frac{-4}{5^2} \times \cdots \times \frac{-4}{(2n+1)^2}.$$

Hill proposes to transform $D(c)$ so that in its expansion, computed to the same approximation as the formula for the infinite product giving $\cos(\pi c)$, the term containing the largest power of c will have this same coefficient.

The transformation consists in multiplying the row of $D(c)$ containing [0] by -4, the rows containing [1] and [−1] by $4/(4^2 - 1)$, and, in general, the rows containing $[j]$ and $[-j]$ by $4/[(4j)^2 - 1] = 4/(2j-1)(2j+1)$. A new determinant, $\nabla(c)$, thus arises, which has the same roots as $D(c)$, since multiplying an equation by a constant does not change its roots.

As a visual aid to the reader, I write out the central 5×5 sub-determinant of $\nabla(c)$:

$$\begin{vmatrix} \frac{4}{63}[-2] & \frac{-4}{63}\theta_1 & \frac{-4}{63}\theta_2 & \frac{-4}{63}\theta_3 & \frac{-4}{63}\theta_4 \\ \frac{-4}{15}\theta_1 & \frac{4}{15}[-1] & \frac{-4}{15}\theta_1 & \frac{-4}{15}\theta_2 & \frac{-4}{15}\theta_3 \\ 4\theta_2 & 4\theta_1 & -4[0] & 4\theta_1 & -4\theta_2 \\ \frac{-4}{15}\theta_3 & \frac{-4}{15}\theta_2 & \frac{-4}{15}\theta_1 & \frac{4}{15}[1] & \frac{-4}{15}\theta_1 \\ \frac{-4}{63}\theta_4 & \frac{-4}{63}\theta_3 & \frac{-4}{63}\theta_2 & \frac{-4}{63}\theta_1 & \frac{4}{63}[2] \end{vmatrix}$$

The product of the five terms in its main diagonal, with the symbols $[j]$ replaced by what they signify, is

$$\frac{4}{7\cdot 9}[(c-4)^2 - \theta_0] \times \frac{4}{3\cdot 5}[(c-2)^2 - \theta_0] \times (-4)[c^2 - \theta_0]$$
$$\times \frac{4}{3\cdot 5}[(c-2)^2 - \theta_0] \times \frac{4}{7\cdot 9}[(c-4)^2 - \theta_0].$$

Note that the factors symmetrically placed on either side of the central factor have the same numerical coefficient. Evidently the coefficient of the highest power of c

[50] See Leonhard Euler, *Opera Omnia*, I.8, 168–169.

in the above product is $(-4)\frac{4}{3^2} \times \frac{4}{5^2} \times \frac{4}{7^2} \times \frac{4}{9^2}$. For larger central sub-determinants of $\nabla(c)$, the new numerical factors added to the product will always have 4 in the numerator and the square of an odd number in the denominator: the pattern is the same as that for $\cos(\pi c)$. Therefore $A = 1$ and Hill can write

$$\nabla(c) = \cos(\pi c) - \cos(\pi c_0).$$

This equation holds for any value of c. Since $\cos(\pi c_0)$ is a constant independent of the value of c, we can determine its value by giving a particular value to c, for instance $c = 0$:

$$\cos(\pi c_0) = \cos(\pi c) - \nabla(c) = \cos(0) - \nabla(0) = 1 - \nabla(0).$$

Our aim is to find a value of c such that $\nabla(c) = 0$, and when $\nabla(c) = 0$, we shall also have $\cos(\pi c) - \cos(\pi c_0) = 0$. Hence, in this case $\cos(\pi c) = 1 - \nabla(0)$. It then follows that $\nabla(0) = 1 - \cos(\pi c)$, or

$$\nabla(0) = 2\sin^2\left(\frac{\pi c}{2}\right). \tag{I.20}$$

Therefore, if we knew the value of $\nabla(0)$, we could solve (I.20) for c. Note that in $\nabla(0)$, c has been set equal to zero, so that it does not occur, and $[j]$ where it appears in the main diagonal is reduced to $(2j)^2 - \theta_0$.

On the way to obtaining a numerical value for $\nabla(0)$, Hill introduces a new determinant, symbolized by $\Box(0)$. He obtains it by dividing the terms in each row of $\nabla(0)$ by the term in that row that is in the main diagonal. Thus the central 5×5 sub-determinant in $\Box(0)$ is

$$\begin{vmatrix} 1 & \frac{-\theta_1}{4^2-\theta_0} & \frac{-\theta_2}{4^2-\theta_0} & \frac{-\theta_3}{4^2-\theta_0} & \frac{-\theta_4}{4^2-\theta_0} \\ \frac{-\theta_1}{2^2-\theta_0} & 1 & \frac{-\theta_1}{2^2-\theta_0} & \frac{-\theta_2}{2^2-\theta_0} & \frac{-\theta_3}{2^2-\theta_0} \\ \frac{-\theta_2}{0^2-\theta_0} & \frac{-\theta_1}{0^2-\theta_0} & 1 & \frac{-\theta_1}{0^2-\theta_0} & \frac{-\theta_2}{0^2-\theta_0} \\ \frac{-\theta_3}{2^2-\theta_0} & \frac{-\theta_2}{2^2-\theta_0} & \frac{-\theta_1}{2^2-\theta_0} & 1 & \frac{-\theta_1}{2^2-\theta_0} \\ \frac{-\theta_4}{4^2-\theta_0} & \frac{-\theta_3}{4^2-\theta_0} & \frac{-\theta_2}{4^2-\theta_0} & \frac{-\theta_1}{4^2-\theta_0} & 1 \end{vmatrix}$$

Then $\nabla(0)$ is equal to $\Box(0)$ multiplied by the product of the elements in the main diagonal of $\nabla(0)$.

The latter product, Hill states, is $1-\cos(\pi\sqrt{\theta_0}) = 2\sin^2\left(\frac{\pi}{2}\sqrt{\theta_0}\right)$. He proves this as follows:

> As, in the particular case, where θ_1, θ_2, etc., all vanish, the proper value of c is $\sqrt{\theta_0}$, it follows that the element of the determinant $\nabla(0)$, formed by the diagonal line of constituents involving θ_0, is
>
> $$1 - \cos(\pi\sqrt{\theta_0}) = 2\sin^2\left(\frac{\pi}{2}\sqrt{\theta_0}\right).$$

In effect, Hill is imagining the following operation as applied to (I.20). On the right-hand side, all the products of $\nabla(0)$ involving θ_i with i other than zero are to vanish; these terms are all and only those that are not in the main diagonal. The determinant $\nabla(0)$, then, is reduced to the product of the elements in the main diagonal. On the left-hand side of the equation, c reduces to $\sqrt{\theta_0}$, since that is the value of c when all the θ_i other than θ_0 vanish. Therefore the product of the elements in the diagonal of $\nabla(0)$ is $2\sin^2\left(\frac{\pi}{2}\sqrt{\theta_0}\right)$, and

$$\sin^2\left(\frac{\pi}{2}c\right) = \sin^2\left(\frac{\pi}{2}\sqrt{\theta_0}\right) \times \square(0). \tag{I.21}$$

Turning to the evaluation of $\square(0)$, Hill remarks that the product of the elements in the main diagonal is 1, and that all the other products in the expanded determinant are much smaller, since all elements of $\square(0)$ other than those in the main diagonal are much less than 1. To obtain these other products, Hill uses the procedure of exchanging columns. Whenever two columns of a determinant are exchanged, the resulting determinant has the same absolute value as the original determinant, but differs in sign, being negative if the original determinant was positive, and *vice versa*. A second exchange of columns reverses the sign again. The product of the elements in the diagonal of the new determinant is thus, when given the appropriate sign, one of the products in the expansion of the original determinant. The columns can be returned to their original positions, and two or more other columns exchanged in order to obtain another product in the expansion of the original determinant.

When two adjacent columns of $\square(0)$ are interchanged, the main diagonal of the resulting determinant will consist of 1's except for two elements, each of which has θ_1 as its numerator. The denominators of these two elements have the form $(2i)^2 - \theta_0$, where i is an integer which can be positive, negative, or zero; but i in the one element will differ from i in the other by 1. Following Hill, we symbolize $(2i)^2 - \theta_0$ by $\{i\}$. Then the product of the elements of the main diagonal of the new determinant will be $-\frac{\theta_1^2}{\{i\}\{i+1\}}$. To obtain the sum of all the terms of this type – they are infinite in number – requires evaluating the sum $-\theta_1^2 \sum_{-\infty}^{+\infty} \frac{1}{\{i\}\{i+1\}}$. Hill develops a formula for doing this; we shall describe its derivation in a moment. In the particular case we are examining it yields

$$-\frac{\theta_1^2 \pi \cot\left(\frac{\pi}{2}\sqrt{\theta_0}\right)}{4\sqrt{\theta_0}(\theta_0 - 1)} = +0.00180\ 46110\ 93422\ 7.$$

Recall that the first product in the determinant $\square(0)$ was equal to 1; the second product, we now see, is less than 0.2% of the first. Yet it is the largest among the remaining products, and in fact 10^4 times larger than any of the others. Hill undoubtedly felt secure about the "convergence" of this determinant.

To derive the formula used in the foregoing calculation, Hill considers the more general sum $\sum_{-\infty}^{+\infty} \frac{1}{\{i\}\{i+k\}} = \sum_{-\infty}^{+\infty} \frac{1}{[(2i)^2 - \theta_0][2^2(i+k)^2 - \theta_0]}$. Here k is to be understood as a fixed integer, later to be assigned the values 1, 2, or 3, while i remains the variable index of the terms in the summation. To convert each of the two factors in the denominator into the difference of two squares, Hill introduces the substitution

$4\theta^2 = \theta_0$, and then factors the factors.[51] The expression under the summation sign can thus be given the form

$$\frac{1}{16(\theta + i)(\theta - i)(\theta + i + k)(\theta - i - k)}.$$

This expression can be resolved into a sum of algebraically irreducible partial fractions:

$$\frac{1}{16}\left[\frac{A}{\theta + i} + \frac{B}{\theta - i} + \frac{C}{\theta + i + k} + \frac{D}{\theta - i - k}\right],$$

where A, B, C, and D are constants. These constants are determined by setting our two expressions equal, clearing them of fractions, then giving i in succession the values $-\theta, +\theta, -\theta - k, \theta - k$, so as to cause, each time, three of the four resulting terms to vanish. We thus obtain four equations for the four constants:

$$\begin{aligned} 2k\theta(2\theta - k)A &= 1, \\ -2k\theta(2\theta + k)B &= 1, \\ -2k\theta(2\theta + k)C &= 1, \\ 2k\theta(2\theta - k)D &= 1. \end{aligned}$$

But, Hill tells us, it is well known that[52]

$$\sum_{-\infty}^{+\infty} \frac{1}{\theta + i} = \sum_{-\infty}^{+\infty} \frac{1}{\theta - i} = \sum_{-\infty}^{+\infty} \frac{1}{\theta + i + k} = \sum_{-\infty}^{+\infty} \frac{1}{\theta - i - k} = \pi \cot \pi\theta.$$

Hence,

$$\begin{aligned} \sum_{-\infty}^{+\infty} \frac{1}{\{i\}\{i+1\}} &= \frac{1}{16}(A + B + C + D)\pi \cot \pi\theta \\ &= \frac{\pi \cot \pi\theta}{8\theta(4\theta^2 - k^2)} \\ &= \frac{\pi \cot\left(\frac{\pi}{2}\sqrt{\theta_0}\right)}{4\sqrt{\theta_0}(\theta_0 - k^2)}. \end{aligned} \tag{I.22}$$

For $k = 1, 2, 3$, respectively, (I.22) yields the coefficients for the products $\theta_1^2, \theta_2^2, \theta_3^2$; and thus Hill obtains the contributions of these three products to the value of $\Box(0)$:

[51] In the article as printed in the *Collected Mathematical Works*, I, 265, this substitution is given incorrectly, as $4\theta = \theta_0$. It is given correctly in *Acta Mathematica*, VIII (1886), 30.

[52] As indicated earlier, this formula was available to Hill in Euler's *Introductio in analysin infinitorum* and in the textbook of Briot & Bouquet, *Théorie des functions doublement périodiques*.

$$\frac{\pi \cot\left(\frac{\pi}{2}\sqrt{\theta_0}\right)}{4\sqrt{\theta_0}}\left[\frac{\theta_1^2}{1-\theta_0}+\frac{\theta_2^2}{4-\theta_0}+\frac{\theta_3^2}{9-\theta_0}\right].$$

Hill could, of course, have included the term for θ_4^2, but this has a value of 3×10^{-15}, and when this factor is multiplied by its coefficient and evaluated numerically, the result proves less than 10^{-15}. He chooses to limit the precision of his calculation to the fifteenth decimal.

Seven more terms, however, must be calculated to bring the overall precision to this level, and they require, in addition to the formula (I.22) given above, two other general formulas derived in the same manner:

$$\sum_{i=-\infty}^{+\infty} \frac{1}{\{i\}\{i+k\}\{i+k'\}}$$

$$= -\frac{1}{16}\frac{3\theta_0 - (k^2 - kk' + k^2)}{\sqrt{\theta_0}(\theta_0 - k^2)(\theta_0 - k^2)[\theta_0 - (k-k')^2]}\pi \cot\left(\frac{\pi}{2}\sqrt{\theta_0}\right), \tag{I.23}$$

$$\sum_{i=-\infty}^{+\infty} \frac{1}{\{i\}\{i+1\}\{i+k\}\{i+k+1\}}$$

$$-\frac{1}{32}\frac{5\theta_0 - (k^2+1)}{\sqrt{\theta_0}(\theta_0 - 1)(\theta_0 - k^2)[\theta_0 - (k+1)^2][\theta_0 - (k-1)^2]}\pi \cot\left(\frac{\pi}{2}\sqrt{\theta_0}\right). \tag{I.24}$$

Consider first the product $\theta_1^2\theta_2$; it is obtained by first exchanging two adjacent columns, then exchanging one of these with its just acquired new neighbor. The term to be calculated is

$$+\frac{3\pi \cot\left(\frac{\pi}{2}\sqrt{\theta_0}\right)}{8\sqrt{\theta_0}(1-\theta_0)(4-\theta_0)}\theta_1^2\theta_2.$$

Somewhat similarly, the product $\theta_1\theta_2\theta_3$ is obtained by first exchanging adjacent columns, then exchanging one of the exchanged columns with the column on the other side of the just acquired new neighbor. Its coefficient is obtained from formula (I.23) by substituting $k = 1, k' = 3$. The term to be calculated is thus

$$+\frac{(7-3\theta_0)\pi \cot\left(\frac{\pi}{2}\sqrt{\theta_0}\right)}{4\sqrt{\theta_0}(1-\theta_0)(4-\theta_0)(9-\theta_0)}\theta_1\theta_2\theta_3.$$

To obtain $\theta_1^3\theta_3$, two adjacent columns are first exchanged, then one of them is moved by two further exchanges, to the right if it is on the right after the first exchange, to the left in the opposite case; thus the initial pattern *abcd* becomes *bcda* or *dabc*. The coefficient is obtained from formula (I.24) above by substituting $k = 2$. The term to be calculated is therefore

$$+\frac{5\pi \cot\left(\frac{\pi}{2}\sqrt{\theta_0}\right)}{16\sqrt{\theta_0}(1-\theta_0)(4-\theta_0)(9-\theta_0)}\theta_1^3\theta_3.$$

The four remaining terms all require double or triple summations. Thus in the case of θ_1^4, we may start from the double summation

$$\sum_{k=2}^{+\infty}\sum_{i=-\infty}^{+\infty}\frac{1}{\{i\}\{i+1\}\{i+k\}\{i+k+1\}}.$$

Here the summation in which i runs from minus to plus infinity gives all interchanges of adjacent columns leading to the product θ_1^2; the summation in which k runs from 2 to plus infinity then gives all interchanges of adjacent columns capable of being combined with the former exchanges so as to yield the product θ_1^4. First we resolve the expression under the summation signs into partial fractions with respect to i as variable, and sum between the indicated limits; the result is

$$-\frac{1}{32}\frac{5\theta_0-(k^2+1)}{\sqrt{\theta_0}(\theta_0-1)(\theta_0-k^2)[\theta_0-(k+1)^2][\theta_0-(k-1)^2]}\pi\cot\left(\frac{\pi}{2}\sqrt{\theta_0}\right).$$

This expression is then to be resolved into partial fractions with respect to k. The summation is most conveniently carried out, not from 2 to infinity, but from 0 to infinity, after which the values of the expression for $k=0$ and $k=1$ can be subtracted. We obtain

$$\frac{\pi\cot\left(\frac{\pi}{2}\sqrt{\theta_0}\right)}{32\sqrt{\theta_0}(1-\theta_0)^2}\left[\frac{\pi\cot(\pi\sqrt{\theta_0})}{\sqrt{\theta_0}}-\frac{1}{\theta_0}+\frac{2}{1-\theta_0}+\frac{9}{2(4-\theta_0)}\right]\theta_1^4.$$

By analogous processes the remaining three summations are obtained:

$$\frac{3\pi cit\left(\frac{\pi}{2}\sqrt{\theta_0}\right)}{32\sqrt{\theta_0}(1-\theta_0)^2(4-\theta_0)}\left[\frac{\pi\cot(\pi\sqrt{\theta_0})}{\sqrt{\theta_0}}-\frac{1}{\theta_0}+\frac{2}{1-\theta_0}+\frac{2}{4-\theta_0}+\frac{20}{3(9-\theta_0)}\right]\theta_1^4\theta_2;$$

$$\frac{\pi\cot\left(\frac{\pi}{2}\sqrt{\theta_0}\right)}{16\sqrt{\theta_0}(1-\theta_0)(4-\theta_0)}\left[\frac{\pi\cot(\pi\sqrt{\theta_0})}{\sqrt{\theta_0}}-\frac{1}{\theta_0}+\frac{2}{1-\theta_0}+\frac{2}{4-\theta_0}+\frac{10}{9-\theta_0}\right]\theta_1^2\theta_2^2;$$

$$\frac{\pi\cot\left(\frac{\pi}{2}\sqrt{\theta_0}\right)}{128\sqrt{\theta_0}(1-\theta_0)^3}\left\{\begin{array}{l}\left[-\frac{1}{\theta_0}+\frac{2}{1-\theta_0}+\frac{9}{2(4-\theta_0)}\right]\frac{\pi\cot(\pi\sqrt{\theta_0})}{\sqrt{\theta_0}}-\frac{25}{8\theta_0}-\frac{1}{\theta_0^2}\\+\frac{2}{1-\theta_0}+\frac{4}{(1-\theta_0)^2}-\frac{9}{8(4-\theta_0)}+\frac{9}{(4-\theta_0)^2}-\frac{4}{9-\theta_0}-\frac{\pi^2}{3\theta_0}\end{array}\right\}\theta_1^6.$$

To obtain his value for $\Box(0)$, Hill evaluated these several expressions numerically to sixteen decimal places – a task which included the computation of $\sqrt{\theta_0}$ and two cotangents to the same precision. Then, adding together these results and the main term of the determinant – which, we recall, was equal to unity – he obtained

$$1.00180\ \ 47920\ \ 21011\ \ 2.$$

This result had then to be introduced into (I.20):

$$\sin^2\left(\frac{\pi}{2}c\right)=\sin^2\left(\frac{\pi}{2}\sqrt{\theta_0}\right)\times 1.00180\ \ 47920\ \ 21011\ \ 2.$$

From this expression Hill derived:

$$c = 1.07158\ 32774\ 16016.$$

To check the accuracy of this result, Hill turned back to the equations from which he had derived the determinant $\nabla(0)$. Each of these equations can be expressed by the formula

$$[j]b_j - \sum_i \theta_{j-i} b_i = 0, \tag{I.25}$$

where $[j] = (c+2j)^2 - \theta_0$, and under the summation sign the term with index $i = j$ is omitted. Using his value for c, Hill computed $[j]$ for the following values:

$$\begin{aligned}
[0] &= -0.01055\ 32191\ 58933,\\
[-1] &= -0.29688\ 63288\ 2300,\\
[1] &= +8.37577\ 98905\ 1,\\
[-2] &= +7.41678\ 05615\ 1,\\
[2] &= +24.56211\ 3,\\
[-3] &= +23.13045,\\
[3] &= +48.85,\\
[-4] &= +46.8.
\end{aligned}$$

Now the central equation of the array – the equation in which [0] occurs – may be written

$$[0]b_0 - \sum_i \theta_{j-i} b_i = 0. \tag{I.26}$$

Hill sets about eliminating from (I.26), successively, the unknowns $b_{-1}, b_1, b_{-2}, b_2, b_{-3}, b_{-4}$, using in each case an equation of the form of (I.25) to eliminate the b having subscript j. Thus to eliminate b_{-1} he solves the equation

$$[-1]b_{-1} - \sum_i \theta_{-1-i} b_i = 0$$

or

$$b_{-1} = \frac{\sum_i \theta_{-1-i} b_i}{[-1]},$$

where in the summation on the right the term in which $i = -1$ is omitted. When this value of b_{-1} is substituted into (I.26), and the terms contributing to the coefficient of each b_i are collected, the result is

$$\left[[0] - \frac{\theta_1^2}{[-1]}\right] b_0 - \sum_i \left[\theta_{-i} + \frac{\theta_1 \theta_{i+1}}{[-1]}\right] b_i = 0.$$

The new coefficient of b_0, which Hill symbolizes by $[0]^{(-1)}$, turns out to be smaller than [0] in absolute value. Hill repeats this eliminative process for the b's we have listed, obtaining the following reductions in the coefficient of b_0:

$$[0] = -0.01055\ 32191\ 58933,$$
$$[0]^{(-1)} = +0.00040\ 72723\ 11650,$$
$$[0]^{(-1,1)} = +0.00001\ 50888\ 08423,$$
$$[0]^{(-2,-1,1)} = +0.00000\ 00253\ 21700,$$
$$[0]^{(-2,-1,1,2)} = +0.00000\ 00009\ 20420,$$
$$[0]^{(-3,-2,-1,1,2)} = +0.00000\ 00000\ 03941,$$
$$[0]^{(-3,-2,-1,1,2,3)} = +0.00000\ 00000\ 00155,$$
$$[0]^{(-4,-3,-2,-1,1,2,3)} = +0.00000\ 00000\ 00008.$$

As the coefficient of b_0 decreases, so, proportionately, must the second term of (I.26), so that the sum of all terms adds to zero.

Can the 8×10^{-15} of $[0]^{(-4,-3,-2,-1,1,2,3)}$ be reduced further by carrying out further eliminations, for instance, of b_4, b_{-5}, etc.? Hill tells us that these further eliminations do not sensibly change the result. Rather than repeating the whole eliminative process with a lower value of c, it will be sufficient, he says, to subtract half of the residual from the value of c he has assumed. To understand this step, notice that $[0] = c^2 - \theta_0$, and that c exceeds 1 by only about 0.072. When the binomial $(c - 4 \times 10^{-15})$ is squared, the result will therefore be less than c^2, very nearly, by $2 \times 4 \times 10^{-15}$. Hence replacing the assumed value of c by $(c - 4 \times 10^{-15})$ will reduce the residual to zero. Hill's final value of c is thus

$$c = 1.07158\ 32774\ 16012.$$

In R.S. Woodward's obituary of Hill published in *The Astronomical Journal, 28* (1914), 161–162, it is stated that Hill made two exploratory trips into Canada, one into the Hudson Bay region and one into the Canadian Northwest.

> It was during journey through the latter territory that he worked out his famous solution of the problem involving an infinite determinant, a solution "aussi originale que hardi," as remarked by Poincaré.

What actually got worked out during the journey – initial steps, main ideas, final steps? – can only be guessed. The solution demanded extensive and complicated paper-and-pencil computations, most easily imagined as performed where paper was plentiful and filing facilities available. In concluding his memoir, Hill remarks:

> It may be stated that all the computations have been made twice, and no inconsiderable portion of them three times.

The value of c that Hill here obtained from gravitational theory differs from the observational value by only one part in 550. To obtain the implied value of dw/dt, the rate of motion of the lunar perigee, it is necessary to substitute in the equation

$$\frac{dw}{dt} = n - c(n - n').$$

With the observational value of c we obtain 0.00194419 radians/day; Hill's calculated value of c gives 0.001971441 radians/day, which exceeds the observational value by $1/72$nd part, or 1.4%. As we affirmed at the start of this section, no earlier computation of dw/dt had approached the observational value anywhere near so closely. A few of Hill's readers, like John Couch Adams, were able to recognize the significance and the wonder of this achievement.

4

Hill's Variation Curve

Hill's second paper, published in the first volume of the *American Journal of Mathematics* in 1878, is divided into three parts, printed on pp. 5–26, 129–147, and 245–260. In our resume, we shall focus on the laying of the groundwork for the Hill–Brown lunar theory. Hill's paper includes treatment of Variation orbits with different values of **m** from over the whole range of possible values; we shall confine our attention to the particular Variation orbit defined by the **m** of Earth's Moon.

The "Researches" consists of an introduction and two chapters, the first on the differential equations and Jacobi's integral, the second on the determination of the inequalities dependent solely on the ratio of the mean motions of the Sun and Moon. In *The Collected Mathematical Works of George William Hill*, Vol. I, the introduction is found on pp. 284–287, and the two chapters on pp. 287–304 and 305–335.

Hill begins by giving reasons for laying a new foundation. Earlier lunar theorists, by making the construction of tables their primary aim, have allowed their choice of variables and parameters to be unduly restricted.

> But the developments having now been carried extremely far, without completely satisfying all desires, one is led to ask whether such modifications cannot be made in the processes of integration, and such coordinates and parameters adopted, that a much nearer approach may be had to the law of the series, and, at the same time, their convergence augmented.

Hill explains his preference for rectangular coordinates over the polar coordinates commonly employed. In the case of elliptical motion, the x and y coordinates are given by series expressible finitely in terms of Bessel functions; these series follow evident laws, and it is a simple matter to calculate their values to any chosen order of approximation. In polar coordinates, the series developments are less obliging by far. Moreover, the differential equations expressed in rectangular coordinates are purely algebraic, whereas their expression in polar coordinates requires trigonometric functions.

As for parameters, Hill sees as unfortunate the choice by lunar theorists from Laplace to Delaunay of the parameter m, the ratio of the sidereal period of the Moon to the sidereal period of the Sun.

C. Wilson, *The Hill–Brown Theory of the Moon's Motion*, Sources and Studies
in the History of Mathematics and Physical Sciences, DOI 10.1007/978-1-4419-5937-9_4,

> Some instances of slow convergence with the parameter m may be given from Delaunay's Lunar Theory; the development of the principal part of the coefficient of the evection in longitude begins with the term $\frac{15}{4}\,me$, and ends with the term $\frac{413,277,465,931,033}{15,288,238,080}m^8ee'$; again, in the principal part of the coefficient of the inequality whose argument is the difference of the mean anomalies of the Sun and Moon, we find, at the beginning, the term $\frac{21}{4}mee'$, and, at the end, the term $\frac{1,207,454,026,843}{3,538,944}m^7ee'$. It is probable that, by the adoption of some function of m as a parameter in place of this quantity, whose numerical value, in the case of our Moon, should not greatly exceed that of m, the foregoing large numerical coefficients might be very much diminished.

Hill will begin by using the ratio of the Moon's *synodic* period to the Sun's sidereal period, the parameter used by Euler in his lunar theory of 1772; it is larger than m (equal to about 1/12 as compared with 1/13), and the series expressed in terms of it converge more rapidly. Hill denotes this ratio by an "**m**" from the Roman font in which his article is printed; we will use this same notation here.

Hill has a further criticism of Delaunay's method:

> Although [it] is very elegant, and, perhaps, as short as any, when one wishes to go over the whole ground of the lunar theory, it is subject to some disadvantages when the attention is restricted to a certain class of lunar inequalities. Thus, do we wish to get only the inequalities whose coefficients depend solely on m, we are yet compelled to develop the disturbing function R to all powers of e.

Hill has the idea of determining, independently of all other inequalities, the inequalities that are a function solely of **m**. In the series obtained by Plana and Delaunay, the convergence problems encountered appeared in almost every case to be produced by the ingression of m into the expressions. A prior determination of the inequalities depending solely on **m** might reduce or eliminate these problems. The ratio of the mean motions of the Sun and Moon was more precisely known than the other parameters; hence the inequalities dependent solely on **m** lent themselves particularly well to a numerical theory. Hill will develop both a literal and a numerical theory of the inequalities dependent solely on **m**. Once the numerical theory was worked out, a *literal* theory of the other inequalities, unencumbered with top-heavy numerical fractions, should be possible. The Hill–Brown theory will in fact be achieved by just this sequence of operations.

In deriving his differential equations, Hill set aside the action of the planets and the influence of the non-spherical figures of the Sun, Earth, and Moon, as also the gravitational action of the Moon on the Sun. He adopted rectangular axes having origin at the Earth's center of gravity, the x-axis directed at the center of the Sun and the y-axis at a point in the ecliptic 90° ahead of the Sun, while the z-axis is perpendicular to the ecliptic. Let

$$r = \sqrt{x^2 + y^2 + z^2} = \text{the Earth-Moon distance,}$$

r' = the Earth-Sun distance,

λ' = the longitude of the Sun,

n' = the mean angular velocity of the Sun about the Earth,

a' = the Sun's mean distance from the Earth.

The Moon's kinetic energy about the Earth's center is

$$T = \frac{1}{2}\left[\frac{dx}{dt} - y\frac{d\lambda'}{dt}\right]^2 + \frac{1}{2}\left[\frac{dy}{dt} + x\frac{d\lambda'}{dt}\right]^2 + \frac{1}{2}\frac{dz^2}{dt^2}$$
$$= \frac{1}{2}\frac{dx^2 + dy^2 + dz^2}{dt^2} + \frac{d\lambda'}{dt}\frac{xdy - ydx}{dt} + \frac{1}{2}\left[\frac{d\lambda'}{dt}\right]^2 (x^2 + y^2). \qquad \text{(II.1)}$$

The potential function yielding by differentiation the forces on the Moon in the coordinate directions is

$$\Omega = \frac{\mu}{\sqrt{x^2 + y^2 + z^2}} + \frac{n'^2 a'^3}{\sqrt{(r' - x)^2 + y^2 + z^2}} - \frac{n'^2 a'^3}{r'^2} x, \qquad \text{(II.2)}$$

where μ is the sum of the masses of the Earth and the Moon and $n'^2 a'^3$ under the restrictions made is equal to the mass of the Sun. (The second and third terms on the right yield, as required, the *difference* between the Sun's force on the Moon and its force on the Earth.) From (II.1) and (II.2) Hill derives differential equations by the well-known Lagrangian algorithm

$$\frac{d}{dt} \cdot \frac{\partial T}{\partial \frac{d\varphi}{dt}} - \frac{\partial T}{\partial \varphi} = \frac{\partial \Omega}{\partial \varphi}, \qquad \text{(II.3)}$$

where φ denotes successively each of the variables x, y, z. First, however, for economy of expression he removes the last term of (II.1) from T and adds it to Ω, denoting the modified potential function by Ω'. The resulting differential equations can then be written

$$\frac{d^2x}{dt^2} - 2\frac{d\lambda'}{dt}\frac{dy}{dt} - \frac{d^2\lambda'}{dt^2} y = \frac{\partial \Omega'}{\partial x},$$
$$\frac{d^2y}{dt^2} + 2\frac{d\lambda'}{dt}\frac{dx}{dt} + \frac{d^2\lambda'}{dt^2} x = \frac{\partial \Omega'}{\partial y},$$
$$\frac{d^2z}{dt^2} = \frac{\partial \Omega'}{\partial z}. \qquad \text{(II.4)}$$

If the solar eccentricity is neglected – that is, if the orbit of the Earth about the Sun is taken as circular so that the Earth's motion is uniform – we shall have

$$\frac{d\lambda'}{dt} = n',$$

$$\frac{d^2\lambda'}{dt^2} = 0,$$
$$r' = a'.$$

This restriction enables us to obtain the Jacobian integral. We multiply the three equations of (II.4) respectively by dx, dy, dz, add the products and integrate, obtaining

$$\frac{dx^2 + dy^2 + dz^2}{2dt^2} = \Omega' + C,$$

where C is the constant of integration.

To arrive at equations whence his Variation orbit can be derived, Hill introduced two further restrictions: he neglected the lunar inclination, thus eliminating the equation for z, and he neglected the solar parallax, which appears in the expansion of the quotient expressing the net solar force on the Moon. The relative kinetic energy and potential function thus became

$$T' = \frac{dx^2 + dy^2}{2dt^2} + n'\frac{xdy - ydx}{dt},$$
$$\Omega' = \frac{\mu}{\sqrt{x^2 + y^2}} + \frac{3}{2}n'^2 x^2. \tag{II.6}$$

The differential equations resulting from the substitution of (II.6) in (II.3) are

$$\frac{d^2x}{dt^2} - 2n'\frac{dy}{dt} + \left[\frac{\mu}{r^3} - 3n'^2\right]x = 0,$$
$$\frac{d^2y}{dt^2} + 2n'\frac{dx}{dt} + \frac{\mu}{r^3}y = 0. \tag{II.7}$$

The Jacobian integral resulting from (II.7) is

$$\frac{dx^2 + dy^2}{2dt^2} = \frac{\mu}{\sqrt{x^2 + y^2}} + \frac{3}{2}n'^2 x^2 - C. \tag{II.8}$$

The neatly linear equations (II.7), together with the Jacobian integral (II.8), are Hill's basis for the derivation of the Variation orbit.

The solution of (II.7) considered by Hill will involve trigonometric series in sines and cosines. To introduce the exponential expression of these functions, and thus facilitate multiplication of such series, Hill transforms (II.7) and (II.8) using the complex conjugate variables

$$u = x + y\sqrt{-1},$$
$$s = x - y\sqrt{-1}.$$

Substitution into (II.7) gives

$$\frac{d^2u}{dt^2} + 2n'\sqrt{-1}\frac{du}{dt} + \frac{\mu}{(us)^{3/2}}u - \frac{3}{2}n'^2(u+s) = 0,$$

$$\frac{d^2s}{dt^2} - 2n'\sqrt{-1}\frac{ds}{dt} + \frac{\mu}{(us)^{3/2}}s - \frac{3}{2}n'^2(u+s) = 0. \qquad \text{(II.7}')$$

(Both in the original publication, *American Journal of Mathematics*, I, 13, and in *The Collected Works*, I, 292, the second term in each of these equations is given incorrectly, with the sign wrong and the factor $\sqrt{(-1)}$ omitted. The later development is free from error, just as if this mistake had not occurred.) With the new variables the Jacobian integral becomes

$$\frac{duds}{2dt^2} = \frac{\mu}{(us)^{1/2}} + \frac{3}{8}n'^2(u+s)^2 - C. \qquad \text{(II.8}')$$

We turn now to Chapter II of the "Researches", in which Hill determines his Variation orbit. Here he is seeking, in the rotating coordinate system, a motion of a moon about the Earth that is periodic, so that the particularities of this motion repeat exactly each time the moon completes a cycle with respect to the line of syzygies (the line connecting the Earth and the Sun). Suppose this moon, moving in accordance with (II.7), cuts the x-axis at right angles. At the moment of intersection, $dx/dt = 0$ and $y = 0$, while y is changing from negative to positive or vice versa. The accelerations of the motion before and after the passage will be the reverse of each other; for if in (II.7) the signs of y and t are reversed, and the sign of x is left unaltered, the differential equations do not change. Again, suppose the orbit intersects the y-axis at right angles. If the signs of x and t are reversed, and that of y left unaltered, the differential equations do not change; and the accelerations of motion before and after the passage of the y-axis will be the reverse of each other.

This moon can be imagined to cross the x-axis at right angles but with different velocities, some too small for it to reach and cross the y-axis perpendicularly, and some too large. As the principle of continuity suggests, there should be an intermediate velocity that would bring the moon to the y-axis so as to cross it at right angles. A moon crossing first the x-axis and then the y-axis at right angles should go on to execute a closed curve symmetrical with both axes. Such is the motion that Hill wished to characterize mathematically, for a moon having the mean synodic angular speed of Earth's Moon (namely, 2π radians per 29.5305889 days, or 0.21276871 rad/day

The coordinates of a moon moving in this way, Hill asserted, can be represented by

$$x = \sum_{i=0}^{\infty} A_i \cos[(2i+1)\nu(t-t_0)],$$

$$y = \sum_{i=0}^{\infty} B_i \sin[(2i+1)\nu(t-t_0)].$$

Here ν is the constant just mentioned (0.21276871 rad/day) and t_0 the time at which the moon crosses the x-axis. The selection of the successive odd integers $(2i + 1)$ as multipliers of $\nu(t - t_0)$ guarantees that when $\nu(t - t_0)$ is $\pi/2$ or an odd integral multiple thereof, x and dy/dt will both be zero, and when $\nu(t - t_0)$ is zero or π or an integral multiple of π, y and dx/dt will both be zero–the conditions we have found to be necessary for a periodic orbit.

To simplify the notation Hill denoted the mean anomaly $\nu(t - t_0)$ by τ, and set $A_i = a_i + a_{-i-1}$ and $B_i = a_i - a_{-i-1}$. Then

$$x = \sum_{0}^{\infty}(a_i + a_{-i-1})\cos(2i + 1)\tau = \sum_{-\infty}^{+\infty} a_i \cos(2i + 1)\tau.$$

Here the summation has been extended to all positive and negative integers and zero. Similarly,

$$y = \sum_{0}^{\infty}(a_i - a_{-i-1})\sin(2i + 1)\tau = \sum_{-\infty}^{+\infty} a_i \sin(2i + 1)\tau.$$

To replace the infinite series of sines and cosines by exponential expressions, Hill now introduced the complex conjugate variables u and s given in (II.7′), and the temporal variable $\zeta = \varepsilon^{\tau\sqrt{(-1)}}$, where ε is the base of natural logarithms (he retained "e" to represent orbital eccentricity). Thus

$$u = x + \sqrt{-1}y = \sum_{-\infty}^{+\infty} a_i[\cos(2i + 1)\tau + \sqrt{-1}\sin(2i + 1)\tau]$$

$$= \sum_{-\infty}^{+\infty} a_i \zeta^{2i+1},$$

and

$$s = x - \sqrt{-1}y = \sum_{-\infty}^{+\infty} a_i[\cos(2i + 1)\tau - \sqrt{-1}\sin(2i + 1)\tau]$$

$$= \sum_{-\infty}^{+\infty} a_i \zeta^{-(2i+1)} = \sum_{-\infty}^{+\infty} a_{-i-1}\zeta^{2i+1}$$

Next, in the differential equations (II.7′) and in the Jacobian integral (II.8′), the variable t is to be replaced by the variable ζ. Since $\tau = \nu(t - t_0)$ and $\zeta = \varepsilon^{\tau\sqrt{(-1)}}$, it follows that $d\tau = \nu dt$ and $d\zeta/d\tau = \sqrt{(-1)}\zeta$. Hence

$$\zeta\frac{d}{d\zeta} = -\sqrt{-1}\frac{d}{d\tau} = -\frac{\sqrt{-1}}{\nu}\frac{d}{dt}, \text{ so that } \frac{d}{dt} = (\sqrt{-1})\nu\zeta\frac{d}{d\zeta}.$$

Hill denoted the operator $\zeta \frac{d}{d\zeta}$ by D, which he treated as if it were a multiplier. Also, he put **m** for n'/ν and κ for μ/ν^2. With these substitutions, (II.7′) and (II.8′) became

$$\left\{ D^2 + 2\mathrm{m}D + \frac{3}{2}\mathrm{m}^2 - \frac{\kappa}{(us)^{3/2}} \right\} u + \frac{3}{2}\mathrm{m}^2 s = 0, \tag{II.7a''}$$

$$\left\{ D^2 - 2\mathrm{m}D + \frac{3}{2}\mathrm{m}^2 - \frac{\kappa}{(us)^{3/2}} \right\} s + \frac{3}{2}\mathrm{m}^2 u = 0. \tag{II.7b''}$$

$$Du \cdot Ds + \frac{2\kappa}{(us)^{1/2}} + \frac{3}{2}\mathrm{m}^2 (u+s)^2 = C. \tag{II.8''}$$

In these equations, all terms in the left members are linear except those terms in (II.7a″) and (II.7b″) having the denominator $(us)^{3/2}$; with the latter terms present, the solution of the equations would presumably require numerical integration. But Hill wanted to obtain a solution in which the parameter **m** is retained in literal form; he therefore eliminated the non-linear terms *pro tem*, as the Jacobian integral (II.8″) permits him to do. First he multiplied (II.7a″) by s and (II.7b″) by u, and took their sum and difference, obtaining

$$uD^2s + sD^2u - 2\mathrm{m}(uDs - sDu) - \frac{2\kappa}{(us)^{1/2}} + \frac{3}{2}\mathrm{m}^2(u+s)^2 = 0,$$

$$uD^2s - sD^2u - 2\mathrm{m}(uDs + sDu) + \frac{3}{2}\mathrm{m}^2(u^2 - s^2) = 0.$$

Then he added (II.8″) to the first of these, while retaining the second as it is:

$$D^2(us) - Du \cdot Ds - 2\mathrm{m}(uDs - sDu) + \frac{9}{4}\mathrm{m}^2(u+s)^2 = C, \tag{II.9}$$

$$D(uDs - sDu - 2\mathrm{m}us) + \frac{3}{2}\mathrm{m}^2(u^2 - s^2) = 0. \tag{II.10}$$

Equations (II.9) and (II.10) are not equivalent to (II.7a″), (II.7b″), and (II.8″), since the constant κ does not appear in them. This constant determines the scale of the solution, and is thus essential to the problem. Integration of (II.9) and (II.10) will introduce an inadmissible constant of integration. Hill will deal with these difficulties after (II.9) and (II.10) have been integrated.

The first and second derivatives of u and s are:

$$Du = \sum_{-\infty}^{+\infty} (2i+1) a_i \zeta^{2i+1}, \quad Ds = \sum_{-\infty}^{+\infty} (2i+1) a_{-i-1} \zeta^{2i+1},$$

$$D^2u = \sum_{-\infty}^{+\infty} (2i+1)^2 a_i \zeta^{2i+1}, \quad D^2s = \sum_{-\infty}^{+\infty} (2i+1)^2 a_{-i-1} \zeta^{2i+1}.$$

In products such as (us), (u^2), (s^2), and $(Du)(Ds)$, each of the two factors is an infinite series, and each term of the one series is to be multiplied by each term of the other. Using i and k for the indices of the two factors, we may write the product us as

$$\sum_i a_i \zeta^{2i+1} \times \sum_k a_k \zeta^{-(2k+1)} = \sum_k \sum_i a_i a_k \zeta^{2(i-k)}.$$

Here as before the indices associated with the summation signs are understood to have a range form $-\infty$ to $+\infty$; that is, to start at 0 and range in both the minus and plus directions, it being assumed that the sequence converges. Note that the exponent of ζ in this product is an even integer–a result to be expected in all the products, since in the factors the exponent of ζ is always odd, and the sum of two odd integers is even. Hill used the method of undetermined coefficients, in which the exponent of ζ needs to be the same in all the products; he made it $2j$. In the value of (us) just calculated, this result can be obtained by putting $i-k=j$ so that $k=i-j$:

$$us = \sum_k \sum_i a_i a_{i-k} \zeta^{2j}.$$

Analogous substitutions yield the other needed formulas:

$$u^2 = \sum_i a_i \zeta^{2i+1} \times \sum_k a_k \zeta^{2k+1} = \sum_i \sum_j a_i a_{j-i-1} \zeta^{2j},$$

$$s^2 = \sum_i a_i \zeta^{-2i-1} \times \sum_k a_k \zeta^{-2k-1} = \sum_i \sum_j a_i a_{j-i-1} \zeta^{-2j},$$

$$Du \cdot Ds = -\sum_i \sum_j (2i+1)(2i-2j+1) a_i a_{j-i-1} \zeta^{2j},$$

$$uDs - sDu = -2 \sum_i \sum_j (2i-j+1) a_i a_{i-j} \zeta^{2j}.$$

In all these expressions the summations with respect to j have the same extension as those with respect to i.

Substituting these expressions into (II.9) and (II.10), and equating the polynomial coefficients of ζ^{2j} to zero, we obtain

$$\sum_i \left\{ (2i+1)(2i-2j+1) + 4j^2 + 4(2i-j+1)\mathrm{m} + \frac{9}{2}\mathrm{m}^2 \right\} a_i a_{i-j}$$

$$+ \frac{9}{4}\mathrm{m}^2 \sum_i (a_i a_{-i+j-1} + a_i a_{-i-j-1}) = 0,$$

$$4j \sum_i (2i-j+1+\mathrm{m}) a_i a_{i-j} - \frac{3}{2}\mathrm{m}^2 \sum_i (a_i a_{-i+j-1} - a_i a_{-i-j-1}) = 0.$$

These equations hold for all positive and negative integral values of j, but when $j = 0$, the right member of the first equation is C rather than 0, and the second equation is identically zero, hence uninformative. For the moment Hill excluded the value $j = 0$ from consideration; he will return to it later.

To obtain somewhat simpler expressions, Hill multiplied the first equation by 2 and the second by 3, and formed first their difference and then their sum. The result is

$$\sum_i \{8i^2 - 8(4j-1)i + 20j^2 - 16j + 2 + 4(4i - 5j + 2)\mathrm{m} + 9\mathrm{m}^2\} a_i a_{i-j}$$
$$+ 9\mathrm{m}^2 \sum_i a_i a_{-i+j-1} = 0, \qquad \text{(II.11)}$$

$$\sum_i \{8i^2 + 8(2j+1)i - 4j^2 + 8j + 2 + 4(4i + j + 2)\mathrm{m} + 9\mathrm{m}^2\} a_i a_{i-j}$$
$$+ 9\mathrm{m}^2 \sum_i a_i a_{-i-j-1} = 0. \qquad \text{(II.12)}$$

These equations are not really distinct; for if in (II.11) $(-j)$ is put for j everywhere, and then, under the sign of summation, $(i-j)$ is substituted for i wherever it appears, the result is identical with (II.12). Evidently a single formula can represent all the equations of condition. Hill derives such a formula as follows.

Putting $i = 0$ in the first summation of both (II.11) and (II.12), he obtains

$$\{20j^2 - 16j + 2 - 4(5j-2)\mathrm{m} + 9\mathrm{m}^2\} a_0 a_{-j}, \qquad \text{(II.11}')$$

$$\{-4j^2 + 8j + 2 + 4(j+2)\mathrm{m} + 9\mathrm{m}^2\} a_0 a_{-j}. \qquad \text{(II.12}')$$

If the substitution $i = j$ is made instead, the result is the same, except that everywhere $-j$ replaces j. (II.II′) and (II.12′) give the terms of principal importance in determining a_{-j} and a_j. Hill next multiplies (II.11) by (II.12′), and (II.12) by (II.11′), and adds the products. The number of terms resulting from this operation is 132; but happily all of them not factored by ij mutually cancel. He then divides the surviving terms by

$$48j^2\{2(4j^2-1) - 4\mathrm{m} + \mathrm{m}^2\}.$$

The result is Hill's general expression for the system of equations determining the coefficients a_i:

$$\sum_i \{[j,i] a_i a_{i-j} + [j] a_i a_{-i+j-1} + (j) a_i a_{-i-j-1}\} = 0, \qquad \text{(II.13)}$$

where

$$[j,i] = -\frac{i}{j} \frac{4(j-1)i + 4j^2 + 4j - 2 - 4(i-j+1)\mathrm{m} + \mathrm{m}^2}{2(4j^2-1) - 4\mathrm{m} + \mathrm{m}^2}, \qquad \text{(II.13a)}$$

$$[j] = -\frac{3\mathrm{m}^2}{16j^2}\frac{4j^2 - 8j - 2 - 4(j+2)\mathrm{m} - 9\mathrm{m}^2}{2(4j^2-1) - 4\mathrm{m} + \mathrm{m}^2}, \tag{II.13b}$$

$$(j) = -\frac{3\mathrm{m}^2}{16j^2}\frac{20j^2 - 16j + 2 - 4(5j-2)\mathrm{m} + 9\mathrm{m}^2}{2(4j^2-1) - 4\mathrm{m} + \mathrm{m}^2}. \tag{II.13c}$$

For each value of j, (II.13) gives an infinity of terms, obtained as i takes successively all integral values between $+\infty$ and $-\infty$. But, Hill assures us, we can nevertheless extract from (H.13) for a particular j, by successive approximations, values of a_j in terms of a_0 to any required level of precision. Both $|a_1|$ and $|a_{-1}|$ prove to be more than two orders of magnitude smaller than a_0. More generally, $|a_{j+1}|$ is more than two orders of magnitude smaller than $|a_j|$, and $|a_{-j-1}|$ more than two orders of magnitude smaller than $|a_{-j}|$. Hill does not demonstrate the general validity of this pattern, but it is confirmed step by step as his calculation proceeds. To illustrate the process:

In determining a_1 in terms of a_0, we start with the terms of (II.13) in which $j = 1$. In a first approximation, how widely should we cast our net, in calculating values of (II.13) for different values of i? Let us try limiting ourselves to the values $i = -2, -1, 0, 1, 2, 3$. Substituting in (II.13), we get

$$\begin{aligned}
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&+ [1,-2]a_{-2}a_{-3} + [1]a_{-2}a_2 + (1)a_{-2}a_0\\
&+ [1,-1]a_{-1}a_{-2} + [1]a_{-1}a_1 + (1)a_{-1}a_{-1}\\
&+ [1,0]a_0a_{-1} + [1]a_0a_0 + (1)a_0a_{-2}\\
&+ [1,1]a_1a_0 + [1]a_1a_{-1} + (1)a_1a_{-3}\\
&+ [1,2]a_2a_1 + [1]a_2a_{-2} + (1)a_2a_{-4}\\
&+ [1,3]a_3a_2 + [1]a_3a_{-3} + (1)a_3a_{-5}\\
&+ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots = 0.
\end{aligned} \tag{II.14}$$

Of the 18 terms here written out, four contain a_1 and five contain a_0. Only two of these are without a_i's other than a_1 or a_0, namely $[1]a_0a_0$ in the third row and $[1,1]a_1a_0$ in the fourth row.

We now give Hill's numerical values for the quantities [1], (1), [1,0], [1,2], etc., in (II.14), as calculated from his value of **m**, namely 0.08084 89338 08312:

$$\begin{aligned}
[1] &= 0.00151, 58491, 71593,\\
(1) &= -0.00109, 74483, 80467,\\
[1,-2] &= 2.34384, 65210,\\
[1,-1] &= 1.11204, 95007,\\
[1,1] &= -1,
\end{aligned}$$

$$[1, 2] = -1.89182, 75672,$$

$$[1, 3] = -2.66324, 029556.$$

Let us try excerpting from (II.14) the two terms that contain no other a_i's than a_0 and a_1, and setting them equal to zero; the result is

$$-a_1 a_0 + [1] a_0 a_0 = 0,$$

whence $a_1 = 0.00151\ 58491\ 71593 a_0$.

Starting with $j = -1$ rather than $j = 1$, we find, analogously,

$$-a_{-1} a_0 + (-1) a_0 a_0 = 0,$$

implying that $a_{-1} = -0.00869\ 58084\ 99634 a_0$.

The smallness of a_1 and a_{-1} relative to a_0 is encouraging. On substituting the values of a_1 and a_{-1} just found into II.14, the resulting equation will give a_2 if we ignore all terms containing a_j other than a_2 and a_0; it will give a_{-2} if we ignore all terms containing a_j other than a_{-2} and a_0. The result is:

$$a_2 = 0.00000\,58793\,35016 a_0,$$

$$a_{-2} = 0.00000\,01636\,69405 a_0.$$

The smallness of these values relative to a_1 and a_{-1} is, again, encouraging. The same procedure, applied to succeeding values of j, leads at length to

$$a_6 = 0.00000\,00000\,00007 a_0,$$

$$a_{-6} = 0.00000\,00000\,00000 a_0.$$

Here the process has gone as far as it can, if with Hill we limit the precision to 15 decimal places.

Taking the a_i thus obtained as first approximations, we can proceed to second approximations by recommencing from the beginning, and re-determining a_1 and a_{-1}, but this time taking account of all terms in which (when $j = 1$) a_1 occurs, or in which (when $j = -1$) a_{-1} occurs. The calculation makes use of the first-approximation values of the a_i for $i > 1$ and for $i < -1$. The new values of a_1 and a_{-1} can then be used in re-determining a_2 and a_{-2}. Step by step, the successive coefficients can be revised. Is the process convergent?

The apparent convergence is remarkably rapid. The corrections from the second approximations are at least four orders of magnitude smaller in absolute value than the first-approximation values. Hill goes on to calculate corrections from the third approximations, and these are at least four orders of magnitude smaller still. Thus in the case of a_1/a_0 Hill finds

First-approximation value:	+0.00151 58491 71593
Second-approximation correction:	−0.00000 01416 98831
Third-approximation correction:	+0.00000 00000 06801
Resulting value adopted:	+0.00151 57074 79563.

By the calculation just described, Hill shows how the ratios a_j/a_0 and a_{-j}/a_0 can be obtained as series in terms of the parameter $\mathbf{m}$ (see II.13 above). The convergence of these series can be improved, Hill remarks, if in them the parameter $\mathbf{m}$ is replaced by a function of $\mathbf{m}$ of the form $\mathbf{m}/(1+\alpha\mathbf{m})$, the constant α being appropriately chosen. "It is easily found," Hill writes, "that α should be $-1/3$." Hill's reasoning is apparently the following. From (II.13a–c) it is seen that the expressions [j,i] [j], and (j), and hence the expression (II.13), all contain $2(4j^2-1)-4\mathbf{m}+\mathbf{m}^2$ as denominator; in the special case where $j=\pm 1$, this becomes $6-4\mathbf{m}+\mathbf{m}^2$. The latter trinomial continues to occur in the formulas (II.13) for j other than ± 1, and its expansion as a power series can be made more convergent if $\mathbf{m}$ is replaced by a function of $\mathbf{m}$ such that the denominator no longer contains a term linear in $\mathbf{m}$. Put $m_1=\mathbf{m}/(1+\alpha\mathbf{m})$ so that $\mathbf{m}=m_1/(1-\alpha m_1)$; substitute this value of $\mathbf{m}$ in the expression $6-4\mathbf{m}+\mathbf{m}^2$, and clear of fractions. The coefficient of m_1 in the resulting expression is $(-12\alpha-4)$, which becomes zero when $\alpha=-1/3$. The denominator becomes $6+m_1^2/3$, whose expansion as a power series in m_1 converges with much greater rapidity than the corresponding power series in $\mathbf{m}$. When numerical values are to be computed from these series, it is found that the number of terms that have to be calculated for a given level of precision is less for the series in m_1 than for the series in $\mathbf{m}$. For the present we continue our account of Hill's calculations as performed with the parameter $\mathbf{m}$; although in fact he checked most of his calculations by performing them in terms of both $\mathbf{m}$ and m_1.

What about the value of a_0, in terms of which the other a_i have been found? To determine it, an equation is needed in which κ is still present. Hill chooses (H.7a″). Into all terms of it, except the term having κ as a factor, he substitutes the expressions for u and s as a function of ζ. The result is

$$\kappa u^{-1/2}s^{-3/2}=\sum_i\left\{\left[(2i+1+\mathbf{m})^2+\frac{1}{2}\mathbf{m}^2\right]a_i+\frac{3}{2}\mathbf{m}^2a_{-i-1}\right\}\zeta^{2i+1}.$$

The right-hand side, if we consider only the terms for which $i=0$, reduces to

$$a_0\left[1+2\mathbf{m}+\frac{3}{2}\mathbf{m}^2+\frac{3}{2}\mathbf{m}^2(a_{-1}/a_0)\right]\zeta.$$

Hill symbolizes this by $a_0\zeta H$, and notes that, since (a_{-1}/a_0) is known to fifteen decimal places, H can be calculated to a like precision. The left-hand side contains the product of two infinite series, each raised to a negative fractional power. It can be written as

$$\kappa u^{-1/2}s^{-3/2}=\kappa a_0^{-2}\zeta[1+(a_1/a_0)\zeta^2+\cdots]^{-1/2}[1+(a_1/a_0)\zeta^{-2}+\cdots]^{-3/2}.$$

The square brackets on the right can then be developed as Taylor series. From the product of these brackets thus developed, let the sum of the terms from which ζ is absent be designated J. Then, equating right- and left-hand sides,

$$\kappa a_0^{-2}\zeta J=a_0\zeta H,\quad \text{or}\quad a_0^3=\kappa(J/H).$$

But

$$\kappa = \frac{\mu}{(n-n')^2} = \frac{\mu}{n^2}(1+\mathrm{m})^2;$$

therefore a_0 has the value

$$\left[\frac{\mu}{n^2}\right]^{1/3}\left[\frac{J(1+\mathrm{m})^2}{H}\right]^{1/3} = \left[\frac{\mu}{n^2}\right]^{1/3}\left[1-\frac{1}{6}\mathrm{m}^2+\frac{1}{3}\mathrm{m}^3+\frac{407}{2304}\mathrm{m}^4-\frac{67}{288}\mathrm{m}^5-\cdots\right]$$

$$= 0.99909, 31419, 62\left[\frac{\mu}{n^2}\right]^{1/3},$$

where Hill has evaluated the expansion to the ninth power of **m**.

The quantity $[\mu/n^2]^{1/3}$ is generally identified in lunar theory, Hill says, with a, the mean distance of the Moon from the Earth; and this is commonly determined empirically.

To determine the constant C of the Jacobian integral, Hill returns to (II.9), which by substitution of the expressions for u and s gives:

$$\sum_i \left\{(2i+1)(2i-2j+1)+4j^2+4(2i-j+1)\mathbf{m}+\frac{9}{2}\mathbf{m}^2\right\} a_i a_{i-j}$$

$$+\frac{9}{4}\mathbf{m}^2\sum_i\{a_i a_{-i+j-1}+a_i a_{-i-j-1}\} = 0 \text{ or } C.$$

The expression equals C if and only if $j=0$, in which case we have

$$C = \sum_i \left\{(2i+2\mathbf{m}+1)^2+\frac{1}{2}\mathbf{m}^2\right\} a_{i^2} + \frac{9}{2}\mathbf{m}^2\sum_i a_i a_{-i-1}.$$

To the eighth order of small quantities, this gives

$$C = a_{0^2}\left[1+4\mathrm{m}+\frac{9}{2}\mathrm{m}^2-\frac{1147}{2^7}\mathrm{m}^4-\frac{1399}{2^5\cdot 3}\mathrm{m}^5-\frac{2047}{2^8}\mathrm{m}^6+\frac{3737}{2^4\cdot 3^3}\mathrm{m}^7\right].$$

This value holds when the differential equations are expressed in terms of u, s, and ζ. When they are expressed in terms of x, y, and t, the preceding value must be multiplied by $\frac{1}{2}V^2 = \frac{1}{2}\frac{n^2}{(1+\mathrm{m})^2}$. Then C, with a_0 replaced by its value, becomes

$$C = \frac{1}{2}(\mu n)^{2/3}\left[1+2\mathrm{m}-\frac{5}{6}\mathrm{m}^2-\mathrm{m}^3-\frac{1319}{288}\mathrm{m}^4-\frac{67}{144}\mathrm{m}^5-\frac{2879}{1296}\mathrm{m}^6-\frac{1321}{1296}\mathrm{m}^7\right].$$

The function κ/r^3 plays an important role in the lunar theory, and Hill therefore takes the trouble to derive an expression for it using the process of "special values" (harmonic analysis – a process promoted byGauss and put to frequent use by

Hansen[53]). The constant κ, we recall, denotes the fraction μ/ν^2, where $\nu = n - n'$. Hill computes the values of κ/r^3 at intervals of 15° over a quadrant:

τ	κ/r^3
0°	1.19699 57017 23421
15°	1.19348 68051 03032
30°	1.18399 66676 76716
45°	1.17125 64904 33157
60°	1.15876 77987 29687
75°	1.149978 07679 95764
90°	1.14652 34925 50570

Then the "special values" procedure yields a Fourier series:

$$\begin{aligned}\frac{\kappa}{r^3} = {} & 1.17150\,80211\,79225 \\ & + 0.02523\,36924\,97860 \cos 2\tau \\ & + 0.00025\,15533\,50012 \cos 4\tau \\ & + 0.00000\,24118\,79799 \cos 6\tau \\ & + 0.00000\,00226\,05851 \cos 8\tau \\ & + 0.00000\,00002\,08750 \cos 10\tau \\ & + 0.00000\,00000\,01908 \cos 12\tau \\ & + 0.00000\,00000\,00017 \cos 14\tau.\end{aligned}$$

53 For a more recent description of the process, see E. Whittaker and G. Robinson, *The Calculus of Observations* (Fourth Edition, New York, NY: Dover Publications 1967), Chapter X.

5

Early Assessments of Hill's Lunar Theory

John C. Adams, writing in the *Monthly Notices of the Royal Astronomical Society* for November 1877, was the first to give public recognition of the importance of Hill's 1877 paper[54]:

> A very able paper has recently been published by Mr. G. W. Hill, assistant in the office of the American Nautical Almanac, on the part of the motion of the lunar perigee which is a function of the mean motions of the Sun and Moon.
>
> Assuming that the values of the Moon's coordinates in the case of no eccentricities are already known, the author finds the differential equations which determine the inequalities which involve the first power of the eccentricity of the Moon's orbit, and, by a most ingenious and skilful process, he makes the solution of those differential equations depend on the solution of a single linear differential equation of the second order, which is of a very simple form. This equation is equivalent to an infinite number of algebraical linear equations, and the author, by a most elegant method, shows how to develop the infinite determinant corresponding to these equations in a series of powers and products of the small quantities forming their coefficients. The value of the multiplier of each of such powers and products as are required is obtained in a finite form. By equating this determinant to zero, an equation is obtained which gives directly, and without the need of successive approximations, the motion of the Moon from the perigee during half of a synodic month. ... The ratio of the motion of the perigee to that of the Moon thus obtained is true to 12 or 13 significant figures. The author compares his numerical result with that deduced from Delaunay's analytical formula, which gives the ratio just mentioned developed in a series of powers of m, the ratio of the mean motions of the Sun and Moon. The numerical coefficients of the successive terms of this series increase so rapidly that the convergence of the series is slow, so that the terms calculated do not suffice to give the

[54] J. C. Adams, *MNRAS, 38* (1877), 43.

C. Wilson, *The Hill–Brown Theory of the Moon's Motion*, Sources and Studies in the History of Mathematics and Physical Sciences, DOI 10.1007/978-1-4419-5937-9_5,

> first four significant figures of the result correctly, although, by induction, a rough approximation may be made in the remaining terms of the series.

Adams goes on to say that his own researches in the Lunar Theory, pursued intermittently since the 1860s, have followed a somewhat parallel course.

> I have long been convinced that the most advantageous way of treating the Lunar Theory is, first, to determine with all desirable accuracy the inequalities which are independent of the eccentricities e and e', and the inclination 2 arcsin γ, and then, in succession, to find the inequalities which are of one dimension, two dimensions, and so on, with respect to those quantities.
>
> Thus the coefficient of any inequality in the Moon's coordinates would be represented by a series arranged in powers and products of e, e', and γ, and each term in this series would involve a numerical coefficient which is a function of m and which may be calculated for any given value of m without the necessity of developing it in powers of m ....
>
> The differential equations which would require solution in these successive operations [after the determination of the inequalities solely dependent on m] would be all linear and of the same form.

The general idea that Adams arrived at for developing the lunar theory was substantially the same as Hill's. In the article from which we are quoting he goes on present his results for the motion of the Moon's node. Here he arrives at a differential equation of the same form as Hill's equation for the motion of the Moon's perigee, and encounters an infinite determinant of the same form. His solution of the latter, he acknowledges, is less skillful and more laborious than Hill's, but – after correcting a small error he has committed in a 12th-order term – he finds the two solutions to be in entire agreement.

On Hill's being awarded the Gold Medal of the Royal Astronomical Society in 1887, the new president of the society, J.W.L. Glaisher, devoted his inaugural address to setting forth the grounds of the award.[55]

> The investigations of Mr. Hill's which the Council have had principally in view are contained in the memoir "On the Part of the Motion of the Lunar Perigee, which is a function of the Mean Motions of the Sun and Moon".... The merits of Mr. Hill's treatment of this question are such, that, even if this memoir stood by itself as his sole contribution to astronomy, the Council would have felt themselves justified in recognizing its value by the highest mark of appreciation which it is in their power to confer. Mr. Hill's object in this memoir is to determine... an absolutely accurate value of that part of c which depends upon m alone.

Hill is the first, Glaisher points out, to obtain c with a numerical precision equal to that of the observational value, and his method is entirely novel.

[55] The address was published in the February issue of *MNRAS, 47* (1887), 203–220.

> It will be observed that the problem of the determination of this quantity is attacked entirely *de novo*, from first principles, by a peculiar analytical method devised for this especial purpose..... [The] object is attained with a degree of precision that sets the problem at rest for ever.
>
> The mathematical process is very peculiar. Not only is it quite different from any of the methods with which the lunar theory is associated, but it even displays novelty from the point of view of the pure mathematician. I am not aware that actual use has ever been previously made of an infinite determinant in any of the applications of mathematics, or that the development of such a determinant (by proceeding outwards from its central constituent, as it were) has ever been the subject of mathematical investigation. One cannot admire too highly the courage and skill with which Mr. Hill has dealt with the new mathematical questions to which his methods have led him.

Glaisher also gives a resumé of Hill's 1878 paper, and emphasizes its departure from established custom in the lunar theory by the introduction of rectangular coordinates and of the imaginary quantity $i[=\surd(-1)]$. He compares Hill's procedures with those that Adams has adopted in dealing with the motion of the lunar node, and in glowing terms expresses his hopes for the effect Hill's and Adam's innovations will have upon the future development of the lunar theory:

> In recent years it has come to be generally believed that a worker had but little chance of performing useful service in the lunar theory unless he was prepared to make it the study of his life. The belief has also been prevalent that the mathematical portion of the treatment of the subject has been worked out, and that there was no scope for the display of mathematical skill or the employment of modern mathematical methods. Until some great discovery should change the face of the whole subject, it has seemed likely that patience and diligence in traversing with greater care the old lines, and extending still further developments already carried to a wonderful extent, would be all that was required to perfect the theory. Considering, on the other hand, the attractiveness of the new and rapidly progressing branches of pure mathematics, and of many recent applications of mathematics, and of many recent applications of mathematics to physical science, it is scarcely to be wondered at that so few of the younger generation of mathematicians should have included the lunar theory within their subjects of research. The papers of Mr. Hill's which I have described, and certain recent papers of Professor Adams's, have invested the lunar theory with a new mathematical interest, and have shown that in the treatment of the special problems included in the subject there is an ample opportunity not only for the application of existing mathematical methods, but even for the discovery of new ones. These papers show also that it is possible for the mathematician to confine himself to these special problems without attempting to cover the whole ground of the lunar theory. I hope that this is the dawn of a new day in the history of the lunar problem, and that, now that the whole territory has been mapped out by Plana and Delaunay,

> it will be found that the special investigations offer a tempting field to the mathematician. So far from the subject having been exhausted by the general methods which have been applied to it as a whole, I believe that the future will show that they have but cleared the ground and disclosed to view the objects to which mathematical investigation may with the greatest advantage be directed.

Poincaré, too, praised Hill's innovations in glowing terms. From his perspective, their importance was not limited to the lunar theory but extended to all of celestial mechanics – to the larger projects of determining whether Newton's inverse-square law was sufficient to account for the motions of the celestial bodies, and whether the solar system was stable. In a memoir submitted to the Société mathématique de France in 1886 he undertook to demonstrate the convergence of Hill's infinite determinant analytically.[56] Hill's bold use of the ordinary rules applicable to finite determinants, wrote Poincaré, was justified by its success – its near agreement with observation. But the analytical substantiation puts "la belle méthode de M. Hill" beyond any possible objection.

In the introduction to the first volume of his *Les méthodes nouvelles de la Mécanique Céleste* (1892), Poincaré refers to Hill's two papers on the lunar theory of 1877 and 1878, and says

> Dans cette oeuvre, malheureusement inachevée, il est permis d'apercevoir le germe de la plupart des progress que la Science a faits depuis.

Here "la Science" undoubtedly means Mécanique Céleste, and the progress Poincaré is referring to is that which he himself has made, in using periodic orbits as "launching pads" for the investigation of the neighboring phase space, as Hill had done in introducing eccentricity into the variation orbit. In his introduction to Vol. I of *The Collected Mathematical Works of George William Hill* (1905), Poincaré asserts that, among Hill's many papers on celestial mechanics, those on the lunar theory constitute his *chef d'oeuvre*:

> . . . c'est lá qu'il a été non seulement un artiste habile, un chercheur curieux, mais un inventeur original et profond.

[56] H. Poincaré, "Sur les determinants d'ordre infini," *Bulletin de la Société mathématique de France*, *XIV*, 77–90.

Part II

Brown Completes the Theory (1892–1908), and Constructs Tables (1908–1919)

6

E. W. Brown, Celestial Mechanician

Ernest Willliam Brown (1866–1938), born into a farming family in Hull, England, attended Christ's College, Cambridge, beginning in 1884, and received the A.B. in 1887. He had been a delicate youth, but during his Cambridge years took to rowing on the Cam and climbing mountains in Switzerland. Like other bright young Cambridge men of his time, he entered upon the strenuous training for the Mathematical Tripos (named, it is said, from the three-legged stool on which, in the earliest times, the competitors sat for the examinations). The value of this training has been variously judged.[57] Stress on applied mathematics, learned by intensive practice in problem-solving, was distinctive of it. Through the last two-thirds of the 19th century, most of Britain's foremost mathematical physicists were top-ranking "wranglers" – those achieving high scores in the examinations. Brown graduated sixth wrangler. He became an assiduous calculator and an able practitioner of applied mathematics, and as the years went on, a knowledgeable inquirer into foundational questions in what became his specialty, celestial mechanics.

Brown in later years had no doubts about the influence he was most indebted to in his career. During 4 years of graduate study at Cambridge (1887–1891), he was the protégé of George Howard Darwin, son of Charles Darwin. This younger Darwin had been second wrangler in 1868, and he became an applied mathematician who achieved recognition for his extended and innovative inquiries into periodic orbits in the three-body problem, and into the tides within the Earth and its oceans. Darwin obtained his results by paper-and-pencil calculations, daunting in their length and intricacy. It was he who proposed to Brown the study of Hill's lunar theory, and guided him in his early study of it. It was Darwin, too, who in 1889 obtained Hill's consent to Brown's undertaking the development of the new lunar theory beyond the point to which Hill had carried it. Darwin proof-read the MS of Brown's *Introductory*

[57] For a favorable view of Cambridge mathematical culture, see A. Warwick, *Masters of Theory*: *Cambridge and the Rise of Mathematical Physics* (The University of Chicago Press: Chicago and London 2003). For a critical view, see D. Lindley, *Degrees Kelvin*. A *Tale of Genius, Invention, and Tragedy* (Joseph Henry Press: Washington, DC 2004) 32 ff.

C. Wilson, *The Hill–Brown Theory of the Moon's Motion*, Sources and Studies in the History of Mathematics and Physical Sciences, DOI 10.1007/978-1-4419-5937-9_6,

Treatise on the Lunar Theory (1896), and critiqued others of his papers. Darwin himself was hard-working, patient, and modest. For over a decade after leaving Cambridge, Brown corresponded with Darwin on a fairly regular basis, and visited with him almost every summer, confiding in him and seeking his counsel. Darwin, we can be pretty sure, was the model that Brown set out to emulate.

In 1889 Brown was made a fellow of Christ's College (Darwin's college); he would retain that position through 1895. In January 1891 he was awarded the A.M. degree. In the same year he received an appointment as instructor in mathematics at Haverford College in Pennsylvania. He took up residence there in the autumn of 1891.

During his first winter in the United States, Brown paid a visit to Hill and Newcomb at the Nautical Almanac Office in Washington, and discussed the lunar project with them.[58] It was at about this time, we believe, that Hill wrote the three-page memorandum reproduced in the Appendix to this study. It refers to Brown's work on the lunar theory, and urges that the Nautical Almanac Office support it by supplying professional computers – a suggestion that was never acted upon. Later Hill checked Brown's results in his first major article on the lunar theory, and Newcomb offered suggestions about the second. Brown came to regard the relative merits of the two men, in their work on celestial mechanics, as "much the same,"[59] but he found Newcomb easier to communicate with.

To Brown, the change in climate – heat and humidity in the summers, blizzards and frigid cold in the winters – was trying. In March of his second academic year at Haverford, he wrote Darwin:

> The weather has almost been too much for me this winter. I have scarcely been well the whole time since I landed in September. This has made me think seriously of leaving and returning to Cambridge, though I have been permanently appointed here, that is, for three years. Socially it is very pleasant indeed here. There is always a good deal going on in the way of receptions, dinners, etc.[60]

On May 19 he reported yet another bout with illness:

> Many thanks for yours of April 9th. I am sorry to have not been able to answer it before. A day or two after receiving it I was taken ill and had three weeks in bed and am only just able to get up to college now. I have had to give up all idea of getting anything done besides my small amount of college lecturing before I sail in four weeks.[61]

Throughout the 1890s Brown spent his summers in England or on the Continent, delaying his return to Haverford till the latest moment. Diffident about his future as

[58] F. Schlesinger and D. Brouwer, "Ernest William Brown" *National Academy Biographical Memoirs, XXI*, 257.

[59] Brown to Darwin, 21 March 1896, Cambridge University Library (hereinafter CUL), MS DAR 251:479.

[60] Brown to Darwin, 10 March 1893, CUL, MS DAR 251:467.

[61] Brown to Darwin, 19 May 1893, CUL, MS DAR 251:468.

a mathematician in America, he kept on the lookout for a job in England or possibly elsewhere. In the job search Darwin did stalwart service, reconnoitering and writing references.

Meanwhile, Brown persevered in his work on the lunar theory. In November 1891 he submitted for publication a first major paper,[62] dealing with the parallactic inequalities, and at nearly the same time, an extended note for publication in the *Monthly Notices* of the Royal Astronomical Society.[63] He had apparently begun his work on the parallactic inequalities earlier in England, under Darwin's supervision.

Brown next turned to the inequalities dependent on the lunar orbit's eccentricity. Eccentricity (displacement of the dynamical center of the orbit from its geometrical center) implied that the orbit would be partially characterized by elliptical elements. In December 1892 and June 1893 he submitted for publication the two parts of a major paper dealing with the "elliptical inequalities."[64] This time he worked without Darwin's supervisory help, carrying out extensive calculations which he characterized as nightmarish. The results were again substantial.

During the academic year 1894–1895 he took a leave of absence from Haverford, partly, as we learn from a subsequent letter to Darwin, to help the college save money.[65] In December 1894 he completed a third major paper, setting forth a plan for systematic development of the whole lunar theory.[66] At the close of this paper he gave his location and the date as "Christ's College, Cambridge, December 24th, 1894." It seems likely that he was in residence at Christ's College, as his fellowship allowed, through all or most of the 1894–1895 academic year. All three of the major papers we have cited were published, like Hill's "Researches in the Lunar Theory" earlier, in the *American Journal of Mathematics*. They supply essential background for understanding the systematic development of the lunar theory which Brown was to embark upon in 1895.

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[62] E. W. Brown, "On the Part of the Parallactic Inequalities in the Moon's Motion which is a Function of the Mean motions of the Sun and Moon," *American Journal of Mathematics, 14* (1892), 141–160.

[63] E. W. Brown, "On the Determination of a certain Class of Inequalities in the Moon's Motion," *Monthly Notices of the Royal Astronomical Society* (hereinafter abbreviated as *MNRAS*), *52* (Dec. 1891), 71–80.

[64] E. W. Brown, "The Elliptic Inequalities in the Lunar Theory," *American Journal of Mathematics, 15* (1893), 244–263, 321–338. As we have seen in Part I, the Moon's orbit cannot be approximated as closely by an ellipse as by Hill's "variation curve."

[65] Brown to Darwin, 21 March 1896, CUL, MS DAR 251:479.

[66] E. W. Brown, "Investigations in the Lunar Theory," *American Journal of Mathematics, 17* (1895), 318–358.

7

First Papers and a Book

The paper on the parallactic inequalities adapts Hill's method so as to include the class of inequalities depending on the ratio of the lunar and solar mean distances (a/a'). Hill, in accounting for the inequalities depending solely on the ratio of the mean motions of the Sun and Moon (n'/n), had left these "parallactic" inequalities out of account, thus implicitly assigning a zero parallax to the Sun. Brown in his introductory paragraph called attention to the special practical import of the *principal* part of the Parallactic Inequality – the part depending solely on the constant $\mathbf{m} = n'/(n - n')$ and the first power of the ratio a/a'. Astronomers employed this part in investigating the Sun's horizontal parallax – a special reason for desiring an accurate value of it. Delaunay's calculation of it was of doubtful accuracy.

Brown in his note "On the Determination of a certain Class of Inequalities in the Moon's Motion" distinguished the two principal methods by which the Lunar Theory had been pursued, one of them *general*, the other *specific* (we earlier called these the *literal* and *numerical* methods). In the first,

> ... we have the theory, worked out to a certain degree of accuracy, immediately applicable to any single Moon in our solar system, and therefore arranged in such a way that any small change which improved data may involve in the values of the constants can be made easily without requiring us to go over the whole of the work again.

A drawback of this method is that

> the number of terms which have been found necessary to secure a degree of accuracy commensurate with that of observation is very large, and it becomes a task of great labor to obtain them with any degree of certainty.

In the second method, the constants are assigned numerical values from the start. The difficulty of slow convergence in the series expansions expressing the coefficients is thereby entirely avoided, and a great increase in accuracy naturally results. But this method has its own drawbacks: if any numerical mistake has been made, it is not easily traceable; and if it is desired to change one of the originally assigned values of the constants, the entire calculation must be begun over again.

C. Wilson, *The Hill–Brown Theory of the Moon's Motion*, Sources and Studies in the History of Mathematics and Physical Sciences, DOI 10.1007/978-1-4419-5937-9_7,

Hill by his method obtained numerical and algebraic results of equal accuracy for the inequalities he dealt with, those dependent solely on the constant **m**. In the end it was Hill's numerical results that Brown employed; by introducing them he was able to eliminate most of the instances of slow convergence encountered earlier by Plana and Delaunay in their literal elaborations of the theory. The other constants of the theory – the eccentricities of the lunar and solar orbits, the inclination of the lunar orbit, and the relative parallax of the two orbits – could be left in literal form, as was advisable in any case, since their numerical values were less securely established than that of the constant **m**. Constants left in literal form in the development of the theory could eventually be determined numerically in a least-squares fitting of the theory to observations.

The chief aim of Brown's note in the *Monthly Notices* was to compare his own results for the parallactic inequalities with those of Delaunay. In later stages of his work, Brown would provide similar comparisons for other classes of inequalities, pitting the two theories, Delaunay's and the Hill–Brown theory, against one another. Delaunay's theory had been left incomplete at the time of Delaunay's accidental death by drowning in 1872, and for some time the further development of it had languished, but by the 1890s the Bureau des Longitudes in Paris had taken it up as an ongoing project. In 1910, Radau acknowledged that the Hill–Brown theory had proven superior, but Delaunay's theory was used as the basis of the lunar tables in the *Connaissance des Temps* till 1923.

In carrying out the comparisons, Brown made use of a discovery of Hill's: Delaunay's series, if expanded not in terms of the constant $m = n'/n$, as Delaunay had done, but in terms of the constant $\mathbf{m} = n'/(n - n')$, were rendered more convergent. Still better for this purpose was the constant $\mu = \mathbf{m}/(1 - \mathbf{m}/3) = m/(1 - 4m/3)$. Thus for the coefficient of the Variation dependent on m Delaunay obtained

$$\begin{aligned}
&\frac{11}{8}m^2 + \frac{59}{12}m^3 + \frac{893}{72}m^4 + \frac{2855}{108}m^5 + \frac{8304449}{165888}m^6 \\
&1586''.8883 + 424''.4474 + 80''.0906 + 12''.7689 + 1''.8087 \\
&\quad + \frac{102859909}{1244160}m^7 + \frac{7596606727}{74649600}m^8 - \frac{8051418262}{1119744000}m^9 \cdots \\
&\quad + 0''.2234 + 0''.0206 - 0.''0001 \qquad (\alpha)
\end{aligned}$$

(The arcseconds corresponding to each term are given beneath the term.) Following a suggestion of Hill's, Brown substituted $m = \mu/(1 + 4\mu/3)$ into the foregoing series, then expanded in powers of μ to obtain

$$\begin{aligned}
&\frac{11}{8}\mu^2 + \frac{5}{4}\mu^3 + \frac{5}{72}\mu^4 - \frac{11}{36}\mu^5 - \frac{82111}{165888}\mu^6 \\
&1957''.9686 + 147''.8944 + 0''.6827 - 0''.2496 - 0''.0336
\end{aligned}$$

$$- \frac{350399}{138240}\mu^7 - \frac{233559113}{74649600}\mu^8 - \frac{10961275281}{1119744000}\mu^9 \cdots$$

$$- 0''.0143 - 0''.0015 - 0.''0004 \qquad (\beta)$$

The convergence in (β) is clearly more rapid than that in (α). Comparing the values of the coefficient determined by (α) and (β) with Hill's value, Brown found:

From (α)	...	...	...	2106″.2478	
From (β)	...	...	...	2106″.2463	
Hill's value	...	...	...	2106″.2463	(γ)

The agreement of Hill's value with the value obtained from (β) showed that Delaunay's series, revised as proposed by Hill, might provide a useful check on derivations in the Hill–Brown theory.

The change in sign in (α), Brown pointed out, occurs between the terms in m^8 and m^9, whereas the change in sign in (β) occurs between the terms in μ^4 and μ^5. Brown commented:

> I have calculated similar expressions in the coefficients of other inequalities, and generally it appears that when we, by any such substitution as that made above, apparently improve the convergency of the series, the change of sign is brought nearer to the beginning of the series. An ideal to aim at would seem to be, firstly, that the numerical multipliers be made as small as possible; secondly, that there be no sudden increase in them in the later part of the series; and thirdly, that these two conditions should involve that there be no long run of powers with the same sign attached to them.... It should be stated that as the series are calculated up to some definite power only, and as the law of progression of the series is not able to be expressed by an algebraical formula, owing to the complicated forms from which they arise, a substitution like that made above *must not be arbitrary*, but must be indicated by theory. An arbitrary substitution... may make the unknown part very slowly convergent, and thus introduce unknown errors into the numerical values of the coefficients.

Brown's derivation of the parallactic inequalities took its start from a point early in Hill's paper of 1878. As the disturbing function Hill had used

$$\Omega = \frac{\mu}{\sqrt{x^2 + y^2 + z^2}} + \frac{n'^2 a'^3}{\sqrt{(r' - x)^2 + y^2 + z^2}} - \frac{n'^2 a'^3}{r'^2} x.$$

The last two terms on the right give, by their derivatives with respect to x, the difference between the force the Sun exerts on the Moon and the force it exerts on the Earth. Hill had expanded the first of these terms, using Taylor's rule for the expansion of a function of three independent variables, and obtained

$$\Omega = \frac{\mu}{\sqrt{x^2+y^2+z^2}} + n'^2 \frac{a'^3}{r'^3}\left[x^2 - \frac{1}{2}(y^2+z^2)\right]$$
$$+ \frac{n'^2}{a'}\frac{a'^4}{r'^4}\left[x^3 - \frac{3}{2}x(y^2+z^2)\right]$$
$$+ \frac{n'^2}{a'^2}\frac{a'^5}{r'^5}\left[x^4 - 3x^2(y^2+z^2) + \frac{3}{8}(y^2+z^2)^2\right]$$
$$+ \frac{n'^2}{a'^3}\frac{a'^6}{r'^6}\left[x^5 - 5x^3(y^2+z^2) + \frac{15}{8}x(y^2+z^2)^2\right]$$
$$+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

He had then simplified the expression by deleting the variable z, setting $r' = a'$, and restricting the further development to the first line of the expansion. He was thus neglecting the terms in which the factors $1/a'$, $(1/a')^2$, and $(1/a')^3$ occur – the terms implicitly involving the ratio of the mean Moon-Earth distance to the mean Earth-Sun distance. In effect, he was assuming the Sun to be of infinite mass and at an infinite distance.

