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William V. Gehrlein

CONDORCET'S PARADOX

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William V. Gehrlein

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Professor William V. Gehrlein

University of Delaware
Department of Business Administration
Newark, Delaware 19716-2710
USA
wvg@udel.edu

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To Barb

Preface

Condorcet's Paradox has been formally studied by an amazing number of people in many different contexts for more than two centuries. Peter Fishburn introduced the basic notion of the Paradox to me in 1971 during a course in Social Choice Theory at Pennsylvania State University. My immediate response to seeing the simple example that he presented was that this phenomenon certainly could not be very likely to ever be observed in reality. Peter quickly suggested that I should work on developing some representations for the probability that the Paradox might occur, and very soon thereafter that pursuit began. It is only after 35 years of effort, with a lot of help from Peter, that I now feel that a good answer can be given to the challenge that was presented in that classroom in 1971. Many people have suggested to me over the years that a book like this should be completed, since the source material is spread over such a wide variety of disciplines of academic journals and books that it is very difficult for people to know what has been done, and has not been done, in this area of determining representations for the probability that Condorcet's Paradox would ever be observed in reality.

The advent of efficient computer search engines that cover large groups of academic journals made the idea of pursuing this project seem much more tractable, but it was only after starting the search that I realized just how difficult this project was going to be, since the number of papers that make a reference to Condorcet's Paradox is truly enormous. As a result, the original scope of this study was significantly reduced to simply focus on the consideration of the existence of the Paradox, and factors that affect the probability that it might be observed in real situations. The availability of the on-line Social Choice Bibliography that Jerry Kelly developed and maintains at Syracuse University has also been extremely helpful in locating sources. A significant effort has been made to find all of the available relevant sources on this specific topic, but some of them undoubtedly have been missed. Apologies are extended in advance to those whose relevant work might have been inadvertently overlooked.

The primary motivation that has led to the continuation of my own work in this area of research over such a long period of time has come from opportunities to present the results of this work at various meetings, seminars and workshops, where very valuable feedback and encouragement have often been obtained. Many different universities and organizations have generously supported this effort, and I am very grateful to all of them. In particular, Dominique Lepelley (currently at University of La Reunion), Vincent Merlin and Maurice Salles have been very generous in sponsoring my visits to University of Caen on more occasions over the years than I can possibly recall.

On a personal note, I am also very grateful to several people who have routinely given positive reinforcement and encouragement to keep me working on this project over the several years during which it has evolved, despite my frequent, disconcerting, and sometimes overwhelming belief that it would never actually be completed during my lifetime. Most notable on this list are Barbara E. Eller, Burton A. Abrams, John F. Preble and Arthur A. Sloane.

It was with great amusement that I accidentally stumbled over a quote from Condorcet regarding the problem that has kept me almost fully occupied for many years [Condorcet (1793a, pg. 7)]:

> "But after considering the facts, the average values or the results, we still need to determine their probability."

William V. Gehrlein
University of Delaware
Newark, Delaware
May 2006

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1.1 Introduction

The problem of considering how a group of individual decision-makers should go about selecting some overall best alternative from a set of available alternatives has been studied in various forms for many years. Our study of this problem begins with a historical overview of the development of early thinking about how this exercise in decision-making should be performed. The ultimate interest of this chapter is to consider some unusual occurrences that can be observed in these decision-making situations and to develop an outline of early work that led to the analysis of this problem with mathematical techniques.

Any group decision-making situation can be viewed in the context of an election in which the available alternatives correspond to the candidates that are being considered for selection, and where the alternative that is selected as the overall best alternative corresponds to the winning candidate in the election. The individual decision-makers within the group are acting as voters in this scenario. In a famous early paper dealing with election procedures, Jean Charles de Borda (1784, pg. 128) clearly makes this point in the concluding statements of a written commentary that summarized a presentation that he made to the French Academy of Science on June 16, 1770:

> "In conclusion, I must stress that everything we have said here about elections also applies to any debate conducted by any company or body of men; these debates are really no more than a type of election between the different options put forward and are therefore subject to the same rules."

Throughout the current study, we examine the process of how groups of individual decision-makers might go about selecting an alternative in the context of election procedures, to try to determine which types of election procedures tend to make the most sense for use in different types of situations.

We typically restrict attention to elections in which all of the voters in a decision-making group have the same input to the voting process. That is, no subgroup of individual decision-makers within the total group has more influence on the outcome of the voting process than does any other subgroup with the same number of voters, once individual voter's preferences on candidates have been formed. This does not preclude the possibility that some individuals might be more persuasive than others in arguing for their particular viewpoint during debate, while individual voters are evaluating the candidates. However, once the individual voters have determined their particular preferences on the candidates, each voter will have the same influence on the outcome.

If all voters have the same most preferred candidate in a given election, then the determination of the winner is a very simple task. The preferred candidate is selected as the winner, and all voters will get their most preferred outcome. The difficulty arises in the much more likely scenario that there is some disagreement among the voters as to which candidate is best. Once this situation arises, all of the individual voters cannot get what they most prefer as an outcome, so the determination of which candidate best represents the overall most preferred candidate of the group becomes an issue.

In the case of only two candidates, a group of decision-makers will almost certainly arrive at the conclusion of applying the notion of majority rule, so that the candidate that is more preferred by the greater number of voters will be selected as the winner. With equal voter influence in the process, a sense of fairness suggests that the group should select that candidate, in order to provide the better outcome for the most voters. Rousseau (1762) presents a detailed analysis of the issue of the fairness of majority rule voting.

Young (1988) summarizes the thoughts of Rousseau (1762) regarding majority rule voting as follows. Rousseau's opinion was that the "general will" of the majority should serve as the legitimate norm for making group decisions. And, any particular individual who is voting in an election can be viewed as trying to decide which candidate is most in conformance with the "general will" of the electorate. If any individuals vote for a candidate that ultimately is not elected, then these individuals are viewed as being incorrect in their view of which candidate is most in conformance of the "general will". The use of majority rule therefore reflects the view of what most voters perceive as conforming to the "general will" of the electorate. This belief in the fairness of majority rule is not held universally, with arguments against it typically attacking it for ignoring the intensity of preferences of voters.

Don Joseph Isadore Morales of Spain wrote a paper after reading about the work in Borda (1784), and submitted a paper to the French Academy of Science. The content of this paper is discussed in Daunou (1803). One of Morales' arguments was that strength of preference must be considered in voting procedures. In particular, situations could exist in which there is a minority group of voters who have a very strong preference that an issue should be adopted, while the majority of voters are marginally opposed to having it adopted. If the sizes of the two voting groups were nearly equal, Morales' arguments would suggest that the strong preference of the minority should outweigh the majority opinion in such a case. In order to account for this, voting procedures would have to ask individual voters to report some measure of their degree of preference for candidates, as opposed to asking for simple approve or disapprove responses.

Daunou (1803, pg. 244) makes his opinion of Morales' arguments about considering intensity of preferences in voting very clear:

"A strong will is already too powerful on its own. While society might owe some outstanding benefits to strong-willed men, it also owes them a greater number of infamous disasters and above all an infinite number of smaller problems. …. The first condition in any debate should therefore be that all votes have equal value, whatever their consistency, scope and strength,"

A much earlier quote that is directly related to this argument is attributed to Pliny the Younger of ancient Rome. Gaertner (2005, pg. 235) reproduces the quote:

"…votes go by number, not by weight; nor can it be otherwise in a public assembly, where nothing is so unequal as the equality that prevails in them."

The debate about giving consideration to the strength of voters' preferences has not been resolved since the early work that has been cited. More recently, Vickery (1960) discusses the difficulties that would be involved in giving different weights to individual voters, based on their strength of preference. It is argued that voters have significant problems simply in correctly determining any actual differences that exist between candidates, without even considering the additional complexity that would result if individual voters also attempted to evaluate their strength of preference. On the other hand, Tullock (1959) and Ward (1961) argue that the use of majority voting in an election can select the wrong outcome if intensity of preference is not considered. The main argument in Tullock (1959) is that a minority of voters with strong preferences, facing a majority of voters with less intense preferences, can only get what it desires by resorting to vote trading, which can lead to irrational outcomes. The notion of vote trading and the possible irrational outcomes that can result from using it will be developed in detail later. Downs (1961) presents arguments that are in opposition to the results in Tullock (1959), suggesting that the assumptions that are used in that study are not realistic.

The notion of using majority rule in two candidate elections is justified on a mathematical basis in the work of Rae and Taylor. Rae (1969) considers the situation in which a group of voters will be faced with a series of votes on policy proposals, and each proposal will be passed or defeated. Voters will have preferences to either support or oppose each of the proposals as they are presented. Rae's analysis determines the size of the majority that should be required to determine if each issue is passed or defeated. It is assumed that the sequence of votes on proposals is unknown, so that it is not known in advance how many voters will support or oppose forthcoming issues. It is also assumed that voters form their preferences independently of all other voters.

Since nothing is known in advance about the issues that are to be presented, it is assumed that each voter has a probability of 0.5 of supporting or opposing any upcoming issue. There are two possible situations that might exist that any given voter would want to avoid. In particular, the voter might oppose an issue that passes, or the voter might support an issue that is rejected. Let P denote the joint probability that either of these events happens to a given voter. Rae (1969) proves that simple majority rule will uniquely minimize P, given the set of assumptions above. Taylor (1969) extends this result to show that simple majority rule uniquely minimizes P for any value of p, where p is the probability that any given voter supports an issue, with 1- p being the probability that the voter opposes an issue. The value of p is assumed to be the same for all voters. Straffin (1977) extends the work of Rae and Taylor to show that majority rule uniquely maximizes the average probability that voters are in agreement with the chosen election outcome for odd n. Fishburn and Gehrlein (1977a) consider the effectiveness of simple majority and other election procedures in two-candidate elections when voters are uncertain of their preferences, and simple majority rule is found to have some attractive properties in such voting situations.

Following the notions behind most of the work in this area, we ignore intensity of preference in the remainder of this study and treat all voters equally. In doing so, we are in agreement with ideas that are proposed by Condorcet (1788a, pg. 155) in his discussion about the necessity of making election procedures as simple as possible:

"We must therefore establish a form of decision-making in which voters need only ever pronounce on simple propositions, expressing their opinions only with a *yes* or a *no*."

1.2 The Case of More than Two Candidates

The problem of selecting the winner of an election becomes significantly more complicated when more than two candidates are being considered, since the concept of majority rule can take on different interpretations in this situation. Much of this work finds its origins in the early studies of Jean Charles de Borda and of Marie Jean Antoine Nicolas Caritat, the Marquis de Condorcet, whose work has already been mentioned. These 18th century French contemporaries were pioneers in the development of formal mathematical studies of election methods, and both found that counterintuitive things could happen when different interpretations of majority rule are considered for elections with more than two candidates. We refer to these unusual occurrences in voting events as voting paradoxes.

Other people obviously considered issues that are related to the process of conducting elections before Borda and Condorcet did so. McLean (1990) discusses observations of Ramon Lull from the14th century and of Nicolas Cusanus from the 15th century that are related to problems of collective decision-making. Similarly, Lagerspetz (1986) and Gaertner (2005) present observations of the writer Pufendorf from the 17th century. All of the notions that are considered in these studies are clearly relevant to issues that are related to conducting elections. However, Borda and Condorcet were the first to formally address these issues from a mathematical perspective, and we begin by developing some of the notions that they brought forward. During this development, we also give a brief outline of the history of the very interesting interaction that took place between these two important figures.

To give formal definitions to the different interpretations of majority rule, we must start by defining the preferences of individual voters. Suppose that we have a set of three candidates, $A, B, C\}$, and that $A \succ B$ denotes that a voter prefers

Candidate *A* to Candidate *B*. A voter's preferences on pairs of candidates from the set of candidates are *complete preferences* if there is a preference on each of the possible pairs. That is, we must have $A \succ B$ or $B \succ A$ for all pairs of candidates like *A* and *B*. When an individual voter's preferences are complete, there is no indifference between any two candidates. We initially assume that all voters have complete preferences on the candidates, and the effect of individual voter indifference between candidates will be considered later. We also assume that each individual voter has *transitive preferences*.

Transitivity is a common requirement in defining rational behavior in the context of the preferences of individual voters. Transitivity requires that if a given voter has $A \succ B$ and $B \succ C$, then that voter must also have $A \succ C$. This prevents situations in which a given voter might respond in a cyclic fashion, such as $A \succ B$, $B \succ C$ and $C \succ A$. The usual argument for transitivity of preference for individual voters falls back on some form of the concept of being able to use such a voter as a 'money pump' if such cyclic preferences exist. Suppose that *A* would be the overall winner of an election in this specific example of cyclic voter preferences. The voter could be given the option to make some small payment to have *C* become the winner instead of *A*. The voter would agree since $C \succ A$. Next, we ask the voter to make a small payment to have *B* become the winner instead of *C*. The voter would agree since $B \succ C$. The voter is then asked to make a small payment to have *A* become the winner instead of *B*. The voter would agree since $A \succ B$. As a result of these transactions, the voter would then have made a series of payments, only to return back to the original situation, with *A* being the winner, to strongly suggest that the voter is not acting rationally.

The notion of using transitivity as one of the standards for rationality for individual voter preferences is nearly universally accepted. However, some studies have considered various models to explain why it might be reasonable to expect intransitivity in individual preferences. Gehrlein (1990a, 1994) presents surveys of much of this work on intransitive individual preferences.

Individual preferences that are complete and transitive are defined as *linear preference rankings*. There are six possible linear preference rankings that each voter might have for three-candidate elections, as shown in Fig. 1.1.

A	*A*	*B*	*C*	*B*	*C*
B	*C*	*A*	*A*	*C*	*B*
C	*B*	*C*	*B*	*A*	*A*
n_1	n_2	n_3	n_4	n_5	n_6

Fig. 1.1 The six possible linear preference rankings on three candidates

Here, n_i denotes the number of voters that have the associated linear preference ranking on the three candidates. That is, n_1 voters all have individual preferences with $A \succ B \succ C$. Of course, we also have $A \succ C$ for these voters, with the assumption of transitivity. If we let *n* define the total number of voters, then $n = \sum_{i=1}^{6} n_i$. Any particular combination of $n_i's$ that sum to *n* will be referred to as a *voting situation*, $\boldsymbol{n}$. Voting situations just report the n_i values that are associated with each possible individual preference ranking for a given election, without specifying the preferences of any individual voter.

Condorcet (1785a) uses the exact same approach in his work. He lists a total of eight possible complete preference structures that individuals might have on three candidates. Six of these structures are the linear preference rankings on candidates that are listed in Fig. 1.1, and two of them represent the cases of cyclic, or intransitive, individual preferences on the candidates. Condorcet notes that these two cyclic preference structures are a "contradiction of terms", to lead to the conclusion that "there really are only six possible options". Condorcet (1788a, pg. 156) later makes his view of the irrationality of individual intransitivity of preference very obvious by stating:

> "Clearly, if anyone's vote was self-contradictory (intransitive), it would have to be discounted, and we should therefore establish a form of voting which makes such absurdities impossible."

We are now able to formally consider two different ways of extending the notion of majority rule to the case of more than two candidates. The most obvious of these extensions is widely known as *plurality rule*. Just as with two candidates, each voter casts a vote for his or her most preferred candidate with plurality rule, and the winner is the candidate who receives the greatest number of votes. Let ***APB*** denote the event that *A* beats *B* by plurality voting. Assuming that all of the voters will cast votes in agreement with their true preferences, *A* will be the winner in a plurality rule election if both ***APB*** $[n_1 + n_2 > n_3 + n_5]$ and ***APC*** $[n_1 + n_2 > n_4 + n_6]$. It is assumed that voters will always vote in accordance with their true preferences throughout this study.

Borda (1784) considers a second extension of majority rule to three-candidate elections, by looking at the basic majority rule relation as applied to pairs of candidates. Let ***AMB*** denote the event that *A* is the majority rule winner over *B* when only *A* and *B* are considered. By ignoring the relative position of *C* in the possible preference ranking for any of the individual voter's rankings in Fig. 1.1, we see that ***AMB*** if $n_1 + n_2 + n_4 > n_3 + n_5 + n_6$, ***AMC*** if $n_1 + n_2 + n_3 > n_4 + n_5 + n_6$, and ***BMC*** if $n_1 + n_3 + n_5 > n_2 + n_4 + n_6$. If ***AMB***, then *A* beats *B* by *Pairwise Majority Rule* (*PMR*). Both Borda (1784) and Condorcet (1785b) refer to *A* as having "plurality support" if both ***AMB*** and ***AMC***. To avoid confusion with the standard definition of plurality rule given above, we refer to *A* as the winner by PMR, or as the *Pairwise Majority Rule Winner* (*PMRW*), for the three-candidate case when both ***AMB*** and ***AMC***. The PMRW is commonly referred to as the *Condorcet Winner* in the literature. If we have ***AMC*** and ***BMC***, then *C* is the *Pairwise Majority Rule Loser* (*PMRL*) for the three-candidate case. These definitions are extended in the obvious fashion when more than three candidates are considered.

Condorcet (1784) comments at length on Borda's earlier work, and the discussion in the paper makes it very clear that Borda (1784) was indeed using unusual terminology while developing his arguments when he makes references to two different forms of "plurality rule", rather than referring to two different forms of "majority rule".

1.3 Borda's Paradox

Borda (1784) makes a very interesting observation regarding a possible outcome of election procedures after developing the notion of using PMR. His original example of the phenomenon uses the voting situation in Fig. 1.2 for 21 voters with linear preferences on three candidates.

$$\begin{array}{cccc} A & A & B & C \\ B & C & C & B \\ C & B & A & A \\ n_1 = 1 & n_2 = 7 & n_5 = 7 & n_6 = 6\,. \end{array}$$

Fig. 1.2 An example voting situation displaying Borda's Paradox from Borda (1784)

The concern that is expressed by Borda in this example is related to the outcome of the election when plurality rule is used to select the winner, versus the outcome when PMR is used. In using plurality rule with the voting situation in Fig. 1.2, ***APB*** (8-7), ***APC*** (8-6) and ***BPC*** (7-6) to give a linear ranking by plurality rule, with ***APBPC***. A very different result is observed using PMR. Here, *BMA* (13-8), *CMA* (13-8) and *CMB* (13-8) to give a linear PMR ranking, with *CMBMA*. With this particular voting situation, plurality rule and PMR reverse the rankings on the three candidates. We refer to this phenomenon as an occurrence of a *Strict Borda Paradox.*

Borda was particularly distressed by the fact that the PMRL would be chosen as the winner by plurality rule, leading to his suggestion that plurality rule should never be used. Borda (1784) also suggests that *C*, the PMRW, "is really the favourite". However, the main concern expressed in Borda's work was the possibility of the negative outcome that the PMRL could be selected as the winner by plurality rule. We define a *Strong Borda Paradox* as a situation in which plurality rule elects the PMRL, without necessarily having a complete reversal in plurality rule and PMR rankings.

Borda blames the possible existence of a Strong Borda Paradox on the failure of plurality rule to allow voters to report their complete preference rankings on all of the possible candidates. An explanation of the phenomenon is given in terms of the particular example that he posed in Fig. 1.2:

> "On reflection, we see that candidate *A* gains the advantage only because candidates *B* and *C* have more or less equally split the 13 votes against him. We might compare them to two athletes who, having exhausted themselves competing against one another, are beaten by a third who is weaker than either."

1.3.1 Actual Occurrences of Various Forms of Borda's Paradox

It is always of interest to know if occurrences of such hypothetical examples like the ones identified by Borda have ever been observed in an actual election, and a number of studies have been conducted to try to find examples in which various forms of Borda's Paradox might have occurred.

Weber (1978a) presents a widely cited example of a Strong Borda Paradox in the 1970 U.S. Senate election in New York State. The three candidates were James Buckley who was endorsed by the Conservative Party, Charles Goodell who was endorsed by both the Liberal Party and the Republican Party, and Richard Ottinger who was endorsed by the Democrat Party. Public opinion polls indicated that the majority of voters were liberal, and that their preferences were split between Goodell and Ottinger. The political conservatives strongly supported Buckley.

There is little doubt that either Goodell or Ottinger would have beaten Buckley by PMR, given the plurality rule percentage votes for the three candidates: Buckley (38.8%), Goodell (24.3%) and Ottinger (36.9%). Buckley would have been the PMRL in this election, but he was elected, based on plurality rule. Riker (1982) presents a similar analysis of this same election and concludes that Ottinger would have been the PMRW.

Riker (1982) performs an analysis of the 1912 U.S. Presidential election, with the three primary candidates being Roosevelt (R), Taft (T) and Wilson (W). Riker reconstructs the probable preference rankings of 93 percent of the voters, with estimates of the percentage of voters that held each of these preference rankings, as shown in Fig. 1.3:

W	R	T
R	T	R
T	W	W
42%	27%	24%

Fig. 1.3 Voting situation for the 1912 U.S. Presidential election. Reprinted from Riker (1982) by permission of Waveland Press, Inc.

Roosevelt would have been the PMRW and Wilson would have been the PMRL, regardless of the preferences of the seven percent of "Other voters" whose preferences could not be determined. However, Wilson was the plurality winner, with 42 percent of the votes, resulting in an example of a Strong Borda Paradox.

Van Newenhizen (1992) gives an example suggesting the possible existence of a Strong Borda Paradox, as observed in the 1988 national elections for Prime Minister of Canada. A critical issue in that election regarded the candidates' stands on the proposition of establishing a free trade agreement between Canada and the United States. Polls showed that approximately 60 percent of Canadian voters were opposed to the establishment of such an agreement, but their votes were divided between two anti-free trade candidates, while the single pro-free trade candidate won the election by plurality rule. Assuming that no other issues of the candidates' platforms dominated the issue of their stand on the free trade agreement, this would seem to constitute an example of a Strong Borda Paradox.

Colman and Poutney (1978) examine survey results of voters' preferences in 261 different three-candidate contests in British General Elections. Complete voter preference rankings are reconstructed from survey results. Two forms of Borda's Paradox are considered in the study. Their 'strong form' is identical to the definition of a Strong Borda Paradox above. Their 'weak form' occurs when ***AMC***, ***BMC***, and either ***CPB*** or ***CPA***. We define this as an occurrence of a *Weak Borda Paradox*. No occurrences of a Strong Borda Paradox were observed, but 14 of the 261 elections exhibited the occurrence of a Weak Borda Paradox.

Forsythe, et al. (1993) conduct an experimental study in which each subject voter in a pool of subjects was given a specified hypothetical preference ranking on fictitious candidates, and elections were then held by plurality rule. The preference rankings that were distributed to subject voters were contrived to force an outcome of a Strong Borda Paradox if voters actually voted according to their given hypothetical preferences. When voters gained information about other voters' preferences through repeated elections, or through the presentation of pre-election poll results, a Strong Borda Paradox occurred infrequently in actual voting by the subjects. Upon observing that the PMRL would win the election, the voters often chose to vote for their second ranked candidate to prevent that outcome. It is therefore concluded that the presence of pre-election polls is likely to significantly reduce the likelihood of actually observing a Strong Borda Paradox in practice, despite the fact that the outcome would result if sincere voting would actually be used.

Bezembinder (1996) considers the possibility that a Strict Borda Paradox, which is called the "Plurality Majority Converse" in that study, might occur with the assumption of a restricting condition on voters' preferences. Statistical analysis of actual voting results is used to consider the possibility that a Strict Borda Paradox might have occurred during voting in the era of the Weimar Germany. The general conclusion is that the election results that were analyzed show a large number of disagreements between plurality rankings and PMR rankings, but they fall short of displaying a Strict Borda Paradox, as it has been defined above. All of these findings lead to the conclusion that Borda's Paradox can exist in its various forms, although it might not be a regularly observed phenomenon.

1.3.2 Borda's Solution to the Possibility of Borda's Paradox

Borda (1784) proposed two election procedures to deal with the possibility that various forms of Borda's Paradox might occur. The first procedure simply calls for using PMR on all pairs of candidates to directly determine the PMRW. With the admission that such a process would be extremely time-consuming, Borda makes a second suggestion. The procedure that he calls "*election by order of merit*" has come to be known as *Borda Rule*. It states that each voter should first rank all of the candidates, and then each voter's most preferred candidate in an m candidate election should receive $a+(m-1)b$ points, the second most preferred candidate should receive $a+(m-2)b$ points, ..., and the least preferred candidate should receive $a+(m-m)b$ points. The winner is determined by summing the points that are received by each candidate from all of the voters, and declaring the candidate with the most points as the winner. Borda suggests using the particular weighting scheme with $a = b = 1$, so that the points awarded to a candidate by a given voter reduces to the rank that the candidate has in that voter's preference ranking on the candidates. Here, a rank of one refers to a voter's least preferred candidate and a rank of m refers to the voter's most preferred candidate.

For a general voting situation as described in Fig. 1.1 with n voters and three candidates, the points scored for A, B and C with Borda Rule with $a = b = 1$ would respectively be $Score(A)$, $Score(B)$ and $Score(C)$ with:

$$\begin{aligned} Score(A) &= 3(n_1+n_2)+2(n_3+n_4)+1(n_5+n_6) \\ Score(B) &= 3(n_3+n_5)+2(n_1+n_6)+1(n_2+n_4) \\ Score(C) &= 3(n_4+n_6)+2(n_2+n_5)+1(n_1+n_3). \end{aligned} \tag{1.1}$$

For the particular example given from Borda (1784) in Fig. 1.2, we obtain $Score(C) = 47$, $Score(B) = 42$, and $Score(A) = 37$. If we let ***ABB*** denote the event that A beats B by Borda Rule, we get a linear ranking on the candidates, with ***CBBBA***. This ranking of candidates in the reverse order of the ranking by plurality rule, and it is in perfect agreement with the ranking that was obtained by PMR. McLean (1990) notes that Nicolas Cusanus suggested a voting rule exactly like Borda Rule, without any mathematical justification for using it, some 400 years before Borda's work.

Some authors have suggested that Borda (1784) was making a claim that Borda Rule would always select the PMRW [see Merlin, et al. (2002), for example]. However, this assertion is not specifically made in Borda (1784). As pointed out in Nurmi (1999, pg. 13):

> "Since this is the only example discussed in Borda's paper, we are left somewhat uncertain about whether Borda at the time of presenting his paper believed that his method would always elect a Condorcet winner (PMRW)."

As stressed before, Borda was primarily concerned with the notion that the PMRL should not be selected as the winner.

Condorcet (1785c) develops the general notion of weighted scoring rules, and Borda Rule is a special case of these types of rules. Weighted scoring rules give some number of points to candidates according to their relative position within individual voter's preference rankings. For three candidates, a general weighted scoring rule assigns three points to a candidate for each most preferred ranking in a voter's preferences, λ points for each second place ranking, and one point for each least preferred ranking. Borda Rule with $a = b = 1$ is a weighted scoring rule with $\lambda = 2$. We restrict $1 \le \lambda \le 3$ since it would not make sense to award more points to the middle ranked candidate in a voter's preference ranking than to the most preferred candidate in the ranking, or to award fewer points to the middle ranked candidate than to the least preferred candidate.

Daunou (1803) presents a simple proof that Borda Rule cannot rank the PMRW in last place in an m-candidate election. Consider the special case of Borda Rule with $a = b = 1$, so that each voter assigns a total of $\frac{m(m+1)}{2}$ points to candidates in an m-candidate election. The total number of points assigned to candidates by all voters is $\frac{nm(m+1)}{2}$, and the average total number of points received by a candidate from all voters is therefore $\frac{n(m+1)}{2}$. It is then argued that if some candidate is the PMRW, that candidate will have the minimum Borda Score for a voting situation in which it is most preferred by $\frac{(n+1)}{2}$ voters and least preferred by $\frac{(n-1)}{2}$ voters for odd n. The PMRW will then have a total Borda Score equal to

$$Score(PMRW) = m\left(\frac{n+1}{2}\right) + \left(\frac{n-1}{2}\right) = \frac{n(m+1)+(m-1)}{2}. \tag{1.2}$$

The score of the PMRW is therefore greater than the average score for all candidates, so some other candidate must have a below average score, and thus the PMRW cannot have the minimum score. A similar argument holds when n is even.

Smith (1973) and Gärdenfors (1973) reproduce this same result with a similar proof, and Smith (1973) shows that for sufficiently large n, voting situations exist such that every weighted scoring rule can rank the PMRW last, except for Borda Rule. Fishburn and Gehrlein (1976a) produce similar proofs and note from symmetry arguments that for sufficiently large n, Borda Rule is the only weighted scoring rule that cannot select the PMRL as the unique winner. As a result of this finding, Borda Rule is the only weighted scoring rule that can meet Borda's criterion of not electing the PMRL as the winner.

Gehrlein (1976) develops a linear programming formulation to show that the PMRL cannot defeat the PMRW for any weighted scoring rule in three-candidate elections when $\frac{2n-4}{n-1} \le \lambda \le \frac{2n}{n-1}$ for odd $n \ge 5$, and with $\frac{2n-8}{n-2} \le \lambda \le \frac{2n}{n-2}$ for even $n \ge 8$. These ranges for λ include Borda Rule for all n. Saari (1992) conducts a more general analysis of weighted scoring rules that use a process of sequential elimination of candidates to obtain a winner to conclude that by using weighted scoring rules, other than Borda Rule, it is possible to give an advantage to the PMRL in winning an election, at the expense of the PMRW.

Borda (1784) also raises the issue of using linearly decreasing points to obtain any given values of a and b in his election by order of merit. In particular, if some voter reports a linear preference ranking with $A \succ B \succ C$, the use of the linearly decreasing 3:2:1 point scale that is suggested by Borda results in the same difference in point values being assigned as we move from any candidate in the voter's preference ranking to the candidate ranked immediately below it. This constant difference in points being given to consecutively ranked candidates inherently assumes that B is not considered as being closer to A in the given voter's true preference spectrum than it is to C. Similarly, we are inherently assuming that B is not considered to be closer to C than it is to A.

This all leads us back to the issue on intensity of preference, when we were considering the use of majority rule in two-candidate elections. If candidates are not actually equally spaced along the preference spectrum of some voter, then one might argue that the use of linearly decreasing weights does not accurately reflect that voter's true preferences in the election outcome. Borda (1784, pg. 124) clearly does not support the notion of making any attempt to account for the intensity of any particular voter's preferences by using a nonlinear system of assigning points to candidates in voters' preference rankings, when he states:

> "Furthermore, because of the supposed equality between the voters, each rank must be assumed to have the same value and to represent the same degree of merit as the same rank assigned to another candidate, or even by another voter."

Laplace (1795) addresses the notion of using linearly decreasing weights with Borda Rule, and justifies the use of such weights from a purely mathematical perspective in general m-candidate elections, with the set of candidates being denoted as $\boldsymbol{C}^m = C_1, C_2,..., C_m\}$. Laplace's arguments are explained here in greater detail than in the original work, since the techniques that he used are directly related to later developments. Consider a model in which each voter represents his or her preference ranking on candidates by assigning points to candidates, with a greater assignment of points to a candidate indicating a greater preference for that candidate. Voters assign as many points to candidates as they wish in order to represent their relative strengths of preference for candidates. Voters then obtain their linear preference rankings on candidates according to the associated ordering of points that they have assigned to the candidates.

Let t_j^i denote the number of points that the i^{th} voter assigns to the j^{th} candidate, and let z define the maximum number of points that a voter might assign to any candidate. Then, each t_j^i can have any real value on the closed interval $[0, z]$, and it is independent of the other t_j^i values. We can assume, without a loss of generality in this argument, that the i^{th} voter has a linear preference ranking on candidates that is given by $C_1 \succ C_2 \succ C_3 \succ ... \succ C_m$. It then follows that we must have a condition on the t_j^i's for that voter such that $t_1^i > t_2^i > t_3^i > ... > t_m^i$. Laplace (1795) proceeds to find the expected value of any given t_j^i when all feasible combinations of t_j^i's with $t_1^i > t_2^i > t_3^i > ... > t_m^i$ are equally likely to be observed.

We begin by considering the "total sum" of the number of combinations of t_j^i's that meet this condition. Given the assumption that all possible combinations of t_j^i's are equally likely to be observed for any voter, we find this "total sum" as $V(t_j^i)$, with

$$V(t_j^i) = \int_{t_1^i=0}^{z} \int_{t_2^i=0}^{t_1^i} \int_{t_3^i=0}^{t_2^i} \cdots \int_{t_m^i=0}^{t_{m-1}^i} dt_m^i dt_{m-1}^i \ldots dt_3^i dt_2^i dt_1^i = \frac{z^m}{m!}. \tag{1.3}$$

As a second step, we find the "total weighted sum" for a given j^{th} value of t_j^i, over the same range of t_j^i's as $V^*(t_j^i)$, with

$$V^*(t_j^i) = \int_{t_1^i=0}^{z} \int_{t_2^i=0}^{t_1^i} \int_{t_3^i=0}^{t_2^i} \cdots \int_{t_m^i=0}^{t_{m-1}^i} t_j^i dt_m^i dt_{m-1}^i \ldots dt_3^i dt_2^i dt_1^i = \frac{z^{m+1}(m-j+1)}{(m+1)!}. \tag{1.4}$$

The expected value of t_j^i with an equally likely distribution over all possible combinations of t_j^i's is then given by the ratio $E(t_j^i) = V^*(t_j^i)/V(t_j^i)$, with

$$E(t_j^i) = \frac{z(m-j+1)}{(m+1)}. \tag{1.5}$$

The result in Eq. 1.5 leads directly to the conclusion that the use of linearly decreasing weights in a scoring rule, like those suggested by Borda with $a = b = \frac{z}{m+1}$, is consistent with using weights that match the expected strengths of preference on candidates for a given voter. This conclusion is, of course, dependent on the model that is developed to describe how individual voters form their preferences on candidates.

Daunou (1803, pages 262-263) does not agree with the analysis that is presented in Laplace (1795) and summarizes his general objections with the following statement:

> "But why substitute this average term for the precise will of the voters? In order to defend Borda's method, we start with the maxim that when the voters are able to express all the nuances in their opinions on the relative merit of the candidates, calculating these nuances reveals the general will, and yet we immediately deprive the voters of this possibility by setting up an average scale and fixing invariable numbers."

Black (1958) argues against the notion of using any scoring rules like those suggested by Borda and Laplace. Black asserts that the concept of rating preferences in scales like 3:1 or 4:1 is plausible for relative evaluations of things like goods in markets. However, Black asserts that the human mind does not operate in the same fashion when performing a relative comparison of candidates in elections. Biswas (1994) addresses the issue of ignoring intensity of preference with

Borda Rule, and considers the resulting number of voters who are adversely affected by the outcome of voting if Borda Rule is used.

Laplace (1795, pg. 286) ends his discussion of Borda Rule by stating

> "This election method would undoubtedly be the best, if considerations other than merit did not often influence the choices of even the most honest voters."

That is, Borda Rule has a problem with being susceptible to the possibility of strategic manipulation by voters. Daunou (1803) gives an example of a voting situation that explains this phenomenon, with 36 voters and m candidates that are denoted by $\boldsymbol{C}^m = C_1, C_2, ..., C_m\}$. There are 22 voters who have C_1 ranked as most preferred, with C_2 ranked as second most preferred, and with the remaining $m-2$ candidates ranked in any order after that. The remaining 14 voters have C_2 ranked as most preferred, with C_1 ranked as least preferred, and with the remaining $m-2$ candidates being ranked in any order between them. Using Borda Rule with $a = b = 1$ we have

$$\begin{gathered} Score(C_1) = 22m + 14 \\ Score(C_2) = 22(m-1) + 14m \\ Score(C_2) - Score(C_1) = 14m - 36\,. \end{gathered} \tag{1.6}$$

Then, the margin by which $C_2 \boldsymbol{B}\, C_1$ increases as m increases, for all $m \geq 3$, despite the fact that $C_1 \boldsymbol{M}\, C_2$ (22-14) for all m. This example verifies the fact that Borda Rule does not necessarily reproduce the rankings that are obtained by PMR. Moreover, it shows the effect that strategic voting could have if the supporters of C_2 were to misrepresent their true preference rankings by falsely reporting that its major competitor, C_1, is ranked as least preferred in their preferences. The problem of strategic voting with Borda Rule was addressed by a number of critics after it was implemented in elections that were held in the French Academy. Steffanson (1991) presents a historical analysis of the actual recorded election results in the French Academy from 1796 through 1803 while Borda Rule was being used to elect new members to the Academy. McLean (1995, pgs, 28-29) notes Borda's response to criticisms of his voting rule being vulnerable to manipulation as: "My election method is only for honest men."

The criticism that Borda Rule is vulnerable to strategic manipulation is somewhat exacerbated by the results of Gibbard (1973) and Satterthwaite (1975) that show that effectively all voting rules are subject to strategic manipulation. Smith (1999) considers the degree to which various common election procedures can be manipulated through strategic misrepresentation of preferences. The results naturally depend upon how the potential for manipulation is measured. Borda Rule is found to have the least potential for manipulation according to one of these measures. In particular, Borda Rule is least susceptible when voters are assumed to randomly select another preference ranking when they misrepresent their preferences.

1.3.3 A Characterization of Borda Rule

Young (1974) examines Borda Rule to consider properties that make it unique among voting rules. In this analysis, voters' preferences are not defined in terms of a voting situation, but in the context of a *voter preference profile*. A voter preference profile associates a specific linear preference ranking on candidates with each particular voter, giving a list of n preference rankings on candidates. The preferences of each specific voter are therefore identifiable in a voter preference profile, while this is not the case in a voting situation. Young analyzes general *social choice functions* that select a subset of winning candidates from a set of all possible candidates.

Let Q_m^n denote a voter preference profile on a set, $\boldsymbol{C}^m = C_1, C_2, ..., C_m\}$, of m candidates, where each of n individual voters has linear preferences on the candidates. A social choice function, f, is an election procedure that selects a subset of winning candidates, $f(Q_m^n)$, given the voters' preferences in the voter preference profile, Q_m^n. Obviously, $f(Q_m^n) \subseteq \boldsymbol{C}^m$.

A social choice function is *anonymous* if the winning candidates in $f(Q_m^n)$ can be determined simply from a knowledge of the voting situation that follows from the voter preference profile. That is, the specific preference rankings that are held by any particular individual voters do not need to be known in order to determine the winning candidates in $f(Q_m^n)$, only the number of voters with each preference ranking must be known.

Suppose that the identities of the candidates in $\boldsymbol{C}^m$ are interchanged according to some permutation, $\sigma(\boldsymbol{C}^m)$. There will be a corresponding change in candidate identities in any associated Q_m^n, to obtain the modified profile $\sigma(Q_m^n)$. A social choice function is *neutral* toward candidates if $f(\sigma(Q_m^n)) = \sigma(f(Q_m^n))$. That is, the subset of winners from the modified profile must be identical to the subset of winners from the original profile, accounting for the interchange of names that is specified by $\sigma(\boldsymbol{C}^m)$.

Assume that we have profiles $Q_m^{n'}$ and $Q_m^{n''}$ on the candidates in $\boldsymbol{C}^m$ for two distinct sets of voters, with n' and n'' members in the respective sets. We also suppose that there is at least one common candidate in the winning subsets from the two profiles, such that $f(Q_m^{n'}) \cap f(Q_m^{n''}) \neq \phi$. The combined profile $Q_m^{n'} + Q_m^{n''}$ is obtained by merging the preference rankings of the voters in the two profiles to obtain a single voter preference profile. A social choice function is *consistent* if the winning subset from the combined profile is identical to the subset of candidates that are common to both of the winning subsets of the individual profiles, with $f(Q_m^{n'} + Q_m^{n''}) = f(Q_m^{n'}) \cap f(Q_m^{n''})$.

Let Q_m^1 represent the preferences for a profile containing only one voter. A social choice function is *faithful* if the winning candidate, $f\left(Q_m^1\right)$, is the most preferred candidate for the individual voter. A social choice function has the *cancellation property* if any given voter's pairwise preference $C_i \succ C_j$ will be offset, or cancelled-out, by any other voter's pairwise preference with $C_j \succ C_i$. It follows that a social choice function with the cancellation property must declare a tie between candidates C_i and C_j if the number of voters having pairwise preferences with $C_i \succ C_j$ is the same as the number of voters with $C_j \succ C_i$.

Young (1974) proves that Borda Rule is the *only* social choice function that is neutral, consistent, faithful, and has the cancellation property. Since each of these properties sounds quite desirable, this finding is a strong endorsement for the use of Borda Rule. Gärdenfors (1973) develops another characterization, and Nitzan and Rubinstein (1981) develop a characterization of Borda Rule for situations in which individual voters do not necessarily have transitive preferences.

Much work has been done on characterizations of Borda Rule and other related voting procedures. Work on characterizations is not the focus of the current study, and the intent here is simply to point out that there are positive characteristics of Borda Rule that make it unique among all voting rules. Saari (1996) and Brams and Fishburn (2002) list many other positive characteristics of Borda Rule.

1.4 Condorcet's Paradox

Condorcet wrote a series of papers that extended some of the ideas in Borda (1784), and criticized others. As the series of articles progressed, Condorcet's tone became more antagonistic toward Borda's work. Condorcet routinely stressed the criterion that the winner of any election should be the PMRW, which has led to the common use of the term *Condorcet Criterion* to refer to this notion. And, this is why the PMRW is commonly referred to as the Condorcet Winner. Condorcet searched tirelessly to find a simple voting procedure that would elect the PMRW. As mentioned before, Borda was the first to suggest that the PMRW should win an election, but Borda was much more concerned about the undesirable possibility of electing the PMRL.

Condorcet (1785b) begins his analysis with an example voting situation on 60 voters with linear preference rankings on three candidates, as shown in Fig. 1.4:

A	C	B	C
C	A	C	B
B	B	A	A
$n_2 = 23$	$n_4 = 2$	$n_5 = 19$	$n_6 = 16$.

Fig. 1.4 A voting situation with a Strict Borda Paradox from Condorcet (1785b)

With plurality rule voting, *A* gets 23 votes, *B* gets 19 votes, and *C* gets 18 votes, so the rank by plurality rule is the linear order ***APBPC***. Condorcet then goes on to note that on the basis of PMR: ***CMA*** (37-23), ***CMB*** (41-19) and ***BMA*** (35-25). The result of PMR voting is then a linear rank, with ***CMBMA***. Thus, the ranking by plurality is the reverse of the ranking by PMR, exactly as in the result given by Borda (1784). Like Borda, Condorcet states that this situation results from ignoring additional information that could be obtained by requiring voters to report preference rankings on candidates "in order of merit". No mention is made of Borda's earlier work in Condorcet (1785b).

Condorcet (1785b) then continues with a famous example of a voting situation with 60 voters on three candidates, as shown in Fig. 1.5:

$$\begin{array}{ccccc} A & B & B & C & C \\ B & A & C & A & B \\ C & C & A & B & A \\ n_1 = 23 & n_3 = 2 & n_4 = 17 & n_5 = 10 & n_6 = 8\,. \end{array}$$

Fig. 1.5 A voting situation showing a PMR cycle from Condorcet (1785b)

Here, Condorcet notes that we have a "*contradictory system*" that represents what has come to be widely known as *Condorcet's Paradox*. In particular, we find that PMR comparison leads to: ***AMB*** (33-27), ***BMC*** (42-18), and ***CMA*** (35-25). There is a cycle in the PMR relation on the three candidates, so that no candidate emerges as being superior to each of the remaining candidates. Given Condorcet's strong arguments that the PMRW should always be selected as the winner, we are left with a difficult question in this case. In particular, "Which candidate should be selected as the winner?"

We noted before that Condorcet was quite adamant in his argument that a lack of transitivity of preference for individual voters was so contradictory, that a system must be used to eliminate "such absurdities". However, after eliminating intransitivity from the preferences of individual voters, we find that collective choice of voters from PMR still might produce intransitive results, suggesting an irrational response in the collective choice of rational voters.

It was stressed before that Borda, Daunou and Laplace were not at all in favor of using anything other than linearly decreasing weights in a weighted scoring rule to account for intensity of voters' preferences. However, Saari (1995a) makes an argument that is based on intensity of preference to justify using Borda Rule, instead of following Condorcet's suggestion and directly using a PMR based approach to find a winner. The argument is that any criterion like the one proposed by Condorcet can result in a lack of "inner consistency" in the form of the PMR cycles that were just observed, since this criterion ignores some dimensions of preference. Specifically, suppose that a given voter has the linear preference ranking on the three candidates $A \succ B \succ C$. Condorcet only accounts for the fact that $A \succ C$ in a PMR comparison between *A* and *C*, thereby ignoring the "intensity" of preference between *A* and *C,* since *B* appears between them in the ranking.

PMR cycles therefore arise because certain information in the preference ranking is being ignored, following previous notions that were given in Borda (1784).

Transitivity of collective choice is often held as a standard of rational behavior for group decisions. The lack of transitivity is typically discussed with a very negative connotation. It is referred to: as reflecting "some uncertainty of opinion" by Condorcet (1785b), as lacking "inner harmony" by Riker (1961), as "discordant" by Fishburn (1973a), as "anarchic" by MacKay and Wong (1979), as "democratically unpalatable" by Riker (1982) and as "chaotic" by Coggins and Perali (1998). The existence of PMR cycles is said: to "lead to inconsistencies" by Sen (1970, pg. 38), to lead to political "incoherence" by Riker and Ordeshook (1973, pg. 84), to lead to "arbitrary" political decisions by Oppenheimer (1975), to lead to a lack of "viability" by Abrams (1976), to lead to "instability" by both Koehler (1975a) and Marhuenda and Ortuño-Ortín (1998), to lead to "pathology" by Brams (1976, pg. 29), and to result in a lack of "stability" by Fishburn and Gehrlein (1980a).

Miller (1983) takes exception to the notion that the existence of PMR cycles is necessarily a bad phenomenon. A number of historical quotes are given to suggest that conditions leading to PMR cycles are likely to result from the electorate having opinions that are "crosscut" in many different ways. This situation results in the electorate routinely forming different factions on many different issues over time, to obtain desired outcomes. The end result of this routine change in factions is argued to lead to political stability and viability, without having long-term total domination of minorities.

Rae (1980) criticizes some work of Riker (1980), regarding the suggestion that PMR cycles reflect "incoherence" within group decision-making. Rae uses a number of quotes, particularly from Dahl (1956), to argue that transitivity of PMR is not a reasonable restriction on group preferences, and concludes with the statement:

> "An understanding of majority rule, of democracy, of liberalism which does without utilitarianism, and which does more than assert that rights are right, must travel a more mysterious space, must walk up odder stairs, and must employ a more intricate altimeter than transitive consistency."

In his famous work in the area of social choice theory, Arrow (1963) argues that the idea of requiring transitivity of group preference from voting procedures, including PMR, is indeed a very important aspect of describing rational behavior in collective choice. However, he acknowledges the work of other researchers in the area [Arrow (1963), pg.118)] who suggest "that a social decision process might well sacrifice transitivity if necessary to satisfy other conditions." Fishburn (1970) presents a number of interesting arguments, with examples, to lead to the conclusion that it is not really reasonable to expect social choice rules, including PMR, to be transitive. Bar-Hillel and Margalit (1988) also perform an analysis of the logic of the assumption of group transitivity to reach a similar conclusion.

Condorcet (1785c) continues with his analysis of intransitive PMR voting situations, to show that there might be a PMRW with more than three candidates, while there is a cycle in the PMR relationship on the remaining candidates. Thus, a distinction is made between the possibility that there is a PMRW and the possibility that the PMR is completely transitive over all candidates. With only three candidates, the existence of a PMRW ensures that the PMR ranking over all candidates is transitive. Condorcet notes that the possible existence of this situation on more than three candidates is of no consequence to the superiority of the PMRW, as long as only one candidate is being elected.

McGarvey (1953) generalizes Condorcet's observation by developing a procedure for constructing voting situations with an arbitrary number of voters with linear preference rankings, to prove that it is possible to have any possible combination of PMR outcomes on pairs for a finite number of candidates. Stearns (1959) obtains McGarvey's result with a procedure for constructing voting situations that requires significantly fewer voters to produce the desired result, and Deb (1976) generalizes McGarvey's result to cover a larger class of voting rules than just PMR.

Condorcet (1785d) does acknowledge the earlier work of Borda (1784), and refers to Borda as "a famous mathematician". Condorcet writes that he had heard of Borda's earlier results, but did not know that anything had been written on the topic. This statement seems a bit confusing, since Condorcet (1784, pg. 121) provided written comments on Borda's work, stating:

> "M. de Borda's observations on the drawbacks of the election method used almost everywhere are very important and totally original."

McLean (1995) suggests that Borda probably played little or no role in the process of having his 1770 paper published. In fact, it was Condorcet who chose to add Borda's 1770 paper to the Proceedings of the French Academy of Science in 1781, which was not actually published until 1784. This could leave a reader with the impression that Borda's work was done much later than when it actually was presented before the French Academy.

McLean (1990) and Gylmour, et al. (1998) note that a voting rule that is very similar to using a procedure to find the PMRW was discussed by Ramon Lull, without any mathematical development, over 500 years before Condorcet's work appeared. The possibility that a PMR cycle might exist has a history of rediscovery by a number of other researchers after Condorcet, including C. L. Dodgson [Lewis Carrol] (1885a), Huntingdon (1938) and in a series of articles by Black (1948a, 1948b, 1948c, 1948d, 1949a, 1949b, 1949c). Thorough reviews of the history of Condorcet's paradox can be found in Granger (1956), Black (1958) and Riker (1961).

1.4.1 A Characterization of PMR

May (1952) examines characteristics of PMR on a pair of candidates to consider properties that make it unique among voting rules. Let Q_2^n denote a voter preference profile for n voters on a pair of candidates, $\boldsymbol{C}^2 = C_1, C_2\}$. Each individual voter in the preference profile has some preference, or indifference, on the candidates in the pair. A social choice function, f, selects a subset of winning candidates, $f(Q_2^n)$, given the voters' preferences in the profile, Q_2^n, with $f(Q_2^n) \subseteq \boldsymbol{C}^2$. A social choice function is *decisive* if $f(Q_2^n)$ is never empty, so that f always selects some winner, even if this results in a tie with $f(Q_2^n) = \boldsymbol{C}^2$.

Assume that we have a profile Q_2^n for the two candidates in $\boldsymbol{C}^2$ and that $C_i \in f(Q_2^n)$. That is, either f selects $C_i \in \boldsymbol{C}^2$ as the single winner based on the preferences in Q_2^n, or there are tied winners with $f(Q_2^n) = \boldsymbol{C}^2$. Now suppose that some voter changes his or her preferences on the candidates within Q_2^n to obtain a modified profile $\hat{Q}_2^n$. Furthermore, that voter changes preference in a manner to show increased support for C_i. As a result, this voter could not have had C_i as the more preferred candidate in Q_2^n. Increased support can result from changing from C_i being the voter's less preferred candidate to having indifference between the two candidates, or from any change that results in C_i becoming the voter's more preferred candidate. A social choice function is *positively responsive* if it must then be true that $f(\hat{Q}_2^n) = C_i$. With increased support, C_i will remain as the single winner if it was the single winner with the original profile, and it will become the single winner if it was tied as the winner with the original profile.

Using the same definitions of anonymous and neutral social choice functions from Young (1974) that were developed in the characterization of Borda Rule earlier in our discussion, May (1952) proves that Simple Majority Rule is the *only* social choice function that is always decisive, anonymous, neutral and positively responsive. Since each of these properties sounds quite reasonable on its own, this finding is a strong endorsement for the use of Simple Majority Rule. This characterization for a pair of candidates can be extended to PMR for general *m*-candidate elections since the general case corresponds to a series of simple majority rule decisions on pairs of candidates. There have been many different characterizations of PMR since the one proposed by May (1952), with a recent one being given in Göksel and Sanver (2002).

Young (1974) notes that PMR exhibits all of the properties in the characterization of Borda Rule when attention is restricted to profiles that have a PMRW. However, the fact that PMR does not necessarily have a PMRW, so that we could have $f(Q_m^n) = \phi$ for some Q_m^n, eliminates PMR from consideration as a true social choice function, as defined in Young's analysis of Borda Rule.

Factors that are related to the determination of the likelihood that Condorcet's Paradox might occur in practice are the primary focus of this study. Any study of this type would therefore be of little significance if Condorcet's Paradox has never been observed in a real situation. Numerous studies have been conducted to find empirical examples of Condorcet's Paradox, and a thorough survey of that work is the topic of the next chapter.

1.5. Condorcet's Other Paradox

Condorcet continued with other papers that considered voting rules like the one that Borda suggested. This is done with an example voting situation, as shown in Fig. 1.6, with 81 voters on three candidates in Condorcet (1785c):

$$\begin{array}{cccccc} A & A & B & C & B & C \\ B & C & A & A & C & B \\ C & B & C & B & A & A \\ n_1 = 30 & n_2 = 1 & n_3 = 29 & n_4 = 10 & n_5 = 10 & n_6 = 1. \end{array}$$

Fig. 1.6 A voting situation showing Condorcet's Other Paradox from Condorcet (1785c)

The use of PMR with this voting situation results in the outcome: *AMB* (41-40) and *AMC* (60-21), so that we have *A* as the PMRW.

Using Borda Rule with $a = b = 1$, we have

$$\begin{aligned} Score(A) &= 3*31 + 2*39 + 1*11 = 182 \\ Score(B) &= 3*39 + 2*31 + 1*11 = 190. \end{aligned} \tag{1.7}$$

Here, we have *BBA* when *A* is the PMRW, to show again that Borda Rule does not always elect the PMRW.

Condorcet (1785c) then goes farther with the example voting situation in Fig. 1.6 to show a phenomenon that Fishburn (1974a) refers to as *Condorcet's Other Paradox*. This argument involves analyzing this voting situation with a general weighted scoring rule with weights 3, λ and 1, as described in earlier discussion. Condorcet computes $Score(A)$ and $Score(B)$ for this general weighted scoring rule:

$$\begin{aligned} Score(A) &= 3*31 + \lambda*39 + 1*11 \\ Score(B) &= 3*39 + \lambda*31 + 1*11. \end{aligned} \tag{1.8}$$

In order for the PMRW, *A*, to be elected by this weighted scoring rule, we must have:

$$\begin{aligned} Score(A) &> Score(B) \\ 104 + 39\lambda &> 128 + 31\lambda \\ 8\lambda &> 24 \\ \lambda &> 3. \end{aligned} \tag{1.9}$$

This contradicts our definition of a weighted scoring rule, and it follows that no weighted scoring rule, including Borda Rule, can elect the PMRW in this example, which is Condorcet's Other Paradox. Fishburn (1974a) generalizes Condorcet's Other Paradox for all $m \geq 3$, to show that there is some voting situation with a PMRW in an *m*-candidate election, such that every weighted scoring rule will have at least $m - 2$ candidates with a greater score than the PMRW.

Condorcet (1788b) considers Borda's work again and becomes more aggressive in his attack on Borda Rule. He gives the example voting situation that is shown in Fig. 1.7 for 30 voters on three candidates:

$$\begin{array}{cccccc} A & A & B & C & B & C \\ B & C & A & A & C & B \\ C & B & C & B & A & A \\ n_1 = 9 & n_2 = 3 & n_3 = 4 & n_4 = 4 & n_5 = 6 & n_6 = 4\,. \end{array}$$

Fig. 1.7 An example voting situation in which Borda Rule does not elect the PMRW and Plurality Rule does from Condorcet (1788b)

In this particular example, ***AMB*** (16-14) and ***AMC*** (16-14), so *A* is the PMRW. With election by plurality: *A* receives twelve votes, *B* receives ten votes, and *C* receives eight votes. We therefore find that plurality rule elects the PMRW. When we use Borda Rule the scores for candidates are:

$$\begin{aligned} Score(A) &= 3*12 + 2*8 + 1*10 = 62 \\ Score(B) &= 3*10 + 2*13 + 1*7 = 63 \\ Score(C) &= 3*8 + 2*9 + 1*13 = 55. \end{aligned} \tag{1.10}$$

For this example, we have ***BBA***, so Borda Rule fails to elect the PMRW, while plurality rule, that Borda discredited, does so. Condorcet (1788b, pg.145) writes:

> "This new method (Borda Rule) is not only no better than the conventional one (plurality), it is actually worse. At least with the conventional method, it is just a possibility that the result was wrong and that we are going against the true will of the plurality. With the new method we can be sure that it was wrong and that we are having to act in accordance with totally erroneous results."

At later times, Condorcet became very antagonistic toward Borda, referring to him as "having abandoned mathematics for petty applied science" [Baker (1975), pg. 42] and as having written papers "that nobody has ever spoken of … or ever will" [McLean (1991), pg. 15]. Young (1995) refers to these comments as showing "a certain amount of personal venom." Baker (1975, pg. 42) also reports a dispute that arose as a result of an attempt by D'Alembert to obtain financial payment for Condorcet when Condorcet was appointed as assistant secretary of the French Academy of Science. On the other hand, Black (1958, pg. 179) claims that Borda and Condorcet remained close friends throughout their lives.

Condorcet was later accused of treason, and he was to be tried as an enemy of the state after it became known that he had anonymously written a pamphlet to urge citizens to reject a constitution that was being proposed during the Reign of Terror. McLean (1995) notes a report that Borda risked his own life in making a plea for clemency on behalf of Condorcet while Condorcet was in hiding during this time. Condorcet made an attempt to escape after being in hiding for some time, but he was ultimately caught in a very unusual episode. He was imprisoned and was found dead in his cell two days later. Thorough biographies of Condorcet can be found in Schapiro (1934), Morley (1965), Baker (1975), Badinter and Badinter (1988), Rosenfield (1989), and McLean and Hewitt (1994).

Given the long history of the debate over which voting rule makes the most sense to use, it is quite fascinating that the debate still continues. Felsenthal and Machover (1992) argue strongly for the use of Condorcet's suggestion of finding the PMRW. Saari (1995b) argues strongly for the direct use of Borda Rule in all elections. Black (1958) suggests that a hybrid model should be used to elect the PMRW when there is one, and to use Borda Rule when a PMRW does not exist.

Saari (1995b) goes well beyond simply supporting the use of Borda Rule in all situations, and goes on to attack the basic concept of using any form of PMR to find the winner in an election. To describe the basis of Saari's argument, we define two different voter preference profiles in a three-candidate election. Voter Profile 1 is shown in Fig. 1.8, and it consists of three rational voters with linear preference rankings:

Voter 1: $A \succ B, B \succ C, A \succ C$
Voter 2: $B \succ C, C \succ A, B \succ A$
Voter 3: $C \succ A, A \succ B, C \succ B$.

Fig. 1.8 Example Voter Profile 1 from Saari (1995b)

By using PMR with the linear preference rankings in Voter Profile 1, we have an example of Condorcet's Paradox, with *AMB* (2-1), *BMC* (2-1) and *CMA* (2-1).

Voter Profile 2 in Fig. 1.9 shows an example in which there are three irrational voters with complete, but intransitive, preferences:

Voter 1: $A \succ B, B \succ C, C \succ A$
Voter 2: $A \succ B, B \succ C, C \succ A$
Voter 3: $B \succ A, A \succ C, C \succ B$.

Fig. 1.9 Example Voter Profile 2 from Saari (1995b)

Using PMR on Voter Profile 2, we obtain the results *AMB* (2-1), *BMC* (2-1) and *CMA* (2-1), which is identical to the results from Voter Profile 1.

Saari (1995b, pg. 48) uses the outcome of obtaining identical results from these two voter preference profiles to claim that PMR procedure has

> "… an inability to distinguish between transitive and intransitive preferences: consequently the pairwise vote (PMR) loses the critical assumption of transitive voters!"

He notes that Condorcet was very careful to impose transitivity on individual voters, but then suggested a system of voting that "surreptitiously drops it" as a condition of aggregated behavior for the electorate. However, Saari (1995b, pg. 46) acknowledges that the notion of seeking the PMRW as the winner in an election does have "nearly universal acceptance". Risse (2005) presents arguments that are very strongly in opposition to the analysis that Saari presents in his criticisms of the notion of considering procedures that are associated with finding the PMRW.

1.6 The Paradox of Multiple Elections

Other types of voting paradoxes can be observed if we consider a sequence of independent majority rule elections, in which a final accept-reject vote is made in each stage, without regard to the outcomes of any other elections. For example we might consider a set of independent proposals that are being voted on by a committee, and each individual proposal would be passed or defeated by majority rule by the committee. The *Paradox of Multiple Elections* can exist in such situations, and it shows inconsistencies that can be observed in voter behavior.

Brams, et al. (1998) consider an example in which a committee is considering three possible proposals that are denoted as $A,B,C\}$. Each proposal will be considered in turn, with the passage or rejection of each proposal being independently determined on the basis of majority rule. Voters will vote yes (*Y*) or no (*N*) on each of the individual proposals, given their preferences on the outcome. A given voter's preferences on the proposals can then be stated in terms of some combination of *Y*-*N* votes over the three proposals. Consider the example of such a situation in Fig. 1.10:

Proposal *A* Preference	Proposal *B* Preference	Proposal *C* Preference	Number of Voters
Y	*Y*	*Y*	3
Y	*Y*	*N*	1
Y	*N*	*Y*	1
N	*Y*	*Y*	0
Y	*N*	*N*	1
N	*Y*	*N*	3
N	*N*	*Y*	3
N	*N*	*N*	1

Fig. 1.10 An example voting situation for the Paradox of Multiple Elections from Brams, et al. (1998)

In this example, there are three voters with preference *YYY* on proposals *A*, *B*, and *C* respectively, one voter with preference *YYN*, and so on. The outcomes of the three different majority rule votes in the series of elections are given by: *NMY* (7-6) for *A*, *YMN* (7-6) for *B*, and *YMN* (7-6) for *C*. Thus, majority rule produces the outcome *NYY*, so that *A* is rejected, while *B* and *C* are both accepted. However, the outcome *NYY* does not represent the preferences of a single voter, which is an example of the Paradox of Multiple Elections.

Brams, et al. (1998) go on to show that the conditions that lead to the existence of the Paradox of Multiple Elections are a generalization of the conditions that lead to the existence of Condorcet's Paradox. Scarsini (1998) develops a more general form of the Paradox of Multiple Elections.

1.7 The Vote Trading Paradox

The *Vote Trading Paradox* can take place in situations like those that are described for the Paradox of Multiple Elections, where a series of independent majority rule elections are taken for separate issues. In this case, each of the voters receives some payoff or loss that is associated with each issue that is being considered. The voters' payoffs or losses depend upon whether the issues are passed or defeated by majority rule, and once a vote has been taken on an issue the outcome is final for that issue. These payoffs or losses could reflect the benefit or loss for the constituency that a voter represents. The issues are independent, so that each is passed or defeated without regard to the outcome of any other issues.

Fig. 1.11 lists the payoffs and losses for an example in which three voters are considering six issues, $A, B, C, D, E, F\}$, following an example from Riker and Brams (1973). Voter 1 will receive a payoff of one if Issue *A* is passed by majority rule, and a loss of two will be incurred if Issue *A* is defeated by majority rule.