Brown began his derivation by re-introducing the three neglected terms, whose sum he labeled Ω_1 (this symbol will later get an extended signification):

$$\Omega_1 = \frac{n'^2}{a'}\left[x^3 - \frac{3}{2}xy^2\right] + \frac{n'^2}{a'^2}\left[x^4 - 3x^2y^2 + \frac{3}{8}y^4\right]$$
$$+ \frac{n'^2}{a'^3}\left[x^5 - 5x^3y^2 + \frac{15}{8}xy^4\right] + \cdots$$

The equations of motion as given by Hill (II.7 in Part I) then had to be modified by the addition of the appropriate partial derivatives of Ω_1 to their right-hand members:

$$\frac{d^2x}{dt^2} - 2n'\frac{dy}{dt} + \left(\frac{\mu}{r^3} - 3n'^2\right)x = \frac{\partial\Omega_1}{\partial x},$$
$$\frac{d^2y}{dt^2} + 2n'\frac{dx}{dt} + \frac{\mu}{r^3}y = \frac{\partial\Omega_1}{\partial y}. \tag{B.1}$$

Whereas Hill's (II.7) admitted of a solution symmetrical with respect to both of the moving axes, the addition of the partial derivatives spoiled the symmetry. If, in the standard test for symmetry, we change the signs of x and t in (B.1) while leaving the sign of y unaltered, the equations are no longer the same; and they are also altered if we change the signs of y and t while leaving the sign of x unchanged. But the terms introduced by the partial derivatives are small compared with the terms originally present. For a range of initial conditions, the resulting orbit remains reentrant or closed.

Multiplying (B.1) by dx/dt, dy/dt respectively, adding them together and integrating the result, Brown obtained the new Jacobian integral:

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 - \frac{2\mu}{r} - 3n'^2x^2 = 2\Omega_1 - 2C.$$

Brown's next steps again paralleled those of Hill in his 1878 paper. He introduced the complex variables u, s and the operator $D = \zeta \frac{d}{d\zeta}$, thus arriving at transformed expressions for the equations of motion and the Jacobian integral:

$$\left[D^2 + 2\mathrm{m}D - \frac{\kappa}{(us)^{3/2}}\right] u = -\frac{3}{2}\mathrm{m}^2(u+s) - \mathrm{m}^2 \cdot \frac{2}{n'^2} \cdot \frac{\partial \Omega_1}{\partial s},$$
$$\left[D^2 - 2\mathrm{m}D - \frac{\kappa}{(us)^{3/2}}\right] s = -\frac{3}{2}\mathrm{m}^2(u+s) - \mathrm{m}^2 \cdot \frac{2}{n'^2} \cdot \frac{\partial \Omega_1}{\partial u},$$
$$Du \cdot Ds + \frac{2\kappa}{(us)^{1/2}} = \frac{3}{4}\mathrm{m}^2(u+s)^2 - \mathrm{m}^2 \cdot \frac{2}{n'^2} \cdot \Omega_1 + C'. \tag{B.2}$$

Next, he eliminated from these equations the non-linear terms in which the product (us) is raised to a negative fractional power, just as Hill had done. He multiplied the first equation by s and the second by u, and added the sum of the products to the third equation. To obtain a second equation, he subtracted the first of the same two products from the second. The two resulting equations were

$$D^2(us) - DuDs - 2\mathrm{m}(uDs - sDu) + \frac{9}{4}\mathrm{m}^2(u+s)^2$$
$$= -\mathrm{m}^2 \cdot \frac{2}{n'^2}\left(u\frac{\partial \Omega_1}{\partial u} + s\frac{\partial \Omega_1}{\partial s} + \Omega_1\right) + C',$$
$$D(uDs - sDu - 2\mathrm{m}us) + \frac{3}{2}\mathrm{m}^2(u^2 - s^2)$$
$$= -\mathrm{m}^2 \cdot \frac{2}{n'^2}\left(u\frac{\partial \Omega_1}{\partial u} - s\frac{\partial \Omega_1}{\partial s}\right). \tag{B.3}$$

If from these equations we delete the terms involving Ω_1 and its derivatives, we obtain the corresponding homogeneous equations given by Hill.

The next stage, for both Hill and Brown, was to introduce a particular integral of the equations. Hill's particular integral, we recall, had the form

$$x = \sum_i \mathrm{a}_i \cos[(2i+1)\nu(t-t_0)], \quad y = \sum_i \mathrm{a}_i \sin[(2i+1)\nu(t-t_0)].$$

Here the index i extended to all positive and negative integers and to zero. The factor $(2i+1)$ in the sinusoidal arguments insured that the orbit would cross the x- and y-axes at right angles. For Brown, such symmetry was no longer possible: the force that flattened the moon's orbit in the direction of the line of syzygies was stronger on the side of the Earth closer to the Sun than on the far side. The resulting orbital asymmetry exhibited itself in new parallactic terms.

Brown proposed the particular integral

$$x = \sum_i a_{i-1} \cos i\nu(t - t_0), \quad y = \sum_i a_i \sin i\nu(t - t_0),$$

where the summation extends to all negative and positive integers and zero. He has here substituted an italic "a" for Hill's roman "a." In his next step, he followed Hill in replacing x, y by the complex variables u, s:

$$u = \sum_i a_i \zeta^{i+1}, \quad s = \sum_i a_{-i} \zeta^{i-1}.$$

Introducing these expressions into the two equations of (B.3), and equating (except when $i = 0$) the coefficients of ζ^i to zero, he obtained

$$\begin{aligned}
&\sum_j \left[i^2 - (j+1)(i-j-1) - 2\mathrm{m}(i-2j-2) + \frac{9}{2}\mathrm{m}^2 \right] a_j a_{j-i} \\
&\quad + \frac{9}{4}\mathrm{m}^2 \sum_j [a_{j-1}a_{i-j-1} + a_{-j-1}a_{-i+j-1}] = -\mathrm{m}^2 L_i, \\
&\sum_j [i(i-2j-2) - 2\mathrm{m}i] a_j a_{j-i} \\
&\quad + \frac{3}{2}\mathrm{m}^2 \sum_j [a_{j-1}a_{i-j-1} - a_{-j-1}a_{-i+j-1}] = -\mathrm{m}^2 M_i,
\end{aligned} \tag{B.4}$$

where $-\mathrm{m}^2 L_1$, $-\mathrm{m}^2 M_i$ are the coefficients of ζ^i on the right-hand sides of the equations. By a short sequence of operations he simplified these equations and obtained a single equation corresponding to Hill's result (equation II.13 in Part I):

$$\begin{aligned}
&\sum_j \{[i, j] a_j a_{-i+j} + [i] a_{j-1} a_{i-j-1} + (i) a_{-j-1} a_{-i+j-1}\} \\
&\quad = -\frac{1}{9}[2L_i\{[i] + (i)\} + 3M_i\{[i] - (i)\}],
\end{aligned} \tag{B.5}$$

where

$$\begin{aligned}
[i, j] &= -\frac{j}{i} \cdot \frac{(i-2)j + i^2 + 2i - 2 - 2(j-i+2)\mathrm{m} + \mathrm{m}^2}{2(i^2-1) - 4\mathrm{m} + \mathrm{m}^2}, \\
[i] &= -\frac{3\mathrm{m}^2}{4i^2} \frac{i^2 - 4i - 2 - 2(i+4)\mathrm{m} - 9\mathrm{m}^2}{2(i^2-1) - 4\mathrm{m} + \mathrm{m}^2}, \\
(i) &= -\frac{3\mathrm{m}^2}{4i^2} \frac{5i^2 - 8i + 2 - 2(5i-4)\mathrm{m} + 9\mathrm{m}^2}{2(i^2-1) - 4\mathrm{m} + \mathrm{m}^2}.
\end{aligned}$$

If, Brown remarked, the terms on the right-hand side of (B.5) are set equal to zero, every coefficient a_k with odd index k vanishes, and with "one or two changes in notation," (B.5) becomes identical with (II.13). The letters i, j are replaced by $2j$, $2i$ respectively, so that Hill's index for a given coefficient is half Brown's index for the same coefficient: Hill's a_1, a_{-1}, become Brown's a_2 and a_{-2}.

Hill's (II.13), we recall, yielded by successive approximations the coefficients $\mathrm{a}_1, \mathrm{a}_{-1}, \mathrm{a}_2, \mathrm{a}_{-2}$, etc., in terms of a_0. For Brown, these coefficients (written, as just explained, with the original indices doubled, and with italicized "a" in place of "a") remained largely unchanged in value. In their series expansions in powers of **m**, the differences emerge only in terms of the 7th and higher orders. The coefficients with odd indices in Brown's theory, on the other hand, were altogether new.

Brown's formulas were more complicated than Hill's. To begin with, he had to find expressions for L_i, which is proportional to $\Omega_1 + u\frac{\partial\Omega_1}{\partial u} + s\frac{\partial\Omega_1}{\partial s}$, and for M_i, which is proportional to $u\frac{\partial\Omega_1}{\partial u} - s\frac{\partial\Omega_1}{\partial s}$. For the right-hand member of (B.5) he obtained

$$
\begin{aligned}
&-\frac{1}{9}[2L_i\{[i]+(i)\}+3M_i\{[i]-(i)\}] \\
&\quad= \frac{1}{a'}[A_i(u^3)_i + A_i'(s^3)_i + B_i(u^2s)_i + B_i'(us^2)_i] \\
&\quad\quad - \frac{1}{a'^2}[C_i(u^4)_i + C_i'(s^4)_i + D_i(u^3s)_i + D_i'(us^3)_i + E_i(u^2s^2)_i] \\
&\quad\quad - \frac{1}{a'^3}[F_i(u^5)_i + F_i'(s^5)_i + G_i(u^4s)_i + G_i'(us^4)_i + H_i(u^3s^2)_i + H_i'(u^2s^3)_i] \\
&\quad\quad - \ldots\ldots\ldots\ldots\ldots
\end{aligned}
$$

Here the letters A_i, A_I', etc. are functions of **m** of the order of $\mathbf{m}^2$ at least, and the symbols $(u^3)_i$, $(s^3)_i$, etc., denote the coefficients of ζ^i in u^3, s^3, etc. For instance, $(u^3)_1 = 3a_0^2 a_{-2}$, which is of the second order with respect to **m**, and since $1/a'$ and A_1 are each of at least the second order, the term $A_1(u^3)_1/a'$ is at least of the sixth order.

A special difficulty arose in obtaining the coefficients a_1 and a_{-1}, the first of the coefficients with odd indices. In this case the denominator in each term of (B.5) reduced to $(-4\mathbf{m} + \mathbf{m}^2)$, thus lowering the order of all terms by one power of **m**. Hence, in calculating these coefficients by (B.5) to a given degree of approximation, the expressions for them needed to be carried one order higher than in the case of the other coefficients. Moreover, when $i = \pm 1$ the process of approximation was especially slow and cumbrous.

To avoid these difficulties, Brown developed a special formula. In the equations (B.4), he set $i = 1$, obtaining

$$
\sum_j \left[\begin{array}{l} a_j a_{j-1}\{j^2 + j + 1 + 2\mathrm{m}(2j+1) + \frac{9}{2}\mathrm{m}^2\} \\ +\frac{9}{4}\mathrm{m}^2\{a_{j-1}a_{-j} + a_{-j-1}a_{j-2}\} \end{array} \right] = -\mathrm{m}^2 L_1,
$$

$$\sum_j \begin{bmatrix} a_j a_{j-1}\{-2j-1-2\text{m}\} \\ +\frac{3}{2}\text{m}^2\{a_{j-1}a_{-j} - a_{-j-1}a_{j-2}\} \end{bmatrix} = -\text{m}^2 M_1.$$

He multiplied the second of these equations by 2**m** and added it to the first to obtain a new equation. In this and in the second equation he substituted values of j such as to yield all values of a_1 and a_{-1} to the seventh order in **m**. The result was

$$\begin{aligned} &a_0a_1\left[3+\frac{1}{2}\text{m}^2+\frac{a_2}{a_0}\left(7+\frac{1}{2}\text{m}^2\right)+\frac{a_{-2}}{a_0}\left(\frac{9}{2}\text{m}^2+6\text{m}^3\right)+\frac{a_{-4}}{a_0}\left(\frac{9}{2}\text{m}^2-6\text{m}^3\right)\right] \\ &\quad + a_0a_3\left[\frac{a_2}{a_0}\left(13+\frac{1}{2}\text{m}^2\right)+\frac{a_4}{a_0}\left(21+\frac{1}{2}\text{m}^2\right)\right] \\ &\quad + a_0a_{-1}\left[1+5\text{m}^2+6\text{m}^3+\frac{a_{-2}}{a_0}(1+5\text{m}^2-6\text{m}^3)\right] \\ &\quad + a_0a_{-3}\left[\frac{9}{2}\text{m}^2-6\text{m}^3+\frac{a_2}{a_0}\left(\frac{9}{2}\text{m}^2+6\text{m}^3\right)+\frac{a_{-2}}{a_0}\left(3+\frac{1}{2}\text{m}^2\right)\right. \\ &\quad \left. +\frac{a_{-4}}{a_0}\left(7+\frac{1}{2}\text{m}^2\right)\right] \\ &\quad = -\text{m}^2(L_1+2\text{m}M_1), \\ &a_0a_1\left[3+2\text{m}+\frac{a_2}{a_0}(5+2\text{m})-3\text{m}^2\frac{a_{-2}}{a_0}+3\text{m}^2\frac{a_{-4}}{a_0}\right] \\ &\quad + a_0a_3\left[\frac{a_2}{a_0}(7+2\text{m})+\frac{a_4}{a_0}(9+2\text{m})\right] \\ &\quad + a_0a_{-1}\left[1+2\text{m}-3\text{m}^2+\frac{a_{-2}}{a_0}(-1+2\text{m}+3\text{m}^2)\right] \\ &\quad + a_0a_{-3}\left[3\text{m}^2-3\text{m}^2\frac{a_2}{a_0}+(-3+2\text{m})\frac{a_{-2}}{a_0}+(-5+2\text{m})\frac{a_{-4}}{a_0}\right] \\ &\quad = \text{m}^2M_1. \end{aligned} \tag{B.6}$$

In this expression the quotients $a_2/a_0, a_{-2}/a_0, a_4/a_0$, and a_{-4}/a_0, good to the seventh order in powers of **m**, had been given by Hill.[67] The right-hand sides of the two equations in (B.6) were given by

$$\begin{aligned} L_1 = \frac{a_0}{a'}&\left[\frac{15}{2}(a_0a_{-2}+a_0a_{-4}+a_{-2}^2)\right. \\ &\left.+\frac{3}{2}(a_0^2+2a_2^2+2a_{-2}^2+2a_2a_{-2}+2a_0a_{-2}+a_0a_2)\right], \end{aligned}$$

[67] See *The Collected Mathematical Works of George William Hill*, *I*, 317, where they are symbolized by $a_{\pm1}/a_0, a_{\pm2}/a_0$.

$$M_1 = \frac{a_0}{a'}\left[\frac{45}{8}(a_0a_{-2} - a_0a_{-4} - a_{-2}^2)\right.$$
$$\left. + \frac{3}{8}(a_0^2 + 2a_2^2 + 2a_{-2}^2 + 2a_2a_{-2} - 2a_0a_{-2} - a_0a_2)\right].$$

The formulas (B.6) for a_1 and a_{-1} contain the factors a_3 and a_{-3}. These when determined from (B.5) are found to depend on a_1 and a_{-1}. Proceeding by successive approximations, one could first determine a_1 and a_{-1} from (B.6) while neglecting the terms involving a_3 and a_{-3}; then use these values in (B.5) to determine a_3 and a_{-3} approximately; next substitute the latter values into (B.6) to obtain improved values of a_1 and a_{-1}, and so on. To avoid such reciprocal substitutions and arrive in a single step at values accurate to the seventh order, Brown wrote

$$a_3 = \alpha a_1 + \beta a_{-1} + \gamma\frac{a_0}{a'}a_0,$$
$$a_{-3} = \alpha' a_1 + \beta' a_{-1} + \gamma'\frac{a_0}{a'}a_0.$$

When these expressions are substituted into (B.6), $\alpha, \beta, \gamma, \alpha', \beta', \gamma'$ prove to be known functions of **m**, dependent on the ratios $a_2/a_0, a_{-2}/a_0, a_4/a_0, a_{-4}/a_0$ as given by Hill.

Having thus found a_1 and a_{-1} to the seventh order, Brown proceeded to determine, to the same order, the other parts of a_i depending on the first power of $1/a'$, namely $a_3/a_0, a_{-3}/a_0, a_5/a_0, a_{-5}/a_0$.

Next to be computed were the increments to the foregoing values depending on $(1/a')^2$ and $(1/a')^3$. Symbolizing these increments by $\delta a_2, \delta a_{-2}, \delta a_3, \ldots, \delta a_5, \delta a_{-5}$ Brown obtained their values in terms of a_0 by means of (B.5) and hitherto unused terms of L and M. For δa_1 and δa_{-1}, he made use of (B.6) in place of (B.5).

Finally, δa_0 had to be determined. For this, it was necessary to start from a differential equation containing $\kappa(us)^{-3/2}$, the nonlinear term eliminated earlier; Brown chose the first of his equations (B.2). Putting in this the substitutions

$$u = \sum_i a_{i-1}\zeta^i, \quad s = \sum_i a_{-i-1}\zeta^i,$$

and taking out the coefficient of ζ, he obtained

$$\kappa(u^{-1/2}s^{-3/2})_1$$
$$= \left(1 + 2\mathrm{m} + \frac{3}{2}\mathrm{m}^2\right)a_0 + \frac{3}{2}\mathrm{m}^2a_{-2} + \frac{\mathrm{m}^2}{a'}\left[\frac{15}{8}(s^2)_1 + \frac{3}{8}(u^2 + 2us)_1\right]$$
$$+ \frac{\mathrm{m}^2}{a'^2}\left[\frac{35}{16}(s^3)_1 + \frac{5}{16}(u^3)_1 + \frac{15}{16}(us^2)_1 + \frac{9}{16}(u^2s)_1\right] + \cdots$$

Hill, working from his form of the same equation but neglecting parallactic terms, had obtained the formula

$$\mathrm{a}_0 = a\left[\frac{J(1+\mathrm{m})^2}{H}\right]^{1/3},$$

where a is the mean Earth-Moon distance, $J = (u^{-1/2}s^{-3/2})_1$, and $a_0 H$ is the value of the right-hand side of the previous equation. Brown substituted $a_0 + \delta a_0$ for a_0, $J + \delta J$ for J, and $H + \delta H$ for H; he thus took δa_0 as arising from the parallactic terms which are contained in δJ and δH. His calculation gave him

$$\frac{\delta a_0}{a_0} = -\left(\frac{a}{a'}\right)^2 \mathrm{m}^2 \left\{\frac{225}{512}\cdot\frac{1+4\mathrm{m}}{\tau^2} + \frac{75}{128}\cdot\frac{\mathrm{m}}{\tau} + \frac{3}{16}(1-2\mathrm{m})\right\},$$

$$\text{where}\quad \tau = 1 - 4\mathrm{m} - \frac{37}{8}\mathrm{m}^2 - \frac{17}{6}\mathrm{m}^3 - \frac{89963}{3^2\cdot 2^{10}}\mathrm{m}^4.$$

Numerically, $\delta a_0/a_0 = -0.00965(a_0/a')^2$.

In the concluding section of his paper Brown transformed his results into polar coordinates, using the formulas

$$r\cos(V - nt) = \frac{1}{2}\sum_i a_i(\zeta^i + \zeta^{-i}), \quad r\sin(V - nt) = \frac{1}{2\sqrt{-1}}\sum_i a_i(\zeta^i - \zeta^{-i}),$$

where V is the longitude of the Moon as modified by the solar perturbations involving the constants **m** and a/a'. The coefficients in the resulting parallactic inequalities in longitude and parallax could then be obtained by introducing the numerical values

$$\mathrm{m} = 0.0808489338, \quad a/a' = 0.00255879,$$

where a is the mean Earth-Moon distance. The most accurate values for the coefficients of the parallactic inequalities, according to Brown, were got by substituting the numerical value of **m** from the outset. For the inequalities in longitude he thus found

$$-128''.070\sin D + 0''.039\sin 2D + 0''.750\sin 3D + 0''.001\sin 4D + 0''.008\sin 5D,$$

where $D = (n - n')(t - t_1)$, the difference between the mean longitudes of the Moon and the Sun. For the inequalities in parallax he found

$$-1''.001\cos D + 0''.008\cos 3D.$$

In his *Monthly Notices* note cited above, Brown gave the comparison of his results with those of Delaunay – both Delaunay unrevised and Delaunay revised by replacing m by μ:

Sin(Arg.)	Brown	Delaunay	Delaunay (μ)
Sin D	$-128''.069$	$-127''.621$	-128.059
Sin $3D$	$+0''.750$	$0''.845$	$\cdots$
Sin $5D$	$+0''.008$	$0''.014$	$\cdots$

In parallax, Brown's and Delaunay's results compared as follows:

Cos(Arg.)	Brown	Delaunay
Cos D	$-1''.0100$	$-0''.9447$
Cos $3D$	$+0''.0096$	$+0''.0158$

Unable to locate a source for the discrepancies between his own and Delaunay's results, Brown suspected small numerical errors in Delaunay's computation.

In his paper on the elliptic inequalities – the first part completed in December 1892, the second part in June 1893 – Brown took for his starting-point Hill's differential equations (II.7) and their Jacobian integral (II.8), leaving the parallactic inequalities aside. (The validity of ignoring inequalities of one class while calculating those of another type is here assumed, the presumptive justification being their smallness relative to the basic Variation orbit of the Moon.) The two equations of (II.7) are both of the second order, and thus a general solution must involve four arbitrary constants. They admit, as we've seen, of Hill's particular solution:

$$x = a_0 \sum_{i=-\infty}^{+\infty} a_i \cos(2i+1)(n-n')(t-t_1),$$

$$y = a_0 \sum_{i=-\infty}^{+\infty} a_i \sin(2i+1)(n-n')(t-t_1).$$

Here the arbitrary constants are two: n (or a_0), determining the mean rate of angular motion, and t_1, fixing the time of crossing of the x-axis. (The constants a_i are functions of **m** rather than arbitrary constants.) If in (II.7) the constant n' is set equal to zero – in effect abolishing the Sun's perturbing force – the equations admit of a general elliptical solution:

$$x = a \sum_{p=-\infty}^{+\infty} \frac{1}{p} J_{pe/2}^{(p-1)} \cos pg(t-t_0),$$

$$y = b \sum_{p=-\infty}^{+\infty} \frac{1}{p} J_{pe/2}^{(p-1)} \sin pg(t-t_0).$$

Here a and b are the semi-major and semi-minor axes of the ellipse, the $J_{pe/2}^{(p-1)}$ are the Bessel functions, **e** is the eccentricity, g is the mean rate of angular motion with respect to the axes of the ellipse, and t_0 is the constant used to make the major axis coincide with the x-axis. The two preceding forms are distinct except when both **e** and n' vanish, in which case the motion is circular and uniform.

Since the particular solution occurs when $\mathbf{e} = 0$ and the general solution when $n' = 0$, Brown assumed that for small values of **e** and n' there exists a solution combining the two and having the form

$$x = a_0 \sum_i \sum_p A_{i,p} \cos\{(2i + 1)(n - n')(t - t_1) + pg(t - t_0)\},$$

$$y = a_0 \sum_i \sum_p A_{i,p} \sin\{(2i + 1)(n - n')(t - t_1) + pg(t - t_0)\}. \quad \text{(B.7)}$$

Here the index i runs through all integral values from $-\infty$ to $+\infty$, and so does the index p. In practical applications, these indices can be restricted to a finite range of values, such as is found sufficient to match the precision of the observations. The foregoing solution involves the necessary four arbitrary constants; it can be viewed as a general solution except insofar as **e** and **m** may need to be limited in size for the resulting series to be convergent. Even when these constants are small, Brown cautioned, convergence is not guaranteed.

To determine the coefficients in the equations, the ratio $g : n - n'$ must be known. Brown put $g = c(n - n')$, so that the argument of the cosine and sine functions could be written

$$\alpha = \{n - n'\}\{(2i + 1)(t - t_1) + cp(t - t_0)\}.$$

Also, in transforming to the complex variables u, s, he followed Hill in defining the operator D as $-\frac{\sqrt{-1}}{n-n'} \cdot \frac{d}{dt}$. Again following Hill, he used the Jacobian integral to eliminate the non-linear terms $\frac{\kappa u}{(us)^{3/2}}, \frac{\kappa s}{(us)^{3/2}}$. He was thus able to arrive at two linear equations formally identical with Hill's (II.9) and (II.10):

$$D^2(us) - Du \cdot Ds - 2\mathrm{m}(uDs - sDu) + \frac{9}{4}\mathrm{m}^2(u + s)^2 = C,$$

$$D(uDs - sDu - 2\mathrm{m}us) + \frac{3}{2}\mathrm{m}^2(u^2 - s^2) = 0. \quad \text{(B.8)}$$

The form of D implies that $D(\alpha\sqrt{-1}) = 2i + 1 + cp$, so that the terms after being subjected to the operation D do not contain the constants t_0 or t_1. Brown temporarily omitted these constants altogether, rewriting the argument α as $(2i+1+cp)(n-n')\tau$, where $dt = d\tau$. Once the coefficients were determined numerically, the constants t_0 and t_1 could be re-introduced.

Brown defined ζ as $\exp[(n - n')\tau\sqrt{(-1)}]$, and so could write $D = \zeta\frac{d}{d\zeta}$, as Hill had done. (Hill's symbol τ, however, included the factor $[n - n']$.) The proposed solution became

$$u = \sum_i \sum_p A_{i,p}\zeta^{2i+1+cp},$$

$$s = \sum_i \sum_p A_{-i-1,-p}\zeta^{2i+1+cp}. \quad \text{(B.9)}$$

This solution, if viable, would constitute the mathematical meld of the variation and elliptic orbits which, two centuries before, Newton had believed prerequisite to solving the lunar problem.

The trigonometric series of (B.9) are of double period. The chief difficulty, according to Brown, lay

> in the fact that the equations of condition between the coefficients of the terms in the series require a relation between the two periods; it is the finding of this relation that entails the trouble.

Given the double periodicity, and assuming with Brown that the ratio of the two periods was irrational (there being no evidence to the contrary), the orbit would be reentrant only after an infinite time.

Substituting the proposed solution (B.9) into the two differential equations (B.8), Brown derived equations of condition for the coefficients. In symbolizing the products us, uDs, sDu, and $DuDs$, where the product of two infinite summations is involved, it was necessary to supplement the indices i and p, already used in expressing u and s, with the additional indices j and q. The resulting equations of condition were:

$$\sum_j \sum_q \left\{ \begin{bmatrix} (2i+cp)^2-(2j+1+cq)(2i+cp-2j-cq-1) \\ -2\mathrm{m}(2i+cp-4j-2cq-2)+\frac{9}{2}\mathrm{m}^2 \end{bmatrix} A_{j,q}A_{j-i,q-p} \atop +\frac{9}{4}\mathrm{m}^2(A_{j,q}A_{i-j-1,p-q}+A_{j,q}A_{-i-j-1,p-q}) \right\}$$

$$= 0$$

$$(2i+cp)\sum_j \sum_q [2i+cp-4j-2cq-2-2\mathrm{m}]A_{j,q}A_{j-i,q-p}$$

$$+\frac{3}{2}\mathrm{m}^2\sum_j \sum_q (A_{j,q}A_{i-j-1,p-q}-A_{j,q}A_{-i-j-1,-p-q})=0. \qquad \text{(B.10)}$$

In these equations j and q take all values from $-\infty$ to $+\infty$, and the equations are true for all values of i and p through the same range, except when $i = p = 0$, in which case the right-hand side of the first equation is the Jacobian constant C.

The next task was to determine the coefficients for the terms containing the first power of $\mathbf{e}$. From the equation for the elliptical solution, Brown knew that the coefficients $A_{i,p}$, $A_{i,-p}$ were at least of the order $\mathbf{e}^p$ in $\mathbf{e}$; hence the coefficients containing the first power of $\mathbf{e}$ would be $A_{i,1}$ and $A_{i,-1}$. (As we shall see later, $A_{i,1}$ and $A_{i,-1}$ will also contain terms proportional to higher powers of $\mathbf{e}$, namely $\mathbf{e}^3$, $\mathbf{e}^5$, etc.) For convenience he set

$$A_{i,1}=\varepsilon_i, \quad A_{i,-1}=\varepsilon_i', \quad A_{i,0}=a_i.$$

By a process exactly parallel to that used by Hill, Brown then combined the two equations of (B.10) to form a single equation of condition. The resulting formula was:

$$\sum_j \sum_q \{(j, i, p, q) A_{j,q} A_{j-i,q-p} + (i, p) A_{j,q} A_{i-j-1,p-q}$$
$$+ [i, p] A_{j,q} A_{-i-j-1,-p-q}\} = 0, \tag{B.1l}$$

where

$$(j, i, p, q) = -\frac{(2j + cq)\{(2i + cp)^2 - (2 + 4\mathrm{m} - \mathrm{m}^2) + (2i + cp)(2 + 2\mathrm{m})\} + (2j + cq)^2(2i + cp - 2 - 2\mathrm{m})}{(2i + cp\{2(2i + cp)^2 - 2 - 4\mathrm{m} + \mathrm{m}^2\})},$$

$$(i, p) = -\frac{3\mathrm{m}^2}{4(2i + cp)^2} \frac{(2i + cp - 2)(2i + cp - 2 - 2\mathrm{m}) - 6 - 12\mathrm{m} - 9\mathrm{m}^2}{2(2i + cp)^2 - 2 - 4\mathrm{m} + \mathrm{m}^2},$$

$$[i, p] = -\frac{3\mathrm{m}^2}{4(2i + cp)^2} \frac{(2 + 10i + 5cp)(2i + cp - 2 - 2\mathrm{m}) + 6 + 12\mathrm{m} + 9\mathrm{m}^2}{2(2i + cp)^2 - 2 - 4\mathrm{m} + \mathrm{m}^2}.$$

(B.11) includes all the equations in which i and p receive positive and negative values. When $i = p = 0$, the right-hand side is a function of the constant C of the Jacobian integral. Because the index i can take all integral values between $-\infty$ and $+\infty$, the number of equations is infinite, each of them containing the infinity of terms obtained as j goes through integral values from $-\infty$ to $+\infty$.

If in (B.11) and its supporting definitions, p and q are set equal to 0, the eccentricity **e** becomes zero, and a formula formally identical with Hill's (II.13) emerges. If, alternatively, i, k, and **m** are set equal to zero, so that $n' = 0$, $n = g$, and $c = 1$, (B.11) reduces to a set of conditions on the Bessel functions for the elliptical solution.

To obtain conditions determining $A_{i,1} = \varepsilon_i$ and $A_{i,-1} = \varepsilon'_I$ from (B.11), Brown substituted first $p = 1$, then $p = -1$. In each term of the two resulting equations, he then substituted a value for q such as to yield one of the constants a_i (= Hill's $\mathrm{a}_i/\mathrm{a}_0$) multiplied by ε_i or ε'_i. The appropriate substitutions proved to be $q = +1, 0,$ and -1. The equations became

$$\sum_j \left\{ \begin{array}{l} (j, i, 1, 1)\varepsilon_j a_{j-i} + (j + i, i, 1, 0)\varepsilon'_j a_{j+i} \\ +2(i, 1)\varepsilon_j a_{i-j-1} + 2[i, 1]\varepsilon'_j a_{-i-j-1} \end{array} \right\} = 0,$$

$$\sum_j \left\{ \begin{array}{l} (j - i, i, -1, 0)\varepsilon_j a_{j-i} + (j, -i, -1, -1)\varepsilon'_j a_{j+i} \\ +2[-i, -1]\varepsilon_j a_{i-j-1} + 2(-i, -1)\varepsilon'_j a_{-i-j-1} \end{array} \right\} = 0. \tag{B.12}$$

Here Brown has multiplied (B.11) by 2, and using two different substitutions in the first term of (B.11), has obtained two different terms in each equation of (B.12). A comparison of corresponding terms in (B.11) and (B.12) identifies the substitutions made in each case. Thus, in the first term of the first equation of (B.12), $p = q = 1$, so that $A_{j,q}$ becomes $A_{j,1}$, which is ε_j, and $A_{j-i,q-p}$ becomes $A_{j-i,0}$, which is a_{j-i}. In the last two terms of both equations, it is only p whose value is explicitly given, but the value of q that is required to complete the formula is easily identified. Thus the last term of the second equation of (B.12) is derived from $(i, p) A_{j,q} A_{i-j-l,p-q}$

in (B.11), with $-i$ replacing i, and p becoming -1. For $A_{-i-j-1,p-q}$ to become a_{-i-j-1}, we must have $p-q=0$, so that $q=-1$ and $A_{j,q}=A_{j,-1}\equiv\varepsilon'_j$.

If the constant c were known, we could determine from (B.12) all the successive values of ε_i and ε'_i, the number of equations then being equal to the number of unknowns. Hill, in his paper of 1877, had computed the part of c that is dependent solely on the ratio of the mean motions of the Sun and Moon – the principal part of c, differing by only a small amount from the observational value. His computation assumed the lunar eccentricity to be small enough so that its square could be neglected. The question remained whether the lunar eccentricity, which averages a little more than 0.05; could have a non-negligible effect on c. How, quantitatively, did c depend on **e**?

In search of a formula for this dependence, Brown turned back to (B.10), and there set $p=1$, and $q=0,1$. Adding and subtracting the results, he obtained

$$\sum_j\{c^2a_{j-i}+G_{j,i}a_{j-i}+G_{j+i,i}a_{j+i}+\frac{3}{4}\mathrm{m}^2(a_{i-j-1}+5a_{-i-j-1})\}X_j$$
$$+c\sum_j(H_{j,i}a_{j-i}-H'_ja_{j+i})Y_j=0,$$
$$\sum_j\{c^2a_{j-i}+G_{j,i}a_{j-i}-G_{j+i,i}a_{j+i}+\frac{3}{4}\mathrm{m}^2(a_{i-j-1}-5a_{-i-j-1})\}Y_j$$
$$+c\sum_j(H_{j,i}a_{j-i}+H'_ja_{j+i})X_j=0, \tag{B.13}$$

where

$$G_{j,i}=\frac{1}{2}(2j+1)^2+(2j+1)(i+2\mathrm{m})+\frac{9}{4}\mathrm{m}^2,$$
$$H_{j,i}=\frac{3}{2}(2j+1)+i+2\mathrm{m},\quad H'_j=\frac{1}{2}(2j+1),$$
$$X_j=\varepsilon_j+\varepsilon'_j,\quad Y_j=\varepsilon_j-\varepsilon'_j.$$

Denoting by ∞ the sequence $1,2,3,\ldots\infty$, Brown observed that there are $4\times\infty+2$ of the foregoing equations and the same number of unknowns X_j and Y_j. The equations are linear, suggesting the possibility of eliminating the unknowns by a determinant. If the determinant were convergent, it could be solved for c^2, just as Hill had done with the determinant (II.32).

The roots of this determinant in c^2, Brown found, are

$$\left.\begin{array}{l}c_0^2,\,(c_0\pm 2)^2,\,(c_0\pm 4)^2,\ldots,(c_0\pm 2i)^2,\ldots\\ 0^2,2^2,2^2,4^2,4^2,\ldots,(2i)^2,(2i)^2,\ldots\end{array}\right\}i=1,2,\ldots\infty.$$

Hill's determinant had not contained the second series.

Brown's determinant, as it stood, proved insoluble. It failed to satisfy one of the conditions for convergence established by Poincaré, namely, that the sum of the non-diagonal elements be finite.[68] No convenient way of following in Hill's footsteps presented itself. The equations (B.13) would later prove useful for verifications, but Brown abandoned for the time being the endeavor to compute c from an infinite determinant, concluding it best to assume Hill's value and calculate the corresponding values of ε_i and ε_i' in terms of ε_0 and ε_0'.

Returning to (B.12), he made an approximate determination of the relation of ε_1 and ε_1' to ε_0 and ε_0'. These four unknowns turn out to be orders of magnitude larger than the other ε_i and ε_i', and thus the latter could be neglected in a first approximation. The determination involved substitutions in (B.12) which yielded two equations, each giving a linear relation between the four quantities, and these equations were then solved simultaneously for ε_1 and ε_1' separately as linear functions of ε_0 and ε_0'.

At the next stage, in computing ε_2 and ε_2', quantities of the next smaller order had to be taken into account, including additional contributions to ε_0, ε_0', ε_1, and ε_1'. Brown computed the additional contributions in successive stages. His results were:

$$\begin{aligned}
\varepsilon_{-1} &= +.01999\ 88763\varepsilon_0 + .20567\ 90112\varepsilon_0' \\
\varepsilon_1' &= -.01054\ 68058\varepsilon_0 - .07779\ 55430\varepsilon_0' \\
\varepsilon_1 &= +.00308\ 02927\varepsilon_0 - .00092\ 80067\varepsilon_0' \\
\varepsilon_{-1}' &= -.00108\ 65960\varepsilon_0 - .00019\ 59999\varepsilon_0' \\
\varepsilon_{-2} &= +.00001\ 15205\varepsilon_0 + .00007\ 34691\varepsilon_0' \\
\varepsilon_2' &= -.00005\ 93876\varepsilon_0 - .00043\ 20782\varepsilon_0' \\
\varepsilon_2 &= +.00001\ 47376\varepsilon_0 - .00000\ 85378\varepsilon_0' \\
\varepsilon_{-2}' &= +.00000\ 01043\varepsilon_0 - .00000\ 08618\varepsilon_0' \\
\varepsilon_{-3} &= -.00000\ 00193\varepsilon_0 - .00000\ 01734\varepsilon_0' \\
\varepsilon_3' &= -.00000\ 04039\varepsilon_0 - .00000\ 29218\varepsilon_0' \\
\varepsilon_3 &= +.00000\ 00843\varepsilon_0 - .00000\ 00708\varepsilon_0' \\
\varepsilon_{-3}' &= +.00000\ 00024\varepsilon_0 - .00000\ 00055\varepsilon_0' \\
\varepsilon_{-4} &= -.00000\ 00001\varepsilon_0 - .00000\ 00012\varepsilon_0' \\
\varepsilon_4' &= -.00000\ 00029\varepsilon_0 - .00000\ 00212\varepsilon_0'
\end{aligned}$$

He checked these computations in several ways. In one such verification he put $i = 0$ in the two equations of (B.12), obtaining two equations, which yielded for Y_0/X_0 the two values $-2.01291\ 56632\ 7$, $-2.01291\ 56634\ 5$. The difference of 1.8 in the tenth place he attributed to accumulated errors. The mean of these two values to ten places could be in error by no more than one unit in the tenth place. The assumed value of c, Brown concluded, must be correct to ten decimal places.

[68] H. Poincaré, *Bulletin de la Société mathématique de France, 14* (1886), 77–90.

Only one of the constants ε_0, ε'_0, X_0, Y_0 could be arbitrary. Brown chose Y_0 for this role. His final values for the new coefficients were thus:

$$\frac{\varepsilon_0}{Y_0} = +.25160\ 40989, \quad \frac{\varepsilon'_0}{Y_0} = -.74839\ 59011,$$

$$\frac{\varepsilon_{-1}}{Y_0} = -.14889\ 75297, \quad \frac{\varepsilon'_1}{Y_0} = +.05556\ 82459,$$

$$\frac{\varepsilon_1}{Y_0} = +.00146\ 95307, \quad \frac{\varepsilon'_{-1}}{Y_0} = -.00012\ 67065,$$

$$\frac{\varepsilon_{-2}}{Y_0} = -.00005\ 20854, \quad \frac{\varepsilon'_2}{Y_0} = +.00030\ 84234,$$

$$\frac{\varepsilon_2}{Y_0} = +.00001\ 00997, \quad \frac{\varepsilon'_{-2}}{Y_0} = +.00000\ 06713,$$

$$\frac{\varepsilon_{-3}}{Y_0} = +.00000\ 01250, \quad \frac{\varepsilon'_3}{Y_0} = +.00000\ 20851,$$

$$\frac{\varepsilon_3}{Y_0} = +.00000\ 00742, \quad \frac{\varepsilon'_{-3}}{Y_0} = +.00000\ 00048,$$

$$\frac{\varepsilon_{-4}}{Y_0} = +.00000\ 00009, \quad \frac{\varepsilon'_4}{Y_0} = +.00000\ 00243.$$

The quantities ε_i and ε'_i become known once Y_0 is determined numerically. Each of them, we recall, contains the first power of the eccentricity **e** as a factor, and in the solution (B.9) of (B.8), multiplies the sine or cosine of the mean motion of the elliptic inequality, which Brown expresses as $\ell = c(n - n')(t - t_0)$. But each of the ε_i's, as we see in (B.11) and (B.12), is multiplied by one of Hill's coefficients a_i, and each of the latter in the solution (B.9) is the coefficient of the sine or cosine of an angle $2iD$, where $i = 0, 1, 2, 3$, etc. The resulting terms in u and s, and hence in x and y, will therefore involve as factors one of the constants ε_i, one of the constants a_i, the sine or cosine of ℓ, and the sine or cosine of $2iD$.

Having obtained x and y, Brown set himself the task of determining the elliptic inequality in longitude. Delaunay by his very different method had also computed the inequalities proportional to the first power of **e**, and Brown wanted to compare his own result with Delaunay's.

Like Delaunay, Brown expressed the mean synodic motion of the Moon by D (replacing the symbol τ used by Hill), so that $D = (n - n')(t - t_1)$. Hill's formulas for the x- and y-coordinates of the Variation orbit could thus be written

$$x = a_0 \sum_{-\infty}^{+\infty} a_i \cos(2i + 1)D,$$

$$y = a_0 \sum_{-\infty}^{+\infty} a_i \sin(2i + 1)D.$$

Now Hill had also expressed x and y as $r\cos\varphi$ and $y = r\sin\varphi$, respectively, where r is the radius vector and φ the true longitude in the Variation orbit. The longitude could thus be obtained from the relation $\varphi = \arctan(y/x)$.

But the true longitude φ in Hill's rotating coordinate system is the sum of the mean synodic motion D plus the excess of the true motion over the mean, which Hill designated by υ (upsilon), so that $\upsilon = \varphi - D$. In order to obtain an expression for υ, Hill had tabulated the numerical values of $r\cos\upsilon$ and $r\sin\upsilon$, and to these quantities Brown assigned the symbols x' and y' :

$$x' = r\cos\upsilon = a_0 \begin{bmatrix} 1 - 0.00718\ \ 00395\cos 2D \\ \quad +0.00000\ \ 60424\cos 4D \\ \quad +0.00000\ \ 00325\cos 6D \\ \quad +0.00000\ \ 00001\cos 8D \end{bmatrix},$$

$$y' = r\sin\upsilon = a_0 \begin{bmatrix} 0.01021\ \ 14544\sin 2D \\ \quad +0.00000\ \ 57149\sin 4D \\ \quad +0.00000\ \ 00276\sin 6D \\ \quad +0.00000\ \ 00002\sin 8D \end{bmatrix}.$$

Thus the excess of the true motion over the mean motion in Hill's rotating orbit – the inequality we call the Variation – could be obtained from the formula $\upsilon = \arctan(y'/x')$.

To symbolize the increments of x' and y' that are produced when eccentricity is introduced, Brown used the symbols $\delta x'$ and $\delta y'$. These increments to x' and y' depend on Y_0. The values obtained above for ε_i and ε_i' in terms of Y_0 yield, when inserted in the terms implied by (B.12), the following expressions for $\delta x'$ and $\delta y'$:

$$\delta x' = a_0 Y_0 \begin{bmatrix} -.49679\ \ 18022\cos\ell \\ \quad -.09332\ \ 92838\cos(2D-\ell) + .00134\ \ 28242\cos(2D+\ell) \\ \quad +.00025\ \ 63380\cos(4D-\ell) + .00001\ \ 07690\cos(4D+\ell) \\ \quad +.00000\ \ 22101\cos(6D-\ell) + .00000\ \ 00790\cos(6D+\ell) \\ \quad +.00000\ \ 00252\cos(8D-\ell) + .00000\ \ 00005\cos(8D+\ell) \\ \quad +.00000\ \ 00003\cos(10D-\ell) \end{bmatrix}$$

$$\delta y' = a_0 Y_0 \begin{bmatrix} +1.00000\ \ 00000\sin\ell \\ \quad +.20446\ \ 57756\sin(2D-\ell) + .00159\ \ 62372\sin(2D+\ell) \\ \quad +.00036\ \ 05088\sin(4D-\ell) + .00000\ \ 94264\sin(4D+\ell) \\ \quad +.00000\ \ 19601\sin(6D-\ell) + .00000\ \ 00694\sin(6D+\ell) \\ \quad +.00000\ \ 00234\sin(8D-\ell) + .00000\ \ 00004\sin(8D+\ell) \\ \quad +.00000\ \ 00003\sin(10D-\ell) \end{bmatrix}$$

To find the increment in υ corresponding to the increments $\delta x'$ and $\delta y'$, Brown took the δ-derivative of $\upsilon = \arctan(y'/x')$:

$$\delta\upsilon = \frac{x'\delta y' - y'\delta x'}{x'^2 + y'^2}. \tag{B.14}$$

Given the values of x', y', $\delta x'$, and $\delta y'$, it was possible to find $\delta\upsilon$ for any particular value of D as a sum of the form $K_1 Y_0 \cos\ell + K_2 Y_0 \sin\ell$, where K_1 and K_2 are numerical coefficients.

Delaunay's expression for $\delta\upsilon$ had the following form:

$$A\sin\ell + B_1\sin(2D-\ell) + B_2\sin(4D-\ell) + B_3\sin(6D-\ell)$$
$$+ C_1\sin(2D+\ell) + C_2\sin(4D+\ell) + C_3\sin(6D+\ell).$$

Brown now sought to derive from his own theory an expression of similar form, but including the additional terms $B_4\sin(8D-\ell)$ and $C_4\sin(8D+\ell)$. This meant finding the numerical values of the coefficients $A, B_1, \ldots, C_1, \ldots$, using (B.14) together with the preceding values of x', y', $\delta x'$, and $\delta y'$. To $2D$ Brown gave the values 0, 30, 90, 150, 180°, and calculated the coefficients of $\sin\ell$ and $\cos\ell$ in these several cases. He then set each of the five results equal to an expression of the above form, substituting in each case the appropriate value of $2D$. Thus he obtained five equations, to be solved simultaneously (if possible!) for the nine unknown coefficients.

Three of the equations, those for $2D = 30$, 90, and 150°, contain both sine and cosine terms on either side; each of them therefore counts as two equations, since sine terms must be set equal to sine terms, and cosine terms to cosine terms. Thus there are eight equations to work from. Moreover, the three equations in which $2D$ is respectively 0, 90, and 180° together yield for the coefficient of $\sin\ell$ the value $0.99972\ 87063Y_0$. Delaunay in his theory had made this coefficient $2e$, where e was his value for the lunar eccentricity, namely 0.05489 930. Brown in his comparison, therefore, set $Y_0 = 2(0.05489930) \div 0.9997287063 = 0.10982\ \ 8395$. Thus in the eight equations the arbitrary Y_0 along with the coefficient A could be replaced by numbers, and the eight equations modified in this way were sufficient to solve for the eight remaining unknowns. Brown's result for the principal elliptic term, $2e\sin\ell$, was thus identical with Delaunay's, with a coefficient equal to $6°17'27''.5870$. The other terms obtained constituted the *evection* in longitude:

$$+\,4607''.984\sin(2D-\ell) + 35''.2200\sin(4D-\ell)$$
$$+\,0''.2906\sin(6D-\ell) + 0.0027\sin(8D-\ell)$$
$$+\,174''.8610\sin(2D+\ell) + 1''.4460\sin(4D+\ell)$$
$$+\,0''.0121\sin(6D+\ell) + 0''.0001\sin(8D+\ell).$$

Delaunay's values were

$$+\,4607''.771\sin(2D-\ell) + 35''.1542\sin(4D-\ell) + 0''.2174\sin(6D-\ell)$$
$$+\,174''.8660\sin(2D+\ell) + 1''.4094\sin(4D+\ell) + 0''.0055\sin(6D+\ell).$$

The differences do not exceed tenths of arcseconds. Brown's numbers were based on Hill's constants a_i, computed with a precision of at least 13 decimal places. Delaunay's calculation, impressive as it was, was less precise.

In the second part of his paper, Brown derived the coefficients of the elliptic inequalities proportional to $\mathbf{e}^2$ and $\mathbf{e}^3$, and also the increments δa_i of a_i and δc of c that are proportional to $\mathbf{e}^2$. As we have seen, his derivation of the elliptic inequalities proportional to the first power of $\mathbf{e}$ was carried out on the basis of Hill's value of c. He now found that the elliptic inequalities proportional to $\mathbf{e}^2$, as well, could be got without invoking the increment δc. The latter was first required in the derivation of the inequalities proportional to $\mathbf{e}^3$.

The increment δc, it turned out, could be computed by the same process of successive approximations that Brown had used in determining the elliptic inequalities. This discovery suggested that it might be unnecessary to invoke a new infinite determinant or face the difficult problem of solving it.

Among the new derivations, Brown first undertook to obtain the increment δa_i, the change in the values of Hill's coefficients a_i required to proceed to approximations of the order of Y_0^2. His starting-point was once again (B.11). Putting $p = 0$ in this equation, he obtained

$$\sum_j \sum_q \{(j, i, 0, q) A_{j,q} A_{j-i,q} + (i, 0) A_{j,q} A_{i-j-1,-q} + [i, 0] A_{j,q} A_{-i-j-1,-q}\} = 0. \tag{B.15}$$

This holds for all integral values of i except $i = 0$. When the subscript q is a positive or negative integer and not zero, it implies the presence of $\mathbf{e}^q$ as a factor, and since there are two A's in each term, each term will contain the factor $\mathbf{e}^{2q}$. The eccentricity thus occurs in the equation only to even powers. To limit the order of the terms to Y_0^2, Brown set q equal to $+1$ and to -1. In addition, he used the value $q = 0$, and put $A_{j,0} = a_j + \delta a_j$, where a_j has the numerical value computed by Hill, and δa_j is the new part of this constant proportional to $\mathbf{e}^2$. For $A_{j,1}$ and $A_{j,-1}$ he put, as before, ε_j and ε'_j. With these substitutions he obtained the equation

$$\begin{aligned}\sum_j \{&(j, i, 0, 0)(a_j + \delta a_j)(a_{j-i} + \delta a_{j-i}) + (j, i, 0.1)\varepsilon_j \varepsilon_{j-i} + (j, i, 0, -1)\varepsilon'_j \varepsilon'_{j-i}\} \\ &+ (i, 0) \sum_j \{(a_j + \delta a_j)(a_{i-j-1} + \delta a_{i-j-1}) + 2\varepsilon_j \varepsilon'_{i-j-1}\} \\ &+ [i, 0] \sum_j \{(a_j + \delta a_j)(a_{-i-j-1} + \delta a_{-i-j-1}) + 2\varepsilon_j \varepsilon'_{-i-j-1}\} = 0.\end{aligned} \tag{B.16}$$

In solving (B.16) for δa_i, the increment δc can be left out of account. For in (B.11) c occurs only in the combinations $2i + cp$ and $2j + cq$. When p and q are zero, as in $(j, i, 0, 0)$, $(i, 0)$, and $[i, 0]$, c is absent. When c occurs in $(j, i, 0, \pm 1)$ it is multiplied by quantities which are themselves of the order of Y_0^2, so that, here too, δc can be ignored. The coefficients $(j, i, 0, 0)$, $(i, 0)$, $[i, 0]$ were already known, being identical with Hill's (j, i), (i), and $[i]$. A two-stage approximation process yielded the following values for the δa_i:

$$\begin{aligned}
\delta a_1 &= +.03938\ \ 170 Y_0^2, & \delta a_{-1} &= +.01376\ \ 519 Y_0^2,\\
\delta a_2 &= +.00046\ \ 113 Y_0^2, & \delta a_{-2} &= +.00002\ \ 216 Y_0^2,\\
\delta a_3 &= +.00000\ \ 473 Y_0^2, & \delta a_{-3} &= +.00000\ \ 026 Y_0^2,\\
\delta a_4 &= +.00000\ \ 005 Y_0^2, & \delta a_{-4} &= +.00000\ \ 000 Y_0^2.
\end{aligned}$$

To obtain δa_0, it was necessary to return to a differential equation containing the constant κ. As in the case of the parallactic inequalities, Brown had recourse to an equation due to Hill (labeled II.7a$''$ in our Part I). Solving it to the order of Y_0^2, he found

$$\frac{\delta a_0}{a_0} = -.13311\ \ 28 Y_0^2.$$

Hill's value of a_0 was $+.99909\ \ 31420(\mu/n^2)^{1/3}$, whence

$$a_0 + \delta a_0 = (+.99909\ \ 31420 - .13299\ \ 21 Y_0^2)\left(\frac{\mu}{n^2}\right)^{1/3}.$$

This value must replace Hill's value of a_0 in all terms where the calculation is of the order of Y_0^2.

Turning next to the determination of the elliptic inequalities proportional to $\mathbf{e}^2$, Brown made use of two equations obtained from (B.11) by putting first $p = +2$ and then $p = -2$. In each, he gave to q the values that yielded terms of the order of Y_0^2, namely 0, 1, 2 in the first equation and 0, -1, -2 in the second. The coefficients sought in this case were $A_{i,2}$ and $A_{i,-2}$, which he denoted by f_i and f'_i. The resulting equations were

$$\sum_j \left\{ \begin{array}{l} (j, i, 2, 2)a_{j-i} f_j + (j, i, 2, 0)a_j f'_{j-i} + (j, i, 2, 1)\varepsilon_j \varepsilon'_{j-i} \\ \quad +2(i, 2)(a_{i-j-1} f_j + \varepsilon_{i-j-1}\varepsilon_j) \\ \quad +2[i, 2](a_{-i-j-1} f'_j + \varepsilon'_{-i-j-1}\varepsilon'_j) \end{array} \right\} = 0,$$

$$\sum_j \left\{ \begin{array}{l} (j-i, -i, -2, 0)a_{j-i} f_j + (j-i, -i, -2, -2)a_j f'_{j-i} \\ \quad +(j-i, -i, -2, -1)\varepsilon_j \varepsilon'_{j-i} \\ \quad +2[-i, -2](a_{i-j-1} f_j + \varepsilon_{i-j-1}\varepsilon_j) \\ \quad +(-i, -2)(a_{-i-j-1} f'_j + \varepsilon'_{-i-j-1}\varepsilon'_j) \end{array} \right\} = 0.$$

Since all the terms are of the order of Y_0^2, Hill's value of c where it occurs in (j, i, p, q), (i, p), and $[i, p]$ could be used without the addition of δc. The two equations, solved together by approximation, yielded the values

$$\frac{f_0}{Y_0^2} = +.09402\ \ 355 \qquad \frac{f'_0}{Y_0^2} = +.03180\ \ 170$$

$$\frac{f_{-1}}{Y_0^2} = +.06517\ 276 \quad \frac{f_1'}{Y_0^2} = +.01564\ 642$$

$$\frac{f_{-2}}{Y_0^2} = +.00132\ 915 \quad \frac{f_2'}{Y_0^2} = +.00428\ 597$$

$$\frac{f_{-3}}{Y_0^2} = +.00000\ 174 \quad \frac{f_3'}{Y_0^2} = +.00004\ 843$$

$$\frac{f_{-4}}{Y_0^2} = +.00000\ 003 \quad \frac{f_4'}{Y_0^2} = +.00000\ 049$$

$$\frac{f_1}{Y_0^2} = +.00112\ 370 \quad \frac{f_{-1}'}{Y_0^2} = +.00006\ 457$$

$$\frac{f_2}{Y_0^2} = +.00001\ 161 \quad \frac{f_{-2}'}{Y_0^2} = +.00000\ 066$$

$$\frac{f_3}{Y_0^2} = +.00000\ 011 \quad \frac{f_{-3}'}{Y_0^2} = +.00000\ 001$$

Next, Brown undertook to determine the parts of $A_{i,1}$ and $A_{i,-1}$ that depend on $\mathbf{e}^3$. They are additions to the constants ε_i and ε_i' already found, and thus Brown denoted them by $\delta\varepsilon_i$ and $\delta\varepsilon_i'$. They figure as terms in the coefficients of $\sin\ell$ and $\cos\ell$. To obtain them, Brown put $p = \pm 1$ in (B.11), obtaining thus two equations; and he let q take the values 0, ± 1, ± 2. The substitution $q = \pm 1$ gave the modified coefficients

$$A_{i,1} = \varepsilon_i + \delta\varepsilon_i, \quad A_{i,-1} = \varepsilon_i' + \delta\varepsilon_i'.$$

The substitutions $q = 0, +2, -2$ gave coefficients we encountered earlier:

$$A_{i,0} = a_i + \delta a_i, \quad A_{i,2} = f_i, \quad A_{i,-2} = f_i'.$$

Most terms in the equation thus came to be of the order of Y_0^3. But in order that every term be of this order, in certain terms account had to be taken of δc, the increment to c of the order of Y_0^2.

The equations that resulted were somewhat complicated. They could be solved in a first approximation for $\delta\varepsilon_{\pm i}$ and $\delta\varepsilon_{\pm i}'$ in terms of Y_0^3, $\delta\varepsilon_0 = \delta\varepsilon_0'$, and $Y_0\delta c$. To obtain the numerical ratios $\delta\varepsilon_0 : Y_0^3$ and $\delta c : Y_0^2$, Brown put $i = 0$ in these first-approximation equations, and for verification added the second equation of (B.13), with the necessary terms of order Y_0^3 included:

$$+ .00425\ 226 Y_0^3 + .02946\ 89 \delta\varepsilon_0 - 1.11898\ 7 Y_0 \delta c = 0,$$

$$+ .00902\ 093 Y_0^3 + .00129\ 30 \delta\varepsilon_0 - 3.33860\ 6 Y_0 \delta c = 0,$$

$$+ .18016\ 812 Y_0^3 + 4.32005\ 66 \delta\varepsilon_0 + .98781\ 5 Y_0 \delta c = 0.$$

Solving the first of these with the third, he obtained

$$\delta c = +.00268\ \ 561 Y_0^2.$$

The second with the third yielded a result for δc differing by only $+1$ in the last decimal place. It followed that

$$\delta\varepsilon_0 = \delta\varepsilon_0' = -.04231\ \ 912 Y_0^3.$$

The final values of the $\delta\varepsilon_i$'s thus became

$$\frac{\delta\varepsilon_{-1}}{Y_0^3} = +.01685\ \ 40 \qquad \frac{\delta\varepsilon_1'}{Y_0^3} = -.00568\ \ 79$$

$$\frac{\delta\varepsilon_1}{Y_0^3} = +.02520\ \ 23 \qquad \frac{\delta\varepsilon_{-1}'}{Y_0^3} = +.000328\ \ 28$$

$$\frac{\delta\varepsilon_{-2}}{Y_0^3} = +.00070\ \ 60 \qquad \frac{\delta\varepsilon_2'}{Y_0^3} = +.00528\ \ 12$$

$$\frac{\delta\varepsilon_2}{Y_0^3} = +.00048\ \ 10 \qquad \frac{\delta\varepsilon_{-2}'}{Y_0^3} = +.00002\ \ 54$$

$$\frac{\delta\varepsilon_{-3}}{Y_0^3} = +.00000\ \ 49 \qquad \frac{\delta\varepsilon_3'}{Y_0^3} = +.00009\ \ 91$$

$$\frac{\delta\varepsilon_3}{Y_0^3} = +.00000\ \ 68 \qquad \frac{\delta\varepsilon_{-3}'}{Y_0^3} = +.00000\ \ 03$$

$$\frac{\delta\varepsilon_4'}{Y_0^3} = +.00000\ \ 14$$

Brown checked these results in several ways, using differential equations other than those employed in the derivations, among them a differential equation involving the Jacobian constant C, with its increment δC proportional to Y_0^2. Concerning the care taken, he remarked:

> All computations once made were gone through a second time. The average error made in the later portions of the work was about one in every four or five hundred figures. It did not seem to be confined to any particular class of operation. In using the new eight-figure tables of the French Government, extra care was exercised for the differences, and the chance of error thus diminished.

In the final section of his paper, Brown combined the preceding results with the Variation as computed by Hill, and by transforming to polar coordinates, obtained the implied inequalities in longitude. Denoting by v the difference between the true and mean longitudes, he found the expressions for $r\cos v$ and $r\sin v$ to be

$$\frac{r\cos\nu}{a_0+\delta a_0} = 1 + (-.00718\ \ 00395 + .05314\ \ 689Y_0^2)\cos 2D$$
$$+ (+.00000\ \ 60424 + .00048\ \ 328Y_0^2)\cos 4D$$
$$+ (+.00000\ \ 00325 + .00000\ \ 499Y_0^2)\cos 6D$$
$$+ (+.00000\ \ 00002 + .00000\ \ 005Y_0^2)\cos 8D$$
$$+ (-.49679\ \ 1802Y_0 - .08463\ \ 82Y_0^3)\cos \ell$$
$$+ (-.09332\ \ 9284Y_0 + .01116\ \ 60Y_0^3)\cos(2D-\ell)$$
$$+ (+.00025\ \ 6338Y_0 + .00598\ \ 71Y_0^3)\cos(4D-\ell)$$
$$+ (+.00000\ \ 2210Y_0 + .00010\ \ 40Y_0^3)\cos(6D-\ell)$$
$$+ (+.00000\ \ 0025Y_0 + .00000\ \ 14Y_0^3)\cos(8D-\ell)$$
$$+ (+.00134\ \ 2824Y_0 + .02848\ \ 51Y_0^3)\cos(2D+\ell)$$
$$+ (+.00001\ \ 0769Y_0 + .00050\ \ 64Y_0^3)\cos(4D+\ell)$$
$$+ (+.00000\ \ 0079Y_0 + .00000\ \ 71Y_0^3)\cos(6D+\ell)$$
$$+ .12582\ \ 524Y_0^2\cos 2\ell$$
$$- .04952\ \ 634Y_0^2\cos(2D-2\ell) + .00118\ \ 827Y_0^2\cos(2D+2\ell)$$
$$+ .00561\ \ 512Y_0^2\cos(4D-2\ell) + .00001\ \ 227Y_0^2\cos(4D+2\ell)$$
$$+ .00005\ \ 017Y_0^2\cos(6D-2\ell) + .00000\ \ 012Y_0^2\cos(6D+2\ell)$$
$$+ .00000\ \ 051Y_0^2\cos(8D-2\ell)$$

$$\frac{r\sin\nu}{a_0+\delta a_0} = +(+.01021\ \ 14544 + .02561\ \ 651Y_0^2)\sin 2D$$
$$+ (+.00000\ \ 57149 + .00043\ \ 897Y_0^2)\sin 4D$$
$$+ (+.00000\ \ 00276 + .00000\ \ 447Y_0^2)\sin 6D$$
$$+ (+.00000\ \ 00002 + .00000\ \ 005Y_0^2)\sin 8D$$
$$+ (+1.00000\ \ 0000Y_0)\sin\ell$$
$$+ (+.20446\ \ 5776Y_0 - .02254\ \ 19Y_0^3)\sin(2D-\ell)$$
$$+ (+.00036\ \ 0509Y_0 + .00457\ \ 52Y_0^3)\sin(4D-\ell)$$
$$+ (+.00000\ \ 1960Y_0 + .00009\ \ 41Y_0^3)\sin(6D-\ell)$$
$$+ (+.00000\ \ 0023Y_0 + .00000\ \ 13Y_0^3)\sin(8D-\ell)$$
$$+ (+.00159\ \ 6237Y_0 + .02191\ \ 96Y_0^3)\sin(2D+\ell)$$
$$+ (+.00000\ \ 9426Y_0 + .00045\ \ 57Y_0^3)\sin(4D+\ell)$$

$$+ (+.00000\ 0069Y_0 + .00000\ 65Y_0^3)\sin(6D+\ell)$$
$$+ .06222\ 185Y_0^2 \sin 2\ell$$
$$+ .08081\ 918Y_0^2\sin(2D-2\ell) + .00105\ 914Y_0^2\sin(2D+2\ell)$$
$$+ .00295\ 682Y_0^2\sin(4D-2\ell) + .00001\ 094Y_0^2\sin(4D+2\ell)$$
$$+ .00004\ 669Y_0^2\sin(6D-2\ell) + .00000\ 011Y_0^2\sin(6D+2\ell)$$
$$+ .00000\ 046Y_0^2\sin(8D-2\ell).$$

The foregoing expressions, arranged in series according to the ascending powers of Y_0, can be denoted by

$$r\cos\nu = S_0 + S_1Y_0 + S_2Y_0^2 + S_3Y_0^3,$$
$$r\sin\nu = S_0' + S_1'Y_0 + S_2'Y_0^2 + S_3'Y_0^3,$$

where S_0, S_1, etc., are the coefficients multiplying the respective powers of Y_0. Now $\tan\nu$ is the quotient of the second of these equations by the first, and from the quotient a series for $\tan\nu$ in powers of Y_0 can be obtained. But the variable ν is itself given by the series

$$\nu = \tan\nu - \frac{1}{3}\tan^3\nu + \frac{1}{5}\tan^5\nu - \cdots .$$

A path thus opens for determining the expressions for ν corresponding to particular values of $2D$. A general expression for the terms in ν proportional to given powers of Y_0 can then be obtained by the "method of special values." Such was the procedure Brown followed.

For the terms proportional to Y_0^2, he obtained the following results, in which Y_0^2 has been replaced by its numerical value (computed on the basis of Delaunay's numerical value for $2e$ or $\mathbf{e}$), and the coefficients have been expressed in arc-seconds:

$$+ 298''.959\sin 2D + 771''.132\sin 2\ell + 212''.610\sin(2D-2\ell)$$
$$+ 5''.217\sin 4D + 13''.240\sin(2D+2\ell) + 31''.055\sin(4D-2\ell)$$
$$+ 0''.067\sin 6D + 0''.169\sin(4D+2\ell) + 0''.532\sin(6D-2\ell)$$
$$+ 0''.001\sin 8D + 0''.002\sin(6D+2\ell) + 0''.007\sin(8D-2\ell).$$

These coefficients, Brown tells us,

> agree with what might have been expected from Delaunay's series, with one exception, the part of the coefficient of the Variation [*i.e.*, the coefficient of $\sin 2D$] which depends on $\mathbf{e}^2$.

Brown put at $298''.84$ the value inferable for this part from Delaunay's series; it differed from his own value, $298''.96$, by $0''.12$. From an examination of Newcomb's comparison of Delaunay's and Hansen's lunar theories,[69] Brown could find

[69] Brown here refers to *Astronomical Papers of the American Ephemeris*, *I*, 92.

> ...no reason to expect that [Delaunay's] value of [this] coefficient is erroneous by so much as one-tenth of a second.

Later, he would seek an independent procedure for confirming his result.

A more serious discrepancy was that in the part of the motion of the Lunar Perigee. Brown's result for δc was $+.00268\ \ 561Y_0^2$. Substituting for Y_0^2 its value in terms of Delaunay's eccentricity, we obtain $\delta c = +.01074\ \ 3023\mathbf{e}^2$. But if dw/dt is the mean sidereal rate of motion of the lunar perigee, then $c = \frac{n-dw/dt}{n-n'}$, so that

$$\delta c = \frac{-\delta(dw/dt)}{n-n'} = -\frac{\mathrm{m}+1}{n}\delta\left(\frac{dw}{dt}\right),$$

whence

$$\delta\left(\frac{1}{n}\frac{dw}{dt}\right) = -\frac{1}{\mathrm{m}+1}\delta c = .00994\ \ 29e^2.$$

Delaunay's value for this same constant, as deduced by Brown from Delaunay's series, was $.00955\ \ 96e^2$. The difference amounts to $0''.055$ per year in the motion, or half a degree in a century. But, as Hill had remarked, this motion could be determined observationally with a precision of about 39 arc-seconds in 125 years, or $0''.312$ per year.

Another difference was discoverable in Brown's and Delaunay's values for the coefficient of $\sin(2D-2\ell)$, Delaunay's value being $212''.318$ and Brown's $212''.610$. The difference amounted to $0''.29$.

Brown in this paper did not compare his results for the terms proportional to Y_0^3 with those of Delaunay. He had computed these terms solely in order to obtain a value for the increment δc. The difference from Delaunay's result meant that further investigation was required.

On 10 March 1893, as Brown was completing the paper, he wrote to Darwin concerning the calculations it had entailed:

> It has been tiring work doing arithmetic day after day. The amount of calculation necessary has been nothing less than terrific. I reckon that I have written about three quarters of a million numerals in connection with it. It has however given exactly all the coefficients depending on **m** and **e**, $\mathbf{e}^2$ together with that part of the motion of the perigee which depends on **m** and $\mathbf{e}^2$. They agree fairly well with Delaunay's values, but if my results are right the latter quantity as given by Delaunay is wrong by about one-twentieth of its value.[70]

Writing to Darwin in October, 1893, Brown still remembered the calculations in the paper as nightmarish:

> Many thanks for your kind letter received a day or two ago. That paper was a most terrible bit of work and I never think of what I went through in doing the calculations without a shudder.[71]

[70] Brown to Darwin, 10 March 1893, CUL, MS DAR.251:467.

[71] Brown to Darwin, 18 October 1893, CUL, MS DAR.251:469.

In his further development of the new lunar theory, Brown would seek to simplify and streamline the calculations, with a view to assigning a considerable portion of them to a human computer other than himself, who could reliably carry out specific computational tasks. Needed as well would be a means of verifying the calculations efficiently, without merely re-doing them.

After finishing his paper on the elliptic inequalities in the spring of 1893, and taking a summer tour in Switzerland to recover from his latest spell of illness, Brown, "quite well again," began a course of reading and writing connected with his teaching at Haverford. As he wrote Darwin in October,

> I have one man who is doing the Lunar Theory and so far all my time has been occupied in writing it out for him. I am hoping to develop it into a work and should very much like your advice on my plan. I have so far given the old method of finding radius, longitude, and latitude in terms of the time using the Perturbing Function and following Pontécoulant further, but in getting the first and second approximations of the principal terms have separated out these latter in Hill's method. As far as I can see this seems the best way to give a student some idea of the problem. The next portion would be Delaunay's method and a third Hill's method. At present I am struggling with Hansen and am very doubtful whether it would be advisable to include him. Hill seems easy in comparison – at any rate as far as method of proceeding goes. I have not worked out Hill's ∞ determinant and do not think it will be necessary when I go on with my own work as Adams has found the principal part of the motion of the node – the only other case in which I think it would occur. But I should be very glad to have the development if you at any time let me see the process.[72]

It was in May, 1893, that Brown had first heard of the computation whereby Adams had obtained the principal part of the motion of the lunar node, using an infinite determinant similar to, but simpler than, the one solved by Hill. As he had then written Darwin,

> I was not very surprised when in reading Glaisher's life of Adams in *Monthly Notices* to see that the latter had been on the same track and had solved the ∞ determinant also. He published I see in *Monthly Notices 1877* but we have not got them here so I have not been able to read what he did.[73]

Brown probably read Adams's *Monthly Notices* article on the lunar node during his stay in Cambridge the following summer. At any rate, he now had the following thought: since Hill and Adams had obtained the principal parts of the apsidal and nodal precessions of the Moon by their infinite determinants, and he himself had been able to obtain the increment δc proportional to $\mathbf{e}^2$ by a process of successive approximation, perhaps all the increments of the apsidal and nodal precessions

[72] Brown to Darwin, 19 May 1893, CUL, MS DAR.251:468.
[73] Brown to Darwin, 18 October 1893, CUL, MS DAR.251:469.

needed for completing the theory could be found in this same way, by successive approximations.

The review of the major lunar theories that Brown wrote out for his student became *An Introductory Treatise on the Lunar Theory*, published in early March of 1896 by Cambridge University Press. Its 292 pages bristle with equations, detailed explanations, and references. In his preface Brown thanks George Darwin, E.W. Hobson, and P.H. Cowell, for suggestions and corrections made during their reading of the proof-sheets. He expressed his special indebtedness to Darwin in a letter of 8 March 1896[74]:

> I am glad to hear from you that the book is out – your letter received last night was the first news of the fact. I want to add something to the very inadequate acknowledgment, in the preface, of all you have done for me. It isn't possible to say there all one would wish and I have always been very grateful – first, that you started me on the Lunar Theory at the point you did, and after that for all the help and encouragement you have given me right along. The book certainly would not have been written but for that and I hope that you may not have great reason to be ashamed of it in the future.... I shall be anxious to hear what Hill and Newcomb have to say about it.

The book contains a full account of Hill's infinite determinant and its solution, as well as of Adams's several discoveries in the lunar theory.

No doubt Brown, in writing this book, was doing his homework: reviewing the groundwork of the lunar theory from Laplace onward, mastering the essentials and much detail, in preparation for the task of carrying Hill's theory to completion. In Chapter XII he gives an assessment of the merits and faults of the several methods, and his reasons for pursuing the procedures pioneered by Hill and Adams.

> There does not appear to be any method which is capable of furnishing the values of the coordinates with a degree of accuracy comparable with that of observation, without great labor. The question to be discussed is mainly the relation between the accuracy obtained and the labor expended.
>
> As regards the inequalities produced by the action of the Sun, the methods may be divided into three classes. The first or algebraical class contains those in which all the constants are left arbitrary; the second or numerical, those in which the numerical values of the constants are substituted at the outset; the third or semi-algebraical, those in which the numerical values of some of the constants are substituted at the outset, the others being left arbitrary: the most useful case of the last class appears to be that in which the numerical value of the ratio of the mean motions is alone substituted. The advantage of an algebraical development will be readily recognized. In a numerical development, slow convergence is to a great extent avoided, but

[74] Brown to Darwin, CUL, MS.DAR.251:478 (8 March 1896). See also MS.DAR.251:471 (30 August 1895), MS.DAR.251:472 (1 Sept. 1895), MS.DAR.251:473 (7 Sept. 1895), MS.DAR.251:475 (20 Oct. 1895).

> the source of an error is traced with great difficulty and any change in the values of the arbitraries can not be fully accounted for without an extended recalculation. The semi-algebraical class, in which the value of **m** is alone substituted, appears to possess an accuracy nearly equal to that of a numerical development, and it has the advantage leaving those constants arbitrary whose values are known with least accuracy.
>
> It is difficult to judge of the labor which any particular method will entail, without performing a considerable part of the calculations by that and by other methods .... For a *complete* algebraical development carried to a greater accuracy than that of Delaunay, none of the methods given up to the present time seem available without the expenditure of enormous labor: Delaunay's calculations occupied him for twenty years. If we may judge from the inequalities computed up to the present time, the methods of Chap. XI [the methods applied by Hill and Adams] seem to be the best suited to a numerical or semi-algebraical development. It is true that they give the results expressed in rectangular instead of in polar coordinates, but the labor of transformation is not excessive in comparison with that expended on the previous computations, while the accuracy obtained far surpasses that of any other method; the transformation of the series, however, would not be necessary for the formation of tables.... Hansen's method labors under the disadvantage of putting the results under a form which makes comparison with those of other methods difficult. Another consideration which is a powerful factor, is the question as to how far the ordinary computer, who works by definite rules only, can be employed in the calculations; and here the methods of Chap. XI ["Method with Rectangular Coordinates"] appear to have an advantage not possessed by any of the earlier theories.[75]

Brown, anxious to hear what Hill and Newcomb would say about his book, could report to Darwin on 21 March 1896:

> I sent [Newcomb] a copy of the Lunar Theory and he wrote me a very nice letter about it.... I have just had a letter from Hill. He thinks it would have been better to leave out Hansen – because he says 'it will probably never be used again!' Otherwise he is complimentary – but I don't think he appreciates what a student beginning the subject wants.[76]

In the same letter Brown also reported that Newcomb was going to Europe in April and would be in Cambridge, and asked Darwin what he thought about the university's giving Newcomb an honorary degree.

> You probably remember that G.W. Hill had one in 1892, and the relative merits of the two men seem much the same. Newcomb has just been decorated by the Paris Academy.[77]

[75] E.W. Brown, *An Introductory Treatise on the Lunar Theory* (New York NY: Dover Publications, 1960) 246–247.
[76] Brown to Darwin, 21 March 1896, CUL, MS.DAR.251:479.
[77] Ibid.

Darwin apparently acted on Brown's suggestion: the University of Cambridge awarded Newcomb an honorary doctorate of science in 1896.[78]

During 1894, while engaged in writing his book, Brown completed a three-part article entitled "Investigations in the Lunar Theory." It was published in the *American Journal of Mathematics* in 1895.[79] Its first part proposed and justified a new way of deriving the lunar inequalities beyond those defining Hill's Variation orbit. Its second part demonstrated certain relations among the constants of the lunar theory. These two parts took their inspiration from two papers published by John Couch Adams in 1877 and 1878.

[78] R.C. Archibald, *A Semicentennial History of the American Mathematical Society, 1888–1938* (New York NY: American Mathematical Society, 1938), 125.

[79] *American Journal of Mathematics*, *17*, 318–358.

8

Initiatives Inspired by John Couch Adams' Papers

Adams' paper of 1877 – we mentioned it earlier for the high praise that Adams there gave to Hill's work on the motion of the lunar perigee – was published in the *Monthly Notices*, and bore the title "The Motion of the Moon's Node in the case when the Orbits of the Sun and Moon are supposed to have no Eccentricities, and when their mutual Inclination is supposed to be indefinitely small."[80] The part of the motion of the node Adams was concerned with depends solely on the ratio of the mean motions of the Sun and Moon; it is the principal part, differing little from the observational value.

In his derivation, Adams encountered an infinite determinant of the same form as the one solved by Hill. Working at intervals, he reduced it to a series of powers and products of small quantities. He obtained the terms of the fourth order – so he reported – on 26 December 1868, and the terms of the twelfth order on 2 December 1875. Thus, a year or so before Hill obtained the principal part in the motion of the perigee, Adams had calculated the principal part in the motion of the node with a precision about equal to Hill's.

As remarked earlier, Hill's and Adams' successes in these calculations suggested that the completion of the lunar theory might require no more than successive approximations. In addition, Adams' formulation of the problem of the Moon's motion in the z-coordinate suggested a new pattern for deriving the lunar inequalities.

Here is Adams' equation for z:

$$\frac{d^2z}{dt^2} + \left(\frac{\mu}{r^3} + \frac{\mu'}{r_1^3}\right) z = 0. \tag{A.1}$$

In the second term μ is the sum of the masses of the Earth and Moon, μ' the Sun's mass, r the radius vector from Earth to Moon, and r_1 the radius (assumed constant) of the Sun's orbit. For the unit of distance Adams chose the mean distance in the orbit a moon would describe in undisturbed motion about the Earth, supposing its period equal to that of the actual Moon. In this case $\mu = n^2$, where n is the Moon's

[80] *MNRAS*, 38 (Nov., 1877), 43–49.

C. Wilson, *The Hill–Brown Theory of the Moon's Motion*, Sources and Studies
in the History of Mathematics and Physical Sciences, DOI 10.1007/978-1-4419-5937-9_8,

mean rate of angular motion. The Sun's orbit being assumed circular, $\mu'/r_1^3 = n'^2$, where n' is the Sun's rate of angular motion. Therefore,

$$\left(\frac{\mu}{r^3} + \frac{\mu'}{r_1^3}\right) = \left(\frac{n^2}{r^3} + n'^2\right) = n^2\left(\frac{1}{r^3} + m^2\right).$$

For Adams, $m = n'/n$; he used Plana's value for m, 0.0748013. Taking r to be the radius vector in the Variation orbit, he expanded the fraction $1/r^3$ as a series:

$$\begin{aligned}\frac{1}{r^3} = {} & 1.00280\ 21783\ 115 + 0.02159\ 98364\ 4\cos 2(n-n')t \\ & + 0.00021\ 53273\ 9\cos 4(n-n')t + 0.00000\ 20644\ 8\cos 6(n-n')t \\ & + 0.00000\ 00192\ 9\cos 8(n-n')t + 0.00000\ 00000\ 3\cos 10(n-n')t.\end{aligned}$$

To the constant term of this series he then added the value of m^2, and proceeded to obtain the expansion

$$\frac{1}{(n-n')^2}\left(\frac{n^2}{r^3} + n'^2\right) = \frac{1}{(1-m)^2}\left(\frac{1}{r^3} + m^2\right) = q_0^2 + 2\sum_{i=1}^{\infty} q_i \cos 2i(n-n')t. \tag{A.2}$$

Given the values of $1/r^3$ and m^2, he found

$$\begin{aligned} q_0 &= 1.08537\ 75828\ 323, \\ q_1 &= 0.01261\ 68354\ 6, \\ q_2 &= 0.00012\ 57764\ 3, \\ q_3 &= 0.00000\ 12059\ 0. \end{aligned}$$

These, he stated, "are all the quantities necessary for finding the motion of the Moon's node, to the order which we require."

Of the further steps in his derivation, Adams reported little. Defining $\mathrm{g}\pi$ as the angular motion of the Moon from its node in half a synodic period, he gave for this quantity, without explanation, the equation

$$\cos \mathrm{g}\pi = C_c \cos q_0\pi + C_s \sin q_0\pi, \tag{A.3}$$

where C_c and C_s are formulas involving the q's. On substituting numerical values for the latter, he obtained

$$\cos \mathrm{g}\pi = -0.96441\ 51972\ 00779.$$

It followed that $\mathrm{g} = 1.08517\ 13927\ 46869$, and the ratio of the Moon's motion from its node to its sidereal motion was

$$\mathrm{g}(1-m) = 1.00399\ 91618\ 46592.$$

The quantity $\mathrm{g}(1-m)$ is usually designated g (italic) in the Lunar Theory. The value of g just given, Adams observed, differs from the true (observational) value in the eighth decimal place. But, he added, if we develop the value of g not in m but in Hill's constant $\mathbf{m}[= m/(1-m)]$, and substitute $\mathbf{m} = 0.08084\ \ 89030\ \ 52$, then we find

$$g = 1.00399\ \ 91591\ \ 1;$$

this, said Adams, "is considerably nearer the truth."

The symbols Brown used in the "Investigations" – inherited in part from Hill – differed from those used by Adams, and so (A.1), as Brown wrote it, had a different look. Brown replaced Adams' q's by M's, where

$$q_0^2 = M_0, \quad \text{and} \quad \frac{1}{2}q_i = M_i = M_{-i} \quad \text{for} \quad i \neq 0.$$

He replaced d/dt by $\surd(-1)(n-n')D$, μ/r^3 by $\kappa(n-n')/r^3$, and m by its equivalent in terms of $\mathbf{m}$, namely $\mathbf{m}/(1+\mathbf{m})$, so that the differential equation took the form

$$D^2z - \left(\frac{\kappa}{r^3} + \mathrm{m}^2\right)z = 0, \ \text{ or } \ D^2z - \left(\sum_{-\infty}^{+\infty} M_i\right)z = 0.$$

Brown's inclusion of the constant $\mathbf{m}^2$ in the constant M_0, paralleling Adams' inclusion of the constant m^2 in the constant q_0^2, supports our supposition that he was here following Adams.