	Voter 1		Voter 2		Voter 3	
Issue	Pass	Defeat	Pass	Defeat	Pass	Defeat
A	1	-2	1	-1	-2	2
B	1	-2	-2	2	1	-1
C	1	-1	-2	2	1	-2
D	-2	2	1	-1	1	-2
E	-2	2	1	-2	1	-1
F	1	-1	1	-2	-2	2

Fig. 1.11 Voter payoffs for vote trading example from Riker and Brams (1973)

Rational voters will decide to vote Yes (*Y*) or No (*N*) on each issue in an effort to maximize their own resulting overall benefit from having that issue pass or fail by majority rule. Voter 1 would therefore vote *Y* on Issue *A*, since the associated payoff of one when that issue passes is greater than a loss of two for this voter when Issue *A* is defeated. Fig. 1.12 shows the votes that the voters would cast on all issues with such sincere voting.

Issue	Voter 1 Vote	Voter 2 Vote	Voter 3 Vote	Election Outcome	Voter 1 Payoff	Voter 2 Payoff	Voter 3 Payoff
A	*Y*	*Y*	*N*	Pass	1	1	-2
B	*Y*	*N*	*Y*	Pass	1	-2	1
C	*Y*	*N*	*Y*	Pass	1	-2	1
D	*N*	*Y*	*Y*	Pass	-2	1	1
E	*N*	*Y*	*Y*	Pass	-2	1	1
F	*Y*	*Y*	*N*	Pass	1	1	-2
				Total Payoff	0	0	0

Fig. 1.12 Total voter payoff with sincere voting for vote trading example from Riker and Brams (1973)

All issues will pass as a result of sincere voting, and each voter would have a total payoff of zero from the election outcomes on the three issues.

Vote trading, or *logrolling*, is a process by which some subset of the voters can act together to misrepresent their true preferences in the series of issues that are being considered. As a result of vote trading, the subset of voters who participate in misrepresenting their preferences can increase their final payoff over that which would be obtained from voting according to their true preferences. The increased payoff for participating voters comes at the expense of voters who were not participating in the vote trading.

For example, given advance knowledge of what the outcomes in this particular example would be with sincere voting, Voter 2 and Voter 3 can consider a trade of votes on Issues *A* and *B*. Both can obtain a higher total payoff from the outcome of voting on these two issues by misrepresenting some of their preferences. If the vote of Voter 2 is changed to *N* for Issue *A*, that issue would then be defeated by majority rule. The payoff to Voter 2 will decrease from a gain of one to a loss of one as a result, but the payoff to Voter 3 will increase from a loss of two to a gain of two. In return, the vote for Voter 3 will be changed to *N* on Issue *B*, so that it will also be defeated. Voter 3 will have a reduction in payoff from a gain of one to a loss of one, but Voter 2 will have an increase in payoff from a loss of two to a gain of two. Both Voters 2 and 3 will then have a net increase of two in the combined payoffs for Issues *A* and *B* as a result of this vote trade.

In the same fashion, Voters 1 and 2 can do a vote trade on issues *C* and *D*, with the vote of Voter 1 being changed to *N* on Issue *C* and the vote of Voter 2 being changed to *N* on Issue *D*. Both Voters 1 and 2 will obtain a net increase in payoff of two as a result of the trade. Voters 1 and 3 can also do a vote trade on Issues *E* and *F*, with the vote of Voter 1 being changed to *N* on Issue *F* and the vote of Voter 3 being changed to *N* on Issue *E*. Both Voters 1 and 3 will then have a net increase in payoff of two as a result of this trade. A summary of the votes that are cast, the election outcome for each issue and total payoff for each voter after these three vote trades are completed is shown in Fig. 1.13.

Issue	Voter 1 Vote	Voter 2 Vote	Voter 3 Vote	Election Outcome	Voter 1 Payoff	Voter 2 Payoff	Voter 3 Payoff
A	*Y*	*N*	*N*	Defeated	-2	-1	2
B	*Y*	*N*	*N*	Defeated	-2	2	-1
C	*N*	*N*	*Y*	Defeated	-1	2	-2
D	*N*	*N*	*Y*	Defeated	2	-1	-2
E	*N*	*Y*	*N*	Defeated	2	-2	-1
F	*N*	*Y*	*N*	Defeated	-1	-2	2
				Total Payoff	-2	-2	-2

Fig. 1.13 Total voter payoff after vote trading in example from Riker and Brams (1973)

The net result of the three vote trades in this example is that all six issues are defeated and the total payoff to each voter is reduced from zero in the sincere voting case to a loss of two with the three vote trade transactions. While each voter who is directly involved in a given vote trade has a net increase in total payoff of two for that particular trade, the voter who is not involved in that trade has a net loss in payoff of six because of the trade. This causes a net overall loss of two for all voters over the three vote trades, since each voter is involved in two vote trades, to gain four, and is not involved in one vote trade, to lose six.

In each of these trades, voters have sequentially chosen to form coalitions in order to improve their respective payoffs during that trade. However, the overall net effect is that each voter has a lower payoff in the end than they would have obtained from sincere voting in the original situation. It is clear that the existence of vote trading can lead to irrational outcomes. McKelvey and Ordeshook (1980) perform extensive experimental analysis to show that the vote-trading phenomenon, as suggested by Riker and Brams (1973), can indeed be induced in an experimental setting with inexperienced players.

Koehler (1975a) shows that the conditions that are required to create a vote-trading outcome like the one in the example above are logically equivalent to the conditions that lead to cycles with PMR, and Bernholz (1973,1974) makes similar observations. Sullivan (1976) discusses conditions that might exist in committee structures that make it likely to lead to vote trading. Koford (1982) considers a model of vote trading when individuals must make trades on groups of votes through a party leader who is attempting to develop an overall balance in vote outcomes. Enelow (1986) develops an explanation of how vote trading can exist in a stable environment. This situation exists for risk-averse voters who are using random variable forecasts of decisions on future issues, in which issues can be brought back for re-voting. These results support similar conclusions in Tullock (1981). Stratmann (1996) considers the possibility of cycles in vote trading by using Markov chain analysis, to conclude that there is little evidence to support the notion that cycling, or unstable coalitions, exists for decisions on the distribution of federal grants to districts in the U.S. House of Representatives.

1.8 The No Show Paradox

Brams and Fishburn (1983a) present an example of a voting paradox in which some subset of voters chooses not to participate in an election, and then prefers the resulting winner to the winner that would have been selected if the subset had actually participated in the election. This phenomenon is referred to as the *No Show Paradox*. We consider an example in which the winner of an election is determined by *Negative Plurality Elimination* in a three-candidate election. This is a two-stage election procedure. In the first stage, voters cast votes for their two most preferred candidates. The candidate that receives the fewest number of votes is then eliminated, and the ultimate winner is selected in the second stage by using majority rule on the remaining two candidates. The voting rule that is used in the first stage is referred to as *Negative Plurality Rule* since it is equivalent to having each voter cast a negative vote against one candidate, with the candidate who receives the most negative votes being eliminated.

Consider a voting situation with 21 voters and three candidates $A,B,C\}$, as shown in Fig. 1.14.

$$
\begin{array}{cccccc}
A & A & B & C & B & C \\
B & C & A & A & C & B \\
C & B & C & B & A & A \\
n_1 = 3 & n_2 = 5 & n_3 = 5 & n_4 = 2 & n_5 = 3 & n_6 = 3\,.
\end{array}
$$

Fig. 1.14 An example voting situation from Brams and Fishburn (1983a)

In the first stage of voting with negative plurality, Candidates *A*, *B*, and *C* receive 15, 14 and 13 votes respectively. Candidate *C* is eliminated in the first stage and then *BMA* by a vote of eleven to ten in the second stage, to select *B* as the overall winner. Voters with the linear preference ranking $A \succ B \succ C$ would not get their most preferred candidate, since *B* is the winner. Suppose that two of these particular voters had not participated in this election for some reason. The resulting voting situation is shown in Fig. 1.15.

$$
\begin{array}{cccccc}
A & A & B & C & B & C \\
B & C & A & A & C & B \\
C & B & C & B & A & A \\
n_1 = 1 & n_2 = 5 & n_3 = 5 & n_4 = 2 & n_5 = 3 & n_6 = 3\,.
\end{array}
$$

Fig. 1.15 The modified example voting situation from Brams and Fishburn (1983a)

In the first stage of voting with negative plurality on this modified voting situation with 19 voters, *A*, *B*, and *C* receive 13, 12 and 13 votes respectively. Candidate *B* is eliminated in the first stage and then *AMC* by a vote of eleven to eight in the second stage. Since the winner in this modified voting situation is *A*, the two voters with linear preferences $A \succ B \succ C$ who did not participate will now have their most preferred candidate chosen as the winner. These two voters have therefore obtained a more preferred outcome by not participating in the election.

Negative plurality elimination does not necessarily elect the PMRW. However, Moulin (1988) proved that any election procedure that does meet the condition that it must select the PMRW, when one exists on four or more candidates, must be subject to the possibility that the No Show Paradox can be observed. Pérez (2001) considers two variations of this paradox.

1.9 Other Voting Paradoxes

Many other voting paradoxes have been discovered, and general surveys of work on voting paradoxes are given in Brams (1976), Niemi and Riker (1976), Petit and Térouanne (1987), and Nurmi (1998). Fishburn (1974a) presents a survey of different voting paradoxes, and gives Monte-Carlo computer simulation estimates of the likelihood that each paradox might occur. The general conclusion is that the most extreme forms of voting paradoxes are probably very rare in practice.

Deb and Kelsey (1987) examine *Ostrogorski's Paradox* to show that the conditions that are necessary for it to exist are similar to, but not identical to, the conditions that are necessary for Condorcet's Paradox to exist, and they conclude that Ostrogorski's Paradox therefore deserves study as a separate phenomenon. So, voting paradoxes do exist with conditions that are not identical to those that lead to Condorcet's Paradox.

1.10 Conclusion

The possible existence of various voting paradoxes has been the focus of numerous investigations. This research has largely been dominated by studies that are associated with Condorcet's Paradox. There is widespread, but not universal, acceptance of the notion that the PMRW is the best candidate for selection in an election process, when such a candidate exists. When Condorcet's Paradox occurs, there is no PMRW in three-candidate elections, and there is a need to find some other voting mechanism to determine a winner in such cases.

There is a resulting interest in determining estimates of the likelihood that a PMRW exists in various situations. This observation is intensified by the fact that we have seen that the conditions that are necessary for Condorcet's Paradox to exist are the same as the conditions that are required for the existence a number of other paradoxes, like the Paradox of Multiple Elections and the Vote Trading Paradox. In some cases there are links between the conditions that are necessary for other paradoxes to exist and the conditions for Condorcet's Paradox, as with the No Show Paradox. Researchers even feel obligated to show that the conditions that are necessary for other paradoxes, like Ostrogorski's Paradox, to exist are not the same as the conditions that are required for Condorcet's Paradox to exist.

2 Condorcet's Paradox

2.1 Introduction

One conclusion of the last chapter is that the conditions on voters' preferences that lead to the possible existence of Condorcet's Paradox are equivalent to the conditions that either directly lead to, or are associated with, a number of other voting paradoxes. It is clearly of interest to have some idea as to whether, or not, Condorcet's Paradox has ever been observed in real voting situations. The practical implications of having this paradox occur are twofold. First of all, one has to deal with the issue of determining who the winner of an election should be if there is no PMRW. No matter which candidate we might select in such a case, it can always be argued that a majority of voters would prefer to have some other candidate selected as the winner.

The second issue relates to the possibility that election outcomes might be manipulated by agenda control. Suppose that we have three options, $A,B,C\}$, before a committee that is going to choose only one of the options. The committee will determine the winner by using *sequential elimination* by majority rule. That is, two issues will be compared by majority rule in the first round, and the loser will then be dropped from further consideration. The winner in the first round will then be carried to the next round for a majority rule vote against the remaining third candidate. The sequential elimination procedure is defined in the obvious way for elections on more than three candidates. When voters' preferences are such that an occurrence of Condorcet's Paradox exists, we would have a PMR cycle like *AMB*, *BMC* and *CMA*. Any chairperson of the committee who knew the preferences of the voters in advance could then set an agenda for the sequence of voting to get any outcome that he or she wanted.

If the chairperson wanted *C* to win, the first round would have *A* versus *B*, with *A* as the first-round winner. The second round would then have *A* versus *C*, with *C* as the ultimate winner. If the chairperson wanted *B* to win, the first round would have *C* versus *A*, with *C* as the first-round winner. The second round would then have *C* versus *B*, with *B* as the ultimate winner. If the chairperson wanted *A* to win, the first round would have *B* versus *C*, with *B* as the first-round winner The second round would then have *B* versus *A*, with *A* as the ultimate winner. This logic can easily be extended to cycles on more than three candidates, and an agenda setting committee chairperson could manipulate the outcome of an election whenever voters' preferences result in an occurrence of Condorcet's Paradox. Davis (1974) discusses "democratic" procedures that can be used to determine the order in which voting should be performed when it is known that the possibility of a PMR cycle exists, and Tullock (1975) points out some difficulties in the proposal presented in Davis (1975).

Theoretical studies have been performed to model processes by which rational voters might choose to misrepresent their preferences in voting on options, once the agenda for voting has been set in such situations. Rubinstein (1980) addresses this issue by considering *farsighted voters* who know what the majority rule outcomes might be in future stages of voting, once an agenda has been established. Suppose that a farsighted voter has the linear preference ranking $A \succ B \succ C$ on three options, and the agenda has been set to start with an initial vote between *A* and *B*, with the first-round winner moving to the next stage to have a majority rule vote against *C* to determine the winner The farsighted voter would want to vote for *A* over *B* in the first-round vote, given the assumed true preferences. However, suppose that this farsighted voter also knows that if the outcome ***AMB*** results in the first round, to send *A* to the second stage, that a majority of voters prefer *C* to *A*. The resulting outcome of ***CMA*** in the second stage would then make the farsighted voter's least preferred candidate, *C*, the ultimate winner of the election. The farsighted voter would be better off to misrepresent his or her true preferences in the assumed preference ranking $A \succ B \succ C$ and vote for *B* in the first round election between *A* and *B*, to possibly lead to ***BMA***, with the hope that the result ***BMC*** would have *B* as the ultimate winner from the second stage PMR vote.

This farsighted voter might then at least get the middle-ranked preference as a winner with this strategy, since it is known that the most preferred candidate can not win, from the knowledge of the fact that voters' preferences are such that ***CMA***. In this example, the election of *B* would result in a "stable outcome" for the farsighted voter. Since this analysis is done on triples of candidates, a farsighted voter is only assumed to have some knowledge of what the outcome of the very next, or second-round, election in the sequential elimination process on a triple of candidates would be. Rubinstein (1980) extends this notion from individual farsighted voters to majority coalitions of farsighted voters, and proves that a "stability set" will never be empty when voters have linear preference rankings on all issues. LeBreton and Salles (1990) and Chakravorti (1999) consider farsighted voters who have a knowledge of future majority rule votes on pairs that extends beyond the very next stage with sequential elimination by majority rule.

As a result of all of this discussion, we find that there are two types of misrepresentation that might take place in the presence of PMR cycles. First, misrepresentation might be used by a chairperson in the determination of the sequence in which issues are introduced for possible elimination when the agenda is being set. Once the agenda has been established, voters might then misrepresent their preferences at some stage of majority rule voting in order to obtain a more desirable outcome than they would get from sincere voting. Black (1958) conjectured that the issue that is entered latest in such a series of sequential elimination votes is most likely to me the ultimate winner of an election. Niemi and Rasch (1987) present an example to show that any given voter's overall utility is maximized by having issues introduced for sequential elimination in the reverse order of that voter's preference ranking on the issues.

It is usually difficult to determine if Condorcet's Paradox actually exists for any situation based only on the reported voting results from most elections, since voters do not normally report their complete preference ranking on candidates, or give their preferences on all pairs of candidates. Marz, et al. (1973) extend and correct results that were reported in Murkami (1968) to consider the number of PMR elections that would have to be held to determine if PMR cycles exist in a specific voting situation. In this work, only the pass-fail outcome is known for each PMR election that is held, and the actual vote count is not revealed from the PMR elections.

Fishburn (1980) considers the restrictions under which it is possible to determine whether, or not, the PMRW has been selected as the winner of an election, based only on the reported vote outcomes from the election. The severity of the restrictions that are required leads Brams and Fishburn (1983b, page 95) to conclude that

> "Because of the varieties of strategies that are allowed and the paucity of detail about how people voted, the likelihood of concluding that the winner is a (PMRW) …., is often small if not zero."

Thus, other factors about voting behavior must typically be assumed in any attempt to reconstruct the preferences of voters from the reported ballot outcome in an election, in order to determine which candidate is the PMRW. Many studies have used various methods to induce the complete preferences of voters from reported votes in actual elections to reconstruct either voting situations or voter preference profiles. William Riker performed several classic studies of this type, and his primary focus was on situations in which there is evidence that PMR cycles were intentionally created by different methods in order to alter the outcome of elections.

2.2 Riker's Empirical Studies

Riker (1982) reconstructs the preferences of 172 members of the U.S. House of Representatives on a vote that took place in 1846 regarding the Wilmot Proviso. The basic vote regarded an appropriation of $2,000,000 to facilitate the negotiation of a territorial settlement with Mexico at the end of the Mexican War. The Wilmot Proviso was an attachment to the appropriation, and it have would prohibited slavery in the land that was acquired from Mexico in the settlement. There were eight political groups within the House of Representatives that were voting on three possible options. The possible options and the reported voting situation results are given in Fig. 2.1.

It is impossible to determine how the Border Democrats and Border Whigs split their votes between the two possible rankings for their respective parties. As a result, we initially ignore the eleven members of these two groups and consider the results of the comparisons on pairs of alternatives for the remaining 161 representatives.

A = Approve the appropriation without the Proviso
B = Approve the appropriation with the Proviso
C = Take no action on either the appropriation or the Proviso

Political Group Ranking	Number of Voters	Preference
Northern Administration Democrats	7	*ABC*
Northern Free Soil Democrats	51	*BAC*
Border Democrats	8	*ABC* or *ACB*
Southern Democrats	46	*ACB*
Northern Pro-War Whigs	2	*CAB*
Northern Anti-War Whigs	39	*CBA*
Border Whigs	3	*BAC* or *BCA*
Southern and Border Whigs	16	*ACB*

Fig. 2.1 Voting situation results for Wilmot Proviso example. Reprinted from Riker (1982) by permission of Waveland Press, Inc.

Under pairwise comparison with the remaining representatives in Fig. 2.1, we find that *AMC* (120-41), *CMB* (103-58) and *BMA* (90-71). There is clearly a PMR cycle in the pairwise preferences of the 161 representatives. The margins of defeat in the pairwise votes show that the eleven members of the Border Democrats and Border Whigs could not have modified the outcome of this PMR cycle, no matter how they might have voted. Brams, et al. (1998) present a similar analysis of this result from Riker (1982).

Having established that Condorcet's Paradox can occur, other studies have considered the issue of the manipulation of election outcomes in these situations. Riker (1958) gives a detailed description of the procedure by which amendments can be proposed for addition to various bills before the U.S. House of Representatives, and the sequential elimination procedure by majority rule that is used to determine the winning outcome. A detailed examination is then made of the Committee of the Whole vote on the Agricultural Appropriation Act of 1953. The original bill provided a $250,000,000 appropriation to the Soil Conservation Service. Four different amendments were put forth by members of the House to modify the appropriation amount to: $142,410,000 (Javits Amendment), $100,000,000 (O'Toole Amendment), $200,000,000 (Andersen Amendment), and $225,000,000 (Whitten Amendment). The nature of the process that is used to vote on amendments makes it impossible to precisely obtain the preference rankings of the voters on the amendments. However, Riker (1958) argues very convincingly to show that some of the budget amendment amounts were involved in a PMR cycle. That cycle very likely contained the winning option, which was the original bill.

Riker (1958) then goes on to argue that the Whitten Amendment was intentionally introduced in order to create that cycle. Representative Whitten was managing the original bill through the House, and there was some uncertainty that it would obtain majority support. By introducing the Whitten Amendment, a PMR cycle was created, and Whitten was thereby able to take advantage of the sequential elimination election process on amendments to be assured that the original bill passed. Riker (1958) estimated that the House of Representatives and the Senate may have voting results that appear to have PMR cycles in more than ten percent of cases when two or more amendments are considered with an original bill. However, some of these cycles are contrived as a result of strategic manipulation, as in the case discussed above.

Riker (1965) considers another example from the U.S. House of Representatives in which an amendment appears to have been added to a bill to intentionally create a PMR cycle to affect the outcome of voting. The particular vote that is considered in the study was a bill in 1956 for appropriations for federal grants-in-aid to build schools. The Powell Amendment was proposed to limit aid to school districts that were segregated, giving three alternatives: The initial bill (*A*), The initial bill with the Powell Amendment (*B*), and the Status quo (*C*). The reported results of the voting were: ***BMA*** (229-197) in the first round, and then ***CMB*** (227-199) in the second round to result in *C* as the winner. Given the outcome and the established voting procedures, the initial bill was defeated, and no vote was taken for *A* versus *C*. However, in the following year the same proposal was considered, and the Powell Amendment was not brought forward. The result of the vote was ***AMC*** (217-209), indicating a PMR cycle with ***AMC***, ***CMB***, and ***BMA***. Riker suggests that there is some evidence that a subset of the voters misrepresented their true preferences in order to intentionally create this PMR cycle, to lead to the defeat the initial bill that was presented.

Jenkins and Munger (2003) discuss "killer amendments" like the Powell Amendment. They identify the conditions that are necessary for such an amendment to successfully defeat a bill, and they provide some other examples in which killer amendments have been successfully used. Their conclusion is that the procedure has only rarely been successful in the U.S. Congress, and that it has typically only been successful when the amendment involves race based issues, like the Powell amendment and the Wilmot Proviso.

Riker (1965) provides another example, to suggest that there are situations in which voters might misrepresent their preferences in voting on amendments so as to create a PMR cycle and obtain the desired outcome as a result of the voting procedure that is used to vote on amendments. The particular example that is considered is somewhat unusual since it regards voting to ratify the Seventeenth Amendment to the Constitution in the U.S. Senate in 1911. The problem is complicated by the fact that a 2/3 majority is required to pass an amendment to the constitution.

Riker (1982) analyzes the results of the U.S. Presidential election of 1860, when the four candidates were Bell, Breckenridge, Douglas and Lincoln. It is concluded that there were 15 feasible preference orders on the four candidates, and estimates of the proportions of voting blocs that held the various preference rankings on candidates are reconstructed. The results suggest that Breckenridge was the PMRL, while a PMR cycle existed on the remaining three candidates.

Given the preferences that were reconstructed, Lincoln was found to be the ultimate winner only because plurality rule was used as the election procedure, with Lincoln receiving only about 40 percent of the vote. Riker suggests that the PMR cycle in this example specifically arose as a result of the intentional introduction of the slavery issue into the election campaign.

Regarding his historical examples, Riker suggests that the existence of PMR cycles were typically created by the introduction of amendments, by the introduction of campaign issues, or by the misrepresentation of voters' preferences to manipulate the outcome. He is quoted [Riker, 1982, pg. 128]

> "So, I conclude that, because of agreement on an issue dimension intransitivities only occasionally render decisions by majoritarian methods meaningless, at least for somewhat homogeneous groups and at least when the subjects for decision are *not* politically important. When, on the other hand, subjects are politically important enough to justify the energy and expense of contriving cycles, Arrow's result (the presence of PMR cycles) is of great practical significance. It suggests that, on the most important subjects, cycles may render social outcomes meaningless."

Levmore (1990) generally agrees with Riker by suggesting that a link exists between the activity level of political interest groups and the level of stability in an election. That is, interest groups are most active in situations in which PMR cycles, or some other voting anomalies, are most likely to exist. The activity of the interest groups can then focus on agenda control, or on bargaining in the formation of winning coalitions with other groups, when the effort is most likely to have an impact on the election outcome.

Rasch (1987) examines the sequential elimination voting procedures that are used in the Norwegian Parliament and considers potential abuse of the process by manipulators. The conclusion is that manipulators of the system are more likely to be successful by misrepresenting preferences while establishing the order of voting for sequential elimination, rather than by misrepresenting preferences during the actual voting on issues during the election.

Chamberlin (1986) reaches a very similar conclusion as Riker did after evaluating several voting rules in the context of how election results might be examined in order to determine if voters had used strategic misrepresentation of preferences to alter the outcome of an election. The conclusion of the study is that the possibility that strategic misrepresentation has altered the outcome of an election is likely to be observed from the existence of PMR cycles in the reported preferences of voters. This connection between PMR cycles and strategic misrepresentation of preferences is based on Chamberlin's belief that PMR cycles in sincere preferences of voters should be very unlikely to be observed under normal conditions.

Tullock (2000) agrees with Chamberlin (1986) in suggesting that Riker's studies never really dealt with any actual voting situations that contained PMR cycles that are based on the true preferences of voters, and goes even father in commenting on the general value of such studies [Tullock (2000, pg. 13)]:

> "Altogether, throughout my life, I have tended to feel that the proof that cycles are present, usually connected with the name of Arrow, is mathematically interesting but has nothing to do with the way government actually operates."

Tullock's main argument seems to be that governments have set up various mechanisms in order to be assured that any potential PMR cycle would be resolved in some manner before it could ever reach an assembly for final voting. This could be accomplished through negotiating amendments that might be added to a bill to make it more appealing to some potential voters. Tullock's claim might therefore be valid for final voting situations in systems like those he describes. However, this simply moves the focus of attention away from finding PMR cycles in final voting situations. The focus would then be on the difficulties of resolving actual PMR cycles that *do* exist in voters' actual preferences before they get to the stage of final voting.

The ability of politicians to create PMR cycles to the degree that Riker suggests is disputed. Critical comments regarding Riker's empirical examples of PMR cycles that have been made by a number of people who share an interest in this area of study are given in Maske and Durden (2003).

Kadane (1972) presents an analysis of another aspect of considering the addition of amendments to bills. In particular, the process of "division of the question" is considered, when a coalition of committee members puts together a package of amendments that will pass by majority rule, when each individual amendment would have failed on its own. This condition is strongly related to the notion of vote trading.

2.3 Other Empirical Studies

Many other empirical studies have been conducted to find possible situations in which Condorcet's Paradox might have occurred. Flood (1955) presents an early result from a simple experiment in which 21 subjects initially gave their individual preference rankings on 16 different objects. The subjects were then divided into subgroups according to their reported preference rankings, in order to observe behavior in the subgroups while they were trying to select a most favored object that might be won from a reduced set of the original 16 objects. The subgroups were intentionally formed so that there were PMR cycles on the subgroup members' preferences for the specific reduced set of objects that each subgroup was considering. The complete initial preference rankings of the 21 subjects on all 16 objects are given in the paper (Table 2, page 4). The interested reader can verify that there is a PMRW (Item 12) for the set of all subjects, and that there is a PMR cycle (on Items 7, 10 and 15), so that PMR is not completely transitive for the set of all subjects.

Flood (1958) discusses a situation that was placed before the West Virginia Legislature in 1939, when elected representatives were attempting to use PMR to select a distribution of state aid for schools among 55 counties within the state. A preliminary study had been conducted to evaluate the possible distribution alternatives. The study evaluated the overall impact that various possible funding alternatives would have on the school system, in order to reduce the number of feasible options that had to be considered by the members of the Legislature. An attempt to select a winning alternative from the final list of alternatives was conducted by using PMR. It became apparent that the representatives were typically selecting the winner from a pair, based solely on a determination of which alternative in the pair would return the greater amount of aid to their respective counties. An attempt was made to find the alternative that would be selected under this apparent voting scenario and no PMRW was found to exist.

A simplified example is given in Fig. 2.2 to illustrate the type of decision-making process that was being used in this particular case, with three counties voting on four allocation alternatives $A,B,C,D\}$ to the counties:

	Allocation			
County	*A*	*B*	*C*	*D*
1	4	1	2	0
2	0	0	1	2
3	0	3	1	2

Fig. 2.2 An example voter preference profile from Flood (1958)

In this example, each allocation has the same total distribution of four units to the three counties. If each county prefers the alternative from any pair of alternatives that returns the greater distribution to itself, then majority rule voting would have *DMC* (2-1), *CMB* (2-1), and *BMD* (2-1). In particular, we would have *DMC* since both Counties 2 and 3 would be better off, with two units, under *D* than with *C*, with one unit. County 1 would prefer *C*, with two units, to *D*, with no payoff. This type of decision-making is identical to the model that led to vote trading cycles in the previous chapter.

Niemi (1970) examines the results from 22 university elections in which individuals were being elected to committees. Voters were not required to rank all candidates in these elections. In the analysis, all unranked candidates on a voter's ballot were assumed to be indifferent to each other, but all were ranked below the least preferred candidate that the voter actually did rank. Given the allowance of a response of indifference between candidates, the possibility of ties exists with PMR for odd n.

Elections were held for situations with three to 36 candidates, with the number of voters ranging from 81 to 463. There were 18 elections on six or fewer candidates, and four of these elections contained some form of Condorcet's Paradox. One of the elections contained a strict PMR cycle, while the remaining three had PMR cycles that included tied PMR votes to form a *tie-cycle*. In each of these four elections, there was no strict PMRW. That is, no candidate could strictly defeat all other candidates on the basis of PMR, given that a PMR tie is not a win. Results indicate that the probability of observing PMR cycles tends to increase as the number of candidates increases.

Blydenburgh (1971) considers tax bills that were voted on in the U.S. House of Representatives when roll call voting was imposed. Tax bills were selected since they were voted on under closed voting rules, so that the attachment of additional amendments by representatives was not allowed. As a result, only a limited number of amendments were being considered in each case. Sequential elimination by majority rule was used for the predetermined sequence of votes on amendments. With roll call voting, the vote that is cast by each representative is recorded at each stage of the procedure, so it is possible to reconstruct most of the preference rankings for the individual representatives.

The Revenue Act of 1932 was considered first, for which three alternative forms of taxes were being considered to raise additional revenue, when only one of the options would to be implemented. Fig. 2.3 shows the three tax options and the voting situation results for the six different voter groups that were identified.

S = Sales Tax
I = Income Tax
E = Excise Tax

	Group 1	Group 2	Group 3	Group 4	Group 5	Group 6
	I	E	S	S	S	$S \sim E \sim I$
	E	S	I	E	(E,I)	
	S	I	E	I		
$n_i =$	162	38	16	69	71	30

Fig. 2.3 Voting situation for Revenue Act of 1932 from Blydenburgh (1971)

The notation $E \sim I$ for Group 6 in Fig. 2.3 indicates that a voter in that group is indifferent between implementing E and I, while the notation (E,I) for Group 5 indicates that the preference on the pair could not be determined for members in that group, based on the way that they voted in the sequential election. If we ignore the representatives in Group 6, since they are completely indifferent between the alternatives, a total of 184 votes are required for a candidate to win by majority rule. The rankings produce the result *EMS* (200-156) and *SMI* (194-162). A PMR cycle exists if we have *IME*, but the known preference relations for representatives only give a vote of 178-107. Blydenburgh goes on to produce strong evidence from other sources to induce the relative preferences on I and E for some of the representatives in Group 5, to obtain the required number of votes for I to obtain support from a majority of all voters. It therefore appears to be very likely that there would have been a PMR cycle in this situation if a PMR vote between I and E had been held.

Blydenburgh (1971) performs a similar analysis on the Revenue Act of 1938. This vote was on three alternatives: the Act in its original form, the Act with the deletion of a corporate tax, and the Act with the addition of an excise tax on pork. Based on the voting results, there did not appear to be a PMR cycle in this particular example.

Fishburn (1973b, pgs. 89-90) presents an example with 175 voters who gave complete preference rankings on five different options for naming a church congregation that was being formed from two separate congregations. Results show that the PMR relationships from the preference rankings were completely transitive.

Bjurulf and Niemi (1978) examine the results of voting in the Swedish Parliament on a number of issues. One particular issue centered on the construction of a hospital in Stockholm in 1931. Three options were considered: Build the hospital as planned (*A*), build the first stage of the hospital according to plan with sharp cuts in expenditures for additional sections (*B*), and do not build the hospital (*C*). The order of presentation of the votes had *B* versus *C* first, with *CMB* (46-41). Option *C* was then carried forward to the second stage versus *A*, with *AMC* (54-16). There was no vote on *A* versus *B*, but the preferences of the voters were reconstructed to indicate that *B* would have defeated *A* under a PMR vote. The results then produce a PMR cycle: *BMA*, *AMC* and *CMB*. The study also presents other examples in which it appears that some members of the parliament strategically misrepresented their preferences in order to create a "strategic cycle" in PMR voting, following the notions of the Riker studies.

Dyer and Miles (1976) perform an analysis of rankings of 36 trajectory options by ten different specialty teams that were working on the Mariner Jupiter/Saturn Project for the U.S. National Aeronautics and Space Agency. The trajectory options were being evaluated under different criteria by the different specialty teams, and the teams responded with indifference on some pairs of options. The analysis in the study gives a PMR relationship on each pair by considering only groups that reported some preference on a pair. This follows from the observation that they report that Option 29 beats Option 31, when the rankings give a pairwise comparison of five to three with two indifferent groups on the pair. The study notes that there is a transitive PMR ranking among a set of the ten trajectory options that were found to be highest ranked by other analysis.

There is indeed a PMRW when all 36 options are considered. However, a further analysis that was not considered in the study (Table 2, page 229) indicates that PMR cycles do exist for some of the trajectory options. For example, a cycle can be found to exist with Option 10 beating Option 4 (7-3), Option 4 beating Option 25 (5-4) with one indifferent group on the pair, and Option 25 beating Option 10 (7-3). It is very interesting to note that Options 10 and 25 were both included in the set of the 10 top ranked trajectories with transitive PMR comparisons, while Option 4 was not.

Dobra and Tullock (1981) examine preference rankings from a committee vote to select a department chair at a university. Six committee members provided thermometer ratings on 37 candidates on each of six different attributes. In using *thermometer scores*, voters give a score to rate each candidate, using a scale of numbers like one through ten, where one is the worst possible rating and ten is the best possible rating. If a particular voter gives a higher thermometer score to Candidate *A* than to Candidate *B*, it follows that the voter has a preference $A \succ B$. Since individuals could give multiple candidates the same thermometer score on any attribute, ties in individual rankings were permitted. After accounting for abstentions and effective abstentions, candidate rankings could be determined for four committee members on two candidate attributes, scholarly competence and an overall evaluation. PMR led to a PMRW for scholarly competence. On the attribute of an overall ranking there was a candidate, *A*, who could beat or tie all other candidates. Candidate *A* tied with *B*, *C* and *D*. However, it is noted that there was a *PMR* tie-cycle since there was a fifth candidate, *E*, such that *AME*, *EMB*, and *B* tied with *A* under PMR.

Toda, et al. (1982) use thermometer scores from survey results of 5,281 subjects in Japan. The survey was designed to measure the relative degree of importance that subjects placed on six different health related issues. The PMR relationships that were reconstructed from the thermometer scores were found to be completely transitive.

Dobra (1983) summarizes the results of 32 different elections in which preference rankings on candidates were obtained from voters. The number of voters ranged from four to 27 and the number of candidates ranged from three to 37. A strict PMR cycle was found in one of the elections, and PMR tie-cycles were found in three of the elections. In further analysis of the specific elections that were considered, Dobra (1983) suggests that these results might tend to overstate the actual likelihood that a PMR cycle would actually be observed. General conclusions from the results suggest that PMR cycles are most likely to occur when the number of candidates is large, relative to the number of voters.

Chamberlin, et al. (1984) examine voters' rankings of candidates in five different annual elections for the position of president of professional societies. In each case there were five candidates being considered. The number of voters ranged from 11,560 to 15,499. Voters were asked to rank all of the candidates in each case, but they did not always do so. Complete rankings were reconstructed in this situation in two different ways. In the *impartial scenario*, the subset of voters who ranked only $k = 1,2,3,4$ candidates were partitioned equally into groups of voters with all possible complete rankings that had the first k candidates ranked the same. In the *proportional scenario*, these voters were partitioned proportionally to complete rankings. The proportion of each ranking in the partition was consistent with the proportions of voters who reported preferences on all candidates, with identical rankings on the first k candidates. No PMR cycles were found in any of the ten different situations that were considered.

Dietz and Goodman (1987) perform an analysis of the 1983 mayoral election in Lima, Peru. Preliminary survey results were used as a basis for partially reconstructing the proportions of the population that had various pairwise preferences on pairs of candidates from the four leading candidates. Different sets of assumptions were used to complete the analysis for the pairwise preference proportions. There was no significant evidence to support the notion that any PMR cycles existed on the four candidates. However, evidence was reported to suggest that the candidate that was elected by plurality rule was not the PMRW, so that some form of Borda's Paradox might have been observed.

Hill (1988) discusses elections by the Royal Statistical Society that were conducted to select one council member to serve as a member for the President Nominating Committee. Reported results show that a clear PMRW and PMRL typically exist, with cycles and ties most likely to be observed among middle ranked candidates. Based on observations from 17 different elections, there was a tie for first place one time, and a "semi-paradox", or PMR tie-cycle, for winner once. No example of a strict paradox that did not involve ties in PMR comparisons was observed.

Rosen and Sexton (1993) discuss a proposed trading of water rights from the Imperial Irrigation District (IID) to the Metropolitan Water District (MWD) in California. The four alternative policy options that were available to members of the IID are shown in Fig. 2.4.

P1 =	Do not participate
P2 =	Participate in the trade, with all trade revenue being invested in IID system conservation to maintain the requirements for water within IID.
P3 =	Apportion the IID water rights to members in proportion to the assessed valuation of their property, and then allow members to sell their individual allotments back to a Board of Directors for transfer to MWD
P4 =	A mix of *P2* and *P3*.

Fig. 2.4 Four policy options from example in Rosen and Sexton (1993)

Survey results from 31 major interests in IID indicate that *P2* was the PMRW and that *P3* was the PMRL, so that there were no PMR cycles.

Radcliff (1994) examines U.S. Presidential elections. Voters' preference rankings were reconstructed from thermometer scores and other information that was based on reported voting intentions for elections in 1972, 1976, 1980 and 1984. Complete and transitive PMR rankings were found in each of the four years. In 1972 the ranking was Nixon ***M*** Humphrey ***M*** McGovern. In 1976 the ranking was Carter ***M*** Ford ***M*** Reagan. In 1980 the ranking was Reagan ***M*** Carter ***M*** Anderson. In 1984 the ranking was Reagan ***M*** Hart ***M*** Mondale.

Abramson, et al. (1995) examine presidential elections in the United States for three recent cases in which significant third-party candidates were present on the ballot. The three cases that were considered include: Wallace in 1968 against Nixon and Humphrey, Anderson in 1980 against Carter and Reagan, and Perot in 1992 against Clinton and Bush. Thermometer score ratings on the candidates were examined to reconstruct the preferences of confirmed voters to indicate that a PMRW existed in all three cases, and that the PMRW did in fact win in the final election. Each third-party candidate was found to be the PMRL in the respective elections. Brams and Merrill (1994) use thermometer scores to obtain the same result for the 1992 election. Kiewiet (1979) also uses thermometer scores to evaluate the 1968 election between Humphrey, Nixon and Wallace to conclude that Nixon was the PMRW.

Gaubatz (1995) analyzes the results of a number of different public opinion polls that were taken for American citizens with regard to various aspects of the use of military force to reverse the Iraqi invasion of Kuwait in 1990. There were four different options that were available for consideration, and the population was partitioned into six possible subgroups, with the preference rankings on the four options being determined for each subgroup. Public opinion poll results were used to obtain estimates of the percentage of the population that fell into each possible subgroup. The results are summarized in Fig. 2.5

W - Withdrawal or do nothing
S - The imposition of multilateral sanctions
U - The use of unilateral military intervention
M - The use of multilateral military intervention

Subgroup	Percentage	Ranking
Unilateral hard-liners	18%	*UMSW*
Soft Unilateralists	7%	*WSUM*
Internationalist multilateralists	26%	*MSUW*
Accommodationist multilateralists	19%	*SWMU*
Forceful isolationists	13%	*WUMS*
Restrained isolationists	17%	*WSUM*

Fig. 2.5 Voting situation for example from Gaubatz (1995)

Given the estimated population proportion values in each of the subgroup types in Fig. 2.5, along with the associated preference rankings on options for each subgroup, it is concluded that there is no PMRW in the set of four options that were available. PMR cycles were found for *WMMMUMW* and *SMMMUMS*.

Browne and Hamm (1996) perform an extensive analysis of a series of votes that led to the passage of the 1951 Electoral Reform Act in Fourth Republic France. The options under consideration were to maintain the existing voting system that was based on proportional representation, to adopt a one-ballot majority list system, and to adopt a two-ballot majority system. The 621 voters in the assembly were partitioned into 17 categories based on the recorded history of their votes on earlier issues. A very thorough analysis is then conducted to conclude that a PMR cycle existed in the preferences of the assembly members, which explains the extreme difficulty that was observed in obtaining the final outcome. Moreover, it is concluded that this PMR cycle actually existed in the true preferences of the assembly members, and that it did not result from strategic voting to manipulate the outcome.

Lagerspetz (1997) analyzes a complex procedure that was used until 1994 in Finland in which a College of Electors elected the President. The preference rankings of voting blocs of the major political parties in ten different elections were partially reconstructed from information that was gained at each step in the multiple stage election procedure. Various other sources of information were also used to further reconstruct the preference rankings of the voting blocs. Of the ten cases that were available for analysis, there were two clear cases, and one possible case, of a PMR cycle.

Beck (1997) examines the results from elections regarding changes to the business curriculum at a university. Twenty faculty members ranked options that were being considered in three different elections. Rankings of four different options that were related to community service requirements for graduation resulted in a transitive ranking by PMR. Rankings on eight different options that were related to additions to core courses resulted in a situation with no PMRW. Rankings on six options that were related to other degree requirements resulted in transitive rankings by PMR.

Flanagan (1997) investigates a series of votes before the Canadian Parliament in 1988, regarding a set of possible bills that were related to abortion rights issues. The bills were presented for consideration in a series of pairwise elimination votes. Three distinct blocs of voters are found: "pro-life", "pro-choice" and "compromise". Any combination of two of the blocs would have been able to form a majority of voters. The preference rankings on the bills that were being presented were reconstructed for each of the blocs to show that a PMR cycle existed. The sequence of the presentation of the options ultimately led to the retention of the status quo situation, and this selection resulted solely from the order of presentation that was used in the sequential elimination process.

Morse (1997) presents an analysis of the political situation that led to the American action to annex the Republic of Texas. The Republic of Texas became independent from Mexico in 1836, and shortly thereafter residents of the Republic petitioned the United States to grant it statehood. Formal action was not taken on the issue until 1844. There were two major issues that were related to the annexation. First, there was a consideration as to whether the annexation would entail a complete annexation of the territory, or a more moderate expansion of the territory. Given the historical background of the time, a second major issue was whether Texas would be admitted to the Union as a slave state or as a free state. The two major political parties consisted of Democrats, who favored significant territorial expansion, and Whigs, who favored more a moderate approach to territorial expansion. Senators from northern states favored admission of free states, while senators from southern states favored admission of slave states.

The composition of the U. S. Senate in 1844 showed a nearly identical split between: Southern-Democrats, Southern-Whigs, Northern-Democrats and Northern-Whigs. It is argued that this balance led to a situation in which no PMRW existed among the options that were available. President John Tyler initially presented the issue of the Annexation of Texas to the Senate as a foreign policy issue. It is argued that Tyler intentionally did this in order to gain authority that is granted to the President by the Constitution on such issues. Tyler was then allowed to set the agenda for voting in order to obtain the outcome that he sought. That is, a complete annexation of the Republic of Texas, with a decision on the slavery issue that was very sympathetic to Southern interests. This outcome led immediately to a war with Mexico and to an unsettled situation regarding the slavery issue in Texas that was not resolved until 1846, as a result of the Wilmot Proviso that was discussed previously in the Riker studies.

Taylor (1997) considers the 1980 election between candidates for the position of U.S. Senator representing the state of New York. The three candidates were Alphonse D'Amato (*D*) who was a conservative candidate, Elizabeth Holtzman (*H*) who was a liberal candidate, and Jacob Javits (*J*) who was a moderate liberal candidate. Results that were obtained from exit polls during the election were used to reconstruct the percentages of voters with various preference rankings on the three candidates. The results are summarized in Fig. 2.6.

H	*D*	*D*	*H*	*J*	*J*
J	*J*	*H*	*D*	*H*	*D*
D	*H*	*J*	*J*	*D*	*H*
29%	23%	22%	15%	7%	4%

Fig. 2.6 Voting situation from example in Taylor (1997)

The results in Fig. 2.6 indicate that *HMD* (51%-49%), *HMJ* (66%-34%) and *DMJ* (60%-40%). Thus, Holtzman is the PMRW, so that Condorcet's Paradox did not occur. However, the winner by plurality rule was D'Amato, with 45% of the first place votes, compared to the 44% obtained by Holtzman, to indicate that some form of Borda's Paradox might have occurred.

Hsieh, et al. (1997) perform an analysis that is based on telephone surveys of voter preferences that were taken before the 1994 elections for mayor of Taipei City. The pre-election telephone survey asked questions that allowed for a partial reconstruction of the ranked preferences of the 450 respondents for the three candidates in the election. The primary observation of interest to our study is that their results indicated that PMR was transitive for the three candidates.

Taplin (1997) evaluated the preference rankings of twelve homeowners on four different species of grass for lawns. The rankings resulted in completely transitive outcomes by PMR.

Regenwetter and Grofman (1998) examine the ballots from elections in seven different professional societies. A procedure was implemented to reconstruct the complete preference rankings for voters from the ballot results that were reported. Results indicate that PMR was completely transitive in six of the cases, while the determination as to whether a PMR cycle existed in the seventh case was not conclusive.

Truchon (1998) considers rankings that were reconstructed from judges' scoring during the evaluation ice skaters in 24 different competitions in Olympic Games, following to the rules of the International Skating Union. PMR cycles were found in the rankings of 15 of the 24 competitions, with cycles involving as many as nine skaters. The cycles typically contained skaters that were ranked near the middle of the competitors. However, one of the cycles contained a skater who was ultimately ranked third in the competition. There was always a PMRW from the rankings, with a tie in one case.

Van Deemen and Vergunst (1998) use results from surveys of approximately 1500 respondents to questions asked in the Dutch Parliamentary Election Study of 1982, 1986, 1989 and 1994, to consider the probability that a PMRW existed for competing parties in the four elections. In that survey, questions were asked in the form of thermometer ratings regarding the probability that the respondent would cast a vote for each of the competing parties in the respective elections. A preference ranking on the parties was reconstructed for each voter, based on these responses. Results showed that respondents' preference rankings gave completely transitive PMR comparisons in each of the four elections: 1982 (13 parties), 1986 (12 parties), 1989 (9 parties) and 1994 (9 parties).

Stensholt (1999a) presents an example of a voting situation that was under consideration by the Norwegian National Assembly (Stortinget) in 1992. Voting was being held to determine the location of a proposed major airport. After the elimination of a number of competing locations, three major options $F,H,G\}$ remained. There were 165 voting members with preference rankings on the options that are reconstructed in the voting situation in Fig. 2.7.

	F	H	H	G	G
	H	F	G	H	F
	G	G	F	F	H
$n_i =$	42	22	37	1	63

Fig. 2.7 Voting situation from example in Stensholt (1999a)

If PMR comparisons are made, we find ***FMH*** (105-60), ***HMG*** (101-64) and ***GMF*** (101-64), to produce a PMR cycle.

Kurrild-Klitgaard (2001) reports survey results of pairwise preference comparisons of three possible candidates, $P,H,U\}$, for the post of Prime Minister of Denmark. The proportions of voters with given pairwise preferences are reported as the percent of voters on a pair for which a preference was expressed, with voters who gave no response being excluded. It was found that *H* defeated *U* (50.6%-49.4%), *U* defeated *P* (51.1%-48.9%) and *P* defeated *H* (52.8%-47.2%), to produce a PMR cycle.

Regenwetter, et al. (2002a) examine survey results from eight different three-candidate elections in three different countries. Results show that no PMR cycles were found in any of the survey results. However, it was discovered that the use of samples of survey respondents could frequently misrepresent the true pairwise preference ranking of the entire population of survey respondents, unless the size of the sample was relatively large.

Regenwetter, et al. (2002b) look at thermometer scores from preference surveys in four different three-candidate U.S. Presidential elections and modify the procedure of determining subject preferences from these scores. Here, higher thermometer scores represent higher approval ratings from subjects on a 0-100 scale. Previous studies have considered subject's responses and have assumed that Candidate *A*, with thermometer score $TS(A)$, is preferred to Candidate *B* in a subject's preference ranking if $TS(A) > TS(B)$. The study makes an interesting modification to this type of analysis to account for the fact that subjects are not perfect discriminators in assigning scores to candidates, and therefore determine a subject's preference for Candidate *A* over Candidate *B* only when $TS(A) - TS(B) > \varepsilon$ for a specified positive ε. Increasing ε reflects a decreased belief in a subject's ability to accurately assign precise scores to candidates.

Results of the 1968 and 1992 election results show transitive PMR rankings Nixon ***M*** Humphrey ***M*** Wallace and Clinton ***M*** Bush ***M*** Perot respectively for all ε. The results from these two elections show a robust definition of the PMR rankings from the survey results. However, the same outcome is not observed in the remaining two elections. The 1980 election results have transitive PMR results in all cases, but the rankings are Carter ***M*** Reagan ***M*** Anderson for $0 \le \varepsilon \le 30$ and Reagan ***M*** Carter ***M*** Anderson for $\varepsilon > 30$. Similarly, the 1996 results have Dole ***M*** Clinton ***M*** Perot for $50 < \varepsilon < 84$ and Clinton ***M*** Dole ***M*** Perot for all other ε. No PMR cycles were found in any case, but the PMR rankings can obviously change with different models.

Wilson (2003) gives voting results from the International Olympic Committee with regard to the selection of the location for the 2010 Winter Olympics. The selection was conducted by using a two-stage plurality elimination procedure. The candidate that would have been the plurality winner in the first stage was not the PMRW, but the PMRW was ultimately selected in the second stage. The proposed locations were Pyeongchang (South Korea), Salzburg (Austria) and Vancouver (Canada). In the first round of voting by plurality rule, the vote outcome was South Korea (51), Canada (40), and Austria (16) so that South Korea would have been a clear winner by simple plurality voting. The second round of the elimination procedure used a majority rule election between South Korea and Canada that resulted in the selection of Canada as the winner by a 56-53 margin over South Korea. This pairwise vote outcome, coupled with the margin of votes in the first stage plurality election, indicate that Canada was the PMRW. A PMRW did exist, but the use of simple plurality rule would not have selected it, indicating that some form of Borda's Paradox would have occurred if simple plurality rule had been used.

Gehrlein (2004a) considers situations in which evaluators were ranking grant proposals that had been submitted by faculty members in the same college at a university. The eight highest ranked proposals were to be awarded research grants of $6,000 each. In one year, a group of five evaluators each ranked 12 proposals and the PMR rankings that resulted were completely transitive. In the second year, a different group of five evaluators each ranked 18 proposals. There were both a PMRW and a PMRL in the second year. However, there were numerous PMR cycles on proposals that were positioned near the center of the PMR rankings in the second year.

Tideman (1992) performs an extensive study that examines the results of 84 different elections that were overseen by the Electoral Reform Society of Great Britain and Ireland, along with the results of three additional elections. Voters were requested to rank all of the candidates in all cases, but they did not always do so. Candidates that were not reported in a voter's ranking were all listed as being indifferent to each other, and they were all ranked at the bottom of the voter's preferences. The number of candidates ranged from three to 29 and the number of voters ranged from nine to 3,500. There was complete transitivity, allowing for ties, by PMR voting for 61 of the 87 elections.

Tideman makes a number of very interesting general observations for the 26 elections that were considered for which PMR was not completely transitive. All of these observations are completely consistent with the results of all of the empirical studies that have been mentioned above:

- Elections with a few candidates almost always have transitive PMR orderings.
- Pairs of candidates that are ranked by a small number of voters are more likely to be involved in a PMR cycle than pairs that are ranked by many voters.
- The size of majorities on pairs that are involved in PMR cycles tends to be small, even after accounting for the fact that these typically involve a small number of voters.
- Candidates involved in PMR cycles tend to be located near the center of the overall PMR ranking. So, candidates that are most preferred, of most disliked, by the electorate are not likely to be involved in PMR cycles.
- PMR cycles typically contain pairs that are ranked relatively close together in the overall PMR ranking.

2.4 Monte-Carlo Simulation Studies

Other empirically based studies have proceeded to analyze actual voting data for observations of the existence of Condorcet's Paradox in a different fashion. These studies proceed in a two-step process. In the first step, a voting situation that shows the number of voters with each possible preference ranking on candidates is reconstructed from actual election data, using some method like the ones described in the survey of empirical studies in the preceding section. The relative proportion of voters with each of the possible preference rankings is then computed to represent the probability that a randomly selected voter from a population of voters will have the associated preference ranking on candidates. In the second step, Monte-Carlo simulation is used to obtain a voter preference profile by sequentially generating random voter preference rankings on candidates for each voter. Each voter's preference ranking on candidates is generated randomly and independently according to the probabilities for voter preference rankings on candidates that were obtained in the first step. After obtaining a number of different voter preference profiles in this fashion, the probability of observing Condorcet's Paradox is then estimated by the proportion of the randomly generated voter preference profiles that exhibit the paradox.

Bowen (1972) does an analysis of this type on roll call votes from the U. S. Senate in the years 1958, 1960, 1962, 1965 and 1966. The determination of the count of senators who had given preference rankings in the first stage for each case followed the same procedure as described above in the work of Blydenburgh (1971). As in the Blydenburgh study, it is not always possible to reconstruct the complete preference rankings for each preference structure type. When there were unknown pairwise comparisons on alternatives, as in Group 5 of the Blydenburgh study, Bowen (1972) splits voters for the preference equally among the complete preference rankings that could be obtained from the preference structure being considered, in general agreement with the notion of the "impartial scenario" in Chamberlin, et al. (1984). For example, Bowen split the 71 voters in Group 5 of the Blydenburgh study into 35.5 voters with the ranking $S \succ E \succ I$ and 35.5 voters with $S \succ I \succ E$.

The Monte-Carlo simulation step of the Bowen (1972) study used the accumulation of all extended ranking results from the first stage, to compute probabilities that randomly selected senators would have an associated complete preference ranking on the alternatives. Random profiles were generated to determine the proportion of randomly generated profiles that did not have a PMRW. A hypothesis test was used to determine that thirteen roll call votes in the set of votes being considered had a significant probability that no PMRW existed. Most notably, the Food and Agriculture Act of 1962, with nine amendments, exhibited Condorcet's paradox with probability 0.965, the Wheat Act of 1960, with three amendments, exhibited the paradox with probability 0.940, the Economic Opportunity Amendments of 1965, with sixteen amendments, exhibited the paradox with probability 0.630, and the Housing Act of 1960, with three amendments, exhibited the paradox with probability 0.460.

Weisberg and Niemi (1972) perform an analysis of roll call votes following the same format as Bowen (1972), but Weisberg and Niemi (1972) use a more sophisticated process for evaluating the probabilities that randomly selected voters would have various preference rankings. Bowen (1972) did an equal splitting of voters over possible preference rankings for preferences on pairs that were not observed from voting, but Weisberg and Niemi (1972) use a model that reconstructs preference rankings for some of the voters, and then does an equal split over possible preference orders for remaining voters. Results give a decreased estimate for the probability that a PMRW does not exist, with the probability that there is no PMRW dropping to 0.198 for the Wheat Act of 1960, as compared to the Bowen (1972) estimate of this probability as 0.940.

Jamison (1975) performs an empirical study to try to determine estimates for the probability that a PMRW exists and for the probability that PMR is completely transitive. There were two groups of subjects in the study, a set of 67 graduate students and a set of 42 undergraduate students. Each subject in both groups was required to rank their preferences, without ties, on: nine potential presidential candidates, 12 types of soup, and 11 European cities that could be toured. The probabilities for given subject rankings on candidates were obtained directly from the reported rankings.

Simulation analysis was then performed in the second stage on the six different combinations that were sampled. The analysis was performed in the same way in each of the six cases. For each number of subjects $n = 3(2)15$, preference rankings were sequentially selected at random, without replacement, from the preferences reported by respondents in the particular case being considered to obtain a voter preference profile for n individuals. Then, for each number of alternatives $m = 3,4,5,6$, a randomly selected set of alternatives was obtained for the particular case. Each of the n individual preference rankings on the m alternatives was then reduced, by removing all alternatives that were not included in the selected set of alternatives. A determination was then made as to whether or not a PMRW existed, and whether or not the PMR relationship was transitive. The process was repeated 4,500 times to obtain the proportion of times that there was a PMRW and the proportion of times that PMR was transitive.

Over all of the cases that were considered, a PMRW existed with a probability ranging between 0.80 and 0.85, while the probability that the PMR relation was transitive ranged between 0.60 and 0.80. Results consistently showed that both probabilities were minimized when there were five voters. For all cases and all numbers of alternatives, both probabilities tended to increase monotonically as the number of voters increased or decreased from five. These observations are consistent with the conclusions from Tideman (1992).

Chamberlin and Featherston (1986) develop a more sophisticated Monte-Carlo simulation model that generates voter preference profiles in a fashion that is more consistent with observed election outcomes. The model randomly generates voter preference profiles in a manner that is similar to the simulation studies described above. However, the probabilities that voters have given preference rankings on candidates are assumed to be random variables themselves, following a Dirichlet probability distribution. The associated parameters of the Dirichlet distribution are determined on the basis of observed voter preference rankings from a given election outcome.

Once the parameters of the Dirichlet distribution have been established for a particular voting situation, it is possible to randomly generate the proportions that are then to be used as a basis for generating a random voter preference profile. A different randomly generated set of proportions is used for the generation of each random voter profile. It would be anticipated that the particular voter preference profiles that are obtained by this process would tend to have the same general type of preference structure as the observed preferences from the election results that were used to obtain the Dirichlet parameters.

The data from the five different elections for president of a professional society, as considered in Chamberlin, et al. (1984), were used as a trial. It is assumed that the same basic type of voter preference profile structure was consistent over all elections, and the data from the five elections were used to compute the parameters of the Dirichlet distribution that would randomly generate proportions of voters with given preference rankings in voter preference profiles. Tests show that the procedure randomly generated profiles that were very similar in nature to the observed election profiles. A sample of 1,000 randomly generated voter preference profiles following this specific scenario found that there were no PMR cycles in any of the profiles. This observation is completely consistent with the fact that no PMR cycles were found in any of the original election outcomes in Chamberlin, et al. (1984).

The line of research that is suggested by these Monte-Carlo simulation studies gives a very different slant to the notion of estimating the likelihood that Condorcet's Paradox might occur. There is typically extreme difficulty involved in trying to reconstruct the preference rankings of voters from the limited information that is given in most elections. By appealing to simulation analysis of the type suggested here, we open the door to the notion of evaluating the impact of varying the propensity of voter preference profiles have different types of characteristics. It is a very interesting idea to rate the characteristics of voter preference profiles on the basis of their probability of causing Condorcet's Paradox to be observed, and that topic will be addressed in detail in later chapters. Our next step is to turn attention to the historical sequence of the research that leads up to that topic.

2.5 Conditions that Prohibit Condorcet's Paradox

It is clearly of interest to ask if there are some natural underlying conditions that are related to the process by which voters form their preferences on candidates that would make it impossible for any PMR cycles to occur. Black (1958) found this to be the case when voters' preferences are restricted to have the property of *single-peaked preferences*. To describe this property, we define a measure of preference or *utility*, $U^i(C_j)$, that a given i^{th} voter associates with candidate C_j in an *m*-candidate election with $\boldsymbol{C}^m = C_1, C_2, ..., C_m\}$. Increased measures of $U^i(C_j)$ indicate that a voter has an increased preference, or utility, for the given candidate, so the given voter's individual preference ranking on candidates will have $C_j \succ C_k$ if, and only if, $U^i(C_j) > U^i(C_k)$.

Consider a simple example voter preference profile with three voters, where each individual voter has a linear preference ranking on six candidates, as shown in Fig. 2.8 with:

Voter 1: $C_6 \succ C_3 \succ C_5 \succ C_1 \succ C_4 \succ C_2$
Voter 2: $C_4 \succ C_3 \succ C_6 \succ C_2 \succ C_5 \succ C_1$
Voter 3: $C_2 \succ C_4 \succ C_3 \succ C_6 \succ C_5 \succ C_1$.

Fig. 2.8 An example preference profile with three voters and six candidates

We can determine if the three voter's preference rankings in the example in Fig. 2.8 meet the definition of single-peaked preferences by trying to find $U^i(C_j)$ values that are consistent with the preference rankings of the individual voters, while simultaneously meeting an additional restriction. This additional restriction can be established by drawing a graph like the one that is shown in Fig. 2.9.

Values of $U^i(C_j)$ are displayed on the vertical axis of the graph in Fig. 2.9, and the horizontal axis of the graph represents the sequence of $C_j's$ that corresponds to some linear overall reference ranking. Let $C_i \boldsymbol{O} C_j$ denote the fact that C_i is ranked as preferred to C_j in this overall reference ranking. The specific overall reference ranking that is used in Fig. 2.9 is $C_2 \boldsymbol{O} C_4 \boldsymbol{O} C_3 \boldsymbol{O} C_6 \boldsymbol{O} C_5 \boldsymbol{O} C_1$. Fig. 2.9 shows a plot of possible $U^i(C_j)$ values for each voter, as associated with specific candidates in the sequence of C_j's in the overall reference ranking, such that the given $U^i(C_j)$ values for a given *i* would reproduce the linear preference ranking of the associated i^{th} voter in Fig. 2.8. The results that are displayed in Fig. 2.9 have $U^1(C_6) > U^1(C_3) > U^1(C_5) > U^1(C_1) > U^1(C_4) > U^1(C_2)$, to correspond with the linear preference ranking $C_6 \succ C_3 \succ C_5 \succ C_1 \succ C_4 \succ C_2$ for Voter 1. We do not claim that the $U^i(C_j)$ values in the graph represent the true utility values that voters have for candidates. The only claim is that they are possible utility values that would result in the voters' preference rankings on candidates.

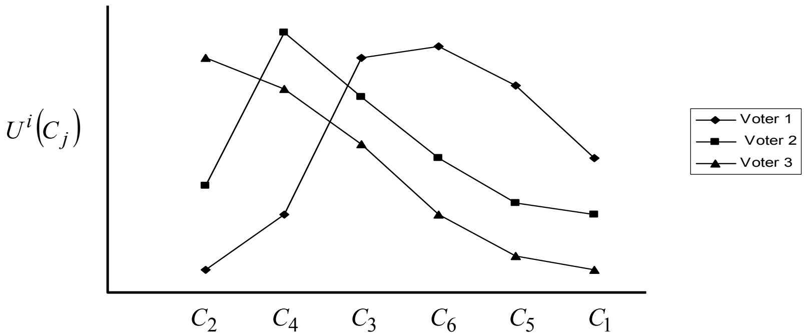

Fig. 2.9 A graph of single-peaked preference curves for three voters

Any of the possible 720 linear rankings on the six candidates could have been used as an overall reference ranking. However, the specific overall reference ranking used for Fig. 2.9 is of particular interest, since it results in plots of the possible $U^i(C_j)$ values that have single-peaked preference curves for each voter. Using the definition in Black (1958, pg. 7), a "single-peaked (preference) curve is one which changes its direction at most once, from up to down."