Given Brown's form of the equation, it is easy to show how the infinite determinant arises. Solving this equation by the method of undetermined coefficients leads to an equation of condition of the form

$$(2j+g)^2 z_j + \left(\sum_{i=-\infty}^{+\infty} M_i\right) z_{j-i} = 0, \tag{A.4}$$

where the z_j are unknown coefficients. (A.4) yields an infinite number of homogeneous equations, a different one for each j, each equation containing an infinite number of terms obtained as i varies from $-\infty$ to $+\infty$. The three equations at the center of the array may be represented by:

$$\begin{array}{lcccl}
\cdots\ [(g-2)^2 - M_0]z_{-1} & -M_1z_0 & -M_2z_1 & \cdots = 0 \\
\cdots\quad -M_1z_{-1} & (g^2 - M_0)z_0 & -M_1z_1 & \cdots = 0 \\
\cdots\quad -M_2z_{-1} & -M_1z_0 & [(g+2)^2 - M_0]z_1 & \cdots = 0
\end{array}$$

For there to be a non-trivial solution of this system of equations, the determinant of the array must equal zero. Setting the determinant equal to zero yields an equation which is formally the same as Hill's II.32; it can be solved for g by the same methods as Hill employed in obtaining c.

Since Adams based his value of g on a value for m differing slightly from that adopted by Hill, Brown ultimately relied on another calculation of g, that due to P.H. Cowell.[81]

We turn now to Brown's new procedure. It differed from Hill's procedure, which Brown had used in his earlier papers. Hill, we recall, started from two second-order differential equations, formulated initially with the variables x and y, and then, more conveniently, with the complex variables u and s. The equations, whether in x, y or in u, s, were linear in these variables, except for the term expressing the gravitational force. They admitted of an integral (the Jacobian integral), with the aid of which Hill freed the two equations of the non-linear term. He solved the resulting linear equations by the method of undetermined coefficients. To obtain the numerical values of the constants of integration, he had to return to an equation containing the gravitational force constant.

The new method that Brown now proposed was more direct. It was applicable to all departures from Hill's Variation orbit, and it avoided the steps of first eliminating and then restoring the non-linear term involving the gravitational force. It employed special symbols for the increments sought: $u_0 + \delta u$ for u and $s_0 + \delta s$ for s, where u_0 and s_0 are the values of u and s in the Variation orbit. The latter orbit Brown expressed by:

$$u_0\zeta^{-1} = a_0 \sum_i a_i \zeta^{2i},$$

$$s_0\zeta = a_0 \sum_i a_{-i} \zeta^{2i}.$$

Here he has multiplied Hill's formulas by a power of ζ such as to put the sum on the right into a standard form, expressible by a cosine series with argument $2i\nu(t - t_0)$. The "a_0" occurring in these formulas outside the summation signs represents the same magnitude as Hill's a_0. As in the paper on the elliptic inequalities, the symbol a_i for $i \neq 0$ represents, in terms of Hill's coefficients, the fraction $\mathrm{a}_i/\mathrm{a}_0$; for the particular case in which $i = 0$, the a_0 under the summation sign is equal to 1.[82]

The departures δu and δs from the Variation orbit can arise from the lunar or solar eccentricity, from the inclination of the lunar orbit, or from the solar parallax. In all such inequalities the treatment is basically the same.

Brown chose one equation for determining the increments to u and s, and another equation for determining the increments to z:

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[81] P.H. Cowell, "On the Inclinational Terms in the Moon's Coordinates," *American Journal of Mathematics, 18* (1896), 99–127.

[82] Brown in his *Theory of the Motion of the Moon, Memoirs of the R.A.S., 53*, 60, will replace a_0 as it occurs outside the summation signs by $\mathbf{a}_0$, or more simply $\mathbf{a}$, specifying that a_0, as it occurs under the summation signs, is equal to 1. Hereafter Brown's *Theory of the Motion of the Moon* will be abbreviated as *TMM*.

$$\left(D^2 + 2\mathrm{m}D + \frac{3}{2}\mathrm{m}^2\right)u + \frac{3}{2}\mathrm{m}^2 s - \frac{\kappa u}{(us+z^2)^{3/2}} = -\frac{\partial \Omega_1}{\partial s}, \tag{B.17a}$$

$$(D^2 - \mathrm{m}^2)z - \frac{\kappa z}{(us+z)^{3/2}} = -\frac{1}{2}\frac{\partial \Omega_1}{\partial z}. \tag{B.17b}$$

(B.17a), with z deleted, is identical with the first equation of (B.2). It governs variations in longitude and radius vector. Both δu and δs are determined by it because u and s, being complex conjugates, contain both x and y. (B.17b) governs the Moon's departures from the ecliptic.

Deleting z from (B.17a), Brown substituted $u_0 + \delta u$ for u and $s_0 + \delta s$ for s, and then subtracted the corresponding equation satisfied by u_0, s_0, namely

$$\left(D^2 + 2\mathrm{m}D + \frac{3}{2}\mathrm{m}^2\right)u_0 + \frac{3}{2}\mathrm{m}^2 s_0 - \frac{\kappa u_0}{(u_0 s_0)^{3/2}} = 0.$$

What remained was an equation for deriving the increments δu and δs:

$$(D+\mathrm{m})^2\delta u + \frac{1}{2}\mathrm{m}^2\delta u + \frac{3}{2}\mathrm{m}^2\delta s - \left[\frac{\kappa}{(u_0+\delta u)^{1/2}(s_0+\delta s)^{3/2}} - \frac{\kappa}{u_0^{1/2}s_0^{3/2}}\right] = \frac{\partial \Omega_1}{\partial s}. \tag{B.18}$$

In each derivation from (B.18), Brown construed Ω_1 as consisting of just the terms required for deriving the particular increment sought. For instance, in seeking the increments to u_0 and s_0 arising from the lunar eccentricity, he set Ω_1 equal to zero, because (B.18) thus modified included all possible eccentric lunar orbits. But in seeking the increments arising from the eccentricity e' of the Sun's (= the Earth's) orbit, he gave Ω_1 the form

$$\Omega_1 = \frac{1}{2}(Au^2 + 2Bus + Cs^2),$$

where A, B, and C are functions of the solar eccentricity e' and the solar mean anomaly $n't + \varepsilon'$. For the derivation of increments due to solar parallax, he set Ω_1 equal to yet other terms, as in his first article on the parallactic inequalities.

In all cases Brown assumed that δu and δs, in relation to u_0 and s_0, were small enough so that the square bracket in (B.18) could be expanded in a Taylor series. To third-order terms this expansion is

$$\frac{\kappa}{u_0^{1/2}s_0^{3/2}}\left[\begin{array}{l} -\frac{1}{2}\frac{\delta u}{u_0} - \frac{3}{2}\frac{\delta s}{s_0} + \frac{3}{8}\frac{(\delta u)^2}{u_0^2} + \frac{15}{8}\frac{(\delta s)^2}{s_0^2} + \frac{3}{4}\frac{\delta u\delta s}{u_0 s_0} \\ \quad -\frac{5}{16}\frac{(\delta u)^3}{u_0^3} - \frac{35}{16}\frac{(\delta s)^3}{s_0^3} - \frac{9}{16}\frac{(\delta u)^2\delta s}{u_0^2 s_0} - \frac{15}{16}\frac{\delta u(\delta s)^2}{u_0 s_0^2} + \cdots \end{array}\right]$$

In any particular derivation, the expansion could be limited to terms from lowest order up to the order of the increment sought. For instance, to obtain the increment proportional to the first power of e', only the first two terms of the expansion were required, but to obtain the terms proportional to e'^2, the three terms involving the squares and product of δu and δs had to be included as well.

Since the coefficients of δu, δs, $(\delta u)^2$, $(\delta s)^2$, $\delta u \delta s$, etc., within the square bracket remained the same in all the derivations, labor could be saved by calculating them once for all. Brown in the present paper gave the numerical series for these coefficients up to and including the third order of powers and products of δu and δs. Within the bracket, the coefficient of δu to the first power is equal to $\kappa/2r_0^3$, for which Hill had provided the series:

$$\frac{\kappa}{2r^3} = \frac{1}{2}\begin{bmatrix} 1.17150\ 80211\ 79225 \\ \quad +.02523\ 36924\ 97860 \cos 2\tau \\ \quad +.00025\ 15533\ 50012 \cos 4\tau \\ \quad + \cdots \end{bmatrix}.$$

Brown – following the pattern adopted by Adams – combined this coefficient of δu with the term $(1/2)\mathbf{m}^2$ – the second term in (B.18) above – to form the single expression:

$$\frac{1}{2}\mathrm{m}^2 + \frac{1}{2}\frac{\kappa}{(u_0 s_0)^{3/2}} = \sum_i M_i \zeta^{2i}.$$

For the terms in (B.18) involving δs to the first power, Brown put

$$\zeta^{-2}\left[\frac{3}{2}\mathrm{m}^2 + \frac{3}{2}\frac{\kappa}{u_0^{1/2} s_o^{5/2}}\right] = \sum_i N_i \zeta^{2i}.$$

The multiplication by ζ^{-2} had the purpose of putting the summation on the right into a standard form expressible by a cosine series. The second term on the left can be re-written in the form

$$\frac{3}{2}\frac{\kappa}{r_0^3} \cdot \frac{u_0^2 \zeta^{-2}}{r_0^2},$$

where the factors κ/r_0^3, $1/r_0^2$, and $u_0^2\zeta^{-2}$ have known series expressions, the last-named being the square of

$$u_0 \zeta^{-1} = a_0 \sum_i a_i \zeta^{2i}.$$

Thus an expression for $\sum N_i \zeta^{2i}$ could be determined by "the method of special values." The coefficients of higher order could be obtained similarly.

By way of illustration, Brown applied this procedure to several of the inequalities to be calculated. The procedure required an appropriate expression for the disturbing function Ω_1, and a correct form for the solution. In the case of the inequality depending on the first power of e', Brown set $\delta u = u_\eta$ and $\delta s = s_\eta$, with the understanding that these increments contain e' to the first power only. He rewrote (B.18) in the form:

$$\zeta^{-1}(D+\mathrm{m})^2 u_\eta + \zeta^{-1} u_\eta \cdot \sum_i M_i \zeta^{2i} + \zeta s_\eta \cdot \sum_i N_i \zeta^{2i} = -B\zeta^{-1}u_0 - C\zeta^{-2} \cdot \zeta s_0. \tag{B.19}$$

The partial derivative of Ω_1 with respect to s on the right of (B.18) is $Bu + Cs$; but since B and C both contain e' as a factor, u must be replaced by u_0 and s by s_0, neglecting δu and δs to avoid producing terms of the second order in e'. B and C must also contain the sine or cosine of the mean anomaly, $n'(t-t_0)$, of the solar orbit, since the Sun's orbital position affects its action on the Moon. This result is obtained by including in B and C the factor $\zeta^{\pm \mathrm{m}}$; for since $\mathbf{m} = n'/\nu$, this implies that B and C contain the factor $\exp[\pm n'(t-t_0)\sqrt{(-1)}] = \cos n'(t-t_0) \pm \sqrt{(-1)} \times \sin n'(t-t_0)$.

As a solution of the foregoing equation, Brown proposed

$$\zeta^{-1}u_\eta = a_0 e' \sum_j [\eta_j \zeta^{2j+\mathrm{m}} + \eta'_{-j}\zeta^{-2j-\mathrm{m}}],$$

$$\zeta s_\eta = a_0 e' \sum_j [\eta'_{-j}\zeta^{2j+\mathrm{m}} + \eta_j \zeta^{-2j-\mathrm{m}}].$$

Substituting these expressions into the equation, he solved it by the method of undetermined coefficients. The terms on the left-hand side proved to be factored either by $\zeta^{2j+\mathrm{m}}$ or $\zeta^{-2j-\mathrm{m}}$, and the same thing had to be true of the terms on the right. Designating the coefficients of the latter by $K_{2j+\mathrm{m}}$ or $K_{-2j-\mathrm{m}}$ according as they contained $\zeta^{2j+\mathrm{m}}$ or $\zeta^{-2j-\mathrm{m}}$, he found them to be

$$K_{2j+\mathrm{m}} = (2j+1+2\mathrm{m})^2\eta_j + \sum_i M_i \eta_{j-i} + \sum_i N_i \eta'_{i-j},$$

$$K_{-2j-\mathrm{m}} = (2j-1)^2\eta'_{-j} + \sum_i M_{-i}\eta'_{i-j} + \sum_i N_{-i}\eta_{j-i}.$$

The final step was to compute the constants η_j and η'_{-j} by successive approximations.

Following on his several illustrations, Brown remarked:

> Sufficient has been said to indicate the manner of treating the whole Lunar Theory after this method.... Nevertheless, should it be considered that in finding the terms of high orders in $\mathbf{e}, e', \gamma, 1/a'$ the method indicated above becomes too troublesome, nothing prevents us from returning to equations (8) [the homogeneous equations], these being available at any stage. No increase of labor results from the mere change of method.[83]

[83] E. W. Brown, "Investigations in the Lunar Theory," *American Journal of Mathematics, 17* (1895), 341–342.

We shall find that, in his *Theory of the Motion of the Moon* (1897–1908), Brown will use the method just described in computing the inequalities of the first and second orders. For the inequalities of the third order, he will modify this method in one particular, and for higher orders he will find it advantageous to return to Hill's method and thus to the homogeneous equations.

In the "Investigations," Brown proceeded on the assumption that all increments to the principal parts of the motions of the perigee and node could be obtained by successive approximations, without recourse to infinite determinants. Yet, after describing the new method, he returned to the question that had stumped him in his essay on the elliptic inequalities: does introduction of the first power of the lunar eccentricity into the theory lead to an infinite determinant that converges? The determinant, this time, would be based on a new equation, and so would be different. Brown was not concerned to solve this determinant, only to learn *whether* it was convergent and would remain so as further increments were introduced into the theory.

In Brown's new method, the equation for finding the increments due to the first power of the lunar eccentricity was similar to (B.19), but with Ω_1 set equal to zero:

$$\zeta^{-1}(D+\mathrm{m})^2 u_e + \zeta^{-1} u_e \sum_i M_i \zeta^{2i} + \zeta s_e \cdot \sum_i N_i \zeta^{2i} = 0.$$

The solution took the form

$$\zeta^{-1} u_e = a_0 \sum_j [\varepsilon_j \zeta^{2j+c} + \varepsilon'_j \zeta^{2j-c}],$$

$$\zeta s_e = a_0 \sum_j [\varepsilon'_{-j} \zeta^{2j+c} + \varepsilon_{-j} \zeta^{2j-c}]. \tag{B.20}$$

If these expressions were substituted into the equation of motion, and the coefficients of ζ^{2j+c} and ζ^{2j-c} equated to zero, the result would be two series of linear homogeneous equations of condition from which, assuming c to be known, $\varepsilon_j, \varepsilon'_j$ could be determined by successive approximations:

$$(2j+1+\mathrm{m}+c)^2 \varepsilon_j + \sum_i M_i \varepsilon_{j-i} + \sum_i N_i \varepsilon'_{i-j} = 0,$$

$$(2j-1-\mathrm{m}+c)^2 \varepsilon_{-j} + \sum_i M_{-i} \varepsilon'_{i-j} + \sum_i N_{-i} \varepsilon_{j-i} = 0. \tag{B.21}$$

But suppose that, without assuming c known, we could eliminate $\varepsilon_j, \varepsilon'_j$ from (B.21); the condition for this elimination was that the determinant of these equations be equal to zero. The determinant was clearly infinite, so its convergence was not obvious. Poincaré in an essay of 1886[84] had proved the relevant theorem: An infinite determinant converges if and only if (1) the non-diagonal elements have a finite sum, and (2) the product of the elements in the main diagonal is finite.

[84] H. Poincaré, "Sur les déterminants d'ordre infini," *Bulletin de la Société mathématique de France, XIV* (1886), 77–90.

Brown represented the determinant of the equations for $\varepsilon_j, \varepsilon'_j$ by

$$\Delta(c) = \left|\frac{d(c)}{d}\middle|\frac{d}{d(-c)}\right|.$$

The vertical and horizontal lines within the determinant delimit the quadrants occupied by d, $d(c)$, and $d(-c)$. The symbol $d(c)$ stands for

$$\left|\begin{array}{lccr} \multicolumn{4}{c}{\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots} \\ \ldots (3+\mathrm{m}+c)^2+M_0 & M_{-1} & M_2 & \ldots \\ \ldots \quad M_1 & (1+\mathrm{m}+c)^2+M_0 & M_{-1} & \ldots \\ \ldots \quad M_2 & M_1 & (-1+\mathrm{m}+c)^2+M_0 & \ldots \\ \multicolumn{4}{c}{\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots} \end{array}\right|.$$

This matrix is to be understood as extending infinitely up, down, and to right and left. The symbol $d(-c)$ stands for the same matrix but with the sign of c negative. The symbol d stands for

$$\left|\begin{array}{ccccc} \multicolumn{5}{c}{\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots} \\ \ldots N_0 & N_{-1} & N_{-2} & N_{-3} & N_{-4} \ldots \\ \ldots N_1 & N_0 & N_{-1} & N_{-2} & N_{-3} \ldots \\ \ldots N_2 & N_1 & N_0 & N_{-1} & N_{-2} \ldots \\ \ldots N_3 & N_2 & N_1 & N_0 & N_{-1} \ldots \\ \ldots N_4 & N_3 & N_2 & N_1 & N_0 \ \ldots \\ \multicolumn{5}{c}{\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots} \end{array}\right|$$

Putting d, $d(c)$, and $d(-c)$ in their places in the expression for $\Delta(c)$, and deleting the cross lines, we have the required determinant.

This determinant, unlike Hill's, is "doubly infinite," in that the rows and columns increase indefinitely toward the cross lines as well as toward the outside of the array. The determinant Brown had obtained in his paper on the elliptic inequality was also "doubly infinite" in this way. The new determinant differs, however, in that by a suitable system of divisors it can be put into a convergent form.

The roots of the determinant $\Delta(c)$, as Brown knew from his earlier paper, were

$$\begin{array}{lll} \pm\, c_1, & \pm(c_1 \pm 2), & \pm(c_1 \pm 4), \ldots, \\ \pm\, c_2, & \pm(c_2 \pm 2), & \pm(c_2 \pm 4), \ldots, \end{array}$$

where either c_1 or c_2 is zero. These two sets of roots are also roots of the equation

$$(\cos \pi c - \cos \pi c_1)(\cos \pi c - 1) = 0.$$

Following a route previously traversed by Hill, Brown showed that

$$(\cos \pi c - \cos \pi c_1)(\cos \pi c - 1) \equiv \Delta(c) \times \left(\frac{4}{1} \cdot \frac{4}{4^2 - 1} \cdot \frac{4}{8^2 - 1} \cdot \cdots \right)^4$$
$$\equiv \nabla(c).$$

The two determinants $\Delta(c)$ and $\nabla(c)$ both meet Poincaré's tests for convergence.

Next, Brown investigated the changes occurring in $\Delta(c)$ when terms dependent on $(e')^2$ and its powers are taken into account. Most of these terms contain the angle $n't + \varepsilon$ but some do not. Brown proposed incorporating the terms not containing functions of angles into the variables u_0 and s_0; thus u_0 became

$$a_0 \sum_j [a_j + e'^2(\eta\eta')_j + e'^4(\eta^2\eta'^2)_j + \cdots]\zeta^{2j}.$$

Secondly, in place of M_i, N_i, he put

$$(M)_i = M_i + \text{terms containing } e'^2 \text{ and its powers},$$
$$(N)_i = N_i + \text{terms containing } e'^2 \text{ and its powers}.$$

The terms arising from A, B, C – the coefficients in the expression for Ω_1 – could be included in (M), (N) as well. The new determinant remained symmetrical and convergent. To express the new variables and the new value of c, Brown wrote

$$(\varepsilon)_j = \varepsilon_j(1 + \text{terms containing } e'^2 \text{ and its powers}),$$
$$(\varepsilon')_j = \varepsilon'_j(1 + \text{terms containing } e'^2 \text{ and its powers}),$$
$$(c)_j = c + \text{terms containing } e'^2 \text{ and its powers}.$$

Satisfied as to the convergence of the infinite determinant, Brown was led to consider (in the short addendum that is Part III of the "Investigations") the problem of "small divisors." This was a besetting difficulty in celestial mechanics, recognized by Euler as a danger but first detected in an important particular case by Laplace. The equations of motion for the Moon, being of the second order, required for their solution a double integration with respect to the time. If a term undergoing integration contained $\sin At$ or $\cos At$, its integral would have, as a divisor, A^2. If A were exceedingly small, the term could come to have a large coefficient – large, that is, in relation to its *order*, as determined by the powers to which the several small parameters entering as factors into its coefficient were raised. To determine its exact value, a computation was required; it could be of daunting length and intricacy. Laplace's famous discovery of "the great inequality of Jupiter and Saturn" depended on computing the effect of just such a small divisor.

Under what circumstances might small divisors arise in the lunar theory? Let the arguments of any series of terms in x, y differing by multiples of $2\nu(t - t_0)$ be

$$(2j + \Lambda)\nu t + \text{const.},$$

where Λ has the form $j_1 + k\mathrm{m} + pc + 2qg$. The coefficient of any such term will contain the factors $(a')^{-j'}(e')^{k'}e^{p'}\gamma^{2q'}$. In Λ, j_1 is either 0 or 1, depending on whether j' is even or odd. The sum $[j' + k' + p' + 2q']$ is called the *order* of the term. Also,

$$k' = \mathrm{k} \ \text{ or } \ k + \ \text{an even positive integer,}$$

$$p' = p \ \text{ or } \ p + \text{an even positive integer,}$$

$$q' = q \ \text{ or } \ q + \text{an even positive integer.}$$

Let the coefficients corresponding to any given value of Λ be λ_j, λ'_j. The equations determining these coefficients are

$$(2j + 1 + \mathrm{m} + \Lambda)^2\lambda_j + \sum_i M_i\lambda_{j-i} + \sum_i N_i\lambda'_{i-j}$$

$$= \text{known terms independent of } \lambda, \lambda',$$

$$(2j - 1 - \mathrm{m} + \Lambda)^2\lambda'_{-j} + \sum_i M_{-i}\lambda'_{1-j} + \sum_i N_{-i}\lambda_{j-i}$$

$$= \text{known terms independent of } \lambda, \lambda'. \qquad \text{(B.22)}$$

These equations have the same form as (B.21), but with Λ replacing c, and λ, λ' replacing $\varepsilon, \varepsilon'$; also, the right-hand members are no longer zero. If (B.22) were to be solved for λ_j, λ'_j by determinants, the denominators in the solutions would be the determinant of the equations. But this determinant is just the infinite determinant of (B.21), namely $\nabla(c)$, with Λ replacing c.

As we have seen, $\nabla(c) \equiv (\cos\pi c - \cos\pi c_1)(\cos\pi c - 1)$. This is an identity holding for any values of c; in it we can therefore put Λ for c, while taking c_1 to be the principal part of the perigee's motion. The determinant of (B.22) will therefore be

$$\nabla(\Lambda) = (\cos\pi\Lambda - \cos\pi c_1)(\cos\pi\Lambda - 1). \qquad \text{(B.23)}$$

This expression will be small if either of the two factors is close to zero. Since c_1 is close to 1, and hence $\cos\pi c_1$ close to -1, the two factors cannot both be close to zero for the same value of Λ. The first factor of (B.23) approaches zero as Λ approaches the sum $[c_1 +$ an even integer]; the even integer can be positive, negative, or zero. In this case the period of the inequality will differ little from that of one of the principal elliptic inequalities, that is, from $2\pi/(2j \pm c_1)$. Thus, short-term inequalities with unusually large coefficients would have periods close to periods of the elliptic inequalities, not to the Moon's sidereal or synodic periods, as previously supposed.

The second factor of (B.23) approaches zero as Λ approaches an even integer, which can be positive, negative, or zero. This is the case of long-period terms.

The foregoing rules, with certain further specifications here omitted, needed to be taken into account in Brown's later systematic development of the theory. In this he will start by computing terms of the first "order," taking this term in its technical sense as the sum $[j' + k' + p' + 2q']$. He will then proceed to terms of the second order, terms of the third order, and so on. This ordering has to be modified where very small divisors cause the size of terms to fall outside the range indicated by their technical order.

The second part of Brown's "Investigations" took its starting-point from the second of Adams' papers, bearing the title "Note on a Remarkable Property of the Analytical Expression for the Constant Term in the Reciprocal of the Moon's Radius Vector."[85] The properties Adams here established were, he said, remarkable "for a degree of simplicity and generality of which the lunar theory affords very few examples." These properties were important to Brown because they supplied ways of verifying certain calculations. We describe them briefly, omitting details.

Let r be the Moon's radius vector, and let a be the Moon's mean distance in the elliptic orbit that the Moon would describe about the Earth if perturbations due to the Sun were absent. Thus $a = (\mu/n^2)^{1/3}$, where μ is the sum of the masses of the Earth and Moon, and n the mean motion of the Moon. If terms depending on the Sun's parallax are omitted, the fraction a/r can be expanded in an infinite series of cosines with arguments of the form $2i\xi \pm j\phi \pm j'\phi' \pm 2k\eta$. Here ξ is the mean elongation of the Moon from the Sun, ϕ is the Moon's mean anomaly, ϕ' is the Sun's mean anomaly, and η is the Moon's mean distance from the ascending node; i, j, j', k are any positive integers or zero. The coefficient of the term with this argument contains $\mathbf{e}^j e'^{j'} \gamma^{2k}$ as a factor, where $\mathbf{e}$ is the mean eccentricity of the Moon's orbit, e' is the mean eccentricity of the Sun's orbit, and γ is the sine of half the mean inclination of the Moon's orbit to the ecliptic. The remaining factor in the coefficient is a function of $\mathbf{e}^2$, e'^2, and γ^2, and of $m = n'/n$, the ratio of the Sun's mean motion to the Moon's mean motion. That this second factor involves the squares of $\mathbf{e}$, e', and γ is a characteristic of the elliptical theory which Laplace was apparently the first to identify, and which holds generally for series giving functions of the longitude or radius vector.

The quotient a/r contains a constant term, corresponding to the case in which i, j, j', and k are all zero. Adams expresses this term in the form

$$A + Be^2 + C\gamma^2 + \mathrm{E}e^4 + 2\mathrm{F}e^2\gamma^2 + \mathrm{G}\gamma^4 + \&c.,$$

where

$$A = A_0 + A_1e'^2 + A_2e'^4 + \&c.,$$
$$B = B_0 + B_1e'^2 + B_2e'^4 + \&c.,$$
$$C = C_0 + C_1e'^2 + C_2e'^4 + \&c., \ldots.$$

Here A_0, A_1, etc., B_0, B_1, etc., C_0, C_1, etc., are all functions solely of m.

[85] *MNRAS, 38* (1878), 460–472.

Adams tells us that Plana, and after him Lubbock, Pontécoulant, and Delaunay, developed the functions of m occurring in the terms of a/r. Plana showed that B_0 and C_0 both vanish when account is taken of terms involving m^2 and m^3. Pontécoulant showed that these coefficients still vanish when account is taken of terms involving m^4 and m^5. Adams reports that,

> Thinking it probable that these cases in which the coefficients had been found to vanish were merely particular cases of some more general property, I was led to consider the subject from a new point of view, and on February 22, 1859, I succeeded in proving, not only that the coefficients B_0 and C_0 vanish identically, but that the same thing holds good of the more general coefficients B and C, so that the coefficients of
>
> $$e^2, e^2e'^2, e^2e'^4, \&c.$$
>
> $$\gamma^2, \gamma^2e'^2, \gamma^2e'^4, \&c.$$
>
> in the constant term of a/r are all identically equal to zero.

To obtain this result, Adams began by imagining two moons, one without orbital eccentricity or inclination, the other with either orbital eccentricity or orbital inclination, but exactly like the first in every other respect. The radius vector and rectangular coordinates of the first he designated by r, x, y, z, and those of the second by r_1, x_1, y_1, z_1. He was able to show that the expression

$$(xx_1 + yy_1 + zz_1)\left(\frac{1}{r_1^3} - \frac{1}{r^3}\right) \tag{A.5}$$

was a complete differential with respect to the time t. It followed that, when developed in cosines of angles proportional to t, it contained no constant terms. A further consequence was that the value of $\left(\frac{1}{r_1} - \frac{1}{r}\right)$ could contain no constant of lower order than the fourth in $\mathbf{e}$ or γ. Thus if one orbit had no eccentricity and the other had a finite eccentricity $\mathbf{e}$, the foregoing difference of reciprocal *radii vectores* could contain no constant term of the order of $\mathbf{e}^2$. Since $1/r$ certainly contained no such constant term, $1/r_1$ didn't either.

The form of reasoning just indicated led Adams on August 14, 1877 to a further result. If the terms of the quantity $c = d\phi/ndt$ involving $\mathbf{e}^2$ and γ^2 are denoted by $H\mathbf{e}^2 + K\gamma^2$, and the terms of the quantity $g = d\eta/ndt$ involving $\mathbf{e}^2$ and γ^2 are denoted by $M\mathbf{e}^2 + N\gamma^2$, where H, K, M, and N are functions of m and $(e')^2$, then

$$\frac{E}{F} = \frac{H}{K} \quad \text{and} \quad \frac{F}{G} = \frac{M}{N}. \tag{A.6}$$

This appears to have been the last of Adams' discoveries concerning the constants in the reciprocal radius vector.

Brown in the first part of his "Investigations," when applying his new method to the derivation of δc, arrived at a formula resembling Adams' expression (A.5). He found, in fact, that

$$2\delta c \sum_j [(2j+1+\mathrm{m}+c)\varepsilon_j^2 + (2j-1-\mathrm{m}+c)\varepsilon_{-j}'^2]$$

$$= \text{const.part of order } e^4, \text{ in expansion of } \frac{\kappa}{a_0^2} \cdot \frac{X_{e^2}x_e + Y_{e^2}y_e}{R_{e^2}^3}, \qquad \text{(B.24)}$$

where $X_{e^2} = x_0 + x_e + x_{e^2}$, $Y_{e^2} = y_0 + y_e + y_{e^2}$, and $R_{e^2}^2 = X_{e^2}^2 + Y_{e^2}^2$.

X and Y with the subscript $\mathbf{e}^2$ are the values of x, y in the Variation orbit as augmented by the increments corresponding to **e** and $\mathbf{e}^2$. To Brown, (B.24) suggested that

> Adams' theorems as to the connection between the constant parts of the Parallax of the Moon and certain parts of the motion of the Perigee and Node must really arise naturally from this mode [Brown's new mode] of development of the lunar theory.

He proceeded to prove Adams' theorems again and to establish them not just in the form of ratios but as exact equations. Thus he was able to define two constants, T_e and T_r, such that $HT_e = 6E$, $KT_e = 6F$, $MT_\gamma = 6F$, and $NT_\gamma = 6G$, where H, K, M, N are Adams' symbols.

What Brown regarded as his chief new discovery in this part of the "Investigations" was this: If the Moon's rectangular coordinates, X, Y, Z, have been calculated to order $2q-2$, the constant part in the expansion of $1/\sqrt{X^2+Y^2+Z^2}$ could be obtained to the order $2q$ without further reference to the equations of motion. This suggested that higher-order increments to c and g might possibly be obtained as functions of terms in the constant part of the reciprocal radius vector, without recourse either to infinite determinants or to successive approximations (the latter being less daunting than infinite determinants but still laborious).

9

Further Preliminaries to the Systematic Development

In 1894 Brown completed calculations yielding the lunar inequalities proportional to $e' f_1(\mathrm{m})$ and to $\mathbf{e}e' f_2(\mathrm{m})$. After finishing work on his *Introductory Treatise on the Lunar Theory*, he also carried out a number of theoretical inquiries resulting in published papers.

His results for the inequalities proportional to the first power of the eccentricity e' of the Sun's (or Earth's) orbit were published in June, 1894.[86] They were:

$$\begin{aligned}
&-659''.2375\sin\ell' + 152''.0828\sin(2D-\ell') - 21''.5942\sin(2D+\ell')\\
&+1''.2550\sin(4D-\ell') - 0''.1800\sin(4D+\ell')\\
&+0''.0105\sin(6D-\ell') - 0''.0015\sin(6D+\ell')\\
&+0''.0001\sin(8D-\ell').
\end{aligned}$$

The first of these terms represents what is known as "the annual equation" in the lunar theory. The coefficients, Brown reported, were close to Delaunay's "when estimate is made for the omitted portions of the series in powers of m, the greatest difference being $0''.06$."

Brown's results for the inequalities proportional to the product $\mathbf{e}e'$ appeared in November, 1894.[87] Because Delaunay's series in powers of m for the coefficients of the several terms converged in some cases very slowly, Delaunay had to resort to estimates as to what the uncalculated terms in the infinite series would contribute to the coefficient. These guesses were often mistaken, as the following table shows. Here the column labeled "B" gives Brown's values, and the column labeled "B–D" gives the difference between Brown's and Delaunay's values. Where no value is given for (B–D), it means that Delaunay did not calculate the term in question.

[86] *MNRAS, 54* (1894), 77.
[87] *MNRAS, 55* (1894-5), 3–5.

C. Wilson, *The Hill–Brown Theory of the Moon's Motion*, Sources and Studies in the History of Mathematics and Physical Sciences, DOI 10.1007/978-1-4419-5937-9_9,

Argument	B	B–D
$\ell + \ell'$	$-110''.1758$	$-0''.0390$
$2D - \ell - \ell'$	+206.8242	−0.2528
$4D - \ell - \ell'$	+4.0861	0.0549
$6D - \ell - \ell'$	+0.0545	+0.0357
$8D - \ell - \ell'$	+0.0006	...
$2D + \ell + \ell'$	−2.6612	+0.0058
$4D + \ell + \ell'$	−0.0370	−0.0080
$6D + \ell + \ell'$	−0.0004	...
$\ell - \ell'$	+149.1996	+0.8228
$2D - \ell + \ell'$	−27.8714	+1.3975
$4D - \ell + \ell'$	−0.5780	+0.1666
$6D - \ell + \ell'$	−0.0078	−0.0038
$8D - \ell + \ell'$	−0.0001	...
$2D + \ell - \ell'$	+13.6294	+0.0016
$4D + \ell - \ell'$	+0.2160	+0.0438
$6D + \ell - \ell'$	+0.0027	...

During 1896, Brown published three papers in the London Mathematical Society's *Proceedings*. All of them had to do with the method of variation of elliptical elements in the lunar theory as applied by Delaunay and later by Newcomb. In the first and second papers Brown showed how these applications could be streamlined and made to reach their goal more directly by a use of Jacobi's principal function. More immediately relevant to the practical development of the lunar theory, Brown in the second and third papers showed how the method of variation of elliptical elements could aid in the determination of certain constants in the theory. The results that Adams and Brown had obtained by successive approximations, as reported in our account of Brown's "Investigations," are here shown to be equally obtainable from Delaunay's starting-point and general method. In addition, the principal parts of the secular accelerations in the lunar theory (of the mean motion, the perigee's advance, and the node's regression) are obtainable from the final values of Delaunay's constants L, G, H when all the periodic terms in R have been removed. Newcomb had calculated them, but Brown now obtained them by a shorter route. Of the importance of these derivations, Brown wrote:

> The basis which furnishes these results is a consideration of the constant parts of the various functions which naturally arise in the solution obtained by varying the arbitrary constants. In all problems of celestial mechanics there are certain arbitrary constants which must be determined from observation; and there are certain others which may also be accurately found by observation, but which depend on those previously found. The comparison of the two sets of values forms so important a test of the sufficiency of the theory that no means which will give tests of the accuracy of the theoretical

> calculations, or which will enable us to obtain the second class of constants without serious risk of error, will be unimportant from a practical point of view, apart from any theoretical interest which they may possess.[88]

We briefly describe Brown's first two papers, but follow the argument of the third in more detail, because of its role in achieving an accurate lunar theory.

The first paper, submitted in April, 1896, was entitled "On the Application of the Principal Function in the Solution of Delaunay's Canonical System of Equations."[89]

Delaunay's canonical system of equations, we recall, was

$$\frac{dL}{dt} = \frac{\partial R}{\partial \ell}, \quad \frac{dG}{dt} = \frac{\partial R}{\partial g}, \quad \frac{dH}{dt} = \frac{\partial R}{\partial h};$$

$$\frac{d\ell}{dt} = -\frac{\partial R}{\partial L}, \quad \frac{dg}{dt} = -\frac{\partial R}{\partial G}, \quad \frac{dh}{dt} = -\frac{\partial R}{\partial H}.$$

To solve the equations, Delaunay wrote

$$R = -B - A\cos\theta + R_1,$$

$$\text{where} \quad \theta = i\ell + i'g + i''h + i'''n't + q.$$

Here $-A\cos\theta$ is any periodic term of the disturbing function R, and $-B$ is the non-periodic part of R; B, A are functions of L, G, H only; i', i'', etc., are positive or negative integers; n' is the solar mean motion; and q is a constant depending on the solar epoch and perigee. Delaunay began by solving the equations with R_1 neglected, and so obtained a solution containing six arbitrary constants. What values, he then inquired, should be given these constants if R_1 were no longer neglected? The values were to be so chosen that the new equations would be, like the original equations, canonical in form.

Delaunay's process, being one of direct transformation, was unavoidably lengthy and tedious.

François Tisserand, in his *Mécanique Céleste*, showed that Delaunay's process of transformation could be greatly shortened by use of a "principal function."[90] Already in December, 1893, Brown had obtained the first volume of Tisserand's work, apparently at Darwin's suggestion, for the help it could give him in preparing his *Introductory Treatise on the Lunar Theory*.[91] The principal function, usually labeled S, had been introduced by William Rowan Hamilton in 1834; Tisserand's account of it derived more immediately from C.G.J. Jacobi.[92] S was not

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[88] E. W. Brown, "On certain Properties of the Mean Motions and the Secular Accelerations of the principal Arguments used in the Lunar Theory," *Proceedings of the London Mathematical Society, 28* (1897), 143.

[89] *Proceedings of the London Mathematical Society, 27* (1896), 385–390.

[90] F. Tisserand, *Traité de Mécanique Céleste*, (Paris: Gauthier-Villars, 1889–1896) III, Chapter 11.

[91] Brown to Darwin, 13 December 1893, CUL, MS.DAR.251:470.

[92] See F. Tisserand, *Traité de Mécanique Céleste*, III, 190.

given explicitly at the start, but constructed in each case so that its partial derivatives represented the integrals of the equations of motion. Brown in the present article devised a different principal function from Tisserand's, bringing a further gain in simplicity and brevity. We shall encounter a variant of it in Brown's third article.

Brown's second article, "On the Application of Jacobi's Dynamical Method to the General Problem of three Bodies," was submitted in October, 1896.[93] Here Brown used the principal function to simplify the derivation of equations in Simon Newcomb's *Theory of the Inequalities in the Motion of the Moon produced by the Action of the Planets* (1895).[94] A further purpose was to exhibit a relation of lunar constants newly discovered by Newcomb. Brown began as follows:

> The connection which exists between the solution of the problem of three bodies, as obtained by varying the elliptic elements and by the ordinary methods of continued approximation, has had new light thrown on it by the appearance of Professor Newcomb's memoir "Action of the Planets on the Moon." ... Mention will be made of the remarkable results which Professor Newcomb obtains for the indirect action of a planet on the Moon, on account of its importance in the calculation of the secular accelerations of the mean motion, the perigee, and the node of the Moon's orbit.

The last-named results are derived again, in Brown's own way, in Brown's third paper, to which we now turn.

This third paper, which Brown submitted at the same time as the second, bore the title "On certain Properties of the Mean Motions and the Secular Accelerations of the principal Arguments used in the Lunar Theory."[95] Employing once more his own version of the principal function, Brown here set out to show three things: how we may pass directly from the elements of the ellipse to Delaunay's final system of constants; how the constant part of the parallax is connected with the motions of the perigee and the node; how by Newcomb's theorem the secular accelerations are obtainable immediately from Delaunay's constants L, G, H. Specifically,

> I shall show that the principal part of the acceleration of the mean motion (that independent of $\mathbf{e}$, γ, a/a') is obtainable directly from the corresponding portion of the constant term in the expression for the lunar parallax; and that, when this is found, the principal parts of the accelerations of the perigee and the node may be deduced from the expressions for their mean motions.[96]

Following Newcomb, Brown began with a change of variables. The disturbing function for Delaunay's equations was

$$R' = \frac{(E+M)^2}{2L^2} + m' \left\{ \begin{array}{c} \frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} \\ -\frac{xx'+yy'+zz'}{r'^3} - \frac{1}{r'} \end{array} \right\},$$

[93] *Proceedings of the London Mathematical Society, 28* (1897), 130–142.

[94] *Astronomical Papers prepared for the Use of the American Ephemeris and Nautical Almanac*, V (Washington, DC: Government Printing Office, 1895), 97–295.

[95] *Proceedings of the London Mathematical Society, 28* (1897), 143–155.

[96] *Proceedings of the London Math. Society, 28*, 144.

E, M, m' being the masses of the Earth, Moon, and Sun, and x, y, z, x', y', z' the coordinates of the Moon and Sun referred to fixed axes through the Earth's center. Delaunay's elliptical elements were

$$L = \sqrt{a(E+M)}, \quad G = L\sqrt{1-e^2}, \quad H = G\cos i = G(1-2\gamma^2);$$
$$\ell = nt + \varepsilon - \pi = \text{mean anomaly},$$
$$g = \pi - \theta = \text{distance from node to perigee},$$
$$h = \theta = \text{longitude of node}.$$

Brown followed Newcomb in putting

$$p_1 = L, \quad p_2 = G - L, \quad p_3 = H - G;$$
$$q_1 = \ell + g + h, \quad q_2 = g + h, \quad q_3 = h.$$

When the meanings of p_1, p_2, p_3 have been assigned, those of q_1, q_2, q_3 are determinate, and *vice versa*. This set of variables has the advantage that p_2, p_3 are small quantities of the order of $\mathbf{e}^2$, γ^2, respectively. A further advantage is that q_1, q_2, q_3 represent familiar variables in the lunar theory – the Moon's longitude, the perigee's longitude, and the node's longitude. The non-periodic parts of these angles are their mean motions.

Replacing R' by $-R$, Brown showed that the new variables satisfy the canonical equations:

$$\frac{dp_i}{dt} = -\frac{\partial R}{\partial q_i}, \quad \frac{dq_i}{dt} = \frac{\partial R}{\partial p_i}, \quad (i = 1, 2, 3). \tag{B.25}$$

Like Delaunay, Brown let $z' = 0$, and gave x', y', the Sun's coordinates, their elliptical values.

The general equations to be satisfied by the principal function were

$$p_i = \frac{\partial S}{\partial q_i}, \quad \frac{\partial S}{\partial t} = -R. \tag{B.26}$$

It was known that, once a solution was obtained for S containing three arbitrary constants (say, $c_1,\ c_2,\ c_3$), all the integrals of the problem would be given by (B.26) together with the three partial derivatives of S with respect to the c_i:

$$\ell_i = \frac{\partial S}{\partial c_i}. \tag{B.27}$$

Here S is to be expressed in terms of the q_i, c_i, and t.

No direct method exists for solving (B.26). Brown built up his solution by assuming general expressions for the p_i as indicated by the practical methods used for the problem, namely,

$$p_i = c_i + \sum_i sj_i \cos N. \tag{B.28}$$

Here the constants s are coefficients dependent only on c_1, c_2, c_3, n', e'; and N is an angle of the form $N = j_1q_1 + j_2q_2 + j_3q_3 + j'n't + \alpha$, the j_i and j' being positive or negative integers or zero, and α a constant.

By (B.26) $\frac{\partial p_i}{\partial t} = \frac{\partial^2 S}{\partial q_i \partial t} = -\frac{\partial R}{\partial q_i}$, and by (B.28) $\frac{\partial p_i}{\partial t} = \sum_i s j_i j'n' \sin N$. Suppose R, like S, to be expressed in terms of q_i, c_i, and t. An expression for R satisfying (B.28) and the first equation of (B.25) can then be obtained by putting

$$R = -B - \sum sj'n' \cos N, \tag{B.29}$$

where $-B$ is the non-periodic portion of R, and depends solely on n', e', and the c_i. Since $\partial S/\partial t = -R$, a solution of (B.26) involving the q_i, the c_i, and the time will be

$$S = c_1q_1 + c_2q_2 + c_3q_3 + Bt + \sum s \sin N. \tag{B.30}$$

The remaining integrals are then given by

$$\ell_i = \frac{\partial S}{\partial c_i} = q_i + t\frac{\partial B}{\partial c_i} + \sum \frac{\partial s}{\partial c_i} \sin N.$$

The three expressions $\ell_i - t\frac{\partial B}{\partial c_i}$ are the non-periodic parts of the variables q_1, q_2, q_3, and therefore

$$-\frac{\partial B}{\partial c_1}, \quad -\frac{\partial B}{\partial c_2}, \quad -\frac{\partial B}{\partial c_3}$$

are the mean motions of the Moon, its perigee, and node respectively. Delaunay used the symbol B with the same meaning as here; it is what remains of his disturbing function after all the periodic terms have been eliminated. It follows, Brown pointed out, that the constants $c_1, c_1 + c_2, c_1 + c_2 + c_3, B$ are the same as Delaunay's L, G, H, R, when these symbols come to have their final values after completion of operations.

Brown's next order of business was to derive the connection between the mean motions of the Moon, its perigee, and node, on the one hand, and the constant term in the lunar parallax on the other. This required demonstrating that

$$b_1c_1 + b_2c_2 + b_3c_3 + B = \frac{3}{2}\left(\frac{\kappa}{r}\right)_0, \tag{B.31}$$

where

$$\kappa = E + M, \quad \text{and} \quad b_i = -\frac{\partial B}{\partial c_i}, \quad (i = 1, 2, 3).$$

The subscript 0 on the right-hand side of (B.31) signifies that only the non-periodic part of the quantity within parentheses is to be considered.

Since S is a function of q_1, q_2, q_3, t, by (B.26)

$$\frac{dS}{dt} = \sum_i \dot{q}_i \frac{\partial S}{\partial q_i} + \frac{\partial S}{\partial t}$$

$$= \sum_i p_i\dot{q}_i - R.$$

Hence

$$\left(\frac{dS}{dt}\right)_0 = \left(\sum_i p_i \dot{q}_i\right)_0 - (R)_0.$$

The left-hand member of this equation is the coefficient of t in the non-periodic part of S. From (B.30) it is evident that the non-periodic part of S must come from the terms $c_1q_1 + c_2q_2 + c_3q_3 + Bt$. The non-periodic part of q_i is $\ell_i + b_i t$, and therefore the non-periodic part of the coefficient of t in S is

$$b_1c_1 + b_2c_2 + b_3c_3 + B.$$

Hence

$$\left(\sum_i p_i \dot{q}_i\right)_0 = b_1c_1 + b_2c_2 + b_3c_3 + B + (R)_0. \tag{B.32}$$

In the second phase of his derivation, Brown introduced rectangular coordinates, writing the equations of motion as follows:

$$\ddot{x} = \frac{\partial}{\partial x}\left(\frac{\kappa}{r} + \Omega\right), \quad \ddot{y} = \frac{\partial}{\partial y}\left(\frac{\kappa}{r} + \Omega\right), \quad \ddot{z} = \frac{\partial}{\partial z}\left(\frac{\kappa}{r} + \Omega\right). \tag{B.33}$$

Both terms within the parentheses contain the variables x, y, z explicitly. The first term does so because $r = (x^2 + y^2 + z^2)^{1/2}$. For Ω, Brown used an approximate expression which neglects the ratio a/a':

$$\Omega = \frac{m'}{r'^3}\left\{\frac{3}{2}\frac{(xx' + yy' + zz')^2}{r'^2} - \frac{1}{2}r^2\right\}.$$

Since in the constants to be calculated the ratio a/a' occurs only as $(a/a')^2$, the error committed is slight, a/a' being approximately 1/390.

Brown multiplied the three equations of (B.33) by x, y, and z respectively, then added the products. Among the terms resulting on the right, the three containing Ω yield, by Euler's homogeneous function theorem,

$$x\frac{\partial \Omega}{\partial x} + y\frac{\partial \Omega}{\partial y} + z\frac{\partial \Omega}{\partial z} = 2\Omega.$$

The other three, after some reductions, give $-(\kappa/r)$, so that the sum is

$$x\ddot{x} + y\ddot{y} + z\ddot{z} = -\frac{\kappa}{r} + 2\Omega.$$

or

$$\frac{1}{2}\frac{d^2}{dt^2}(r^2) - 2T = -\frac{\kappa}{r} + 2\Omega, \tag{B.34}$$

where $T\left(= \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)\right)$ is the kinetic energy. The first term on the left of (B.34) represents only periodic terms. Taking the non-periodic parts of (B.34), therefore, we have

$$(T)_0 = \frac{1}{2}\left(\frac{\kappa}{r}\right)_0 - (\Omega)_0. \tag{B.35}$$

In elliptic motion the kinetic energy is

$$T = \frac{\kappa}{r} - \frac{\kappa^2}{2p_1^2},$$

where the second term on the right is equal to $-\kappa/2a$; this term is also found in the force function for elliptic motion, namely

$$R = -\Omega - \frac{\kappa^2}{2p_1^2} \tag{B.36}$$

The last two equations together yield

$$T = \frac{\kappa}{r} + \Omega + R,$$

so that

$$(T)_0 = \left(\frac{\kappa}{r}\right)_0 + (\Omega)_0 + (R)_0. \tag{B.37}$$

Eliminating $(T)_0$ between (B.35) and (B.37), Brown obtained

$$(R)_0 + 2(\Omega)_0 = -\frac{1}{2}\left(\frac{\kappa}{r}\right)_0. \tag{B.38}$$

The final step in the derivation involved using (B.36) and (B.25) to evaluate the left-hand member of (B.32):

$$\begin{aligned}
\sum_i p_i\dot{q}_i &= \sum_i p_i \frac{\partial R}{\partial p_i} \\
&= \sum_i p_i \frac{\partial}{\partial p_i}\left(-\Omega - \frac{\kappa^2}{2p_1^2}\right) \\
&= -\left(p_1\frac{\partial\Omega}{\partial p_1} + p_2\frac{\partial\Omega}{\partial p_2} + p_3\frac{\partial\Omega}{\partial p_3}\right) - p_1\frac{\partial}{\partial p_1}\left(\frac{\kappa^2}{2p_1^2}\right).
\end{aligned} \tag{B.39}$$

Since Ω is a function of the squares or products of two dimensions in the variables x, y, z, it has the dimension [length]^2 in these variables. When x, y, z are expressed in terms of the variables p_i, q_i, only the p's figure in the coefficients of the resulting terms. As shown by their definitions, the p_i's have the dimensions [length]$^{1/2}$ [mass]$^{1/2}$. Therefore Ω considered as a function of the p's must be of the order $[p]^4$. By Euler's homogeneous function theorem it then follows that the first parenthesis on the right-hand side of (B.39) is equal to -4Ω. The remaining term reduces to $+\kappa^2/p_1^2$; by (B.36) this is equal to $-2R - 2\Omega$. (B.39) thus becomes

$$\sum_i p_i\dot{q}_i = -6\Omega - 2R.$$

Taking the non-periodic parts of this equation, and combining them with three times (B.38) so as to eliminate $(\Omega)_0$, Brown obtained

$$\left(\sum_i p_i \dot{q}_i\right)_0 = (R)_0 + \frac{3}{2}\left(\frac{\kappa}{r}\right)_0.$$

The right-hand side, substituted into (B.32), yields the result sought:

$$b_1 c_1 + b_2 c_2 + b_3 c_3 + B = \frac{3}{2}\left(\frac{\kappa}{r}\right)_0. \tag{B.31}$$

(B.31) can be used to prove Adams's theorems. Recall that the constants

$$b_i = -\frac{\partial B}{\partial c_i}, \quad (i = 1, 2, 3)$$

are n, π_1, θ_1, the mean motions, respectively, of the Moon, its perigee, and its node. We are to differentiate (B.31) partially with respect to n, e^2, and γ^2 in succession. Note also that

$$\begin{aligned}\frac{\partial B}{\partial n} &= \frac{\partial B}{\partial c_1}\frac{\partial c_1}{\partial n} + \frac{\partial B}{\partial c_2}\frac{\partial c_2}{\partial n} + \frac{\partial B}{\partial c_3}\frac{\partial c_3}{\partial n}\\ &= -n\frac{\partial c_1}{\partial n} - \pi_1\frac{\partial c_2}{\partial n} - \theta_1\frac{\partial c_3}{\partial n},\end{aligned}$$

with analogous equations for $\partial B/\partial \mathbf{e}^2$ and $\partial B/\partial\gamma^2$. These equivalences, once we have carried out the partial differentiations of (B.31), enable us to cancel four terms from each of the resulting equations. We thus obtain

$$\begin{aligned}c_1 + c_2\frac{\partial\pi_1}{\partial n} + c_3\frac{\partial\theta_1}{\partial n} &= \frac{3}{2}\frac{\partial}{\partial n}\left(\frac{\kappa}{r}\right)_0,\\ c_2\frac{\partial\pi_1}{\partial e^2} + c_3\frac{\partial\theta_1}{\partial e^2} &= \frac{3}{2}\frac{\partial}{\partial e^2}\left(\frac{\kappa}{r}\right)_0,\\ c_2\frac{\partial\pi_1}{\partial\gamma^2} + c_3\frac{\partial\theta_1}{\partial\gamma^2} &= \frac{3}{2}\frac{\partial}{\partial\gamma^2}\left(\frac{\kappa}{r}\right)_0.\end{aligned} \tag{B.40}$$

The partial derivatives $\partial n/\partial\mathbf{e}^2$ and $\partial n/\partial\gamma^2$ are both zero, since n is not a function of $\mathbf{e}^2$ or γ^2. Each term of c_2 contains the factor $\mathbf{e}^2$, and each term of c_3 the factor γ^2.

Suppose a to be defined by $\kappa = n^2a^3$. Brown, following Adams, expressed the dependence of $(\kappa/r)_0$, π_1, and θ_1 on $\mathbf{e}^2$ and γ^2 by series:

$$\begin{aligned}\left(\frac{\kappa}{r}\right)_0 &= n^2a^2\left(\frac{a}{r}\right)_0 = n^2a^2(A + Be^2 + C\gamma^2 + Ee^4 + 2Fe^2\gamma^2 + G\gamma^4 + \cdots),\\ \pi_1 &= n(P + He^2 + K\gamma^2 + \cdots),\\ \theta_1 &= n(T + Me^2 + N\gamma^2 + \cdots),\end{aligned}$$

where $A, B, \ldots P, H, \ldots T, M, \ldots$ are functions of $\mathbf{m}$, e' only. If we substitute these expressions into the second and third equations of (B.40), carry out the indicated differentiations, and divide all terms by n, we obtain

$$c_2(H + \cdots) + c_3(M + \cdots) = \frac{3}{2}na^2(B + 2Ee^2 + 2F\gamma^2 + \cdots),$$
$$c_2(K + \cdots) + c_3(N + \cdots) = \frac{3}{2}na^2(C + 2Fe^2 + 2G\gamma^2 + \cdots).$$

Since these equations are identities, we can equate the coefficients of like powers of $\mathbf{e}^2$ and γ^2 to zero. Because c_2 contains the factor $\mathbf{e}^2$ and c_3 the factor γ^2, the terms on the right containing B and C cannot be equated to any terms on the left, and so must be zero: $B = 0$, $C = 0$. In the first equation, on the other hand, the term on the right containing $2E\mathbf{e}^2$ can be equated to c_2H, and the term containing $2F\gamma^2$ to c_3M:

$$c_2H = 3na^2Ee^2,$$
$$c_3M = 3na^2F\gamma^2.$$

In the second equation, similarly, the term containing $2F\mathbf{e}^2$ can be equated to c_2K, and the term containing $2G\gamma^2$ to c_3N:

$$c_2K = 3na^2Fe^2,$$
$$c_3N = 3na^2G\gamma^2.$$

It follows that $H/K = E/F$, and $M/N = F/G$, which constitute Adams's second theorem.[97]

In the final section of the paper, Brown shows how the secular accelerations in the lunar theory can be obtained from (B.40) together with a theorem due to Newcomb.[98]

Newcomb's theorem, expressed in terms of Brown's constants, is

$$\delta c_1 = 0, \quad \delta c_2 = 0, \quad \delta c_3 = 0. \tag{B.41}$$

Here $c_1,\ c_2,\ c_3$ are understood to have been expressed in terms of $n,\ \mathbf{e}^2,\ \gamma^2,\ n',\ e'^2,\ \kappa$.

When the solar eccentricity e' undergoes a variation $\delta e'$ due to planetary perturbation of the Earth's orbit, the constants n, $\mathbf{e}$, and γ undergo variations δn, $\delta\mathbf{e}$, $\delta\gamma$, which can be determined from (B.40) and (B.41). The secular accelerations of the Moon's mean motion and apsidal and nodal precessions can then be obtained from

$$\int \delta n dt, \quad \int \delta\pi_1 dt, \quad \int \delta\theta_1 dt.$$

[97] Brown's results in equations (16), on p. 152 of his article, contain an erroneous factor 1/2.

[98] *Astronomical Papers prepared for the Use of the American Ephemeris and Nautical Almanac*, 5 (1895), 191.

Applying the variation δ to (B.40), with the simplifications that (B.41) permits, yields

$$c_2\delta\frac{\partial\pi_1}{\partial n}+c_3\delta\frac{\partial\theta_1}{\partial n}=\frac{3}{2}\delta\frac{\partial}{\partial n}\left(\frac{\kappa}{r}\right)_0,$$
$$c_2\delta\frac{\partial\pi_1}{\partial e^2}+c_3\delta\frac{\partial\theta_1}{\partial e^2}=\frac{3}{2}\delta\frac{\partial}{\partial e^2}\left(\frac{\kappa}{r}\right)_0,$$
$$c_2\delta\frac{\partial\pi_1}{\partial \gamma^2}+c_3\delta\frac{\partial\theta_1}{\partial \gamma^2}=\frac{3}{2}\delta\frac{\partial}{\partial \gamma^2}\left(\frac{\kappa}{r}\right)_0. \tag{B.42}$$

Brown began by neglecting, in the first equation of (B.42), all powers of $\mathbf{e}^2$ and of γ^2. Since $\mathbf{e}^2$ occurs as a factor in c_2 and γ^2 as a factor in c_3, the two terms on the left were thus eliminated. What remained was

$$\delta\frac{\partial}{\partial n}\left(\frac{\kappa}{r}\right)_0=0.$$

The variation operator δ introduces variations in e'^2 and n:

$$\frac{\partial^2}{\partial n\partial n}\left(\frac{\kappa}{r}\right)_0\delta n+\frac{\partial^2}{\partial n\partial e'^2}\left(\frac{\kappa}{r}\right)_0\delta e'^2=0,$$

$$\text{whence}\quad \delta n=-\frac{\left\{\frac{\partial^2}{\partial n\partial e'^2}\left(\frac{\kappa}{r}\right)_0\right\}\delta e'^2}{\left\{\frac{\partial^2}{\partial n\partial n}\left(\frac{\kappa}{r}\right)_0\right\}}.$$

The part of δn expressed here is its *principal* part, that is, the part arising from $\delta e'^2$ alone, the variations in the lunar eccentricity and inclination being left out of account. To compute the principal part of δn from the foregoing equation, it is necessary to know how $(\kappa/r)_0$ depends on n and e'^2, and to have a value for $\delta e'^2$. Since the variation $\delta e'^2$ is negative in the present age (e' is decreasing), the change in n is positive, constituting a secular acceleration.

With a value of δn in hand, the principal parts of $\delta\pi_1$ and $\delta\theta_1$ can be obtained from

$$\delta\pi_1=\frac{\partial\pi_1}{\partial n}\delta n+\frac{\partial\pi_1}{\partial e'^2}\delta e'^2,$$
$$\delta\theta_1=\frac{\partial\theta_1}{\partial n}\delta n+\frac{\partial\theta_1}{\partial e'^2}\delta e'^2.$$

Turning next to the terms in δn containing the factors $\mathbf{e}^2$ and γ^2, Brown designated them by $(\delta n)_{e^2}$, $(\delta n)_{r^2}$. He proposed that these new terms be computed from the following equations, in which the previously found principal part of δn is symbolized by $(\delta n)_0$.

$$c_2\left\{\frac{\partial^2\pi_1}{\partial n\partial n}(\delta n)_0+\frac{\partial^2\pi_1}{\partial n\partial e'^2}\delta e'^2\right\}=\frac{3}{2}(\delta n)_{e^2}\frac{\partial^2}{\partial n\partial n}\left(\frac{\kappa}{r}\right)_0,$$

$$c_3\left\{\frac{\partial^2\theta_1}{\partial n\partial n}(\delta n)_0+\frac{\partial^2\theta_1}{\partial n\partial e'^2}\delta e'^2\right\}=\frac{3}{2}(\delta n)_{\gamma^2}\frac{\partial^2}{\partial n\partial n}\left(\frac{\kappa}{r}\right)_0.$$

To compute the parts of $\delta\pi_1$ and $\delta\theta_1$ containing the factors $\mathbf{e}^2$ and γ^2, he proposed finding the variations $\delta\mathbf{e}^2$ and $\delta\gamma^2$ from the equations

$$\delta c_2=0,\quad \delta c_3=0,$$

then substituting them into the equations

$$\delta\pi_1=\frac{\partial\pi_1}{\partial e^2}\delta e^2+\frac{\partial\pi_1}{\partial\gamma^2}\delta\gamma^2+\frac{\partial\pi_1}{\partial n}(\delta n)_0+\frac{\partial\pi_1}{\partial e'^2}\delta e'^2,$$

$$\delta\theta_1=\frac{\partial\theta_1}{\partial e^2}\delta e^2+\frac{\partial\theta_1}{\partial\gamma^2}\delta\gamma^2+\frac{\partial\theta_1}{\partial n}(\delta n)_0+\frac{\partial\theta_1}{\partial e'^2}\delta e'^2.$$

In an article in the *Monthly Notices* for March, 1897, Brown compared his own theoretical results for the Moon's mean motion and apsidal and nodal precessions with the corresponding results of Hansen and with the observational values.[99] This comparison was provisional in that he had carried the development of the theory, as yet, only to the second-order terms in the eccentricities and inclination. Where higher-order terms were significant he had to use estimates of them, inevitably doubtful, from Delaunay's theory.

For the annual precession of the node the numbers were

Brown's theoretical value:	−69 679″.5
Hansen's theoretical value:	−69 677″.3
The observational value:	−69 679″.5

The difference 2″.2 in Hansen's result, Brown believed, was at least partly due to some error.

For the annual precession of the perihelion, the numbers were

Brown's theoretical value:	+146 435″.3
Hansen's theoretical value:	+146 434″.9
The observational value:	+146 435″.6

Here Brown had to acknowledge an uncertainty in his theoretical value of about 2″. It could be removed only by the further development of the theory.

99 *MNRAS*, 57 (1897), 332–349.

The question of the accuracy of this theoretical value was, at just this time, of special interest. In 1859 Le Verrier had found that some 38 arc-seconds per century in the precession of Mercury's perihelion could not be accounted for on the basis of Newton's law of gravitation, and in 1882 Newcomb had revised this estimate upward to 43 arc-seconds per century.[100] In 1894 Asaph Hall proposed altering the law of gravitation so as to accommodate this discrepancy: it was only necessary to change the exponent of the distance from -2 to $-2.000\,000\,16$.[101] Newcomb in his book of 1895 on the fundamental constants of astronomy refined Hall's value of the exponent to $-2.000\,000\,1574$, using a more precise value for the anomaly in Mercury's precession[102]; he also computed the effects that this formula would produce in the precessions of the perihelia of Venus, the Earth, and Mars. In planets other than Mercury, however, the additional precession was too small and the observational evidence too imprecise to provide a clear confirmation. He called for an independent test of the hypothesis in the case of the Moon:

> An independent test of this hypothesis in the case of other bodies is very desirable. The only case in which there is any hope of determining such an excess is that of the Moon, where the excess would amount to about $140''$ per century. This is very nearly the hundred-thousandth part of the total motion of the perigee.

Brown accepted the challenge:

> The question awaits the determination of the higher terms due to the Sun's action – a determination which I hope to make in the course of a year or two.[103]

[100] U.J.J. Le Verrier, "Théorie du mouvement de Mercure," *Annales de l' Observatoire Impériale de Paris, V* (1859), 98–106; S. Newcomb, "Discussion and results of observations on transits of Mercury from 1677 to 1881," *Astronomical Papers prepared for the use of the American Ephemeris and Nautical Almanac, I* (1882), 473.

[101] A. Hall, "A suggestion in the theory of Mercury," *The astronomical journal, xiv* (1894), 49–51.

[102] S. Newcomb, *The Elements of the Four Inner Planets and the Fundamental Constants of Astronomy*, (Washington DC, Government Printing Office, 1895), 118–120.

[103] *MNRAS*, 57 (1897), 332–333.

10

Brown's Lunar Treatise: *Theory of the Motion of the Moon; Containing a New Calculation of the Expressions for the Coordinates of the Moon in Terms of the Time*

This treatise, published in Volumes 53, 54, 57, and 59 of the *Memoirs* of the Royal Astronomical Society (1897–1908), embodies some 18 years of calculative labor, from 1890 to 1907, on the development of the lunar theory. Initially Brown worked alone, but after 1895 he was assisted by Ira I. Sterner, A.B., of Haverford College, as a computer. The treatise incorporated the results of Brown's earlier papers, and carried to completion the task he had set for himself. This consisted of two sub-tasks. In the first and more extensive of these, he developed the theory under idealizing restrictions: the assumption of a strictly elliptical orbit for the Sun relative to the center of gravity of the Earth and Moon, and the representation of the bodies of the Moon, Earth, and Sun as point-masses – equivalent to assuming the mass-distribution within each of them to be spherically symmetrical. The second sub-task consisted in correcting for these idealizations. This required, principally, the introduction of planetary perturbations of the Sun's (or Earth's) and Moon's orbits, and of non-spherical shapes for the Moon, Earth, and Sun. The corrections were to be carried to a degree of approximation that would permit predictions accurate to 0.01 arc-seconds.

In this calculation, efficiency as well as accuracy was crucial, and Brown experimented extensively in seeking appropriate routes. He sought to regularize the calculations, reducing them insofar as possible to simple, exactly specifiable processes. Thus they could be assigned to a (human) computer, who during some seven and a half years would be Mr. Sterner. Brown also sought out independent paths for checking the accuracy of these calculations – paths different from mere repetition. The task as a whole was demanding in its intricacy and enormous in its scope; Brown made it manageable. On completing the first of the two sub-tasks in December, 1904, he reported that

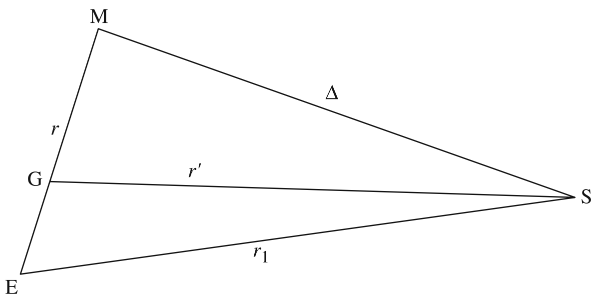
> [Mr. Sterner] has in all spent some three thousand hours on these calculations, extended over seven and a half years; my own share I estimate at five or six thousand hours since the work was begun on a complete plan in 1895.[104]

[104] E. W. Brown, *TMM*, Part IV, in *Memoirs of the R.A.S.*, Vol.57 (1905), p.53.

C. Wilson, *The Hill–Brown Theory of the Moon's Motion*, Sources and Studies in the History of Mathematics and Physical Sciences, DOI 10.1007/978-1-4419-5937-9_10,

For Sterner's knowledge of computation, and his speed and accuracy, Brown had high praise. The expense of employing Sterner was met in part by grants from the Royal Society of London.

In Chapter I of his treatise, Brown described the general features of the theory, and also the particular processes by which he proposed to compute the terms of the theory in their successive orders. In the later chapters dealing with orders higher than the second, he would find it expedient to introduce changes in the procedures described in Chapter I.



The Problem of Three Bodies and the Disturbing Function

What we have called the first sub-task is the construction of an approximate solution of the three-body problem. Apart from its not taking account of planetary perturbations and non-spherical distributions of mass, this solution deviated from strict truth in employing a number of approximations. One of these was caused by the selection of a strictly elliptical orbit for the center of mass of the Earth and Moon about the Sun. What correction, Brown asked, is required to compensate for it?

Let the masses of the Moon, Earth, and Sun be M, E, and m'. In plotting the motions of these masses, Brown used three coordinate systems (see the diagram above).[105]

(1) The first system, used for plotting the Moon's motion, had its origin at the Earth's center of mass. The Moon's distance from the Earth's center was r, and its coordinates in this frame were X, Y, z, where X, Y were non-rotating. Since the Earth was in accelerated motion, this frame was not inertial.

(2) The second system was for plotting the Sun's relative motion, and had its origin in the center of mass G of the Earth and the Moon; its coordinate axes were parallel to those of the first coordinate frame. The Sun's distance from G was

105 As Brown acknowledges, the analysis that follows is very close to that given by S. Newcomb, "Theory of the Inequalities in the Motion of the Moon produced by the Action of the Planets," *Astronomical Papers prepared for the use of the American Ephemeris and Nautical Almanac, V* (1895), 105ff.

r', and its coordinates were x', y', z'. Since this system, too, was in accelerated motion, it was not inertial.

(3) The third system had its origin at the center of mass of M, E, and m', on the line connecting G with m'. The coordinate axes in this frame were parallel to those in the earlier two frames. In a universe consisting solely of the three bodies M, E, and m' – the universe assumed in the first sub-task – this third frame would be inertial.

Brown now asked: how much does the Moon's motion cause the Sun to deviate from the assumed elliptic orbit in the X-Y plane? To answer this question he derived the Lagrangian equations of motion for the Sun in the third frame of reference. Let r_1' be the Sun's distance from the Earth and $\triangle$ its distance from the Moon; then the potential energy F of the system is

$$F = \frac{EM}{r} + \frac{Em'}{r_1'} + \frac{Em'}{\Delta}, \tag{B.43}$$

$$\text{where} \quad r_1'^2 = r'^2 + 2\frac{M}{E+M}r'rS + \frac{M^2}{(E+M)^2}r^2,$$

$$\text{and} \quad \Delta^2 = r'^2 - 2\frac{E}{E+M}r'rS + \frac{E^2}{(E+M)^2}r^2.$$

Here $r_1'^2$ and $\triangle^2$ are given by means of the cosine law, S being the cosine of the angle subtended at G by $\triangle$.

The kinetic energy T, if expressed in terms of the coordinates of the Moon in the first frame of reference and the coordinates of the Sun in the second frame of reference, is given by

$$2T = \mu_1(\dot{X}^2 + \dot{Y}^2 + \dot{z}^2) + \mu_2(\dot{x}'^2 + \dot{y}'^2 + \dot{z}'^2), \tag{B.44}$$

$$\text{where} \quad \mu_1 = \frac{EM}{E+M} \quad \text{and} \quad \mu_2 = \frac{m'(E+M)}{m'+E+M}.$$

By the Lagrangian algorithm, the equations of motion for the Sun are then found to be

$$\mu_2\frac{d^2x'}{dt^2} = \frac{\partial F}{\partial x'}, \quad \mu_2\frac{d^2y'}{dt^2} = \frac{\partial F}{\partial y'}, \quad \mu_2\frac{d^2z'}{dt^2} = \frac{\partial F}{\partial z'}. \tag{B.45}$$

For computing $\partial F/\partial x'$, $\partial F/\partial y'$, $\partial F/\partial z'$, the relevant part of F is given by

$$\frac{F}{\mu_2} = \frac{m'+E+M}{E+M}\left(\frac{E}{r_1'} + \frac{M}{\Delta}\right).$$

Here $1/r_1'$ and $1/\triangle$ can be expanded in powers of r/r', to yield

$$\frac{F}{\mu_2} = (m'+E+M)\left[\frac{1}{r'} + \frac{EM}{(E+M)^2}\frac{r^2}{r'^2}\left(\frac{3}{2}S^2 - \frac{1}{2}\right) + \cdots\right]. \tag{B.46}$$

The first term within the square brackets yields the Sun's elliptical orbit; the second and further terms represent perturbations of the ellipse. The ratio of the second term to the first is, approximately,

$$Mr^2 : Er'^2 \cong 1 : 12{,}000{,}000.$$

Here Brown took $M{:}E$ to be about 1:81, and $r{:}r'$ to be about 1:390. The third term within the square brackets (not shown) was about $1/400^{\text{th}}$ smaller still, and Brown proposed to neglect it. Thus, he claimed, a sufficient correction to the elliptic motion of the Sun about G could be obtained by using the second term of (B.46),

$$(m' + E + M)\frac{EM}{(E + M)^2}\frac{r^2}{r'^3}\left(\frac{3}{2}S^2 - \frac{1}{2}\right), \tag{B.47}$$

as a disturbing function, and substituting for the Moon's coordinates their elliptic values modified by the principal inequalities due to the Sun.

Further inaccuracies in Brown's procedure for the first sub-task arose from the disturbing function he used for the Moon's motion. A strictly correct disturbing function in this case would be

$$\begin{aligned}\frac{F}{\mu_1} &= \frac{E + M}{r} + \frac{m'(E + M)}{EM}\left(\frac{E}{r_1'} + \frac{M}{\Delta}\right)\\ &= \frac{E + M}{r} + \frac{m'r^2}{r'^2}\left[\begin{array}{l}\left(\frac{3}{2}s^2 - \frac{1}{2}\right) + \frac{E-M}{E+M}\frac{r}{r'}\left(\frac{5}{2}s^3 - \frac{3}{2}s\right)\\ \quad + \frac{E^2-EM+M^2}{(E+M)^2}\frac{r^2}{r'^2}\left(\frac{35}{8}s^4 - \frac{15}{4}s^2 + \frac{3}{8}\right)\\ \quad + \frac{E^3-E^2M+EM^2-M^3}{(E+M)^3}\frac{r^3}{r'^3}\left(\frac{63}{8}s^5 - \frac{35}{4}s^3 + \frac{15}{8}s\right) + \cdots\end{array}\right].\end{aligned} \tag{B.48}$$

Here the several quotients involving E and M within the square brackets posed a difficulty, since at this stage the relative values of E and M had not yet been precisely evaluated. Brown employed, instead, a disturbing function from which these quotients have been removed:

$$\begin{aligned}\Omega &= \frac{E + M}{r} + m'\left[\frac{1}{\sqrt{r'^2 - 2rr'S + r^2}} - \frac{1}{r'} - \frac{rS}{r'^2}\right]\\ &= \frac{E + M}{r} + n'^2a'^3\frac{r^2}{r'^3}\left[\begin{array}{l}\left(\frac{3}{2}S^2 - \frac{1}{2}\right) + \frac{r}{r'}\left(\frac{5}{2}S^3 - \frac{3}{2}S\right)\\ \quad + \frac{r^2}{r'^2}\left(\frac{35}{8}S^4 - \frac{15}{4}S^2 + \frac{3}{8}\right)\\ \quad + \frac{r^3}{r'^3}\left(\frac{63}{8}S^5 - \frac{35}{4}S^3 + \frac{15}{8}S\right) + \cdots\end{array}\right].\end{aligned} \tag{B.49}$$

The expansion within the square brackets, Brown believed, had been carried as far as necessary.

To compensate for the inaccuracies introduced by using (B.49) in place of (B.48), Brown introduced the following corrections:

(a) Comparing (B.49) with (B.48), we see that m' has been replaced by $n'^2a'^3$, whereas in fact $n'^2a'^3 = m' + E + M$. A sufficient correction for this inaccuracy, Brown tells us, can be obtained by multiplying all the lunar inequalities due to the Sun by

$$1 - \frac{E+M}{m'} \cong 1 - \frac{1}{330000}.$$

For example, since the largest solar inequality in the Moon's motion is the Variation, with a coefficient of some 39.5 arc-minutes, the correction in this case will amount to subtracting $39.5 \times 60/330000 = 0.0072$ arc-seconds. Thousandths of arc-seconds have to be taken into account in the calculations, if the final result is to be accurate to 0.01 arc-seconds.

(b) Correction for use of the elliptic values instead of the true values of the Sun's coordinates in (B.49) is necessary only in the largest term within the square brackets. The correction consists in adding to Ω the term

$$n'^2a'^3\delta\left[\frac{r^2}{r'^3}\left(\frac{3}{2}S^2 - \frac{1}{2}\right)\right],$$

where δ operates on x', y', z', and $\delta x'$, $\delta y'$, $\delta z'$ are obtained by using (B.47) as a disturbing function as previously described.

(c) Terms within the square brackets of (B.49) having the factor $(r/r')^j$ lead to inequalities having the factor $(\mathrm{a}/a')^j$, where a is the constant of the Moon's distance. These same terms are those in (B.48) which were multiplied by one of the quotients involving E and M. A partial correction for the omission of these quotients will be obtained if the resulting inequalities are multiplied by

$$\left(\frac{E-M}{E+M}\right)^j = \left(1 - 2\frac{M/E}{1+M/E}\right)^j,$$

where the fraction M/E is approximately 1/81. To the order of precision that Brown is aiming to achieve, it is then necessary to correct further the third term within the square brackets by adding to Ω the term

$$n'^2a'^3\frac{r^2}{r'^3}\left[\frac{EM}{(E+M)^2}\cdot\frac{r^2}{r'^2}\left(\frac{35}{8}S^4 - \frac{15}{4}S^2 + \frac{3}{8}\right)\right].$$

The Equations of Motion

In this section Brown introduces rotating coordinates for the Moon:

x, y, z are the lunar coordinates, referred to rectangular axes through the Earth's center; x, y are in the plane of the Sun's orbit (supposed constant), with the positive x-axis constantly directed to the Sun's mean place;

$r^2 = x^2 + y^2 + z^2$; $\rho^2 = x^2 + y^2$;

n, n' are the observed mean motions of the Moon and Sun;

r', e', a' are the radius vector, eccentricity, and semi-axis major of the Sun's orbit;

υ is the Sun's equation of center;

$S_1 = x \cos \upsilon + y \sin \upsilon = rS.$

Also, using ι for $\surd(-1)$, Brown stipulates that

$u = x + y\iota, \quad s = x - y\iota, \quad us = \rho^2;$

$\mathrm{m} = n'/(n - n'), \quad \kappa = (E + M)/(n - n')^2;$

$\zeta = \exp \cdot (n - n')(t - t_0)\iota, \quad D = \zeta(d/d\zeta);$

where t_0 is a constant to be evaluated later.

The rotation of the coordinates causes a new term to appear in the equations; Brown follows Hill in including it in the disturbing function:

$$\Omega' = \Omega + \frac{1}{2}n'^2(x^2 + y^2) = \Omega + \frac{1}{2}n'^2 us.$$

The equations of motion thus become

$$D^2u + 2\mathrm{m}Du = -\frac{2}{(n - n')^2}\frac{\partial \Omega'}{\partial s},$$

$$D^2s - 2\mathrm{m}Ds = -\frac{2}{(n - n')^2}\frac{\partial \Omega'}{\partial u},$$

$$D^2z = -\frac{1}{(n - n')^2}\frac{\partial \Omega'}{\partial z},$$

where, by (B.49),

$$\Omega' = \frac{E + M}{(us + z^2)^{1/2}} + n'^2 a'^3 \left[\frac{1}{(r'^2 - 2r'S_1 + us + z^2)^{1/2}} - \frac{1}{r'} - \frac{S_1}{r'^2}\right] + \frac{1}{2}n'^2 us.$$

Expansion of Ω' in powers of $1/r'$ yields

$$\frac{2}{(n - n')^2}\Omega' = \frac{2\kappa}{(us + z^2)^{1/2}} + \frac{3}{4}\mathrm{m}^2(u + s)^2 - \mathrm{m}^2 z^2 + \Omega_1,$$

where

$$\Omega_1 = 3\mathrm{m}^2\left[\frac{a'^3}{r'^3}S_1^2 - \frac{1}{4}(u+s)^2\right] - \mathrm{m}^2(us+z^2)\left(\frac{a'^3}{r'^3} - 1\right)$$
$$+ \frac{\mathrm{m}^2}{a'}\cdot\frac{a'^4}{r'^4}[5S_1^3 - 3S_1(us+z^2)]$$
$$+ \frac{\mathrm{m}^2}{a'^2}\cdot\frac{a'^5}{r'^5}\left[\frac{35}{4}S_1^4 - \frac{15}{2}S_1^2(us+z^2) + \frac{3}{4}(us+z^2)^2\right]$$
$$+ \frac{\mathrm{m}^2}{a'^3}\cdot\frac{a'^6}{r'^6}\left[\frac{63}{4}S_1^5 - \frac{35}{2}S_1^3(us+z^2) + \frac{15}{4}S_1(us+z^2)^2\right]$$
$$+ \cdots\cdots\cdots\cdots\cdots\cdots$$
$$= \omega_2 + \omega_3 + \omega_4 + \omega_5 + \cdots\cdots\cdots\cdots . \tag{B.50}$$

The equations may now be written in the form

$$(D+\mathrm{m})^2u + \frac{1}{2}\mathrm{m}^2u + \frac{3}{2}\mathrm{m}^2s - \frac{\kappa u}{(us+z^2)^{3/2}} = -\frac{\partial\Omega_1}{\partial s},$$
$$(D-\mathrm{m})^2s + \frac{1}{2}\mathrm{m}^2s + \frac{3}{2}\mathrm{m}^2u - \frac{\kappa s}{(us+z^2)^{3/2}} = -\frac{\partial\Omega_1}{\partial u},$$
$$(D^2-\mathrm{m}^2)z - \frac{\kappa z}{(us+z^2)^{3/2}} = -\frac{1}{2}\frac{\partial\Omega_1}{\partial z}. \tag{B.51a,b,c}$$

Either the first and third of these equations, or the second and third, are sufficient for developing the theory; Brown will employ the first and the third in the initial phases of the systematic development. Later, considerations of efficiency will lead him to return to the homogeneous equations that Hill had used. These are derived from (B.51a,b,c) by means of the Jacobian integral, and may be written as follows:

$$D^2(us+z^2) - Du\cdot Ds - (Dz)^2 - 2\mathrm{m}(uDs - sDu) + \frac{9}{4}\mathrm{m}^2(u+s)^2 - 3\mathrm{m}^2z^2$$
$$= C' - \sum_{q=2}^{\infty}(q+1)\omega_q + D^{-1}(D'\Omega_1),$$
$$D(uDs - sDu - 2\mathrm{m}us) + \frac{3}{2}\mathrm{m}^2(u^2-s^2) = s\frac{\partial\Omega_1}{\partial s} - u\frac{\partial\Omega_1}{\partial u},$$
$$D(uDz - zDu) - 2\mathrm{m}zDu - \mathrm{m}^2uz - \frac{3}{2}\mathrm{m}^2z(u+s) = z\frac{\partial\Omega_1}{\partial s} - \frac{1}{2}u\frac{\partial\Omega_1}{\partial z}. \tag{B.52a,b,c}$$

Here C' is the Jacobian constant of integration; D^{-1} is the operation inverse to D, *viz.*, integration with respect to ζ followed by division by ζ; and $D'\Omega_1$ signifies the operation D performed on Ω_1 only insofar as ζ occurs in r', υ.