Single-peaked preferences curves would tend to suggest a situation in which all voters would tend to agree that some underlying characteristics of candidates would allow for the sequencing of the candidates in some natural order from left to right, according to their rankings in the overall reference order. Each voter would then have some particular most preferred candidate in the sequence, with decreasing preferences on candidates as they are ranked farther away, to the left or to the right, from their most preferred candidate within the sequence of candidates in the overall reference order. Black (1958, pgs. 8-9) suggests that this scenario might be plausible in many situations.

"While in practice a (committee) member's preference curve may be of any shape whatsoever, there is reason to expect that, in some important practical problems the (preference) valuations actually carried out will tend to take the form of isolated points on single-peaked curves. This would be particularly likely to happen if the committee were considering different possible sizes of a numerical quantity and choosing one size in preference to the others. It might, for example, be reaching a decision with regard to the price of a product to be marketed by a firm, or to the output for a future period, or the wage rate of labor, or the height of a particular tax, or the legal age of leaving school, and so on."

Buchanan (1970) and Browning (1972) also consider natural conditions that are likely to lead to the existence of single-peaked preferences. Gaertner (2005) notes that arguments that are based on the notions that lead to the definition of single-peaked preferences are found as far back as Pufendorf in the 17^{th} century.

Black (1958) then goes on to develop arguments that can be used to show that PMR will be transitive for odd n if *any* overall reference order and possible $U^i(C_j)$ values that are consistent with voters' preference rankings can be found to result in single-peaked preference curves for all voters. That is, all voters' preference curves must be single-peaked relative to the same overall reference order.

Arrow (1963) approaches the concept of single-peaked preferences by considering only the ordinal relationships between candidates in rankings, without using Black's $U^i(C_j)$ values. We begin by defining a relationship of having a candidate being located between two other candidates in the linear overall reference ranking. For any triple of candidates, $C_i, C_j, C_k\}$, let $B(C_i, C_j, C_k)$ denote that C_j is ranked between candidates C_i and C_k in the overall reference ranking, so that either $C_i\boldsymbol{O}C_j\boldsymbol{O}C_k$ or $C_k\boldsymbol{O}C_j\boldsymbol{O}C_i$.

Voters' preference rankings are single-peaked if there exists a complete and transitive overall reference ranking, such that for each triple of candidates, $C_i, C_j, C_k\}$, if a voter has an individual preference $C_i \succ C_j$ and $B(C_i, C_j, C_k)$ in the overall reference ranking, then the voter must have $C_j \succ C_k$ as an individual preference. Arrow (1963) goes on to prove that if voters' preference rankings on candidates meet this definition of single-peaked preferences, then PMR will be transitive for odd n.

It is easily verified that these two definitions of single-peaked preferences are equivalent. In addition, we find that the definition provided by Arrow (1963) leads to a simple way of determining if voters' preferences are single-peaked. If we take any triple of candidates, $C_i, C_j, C_k\}$, suppose that $B(C_i, C_j, C_k)$ in the overall reference ranking, and let C^* denote the candidate that a given voter ranks as most preferred from the set of all candidates. If C^* is C_i or if $C^*\boldsymbol{O}C_i$ in the overall ranking, then the given voter will have preferences on the triple with $C_i \succ C_j \succ C_k$, with C_k being ranked last in the triple. If C^* is C_k or if $C_k\boldsymbol{O}C^*$ in the overall ranking, then the given voter will have preferences on the triple with $C_k \succ C_j \succ C_i$, with C_i being ranked last in the triple. If $B(C_i, C^*, C_k)$ in the overall ranking, the given voter could have preference rankings that include any of any of $C_i \succ C_j \succ C_k$, $C_k \succ C_j \succ C_i$, $C_j \succ C_k \succ C_i$ or $C_j \succ C_i \succ C_k$, so that either C_i or C_k is ranked last in the triple. However, it is impossible to ever have either $C_k \succ C_i \succ C_j$ or $C_i \succ C_k \succ C_j$, with C_j being ranked last in the triple, if $B(C_i, C_j, C_k)$ and the given voter's preferences are single-peaked.

This leads to the following alternative definition that voters' preferences are single-peaked if for every triple of candidates, at least one candidate is never ranked last among the three by any voter. Arrow's definition lacks the intuitive appeal of Black's definition of single-peaked preferences, but it is an equivalent definition, and it will be very useful in later discussion. This is particularly true when attention is restricted to three-candidate elections. Numerous studies have been conducted to establish similar restrictions on voters' preferences that require PMR to be transitive.

Vickery (1960) discusses another condition on voter preference profiles that will guarantee the existence of a transitive PMR relationship for odd *n*. This condition exists when voters have *single-troughed preferences*. Like the condition of single-peaked preferences, this condition assumes that all voters agree on some common overall reference ranking of candidates along some dimension from left to right. Each voter would then have some particular least preferred candidate in the sequence, with increasing preferences on candidates as they are ranked farther away, to the left or right, from their least preferred candidate within the sequence of candidates in the overall reference order. Vickery (1960) points out that the condition of single-troughed preferences is equivalent to the assumption of single-peaked preferences, when all of the voter preference rankings are inverted. It then follows directly from Arrow's arguments that preferences are single-troughed if some candidate in every triple of candidates is never the most preferred candidate among the three for any voter.

Ward (1965) develops another restriction on profiles that will assure transitivity of PMR for odd *n*. Following earlier discussion, This condition requires that some candidate in every triple of candidates is never the middle-ranked candidate among the three candidates for any voter. Ward then combines all three conditions that we have discussed to develop an overall condition on voter profiles that assures the existence of a PMRW with odd *n*.

Ward's Condition is developed here, since it is relevant to later discussion. This condition holds on all possible triples from the *m* candidates, and is described in terms of "Condorcet triples" or in terms of a *Latin Square*. A Latin Square exists on a triple of candidates *A*, *B* and *C* if there are voter preference rankings within the profile that contain $A \succ B \succ C$ for some voters, $C \succ A \succ B$ for some voters and $B \succ C \succ A$ for some voters. It is clear why this is called a Condorcet triple, since a situation in which equal numbers of voters have each of these rankings on the three candidates, forming a *perfect Latin Square*, would result in the PMR cycle with *AMB*, *BMC*, and *CMA*. However, the numbers of voters with each ranking need not be equal for a Latin Square to exist. Ward's Condition requires that there are no Latin Squares on any triples of candidates in a profile.

This condition is clearly much less restrictive than any of the three previous conditions that have been defined. Ward's Condition only requires that some candidate either not be ranked first, or not be ranked second, or not be ranked third among voters' preferences on any triple. Each candidate in a triple could be ranked in third place by some voter in a voter preference profile, to violate Arrow's Condition. But, Ward's Condition could still be met for that profile as long as one of the three candidates was either not ranked first of the three candidates by any voter, or was not ranked second of the three candidates by any voter.

A number of papers have been written on the topic of determining the maximum number of different linear preference rankings, of the $m!$ possible rankings, on candidates that could be included in a profile and still meet Ward's Condition. Most of this work is based on *Craven's Conjecture* [Kelly (1991)], which is related to bounds on this number, and most of this work is summarized in Fishburn (1997). Sen (1966) extends the notion of Ward's Condition to situations in which each voter's preferences do not have to be complete, so that some voter indifference between candidates is permitted. In particular, Sen's condition of *Value-restricted Preferences* allows voters to have *weak ordered preferences*. An individual voter has weak ordered preferences if transitivity holds, and if the set of candidates can be partitioned into k equivalence classes $E_1, E_2, ..., E_k\}$ such that the voter is indifferent between all candidates within the same equivalence class, and if $C_x \in E_i$ and $C_y \in E_j$ with $i \neq j$ then the voter must have a preference on the pair with either $C_x \succ C_y$ or $C_y \succ C_x$. In addition, if $C_x \in E_i$ and $C_y \in E_j$ with $C_x \succ C_y$ for any $i \neq j$, then $C_z \succ C_w$ for all $C_z \in E_i$ and all $C_w \in E_j$. Inada (1964) proves that PMR must be transitive when individual voters have *dichotomous preferences*, which corresponds to the case of weak ordered preferences with $k = 2$.

Niemi (1983) describes a *semi-single-peaked condition* that is much less restrictive than the single-peaked condition. The semi-single-peaked condition will not guarantee that PMR is transitive, but it is sufficient to guarantee that a PMRW exists. The definition of the semi-single-peaked condition follows from the logic of our earlier discussion of the definition of the single-peaked condition. Suppose that all voters agree on a common overall linear reference ranking on the candidates, $C_1 \boldsymbol{O} C_2 \boldsymbol{O} C_3 \boldsymbol{O} .... \boldsymbol{O} C_{m-2} \boldsymbol{O} C_{m-1} \boldsymbol{O} C_m$, on some dimension. A profile is semi-single-peaked if there is some C_j such that a majority of voters have $U^i(C_{\ell-1}) < U^i(C_\ell)$ for each $\ell \le j$ and a majority of voters have $U^i(C_\ell) > U^i(C_\ell + 1)$ for each $\ell \ge j$. It is not necessary for the majority of voters in both cases to be composed of the same subset of voters. A profile can be semi-single-peaked, while very few individual voters have preferences that are perfectly single-peaked. Demange (1982) presents a condition in which candidates have preferences that are single-peaked on a tree, which is also sufficient to guarantee the existence of a PMRW, but not guarantee complete transitivity of PMR.

Summaries of studies of other conditions that require transitivity of PMR can be found in Chapters 10 and 10* in Sen (1970), in Chapter 9 in Fishburn (1973b), in Arrow and Raynaud (1986), and Chapter 5 in Gaertner (2001). DeDonder (2000) summarizes work regarding conditions that guarantee the existence of a PMRW when PMR is treated as a series of voting games, and discusses the economic implications of each of these conditions. Saari and Valognes (1999) consider the impact that single-peaked preferences and single-troughed preferences have on the existence of voting paradoxes in general. They conclude that while each prohibits the existence of Condorcet's Paradox, they still do allow for the existence a number of other voting paradoxes.

Fishburn (1974b) and Tullock (1967) use very different approaches to consider the extension of the notion of single-peaked preferences to more than one dimension, or criterion, for ranking candidates. Both studies generally conclude that the probability that there is no PMRW will be exceedingly small when the number of voters is much greater than the number of candidates in the case of two dimensions.

2.6 Variations of Condorcet's Paradox

The impact that variations of Condorcet's Paradox has on group decision-making has been developed in the context of a number of different types of decision-making scenarios. Brennan (2001) presents a view of Condorcet's Paradox from a perspective of group decision versus group coherence. To explain this situation, we consider a legal example in which three judges are evaluating a case in which a plaintiff has brought suit against a defendant for breach of contract. The three judges have individually weighed the evidence to conclude:

Judge *X*: A legal contract was formed between the two parties, and a breach of contract did occur on the part of the defendant.

Judge *Y*: If a legal contract had been formed between the two parties, then a breach of the contract would have occurred. However, a legal contract had not been formed.

Judge *Z*: A legal contract was formed between the two parties, but no breach of contract occurred.

In order to find the defendant guilty, a judge must conclude that a legal contract had been agreed to by both parties and that the defendant did not meet the specifications of that contract. Both Judges *Y* and *Z* will vote on the side of the defendant in this example, since neither finds that both necessary conditions occurred. So there is a clear majority decision to find the defendant not guilty. But, the question of the coherence of that decision might well be attacked, since a majority of Judges (*X* and *Z*) agrees that a legal contract had been formed, and a majority of Judges (X and Y) agrees that a breach had occurred. Thus, a majority of judges would have voted on the side of the plaintiff in terms of each of the necessary conditions. As a result, observers might find that the final decision of the judges was incoherent.

Stearns (1995) addresses a similar issue related to the context of a courtroom scenario in which a panel of judges decides the outcomes of cases by majority rule. An example is presented to show how the outcome of a decision might change, based upon the sequence in which different aspects of a hypothetical case are addressed, based on actual rulings that were made on cases related to the specific aspects. Realizing that the distinct possibility that PMR cycles exist, the study describes how specific decision rules have been established to determine how the U. S. Supreme Court and lower federal courts must consider cases. These rules, as established, prevent the possibility of observing court decisions with PMR cycles within and across case decisions. Easterbrook (1982) and Block (1998) present a background into studies in which the existence of PMR cycles might have had an impact on legal decisions, and on the increased level of awareness that is being given to the possible problems that their existence might pose in making legal and political decisions.

List and Petit (2004) describe a generalization of the decision-making scenario in a courtroom as described in Brennan (2001) in the context of the "Doctrinal Paradox" of logical decision-making, and then draw parallels between the Doctrinal Paradox and Condorcet's Paradox. List (2003) compares the relative likelihoods of the occurrence of the Doctrinal Paradox and Condorcet's Paradox.

May (1954) considers the relationship between collective group intransitivity and individual preference intransitivity. The candidates in the group voting model correspond to different alternatives that a subject is comparing on the basis of individual preference in *May's Model* of individual preference. The preference ranking that a given voter has on candidates then corresponds to the subject's preference ranking of alternatives according to some attribute of comparison. Then each voter preference ranking on candidates in a voter preference profile would correspond to the subject's ranking of alternatives by a different attribute of comparison. In May's Model, the subject will prefer alternative X to alternative Y if X is rated as better than Y on a majority of attributes of comparison. This is equivalent to XMY in the group voting model. A PMR cycle in the group voting model then corresponds to a subject having an intransitive preference cycle on alternatives in May's Model. This analogy is also developed by Packard (1975) and Gehrlein (1990b), and it will be studied in detail as the subject of Chapter 7.

Lagerspetz (2004) develops a *Meta-Paradox* by considering a set of rankings of three different voting procedures. Three different rankings of these voting procedures are generated, based on the effectiveness of the voting rules at performing effectively on three different criteria. One criterion that was used for a ranking was the Condorcet efficiency, or the propensity of the voting rules to select the PMRW when one exists, of the voting procedures. The Meta Paradox results when a PMR cycle exists with PMR comparisons of the three voting procedures over the three different rankings. So, the cycle exists not on candidates, but on the voting procedures that can be used to select the winner in an election.

Finally, we note an analysis of Condorcet's Paradox that was developed by Fischel (1972) in an examination of one of Aesop's fables, "The Miller, His Son and Their Ass". The study reconstructs the preferences of various different observers to social situations that occur in the development of the fable, to determine that the use of PMR voting by the observers to determine the most socially preferred situation would lead to a PMR cycle.

2.7 Conclusion

Numerous empirical studies have been conducted to determine if Condorcet's Paradox is ever observed in actual elections. After surveying these studies, we must conclude that the evidence does not suggest that the phenomenon is widespread in voting situations. However, there clearly are cases in which the evidence shows that Condorcet's Paradox has occurred in actual elections. The most typical observations of the phenomenon occur when there are a large number of candidates in an election, but there are cases in which it has been observed in three-candidate elections. Following notions that are suggested in some theoretical studies, examples have also been found to show that various means occasionally have intentionally been used to create PMR cycles in election settings to gain a political advantage. It is clearly of interest to determine what the characteristics of voting situations are that make the possible existence of Condorcet's Paradox most likely to occur.

3 The Cases of Two and Three Candidates

3.1 Introduction

The results of the previous chapter support the fact that we can indeed find instances of Condorcet's Paradox in real situations, but that they are not necessarily pervasive. The next step of our study is suggested by Condorcet (1793a, pg. 7) himself:

> "But after considering the facts, the average values or the results, we still need to determine their probability."

Condorcet did a number of studies that applied probability analysis to aspects of election procedures, and he discovered a number of very important concepts while doing so. However, the arguments behind his explanations of these ideas can be extremely difficult to unravel. A quote from Todhunter (1931, pg. 352) suffices to give an idea of the extreme difficulty that can be involved with following Condorcet's logic in developing probability models:

> "We must state at once that Condorcet's work is excessively difficult; but the difficulty does not lie in the mathematical investigations, but in the expressions that are employed to introduce these investigations and to state their results: it is in many cases almost impossible to discover what Condorcet means to say. The obscurity and self contradiction are without parallel. … We believe that the work has been very little studied, for we have not observed any recognition of the repulsive peculiarities by which it is so undesirably distinguished."

Todhunter (1931) then recreates many of the results that Condorcet worked on by using a much more transparent style. The first of Condorcet's probability studies resulted in an observation that has become known as *Condorcet's Jury Theorem*. The written presentation of this result in Condorcet (1785a) is not an exception to Todhunter's description. The proof of the Jury Theorem is developed here in some detail, significantly elaborating on the proof of the Jury Theorem that was presented in Black (1958).

Consider a situation in which a group of n jurors, or voters, is deliberating the truth of a statement, and the combined opinion of the jury will be used to reach a verdict, or election outcome. In the context of an election, we let Candidate A represent a juror's belief that the statement under deliberation is true, and B represent a juror's belief that the statement is false. If an individual juror's preference is $A \succ B$, that juror would vote that the statement is true, and if $B \succ A$ that juror would vote that the statement is false. No juror will know with certainty if the statement under deliberation is true or false. There is some probability, p, with which each juror will reach the correct decision, and each juror will be wrong with probability $1-p$. The value of p is assumed to be the same for each voter, and we are assuming that all voters will reach a decision regarding the statement under deliberation without abstaining from the election.

Suppose that h jurors believe that the statement under deliberation is true and vote for A, while $n-h$ jurors believe that the statement is false and vote for B. Let $P(h \mid A)$ denote the conditional probability that we have an outcome in which h of the jurors vote for A, given that the statement under deliberation is true. If each voter's decision is formed independently of other voters, binomial probabilities apply with h voters being correct and $n-h$ voters being incorrect, and

$$P(h \mid A) = \frac{n!}{h!(n-h)!} p^h (1-p)^{n-h}. \tag{3.1}$$

If the statement under deliberation is false, then h jurors are incorrect and $n-h$ are correct with conditional probability

$$P(h \mid B) = \frac{n!}{h!(n-h)!} p^{n-h} (1-p)^h. \tag{3.2}$$

Let $P(A)$ denote the probability that the statement under deliberation is true. We assume that it is equally likely that A is true or that B is true, so $P(A) = P(B) = 1/2$. The probability, $P(h)$, that h of the n jurors vote for A is then given by

$$P(h) = P(h \mid A)P(A) + P(h \mid B)P(B) = \frac{n!}{2h!(n-h)!}\left[p^h (1-p)^{n-h} + p^{n-h}(1-p)^h\right] \tag{3.3}$$

Bayes' Rule can then be used to find the conditional probability, $P(A|h)$, that the statement under consideration is true, given that h jurors voted for A as

$$P(A \mid h) = \frac{P(h \mid A)P(A)}{P(h)} = \frac{p^h (1-p)^{n-h}}{p^h (1-p)^{n-h} + p^{n-h}(1-p)^h}. \tag{3.4}$$

If we assume that $h \geq n/2$, this can be reduced to

$$P(A \mid h) = \frac{p^{2h-n}}{p^{2h-n} + (1-p)^{2h-n}} = \frac{1}{1+X}, \tag{3.5}$$

where

$$X = \left(\frac{1-p}{p} \right)^{2h-n} . \tag{3.6}$$

We note here that the exponent term $2h-n$ in Eq. 3.6 expresses the difference between the number of jurors voting for *A*, there are *h* of them, and the number voting for *B,* there are $n-h$ of them. Clearly, $P(A \mid h)$ increases as *X* decreases, and this will happen for any fixed *n* and *h*, as *p* increases. This will also happen for any fixed *n*, as *h* increases up to *n*, as long as $1-p < p$, or $p > 1/2$.

It is therefore possible to increase the probability that a jury of *n* members makes the correct decision either by increasing the probability that individual jurors are correct in their perceptions, or by increasing the value of *h* that is required to ultimately determine a final decision. Increasing the value of *h* is the same as increasing the difference between the number of voters selecting *A* and *B* that is requires before a final decision is reached. These results constitute the basis of Condorcet's Jury Theorem.

Grofman and Feld (1988) elaborate on the relationship between the analysis that Condorcet presents here and Rousseau's notion of using majority rule voting to determine the "general will" of the electorate, as discussed in Chapter 1. The basic logic of both writers is found to be consistent. Numerous papers have been written about various other aspects of Condorcet's Jury Theorem and related topics. The interested reader is directed to Nagel (1981), Gehrlein (1981a), Grofman, et al. (1983), Ladha (1993), Berg (1993,1996). Karotkin (1993) does an analysis of work related to Condorcet's Jury Theorem when decisions are being made by committees from the set of voters.

3.2 The Problem with Three Candidates

Condorcet (1785a) extends the same type of analysis that is used in the proof of the Jury Theorem, to consider the probabilities that each candidate is the *true PMRW* in the overall preference of society, given PMR voting situations in a three-candidate election. The extension of this analysis to three-candidate elections led Condorcet into a dilemma. The study assumes that voters are using PMR with three candidates, $A, B, C\}$. Let ***ASB*** denote that *A* is preferred to *B* according to the true preference of society. Following the logic of the Jury Theorem, when individual voters make their pairwise comparison between *A* and *B*, there is a probability, *p*, that each voter will respond correctly with an individual preference of $A \succ B$ that is in agreement with the social preference ***ASB***, rather than responding incorrectly with $B \succ A$.

Condorcet's analysis of this problem is nearly unintelligible, with written comments that do not match numerical examples. Young (1988) recreates the logic that Condorcet (1785a) seemed to be using, and we present an extended version of the analysis that is given in Young (1988). Similar efforts to reconstruct the logic of Condorcet (1785a) are given in Monjardet (1976) and Michaud (1985).

Young considers an example with 60 voters with candidates $A, B, C\}$ as shown in Fig. 3.1, where the results of an election are given in terms of the number of voters with a reported preference on pairs of candidates.

$A \succ B$	23	$B \succ A$	37
$A \succ C$	29	$C \succ A$	31
$B \succ C$	29	$C \succ B$	31

Fig. 3.1 Pairwise preference example for 60 voters from Young (1988)

As mentioned above, we do not have the complete preference rankings for the voters in Fig. 3.1, and each voter has a probability, *p,* of having voted in agreement with the true social preference on each pair of candidates. Candidate *C* would be the PMRW based on the simple vote outcomes in this example, since it beats each of *A* and *B* by a vote of 31-29. However, this observed set of voter responses might be leading to the wrong conclusion with regard to the true social preference if too many voters have given an incorrect response. The numerical example that is used by Young (1988) in Fig. 3.1 is different than the one presented in Condorcet (1785a), but all of Condorcet's discussion about the example that he used is still valid for the current example. McLean (1995) explains the confusion over this issue.

We wish to compute the joint conditional probability, $P(C \mid Vote)$, that *C* is the true PMRW, with both *CSA* and *CSB*, given the voting situations for the 60 voters that is listed above. To begin, we compute a preliminary probability $P(C - A)$ that *C* beats *A* by the vote of 31-29, as observed in the results. If *CSA*, then some combination of 31 of the 60 voters have voted correctly, and the 29 remaining voters have voted incorrectly. Assuming that we have independent voters with an identical probability *p* of voting correctly, the likelihood of a 31-29 vote for *C* over *A* would be given by the binomial probability

$$P(C - A|C\boldsymbol{S}A) = \frac{60!}{31!29!} p^{31}(1-p)^{29} . \tag{3.7}$$

If *ASC*, then some combination of 31 of the 60 voters voted incorrectly and the remaining 29 voted correctly, with the binomial probability

$$P(C - A \mid A\boldsymbol{S}C) = \frac{60!}{31!29!} p^{29}(1-p)^{31} . \tag{3.8}$$

If we assume that the probability of the events *ASC* and *CSA* are equally likely, with probabilities *P*(*ASC*) = *P*(*CSA*) = 1/2 for each outcome, we obtain

$$P(C - A) = P(C - A \mid A\boldsymbol{S}C)P(A\boldsymbol{S}C) + P(C - A \mid C\boldsymbol{S}A)P(C\boldsymbol{S}A) . \tag{3.9}$$

$$P(C-A)=\frac{60!}{31!29!}\left[\frac{p^{31}(1-p)^{29}}{2}+\frac{p^{29}(1-p)^{31}}{2}\right]. \tag{3.10}$$

Bayes' Rule can be used to obtain the conditional probability, $P(C\boldsymbol{S}A \mid Vote)$, that $C\boldsymbol{S}A$, given the pairwise majority votes from the 60 voters with

$$P(C\boldsymbol{S}A|Vote)=\frac{P(C-A|C\boldsymbol{S}A)P(C\boldsymbol{S}A)}{P(C-A)}. \tag{3.11}$$

Using arguments from above and algebraic reduction, we obtain

$$P(C\boldsymbol{S}A|Vote)=\frac{p^{31}(1-p)^{29}}{p^{31}(1-p)^{29}+p^{29}(1-p)^{31}}. \tag{3.12}$$

The same arguments can be used to develop a representation for $P(C\boldsymbol{S}B \mid Vote)$ as:

$$P(C\boldsymbol{S}B|Vote)=\frac{p^{31}(1-p)^{29}}{p^{31}(1-p)^{29}+p^{29}(1-p)^{31}}. \tag{3.13}$$

With the assumption that the events $C\boldsymbol{S}A$ and C$\boldsymbol{S}$B are statistically independent,

$$\begin{aligned} P(C \mid Vote) &= P(C\boldsymbol{S}A \mid Vote)P(C\boldsymbol{S}B \mid Vote) \\ &= \left[\frac{p^{31}(1-p)^{29}}{p^{31}(1-p)^{29}+p^{29}(1-p)^{31}}\right]^2 \\ &= \left[\frac{p^2}{p^2+(1-p)^2}\right]^2. \end{aligned} \tag{3.14}$$

Similarly,

$$\begin{aligned} P(B|Vote) &= \frac{p^{14}(1-p)^2}{\left\{p^{14}+(1-p)^{14}\right\}\left\{(1-p)^2+p^2\right\}} \\ P(A|Vote) &= \frac{(1-p)^{14}(1-p)^2}{\left\{(1-p)^{14}+p^{14}\right\}\left\{(1-p)^2+p^2\right\}}. \end{aligned} \tag{3.15}$$

Young (1988) contains a minor typographical error in the representation of $P(A \mid Vote)$ in Eq. 3.15.

We consider these probability representations with $p=1/2+\varepsilon$ for $\varepsilon>0$. When ε is arbitrarily small, the probability representations can be reduced for this case, and we can ignore all terms containing powers of ε^i with $i>1$. After reduction,

$$P(C \mid Vote) = \frac{1+8\varepsilon}{4} \tag{3.16}$$

$$P(B \mid Vote) = \frac{1+24\varepsilon}{4}$$

$$P(A \mid Vote) = \frac{1-32\varepsilon}{4}.$$

When ε is arbitrarily small, these results are valid, and the maximum probability that a candidate is the true PMRW occurs for Candidate *B*. This result would have been quite disconcerting for Condorcet, since *C* is the PMRW according to the reported votes. The fact that *B* has the maximum probability of being the true PMRW with this analysis follows from the fact that it wins in more PMR comparisons, with 66 total votes over *A* and *C*, than does either *A*, with 52 votes, or *C*, with 62 votes.

Condorcet (1785b, pg. 76) addresses the issue of what should be done in this example in which computed probabilities disagree with observed election outcomes, by appealing to "simple reason" when he writes:

> "Candidate *A* clearly does not have the preference, because there is a plurality (majority) of votes against him whether he is compared to *B* or to *C*, and this is always the case in such situations. The choice is therefore between *B* and *C*. As the proposition '*B* is better than *C*' has only minority support, we must conclude that *C* has plurality (majority) support."

So, Condorcet now argues that we should rely on "simple reason" with the actual vote count, rather than relying on the computed probabilities, to resolve this dilemma.

Young (1988) generalizes this result to state for any number of voters and candidates, if *p* is sufficiently close to 1/2, then the candidate that receives the most pairwise votes from all voters is most likely to be the candidate that is the true majority rule winner in the societal preference. It follows easily from arguments in Chapter 1 that this candidate would always win by Borda Rule with $a = b = 1$, if the voters had reported complete preference rankings on the candidates. This outcome is, of course, dependent upon independence between voters' preferences and the assumption that each voter has a probability of 1/2 of responding with an individual preference that is in agreement with the societal preference on any pair of candidates.

Condorcet (1785c) was clearly thinking along the same line of reasoning that is presented by Young (1988), since he makes specific reference to situations in which voters are making decisions in which they have individual probabilities of preferences on pairwise comparisons that are only slightly greater than 1/2. Given the resulting support of Borda Rule that results from Young's analysis above, it is quite possible that Condorcet made the same observation, leading to his appeal to use "simple reason" instead of relying on the computed probabilities.

Condorcet (1785d) develops a third analysis that is related to the probability that a PMRW might exist for the general case of *m* candidates. This study also considers situations in which there is some probability that voters will be making a correct decision. The particular situation that has received the most attention from this work is when the PMR vote on any pair of candidates has an equal likelihood of having either candidate win. To be more specific, this assumption suggests that there is an equal likelihood that any pair of candidates, *A* and *B*, will have ***BMA*** or ***AMB*** by PMR, regardless of the PMR voting results on any other pairs of candidates.

We follow Condorcet's rather indirect logic, and suppose that there are *m* candidates, and that *A* is the PMRW. Any outcomes of voting are allowed on the social relations on the pairs of candidates among the remaining $m-1$ candidates. There are $\frac{(m-1)(m-2)}{2}$ PMR comparisons on the remaining candidates, and there are two possible outcomes on each comparison. The total number of social relations in which any one of the *m* candidates is the PMRW over the remaining candidates is then given by

$$m2^{\frac{(m-1)(m-2)}{2}}. \tag{3.17}$$

Each of these social relations is assumed to be equally likely to be observed.

The total number of possible social outcomes by PMR on all *m* candidates is given by $2^{\frac{m(m-1)}{2}}$, and each of these is assumed to be equally likely to be observed. Given all of the above, the probability that there is a PMRW is given by the ratio

$$\frac{m2^{\frac{(m-1)(m-2)}{2}}}{2^{\frac{m(m-1)}{2}}} = \frac{m}{2^{m-1}}. \tag{3.18}$$

This relationship is stated incorrectly in the original paper by Condorcet (1785d), and Sommerlad and McLean (1989) therefore state it incorrectly in their translation. Riker (1961) develops a representation for this probability that is identical to the one given in Eq. 3.18.

May (1971) develops an identical representation to Eq. 3.18 for the probability that a PMRW exists, while considering societies in which voters have random responses on pairwise comparisons, and there are absolutely no requirements for any consistency on their individual preferences for candidates. Clearly, May's assumptions are in sharp contrast to the notions in the earlier comments from Condorcet (1788a) regarding the necessary requirement for transitivity of individual preference. However, we find that both studies result in the development the same representation for the probability that a PMRW exists.

If we further require that the social relation has completely transitive PMR relations, then similar arguments to those used above can be used to show that this probability is given as

$$\frac{m!}{2^{\frac{m(m-1)}{2}}}. \tag{3.19}$$

An identical representation for this probability was developed by Klahr (1966).

Both of these probabilities become quite small as m gets at all large, as seen in the computed values in Table. 3.1, which led Condorcet to argue in several articles that some type of elimination procedure must be implemented to remove candidates that are not serious contenders from consideration. If we were to believe that these probabilities represent a realistic situation, we would then expect to be observing PMR cycles on a regular basis, which is not the case.

Table 3.1 Computed probabilities with equally likely social outcomes from Condorcet's representations

Candidates m	A PMRW Exists	PMR is Transitive
3	.7500	.7500
4	.5000	.3750
5	.3125	.1172
6	.1875	.0220
7	.1094	.0024

The underlying notion of considering an equal likelihood of social outcomes on all pairs of candidates under PMR ignores the assumption that voters will have some type of coherent preferences, as indicated by the interpretation in May (1971). Consider an example in an m candidate election, and suppose that m is large and that Candidate A has sequentially defeated the first $m-2$ of the $m-1$ candidates that it must defeat by PMR to become the PMRW. This would suggest that A tends to be a highly preferred candidate in the preference rankings of the voters. It is not plausible to then assume that A has only a 50-50 chance of defeating the one remaining candidate by PMR, as assumed in Condorcet's model.

Neither of Condorcet's last two probability studies is based on assumptions that rely on the notion that there is some underlying coherence in the preferences of voters. An increased level of coherence among all of the individual voter's pairwise preferences would seem likely to reduce the likelihood of PMR cycles. This leads to the conclusion that the probabilities in Table 3.1 grossly underestimate both the probability that a PMRW exists and the probability that PMR is transitive.

3.3 Probabilities with Balanced Preferences

Tangian (2000) does an analysis of conditions on voters' preferences that tend to maximize the likelihood that PMR cycles might occur. The arguments follow from using utility measures of voter preferences, as developed in Chapter 2, where $U^i(C_j)$ measures the degree of preference, or utility, that a given i^{th} voter associates with each C_j in an m-candidate election with candidates in the set $\boldsymbol{C}^m = C_1, C_2, ..., C_m\}$. The i^{th} voter's individual preference ranking on candidates will have $C_j \succ C_k$ if, and only if $U^i(C_j) > U^i(C_k)$.

These $U^i(C_j)$ values represent *Cardinal Utilities,* since they express voter's preferences as precisely measurable values of utilities for candidates, as associated with points along a number line within the closed interval $[0, z]$. The relative differences between utilities for candidates can then be used to precisely represent the relative degrees of preference between them. The aggregated social utility, $SU(C_j)$, for any candidate C_j within a society of n individuals is simply obtained in such situations as the sum of the individual utilities, with

$$SU(C_j) = \sum_{i=1}^{n} U^i(C_j). \tag{3.20}$$

The socially preferred candidate, or election winner, is that candidate that has the maximum value of $SU(C_j)$. Tangian asserts that the exact precision of cardinal utility representations gives them a "solvability" that prevents them from suffering from problems like Condorcet's Paradox while determining the winning candidate.

Ordinal Utility only expresses a voter's preferences according to relative rankings of candidates, without an association of degree of preference. They are most appropriate in the context of voting situations, where only the relative rankings of preferences are reported. Tangian considers utility-based representations for voters' ordinal preferences, with the assumption that individual voter's preferences are independent. The main conclusion of the study is that the probability that Condorcet's Paradox occurs will typically vanish for large electorates. The probability of observing Condorcet's Paradox vanishes, except when the individual $U^i(C_j)$'s have some underlying probability distributions that describe the likelihood that they are observed that exhibit the condition of *balanced preferences*. This condition of balanced preferences exists when the probability distributions over $U^i(C_j)$'s are such that the "average voter" is indifferent between candidates on an expected value basis, and is equally likely to vote for or against any particular candidate in pairwise votes.

Tangian (2000) obtains this outcome by considering the probability that the election winner that is obtained by summing individual cardinal utilities is the same as the winner that is obtained by PMR voting. It is shown that this probability goes to unity as the number of voters becomes infinite, unless the condition of balanced preferences exists. Since we know that the probability of observing

Condorcet's Paradox will vanish when using a cardinal utility representation in this case, the probability of observing Condorcet's Paradox must therefore also vanish unless the condition of balanced preferences exists. The notion that cardinal utilities have a "solvability" that is not present in ordinal utilities was suggested earlier by Waldner (1973).

A somewhat different approach to this problem is taken by Gehrlein (1983, 1997, 2002a), Stensholt (1999b), and Grofman, et al. (2003). In these studies, models are developed to generate random voting situations, and every possible voting situation could be observed with some probability. For any pair of candidates, A and B, some proportion of the voters in each voting situation will have the preference $A \succ B$. This proportion is not required to be equal to 1/2 in every voting situation to have balanced preferences. However, if this proportion is calculated for every possible voting situation, and each proportion is weighted by the respective probability that its associated voting situation is observed, then that weighted sum, or expected value of this proportion, must be equal to 1/2. This condition must hold for every pair of candidates to have balanced preferences. All of these studies find that an expected balance in voter preferences on pairs of candidates tends to maximize the probability that Condorcet's Paradox occurs.

The assumption of balanced preferences is also consistent with the notion of the principle of insufficient reason, given that nothing is known *a priori* about voters' preferences on pairs of candidates in any particular voting situation. We shall see later that it is possible to enter a bias into models for obtaining voting situations that will increase the likelihood of observing Condorcet's Paradox, compared to models with balanced preferences. However, entering such a bias is equivalent to assuming that we know something about the way in which voter preferences are being formed. This is the reverse of the situation that was discussed in Chapter 2, where it was shown that restricting voters to having single-peaked preferences leads to the required existence of transitive PMR. By restricting voter's preferences in other ways, it is also possible to greatly increase the probability that Condorcet's Paradox is observed. No such bias, or restriction, is implicit to the basic notion of balanced preferences.

A number of different models that exhibit the condition of balanced preferences have been developed for generating random voting situations, or voter preference profiles. Each of the models generates voting situations, or voter preference profiles, with linear voter preference rankings on candidates. We develop each of these models for three-candidate elections with n voters, using the terminology from Chapter 1. Voting situations are defined following the discussion that led to of Fig. 1.1, which is repeated here for convenience as Fig. 3.2.

A	A	B	C	B	C
B	C	A	A	C	B
C	B	C	B	A	A
n_1	n_2	n_3	n_4	n_5	n_6

Fig. 3.2 Voting situations with linear preference rankings on three-candidates

3.4 Expected Balance for Voting Situations

3.4.1 A Fixed Number of Voters

Kuga and Nagatani (1974) and Gehrlein and Fishburn (1976a) developed the notion of *Impartial Anonymous Culture* (*IAC*). With IAC, it is assumed that all possible voting situations for a specified number of voters, n, are equally likely to be observed. This process is "anonymous" in the sense that we only know the values on the n_i terms for a voting situation, and have no knowledge of the preferences of any particular voter. IAC produces a balance in the expected preferences on pairs of candidates over all possible voting situations. This balance with IAC follows from partitioning the set of all possible voting situations into pairs. To form a pair of voting situations in the partition, we match each voting situation to the voting situation that interchanges rankings according to: $n_1 \leftrightarrow n_6$, $n_2 \leftrightarrow n_5$, and $n_3 \leftrightarrow n_4$.

This transformation matches every voting situation with its *dual voting situation*, in which the linear preference ordering of candidates is reversed for every voter. Thus, for any two candidates, A and B, the number of voters with $A \succ B$ in one of the voting situations will have the same number of voters with $B \succ A$ in the matching voting situation. Since both voting situations are equally likely to be observed under IAC, there is an expected balance between the number of voters with $A \succ B$ and with $B \succ A$ within the pair of voting situations. This observation extends to all of the pairs of voting situations in the partition, since all voting situations are equally likely to be observed with IAC. In the event that $n_1 = n_6$, $n_2 = n_5$, and $n_3 = n_4$, the interchange of rankings matches the voting situation with itself. In this case, the difference in the number of rankings with $A \succ B$ and with $B \succ A$ is not cancelled out over a pair of equally likely voting situations, but within this particular voting situation itself.

We begin by developing a simple closed-form representation for the probability, $P^S_{PMRW}(m,n,IAC)$, that a *Strict PMRW* exists for m candidates under IAC with m = 3, following the derivation in Gehrlein and Fishburn (1976a). The general restrictions on the n_i terms in a voting situation to have A as the strict PMRW for the case of odd n can be restated from Chapter 1 as:

$$\begin{aligned} n_3 + n_5 + n_6 &\le \frac{n-1}{2} \Rightarrow AM\boldsymbol{B} \\ n_4 + n_5 + n_6 &\le \frac{n-1}{2} \Rightarrow AM\boldsymbol{C}. \end{aligned} \qquad (3.21)$$

The restrictions on the individual n_i terms that result in the conditions in Eq. 3.21 are given by:

$$
\begin{aligned}
0 \le n_6 &\le \frac{n-1}{2} \\
0 \le n_5 &\le \frac{n-1}{2} - n_6 \\
0 \le n_4 &\le \frac{n-1}{2} - n_6 - n_5 \\
0 \le n_3 &\le \frac{n-1}{2} - n_6 - n_5 \\
0 \le n_2 &\le n - n_6 - n_5 - n_4 - n_3 \\
n_1 &= n - n_6 - n_5 - n_4 - n_3 - n_2.
\end{aligned}
\tag{3.22}
$$

Computing probabilities with IAC is accomplished by using a simple process of counting the number of voting situations that meet these given conditions, since all voting situations are equally likely. In doing this, the notions that Condorcet (1785d) applied to counting the number of possible social outcomes to obtain the probability representation in Eq. 3.18 are applied instead to counting voting situations. The number of voting situations that meet the restrictions on the n_i's in Eq. 3.22 to have A as the PMRW for odd n values can be computed as $N_{PMRW}^{A\}}(3,n,IAC)$, with

$$
N_{PMRW}^{A\}}(3,n,IAC) = \sum_{n_6=0}^{\frac{n-1}{2}} \sum_{n_5=0}^{\frac{n-1}{2}-n_6} \sum_{n_4=0}^{\frac{n-1}{2}-n_6-n_5} \sum_{n_3=0}^{\frac{n-1}{2}-n_6-n_5} \sum_{n_2=0}^{n-n_6-n_5-n_4-n_3} 1, \quad \text{for odd } n.
\tag{3.23}
$$

Gehrlein and Fishburn (1976a) algebraically reduce this representation for $N_{PMRW}^{A\}}(3,n,IAC)$ by sequentially using known relations for sums of powers of integers [Selby (1965)]. This is a cumbersome, but simple, process. For example, the first step of the process is to evaluate the summation $\sum_{n_2=0}^{n-n_6-n_5-n_4-n_3} 1$, which is equivalent to the determining the number of distinct integer values that n_2 can have within the range $0 \le n_2 \le n - n_6 - n_5 - n_4 - n_3$. This general value is given quite simply as $n - n_6 - n_5 - n_4 - n_3 + 1$. Eq. 3.23 can then be reduced to:

$$
N_{PMRW}^{A\}}(3,n,IAC) = \sum_{n_6=0}^{\frac{n-1}{2}} \sum_{n_5=0}^{\frac{n-1}{2}-n_6} \sum_{n_4=0}^{\frac{n-1}{2}-n_6-n_5} \sum_{n_3=0}^{\frac{n-1}{2}-n_6-n_5} \left[(n - n_6 - n_5 - n_4 + 1) - n_3 \right].
\tag{3.24}
$$

The reduction of Eq. 3.24 for the n_3 summation has two components. The first of these two components is given by $\sum_{n_3=0}^{\frac{n-1}{2}-n_6-n_5} (n - n_6 - n_5 - n_4 + 1)$ which is equivalent to $(n - n_6 - n_5 - n_4 + 1)\sum_{n_3=0}^{\frac{n-1}{2}-n_6-n_5} 1$. Following the discussion for the reduction of the n_2 summation, this first component of the n_3 summation reduces to $(n - n_6 - n_5 - n_4 + 1)\left(\frac{n-1}{2} - n_6 - n_5 + 1\right)$. The second component of the n_3 summation is $\sum_{n_3=0}^{\frac{n-1}{2}-n_6-n_5} n_3$, which is the sum of the integer values for all integers in the range $0 \le n_3 \le \frac{n-1}{2} - n_6 - n_5$. In general, $\sum_{n_3=0}^{k} n_3 = \frac{k(k+1)}{2}$, so after substitution Eq. 3.24 reduces to:

$$N_{PMRW}^{A\}}(3,n,IAC) = \sum_{n_6=0}^{\frac{n-1}{2}} \sum_{n_5=0}^{\frac{n-1}{2}-n_6} \sum_{n_4=0}^{\frac{n-1}{2}-n_6-n_5} \left[\begin{array}{c} (n - n_6 - n_5 - n_4 + 1)\left(\frac{n+1}{2} - n_6 - n_5\right) - \\ \frac{1}{2}\left(\frac{n-1}{2} - n_6 - n_5\right)\left(\frac{n+1}{2} - n_6 - n_5\right) \end{array} \right]. \tag{3.25}$$

The process continues in the same fashion to sequentially reduce Eq. 3.25 for the n_4, n_5 and n_6 summations, using known representations for sums of higher order powers of integers to obtain:

$$N_{PMRW}^{A\}}(3,n,IAC) = \frac{45}{128} + \frac{99n}{128} + \frac{39n^2}{64} + \frac{43n^3}{192} + \frac{5n^4}{128} + \frac{n^5}{384}. \tag{3.26}$$

This can be further reduced to

$$N_{PMRW}^{A\}}(3,n,IAC) = \frac{(n+1)(n+3)^3(n+5)}{384}, \text{ for odd } n. \tag{3.27}$$

The total number of possible voting situations, $K(3,n,IAC)$, for three candidates with n voters is given by

$$K(3,n,IAC) = \sum_{n_6=0}^{n} \sum_{n_5=0}^{n-n_6} \sum_{n_4=0}^{n-n_6-n_5} \sum_{n_3=0}^{n-n_6-n_5-n4} \sum_{n_2=0}^{n-n_6-n_5-n_4-n_3} 1. \tag{3.28}$$

It follows from Feller (1957) that

$$K(3,n,IAC) = \frac{\prod_{i=1}^{5}(n+i)}{120}. \tag{3.29}$$

Using the definition of IAC and its symmetry with respect to candidates, it follows that $N_{PMRW}^{A\}}(3,n,IAC) = N_{PMRW}^{B\}}(3,n,IAC) = N_{PMRW}^{C\}}(3,n,IAC)$, so we then find $P_{PMRW}^{S}(3,n,IAC) = 3\,N_{PMRW}^{A\}}(3,n,IAC)/K(3,n,IAC)$, to lead to

$$P_{PMRW}^{S}(3,n,IAC) = \frac{15(n+3)^2}{16(n+2)(n+4)}\text{, for odd } n. \tag{3.30}$$

McNutt (1993) and Chen (2002) discuss minor variations of the methods that were used to develop this representation for $P_{PMRW}^{S}(3,n,IAC)$.

When n is even, a representation for $P_{PMRW}^{S}(3,n,IAC)$ is obtained from

$$P_{PMRW}^{S}(3,n,IAC) = \frac{3\sum_{n_6=0}^{\frac{n-2}{2}}\sum_{n_5=0}^{\frac{n-2}{2}-n_6}\sum_{n_4=0}^{\frac{n-2}{2}-n_6-n_5}\sum_{n_3=0}^{\frac{n-2}{2}-n_6-n_5}\sum_{n_2=0}^{n-n_6-n_5-n_4-n_3} 1}{K(3,n,IAC)}. \tag{3.31}$$

Lepelley (1989) uses this to obtain

$$P_{PMRW}^{S}(3,n,IAC) = \frac{15n(n+2)(n+4)}{16(n+1)(n+3)(n+5)}\text{, for even } n. \tag{3.32}$$

Fishburn, et al. (1979a,b) introduce a different notion that is related to the probability that a PMRW exists. In particular, they consider the probability $P_{PMRW}^{X\}}(m,n,IAC)$ that a given set, $X\}$, of candidates is included in the set of PMRW's, for n voters with m candidates under IAC. Here, each pair of candidates in $X\}$ is tied by PMR, and each candidate in $X\}$ beats or ties all candidates that are not included in $X\}$ by PMR.

Let $P_{PMRW}^{\#i}(m,n,IAC)$ denote the value of $P_{PMRW}^{X\}}(m,n,IAC)$ when the cardinality of a specified $X\}$ is equal to i. When n is odd, there can be no ties with PMR, and it follows from Eq. 3.30 that

$$P_{PMRW}^{\#1}(3,n,IAC) = \frac{P_{PMRW}^{S}(3,n,IAC)}{3} = \frac{5(n+3)^2}{16(n+2)(n+4)}\text{, for odd } n. \tag{3.33}$$

We continue this logic and develop a representation for $P_{PMRW}^{\#1}(3,n,IAC)$ as the probability that A beats or ties all other candidates by PMR when n is even. Following earlier notation, the number of voting situations that meet this condition is denoted by $N_{PMRW}^{A\}}(n,3,IAC)$ with

$$N_{PMRW}^{A\}}(3,n,IAC) = \sum_{n_6=0}^{\frac{n}{2}} \sum_{n_5=0}^{\frac{n}{2}-n_6} \sum_{n_4=0}^{\frac{n}{2}-n_6-n_5} \sum_{n_3=0}^{\frac{n}{2}-n_6-n_5} \sum_{n_2=0}^{n-n_6-n_5-n_4-n_3} 1\text{, for even } n. \tag{3.34}$$

Gehrlein (2002a) uses algebraic reduction techniques like those described previously to obtain

$$N_{PMRW}^{A\}}(3,n,IAC) = \frac{(n+2)^2(n+4)^2(n+6)}{384}\text{, for even } n. \tag{3.35}$$

Following earlier arguments related to IAC, we then find $P_{PMRW}^{\#1}(3,n,IAC)$ as the ratio $N_{PMRW}^{A\}}(3,n,IAC)/K(3,n,IAC)$, with

$$P_{PMRW}^{\#1}(3,n,IAC) = \frac{5(n+2)(n+4)(n+6)}{16(n+1)(n+3)(n+5)}\text{, for even } n. \tag{3.36}$$

Kelly (1974) introduced the notion of a *Weak PMRW*. A given profile has a weak PMRW if some candidate beats or ties all other candidates under PMR. Let $P_{PMRW}^{W}(m,n,IAC)$ denote the probability that a weak PMRW exists for n voters with m candidates under IAC. If n is odd, there can be no ties by majority rule, so

$$P_{PMRW}^{W}(3,n,IAC) = 3\,P_{PMRW}^{\#1}(3,n,IAC) = P_{PMRW}^{S}(3,n,IAC)\text{, for odd } n. \tag{3.37}$$

Our next step is to develop a representation for $P_{PMRW}^{W}(3,n,IAC)$, for even n. To do this we use a relationship that follows from our definitions

$$\begin{aligned} &P_{PMRW}^{W}(3,n,IAC) = \\ &P_{PMRW}^{A\}}(3,n,IAC) + P_{PMRW}^{B\}}(3,n,IAC) + P_{PMRW}^{C\}}(3,n,IAC) \\ &- \left\{P_{PMRW}^{A,B\}}(3,n,IAC) + P_{PMRW}^{A,C\}}(3,n,IAC) + P_{PMRW}^{B,C\}}(3,n,IAC)\right\} \\ &+ P_{PMRW}^{A,B,C\}}(3,n,IAC). \end{aligned} \tag{3.38}$$

Due to the symmetry of IAC with respect to candidates

$$\begin{aligned} &P_{PMRW}^{W}(3,n,IAC) = \\ &3P_{PMRW}^{\#1}(3,n,IAC) - 3P_{PMRW}^{\#2}(3,n,IAC) + P_{PMRW}^{\#3}(3,n,IAC). \end{aligned} \tag{3.39}$$

Gehrlein (2002a) uses algebraic reduction techniques to obtain:

$$P_{PMRW}^{W}(3,n,IAC) = \frac{15(n+2)(n^2+8n+8)}{16(n+1)(n+3)(n+5)}\text{, for even } n. \tag{3.40}$$

This representation has been verified by computer enumeration, and it is not in agreement with a representation for $P^W_{PMRW}(3,n,IAC)$ with even n in Berg and Bjurulf (1983).

Given all of the representations above, the following results follow directly from taking derivatives:

Theorem 3.1 (IAC). $P^S_{PMRW}(3,n,IAC) > P^S_{PMRW}(3,n+2,IAC)$, for all odd $n \geq 1$.

Theorem 3.2 (IAC). $P^S_{PMRW}(3,n,IAC) < P^S_{PMRW}(3,n+2,IAC)$, for all even $n \geq 2$.

Theorem 3.3 (IAC). $P^S_{PMRW}(3,n,IAC) = 3\, P^{\#1}_{PMRW}(3,n,IAC) = P^W_{PMRW}(3,n,IAC)$, for all odd $n \geq 1$.

Theorem 3.4 (IAC). $P^{\#1}_{PMRW}(3,n,IAC) > P^{\#1}_{PMRW}(3,n+2,IAC)$, for all even $n \geq 2$.

Theorem 3.5 (IAC). $P^W_{PMRW}(3,n,IAC) > P^W_{PMRW}(3,n+2,IAC)$, for all even $n \geq 2$.

Table 3.2 shows computed values of $P^S_{PMRW}(3,n,IAC)$, $P^{\#1}_{PMRW}(3,n,IAC)$, and $P^W_{PMRW}(3,n,IAC)$ for various values of n. The results show very different behaviors for odd and even n. With odd n, each of the probabilities approaches its limiting values quickly for relatively small values of n. For even n, we find a much slower rate of convergence to the limiting probabilities as n increases, with a rather small probability for $P^S_{PMRW}(3,n,IAC)$ when n is small.

Table 3.2 Probabilities with Impartial Anonymous Culture Condition (IAC)

n	$P^S_{PMRW}(3,n,IAC)$	$P^{\#1}_{PMRW}(3,n,IAC)$	$P^W_{PMRW}(3,n,IAC)$
3	.9643	.3214	.9643
4	.5714	.4762	1.0000
5	.9524	.3175	.9524
6	.6494	.4329	.9957
7	.9470	.3157	.9470
8	.6993	.4079	.9907
9	.9441	.3147	.9441
10	.7343	.3916	.9860
11	.9423	.3141	.9423
20	.8199	.3553	.9702
21	.9391	.3130	.9391
40	.8735	.3348	.9569
41	.9380	.3127	.9380
100	.9105	.3217	.9462
101	.9376	.3125	.9376
∞	.9375	.3125	.9375

3.4.2 A Variable Number of Voters

Another model has been developed that is similar to the notion of IAC, in which each possible voting situation is considered to be equally likely to be observed. However, the *Maximal Culture Condition* (*MC*) does not require that the number of voters in voting situations is fixed. MC was first used in a Monte-Carlo simulation study in Fishburn and Gehrlein (1976b), and it was first called MC in Fishburn and Gehrlein (1977b). MC fixes some positive integer, *L*, and the associated n_i for each linear preference ranking is equally likely to have any integer value in the closed interval $[0,L]$. With the assumption of MC on three candidates, there are a total of $(L+1)^6$ possible voting situations that are equally likely to be observed. The expected total number of voters in a voting situation, $E(n)$, with MC is given by $E(n)=6(L/2)=3L$. The same matching of voting situations that was used in the IAC case can be used here to show that there is an expected balance of preference on all pairs of candidates for voting situations with MC.

Gehrlein and Lepelley (1997) follow the development of the representation for $N^{A\}}_{PMRW}(3,n,IAC)$ in Eq. 3.27 to obtain a representation for $N^{A*\}}_{PMRW}(3,L,MC)$. Here, *A* is required to be a strict PMRW when the total number of voters in a voting situation is even with MC. To start, the restrictions on the n_i's that result in Candidate *A* being the strict PMRW with MC are given by:

$$
\begin{gathered}
0 \le n_3 \le L \\
0 \le n_4 \le L \\
0 \le n_1 \le L \\
Max\begin{Bmatrix} 0 \\ n_4 - n_3 - n_1 + 1 \\ n_3 - n_4 - n_1 + 1 \end{Bmatrix} \le n_2 \le L \\
0 \le n_5 \le Min\begin{Bmatrix} L \\ n_1 + n_2 + n_3 - n_4 - 1 \\ n_1 + n_2 + n_4 - n_3 - 1 \end{Bmatrix} \\
0 \le n_6 \le Min\begin{Bmatrix} L \\ n_1 + n_2 + n_3 - n_4 - n_5 - 1 \\ n_1 + n_2 + n_4 - n_3 - n_5 - 1 \end{Bmatrix}
\end{gathered} \tag{3.41}
$$

Here, $Min\begin{Bmatrix} a \\ b \end{Bmatrix}$ and $Max\begin{Bmatrix} a \\ b \end{Bmatrix}$ respectively denote the minimum and maximum of arguments *a* and *b*. The *Min* and *Max* functions in the summation limits significantly complicate the problem of obtaining a closed-form representation for $N^{A*\}}_{PMRW}(3,L,MC)$. This is dealt with by partitioning the set of all voting situations that meet the restrictions in Eq. 3.41 into 13 subspaces, such that none of the restrictions for inclusion in these subspaces contain any *Min* or *Max* arguments.

We develop the initial stages of the partitioning process that lead to a representation for $N_{PMRW}^{A*\}}(3,L,MC)$, since the same basic procedure is also used to obtain a number of results that follow. To begin, it is observed that the number of voting situations in $N_{PMRW}^{A*\}}(3,L,MC)$ with $n_4 > n_3$ is identical to the number of voting situations with $n_3 > n_4$. This follows from the simple fact that Candidate *A* is the PMRW if it is included among the voting situations in $N_{PMRW}^{A*\}}(3,L,MC)$ with both $n_4 + n_5 + n_6 < n_1 + n_2 + n_3$ for ***AMB*** and $n_3 + n_5 + n_6 < n_1 + n_2 + n_4$ for ***AMC***. The interchange of n_3 and n_4 simply changes a voting situation in which ***AMB*** and ***AMC*** to a voting situation in which ***AMC*** and ***AMB***, along with the converse.

We begin by developing a relationship for the number, $N_{PMRW}^{S(n_4>n_3)}(3,L,MC)$, of voting situations that are included in $N_{PMRW}^{A*\}}(3,L,MC)$ with $n_4 > n_3$. The situation with $n_4 = n_3$ will be considered as a separate issue later. The restrictions on n_i's in Eq. 3.41 for a voting situation to be included in $N_{PMRW}^{A*\}}(3,L,MC)$ are reduced when we add the restriction $n_4 > n_3$ to the conditions:

$$
\begin{gathered}
0 \le n_3 \le L-1 \\
n_3 + 1 \le n_4 \le L \\
0 \le n_1 \le L \\
Max\begin{Bmatrix} 0 \\ n_4 - n_3 - n_1 + 1 \end{Bmatrix} \le n_2 \le L \\
0 \le n_5 \le Min\begin{Bmatrix} L \\ n_1 + n_2 + n_3 - n_4 - 1 \end{Bmatrix} \\
0 \le n_6 \le Min\begin{Bmatrix} L \\ n_1 + n_2 + n_3 - n_4 - n_5 - 1 \end{Bmatrix}.
\end{gathered} \tag{3.42}
$$

To begin the process of further partitioning the voting situations that meet the conditions in Eq. 3.42, we start by removing the *Min* argument on the upper bound for n_6. This is done by creating two disjoint subspaces. *Subspace I* contains voting situations in which $L \le n_1 + n_2 + n_3 - n_4 - n_5 - 1$, and *Subspace II* contains voting situations in which $L > n_1 + n_2 + n_3 - n_4 - n_5 - 1$.

The $n_1 + n_2 + n_3 - n_4 - n_5 - 1$ term in the *Max* argument for the upper bound on n_6 can then be removed from the bounds on *Subspace I* if we add the restriction that $n_5 \le n_1 + n_2 + n_3 - n_4 - 1 - L$, to create the upper range limit on n_5 with

$$0 \le n_5 \le Min\left\{\begin{matrix} L \\ n_1 + n_2 + n_3 - n_4 - 1 \\ n_1 + n_2 + n_3 - n_4 - 1 - L \end{matrix}\right\}. \tag{3.43}$$

It is easily shown that $n_1 + n_2 + n_3 - n_4 - 1 - L$ is the minimum of the three *Min* arguments in Eq. 3.43 when $n_4 > n_3$, so the upper range limit of n_5 reduces to $n_1 + n_2 + n_3 - n_4 - 1 - L$.