Development of Ω_1 According to Powers of e' and z

The development of Ω_1 according to powers of $1/a'$ was given in (B.50). Brown now develops it according to the powers of the solar eccentricity and the motion in the coordinate z, both of them small quantities of the first order. He carries this development to quantities of the orders

$$\frac{\mathrm{a}^3}{a'^3}, \quad \frac{\mathrm{a}^2}{a'^2}e', \quad \frac{\mathrm{a}^2}{a'^2}z^2, \quad \frac{\mathrm{a}}{a'}e'^3, \; e'^5.$$

The first step is to introduce an exponential expression for S_1:

$$S_1 = x\cos\upsilon + y\sin\upsilon = \frac{1}{2}(ue^{-\upsilon\sqrt{-1}} + se^{\upsilon\sqrt{-1}}) = \frac{1}{2}(ue^{-\upsilon\iota} + se^{\upsilon\iota}),$$

where e is the base of natural logarithms. Substituting this expression for S_1 into (B.50), Brown finds the successive terms of Ω_1 to be

$$\omega_2 = \mathrm{m}^2\left[\frac{3}{4}(u^2\mathrm{a}_2 + s^2\overline{\mathrm{a}_2}) + \frac{1}{2}us\mathrm{b}_2 - z^2\mathrm{b}_2\right],$$

$$\omega_3 = \frac{\mathrm{m}^2}{a'}\left[\frac{5}{8}(u^3\mathrm{a}_3 + s^3\overline{\mathrm{a}_3}) + \frac{3}{8}(u^2s\mathrm{c}_3 + us^2\overline{\mathrm{c}_3}) - \frac{3}{2}uz^2\mathrm{c}_3 - \frac{3}{2}sz^2\overline{\mathrm{c}_3}\right],$$

$$\omega_4 = \frac{\mathrm{m}^2}{a'^2}\left[\begin{array}{l}\frac{35}{64}(u^4\mathrm{a}_4 + s^4\overline{\mathrm{a}_4}) + \frac{5}{16}(u^3s\mathrm{c}_4 + us^3\overline{\mathrm{c}_4}) + \frac{9}{32}u^2s^2\mathrm{b}_4\\ \quad -z^2\left(\frac{15}{8}u^2\mathrm{c}_4 + \frac{15}{8}s^2\overline{\mathrm{c}_4} + \frac{9}{4}us\mathrm{b}_4\right)\end{array}\right],$$

$$\omega_5 = \frac{\mathrm{m}^2}{a'^3}\left[\frac{63}{128}(u^5 + s^5) + \frac{35}{128}(u^4s + us^4) + \frac{15}{64}(u^3s^2 + u^2s^3)\right].$$

Here, with e again as the base of natural logarithms,

$$\mathrm{a}_2 = \frac{a'^3}{r'^3}e^{-2\upsilon\iota} - 1, \quad \mathrm{a}_3 = \frac{a'^4}{r'^4}e^{-3\upsilon\iota}, \quad \mathrm{a}_4 = \frac{a'^5}{r'^5}e^{-4\upsilon\iota},$$

$$\mathrm{b}_2 = \frac{a'^3}{r'^3} - 1, \quad \mathrm{b}_4 = \frac{a'^5}{r'^5},$$

$$\mathrm{c}_3 = \frac{a'^4}{r'^4}e^{-\upsilon\iota}, \quad \mathrm{c}_4 = \frac{a'^5}{r'^5}e^{-2\upsilon\iota},$$

and $\overline{\mathrm{a}_2}, \overline{\mathrm{b}_2}, \ldots$ are the values of $\mathrm{a}_2, \mathrm{b}_2, \ldots$ when $-i$ is put for i.

The quantities a_2, b_2, etc., are then expanded in powers of e', the solar eccentricity. These elliptic expansions were well-known and had been given by several authors, e.g., Richard Cayley.[106]

Form of the Solution

Three variables dependent on the time are necessary and sufficient to determine the Moon's position. The variables Brown chose were V, the true longitude of the Moon in the X-Y plane, measured from the (fixed) X-axis; r, the radius vector; and ψ, the Moon's latitude above the X-Y plane. Following the practice of earlier lunar theorists, he assumed that r, ψ, and $V - n't - \varepsilon'$ (where $n't + \varepsilon'$ is the Sun's mean longitude at time t) can be expressed as functions of periodic terms whose arguments are algebraic sums of multiples of the following four angles ("D" here is to be distinguished by context from "D" used as a differential operator)

$$D = (n - n')t + \varepsilon - \varepsilon' = \text{Half argument of the "Variation,"}$$
$$\ell = cnt + \varepsilon - \omega = \text{Argument of the Principal Elliptic Term,}$$
$$\ell' = n't + \varepsilon' - \omega' = \text{Argument of the "Annual Equation,"}$$
$$F = gnt + \varepsilon - \theta = \text{Argument of the Principal Term in Latitude.}$$

Here ε, ω are the mean longitudes of the Moon and its perigee when t is zero in the equation defining ℓ; θ is the mean longitude of the Moon's orbital node when t is zero in the equation defining F; ε' and ω' are the mean longitudes of the Sun and its perigee when t is zero in the equation defining ℓ'; and $(1 - c)n$, $(1 - g)n$ are the mean motions of the lunar perigee and node. Brown assumes D, ℓ, ℓ', and F and their multiples are the only angles needed in deriving the Moon's solar perturbations; the proof will be in the theory's success.

The stationary coordinates of the Moon, expressed in terms of V, ρ, r and ψ, are

$$X = \rho \cos V, \quad Y = \rho \sin V, \quad z = \rho \tan \psi = r \sin \psi.$$

The corresponding rotating coordinates are

$$x = \rho \cos(V - n't - \varepsilon') = \rho \cos(V - nt - \varepsilon + D),$$
$$y = \rho \sin(V - n't - \varepsilon') = \rho \sin(V - nt - \varepsilon + D).$$

The complex variables u, s, expressed in exponential form, are:

$$u = \rho \exp \cdot (V - nt - \varepsilon + D)\iota, \quad s = \rho \exp \cdot [-(V - nt - \varepsilon + D)\iota]$$
$$\text{or} \quad u\zeta^{-1} = \rho \exp \cdot (V - nt - \varepsilon)\iota, \quad s\zeta = \rho \exp \cdot [-(V - nt - \varepsilon)\iota].$$

As before, the variable ζ in the last line denotes $\exp \cdot D\iota$, where $\iota = \sqrt{(-1)}$.

[106] *Memoirs of the R.A.S., 29*, and also Cayley, *Collected Works, 3*.

The assumption of sufficiency previously stated is expressed in the equations

$$\left.\begin{matrix} x \\ y \\ z \end{matrix}\right\} = \mathrm{a} \sum A_{i,p,q,r} \left.\begin{matrix} \cos \\ \sin \\ \sin \end{matrix}\right\} (iD + p\ell + r\ell' + qF), \quad i, p, q, r = 0, \pm 1, \pm 2, \ldots.$$

Here "a" refers to a constant of distance in the lunar theory, A is a coefficient, and the summation is to be extended to all terms of the form shown. The corresponding complex variables $u, s, z\iota$ are

$$u, s, z\iota = \mathrm{a} \sum A_{i,p,q,r} \exp \cdot (iD + p\ell + r\ell' + qF)\iota. \tag{B.53}$$

The additive constants $\varepsilon - \varepsilon'$, $\varepsilon - \omega$, $\varepsilon' - \omega'$, $\varepsilon - \theta$, contained in D, ℓ, ℓ', and F are as yet undetermined. The previously given definitions are equivalent to

$$D = (n - n')(t - t_0),$$

$$\ell = c(n - n')(t - t_1),$$

$$\ell' = m(n - n')(t - t_3),$$

$$F = g(n - n')(t - t_2),$$

where t_0, t_1, t_3, t_2 are constants, different from one another and as yet undetermined. The constants c and g are related to c and g by $\mathrm{c} = c\mathrm{m}$ and $\mathrm{g} = g\mathrm{m}$. The replacing of c by c and g by g enables Brown to express the angles ℓ and F as powers of ζ.

The constants $A_{i,p,q,r}$ in (B.53) can be determined by the method of undetermined coefficients. One of the equations (B.53) is substituted into (B.51) or (B.52), and the sum of all terms with identical arguments is then set equal to zero. In making these substitutions, it is necessary to apply the operator D to the variables u, s, and $z\iota$. Suppose, for instance, that we are to form the expression Du in the particular case where

$$\begin{aligned} u &= \mathrm{a}A_{i,p,0,0} \exp \cdot [(iD + p\ell)\iota] \\ &= \mathrm{a}A_{i,p,0,0} \exp \cdot \{[i(n - n')(t - t_0) + pc(n - n')(t - t_1)]\iota\}. \end{aligned}$$

Remembering that $D = -\frac{\iota}{n-n'} \frac{d}{dt}$, we find that

$$Du = \mathrm{a}A_{i,p,0,0}(i + pc) \exp \cdot [(iD + p\ell)\iota].$$

More generally, let

$$\zeta_{\mathrm{c}}^{\mathrm{c}} = \exp \cdot [\mathrm{c}(n - n')(t - t_1)\iota].$$

Then

$$D^j (\zeta^i \zeta_{\mathrm{c}}^{p\mathrm{c}}) = (i + p\mathrm{c})^j \zeta^i \zeta_{\mathrm{c}}^{p\mathrm{c}},$$

where i, j, p are positive or negative integers. Despite the difference between t_0 and t_1, the result is the same as if $\zeta_{\mathrm{c}}^{\mathrm{c}}$ were equal to ζ^{c}. The differentiation of $\zeta^i \zeta_{\mathrm{c}}^{p\mathrm{c}}$

may thus be carried out as if it were the differentiation of $\zeta^{i+p\mathrm{c}}$. Only when the Moon's coordinates are to be calculated numerically, need the numerical values of the additive constants (t_0, t_1, t_3, t_2) be introduced.

After the substitutions have been made in all terms, the exponential factor $\exp(iD + p\ell)\iota$ can be divided out. What remains will give $A_{i,p,0,0}$ in terms of c, i, p, and other constants.

The general term in $u\zeta^{-1}$ or $z\iota$ can be expressed by

$$\mathrm{a}(\varepsilon^{p+p'}\varepsilon'^{p'}\eta^{r+r'}\eta'^{r'}k^{q+q'}k'^{q'}\alpha^{s'})_i \; e^{p+2p'} \; e'^{r+2r'} \; k^{q+2q'} \; a^{s'} \; \zeta^{2i\pm p\mathrm{c}\pm r\mathrm{m}\pm q\mathrm{g}}. \quad \text{(B.54)}$$

To explain the several factors, we begin at the right-hand side: $\zeta^{2i\pm p\mathrm{c}\pm r\mathrm{m}\pm q\mathrm{g}}$ is a sine or cosine, with argument given by the exponent. The letters p, r, q are positive integers or zero; $2i$ is a positive or negative integer or zero. (When Brown appends a subscript "1" to i, it signifies that $2i_1$ is restricted to the odd values ± 1, ± 3, ± 5, etc.; this restriction applies whenever the parameter a is raised to an odd power.) When the lower sign of pc is taken, the equations of motion require that the superscripts (not exponents!) of ε and ε' be interchanged; similarly, when the lower signs of $r\mathbf{m}$ and qg are taken, the superscripts of η and η', and those of k and k' must be interchanged. (The origin of these quantities will be explained shortly.) The integer r as used here is an angle-multiplier, distinguishable by context from r used to denote the Moon's radius vector.

The lunar theory is to be expanded in powers of the parameters $\mathbf{e}$, e', k, $a = \mathbf{a}/a'$. Suppose, for instance, that pc is present as a term in the exponent of ζ; this means that, in the argument of the periodic term, the angle $\mathrm{c}(n - n)(t - t_1)$ is multiplied by the positive integer p. At the same time the parameter $\mathbf{e}$ will be raised to the power $p + 2p'$, where p' takes the values $0, 1, 2$, etc., successively. Thus the coefficient of a periodic term with the argument $p\mathrm{c}(n - n')(t - t_1)$ will be a series in whose successive terms the factors $\mathbf{e}^p$, $\mathbf{e}^{p+2}$, $\mathbf{e}^{p+4}$, etc., appear. Analogous statements apply to the powers $r + 2r'$, $q + 2q'$ to which e', k respectively are raised.

Brown defines the parameters $\mathbf{e}$, e', $\mathbf{k}$, a as follows. The constant $\mathbf{e}$, relating to the lunar eccentricity, is the observational value of the coefficient of the term $(\mathbf{a}\sin\ell)$ in the longitude – a coefficient which in the ordinary elliptical theory would be given as $2e$; Brown's $\mathbf{e}$ is thus about twice the constant used by Delaunay, and is equal to the constant Y_0 in Brown's essay, "The Elliptical Inequalities in the Lunar Theory".[107] The constant e' is the eccentricity of the assumed elliptical orbit of the Sun. The constant $\mathbf{k}$, which fixes the mean inclination of the lunar orbit to the ecliptic, is defined as half the empirical value of the coefficient of $(\mathbf{a}\sin F)$ in the expression of z as a sum of periodic terms.

The remaining parameter, $a = \mathbf{a}/a'$, is the constant of parallax, giving the ratio of the mean Earth-Moon distance to the mean Earth-Sun distance. In the elliptical theory here assumed for the Sun, $n'^2 a'^3 = m' + E + M$, where n' is the observed mean motion of the Sun in longitude; the distance a' is thereby defined. Similarly, $\mathbf{a}$ could be defined by the relation $n^2\mathbf{a}^3 = E + M$; but Brown chose another definition

107 *American Journal of Mathematics, 15* (1893), 261.

which shortens the calculations. The variable u_0 expressing the Variation curve, we recall, is given by

$$u_0\zeta^{-1} = \mathrm{a}\sum_i a_i\zeta^{2i}, \quad i = 0, \pm 1, \pm 2, \ldots \tag{B.55}$$

Here either **a** or a_0 can be arbitrary; Brown put $a_0 = 1$. Thus **a** became the coefficient of ζ^0 in $u_0\zeta^{-1}$. It turned out to be slightly smaller than the lunar mean distance (call it a) given by the elliptical formula, the ratio **a** : a being a function of **m**:

$$\begin{aligned}\mathrm{a} &= \left(\frac{E+M}{n^2}\right)^{1/3} f(\mathrm{m}) \\ &= a \cdot f(\mathrm{m}) = a(0.99909\,31419\,75298).\end{aligned}$$

(The numerical coefficient here differs from Hill's value for the same coefficient in the eleventh decimal place.) Brown stipulated that **a** have this constant value throughout his theory. However, as inequalities involving **e**, e', **k**, and a are introduced, the coefficient of ζ^{2i} receives certain small augmentations; they take the form $\mathbf{a}(1+\nu)$, as will be explained shortly.

Brown called the factor $\mathbf{e}^{p+2p'}e'^{r+2r'}k^{q+2q'}a^{s'}$ of (B.54) the *characteristic* of the coefficient. The *order* of a coefficient is given by the sum of the exponents in the characteristic, namely, $p + 2p' + r + 2r' + q + 2q' + s'$. This order is independent of the constant **m**: since the numerical value of **m** was substituted at the outset, the power to which **m** is raised, or the number of terms of the series in **m** that would have to be taken into account in a purely literal development of the theory, is irrelevant to the determination of the precision of the calculation.

Finally, in (B.54) we have a set of factors enclosed in parentheses:

$$(\varepsilon^{p+p'}\varepsilon'^{p'}\eta^{r+r'}\eta'^{r'}k^{q+q'}k'^{q'}a^{s'})_i. \tag{B.56}$$

In Brown's original plan for the development of the theory, these factors were to be obtained by successive approximations in the solving of the equations of motion for particular values of the coefficients $A_{i,p,r,q}$ (see B.53). Six of them come in pairs: $\varepsilon, \varepsilon'$ associated with **e**; η, η' associated with e'; k, k' associated with **k**. The seventh, a, is associated with $a = \mathrm{a}/a'$. The symbols $\varepsilon_i, \varepsilon'_i$ occurred in Brown's essay on the elliptical inequalities, but are not the same here, being equal, respectively, to the quantities designated ε_i/Y_0 and ε'_i/Y_0 in the earlier essay. The calculation of η, η' and that of k, k' are similar to that of $\varepsilon, \varepsilon'$. The superscripts appearing in (B.56) are not exponents, but indices signifying that the symbol is associated with a parameter raised to the power indicated. Thus the symbols f_i/Y_0^2, f'_i/Y_0^2 in the earlier essay here become $(\varepsilon^2)_i$, $(\varepsilon'^2)_i$.

In (B.54) suppose that $p = r = q = 0$. The periodic factor will become ζ^{2i}, as in (B.55). The coefficients of this factor, however, can no longer be written simply as $\mathbf{a}a_i$. Instead we shall have

$$u\zeta^{-1} = \mathrm{a}\sum_i \left[a_i + \sum(\varepsilon^{p'}\varepsilon'^{p'}\eta^{r'}\eta'^{r'}k^{q'}k'^{q'}a^{2s'})_j e^{2p'}e'^{2r'}k^{2q'}a^{2s'}\right]\zeta^{2i}.$$

When $i = j = 0$, we obtain for the coefficient of ζ^0, since $a_0 = 1$,

$$\mathrm{a}\left[1 + \sum (\varepsilon^{p'} \varepsilon'^{p'} \eta^{r'} \eta'^{r'} k^{q'} k'^{q'} a^{2s'})_0 \, e^{2p'} e'^{2r'} k^{2q'} a^{2s'}\right] = \mathrm{a}(1 + \nu),$$

where ν is a very small second-order augmentation.

The Solution Process

Brown will begin by determining the terms of order zero, then go on to terms of the first, second, third, etc., orders in succession.

Terms of Order Zero

These terms are given in Chapter II of Brown's *Theory of the Motion of the Moon*.[108] They are functions of **m** alone. The orbit they define in the rotating coordinates x, y is a closed orbit; Brown like Hill called this orbit the *Variation curve*. In the notation of Brown's *Theory*, it is given by

$$u_0 \zeta^{-1} = \mathrm{a} \sum a_i \zeta^{2i}, \quad s_0 \zeta = \mathrm{a} \sum a_{-i} \zeta^{2i}. \tag{B.57}$$

The variables u_0, s_0 constitute a particular solution of the differential equation

$$(D + \mathrm{m})^2 u + \frac{1}{2}\mathrm{m}^2 u + \frac{3}{2}\mathrm{m}^2 s - \frac{\kappa u}{\rho^3} = 0, \tag{B.58}$$

or of the pair of homogenous equations

$$D^2(us) - Du \cdot Ds - 2\mathrm{m}(uDs - sDu) + \frac{9}{4}\mathrm{m}^2(u + s)^2 = C',$$

$$D(uDs - sDu - 2\mathrm{m}us) + \frac{3}{2}\mathrm{m}^2(u^2 - s^2) = 0. \tag{B.59}$$

Equation (B.58) is obtained from (B.51a); the two equations of (B.59) are obtained from (B.52a), (B.52b), by setting $z = 0$, $\Omega_1 = 0$. Substitution of (B.57) into either (B.58) or (B.59) yields equations of condition from which the constants $\mathbf{a}a_i$ can be determined.

Brown took over the whole theory of these terms from Hill's "Researches in the Lunar theory." He also used Hill's numerical results, with a single exception: Hill had put

$$\mathrm{a} = 0.99909\ 31419\ 62 \left[\frac{\mu}{n^2}\right]^{1/3};$$

Brown revised the numerical coefficient from the eleventh decimal onwards:

$$\mathrm{a} = 0.99909\ 31419\ 75298 \left[\frac{\mu}{n^2}\right]^{1/3}.$$

[108] *TMM, Memoirs of the R.A.S., 53*, 88–92.

Terms of the First Order

These are given in Chapter III of Brown's *Theory of the Motion of the Moon.*[109] These terms contain a first power of **e**, e', **k**, or a, but no higher power and no product of these parameters. Suppose $u = u_0 + u_1$, $s = s_0 + s_1$, $z = z_1$, where u_1 or s_1 signifies terms of u or s dependent on **e**, e', or a; and z_1 signifies terms of z dependent on **k**. Substituting these expressions into the equations of motion, then subtracting out the terms for the Variation curve, we obtain an equation for a particular term of u_1 (u_e or $u_{e'}$ or u_a) or of z_1 (namely $z_{\mathbf{k}}$), and this equation can be solved by successive approximations. The s corresponding to u can be obtained in all cases from the defining equation for u by substituting $1/\zeta$ for ζ.

The terms containing the first power of **e** are obtained from (B.51a) with z and Ω_1 set equal to zero:

$$\zeta^{-1}(D + \mathrm{m})^2 u_e + M u_e \zeta^{-1} + N s_e \zeta = 0.$$

Here as earlier

$$M = \frac{1}{2}\mathrm{m}^2 + \frac{1}{2}\frac{\kappa}{(u_0 s_0)^{3/2}} = \sum_i M_i \zeta^{2i},$$

$$N = \frac{3}{2}\mathrm{m}^2 \zeta^{-2} + \frac{3}{2}\frac{\kappa \zeta^{-2}}{u_0^{1/2} s_0^{5/2}} = \sum_i N_i \zeta^{2i}.$$

The solution is of the form

$$u_e \zeta^{-1} = \mathrm{a}e \sum_i (\varepsilon_i \zeta^{2i+c} + \varepsilon_i' \zeta^{2i-c});$$

the exponents $2i \pm c$ modify the Variation curve, introducing eccentricity.

Terms containing the first power of e' are obtained from (B.51a) by setting $z = 0$ and $\Omega_1 = \omega_2$, this being the only part of Ω_1 varying as the first power of e'. The result is

$$\zeta^{-1}(D + \mathrm{m})^2 u_{e'} + M u_{e'} \zeta^{-1} + N s_{e'} \zeta = -\frac{\partial \omega_2}{\partial s} \zeta^{-1}.$$

The right-hand side, with ω_2 expanded in accordance with Cayley's formulas (see Section "Development of Ω, according to powers of e' and z" above for reference), but with z and e' set equal to zero and u, s replaced by u_0, s_0, becomes

$$\frac{3}{4}\mathrm{m}^2 e' [(u_0 \zeta^{-1} + 7 s_0 \zeta \cdot \zeta^{-2}) \zeta^{\mathrm{m}} + (u_0 \zeta^{-1} - s_0 \zeta \cdot \zeta^{-2}) \zeta^{-\mathrm{m}}].$$

Here u_0, s_0 are functions of known quantities, and **m** and e' are known numerical constants. The solution has the form

$$u_{e'} \zeta^{-1} = \mathrm{a}e' \sum_i (\eta_i \zeta^{2i+\mathrm{m}} + \eta_i' \zeta^{2i-\mathrm{m}}).$$

[109] *TMM, Memoirs of the R.A.S.*, *53*, 92–98.

Terms containing the first power of a are obtained from (B.51a) by setting $z = 0$ and $\Omega_1 = \omega_3$, this being the only part of Ω_1 that contains the fraction $a = \mathbf{a}/a'$ to the first power. The result is

$$\zeta^{-1}(D + \mathrm{m})^2 u_a + M u_a \zeta^{-1} + N s_a \zeta = -\frac{\partial \omega_3}{\partial s}\zeta^{-1}.$$

In evaluating the right-hand side we are to replace u, s by u_0, s_0 and to set $z = 0$ and $e' = 0$. Each of the Cayley expansions involved is reduced to the single term 1, and we find

$$\frac{\partial \omega_3}{\partial s}\zeta^{-1} = \mathrm{a}a \cdot \frac{3}{4}\mathrm{m}^2 \cdot \frac{1}{\mathrm{a}^2}\left[\frac{5}{2}(s_0\zeta)^2\zeta^{-3} + \frac{1}{2}(u_0\zeta^{-1})^2\zeta + (u_0 s_0)\zeta^{-1}\right].$$

The solution has the form $u_a\zeta^{-1} = \mathrm{a}a \sum (a)_i \zeta^{2i}$ with $i = i_1$ and $2i_1 = \pm 1, \pm 3, \pm 5$, etc.

Finally, the terms of the first order with respect to **k** are obtained from (B.51c) by setting $\Omega_1 = 0$:

$$D^2 z_k - 2M z_k = 0.$$

Brown's solution has the form $z_k\iota = \mathbf{ak}\sum k_i(\zeta^{2i+\mathrm{g}} - \zeta^{2i-\mathrm{g}})$, where $k_i' = -k_{-i}$, and **g** has the value of $\mathbf{g}_0$ given by P.H. Cowell.[110]

The arguments and types of coefficients calculated for each of the first-order characteristics are listed in the following table:

λ	Arguments	Types of coefficients
E	$2i + \mathrm{c}$	$\varepsilon_i, \varepsilon_i'$
E'	$2i \pm \mathrm{m}$	η_i, η_i'
a	$2i_1$	$(a)_i$
K	$\pm(2i + \mathrm{g})$	$k_i, k_{-i}'(= -k_i)$

Terms of the Second and Higher Orders

We now describe the procedure that Brown expected to use for the terms of the second and higher orders, as set forth in Chapter I of his *Theory*.[111] Later he introduced modifications for terms of the third and higher orders.

Let λ be a characteristic of order 2 or higher. In seeking the terms in u with this characteristic (let them be u_λ), we must take into account all terms of lower order that can contribute to u_λ. Let $\sum u_\mu$ be the sum of the terms of u of orders greater than 0 but less than λ, so that $u = u_0 + \sum u_\mu + u_\lambda$. Similarly, let $\sum z_\mu$ be the sum of the terms of orders less than λ (there are no terms of zero order in z). If these

[110] P.H.Cowell, *American Journal of Mathematics, 18.*

[111] *TMM, Memoirs of the R.A.S., 53*, 64–69.

expressions are substituted into (B.5la), the result is $\zeta^{-1}(D+\mathrm{m})^2 u_\lambda + M u_\lambda \zeta^{-1} + N s_\lambda \zeta =$ the part with characteristic λ in

$$\left[\begin{array}{l} -\zeta^{-1}(D^2+2\mathrm{m}D)\left(\sum u_\mu\right) - \frac{\partial\Omega_1}{\partial s}\zeta^{-1} \\ +\frac{\kappa u_0 \zeta^{-1}}{\rho_0^3}\left\{\begin{array}{l} \frac{3}{8}\left(\frac{\sum u_\mu}{u_0}\right)^2 + \frac{15}{8}\left(\frac{\sum s_\mu}{s_0}\right)^2 + \frac{3}{4}\frac{\sum u_\mu \cdot \sum s_\mu}{u_0 s_0} - \frac{3}{2}\left(\frac{\sum z_\mu}{\rho_0}\right)^2 \\ -\frac{5}{16}\left(\frac{\sum u_\mu}{u_0}\right)^3 - \frac{35}{16}\left(\frac{\sum s_\mu}{s_0}\right)^3 - \frac{9}{16}\left(\frac{\sum u_\mu}{u_0}\right)^2 \frac{\sum s_\mu}{s_0} \\ -\frac{15}{16}\left(\frac{\sum s_\mu}{s_0}\right)^2 \frac{\sum u_\mu}{u_0} + \frac{9}{4}\left(\frac{\sum z_\mu}{\rho_0}\right)^2 \frac{\sum u_\mu}{u_0} + \frac{15}{4}\left(\frac{\sum z_\mu}{\rho_0}\right)^2 \frac{\sum s_\mu}{s_0} \end{array}\right\} \end{array}\right] \tag{B.60}$$

What is here sought, namely u_λ which is a term of order λ having a particular characteristic, appears only in the left-hand member: every variable and constant on the right-hand side is known. The first term on the right involves the operators D^2 and D applied to $\sum u_\mu$; it contributes nothing to the coefficients determining u_λ, but introduces the factors $(2i + p\mathrm{c} + r\mathrm{m} + q\mathrm{g})$, $(2i + p\mathrm{c} + r\mathrm{m} + q\mathrm{g})^2$ – factors determining increments to **c** and **g** that are proportional to the second and higher powers of the parameters. In the next term, $-(\partial\Omega_1/\partial s)\zeta^{-1}$, we are to substitute $u_0 + \sum u_\mu$ for u and $\sum z_\mu$ for z. The rest of the terms arise from the expansions of $\kappa u \zeta^{-1}/r^3$, $\kappa z/r^3$.

The terms z_λ are given by an equation similar to (B.60):

$D^2 z_\lambda - 2M z_\lambda =$ the part having characteristic λ in

$$\left[\begin{array}{l} -D^2\left(\sum z_\mu\right) - \frac{1}{2}\frac{\partial\Omega_1}{\partial z} \\ +\frac{\kappa}{\rho_0^2}\left\{\begin{array}{l} -\frac{3}{2}\frac{\sum z_\mu}{\rho_0}\left(\frac{\sum u_\mu}{u_0} + \frac{\sum s_\mu}{s_0}\right) \\ +\frac{\sum z_\mu}{\rho_0}\left\{\frac{15}{8}\left(\frac{\sum u_\mu}{u_0}\right)^2 + \frac{15}{8}\left(\frac{\sum s_\mu}{s_0}\right)^2 + \frac{9}{4}\frac{\sum u_\mu}{u_0}\frac{\sum s_\mu}{s_0}\right\} - \frac{3}{2}\left(\frac{\sum z_\mu}{\rho_0}\right)^3 \\ -\ldots\ldots\ldots\ldots\ldots \end{array}\right\} \end{array}\right]. \tag{B.61}$$

To solve (B.60), we write

$$u_\lambda \zeta^{-1} = a\lambda \sum_i (\lambda_i \zeta^{2i+\tau} + \lambda_i' \zeta^{2i-\tau}) = a\lambda \sum_i (\lambda_i \zeta^{2i+\tau} + \lambda_{-i}' \zeta^{-(2i+\tau)}), \tag{B.62}$$

where τ is one of the values of $\pm p\mathrm{c} \pm r\mathrm{m} \pm 2q\mathrm{g}$. In the first term on the left-hand side of (B.60), an expression is needed for u_λ; we obtain it by multiplying (B.62) by ζ:

$$u_\lambda = a\lambda \sum_i (\lambda_i \zeta^{2i+\tau+1} + \lambda_{-i}' \zeta^{-2i-\tau+1}).$$

In the third term on the left of (B.60), we need an expression for $s_\lambda \zeta$; we obtain it by changing ζ to $1/\zeta$ in (B.62):

$$s_\lambda \zeta = a\lambda \sum_i (\lambda_i \zeta^{-(2i+\tau)} + \lambda_{-i}' \zeta^{2i+\tau}).$$

On the right-hand side of (B.60), the terms involving $\zeta^{\pm(2i+\tau)}$ can be put in the form $a\lambda A = a\lambda \sum [A_i \zeta^{2i+\tau} + A'_{-i}\zeta^{-(2i+\tau)}]$. With these substitutions, and dividing the equation by $a\lambda$, we obtain

$$\begin{aligned}
&\sum_i \lambda_i (2i+\tau+1+\mathrm{m})^2 \zeta^{(2i+\tau)} + \sum_i \lambda'_{-i}(2i+\tau-1-\mathrm{m})^2 \zeta^{-(2i+\tau)} \\
&\quad + \sum_j M_j \zeta^{2j} \times \sum_i (\lambda_i \zeta^{(2i+\tau)} + \lambda'_{-i}\zeta^{-(2i+\tau)}) \\
&\quad + \sum_j N_j \zeta^{2j} \times \sum_i (\lambda_i \zeta^{-(2i+\tau)} + \lambda'_{-i}\zeta^{(2i+\tau)}) \\
&= \sum_i (A_i \zeta^{(2i+\tau)} + A'_{-i}\zeta^{-(2i+\tau)}).
\end{aligned} \tag{B.63}$$

In each of the two products of infinite series in the second line, every term of one series multiplies every term of the other. However, in the series M_j, N_j, the decrease in size from one term to the next is rapid; thus M_1 and M_{-1} (which are equal) are about 100 times smaller than M_0, and successive terms continue to diminish in about this same ratio. The N_j converge in absolute value at a similar rate as j goes from 0 to the values ± 1, ± 2, etc. Following the pattern of Hill's and his own earlier solutions of differential equations by the method of undetermined coefficients, Brown proposed solving these equations by successive approximations.

In z, the terms with characteristic λ are given by

$$D^2 z_\lambda - 2M z_\lambda = a\lambda\iota \sum_i A_i (\zeta^{2i+\tau} - \zeta^{-(2i+\tau)}). \tag{B.64}$$

Into this we substitute $z_\lambda \iota = a\lambda \sum_i \lambda_i (\zeta^{2i+\tau} - \zeta^{-(2i+\tau)})$, since in (B.64) λ'_{-i} is always equal to $-\lambda_i$. The resulting equations of condition are

$$(2i+\tau)^2 \lambda_i - 2\sum_j M_j \lambda_{i-j} = A_i, \tag{B.65}$$

where $j = 0, \pm 1, \pm 2, \ldots$, and $2i$ either $= 0, \pm 2, \pm 4, \ldots$ or $= \pm 1, \pm 3, \pm 5, \ldots$.

For the coefficients determining z_λ, the equations of condition turn out to be easier to resolve than those for the coefficients determining u_λ. If in (B.63) we set equal to zero the coefficients of $\zeta^{2i+\tau}$ and also those of $\zeta^{-(2i+\tau)}$, understanding i to have the same value in the two cases, we obtain

$$\begin{aligned}
&(2i+\tau+1+\mathrm{m})^2 \lambda_i + \sum_j M_j \lambda_{i-j} + \sum_j N_j \lambda'_{j-i} = A_i, \\
&(2i+\tau-1-\mathrm{m})^2 \lambda'_{-i} + \sum_j M_j \lambda'_{-i-j} + \sum_j N_j \lambda_{j+i} = A'_{-i}.
\end{aligned} \tag{B.66}$$

Evidently, if we wish to solve (B.66) simultaneously for λ_0 and λ'_0, we will need as well, in evaluating the summations, values of both λ and λ' with subscripts equal to $\pm 1, \pm 2, \pm 3$, and so on. Suppose that, for a given i, I take from the summations only the terms in which $j = 0$; (B.64) reduce to the approximations:

$$[(2i + \tau + 1 + \mathrm{m})^2 + M_0]\lambda_i + N_0\lambda'_{-i} \approx A_i,$$
$$[(2i + \tau - 1 - \mathrm{m})^2 + M_0]\lambda'_{-i} + N_0\lambda_i \approx A'_{-i}. \quad \text{(B.67)}$$

These could be solved simultaneously for approximate values of λ_i, λ'_{-i}. But can successively closer approximations be obtained by including further approximate values of terms in the summations $\sum M_j\lambda_j$ and $\sum N_j\lambda'_{-j}$, with $\pm j$ taking values other than zero? Brown apparently proposed a process of this kind to his computer, Mr. Sterner, but, as he reported, the exact route to be followed could not be specified once for all, and the calculations did not proceed smoothly:

> This method is troublesome to put into a form which a computer can use easily and is besides peculiarly liable to chance errors; a large number of processes would have to be learnt before the computer could proceed quickly and securely.[112]

For each second-order inequality, the equations to be solved simultaneously turned out to be about 20 in number, with the index i taking values from -5 to $+5$. Of second-order inequalities, there were 10 to be calculated: those proportional to the squares of the four parameters **e**, e', **k**, a, and those proportional to their combinations in pairs, in number $4!/2!2! = 6$. Hence the total number of equations to be dealt with, and of coefficients to be solved for, was about 200. Of third-order inequalities there would be 20, hence 400 coefficients to be solved for; and of fourth-order inequalities there would be 35, hence 700 coefficients to be solved for. Clearly a new method, avoiding solutions of simultaneous equations by successive approximations, was desirable.

Before turning to Brown's new method, we mention the effect of the small-divisor problem on the calculation of the second-order inequalities.[113] We summarized Brown's earlier and more general discussion of this problem in connection with (B.22) and (B.23). In the simultaneous solution of the approximate equations (B.67), the common divisor is

$$[(2i + \tau + 1 + \mathrm{m})^2 + M_0][(2i + \tau - 1 - \mathrm{m})^2 + M_0] - N_0^2. \quad \text{(B.68)}$$

This expression has nearly the form of, and is nearly equal to $(2i+\tau)^2[(2i+\tau)^2 - \mathrm{c}_0]$. The near-equality gives rise to cases requiring special devices or the calculation of additional decimal places. When $2i + \tau$ is equal to zero (implying, since **c**, **m**, and

[112] E.W. Brown, "On the Solution of a Pair of Simultaneous Linear Differential Equations, which occur in the Lunar Theory," *Transactions of the Cambridge Philosophical Society, 18* (1900), 94ff.

[113] See *TMM*, *Memoirs of the R.A.S., 53*, 75–79 for Brown's treatment of this problem.

g are assumed incommensurable with the unit, that $i = \tau = 0$), the denominator of (B.66) becomes very small (equal approximately to 0.00024), and the two equations (B.65) coalesce into one equation, which is of principal importance in determining λ_0. When $2i + \tau = \pm \mathbf{c}_0$, A turns out to contain part of the motion of the perigee, which Brown computes by a special route. When $2i+\tau$ is small compared with unity, inequalities arise that are of long period compared with the lunar month. When $2i + \tau \pm \mathbf{c}_0$ is small compared with unity, numerous short-term inequalities arise which have arguments nearly equal to the principal elliptic term, for instance the Evection and Parallactic Inequality. In both the latter cases, extra terms have to be calculated to attain the requisite precision.

In the case of z_λ, the coefficients λ_i are to be determined by (B.65). Here the coefficient of λ_i is $(2i+\tau)^2 - 2M_0$; this, if we had eliminated all the other unknowns, would have been $(2i+\tau)^2 - g_0^2$ multiplied by a numerical factor close to 1. The only special cases requiring to be considered are those in which $2i + \tau \pm g_0$ is zero or small compared with the unit. In the first case, A contains an unknown part of the motion of the node, which Brown computes by a special route. The second case leads to inequalities with periods nearly equal to that of the principal term in latitude.

By the procedures described above, Brown obtained the second-order inequalities, completing (with Mr. Sterner's aid) the computation by late March, 1897.[114] Each inequality was given by a sum of terms, in each of which a coefficient was multiplied by a power of ζ. For instance, the second-order inequality proportional to $\mathbf{e}^2$ was given by

$$u_{e^2}\zeta^{-1} = ae^2 \sum_i [(\varepsilon^2)_i \zeta^{2i+2c} + (\varepsilon'^2)_i \zeta^{2i-2c} + (\varepsilon\varepsilon')_i \zeta^{2i}].$$

But as mentioned above, for the third-order terms, the procedure of solving for the coefficients by successive approximations was too troublesome; a new method was needed.

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114 Brown to Darwin, 24 March 1897, CUL, MS.DAR.251:488.

11

A Solution-Procedure Without Approximations

During the 1890s, at the invitation of George Darwin, Brown prepared a paper for inclusion in the volume of the *Transactions of the Cambridge Philosophical Society* commemorating the jubilee of Sir George Gabriel Stokes.[115] As his topic, he chose the problem of getting an exact solution for the equations

$$(D + \mathrm{m})^2 u + Mu + Ns = A,$$
$$(D - \mathrm{m})^2 s + Ms + \bar{N}s = \bar{A}. \tag{B.69}$$

Here A is of the form $\sum p_i \zeta^{2i+1+\tau} + \sum p_i' \zeta^{2i+1-\tau}$. The bar over a letter means that ζ has been replaced by ζ^{-1}. These equations are essentially the same as those Brown and Sterner had been solving by successive approximations. Brown's initial interest was theoretical: to know the form of the exact solution. Only later did the exact solution become important practically, as providing a means to obtain the higher-order inequalities without relying on successive approximations.

For exact solutions of (B.69), the first requisite is a *general* solution of the homogeneous equations

$$(D + \mathrm{m})^2 u + Mu + Ns = 0,$$
$$(D - \mathrm{m})^2 s + Ms + \bar{N}u = 0. \tag{B.70}$$

A solution of such a system is general or complete if it contains as many arbitrary constants as there are independent *particular* solutions of the differential equations. Since (B.70) are two in number, and each of the second order, four independent particular solutions must exist. Three such solutions were already known:

$$u_1 = \sum_i \varepsilon_i \zeta^{2i+1+\mathrm{c}}, \quad s_1 = \sum_i \varepsilon'_{-i} \zeta^{2i-1+\mathrm{c}};$$

$$u_2 = \sum_i \varepsilon'_i \zeta^{2i+1-\mathrm{c}}, \quad s_2 = \sum_i \varepsilon_{-i} \zeta^{2i-1-\mathrm{c}};$$

[115] Brown, *Transactions of the Cambridge Philosophical Society, 18* (1900), 94–106.

C. Wilson, *The Hill–Brown Theory of the Moon's Motion*, Sources and Studies in the History of Mathematics and Physical Sciences, DOI 10.1007/978-1-4419-5937-9_11,

$$u_3 = \sum_i (2i+1)a_i \zeta^{2i+1}, \quad s_3 = \sum_i (2i-1)a_{-i}\zeta^{2i-1} \tag{B.71}$$

The first two of these, u_1, s_1 and u_2, s_2, combine the variation curve with eccentricity, but with opposite signs of **c**; this difference makes them independent, since no linear combination of ζ^{c} and $\zeta^{-\mathrm{c}}$ can be zero. The solution u_3, s_3 gives the variation curve; here the exponents of ζ are integers, and so the successive terms are independent of the terms contributing eccentricity because $+\mathbf{c}$ and $-\mathbf{c}$ are incommensurable with unity.

Any linear combination of the solutions (B.71) is also a solution. Thus

$$u = \sum_j Q_j u_j, \quad s = \sum_j Q_j s_j, \quad j = 1, 2, 3, \tag{B.72}$$

where the Q_j are arbitrary constants, is a solution of (B.70), but not linearly independent of the solutions (B.71). To obtain a fourth particular solution that is linearly independent of (B.71), Brown employed Lagrange's method of varying the Q_j of (B.72) so that they become functions of the independent variable t. Applying this method required that

$$u_1 DQ_1 + u_2 DQ_2 + u_3 DQ_3 = 0. \tag{B.73}$$

Substituting (B.72) into the differential equations (B.70), and making use of (B.73), he found that

$$\begin{aligned} &Du_1 \cdot DQ_1 + Du_2 \cdot DQ_2 + Du_3 \cdot DQ_3 = 0, \\ &\sum_j \{s_j D^2 Q_j + 2Ds_j \cdot DQ_j - 2\mathrm{m} s_j DQ_j\} = 0. \end{aligned} \tag{B.74}$$

To satisfy these conditions, Brown put

$$\left\{ \begin{aligned} u_2 Du_3 - u_3 Du_2 &= \alpha_1, \\ u_3 Du_1 - u_1 Du_3 &= \alpha_2, \\ u_1 Du_2 - u_2 Du_1 &= \alpha_3. \end{aligned} \right\},$$

$$\frac{DQ_1}{\alpha_1} = \frac{DQ_2}{\alpha_2} = \frac{DQ_3}{\alpha_3} = L. \tag{B.75}$$

From (B.74) and (B.75), he derived a differential equation for L:

$$\left(\sum \alpha s\right) DL + 2LD\left(\sum \alpha s\right) - L\left[\sum (sD\alpha) + 2\mathrm{m}\sum \alpha s\right] = 0, \tag{B.76}$$

where $\sum \alpha s = \alpha_1 s_1 + \alpha_2 s_2 + \alpha_3 s_3$.

(B.76) can be simplified. Introducing u_1, s_1 and u_2, s_2 from (B.71) into (B.70), Brown showed that

$$(D + 2\mathrm{m})(s_2 Du_1 - s_1 Du_2) + (\mathrm{m}^2 + M)(s_2 u_1 - u_2 s_1) = 0,$$
$$(D - 2\mathrm{m})(u_2 Ds_1 - u_1 Ds_2) + (\mathrm{m}^2 + M)(u_2 s_1 - s_2 u_1) = 0.$$

The sum of these two equations can be integrated; its integral is

$$\begin{aligned} C_{12} &= s_2 Du_1 - u_1 Ds_2 + u_2 Ds_1 - s_1 Du_2 + 2\mathrm{m}(s_2 u_1 - u_2 s_1) \\ &= 2\sum(2i + 1 + \mathrm{m} + \mathrm{c})\varepsilon_i^2 + 2\sum(2i - 1 - \mathrm{m} + \mathrm{c})\varepsilon'^2_{-i}. \end{aligned} \tag{B.77}$$

or, for brevity, $f_{12} = C_{12}$. Here C_{12} is a constant, but not arbitrary, since u_1, s_1 and u_2, s_2 are functions fully defined by (B.71), and the substitution of these functions in (B.77) yields a value free of variables. By entirely parallel processes, with rotation of indices, Brown derived the integrals $f_{23} = C_{23}$, $f_{31} = C_{31}$. Multiplying C_{23} by u_1, C_{31} by u_2, and C_{12} by u_3, he showed that

$$u_1 C_{23} + u_2 C_{31} + u_3 C_{12} = \sum \alpha s,$$

and

$$u_1 Df_{23} + u_2 Df_{31} + u_3 Df_{12} = \sum s D\alpha + 2\mathrm{m} \sum \alpha s = 0.$$

This last result reduces (B.75) to

$$\frac{DL}{L} + 2\frac{D\left(\sum \alpha s\right)}{\sum \alpha s} = 0.$$

The integral of this equation is

$$\ln L + \ln \left(\sum \alpha s\right)^2 = \ln L_0,$$

whence

$$L = \frac{L_0}{\left(\sum \alpha s\right)^2} = \frac{L_0}{(u_1 C_{23} + u_2 C_{31} + u_3 C_{12})^2}, \tag{B.78}$$

where L_0 is a new arbitrary constant. Since by (B.75) $DQ_j = \alpha_j L$ for $j = 1, 2, 3$, it follows that, for the same three indices,

$$Q_j = (Q_j) + L_0 D^{-1} \frac{\alpha_j}{(u_1 C_{23} + u_2 C_{31} + u_3 C_{12})^2}. \tag{B.79}$$

Here D^{-1} denotes integration; (Q_j) for each j is an arbitrary constant, which can be zero.

Having thus identified the Q_j that will make (B.72) a linearly independent solution of (B.70), Brown introduced four *new* arbitrary constants, Q_1, Q_2, Q_3, Q_4, which enabled him to express the general solution of (B.70) as

$$
\begin{aligned}
u &= Q_1u_1 + Q_2u_2 + Q_3u_3 + Q_4u_4, \\
s &= Q_1s_1 + Q_2s_2 + Q_3s_3 + Q_4s_4,
\end{aligned}
\tag{B.80}
$$

where

$$u_4 = \sum_j u_j D^{-1} \frac{\alpha_j}{(u_1C_{23} + u_2C_{31} + u_3C_{12})^2}, \quad j = 1, 2, 3,$$

with s_4 given by $\overline{u_4}$.

The expressions for u_4, s_4 can be further simplified, because $C_{31} = C_{23} = 0$. For as is evident from (B.71), (u_1, s_1) contains ζ^{c}, (u_2, s_2) contains $\zeta^{-\mathrm{c}}$, and (u_3, s_3) contains neither ζ^{c} nor $\zeta^{-\mathrm{c}}$. Therefore f_{23} contains the factor $\zeta^{-\mathrm{c}}$, and f_{31} the factor ζ^{c}, while f_{12} contains the product of these factors, which equals 1. Since **c** is assumed to be incommensurable with unity, f_{31} and f_{23} can be constants only if each of them is zero. The detailed algebra corroborates this conclusion. (B.79) thus becomes

$$
\begin{aligned}
u_4C_{12}^2 = {} & u_1D^{-1}\frac{u_2Du_3 - u_3Du_2}{u_3^2} + u_2D^{-1}\frac{u_3Du_1 - u_1Du_3}{u_3^2} \\
& + u_3D^{-1}\frac{u_1Du_2 - u_2Du_1}{u_3^2}.
\end{aligned}
\tag{B.80a}
$$

The first two terms on the right can be integrated, yielding

$$-u_1\frac{u_2}{u_3} + u_2\frac{u_1}{u_3} = 0.$$

Thus, if we let C_{12}^2 be absorbed into Q_4, (B.80) reduces to

$$\frac{u_4}{u_3} = D^{-1}\left(\frac{u_1Du_2 - u_2Du_1}{u_3^2}\right). \tag{B.80b}$$

Similarly,

$$\frac{s_4}{s_3} = D^{-1}\left(\frac{s_1Ds_2 - s_2Ds_1}{s_3^2}\right).$$

According to Brown, these are probably the simplest forms for $u_4,\ s_4$. However, the values of u_1, s_1, u_2, s_2, and u_3, s_3 are all of the form {(sum of cosines) + $\sqrt{(-1)}$ (sum of sines)}. For ease of calculation, u_4 and s_4 need to be expressed in the same form. By a bit of algebraic legerdemain Brown showed that

$$
\begin{aligned}
\frac{u_4}{u_3} &= D^{-1}\left(\frac{u_1Du_2 - u_2Du_1}{u_3^2}\right) \\
&= \frac{1}{2}\frac{s_1u_2 - u_1s_2}{u_3s_3} + \frac{1}{2}D^{-1}\left\{\frac{C_{12}}{u_3s_3} - \frac{s_1u_2 - u_1s_2}{u_3s_3}\left(2\mathrm{m} + \frac{Du_3}{u_3} - \frac{Ds_3}{s_3}\right)\right\}.
\end{aligned}
\tag{B.80b}
$$

Here the first term proves to be real and the second term a pure imaginary, as required.

By a further transformation Brown showed u_4 to be of the form

$$u_3[iBt(n-n') + \text{ a power series in } \zeta^2].$$

The solution (u_3, s_3) is also a power series in ζ^2, but lacks the term with factor t. The latter term turns out to have no role in the lunar theory.

The solution (u_4, s_4) is a solution of the homogeneous equations but not of the non-linear equations. In itself, it supplies no new information concerning the Moon's motions. But in conjunction with the other solutions it makes possible the general solution (B.80) of the homogeneous equations, and the latter make possible a solution, free of any reliance on successive approximations, of the non-linear equations.

The non-linear equations, we recall, were

$$(D+\mathrm{m})^2u + Mu + Ns = A,$$
$$(D-\mathrm{m})^2s + Ms + \bar{N}u = \bar{A},$$

where A, $\bar{A}$ are functions, already known, of the time t. The standard procedure is to vary the new arbitraries under the restricting conditions given by the 16 equations

$$\sum Du_j \cdot DQ_j = A, \quad \sum Ds_j \cdot DQ_j = \bar{A},$$
$$\sum u_jDQ_j = 0, \quad \sum s_jDQ_j = 0. \tag{B.82}$$

These equations are to be solved simultaneously for the DQ_j. The solution for DQ_j is given by the quotient of the determinants Δ_j/Δ, where

$$\Delta = \begin{vmatrix} Du_1 & Du_2 & Du_3 & Du_4 \\ Ds_1 & Ds_2 & Ds_3 & Ds_4 \\ u_1 & u_2 & u_3 & u_4 \\ s_1 & s_2 & s_3 & s_4 \end{vmatrix}$$

and

$$\Delta_1 = \begin{vmatrix} A & Du_2 & Du_3 & Du_4 \\ \bar{A} & Ds_2 & Ds_3 & Ds_4 \\ 0 & u_2 & u_3 & u_4 \\ 0 & s_2 & s_3 & s_4 \end{vmatrix}, \text{ etc.}$$

The four Δ_j can be shown to be

$$\Delta_1 = -(s_2A + u_2\bar{A})C_{12},$$
$$\Delta_2 = (s_1A + u_1\bar{A})C_{12},$$
$$\Delta_3 = (s_4A + u_4\bar{A})C_{12},$$

$$\Delta_4 = -(s_3 A + u_3 \bar{A}) C_{12}.$$

The determinant Δ is equal to

$$-(s_2 D u_1 - s_1 D u_2 - s_4 D u_3 + s_3 D u_4) C_{12} = -2C_{12}^2.$$

Brown thus obtained expressions for the DQ_i and hence (by integration) for the Q_i. The particular integral of (B.69) corresponding to the right-hand members $A, \bar{A}$ was thus found to be

$$u = \frac{1}{C_{12}} \left\{ \begin{matrix} u_1 D^{-1}(s_2 A + u_2 \bar{A}) - u_2 D^{-1}(s_1 A + u_1 \bar{A}) \\ -u_3 D^{-1}(s_4 A + u_4 \bar{A}) + u_4 D^{-1}(s_3 A + u_3 \bar{A}) \end{matrix} \right\},$$

$$s = \frac{1}{C_{12}} \left\{ \begin{matrix} s_1 D^{-1}(s_2 A + u_2 \bar{A}) - s_2 D^{-1}(s_1 A + u_1 \bar{A}) \\ -s_3 D^{-1}(s_4 A + u_4 \bar{A}) + s_4 D^{-1}(s_3 A + u_3 \bar{A}) \end{matrix} \right\}. \tag{B.83}$$

Calculation of Terms from Third to Sixth Order

Thus far in describing Brown's new method we have followed his essay for the Stokes volume. In *Theory of the Motion of the Moon*, we find him introducing new symbols, rearranging equations, and changing procedures as the computation proceeds, all with a view to efficiency.

In the case of the third-order terms, Brown transformed the equations so as to obtain u_λ/u_0 instead of u_λ. The formulas for determining A', except for those deriving from Ω, had u_0 as denominator. Calculating u_λ/u_0 from the start, then multiplying the parts due of Ω by u_0, was easier than first finding u_λ, then multiplying almost all of the terms by $1/u_0$. To the quotient u_λ/u_0 Brown gave the general form

$$\frac{1}{\lambda}\frac{u_\lambda}{u_0} = U_1 Q_\lambda + U_2 \bar{Q}_\lambda + U_3 T_\lambda + U_4 V_\lambda, \tag{B.84}$$

where the products on the right are of the form $\sum p_i \zeta^{2i} \cdot \sum q_j \zeta^{2j}$. He thus reduced a large part of the computation to uniform processes, performable by a reliable and competent computer.

For the third-order terms and those of higher orders, we shall not list the arguments and types of coefficients for each characteristic. The length of the lists increases drastically from the second to the third and fourth orders, diminishes somewhat in the fifth order and still more in the sixth order. The general "look" of these lists can be inferred from that of the lower-order lists.

In calculating the fourth-order terms, Brown returned to computing u_λ directly. He had concluded that the fifth-order terms should be computed from the homogeneous equations (our B.51a, b, c) rather than from the non-homogeneous equations (our B.60 and B.61), and the former require the results for u_λ rather than u_λ/u_0.

Another change made in computing the fourth-order terms was to put

$$A = \frac{3}{4}\frac{\kappa u_0 \zeta^{-1}}{\rho_0^3} A_1,$$

and instead of the series u_2, s_2, s_3, s_4, to use each these series multiplied by $3\kappa u_0 \zeta^{-1}/4\rho_0^3$. The four new series could be computed once for all. As is apparent from (B.60), A consists of two parts, a small part in the first line whose terms do not contain the above factor, and the terms in the succeeding lines, all of which do contain it. The terms in the first line then have to be multiplied by the reciprocal of the above factor, but this, Brown wrote, would be short work.[116]

In computing the fifth-order terms, as indicated above, Brown returned to the homogeneous equations (our B.51a, b, c). The non-homogeneous equations (our B.60 and B.61) would have required expanding $\kappa u/\rho^3$ to the fifth order, an enormous piece of work. This expression was easy enough to expand to lower orders, but with each passage to a higher order the number of terms and the complexity of the calculation increased. On arriving at the fifth-order terms, Brown judged that these expansions would be too costly in time and labor. The homogeneous equations, in contrast, required calculation only of such expressions as u^2, uDs, etc., to the fifth order, and this operation was far less labor-intensive.[117]

This change, however, introduced wrinkles of its own. The unknowns occur in the homogeneous equations either as squared or in products of the second degree. These equations, given earlier as (B.52a, b), are repeated here with the constants of integration omitted and with the disturbing function limited to the parts required for computing fifth-order terms:

$$D^2(us+z^2) - Du \cdot Ds - (Dz)^2 - 2\mathrm{m}(uDs - sDu) + \frac{9}{4}\mathrm{m}^2(u+s)^2 - 3\mathrm{m}^2 z^2$$
$$+ 3\omega_2 + 4\omega_3 - D^{-1}(D'\omega_2 + D'\omega_3) = 0; \qquad \text{(B.85)}$$

$$uDs - sDu - 2\mathrm{m}us + D^{-1}\left[\frac{3}{2}\mathrm{m}^2(u^2 - s^2)\right]$$
$$+ D^{-1}\left[u\frac{\partial\omega_2}{\partial u} + u\frac{\partial\omega_3}{\partial u} - s\frac{\partial\omega_2}{\partial s} - s\frac{\partial\omega_3}{\partial s}\right] = 0. \qquad \text{(B.86)}$$

The operator D' in (B.85) signifies the operation D performed on Ω_1 only insofar as ζ occurs in r' or u.

Suppose it is required to determine the terms in $u\zeta^{-1}$ with characteristic λ and arguments $2i \pm \tau$, where τ is one of the fifth-order values of $\pm p\mathrm{c} \pm r\mathrm{m} \pm 2q\mathrm{g}$, the values of lower order having been previously determined. The terms sought will have the form

$$\mathrm{a}\lambda \sum_i (\lambda_{\tau,i}\zeta^{2i+\tau} + \lambda_{-\tau,i}\zeta^{2i-\tau}),$$

[116] *TMM, Memoirs of the R.A.S., 53* (1896–1899), 170.

[117] Ibid., *54* (1900), 1.

where the coefficients $\lambda_{\tau,i}$, $\lambda_{-\tau,i}$ are unknowns. How do terms with characteristic λ and argument $2i \pm \tau$ arise in such expressions as $D^2(us)$, u^2, uDs, etc.? Evidently from two factors. Let one have characteristic μ and argument $\pm(2i + \sigma)$, and the other characteristic ν and argument $\pm(2i + \tau - \sigma)$, where μ and ν are such that $\mu\nu = \lambda$. The factors that combine can be expressed by

$$u_\mu \zeta^{-1} = \mathrm{a}\mu \sum_{\sigma,i} (\mu_{\sigma,i}\zeta^{2i+\sigma} + \mu_{-\sigma,i}\zeta^{2i-\sigma}),$$

$$u_\nu \zeta^{-1} = \mathrm{a}\nu \sum_{\sigma,i} (\nu_{\tau-\sigma,i}\zeta^{2i+\tau-\sigma} + \nu_{\sigma-\tau,i}\zeta^{2i-\tau+\sigma}).$$

Most of the exponents of ζ contain either the constants c or g or both. In computing terms of the fifth order, account must be taken of the increments that c and g incur. Here we change notation, replacing c by c_0, and symbolizing the increments to c_0 by c_2, c_4; similarly, we replace g by g_0, and symbolize the increments to g_0 by g_2, g_4. These increments are introduced by the operators D, D^2, D^{-1}. Brown carried out the main computation with the principal values of c and g, namely c_0 and g_0, and computed the effects of the increments separately. If the first lines of (B.85) and (B.86) are labeled $\mathbf{f}$ and $\mathbf{f}'$, respectively, then the parts due to $\mathrm{c} - \mathrm{c}_0$, $\mathrm{g} - \mathrm{g}_0$ can be designated $\delta\mathbf{f}$ and $\delta\mathbf{f}'$, and computed by successive approximations.

When the two equations for λ_i, λ'_{-i} possess a small divisor, the approximations proceed slowly. In such cases Brown found that labor could be saved by first solving the equations for $\lambda_{i\pm1}$, $\lambda'_{i\pm1}$ so as to obtain the latter coefficients in terms of λ_i, λ'_{-i} and the known quantities, then substituting the results in the equations for λ_i, λ'_{-i} before solving them.

Brown retained the non-homogenous form of equation for z, since many of the required products and sums of series had already been obtained in computing the terms of lower orders. He put the required expansion of $\kappa z/r^3$ in the form

$$\frac{\kappa z}{r^3} = \kappa''_1 B_4 + \kappa''_2 B_3 + \kappa''_3 B_2 + \kappa''_4 B_1, \tag{B.87}$$

where $\kappa''_i = -3\kappa z_i/2\rho_0^3$ and the B_i have expressions whose complication increases with i.

Among sixth-order terms in u_λ, only those with characteristics $\lambda = \mathbf{e}^4 k^2, \mathbf{e}^2 k^4$ were calculated. For those in $\mathbf{e}^6$, the elliptic values could be substituted; those in k^6 proved insensible. The same method was used as for the fifth-order terms, with $\Omega_1 = L' = \Lambda' = 0$. The main difference arose in the development of $\delta\mathbf{f}$, $\delta(D\mathbf{f}')$. Thus

$$\delta\mathrm{f} = \mathrm{c}_2\frac{\partial \mathrm{f}_4}{\partial \mathrm{c}} + \mathrm{g}_2\frac{\partial \mathrm{f}_4}{\partial \mathrm{g}} + \frac{1}{2}\mathrm{c}_2^2\frac{\partial^2 \mathrm{f}_2}{\partial \mathrm{c}^2} + \frac{1}{2}\mathrm{g}_2^2\frac{\partial^2 \mathrm{f}_2}{\partial \mathrm{g}^2} + \mathrm{c}_4\frac{\partial \mathrm{f}_2}{\partial \mathrm{c}} + \mathrm{g}_4\frac{\partial \mathrm{f}_2}{\partial \mathrm{g}},$$

with a similar expression for $\delta(D\mathbf{f}')$. In most of the products of series, only three significant figures were required; the computation was thereby considerably shortened.

The calculation of the sixth-order terms in z, had it been carried out with the non-homogeneous equations, as were the lower-order terms in z, would have required the expansion of $\kappa z/r^3$ to the fifth order. So Brown returned, here too, to the homogeneous equation:

$$D(uDz - zDu) - 2\mathrm{m}zDu - \mathrm{m}^2uz - \frac{3}{2}\mathrm{m}^2z(u + z) = 0.$$

12

The "Main Problem" Solved

What Brown called "the main problem" in the lunar theory consisted in the deduction, from Newton's law of gravitation alone, of the motions of the Moon under the restrictions that the Moon, Earth, and Sun be regarded as point-particles, and the center of gravity of the Earth and Moon be assumed to move about the Sun in a fixed elliptic orbit. Brown set the accuracy to be achieved at $0''.01$ arc-second. With the calculation of the sixth-order terms toward the end of 1904, he could announce that the solution had now been obtained.[118] He took the occasion – "the completion of a laborious piece of work which has occupied many years for its execution" – to explain why the task had been undertaken and by what method it had been carried out.

To Euler he credited the idea of the method. It consisted in taking as starting-point an accurate calculation of the "Variation curve," and developing the theory along the powers of four small parameters, namely: the orbital eccentricities of the Moon and Sun, (or Earth), the inclination of the Moon's orbit to the ecliptic, and the ratio of the lunar and solar parallaxes.

He credited G.W. Hill with putting the early steps in the development into a form such that high accuracy could be obtained without excessive labor. Economy was as crucial as accuracy:

> The working value of a method of treatment is not really tested by the closeness with which the first or second approximation will make the further approximations converge quickly to the desired degree of accuracy; the real test is, perhaps, the ease with which the final approximation can be obtained. Here we have the essential difference between the present method and all other methods. The approximations of the latter proceed along powers of the disturbing force. Euler's idea was to approximate along powers of the other small constants present. This gives a more rapid convergence and a degree of certainty in knowing the limits of error of the final results which no other method approaches.

[118] *MNRAS, 65* (Dec., 1904), 104–108.

C. Wilson, *The Hill–Brown Theory of the Moon's Motion*, Sources and Studies in the History of Mathematics and Physical Sciences, DOI 10.1007/978-1-4419-5937-9_12,

The one further small parameter involved in the theory was the ratio of the mean motions of the Sun and the Moon. Its numerical value had been assumed from the outset:

> [This ratio] is known with a degree of certainty which satisfies all the possible needs of the theory, and the effect of any possible change which may be made in its observed value can be easily deduced from Delaunay's purely literal theory. The chief advantage gained is due to the fact that slow convergence (perhaps divergence) occurs only along powers of this ratio, while there is little loss of theoretical interest in using its numerical value. Moreover it is not difficult to find out how many places of decimals are necessary at the outset in order to secure a given number of places in the results.

Every coefficient in longitude, latitude, and parallax as great as $0''.01$, Brown claimed, had been computed to at least this accuracy. To avoid calculative errors, he had taken exceptional precautions; each page of manuscript work has been checked by, on the average, two test equations. "Very searching final tests, eleven in number," were furnished by the relations existing between the mean motions of the perigee and node and the constant term of the parallax – relations discovered by J.C. Adams and Newcomb, with refinements by Brown himself.

It was now possible to carry out some important comparisons. Already, in the *Monthly Notices* for May, 1903, Brown had believed himself in a position to announce that Asaph Hall's hypothesis – the proposal that the exponent in the gravitational law be changed to accommodate the anomalous motion of Mercury's perihelion – was untenable at the Moon's distance from the Earth. In 1897, this point had remained undecided, because of a possible remaining error of $1''.8$ in his calculated value of the annual motion of the lunar perigee. With the sixth-order terms known, Brown could now re-do the calculation more precisely; the result showed a remarkable agreement with the observed values. In addition, Hansen's values were seen to be of less accuracy than Brown's.

	For the perigee	For the node
Calculated	$+146,434''.5 \pm 0''.2$	$-69,679''.6 \pm 0''.2$
Hansen	$+146,434''.0$	$-69,676''.8$
Observed	$+146,435''.6$	$-69,679''.5$

Hall's hypothesis would give the gravitational law as $r^{-2-\delta}$, and Newcomb in seeking to account for Mercury's perihelion motion had put $\delta = 0.0000001574$. This value of δ would cause a correction of $1''.4$ in the motion of the Moon's perigee. Brown's calculated values given above differ from the observational values by less than $0''.3$ arc-second, making $\delta < 0.00000004$, a value quite insufficient to account for the anomalous deviation in the motion of the perihelion of Mercury.

This conclusion, however, proved to be premature. In April, 1904, Brown in effect acknowledged that his previous conclusion was unwarranted:

> In order to make the comparison complete to this degree of accuracy [$0''.01$] it was found necessary to undertake an examination into the numerical values of the constants used and into the effects produced by sources other than the direct solar action. This inquiry revealed several differences which had a perceptible effect. It was first necessary to develop a general method of dealing with the effects of planetary and other perturbations on these mean motions which would permit of their being found easily and accurately.... It was found necessary to make several changes in the values of these non-solar perturbations as collected in my papers in the *Monthly Notices* for 1897 March and June: four of these caused alterations of about half a second each in the annual motions.[119]

We will take up Brown's methods in this investigation in a later section of Part II; here we give his results as he reported them in April, 1904. Only one constant, at this date, seemed so far doubtful as to affect the results by as much as $0''.1$; this was f, the ratio of the thickness of the Earth's equatorial bulge to its equatorial radius. For f the values 1/292.9 and 1/296.3 were in competition.[120] Brown carried out the calculation first with the value $1/292.9 (= \alpha)$, then with the value $1/296.3 (= \beta)$:

Annual Mean Motions, Epoch 1850

	Perigee	Node
Calculated (α)	$+146,435''.27 \pm 0''.10$	$-69,679''.37 \pm 0''.05$
Calculated (β)	$+146,435''.11 \pm 0''.05$	$-69,679''.22 \pm 0''.05$
Observed	$+146,435''.23$	$-69,679''.45$
C – O (α)	+0.04	+0.08
C – O (β)	−0.12	+0.23

Taking into account possible errors – due to neglect of terms of higher orders than those calculated and to a questionable value of the mass of Venus, Brown judged errors as high as $\pm 0''.10$, $\pm 0''.05$ to be extreme. Thus α appeared to be better than β. (Today we know that α is more erroneous.)

From the sixth-order terms in rectangular coordinates, Brown proceeded to compute the final values of the terms in polar coordinates, using "the method of special values." He then carried out a comparison between the resulting coefficients and those obtained by Hansen; the report of it appeared in the *Monthly Notices* for January, 1905. He claimed to have obtained all terms in longitude and latitude equal to or greater than $0''.01$, and all terms in parallax equal to or greater than $0''.001$. In the following table we give the total number of terms in longitude and latitude, then the number of terms in which Hansen's coefficients (H) differed from Brown's (B) by more than $0''.02$, and by $0''.10$ or more. For Hansen's coefficients Brown used

119 "On the Degree Accuracy of the New Lunar Theory and on the Final Values of the Mean Motions of the Perigee and Node," *MNRAS, 64* (April, 1904), 524–534.

120 The present-day value is 1/298.297 (*Explanatory Supplement to the Astronomical Almanac*, ed. P. Kenneth Seidelmann, 1992, 700).

the values given by Newcomb, who had transformed them to make them comparable to Delaunay's coefficients, and so to Brown's.

	No. terms	$B - H > 0''.02$	$B - H \geq 0''.10$
Longitude	275	34	6
Latitude	237	10	3

The analogous comparisons in parallax are

	No. of terms	$B - H > 0''.002$	$B - H \geq 0''.010$
Parallax	148	11	4

Brown made another synoptic comparison of his theory with Hansen's, showing the sum of the absolute values of the differences $B - H$ in each coordinate:

In longitude ....$3''.61$
In latitude .... $1''.90$
In parallax .... $0''.242$

These are the maximum differences which tables constructed on the two theories would show. As Hill had suggested long before, differences of this smallness are inconsequential from a practical point of view (i.e. for the ordinary uses of the *Nautical Almanac*). Brown was striving for results that would be both certain and exact.

The astronomers Frank Schlesinger and Dirk Brouwer assess as follows the importance of Brown's solution of the "main problem:"[121]

> Both as to completeness and accuracy this solution surpassed the work of Brown's predecessors to a remarkable degree. Few terms having coefficients in longitude and latitude exceeding $0''.001$ were not included, and in the great majority of terms the uncertainty did not exceed $0''.001$. In Hansen's theory some coefficients were in error by some tenths of a second of arc; Delaunay's theory, on account of the slow convergence peculiar to his development, contained a few terms that were in error by as much as a whole second of arc.

The accuracy of Brown's computation was confirmed by a numerical verification of this part of the lunar theory, carried out by his former pupil, Dr. W.J. Eckert, during the last few years of Brown's life.[122]

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[121] *National Academy Biograpical Memoirs*, *XXI*, 245–246.

[122] See the final paragraph of E.W. Brown, "The equations of motion of the Moon," *American Journal of Mathematics, 60* (1938), 792, and also W.J. Eckert and Harry F. Smith, Jr., "The Solution of the Main Problem of the Lunar Theory by the Method of Airy," *Astronomical Papers prepared the Use of the American Ephemeris and Nautical Almanac, 19*, Part II, 196.

13

Correcting for the Idealizations: The Remaining Inequalities

We recall once more that solution of "the main problem" invoked two idealizations: the center of gravity of the Earth and Moon was to move in a perfect ellipse about the Sun; and the Sun, Earth, and Moon were to be point-masses or, equivalently, spherical bodies with mass-distributions symmetrical about their centers. To correct for these idealizations meant determining the direct and indirect actions of the planets on the Moon (the indirect actions are effects of the planetary perturbations of the Earth's motion, transmitted from the Earth to the Moon), and the effects of the non-spherical shapes of the Earth and Moon on the Moon's motions.

By the time Brown began preparations for computing these inequalities (*ca.* 1903), the hope, expressed by Laplace a century earlier, that all perceptible inequalities of the Moon would prove deducible from the law of gravitation alone, had dimmed. True, deductions from the theory had been extended and refined, and the precision with which the Moon's positions could be measured had been remarkably improved. But the problem-situation had complicated itself.

In 1853 John Couch Adams discovered that Laplace's value for the Moon's secular acceleration erred in excess by about $5''$ – nearly half its value. The Moon's secular acceleration, first detected by Halley in a comparison between ancient and 17th-century solar eclipses, assumed, like all astronomy before the 20th century, the Earth's diurnal rotation as measure of time, the clock. Mid-18th-century astronomers assigned various values to the empirical centennial increase in the Moon's motion: Dunthorne and after him Lalande gave values of about $10''$. (This value is equivalent to an angular acceleration of $20''$ per century per century.) The increase remained unexplained till 1787, when Laplace showed that the secular variation in the Earth's orbital eccentricity – a known periodic phenomenon caused by the gravitational action of the other planets on the Earth – implied such an effect during the present age, when this eccentricity is decreasing. Reduction in the Earth's orbital eccentricity leads to a reduction in the Sun's average gravitational pull on the Moon; the Earth's mean attraction of the Moon, thus enhanced relative to the Sun's mean attraction, pulls the Moon into an orbit closer to the Earth, in which its mean motion is greater. Laplace's theoretical value for the increase was slightly less than

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11″ arc-seconds of mean motion per century, a value in satisfactory agreement with the observed increase.

What Adams showed in 1853 was that Laplace's deduction, properly carried out, led to a smaller value. Laplace had treated e', the Earth's orbital eccentricity, as a constant when integrating the equations of motion, then substituted the variable value of e' in the *result* of the integration.[123] But this variable value should have been introduced into the differential equations from the start; the integration would then have yielded only 6″.05 of increase in a century (= an angular acceleration of 12″.10 per century-squared).

The correctness of this correction was hotly disputed by prominent astronomers – Hansen at first, Pontécoulant and Le Verrier for a longer time.[124] In 1859 Delaunay came to Adams' support, proving again that gravitational theory would yield an increase of only 6″ rather than 11″, or the 12″.18 value that Hansen had derived from ancient solar eclipses and used in his tables.

In 1863 Delaunay went on to suggest that the difference between the theoretical and observed values could be due to tidal friction, slowing the Earth's rotation. Just how the energy would be dissipated was not immediately and in detail clear.

Meanwhile, in completing his lunar tables in 1857, Hansen had fitted his theory to the Greenwich lunar observations for the century from 1750 to 1850. To the larger of the two Venus inequalities, which has a period of 239 years, he assigned a coefficient of 21″.47. The theory, Delaunay showed in 1863, could yield no more than 0″.272 for this coefficient. In effect, Hansen had inserted an empirical term of 21″.20. When Simon Newcomb sought to correct Hansen's theory in the early 1880s, he included in his solution for the long-term mean motion of the Moon a similar periodic empirical term. Brown would follow him in this. The inclusion of a periodical empirical term unavoidably influenced the value of the secular acceleration; neither of them could be separately ascertained with high precision from the observations. But the two terms could always be mutually adjusted so as not to disturb the theory's agreement with modem observations.

Newcomb, beginning in 1870, had undertaken a study of lunar observations, with a view to correcting Hansen's lunar tables for use in the U.S. *Nautical Almanac* (the tables used earlier were those devised by Benjamin Peirce). Hansen's larger Venus term, with a period of 239 years, might fit the Greenwich observations from 1750 to 1850, but it failed, Newcomb found, to fit observations before and after this period. It occurred to him that these divagations could be due to variations in the Earth's speed of rotation.[125] But this hypothesis implied that similar variations should occur in the observed motions of the planets; and this corroboration Newcomb was unable to obtain. He carried to his grave in 1909 the frustration of an unsolved mystery:

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[123] *Philosophical Transactions of the Royal Society*, 1853, 397–406. I am here following the resumé given by E.W. Brown in his *Introductory Treatise on the Lunar Theory*, 243.

[124] See D. Kushner, "The Controversy Surrounding the Secular Acceleration of the Moon's Mean Motion," *Archive for History of Exact Sciences, 39* (1988/89), 291–316.

[125] Newcomb, "Researches on the Motion of the Moon, Part I," *Washington Observations for 1875*, Appendix, 1878.

whence the Moon's divagations? We shall take up the later history of this problem in Part III of this study.

Brown in seeking to complete his theory knew that he did not and perhaps could not know the source of the empirical term or terms. He knew also that an empirical value of the secular acceleration could not be obtained independently of the periodic empirical term. He had to set this problem aside in order to define a clear immediate task for himself. The goal he set himself was to discover and evaluate quantitatively all *gravitational* effects on the Moon's motion.

The next part of our review, therefore, will deal with the inequalities remaining after solution of the "main problem": those arising from the direct and indirect actions of the planets, the deviations of the bodies of the Earth and Moon from mechanical sphericity, and second-order perturbations previously ignored. These are dealt with in Part V of *Theory of the Motion of the Moon*.[126]

At the beginning of Part V, Brown reported that he had found no new terms large enough to account for the empirical term or terms:

> . . . the search has led more and more to the conclusion that no such terms can possibly arise with the laws of motion and of gravitation on which this theoretical investigation is based. If these inequalities have a real existence, it would seem that the cause must be sought in some action not purely gravitational.[127]

The main difficulty of this final phase of Brown's undertaking did not lie in numerical work – the detailed computation of inequalities. There were few such computations he could turn over to other computers. Many terms and classes of terms had special peculiarities permitting the calculations to be abbreviated. To devise a general set of directions that a computer could follow would have meant ignoring such peculiarities, and, if the accuracy Brown was aiming at was to be achieved, would have entailed an amount of computation out of all proportion to the final results. No more than one-third of the time occupied in these investigations, by Brown's estimate, was spent in accurate numerical work. A large portion of it went into the construction of *sieves* – approximate tests which could identify which terms needed to be calculated. These rough calculations often required days or weeks. Brown in writing up Part V found it difficult to describe this part of the work; he settled for a general characterization.

His procedure was an application of the method of variation of arbitrary constants as developed by Lagrange. Laplace had lacked a systematic procedure for computing higher-order perturbations, and by the 1830s the superiority of Lagrange's procedure was generally recognized. It was fully systematic, and could be applied consecutively to all orders of perturbations. Hansen followed it in essentials, applying it in a numerical rather than a literal form, first to Saturn and then to the Moon. Delaunay applied it in a literal form to the Moon, but was kept from completing his computations by his accidental death in 1872. G.W. Hill in 1884 extended Delaunay's

126 *TMM, Memoirs of the R.A.S., 59* (1908), 1–103.
127 Ibid., *59*, 2–3.

computation to the lunar inequalities deriving from the oblate shape of the Earth. [128] J.C. Rodolph Radau in 1892 extended it to the planetary inequalities of the Moon.[129]

In the 1870s Simon Newcomb had also begun an investigation of the planetary perturbations of the Moon by the Lagrangian method. Lack of time prevented him from completing it, but he published his results in 1895, believing that some of them would prove useful.[130]

Brown thus had precedent for applying the Lagrangian procedure to the remaining inequalities. In the solution of the main problem, he had started with the variation orbit, then added perturbations of that orbit varying with the powers and products of four small parameters – lunar eccentricity, solar eccentricity, orbital inclination, and parallax. The new inequalities involved factors of a quite different kind: positions of planets, the shapes of the Earth and Moon, etc. These factors could be introduced as new terms in a disturbing function, and their effects systematically determined by the algorithms of the Lagrangian procedure.

This method, however, seemed especially difficult to apply in the case of the perturbations of a satellite like the Moon, where characteristics pertaining to the satellite, its primary, and several perturbing bodies had all to be taken into account. Hansen had complained about this difficulty. In response, Hill in an article of 1883 showed that the application was easy enough if the shortest computational routes were chosen.[131] The gist of his suggestion was this:

> The work may be divided into two portions, independent of each other. In one the object is to develop, in a periodic series, certain functions of the Moon's coordinates, which in number do not exceed five. This portion is the same whatever planet may be considered to act, and hence may be done once for all. In the other portion we seek the coefficients of certain terms in the periodic development of certain functions, five also in number, which involve the coordinates of the Earth and planet only. And this part of the work is very similar to that in which the perturbations of the Earth by the planet in question are the things sought.[132]

Radau followed Hill's proposal in his "Recherches concernant les inégalités planétaires de la lune" of 1892,[133] and Brown also followed it in his derivations of the Moon's planetary perturbations.

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[128] G.W. Hill, *Determination of the Inequalities of the Moon's Motion which are produced by the Figure of the Earth, Astronomical Papers of the American Ephemeris*, III (1884), 201–344; *The Collected Mathematical Works of George William Hill*, II, 179–320.

[129] J.C.R. Radau, *Recherches concernant les inégalités planétaires de la lune, Annales de l'Observatoire de Paris, 21* (1892), B1–B114.

[130] *Astronomical Papers prepared for the Use of the American Ephemeris and Nautical Almanac, 5*, 97–205.

[131] "On Certain Possible Abbreviations in the Computation of the Long-Period Inequalities of the Moon's Motion due to the Direct Action of the Planets," *American Journal of Mathematics, 6* (1883), 115–130.

[132] Ibid., 115.

[133] *Annales de l' Observatoire de Paris, 21*, B.1–B.114.

But there was still another difficulty, arising from the manner of Brown's solution of the "main problem." Lagrange's method presupposed a *literal* theory, in which the derivatives of the coordinates and velocity components with respect to the arbitrary constants of the solution (the orbital elements) could be written out as algebraic formulas. The lunar theory Brown was developing was a *semi-numerical* rather than a literal theory, in that the numerical value of the constant $m = n'/n$, or $\mathbf{m} = n'/(n - n')$, was introduced from the start. Consequently, no direct route was available for obtaining the algebraic formulas for the derivatives of the coordinates and velocity components with respect to n, the mean rate of motion of the Moon.

Brown had foreseen this difficulty. In a paper published in 1903,[134] he showed how the required derivatives could be obtained, not with complete precision but to a sufficient approximation. The following account gives a general idea of the method.

Derivatives of the Coordinates with Respect to *n*

The six arbitrary constants of the lunar theory may be taken as ε, π, θ, (the epochs of the Moon's mean longitude, perigee, and node), and e (the Moon's orbital eccentricity), γ (the sine of half the Moon's mean orbital inclination to the Ecliptic), and a (the Earth-Moon distance or reciprocal of the Moon's parallax). Of the six constants, the first three mentioned, ε, π, θ, occur only as parts of angular arguments of sines and cosines, whether the theory has been worked out literally or with numerical values. The derivative of a coordinate with respect to ε, π, or θ can therefore be calculated by an elementary application of the chain rule; the result will be as accurate as the terms differentiated.

The arbitraries e, γ, a occur in the *coefficients* of the periodic terms. Obtaining exact expressions for the derivatives of the coordinates and velocity components with respect to these parameters would require having explicit and exact algebraic relations between the coordinates and the coefficients – hence a literal theory. But given the slow convergence of Delaunay's theory, Brown had concluded that obtaining a sufficiently precise literal theory for the Moon's motion was a practical impossibility.

By 1903, however, he had obtained the fourth-order terms in the semi-numerical theory. From the theory developed this far, he believed he could obtain, with sufficient accuracy, the derivatives of the coordinates with respect to n.

Every coefficient of a periodic term was of the form $aA\lambda$, where λ is the product of the highest positive powers of e, γ, e', a that are factors of the coefficient, and A could be expanded in powers of m or $\mathbf{m}$, e^2, γ^2, e'^2, a^2. The convergence of A with respect to powers of e^2, γ^2, e'^2, a^2 appeared sufficiently rapid for practical purposes. With respect to m or $\mathbf{m}$, however, the convergence was often too slow – about as slow as that of a series in which the ratio of successive terms is 1/2. This was the case with the derivatives of the coordinates with respect to n. Brown was thus led to pose the following problem:

[134] E.W. Brown, "On the Formation of the Derivatives of the Lunar Coordinates with Respect to the Elements," *Transactions of the American Mathematical Society, 4* (1903), 234–248.

Given the derivatives of the various functions with respect to e, e', γ, a, *to find those with respect to n from a theory in which the numerical value of m has been substituted.*[135]

To resolve this problem, Brown had recourse to certain relations intrinsic to the Lagrangian method. To give an idea of the new method, we begin by imagining that the "main problem" of the lunar theory has been solved by the Lagrangian method. The equations of motion solved by this solution would be of the form

$$\frac{d^2x}{dt^2} = \frac{\partial F}{\partial x}, \quad \frac{d^2y}{dt^2} = \frac{\partial F}{\partial y}, \quad \frac{d^2z}{dt^2} = \frac{\partial F}{\partial z}, \tag{B.93}$$

where F is the force-function, and x, y, z are referred to fixed axes. The solution would involve six arbitrary constants, which might, for instance, be $e, \gamma, a, \varepsilon, \pi, \theta$ as listed above. Brown designated them generically by $a_p (p = 1, 2, \ldots, 6)$. He designated the derivatives of the coordinates and of their velocities with respect to the a_p by

$$x_p = \frac{dx}{da_p}, \quad \dot{x}_p = \frac{d^2x}{dt da_p} = \frac{d^2x}{da_p dt} = \frac{d\dot{x}}{da_p}, \ldots, \ldots.$$

The equations for the variations, which are deducible from (B.93), are then

$$\ddot{x}_p = \frac{\partial^2 F}{\partial x^2} x_p + \frac{\partial^2 F}{\partial x \partial y} y_p + \frac{\partial^2 F}{\partial x \partial z} z_p, \ddot{y}_p = \ldots, \ddot{z}_p = \ldots. \tag{B.94}$$

These equations possess $6!/2!4! = 15$ integrals of the form

$$(p, q) = \dot{x}_p x_q - \dot{x}_q x_p + \dot{y}_p y_q - \dot{y}_q y_p + \dot{z}_p z_q - \dot{z}_q z_p = C_{pq} (p, q = 1, 2, \ldots, 6), \tag{B.95}$$

where C_{pq} is a constant, with $C_{pq} = -C_{qp}$ and $C_{pp} = 0$. The left-hand members of (B.95) are called "Lagrange brackets."

Brown stipulated that the solution of (B.94) be such that x, y, z each consists of sines or cosines of sums of multiples of angles

$$w_j = b_j t + a_j \quad (j = 4, 5, 6).$$

The b_j and the coefficients of the periodic terms are functions of the arbitraries a_1, a_2, a_3. It is then well-known, Brown tells us, that

$$C_{ii'} = 0, C_{jj'} = 0, C_{ij} = \frac{dc_{j-3}}{da_i}, b_j = -\frac{dB}{dc_{j-3}} \quad (i, i' = 1, 2, 3;\ j, j' = 4, 5, 6),$$

where B is a constant expressible in terms of c_1, c_2, c_3, and the latter are functions of a_1, a_2, a_3.

Considering the sixth-order determinant

$$\Delta = |\dot{x}_p, x_p, \dot{y}_p, y_p, \dot{z}_p, z_p| \quad (p = 1, 2, \ldots, 6) \tag{B.96}$$

[135] Ibid., 239.

Brown formed the derivative $d\Delta/dt$. This derivative is got by differentiating in turn the elements in each of the six columns with respect to t and taking the sum of the six resulting determinants. Each of the six determinants turns out to be zero. Consequently, $d\Delta/dt = 0$ and $\Delta = K$, a constant.

Brown then obtained the value of this constant as the square-root of the 6×6 determinant

$$\Delta = |C_{pq}|^{1/2} = \begin{vmatrix} 0 & 0 & 0 & C_{14} & C_{15} & C_{16} \\ 0 & 0 & 0 & C_{24} & C_{25} & C_{26} \\ 0 & 0 & 0 & C_{34} & C_{35} & C_{36} \\ C_{41} & C_{42} & C_{43} & 0 & 0 & 0 \\ C_{51} & C_{52} & C_{53} & 0 & 0 & 0 \\ C_{61} & C_{62} & C_{63} & 0 & 0 & 0 \end{vmatrix}^{1/2}$$

$$= \begin{vmatrix} C_{14} & C_{15} & C_{16} \\ C_{24} & C_{25} & C_{26} \\ C_{34} & C_{35} & C_{36} \end{vmatrix} = K. \tag{B.97}$$

Designating the first minor of C_{ij} in the 6×6 determinant by k_{ij}, Brown introduced the definitions

$$X_i = k_{i4}x_4 + k_{i5}x_5 + k_{i6}x_6, \quad X_j = k_{1j}x_1 + k_{2j}x_2 + k_{3j}x_3, \ (i = 1, 2, 3;\ j = 4, 5, 6)$$

where x_i, x_j, we recall, are the partial derivatives of x with respect to the elements a_i, a_j. He was then able to prove that

$$\Delta = \sum_i \dot{x}_i X_i - \sum_j \dot{x}_j X_j,$$

or equivalently,

$$\sum_i (\dot{x}_i X_i - x_i \dot{X}_i) = K \quad (i = 1, 2, 3). \tag{B.98}$$

Defining Y_i, Z_j with respect to y_j, z_j in the same way as X_i was defined with respect to x_j, he obtained the two equations $K = \sum(\dot{y}_i Y_i - y_i \dot{Y}_i) = \sum(\dot{z}_i Z_i - z_i \dot{Z}_i)$.

Now suppose that all the derivatives of the coordinates x, y with respect to the orbital elements a_p are known except for x_1, y_1. (a_1 could be n.) Can (B.98) be solved for x_1, and the corresponding equation involving y_i, Y_i for y_1? To see that this can be done, we need to know that

$$\dot{x}_1 X_1 - x_1 \dot{X}_1 = \frac{d}{dt}(x_1 X_1).$$

To understand the signs in the left-hand member of this equation, recall that the coordinate x is given by a cosine term whose argument is a sum of angles of the form $b_j t + a_j$. The derivatives $x_i (i = 1, 2, 3)$ are to be understood as partial derivatives with respect solely to elements occurring in the *coefficient* of this cosine term; hence the derivative x_1 is a positive cosine term. The derivatives x_j in $X_i (= k_{i4}x_4 + k_{i5}x_5 + k_{i6}x_6)$, in contrast, are with respect to orbital elements occurring in the *arguments* of cosine terms; hence X_i is a sum of negative sine terms. However, in the first term of the left-hand member, the first factor is dx_1/dt; since t occurs in the *argument* of x_1, the time-derivative with respect to t will be a negative sine; the first term, therefore, is a product of negative factors, hence positive. In the second term of the left-hand member, the differentiation of X_1 (a sum of negative sine terms) with respect to t will not change the sign, but produce a sum of negative cosine terms; the sign of the second term is therefore negative.

The solution of (B.98) for x_1 is therefore

$$x_1 = X_1 \int \frac{K - \dot{x}_2 X_2 + x_2 \dot{X}_2 - \dot{x}_3 X_3 + x_3 \dot{X}_3}{X_1^2} dt; \tag{B.99}$$

the equation in y_i, Y_i analogous to (B.98) gives the solution for y_1.

Applying (B.99) directly to the lunar theory leads to difficulties, among them the emergence of terms proportional to the time, and the possibility that the principal term of X_1, which appears as a denominator in the integrand of (B.99), may vanish, since it is the derivative of $a \cos(nt + \varepsilon)$ with respect to n. Brown avoided these difficulties by using a *canonical* system of orbital elements. "Canonical" here means, as earlier, that the elements are related to one another in pairs through a force-function, the time-derivative of each element being given by a partial derivative of the force-function with respect to the element with which it is paired.

For the three angular canonical parameters, Brown chose the Moon's mean motion and the mean motions of its perigee and node, designating them by

$$w_i = b_i t + a_{i+3} \quad (i = 1, 2, 3).$$

These are the same angles as used previously, but with changed subscripts. Their values are understood to be those that the w_i have after the "main problem" has been solved.

These three parameters being chosen, the remaining three parameters, $c_i (i = 1, 2, 3)$, in order to be canonical, had to be such that

$$\frac{dc_i}{dt} = \frac{\partial F}{\partial w_i}, \quad \frac{dw_i}{dt} = -\frac{\partial F}{\partial c_i} + b_i \quad (i = 1, 2, 3).$$

The c_i can be derived in more than one way; Brown, for instance, had shown[136] that

$$c_i = \text{const.term in } \dot{x}x_{i+3} + \dot{y}y_{i+3} + \dot{z}z_{i+3}.$$

The b_i and the coefficients of the periodic terms in the theory depend only on the arbitraries c_1, c_2, c_3. These assumptions turn out to imply that

136 "On certain Properties of the Mean Motions, etc.," *Proceedings of the London Mathematical Society, 28* (1896), 150.

$$b_i = -\frac{dB}{dc_i}(i = 1, 2, 3), \quad \frac{dc_1}{dc_2} = -\frac{db_2}{dn}, \quad \frac{dc_1}{dc_3} = -\frac{db_3}{dn},$$

where B, as before, is expressed in terms of c_1, c_2, c_3 and the known constants.

Given the values of c_1, c_2, c_3, Brown found that, for substitution in (B.99),

$$K = 1, \quad k_{i,i+3} = 1, \quad k_{ij} = 0 \ \ (j \neq i + 3), \quad X_i = x_{i+3}.$$

The derivatives of x or y with respect to any of the c_i have two terms, since the c_i occur both in the coefficients of the sines and cosines, and in b_j. Using the operator $\partial/\partial c_i$ to denote differentiation with respect to c_i solely where it occurs in the *coefficients* of the periodic terms on which it operates, Brown obtained

$$\frac{dx}{dc_i} = \frac{\partial x}{\partial c_i} + t\sum_j \frac{db_j}{dc_i}x_j,$$

$$\frac{d\dot{x}}{dc_i} = \frac{d}{dt}\frac{\partial x}{\partial c_i} + \left(1 + t\frac{d}{dt}\right)\left(\sum_j \frac{db_j}{dc_i}x_j\right)$$

$$= \frac{\partial}{\partial c_i}\frac{dx}{dt} + t\frac{d}{dt}\left(\sum_j \frac{db_j}{dc_i}x_j\right).$$

Substituting these expressions into (B.99), and equating separately to zero the terms which do and do not contain t as a factor, he obtained in the first case

$$\frac{db_j}{dc_i} = \frac{db_{i+3}}{dc_{j-3}} \ (i = 1, 2, 3;\ j = 4, 5, 6), \tag{B.100}$$

and in the second case,

$$\sum_i\left(x_{i+3}\frac{d}{dt}\frac{\partial \dot{x}}{\partial c_i} - \dot{x}_{i+3}\frac{\partial x}{\partial c_i}\right) = 1,$$

$$\text{or} \quad \sum_i (x_{i+3}\dot{x}_i - \dot{x}_{i+3}x_i) + \sum_j\sum_i \frac{db_j}{dc_i}x_j x_{i+3} = 1, \tag{B.101}$$

according as the first or second form of $d^2x/dtdc_i$ is used. He employed the first equation of (B.101) when substituting $i = 1$, and the second when substituting $i = 2, 3$, and so found

$$x_4\frac{\partial \dot{x}}{\partial c_1} - \dot{x}_4\frac{\partial x}{\partial c_1} = 1 - x_5\frac{\partial \dot{x}}{\partial c_2} + \dot{x}_5\frac{\partial x}{\partial c_2} - x_6\frac{\partial x}{\partial c_3} + \dot{x}_6\frac{\partial x}{\partial c_3} - x_4\sum_j \frac{db_j}{dc_i}x_j$$

$$= Q - x_4^2\frac{db_4}{dc_1}.$$

Putting $db_5/dc_1 = db_4/dc_2, db_6/dc_1 = db_4/dc_3$ in accordance with (B.100), then integrating and dividing by x_4^2, Brown had at last the result

$$\frac{\partial x}{\partial c_1} = x_4 \int \left(\frac{Q}{x_4^2} - \frac{db_4}{dc_1} \right) dt. \tag{B.102}$$

The derivative db_4/dc_1 on the right-hand side is a constant since b_4 and c_1 are constants; hence this term on integration will lead to a term in $\partial x/\partial c_1$ factored by t, which is impossible. Therefore db_4/dc must be set equal to the constant term in the expansion of Q/x_{42}.

Although (B.102) solves the problem Brown had set for himself, he did not leave his result in this form, but chose to adapt the results to the semi-canonical system $b_4(= n), c_2, c_3$, so that c_1 becomes a function of the independent constants n, c_2, c_3. In this semi-canonical system, he found the new form of (B.102) to be

$$\frac{\partial x}{\partial n} = X_1 \int \left(\frac{dc_1}{dn} \frac{Q'}{X_1^2} - 1 \right) dt \tag{B.103}$$

$$\text{where} \quad Q' = 1 - x_5 \frac{\partial \dot{x}}{\partial c_2} + \dot{x}_5 \frac{\partial x}{\partial c_2} - x_6 \frac{\partial \dot{x}}{\partial c_3} + \dot{x}_6 \left(\frac{\partial x}{\partial c_3} \right).$$

The only derivative with respect to n in the integrand of (B.103) is dc_1/dn, and this is determined by the vanishing of the constant term under the integral sign. (B.103) could have been obtained, Brown tells us, from (B.99) by direct transformation of the set c_1, c_2, c_3 into the set n, c_2, c_3.

14

Direct Planetary Perturbations of the Moon (The Adams Prize Paper)

Brown gave his full account of the derivation of the direct planetary perturbations of the Moon, not in his *Theory of the Motion of the Moon*, but in a separate treatise entitled *The Inequalities in the Motion of the Moon due to the Direct Action of the Planets*. This essay was awarded the Adams Prize of the University of Cambridge for 1907, and was published by the Cambridge University Press in 1908.[137] Brown's dedication of the work reads

> To George Howard Darwin, at whose suggestion the study of the Moon's motion was undertaken by the author, and whose advice and sympathy have been freely given during the past twenty years, this essay is gratefully dedicated.

Another award came to Brown early in 1907: on February 8 he became the seventh recipient of the Gold Medal of the Royal Astronomical Society. The president of the society, William H. Maw, devoted his presidential address that year to the presentation of the award. (Brown could not be present.) This award, from its inception, had been given exclusively for contributions to the lunar theory. The previous recipients had been the Baron Damoiseau (1831), Giovanni Plana (1840), Peter Hansen (1860), J.C. Adams (1866), Charles Delaunay (1870), and G.W. Hill (1887). With these illustrious workers, Maw stated, "our present medalist is well qualified to rank".[138] According to P.H. Cowell as quoted by Maw, Brown was "the first Lunar theorist to use independent equations of verification," his device being to form a small variation of the solution of Hill's equations:

> The numerical application of this device was rendered possible by calculating series for various complicated fractions of the coordinates in Hill's variation curve. The utility of the plan is obvious as soon as it is got into working order, and its conception implies rare insight on the part of our medalist. It lies at the root of his success in obtaining more accurate results,

[137] We shall reference this treatise hereafter simply as "the Adams prize paper."
[138] "The President's Address," *MNRAS, 67* (1907), 300.

C. Wilson, *The Hill–Brown Theory of the Moon's Motion*, Sources and Studies in the History of Mathematics and Physical Sciences, DOI 10.1007/978-1-4419-5937-9_14,

> with less labor than his predecessors. He has also obtained theorems by which the higher parts of the motion of the perigee and the node may be calculated in advance of the corresponding group of periodic terms.[139]

The following passage, also quoted by Maw, is from a letter by G.W. Hill:

> Much as we rightly welcome the results of Professor Brown's devoted labors, we should be unwarranted in assuming that their employment in the Lunar tables would give rise to a marked improvement in the representation of observations. A slight one might indeed be expected; but it has been evident for some time that the Moon deviated from its calculated orbit more because it is subject to irregular forces, which we have not yet the means of estimating, than because the tables are affected by slight defects in the mathematical treatment of the forces which are already recognized. This circumstance in no sense diminishes the credit due to Professor Brown's work.[140]

Hill is referring to the troublesome "empirical term." Maw also mentions receiving a letter from Brown in which

> he modestly states that the only portions of his work presenting real difficulties were those arising from the *direct* and *indirect* actions of the planets.[141]

From this we gather that, toward the end of Brown's extended assault on the lunar inequalities, the terrain to be traversed became more difficult.

Section I of the Adams Prize Paper

Brown took his start from the canonical equations for determining the direct planetary perturbations of the Moon. We gave the same equations earlier, using the symbol F for the force-function. Brown now proposed to add to F the additional terms necessary to account for the planetary perturbations, symbolizing these terms by R, so that the equations now to be solved were

$$\frac{dc_i}{dt} = \frac{\partial R}{\partial w_i}, \quad \frac{dw_i}{dt} = -\frac{\partial R}{\partial c_i} + b_i \quad (l = 1, 2, 3). \tag{B.104}$$

Here b_1, b_2, b_3 are the mean motions of the Moon, of its perigee, and of its node, and the c_i are the canonical constants corresponding to the w_i, and are functions of n, $\mathbf{e}$, $\mathbf{k}$, the constants of mean motion, eccentricity, and orbital inclination in Brown's lunar theory. The c_i also contain e', n', which characterize the Sun's (or Earth's) motion.

[139] Ibid., 310.
[140] Ibid., 308.
[141] Ibid., 309.

But rather than proceeding to solve (B.104), Brown (as in his paper of 1903) first shifted to the semi-canonical system n, c_2, c_3, while retaining the w_i unchanged. Putting

$$\frac{dc_1}{dn} = -a^2\beta,$$

and remembering that

$$\frac{dc_1}{dc_2} = -\frac{db_2}{dn}, \quad \frac{dc_1}{dc_3} = -\frac{db_3}{dn},$$

he found

$$\frac{dn}{dt} = \frac{1}{a^2\beta}\left(-\frac{\partial R}{\partial w_1} - \frac{db_2}{dn}\cdot\frac{dc_2}{dt} - \frac{db_3}{dn}\cdot\frac{dc_3}{dt}\right), \quad \frac{dw_1}{dt} = \frac{1}{a^2\beta}\frac{\partial R}{\partial n} + b_1,$$

$$\frac{dc_2}{dt} = \frac{\partial R}{\partial w_2}, \quad \frac{dw_2}{dt} = -\frac{\partial R}{\partial c_2} + b_2 + \left(\frac{dw_1}{dt} - b_1\right)\frac{db_2}{dn},$$

$$\frac{dc_3}{dt} = \frac{\partial R}{\partial w_3}, \quad \frac{dw_3}{dt} = -\frac{\partial R}{\partial c_3} + b_3 + \left(\frac{dw_1}{dt} - b_1\right)\frac{db_3}{dn}. \tag{B.105}$$

Here b_2, b_3, c_1 are understood to be expressed in terms of n, c_2, c_3, and R in terms of $n, c_2, c_3, w_1, w_2, w_3$.

A periodic term in R has the form

$$R = n'^2\mathrm{a}^2 A\cos(qt + q') = n'^2\mathrm{a}^2 A\cos(i_1w_1 + i_2w_2 + i_3w_3 + q''t + q'''),$$

where **a** is the linear constant of Brown's lunar theory; A is a dimensionless numerical coefficient; i_1, i_2, i_3 take integral values; and $q''t + q'''$ is a combination of the solar and planetary arguments. The time-derivatives of n, c_2, c_3 in (B.105) are thus

$$\frac{dn}{dt} = \frac{n'^2}{\beta}\cdot\frac{\mathrm{a}^2}{a^2}A\frac{dq}{dn}\sin(qt + q'),$$

$$\frac{dc_2}{dt} = -i_2n'^2\mathrm{a}^2A\sin(qt + q'),$$

$$\frac{dc_3}{dt} = -i_3n'^2\mathrm{a}^2A\sin(qt + q'). \tag{B.106}$$

Among the planetary perturbations of the Moon, Brown knew that there were many long-period terms with tiny coefficients. Inequalities with periods greater than 3500 years, he decided, could be safely ignored. He initially assumed that the planetary perturbations were small enough so that they could be treated as first-order variations (later he would investigate whether second-order variations would need to be computed). Equations (B.106) could then be integrated with respect to t so as to yield the increments δn, δc_2, and δc_3:

$$\frac{\delta n}{n} = -\frac{m}{\beta} \cdot \frac{\mathrm{a}^2}{a^2} \cdot \frac{dq}{dn}\frac{n'}{q} A\cos{(qt+q')},$$

$$\frac{\delta c_2}{na^2} = i_2 m \cdot \frac{\mathrm{a}^2}{a^2}\frac{n'}{q} A\cos{(qt+q')},$$

$$\frac{\delta c_3}{na^2} = i_3 m \cdot \frac{\mathrm{a}^2}{a^2}\frac{n'}{q} A\cos{(qt+q')}. \tag{B.107}$$

The corresponding equations for $\delta w_1, \delta w_2, \delta w_3$ proved to be

$$\delta w_1 = \frac{1}{\beta}\frac{\mathrm{a}^2}{a^2}\left(m\frac{n'}{q}A_1 - \frac{dq}{dn}\frac{n'^2}{q^2}A\right)\sin(qt+q'),$$

$$\delta w_2 = \frac{\mathrm{a}^2}{a^2}\left\{m\frac{n'}{q}\left(A_2 + \frac{A_1}{\beta}\frac{db_2}{dn}\right) - \frac{n'^2}{q^2}A\left(\frac{q_2}{n} + \frac{1}{\beta}\frac{db_2}{dn}\frac{dq}{dn}\right)\right\}\sin(qt+q'),$$

$$\delta w_3 = \frac{\mathrm{a}^2}{a^2}\left\{m\frac{n'}{q}\left(A_3 + \frac{A_1}{\beta}\frac{db_3}{dn}\right) - \frac{n'^2}{q^2}A\left(\frac{q_3}{n} + \frac{1}{\beta}\frac{db_3}{dn}\frac{dq}{dn}\right)\right\}\sin(qt+q'),$$
$$\tag{B.108}$$

$$A_1 = \frac{n}{\mathrm{a}^2}\frac{d}{dn}(\mathrm{a}^2 A), \quad A_2 = -a^2 n\frac{dA}{dc_2}, \quad A_3 = -a^2 n\frac{dA}{dc_3},$$

where

$$q_2 = -na^2\frac{dq}{dc_2}, \quad q_3 = -na^2\frac{dq}{dc_3}.$$

Equations (B.107) and (B.108) constituted Brown's solution of the problem.

Numerical values had next to be substituted for the derivatives

$$\frac{dc_1}{dn}, \quad \frac{db_2}{dn}, \quad \frac{db_3}{dn}, \quad \frac{db_2}{dc_2}, \quad \frac{db_2}{dc_3} = \frac{db_3}{dc_2}, \quad \frac{db_3}{dc_3},$$

and the other constants in (B.107) and (B.108). Brown relied on his own earlier evaluations of constants as compared with the constants adopted by Delaunay, Hill, Newcomb, and Radau. The functions c_1, c_2, c_3 were the same as Delaunay's L, $G - L$, $H - G$, after the transformations to Delaunay's final system of arbitraries had been made These same elements were used by Newcomb and Radau. However, Brown in his theory had adopted the constants **e** and **k** rather than Delaunay's e and γ, and to make comparisons with Delaunay's, Newcomb's and Radau's numbers, he had to convert their constants into his own system. The calculations involved were not intricate, but Brown invested enormous care to insure that his results were accurate to four significant figures. With the introduction of numbers, (B.107) and (B.108) took the following form, where the numbers in square brackets are common logarithms plus 10.

$$\frac{\delta n}{n} = (-i_1 + .01486 i_2 - .003744 i_3) f' \frac{A}{s} \cos(qt + q'),$$

$$\frac{\delta c_2}{na^2} = +[9.51801] i_2 f' \frac{A}{s} \cos(qt + q'),$$

$$\frac{\delta c_3}{na^2} = +[9.51801] i_3 f' \frac{A}{s} \cos(qt + q'); \tag{B.107a}$$

$$\delta w_1 = \left\{(-i_1 + .01486 i_2 - .003744 i_3) f \frac{A}{s^2} + f' \frac{A_1}{s}\right\} \sin(qt + q'),$$

$$\delta w_2 = \left\{\begin{array}{l} (+.01486 i_1 - .007066 i_2 - .008148 i_3) f \dfrac{A}{s^2} \\ + \left(-[8.1720] A_1 + [11.0999] \dfrac{dA}{d\mathrm{e}} - [7.9422] \dfrac{dA}{d\mathrm{k}}\right) \dfrac{f'}{s} \end{array}\right\} \sin(qt + q'),$$

$$\delta w_3 = \left\{\begin{array}{l} (-.003744 i_1 - .008148 i_2 + .001210 i_3) f \dfrac{A}{s^2} \\ + \left(+[7.5733] A_1 + [10.2620] \dfrac{dA}{d\mathrm{k}} - [8.2962] \dfrac{dA}{d\mathrm{e}}\right) \dfrac{f'}{s} \end{array}\right\} \sin(qt + q'). \tag{B.108a}$$

Here

$$A_1 = \frac{n}{\mathrm{a}^2} \frac{d}{dn} (Aa^2),$$

s = no. of arc-seconds in the daily mean motion of the argument $qt + q'$,

s' = no. of arc-seconds in the daily mean motion of the Sun = $3548''.19$,

$$\beta = -\frac{1}{a^2} \cdot \frac{dc_1}{dn}, \quad f = \frac{1}{4} \frac{m''}{m'} \frac{\mathrm{a}^2}{a^2} \frac{s'^2}{\beta} 2{,}06{,}265, \quad f' = \frac{1}{4} \frac{m''}{m'} \frac{\mathrm{a}^2}{a^2} \frac{ms'}{\beta} 2{,}06{,}265.$$

The number 2,06,265 is $180 \cdot 3600/\pi$, the factor for turning radians into arc-seconds.

Section II of the Adams Prize Paper

Brown now transformed the disturbing function. This had been given initially as

$$\frac{R}{m''} = \frac{1}{[(\xi - x)^2 + (\eta - y)^2 + (\zeta - z)^2]^{1/2}} - \frac{x\xi + y\eta + z\zeta}{\Delta^3}, \tag{B.109}$$

where m'' is the mass of the perturbing planet, ξ, η, ζ are its coordinates measured from the Earth's center, and Δ is its distance from that center; x, y, z are the Moon's coordinates, again with the same center as origin. The first term of (B.109) yields by differentiation the components of the planet's action on the Moon; the second term yields the planet's action in drawing the Earth in a direction opposite to that in which the Earth draws the Moon. Brown expanded the first term by Taylor's theorem for three variables, using the operator

$$x\frac{\partial}{\partial\xi}+y\frac{\partial}{\partial\eta}+z\frac{\partial}{\partial\zeta}=\frac{\partial}{\partial Q},$$

and obtained

$$\begin{aligned}&\frac{1}{[(\xi-x)^2+(\eta-y)^2+(\zeta-z)^2]^{1/2}}\\&\quad=\left[1-\frac{\partial}{\partial Q}+\frac{1}{2!}\left(\frac{\partial}{\partial Q}\right)^2-\frac{1}{3!}\left(\frac{\partial}{\partial Q}\right)^3+\cdots\right]\frac{1}{\Delta}\\&\quad=\frac{1}{\Delta}+\frac{x\xi+y\eta+z\zeta}{\Delta^3}+\left[\frac{1}{2!}\left(\frac{\partial}{\partial Q}\right)^2-\cdots\right]\frac{1}{\Delta}.\end{aligned}\tag{B.110}$$

The second term in this expansion will cancel the second term of (B.109). Since R will be used only in the form of its derivatives with respect to the lunar orbital *elements*, and these elements are absent from Δ, the effective disturbing function is

$$\frac{R}{m''}=\left[\frac{1}{2!}\left(\frac{\partial}{\partial Q}\right)^2-\frac{1}{3!}\left(\frac{\partial}{\partial Q}\right)^3+\cdots\right]\frac{1}{\Delta}.\tag{B.111}$$

The separation of the terms of R into a sum of products, one factor of which involves the lunar coordinates and the other the planet's coordinates, as proposed by Hill and adopted by Radau, is implicit here.

The next step was to substitute complex coordinates:

$$\begin{aligned}&x+y\sqrt{-1}=\mathrm{u},\quad \xi+\eta\sqrt{-1}=\mathrm{u}_1,\\&x-y\sqrt{-1}=\mathrm{s},\quad \xi-\eta\sqrt{-1}=\mathrm{s}_1,\\&r^2=\mathrm{us}+z^2,\quad \Delta^2=\mathrm{u}_1\mathrm{s}_2+\zeta^2,\\&\frac{\partial}{\partial Q}=\mathrm{u}\frac{\partial}{\partial \mathrm{u}_1}+\mathrm{s}\frac{\partial}{\partial \mathrm{s}_1}+z\frac{\partial}{\partial\zeta}.\end{aligned}$$

With these substitutions, $1/\Delta$ became a function of the complex variables u, s, u_1, s_1, so that the Cauchy-Riemann conditions applied to its real and imaginary parts. Those conditions implied two second-order differential equations that Laplace had been the first to derive. The formulas for these equations, given $u+iv=f(\xi+\eta i)=f(z)$, are

$$\frac{\partial^2u}{\partial\xi^2}+\frac{\partial^2u}{\partial\eta^2}=0,\quad \frac{\partial^2v}{\partial\xi^2}+\frac{\partial^2v}{\partial\eta^2}=0.$$

Thus both the real and the imaginary parts of the function $f(z)$ satisfy the condition that their second-order derivatives add to zero. Applied to $1/\Delta$, this result gave

$$\frac{\partial^2}{\partial\zeta^2}\frac{1}{\Delta} = -\left(\frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\eta^2}\right)\frac{1}{\Delta} = -4\frac{\partial^2}{\partial \mathrm{u}_1 \partial \mathrm{s}_1}\frac{1}{\Delta}.$$

From this equivalence Brown derived a rule for expanding the disturbing function: expand $\left(\frac{\partial}{\partial Q}\right)^n$ and replace $\left(\frac{\partial}{\partial\zeta}\right)^{2m}$ by $\left(-4\frac{\partial^2}{\partial \mathrm{u}_1 \partial \mathrm{s}_1}\right)^m$ and us by $r^2 - z^2$.

Next, Brown replaced the derivatives with respect to the coordinates by the derivatives with respect to a' (the Earth's mean solar distance), T (the Earth's mean longitude), h'' (mean longitude of the node of the planet on the ecliptic), and γ'' (the sine of half the inclination of the planet's orbit to the ecliptic). This enabled him to exploit an already available expansion of $1/\Delta$, given by Le Verrier in Volume I of the *Annales* of the Paris Observatoire. Le Verrier had calculated it to terms of the seventh order in the eccentricities and mutual inclination of the Earth and perturbing planet.[142]

The algebra here, lengthy and complex, we omit. According to Brown, the resulting formulas are easy and rapid for numerical calculation. (As indicated earlier, the computations beyond those used in solving the "main problem" were performed largely by Brown himself.) The main features of Brown's transformation may be characterized as follows.

Each term of R is made to consist of two main parts together with a constant factor. The first part is a function of the Moon's coordinates, with which are combined coefficients involving the Earth's solar distance and true longitude: $1/r'^2$, $1/r'^3$, $e^{(V'-h'')j}$, where j denotes $\sqrt{(-1)}$. The second part involves the derivatives of $1/\Delta$, Δ being a function of the Earth's and the disturbing planet's elements only. Let θ be an angle present in the first part, and ϕ an angle present in the second part; $\theta \pm \phi$ will then be the argument of a term in R.

Brown expressed the first part in terms of certain coefficients M_i which could be computed once for all:

$$M_1 = \text{coef. of } e^{j\theta} \text{ in } \frac{a'^2}{r'^2}\cdot\frac{r^2 - 3z^2}{\mathrm{a}^2},$$

$$\frac{1}{2}(M_2 \pm M_3) = \text{coef. of } e^{\pm j\theta} \text{ in } \frac{a'^2}{r'^2}e^{\mp 2j(V'-T)}\cdot\frac{\mathrm{u}^2}{\mathrm{a}^2},$$

with similar formulas for $M_4, M_5, \ldots, M_{12}$.

He expressed the planet portions of R in terms of derivatives of $1/\Delta$ with respect to α, T, h'', l', the quantities present explicitly in Le Verrier's expansion. Suppose a_1, a_2 represent the mean solar distances of two planets, with $a_1 < a_2$. Brown put $\alpha = a_1/a_2$ along with

$$\alpha\frac{\partial}{\partial\alpha} = D.$$

Then, for the Earth and an inner planet, $a_2 =$ the Earth's mean distance (a'), $a_1 =$ the planet's mean distance (a''), and (because $1/\Delta$ is homogeneous in α with degree -1),

[142] *Annales de l'Observatoire de Paris*, I. Brown notes that Boquet, ibid., XIX, had given the terms of the eighth order.

$$r'\frac{\partial}{\partial r'} = -\alpha\frac{\partial}{\partial\alpha} - 1 = -D - 1.$$

For an outer planet and the Earth, $a_1 = a'$, $a_2 = a''$, and

$$r'\frac{\partial}{\partial r'} = \alpha\frac{\partial}{\partial\alpha} = D.$$

For the second part Brown then put

$$\frac{1}{\Delta} = P\cos\phi = P\cos(iT + 2i_1h'' + \phi'),$$

where i and i_1 are integers and ϕ' is independent of T, h''. The planetary factors in the several terms of R are obtained as derivatives of P, for example,

$$P_1 = [(D+1)^2 - i^2]P,$$

$$P_2 = \frac{1}{2}P_1 + (D+2)P + (i^2-1)P,$$

$$P_3 = (D+2)iP,$$

$$P_4 + P_5 = (D+1-i)\left\{i + \frac{i_1}{\gamma''^2} + (1-\gamma''^2)\frac{\partial}{\partial\gamma''^2}\right\}P,$$

$$P_4 - P_5 = (D+1+t)\left\{-i - \frac{i_1}{\gamma''^2} + (1-\gamma''^2)\frac{\partial}{\partial\gamma''^2}\right\}P, \text{ etc.}$$

Dividing R into two parts, R_1 and R_2, Brown gave for R_1 the formula

$$R_1 = \frac{m''}{4m'}n'^2\mathrm{a}^2a'\left[M_1P_1 + M_2P_2 \mp M_3P_3 - \frac{2\gamma''}{\sqrt{1-\gamma''^2}}M_4(P_4 \pm P_5)\right]\cos(\theta\pm\phi), \tag{B.112}$$

all upper signs being taken for the sum $\theta + \phi$, and the lower signs for the difference $\theta - \phi$.

R_2 is similarly formed from products of additional coefficients M_i and P_p:

$$R_2 = \frac{1}{16}\frac{m''}{m'}\cdot n'^2\mathrm{a}^2\frac{\mathrm{a}}{a'}a'\left[\begin{array}{l} M_6P_6 \mp M_7P_7 + M_8P_8 \mp M_9P_9 \\ -\dfrac{2\gamma''}{\sqrt{1-\gamma''^2}}\left\{\begin{array}{l} M_{10}(P_{10} \pm P_{11}) \\ \quad +M_{12}(P_{12} \pm P_{13}) \end{array}\right\} \end{array}\right]\cos(\theta\pm\phi). \tag{B.113}$$

The formulas (B.112) and (B.113) are for inner planets. For exterior planets the corresponding formulas are obtained by substituting $(-D-1)$ for D, and the factor $\alpha a''$ for a'.

Section III of the Adams Prize Paper

This section concerns the computation of the coefficients in the expansion just described. The distance Δ between disturbing planet and the Earth is given by

$$\Delta^2 = 1 + \alpha^2 - 2\alpha \cos(T - P),$$

where T and P are the heliocentric longitudes of the Earth and disturbing planet. The successive inverse odd powers of Δ, with $\sum$ denoting summation for integral values of i from $-\infty$ to $+\infty$, can be represented generically by

$$\frac{\alpha^{\frac{s-1}{2}}}{\Delta^s} = \frac{1}{2}a' \sum \beta_s^{(i)} \cos i(T - P),$$

where s is an odd integer. Brown obtained these expansions from Tisserand's *Traité de Mécanique Céleste*. To form the functions P_p, the derivatives $\left(\alpha \frac{d}{d\alpha}\right)^p$ of the coefficients $\beta_s^{(i)}$ were required; Brown represented them by

$$\beta_{s,p}^{(i)} = \frac{1}{p!}\alpha^p \frac{d^p}{d\alpha^p}\beta_s^{(i)}.$$

The coefficients in Le Verrier's expansion of $1/\Delta$ are all functions of e', e'', γ'' and the $\beta_{s,p}^{(i)}$. Brown's method of deriving the P_p, he tells us

> ...does not necessarily give the shortest algebraical expressions for the coefficients, but for numerical computation, which is the principal end in view, these expressions have this advantage – that they require very little use of logarithmic tables. The calculations consist mainly of additions, subtractions and multiplications by integers less than 100, and the functions are read straight from Le Verrier's expansion. Moreover, it is possible to see almost immediately when the terms in a given coefficient become insensible.[143]

In illustration of the last-mentioned assertion, Brown turned to the largest planetary perturbation of the Moon, a Venusian perturbation. It has the argument $\ell + 16T - 18V$, in which ℓ is the mean anomaly of the Moon and V the heliocentric longitude of Venus. The principal term has the factor γ''^2 and the argument $\ell + 16T - 18V - 2h''$. The contributions to it that are greater than $0''.01$ are shown in the following table, with their orders indicated in the left-hand column.

Order	P_1	$(P_2 + P_3)/2$	$(P_2 - P_3)/2$
γ''^2	$-15''.89$	$-1''.28$	$+1''.19$
γ''^4	$+1.65$	$+.17$	$-.09$
γ''^6	$-.11$	$+.01$	$.00$
γ''^8	$+.01$		
$e'^2\gamma''^2$	$+.27$	$.00$	$-.01$
$e''^2\gamma''^2$	$+.12$	$+.01$	$-.01$
$e'^4\gamma''^2$	$.00$		
$e'^2e''^2\gamma''^2$	$-.01$		
Sums	$-13''.96$	-1.11	$+1.08$

[143] The Adams Prize paper, 27–28.

The sum of the sums is $-13''.99$; Brown stated it to be accurate to within $0''.05$, or 0.36%, and explained why he had had to accept this upper bound, greater than $0''.01$:

> [T]he additional computations necessary to obtain the final coefficient within $0''.005$ are not very long, but in view of the uncertainty in the mass of Venus, which is doubtful within one per cent, and of the length of the period of the term [273 years], the present results are sufficient.[144]

None of the other planetary perturbations of the Moon are as great as $2''$. Since, in general, the degree of accuracy aimed at was $0''.01$, Brown concluded that, where the computations required logarithms, it would be sufficient to use four-place tables and to retain four significant figures in the quantities involved, so that the final results would be accurate to three significant figures.

The most arduous part of the work, according to Brown, consisted in the computation of the derivatives of r^2, $x^2 - y^2 + 2ixy = \mathrm{u}_0^2$ with respect to n, to the accuracy required. He used the method described in his paper of 1903.[145]

Section IV – A Sieve for the Rejection of Insensible Coefficients

Brown divided the planetary perturbations of the Moon into two categories, which he called *primary* and *secondary*. The primary inequalities were those arising from the substitution of $w_1 + \delta w_1$ for the non-periodic term of V, the Moon's true longitude; they were thus given by

$$\delta V = \delta w_1.$$

The secondary inequalities were those arising from the substitution of the variations of the elements in the periodic terms, and so were obtained from

$$\delta V = \left(\frac{dV}{dw_1} - 1\right)\delta w_1 + \frac{dV}{dw_2}\delta w_2 + \frac{dV}{dw_3}\delta w_3 + \frac{dV}{dn}\delta n + \frac{dV}{dc_2}\delta c_2 + \frac{dV}{dc_3}\delta c_3.$$

(An analogous formula gives the variations of the radius vector r.) The majority of the primary inequalities have periods of a year or more. Nearly all the secondary inequalities have periods of a month or less.

Among the primary inequalities, only those with a period of a year or more were in need of a sieve. If any part of the Great Empirical Term was to prove explicable gravitationally, it was most likely to be found amongst these long-period primary inequalities. There were thousands of inequalities whose periods suggested that their coefficients might be sensible, but which on computation were found to be smaller than $0''.01$. Even the roughest approximation, Brown tells us, could be a laborious process. Formulas that could be rapidly applied were thus a desideratum.

Fortunately, for the two most troublesome planets, Venus and Mars, Newcomb in his treatise of 1895 had given expansions of $1/\Delta^3$ and $1/\Delta^5$ in sines and cosines

[144] The Adams Prize paper, 28.

[145] *Transactions of the American Mathematical Society*, *IV*, 234–248.

of $iT - jP$ up to $i = 29$, with a considerable number of values of $i - j$.[146] Using Newcomb's expansions, Brown was able to construct an approximate formula for the coefficient in longitude. Numerical values were introduced from the start. Here Brown sought only rough values for the coefficients, his purpose being to identify the terms that needed to be computed accurately. This sieve stood Brown in good stead, leading to a considerably more complete computation of the direct planetary perturbations of the Moon than the only other computations comparable with it, those by Radau[147] in 1992 and by Newcomb[148] in 1907.

Frank Schlesinger and Dirk Brouwer gave the following assessment of Brown's computation:[149]

> When Brown began his work on the lunar theory it was known by Newcomb's researches that large unexplained differences existed between Hansen's theory and the Moon's observed motion. The question whether these differences could be ascribed to imperfections of the gravitational theory thus became one of the most urgent problems in gravitational astronomy. Its solution required a reliable determination of the planetary perturbations in the Moon's motions. This work was done independently by Radau (1835–1911), Newcomb (1835–1909), and Brown. Of these determinations Brown's was the most complete; moreover, his comparison [in *Monthly Notices of the R.A.S.*, vol. 68, pp. 148–170] of the three results left very few discrepancies unexplained.

In the comparison just referred to, inserted in the *Monthly Notices* for January of 1908, Brown listed 441 direct planetary perturbations in the longitude of the Moon with coefficients exceeding $0''.003$. (He had in fact computed almost all those with coefficients greater than $0''.001$, but Newcomb had chosen $0''.003$ for the lower bound in his list, and Brown in his comparison followed suit.) Most of these terms were of short period, and such terms were most conveniently added to the true longitude. The terms of long period, Brown decided, were best added to the mean longitude. Of the 441 inequalities, Newcomb omitted 238, and Radau 318. Among those listed by Newcomb, Brown found 41 whose coefficients differed from his own values by more than $0''.02$. In his conclusion, Brown discussed with great care the likely causes of these differences; for most of them he was able to provide a satisfactory account.

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146 *American Ephemeris Papers*, V, pt. 3.

147 J.C. Rodolphe Radau, "Recherches concernant les inégalités planétaires du movement de la lune," *Paris Observatoire, Annales*, *XXI* (1892).

148 S. Newcomb, "Investigation of Inequalities in the Motion of the Moon Produced by the Action of the Planets," *Carnegie Institute Publication*, 72 (1907).

149 F. Schlesinger and D. Brouwer, "Ernest William Brown," *National Academy Biographical Memoirs*, *XXI*, 246–247.

15

Indirect Planetary Perturbations of the Moon

In 1905 Brown published a paper "On a general method for treating transmitted motions and its application to indirect perturbations."[150] This paper begins with a generalized description of the process he was planning to employ in deriving the indirect planetary perturbations of the Moon:

> The mathematical treatment of any physical problem demands the construction of an ideal problem in which the conditions are different from those of the actual problem. It is assumed that the same general laws hold for the actual and ideal problems, but the complexity of the circumstances surrounding the former makes simplifications of some kind necessary in order that the analysis should not be unreasonably tedious. The ideal problem is therefore usually constructed by neglecting at first some of the influences which form a part of it but which are assumed to affect the results to a much smaller extent than those we retain. The simplified problem, which I call problem A, is then solved. The second step consists in finding what changes are necessary in the solution of problem A when some or all of the neglected influences are included; this second problem I call B. The question under consideration here is the deduction of the solution of B when that of A has been found.
>
> One of the methods for solving B is that known as the *Variation of Arbitrary Constants*.

Using Jacobi's formulation of Lagrange's method, Brown derived a number of results applicable to the indirect planetary perturbations of the Moon. The chief of these was the following theorem, in which it is supposed that the disturbing function R contains a term of long period, causing the factor α to occur in the Moon's δc and the factor α^2 to appear in the corresponding δw. The theorem states:

> When squares and higher powers of the ratio of the mass of a planet to that of the Sun are neglected, the large factor α due to a long-period inequality

[150] *American Mathematical Society Transactions, 6* (1905), 332–343.

C. Wilson, *The Hill–Brown Theory of the Moon's Motion*, Sources and Studies in the History of Mathematics and Physical Sciences, DOI 10.1007/978-1-4419-5937-9_15,

can never occur in the corresponding term in the Moon's motion to a power higher than α^2, even if its square is present in the corresponding inequality of the Earth's motion.

This result enabled Brown to reject in advance many terms of long period, which, in the absence of the theorem, would have had to be examined in order to find out whether their coefficients were sensible in the Moon's motion.

Brown gave a more concrete description of his procedures for the indirect planetary perturbations in Chapter XII of his *Theory of the Motion of the Moon*.[151] He began with an approximate expression of the disturbing function for the action of the Sun on the Moon (this expression had originally been derived in Chapter I):

$$\frac{m'}{r'^3}\left\{\frac{r^2-3z^2}{4}+\frac{3}{4}\rho^2\cos(V-V')+\frac{5}{8}\frac{\rho^3}{r'}\cos 3(V-V')\right.$$
$$\left.+\frac{3}{8}\frac{(r^2-5z^2)}{r'}\rho\cos(V-V')\right\}.$$

Here as before the unprimed coordinates belong to the Moon, the primed coordinates to the Sun. Brown let $\delta r'$, $\delta V'$ represent the Earth's departures from elliptical motion due to the action of planets, the effect being to displace the Sun as seen from the Earth. Substituting $r'+\delta r'$ for r', and $V+\delta V'$ for V' in the preceding expression, then putting $\delta r'/r'=\delta\rho'$ and neglecting powers of $\delta p'$, $\delta V'$ above the first, he thus found for the disturbing function due to $\delta r'$, $\delta V'$,

$$R=\frac{3m'}{4r^3}[-\delta\rho'\{r^2-3z^2+3\rho^2\cos 3(V-V')\}+\delta V'\{2\rho^2\sin 2(V-V')\}]$$
$$+\frac{3m'}{2r^4}\begin{bmatrix}-\delta\rho'\left\{\frac{5}{3}\rho^3\cos 3(V-V')+(r^2-5z^2)\rho\cos(V-V')\right\}\\ +\delta V'\left\{\frac{5}{4}\rho^3\sin 3(V-V')+\frac{1}{4}(r^2-5z^2)\rho\sin(V-V')\right\}\end{bmatrix}.$$

In this formula he replaced the functions of the coordinates of the Moon and the Sun by the series M_i used in the disturbing function for the direct planetary perturbations, so that R would now denote that part of the disturbing function that depends on the lunar angle θ. Also, he multiplied all the M_i of R by a'/r', marking them with a prime mark to indicate the fact. Taking ϕ as an angle in $\delta\rho'$, $\delta V'$, he substituted

$$\delta\rho'=\rho_c\cos\phi,\quad \delta V'=\nu_s\sin\phi,\quad m'=n'^2a'^3.$$

[151] *TMM, Memoirs of the R.A.S., 59* (1908), 39–77.

Finally, putting $a_1 = \mathrm{a}/a'$, he obtained

$$R = \frac{1}{4}n'^2\mathrm{a}^2(-3)\left[\left(M_1' + \frac{3}{2}M_2'\right)\rho_c \pm M_3'\nu_s + a_1(M_6' + 5M_8')\rho_c \pm a_1\left(\frac{1}{4}M_7' + \frac{15}{4}M_9'\right)\nu_s\right]\cos(\theta \pm \phi).$$

Brown took his values of $\delta p'$, $\delta V'$ from Newcomb's tables for the Sun,[152] with certain modifications. Of Newcomb's tabulated perturbations, Brown retained only those of the first order relative to the masses of the disturbing bodies.

The action of the planets on the Earth causes the plane of the ecliptic or the plane of the Earth's orbit, to move; this motion can be expressed as secular and periodic variations of the inclination and node of the ecliptic with reference to some fixed plane. As fixed plane Brown chose the ecliptic of date 1850.0; he then referred the motion of the Moon to the *mean ecliptic* of time t. He described the derivation of these variations both in his *Theory of the Motion of the Moon*[153] and in the *Monthly Notices* for April, 1908;[154] we follow the latter account here.

Let $\theta_1, \theta_2, \theta_3$ be the angular velocities of a set of rectangular axes rotating about themselves; x, y, z the coordinates; u, v, w the velocities of a particle with respect to these axes; F the force-function divided by the mass of the moving particle. The equations of motion will be

$$\frac{du}{dt} - v\theta_3 + w\theta_2 = \frac{\partial F}{\partial x};$$

$$\frac{dv}{dt} - w\theta_1 + u\theta_3 = \frac{\partial F}{\partial y};$$

$$\frac{dw}{dt} - u\theta_2 + v\theta_1 = \frac{\partial F}{\partial z}.$$

Here

$$u = \frac{dx}{dt} - y\theta_3 + z\theta_2,$$

$$v = \frac{dy}{dt} - z\theta_1 + x\theta_3,$$

$$w = \frac{dz}{dt} - x\theta_2 + y\theta_1.$$

The total energy of the system may be written

$$H = \frac{1}{2}(u^2 + v^2 + w^2) - F - R,$$

152 *American Ephemeris Papers*, VI, pt. 1.

153 *TMM, Memoirs of the R.A.S., 59* (1908), 45–48.

154 *MNRAS, 68* (1908), 450–455.

where

$$R = vx\theta_3 - wx\theta_2 + wy\theta_1 - uy\theta_3 + uz\theta_2 - vz\theta_1.$$

Assuming that $\theta_1, \theta_2, \theta_3$ are independent of x, y, z, u, v, w, we can write the equations of motion in canonical form:

$$\frac{du}{dt} = -\frac{\partial H}{\partial x}, \quad \frac{dx}{dt} = \frac{\partial H}{\partial u},$$
$$\frac{dv}{dt} = -\frac{\partial H}{\partial y}, \quad \frac{dy}{dt} = \frac{\partial H}{\partial v},$$
$$\frac{dw}{dt} = -\frac{\partial H}{\partial z}, \quad \frac{dz}{dt} = -\frac{\partial H}{\partial w}.$$

If R is neglected in H, these equations become identical with those for the motion of the particle with respect to fixed axes; hence R is the disturbing function for the motions of the axes. The latter motions are so small that the squares and products of the terms in R can be neglected.

Brown put i' for the inclination of the moving ecliptic (the xy plane) to the fixed ecliptic of 1850.0, τ for the longitude of its ascending node on the fixed ecliptic, and L for the angle which the zx plane makes with the plane containing the poles of the fixed and moving ecliptics. Euler's equations for the angular velocities of the coordinates about themselves are then

$$\theta_1 = \frac{di'}{dt}\sin L - \sin i' \cos L \frac{d\tau}{dt},$$
$$\theta_2 = \frac{di'}{dt}\cos L + \sin i' \sin L \frac{d\tau}{dt},$$
$$\theta_3 = \frac{d\tau}{dt}\cos i' + \frac{dL}{dt}.$$

L can be so taken that the distances of the origins of reckoning on the fixed and moving ecliptics from their common node are the same; then $L = 90° - \tau$. As a result the moving axis of x passes through what Cayley dubbed "a departure point."

The quantities $i', di'/dt, d\tau/dt$ are small enough so that their squares and products can be neglected, and the following approximations are valid:

$$\sin i' = i', \quad \cos i' = 1, \quad \tau = \text{const}.$$

The expressions for the angular velocities thus reduce to

$$\theta_1 = \frac{di'}{dt}\cos\tau, \quad \theta_2 = \frac{di'}{dt}\sin\tau, \quad \theta_3 = 0.$$

R can then be written

$$R = \frac{di'}{dt}\{(wy - vz)\cos\tau + (uz - wx)\sin\tau\}.$$

Alternatively, since the differences between u and dx/dt, between v and dy/dt, and between w and dz/dt are quantities of the same order of smallness as i',

$$R = \frac{di'}{dt}\left\{\left(y\frac{dz}{dt} - z\frac{dy}{dt}\right)\cos\tau - \left(x\frac{dz}{dt} - z\frac{dx}{dt}\right)\sin\tau\right\}$$
$$= \frac{di'}{dt}Q. \tag{B.117}$$

In accordance with the method of variation of arbitrary constants, into Q can be substituted the values of $x, y, z, dx/dt, dy/dt, dz/dt$ found in the solution of "the main problem" of the lunar theory. The value of i' is of the form $pt + P$, where p is a constant and P a sum of periodic terms whose arguments depend on the motions of the Earth and planets. From his solution of "the main problem," Brown found that P contains a few long-period terms and more numerous short-period terms. To include these inequalities amounted to referring the Moon's motions to the *actual* ecliptic. Setting P equal to zero was equivalent to referring these motions to a *mean* ecliptic. If the latter choice were adopted, the latitude of the Sun above the mean ecliptic would have to be introduced into F. In this case, Brown discovered, all the short-period terms became so small that they could be neglected, and the long-period terms had smaller coefficients than with the actual ecliptic as the plane of reference. Brown therefore adopted the mean ecliptic, and recognized a few very small terms which were of long period relative to the period of the Moon's node, but which did not give rise to any terms in the Moon's coordinates. He wrote $R_1 = pQ$.

These choices required the introduction of an additional part of R which Brown called R_2. The principal part of the force-function for the Moon's motion under the influence of the Earth and the Sun is

$$F = \frac{\mu}{r} + \frac{m'}{r^3}\left\{\frac{3}{2}\frac{(xx' + yy' + zz')^2}{r^2} - \frac{1}{2}(x^2 + y^2 + z^2)\right\}. \tag{B.118}$$

The "main problem" was solved with $z' = 0$, that is, the Sun was assumed to remain in the reference ecliptic. The force-function was in effect

$$F = \frac{\mu}{r} + \frac{m'}{r^3}\left\{\frac{3}{2}\frac{(xx' + yy')^2}{r^2} - \frac{1}{2}(x^2 + y^2 + z^2).\right\} \tag{B.118a}$$

No longer taking z' to be zero, but stipulating that it is small enough so that its square can be neglected, Brown introduced the following additional portion into R:

$$R_2 = \frac{m'}{r^3}\cdot\frac{3(xx' + yy')zz'}{r^2}. \tag{B.119}$$

With the notation and restrictions previously accepted, he set

$$z' = i'(y'\cos\tau - x'\sin\tau). \tag{B.120}$$

Substituting (B.120) into (B.119), Brown stated that "it is easy to show that" $R_2 = -i'(dQ/dt)$. To assist readers in seeing that this is so, we indicate the steps.

From (B.118a) we have that

$$\frac{\partial F}{\partial x} = -\frac{\mu x}{r^3} + \frac{m'}{r^3}\left\{\frac{3(xx' + yy')x'}{r^2} - x\right\} = \frac{d^2x}{dt^2},$$

$$\frac{\partial F}{\partial y} = -\frac{\mu y}{r^3} + \frac{m'}{r'^3}\left\{\frac{3(xx' + yy')y'}{r^2} - y\right\} = \frac{d^2y}{dt^2},$$

$$\frac{\partial F}{\partial z} = -\frac{\mu z}{r^3} - \frac{m'z}{r'^3} = \frac{d^2z}{dt^2}. \tag{B.121}$$

Introducing (B.120) into (B.119) we obtain

$$R_2 = \frac{3m'i'}{r'^5}\{(xzx'y' + yzy'^2)\cos\tau - (xzx'^2 + yzx'y')\sin\tau\}. \tag{B.122}$$

From the second equation of (B. 121), multiplying through by z, we can write

$$\begin{aligned}\frac{3m'}{r'^5}(xzx'y' + yzy'^2) &= z\frac{d^2y}{dt^2} + \frac{\mu yz}{r^3} + \frac{m'yz}{r'^3}\\ &= z\frac{d^2y}{dt^2} - y\frac{d^2z}{dt^2}\end{aligned}$$

where the last step is obtained from the third equation of (B.121). From the first equation of (B.121), we obtain, in the same way,

$$\begin{aligned}\frac{3m'}{r'^5}(xzx'^2 + yzx'y') &= z\frac{d^2x}{dt^2} + \frac{\mu xz}{r^3} + \frac{m'xz}{r'^3}\\ &= z\frac{d^2x}{dt^2} - x\frac{d^2z}{dt^2}.\end{aligned}$$

Substituting these results into (B.122), we obtain

$$\begin{aligned}R_2 &= i'\left\{\left(z\frac{d^2y}{dt^2} - y\frac{d^2z}{dt^2}\right)\cos\tau - \left(z\frac{d^2x}{dt^2} - x\frac{d^2z}{dt^2}\right)\sin\tau\right\}\\ &= i'\frac{d}{dt}\left\{\left(z\frac{dy}{dt} - y\frac{dz}{dt}\right)\cos\tau - \left(z\frac{dx}{dt} - x\frac{dz}{dt}\right)\sin\tau\right\},\end{aligned} \tag{B.123}$$

where τ has been treated as a constant. Comparing (B.123) with (B.117) we see that (B.123) is

$$\begin{aligned}R_2 = -i'\frac{dQ}{dt} &= Q\frac{di'}{dt} - \frac{d}{dt}(i'Q)\\ &= R_1 - \frac{d}{dt}(i'Q).\end{aligned} \tag{B.124}$$

As Brown put it, the effect of the rotation of the axes is to introduce a term $R_1 - R_2 = (di'/dt)(i'Q)$ into the force-function.

Taking w_1, w_2, w_3 for the mean longitudes of the Moon, its perigee, and its node, and ℓ, D, F, n, e, γ for the quantities defined by Delaunay, Brown obtained the following variations of the Moon's orbital elements for the principal inequality (with argument ϕ) arising from R_1:

$$\delta w_1 = -0''.289 \sin\phi, \quad \delta n = +0''.0014 n \cos\phi,$$
$$\delta w_2 = +0''.840 \sin\phi, \quad \delta e = 0.000 \cos\phi,$$
$$\delta w_3 = -15''.59 \sin\phi, \quad \delta\gamma = +0''.698 \cos\phi.$$

These values, he reported, agree closely with those found by Newcomb.[155] They differed a little from those found by Hill;[156] the differences, Brown suggested, were due to Hill's use of a literal development having slow convergence.

For the new terms in longitude and latitude arising from R_2, Brown obtained (omitting terms with coefficients less than $0''.010$, and putting T, V, J for the mean longitudes of Earth, Venus, and Jupiter),

In Longitude.	In Latitude.
$+0''.019 \sin(\phi + 5T - 3V + 119^\circ)$,	$+0''.077 \sin(\phi' + 5T - 3V + 119^\circ.4)$,
$+0.003 \sin(\phi + 2w_2 - 2J + 90^\circ)$,	$+0''.030 \sin(\phi' - 5T + 3V - 119^\circ.4)$,
	$+0.035 \sin(\phi' + 2J + 72^\circ)$,
	$+0''.018 \sin(\phi' - 2J - 72^\circ)$.

The second term in longitude, Brown remarked, had a period equal to that of "the great empirical term," some 280 years, but its coefficient was much too small to account for this discrepancy between theory and observation.

Brown constructed a sieve for the indirect planetary terms, just as he had for the direct planetary terms. As before, he found that he could restrict consideration to the terms of long period; and he considered separately those that do and do not contain w_1 in their arguments (the primary and secondary terms, respectively). For the primary terms depending mainly on $\delta\rho'$, he showed that the order of the indirect terms is to that of the direct terms approximately as m^2:1, that is, as 1:178.7. It followed that any term found by calculation to have a coefficient less than $1''$ due to direct action would not be sensible in the indirect action. Only one term was left, the Venusian term with argument $\ell + 16T - 18V$, and coefficient $14''.55$. The order of the coefficient for the indirect action is thus $14''.55m^2 = 14''.55/178.7 = 0''.08$. When Brown computed the exact value for this coefficient he obtained $0''.06$.

The primary terms due to $\delta V'$ gave similar results: the direct inequalities proved to be small, and there were no sensible ones arising from the indirect action.

155 Carnegie Institute publication no. 72, 132.

156 *Collected Mathematical Works of G.W. Hill, II*, 77.

The sieve selected the terms that needed to be computed precisely. Brown's procedure was to take one lunar argument with all multiples of $T - \varpi'$ (the Earth's mean anomaly), and form the products for all the planetary arguments. As he remarked,

> The experience gained in computing the direct inequalities suggested that the work could be much abbreviated by considering the peculiarities of each lunar argument... [157]

Brown provided some of the details toward the end of the text of Chapter XII,[158] then went on to list the indirect planetary inequalities of the Moon greater than $0''.003$, most of them very small. The most numerous were those due to Venus; Brown listed 169 indirect Venusian perturbations of the Moon, of which 71 were greater than $0''.01$, and 12 as great as or greater than $0''.10$.

[157] *TMM, Memoirs* of the R.A.S., *59* (1908), 54.
[158] Ibid., 54–57.

16

The Effect of the Figures of the Earth and Moon

In 1884 G.W. Hill had published a long paper entitled "Determination of the inequalities of the Moon's motion which are produced by the figure of the Earth: a supplement to Delaunay's lunar theory."[159] Hill explained his purpose as follows:

> The sensible character of these inequalities [the lunar inequalities due to the Earth's figure] was discovered by Laplace; but he and his immediate successors contented themselves with determining the coefficients of two periodic terms; one of the fourth order in the longitude, the other in the latitude and of the third order, whose periods depend on the position of the Moon's node with reference to the equinox. The most elaborate treatment of this subject, we at present have, is by Hansen.... The coefficients of about twenty terms are computed, and all that can be of utility for the formation of the most exact tables are supposed to be there contained. But these coefficients appear in the work only as numbers; hence it is impossible to see to what cause they owe their magnitude. Moreover, no regard has been paid to the algebraic order of magnitude in retaining or rejecting terms. Thus it will be seen that, in this portion of the subject, we have nothing to compare with Delaunay's splendid treatment of the solar perturbations.
>
> The problem, then, which I propose to solve in this memoir is to determine, in a literal form, all the inequalities of the Moon which arise from the figure of the Earth, to the same degree of algebraical approximation as Delaunay has adopted in determining the solar perturbations, viz., to terms of the seventh order inclusive.[160]

We recall that, only a few years before, in the late 1870s, after praising Delaunay's procedure to the skies, Hill had lost all hope for it as a way of developing the lunar theory, on account of the slow convergence of the series that resulted for the coefficients. This difficulty had led him to initiate the alternative whose development we

[159] *Astronomical Papers of the American Ephemeris*, *III* (1891), 201–344; also *The Collected Works of George William Hill*, *II*, 180–320.

[160] *The Collected Works of George William Hill*, *II*, 181.

C. Wilson, *The Hill–Brown Theory of the Moon's Motion*, Sources and Studies in the History of Mathematics and Physical Sciences, DOI 10.1007/978-1-4419-5937-9_16,

have been tracing in the present study. Yet here we find him speaking of "Delaunay's splendid treatment of the solar perturbations!" Delaunay's attempt to derive the Moon's motions was indeed a splendid, if failed, enterprise. Was the task of carrying out a Delaunay-style derivation of the perturbations of the Moon's motions due to the Earth's figure perhaps imposed by Simon Newcomb, Hill's boss?

Whatever its motive, the undertaking involved a time-consuming calculation. In Chapter 4 of his essay, Hill obtained literal expressions for all perturbations of the Moon due to the Earth's figure, accurate to algebraic orders up to and inclusive of the seventh. He found 165 in the Moon's longitude, 209 in its latitude. In Chapter 6 he gave the numerical values of the coefficients of these terms, accurate to one-ten-thousandth of an arc-second. Aside from one term in the longitude with a coefficient of $7''.6708$, and another in the latitude with a coefficient of $8''.7356$, all the terms had coefficients less than $1''.0$, and most of them were much smaller; Hill gave 39 of them in the longitude, and 27 in the latitude, as $0''.0000$.

From Hill's Chapter 4 we also learn that most of the coefficients involved the parameter $m = n'/n$, the gremlin responsible for the slow convergence in Delaunay's theory; all but 16 terms in the longitude, and 27 terms in the latitude, contain it. Terms containing this parameter are, of course, incomparable with any that Brown will find, since in the Hill–Brown theory the constant m is introduced numerically from the start.

Like Hill before him, Brown began his own account of the effect of the Earth's figure (as reported in Chapter 13 of *Theory of the Motion of the Moon*) by introducing the term in the disturbing function on which it depended. To a sufficient approximation it was

$$R = (E + M)\frac{A + B + C - 3\mathrm{I}}{2r^3 E},$$

where E and M are the masses of the Earth and Moon; A, B, C are the moments of inertia of the Earth about its three principal axes; I is its moment of inertia about the line joining the Earth's center of mass with the Moon (the line designated r).

The "principal axes" had been so-named by Euler; they are axes about which the body can rotate freely without wobbling. The fact that there are at least three such axes in any body, whatever its shape and density distribution, was first proved by J.A. Segner in 1755, and again by Euler a few years later.[161] The moments of inertia A, B, C are present in Brown's formula not for their role in governing the Earth's rotation but because they reflect the distribution of mass within the Earth. The symbol I is expressible in terms of A, B, C: if α is the right ascension of the Moon along the celestial equator, reckoned from the A-axis, and δ the Moon's declination, or angular distance above the celestial equator, then I is given by

$$I = A\cos^2\alpha\cos^2\delta + B\sin^2\alpha\cos^2\delta + C\sin^2\delta.$$

[161] See C. Wilson, "D'Alembert *versus* Euler on the Precession of the Equinoxes and the Mechanics of Rigid Bodies," *Archive for History of Exact Sciences, 73* (1987), 266–267.

Substituting, Brown showed – again like Hill before him – that

$$A + B + C - 3I = 3\left(C - \frac{A+B}{2}\right)\left(\frac{1}{3} - \sin^2\delta\right) - \frac{3}{2}(A - B)\cos 2\alpha \cos^2\delta. \tag{B.125}$$

This formula comes out of spherical harmonic analysis, invented by Legendre and Laplace to express in converging series the attraction of a body of nearly spherical figure. Hill can have learned it from Thomson and Tait, *Treatise on Natural Philosophy*,[162] to which he refers. (B.125) gives the first two terms of a series, but its second term in the case of the Earth is negligible, $A - B$ being very small compared with $C - 1/2(A + B)$.

How is the first term of (B.125) best evaluated numerically? Hill used 73 measurements of the lengths of the seconds-pendulum at different locations on the Earth's surface, adjusted in all cases to sea level. Combining them by the method of least squares, he computed a value for $\frac{3}{2D^2}\left(C - \frac{A+B}{2}\right)$, where D is the Earth's mean equatorial radius. He gave his result as 0.001759484.

Brown either did not trust the precision Hill claimed for this result, or regarded the carrying of so many significant figures through the calculations as unnecessary. He employed a more traditional procedure which depended on the empirical value of the coefficient of the principal term in the latitude arising from the figure of the Earth. The argument of this term is $w_1 + \psi$, ψ being the precession of the equinoxes. The observational value Brown cited for the coefficient of this term was $-8''.382$; he attributed it to Hansen – mistakenly, as we learn from a later correction in the *Monthly Notices*.[163] The correction was due to Frank E. Ross, who pointed out that Brown had used two different values for the coefficient, one due to Faye, the other due to Hansen.

The empirical value of the term with argument $w_1 + \psi$ contains a part due to the motion of the ecliptic. This must be subtracted out to give the part that is due solely to the figure of the Earth. Hill's theoretical value for the former part was $-0''.226$, leading Brown to assign, to the part due to the Earth's figure, the value $-8.''382 - (-0''.226) = -8''.156$.[164] He took this value as a basis for determining the difference between the Earth's polar and equatorial radii of gyration.

It is not immediately clear which of the two values for the empirical term, Faye's or Hansen's, Brown used in the several parts of his calculation. He computed the terms in the Moon's longitude and latitude arising from the Earth's figure and having coefficients greater than or equal to $0''.003$; there were only three of the former, and five of the latter. He also computed the changes in the Moon's orbital elements arising from the same cause; the resulting theoretical value for the principal term in latitude, he reported, was $-8''.355\sin(w_1 + \psi)$, which disagrees with the empirical value given earlier.

[162] See W. Thomson and P. G. Tait, *Principles of Mechanics and Dynamics* (Dover Edition, a republication of the 1912 edition), Part Two, 87, section 539.

[163] "On an Error in the New Lunar Theory," *MNRAS, 70* (1909), 3.

[164] E.W. Brown, "On the Degree Accuracy of the new Lunar Theory," *MNRAS, 64* (1904), 530.

The disagreement, Brown expected, would be resolved in the final fitting of theory to observation:

> It is immaterial at the moment which constant is used for the preliminary tables of the Moon's motion, since it will have to be corrected by comparison with observation.[165]

We turn now to the inequalities due to the figure of the Moon. The disturbing function for these inequalities is of the same form as that for the action of the Earth's figure, but in this case the second term in the expansion must be retained, since the difference in the equatorial principal axes, A' and B', cannot be neglected. Let α' be the angle from the A'-axis on the Moon's equator to the projection of r onto the equatorial plane, and δ' the inclination of r to the same plane. In strict analogy to the case for the Earth's figure, we shall have

$$A' + B' + C' - 3\mathrm{I}' = 3\left(C' - \frac{A' + B'}{2}\right)\left(\frac{1}{3} - \sin^2\delta'\right) + \frac{3}{2}(B' - A')\cos 2\alpha'\cos^2\delta'.$$

The Moon always turns very nearly the same face to the Earth, such difference as occurs being due to a clearly observable apparent libration together with a tiny real libration about the C'-axis, attested by theory but not empirically detectable. Neglecting the latter, we can set the Moon's speed of axial rotation equal to n, the Moon's mean orbital speed. The Moon's equator and the ecliptic intersect in a line whose longitude is w_3; Brown designates this point on the celestial sphere by Ω. Then the mean angular distance between the A'-axis and Ω is $w_1 - w_3$. Dropping a perpendicular from M, the position of the Moon, onto the ecliptic, and another such onto the equatorial plane, we obtain two right spherical triangles with the common hypotenuse ΩM. A rule of spherical trigonometry then gives

$$\cos\delta'\cos(\alpha + w_1 - w_3) = \cos U\cos(V - w_3).$$

Neglecting δ', U, we can write $\alpha' = V - w_1$; if we put $\alpha' = V - w_1 + \delta\alpha'$ then $\delta\alpha'$ will depend on the squares of δ', U, and

$$\cos 2\alpha' = \cos 2(V - w_1) - 2\delta\alpha'\sin(V - w_1).$$

Brown neglected quantities of an order higher than the second in the eccentricities and inclination, and also the inclination multiplied by m^2; thus he could neglect $2\delta\alpha'\sin(V - w_1)$ in the last equation. Also, if $\gamma = \sin(i/2)$, and $-i_1$ is the inclination of the lunar orbit to the ecliptic, then to a sufficient approximation

$$\sin\delta' = \sin(i + i_1)\sin(V - w_3) = \sin(i + i_1)\sin(w_1 - w_3).$$

Substituting these expressions, Brown obtained for the disturbing function

$$R = (E + M)\frac{a^2\mu'}{r^3}\left[\begin{array}{l}\frac{1}{3} - \sin^2(i + i_1)\sin^2(w_1 - w_3)\\ +(\mu''/\mu')\cos 2(V - w_1)\{1 - \sin^2(i + i_1)\sin^2(w_1 - w_3)\}\end{array}\right],$$

165 *MNRAS, 70* (1909), 3.

where

$$a^2\mu' = \frac{3}{2M}\left(C' - \frac{A' + B'}{2}\right), \quad a^2\mu'' = \frac{3}{4M}(B' - A').$$

All the periodic terms turned out to have very short periods. Brown therefore neglected them and focused on the constant parts of R, which gave additions to b_2, b_3, the mean motions of the perigee and node. The portions of R depending solely on e^2 affected b_2, and those depending solely on γ^2 affected b_3. For the former, R reduced to

$$R = (E + M)\frac{a^2}{\rho^3}\left[\frac{1}{3}\mu' + \mu'' \cos 2(V - w_1)\right];$$

and for the latter, to

$$R = (E + M)\frac{a^2}{r^3}\left[-\frac{1}{2}\mu' - \frac{1}{2}\mu''\right]\sin^2(i + i_1).$$

Brown next put $\delta n = \delta_0 n$, $\delta c_2 = 0$, $\delta c_3 = 0$. These stipulations led to

$$\delta w_1 = \left(\frac{1}{\alpha^2\beta}\frac{\partial R}{\partial n} + \delta_0 n\right)t, \quad \delta w_2 = \left(-\frac{\partial R}{\partial c_2}t + \frac{db_2}{dn}\delta w_1\right),$$

$$\delta w_3 = \left(-\frac{\partial R}{\partial c_3}t + \frac{db_3}{dn}\delta w_1\right).$$

Since the mean longitude is a quantity observed directly, he chose $\delta_0 n$ so that w_1 was still represented by $nt + \varepsilon$; consequently $\delta w_1 = 0$, so that from the first of the foregoing equations

$$\delta_0 n = -\frac{1}{\alpha^2\beta}\frac{\partial R}{\partial n},$$

where $-\alpha^2\beta = dc_1/dn$. The changes in the angles w_2, w_3 were obtained by adding to their motions the quantities

$$\delta b_2 = -\frac{\partial R}{\partial c_2}, \quad \delta b_3 = -\frac{\partial R}{\partial c_3}.$$

The change $\delta_0 n$ in n was to be substituted only in the *coefficients* of terms representing the Moon's coordinates. These included the principal elliptic term in longitude, with $2e$ as coefficient, and the principal term in latitude, with 2γ as coefficient. Since the coefficients $2e$ and 2γ were to be obtained directly from observation, and Brown wished to retain these very expressions to designate them, he found it necessary to add to e, γ where they occur in all other terms the amounts

$$\delta_0 e = -\frac{de}{dn}\delta_0 n, \quad \delta_0 \gamma = -\frac{d\gamma}{dn}\delta_0 n.$$

He neglected the further changes thus produced in b_2, b_3, since they produced less than $0''.01$ of difference in any coefficient.

We shall not follow in detail Brown's processes in arriving at his numerical results. By a strict deduction he obtained the ratios $db_2(M) : db_2(E)$, $db_3(M) : db_3(E)$ for the figures of the Moon and Earth:

$$\mu'\rho_e + 3\mu''\rho_c : \mu\left(1 - \frac{3}{2}\sin^2\varepsilon_1\right)\rho_e,$$

$$(\mu' + \mu'')\frac{d}{di}\sin^2(i + i_1) : \mu\left(1 - \frac{3}{2}\sin^2\varepsilon_1\right)\frac{d}{di}\sin^2 i.$$

Introducing the already known numbers,

$$1 - \frac{A + B}{2C} = 0.00328, \quad i = 5^\circ.1, \quad i_1 = 1^\circ.5,$$

he found for the values of db_2, db_3 attributable to the Moon's figure, in arc-seconds per year,

$$db_2 = 191''\left(1 - \frac{A' + B'}{2C'}\right) - 503''\left(\frac{B' - A'}{C'}\right), \quad db_3 = -230''\left(1 - \frac{A'}{C'}\right).$$

From a study of the lunar librations, F. Hayn had found[166]

$$B' - A' = +.000157C', \quad C' - A' = +.000629C'.$$

Adopting these values, Brown obtained for the annual mean motions due to the Moon's figure, $db_2 = +0''.03, db_3 = 0''.14$. To know these quantities accurately to $0''.01$, he pointed out, would have required having the mechanical ellipticities accurate to within 5%.

[166] *Abh. der Math.-Phys. Kl. der K.-Sächs. Gess. der Wiss.*, *XXX* (1907), 69.

17

Perturbations of Order $(\delta R)^2$

Brown, we recall, in commencing his computation of the inequalities due to the direct and indirect actions of the planets and the figures of the Earth and Moon, proposed to neglect, *pro tem*, quantities of the order $(\delta R)^2$. With the initial computation now completed, he undertook, in Chapter 14 of his *Theory of the Motion of the Moon*, an investigation to discover whether any of these second-order perturbations were non-negligible.

The complete disturbing function for all actions other than those dealt with in the main problem was

$$\delta R = R(r' + \delta r' + \delta^2 r', V' + \delta V' + \delta^2 V', z' + \delta z') + \sum(R_P + R_E + R_e).$$

Here $\delta^2 r'$, $\delta^2 V'$, $\delta z'$ are the terms of the second order in the motion of the Sun, and $R(r', V', z')$ is the disturbing function for the Sun's action; R_P, R_E, R_e are, respectively, the parts of the disturbing function for the actions of a planet, the figures of the Earth and Moon, and the motion of the ecliptic. Having derived $\delta^2 R$ from the above expression, Brown examined in detail eight possible second-order results, and found most of them negligible. Two that were not were the indirect effects of solar terms with arguments $4M - 7T + 3V$ and $3J - 8M + 4$; they yielded the terms

$$\delta^2 w_1 = +0''.04 \sin(152^\circ + 119^\circ.0 t_c) + 0''.84 \sin(41^\circ.1 + 20^\circ.2 t_c),$$

where t_c is the number of centuries since 1850. Among additions due to periodic perturbations of the solar and planetary coordinates in ΣR_P due to the Earth's action, Brown found a single term with a sensible coefficient; it had the argument $\ell + 3T - 10V$, and a period of 1900 years. The yield in included terms was small; its main result was to establish that other possible terms in $\delta^2 R$ were negligible.

C. Wilson, *The Hill–Brown Theory of the Moon's Motion*, Sources and Studies in the History of Mathematics and Physical Sciences, DOI 10.1007/978-1-4419-5937-9_17,

18

The Tables

Chapters X–XV of Brown's *Theory of the Motion of the Moon* reached the editorial office of the Royal Astronomical Society in April 1908, signaling the completion of the task that Brown had begun in 1895: the systematic computation of all terms produced by gravitational attraction in the Moon's coordinates, with coefficients equal to or greater than a pre-specified minimum. In the longitude and latitude, the minimum was $0''.01$; in the sine parallax it was $0''.001$. At this point, abruptly, Brown turned to the construction of *lunar tables*.[167]

Astronomical tables, from antiquity down to Brown's time, were used for making predictions and computing ephemerides – positions of a celestial body at given intervals during a given year. In the 17th century European governments began investing funds in the construction of lunar tables and ephemerides, seeking thereby to supply navigators with a means of finding the longitude at sea. The ancient lunar tables had been founded on observational findings, and this continued to be the case into modern times. Newton's *Principia* introduced a new basis from which the motions of the Moon might be deduced: the law of gravitation; but it was not at first clear how the deduction could be carried out systematically. An important first step was taken by the 18th-century mathematicians Euler, Clairaut, and d'Alembert, in applying the Leibnizian calculus to the problem; but it was Lagrange who first opened the way to carrying out second- and higher-order approximations systematically.

Hansen's lunar tables, constructed by a development of Lagrange's methods and published in 1857, were the first tables purporting to be strictly in accordance with Newton's law and also accurate enough for navigational use. (They were not quite what they seemed: as mentioned earlier, they surreptitiously included an empirical term.) They became the basis of the British and the French nautical almanacs in 1862.[168] These tables embodied 300 sinusoidal terms, far more than any earlier astronomical tables. Requiring ephemeris-computers to derive lunar ephemerides

[167] Ernest W. Brown, "On the Plans for New Tables of the Moon's Motions," *MNRAS, 70* (1909), 148–175.

[168] Hansen's theory was used by the British from 1862 to 1922; by the French from 1862 to 1910; and by the U.S.A. from 1883 to 1922.

C. Wilson, *The Hill–Brown Theory of the Moon's Motion*, Sources and Studies in the History of Mathematics and Physical Sciences, DOI 10.1007/978-1-4419-5937-9_18,

from a set of 300 tables, one for each term, was not practical; Hansen exercised much ingenuity and labor in devising tables that would incorporate multiple terms.

The exfoliation of terms out of Newton's law reached a still higher point in Brown's elaboration of the lunar theory. To compute the coordinates of the Moon for a given time, Brown's theory required taking some 1500 terms into account. Electronic computers with their enormous speed and capacity had not yet been dreamed of, and would not begin to replace human ephemeris-computers till the middle years of the 20th century. The construction of lunar tables therefore constituted a problem of major proportions. Brown took Hansen's tables as setting the standard: the new tables should require no more labor of the ephemeris-computer than had Hansen's tables.

When Brown joined the Department of Mathematics at Yale in 1908, an important part of the prior agreement was that the Yale Observatory would absorb the expense of constructing, printing and publishing the tables. Henry B. Hedrick was recruited from the U.S. Nautical Almanac Office to take chief responsibility for devising the new tables; he took up his post at Yale in 1909. From his earlier work Hedrick had gained expertise in assessing the merits and drawbacks of different types of tables.[169] According to Brown,

> Much the heaviest part of the arrangement and performance of the calculations has been borne by Dr Henry B. Hedrick, whose services were secured at the outset and who has spent his whole time on the work for nearly nine years. Every part of it has passed through his hands. He has prepared and tested all calculations which were performed by others. Many of the devices which have been employed to simplify the use of the Tables are due to him, and no decisions have been made without frequent discussions in which his suggestions have given valuable aid.[170]

The Brown-Hedrick Tables were an achievement for their time. They were fated, however, to be outmoded in the 1950s, electronic computers rendering them, along with all astronomical tables, obsolete. Yet the artistry and hard work that Brown, Hedrick and their assistants invested in the construction of their Tables has an interest of its own, and a merited place in the history. We limit our account to general features, with a few illustrations.

The labor of the ephemeris computer consisted chiefly in extracting numbers from single- or double-entry tables, and the extraction of almost every number required time-consuming interpolation. A primary goal for Brown and company, therefore, was to render the interpolation easy. The intervals between adjacent

[169] Hedrick had produced the "Catalogue of Stars for the Epochs 1900 and 1920 Reduced to an Absolute System," *Astronomical Papers for the American Ephemeris*, 8, Part 3 (Nautical Almanac Office, Washington, 1905). It was adopted by all national almanacs as providing the reference stars against which the motions of the Moon and planets could be measured, and served in this role for 35 years.

[170] E. W. Brown with the assistance of H. B. Hedrick, *Tables of the Motion of the Moon*, (New Haven, CT: Yale University Press, 1919). Preface, xi.

numbers needed to be small, and the tables correspondingly large. The *number* of tables, on the other hand, needed to be kept to the least possible.

Hansen in his single-entry tables for the longitude had given the argument in the horizontal direction for successive half-day intervals, this being the interval used for tabulating lunar positions in the nautical almanacs. But interpolation over this interval to the accuracy required would have required forming sixth- or seventh-order differences – far too costly in time. Hansen therefore divided the argument vertically into hundredths of a day, to reduce the difficulty.

Brown and Hedrick accepted Hansen's idea of converting single-entry tables into two-dimensional arrays, but with differences. These we illustrate from their Table 30, which incorporates six terms:

$$\sum_{30} = +22639''.500 \sin \ell + 769''.016 \sin 2\ell + 36''.124 \sin 3\ell$$
$$+ 1''.938 \sin 4\ell + 0''.113 \sin 5\ell + 0''.007 \sin 6\ell.$$

Here ℓ is the Moon's mean anomaly, its mean longitude minus the longitude of its perigee. Brown derived these six terms in his *Theory of the Motion of the Moon*.[171] For Brown-Hedrick as for Hansen, the inclusion of multiple terms in a single table was desirable, but the feasibility of combining particular terms had to be assessed: a particular combination could prove unwieldy, requiring a division of the argument into excessively small intervals and an increase in the size of the table out of proportion to its relative importance. In Table 30, incorporating all six terms in a single table caused no increase in size over a table for $\sin \ell$ alone. By allowing entries to be negative as well as positive, and making the tabulation *reversible* and *reentrant* (properties explained farther on), Brown-Hedrick more than halved the size of the table to 12 folio pages.

Brown-Hedrick first decided to rotate Hansen's array through 90°, and so to list the half-day divisions vertically. Interpolations would now be made between adjacent columns rather than adjacent rows; errors due to slipping from column to column were less likely than errors due to slipping from row to row. Next, they sought a substitute for Hansen's division of the argument into hundredths of a day. A division into smaller intervals was sometimes desirable, and decimal fractions were best avoided, as they could occasion rounding errors.

They substituted a whole-number measure of the argument. The unit was to be such that the period and the half-day intervals would be represented by whole numbers. The argument ℓ of $\sin \ell$ goes through a full cycle of 360° in 27.55455 ... days – the anomalistic month – and the arguments $2\ell, 3\ell, \ldots, 6\ell$ go through their full cycles in sub-multiples of this same period, so that the sum Σ_{30} becomes zero

[171] See *TMM, Memoirs of the R.A.S., 57* (1909), 130–132 and 135. The first two terms as given in our text above have slightly smaller coefficients than in the passage of the *Memoirs* just cited. The tables were calculated with the value 22639″.500 for the coefficient of the principal elliptic term, whereas the final value of this coefficient was 22639″.550. The method for changing to the final value is given in Chapter IV of Section I the *Tables*.

whenever ℓ equals 0°, 180°, or 360°. A whole-number representation was thus required for the ratio

$$\frac{27.55455}{0.5} = \frac{55.10910}{1}.$$

Whole-number expressions of this ratio can be got by developing 55.10910 as a continued fraction, and breaking off the development at a convenient stage (the result at any stage is called a *convergent*). The convergent for 55.10910 after two stages is

$$55.10910 = 55 + \frac{1}{9.1659} \approx 55 + \frac{1}{9 + \frac{1}{6}} = \frac{3031}{55}.$$

This division was deemed accurate enough, but for greater precision without introduction of decimal fractions, Brown-Hedrick sextupled the number of units in the half-day to $55 \times 6 = 330$; the full period thus became $3031 \times 6 = 18186$. The units were now less than a third as large as Hansen's, being equal to $71''.264$. Any discrepancy between the adopted period/half-day ratio, 18186/330, and its observational value, could be incorporated in the expression for the secular variation of the anomalistic period. This policy, a suggestion of Hedrick's, was adopted in all the tables: numerical ratios were chosen for their convenience in computation, while strict empirical accuracy was obtained by adjusting the "secular variation."