To maintain consistency between the upper and lower range limits on n_5, it is now necessary to require $n_1 + n_2 + n_3 - n_4 - 1 - L \ge 0$, which in turn requires that $n_4 - n_1 - n_3 + 1 + L \le n_2$. It is easily shown that $n_4 - n_1 - n_3 + 1 + L > 0$, so the *Max* argument in the lower range limit of n_2 becomes $n_4 - n_1 - n_3 + 1 + L$. For consistency between the lower bound limit and the upper range limits on n_2, we now require $n_4 - n_1 - n_3 + 1 + L \le L$, which in turn requires $n_1 \ge n_4 - n_3 + 1$. After completing the remaining consistency requirements for n_4 and n_3, with the restriction $n_4 > n_3$, the limits on n_i's in *Subspace I* are ultimately given by:

$$\begin{gathered} 0 \le n_3 \le L - 1 \\ n_3 + 1 \le n_4 \le L \\ n_4 - n_3 + 1 \le n_1 \le L \\ n_4 - n_3 - n_1 + 1 + L \le n_2 \le L \\ 0 \le n_5 \le n_1 + n_2 + n_3 - n_4 - 1 - L \\ 0 \le n_6 \le L. \end{gathered} \tag{3.44}$$

The only remaining possible conflict in the bounds on the n_i's in *Subspace I* in Eq. 3.44 is for the bounds on n_1, for the specific case in which both $n_3 = 0$ and $n_4 = L$. This conflict is dealt with by further partitioning *Subspace I* into *Subspace* #1 which has $n_3 = 0$ and *Subspace* #2 which has $1 \le n_3 \le L - 1$, as shown in Eq. 3.45.

$$\begin{array}{cc} \textit{Subspace}\ \#1 & \textit{Subspace}\ \#2 \\ n_3 = 0 & 1 \le n_3 \le L - 1 \\ 1 \le n_4 \le L - 1 & n_3 + 1 \le n_4 \le L \\ n_4 + 1 \le n_1 \le L & n_4 - n_3 + 1 \le n_1 \le L \\ L + 1 + n_4 - n_1 \le n_2 \le L & L + 1 + n_4 - n_1 - n_3 \le n_2 \le L \\ 0 \le n_5 \le n_1 + n_2 - n_4 - 1 - L & 0 \le n_5 \le n_1 + n_2 + n_3 - n_4 - 1 - L \\ 0 \le n_6 \le L & 0 \le n_6 \le L \end{array} \tag{3.45}$$

The same general procedure is used to partition *Subspace II*. However, the process becomes more cumbersome to work with, and it requires seven subspaces, denoted as *Subspace #3* through *Subspace #9*, to partition it to remove all *Max* and *Min* arguments from bounds on the n_i's, as shown in Eqs. 3.46 through 3.49

$$\begin{array}{ll} \textit{Subspace}\ \#3 & \textit{Subspace}\ \#4 \\ n_3 = 0 & 1 \le n_3 \le L-1 \\ 1 \le n_4 \le L-1 & n_3+1 \le n_4 \le L \\ n_4+1 \le n_1 \le L & n_4-n_3+1 \le n_1 \le L \\ L+1+n_4-n_1 \le n_2 \le L & L+1+n_4-n_1-n_3 \le n_2 \le L \\ n_1+n_2-n_4-L \le n_5 \le L & n_1+n_2+n_3-n_4-L \le n_5 \le L \\ 0 \le n_6 \le n_1+n_2-n_4-n_5-1 & 0 \le n_6 \le n_1+n_2+n_3-n_4-n_5-1 \end{array} \tag{3.46}$$

$$\begin{array}{ll} \textit{Subspace}\ \#5 & \textit{Subspace}\ \#6 \\ n_3 = 0 & 1 \le n_3 \le L-1 \\ 1 \le n_4 \le L-1 & n_3+1 \le n_4 \le L \\ n_4+1 \le n_1 \le n_4 & 0 \le n_1 \le n_4-n_3 \\ n_4-n_1+1 \le n_2 \le L & n_4-n_3-n_1+1 \le n_2 \le L \\ 0 \le n_5 \le n_1+n_2-n_4-1 & 0 \le n_5 \le n_1+n_2+n_3-n_4-1 \\ 0 \le n_6 \le n_1+n_2-n_4-n_5-1 & 0 \le n_6 \le n_1+n_2+n_3-n_4-n_5-1 \end{array} \tag{3.47}$$

$$\begin{array}{ll} \textit{Subspace}\ \#7 & \textit{Subspace}\ \#8 \\ n_3 = 0 & n_3 = 0 \\ n_4 = L & 1 \le n_4 \le L-1 \\ 1 \le n_1 \le L & n_4+1 \le n_1 \le L \\ L-n_1+1 \le n_2 \le L & 0 \le n_2 \le L-n_1+n_4 \\ 0 \le n_5 \le n_1+n_2-L-1 & 0 \le n_5 \le n_1+n_2-n_4-1 \\ 0 \le n_6 \le n_1+n_2-n_5-L-1 & 0 \le n_6 \le n_1+n_2-n_4-n_5-1 \end{array} \tag{3.48}$$

$$\begin{array}{c} \textit{Subspace}\ \#9 \\ 1 \le n_3 \le L-1 \\ n_3+1 \le n_4 \le L \\ n_4-n_3+1 \le n_1 \le L \\ 0 \le n_2 \le L+n_4-n_1-n_3 \\ 0 \le n_5 \le n_1+n_2+n_3-n_4-1 \\ 0 \le n_6 \le n_1+n_2+n_3-n_4-n_5-1 \end{array} \tag{3.49}$$

The number of voting situations in each of the nine subspaces in this partition can then be computed by using algebraic relations for sums of powers of integers, as described in the development of the representation for $N_{PMRW}^{A\}}(3,n,IAC)$ in Eq. 3.26. After doing this for each of the subspaces and accumulating the results:

$$N_{PMRW}^{S(n_4>n_3)}(3,L,MC) = \frac{L\left(109L^5 + 375L^4 + 415L^3 + 45L^2 - 164L - 60\right)}{720}. \tag{3.50}$$

A similar procedure is used to account for the number of voting situations, $N_{PMRW}^{S(n_4=n_3)}(3,L,MC)$, in $N_{PMRW}^{A*\}}(3,L,MC)$ with $n_3 = n_4$. After all partitioning is done to remove *Max* and *Min* arguments from the lower and upper bounds on the restrictions on n_i's to obtain $N_{PMRW}^{S(n_4=n_3)}(3,L,MC)$, we require four subspaces, denoted *Subspace #10* through *Subspace #13*, as shown in Eqs. 3.51 and 3.52.

$$\begin{array}{cc} \textit{Subspace}\ \#10 & \textit{Subspace}\ \#11 \\ 0 \le n_5 \le L-1 & 0 \le n_5 \le L-1 \\ 0 \le n_6 \le L-1-n_5 & 0 \le n_6 \le L-1-n_5 \\ n_5+n_6+1 \le n_1 \le L & 0 \le n_1 \le n_5+n_6 \\ 0 \le n_2 \le L & n_5+n_6-n_1+1 \le n_2 \le L \end{array} \tag{3.51}$$

$$\begin{array}{cc} \textit{Subspace}\ \#12 & \textit{Subspace}\ \#13 \\ n_5 = L & 0 \le n_5 \le L-1 \\ 0 \le n_6 \le L-1 & L-n_5 \le n_6 \le L \\ n_6+1 \le n_1 \le L & n_5+n_6+1-L \le n_1 \le L \\ L+1+n_6-n_1 \le n_2 \le L & n_5+n_6-n_1+1 \le n_2 \le L \end{array} \tag{3.52}$$

Gehrlein and Lepelley (1997) contains a minor typographical error for the bounds for *Subspace* #13. After developing representations for each of these four subspaces and accumulating the results

$$N_{PMRW}^{S(n_4=n_3)}(3,L,MC) = \frac{L\left(3L^3 + 10L^2 + 12L + 5\right)}{6}. \tag{3.53}$$

It was noted above that there are the same number of voting situations in $N_{PMRW}^{A*\}}(3,L,MC)$ that have $n_4 > n_3$ as there are with $n_3 > n_4$. Using all of this, along with the fact that there are $L+1$ different values that each of n_3 and n_4 can have when $n_3 = n_4$, we obtain a representation for $N_{PMRW}^{A*\}}(3,L,MC)$ for each $L \ge 3$ as:

$$N_{PMRW}^{A*\}}(3,L,MC) = 2N_{PMRW}^{S(n_4>n_3)}(3,L,MC) + (L+1)N_{PMRW}^{S(n_4=n_3)}(3,L,MC). \tag{3.54}$$

After substitution and algebraic reduction,

$$N_{PMRW}^{A^*\}}(3,L,MC)=\frac{2L}{3}+\frac{107L^2}{45}+\frac{91L^3}{24}+\frac{239L^4}{72}+\frac{37L^5}{24}+\frac{109L^6}{360}. \tag{3.55}$$

Due to the symmetry of MC with respect to candidates

$$P_{PMRW}^{S}(3,L,MC)=\frac{3N_{PMRW}^{A^*\}}(3,L,MC)}{(L+1)^6}, \tag{3.56}$$

which reduces to

$$P_{PMRW}^{S}(3,L,MC)=\frac{L(109L^4+446L^3+749L^2+616L+240)}{120(L+1)^5}, \text{ for } L\geq 3. \tag{3.57}$$

Following the logic of previous development of representations for IAC probabilities, it is possible to obtain representations for the probabilities $P_{PMRW}^{\#1}(3,L,MC)$ and $P_{PMRW}^{W}(3,L,MC)$ for the case of MC. Gehrlein (2002a) obtains these representations for $L\geq 3$ as:

$$P_{PMRW}^{\#1}(3,L,MC)=\frac{109L^5+644L^4+1541L^3+1894L^2+1212L+360}{360(L+1)^5} \tag{3.58}$$

$$P_{PMRW}^{W}(3,L,MC)=\frac{109L^5+578L^4+1157L^3+1168L^2+588L+120}{120(L+1)^5} \tag{3.59}$$

Since the number of voters is not fixed with MC, the odd-even effects that were observed with IAC do not occur with MC. The following results are obtained directly by taking derivatives of the representations in Eqs. 3.57 through 3.59:

Theorem 3.1 (MC). $P_{PMRW}^{S}(3,L,MC)<P_{PMRW}^{S}(3,L+1,MC)$, for all $L\geq 3$.

Theorem 3.2 (MC). $P_{PMRW}^{\#1}(3,L,MC)>P_{PMRW}^{\#1}(3,L+1,MC)$, for all $L\geq 3$.

Theorem 3.3 (MC). $P_{PMRW}^{W}(3,L,MC)>P_{PMRW}^{W}(3,L+1,MC)$, for all $L\geq 3$.

Table 3.3 lists computed values of $P_{PMRW}^{S}(3,L,MC)$, $P_{PMRW}^{\#1}(3,L,MC)$ and $P_{PMRW}^{W}(3,L,MC)$ for various values of L. The computed probability values approach their limiting values slowly as L increases, to suggest that the convergence to the limiting probability values is quite slow as $E(n)$ increases. $P_{PMRW}^{S}(3,L,MC)$ is also found to be relatively small for small value of $E(n)$.

Table 3.3 Probabilities with Maximum Culture Condition (MC)

L	$P^{S}_{PMRW}(3,L,MC)$	$P^{\#1}_{PMRW}(3,L,MC)$	$P^{W}_{PMRW}(3,L,MC)$
3	.7251	.3833	.9517
4	.7588	.3650	.9461
5	.7819	.3535	.9417
6	.7988	.3456	.9382
7	.8117	.3398	.9354
8	.8218	.3354	.9330
9	.8301	.3319	.9310
10	.8368	.3291	.9293
11	.8426	.3268	.9278
20	.8700	.3162	.9203
40	.8885	.3096	.9147
50	.8923	.3082	.9135
∞	.9083	.3028	.9083

As we have observed, the calculations that are required to develop representations with IAC and MC can be very cumbersome to perform, despite the fact that the logic behind the computational process is quite simple. It will be seen later that these procedures can be dramatically simplified. However, an understanding of the logic that is employed by these simplified techniques requires an understanding of the algebraic techniques that we have just developed.

3.5 Expected Balance for Individual Preferences

Another view of balanced preferences considers an expected balance of preferences between pairs of candidates within the preference rankings of each individual voter within a population. In this situation, we let $\boldsymbol{p}$ denote a six-dimensional vector for the three-candidate case, where p_i denotes the probability that a voter who is selected at random from the population of voters will have the corresponding linear preference ranking on candidates that is shown in Fig. 3.3, That is, a randomly selected voter will have the linear preference ranking $A \succ B \succ C$ with probability p_1. We also assume that each voter's preferences are independent of the other voters' preferences.

A	A	B	C	B	C
B	C	A	A	C	B
C	B	C	B	A	A
p_1	p_2	p_3	p_4	p_5	p_6

Fig. 3.3 The probability that a randomly selected voter will have a given linear preference ranking

The existence of any given voter preference profile can be considered to be the result of a random selection of n individual voter's preferences. Following classic analysis from probability modeling, we start with an urn that contains some total number of balls, with each ball being one of six different colors. Each color corresponds to one of the six possible linear preference rankings on three candidates. The proportions of the total number of balls of each color in the urn are equal to their associated probabilities that are specified in $\boldsymbol{p}$. Then, n balls are sequentially drawn at random from the urn, with replacement. The color of the ball that is drawn during the i^{th} step of the sequential drawing is used to assign the associated linear preference ranking to the i^{th} voter before the ball is placed back in the urn. Following previous discussion, this procedure is used to obtain voter preference profiles in which the preferences of each individual voter are identifiable, so that the voter's preferences are not anonymous.

The random selection of balls is being done with replacement during the experiment, so that the probability of observing any given preference ranking for an individual voter does not change from draw to draw. A multinomial probability model is appropriate for developing representations for observing any particular given event under such an experiment. The voting situation that results from any given voter preference profile with these identifiable voters can be obtained simply by determining the number of voters that have each of the six possible linear preference rankings. The probability that any such voting situation is observed from the identifiable voters is then given by $n!\prod_{i=1}^{6}\frac{p_i^{n_i}}{n_i!}$. We can then directly obtain a representation for the probability, $P_{PMRW}^{A\}}(3,n,\boldsymbol{p})$, that A is the strict PMRW for odd n for any given $\boldsymbol{p}$ from the discussion that led to Eq. 3.23, with

$$P_{PMRW}^{A\}}(3,n,\boldsymbol{p}) = \sum_{n_6=0}^{\frac{n-1}{2}} \sum_{n_5=0}^{\frac{n-1}{2}-n_6} \sum_{n_4=0}^{\frac{n-1}{2}-n_6-n_5} \sum_{n_3=0}^{\frac{n-1}{2}-n_6-n_5} \sum_{n_2=0}^{n-n_6-n_5-n_4-n_3} n!\prod_{i=1}^{6}\frac{p_i^{n_i}}{n_i!}, \tag{3.60}$$

where $n_1 = n-n_6-n_5-n_4-n_3-n_2$. Similar logic can then be used to find representations for the probability that each of B and C is the PMRW. The probability, $P_{PMRW}^{S}(3,n,\boldsymbol{p})$, that a PMRW exists for a given $\boldsymbol{p}$ with n voters for three-candidate elections would then be obtained as the sum of these three representations.

Gehrlein and Fishburn (1976b) develop a much simpler form of the resulting probability representation for $P_{PMRW}^{S}(3,n,\boldsymbol{p})$ that only requires a three-summation function as:

$$P_{PMRW}^{S}(3,n,\boldsymbol{p}) = \sum_{m_1=0}^{\frac{n-1}{2}} \sum_{m_2=0}^{\frac{n-1}{2}-m_1} \sum_{m_3=0}^{\frac{n-1}{2}-m_1} \frac{n!}{m_1!m_2!m_3!m_4!} \left\{ \begin{array}{l} (p_5+p_6)^{m_1} p_3^{m_2} p_4^{m_3} (p_1+p_2)^{m_4} + \\ (p_2+p_4)^{m_1} p_1^{m_2} p_6^{m_3} (p_3+p_5)^{m_4} + \\ (p_1+p_3)^{m_1} p_5^{m_2} p_2^{m_3} (p_4+p_6)^{m_4} \end{array} \right\}. \tag{3.61}$$

Here, $m_4 = n - m_1 - m_2 - m_3$. The logic that leads to this representation is quite straightforward, and it will be extended in later analysis. Gillett (1976, 1978) independently developed the same representation for $P_{PMRW}^{S}(3,n,\boldsymbol{p})$.

In previous discussion related to IAC, we observed an expected balance of preference on all pairs of candidates by matching voting situations. We now consider special cases of $\boldsymbol{p}$ in the representation for $P_{PMRW}^{S}(3,n,\boldsymbol{p})$ that result in an expected balance of individual preference on all pairs of candidates for a randomly selected voter. Let $\Delta(A,B)$ denote the difference between the sum of the p_i values for linear preference rankings with $A \succ B$ in Fig. 3.3 and the sum of the p_i values for linear preference rankings with $B \succ A$. The same definition applies to all pairs of candidates in the same fashion, so that

$$\begin{aligned} \Delta(A,B) &= p_1 + p_2 + p_4 - p_3 - p_5 - p_6 \\ \Delta(A,C) &= p_1 + p_2 + p_3 - p_4 - p_5 - p_6 \\ \Delta(B,C) &= p_1 + p_3 + p_5 - p_2 - p_4 - p_6 \,. \end{aligned} \tag{3.62}$$

Since voter's preferences are independent of the preferences of other voters, it follows from these definitions that a randomly selected voter will be more likely to have a preference ranking with $A \succ B$ than $B \succ A$ if $\Delta(A,B) > 0$. A total balance in preference on all pairs of candidates will exist for voters when $\Delta(A,B)$ = $\Delta(A,C)$ = $\Delta(B,C)$ = 0. Gehrlein (1978) proves that this complete balance exists only when each possible linear preference ranking has the same probability of being observed as it reversed, or *dual preference ranking*. That is, when $p_1 = p_6$, $p_2 = p_5$ and $p_3 = p_4$, which is defined as the *Dual Culture Condition* (*DC*) in Gehrlein (1978).

3.5.1 Dual Culture Condition

Sen (1970) considers an example that is very much in the spirit of an extreme case of DC. That example considers a two-class society in which the classes have radically different interests. For this "class war" condition we would expect to have voter profiles containing only two different rankings on alternatives. One class would have some preference ranking and the other class would have dual preference ranking. It is noted that PMR would always be transitive for an odd number of voters in this particular situation with two possible rankings. DC would further assume that the two classes contain the same number of members in Sen's example.

Balasko and Crès (1998) perform an evaluation of the probability that PMR cycles exist when voter profiles are restricted to populations that are of a general form of a "class war" type model. They define "bipartite populations" in which voters have overall preferences that partition them into two groups. Voters within one group have preferences that cluster about a "north pole" and the other group has preferences that cluster about a "south pole". With PMR, the group of voters that is associated with the "heaviest pole" will then dominate all decisions. It is shown that the probability that a PMR cycle exists in such situations becomes quite small.

Gehrlein (1978) gives a closed form representation for the limiting probability, $P^S_{PMRW}(3,\infty,DC)$, that a strict PMRW exists for three alternatives in the limiting case of voters, as $n \to \infty$, for any $\boldsymbol{p}$ meeting DC. The representation is developed by using a procedure that is a direct application of the Central Limit Theorem. The background of this procedure is described in detail here, since it is also used to develop a number of other results that follow.

Consider the experiment in which a random voter preference profile is being obtained by sequentially drawing balls at random from an urn to represent individual voter's preference rankings in the profile. We start by considering the probability that A will be the PMRW in such a random voter preference profile. We define two discrete variables X_B^i and X_C^i that describe two joint events that can result as each ball is drawn in the experiment. The probabilities that are associated with the discrete outcomes for the two events for the i^{th} ball that is drawn are given by:

$$X_B^i = \begin{matrix} +1: p_1+p_2+p_4 \\ -1: p_3+p_5+p_6 \end{matrix} \qquad (3.63)$$

$$X_C^i = \begin{matrix} +1: p_1+p_2+p_3 \\ -1: p_4+p_5+p_6. \end{matrix}$$

Based on the definitions of these variables, $X_B^i = +1$ if $A \succ B$ in the i^{th} voter's preference ranking, and $X_B^i = -1$ if $B \succ A$ for the i^{th} voter. Then, $\boldsymbol{AMB}$ for the n voters in the random voter preference profile if $\sum_{i=1}^{n} X_B^i > 0$. Similarly, $\boldsymbol{AMC}$ for the n voters if $\sum_{i=1}^{n} X_C^i > 0$. Let $\overline{X}_B$ denote the average value of X_B^i, with $\overline{X}_B = \left[\sum_{i=1}^{n} X_B^i\right]/n$. Then, A will be the strict PMRW with the joint probability that $\overline{X}_B > 0$ and $\overline{X}_C > 0$. This can be restated in the form that A will be the PMRW in a randomly drawn profile with the joint probability that $\overline{X}_B\sqrt{n} > 0$ and $\overline{X}_C\sqrt{n} > 0$.

As the number of voters gets very large, with $n \rightarrow \infty$, the Central Limit Theorem applies [Wilks (1962)] and the limiting joint distribution of $\overline{X}_B\sqrt{n}$ and $\overline{X}_C\sqrt{n}$ has a bivariate normal distribution. The probability that $\overline{X}_B\sqrt{n}$ and $\overline{X}_C\sqrt{n}$ take on any specific value, including zero, in this bivariate normal distribution is zero, so the probability that A is the PMRW in a randomly drawn preference profile can be restated as the joint probability that $\overline{X}_B\sqrt{n} \geq 0$ and $\overline{X}_C\sqrt{n} \geq 0$. The Central Limit Theorem also states that the correlation between $\overline{X}_B\sqrt{n}$ and $\overline{X}_C\sqrt{n}$ in this bivariate normal distribution is identical to the correlation between the original variables X_B^i and X_C^i.

In order to obtain the correlation between X_B^i and X_C^i, we start by obtaining representations for the expected values, $E\left(X_B^i\right)$ and $E\left(X_C^i\right)$ of these variables:

$$E\left(X_B^i\right) = +1p_1 + 1p_2 - 1p_3 + 1p_4 - 1p_5 - 1p_6 \tag{3.64}$$
$$E\left(X_C^i\right) = +1p_1 + 1p_2 + 1p_3 - 1p_4 - 1p_5 - 1p_6\,.$$

Since DC requires that $p_1 = p_6, p_2 = p_5$ and $p_3 = p_4$, it follows from Eq. 3.64 that $E\left(X_B^i\right) = E\left(X_C^i\right) = 0$. The variance terms, $Var\left(X_B^i\right)$ and $Var\left(X_C^i\right)$, are then obtained by definition from

$$Var\left(X_B^i\right) = E\left[\left(X_B^i - E\left(X_B^i\right)\right)^2\right] = E\left[\left(X_B^i\right)^2\right] = \tag{3.65}$$
$$(+1)^2 p_1 + (+1)^2 p_2 + (-1)^2 p_3 + (+1)^2 p_4 + (-1)^2 p_5 + (-1)^2 p_6 = 1$$
$$Var\left(X_C^i\right) = E\left[\left(X_C^i - E\left(X_C^i\right)\right)^2\right] = E\left[\left(X_C^i\right)^2\right] =$$
$$(+1)^2 p_1 + (+1)^2 p_2 + (+1)^2 p_3 + (-1)^2 p_4 + (-1)^2 p_5 + (-1)^2 p_6 = 1.$$

The covariance, $Cov\left(X_B^i, X_C^i\right)$, between X_B^i and X_C^i is obtained directly by definition from

$$Cov\left(X_B^i, X_C^i\right) = E\left\{\left(X_B^i - E\left(X_B^i\right)\right)\left(X_C^i - E\left(X_C^i\right)\right)\right\} = E\left\{X_B^i X_C^i\right\} = \tag{3.66}$$
$$(+1)(+1)p_1 + (+1)(+1)p_2 + (-1)(+1)p_3 + (+1)(-1)p_4 + (-1)(-1)p_5 + (-1)(+1)p_6.$$

The symmetry of DC, with $p_1 = p_6, p_2 = p_5$ and $p_3 = p_4$, requires that $p_1 + p_2 + p_3 = 1/2$, and after algebraic reduction of Eq. 3.66 we obtain

$$Cov\left(X_B^i, X_C^i\right) = 1 - 4p_3\,. \tag{3.67}$$

The coefficient of correlation, $Cor\left(X_B^i, X_C^i\right)$, between X_B^i and X_C^i is obtained directly by definition from

$$Cor\left(X_B^i, X_C^i\right) = \frac{Cov\left(X_B^i, X_C^i\right)}{\sqrt{Var\left(X_B^i\right)Var\left(X_C^i\right)}} = 1 - 4p_3 . \tag{3.68}$$

The probability that *A* is the PMRW in a randomly drawn voter preference profile is therefore given as the joint probability that $\overline{X}_B\sqrt{n} \geq 0$ and $\overline{X}_C\sqrt{n} \geq 0$, in a bivariate normal distribution with a coefficient of correlation that is equal to $1-4p_3$.

We make an additional observation regarding this probability. It was shown that $E\left(X_B^i\right) = E\left(X_C^i\right) = 0$ as a result of Eq. 3.64, and it therefore follows directly that $E\left(X_B^i\sqrt{n}\right) = E\left(X_C^i\sqrt{n}\right) = 0$. So, the probability that *A* is the PMRW in a randomly drawn preference profile under DC as $n \to \infty$ is the same as the joint probability that $\overline{X}_B\sqrt{n} \geq E\left(\overline{X}_B\sqrt{n}\right)$ and $\overline{X}_C\sqrt{n} \geq E\left(\overline{X}_C\sqrt{n}\right)$, in a bivariate normal distribution with a coefficient of correlation equal to $1-4p_3$. The probability that both variables in a bivariate normal distribution are greater than, or equal to, their respective expected values is defined as a bivariate normal positive orthant probability [Johnson and Kotz (1972)].

Sheppard's 1898 Theorem of Median Dichotomy [Johnson and Kotz (1972), pg. 92] shows that the bivariate normal positive orthant probability for a distribution with a coefficient of correlation equal to ρ is $\frac{1}{4} + \frac{1}{2\pi}Sin^{-1}(\rho)$. A representation for the limiting probability that *A* is the PMRW in a randomly drawn preference profile under DC as $n \to \infty$ then follows directly from Sheppard's Theorem. Exactly the same process can be used to develop representations for the probability that *B* is the PMRW and that *C* is the PMRW. After accumulating all of the results, we find

$$P_{PMRW}^S(3,\infty,DC) = \frac{3}{4} + \frac{1}{2\pi}\sum_{j=1}^{3} Sin^{-1}\left(1-4p_j\right). \tag{3.69}$$

Table 3.4 lists computed values of $P_{PMRW}^S(3,\infty,DC)$ for each value of p_1, p_2 and p_3 = 0.00(.025).50 from Gehrlein (1999a). Columns of entries have been truncated in this table to account for the fact that $P_{PMRW}^S(3,\infty,DC)$ is invariant under permutations of p_1, p_2 and p_3. There is a significant probability that a Condorcet winner exists for many entries in Table 3.4. The results in Table 3.4 also lead to the following observations from Gehrlein (1999a) that will be useful in discussion later.

Table 3.4 The Limiting Probability that there is a PMRW for Large Electorates with the Dual Culture Condition (DC)

	p_2										
p_1	*0.000*	*0.025*	*0.050*	*0.075*	*0.100*	*0.125*	*0.150*	*0.175*	*0.200*	*0.225*	*0.250*
0.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.025	1.000	0.959	0.952	0.949	0.947	0.946	0.945	0.945	0.944	0.944	0.944
0.050	1.000	0.952	0.943	0.938	0.935	0.932	0.931	0.930	0.930	0.929	0.930
0.075	1.000	0.949	0.938	0.931	0.927	0.925	0.923	0.922	0.921	0.921	0.922
0.100	1.000	0.947	0.935	0.927	0.923	0.920	0.918	0.917	0.917	0.917	0.918
0.125	1.000	0.946	0.932	0.925	0.920	0.917	0.915	0.914	0.914	0.915	0.917
0.150	1.000	0.945	0.931	0.923	0.918	0.915	0.913	0.912	0.913	0.915	0.918
0.175	1.000	0.945	0.930	0.922	0.917	0.914	0.912	0.912	0.914	0.917	0.922
0.200	1.000	0.944	0.930	0.921	0.917	0.914	0.913	0.914	0.917	0.921	0.930
0.225	1.000	0.944	0.929	0.921	0.917	0.915	0.915	0.917	0.921	0.929	0.944
0.250	1.000	0.944	0.930	0.922	0.918	0.917	0.918	0.922	0.930	0.944	1.000
0.275	1.000	0.944	0.930	0.923	0.920	0.920	0.923	0.930	0.944	1.000	
0.300	1.000	0.945	0.931	0.925	0.923	0.925	0.931	0.945	1.000		
0.325	1.000	0.945	0.932	0.927	0.927	0.932	0.945	1.000			
0.350	1.000	0.946	0.935	0.931	0.935	0.946	1.000				
0.375	1.000	0.947	0.938	0.938	0.947	1.000					
0.400	1.000	0.949	0.943	0.949	1.000						
0.425	1.000	0.952	0.952	1.000							
0.450	1.000	0.959	1.000								
0.475	1.000	1.000									
0.500	1.000										

Lemma 3.1 $P^S_{PMRW}(3,\infty,DC) = 1$ if p_1, p_2 or p_3 is equal to zero.

Proof: $P^S_{PMRW}(3,\infty,DC)$ is invariant under permutations of p_1, p_2 and p_3, so we assume arbitrarily that $p_3 = 0$. Substitute $p_3 = 0$ and $p_2 = 1/2 - p_1$ in the representation for $P^S_{PMRW}(3,\infty,DC)$ in Eq. 3.69, and the result follows directly from basic trigonometric identities. **QED**

Gehrlein (1978) also proves that $P^S_{PMRW}(3,\infty,DC)$ is minimized for DC for the special case in which $p_j = 1/6$ for all j = 1,2,3,4,5,6, and entries in Table 4.3 verify that result.

This special case in which $p_j = 1/6$ for all j = 1,2,3,4,5,6 has been widely referred to as the *Impartial Culture Condition* (*IC*) and it has received a great deal of attention in the literature on voting models.

Lemma 3.2 $P^S_{PMRW}(3,\infty,DC)$ is minimized by IC.

Proof: It can assumed without loss of generality that $p_1 \geq p_2 \geq p_3$. We are interested in considering the effect of increasing p_1, while decreasing p_3, with p_2 remaining fixed. Using the fact that $p_1 + p_2 + p_3 = 1/2$ with DC, $P^S_{PMRW}(3,\infty,DC)$ can be written as

$$P^S_{PMRW}(3,\infty,DC) = \tag{3.70}$$
$$\frac{3}{4} + \frac{1}{2\pi}\ Sin^{-1}(1-4p_1) + Sin^{-1}(1-4p_2) + Sin^{-1}(4p_1 + 4p_2 - 1)\}.$$

By taking the derivative with respect to p_1, we obtain

$$\frac{\sqrt{2p_1(1-2p_1)} - \sqrt{2p_1(1-2p_1) + 2p_2(1-4p_1-2p_2)}}{\sqrt{2p_1(1-2p_1)}\sqrt{2p_1(1-2p_1) + 2p_2(1-4p_1-2p_2)}}. \tag{3.71}$$

This derivative is positive as long as

$$2p_2(1-4p_1-2p_2) < 0. \tag{3.72}$$

Using the fact that $p_1 + p_2 + p_3 = 1/2$ again, this requires that

$$\begin{gathered} 2p_2(2p_1 + 2p_2 + 2p_3 - 4p_1 - 2p_2) < 0 \\ 2p_2(2p_3 - 2p_1) < 0 \\ p_3 < p_1. \end{gathered} \tag{3.73}$$

Since $p_3 < p_1$ in our basic assumption, it follows that $P^S_{PMRW}(3,\infty,DC)$ will always be reduced in value if the maximum value of p_1, p_2 and p_3, which is p_1 in this case, is decreased while increasing the minimum value of p_1, p_2 and p_3, which is p_3 in this case, with the remaining value of p_1, p_2 and p_3 being kept fixed. This reduction will continue until the term that starts out with the maximum value is made equal to the term that starts out with the minimum value, while the remaining term stays fixed.

By sequentially applying this operation to the p_1, p_2 and p_3 terms, $P^S_{PMRW}(3,\infty,DC)$ will be reduced in each step as we converge toward the situation in which p_1, p_2 and p_3 stabilize at IC. **QED**

Weisberg and Niemi (1973) produce many of these limiting value results with the assumption of DC, after starting with a different set of assumptions.

3.5.2 Impartial Culture Condition

The basic notion behind IC was presented in Chapter 1 in a very different form, while discussing the work in Laplace (1795). General m-candidate elections, with candidates $\boldsymbol{C^m} = C_1, C_2, ..., C_m\}$ were considered. A model was developed in which voters represent their individual preference rankings on candidates by assigning points to candidates. Let t_j^i denote the number of points that the i^{th} voter assigns to the j^{th} candidate, with a greater assignment of points to a candidate indicating a greater preference for that candidate. Then, each t_j^i has some real value on the closed interval $[0,z]$ and it is independent of all other t_j^i values. Each voter then has a linear preference ranking on candidates, according to the ordering of points that have been assigned to the candidates. Given Laplace's assumption that all possible combinations of t_j^i 's are equally likely to be observed, it immediately follows that all possible orderings on the t_j^i 's are equally likely to be observed. Consequently, all possible linear preference orders on candidates must be equally likely to be observed, which is completely consistent with the underlying notion behind IC.

Weber (1978a, 1978b, 1978c) develops a model like the one considered by Laplace (1795), in which the t_j^i values represent the utilities that voters have for candidates in a *random society*. Otherwise, the nature of the models is identical. Maassen and Bezembinder (2000) also develop the same basic model to compute probabilities that a PMRW exists with the assumption of IC.

Klahr (1966) presents two different versions of the IC assumption that produce identical results. The first version is consistent with discussion to this point, with all voters having their own specific linear preference ranking on candidates. Then voters are selected at random from the population to obtain a voter preference profile, with IC representing the case in which all linear rankings are equally likely to represent the true preferences of the randomly selected voter. The second version describes a situation in which every voter is completely indifferent between all candidates. When voters are randomly selected from the population, their complete indifference between all candidates will lead them to randomly select any one of the possible linear rankings on the candidates with equal likelihood, when they are asked for their linear preference ranking.

A representation for $P_{PMRW}^S(3,n,IC)$ follows directly from the representation for $P_{PMRW}^S(3,n,\boldsymbol{p})$ in Eq. 3.61, as stated in Gehrlein and Fishburn (1976a):

$$P_{PMRW}^S(3,n,IC) = \frac{3n!}{6^n} \sum_{s_1=0}^{\frac{n-1}{2}} \sum_{s_2=0}^{\frac{n-1}{2}-s_1} \sum_{s_3=0}^{\frac{n-1}{2}-s_1} \frac{2^{n-s_2-s_3}}{s_1!s_2!s_3!(n-s_1-s_2-s_3)!}. \tag{3.74}$$

Many earlier attempts were made to develop a simple representation for $P_{PMRW}^S(3,n,IC)$, including Campbell and Tullock (1966), Garman and Kamien (1968), Niemi and Weisberg (1968) and DeMeyer and Plott (1970). However, each of these representations is significantly more complicated than the form given in Eq. 3.74.