The columns of Table 30 are arranged in the order of successive starting values of the argument, 0, 1, 2, ..., 329, and the latter numbers are placed as identifiers at the head of the column to which they apply. The first 19 columns are numbered from 0 to 18, and each of these columns contains 56 rows, labeled in the left-hand margin by half-days from $0^d.0$ to $27^d.5$. In any column, the argument increases by 330 units (a half-day's worth) from one row to the next. The entry in each row is the value of Σ_{30} for the value of the argument in that row. In the final row of column 0, the argument is $(2 \times 27.5) \times 330 = 18150$, just 36 units shy of a full period. In all columns, owing to the dominant role of $\sin \ell$ in Σ_{30}, the entries in the upper half are positive, and those in the lower half are negative. In column 18 the entries are "antisymmetric": the column when read from top to bottom is the same as the column read from bottom to top with the signs of all entries reversed. If column 17 is read from bottom to top with the signs of all entries reversed, it turns out to be column 19, and is so labeled at the bottom. Each of the columns thus serves as two columns, depending on whether it is read from top to bottom, or from bottom to top with reversal of signs. Column 0 read from bottom to top with reversal of signs is Column 36. The table is in this sense *reversible*.

A column with a higher starting value of the argument than 36 must have only 54 rows, corresponding to 54 half-days or 27 days, in order that the argument in its final row not exceed a full period. In column 37, the value of the argument in the final row is $37 + (2 \times 27) \times 330 = 17857$, which is 329 shy of a full period. And just as with the earlier columns, column 37 read from bottom to top with reversal of signs is column 329. The last column in the table, numbered 183 at both top and bottom, is the same when read downward as when read upward with reversal of signs.

Brown-Hedrick wanted to be able to *print* the interpolation factor, or variation of the function Σ_{30} per unit change of the argument – an aid to the computer that Hansen had not supplied. Given the arrangement of the successive columns in the order of the starting values of their arguments, the interpolation factor remains nearly constant in any row. Thus in row $0^d.0$, the variation per unit of the argument is $8''.39$ for columns 0–120, $8''.38$ for columns 121–176, $8''.37$ for columns 177–210, $8''.36$ for columns 211–252, $8''.35$ for columns 253–280, $8''.34$ for columns 281–308, and finally $8''.33$ for columns 309–329. Each interpolation factor is printed at the left of the row to which it applies.

Table 30, we now see, has no real beginning or end, and wherever a start is made, the values for the half-day intervals can be continued indefinitely without re-computation of the argument. The end of Table 30 connects up with its beginning; the table is thus *reentrant*.

Among the Brown-Hedrick tables for the longitude, 25 are, like Table 30, single-entry tables; of these, 24 are reentrant and reversible. All but three of the 25 tables have periods of about a month or half-month, and relatively large coefficients, ranging from $22639''.50$ in Table 30 to $1''.979$ in Table 39. Table 30 is the only table containing negative terms.

We consider next the *double-entry* tables for the longitude. Double-entry tables allowed inclusion of large numbers of small terms in a single table. They tabulated expressions of the form

$$\sum_{i,j} a_{i,j} \cos(iA + jB + \alpha),$$

where $i, j = 0, \pm 1, \pm 2, \ldots$, and a and α are constants. The general idea is that the arguments A have a common period and vary stepwise through that period in one dimension of the table, while the arguments B share another period and vary stepwise through their period in the other dimension.

Calculating positions from tables of this form can entail two interpolations, one for each argument. To avoid the laboriousness of double interpolations, Hansen in his *Tables de la lune* grouped together double-entry tables having a common argument A, but different arguments B. Extractions were made with a particular value of A, the interpolations being carried out with respect to B alone. Results for the several tables with the same value of A were then summed. Interpolations between two values of A were performed solely on the sums.

Brown-Hedrick adopted Hansen's procedure here, but with some changes. Hansen had taken the Moon's mean anomaly ℓ for the common argument A, a natural choice since he was calculating the Moon's *longitude in orbit* and regarding the orbit as a perturbed ellipse. Brown-Hedrick chose other variables, and chiefly $D = L - L'$, the synodic variable, or difference between the mean longitudes of the Moon and the Sun. $2D$, we recall, is the variable of the Variation Curve, and thus basic to the Hill–Brown theory. The Brown-Hedrick Tables 1–22 for the longitude are all double-entry tables in which the functions summed are sinusoidal terms having for their arguments various linear combinations of one or more of the variables $\ell, \ell', 2F (= 2\ell - 2\Omega,$

where Ω is the longitude of the lunar node) with some multiple of D. Table 1, for example, contains 24 terms of the form

$$\alpha_{i,j} \sin(i\ell' \pm jD),$$

where $i = 1, 2, 3$ or 4, and $j = 0, 1, 2, 4, 5$ or 6. The absolute values of the coefficients of these 24 terms add up to nearly $40''.00$. In the 22 double-entry tables taken together the total number of included terms is 218; the absolute values of their coefficients add up to about $173''$.

In forming his double-entry tables, Hansen had tabulated horizontally the successive values of ℓ, his choice for A, using quarter-day intervals to lessen the numerical differences over which interpolations had to be performed; this arrangement required the computer to work from alternate rather than adjacent columns – a source of frequent trouble and complaint. Brown-Hedrick eliminated the quarter-day divisions, and tabulated the consecutive half-days vertically rather than horizontally. Thus D, the Brown-Hedrick choice for A, in any column takes on the successive values $-15^d.5$, $-15^d.0, \ldots, +15^d.0$, $+15^d.0$, sixty-three values in all. The period of D is the synodic month, $29^d.530588$; the range from $-15^d.5$ to $+15^d.5$ exceeds the period by a half-day at either end in order to furnish the second differences needed in interpolating between different values of the function. To avoid negative values of the vertical variable, the variable $\mathrm{D} = D + 15^d.0$ was substituted for D.

In any one of these double-entry tables, consider a given horizontal row, say the row in which $\mathrm{D} = 15^d.0$, so that $D = 0$. The terms whose sum forms the B argument are chosen so as to have a common period after which they return to their initial values. Brown-Hedrick divided this time-period into an integral number ν of parts sufficient to allow for easy interpolation, a column being devoted to each value of ν. The B argument is thus constant in a given column. In the horizontal variation in a single row, proceeding from column to column, D remains constant while the B argument steps through its cycle of values.

For illustration consider Table 3. The argument tabulated is of the form $j(\ell' - \ell)$ $\pm iD$, where $(\ell' - \ell)$ is the difference between the solar and lunar mean anomalies; j is either 1 or 2, and $i = \pm 1, \pm 2, \pm 3, \pm 4, +5$, or ± 6. Different combinations of the factors i and j yield the arguments of the 14 terms incorporated in the table. The period of the B argument is divided into 58 parts, thus identifying 58 successive values of the argument beginning with zero. Table 3 thus has 58 columns, one for each of the 58 selected values of $\ell' - \ell$.

Computing an entry at any position in Table 3 consists in adding up all the terms included in the table, when in each the variable $\ell' - \ell$ has the value characteristic of its column, and D has the value designated for its row. In a given column, there are 63 entries, therefore 63 summations to be carried out, each involving 14 addends. Since Table 3 has 58 columns, the total number of these summations is $63 \times 58 = 3654$.

Brown designed and built a machine to assist in carrying out these summations.[172] It consisted of tapes on which the numbers to be added were written, a

[172] *MNRAS, 72* (April, 1912), 454–463. Brown expresses his indebtedness to Sir George Darwin "for criticisms on the method of presentation of the device and its applications,

carrier holding the tapes, and a frame on which the carrier was placed, with guides to prevent the tapes from getting out of position or becoming entangled. The carrier was a flat piece of brass 1/16 in. by 3/4 in. in cross section, and about 18 in. long. A photograph of the machine shows 14 tapes looped over the carrier, just the number of terms summed in Table 3. Each tape gives the values of one of the 14 terms corresponding to the 63 values of D in a given column. Rotating the carrier about its length advances all 14 tapes to the values of their terms corresponding to the next value of D. As the value of $\ell' - \ell$ changes from column to column, a different set of 14 tapes is required for each column. The whole purpose of the machine is to *present* together the numbers to be added; the actual addition was performed using an adding machine.

Of the 22 double-entry tables, Tables 4–15 and 18–22 are both reversible and reentrant; Tables 1, 2, 3, 16, and 17 are neither.

We turn next to the many planetary terms, most of them small. Most of the errors of Hansen's tables were here: omission of some terms, erroneous values assigned to others.

Brown-Hedrick incorporated a considerable number of the planetary terms in three tables of double entry with ℓ', the mean anomaly of the Sun, as common argument. In Table P1, $V - T$, the mean heliocentric longitude of Venus less the mean heliocentric longitude of the Earth, is tabulated horizontally against ℓ' as vertical variable. In Tables P2 and P3, the analogous job is done for $T - J$, where J is the mean heliocentric longitude of Jupiter, and for $T - M$, where M is the mean heliocentric longitude of Mars.

Besides the terms included in Tables P1, P2, and P3, there are many more planetary terms in the longitude which depend on T, on one of the three arguments V, J, M, and also on one of the three arguments ℓ, $2D$, $2D - \ell$. Rather than introducing separate tables for these terms, Brown-Hedrick added their contributions to Table 30 for the Equation of Center, Table 31 for the Variation, and Table 32 for the Evection.

For illustration consider the incorporation of terms of the form $a \sin(\ell + A)$ into Table 30. Here a is a small coefficient, and A an argument composed of ℓ', one of the three arguments V, J, M, and a constant. The term $a \sin(\ell + A)$ is equivalent to

$$a \sin A \cdot \cos \ell + a \cos A \cdot \sin \ell.$$

In the first of these terms, $a \sin A$ can be treated as a variation of ℓ, since $\delta(\sin \ell) = \delta\ell \cdot \cos \ell$. The variation $\delta\ell$ can be added to ℓ, which is the argument of the largest term in Table 30, *viz.*, $22639''.5 \sin \ell$. But $(a \sin A)$, to be added, must be expressed in appropriate units, namely, the parts into which the period of ℓ has been divided, 18186 of them. Both a and 22639.5 are in arc-seconds. The correct expression for $\delta\ell$ is, therefore,

$$(a \sin A) \cdot \left(\frac{18186}{22639.5 \cdot 2\pi} \right).$$

which led to this paper being almost entirely rewritten." Darwin had described a different apparatus for the analysis of large numbers of observations in *Proceedings of the Royal Society 52* (1892), 345.

The second of the above terms, $(a \cos A) \sin \ell$, may be treated as an addition to the *coefficient* of $\sin \ell$. It is merely necessary to multiply the entries of Table 30 by

$$1 + \frac{a \cos A}{22639.5}.$$

In an entirely similar way, terms of the form $a \sin(2D + A)$ may be attached to the term $2369'' \sin 2D$ of Table 31, and terms of the form $a \sin(2D - \ell + A)$ to the term $4586''.4 \sin(2D - \ell)$ of Table 32.

After the larger terms (those over about $0''.4$) had been included in various tables, along with such smaller terms as could be included without altering the forms of the tables, there remained a host of minute terms which, according to Brown,

> it seemed desirable not to neglect but which would have required many tables. The plan adopted was their summation in blocks for a period of years sufficient to satisfy the needs of the ephemeris up to the year 2050.

These "Remainder Terms," as they were called, were placed in Tables P39–P49. Each of these tables has 150 columns, numbered successively for the 150 years from 1900 through 2049. The vertical variable was tabulated at intervals of 10 or 14 days. An approximate interpolation was deemed accurate enough for furnishing the values for the half-days.

The preceding examples exemplify the inventiveness that went into the Brown-Hedrick Tables. We should perhaps also mention a particularly difficult problem that arose in the construction of the latitude tables.

In his *Theory of the Motion of the Moon*[173], Brown had obtained the latitude terms in the form

$$U = \sum u \sin(F + a) + \sum u' \sin(3F + a) + \sum u'' \sin(5F + a), \qquad \text{(B.126)}$$

where a is an argument formed of multiples of ℓ, ℓ', D; F is $L - \Omega$; and u, u', u'' are coefficients in arc-seconds. U includes about 300 terms, the principal one with a coefficient of $18461''$, the 44 next smaller terms ranging from $1010''$ to $1''$. The sum U, with its many still smaller terms, was believed accurate to $0''.001$.

To make possible the incorporation of the 300 terms of U into a small number of tables, Brown undertook to transform (B.126) into an expression of the form

$$U = (k + C)\{\sin(F + S) + \mu \sin 3(F + S) + \nu \sin 5(F + S) + N\}. \qquad \text{(B.127)}$$

Here N was to consist of a few terms with large coefficients, chosen so that the terms in C would be small and could be put into double-entry tables; C was to be independent of F and contain no constant term; and S was to consist solely of periodic terms, including the terms with large coefficients already incorporated in single-entry tables for the longitude. The terms in S were to be transformed into additions to arguments of sine terms; this turned out to require the expansion of (B.127) in powers of S up to S^5.

173 *TMM, Memoirs of the R.A.S., 57* (1905), 136–141.

The transformation of (B.126) into (B.127), Brown tells us, was an indeterminate problem until μ, ν, and N had been chosen.[174] Choosing them involved a succession of mutual adjustments. Even then, a straightforward method of proceeding further did not present itself; k, C, and S were obtained at last by a sequence of approximations.

[174] *MNRAS*, *72* (1911), 651.

19

Determining the Values of the Arbitrary Constants

The final values of the arbitrary constants (constants of integration arising from the solution of the differential equations) were to be obtained from a least-squares analysis of observations. Brown undertook this task, choosing the Greenwich meridian observations as a basis:

> The Greenwich observations for the past 160 years [1750–1910] present advantages for this purpose which much outweigh the objection to the use of material gathered from one source only. They are continuous, and have been reduced on a consistent plan, which renders their correction and comparison a matter of far greater ease and certainty than observations gathered from a variety of sources.[175]

As will be reported in Part III, the meridian observations at Greenwich and elsewhere were later discovered to be severely subject to systematic errors, and hence problematic.

For Brown, a special advantage of the Greenwich meridian observations was that P.H. Cowell of the British Nautical Almanac Office had already used them for a comparison with Hansen's Tables as revised by Newcomb; Cowell's results were published in the *Monthly Notices* between 1903 and 1905. Cowell had assembled the observations, not into successive years as done previously, but into successive periods of 400 lunar days each. A lunar day is the time between two consecutive meridian transits of the Moon – on average 1.03505 mean solar days. The period of 400 mean lunar days (= 414.02 solar days) is equal, very nearly, to 14 synodic and 15 anomalistic periods of the Moon.

The lunar observations from 1750 to 1901 constituted 134 Cowell periods. For the first 89 of these periods, from 1750 to 1851, Cowell had used the reductions of the observations by G.B. Airy, Astronomer Royal from 1835 to 1881.[176]

[175] *MNRAS, 73* (Suppl., 1913), 692.

[176] G.B. Airy, *Reduction of the Observations of the Moon made at the Royal Observatory, Greenwich, from 1750 to 1830, I* (London, 1848), and *Reduction of the Observations of the Moon made at the Royal Observatory, Greenwich, from 1831 to 1851, . . . , forming a*

C. Wilson, *The Hill–Brown Theory of the Moon's Motion*, Sources and Studies in the History of Mathematics and Physical Sciences, DOI 10.1007/978-1-4419-5937-9_19,

"Reducing the observations" meant computing, from the timed meridian transits and zenith distances of the Moon, the successive positions of the Moon in longitude and latitude. Airy had, as well, compared these positions with Damoiseau's lunar theory, but with Damoiseau's coefficients replaced by values derived from Plana's and Pontécoulant's literal theories. Cowell found many errors in these coefficients.

For the years from 1847 to 1900, Cowell compared the Greenwich observations with Hansen's tabular values as corrected by Newcomb. The latter values still needed correction in order to agree fully with Brown's theory.

These final corrections having been made, it was Brown's and Hedrick's idea to use means of the Moon's observed positions over each of the Cowell periods to obtain the Moon's mean longitudes free of inequalities that were functions solely of the variable D. Special corrections had also to be made for other short-period inequalities. A further difficulty to be surmounted, before the means could be employed in determining mean motion, concerned the Moon's secular acceleration and its fluctuations or "Great Empirical Term." As Newcomb had been pointing out since the late 1870s, the determination of the Moon's average motion and longitude at epoch involved – inextricably – the magnitudes of both these effects.[177]

In 1908, toward the end of his *Theory of the Motion of the Moon*, Brown gave $5''.8$ as his theoretically derived value for the Moon's secular acceleration, in agreement with the value endorsed by Newcomb in 1878.[178] But in 1909 Newcomb announced that, to take account of the effects of the Earth's oblateness and the diminution of the obliquity of the ecliptic, $0''.27$ should be added to this earlier value, yielding $6''.08$.[179] He also gave an observational value, $7''.96$, derived from ancient eclipses along with modern observations. The excess of this value over the theoretical value he attributed to the effect of the tides. (The tidal effect is twofold: the Earth's rotation is slowed by tidal friction, causing an *apparent* acceleration in the Moon's motion, and the Moon receives energy from the tides, causing it to rise into a higher orbit with a diminished mean motion. Newcomb's $1''.88$ is the excess of the first of these effects over the second.)

In April, 1913 Brown received Simon Newcomb's posthumously published "Researches on the Motion of the Moon, Part II."[180] Here Newcomb had employed the Moon's occultations of stars as measures of the Moon's positions before 1750, and also as a check on the accuracy of theGreenwich observations. The new data

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continuation to the Reduction of the Observations of the Moon from 1750 to 1830 (London, 1859).

177 Newcomb, "Researches on the Motion of the Moon, Part I," in Appendix II of *Washington Observations for 1875*.

178 *TMM, Memoirs of the R.A.S.*, *59*, 93.

179 *MNRAS, 69* (1909), 167.

180 *Astronomical Papers for the American Ephemeris 9, Pt.I* (1912), 249 pp. The subtitle reads "The mean motion of the Moon and other astronomical elements derived from observations extending from the period of the Babylonians until A.D. 1908."

caused Brown to withdraw and revise a paper he had already submitted to the *Monthly Notices*. The revised version, entitled "The Longitude of the Moon from 1750 to 1910," appeared in late 1913.[181]

The original object of Brown's paper had been to determine the Moon's mean motion and mean longitude at epoch 1900.0 for use in the new tables. But, he now allowed, "the actual values to be adopted are of less importance than a knowledge of what those values represent and whence they are derived." An exact, unique solution no longer appeared within reach. He reviewed and corrected all the modern data used by Cowell and Newcomb – a considerable task. From these he then re-determined the mean motion and great empirical term, using the theoretical value of the secular acceleration leaving aside Newcomb's observationally derived "tidal excess" of $1''.88$. Comparing his formula with Newcomb's for 36 epochs from 1620 to 1980, he showed that the differences were small: "between 1710 and 1930 there is no error as great as $0''.1$, and…, for the century before, the errors are all less than $0''.5$." Brown was in effect admitting that errors of this magnitude in these constants were unavoidable. The great empirical term and mean motion values obtained in this paper differed but slightly from those later adopted for the Tables.

The puzzle of the great empirical term will re-appear in our Part III. The supposition of a periodical term was really without empirical support. In fact, for the second half of the 19th century, the observational differences from theory were well represented by four straight lines with changes in inclination in 1863, 1877, and 1897. Brown commented:

> These apparently sudden changes may be the combined effect of several periodic terms, but the possibility of their real existence should not be excluded in an attempt to account for the differences between theory and observation.[182]

With regard to the remaining arbitrary constants, Brown was now facing a deadline: a printing schedule for the tables had been established, requiring that he decide on the final values for these constants by early 1915. A series of papers in the *Monthly Notices* record his decisions.

In January 1914 appeared his paper on "The Mean Latitudes of the Moon and Sun."[183] Hansen, in order to bring the Moon's mean orbit into coincidence with the ecliptic, had subtracted $1''.0$ from all the observed declinations, and hypothesized that the Moon's center of mass was below its geometrical center. Newcomb, from the immersions and emersions of stars in occultations, found the subtractive difference to be only $0''.36$; he thought Hansen's hypothesis unwarranted. Brown obtained $0''.51$ from the Greenwich meridian transits for the years 1847–1911, and added that, if Newcomb had used immersions only, his result would have been about the same.

[181] *MNRAS*, 73(1913), 692–714.
[182] *MNRAS, 73* (1913–1914), 713.
[183] *MNRAS, 73* (1913–1914), 156–167.

In March, 1914, appeared Brown's paper on the perigee's longitude and motion and the Moon's orbital eccentricity.[184] After correcting Cowell's data,[185] Brown first removed two periodicities therein contained – possibly due to the irregular shape of the Moon's limbs near the equator – then did a least-squares analysis to determine corrections to the previously adopted values of the perigee's mean motion and secular acceleration. Each of these constants was derivable from both theory and observation. He found $(-37''/1t_c^2)$ for the theoretical value of the secular acceleration and $(-35''.7t_c^2)$ for its observed value, t_c being the number of centuries since 1800. The difference, $1''.4t_c^2$, was smaller than the probable error in the observed value; Brown therefore adopted the theoretical value.

The mean motion of the perigee depends theoretically on six constants: the Moon's mean motion in longitude, its orbital eccentricity and inclination, planetary action, and the flattening ratios for the Earth and Moon. The magnitudes of the first four of these factors could be deduced from either observation or theory more accurately than the perigee's mean motion could be determined observationally, but the values of the flattening ratio for Earth and Moon were known less accurately. For the perigee's mean motion the observational value had therefore to be adopted. Brown found it to be $14643536t_c$ at epoch 1850, $17''$ greater than the value derivable from theory if the Earth's flattening ratio were set at 1/297. Increasing the flattening ratio to 1/294 eliminated the discrepancy here as well as a similar discrepancy in the mean motion of the node. The increase in the flattening ratio, however, proved unacceptable. Years later, in 1936, Brown discovered that a portion of the perigee's motion he had neglected as insensible in 1914 – the term in e'^2 – was large enough to account for most of the previously unexplained difference between the observed and theoretical values.[186] In the case of the node the discrepancy remained unexplained.

Brown determined the Moon's orbital eccentricity from the coefficient of $\sin \ell$ given by the Greenwich meridian observations. This coefficient was given by the Airy series (1750–1850) as $22639''.542$, and by the Hansen series (1847–1910) as $22639''.549$. The probable error of the mean of these two results was about $0''.02$. Brown-Hedrick adopted $22639''.550$ for the Tables. The corresponding value of the orbital eccentricity was got by turning this coefficient into radian measure and setting it equal to the theoretical value, $2e - \frac{1}{4}e^3 + \frac{5}{96}e^5 + \frac{107}{4608}e^7$; solving the equation for e by approximation, Brown obtained 0.054900489.

Brown's fourth and final paper on the constants dealt with the node, inclination, flattening ratio for the Earth, and obliquity of the ecliptic.[187] Here he was seeking constants determinable from an analysis of the coefficients of $\cos F$, $\sin F$, where F is "the argument of the latitude," or $L - \Omega$, L being the Moon's mean longitude, and Ω the longitude of the node. Brown's analysis was based on Cowell's means over 414 days for these coefficients in the Hansen series (1847–1901); the results were found to agree in all cases with the occultations given by Newcomb in 1912.

184 *MNRAS, 74* (1914), 156–167.

185 *MNRAS, 65* (1905), 269ff.

186 *MNRAS, 97* (1936), 116–127; *MNRAS, 98* (1937), 170–171.

187 *MNRAS, 74* (May, 1914), 552–568.

For the inclination, Brown obtained $18461''.39 \pm 0''.03$. Newcomb's value, corrected for certain terms Newcomb had omitted, was $18461''.44$. Brown settled on $18461''.400$.

For the node's longitude in 1800.0, Brown obtained

$$33^\circ 16' 28''.26 - (5^r + 134^\circ 8' 46''.98)t_c + 7''.47t_c^2.$$

Newcomb's value as corrected by Brown was, happily, in good agreement:

$$33^\circ 16' 27''.50 - (5^r + 134^\circ 8' 47''.00)t_c + 7''.47t_c^2.$$

But Brown found the observed value of the node's mean motion in excess of the theoretical value by $12''$ per century, and concluded

> It appears then that we must adopt the observed value and attribute the difference to inaccurate values of the ellipticities of the Earth and Moon (which are uncertain to at least this extent), to unknown perturbations, or to a combination of all three.[188]

W.J. Eckert would be finding a similar discrepancy, still unresolved, in 1965.[189]

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188 *MNRAS, 74* (1914), 563.

189 W. J. Eckert, "On the Motions of the Perigee and Node and the Distribution of Mass in the Moon," *Astronomical Journal, 70* (1965), 787–792.

20

Ernest W. Brown as Theorist and Computer

Brown was primarily an *applied* mathematician. He developed an expertise in choosing or devising maximally efficient routes, coaxing the lunar theory to yield its consequences with a precision comparable to that attained in observations.

Tisserand's *Traité de mécanique céleste*, when it came into his hands in the mid-1890s, acquainted him with Jacobi's streamlined procedure for deducing consequences by the Lagrangian method of varying arbitrary constants. He applied it in instituting certain economies in Delaunay's and Newcomb's treatments of the secular accelerations of the Moon's mean motion, node and perigee.[190] Newcomb had obtained a theorem that gave him the theoretical values of these accelerations. However, he had used Delaunay's developments exclusively, and their slow convergence left an uncertainty amounting to about five percent in the secular acceleration of the Moon's mean longitude. Brown managed to derive a new theorem which allowed him to abbreviate the computations and reduce the uncertainty to one-third of one percent. Of this achievement he was especially proud.[191]

The third and final volume of Poincaré's *Les méthodes nouvelles de la mécanique céleste* appeared in 1899. As a lunar theorist, Brown felt the need to acquaint himself with Poincaré's three-volume work. When in late 1900 he was asked to give one of two annual courses of six lectures for the summer meeting of the American Mathematical Society in 1901, he chose Poincaré's volumes as his topic. As he reported to Darwin in a letter of December 30, 1900[192]:

> [I] have been spending a good deal of time over [Poincaré] and expect to take most of the next six months. I find that when one goes into details it is fearfully hard reading – so many small errors, typographical and otherwise – though I suppose that the general results are right. But I am gradually getting into the heart of it now.

[190] See Brown's articles in *Proceedings of the London Mathematical Society, 27* (1896), 385–390, *28* (1896), 130–142, 143–155.

[191] See F. Schlesinger and D. Brouwer, "Ernest William Brown, 1866–1938," *Biographical Memoirs of the National Academy of Sciences, XXI*, 246.

[192] CUL, MS.DAR.251:1593.

C. Wilson, *The Hill–Brown Theory of the Moon's Motion*, Sources and Studies in the History of Mathematics and Physical Sciences, DOI 10.1007/978-1-4419-5937-9_20,

In the following April he added[193]:

> I haven't touched calculations this year. My whole time has been put on Poincaré's *Mécanique Céleste* and I am now becoming a little more at home in his methods. My respect for him grows with every chapter I read and annoyance at the same time. He is very careless – sometimes proofs are faulty – sometimes incomprehensible, and the number of misprints – some of them misleading – is fearful. I have occasionally spent three or four hours on a single page and then found that the difficulty arose from some misprint or misstatement. But it reads beautifully if one doesn't try to go too much into details.

Brown was in England in 1901 but restricted his stay to the early summer, since he had to be in Cornell by early August for the Mathematical Society meetings. He promised Darwin an account of the lectures, and when he next wrote (on October 19), sent along the syllabus he had drawn up[194]:

> The voyage across in August was good, but the climate on arrival [in the U.S.A.] decidedly steamy. However, Ithaca – or rather Cornell – is on the top of a hill above the Lake. We had a good time there. Over 45 members of the Society attended and 25 stayed for the lectures. As I feared, I had to cover rather too much ground and they became somewhat sketchy. However my main purpose, I think, was achieved – the interesting of pure mathematicians in the subject. It was rather hard work – four successive days with nearly two hours talking each session – two or three hours in the morning spent in preparation – talking mathematics and mathematical business between times and formal or informal committee meetings about the *Transactions* [of the American Mathematical Society; Brown had been chosen as one of the editors] in the evenings. I enclose the syllabus. Poincaré's classification at the end of it may interest you. I had a lot of trouble finding out what his various divisions meant. I am trying to write out now a fuller account for the *Bulletin* of the American Mathematical Society which I will send when published.[195]

Brown's intensive preparation for the lectures did not lead him into Poincaré-style investigations of his own. Poincaré's new topological methods were not needed in developing the successive approximations of the lunar theory. The papers Brown published between 1891 and 1908 were, with few exceptions, dedicated to achieving an accurate lunar theory.

Brown's concern with computational efficiency emerged early. The computations required for a sufficiently precise lunar theory, he realized, would be numerous and

[193] CUL, MS.DAR.251:4874.

[194] CUL, MS.DAR.251:4942.

[195] The *Bulletin* article, "Modern Methods of treating Dynamical Problems and in particular the Problems of Three Bodies," was published in *Bulletin of the American Mathematical Society*, series 2, **8**, 103–113.

complicated. In his letters to Darwin he used such adjectives as "tedious," "everlasting," "soul-destroying," in describing these computations.

A little over 2 years later, as he was finishing corrections to the proof sheets for his *Introductory Treatise on the Lunar Theory*, he reported to Darwin that

> During the intervals of proof sheets in this last month I have been trying to reduce the rectangular coordinate method [Hill's method] to a form in which all the inequalities can be practically reduced to the multiplication of series with an occasional quadrature – i.e. so that the grind can nearly all be done by a professional computer. I have succeeded with the latitude inequalities – no solving of linear equations needed and the higher parts of the motion of the node come out as nearly as possible. I think I can manage to do the same with the inequalities in X, Y – at any rate for those independent of the polar parallax.[196]

In March, 1896, in a further letter to Darwin, he was able to add:

> I told you in my last that I was making an attempt to get the Lunar Theory reduced to the form where a practical computer may do nearly all the work. Shortly after, it came out in a form simple beyond my utmost expectations. It depends on the 'complementary function' and 'particular integrals' of Differential Equations.[197]

In the last sentence Brown refers to topics dealt with in the essay published in the Stokes Memorial volume, which appeared in 1900.[198] This essay had been largely completed in 1896. Its original aim, it appears, was to construct *the general solution* of Hill's equations in their homogeneous form (see, for example, equations B.52a, b, c), rather than to achieve computational efficiency. But the reduction of the development of the theory to the multiplication of series of sinusoidal terms expressed exponentially needed to be justified: it was necessary to know that non-periodic terms – terms proportional to t – were absent. These equations had yielded Hill's variation curve as a particular periodic solution, and Brown's "solution of the main problem" was also a solution of them. Since each of Hill's equations was of the second order, the two of them formed a system of the fourth order, and the general solution consequently required knowing four independent particular solutions. Once having obtained the general solution, Brown was able to argue that,

> ... it being granted that the series forming the 'Variation' inequalities and the elliptic inequalities depending on the first power of the Moon's eccentricity are convergent, it is not difficult to demonstrate ... that all the terms multiplied by a given combination of powers of the eccentricities, inclination and

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[196] CUL, MS DAR.251:477.

[197] CUL, MS DAR.251:478.

[198] "On the Solution of a Pair of Simultaneous Linear Differential Equations, which occur in the Lunar Theory," *Transactions of the Cambridge Philosophical Society, 18*, 94–106.

> ratio of the parallaxes, that is, all the terms with a given characteristic, form a convergent series.[199]

The quest for computational efficiency had thus led Brown to a conditional confirmation of the theoretical legitimacy of the development.

The superior accuracy that Brown was able to achieve, as compared with earlier lunar theorists, may be attributed to the reduction of much of the theory's development to the multiplication of series, as well as to Brown's use of independent equations of verification (mentioned in Cowell's letter as quoted by Maw in his Presidential Address of February, 1907). A still more fundamental cause lay in Hill's "variation curve" itself. This was the "intermediate orbit" employed in Brown's calculation; all earlier theorists except Euler had assumed an elliptical orbit, a solution of the two-body problem applied to the Moon moving about the Earth, as starting-point for the successive approximations. Hill's variation curve was a better approximation to the Moon's actual motions. It led to steeper convergence in the series for the coefficients of the "perturbation" terms – the terms added to tweak the variation curve into conformity with reality.

During the first 11 of his 16 years at Haverford, Brown found himself, a number of times, considering the possibility of seeking a different post elsewhere. Impelling him in this direction during the early years was the American climate with its extremes, which appeared to have a very adverse effect upon his health and made him think of returning to England. "When," he asked Darwin in December, 1893, "are they going to appoint a man in astronomy at Oxford?"[200]

Haverford College went through spells of financial stringency during the 1890s. Brown took a leave of absence for the 1894–1895 academic year, in part to help the College save money. In the spring of 1896 the College's situation worsened, and the College President gave Brown a bit of a nudge toward moving on. As Brown reported to Darwin,

> I am in rather a fix just now. The College is financially in rather low water – pending the death of an old lady (the College will have an extra income of about £4000 a year when she dies.) They've been reducing expenses to the least possible minimum, and not long ago the President told me that they couldn't keep two men in Mathematics. The fact is that there isn't really enough work for both of us. Morley (formerly of King's) is a married man with a family and has been here nine years longer than I have, so, of course, when the President said he didn't know which of us to keep, I offered my resignation. He told me that if I like to stay on for the present on a small salary and very little work, he would only be too glad, but that if I wanted to get another post, he would give me the best testimonials. I've told you all these details because I don't want you to think I've been a failure here.[201]

[199] Ibid., 94.
[200] CUL, MS.DAR.251:470.
[201] CUL, MS.DAR.251:479.

Darwin proceeded to apprise Brown of an opening (unidentified) in Bombay, but Brown believed he shouldn't apply for it[202]:

> ...the climate is an obstacle which will prevent me thinking of it. I can't stand heat and, from what I hear, that is about the warmest place on the globe. It would probably kill me in 12 hours. The warmest days here are comparatively cool and they have a sufficiently bad effect. If nothing turns up, I can make an arrangement to stop here next year at 500 dollars a year with five hours work per week. It isn't much but with that and my own income I can at least keep myself. I need not decide about staying until it is time to return [to Haverford from England] in September, so if anything turns up, I shall be free. I suppose there is no chance of a post in the Nautical Almanac Office?

Darwin sent Brown a letter of recommendation for use in seeking a job, and also advised him to apply forthwith for Cambridge's Doctor of Science degree (Brown would receive the degree in 1897). Brown in his next letter reported that he was sending Darwin's letter to Seth Low, President of Columbia University[203]:

> Of course I should immensely like work in England and especially at Oxford if there were any probability of its leaving me a fair part of the time for my own work. I am sending the letter you were good enough to write, to Seth Low. I don't know what chances there are there – but it can't do any harm to let him know that I am open to offers and your letter will have weight with him.

No opening emerged, and in the autumn Brown was back at Haverford, on a low salary and light teaching load. On December 7, 1896, he wrote Darwin as follows[204]:

> I must say I like the present arrangement – it is delightful to have nearly all one's time at liberty and I get lots of work done. Sometime in the Spring I expect to have read a paper which will contain the complete solution of the lunar theory as far as the second order of the eccentricities, inclinations and parallaxes together with the motions of perigee and node to same order and some inequalities of the third order – <u>all</u> powers of *m* being included numerically. I don't in the least know where to put it – there will be so much arithmetic in it and it will run to perhaps sixty or seventy quarto pages. The *American Journal* [*of Mathematics*] does not seem quite the right place. Do you think that the Royal Society would publish it or that the Astronomical Society *Memoirs* would be a more fitting place?

In March 1897 Brown informed Darwin that he was submitting the paper for publication in the *Memoirs* of the Royal Astronomical Society.[205]

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[202] Brown to Darwin, April 30, 1896, CUL, MS.DAR.251:480.

[203] Brown to Darwin, May 8, 1896, CUL, MS.DAR.251:481.

[204] CUL, MS.DAR.251:487.

[205] CUL, MS.DAR.251:488.

In his letter to Darwin of December 30, 1900, Brown reported having had a visit from Stimson J. Brown, newly appointed Astronomical Director of the Naval Observatory and Director of the Nautical Almanac Office. Stimson Brown proposed supplying our Brown, E.W. Brown, with a team of computers who would assist in completing his lunar theory. The team would then proceed to construct lunar tables from the theory. Our Brown would be in charge of the whole operation, with Hill and Newcomb serving as an advisory board. To support this plan financially, Stimson Brown would seek a grant from Congress. To Darwin our Brown wrote[206]:

> ...there are so many [obstacles] in the way that I doubt whether he [Stimson Brown] will be able to do anything for a long time yet .... But if he manages to get what he wants and does ask me to undertake them [the tables] – of course for a proper remuneration – I don't know whether to accept. I should probably be able to continue my work here just as before but [not] if it would mean giving up my whole spare time to the tables for perhaps eight or ten years, thus doing work which would get very tedious and mean the stoppage of every other kind of research and mathematics. Of course, there would not be a large amount of actual calculation to do as the Office would furnish the computers.... If you can with this very rough outline give me any advice, I should be very grateful. I am not very sure of my ability to advance the subject in a mathematical direction and in that case it would be much the best to undertake the work.

Brown, we gather, was diffident about his ability to establish himself as a successful mathematician in the American mathematical community. This sense of inadequacy appears to have diminished in the following years as he completed the theory and gained recognition for his achievement. But a great deal of modesty remained. In an address of 1914 he said:

> My own theory [of the Moon], which was completed a few years ago, is rather a fulfillment to the utmost of the ideas of others than a new mode of finding the Moon's motion. Its object was severely practical – to find in the most accurate way and by the shortest path the complete effect of the law of gravitation applied to the Moon. It is a development of Hill's classic memoir of 1877.[207]

Schlesinger and Brouwer, commenting on this statement, urged that "it does not give the emphasis that it should to [Brown's] own resourcefulness in finding the most accurate solution by the shortest path." We suspect in Brown a certain pride in his practical achievement as a computer, though it long remained covert. Toward the end of his life he will show that Hill's invocation of the infinite determinant was an unnecessary detour into mathematical sophistication: the same result could be obtained by the lowly computer, without resort to highfalutin mathematics.

[206] CUL, DAR.MS.251:1593.

[207] F. Schlesinger and D. Brouwer, "Ernest William Brown," *National Academy Biographical Memoirs, XXI*, 244.

As for the dilemma posed by Stimson Brown in December, 1900, no decision proved necessary, as we learn from Brown's letter to Darwin of April 21, 1901[208]:

> As it has turned out, . . . it looks as though the matter would be postponed for a long time even if it is undertaken at all. There has been another row in the Naval Observatory. The nominal chief is a retired navy sea captain and the working head under him a scientific man but a government servant. Stimson Brown who was in the latter position when I wrote (the position carries also that of director of the Nautical Almanac) criticized the sea-captain in rather too free a manner – result – threats of a court-martial which was averted by the Secretary for Agriculture removing Brown from duty at the Observatory and putting a fellow called Harschman in his place. . . . So what will happen I don't know but personally I am relieved that the work won't come on at present – if it comes my way at all. They are trying to reorganize the Almanac and Observatory and put them on a proper basis – many rows and more wire-pulling will result for at least a year.[209]

Brown in the same letter thanked Darwin for writing (on February 1) about "the question of the Lunar Tables:"

> . . . it helped me a great deal and I felt that when the time came I should have less difficulty in coming to a decision.

The question of how and by whom the tables would be constructed was resolved later, Yale University offering major assistance.

In late October of 1902 Brown submitted his name for possible nomination to take charge of a meteorological office with headquarters at Simla, India, in the mountains north of Delhi. Darwin and Larmor were on the selection committee. The notification of the opening had reached Brown late – the deadline for applying had already arrived, but Brown cabled his willingness to be considered. On November 3 Brown sent off a long letter to Darwin explaining what his thoughts were in deciding to apply[210]:

> Of course the most important matter to me was the change in my work. It was clear to me that the Lunar Theory would have to be definitely and finally abandoned. It might be possible, in intervals of leisure and holidays to finish the remaining work though I doubt much whether I should be able to do so. It was also perfectly clear that if the post came to me I should have to devote myself to meteorology. You were quite right in telling Eliot that I should do so. In fact I want to say this in the strongest way possible – all my energies would be given to the subject. It is hard to leave the numerical Lunar Theory just now when it is so near completion – I could finish it in

[208] CUL, DAR.MS.251:4874.

[209] For an historian's account this episode, see S. J. Dick, *Sky and Ocean Joined*: *The U.S. Naval Observatory, 1830–2000* (Cambridge: Cambridge University Press, 2003), 329ff.

[210] CUL, DAR.MS.251:5051.

> a year's steady work – and nearly as hard to leave the subject itself. I have been very much interested – more than usual – lately in working up the literature for the Encyclopedia article, which of course would have to be dropped. [Brown had agreed to write an article on the Lunar Theory for the German *Encyklopädie der Mathematischen Wissenschaften*. It appeared in 1915 under the title "Theorie des Erdmondes," Brown's English version having been translated into German by A.v. Brunn.] On the other hand, I am not very certain whether the L.T. [Lunar Theory] has very much more in store for me apart from the calculations part. If it has [hasn't?] it will mean that I shall soon have to take up other subjects, e.g. theory of differential equations, and apply them to celestial mechanics. On the other side [the side of Meteorology] lies an almost untouched field from the mathematical standpoint. The fact that the Meteorological Re. [?] would be relieved of much of the routine work pointed to a certain degree of leisure for going into the subject mathematically and all round. Of course one naturally feels doubtful of any success in tackling such difficult problems but something might be done to at least make a start.

Brown believed that questions of health would not be a problem: Simla, he had heard, was "one of the healthiest places in the world." Moreover, he had been

> growing stronger every year by finding out how to take care of myself, and now I rarely have anything the matter with me and can do almost as much work in the day as I want to.

In a final paragraph, Brown revealed what may have been the deciding motive for his allowing his name to go forward:

> Another reason which made me anxious to see my way clear to accepting was the fact that you especially (and I fancy one or two others) have been on the lookout for me and I have done nothing on my part. . . . I do want you to know that I appreciate what you have done for me and the best way I can show it is by doing my best to second your efforts.

A letter from Darwin, a few days later, informed Brown that the meteorological post had gone to another man. Brown's response, dated November 10, 1902, is preserved only in part.[211] A month later Brown was once more at work on his *Encyklopädie* article.[212]

In late 1900 Brown had decided to build a house on the Haverford campus.

> I get the land for nothing and the College agrees to take the house off my hands, should I leave, at a valuation.[213]

[211] CUL, DAR.MS.251:5052.
[212] CUL, DAR.MS.251:5053.
[213] CUL, DAR.MS.251:1593.

Ground for the house was broken on May 1, and construction completed by October. An important part of the plan was that his younger sister, Mildred, was to come from England to live with him. She was in Haverford by September, and she and Brown labored through the autumn to get the house furnished and in working order. Of her role in his life, Schlesinger and Brouwer in their *Biographical Memoir* of Brown have this to say:

> [The] household was presided over for many years by [Brown's] maiden sister Mildred, junior by two years. For most of her adult life she made it her chief, almost her sole, concern to see to his comfort and shield him from cares and disturbances. She succeeded in utterly spoiling him. She died a few years before her brother.[214]

Mildred's ministrations are probably responsible for the thermos bottle of coffee, mentioned in our biographers' account of Brown's rather unusual daily routine[215]:

> He would retire rather early in the evening and as a consequence would awaken usually from three o'clock to five o'clock in the morning. Having fortified himself with a number of cigarettes and a cup of strong coffee from a thermos bottle, he would then set to work in earnest without leaving his bed. At nine o'clock he would get up and have his breakfast. Unless he had something especially exciting on hand, he would not return to mathematical work until the next morning, devoting the intervening time to correspondence, teaching and other similar duties. This program he carried out whenever possible, at home, at the houses of friends he was visiting, and even on board ship.

Once the new pattern in his life was established, we can imagine that Brown settled down to his task of finishing the lunar theory with a new confidence and determination. Each successive stage of the computation was prepared for carefully and systematically executed.

How expert was Brown as a computer? In the 1960s, Walter J. Eckert and Harry F. Smith carried out, by means of the electronic computers that by then had come into use, a computation of the main problem of the lunar theory, some 9600 terms in all, to an accuracy of $2'' \times 10^{-7}$. Comparing their results with Brown's, they concluded:

> Our results show that Brown's solution is even better in many respects than he had hoped when he made it, and the freedom from error in his work is truly phenomenal.... In our comparison we found only one correction as large as $0''.01$, eight as large as $0''.005$, and 51 as large as $0''.002$. The

[214] Schlesinger and Brouwer, "Ernest William Brown," *National Academy Biographical Memoirs, XXI*, 258.

[215] Ibid., 259.

outstanding correction is that in y with argument $2F - 2\ell$ which corresponds to a correction in the longitude of approximately $0''.072 \sin(2F - 2\ell)$.[216]

Events that we are unable to trace led to Brown's appointment as professor of mathematics at Yale University beginning in 1907. A major factor leading to Brown's acceptance of this appointment was Yale's commitment to support the construction of lunar tables by establishing a fund for the purpose, which eventually expended $34,000 on the project. The tables, as we have seen, required the solution of a difficult problem presented by the vastly larger number of sinusoidal terms in Brown's theory than in any earlier theory.

As we have also seen, the Tables had to be devised despite the puzzle presented by the unexplained fluctuations in the Moon's motion. What was their cause, and how were they to be dealt with? To these questions we shall turn in Part III.

[216] W.J. Eckert and H.F. Smith, Jr., "The Solution of the Main Problem of the Lunar Theory by the Method of Airy," *Astronomical Papers prepared for the Use of the American Ephemeris and Nautical Almanac, 19*, Part II, 196.

Part III

Revolutionary Developments in Time-Measurement, Computing, and Data-Collection

21

Introduction

Parts I and II of this study traced the development of the Hill–Brown lunar theory (1) starting from George W. Hill's determination of the parameters of the Variation Curve in 1877–78, (2) continuing with Ernest W. Brown's computation in the years 1891–1908 of the 3000 or so additional terms required to yield the coordinates accurately to $0''.01$ in latitude and longitude and to 0.001 in sine parallax, and (3) ending with the incorporation of these terms in the Brown-Hedrick *Tables of the Motion of the Moon*, published in 1919.

Beginning with the year 1923, the lunar ephemerides in the nautical almanacs of the United States, Great Britain, France, Germany, and Spain were computed from the Brown-Hedrick *Tables*.[217] This practice might have continued through the century had it not been for (a) questions left unanswered when the *Tables* were published and (b) the development of automatic-sequenced computers, atomic clocks, and remarkable new modes of collecting data.

Brown had omitted from the *Tables* any reference to a *tidal* acceleration of the Moon, as being uncertain. But in 1920 evidence emerged to show that a tidal acceleration was sizable (the value then newly given for the increment per century was $4''.75$ per century). A tidal acceleration of the Moon would entail a gradual slowing of the Earth's rotation – the clock that astronomers had relied on since the dawn of astronomy.

Another kind of departure of the Moon from the motion implied by gravitational theory had also emerged. These were "fluctuations," which, unlike the tidal acceleration, involved changes in mean motion lasting for days or months or years, but not steadily persisting as did the secular acceleration. Fluctuations had first been detected in the 1860s. Brown's computation of the Moon's planetary perturbations, finished in 1908, and clearly more complete than any earlier computation of these perturbations, excluded any lingering hope that a gravitational source for these departures would be found. To represent a large, long-term component of the fluctuations, Brown included

[217] *Explanatory Supplement to the Astronomical Almanac*, ed. P. Kenneth Seidlemann. Sausalito, California: University Science Books, 1992, 640. See ibid., 616ff. on cooperation of the principal ephemeris offices.

C. Wilson, *The Hill–Brown Theory of the Moon's Motion*, Sources and Studies in the History of Mathematics and Physical Sciences, DOI 10.1007/978-1-4419-5937-9_21,

in his *Tables* a sinusoid, $10''.71 \sin[140^\circ.0T + 240^\circ.71]$. This was a makeshift, since there was no guarantee that such a term would continue to fit observations in the future, or that the fluctuations were periodic as the sinusoidal term implied. In the 1870s Newcomb had already proposed the hypothesis that the fluctuations arose from variations in the rotation of the Earth,[218] but he was unable to confirm this hypothesis. If the hypothesis should prove true, then clearly the Earth's rotation would no longer be able to serve as the astronomer's clock. A major re-thinking of astronomical time-keeping would be required.

Investigations into the tidal acceleration and fluctuations continued after the publication of the Brown-Hedrick *Tables*. In 1939 these investigations at last yielded persuasive evidence that the anomalies in the Moon's motion could be accounted for by variations in the Earth's rotation. No other interpretation was plausible. The empirical term in the *Tables* had therefore to be deleted. The time used in the ephemerides of Sun, Moon, and planets, at first called Newtonian Time, was now recognized as distinct from the time in which astronomical observations were made, namely Universal Time. The latter was dependent on the rotation of the Earth, and therefore variable. New rules needed to be established for correcting Universal Time to Newtonian Time.

Meanwhile the powers of the electronic computer began to be developed, partly in connection with the further investigation of the Moon's motion. Beginning in the 1920s, automated ways of computing were introduced into astronomical practice, first by Leslie J. Comrie in Britain, then by Wallace J. Eckert in the United States. By 1948 Eckert had developed an automatic-sequenced computer capable of calculating the Moon's celestial coordinates directly from the trigonometric series of Brown's theory and independent of the *Tables*. The improvement over the accuracy and precision attainable using the *Tables* was significant. Further improvements in accuracy and precision soon followed. These developments are summarized in this final part of our study.

218 See S. Newcomb, "Fluctuations in the Moon's Mean Motion," *Monthly Notices of the Royal Astronomical Society* (hereinafter *MNRAS*), *69* (1909), 164.

22

Tidal Acceleration, Fluctuations, and the Earth's Variable Rotation, to 1939

The proposition that the Moon's motion was slowly accelerating over the centuries was first put forward by Edmond Halley in 1692.[219] Determinations of the size of this acceleration during the 18th and early 19th centuries put it at about $10''$ of increase per century. This meant that the Moon's mean motion contained the term $10''T^2$, where T represents 100 Julian years of 36525 days each; the increment is equal numerically to half the angular acceleration. The 18th-century astronomers referred to the increment as the "secular acceleration" of the Moon, and this usage continued into the 20th century.

The cause of the Moon's acceleration was much debated. At last in November, 1787, Pierre Simon Laplace announced that he had succeeded in deriving it from the law of gravitation.[220] He found the Moon's acceleration to be the indirect effect of planetary perturbations of the Earth's motion. These produced a sinusoidal variation in the Earth's orbital eccentricity, cycling through its values over a period of some hundreds of thousands of years. In the present age the eccentricity is decreasing, causing an increase in the ratio of the Earth's to the Sun's mean gravitational action on the Moon. The Moon therefore falls into an orbit closer to the Earth, with a consequent increase in its mean angular motion.

In his *Mécanique Céleste*, Laplace gave $10''.181621T^2 + 0''.01853844T^3$ for the theoretical increase in the Moon's mean angular motion per century during the present age. (The number of significant figures that Laplace claimed here was outrageously excessive, such excess being customary in the 18th century. The whole

[219] Information regarding the Moon's secular acceleration and its discovery and interpretation can be found in R. Grant, *History of Physical Astronomy* (Johnson Reprint Corporation, 1966), 61–62, and in J.P. Britton, *Models and Precision* (New York: Garland Publishing, 1992), 153–171.

[220] On 23 November 1785 Laplace read "Un mémoire sur les inégalités séculaires des planets et des satellites." This was a draft of "Mémoire sur les inégalités des planets et des satellites" published in the *Mémoires de l'Académie royale des Sciences de Paris*, 1784/1787. The latter memoir is also found in *Oeuvres completes de Laplace*, XI, 49–92.

C. Wilson, *The Hill–Brown Theory of the Moon's Motion*, Sources and Studies
in the History of Mathematics and Physical Sciences, DOI 10.1007/978-1-4419-5937-9_22,

matter of significant figures was first clarified by Karl Friedrich Gauss in 1809.[221]) Values close to Laplace's were found by J.L. Lagrange in the 1790s and by G.A. Plana and M.C. Baron de Damoiseau in the 1820s. The theoretical values agreed as well with the empirical values as ancient eclipse reports, unavoidably vague and uncertain, allowed.

But in 1853 John Couch Adams showed this agreement to be specious.[222] Certain terms in the theoretical derivation, judged negligible by Laplace, were in fact sizable. Including these terms in the computation, Adams obtained 5″.70 of increase per century, considerably less than Laplace's 10″.18. Adams' theoretical result for the Moon's centennial acceleration was initially disputed by Giovanni Plana, Philippe de Pontécoulant and Peter A. Hansen. Adams showed that the theoretical derivations of Plana and de Pontécoulant were fallacious.[223] Hansen's value of 12″.18 for the centennial acceleration was derived from the paths of totality of certain ancient solar eclipses described by George B. Airy.[224] This value was observational, and hence not relevant to the theoretical dispute. Adams' theoretical result was substantiated in re-calculations by Richard Cayley and by Charles Delaunay. William Ferrel in 1864 and Delaunay in 1865 suggested that the difference between the theoretical and observational values could be due to tidal friction.[225] Tidal friction would cause a slowing of the Earth's rotation, leading to an *apparent* acceleration of the Moon. At the same time, there would be a variation in the Moon's orbital velocity, as required by conservation of angular momentum.

This topic was discussed by William Thomson and Peter G. Tait in their *Treatise on Natural Philosophy* of 1867, and it has been dealt with in more recent discussions.[226] Tidal friction and imperfect elasticity in the solid parts of the Earth cause the tidal bulge to be carried eastward by the Earth's rotation. The attraction between this bulge and the Moon gives the Moon added energy in the direction of its forward

[221] See K.F. Gauss, *Theory of the Motion of the Heavenly Bodies Moving about the Sun in Conic Sections*, (tr. C.H. Davis: New York: Dover publications, 1963), Second Book, *passim.*

[222] J.C. Adams, "On the Secular Variation of the Moon's Mean Motion," *Philosophical Transactions of the Royal Society London*, *143* (1853), 397–406. Reprinted in *MNRAS*, *14* (1854), 59–62.

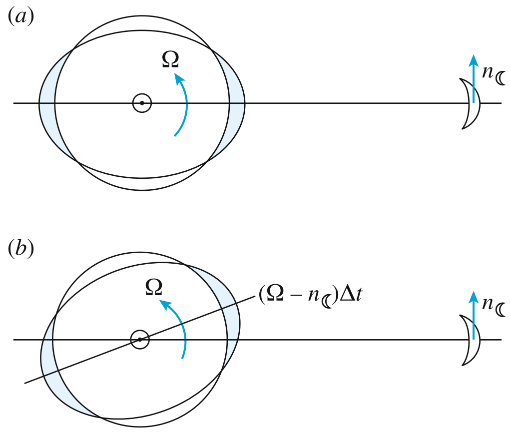
[223] See S. Newcomb, *Popular Astronomy* (New York: Harper & Brothers, 1878), 97.

[224] G.B. Airy, "On the Eclipses of Agathocles, Thales, and Xerxes," *Philosophical Transactions of the Royal Society of London*, *143* (1853), 179–200. See also *MNRAS*, *17* (1857), 234. The paths of totality comprise the successive areas on the Earth's surface from which the Sun is seen as totally eclipsed.

[225] W. Ferrel, "Note on the Influence of the Tides in Causing an apparent Secular Acceleration of the Moon's Mean Motion," *Proceedings Of the American Academy Arts and Sciences*, *VI* (1864); C.E. Delaunay, "Sur l'existence d'une cause nouvelle ayant une action sensible sur la valeur de l'equation séculaire de la Lune," *Compte Rendu des Séances de l'Académie des Sciences*, *61* (1865), 1023–1032.

[226] Sir W. Thomson and P.G. Tait, *Treatise on Natural Philosophy*, (Oxford, 1867), Part I, section 276. A more recent account is in Kurt Lambeck, *The Earth's Variable Rotation* (Cambridge University Press, 1980), Chapter 6.

motion (eastward with respect to the stars). As a consequence, the Moon rises into a higher orbit where its mean motion is less. The accompanying diagram[227] shows the relevant geometrical relations. Here Ω is the Earth's rate of rotation about its axis, and $n_☾$ is the Moon's orbital mean motion. In Figure *(a)* the Earth responds elastically to the Moon's gravitational force, in *(b)* it responds with a time delay due to friction or anelasticity.



The amount of energy transferred from the Earth to the Moon during any time interval depends on where the Moon is in its orbit and the force of attraction exerted by the tidal bulge at that time. These quantities cannot be obtained directly from general theory. In the late 20th century, laser-ranging made possible the verifying of the Moon's recession from the Earth, about 3.82 cm. per year, or 3.82 m. per century.

During the 1870s, Simon Newcomb set out to review and correct Hansen's *Tables*. He would later use his corrected version for the lunar ephemerides in the *American Ephemeris and Nautical Almanac*.[228] In 1878 he published revised values for the Moon's secular acceleration.[229] He gave 5″.80 for the part of this acceleration caused by planetary perturbations of the Earth and Moon, and he gave 8″.32 for the total acceleration, comprising the theoretically derived part just mentioned and the part ascribed to tidal friction. Newcomb derived the value 8″.32 from several ancient

227 From Lambeck, *The Earth's Variable Rotation* as cited in the preceding note, p. 119.

228 The lunar ephemerides in the *American Ephemeris and Nautical Almanac* from its inception in 1855 were based on the lunar theory of Benjamin Peirce. The corrected Hansen *Tables* became the basis beginning with the year 1883.

229 Simon Newcomb, *Researches on the Motion of the Moon, Part I*, in Appendix II of "Astronomical and Meteorological Observations Made during the Year 1875 at the United States Naval Observatory," 1878.

solar and lunar eclipses; he distrusted the ancient reports of total solar eclipses from which Hansen had derived his larger value. His results implied a tidal part of the Moon's secular acceleration of $8''.32 - 5''.80$ or $2''.52$.

Shortly before his death in 1909, Newcomb published a further revision in his values for the Moon's observed secular acceleration and its theoretically derivable part[230]:

> The observed secular acceleration is now found to be less by $0''.37$ than that which I derived in 1876. As for the theoretical value, I have added $0''.27$ to the value found by Brown and myself, on account of the effect due to the combination of the Earth's oblateness with the secular diminution of the obliquity of the ecliptic. This carries the theoretical acceleration up to $6''.08$. The value now found from all observations is

Secular acceleration from mean equinox	$9''.07$
Sidereal value	$7''.96$
Tidal excess	$1''.88$

Newcomb's "tidal excess" refers to the difference between the new observational value of $7''.96$ and the new theoretical value of $6''.08$.

The Moon's Fluctuations, from Their Discovery to the Publication of Brown's Tables

That the Moon's longitudinal motion was fluctuating in ways not derivable from gravitational theory became evident in the 1860s. It emerged, for instance, in an examination of the planetary terms which Hansen had included in his *Tables de la lune* (1857).[231] Two of them were relatively large, long-term perturbations of the Moon by Venus. For one them Hansen gave the formula $15''.34 \sin(18V - 16E - g)$, where V is the mean longitude of Venus, E that of the Earth, and g the mean anomaly of the Earth. The implied period is 273 years. Hansen's coefficient and argument for this term were refined in the 1890s, but the corrections were small. For the other term Hansen gave the formula $21''.47 \sin(8V - 13g + 4°44')$, with an implied period of 239 years. In 1863 Charles Delaunay carried out a careful derivation of this second term and found that the coefficient could not exceed $0''.272$.[232] According to Newcomb, "Hansen himself admitted that he had been unable to determine the

[230] *MNRAS*, *69* (1909), 167.

[231] Newcomb in his *Popular Astronomy* (1878, pp. 98–99) states that Laplace had found in the years just before 1800 that the Moon was falling behind its calculated place. He proposed two conjectural explanations of the inequality, but both were disproved by later investigators.

[232] C.E. Delaunay, "Sur l'inégalité lunaire à longue période due à l'action perturbatrice de Vénus et dependant de l'argument, $13L' - 8L''$," *Additions à la connaissance des temps* 1–56, 1863.

amount of this inequality in a satisfactory manner from the theory of gravitation, and had therefore made it agree with observation...."[233]

Hansen's mistaken Venus term, combined with Hansen's gravitational theory, produced agreement with the observations for the period 1750–1850. But Newcomb found that this same combination failed to produce agreement with occultations observed before 1750 and after 1850. The Moon was deviating from the path and motion that gravitational theory implied, and this deviation was large and centuries long.[234] To represent this deviation, Newcomb found no better way than to employ a sinusoidal term similar to Hansen's, but of different period and coefficient: $15''.5\sin(1^\circ.32t + 93^\circ.9)$. The implied period was 273 years. Newcomb thought that variability in the Earth's rotation was the most likely cause of the deviation.[235]

In 1882 Newcomb published a study entitled "Discussion and Results of Observations on Transits of Mercury, from 1677 to 1881."[236] Among other aims, this study was designed to test whether the fluctuations of the Moon were caused by variations in the axial rotation of the Earth. This idea could best be tested on relatively rapidly moving celestial bodies (e.g., Mercury and the satellites of Jupiter) to see whether they showed the same apparent inequalities as the Moon.

In a study of the Moon's motion he had published in 1878, Newcomb included a computation of the errors, Δt, with which the astronomical determinations of time would have to be affected in order to explain the non-gravitational inequalities detected in the Moon's motion.[237] Supposing that the motion of Mercury was affected by errors due to the same cause, he interpolated between the tabulated values of the Moon's errors to find values for Δt at the times of the 23 transits of Mercury observed since 1677. Before inserting these values into the equations of condition, he multiplied them by a constant k. The value of this constant was to be determined, along with ten other quantities including the orbital elements of Mercury, by the method of least squares. If the hypothesis of perfect uniformity in the Earth's rotation was correct, k would be zero or close to zero. If the observed inequalities in the Moon's mean motion arose from the errors Δt, the value of k would come out to be equal, or nearly equal, to unity.

In his least-squares analysis, Newcomb found k to be 0.295. He inferred that the probability of $k = 1$ was less than that of $k = 0$. But he regarded $k = 0$ as also improbable, given the systematic character of the residuals. He was puzzled that a value near $k = 0.3$ should nearly satisfy the whole series of observations:

> ...we must regard it as quite improbable that the inequalities in the mean motion of the Moon are entirely to be accounted for by changes in the earth's

[233] *Popular Astronomy* (1878), 99.

[234] *Researches on the Motion of the Moon, Part I*, in Appendix II of "Astronomical and Meteorological Observations Made during the Year 1875 at the United States Naval Observatory," 1878.

[235] Newcomb, *Popular Astronomy* (1878), 99.

[236] *Astronomical Papers prepared for the use of the American Ephemeris and Nautical Almanac*, *I* (1882), 363–487.

[237] *Researches on the Motion of the Moon*, Part I, 1878, 266.

> rotation. One of the conclusions of the present discussion is therefore this: *Inequalities in the motion of the Moon not accounted for by the theory of gravitation really exist, and exist in such a way that the mean motion of the Moon between* 1800 *and* 1875 *was really less than it was between* 1720 *and* 1800.

Newcomb endorsed the idea that the Moon's fluctuations were due (at least in part) to real changes in the Moon's mean motion and he dismissed the possibility that the anomalous value of k might be due to systematic observational error.

On the other hand, Newcomb did not relinquish the idea that the Moon's fluctuations were in part due to errors in timing. In the *Comptes Rendus* of the Paris Académie des Sciences for 1896, he asserted that

> Les observations des passages de Mercure accusent nettement de petite variations dans la rotation de la Terre dont le montant integer, pendant de longues périodes de temps s'élève probablement à cinque, ou meme à dix secondes. En particulier il semble que, entre 1769 et 1789, un ralentissement de la rotation avait lieu et que, entre 1840 et 1861, encore un autre. Vers 1862, ce ralentissement était suivi brusquement d'une acceleration bien accentuée, qui a persisté peut-être jusqu'à 1870. Ce qui est remarquable, c'est que cette dernière conclusion est confirmée par le movement observé de la Lune.[238]

Apparently, he attributed the discrepancies he found in Mercury's motion to clock errors Δt, but he also believed that at least part of the Moon's errors arose from an unidentified source.

In a 1903 paper entitled "On the Desirableness of a Re-investigation of the Problems growing out of the Mean Motion of the Moon,"[239] Newcomb characterized the discrepancies between observed and theoretical values of the Moon's mean motion as "the most important unsolved problem growing out of the celestial motions." Concerning his earlier investigation as to whether Mercury's deviations from its tables bore to the Moon's deviations the ratio of their respective mean motions, he reported that they were in almost all cases in the right direction but too small.

> The evidence seems almost conclusive that the very improbable deviations in the Earth's rotation inferred from the observation of the Moon are unreal, and that the motion of our satellite is really affected by causes which have, up to the present time, eluded investigation.

In a 1909 paper, "Fluctuations in the Moon's Mean Motion," Newcomb assigned a new, reduced value to the main term in the fluctuation[240]:

$$12''.95 \sin(1^\circ.31t + 100^\circ.6).$$

[238] Académie des Sciences, *Comptes Rendus*, t.cxxii (1896), 1238.
[239] *MNRAS*, *63* (1903), 8.
[240] *MNRAS*, *69* (1909), 165.

As his final view of the fluctuations, he stated that

> Taken in connection with the recent exhaustive researches of Brown, which seem to be complete in determining with precision the action of every known mass of matter upon the Moon, the present study seems to prove beyond serious doubt the actuality of the large unexplained fluctuations in the Moon's mean motion to which I have called attention at various times during the past forty years.... The feature of most interest is the great fluctuation with a period of between 250 and 300 years.... In the absence of any physical cause for its continuance, there is no reason to suppose that it will continue in the future in accordance with the law followed in the past.[241]

Newcomb presented these deviations from theory in a graph reproduced below.[242] The straight medial line in each of the three sections of the graph represents the motion derived from pure gravitational theory. The fine, sharp curve represents the large sinusoidal term of the great fluctuation.

> The curve of actual longitude is bounded on each side by a shaded area showing the mean error at each point.... In this way not only the fluctuations as shown by observations are exhibited, but also the error to which the curve may be subject, the probability being 2/3 that at any point the true curve lies inside the shaded area, and 1/3 that it lies without it.... [B]efore 1750 the observations are not sufficiently continuous, numerous, and accurate to show any fluctuation with certainty. The first minor fluctuations fairly well shown began about 1760. During the years 1765–1784 the Moon ran ahead by about 1″. Then the excess of motion ceased, and became temporarily reversed.[243]

Newcomb had divided the entire deviation into two parts, the great fluctuation of long period, and the minor fluctuations superimposed upon the great one. He acknowledged that this division was made purely for convenience in representing past observations, and might not serve to represent future observations.

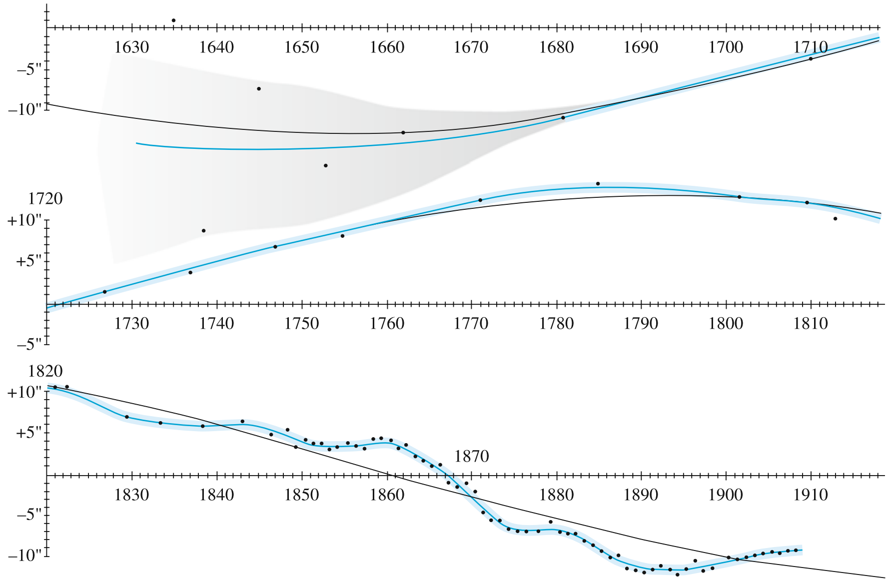
In setting out to investigate the Moon's fluctuations and secular acceleration, Brown appears to have taken the views Newcomb expressed in his 1909 paper as a point of departure. His first concern was to explore possible non-gravitational causes for Newcomb's long-period sinusoidal term. In a paper published in 1910,[244] Brown considered the following possible causes of this long-period, sinusoidal term: (1) the flattening of Jupiter's body as it affects the Moon, (2) the gravitational attraction

[241] Ibid., 164.

[242] Ibid., Plate 11.

[243] Ibid., 167.

[244] E.W. Brown, "On the Effects of Certain Magnetic and Gravitational Forces on the Motion of the Moon," *American Journal of Science*, *29* (1910), 529–539.



of the Moon by asteroids between Mars and Jupiter, (3) the non-sphericity of the Sun as it affects the Moon, (4) a magnetic attraction of the Moon by the Earth, and (5) a physical libration of the Moon's axes. Brown found that some of these proposed causes were improbable, others could not be substantiated in detail because crucial information was lacking, and still others he described as "difficult." Addressing the British Association in 1914, he commented on the causes he had examined as follows:

> The main objection to all these ideas consists in the fact that they stand alone: there is as yet little or no collateral evidence from other sources. The difficulty, in fact, is not that of finding a hypothesis to fit the facts, but of selecting one [hypothesis] out of many.[245]

Brown mentioned favorably the conjecture that bursts of magnetic flux, issuing from the Sun, caused the Moon's fluctuations, as well as fluctuations in nearby planets. But neither in his paper of 1910 nor in his address of 1914 did he mention variability in the Earth's rotation as a possible cause of the fluctuations. He can hardly have been unfamiliar with this proposal. His silence about it was curious, and it provoked an early reaction.

Meanwhile, during the years 1912–1915, Brown faced the task of determining final values of the Moon's orbital elements for inclusion in the *Tables*. He began with the idea of using the Greenwich meridian observations. P.H. Cowell of the British Nautical Almanac Office had previously grouped and analyzed the Greenwich meridian observations in a convenient way. Cowell's results needed only to be corrected

[245] *British Association Report*, Australia, September, 1914, "Address on Cosmical Physics," printed in the journal Science for Sept. 18, 1914, 389–401. The quotation is from p. 399.

for the differences between Hansen's and Brown's theories.[246] When Newcomb's "Researches on the motion of the Moon, Part II"[247] appeared posthumously in 1912, Brown decided to expand the basis of his determination to include Newcomb's extensive data on occultations. Comparing the occultations with the meridian observations, Brown found good agreement in observations after 1830, but large discrepancies before 1830. He concluded that the meridian observations made before 1830 were unreliable for the determination of constants.

The first constants requiring determination were those used to express the Moon's mean motion. These included the sizes of the Moon's secular acceleration and long-term fluctuation, which had to be determined simultaneously since any choice for the one influenced the value for the other. Newcomb on reducing the tidal part of the Moon's secular acceleration from $2''.52$ to $1''.88$ found he could reduce the coefficient of the fluctuation from $15''.5$ to $12''.95$. The tidal part of the secular acceleration, because of its dependence on ancient astronomical reports, was dubious, and Brown wanted to avoid publishing questionable constants in his *Tables*. As an experiment, he eliminated the tidal acceleration altogether, and found that he could then reduce the secular acceleration to its theoretical value of $6''.08$ per century^2. With this value, he then solved the observational equations of condition for the mean motion by least squares. For the long-period term in the fluctuation, he obtained

$$10''.71 \sin[140^\circ.0T + 240^\circ.71].$$

This solution for the Moon's secular acceleration and fluctuation yielded residuals that differed from those obtained with Newcomb's solution by only $1''$ or $2''$. Brown believed that uncertainties of this magnitude were unavoidable in the tidal acceleration and fluctuation. The Tables were scheduled to go to press in 1915, and a decision was required. Brown chose the reduced values. But in 1922, new information would cause him to retract this decision.

For the fluctuations, a single sinusoidal term did not suffice. Smaller, shorter-term wobbles, called "minor fluctuations," were present. Cowell, Brown, and other astronomers proposed sinusoidal formulas for these smaller wobbles, but all these proposals were later disconfirmed. In his *Tables*, Brown offered no formulas for the minor fluctuations.

Joseph Larmor, a well-known physicist at the University of Cambridge, was one of the first to react to Brown's 1914 address. The Moon's fluctuations, Larmor felt, created "an intolerable discrepancy."[248]

> The results of the application of the law of gravitation to the lunar motion have now been summed up in magisterial manner by Prof. E.W. Brown,

[246] See articles by Cowell in *MNRAS* for the years 1903–1905, *passim.*

[247] *Astronomical Papers prepared for the use of the American Ephemeris and Nautical Almanac*, *9* (1912), 249p.

[248] J. Larmor, "On Irregularities in the Earth's Rotation, in Relation to the Outstanding Discrepancies in the Orbital Motion of the Moon," *MNRAS*, *75* (1915), 211–219. Larmor's values for the coefficients of the long-period sinusoidal term and a minor fluctuation are taken from Brown's 1914 address.

> as the culmination of his classical investigation on the Lunar Theory. The circumstance that there remains an outstanding irregularity in the orbital motion, composed roughly of a fluctuation of 13 seconds of arc on each side of a mean in a half-period of about 140 years, combined with a like fluctuation of about 3 seconds of arc in the shorter half-period of about 35 years, has been felt to create an intolerable discrepancy, which demands every effort of the gravitational astronomer to resolve. No higher tribute than this could be paid to the extreme refinement and exactness of the gravitational explanation of the celestial motions, a department of knowledge which leaves far behind the highest amounts of precision that the presence of intractable disturbing agencies allows to us in other branches of physical science.

Larmor went on to ask whether the empirical terms in the Moon's motion could arise from vertical motions of the Earth's surface masses – an idea that William Thomson (Lord Kelvin) had proposed earlier.[249] Larmor imagined the case of an earthquake that would lower the sea bottom with a counterbalancing rise of a land surface, not of the adjacent ocean floor. The level of the ocean would be lowered over the entire Earth, and the Earth's rotation would speed up. To produce a change in the apparent mean motion of the Moon of $0''.2$ per year, he found that an area 400-miles-square of the ocean floor would need to rise or fall 420 feet – a displacement perhaps excessive to postulate. Still, he believed this type of explanation might prove partially adequate.

Arthur S. Eddington, another Cambridge physicist, was also aware of the problem that Brown had raised in his 1914 address. Eddington supervised H. Glauert, an honors student at Cambridge, in a study of the Greenwich meridian observations of Mercury, Venus, and the Sun.[250] Glauert attempted to show that the deviations of Mercury, Venus, and the Earth from their gravitationally derived paths were similar to the Moon's deviations, with their maxima occurring at the same times as the Moon's. He assumed, rather than attempting to establish, the proportionality of the deviations to the mean motions of the three bodies – the relation that would most strongly argue for the deviations originating in variations in the Earth's rotation. Since the error in the Sun's longitude entered as an element in the geocentric error of Venus and especially Mercury, a similarity of form was to be expected, and did not, by itself, constitute a sufficient proof that the residuals in longitude of the Sun, Mercury, and Venus were closely correlated. Glauert did not investigate whether the orbital elements of the three planets required revision, nor did he take into account

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[249] Kelvin had discussed it in his address to the British Association in 1876, as a possible explanation for the fluctuation that Newcomb had discovered. Earlier, in Thomson and Tait, *Natural Philosophy*, ed. *i*, 1867, §§276, 830, Kelvin had dealt with the frictional slowing of the Earth's axial rotation, as causing both a real and a merely apparent acceleration of the Moon.

[250] H. Glauert, "The Rotation of the Earth," *MNRAS*, *75* (April, 1915), 489–495, 685–687.

the systematic errors which affect observations of Mercury and Venus when they are near their conjunctions with the Sun.[251]

The efforts of Larmor and Glauert, though inconclusive, posed two questions that henceforth would not go away: (1) *Was* the Earth's rotation varying? (2) If so, *what* were the changes in the Earth's constitution that would cause such variations?

In 1916 Frank E. Ross published two papers relevant to the fluctuations.[252] In the first paper, entitled "The Sun's Mean longitude,"[253] he compared Newcomb's *Tables of the Sun* with meridian observations made at Greenwich, Paris, and Washington:

> ...the Washington observations of the Sun for 60 years have given a result $0^s.05$ [i.e., $0''.75$ arcseconds] less in right ascension than Greenwich, and $0^s.08$ [i.e., $1''.20$ arcseconds] less than Paris for the same period. On account of the great number of observers which must have taken part in the observations over such an extended period, it is inconceivable that this is a result of personal equation. It must accordingly be considered as of instrumental or housing origin, or as a local systematic refraction.[254]

By combining the data supplied from the three observatories, and comparing the resulting mean values with Newcomb's *Tables of the Sun*, Ross obtained a set of differences in the sense of [observation minus tabular value]. He compared this set with the *minor* residuals in the Moon's longitude (those residuals remaining after the Great Empirical Term had been subtracted out) reduced to one-fifth. In effect, by choosing the minor residuals for this comparison, Ross was taking the distinction between the G.E.T. and the minor fluctuations as theoretically significant. This was a questionable step. In observations made after 1830, he found a well-marked correlation between the minor residuals in the Moon's longitude and the differences [observation minus tabular value] for the Sun. But in the years before 1830, there was no correlation: the Moon's fluctuations were very small, while those of the Sun were very large. The result could not, therefore, support the view that the Sun's fluctuations were caused by changes in the Earth's rate of rotation.

In a second paper, "Investigations on the Orbit of Mars,"[255] Ross was primarily concerned to correct Newcomb's *Tables of Mars*, which he found to be in error by $3''$ for the years 1902–1903, and by $6''$ for 1906–1907. In a final section, Ross compared the deviations of Mars and the Sun from their tables during the interval 1754–1912, relying throughout on meridian observations. Between the two sets of deviations he found a considerable similarity.

[251] These flaws in Glauert's procedure were pointed out later by H. Spencer Jones in *MNRAS*, *87* (1926), 5.

[252] Ross had served as Newcomb's assistant in assembling the "Researches on the Motion of the Moon, Part II."

[253] *The Astronomical Journal*, *29* (1916), 152–156.

[254] Ibid., 152–156.

[255] *The Astronomical Journal*, *29* (1916), 157–163.

The Work of J.K. Fotheringham

In 1915, J.K. Fotheringham, a classicist turned historian of astronomy, set out to derive the Moon's secular acceleration from the lunar occultations of stars reported by Ptolemy.[256] Newcomb had rejected these observations as untrustworthy, on the grounds that they were chosen by Ptolemy to support Hipparchus's erroneous value of the precession of the equinoxes (1° per century).[257] Fotheringham attempted to refute this objection. Brown and other leading astronomers up to the 1950s accepted Fotheringham's conclusions. Britton in his 1992 study, however, sides with Newcomb.[258] In recent decades, investigators of the Moon's secular acceleration have come to avoid reliance on Ptolemy's reports.

For the increment in the Moon's mean motion per century Fotheringham found $10''.8 \pm 0''.70$.[259] In 1923 he discovered an error in his 1915 calculation, and revised his value to $10''.3$. Subtracting the gravitational increase, $6''.1$, he obtained $4''.7 \pm 0''.70$ in the first case, and $4''.2 \pm 0''.70$ in the second case, as the non-gravitational or tidal increase. These numbers are to be understood as mean values over the previous 2000 years.

The value of the Moon's secular acceleration derived from lunar eclipses turned out smaller. Fotheringham attributed this difference to a secular acceleration in the Sun's mean motion. Previous authors, except for Cowell, had explained it by an acceleration of the lunar node in the opposite direction. By a least-squares analysis, Fotheringham found that the non-gravitational acceleration of the node was $1''.1 \pm 0''.94$. The large probable error made this acceleration doubtful, and Fotheringham chose to regard it as non-existent.

In 1918 Fotheringham derived an estimate of the Sun's secular acceleration from Hipparchus's observations of equinoxes as reported by Ptolemy, and thus independently of lunar data.[260] From Newcomb's *Tables of the Sun* he extracted the Sun's longitude for each of the dates of these equinoxes, of which there were 20. (A difficulty here was that Hipparchus had reported the times of the equinoxes only to the nearest quarter-day.) By correcting the Sun's declinations for neglect of refraction and errors in the setting and graduation of instruments (as obtained from a least-squares analysis of Hipparchus's values for the declinations of seven stars), Fotheringham obtained $1''.93T^2 \pm 0''.27$ for the Sun's centennial or secular acceleration.

256 "The Secular Acceleration of the Moon's Mean Motion as determined from the Occultations in the Almagest," *MNRAS*, *75*, 377–396.

257 *Astronomical Papers for the American Ephemeris*, Vols. 1 (1878) and 9 (1912).

258 See J.P. Britton, *Models and Precision*: *the quality of Ptolemy's observations and parameters*, (New York NY: Garland, 1992), 77–98, for an analysis of the occultation-reports used by Fotheringham.

259 *MNRAS*, *75* (1915), 394.

260 *MNRAS*, *78* (1918), 407.

In 1920,[261] Fotheringham reduced his estimate to $1''.50T^2$. More recent estimates have varied between $1''.01T^2$ and $1''.88T^2$. Britton in his 1992 study[262] reviewed these values and judged the best estimate to be $(1''.15 \pm 0''.15)T^2$. Celestial mechanics supplied no cause for a real speed-up in the Sun's (or Earth's) orbital motion. The most plausible explanation was a deceleration of the Earth's rotation. The evidence Fotheringham presented for the Sun's apparent secular acceleration was a strong argument that a tidal deceleration of the Earth's rotation was occurring.

Recall that Brown applied in his tables a value of $6''.08T^2$ for the Moon's secular acceleration limited to the gravitational part. In December 1919 Fotheringham published a study entitled "The Longitude of the Moon from 1627 to 1918."[263] Here he gave a new value for the non-gravitational acceleration of the Moon as determined from modern observations, and a revised formula for Brown's G.E.T. He took Newcomb's work on occultations as a basis, and added to it the Greenwich meridian observations for the years 1908–1918. He excluded the ancient, medieval, and 17th-century eclipses, because these eclipses involved the Sun's apparent motion, which by this time had to be understood as involving an apparent acceleration.

Fotheringham's new value for the Moon's mean longitude, reduced to the epoch 1800.0, was $-2''.29 + 1''.96T + 10''.53T^2$, where the secular acceleration includes a non-gravitational part, $4''.52T^2$, in addition to Brown's theoretical part, $6''.08T^2$. Fotheringham showed that the inclusion of data from 1908–1918 was crucial to the discovery of the increase, for in 1908, the last year of Newcomb's data, the Moon's fluctuation had been in the negative direction for some years, and after 1908 it turned in the positive direction. Fotheringham's results, derived from modem data alone without any reliance on Ptolemaic reports, strongly supported a significant tidal deceleration of the Earth's rotation.

Fotheringham's new expression for the Great Empirical Term was

$$+13''.60 \sin(139^\circ T + 104^\circ.2).$$

The coefficient is $2''.9$ larger than Brown's coefficient.

Harold Jeffreys on the Deceleration of the Earth's Rotation from Tidal Friction

In 1920, the geophysicist Harold Jeffreys set out to provide a dynamical account for the Earth's slowing rotation.[264] He took the Moon's and Sun's non-gravitational accelerations as established, and accepted tentatively Fotheringham's most recent

[261] *MNRAS*, *81* (1920), 104–126.
[262] Britton, *Models and Precision*, 176.
[263] *MNRAS*, *80* (1919), 289–307.
[264] *MNRAS*, *80*, 309–317.

values for them. He expressed the non-gravitational accelerations as *angular accelerations* rather than centennial increments, the accelerations being the numerical doubles of the increments. Thus he gave Fotheringham's value for the Moon as $(9''.0 \pm 1''.40)/T^2$, and his value for the Sun as $(3''.86 \pm 0''.54)/T^2$. From his analysis Jeffreys judged that about half Fotheringham's value for the secular acceleration of the Sun was in error.[265]

Jeffreys assumed that the Earth was losing angular momentum due to some form of tidal friction. He wrote an equation to express how that loss would be compensated by gains in the Moon's and Sun's orbital angular momentum, as conservation of angular momentum requires. Letting Ω be the Earth's angular velocity of rotation, C its principal moment of inertia, and N, N_1 the Moon's and the Sun's rates of change of angular momentum, Jeffreys obtained the equation

$$C\frac{d\Omega}{dt} = -N - N_1.$$

Here N and N_1 are positive rates of increase. For terrestrial observers, the slowing of the Earth's rotation produces *apparent* accelerations of the Moon and Sun, in the amounts

$$\frac{n}{\Omega}\frac{d\Omega}{dt}, \quad \frac{n_1}{\Omega}\frac{d\Omega}{dt},$$

where n and n_1 are the mean motions of the Moon and Sun. Consequently, the total *observed* accelerations would be

$$\nu = \frac{dn}{dt} - \frac{n}{\Omega}\frac{d\Omega}{dt}, \quad \nu_1 = \frac{dn_1}{dt} - \frac{n_1}{\Omega}\frac{d\Omega}{dt},$$

where dn/dt and dn_1/dt are the *true* rates of change in the mean motions of the Moon and the Sun.

That the differential coefficients dn/dt and dn_1/dt are negative can be shown as follows. The increments in angular momentum in the Moon and Sun are proportional to increments in the products c^2n, $c_1^2n_1$, where c, c_1 are the distances of the Moon and Sun from the Earth. The variables c, n and c_1, n_1 must vary in accordance with Kepler's third law, so that the products c^3n^2 and $c_1^3n_1^2$ remain constant. Therefore, by differentiation,

$$\frac{2}{n}\frac{dn}{dt} + \frac{3}{c}\frac{dc}{dt} = 0, \quad \frac{2}{n_1}\frac{dn_1}{dt} + \frac{3}{c_1}\frac{dc_1}{dt} = 0.$$

Thus dn/dt is opposite in sign to dc/dt, and dn_1/dt to dc_1/dt. Jeffreys found dn_1/dt to be negligible. But the increment in the Moon's angular momentum is positive: $\Delta(c^2n) > 0$, or $(c + \Delta c)^2(n + \Delta n) > 0$. If Δc and Δn expressed as $(dc/dt)\Delta t$ and $(dn/dt)\Delta t$ are substituted into this inequality, dc/dt proves to be positive and dn/dt negative. The increment in angular momentum leads to the Moon's orbit having a greater mean radius c, and a smaller mean motion n about the Earth.

[265] Ibid., 317.

For the rate of dissipation of energy due to tidal friction, Jeffreys gave the formula

$$-\frac{dE}{dt} = (N + N_1)\Omega - Nn - N_1 n_1.$$

Using Fotheringham's values for the non-gravitational acceleration of the Moon, he evaluated this rate when the tidal friction is primarily bodily and obtained

$$-\frac{dE}{dt} = 1.41 \times 10^{19} \text{ ergs per second.}$$

For the case of primarily liquid friction, he obtained

$$-\frac{dE}{dt} = 1.38 \times 10^{19} \text{ ergs per second.}$$

Jeffreys thought it improbable that bodily friction was the chief force in slowing the Earth's rotation, for it would have to arise through an imperfection of elasticity, either plasticity (incompleteness of elastic recovery) or elastic afterworking (slowness of the recovery).

> Now, few things in geophysics are more certain than that the outer two thousand miles or so of the crust must be practically free from both these qualities when small stresses alone are considered.[266]

Likewise he judged that tides in mid-ocean and movements of the Earth's atmosphere were incapable of accounting for the Moon's secular acceleration. He thought that the only cause capable of producing an effect of the correct order of magnitude was tidal friction in shallow seas, as considered by G.I. Taylor in a 1919 paper on this subject.[267] By two different calculative routes, Taylor obtained for the rate of dissipation of energy in the Irish Sea, 2.5×10^{17} and 3.0×10^{17} ergs per second. The former is about 1/56th of the amount that Jeffreys had calculated as necessary to produce Fotheringham's value of the Moon's non-gravitational secular acceleration. The areas contributing most to the dissipation should thus be seas of moderate size that are partly enclosed, such as the English Channel, the Bay of Fundy, Behring Strait, the Mozambique Channel, and probably several areas in the Sea of Japan and the East and West Indies.

Jeffreys' 1920 paper was at first considered to have solved the problem of the Moon's tidal secular acceleration, but Jeffreys himself later acknowledged that tidal friction in shallow seas was an inadequate source for the dissipation of energy required. It has not as yet been possible to identify the locales of dissipation in the present geologic era with certainty, but they are believed to be largely oceanic and may include deep-sea bottom friction and the breaking of waves against shores.[268]

[266] Ibid., 310.

[267] The article appeared in *Phil. Trans.*, *220* (1919), 1–33. Taylor provided a brief summary in *MNRAS*, *80*, 308–309.

[268] Ibid., Ch. 10, "Tidal Dissipation," p. 286ff.

Brown on the Results of Fotheringham and Jeffreys

Writing in the *Astronomical Journal* in February, 1922, Brown stated:

> The memoirs of Dr. J.K. Fotheringham on the ancient eclipses and those of C.I. Taylor and H. Jeffreys on tidal friction in shallow seas have largely cleared away the doubts that surrounded the old hypothesis that the Moon's apparent residual acceleration is, in reality, due to a retardation of the Earth's rate of rotation. While the importance of their work in clearing up a difficulty in the recorded observations of the Moon is not to be minimized, my immediate object in this note is to give briefly the numerical consequences as far as predictions of the Moon's place by means of the new tables are concerned, and to indicate how predictions for the unexplained minor fluctuations can best be made when, for example, it is desired to predict the time and terrestrial path of a solar eclipse with the best possible accuracy.[269]

Brown wrote, for the Moon's mean motion to be used in the new tables,

$$\begin{aligned} T_0 = {} & 335^\circ 43' 27''.81 + 1336^r 307^\circ 53' 11''.80T + 7''.12T^2 + 0''.0068T^3 \\ & + 10''.71 \ \sin(140^\circ T + 100^\circ.7), \end{aligned}$$

and for Fotheringham's value,

$$\begin{aligned} T_0 + \delta T = {} & 335^\circ 43' 25''.26 + 1336^r 307^\circ 53' 13''.82T + 11''.91T^2 + 0''.0068T^3 \\ & + 13''.60 \ \sin(139^\circ T + 104^\circ.2), \end{aligned}$$

where T was the number of Julian centuries from 1800.0. With sufficient accuracy from 1800 on, Brown put the difference at

$$\begin{aligned} \delta T = {} & -2''.55 + 2''.02T + 4''.79T^2 - 2''.90 \cos(139^\circ T - 166^\circ) \\ & + (3.5 - T)0''.187 \ \sin(139^\circ T - 166^\circ). \end{aligned}$$

The change δT proved to make little difference before 1890, but after 1890 it yielded considerable improvement. Brown found $+4''.65$ for its value in 1925, and $+9''.42$ for its value in 1950. He cautioned against changing the hourly ephemeris of the Moon for δT "until it [δT] has been well established by further observations."[270]

After first applying δT, Brown carried out a new analysis of the data for minor fluctuations since 1750. This analysis suggested that a term with a period of about 40 years and a coefficient of $1''$ had persisted in recent years, but that any additional terms would be extremely doubtful. Brown called attention once again to the fact that the fluctuations appeared to proceed by sudden rather than gradual changes of the mean motion.

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269 E.W. Brown, "The Moon's Mean Motion and the New Tables," *Astronomical Journal*, *34* (1922), 52.

270 Ibid., 53.

Dyson and Crommelin on the Greenwich Meridian Observations to 1923

In 1923 Dyson and Crommelin of the Royal Greenwich Observatory updated Cowell's earlier comparison between the Greenwich observations and tabular values, and continued the comparison up through 1922.[271] They achieved a better accord between theory and observation by substituting the formula $\delta T - 0''.16$ for Brown's formula δT:

$$\delta T - 0''.16 = -2''.71 + 2''.02T + 4''.79T^2 + 13''.60 \sin(139^\circ T + 104^\circ.2) - 10''.71 \sin(140^\circ T + 100^\circ.7).$$

They also obtained a new formula for the Great Empirical Term:

$$+13''.28 \sin(138^\circ.3T + 104^\circ.11).$$

They then went on to seek a two-term sinusoidal formula that would represent the remaining residuals, while recognizing that the attempt to represent these residuals by periodic terms was a doubtful enterprise.

R.T.A. Innes on Fluctuations in Mercury, Satellites of Jupiter, and the Sun

In 1925, R.T.A. Innes, director of the Union Observatory in Johannesburg, South Africa, published two notes under the title "Variability of the Earth's Rotation."[272] Like Newcomb earlier, he carried out a comparison between the time-errors (observation minus theory-based prediction) in transits of Mercury from 1677 to 1924 and the fluctuations in the Moon's motion from 1680 to 1909 as determined by Newcomb. To make the two sets of data comparable, Innes multiplied Mercury's time-errors by the Moon's mean motion ($0''.55$ per 1^s) with the sign changed. The correlation was inexact, but according to Innes this inexactness was to be expected, since the times in the two series were determined differently. Innes also showed that the time-errors in the eclipses of the first two Medicean satellites of Jupiter from 1910 to 1923 closely mirrored those in the transits of Mercury during the same period.

Innes also determined the longitude errors in the Greenwich observations of the Sun as compared with the Nautical Almanac (the latter being derived from Newcomb's *Tables of the Sun*) for each year from 1901 to 1921, again in the sense (Observed minus Calculated Value). Using means over 7-year periods, he found that the Sun's position was $0''.12$ ahead (eastward) of the ephemeris value in 1904, and $1''.45$ behind the ephemeris value in 1918. Thus it had fallen back a total of $1.''57$ in 14 years. Motion of the Sun through $1.''57$ requires $38^s.23$ of time. Innes interpreted the $38^s.23$ as a gain in clock-time (with the rotating Earth as clock) during

[271] "The Greenwich Observations of the Moon (1751–1922)," *MNRAS*, *83* (1923), 359–370.

[272] *Astronomische Nachrichtung*, 1925, cols. 109–110. The two notes occupy a single page.

these 14 years. The average gain per year was $2^s.7$. For the interval 1908–1921 the average gain was $2^s.1$ per year. From the transits of Mercury between 1908 and 1924 Innes found a gain of $+1^s.1$ per year, and from the eclipses of Jupiter's satellites I and II, a gain of $+1^s.9$ per year. With regard to the fluctuations of the Moon, Innes added,

> [The lunar fluctuations] indicate a similar result, but the large empirical terms used in the lunar theory make an exact comparison too onerous.... [W]e have to be contented with rather crude results. It is however satisfactory to find that each of the four available tests gives qualitatively the same result.

Innes was wholeheartedly opposed to the use of empirical terms:

> When allowance is made for the variability of rotation of the Earth, the Moon's motion will probably be found to be purely gravitational. The inclusion of empirical terms confuses.

Brown on the Variability of the Earth's Rotation, 1926

In this year Brown published a major paper entitled "The evidence for changes in the rate of rotation of the Earth and their geophysical consequences, with a summary and discussion of the deviations of the Moon and Sun from their gravitational orbits."[273] In this paper he unequivocally endorsed the hypothesis that the Moon's fluctuations were due to changes in the Earth's rate of rotation. Taking the non-gravitational acceleration of the Moon due to tidal friction as established, Brown focused on the fluctuations:

> I am not here mainly concemed with the secular changes due to tidal friction, but with the considerable fluctuations which are exhibited in the difference between observed and calculated longitudes of the Moon when all known causes of variation have been eliminated. The numerous investigations into ancient eclipses culminating in the results of Fotheringham, and the work of Taylor and Jeffreys on tidal friction in shallow seas, are in substantial agreement as to the amount of the frictional effect, so that it may be regarded as known. The tabular place of the Moon due to gravitational theory is therefore first corrected for this effect, the fluctuations referred to being the differences between this corrected theory and observation.[274]

The "substantial agreement" mentioned here consisted in Jeffrey's having accepted Fotheringham's value for the tidal part of the Moon's secular acceleration, i.e., $4''.5T^2$. Fotheringham and Jeffreys also agreed that the Sun was subject to an

[273] *Transactions of the Astronomical Observatory of Yale University*, *3* (1926), 205–235 + three plates.
[274] Ibid., 209.

apparent secular acceleration. For this increment, Fotheringham gave a value of $1''.5T^2$. But Jeffreys showed that Fotheringham's value for the tidal part of the Moon's secular acceleration necessarily limited the tidal acceleration in the Sun's longitude to a maximum of $0''.9T^2$. Brown adopted the round number $1''.0T^2$, with a possible error of $0''.2$.

If the Moon's fluctuations inversely reflect fluctuations in the Earth's rate of rotation, then the Earth's rotational fluctuations should affect the apparent motions of other celestial bodies in a similar way. Earthlings see the Earth's heliocentric motion as reflected in the Sun's eastward motion about the Earth. Brown multiplied the Moon's fluctuations by $0.075 = 1/13.3$, the ratio of the mean apparent motion of the Sun to that of the Moon, and then asked whether these reduced fluctuations figured as a component in the Sun's observed motion.

In his 1916 paper on the Sun's mean longitude, Frank Ross had tabulated the differences (mean longitude from the Greenwich observations minus Newcomb's tables), giving the averages for successive groups of 4–7 years from 1750 to the 1890s. Brown continued this same tabulation down to 1924, using the records of the more recent Greenwich observations. He then plotted these results in the solid line of the figure reproduced below.[275] The dashed line in the figure gives 1/13.3 of the lunar fluctuations. The agreement is extremely good from the 1840s to 1924 where the dashed line and the solid line never differ by as much as $1''$. The plausible conclusion is that the fluctuations appearing in the Moon's motion are due to fluctuations in the Earth's rotation, and that the latter fluctuations also appear in observations of the Sun.

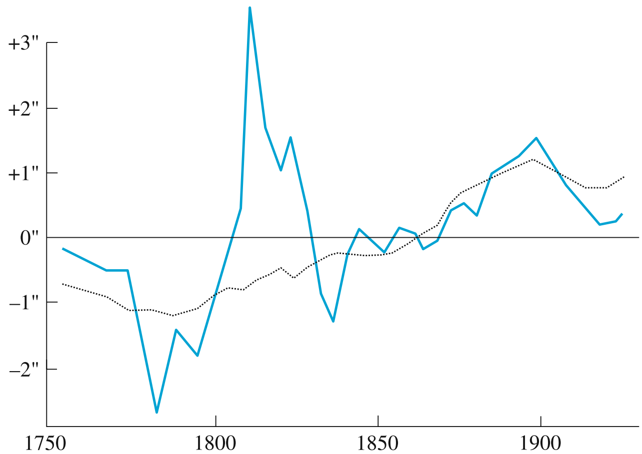
With respect to the earlier observations plotted in the figure, Brown remarked that

> ...the large deviations in the Greenwich series from 1810 to 1825 appear to be due to systematic errors of observation in this period. Partly or wholly within these fifteen years we have results from Paris, Königsberg and Dorpat, and, except for the first Paris group, none of them show any such large differences.... These also are within the period when the differences between the Greenwich meridian observations of the moon and the occultations become large. Hence it would seem that it is scarcely safe to use the Greenwich observations with full weight during these years in any discussion that involves theory. Before this time, both sun and moon indicate that the observations are sufficiently good for use in obtaining the mean motion and epoch of the Sun but are of doubtful value for any other purpose.

The evidence presented in the figure supports the hypothesis that the Moon's fluctuations in longitude are caused by variations in the rotation of the Earth.

Besides presenting evidence that the Earth's rate of rotation was fluctuating, Brown outlined a hypothesis to account for the larger variations in the Earth's rotation. A plot of the Moon's longitude over the last two and a half centuries

[275] Brown's figure, captioned "Figure 4", is on p. 225 of his paper.



Full line curve: the Greenwich tabular minus observed errors of the Sun including the secular acceleration. *Dash line*: 1/13.3 of the lunar fluctuations.

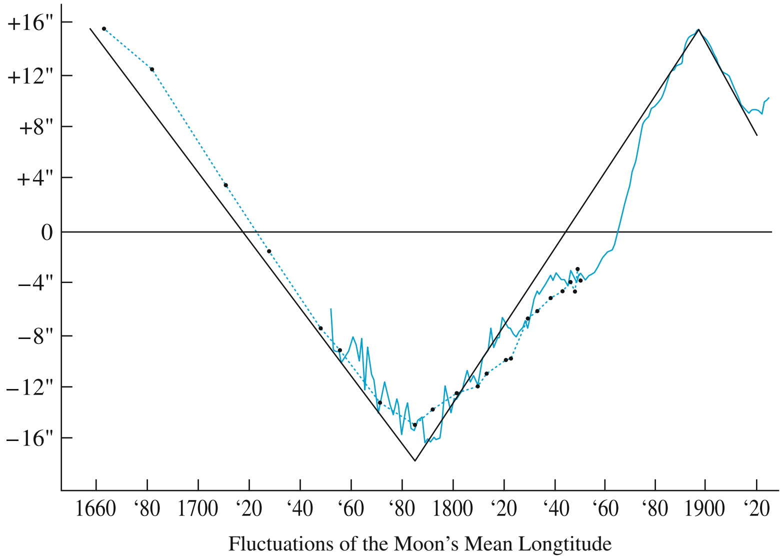
shows large, sudden changes in the rate of the Moon's mean motion. In the figure reproduced below, Brown plotted the Moon's mean longitude since 1660, averaged over successive time-intervals of 400 lunar days (= 414 solar days) or (during the 20th century) a year. The dots in the figure represent longitudes determined from occultations.

As Brown explained, this curve

> ...exhibits the apparent variations which the Moon's longitude has shown during the past 260 years referred to the Earth as a clock having a constant change of rate (the frictional retardation). If we regard these variations as due solely to further changes in the rate of the clock, the ordinates are proportional to the errors of the clock at any time; the slope measures the rate of the clock, and the curvature measures the change of rate.

It is striking that from 1660 to 1920 this plot can be approximated by three straight lines, with sharp changes of slope around 1785 and 1898. (Brown inserted the straight lines to show the goodness of the approximation.) The rapid change of slope occurring near 1898 was strongly confirmed by evidence. A rate increase of $0''.4$ per year diminished to zero within 5 years and then changed to a decrease in rate of $0''.4$ per year within 2 years. On the hypothesis that these changes are reflections of changes in the Earth's rate of rotation, Brown set out to explain how the Earth's annual rate of rotation might increase by $0''.8$ within 7 years or less, a rate far greater than that produced by tidal friction.

If the changes in the Earth's rotational velocity around 1898 were to be brought about by the Moon or Sun acting gravitationally on the Earth's atmosphere or ocean or crust, the effect would have to be less than that produced by tidal friction,



Fluctuations of the Moon's Mean Longitude.

and would have to be periodic with periods formed from combinations of the periods in the Sun's and Moon's motion. None these conditions obtained. Brown argued that the source of the effect must therefore be internal to the Earth. In this case, the angular momentum of the Earth's rotation, $I\omega$, must remain constant. Here ω is the Earth's angular velocity, and I is its moment of inertia. I is given by the integral $I = \int r^2 dm$, where r is the distance of the mass dm from the axis of rotation, and the integral permits taking into account the variation of density as a function of radius r. Because $I\omega$ is constant, any increase in ω must be compensated by a decrease in I. Assuming that the total mass remains constant, any decrease in I implies a displacement of mass toward the axis of rotation.

If the observed change in the Earth's rate of rotation in the years around 1898 was brought about by a local shift of a mass near the Earth's surface, then a staggeringly large mass must have been transported a good many miles closer to the Earth's axis, with catastrophic results that could not go undetected. As an alternative hypothesis, Brown proposed expansions or contractions extending through a large part or the whole of the Earth's body. If the expansion or contraction were uniform throughout the Earth's body, we would have

$$\frac{\delta\omega}{\omega} + \frac{2}{r}\delta r = 0.$$

Since $\delta\omega/\omega$ in 1898 was approximately 4×10^{-8}, $\delta r/r$ was approximately -2×10^{-8}, which gives about a five-inch decrease of the Earth's radius. If the change were produced by a contraction taking place in a layer about 50 miles below the surface, lowering the crust but not affecting the nucleus farther down, Brown found for δr a value of about -12.5 feet.

In more recent investigations of the larger variations in the Earth's rotation, geophysicists have agreed with Brown in seeking a cause beneath the Earth's surface. The hypothesis now favored posits some form of coupling between the Earth's core and mantle. Such a coupling is required by the observed westward drift of the Earth's magnetic field. The most plausible mechanism, given the elaboration of geomagnetic dynamo theories by E.C. Bullard and E. Elsasser in the early 1950s, appears to be an electromagnetic one.[276]

H. Spencer Jones on the Variable Rotation of the Earth, 1926

In the same year as Brown's publication, another major paper on the Earth's rotation appeared. Its author was H. Spencer Jones, H.M. Astronomer at the Cape of Good Hope.[277] Jones makes no mention of Brown's paper, and was presumably unaware of it. In contrast to Brown, he focused exclusively on supplying evidence that the Earth's rotation was varying, and set aside entirely the question of the imaginable causes for such variation. In two earlier papers he had derived elements of the Moon's orbit from occultations of stars by the Moon as observed between 1880 and 1922 at the Cape of Good Hope, for comparison with the elements derived by Brown.[278] He now turned to the fluctuations, the most troublesome problem in the Moon's motions. As he explained,

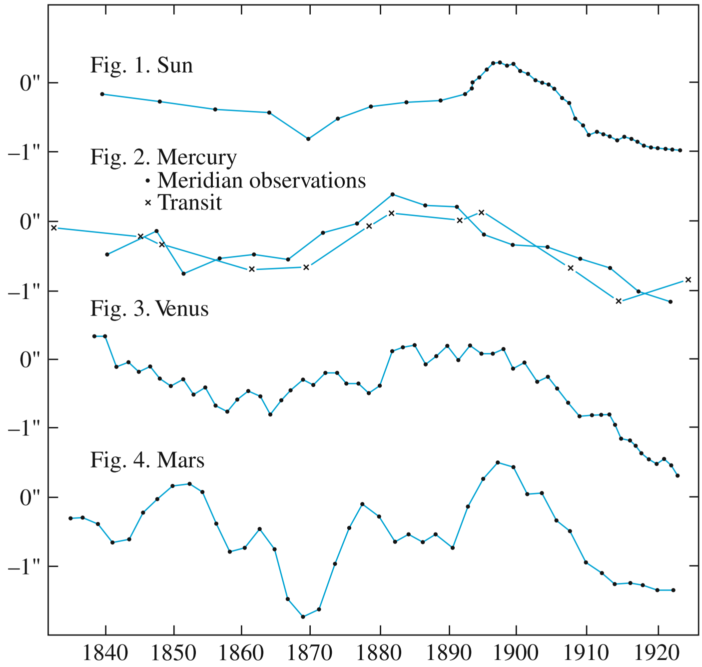
> The present paper gives the results of a reexamination of the information obtainable from observations of the Sun, Mercury, Venus, and Mars as to the variability of the rate of rotation of the Earth. The discussion was limited to observations made from 1836 onwards. It was not judged expedient to extend the investigation to an earlier period than this, on account of the large accidental errors of the earlier observations....

Jones based his discussion primarily on the Greenwich meridian observations. In the case of Mercury, he employed as well the transits of Mercury across the Sun's disk as discussed by Innes. In the four graphs shown below, Jones plotted the errors of

[276] Lambeck, *The Earth's Variable Rotation* (Cambridge University Press, 1980), pp. 246–254.

[277] H. Spencer Jones, "The Rotation of the Earth," *MNRAS*, *87* (Nov., 1926), 4–31.

[278] H. Spencer Jones, "The Moon's Mean Longitude, Longitudes of Perigee and Node, Semi-Diameter and Parallactic Inequality derived from Occultations of Stars observed at the Royal Observatory, Cape of Good Hope, 1880–1922," *MNRAS*, *85* (1924), 11–34; "Determination of the Elements of the Moon's Orbit, the Parallactic Inequality, and the Moon's Semidiameter from Occultations of Stars by the Moon...," *Annals of the Cape Observatory*, VIII, Part VIII, IH–47H.



the tables for the four bodies Sun, Mercury, Venus, and Mars, over the interval from 1840 to 1923. In each case the errors were taken in the sense (tabular value minus observation). Before being plotted, the residuals in the case of Mercury, Venus and Mars were first multiplied by the ratio of the mean motion of the Earth to the mean motion of the planet. While relying mainly upon the Greenwich observations from 1836 onwards, Jones strove to insure that the planetary theories employed were of uniformly high quality, and that the observations were free from systematic error. The Sun's errors had to be determined with special care, since they were presupposed in the derivation of the fluctuations of Mercury, Venus, and Mars.

Interpreting the graphs, Jones remarked:

> If these residuals are due to slight changes in the rotation period of the Earth, the four curves should theoretically be identical, assuming that the tabular longitudes and mean motions are free from error. But observations both of the Sun and of the planets are peculiarly liable to errors of a systematic nature, and the curves will be modified by these errors. Considering the smallness of the quantities under discussion, the general similarity between the four curves is very marked. They all show very clearly a gradual fall of about the same amplitude commencing near 1896, the rate of fall slowing down during recent years. All the curves agree in showing a minimum somewhere near 1870, preceded by a gradual fall from the beginning of the period discussed and followed by a gradual rise to about 1896, the rise being at first fairly rapid, then slowing down for a time and then again becoming more

> rapid a few years before the maximum. Not only are the curves qualitatively similar, but also the amplitude changes are approximately equal.[279]

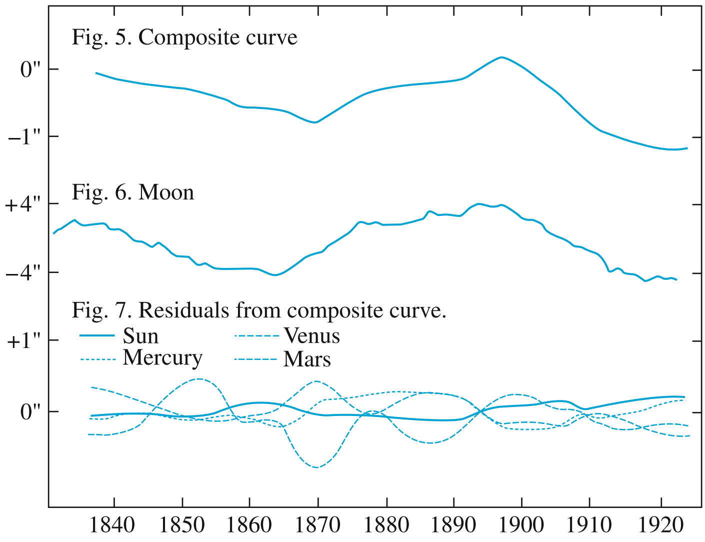
Jones computed correlation coefficients to express the similarity of the four curves numerically:

Mercury (meridian observations) and Sun	$+0.54 \pm 0.11$
Mercury (transits) and Sun	$+0.85 \pm 0.05$
Mercury (transits) and Venus	$+0.82 \pm 0.06$
Venus and Sun	$+0.83 \pm 0.03$
Mars and Sun	$+0.86 \pm 0.03$

The high values of the correlation coefficients, Jones concluded, gave strong support to the idea that the longitude fluctuations of the Sun, Mercury, Venus, and Mars had a common cause. The approximate equality in amplitude of the four graphs, given that the actual fluctuations in each case had been increased or decreased in the ratio of the Earth's mean motion to the other planet's mean motion, favored the supposition that the common cause was variation in the Earth's rate of rotation.[280]

Toward the end of his paper, Jones formed a composite curve of the residuals of the three planets and the Sun. He then compared this curve with "the minor fluctuations" of the Moon, the fluctuations remaining after the Great Empirical Term is removed.

The two curves were remarkably similar in shape and amplitude. Once more, from a similarity of shape and amplitude, Jones inferred a sameness of cause. The



279 Ibid., 25–26.

280 Ibid., 26.

minor longitude fluctuations of the Moon could be attributed, within the limits of error of the observations, to changes in the rate of rotation of the Earth.[281]

Why did Jones treat the minor fluctuations as a special class, separate from the fluctuations as a whole? The short answer is that Newcomb had made this separation earlier, when for convenience he introduced a long-period sinusoidal term, later called the Great Empirical Term. The minor fluctuations were simply the remainder of the fluctuations after the Great Empirical Term was subtracted out. Brown in his 1926 paper suggested that the distinction was artificial. Our next author will prove it so.

Willem de Sitter on Secular Accelerations and Fluctuations, 1927

De Sitter's 1927 paper "On the secular accelerations and the fluctuations of the longitudes of the Moon, the Sun, Mercury and Venus" advanced the argument for the variability of the Earth's rotation in important ways.[282] He addressed two questions: (1) whether the fluctuations in the longitudes of the Sun and planets are equal to those of the Moon diminished in the exact ratio of the mean motions, or to those fluctuations, thus diminished, multiplied by a factor, for which values from unity to about 2.5 are found; and (2) whether the fluctuations of the Sun and planets agree with the total fluctuations of the Moon, or with the "minor fluctuations" which remain after the removal of the "great empirical term."

To address his first question, de Sitter constructed, for each of the bodies Moon, Sun, Mercury and Venus, a formula to express the excess of its observed longitude over its theoretical longitude as deduced from gravitational theory. The excess consisted of a non-gravitational secular acceleration and a remainder constituting the fluctuation. He suspected that both were caused by changes in the Earth's rotation. In the case of the Moon, he expressed the excess as

$$\begin{aligned}\Delta L = \Delta L_0 + T\Delta n + 5''.92(1+\kappa')S + B' \\ - 10^\circ.71 \sin(140^\circ.0T + 240^\circ.7).\end{aligned} \tag{1}$$

with T as the number of centuries since the beginning of January, 1900 (midnight before Jan. 1, written 1900.0). The term $-10^\circ.71 \sin(140^\circ.0T + 240^\circ.7)$ is Brown's Great Empirical Term, subtracted out from ΔL_0 as given in Brown's tables, since it was not deducible from gravitational theory. The term B' is the Moon's fluctuation proper, the part of the excess in the Moon's longitude not attributable to gravitational action of the Sun or planets, *or* to tidal friction.

The term $5''.92(1+\kappa')S$ introduces the non-gravitational part of the secular acceleration. The number κ' is an unknown, $5''.92$ is an arbitrary starting-value for the calculation of this part of the acceleration, and S is $T^2+1.33T-0.26$, a quadratic expression that is zero for $T = -1.50$ and 0.17, i.e., for the years 1750 and 1917.

[281] Ibid., 31.

[282] *Bulletin of the Astronomical Institutes of the Netherlands*, *4*, no. 124.

The expression S minimizes the effect of the corrections on the agreement between theory and modern observations, and makes the effective epoch of the mean motions 1833.5, midway between 1750, when the Greenwich observations began, and 1917, the last year for which de Sitter (at the time of writing) had access to Greenwich observations in reduced form.[283] Once κ' is known, the non-gravitational secular acceleration can be obtained from $5''.92(1+\kappa')T^2$. The other terms resulting from $5''.92(1+\kappa)S$ are $7''.87(1+\kappa')T - 1''.54(1+\kappa')$. The first of these is to be added to $T\Delta n$, and the second to ΔL_0, to correct the mean motion per century and the mean longitude at epoch. (Such corrections have to be introduced whenever the secular acceleration is changed.) The unknowns in equation (1) are κ', ΔL_0 and Δn.

For the Sun de Sitter wrote the equation

$$\Delta L' = \Delta L'_0 + T\Delta n_0 + S(1+\kappa) + Q\frac{n_0}{n}B'. \tag{2}$$

For Mercury and Venus he wrote the similar equations

$$\Delta\lambda_i = \Delta\lambda_{i,0} + T\Delta n_i + \frac{n_i}{n_0}S(1+\kappa_i) + Q_i\frac{n_i}{n}B', \tag{3,4}$$

where $i = 1$ for Mercury and $i = 2$ for Venus. In these equations, n and n_0 are the mean motions of the Moon and the Sun, n_1 is the mean heliocentric motion of Mercury, and n_2 the mean heliocentric motion of Venus. In each of the equations (2), (3) and (4) there are four unknowns: three analogous to those in (1), and a factor Q or Q_i which would be unity if the fluctuations in all cases simply reflected variations in the Earth's rotation.

De Sitter states that the values for Q or Q_i are found "ranging from unity to about 2.5." A value of Q or Q_i differing from unity seems difficult to interpret, but de Sitter, as we shall see, nevertheless presented a way of interpreting such values.

Not all of the fifteen unknowns in (1), (2), (3), and (4) could be determined independently. The Moon's secular acceleration could not be separated from its fluctuations in the modern observations. Among the modern observations of the Sun, de Sitter followed Brown and Jones in deeming trustworthy only those made since about 1835. The non-gravitational secular accelerations of the Sun and Moon could only be determined from ancient observations, in which the fluctuations were not discernible.

To determine the non-gravitational secular accelerations, therefore, de Sitter left the fluctuations out of account, and reduced (1) and (2) to

$$\Delta L = \Delta L_0 + T\Delta n + 5''.92S(1+\kappa'),$$

$$\Delta L' = \Delta L'_0 + T\Delta n_0 + S(1+\kappa).$$

Using ancient observations discussed by Fotheringham and Schoch,[284] with weights matching the probable errors that these two authors assigned, de Sitter obtained seven

[283] As pointed out by Britton, *Models and Precision*, 167.

[284] See P.V. Neugebauer, 1930, ed., *Neudruck der im Selbstverlag von V. Schoch erschienenen Schriften, Die Verbesserten Syzygientafeln von C. Schoch, Astronomische Abhandlungen, Ergänzungshefte zu den Astronomischen Nachrichten* 8.2: B2–B5.

equations for $\Delta L'$ and two for ΔL, and solved them by least squares for κ and κ'. His final results were $\kappa = +0.80 \pm 0.16$ and $\kappa' = -0.12 \pm 0.05$. The values he thence deduced for the non-gravitational secular accelerations were,

for the Sun, $+(1''.80 \pm 0''.16)S$,

for the Moon, $+(5''.22 \pm 0''.30)S$.

Britton in his recent study found certain steps in de Sitter's procedure objectionable:

> In the first place, he treats a number of Fotheringham's results – e.g., the accelerations of the Sun and Moon derived from solar eclipses, and the relation between them derived from the solar eclipse of Hipparchus (-128) – as independent determinations, when in fact they are independent neither of each other nor of the rest of Fotheringham's results...
>
> [M]ost significantly, de Sitter's results are vitiated by important numerical errors. In deriving the equation of condition for the Moon's secular acceleration as determined from the occultations..., de Sitter not only disregards Fotheringham's subsequent correction of his first determination, he also computes ΔL incorrectly, arriving at a figure $610''$ too large. Even worse, in his equations derived from the accelerations of the Moon's elongation found by Fotheringham, he includes the total difference, $S_D = S_m - S_s$ [the difference between the total lunar acceleration and the Sun's acceleration], into the computation, although the rest of his equations and his solution are for only the non-gravitational component, S'_D.[285]

By correcting these errors, Britton obtained considerably lower values for the non-gravitational secular increments: $+3''.62 \pm 0''.5$ for the Moon and $+1''.14 \pm 0''.3$ for the Sun.[286] Investigators after de Sitter (most importantly, H. Spencer Jones) accepted, without critical review, de Sitter's results as correctly derived from ancient observations – a negligence Britton finds hard to understand.

After his determination of κ and κ', De Sitter turned to the question whether the fluctuations in the planets and Sun correspond to the Moon's 'minor fluctuations,' or to its total fluctuation, including both the Great Empirical Term and the "minor fluctuations." De Sitter's analysis showed that the values of Q_1 determined from the sine term and from the residuals were the same. It followed that the distinction between the Great Empirical Term and the minor fluctuations had no basis in nature, and could not be regarded as theoretically significant. He also urged, in agreement with Brown, that the observed lunar fluctuations were not so well represented by a sinusoid as by a sequence of straight lines of different slopes. In addition, he showed that the fluctuations are reflected proportionately in the motions of Mercury, Venus, and the Sun.

In the case of the Sun and Venus, the Great Empirical Term during the period of trustworthy observations (from 1835 to 1925) differed so little from a straight line

285 Britton, *Models and Precision*, 166–168.
286 Ibid., 168.

that any satisfactory representation of the total fluctuations could be transformed into an equally satisfactory representation of the minor fluctuations, and *vice versa*, by corrections to mean motion and epoch. A longer series of trustworthy planetary observations with discernible fluctuations was available only for the transits of Mercury across the Sun's disk. Accordingly, de Sitter set about deriving an empirical sine term for the Moon and another for the transits of Mercury, to see whether these two terms agreed in period and phase.

For the Moon, he expressed the total difference between observed and theoretical longitude by the formula

$$B_0' = \Delta L_0 + T\Delta n + cS + K\sin(\beta T + \gamma). \tag{5}$$

To determine the constants in this formula, de Sitter used right ascensions observed from 1621 to 1925, as listed by Brown.[287]

De Sitter included the term cS in (5), but it was not possible to determine c independently of β and γ, and therefore he set c equal to zero. The values found for β and γ proved to depend largely on the weights assigned to the early observations. De Sitter concluded that his best option was to adopt Brown's values, slightly rounded, for these constants. The least-squares solution could thereby be limited to the three unknowns ΔL_0, Δn, and K. De Sitter obtained

$$B_0' = 0''.66 + 0''.79T + 14''.42\sin(140^\circ.0T + 240^\circ.0).$$

From the residuals F he derived for the probable errors of ΔL_0, Δn, K the values $\pm 0''.02$, $\pm 0''.02$, and $\pm 0''.03$ respectively.

For the transits of Mercury the equations of condition had the form

$$O - C = a_1 + b_1T + c_1S + K_1\sin(\beta_1T + \gamma_1). \tag{6}$$

Here $O - C$ is the excess of the observed difference between the longitudes of Mercury and the Sun, over the same difference as calculated from Newcomb's tables. To find the constants in (6), de Sitter had recourse to Innes' discussion of the observed transits of Mercury since 1677. He again found it impossible to determine the sine term independently of the secular acceleration. Hence for c_1 he chose the value corresponding to $\kappa_1 = +0.80$, the same as the value of κ previously found for the Sun. The sine term he then found to be

$$4''.57\sin(136^\circ.0T + 236^\circ.4).$$

The argument of this sine function was so close to that found by Brown for the Moon's Great Empirical Term, that de Sitter assumed their equality and proceeded to a new least-squares solution, introducing as unknowns only a_1, b_1, and K_1. For K_1 he obtained $4''.39 \pm 0''.21$. The ratio of K_1 to the coefficient of the Moon's Great Empirical term was

[287] In tables I–III of his 1926 paper in *Transactions of Yale University Observatory*, III, Tables 1–3, 216–218.

$$\frac{K_1}{K} = 0.304 = 1.32\frac{n_1 - n_0}{n}.$$

For Q_1 in (3) he thus found the value $1.32 \pm .07$.

The residuals F_1 had now to be compared with the corresponding residuals F, or "minor fluctuations," found in the Moon's longitude. The value of Q_1 obtained from the residuals proved to be

$$Q_1 = \frac{F_1}{F} = (1.34 \pm .02)\frac{n_1 - n_0}{n}.$$

While allowing that the uncertainties in the two determinations were probably greater than those suggested by the probable errors (± 0.07, ± 0.02), de Sitter concluded that the distinction between Great Empirical Term and the minor fluctuations was simply artificial and should be dropped.

To define B', the Moon's fluctuation, de Sitter now wrote

$$B' = \text{observed mean longitude of Moon} - C, \tag{7}$$

$$\text{where} \quad C = \text{Brown's tables} - 10''.71 \sin(140^\circ.0T + 240^\circ.7) + 5''.22S + 4''.00T + 6''.70.$$

Here C is the theoretical value of the Moon's mean longitude, computed from Brown's Tables with the Great Empirical Term subtracted out, and with de Sitter's corrections (shown in the second line) inserted.

Comparing this formula with observations of the Moon's longitude from 1621 to 1925, as derived by Newcomb from eclipses and occultations up to 1835, and given by the Greenwich meridian observations after that date, de Sitter found that the probable errors of the successive normal points steadily decreased, from $\pm 14''$ in 1621, to $\pm 1''$ in 1681, $\pm 0''.3$ around 1800, and $\pm 0''.04$ or $\pm 0''.05$ after 1900.

He next turned to the determination of the value of Q from meridian observations of the Sun, Venus, and Mercury, choosing in each case the observations he believed reliable. In the case of the Sun, relying on Greenwich meridian observations from 1839 to 1863 and from 1896 to 1922, and assuming a secular acceleration of $1''.80S$, he found the correction to Newcomb's tables to be

$$\Delta L' = +1''.89 + 1''.41T + 1''.80S + 0.098B'.$$

The coefficient of B' corresponds to

$$Q = 1.31 \pm 0.13.$$

Assuming that the resulting fluctuation in longitude in the Sun's motion was a reflection of the very same variation in the Earth's rotation that caused B' to appear in the longitude of the Moon, de Sitter identified a multiple of this fluctuation that would make it comparable to B', namely

$$B'_0 = 10.72(O - C),$$
$$\text{where} \quad C = \text{Newcomb's tables} + 1''.89 + 1''.41T + 1''.80S.$$

Plotting observational values of B'_0 and observational values of B', de Sitter found that the two variations were practically coincident.

The corresponding quantities in the case of Venus were

$$Q_2 = 1.262 \pm .062,$$
$$B'_2 = 6.58[(O - C)_0 - \Delta\lambda_2],$$

where $(O - C)_0$ signifies the difference between observation and tabular value in the Sun's case, and $\Delta\lambda_2$ the corresponding difference in the case of Venus. Here again, comparing B'_2 with B', the lunar fluctuations, de Sitter found the variations to be practically coincident.

The transits of Mercury gave

$$Q_1 = 1.19 \pm .075.$$

B'_1, like B'_0 and B'_2, closely tracked B'.

The values found for Q, Q_1, and Q_2 clustered closely enough to support the conclusion that they represented a single constant, to which de Sitter assigned the mean value $1.25 \pm .02$. If Q, Q_1, Q_2 had averaged to 1.0, the fluctuations in the motions of the Moon, Sun, Mercury, and Venus would all have been interpretable as direct reflections of the variations in the Earth's rotation. To interpret the deviation of the Q's from unity, de Sitter turned to mechanical theory.

With certain simplifications (e.g., de Sitter neglected the axial rotation of the Moon and orbital motion of the Earth), the projection of the angular momentum of the Earth-Moon system onto the ecliptic gave

$$I\omega \cos\varepsilon + \mu a^2 n(1 - e^2)^{1/2} \cos i = c_3. \tag{8}$$

Here I is the Earth's moment of inertia about its axis of rotation,[288] ω is its rotational velocity, and c_3 a constant. The other constants pertain to the Moon: μ is its mass, a its mean distance from the Earth, n its mean orbital speed about the Earth, i the inclination of its orbit to the ecliptic, and e the eccentricity of its orbit.

De Sitter took the variation of (8), finding (with some simplifying)

$$\frac{d(I\omega)}{I\omega} - k\cos\varepsilon\left[\frac{dn}{n} + 3(e\,de + \sin i\,di)\right] = 0, \tag{9}$$

where $k = \mu a^2 n/3I\omega$. This equation says that a change in the Earth's angular momentum of rotation is compensated by changes in three of the Moon's orbital

[288] De Sitter symbolizes the moment of inertia by C. Having used "C" for "calculated value," we substitute "I" for De Sitter's "C."

elements: its mean motion n, its orbital eccentricity e, and the inclination i of its orbit to the ecliptic. For the factor $3(ede + \sin i di)$ de Sitter substituted – $f(dn/n)$, signifying that changes in orbital eccentricity and inclination, for a given change $d(I\omega)$, reduce dn/n.

A change in $I\omega$ can include changes in both I and ω:

$$\frac{d(I\omega)}{I\omega} = \frac{dI}{I} + \frac{d\omega}{\omega}. \tag{10}$$

For I to change, terrestrial masses must move outward from or inward toward the Earth's rotational axis. With any change of I, ω also changes. But ω can also change due to tidal friction, in which case no change is produced in I. In this case angular momentum is transferred to the Moon, where it will appear as an increase in the orbital mean motion (dn) or in the eccentricity (de) or in the Moon's orbital inclination (di). How the transfer is divided up among these different effects depends on the Moon's instantaneous orbit and the direction and magnitude of the acting force from the tides. The quantity of these effects cannot be predicted from general theory alone.

In an attempt to account for the excess of Q over unity, de Sitter now introduced a complication: he supposed the *observed* rotation of the Earth to be different from its rotation as a whole. The observed rotation, which he labeled ω', is what is detected by astronomers at Greenwich. Other parts of the Earth may be supposed to rotate at a different rate ω. To allow for this, de Sitter supposed the Earth to consist of two parts with moments of inertia $I(1 - p)$ and Ip. Equation (10) had then to be altered to read

$$\frac{d(I\omega)}{I\omega} = \frac{dI}{I} + \frac{(1-p)d\omega + pd\omega'}{\omega} = \frac{dI}{I} + (1+\Theta)\frac{d\omega'}{\omega}. \tag{10a}$$

The constant Θ could be positive or negative, but according to de Sitter, it was probably small.

The observed change in the Moon's apparent mean motion, as measured using the Earth as clock, is

$$\frac{dn'}{n} = \frac{dn}{n} - \frac{d\omega'}{\omega} \tag{11}$$

The differential dn' is an observed change in the Moon's mean motion. The variation dn is a change in the Moon's mean motion, ultimately due to tidal friction and the transfer of angular momentum from the Earth's rotation to the Moon's orbital motion, but rather small, and so not measurable in de Sitter's time. The variation $d\omega'$ is a positive or negative change in the Earth's rotational rate owing to a change in I. It causes the Moon to *appear* to accelerate or decelerate. De Sitter put

$$\frac{d\omega'}{\omega} = -Q\frac{dn'}{n}, \tag{lla}$$

where Q is a factor to be determined observationally. Substituting it into (11) he obtained

$$\frac{dn}{n} = (1-Q)\frac{dn'}{n}. \tag{llb}$$

From (9), (10a), (lla), and (llb), he deduced that

$$\frac{dI}{I} = Q(1+\Theta)\frac{dn'}{n} - (Q-1)(1-f)k\cos\varepsilon\frac{dn'}{n}. \tag{12}$$

De Sitter applied (12) both to the Moon's non-gravitational secular acceleration and to its fluctuations, obtaining different values for Q in the two cases.

In the first of these applications, he set $dI = 0$, and also $\Theta = 0$ because

> . . . it is inconceivable that the rotation of the crust, or of any part of it, should be secularly different from that of the whole Earth.

Given that $k\cos\varepsilon = 1.49$, (12) reduced to

$$(0.49 - 1.49f)(Q-1) = 1. \tag{12a}$$

In non-gravitational secular acceleration, therefore, a value for Q determined f. The observational value of Q in the secular acceleration is $(dn_0/n_0)/(dn/n)$, where n/n_0 is the ratio of the Moon's mean motion to the Sun's ($= 13.369/1$), and dn_0/dn is the ratio of the Sun's to the Moon's non-gravitational secular acceleration. According to de Sitter, the latter ratio was 1.80/5.22 ($= 0.3448/1$). These numbers gave for Q the value

$$Q_s = 0.3448 \times 13.3687 = 4.6.$$

Britton, re-doing these calculations in 1992, found for the average non-gravitational secular accelerations over the period from 300 B.C. to A.D.1900, the values $1''.15 \pm 0''.15$ for the Sun and $4''.00 \pm 0''.6$ for the Moon. These give

$$Q_s = 3.84.$$

Both results are considerably larger than the value $Q = 1.25$ which de Sitter had obtained from the fluctuations. With $Q_s = 4.6$, de Sitter found $f = 0.14$. Britton would have obtained $f = 0.94$.

As applied to the fluctuations, (12) yielded a more complicated variation. As before, $k\cos\varepsilon = 1.49$. Q must now be 1.25, and de Sitter assumed that f remained equal to 0.14. For finding dI/I, (12) also required values for dn'/n and Θ. Observations could not yield a value of Θ directly. But changes in dn' compared to the long-term mean rate n were easily detectable.

Brown had plotted the excesses and deficits in the Moon's advances in longitude since 1750, as compared with its long-term mean rate. The plot appeared approximately as a series of straight lines with rather sharp changes in slope (i.e., in rate of motion) at the junctions. De Sitter's plots were similar but inverted, with the up and down directions interchanged. They represented the changes in the Earth's rotation which de Sitter took to be the causes of the changes dn'. To both investigators, the changes at the junctions were the puzzle. They occurred either suddenly, or at most within a few years, and yet they were of the same order of magnitude as the changes produced by secular acceleration over a century. Thus in 1897 the change came to $dn'/n = +4.10^{-8}$. With this value, de Sitter reduced (12) to

$$10^8 \cdot \frac{dI}{I} = (1 + 0.25)(1 + \Theta)(4) - (0.25)(0.86)(1.49)(4).$$

Here terms in which the factor (0.25) appears explicitly are those attributable to the excess of Q over 1.

De Sitter produced a chart of the values taken by $\delta I/I$ and also by $\delta E/(I\omega^2)$, the change in energy, for three different values of Θ:

$\Theta =$	-0.20	0	$+0.20$
$10^8(\delta I/I) =$	$+3.2 - 0.5$	$+4.0 - 0.3$	$+4.8 - 0.1$
$10^8(\delta E/(I\omega^2) =$	$-1.6 - 1.0$	$-2.0 - 1.2$	$-2.4 - 1.2$

Here the first number in each pair corresponds to $Q = 1$, and the second number to the excess of Q over unity. The assumption $Q = 1$ implies that the Moon's true mean motion does not change ($dn = 0$).

De Sitter remarked that the changes δI and δE for $dn'/n = +4 \cdot 10^{-8}$ were very great. If $Q = 1$ so that the Moon's true mean motion does not change, the dissipation of energy is $\delta E = -8 \cdot 10^{28}$ ergs, and if $Q = 1.25$ so that the Moon's true mean motion increases, the dissipation increases by 160% to $-1.28 \cdot 10^{29}$. To give an idea of the order of magnitude of $\delta I/I$, de Sitter imagined it as produced by local displacement of a mass μ, originally at distance r from the Earth's polar axis, to the distance $r + \delta r$. This would give

$$\frac{\delta I}{I} = 6\frac{\mu}{M}\cos^2\phi\frac{\delta r}{r},$$

where M is the mass of the Earth and ϕ is the terrestrial latitude.

> The effect of a displacement of the whole of the central Asian highlands, including the Himalaya and the Kven Lin [= Kwenlun Shan], over its own height would produce a change of the order of $\delta I/I = 10^{-8}$, i.e. about a fourth of the change in 1897.

De Sitter acknowledged the improbability that catastrophes of this order of magnitude had happened in historical times without producing effects that geologists would notice. Like Brown, he concluded that the change was not local, but distributed through the whole or a large portion of the body of the Earth, in which case it would correspond to an expansion or contraction produced by a change of temperature of a fraction of a degree. De Sitter commented:

> We are compelled, as Brown has convincingly shown, to ascribe the changes of I to some deep-seated origin, however difficult it may be to imagine a cause which can produce such enormous effects in so short a time.

Given that Q's excess over unity came with an increment dn in the Moon's true mean motion, the excess had to involve a dynamic interaction between Earth and Moon. The only likely candidate was tidal friction. But tidal friction always produces a *deceleration* of the Earth's rotation, hence an apparent *acceleration* in the Moon's motion, whereas the fluctuations involve both apparent accelerations and decelerations. De Sitter resolved this difficulty by suggesting that tidal friction, and the secular accelerations produced by it, might vary irregularly, the value found for Q_s, like that for Q, being merely an average value over 2000 or 2500 years.

The fluctuations, then, could be the combined effect of changes in the Earth's moment of inertia, and of the variability of tidal friction. The two causes would act independently. For the part of the fluctuation produced by changes in I, the effect in the Earth's rotation and in the apparent mean motion of the Moon would be given by $d\omega'/\omega = -dn'/n$. For the part produced by tidal friction, the effect would be given by $d\omega'/\omega = -Q_s dn'/n$. Let the ratio of the actions produced by the two causes be $p : q$. Then, necessarily,

$$p + q = 1,$$

$$p + Q_s q = Q.$$

Using his values 4.60 for Q_s and 1.25 for Q, de Sitter found $p = 0.93$ and $q = 0.07$. This was his explanation for the otherwise puzzling result that $Q = 1.25$.

De Sitter considered the excess of Q over unity as solidly established[289]:

> From the material discussed in this paper I would judge the true probable error to be about ±.08, so that e.g. the chance of the true value being inside the limits 0.95 and 1.05 would be about 1/25.

He also considered well established the explanation of the Moon's fluctuations and non-gravitational acceleration as due to changes in the Earth's rate of rotation[290]:

> The striking parallelism between the fluctuations of the different bodies, and the equality of the factor Q derived independently from the sun and the two planets, make it very difficult to escape the conclusion that the origin of the fluctuations, as well as of the secular acceleration, is in the rotation of the earth… [W]e have seen that all observed facts can be satisfactorily explained by the hypothesis that the actual fluctuations arise from the superposition of the effects of two causes. The first of these is a series of abrupt changes in the rate of rotation of the earth caused by changes of the moment of inertia due perhaps to expansions and contractions of the earth, and the other a variability of the coefficient of tidal friction. The first cause corresponds to the factor $Q = 1$, the other to $Q = Q_s$. The combination of the two causes gives rise to an apparent factor $Q = 1.25$.

[289] *Bulletin of the Astronomical Institutes of the Netherlands*, *4*, 37.
[290] Ibid., 38

In the final paragraphs of his paper, de Sitter addressed the question: How is "astronomical time" – time as astronomers had always measured it, namely by the apparent sidereal motion of the Sun (corrected for the "equation of time" from Ptolemy onwards) – to be corrected so as to yield "uniform" or "Newtonian" time, the independent variable in the equations of celestial mechanics? The correction must consist of two parts:

$$\Delta t = \Delta_1 t + \Delta_2 t.$$

$\Delta_2 t$ is a secular term, given by $+43^s.8S$, where $S = T^2 + 1.33T - 0.26$. As we have seen, S is zero for $T = -1.5$ and for $+0.17$, that is, for 1750 and 1917. This term takes account of the deceleration of the Earth's rotation due to tidal friction. $\Delta_1 t$ is an irregular correction which has to be computed from the Moon's observed fluctuations. De Sitter provided a table of these corrections; we excerpt a few values:

$-\ 38^s.5$ in 1640,

$-\ 13^s.4$ in 1700,

$+\ 29^s.7$ in 1800,

$-\ 35^s.9$ in 1900,

$-\ 28^s.2$ in 1926.5.

De Sitter gave no indication of his level of confidence in the accuracy of these values.

A Revision of Newcomb's "Researches on the Motion of the Moon, Part II", by H. Spencer Jones, 1932

In "Researches, Part II," Newcomb reduced and discussed a large number of the Moon's occultations of stars. The data thus made available would be of key importance in establishing the variability of the Earth's rotation. Jones introduced his revision of this work with praise for Newcomb's achievement[291]:

> In the year 1912 was published the last of a series of papers on the Moon by Simon Newcomb, entitled "Researches on the Motion of the Moon, Part II: The Mean Motion of the Moon and Astronomical Elements, based on Observations extending from the Era of the Babylonians until A.D. 1908." The main and most important portion of this great work consists of the reduction and discussion of a large number of observations of occultations of stars by the Moon from 1672 to 1908. The reduction and discussion of

[291] H. Spencer Jones, *Discussion of observations of occultations of stars by the Moon, 1672–1908, being a revision of Newcomb's "Researches on the Motion of the Moon, Part II. "* (London: His Majesty's Stationery Office, 1932) pp. 1–70.

> the material occupied a period of 30 years and was completed during the author's last illness, the copy for the printer being finished only a month before his death in 1909.

Newcomb had two motives for turning to reports of occultations: (1) to obtain positions of the Moon for years in which the regular meridian observations were not available, and (2) to check on the accuracy of the meridian observations themselves. He had come to suspect that meridian observations were subject to systematic errors from which it might be impossible to free them. Jones shared this suspicion, as his paper on the Cape occultations for the period 1880–1922 testifies.

Of the occultations reduced by Newcomb, those dating from 1750 to about 1835 were sharply discordant with the corresponding Greenwich meridian observations. Brown had remarked on these discrepancies, concluding that the meridian observations for those years were unreliable. Jones posed the question: Might this discordance be due to the deficiencies of the theoretical basis Newcomb had used, i.e., Hansen's *Tables de la lune*, supplemented by only some of the principal terms omitted by Hansen? In 1925, with the aid of the more complete theory embodied in Brown's tables, Jones set out to answer this question.

Newcomb had anticipated that later investigators would be in a position to improve on his "Researches," and had attempted to present them in such a way that revision would be straightforward. Unfortunately, as Jones found, Newcomb had failed to specify crucial details, and it was thus impossible to reconstitute Newcomb's original normal equations. Jones was forced to begin over and construct normal equations from the beginning. He also found that Newcomb had committed errors of both sign and magnitude, so that, on this score alone, a revision was necessary. Newcomb, apparently aware that time was running out for him, had worked with excessive haste.

Jones modified the theoretical basis adopted by Newcomb in three principal respects:

(1) He inserted 18 additional terms, most of them representing perturbations of the Moon by the planets. The largest coefficient among these was $1''.07$; the smallest coefficient was $0''.02$, and the average was $0''.23$. Jones was thus omitting many terms in Brown's theory with yet smaller coefficients. He believed, however, that he had included all the terms required to yield results precise to about $0''.02$.
(2) He corrected the Hansen-Newcomb mean longitude to reduce it to the mean longitude of Brown's tables.
(3) He corrected Brown's values for the terms dependent on the Earth's flattening. Brown had used 1/294 for the flattening ratio, whereas a smaller value, 1/297, was now accepted. The corrections included $+1''.08\sin(\Omega + 176^\circ.8)$ in the Moon's longitude, $+0''.04\sin(\Omega + 115^\circ)$ in its perigee, and $+0''.06\sin(\Omega + 325^\circ)$ in the node. Here Ω is the longitude of the lunar node.
(4) Jones also replaced Newcomb's value for the Moon's mean semi-diameter, $932''.58$, by the larger value $932''.70$, derived from his work with the Cape occultations. The reduction of the occultations required an accurate value of this constant.

Newcomb's equations of condition contained nine unknowns:

λ, the correction to the mean longitude of the Moon
$\kappa = -2e\,d\Pi$, where Π is the longitude of the perigee
$i\theta = \sin i\,d\Omega$, where Ω is the longitude of the Moon's node
i, the correction to the inclination of the Moon's orbit
b_o, the correction to the Moon's tabular latitude
α_o, the mean correction to the adopted position of the equinox
δ_o, the mean correction to the declination of all the stars
ε, the correction to the tabular obliquity of the ecliptic
P, the correction to the coefficient of the principal parallactic term

The numbers κ, $i\theta$, α_o, δ_o, and ε were assumed to be varying slowly over the centuries. To determine their secular variations, Newcomb had divided his observations chronologically into 13 groups, and assumed that all the unknowns except λ were constant within each group. He then determined the secular variations from the change in the mean value from one group to another. The groups, with the number of occultations included in each, were:

Group	Years	No. of occultations
I	1672–1686	24
II	1699–1720	50
III	1725–1729	12
IV	1736–1739	25
V	1746–1747	21
VI	1753–1779	59
VII	1783–1801	82
VIII	1801–1820	126
IX	1821–1838	335
X	1839–1856	517
XI	1857–1873	589
XII	1874–1890	952
XIII	1891–1908	1586

Jones sought to improve the rigor of Newcomb's procedure. In deriving longitude corrections Newcomb had neglected altogether the corrections to other orbital elements. When, as happened in some years, there were several nights on which a large number of occultations had been observed, the effect of the neglected terms could amount to as much as $0''.5$. In re-deriving the longitude correction, Jones took account of all factors required to achieve an accuracy of about $0''.02$.

From the observed occultations, Jones computed average longitudes for the mean date of the occultations in each group, i.e., for the years 1681, 1710, 1720, 1727, 1738, 1744, 1747, for every Cowell period from 1750 to 1850, and for every mid-year date from 1850.5 to 1908.5. Since the weights assignable to these 167 mean longitudes varied considerably, he constructed a smooth curve to represent the observed values as closely as possible. In computing the difference (observed longitude minus theoretical longitude), he took the observed longitude from the curve for a sequence of dates, and extracted the theoretical longitude for these same dates from Brown's Tables, neglecting the small terms he had chosen to neglect. These differences provided the residuals for Jones's least-squares solution for the longitude. As in all such operations, the residuals presupposed approximate values of the constants, and the least-squares solution refined these approximations.

For the correction to Brown's tabular value of the Moon's longitude, Jones obtained

$$-3''.09 + 2''.52T + 5''.22T^2 - 10''.71\sin(140^\circ T + 100^\circ.7),$$

where T stands for centuries since 1800. He included here a term for the Moon's non-gravitational secular acceleration, using de Sitter's value, $(5''.22 \pm 0''.30)T^2$. He remarked that this value differed little from Fotheringham's final value, $4''.79T^2$, which Brown had adopted in his 1926 paper on the Earth's rotation. (Fotheringham's final value was actually $4''.52T^2$.) This inclusion had the purpose of bringing the tables into agreement with ancient observations.

In his longitude corrections Jones also included a term which subtracted out Brown's Great Empirical Term. The Tables thus corrected gave the Moon's longitude as derived from gravitational theory together with the tidal part of the secular acceleration, an appropriate adjustment being made of the epoch of mean longitude.

For the years from 1840 onward, Jones found the occultations to be in close agreement with the meridian observations. It was on the basis of these same meridian observations that Brown had determined the constants of his theory. As was to be expected, Jones's analysis closely confirmed the accuracy of Brown's values for these constants.

As for the original aim of Jones's revision, the analysis confirmed Newcomb's earlier conclusion. The differences between the corrections (occultations minus theoretical values) found by Jones and those found by Newcomb were seldom as great as $1''$. Therefore, systematic differences remained between the occultations and the Greenwich meridian observations for the period 1785–1835. Jones obtained his values for these differences by subtracting the positions given by the meridian observations from the corresponding positions taken from the smoothed curve for the occultations. The differences increased from $-2''$ in 1788 to $+3''$ in 1814 and nearly $+4''$ in 1819, but then fell to zero in 1829.

Similarly, the Sun's theoretical longitude (given by Newcomb's tables) minus its longitude as observed at Greenwich increased from about $-2''$ in 1785 to $+4''$ in 1814, but then fell to zero in 1830. Evidently the Greenwich meridian observations during this time-period were affected by varying systematic errors, approximately the same for the Moon and the Sun. Occultations, though lacking the precision of the meridian observations, were free of such systematic errors, and were therefore more trustworthy for the years 1785–1830.

For the Moon's fluctuations, Jones obtained a series of values by subtracting the new theoretical longitude (given by the above formula) for each date from the corresponding position given by the smoothed curve of the occultations. Thus he obtained values for 1681, 1710, 1720, 1727, 1738, 1744, 1747, for each Cowell period from 1750 to 1849, and each year from 1850 to 1908. He provided the following table[292]:

T	B''	T	B''	T	B''	T	B''
1681.0	−12.72	1809.1	+11.88	1867.5	−1.57	1906.5	−13.43
1710.0	−3.92	1813.6	+11.23	1872.5	−6.38	1909.5	−12.78
1727.0	+2.15	1821.8	+10.02	1877.5	−9.38	1912.5	−11.62
1738.0	+5.97	1831.5	+6.85	1882.5	−11.35	1915.5	−10.35
1747.0	+8.49	1837.4	+4.91	1887.5	−13.05	1918.5	−10.20
1755.0	+10.34	1843.1	+4.31	1891.5	−14.34	1921.5	−10.18
1771.0	+13.54	1848.8	+3.97	1894.5	−15.23	1924.5	−11.82
1785.0	+14.84	1852.5	+3.37	1897.5	−15.99	1925.5	−12.21
1792.0	+14.53	1857.5	+2.40	1900.5	−15.87	1926.5	−12.20
1801.5	+13.09	1862.5	+0.91	1903.5	−14.50		

These values differ somewhat from de Sitter's values. Jones remarked that the series of straight lines by which de Sitter had proposed representing the fluctuations gave only a rough picture of the fluctuations as derived from the occultations.

Jones revised his own previous results for the fluctuations of the Sun, Mercury, and Venus.[293] Re-calculating de Sitter's Q with his new values for the fluctuations of these three bodies, Jones obtained

$$\begin{array}{ll} \text{Sun} & Q = 1.17 \pm 0.07 \\ \text{Venus} & Q = 1.35 \pm 0.07 \\ \text{Mercury} & Q = 1.11 \pm 0.05 \end{array}$$

The mean of these three values was 1.19 ± 0.04, smaller than de Sitter's mean, 1.25.

Brown, we recall, had reconciled the differences between the theoretical and observational values of the secular motions of the Moon's node and perigee by increasing the flattening ratio of the Earth's shape from 1/297 to 1/294. An objection raised by de Sitter and now repeated by Jones was that the larger value (1/294) disagreed with the value derived from the well-confirmed precessional constant of $50''.2500$ per year (the flattening ratio of the Earth and the precessional constant imply each other). Jones reinstated the value 1/297, and added to the theoretical

[292] Ibid., 31.

[293] In his paper on "The Rotation of the Earth" of 1926, *MNRAS*, *87*, 4–31.

values of the secular motions of the perigee and node the relativity corrections proposed by de Sitter, $+1''.97$ per century for the perigee and $+1''.91$ per century for the node. He then compared the observational values obtained from the occultations with the theoretical values, obtaining the following results:

	Perigee	Node
Theoretical	$+14643521''$	$-6967931''$
From meridian observations	$+14643535''$	$-6967944''$
From occultations	$+14643535''$	$-6967943''$
Occ. – Theor.	$+14''$	$-12''$

In 1938 E.W. Brown announced that he had reduced the discrepancy in the case of the perigee by $11''$.[294] The discrepancy between the theoretical and observational values of the secular motion of the lunar node, however, was still unresolved in 1965.[295]

R.T.A. Innes, E.W. Brown and the Occultation Program

From 1923 to 1952, Brown's Tables served as a basis for the lunar ephemerides in the *Nautical Almanac*. Brown was naturally interested in how well observations confirmed his new tables. In 1926 he published a "Comparison of the Washington and Greenwich Observations of the Moon for 1923, 4, 5 with the New Tables."[296] He also devoted a good deal of time and energy to organizing observers, professional and amateur, to observe the solar eclipse of January 24, 1925, and the positions of the Moon at or near this eclipse.[297]

Meanwhile, R.T.A. Innes of the Union Observatory in Johannesburg set about observing, collecting, and reducing reports of lunar occultations of stars. His purpose was to monitor the Moon's fluctuations by observations independent of the meridian observations. By the end of 1926, he and his staff had assembled a list of 560 reduced occultations (immersions rather than emersions) for the years 1923–1926, against which to judge the performance of Brown's Tables.[298]

Brown enthusiastically endorsed Innes's project. He persuaded the American Association of Variable Star Observers to adopt the collecting and reduction of lunar occultations as a project, an "infant industry," as he called it. Special forms were

[294] E.W. Brown, "Calculation of the term in the motion of the lunar perigee with characteristic e'^4", *MNRAS*, 98 (1937), 170–171.

[295] See W.J. Eckert, "On the Motions of the Perigee and Node and the Distribution of Mass in the Moon," *Astronomical Journal*, *70* (1965), 788.

[296] *Astronomical Journal*, *37* (1926), 29–32.

[297] E.W. Brown, "Discussion of Observations of the Moon at and near the Eclipse of 1925, January 24," *Astronomical Journal*, *37*, 9–19.

[298] E.W. Brown, "Occultations: A Report of Progress," *Popular Astronomy*, *36* (1928), 282–284.

drawn up for reporting immersions. Advanced students in astronomy at colleges and universities were invited to participate. In the *Astronomical Journal*, Brown, with the assistance of Dirk Brouwer, published reports of the occultations collected and reduced for each year from 1927 to 1935. For the year 1927 the number of reduced occultations reported was 418. The number reported rose in successive years, and for the year 1935 reached 1405. After Brown's death, and in accordance with his request, the program of collecting and publishing lunar occultations was continued by the Yale Observatory staff under the leadership of Brouwer.[299] The occultation reports thus assembled, along with the Cape occultations published by Jones in 1925, and the occultations reported by Newcomb and revised by Jones in 1932, provided the data for Jones's definitive demonstration that the fluctuations observed in the Moon, Sun, Mercury, and Venus were proportional to the mean motions of these bodies.

H. Spencer Jones's New Value of *Q*, 1939

Jones's proof that the fluctuations of the Moon, Sun, Mercury, and Venus were proportional to the mean motions of these bodies consisted in showing that *Q*, the constant introduced by de Sitter,[300] was equal to unity. De Sitter's value for *Q* was 1.25. In 1932, Jones obtained a lower value, 1.19. Jones regarded de Sitter's explanation of how *Q* could have a value greater than 1.0 as unsatisfactory:

> There are great difficulties in interpreting a value of *Q* that is greater than unity and de Sitter's suggested explanation, involving sudden and very great changes in tidal friction, is artificial and not convincing.[301]

Jones began by reviewing the differences between the two kinds of process hypothesized as causing changes in the Earth's rotation: (1) frictional resistance to the tides, and (2) the fluctuations.

Tides in the Earth's oceans and solid matter are raised by the gravitational attraction of both the Moon and the Sun. These tides can give rise to friction which slows the Earth's rotation. The evidence for such deceleration first appeared in the apparent secular acceleration of the Moon, which was larger than the amount of secular acceleration deducible from planetary perturbations of the Moon. Further evidence showed up later in an apparent secular acceleration of the Sun. The non-gravitational secular accelerations of both the Moon and the Sun had to be apparent only, produced as a projection of the Earth's slowing rotation. Tidal friction also caused a transfer of angular momentum to the Moon and the Sun, to compensate for the Earth's loss of angular momentum. In the case of the Moon, this added angular momentum pushes the Moon into a higher orbit in which its mean motion

299 Dirk Brouwer, "The Occultation Campaign. Outline of a Revised Program," *Astronomical Journal*, *47* (1939), 191–192.

300 H. Spencer Jones, "The Rotation of the Earth, and the Secular Accelerations of the Sun, Moon, and Planets," *MNRAS*, *99* (1939), 541–558.

301 Ibid., 543.

is less. The detected non-gravitational secular acceleration of the Moon is therefore the difference between the apparent-only acceleration due to the slowing of the Earth's rotation, and the actual slowing due to the transfer of angular momentum to the Moon's orbital motion. As the average value of this difference over the past 2000 years, Jones followed de Sitter (who claimed to be following Fotheringham) in choosing the value $5''.22$ per century. As made clear earlier, this value is excessive. In the case of the Sun, the transfer of angular momentum has too small an effect on the Sun's mean motion to be measurable.

Jones followed de Sitter in defining the Moon's fluctuation as

$$B = \text{Observed Longitude} - C$$

$$\text{where} \quad C = \text{Brown's Tables} - 10''.71 \sin(140^\circ.0T + 240^\circ.7) + 5''.22T^2 + 12''.96T + 4''.65.$$

T represents centuries since 1900. The effect of the fluctuation can be either an apparent acceleration or an apparent deceleration of the Moon's motion. In both cases it is only apparent, and due entirely to a slowing or speeding-up of the Earth's rotation. Since no dynamic interaction with extra-terrestrial bodies is involved, the Earth's angular momentum must remain constant, and the changes in angular speed must be accompanied by changes in the Earth's moment of inertia. Leaving aside the inquiry into such changes, Jones turned to re-doing the calculation of the proportionality constant Q.

For the corrections to Newcomb's tables of the motions of the Sun, Mercury, and Venus, Jones wrote

$$\Delta L = a + bT + cT^2 + Q(0.0747)B, \tag{1}$$

$$\Delta l_1 = a' + b'T + 4.15cT^2 + Q(0.310)B, \tag{2}$$

$$\Delta l_2 = a'' + b''T + 1.63cT^2 + Q(0.122)B. \tag{3}$$

These equations are the same as de Sitter's, but with numbers substituted for the mean motions. In place of de Sitter's value for the non-gravitational acceleration of the Sun (viz., $1''.80T^2$), Jones put the constant c, to be determined by his least-squares analysis. According to Jones, Mercury's non-gravitational secular acceleration had been found to be to the Sun's as the mean motions of the two bodies, namely 4.15:1. The available data were as yet insufficient to establish the analogous proportion for Venus, but Jones assumed nevertheless that it held. And he took the fluctuations of the Sun, Mercury, and Venus to be to those of the Moon as their respective mean motions, that is, as 0.0747:1, 0.310:1, and 0.122:1.

In his new attempt to determine Q, Jones chose his data with exceeding care. He hypothesized that the earlier determinations were misleading because of their dependence on the Greenwich meridian observations. He chose a procedure that would test the correctness of this hypothesis.

For the Moon, Jones chose to avoid dependence on the meridian observations altogether. He replaced them with lunar occultations of stars. These included the occultations assembled by Newcomb in "Researches, Part II" as re-discussed and corrected by Jones in 1932, the Cape observations published by Jones in 1925, and the occultations collected and discussed by Innes, Brown, and Brouwer in more recent years. He believed that these could be considered free of systematic errors, though subject to accidental errors which limited the precision of the results.

In the case of the Sun, Jones proposed to use both meridian observations of right ascension and observations of declinations, but he first determined Q by means of the declinations alone, and then he used both kinds of solar observation together, so that the results obtained in the two cases could be compared. Of the two kinds of observation, right ascension observations carried more weight for determining longitudes because of internal accordance between the observations over short time-intervals. But the declination observations were far less subject to errors of a systematic nature. Moreover, for the years 1785–1830, when the right ascension observations at Greenwich showed large and erratic variations, the declination observations appeared to be quite free from such variations. Jones found he could use the Greenwich declination observations back to 1750.

In order to determine the corrections in longitude for Mercury and Venus, the correction ΔL for the Sun's longitude was required. For this reason, Jones decided against using either smoothed values of the Sun's longitude taken from a graph or the representation of the observed longitudes by formula because these procedures could introduce systematic errors. Instead, he used the quantities directly furnished by observations of the Sun.

In the case of Mercury, he used the original data on transits of Mercury that Innes had rediscussed and de Sitter had summarized. These observations together with Newcomb's tables gave the quantities Newcomb had denoted by V and W:

$$\text{November transits:}\quad V = 1.487\Delta l_1 - 1.01\Delta L + \text{corrections to other elements}$$
$$= a_1 + b_1 T + 5.16cT^2 + Q(.385)B, \tag{4}$$
$$\text{May transits:}\quad W = 0.716\Delta l_1 - 0.97\Delta L + \text{corrections to other elements}$$
$$= a_2 + b_2 T + 1.97cT^2 + Q(.147)B. \tag{5}$$

For Venus, Jones used data he had assembled in a paper of 1926,[302] along with more recent observations. The data were presented in the form

$$\Delta l_2 - \Delta L = a_3 + b_3 T + 0.63cT^2 + Q(.0471))B. \tag{6}$$

In presenting the observed data, Jones used expressions (1), (4), (5), and (6). He combined the observations of Sun, Mercury, and Venus and formed normal equations in order to solve for c and Q. He carried out two such solutions. In the first solution he excluded the data from right ascensions, and in the second solution he included the data from right ascensions. The two solutions were as follows:

[302] *MNRAS*, *97* (1926), 4.

I. Sun's R.A.'s Excluded

$a + 1''.02$	$a_1 + 5''.90$	$a_2 + 2''.78$	$a_3 + 1''.00$
$b + 3''.02$	$b_1 + 16''.26$	$b_2 + 6''.94$	$b_3 + 2''.72$
	$c = +1''.25$	$Q = 1.025$	

II. Sun's R.A.'s Included

$a + 1''.04$	$a' + 1''.32$	$a_1 + 6''.03$	$a_2 + 2''.85$	$a_3 + 1''.02$
$b + 3''.09$	$b' + 2''.67$	$b_1 + 16''.55$	$b_2 + 7''.10$	$b_3 + 2''.76$
	$c = +1''.26$	$Q = +1.062$		

In Solution II, a and b denote as before the Sun's longitude correction in epoch and mean motion derived from declinations, whereas a' and b' denote this correction as derived from right ascensions.

Jones found that the probable error of the value of Q in these determinations was approximately $\pm.033$. Thus, in Solution I, where the right ascensions were excluded, the value of Q differed from unity by less than the probable error. In Solution II, with the right ascensions included, the value of Q increased so as to differ from unity by more than the probable error. Jones believed that this increase was caused by systematic errors in the observations of the Sun's right ascensions. On the strength of these determinations, and from the difficulty of interpreting any value of Q other than $Q = 1$, Jones concluded that Q should be assumed to be equal to unity. This conclusion implied that the fluctuations of the Moon, Sun, Mercury, and Venus were all simply reflections of variations in the Earth's rotation.

Assuming $Q = 1$, and seeking now to refine the other constants, Jones obtained his final least-squares solution:

III. Adopted Solution

$a + 1''.00$	$a' + 1''.28$	$a_1 + 5''.81$	$a_2 + 2''.74$	$a_3 + 0''.98$
$b + 2''.97$	$b' + 2''.69$	$b_1 + 16''.01$	$b_2 + 6''.82$	$b_3 + 2''.70$
	$c = 1''.23$	$Q = +1.00$		

The values of the constants in the adopted solution, together with the values of expressions (1), (4), (5), and (6) obtained from the series of observations of the Sun, Mercury and Venus employed in all three solutions, could be used to obtain values of B at the times of these observations. For selected dates Jones derived the Moon's fluctuations from observations of the Sun, Mercury, and Venus. Plotting these results on a graph of the Moon's fluctuations as determined from observations of the Moon, he found an impressive agreement between the values of B obtained from these two disparate sources. Solution III was therefore consistent with a very considerable body of observations of the Moon, Sun, and Mercury. The observations of Venus were of less weight and hence of less value in this comparison.

For the secular acceleration of the Sun, Solution III gives the value $c = +1''.23 \pm 0''.04$. The difference between this new value and de Sitter's value is $1''.80 - 1''.23 =$

$0''.57$, nearly three times the sum of the probable errors of the two determinations, $0''.16 + 0''.04 = 0''.20$. Was the difference significant? Jones thought it probable.

The two values did not contradict each other. The larger value was an average over 20 centuries, the smaller value an average over only the last two and a half centuries. It was not necessary to assume that the effects of tidal friction and consequent deceleration of the Earth's rotation had remained absolutely constant over 2000 years. The two values were compatible with a slow progressive change in the effects of tidal friction. Jones saw no evidence supporting the abrupt changes in tidal friction that de Sitter had hypothesized to account for the larger shifts in the Moon's mean motion.

On the other hand, Jones had assumed that the Moon's non-gravitational secular acceleration was $+5''.22$ per century. If the effects of tidal friction had changed as appeared from the Sun's case, the Moon's non-gravitational secular acceleration should have had a different average in recent centuries.

> If we suppose that during the period covered by the present investigation the true [non-gravitational] secular acceleration of the Moon is not $+5''.22$, as we have supposed, but $+5''.22 + s$, then we must replace B by a quantity B', which we may define by
>
> $$B' = B - sS,$$
>
> where $S = T^2 + 1.3T - 0.3$. The quantity S, which has zero values at 1750 and 1920, is introduced instead of T^2 in order to secure agreement with modem observations; in other words, it automatically takes account of the necessary adjustments to longitude at epoch and to mean motion.
> The term in T^2 and B in the Sun and planets then becomes
>
> $$\text{for the Sun,} \quad \frac{n_0}{n}(B - sS) + \left(c + \frac{n_0}{n}\right) T^2,$$
>
> $$\text{for the planets,} \quad \frac{n_i}{n_0}\left[\frac{n_0}{n}(B - sS) + \left(c + \frac{n_0}{n}s\right) T^2\right].$$
>
> The analysis therefore proceeds as before, but corresponding to a secular acceleration for the Moon of $+5''.22+s$ we will derive a secular acceleration for the Sun of amount $(c + n_0 s/n)$, which is equal to $+1''.23 + (.0747)s$.[303]

To determine s, Jones needed a further equation. To obtain it, he assumed that, as tidal friction slowly changed, its effects on the secular accelerations of the Moon and Sun remained in a constant ratio. The ratio could not be determined either from theory or from empirical observations, because the changes in the Moon's orbital elements caused by tidal action could not be calculated from general theory and were too small to be measured observationally. Jones thought it at least plausible that, if there was a slow, gradual change in tidal friction, then the ratio would remain approximately constant. In that case, Jones could set the ratio of the Sun's and

[303] *MNRAS*, *99* (1939), 555.

Moon's secular accelerations determined over 2000 years equal to the ratio involving s for the more recent period:

$$\frac{1.80 \pm .16}{5.22 \pm .30} = \frac{(1.23 \pm .04) + .0747s}{5.22 + s}.$$

From this equation he found $s = -2.11 \pm .57$. Inserting this value of s into the expressions for the two non-gravitational secular accelerations, he obtained

$$\begin{aligned} \text{for the Moon} \quad & +3''.11 \pm 0''.57, \\ \text{for the Sun} \quad & +1''.07 \pm 0''.06. \end{aligned}$$

He described these as "the best average values for the past 250 years."

The reader will recall that Britton, in reviewing de Sitter's calculations, found errors, as well as improbabilities in Fotheringham's estimates of probable error, on which Fotheringham's and hence de Sitter's results depended. Correcting the errors, and resolving anew de Sitter's equations for the tidal secular accelerations of the Moon and the Sun, Britton found the values[304]

$$\begin{aligned} \text{for the Moon} \quad & +3''.62 \pm 0''.5, \\ \text{for the Sun} \quad & +1''.14 \pm 0''.3. \end{aligned}$$

These values are to be understood as the averages over 2000 years. When compared with Jones's "best average values for the last 250 years," they give no grounds for supposing that the effects of tidal friction have measurably changed from ancient to modem times.[305] Jones's trust in de Sitter's values burdened him with a false problem.

[304] J.P. Britton, *Models Precision* (New York, NY: Garland Publishing, 1992), 168.

[305] For recent discussions, see K. Lambeck, *The Earth's Variable Rotation*, (Cambridge: Cambridge University Press, 1980) Chapter 10, and in particular p. 288.

23

The Quest for a Uniform Time: From Ephemeris Time to Atomic Time

In his 1939 paper H. Spencer Jones presented compelling evidence that de Sitter's proportionality constant Q did not differ from unity. This meant that the fluctuations of the Moon and Mercury were proportional to their respective mean motions. Jones also verified that the secular accelerations of the Sun and Mercury were proportional to their mean motions. (The secular accelerations of the Moon and Mercury, on the other hand, were *not* proportional to their respective mean motions, because tidal friction not only slows the Earth's rotation but also changes the Moon's mean motion.) The two proportionalities – of the fluctuations of the Moon and Mercury to their mean motions, and of the secular accelerations of the Sun and Mercury to their mean motions – were straightforwardly interpretable as due to variations in the Earth's rotation, that is, as errors in the accepted measure of time. Other hypotheses left these proportionalities as unexplained coincidences. The general conclusion in the astronomical community was that the Earth's rotation was slowing and fluctuating.

Up to the middle of the 20th century, astronomers measured time by the rotation of the celestial sphere, understood as a reflection of the Earth's rotation. The return of a star to the meridian signified the passage of a sidereal day, provided that the star was free from any "proper" motion – detectable motion with respect to the general stellar background. (In practice, the meridian transits of many stars were clocked, proper motions were taken into account, and the passage of the sidereal day was taken as the return to the meridian of the first point of Aries, or Vernal Equinox.) The return of the Sun to the meridian signified the passage of the solar day, a little longer than the sidereal day because the Sun on average moves $0^\circ.9855$ or $3547''.8$ eastward each day with respect to the stars, and the celestial sphere in its apparent westward rotation requires about $3^m.94$ to turn through this angle. The Sun's motion eastward is not quite equable, but subject to an inequality which had already been recognized in ancient times. It was caused by the inclination of the Sun's eastward path (the ecliptic) to the celestial equator, and by the eccentricity of its orbit about the Earth. Claudius Ptolemy in the *Almagest* explained how to correct for this inequality.[306]

[306] See G.J. Toomer, *Ptolemy's Almagest*, (New York, NY: Springer-Verlag, 1984), 169–172.

C. Wilson, *The Hill–Brown Theory of the Moon's Motion*, Sources and Studies in the History of Mathematics and Physical Sciences, DOI 10.1007/978-1-4419-5937-9_23,

Astronomers were thus able to relate their observations to *mean solar time*, time as given by a fictitious mean Sun, moving uniformly around the celestial equator in the time that the true Sun takes to pass round the ecliptic. Mean solar time was assumed to be a dependable measure of time, advancing uniformly without fluctuations in speed.

Spencer Jones's empirical determinations of 1939 made evident that the Earth's axial rotation could no longer serve as a uniform measure of time. The astronomers who first accepted the non-uniformity of the Earth's rotation did so because of departures of celestial bodies, in particular the Moon, from their ephemerides. In his *Researches on the Motion of the Moon*, Part I (1778), Newcomb calculated the time-errors in the Earth's rotation by comparing the Moon's observed positions with a lunar ephemeris computed from a corrected version of Hansen's *Tables de la lune*. Similarly, in 1925, R.T.A. Innes computed the approximate errors of the Earth as a clock from five sets of residuals or differences of the form (Observation minus Ephemeris Value), namely, for the transits of Mercury, the fluctuations of the Moon's motion, the eclipses of Jupiter's satellites I and II, and the Sun's motion from 1901 to 1921. The apparent errors in mean solar time from these different lines of evidence were in fairly good agreement; failure of precise agreement could be due to observational errors or errors in the ephemerides.[307]

In 1927, W. de Sitter provided formulas that were more precise, though of questionable accuracy. He defined "astronomical time" as the time given by the Earth's rotation, affected by both a secular deceleration of the rotation and by the fluctuations. The correction of "astronomical time" to uniform or "Newtonian" time was to consist of two parts:

$$\Delta t = \Delta_1 t + \Delta_2 t.$$

$\Delta_2 t$ was the secular term due to the Earth's rotational deceleration. De Sitter estimated it to be $43^s.8S$, where $S = T^2 + 1.33T - 0.26$ and T is number of centuries since 1900. (De Sitter's value of the secular slowing, $43^s.8I^2$, is a good deal larger than the value later accepted.) $\Delta_1 t$ was a correction for the irregular fluctuations in the Earth's rotation, for which de Sitter provided a table running from $-38^s.5$ in 1640 to $-28^s.2$ in 1926.5, with several ups and downs in between. He entitled this table "Corrections from astronomical to uniformly accelerated time."[308] (In a later article, he used the term "astronomical time" for what he here called "uniformly accelerated time."[309]) If the tidal secular acceleration could be regarded as constant, $\Delta_2 t$ could be specified in advance. Since the fluctuations were not specifiable in advance, $\Delta_1 t$ was ascertainable only for the past. De Sitter's values for the tidal secular accelerations of both the Moon and Sun, we again note, were exaggerated.

[307] *Astronomische Nachrichten*, 1925, cols.109–110.

[308] W. de Sitter, "On the Secular Accelerations and the Fluctuations of the Longitudes of the Moon, the Sun, Mercury and Venus," *Bulletin of the Astronomical Institutes of the Netherlands, IV*, June 8 (1927), 38.

[309] W. de Sitter, "On the System of Astronomical Constants," *Bulletin of the Astronomical Institutes of the Netherlands, VIII*, July 8 (1938), note 1, p. 219.

Like de Sitter, H. Spencer Jones sought to quantify the errors in the Earth's time-keeping. In his Halley Lecture of 1939, he estimated that the length of the day had increased by about $0^s.002$ per century over the preceding 25 centuries as a result of tidal friction.[310] The increase in the Moon's potential energy due to tidal friction had caused the Moon's mean distance to increase by about five feet per century. For the effect of the fluctuations on the Earth's time-keeping he gave a table for the approximate excesses of the length of the day over its average value during the last 250 years. These numbers were approximate only. Jones treated the issue of corrections as a practical matter. He did not address questions of principle, for instance the relevance of relativity theory.

Nevertheless, the new situation revealed by Jones's *Monthly Notices* paper of 1939 demanded a careful re-examination of time-measurement in astronomy, from the standpoint of both astronomical practice and astronomical theory. G.M. Clemence, director of the U.S. Nautical Almanac Office from 1945 to 1958, incorporated such a re-examination in a 1948 paper on the system of astronomical constants. The central theme of the paper was the necessity of self-consistency in the solar, lunar, and planetary theories.[311] This aim, he stated, had never been completely attained, but during the 20th century had been approached more closely than ever before. If the theory lacked self-consistency, discrepancies between theory and observation became difficult or impossible to interpret. The chief *scientific* value of ephemerides was to permit comparisons with observations, whereby the correctness of the theories could be tested. For this purpose, the ephemerides needed to represent the theories to within amounts smaller than the errors of the observations, and the theories needed to be logically self-consistent in order to have an unambiguous interpretation.

According to Clemence, a major inconsistency that needed attention was the discrepancy between the time used in the ephemerides and the time used in astronomical observations. The time used in the ephemerides was the independent variable in the equations of motion of celestial mechanics. De Sitter and Jones had called it "Newtonian Time," and Clemence in his 1948 paper continued this practice, although he was aware that there were relativistic effects in the planetary motions that needed to be taken into account.[312] Until 1925, the time to be used in clocking astronomical observations was officially stipulated by the International Astronomical Union to be Greenwich Mean Time (GMT). After 1925, it became Universal Time (UT), identical with GMT except that its epoch was 12 h earlier, at midnight, January 0, 1900, instead of the following noon, January 0.5. Universal Time and Greenwich Mean Time were both variable because they depended on the Earth's rotation. This variability

[310] H. Spencer Jones, "The Earth as a Clock, being the Halley Lecture delivered on 5 June 1939" (Oxford: Clarendon Press), 1939.

[311] G.M. Clemence, "On the System of Astronomical Constants," *Astronomical Journal 53* (1948), 169–179.

[312] See G.M. Clemence, "Relativity Effects in Planetary Motion," *Proceedings of the American Philosophical Society*, *93* (1949), 532–534.

defeated the purpose of the tables of the Sun, Moon, and planets, namely, accurate prediction.[313]

The difficulty could not be resolved by ordaining that astronomers employ "Newtonian Time" in place of Universal Time in their observations. Newtonian Time could be accessed only in retrospect. For this retrospective access, astronomical measurements in Universal Time were necessary. What was needed was a regular and precise way of ascertaining the *difference* between the two times.

In 1950, Dirk Brouwer proposed the name "Ephemeris Time" to designate the independent variable of the equations of motion in celestial mechanics. This name had the merit of not implying a restriction to pre-relativistic dynamical theory.[314] The name "Ephemeris Time" was adopted by the International Astronomical Union at its eighth General Assembly held in Rome in September, 1952. The difference between the time used in theoretical astronomy and the time used in observational astronomy therefore became the difference between Ephemeris Time (ET) and Universal Time (UT). In the discussion which follows we shall represent this difference by $\Delta T = \mathrm{ET} - \mathrm{UT}$.

In his 1948 paper, Clemence turned to Newcomb's *Tables of the Sun*, the official solar tables since 1900, in order to determine ΔT. The Sun's apparent eastward motion is a reflection of the Earth's motion about the Sun. This motion could be assumed to be in strict accordance with the theory of gravitation, because no cause or evidence for the Earth's departing from its gravitational orbit was known. On the other hand, the Sun as observed in Universal Time was subject to a secular acceleration and to fluctuations, both of which had now come to be attributed to time-errors in Universal Time caused by variations in the Earth's rotation. In his 1939 paper, Jones had obtained from his least-squares solution for the Sun a correction to Newcomb's tables for these departures from a uniform time[315]:

$$\Delta L_S = +1''.00 + 2''.97T + 1''.23T^2 + 0.0748B. \tag{1}$$

Here T is measured in Greenwich Mean Time or Universal Time. Its unit is the Julian century of 36525 days, counted from 1900 January 0, Greenwich Mean Noon (or equivalently, Jan. 0.5 UT). Its actual measurement is by timing meridian transits of stars in order to clock the passage of the sidereal day, then correcting sidereal time to mean solar time by formula.

The first two terms on the right of (1) are corrections to the Sun's epoch and mean motion in Newcomb's *Tables*. These terms are chosen so as to make the Ephemeris day approximately identical with the mean solar day in epoch and duration on Jan. 0, 1900.

The third and fourth terms introduce the effects of the variable rotation of the Earth. The term $1''.23T^2$ gives the Sun's non-gravitational secular acceleration. It is the reflection in the Sun of the gradual slowing of the Earth's rotation. The fourth

313 Ibid., 171.

314 See G.M. Clemence, "The Concept of Ephemeris Time: A Case of Inadvertent Plagiarism," *Journal for the History Astronomy*, *ii* (1971), 76.

315 *MNRAS*, *99* (1939), 556.

term gives the reflection in the Sun's motion of fluctuations in the Earth's rotation. B is the Moon's fluctuation. How it can be determined will be explained below. The number 0.0748 (equal to $1 \div 13.369$) is the ratio of the Sun's mean motion to the Moon's mean motion; it reduces the fluctuation from the size it has in the Moon's motion to the size it has in the Sun's motion. The fluctuation is best observed in the Moon, because among easily observable bodies in the solar system the Moon has the most rapid mean motion, and therefore reflects any departure from uniformity in the Earth's rotation with a larger and hence more precisely measurable departure from its tabular mean longitude.

The correction ΔL_S can be converted into a change in time Δt by a change of the unit in which (1) is expressed, from the arc-second to the second of time. The Sun in its mean motion from equinox requires 24.349 seconds of time to traverse one arc-second ($1''.0$). Multiplying (1) by 24.349 seconds/arc-second gives

$$\Delta t = +24^s.349 + 72^s.3165T + 29^s.949T^2 + 1.821B. \tag{2}$$

If B were accurately known, and if in addition the other terms in Δt were accurate, (2) would furnish the difference between Ephemeris Time and Universal Time.

But Clemence saw that a better course would be to determine empirically the whole difference between Ephemeris Time and Universal Time, ET − UT, including both non-gravitational secular acceleration and fluctuation. In the least-squares solutions for these two effects, they were not sharply separable. The whole difference ET − UT could be determined with higher precision and less uncertainty. The procedure would be to make multiple comparisons between a corrected version of the Brown-Hedrick *Tables of the Motion of the Moon* on the one hand, and lunar observations on the other, and then to average the results.

The necessary correction to bring the Brown-Hedrick tables into agreement with Ephemeris Time was a correction to the mean motion. Jones had already proposed its value:

$$\begin{aligned} \Delta L_M = &+4''.65 + 12''.96T + 5''.22T^2 + B \\ &- \text{Brown's empirical term (G.E.T.).} \end{aligned} \tag{3}$$

The subtraction in (3) of Brown's empirical term was necessary to insure that the tables would be in strict accordance with the equations of motion. The non-gravitational secular acceleration, here given as $5''.22T^2$, had to be included, since it represented a difference from uniform time stemming from the Earth's variable rotation. As earlier explained,[316] the value $5''.22$ was too large, but it had been accepted by astronomers on de Sitter's authority. Correction (3) like correction (1) contains the fluctuation B, another timing-error. Clemence warned that (3) could not be used as a definition of B if B exceeded a few arc-seconds. If the time-error due to retardation and fluctuation in the Earth's rotation is Δt, then during Δt the Moon's mean longitude and all the other arguments on which the Moon's true longitude depends

316 In our discussion of de Sitter's 1927 paper.

are changing, and all these changes need to be taken into account. During Δt the Moon's mean longitude increases by

$$\Delta L_{\Delta t} = +13''.37 + 39'.71T + 16''.44T^2 + B. \qquad (4)$$

When the observations are referred to Ephemeris Time, the correction to Brown's tables becomes the difference between (3) and (4):

$$\Delta L_M - \Delta L_{\Delta t} = -8''.72 - 26''.75T - 11''.22T^2 - \text{Brown's empirical term (G.E.T.)}. \qquad (5)$$

Correction (5) introduces into the lunar theory the same unit of Ephemeris Time that corrections (1) and (2) introduce into solar theory. Clemence proposed that these corrections be accepted at a future date when the entire official system of astronomical constants was revised. After that time (which turned out to be 1960.0), the difference ΔT could be determined as follows. As before, observations of the Moon's longitude would be timed in UT. The lunar ephemeris corrected by (5) would then yield, for the clock time of the observation, a longitude differing from the observed longitude. The difference, ΔL, can be converted into time by multiplying by 1.821. This number is the product of 0.0748, the ratio of the Sun's mean motion to the Moon's mean motion, by $24^s.349$, the number of seconds of time the Sun requires to traverse one arc-second in its mean motion. Thus $1.821\Delta L$ furnishes a value of ΔT, or ET − UT. Clemence proposed that many values of ΔT be determined and averaged over some number of weeks or months. The average value would be taken as the definitive value of ΔT for the mean date of the observations used. This procedure depends on the fact that ΔT changes rather slowly over time. It has the great merit that it can be freed from any dependence on estimates of the non-gravitational acceleration and fluctuation.

Clemence's 1948 paper was the first to set forth in detail the concept that would be called Ephemeris Time. Clemence made the concept practical by explaining how it could be realized. Observations of the Moon were to be carried out as in the past, and timed in UT. ΔT was to be determined by multiple comparisons between lunar observations and a corrected lunar ephemeris. For this purpose, the lunar ephemeris needed to be free of empirical terms and in strict agreement with the gravitational equations of motion. Satisfaction of the second condition was inevitably provisional, since observational precision is ever on the increase, and the number of smaller terms derivable in the theory is potentially infinite

Clemence was not in a rush to see these changes adopted. He urged that circumspection be exercised in introducing changes in fundamental constants and procedures. All proposed corrections of constants should be checked carefully to avoid introducing inconsistencies or needless labor for the astronomer.

Steps leading to the adoption of Ephemeris Time, however, came on apace. In 1950, a conference on the fundamental constants of astronomy was held at the Observatoire de Paris under the chairmanship of André Danjon, the Observatoire's director. Danjon, as Clemence learned some years later, had proposed already in 1929 that time be measured by the circumsolar motions of planets rather than by the

Earth's rotation.[317] At the 1950 conference the measurement of time was discussed at length, and a recommendation was formulated: in all circumstances where the second of mean solar time was unsatisfactory because of its variability, the unit adopted should be the sidereal year at 1900.0, and time measured in this unit was to be called Ephemeris Time. The recommendation specified the formula for translating mean solar time into Ephemeris Time (formula (2) above, due to Jones). This recommendation was forwarded to the International Astronomical Union, and adopted at the eighth General Assembly of the International Astronomical Union, meeting in Rome in September, 1952.[318] The recommendation was to go into effect in 1960.

In one respect, the unit of time adopted in 1960 differed from the one decided on in 1952. It was based on the tropical rather than the sidereal year. Thus the official second adopted in 1960 was "the fraction 1/31, 556, 925.9747 of the tropical year 1900 January 0 at 12 h Ephemeris Time."[319] The tropical year was substituted for the sidereal year because it could be deduced from observation without an assumed knowledge of the precession of the equinox. It was therefore more fundamental.[320]

In the 1950s it was also realized that Spencer Jones's values for the non-gravitational secular accelerations of the Sun and the Moon ($1''.23T^2$ and $5''.22T^2$ respectively) were questionable. Writing in 1954, D.H. Sadler reported that more recent investigations suggested "a reduction of the coefficients to $1''.01$ and $2''.2$, with the very large mean errors $\pm 0''.70$ and $\pm 9''.5$, the changes ($0''.22$ and $3''.0$) being in the ratio of the mean motion." This change would not alter ΔT, since the difference would be incorporated directly into B.[321]

The discoveries and discussions leading to the adoption of Ephemeris Time brought into clear light the disconnection between Universal Time and the motion of the Sun. Universal Time was defined as Sidereal Time minus certain additional terms, which were intended to put Universal Time into agreement with the motion of the mean Sun:

$$\mathrm{UT} = \mathrm{ST} - 12^h - 18^h 38^m 45^s.836 - 8,640,184^s.542T - 0^s.0929T^2.$$

Sidereal Time (ST) was and is based on multiple meridian transits of stars; 0^h of the sidereal day is identified with the transit of the first point of Aries. The last three terms in the foregoing expression for UT were Newcomb's expression for the right ascension of the Mean Sun, with T measured in Universal Time. (The circularity does not make the formula unusable, since T can be derived approximately from the

317 See Clemence, "The Concept of Ephemeris Time: A Case of Inadvertent Plagiarism," *Journal for the History Astronomy*, *ii* (1971), 76–78.

318 Ibid., 73–79.

319 Ibid., 76.

320 D.H. Sadler, "Ephemeris Time," *Occasional Notes of the Royal Astronomical Society*, *3*, No. 17 (October, 1954), 103–113; 105.

321 Ibid., 106.

day of the century and the local time.) But Ephemeris Time was introduced in order to bring Newcomb's Tables of the Sun into agreement with the Sun's true motion. The true Mean Sun has not transited at noon UT since some time between 1900 and 1905. In practice Mean Solar Time was defined by formula, and had no rigorous relation to the actual true Sun. Universal Time was not a fundamental measure of time, but an artificial measure of the Earth's rotation, chosen to be a close approximation to mean solar time at the meridian of Greenwich.[322]

The accuracy and precision with which ΔT and hence Ephemeris Time could be determined depended on (1) the accuracy and precision of the ephemeris of the Moon's motion, and (2) the accuracy and precision of observational determination of the Moon's celestial positions. During the 1950s, important advances were made in the precision and accuracy of both the ephemeris and of lunar observations.

Observational accuracy and precision were improved chiefly through introduction of a dual-rate, Moon-position camera for determining the Moon's positions with respect to the stars. It was developed by William Markowitz of the U.S. Naval Observatory, and it was put to work in a regular program of observation beginning in June, 1952.[323]

Previous ways of determining the Moon's position were subject to severe disadvantages and restrictions. As indicated several times before, meridian transit observations were exposed to systematic errors that were difficult to identify and eliminate. Occultations were free of this difficulty, but required that the Moon be within a degree or two of first quarter for optimal observing conditions. Attempts at photographically determining the Moon's position had been hampered by the Moon's size, brilliance, and motion.[324]

Markowitz's instrument permitted a photographic exposure of the Moon and background stars for 20 s, during which time the Moon was held fixed in frame relative to the stars. A synchronous motor and micrometer moved the carriage holding the photographic plate so as to keep the images of the stars fixed on the plate. The Moon's image was intercepted by a dark filter consisting of a plane-parallel glass plate 1.8 mm thick, having a transmission factor of 0.001. This filter was attached to a lever arm, which was slowly rotated by a second motor so as to alter the tilt of the filter with respect to the photographic plate. The change in tilt was adjusted to hold the Moon's image fixed on the photographic plate for the duration of the exposure. A light yellow filter, having at its center a hole which accommodated the dark Moon filter, cut out the blue rays of the stars. An electric contact was set to record the instant when the two filters were parallel, at which moment the Moon's image was not shifted with respect to the stars. This moment defined the epoch of

[322] Ibid., 107.

[323] W. Markowitz, "Photographic Determination of the Moon's Position, and Applications to the Measure of Time, Rotation of the Earth, and Geodesy," *Astronomical Journal*, *59* (1954), 69–73.

[324] The only photographic program of lunar observations previously executed appears to have been that due to A.S. King carried out at Harvard Observatory from 1911 to 1917; see *Annals of the Harvard College Observatory*, *72* (1913), 1; *76* (1916), 127.

the observation. The camera was attached to a 12-inch refractor at the U.S. Naval Observatory.

On each photographic plate, Markowitz determined the x- and y-coordinates of some 10 stars and 30 or 40 points on the Moon's bright limb, using a double-screw measuring machine. He also fitted a circle to the bright limb by the method of least-squares, thus determining the coordinates of the Moon's center. Comparing observations from night to night, Markowitz estimated a probable error in either right ascension or declination of about $0.''15$. A determination of the Moon's position derived from 100 observations would therefore have a probable error of $0.''015$, provided no systematic errors were present. The Markowitz camera required considerable expertise to operate. Its use was terminated in the mid-1970s, after laser-ranging proved superior for determining the Moon's position.[325]

On the side of theory, a major advance envisaged and promoted by Walter J. Eckert was on its way to actualization by 1948. One of its goals was to refine the measurement of Ephemeris Time, but its significance was more far-reaching. It would reduce the cost in human time and labor of developing planetary and lunar theories and extracting from them the necessary ephemerides.

Wallace John Eckert (1902–1971) after receiving a bachelor's degree from Oberlin College in 1925 and a master's degree from Amherst College in 1926, began work on a doctorate in astronomy under E. W. Brown at Yale. Simultaneously he became an assistant in the Astronomy Department of Columbia University. He completed the Ph.D. degree in 1931.[326]

In 1926, Brown and Eckert received a visit from J. Leslie Comrie who was just then introducing machine computation into the operations of Great Britain's Nautical Almanac Office. Comrie employed commercially available machines, principally the Hollerith tabulating machine. This invention of Herman Hollerith (1860–1929) was used to tabulate the U.S. census of 1890. It depended on a system of punched holes in a nonconducting material, and counted the items by the passage of an electric current through the holes. Comrie employed another machine to carry out mechanical integrations by the building up of a function from its finite second differences. On the basis of the Brown-Hedrick Tables, the British Nautical Almanac Office completed by 1932 a computed ephemeris of the Moon for every year from the 1930s to the year 2000.[327] Comrie expressed the opinion that there was little likelihood of the Brown-Hedrick *Tables* being superseded before the end of the century. Any acquisition of knowledge of the Moon during the next seven decades, he opined, was almost certain to be expressed in the form of corrections to Brown's Tables, not in the form of new tables. Comrie's confidence on this matter was mistaken.

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[325] S.J. Dick, *Sky and Ocean Joined* (Cambridge: Cambridge University Press, 2003), 481

[326] The account of Eckert that follows is drawn from Henry S. Tropp, art. "Wallace John Eckert," *Dictionary of Scientific Biography, XV*, suppl. I, 128–130, and Martin C. Gutzwiller, "Wallace Eckert, Computers, and the Nautical Almanac Office," *Proceedings of the Nautical Almanac Office*: *Sesquicentennial Symposium*, ..., *March 3–4, 1999* (eds. A.D. Fiala and S.J. Dick, U.S.: Washington, DC, Naval Observatory, 1999.

[327] See Comrie's reports in *MNRAS*, *92* (1932), 523, 694.

During his early years as an assistant in Columbia's Astronomy Department, Eckert borrowed time on the calculating equipment in the Columbia Statistical Bureau for the performance of astronomical calculations. The equipment had been a gift of IBM, and was used primarily for educational research. In 1933, Eckert asked Benjamin D. Wood, head of the Statistical Bureau, to approach T.J. Watson, Sr., of IBM, with a wish list of equipment. Eckert wanted some of the machines modified from their commercial form to improve their suitability for the scientific applications he had in mind. The deal was closed, and the equipment installed in a room of the Astronomy Department. Thus was the Thomas J. Watson Astronomical Computing Bureau established, with Eckert as director. It was operated as a joint enterprise of Columbia, the American Astronomical Society, and IBM.[328]

At Brown's request, Eckert applied the new equipment to check Brown's earlier, years-long, paper-and-pencil computation of the terms of the lunar theory. The idea had occurred to Brown of trying the result of referring the Moon's motion to x- and y-axes that followed the mean motion of the Moon rather than that of the Sun.[329] Later he discovered that Euler in his last lunar theory (of 1772) had used such axes. If the coordinates in the ecliptic plane followed the Moon's mean motion, they took the form $a+x$, y, where x, y are small enough so that expressions in powers of x and y are possible. An unexpected outcome was that the equations for x, y could be put into a form such that the first gave x and then the second gave y to the same degree of approximation as had been obtained in Hill's method only by a troublesome solution of two simultaneous equations. Moreover, the homogeneous equation for x contained fewer large terms than the earlier equation it superseded. The extent to which small divisors caused loss of accuracy was easier to assess and compensate for than in the earlier form of the equations. It was the new set of equations that Brown proposed employing for the verification of his earlier calculations. Brown tells the story in an article of 1938:

> As the coordinates consist of series of harmonic terms, by far the greatest part of the work consists of the multiplication of pairs of harmonic series. If then a technique could be developed for the multiplication of harmonic series by machinery, practically the whole of the work of calculating the action of the Sun on the Moon could be done in perhaps a tenth of the time it originally required. I was fortunate in interesting my friend and former pupil, Professor W. J. Eckert of Columbia University, in this problem. He had already adapted several commercial machines of the Hollerith type to the solution of problems of celestial mechanics, and had developed a computing laboratory for their effective use. With the aid of the facilities of this laboratory we have been able in a year or two to test and extend calculations

328 W.J. Eckert, "The Astronomical Hollerith-Computing Bureau," *Publications of the Astronomical Society of the Pacific*, *49* (1937), 249–253.

329 Our account is extracted from E.W. Brown, "The Equations of Motion of the Moon," *American Journal of Mathematics*, *60* (1938), 785–792. Brown received the proof sheets of this article some days before his death on July 22, 1938.

> which took me nearly twenty years to carry out with the old-fashioned methods.[330]

In the final paragraph of his article, Brown reported the result of the verification to date:

> The verification…has proceeded sufficiently far to enable me to state that amongst the terms containing even multiples of D, F – about half the whole number – there appears to be no error in the earlier work greater than 0.″01 in longitude, on the assumption that the transformation from rectangular to polar coordinates has been correctly computed. Two of the terms containing odd multiples of D apparently have errors of 0.″03, 0.″04. But it is not certain whether these are apparent only, some further work being necessary to decide the matter.[331]

In the Thomas J. Watson Astronomy Computing Bureau, Eckert worked closely with engineers. He may have been the first to develop a rudimentary mechanical programmer, in this early form a box of pluggable relays with about twenty switch settings, permitting him to coordinate the functions of his tabulating machines. In 1940 he published a slender volume entitled *Punched Card Methods in Scientific Computation*. Eckert there describes the machines available from IBM with their various capabilities, and discusses the performance of special tasks like mechanical quadrature, harmonic analysis, interpolation, and numerical solution of differential equations. Special attention is given to astronomical applications. On the basis of this publication, Eckert was promoted to a full professorship at Columbia University in 1940.

In late 1939, Eckert was invited to become director of the U.S. Nautical Almanac Office in Washington, DC, and he began his work there in early 1940. He later recalled that the Almanac Office "had no automatic equipment. Every digit was written by hand and read and written repeatedly…. They had desk calculators."[332] One of his accomplishments was to bring the office up to speed in computing equipment. His most important publication during his tenure there was the *Air Almanac*, soon imitated by the Germans, the French, and the British. It was a large annual volume giving positions for the Sun, Venus, Mars, Jupiter, and the Moon for every 10 min. of each day, in degrees with an accuracy of one arc-minute. The first volume appeared in time for use by the U.S. Army and Navy airplane pilots in World War II.

In 1944, Watson invited Eckert to join IBM as director of a new department of pure science. Eckert persuaded Watson that the goals of such a department could best be met by establishing a research center at Columbia University. In March

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[330] Op.cit., n.115, Ibid., 786.

[331] Ibid., 792.

[332] Quoted by M. Gutzwiller, "Wallace Eckert, Computers, and the Nautical Almanac Office," in Proceedings, Nautical Almanac Office Sesquicentennial Symposium, U.S. Naval Observatory, 1999, p. 151, from Jean F. Brennan, *The IBM Watson Laboratory at Columbia University: A History* (Armonk, NY: IBM Corporation, 1971), 10

1945 Eckert returned to Columbia as director of the Watson Scientific Laboratory.[333] An entire building was dedicated to this research center. It became a hub where the activities of many people, from IBM, from government, universities, and industry, intersected. The informal atmosphere fostered interactions.[334]

At the Computer Laboratory, Eckert supervised the design and construction of new computers.[335] Among them was a large-scale, general-purpose computer called the Selective Sequence Electronic Calculator (SSEC), which was dedicated in January, 1948. Eckert was responsible for its logical design, and Frank Hamilton of IBM for its hardware. Its components were 13,000 vacuum tubes and 21,000 electromagnetic relays.

One of the first tasks performed by the SSEC was the computation of the heliocentric coordinates of Jupiter, Saturn, Uranus, Neptune, and Pluto at forty-day intervals from 1653 to 2060. This work was supervised by Eckert along with Dirk Brouwer, Director of the Yale University Observatory, and G.M. Clemence, successor to Eckert as director of the Nautical Almanac Office. The computation was performed by numerically solving a set of simultaneous, non-linear, differential equations of the thirtieth order with an accuracy of 14 decimals. The starting positions and velocities for the integration were determined from about 25,000 observations covering the period 1780–1940. This computation was needed because of the defectiveness of all earlier theories for the outer planets. Hill's tables for Saturn (published in 1900) were reduced in accuracy to five significant figures by the 1940s, and soon they would not suffice for the most routine applications. Jupiter, Uranus, and Neptune were also departing from their predicted positions by increasing amounts. The new computation resulted in a volume containing one and a half million figures.[336]

From the start, one of the major projects planned for the SSEC was a new ephemeris of the Moon, computed directly from Brown's harmonic series for the coordinates. This project was realized in the *Improved Lunar Ephemeris, 1952–1959*, published in 1954.[337] A preliminary study, carried out by Edgar W. Woolard of the U.S. Nautical Almanac Office, showed that an ephemeris drawn from Brown's *Tables* differed significantly from an ephemeris computed directly from Brown's theoretical series for the coordinates (longitude, latitude, and sine parallax). In a test run, Woolard compared positions computed in these two ways at intervals of $0^d.5$ for the month from April 24 to May 24, 1948. In the extracting of positions from Brown's *Tables*, Brown's published precepts were followed without special refinements, since the object was to investigate the discrepancies between the SSEC values

[333] For a history of this institution, see the work by Brennan cited in the preceding note.

[334] P. 19 of the work by Brennan cited in n.332.

[335] W.J. Eckert, "Electrons and Computations," *Scientific Monthly*, *67* (Nov., 1948), 315–323.

[336] *APAE*, XII: *Coordinates of the Five Outer Planets*, 1653–2060, U.S. Government Printing Office, 1951. The information reported here is from the Preface, page v.

[337] The title page announces that this is "A Joint Supplement to the American Ephemeris and the (British) Nautical Almanac, Prepared jointly by the Nautical Almanac Offices of the United States of America and the United Kingdom." The volume of xii + 422 pages was published by the U.S. Government Printing Office, Washington, D.C., in 1954.

and values obtained by routine use of the *Tables*. Systematic differences made their appearance in both longitude and latitude. In the latitude, a large part of the discrepancy arose from a mistake in the *Tables*: the effect of long-period variations in the Moon's orbital inclination on several large terms in the latitude had inadvertently been included twice. In addition, a large number of small, short-period terms given in Brown's theoretical expressions were either entirely or partly omitted from the *Tables*. The effect known as "aberration" (the discovery of James Bradley in 1729) had been entirely overlooked in the *Tables*.

Woolard plotted the differences found between the tabular values and the SSEC values in longitude and latitude in the figure reproduced below. The errors in the tabular ephemeris vary between $-0''.10$ and $+0''.05$ in longitude, and between $-0''.20$ and $+0''.20$ in latitude. The final values of the longitude and latitude calculated by the SSEC are computationally accurate to three decimal places – an order of accuracy not obtainable from the *Tables*.

The *Improved Lunar Ephemeris* was a considerable improvement over lunar ephemerides extracted from the Brown-Hedrick *Tables*. It made possible a more accurate determination of Ephemeris Time. Further improvements were possible, and Eckert pursued them as we shall see. But we must first address a new development that emerged in the summer of 1955, the advent of Atomic Time.

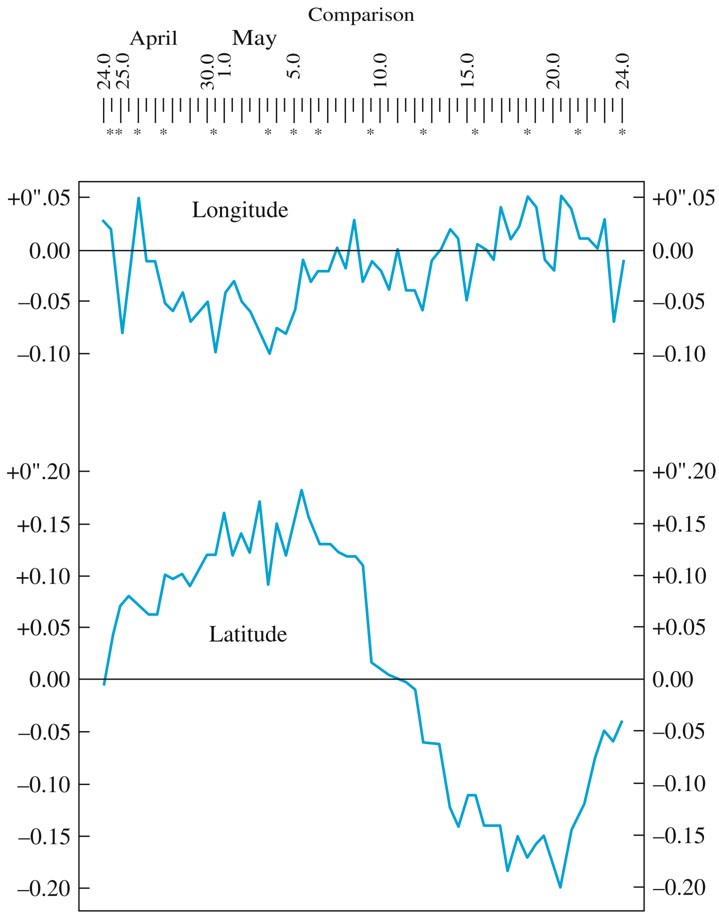
In August, 1955, L. Essen and J.V.L. Parry of the National Physical Laboratory in Teddington, Middlesex, announced their construction of a frequency standard based on a natural resonant frequency of the cesium atom.[338] Quartz clocks could be calibrated by means of it. Quartz clocks, dependent on the piezo-electric effect, were already the most precise and stable of clocks. The new calibration would make them many orders of magnitude more precise. The calibration apparatus depended on the atomic beam magnetic resonance technique that Isidore Rabi had invented in 1937. Rabi and his co-workers at Columbia University had perfected this technique in later years. In 1945, Rabi suggested the possibility of applying such a standard to the calibration of clocks.[339] When Essen and Parry made their announcement in August of 1955, the accuracy so far obtained was ± 1 part in 10^9 per day, or 0.1 milliseconds per day. By 1970 the achieved accuracy would be ± 1 part in 2×10^{13} per day (5 nanosec/day).[340]

A major advantage of the new standard was that calibrating a quartz clock could be carried out in minutes. Time measured by such a clock ("Atomic Time") was quickly accessible. Ephemeris Time, in contrast, could be known with a similar precision only after the lapse of a year or more, and would be an average value over that year. The idea was not far to seek that the second of time measured by quartz clocks calibrated against the cesium frequency could advantageously be adopted as

338 Essen and Parry, "An atomic standard of frequency and time interval," *Nature*, *176* (1955), 280–282.

339 I.I. Rabi, Richtmeyer Lecture to the American Physical Society, New York, 1945 (unpublished).

340 Winkler, G.M.R., Hall, R.G., and Percival, D.B., *Metrologia*, *6* (1970), 126.



Differences, **Tab.**–SSEC in longitude and latitude for every half-day. Asterisks indicate dates selected for detailed analysis.

the standard second, replacing the ephemeris second. This suggestion was made by E.C. Bullard, director of the National Physical Laboratory in Teddington, in a note accompanying the Essen-Parry announcement. [341]

A month later, in September, 1955, the General Assembly of the International Astronomical Union, at its meeting in Dublin, passed a resolution approving the following definition of the ephemeris second:

> The second is the fraction 1:31, 556, 925.975 of the length of the tropical year for 1900.0.[342]

[341] E.C. Bullard, "Definition of the Second of Time," *Nature*, *176* (1955), 282.

[342] H. Spencer Jones, "Definition of the Second of Time," *Nature*, *176* (1955), 669–670.

In his report of the passage of this resolution, H. Spencer Jones remarked that this fraction is accurate enough for practical purposes, but that the value required for exact agreement with Newcomb's *Tables of the Sun* was 1:31, 556, 925.97474. He gave a step-by-step account of how to obtain a frequency v_E corresponding to the invariable unit of Ephemeris Time defined in the resolution. The steps included a determination of the difference Δt between Ephemeris Time and Universal Time, using observations with the Markowitz dual-rate, Moon-position cameras. According to Jones, the cesium frequency standard would have a role in determining the frequencies that correspond to the varying second of Universal Time, and in relating the mean solar second at any future time to the fundamental second of 1900.0 defined in the resolution.

A back-and-forth discussion now commenced concerning the relative merits of Ephemeris Time and Atomic Time.[343] Ephemeris Time had been invented to supply the need for a "more uniform" time than the Earth-clock could supply. Uniformity in the measurement of time could not be determined by direct empirical test. A clock was judged to furnish a uniform measure of time if phenomena timed by it agreed with the accepted theory of dynamics. Uniformity here was a theory-dependent concept. In mid-20th century, the theory of dynamics used in planetary and lunar astronomy was in the main Newtonian, but astronomers knew that certain relativistic effects were observationally detectable. The motions of the planetary perihelia could in large part be derived from Newtonian gravitation, but the full motion could be derived only by calling on General Relativity. These relativistic increments in the motions of the planetary perihelia, astronomers believed, were the only detectable relativistic effects.[344] In a few years, however, further relativistic effects would be detectable.

Explanations of Ephemeris Time frequently made untenable claims. For instance, the *Explanatory Supplement to the Ephemeris* of 1961 described Ephemeris Time as follows:

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343 See for instance G.M. Clemence, "Definition of the Second of Time," *Nature*, *176* (Dec. 24, 1955), 1230 and "Standards of Time and Frequency," *Science*, *123* (April 6, 1956), 567–573; A. Perard, "The Standard of Length and the Standard of Time," *Nature*, *177* (May 5, 1956), 850–851; L. Essen, "Atomic Time and the Definition of the Second," *Nature*, *178* (July 7, 1956), 34–35.

344 G.M. Clemence, "The Relativity Effect in Planetary Motion," *Reviews of Modern Physics*, *19* (1947), 361–364; "Relativity Effects in Planetary Motion," *Proceedings of the American Philosophical Society*, *93* (1949), 532. It is worthy of remark that Clemence carried out a very careful determination of the observed motion of Mercury's perihelion, based on observations covering the period 1790–1940. He found that planetary perturbation accounted for $531.''47$ of Mercury's centennial perihelion motion. Added to the precession this gave a motion of the perihelion from the equinox of $5557''.18$ per century. The observed motion per century from equinox was $5599.''74$, greater by $42.''56$. The effect derived from relativity was $43.''03$. The difference, $0.''47$, was smaller than its probable error, $\pm 0''.97$.

> Ephemeris time is a uniform measure of time depending on the laws of dynamics. It is the independent variable in the gravitational theories of the sun, moon, and planets, and the argument for the fundamental ephemerides in the Ephemeris.[345]

This description assumes consistency between the laws of dynamics, the gravitational theories of the Sun, Moon, and planets, and the argument of the fundamental ephemerides. But the ephemerides were at best *approximations* to the gravitational theories, and the gravitational theories of planets and the Moon were at best *approximate* instantiations of the dynamical principles. In both cases the approximations were works in progress, using infinite series with doubtful convergence. Consistency was a goal rather than an assured achievement.

To specify Ephemeris Time in terms of rate and epoch was also troublesome. The epoch and rate were chosen with the intention of making Ephemeris Time the independent variable in Newcomb's *Theory of the Sun*. The ephemeris second was defined as the tropical second at 1900 January 0.5 ET, and corresponded to a geometric mean longitude of the Sun equal to $279^\circ 41' 48''.04$.[346] But the geometric longitude of the Sun is not an angle that astronomers can measure by observing the Sun. The Sun is too bright, and its motion in right ascension is too slow, to obtain its precise position by direct observation. The specification of its position depends on the system of astronomical constants, and these constants are refined from time to time. When a new value of the constant of aberration was introduced in 1964, this change called for an alteration of the Sun's longitude on 1900 January 0.5.

In practice, Ephemeris Time was determined by comparing observations of the Moon with a lunar ephemeris. This assumed that the independent variable in Newcomb's solar theory bore the same relation to uniform time as did the independent variable in Brown's lunar theory; both were assumed to be in strict accord with dynamical theory. Repeated improvements in the lunar ephemerides during the 1950s and 1960s contradicted this assumption. Ephemeris Time, determinable only for the past, was subject to repeated changes.

Ephemeris Time was a questionable way of measuring time, and critics decried its questionable features. At a colloquium of the International Astronomical Union held in August, 1970, I.I. Shapiro pronounced the then current definition of ephemeris time to be "philosophically repugnant, aesthetically horrifying, and completely inadequate."[347]

Already in 1967 the General Conference of Weights and Measures had tentatively defined the second of time on the basis of an atomic frequency. The second was to be "the duration of 9, 192, 631, 770 periods of the radiation corresponding to the

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[345] Quoted from J. Derral Mulholland, "Measures of Time in Astronomy," *Publications of the Astronomical Society of the Pacific*, *84* (June, 1972), 357–364. My account owes much to Mulholland's analysis.

[346] Ibid., 361.

[347] Ibid., 357.

transition between the two hyperfine levels of the ground state of the cesium-133 atom."[348] This second was to replace the second of Ephemeris Time in the International System of Units (SI). The duration was chosen to be as near as possible to the definition of the ephemeris second. The ephemeris second, meanwhile, remained in the System of Astronomical Constants of the International Astronomical Union.

New developments increased the accessibility of Atomic Time. An international standard of atomic time (TAI) was set by the Bureau International de l'Heure. Coordinated Universal Time (UTC) was disseminated by radio starting in 1964. Its second was the TAI second, while its epoch was defined relative to UT. UTC became the most readily available measure of time, the time scale against which astronomical observations were customarily made.[349]

Was measurement of time by the new clocks subject to General Relativity? In late 1971, J.C. Hafele and Richard E. Keating carried out an experiment to resolve the much debated question of the difference in aging of two twins with diverse histories of space travel. Four quartz clocks calibrated to the cesium frequency were flown twice around the world on regularly scheduled commercial jet flights, once eastward and once westward.[350] During flight, the clocks were miles above the Earth's surface, hence in a weaker gravitational field than similarly calibrated quartz clocks on the Earth's surface. According to relativity theory, the altitude of these clocks should cause them to gain time relative to reference clocks at the U.S. Naval Observatory. In addition, the clocks carried in the jet flights were in coordinate frames rotating at different rates than clocks at the Naval Observatory; clocks in the westward flight were rotating less rapidly than the Naval Observatory clocks, and clocks in the eastward flight were rotating more rapidly. Relativity theory required that the clocks in the eastward flight should lose time, and those in the westward flight should gain time, compared with the Naval Observatory clocks. The predicted relativistic time differences in nanoseconds (10^{-9} sec.) were as follows:

Effect	Eastward flight	Westward flight
Gravitational	144 ± 14	179 ± 18
Kinematic	-184 ± 18	96 ± 10
Net	-40 ± 23	275 ± 21

The estimated error ranges were derived from uncertainties as to the average speeds and altitudes of the flights. The observed times measured by the four clocks in the eastward and westward flights were averaged. The mean times with their standard deviations were:

[348] G.M. Clemence, "The Concept of Ephemeris Time," *Journal for the History of Astronomy*, *ii* (1971), 76

[349] Mulholland, Op.cit. in n.345, p. 364.

[350] Keating, R.E. and Hafele, J.C., "Around-the-World Atomic Clocks," *Science*, *177* (1972), 166–170.

	Eastward flight	Westward flight
Mean ± S.D.	−59 ± 10	273 ± 7

Thus the measured times were in satisfactory agreement with the predictions of relativity theory.

This experiment demonstrated that, in order to achieve nanosecond accuracy in timing, relativistic effects influencing clocks on the rotating Earth or on artificial satellites had to be taken into account. Thus the distinctions and formalism of General Relativity had to be applied. Ephemeris Time could no longer be taken as a true or uniform measure of time, because the pre-relativistic equations of motion on which it was based were imperfect.

In 1977, Ephemeris Time was finally retired in favor of two new dynamical scales of time, called "Proper Dynamical Time" (TDP) and "Coordinate Dynamical Time" (TDC). The TDP second was defined by TDP = TAI + $32^s.184$, where TAI is International Atomic Time. Earlier, the offset of $32^s.184$ had been used with Ephemeris Time to keep ET and TAI distinct. It was continued in TDP to make it distinct from TAI but continuous with its predecessor ET. TAI was the time given by an atomic clock on the Earth's surface, and hence was situated in a rotating, non-inertial frame of reference. The Earth's rotation was known to vary with annual, monthly, and diurnal periodicities; therefore TDP would necessarily have periodic differences from a clock in an inertial system. Coordinate Dynamical Time (TDC) was stipulated to be identical with TDP except in being free from the periodic variations to which TDP was known to be subject. The periodic variations included an annual term with a 1658-microsecond amplitude, and monthly and diurnal terms with 2-microsecond amplitudes.[351]

To illustrate the use of these new dynamical time scales, consider the construction of an ephemeris for a planet X.[352] Equations of motion for X are selected to be in accordance with General Relativity. They contain an independent variable t which may be identified with TDC. The equations are integrated numerically or analytically from initial conditions to yield an ephemeris of X as a function of TDC. This ephemeris may be labeled "geometric" to indicate that it takes no account of the time required for light to come from planet X to an observer on planet Earth. A similar geometric ephemeris is also constructed for planet Earth, using the same independent variable TDC.

It is next necessary to develop an *apparent geocentric ephemeris* of X. From selected space-time "points" in the geometric ephemeris of X, light-paths are traced to the Earth. According to General Relativity, the time for light to travel from X to the Earth is a function of distance, velocity, and the gravitational potential along the path. The arrival of the light on Earth is an observable phenomenon. The apparent position of X at this instant can be calculated from the Earth's geometric position

351 The foregoing account follows closely that in G.M.R. Winkler and Thomas C. Van Flandern, "Ephemeris Time, relativity, and the problem of uniform time in astronomy," *Astronomical Journal*, *82* (1977), 90.

352 The illustration is from the Winkler and Van Flandern article, 90.

together with $X's$ geometric position at the instant of the departure of the light from X. Thus a geocentric ephemeris of apparent positions of X can be built up and fitted to observations. This makes possible an adjustment of the initial conditions for the original integration of the equations of motion for X. The independent variable of the apparent geocentric ephemeris of X will then be a proper dynamical time, TDP. Let us suppose that the observer's clock time is Coordinated Universal Time (UTC) as described earlier, a broadcast time available at all observatories. The difference TAI – UTC is an integral number of seconds which changes whenever a leap second is introduced. And TAI differs from TDP only by the $32^s.184$ offset. Therefore, TDP will be knowable.

This procedure is required in order to achieve the new level of precision in time-keeping made possible by Atomic Time. Atomic Time, while supplying a new level of precision, was looked to for resolution of a cosmological question. The expansion of the Universe suggested that the Universal Gravitational Constant, G, might be decreasing, either as a cause or as an effect of the expansion. E.C. Bullard in his note accompanying the Essen-Parry announcement of August 14, 1955, remarked:

> Dirac, Milne, Jordan and others have suggested that what are usually regarded as the 'constants of physics' may change by amounts of the order of $1/T$ per year, where T is the 'age of the universe' in years. If T is about 4×10^9 years, changes of this order may soon be measurable over intervals of a few years. This is a matter that can only be settled by experiment...[353]

During the 1970s several researchers obtained positive values for the diminution of G.[354] T.C. Van Flandern found a higher value for the Moon's secular deceleration when using an atomic time scale than when using Ephemeris Time. From this difference he derived a value of $(1/G)dG/dt$.[355] The difference, however, has not been confirmed. In the absence of firm evidence to the contrary, it appears reasonable to take what has been called "the optimistic and conservative stand" and to assume that G remains unchanging.[356]

[353] E.C. Bullard, "Definition of the Second of Time," *Nature*, *176* (August 13, 1955), 282.

[354] See H.C. Ohanian, *Gravitation and Spacetime* (New York: W.W. Norton, 1976), 187–188, 216.

[355] *MNRAS*, *170* (1975), 333–342.

[356] Ohanian as cited in n.354. See also E. Myles Standish, "Numerical Planetary and Lunar Ephemerides," *Relativity in Celestial Mechanics Astrometry* (eds. J. Kovalesky and V.A. Brumberg; Reidel, Dordrecht, Holland), pp. 82–83.

24

1984: The Hill–Brown Theory is Replaced as the Basis of the Lunar Ephemerides

During the 1960s and until his death in August, 1971, Walter Eckert devoted his efforts to improving the precision with which the Moon's position could be derived from the Hill–Brown theory.

One of these efforts was *The Solution of the Main Problem of the Lunar Theory by the Method of Airy* in 1966.[357] The "Main Problem of the lunar theory," we recall, treats the Sun, Moon, and Earth as point-masses interacting in accordance with Newton's inverse-square law. The center of mass of the Earth and Moon is assumed to move about the Sun in an ellipse. All other effects are set aside, to be treated later as perturbations of the solution to the Main Problem.

The "method of Airy" corrects a theory by comparison with observations. The best available earlier solution is substituted into the differential equations, with numerical values assigned to the arbitrary constants. A set of linear variation equations is then formulated and solved on the assumption that the squares and products of the residuals can be neglected. The astronomer G.B. Airy during fourteen years toward the end of his life attempted to apply this process to Delaunay's theory. He abandoned the effort at age 88, concluding that his declining powers were unequal to the demanding computations required.[358]

Brown in the late 1930s proposed to Eckert that Airy's method be applied by electronic computer to check his own earlier elaboration of the lunar theory. He hoped that greater accuracy could be achieved in the centennial motions of the perigee and node. This project was not completed in the 1930s, but was carried far enough to assure Brown that his series developments were free from serious errors. With Brown's death and the coming of WWII, the project was set aside.

In 1957 Eckert revived the project, and with the assistance of Harry F. Smith, Jr., carried it to completion. The progress in automatic computing since the 1930s

[357] W.J. Eckert and H.F. Smith, Jr., "The Solution of the Main Problem of the Lunar Theory by the Method of Airy," *Astronomical Papers prepared for the use of the American Ephemeris and Nautical Almanac* (hereinafter *APAE*), *19*, Part II, U.S. Government Printing Office, Washington, DC: 1966.

[358] G.B. Airy, "The Numerical Lunar Theory," *MNRAS*, *49* (1988), 2.

C. Wilson, *The Hill–Brown Theory of the Moon's Motion*, Sources and Studies in the History of Mathematics and Physical Sciences, DOI 10.1007/978-1-4419-5937-9_24,

permitted a considerably higher level of accuracy.[359] In the re-computation, the rectangular coordinates, consisting of about 9,600 terms, were given with a precision of 1×10^{-12}, corresponding to a precision of $2'' \times 10^{-7}$ in angular measure. Eckert elaborated:

> The great majority of the terms are believed accurate to a few units in the last place. Perhaps 30 or 40 terms have errors as large as 10^{-11}, some as large as 10^{-10}, and a very few terms of long period have larger errors.... Our results show that Brown's solution is even better in many respects than he had hoped when he made it, and the freedom from error in his work is truly phenomenal.... In our comparison we found only one correction as large as $0''.01$, eight as large as $0''.005$, and 51 as large as $0''.002$. The outstanding correction is that in y with argument $2F - 2\ell$ which corresponds to a correction in the longitude of approximately $0''.072 \sin(2F - 2\ell)$.[360]

The *Improved Lunar Ephemeris* or ILE was derived from Brown's theory, but with certain losses of precision. Therefore, it did not reflect the full accuracy of Brown's solution. Losses in precision occurred in Brown's transformation from rectangular to polar coordinates, in his special transformation of the longitude series for use in obtaining the terms in the latitude, in changes he made in the arbitrary parameters, and in his omission of some small terms in the parallax.

In a paper published in 1966, Eckert and colleagues at the Watson Laboratory replaced Brown's transformations with others more precise, with a view to making the accuracy of Brown's theory more fully available for comparison with observation.[361] The improved transformations led to corrections to the ILE values for longitude, latitude, and sin parallax. The corrections brought Brown's theory and the Eckert-Smith solution of the Main Problem more closely into agreement. The improvement in the accuracy and precision of sin parallax was particularly notable. Earlier, parallax had been used only in the correction of angular coordinates for the position of the observer, and Brown had relaxed the standard of precision in his formulas for determining this coordinate.

In December of 1967, W.J. Eckert and Dorothy Eckert published an article on "The Literal Solution of the Main Problem of the Lunar Theory."[362] Their purpose was to carry the solution of the main problem to a higher level of precision than Hill and Brown had attempted. In the Hill–Brown procedure, the forces containing the ratio of the mass of the Earth-Moon system to the mass of the Sun, and the small terms depending upon the ratio of the mass of the Moon to the mass of the Earth, had been neglected in the solution of the main problem, and treated instead as

[359] W.J. Eckert, "Improvement by Numerical Methods of Brown's Expressions for the Coordinates of the Moon," *Astronomical Journal*, *63* (1958), 415–418.

[360] W.J. Eckert and H.F. Smith, Jr., "The Solution of the Main Problem of the Lunar Theory by the Method of Airy," *APAE*, *19*, Part II, 196.

[361] W.J. Eckert, M.J. Walker, and D. Eckert, "Transformation of the Lunar Coordinates and Orbital Parameters," *Astronomical Journal*, *71* (1966), 314–332.

[362] *Astronomical Journal*, *72* (1972), 1299–1308.

perturbations of this solution. But these effects could be treated by exactly the same method as that used in solving the Main Problem. The results were merely additions to the coefficients of terms with the same arguments. The same machine programs could be used for both the complete and the simplified equations, and the inclusion of the added terms would yield high accuracy with little work. The new equations of motion incorporated two new numerical parameters,

$$q = \frac{E + M}{m' + E + M}, \quad \text{and} \quad p = \frac{EM}{(E - M)^2} = \frac{M/E}{(1 - M/E)^2}.$$

Both q and p are small relative to unity. When they are deleted from the new Eckert equations the latter reduce to the Hill–Brown form. The coefficients a_i of the Variation Curve were computed by an iterative process carried out to the stage of reducing the corrections to less than 1×10^{-20}. The motions of the perigee and node were also derived from the new solution and compared with Brown's values.

In what would be his last major project, Eckert in 1967 undertook the construction of a new solution for the "Main Problem" of the lunar theory, aiming for a higher level of accuracy than ever before. He was assisted by Sarah Bellesheim, who wrote the computer programs under Eckert's direction. After Eckert's death in 1971, Bellesheim carried this project forward, and completed new tables in 1975.

The Eckert-Bellesheim tables had a sequel to which we shall return later. At this point our story shifts to a different development, taking place on the opposite side of the American continent. In Pasadena, California, Jet Propulsion Laboratory during the 1960s was planning and executing spacecraft missions to the Moon, Venus, Mars, and Mercury. This organization had begun under Army jurisdiction in the 1930s as a research group concerned with rocket launches and guided missiles. Its regular mode of trajectory computation was numerical integration of differential equations.

In the issue of *Science* for May, 1968, three consecutive reports appeared, two from JPL and one from MIT. These reports gave evidence of worrisome errors in the lunar ephemeris supplied by the Nautical Almanac Office – worrisome, at least, to engineers concerned with sending out spacecraft to orbit around, or land on, the Moon. The critiqued ephemeris was an updated version of ILE, designated LE4 by JPL.[363] The first report, by J.D. Mulholland and C.J. Devine of JPL, compared LE4 to a numerical integration, designated LE5, of the equations of the Moon over the period from 25 April 1966 to 26 April 1968.[364] They found maximum residuals of 0.16 arcseconds in longitude and 0.12 arcseconds in latitude, corresponding to errors in LE4 of roughly 300 and 225 m. The standard deviations for these coordinates were 0.057 and 0.053 arcseconds, corresponding to 143 and 133 m. The results in sine parallax showed a maximum error of 0.0047 arcseconds, with a standard deviation of

[363] "Supplement to the A.E. 1968" in *The American Ephemeris and Nautical Almanac for 1968* (U.S. Govemment Printing Office, Washington, DC, 1966). The improvements included those given in "Transformation of the Lunar Coordinates and Orbital Parameters," *Astronomical Journal*, *71* (1966), 314–332.

[364] J.D. Mulholland and C.J. Devine, "Gravitational Inconsistency in the Lunar Theory: Numerical Determination," *Science*, *160* (May, 1968), 874–875.

0.0018 arcseconds, corresponding roughly to 500 and 200 m in range. The residuals contained distinct periodicities, closely resembling several planetary arguments, such as Earth + Venus (140 days) and the synodic periods of Venus (600 days) and Jupiter (400 days). The authors concluded that the planetary portion of the lunar theory needed to be recalculated. The error of 0.16 arcseconds in the Moon's longitude corresponded to an error in the determination of Ephemeris Time of 0.30 s.

Lunar laser-ranging would not be available until retro-reflectors were in place on the Moon's surface. The first of these was installed during the lunar landing from Apollo 11 in the summer of 1969. The maximum errors of lunar laser-ranging would be of the order of a few centimeters.

The second report confirmed evidence of error in the corrected Hill–Brown theory as compared with range and Doppler observations of space probes near or at the Moon's surface.[365] The residuals of the predictions from LE4 were as large as 440 m in position and 1.5 mm per second in velocity.

The third report compared the Doppler shift of radar waves reflected from the Moon with radial velocities derived from LE4.[366] The comparison again disclosed large discrepancies, averaging about 0.6 cm per second, between the observed radial velocities and the predictions derived from LE4.

The main conclusion to be drawn was that lunar ephemerides, as derived from Brown's theory and improved upon by Eckert up to the mid-1960s, were not accurate enough for spacecraft navigation. The greater accuracy of JPL's L5 was due to the precision and accuracy of new empirical data on the Moon's *distance*. In the past, all observations of celestial bodies were optical, and the dimension of depth or distance had to be determined by triangulation. Distances were a good deal less precisely measurable than right ascensions or declinations. But now the situation was reversed: radar ranging, very-long-baseline interferometry, spacecraft ranging, Doppler techniques, and finally lunar laser-ranging, were more precise by several orders of magnitude than observations of right ascension and declination. The Nautical Almanac Office had not sought to acquire these new types of data, so crucial to obtaining precise initial conditions for JPL's numerical integrations. Account could be taken of thousands of data-points. The engineers at JPL had long been using numerical integrations to compute trajectories of rockets and missiles, and they applied the same technique when it came to determining the paths of spacecraft. The newer high-speed computers made possible the extension of the numerical integrations to an arbitrary number of simultaneously interacting bodies. Such were JPL's advantages when it came to producing ephemerides. Moreover, accurate prediction was now a necessity, given the mission of sending astronauts into space. It was no surprise that JPL initiated its own work on ephemeris production.

At the Naval Observatory, G.M. Clemence, Scientific Director from 1958 to 1963, made the decision to limit the institution's role in space-age projects. To have

365 C.N. Cary and W.L. Sjogren, "Gravitational Inconsistency in the Lunar Theory: Confirmation by Radio Tracking," *Science*, *160* (May, 1968), 875–876.

366 C.R. Smith, G.H. Pettengill, I.I. Shapiro, and F.S. Weinstein, "Discrepancies between radar Data and the Lunar Ephemeris," *Science*, *160* (May, 1968), 876–878.

decided otherwise would have meant a radical change in a tradition-sanctioned style and scope of operations. The observatory was cooperative in many phases of the new developments spearheaded by NASA. But one consequence of Clemence's decision was that in 1984 the responsibility for planetary and lunar ephemerides in the national almanacs passed from the Nautical Almanac Office at the Naval Observatory to JPL in California.[367]

In the years leading up to this transfer, JPL produced a succession of ephemerides of the Moon and planets. The integrations were fitted primarily to recent measurements of position: radar ranges, planetary spacecraft positions, and lunar laser ranges. Radar ranges to the surface of the terrestrial planets (Mars, Venus, Mercury) had been measured since 1964. Laser ranges to the Moon, as already remarked, began in 1969. In 1976 the Viking landers on Mars began returning ranges with accuracies better than 10 m. The transition from optical angles to Viking ranges improved the accuracy of position determinations for Mars by more than four orders of magnitude. In the case of the Moon, the shift from optical angles to laser ranges improved precision by nearly as much. These improvements drove corresponding improvements in the accuracy of planetary and lunar ephemerides.

In 1977, JPL completed Development Ephemeris 102, the result of a simultaneous numerical integration of equations for the Moon and nine planets covering the time-span from 1411 B.C. to A.D. 3002. An account of this ephemeris was published in 1983.[368] The authors made the following claim:

> While the fitting of optical data was long accomplished with analytical theories for the Moon and planets, the newer data types required the development of numerical integration techniques and more comprehensive physical models. The numerical integrations are necessary to match the accuracy of the modern data types.[369]

The point made here is a practical one, having to do with the benefits of an extended process of trial and error, hypothesis testing, and adaptive refinement, to which JPL's procedures were especially well suited.

A case in which experiment played a decisive role was the decision to compute perturbations due to the asteroids Ceres, Pallas, Vesta, Iris, and Bamberga. "These were the five asteroids found to have the most pronounced effect on the Earth-Mars range in an integration from the standard 1969 epoch of initial conditions to 1985."[370] If the Earth and Mars were perturbed detectably by these asteroids, the same must be true of the Moon.

The main line of attack in the integrations was relativistic. The equations of motion for the Moon and planets were post-Newtonian:

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367 For an account of this transition, see S. J. Dick, *Sky and Ocean Joined*: *the U.S. Naval Observatory, 1830–2000* (Cambridge: Cambridge University Press, 2003), 530–536.

368 X.X. Newhall, E.M. Standish, Jr., and J.G. Williams, "DE 102: A Numerically Integrated Ephemeris of the Moon and Planets Spanning Forty-Four Centuries," *Astronomy and Astrophysics*, *125* (1983), 150–167.

369 Ibid., 150.

370 Ibid., 151.

> The principal gravitational force on the nine planets, the sun, and the Moon is modeled by considering those bodies to be point-masses in the isotropic, Parameterized Post-Newtonian (PPN) n-body metric.... The n-body equations were derived from the variation of a time-independent Lagrangian action integral formulated in a non-rotating solar-system barycentric Cartesian coordinate frame.[371]

In each equation of motion for the interacting bodies taken as point-masses, the terms giving the gravitational forces look familiarly pre-relativistic:

$$\ddot{r}_i = \sum_{J \neq i} \frac{\mu_j (r_j - r_i)}{r_{ij}^3},$$

where r_i is the position vector of body i with respect to the solar-system barycenter, μ_j is Gm_j where G is the gravitational constant and m_j is the mass of body j, and r_{ij} is $|r_j - r_i|$. But the terms in this sum are then modified by multiplicative factors and additive terms which have c^2, the square of the speed light, as denominators. These modifying factors and terms introduce relativistic space curvature and non-linearity in the superposition of gravity. Relativity thus enters into the basic structure of the theory.

Figure effects were taken into account as zonal harmonics of the Earth acting on the point-masses of the Moon and Sun, and as zonal and tesseral harmonics of the Moon acting on the point-masses of Earth and Sun.

Tidal acceleration of the Moon was modeled by taking into account the attraction of a tidal bulge leading the Earth-Moon line by a phase angle δ. In DE 102 the resulting inertial acceleration of the Moon was derived quantitatively from the conservation of the center of mass in the system of bodies considered.

The equations of DE 102 included differential equations for the physical librations of the Moon. In an earlier integration, in which the lunar librations were modeled by an analytic formula, the JPL investigators found a secular runoff in the Moon's longitude. Computing the librations from the differential equations caused the runoff to disappear. But the secular instability turned out to be ultimately traceable to imperfections in the modeling of long-period additive and planetary terms. When these imperfections were removed, analytic modeling of the physical librations no longer produced secular instability.

DE 102 was the end-result of a series of integrations and least-squares fits. Separate least squares fits for planetary and lunar data were carried out in alternation. First the planetary data were used to obtain new planetary starting conditions. Then the latter conditions were integrated with old lunar starting conditions. The planetary initial conditions were then held fixed while several iterations of lunar fits and joint integrations were performed until the lunar orbit had converged. In one of the intermediate iterations, a simultaneous rotation of the lunar and planetary orbits was included to bring the Earth's equator into alignment with the ephemerides.[372]

[371] Ibid., 151.
[372] Ibid., 159.

In the case of the planets, the orbital elements implicit in DE 102 were necessarily osculating or instantaneous rather than mean values. For the Moon, the mean orbital elements affecting its geocentric distance were strongly determined by the laser ranging data.

The mean motions of the planets could be obtained from distance-determinations because the product a^3n^2, where a is the semimajor axis of the ellipse and n the mean motion, is a solar system constant. The first range data sent back from Mars by the Viking orbiter made possible corrections to the mean motions of the four inner planets by $-0''.5\,\mathrm{cy}^{-1}$ [per century]. The astronomical unit was reduced by 700 m, with a resultant decrease in the planetary radii by several 100 m.

The planets in their elliptical orbits undergo periodic displacements from circular motion. If e is the orbital eccentricity, the displacements have amplitude ae in the radial direction, and amplitude $2ae$ in the longitude direction. The projections of these displacements on the range direction could be measured, permitting determination of both the amplitude and the phase of the displacements. The phase gives the mean anomaly (the difference between the planet's longitude and the longitude of its perigee), and the amplitude gives ae.

The foregoing account will give some sense of the impetus of the ongoing ephemerides program at JPL. JPL's ephemerides were of unprecedented accuracy and precision, but their accuracy depended on close control by contemporaneous data. Displacement of the epoch from the center of the data-span led to error.

In the production of ephemerides for the guidance of space flights, the Nautical Almanac Office was hardly in a position to compete with JPL. Eckert had worked steadily toward improving the accuracy of the Hill–Brown theory and of the ephemerides derived it, and this effort was continued after his death in 1971. But the new and more accurate data available to JPL were not accessed in this effort, and the accuracy achieved was orders of magnitude less than that achieved by JPL. Organizationally, the two enterprises were disparate: JPL's was the larger, with more personnel, and it ran on a tight schedule tied to a projected program of spacecraft missions. Given the divergence in technical traditions at the two institutions, melding did not look like a viable option. Clemence as Scientific Director of the Naval Observatory chose to focus the institution on what he believed it could do best and most helpfully in the changed circumstances of the Space Age: positional astronomy and Time service.[373]

[373] See S. J. Dick, *Sky and Ocean Joined* (Cambridge: Cambridge University Press, 2003), Section 10.3: Positional astronomy in the Space Age, 414–450; Section 11.3: Time service in the Space Age, 487–503

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The Mathematical and Philosophical Interest in an Analytic Solution of the Lunar Problem

If an analytic or semi-analytic solution is sought for the motion of the Moon, the problem, mathematically considered, is typical of problems in which the series obtained are not known to be convergent, but yet appear to offer useful results. Not differently than the numerical lunar theory pursued by JPL, analytical lunar theory thus presents itself as an essentially pragmatic enterprise. A provocative question has been whether it is possible by analytical or semi-analytical integrations to achieve the same level of accuracy as JPL had achieved in its numerical integrations.

In the late 1960s, André Deprit developed computer software for managing symbolic operations; he was one of the first to do so. Using Lie transforms, he carried out an analytic integration of the lunar problem in the manner of Delaunay.[374] Whereas Delaunay had computed all terms to the seventh order of small quantities, and some to the eighth and ninth order, Deprit aided by the computer carried the development of all terms to order 20. Comparing his own results with Delaunay's, he provided the first detailed list of Delaunay's errors, showing that they were few. His aim had been to compute all terms with an accuracy of $0''.0005$. For a number of terms, the computation to order 20 was insufficient to reach this goal; it left uncertainties between $0''.001$ and $0''.005$, corresponding to between 2 and 10 m on the orbit of the Moon. In these cases Deprit provided estimates of the uncertainty. An ephemeris computed from Deprit's theory received the acronym ALE ("Analytical Lunar Ephemeris"). The slow convergence that Delaunay had encountered in the 1870s was thus re-encountered by Deprit in his more advanced, computer-aided integration. The result of iterations to order 20 still failed to match the precision of JPL's lunar ranges. Deprit and his co-workers Jacques Henrard and Arnold Rom decided it would be useful to combine the advantages of the analytical method using Lie transforms with the advantages of Hill's proposal to start from a solution of a simplified form of the problem. They proposed starting with an analytical solution of "the non-planar Hill's problem." By this they meant a solution that took account of the eccentricity (e) and inclination (γ) of the lunar orbit, but left unaccounted for the

[374] A. Deprit, J. Henrard, and A. Rom, "Lunar Ephemeris: Delaunay's Theory Revisited," *Science*, *168* (1970), 1569–1570, and "Analytical Lunar Ephemeris: Delaunay's Theory," *Astronomical Journal*, *76* (1971), 269–272.

C. Wilson, *The Hill–Brown Theory of the Moon's Motion*, Sources and Studies in the History of Mathematics and Physical Sciences, DOI 10.1007/978-1-4419-5937-9_25,

solar eccentricity (e') and parallax ($\alpha = a/a'$) of the lunar orbit.[375] The Sun was in effect removed to an infinite distance, but endowed with a gravitational force equal to that which it actually exerts on the Moon. In a second step, the difference between the solution of the non-planar Hill's problem and a solution to the Main Problem was to be expanded around mean values of m, e, γ and the corrections to the solution would be sought in the form of formal power series in $e', \alpha, \delta m, \delta e, \delta\gamma$. Thus the expansions in m would be replaced by expansions in δm, which should converge much more rapidly.[376] An ephemeris derived from this development of the theory received the acronym SALE ("Semi-analytical Lunar Ephemeris").

Comparing SALE with ALE, Henrard found 13 terms in the longitude and 7 in the latitude with differences equal to or greater than $0''.0005$. These differences, he believed, were attributable to truncation errors in ALE.

In 1979, Martin Gutzwiller published "The Numerical Evaluation of Eckert's Lunar Ephemeris."[377] The ephemeris in question was derived from the lunar theory that Eckert and Bellesheim had begun constructing in 1967, and that Bellesheim had completed in 1975. In method, the theory followed along the lines of Brown's theory as completed in 1908. The ratio m of the mean motions was treated as a numerical parameter, and so were the two mass-ratios

$$\frac{M}{M+E}, \quad \frac{M+E}{M+E+S}.$$

The two eccentricities (e and e'), the orbital inclination (γ), and the ratio of the mean distances ($\alpha = a/a'$) were treated as literal parameters. Each polynomial coefficient was calculated for seven different sets of the three numerical parameters, so that the observational values of these parameters could be easily interpolated. The calculations were carried consistently to the sixth order in the four literal parameters; that is, supposing the characteristic of a term to be $e^p(e')^q\gamma^r\alpha^s$, all terms with characteristics such that $p+q+r+s \leq 6$ were computed with high accuracy.

From this theory Gutzwiller calculated an ephemeris, to which he gave the acronym ELE. The chief labor here was the derivation, from the rectangular coordinates given by the Eckert-Belleshim theory, of the corresponding terms in polar coordinates. Gutzwiller obtained the longitude and latitude terms to $0''.0001$ and the sine parallax terms to $0''.000001$. These levels of precision were commensurate to those in ALE, SALE, and ILE (Eckert's *Improved Lunar Ephemeris* of 1954), except that Henrard had given the longitude and latitude in SALE to $0''.00001$. Gutzwiller compared ELE with ALE and SALE. First, he showed that the differences (ELE minus ALE) were fewer and smaller than the differences (ILE minus ALE). Henrard had already carried out a detailed comparison of SALE with ALE.[378] Gutzwiller

375 J. Henrard, "Hill's Problem in Lunar Theory," *Celestial Mechanics*, *17* (1978), 195–204.

376 J. Henrard, "A New Solution of the Main Problem of Lunar Theory," *Celestial Mechanics*, *19* (1979), 337–355.

377 *Astronomical Journal*, *84* (1979), 889–899.

378 J. Henrard, *A New Solution to the Main Problem of Lunar Theory* (Namur, Facultés Universitaires de Namur, 1978).

concluded that ELE's agreement with SALE was as good as ALE's agreement with SALE.[379]

In the early 1980s, the Chapront theory made its appearance. In 1974, Michelle Chapront-Touzé had set forth a plan for developing lunar theory in the form

$$\sum_{i_1,i_2,i_3,i_4} \left[A_{i_1,i_2,i_3,i_4} + \sum_j B^j_{i_1,i_2,i_3,i_4} \delta x^0_j \right] \cdot \begin{pmatrix} \cos \\ \sin \end{pmatrix} (i_1 \bar{D} + i_2 \bar{F} + i_3 \bar{\ell} + i_4 \bar{\ell}')$$

> where A_{i_1,i_2,i_3,i_4} and $B^j_{i_1,i_2,i_3,i_4}$ are numerical coefficients, and δx^0_j are literal variations of the constants used for constructing the theory.

The execution of this plan led to a semi-analytical solution from which a century-long ephemeris was deduced. This ephemeris was compared to JPL's numerical integration LE 200. From the comparison, M. Chapront-Touzé and J. Chapront derived a new set of lunar and solar elements, S_{200}, leading to a new semi-analytical lunar theory, ELP 2000-82. ELP stands for *Ephemeride Lunaire Parisienne*, and the number 2000 indicates that the epoch of the ephemeris is the year 2000. The lunar ephemerides derived from ELP 2000-82 were introduced into the *Connaissance des Temps* (the French equivalent of the *Nautical Almanac*) beginning with the year 1984.

Several versions of ELP 2000 had been developed, and in 1984, ELP 2000-82 was the latest and most accurate of them. As its authors pointed out, ELP 2000 could be numerically improved by adding the difference ρ between ELP 2000 and a given numerical integration over one century. The semi-analytical theory differs from an ephemeris in that it gives explicit formulas for the terms in longitude, latitude, and sine parallax. It owes its refined numerical accuracy, however, to comparisons with the JPL ephemerides. In comparisons of ALE, SALE, and ELE with ELP 2000, all three came up short. The accuracy of ELP 2000, as refined by comparisons with the JPL ephemerides, was superior. It became a standard of comparison for checking the results of other methods.

According to Dieter Schmidt writing in the 1990s, an important advantage of an analytical or semi-analytical integration of the lunar theory over the numerical integration lies in its providing a better understanding of the Moon's dynamical behavior.[380] The theory makes possible the detection of effects that are sensitive to changes in the physical parameters:

> Many of these parameters could be determined more accurately if a better analytic solution were available. The solution of Chapront may satisfy this need some day, but at the moment it requires an independent verification of its own. This can be accomplished with another analytic solution which is developed independently, preferably by different methods and on computers with different hardware.

[379] *Astronomical Journal*, *84* (1979), 896.

[380] D.S. Schmidt, "Computing the Motion of the Moon Accurately," *Hamiltonian Dynamical Systems*: *History, Theory, and Applications*, (eds. H.S. Dumas, K.R. Meyer, and D.S. Schmidt, Springer-Verlag, 1995) 342.

In fact, Schmidt, with the encouragement of Deprit, had already in the late 1870s set out to develop a semi-analytical integration of the lunar theory along the lines of the Hill–Brown theory. In preparation, Schmidt had developed a package of computer programs ("POLYPAK") for the formal manipulation of power series in several real or complex variables. Again with Deprit's encouragement, he joined with Gutzwiller in undertaking a solution of the Main Problem of the lunar theory. Publication of the Eckert-Bellesheim theory and of ELE was abandoned, because Schmidt in his development had already surpassed the precision of ELE. The joint report of Schmidt and Gutzwiller carried the title "The Motion of the Moon as Completed by the Method of Hill, Brown, and Eckert."[381]

In this new solution of the Main Problem, two constants were inserted with their numerical values from the start. One was m$[= n'/(n - n')]$, the ratio of the solar mean motion to the lunar mean synodic motion. The insertion of the numerical value of this constant goes back to Hill–Brown and was continued by Eckert. The other such constant was $M/(E + M)$, which had been measured with high accuracy in the lunar orbiter program of the early 1970s. The constants that were to remain literal in the theory-development were the lunar eccentricity (e), the solar eccentricity (e'), the lunar inclination (γ), and a new constant β now to be defined. Brown and Eckert had failed to realize that the mass-ratio $(E + M)/(S + E + M)$ was not an independent constant. This was so because

$$\frac{E + M}{S + E + M} = \frac{n^2 a^3}{n'^2 a'^3} = \frac{(1 + \mathrm{m})^2 a^3}{\mathrm{m}^2 a'^3},$$

where $a/a' = \alpha$ is the ratio of the lunar and solar mean distances. Convenience now dictated that this mass-ratio be set equal to β^3:

$$\frac{E + M}{S + E + M} = \beta^3.$$

This stipulation makes β approximately 1/70, hence comparable in size to another of the constants, $e'(\sim 1/60)$.

Eckert's original program was modified in other respects. The criterion for including a term in the calculation was no longer its having an abstract order of 6 or less, but rather its numerical size after the values of e, γ, e', and β had been inserted. The new aim was to guarantee 10^{-10} in the complete calculation of a lunar position. Calculation to a still higher order, however, proved necessary because of the rapid accumulation of small terms. In ELE, for instance, Gutzwiller found that the least of the 609 largest terms in longitude was $0''.00012$, but the square root of the sum of the squares of all the smaller terms was $0''.001$, ten times larger.[382] The new criterion for retaining terms in the calculation was that they exceed 10^{-14}. In certain cases, problems with small denominators necessitated keeping terms down to 10^{-17}.

[381] *Astronomical Papers prepared for the use of the American Ephemeris and Nautical Almanac*, U.S. Government Printing Office, Washington, 1986.

[382] M.C. Gutzwiller, "The Numerical Evaluation of Eckert's Lunar Ephemeris," *Astronomical Journal*, *84* (1979), 889–899.

In the solution of the Main Problem, complete agreement with Chapront was at length achieved. Schmidt has stressed the surprising character of this outcome, given that a formal solution of the equations of motion diverges. Ultimate divergence of solutions to the three-body problem had been suggested by Poincaré. The more recent Kolmogorov-Arnold-Moser theorem implied the same thing.[383] Gutzwiller and Schmidt, in their work on the Main Problem, found numerical instabilities cropping up when they sought to deal with small divisors – a sign of the ultimate divergence of the theory. Non-convergence was the 'beast in the jungle' of the lunar problem. But, according to Schmidt,

> What should be more surprising is the fact that one is actually able to calculate a solution to a very high degree of accuracy. This is a consequence of the judicious choice of Hill's intermediate orbit as a starting point and the nature of the phase space in the vicinity of this orbit.[384]

It was familiarity with Euler's last lunar theory that led Hill to his "judicious choice" of the variational orbit as a starting point. The further development of the theory – by Brown, by Eckert, by Gutzwiller and Schmidt – was an experiment requiring mathematical skill and tact, and success was not guaranteed. As Schmidt sums up the situation,

> Despite these difficulties, it is possible to compute a formal solution in the sense of an asymptotic approximation which matches the observational accuracy of the moon today.[385]

Gutzwiller and Schmidt see this result as confirming that the Hill–Brown theory of lunar motions is "one of the great achievements in Celestial Mechanics."[386]

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[383] The Kolmogorov-Amold-Moser theorem is a result in the topological study of Hamiltonian dynamics. For an account of it, see V.I. Arnold, *Mathematical Methods of Classical Mechanics*, (New York NY: Springer-Verlag, 1989) 405–406.

[384] D.S. Schmidt, "Computing the Motion of the Moon Accurately," *Hamiltonian Dynamical Systems*: *History, Theory, Applications* (New York, NY Springer-Verlag, 1995), 359.

[385] Ibid., 361.

[386] P. 13 of "The Motion of the Moon as Computed by the Method of Hill, Brown, and Eckert," as referenced in note 381.

Appendix

"Observations on the Desirability of New Tables of the Moon" (undated typescript of 3 pages, possibly intended for Newcomb; Naval Observatory Library, file of George William Hill)

The tables now in use are those of Hansen modified by the addition of certain corrections due to Prof. Newcomb. As far as practical considerations are concerned, these tables might be used for an indeterminate length of time, without the occurring of errors of a serious character. But the comparison which Prof. Newcomb has made of Hansen with Delaunay shows discrepancies in the values of the coefficients amounting in some cases to half a second of arc. Although, in many cases, these differences are evidently due to the slow convergence of the literal series employed by Delaunay, others remain which are possibly to be attributed to numerical mistakes by Hansen. It is not creditable to the advanced science of the present day that we should be in any uncertainty in this respect. Therefore I think that Professor E.W. Brown should be encouraged to carry on the new computation he has commenced in the Polar Perturbations. Aid should be given in order that we may have the results sooner. In order to reach a degree of approximation which would satisfy all reasonable wishes, I think the terms we should stop at should be of the order of e^6 and γ^6.

As regards the planetary perturbations, M. Radau's work seems well done and may be adopted; but it might be well to subject his selection of arguments to be treated to tests to see whether he may not have overlooked terms of importance. The terms due to the Figure of the Earth might be taken from my memoir [Hill is referring to his paper, "Determination of the Inequalities of the Moon's Motion which are produced by the Figure of the Earth...," *Astronomical Papers for the use of the American Ephemeris and Nautical Almanac*, Vol. III (1884), pp. 201–344]; but in a few cases slight imperfections have crept in, which I will gladly remove when called upon.

It seems to me desirable that in treating this subject, we should start from a foundation reasonably certain in its details, all known forces being taken correctly into account. The comparison of such a theory with observation will give residuals which are the combined effects of the necessary changes in the values of the arbitrary constants and the action of the unknown forces. The latter undoubtedly exist, and I am afraid the period of observation is too short to show their real law. We will probably be driven to resort to empirical formulae. The latter, however, should be of

C. Wilson, *The Hill–Brown Theory of the Moon's Motion*, Sources and Studies in the History of Mathematics and Physical Sciences, DOI 10.1007/978-1-4419-5937-9,

as simple a nature as possible, and should not contradict our present knowledge. Prof. Newcomb's modification of the Venus-inequality of 273 years by altering the argument by 70° is open to serious objection. However well it may answer at present, it is certain that, in the near future, the observations are going to march away from such a theory.

The most probable cause that can be assigned to the outstanding residuals of the above mentioned presumably accurately determined theory is the mass of meteors passing through the regions immediately surrounding the Tellurian System. It may be difficult to arrive at formulae giving the effect of these in mass on the position of the Moon relatively to the Earth; but in the rough, some kind of periodicity must shape it. If we could find the periods of the terms compassing it, the observations might be capable of furnishing the coefficients and phases of these terms.

Further tinkering of Hansen seems not desirable as the result would be most likely only a temporary makeshift to be supplanted by something better.

I estimate that on this plan new tables could be prepared and ready for use in ten years. Of course, sufficient computing force must be given to the undertaker of this project, perhaps three persons might suffice. The heaviest part of the work is the comparison of the theory with observation. To pass from Hansen to a theory absolutely unencumbered with empiricism is a matter of difficulty. It is not even certain that the figures in the tables are actually founded on the formulae of the introduction.

The form to be given to the tables is a matter of some moment. M. Radau's investigation shows the planetary perturbations are composed of numerous small terms with the most various arguments and which it would be impossible to tabulate singly; tables to double entry would scarcely succeed better. The only course open would seem to be to tabulate them in lump specially for given times.

G.W. Hill

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