A study that takes a very different approach to this problem is due to Hansen and Prince (1973). Each study mentioned before attempted to obtain a representation for $P^S_{PMRW}(3,n,IC)$ by finding summation functions that would enumerate all profiles with a PMRW. Hansen and Prince (1973) developed a representation for the probability, $P^S_{PMRC}(3,n,IC)$, that a PMR cycle exists for three candidates under the assumption of IC. Their summation functions implicitly enumerate all profiles on three candidates that have a PMR cycle, with

$$P^S_{PMRC}(3,n,IC) = \sum_{m_1=0}^{\frac{n-3}{2}} \sum_{m_2=0}^{\frac{n-3}{2}-m_1} \sum_{m_3=0}^{\frac{n-3}{2}-m_1-m_2} \sum_{m_4=0}^{\frac{n-3}{2}-m_1-m_2-m_3} \sum_{m_5=0}^{\frac{n-3}{2}-m_1-m_2-m_3-m_4} \frac{2F(m_1,m_2,m_3,m_4,m_5)}{6^n}, \tag{3.75}$$

where

$$F(m_1,m_2,m_3,m_4,m_5) = \frac{n!}{\left[\begin{array}{c}(m_1+m_2+m_4+1)!(m_1+m_3+m_5+1)!\left(\frac{n-1}{2}-m_1-m_4-m_5\right)!\times \\ m_4!m_5!\left(\frac{n-3}{2}-m_1-m_2-m_3-m_4-m_5\right)!\end{array}\right]}. \tag{3.76}$$

Guilbaud (1952) was the first to develop a representation for the limiting probability $P^S_{PMRW}(3,\infty,IC)$ as

$$P^S_{PMRW}(3,\infty,IC) = \frac{3}{4} + \frac{3}{2\pi} Sin^{-1}\left(\frac{1}{3}\right), \tag{3.77}$$

which could be obtained from the representation for $P^S_{PMRW}(3,\infty,DC)$ in Eq. 3.69. Niemi and Weisberg (1968) and Garman and Kamien (1968) reproduce Guilbaud's result by using the normal approximation to the binomial probabilities in Eq. 3.60. Guilbaud's result is stated without any discussion, and it was presumably developed with the same approach. Krishnamoorthy and Raghavachari (2005) reproduce the representation in Eq. 3.77 by using an approach that is based on the Central Limit Theorem, as discussed in the arguments leading to Eq. 3.69, and they refer to this approach as using a "statistical perspective."

Representations for $P^S_{PMRW}(3,\infty,IC)$ have been developed by different techniques and in different forms. Berg and Bjurulf (1983) and Gehrlein (2004b) use similar techniques to those discussed above to find

$$P^S_{PMRW}(3,\infty,IC) = \frac{3}{\pi} Cos^{-1}\left(\sqrt{\frac{1}{3}}\right). \tag{3.78}$$

Stensholt (1996) uses an analysis that is based on circle pictograms to find

$$P^{S}_{PMRW}(3,\infty,IC) = 1 - \frac{1}{2\pi} Cos^{-1}\left(\frac{23}{27}\right). \tag{3.79}$$

All of these different representations must obviously lead to the same numerical value, with $P^{S}_{PMRW}(3,\infty,IC) \approx .91226$. Kalai (2002) uses arguments that are based on Fourier-theoretic analysis to obtain bounds on the probability that election outcomes on pairs of candidates will cycle, while considering the set of all election procedures that are neutral toward candidates, rather than just considering PMR.

Kelly (1974,1986) and Buckley and Westen (1979) consider the general behavior of various aspects of the probability that a PMRW exists under IC. Some of the results from these observations were proved, while a number of conjectures remained regarding this general behavior. Fishburn, et al. (1979a,b) later proved some of the conjectures for the special case of three candidates, and a summary of known results is given by:

Theorem 3.1 (IC). $P^{S}_{PMRW}(3,n,IC) > P^{S}_{PMRW}(3,n+2,IC)$, for all odd $n \geq 1$.

Theorem 3.2 (IC). $P^{S}_{PMRW}(3,n,IC) < P^{S}_{PMRW}(3,n+2,IC)$, for all even $n \geq 2$.

Theorem 3.3 (IC). $P^{S}_{PMRW}(3,n,IC) = 3\, P^{\#1}_{PMRW}(3,n,IC) = P^{W}_{PMRW}(3,n,IC)$, for all odd $n \geq 1$.

Theorem 3.4 (IC). $P^{\#1}_{PMRW}(3,n,IC) > P^{\#1}_{PMRW}(3,n+2,IC)$, for all even $n \geq 2$.

Theorem 3.5 (IC). $P^{W}_{PMRW}(3,n,IC) > P^{W}_{PMRW}(3,n+2,IC)$, for all even n greater than some integer N.

Table 3.5 lists computed values for each of $P^{S}_{PMRW}(3,n,IC)$, $P^{\#1}_{PMRW}(3,n,IC)$ and $P^{W}_{PMRW}(3,n,IC)$ for various values of n. Values of $P^{S}_{PMRW}(3,n,IC)$ for odd n were computed directly from Eq. 3.74, and values of $P^{S}_{PMRW}(3,n,IC)$ for even n were computed with a simple modification of Eq. 3.74. Values of $P^{\#1}_{PMRW}(3,n,IC)$ and $P^{W}_{PMRW}(3,n,IC)$ were computed in the same fashion as that used in the development of $P^{\#1}_{PMRW}(3,n,IAC)$ and $P^{W}_{PMRW}(3,n,IAC)$ in Eqs. 3.36 and 3.40, following Gehrlein (2002a) with multinomial probabilities for IC replacing the simple counting techniques of IAC.

The computed values in Table 3.5 are close to the Monte-Carlo simulation estimates of these probabilities that are given in Buckley and Westen (1979). As with the IAC results, we observe different behavior for these probabilities, depending upon whether n is odd or even valued. These probabilities converge to their limiting values quite quickly for odd n, while the rate of convergence is much slower for even n, with rather small values of $P^{S}_{PMRW}(3,n,IC)$ for small even values of n.

Table 3.5 Probabilities with Impartial Culture Condition (IC)

n	$P^{S}_{PMRW}(3,n,IC)$	$P^{\#1}_{PMRW}(3,n,IC)$	$P^{W}_{PMRW}(3,n,IC)$
3	.9444	.3148	.9444
4	.4444	.5231	1.0000
5	.9306	.3102	.9306
6	.5087	.4821	.9961
7	.9250	.3083	.9250
8	.5519	.4574	.9920
9	.9220	.3073	.9220
10	.5834	.4406	.9882
11	.9202	.3067	.9202
20	.6686	.3991	.9750
21	.9163	.3054	.9163
40	.7346	.3702	.9616
41	.9143	.3048	.9143
∞	.9123	.3041	.9123

3.6 Uniform Culture Condition

Buckley (1975) develops the notion of conditional probabilities for observing voting events. For example, the representation for $P^{S}_{PMRW}(3,n,\boldsymbol{p})$ in Eq. 3.61 is conditional in the sense that it computes the probability that a PMRW exists, given the specified probability vector $\boldsymbol{p}$. Buckley suggests that it might be more appropriate to consider the *unconditional probability* that a PMRW is observed. To do this, we would define the set of all possible $\boldsymbol{p}$ vectors as Ω, with a probability density function $\boldsymbol{F}(\boldsymbol{p})$ denoting the probability that each possible $\boldsymbol{p}$ in Ω is observed.

Gehrlein (1981b) uses this same basic notion to develop a representation for the *expected probability* that a PMRW exists. This approach takes a different perspective on an expected balance between all pairs of candidates within a voter's' preferences on pairs of candidates by using the *Uniform Culture Condition* (*UC*). In particular, all $\boldsymbol{p}$ vectors with $\sum_{i=1}^{6} p_i = 1$ are assumed to be equally likely to represent the preferences of a population of voters. This condition sounds very similar to the assumption of equally likely combinations of t_j^i's in Laplace's analysis that was shown to be equivalent to IC. However, Laplace's arguments lead to an expected balance for each individual voter's preferences. With UC, the expected balance will be seen to refer to the overall preferences for all voters in a voting situation, making it more closely linked to IAC, than to IC.

As in the case of IAC, Ω can be partitioned into pairs of vectors according to the matching: $p_1 \leftrightarrow p_6$, $p_2 \leftrightarrow p_5$, $p_3 \leftrightarrow p_4$. If both vectors in the matching pair are equally likely to be observed, then the expected probability that $A \succ B$ in a voter's preference ranking is the same as the expected probability that $B \succ A$.

The expected probability that there is a strict PMRW under UC is then denoted by $E\{P^S_{PMRW}(3,n,UC)\}$.

The first step in developing a representation for $E\{P^S_{PMRW}(3,n,UC)\}$ is done in the same fashion that was used in Laplace's development of the representation in Eq. 1.5. The "total sum" that corresponds to $V\left(t^i_j\right)$ in Eq. 1.3 is $f(\boldsymbol{p})$ in the current situation, with

$$f(\boldsymbol{p}) = \int_{p_6=0}^{1} \int_{p_5=0}^{1-p_6} \int_{p_4=0}^{1-p_6-p_5} \int_{p_3=0}^{1-p_6-p_5-p_4} \int_{p_2=0}^{1-p_6-p_5-p_4-p_3} dp_2\,dp_3\,dp_4\,dp_5\,dp_6\,, \tag{3.80}$$

and Gehrlein (1982a) shows that $f(\boldsymbol{p}) = 1/120$.

To obtain a representation for the "total weighted sum" $f^*(\boldsymbol{p})$ that corresponds to $V^*\left(t^i_j\right)$ in Eq. 1.4, we first substitute $1-p_2-p_3-p_4-p_5-p_6$ for p_1 in the representation for $P^{A\}}_{PMRW}(3,n,\boldsymbol{p})$ in Eq. 3.60, and then find $f^*(\boldsymbol{p})$ from

$$f^*(\boldsymbol{p}) = \int_{p_6=0}^{1} \int_{p_5=0}^{1-p_6} \int_{p_4=0}^{1-p_6-p_5} \int_{p_3=0}^{1-p_6-p_5-p_4} \int_{p_2=0}^{1-p_6-p_5-p_4-p_3} P^{A\}}_{PMRW}(3,n,\boldsymbol{p})\,dp_2\,dp_3\,dp_4\,dp_5\,dp_6\,. \tag{3.81}$$

Gehrlein (1981b) sequentially reduces the integral in Eq. 3.81, and uses the symmetry of UC with respect to candidates to show that

$$E\{P^S_{PMRW}(3,n,UC)\} = 3f^*(\boldsymbol{p})/f(\boldsymbol{p}) = P^S_{PMRW}(3,n,IAC)\,. \tag{3.82}$$

In a later study, Gehrlein (1984) shows that this result can easily be generalized, so that we have

$$\begin{aligned} E\{P^{\#1}_{PMRW}(3,n,UC)\} &= P^{\#1}_{PMRW}(3,n,IAC) \\ E\{P^{W}_{PMRW}(3,n,UC)\} &= P^{W}_{PMRW}(3,n,IAC)\,. \end{aligned} \tag{3.83}$$

Thus, the same relationships that hold for IAC and IC regarding the probability that a PMRW exists as n changes for three-candidate elections are valid on an expected value basis with UC.

Tovey (1997) performs some analysis that is related to the probability of observing other election outcomes based on the notions of IAC, without using that term. Tovey's general conclusion about the idea of using IAC as a basis of analysis for such studies fits the expected value nature of the findings in Eqs. 3.81, 3.82 and 3.83 very closely [Tovey (1997), page 271]:

> "Any result about a particular distribution is open to the doubt about its significance or applicability. … (IAC) can get around this difficulty by establishing a result about 'most' distributions. … following a very natural model."

3.7 Other IC-IAC Connections

Berg (1985a) uses *Pólya-Eggenberger* (*P-E*) models [Johnson and Kotz (1977)] to evaluate the probability that a PMRW exists. These models are best described in the context of constructing random voter preference profiles by drawing colored balls from an urn, following earlier discussion. The experiment starts with balls of six different colors being placed in the urn. For each possible individual preference ranking, there are A_i balls of the particular color that corresponds to the i^{th} possible individual preference ranking. A ball is drawn at random and the corresponding individual preference ranking is assigned to the first voter. The ball is then replaced, along with α additional balls of the same color. A second ball is then drawn, the corresponding ranking is assigned to the second voter, and the ball is replaced along with α additional balls of the same color. The process is repeated n times to obtain an individual preference ranking for each of the n voters. When $\alpha > 0$, the color of the ball that is drawn for the first voter will have an increased likelihood of representing the color of the ball that is drawn for the second voter, and so on. These are *contagion models* that create an increasing degree of dependence among the voters' preferences as α increases. However, there is no dependence among voters' preferences for the particular case with $\alpha = 0$.

With P-E models, the probability, $P(\boldsymbol{n},\alpha)$, of observing a given voter preference profile, with associated voting situation $\boldsymbol{n}$, in a three-candidate election is given by

$$P(\boldsymbol{n},\alpha) = \frac{n!}{\boldsymbol{A}^{[n,\alpha]}} \prod_{i=1}^{6} \frac{A_i^{[n_i,\alpha]}}{n_i!}. \tag{3.84}$$

Here, $\boldsymbol{A} = \sum_{i=1}^{6} A_i$ and $A^{[k,\alpha]}$ is the generalized ascending factorial with

$$\boldsymbol{A}^{[k,\alpha]} = \boldsymbol{A}(\boldsymbol{A}+\alpha)(\boldsymbol{A}+2\alpha)....(\boldsymbol{A}+(k-1)\alpha). \tag{3.85}$$

By definition, $\boldsymbol{A}^{[k,\alpha]} = \boldsymbol{A}$, for $k = 0$ and $k = 1$.

We give particular attention to the P-E probability $P^1(\boldsymbol{n},\alpha)$ which has $A_i = 1$ for all i = 1,2,3,4,5,6. When we consider the special cases of $\alpha = 0$ and $\alpha = 1$, we obtain

$$P^1(\boldsymbol{n},0) = \frac{n!}{n_1!n_2!n_3!n_4!n_5!n_6!}\frac{1}{6^n} \tag{3.86}$$

$$P^1(\boldsymbol{n},1) = \frac{120}{(n+1)(n+2)(n+3)(n+4)(n+5)}.$$

Thus, we find that P-E probability model with $\alpha = 0$ is equivalent to an independent voter model with a multinomial probability for profiles, with equally likely preference rankings. That is, when α = 0 we have the equivalent of IC. The combined results of Eq. 3.29 and the representation for $P^1(\boldsymbol{n},1)$ in Eq. 3.86 lead to the conclusion that each possible voting situation is equally likely to be observed, given n, for a P-E model with α = 1. That is, when α = 1 we have the equivalent of IAC, and the direct implication follows that IAC represents a situation in which there is some dependence among voters' preferences. Berg (1985a) and Stensholt (1999b) give various other interpretations of the IAC assumption, particularly with regard to the small degree of dependence between voters' preferences that it implies. Kara (2005) makes a similar observation regarding the implied dependence of voters' preferences with IAC by showing that IAC tends to give greater probability to voting situations that are closer to unanimity than IC does.

Berg and Bjurulf (1983) do a study of the probability that a PMRW exists with IAC and they make a number of observations. An analogy is drawn with the subject of statistical mechanics in physics, which considers the behavior of collections of particles. In particular, physicists do computations in statistical mechanics, and the approach that is used to perform these computations depends upon whether or not it is possible to distinguish one particle from another. When particles are indistinguishable, the use of Bose-Einstein statistics is applicable. When particles are distinguishable, the use of Maxwell-Bolzmann statistics is applicable. In the study of probabilities the assumption of IC is equivalent to the use of Maxwell-Bolzmann statistics, and the assumption of IAC is equivalent to using Bose-Einstein statistics. As a result, the term Impartial 'Anonymous' Culture, as coined by Gehrlein and Fishburn (1976a), is very appropriate, since 'anonymous' voters are equivalent to the concept of dealing with 'indistinguishable' particles in statistical mechanics. The connection between the use of IAC and IC for computing probabilities of voting events and their link to the notions of statistical mechanics is also discussed in Meyer and Brown (1998) and Feix and Rouet (1999).

Berg and Bjurulf (1983) show results to suggest that any differences between IC and IAC should become small for m = 4, and insignificant for $m \geq 5$. This result can be explained on an intuitive basis be considering the experiment of generating voter preference profiles for m candidates with a P-E procedure which has A_i = 1 for all $1 \leq i \leq m!$. The probability that the second ball that is drawn in this case will have a color that is different than the first ball that was drawn is $1-\frac{1+\alpha}{m!+\alpha}$. In the limit that $m \to \infty$, this probability approaches one, so the dependence among voters' preferences vanishes for all small α, since no selection bias has been introduced for the $m!-1$ colors that were not drawn for the first voter's preference ranking. That is, each of these $m!-1$ colors has a probability $\left(1-\frac{1+\alpha}{m!+\alpha}\right)/(m!-1)$ for selection in the second draw. For any given large m, these P-E models will therefore generate a given voter preference profile with approximately the same probability for all small values of α. The question remains open as to how fast the rate of convergence between IC and IAC is as m increases.

Tovey (1997) develops a simple procedure that is useful to generate random voting situations under IAC as $n \to \infty$. For m candidates, there are $m!$ possible linear preference rankings on the candidates, and the procedure starts by generating $m!$ - 1 random numbers from a uniform distribution over the closed interval [0,1] and ranking them in increasing value. Let O_i denote the value of the i^{th} number in the ranking, and let q_j denote the proportion of voters with the j^{th} preference ranking in a random voting situation. Then, $q_1 = O_1$, $q_{m!} = 1 - O_{m!-1}$ and for all $0 < j < m!$, $q_j = O_j - O_{j-1}$.

Feix and Rouet (1999) use Tovey's procedure to obtain Monte-Carlo simulation estimates for $P^S_{PMRW}(m,\infty,IAC)$ for each m = 3(1)8, as shown in Table 3.6. Monte-Carlo simulation estimates of $P^S_{PMRW}(m,\infty,IC)$ are also included in Table 3.6 for each m = 3(1)8. A comparison of these probabilities strongly supports the conjecture from Berg and Bjurulf (1983) that IC and IAC probabilities converge quite quickly as m increases..

Table 3.6 Simulation estimates of $P^S_{PMRW}(m,\infty,IAC)$ and $P^S_{PMRW}(m,\infty,IC)$ from Feix and Rouet (1999)

m	$P^S(m,\infty,IAC)$	$P^S(m,\infty,IC)$
3	.9376	.9123
4	.8384	.8244
5	.7523	.7484
6	.6857	.6848
7	.6309	.6306
8	.583	.586

The simulation results in Table 3.6 compare very closely to results from a similar simulation study by Lepelley, et al. (2000) and to exact computations that are already known for the IC case for $m > 3$ [Gehrlein and Fishburn (1979a)], and for the IAC case with four candidates [Gehrlein (2001)].

Gehrlein (2004a) develops representations for the probability that a specified candidate in given a triple of candidates is the PMRW for that triple, using the rankings on the triple that are embedded within the linear preference rankings of a voter preference profile on m candidates. Let $Q^S_{PMRW}(m,n,IC)$ denote this probability with IC. Define the set of candidates as $\boldsymbol{C^m} = C_1, C_2, ..., C_m\}$, and consider the probability that the specific candidate C_1 is the PMRW for a given triple $C_1, C_j, C_k\}$. Let $\boldsymbol{Z^m}$ define the set of all possible linear preference rankings that individual voters might have on the candidates, where $\#\boldsymbol{Z^m} = m!$ and $\succ_i^m$ is the

i^{th} linear ranking in $\boldsymbol{Z^m}$. We partition $\boldsymbol{Z^m}$ into four subsets, according to individual preferences on pairs from $C_1, C_j, C_k\}$ within each ranking:

$$\begin{aligned} Z_1^m &= \succ_i^m: C_j \succ C_1 \text{ and } C_k \succ C_1\} \\ Z_2^m &= \succ_i^m: C_j \succ C_1 \text{ and } C_1 \succ C_k\} \\ Z_3^m &= \succ_i^m: C_k \succ C_1 \text{ and } C_1 \succ C_j\} \\ Z_4^m &= \succ_i^m: C_1 \succ C_j \text{ and } C_1 \succ C_k\}. \end{aligned} \tag{3.87}$$

Obviously, $\#Z_1^m = \#Z_4^m = m!/3$ and $\#Z_2^m = \#Z_3^m = m!/6$.

The set of all possible voter preference profiles, $\boldsymbol{R_n^m}$, consists of every possible assignment of the linear preference rankings in $\boldsymbol{Z^m}$ to n identifiable voters. Let s_i denote the total number of linear preference rankings in a given voter preference profile that are included in Z_i^m. We begin by determining the total number of different voter preference profiles that exist in $\boldsymbol{R_n^m}$ that are consistent with a specified combination of values of s_1, s_2, s_3, s_4, where $s_4 = n - s_1 - s_2 - s_3$. There are $\binom{n}{s_1}$ combinations of s_1 of the n identifiable voters whose preferences agree with linear rankings in Z_1^m, and each of these identifiable voters can have any of the $m!/3$ linear preference rankings in Z_1^m. Then, there are $\binom{n-s_1}{s_2}$ combinations of s_2 of the remaining $n - s_1$ identifiable voters whose preferences agree with linear rankings in Z_2^m, and each of these identifiable voters can have any of the $m!/6$ linear preference rankings in Z_2^m. The process continues in the same fashion for s_3 and s_4 to arrive at the total number of voter preference profiles with a specified combination of values of s_1, s_2, s_3, s_4 being equal to $K_{PMRW}^{C_1\}}(m,n,IC)$, with

$$K_{PMRW}^{C_1\}}(m,n,IC) = \tag{3.88}$$

$$\binom{n}{s_1}\left(\frac{m!}{3}\right)^{s_1}\binom{n-s_1}{s_2}\left(\frac{m!}{6}\right)^{s_2}\binom{n-s_1-s_2}{s_3}\left(\frac{m!}{6}\right)^{s_3}\left(\frac{m!}{3}\right)^{n-s_1-s_2-s_3}.$$

With the assumption of IC, each of these voter preference profiles has a probability equal to $\left(\frac{1}{m!}\right)^n$ of being observed, and C_1 is the PMRW for a given triple $C_1, C_j, C_k\}$ as long as the values of the s_i's are restricted according to:

$$0 \le s_1 \le \frac{n-1}{2} \tag{3.89}$$
$$0 \le s_2 \le \frac{n-1}{2} - s_1$$
$$0 \le s_3 \le \frac{n-1}{2} - s_1 .$$

All of this leads to a representation for the probability, $Q^{C_1\}}_{PMRW}(m,n,IC)$, that C_1 is the PMRW for a given triple $C_1, C_j, C_k\}$ with IC being given as

$$Q^{C_1\}}_{PMRW}(m,n,IC) = \tag{3.90}$$
$$\left(\frac{1}{m!}\right)^n \Sigma_1 \binom{n}{s_1}\left(\frac{m!}{3}\right)^{s_1}\binom{n-s_1}{s_2}\left(\frac{m!}{6}\right)^{s_2}\binom{n-s_1-s_2}{s_3}\left(\frac{m!}{6}\right)^{s_3}\left(\frac{m!}{3}\right)^{n-s_1-s_2-s_3} .$$

Here Σ_1 is a triple summation function with indexes that are consistent with Eq. 3.89. After algebraic reduction is performed on Eq. 3.90, along with the result in Eq. 3.74, we find

$$Q^{C_1\}}_{PMRW}(m,n,IC) = P^{S}_{PMRW}(3,n,IC)/3 . \tag{3.91}$$

As a result, $Q^{C_1\}}_{PMRW}(m,n,IC)$ is not a function of m with IC, but the same result is not found to be true with IAC.

The derivations to obtain a representation for $Q^{C_1\}}_{PMRW}(m,n,IAC)$ are largely based on a result from Feller (1957). Suppose that v voters are to be assigned into w different categories, with v_i voters in each category so that $\sum_{i=1}^{w} v_i = v$. The total number of such possible voting situations is given by $H(v,w)$, with

$$H(v,w) = \binom{v+w-1}{w-1} . \tag{3.92}$$

The use of this identity with $w = 6$ and $v = n$ led to the result in Eq. 3.29.

Following the logic that led to Eq. 3.88 with anonymous, or unidentifiable, voters, the total number of voting situations with specified combination of values of s_1, s_2, s_3, s_4 is given by $K^{C_1\}}_{PMRW}(m,n,IAC)$, with

$$K^{C_1\}}_{PMRW}(m,n,IAC) = \tag{3.93}$$
$$\binom{s_1+\#Z_1-1}{\#Z_1-1}\binom{s_2+\#Z_2-1}{\#Z_2-1}\binom{s_3+\#Z_3-1}{\#Z_3-1}\binom{n-s_1-s_2-s_3+\#Z_4-1}{\#Z_4-1} .$$

There are a total of $H(n,m!)$ equally likely voting situations with IAC, and after performing some algebraic manipulation, it follows that

$$Q_{PMRW}^{C_1\}}(m,n,IAC) = \tag{3.94}$$

$$\frac{n!(m!-1)!}{(n+m!-1)!}\sum_{s_1=0}^{\frac{n-1}{2}}\frac{\left(s_1+\frac{m!}{3}-1\right)!}{\left(\frac{m!}{3}-1\right)!s_1!}\sum_{s_2=0}^{\frac{n-1}{2}-s_1}\frac{\left(s_2+\frac{m!}{6}-1\right)!}{\left(\frac{m!}{6}-1\right)!s_2!}\sum_{s_3=0}^{\frac{n-1}{2}-s_1}\frac{\left(s_3+\frac{m!}{6}-1\right)!}{\left(\frac{m!}{6}-1\right)!s_3!}\frac{\left(n-s_1-s_2-s_3+\frac{m!}{3}-1\right)!}{\left(\frac{m!}{3}-1\right)!(n-s_1-s_2-s_3)!}.$$

Table 3.7 lists computed value of $Q_{PMRW}^{S}(3,n,IC)$, $Q_{PMRW}^{S}(3,n,IAC)$, $Q_{PMRW}^{S}(4,n,IAC)$ and $Q_{PMRW}^{S}(5,n,IAC)$ for each n = 3(8)51.

Table 3.7 Computed values of $Q_{PMRW}^{S}(3,n,IC)$ and $Q_{PMRW}^{S}(m,n,IAC)$ from Gehrlein (2004a) and Stensholt (1999b).

n	$Q_{PMRW}^{S}(3,n,IC)$	$Q_{PMRW}^{S}(3,n,IAC)$	$Q_{PMRW}^{S}(4,n,IAC)$	$Q_{PMRW}^{S}(5,n,IAC)$
3	.3148	.3214	.3169	.3153
11	.3067	.3141	.3084	.3070
19	.3056	.3131	.3073	.3059
27	.3051	.3128	.3069	.3054
35	.3049	.3127	.3068	.3052
43	.3047	.3126	.3067	.3050
51	.3046	.3126	.3066	.3050
∞	.3041	.3125	.3064	.3046

Table 3.7 also contains values of the limiting values for $Q_{PMRW}^{S}(3,\infty,IC)$ and $Q_{PMRW}^{S}(3,\infty,IAC)$ as $n \to \infty$ that follow directly from Eq. 3.77 and Eq. 3.32 respectively. The estimates for $Q_{PMRW}^{S}(4,\infty,IAC)$ and $Q_{PMRW}^{S}(5,\infty,IAC)$ were obtained by Monte-Carlo simulation in Stensholt (1999b). All of these results give additional credence to the claim by Berg and Bjurulf (1983) that IC and IAC probabilities converge rapidly to very similar values for m greater than four. These results, coupled with previous discussion, support the conjecture in Stensholt (1999b) that in the limit $m \to \infty$, $Q_{PMRW}^{S}(\infty,\infty,IAC) = Q_{PMRW}^{S}(3,\infty,IC)$.

3.8 The Impact of Unbalanced Preferences

The introduction to this chapter suggested that conditions that produce balanced voter preferences generally tend to maximize the probability PMR cycles exist. It was also noted that the introduction of bias to individual voter's preferences could lead to situations with a greater probability of having PMR cycles.

The result of unbalanced preferences when $n \to \infty$ can be described in terms of the $\Delta(A,B)$ terms that are defined in Eq. 3.62. The representation for $P_{PMRW}^{S}(3,\infty,DC)$ in Eq. 3.69 results if $\Delta(A,B) = \Delta(A,C) = \Delta(B,C)$. The law of large numbers requires that a randomly generated voter preference profile with $n \to \infty$ must have *AMB* if $\Delta(A,B) > 0$ for any pair of candidates like *A* and *B*. As a result, *A* will be the PMRW with probability approaching one if $\Delta(A,B) > 0$ and $\Delta(A,C) > 0$, *B* will be the PMRW with probability approaching one if $\Delta(A,B) < 0$ and $\Delta(B,C) > 0$, and *C* will be the PMRW with probability approaching one if $\Delta(A,C) < 0$ and $\Delta(B,C) < 0$. There will be a PMR cycle *AMBMCMA* with probability approaching one if $\Delta(A,B) > 0$, $\Delta(B,C) > 0$ and $\Delta(A,C) < 0$, and the reverse PMR cycle with *AMCMBMA* will exist with probability one if $\Delta(A,C) > 0$, $\Delta(B,C) < 0$ and $\Delta(A,B) < 0$. Gehrlein (1983) gives representations for $P^S_{PMRW}(3,\infty,\boldsymbol{p})$ when one or two of the $\Delta(A,B) = 0$. The observation that $P^S_{PMRW}(3,\infty,\boldsymbol{p})$ goes to a value of either zero or one with probability approaching one for three candidates when $\Delta(A,B) \neq 0$ for any pair of candidates has been pointed out by Todhunter (1931), Gleser (1969), May (1971), Weisberg and Niemi (1972) and Gehrlein (1983). Tideman (1985) proves that the probability that $0 < P^S_{PMRW}(3,\infty,\boldsymbol{p}) < 1$ is of measure zero with the assumption of UC.

Analysis has also been done to try to determine the $\boldsymbol{p}$ vectors that will minimize $P^S_{PMRW}(m,n,\boldsymbol{p})$ for small n. Gillett (1979) performs an analysis of the behavior of $P^S_{PMRW}(3,n,\boldsymbol{p})$ for some special cases of $\boldsymbol{p}$, and Gillett (1980a) uses Monte-Carlo simulation analysis to arrive at a conjecture that $P^S_{PMRW}(3,n,\boldsymbol{p})$ is minimized by the vector $\boldsymbol{p}_1^*$ with $p_1 = p_4 = p_5 = 1/3$ or by $\boldsymbol{p}_2^*$ with $p_2 = p_3 = p_6 = 1/3$. Buckley (1975) proved this conjecture to be true with n = 3, where $P^S_{PMRW}(3,3,\boldsymbol{p}_1^*) = P^S_{PMRW}(3,3,\boldsymbol{p}_2^*) = 7/9$, while $P^S_{PMRW}(3,3,IC) = 17/18$. Krishnamoorthy and Raghavachari (2005) prove the conjecture to be true for general n.

The linear individual preference rankings in Fig. 3.3 show that $\boldsymbol{p}_1^*$ and $\boldsymbol{p}_2^*$ will only include individual preference rankings in a profile if they are consistent with one of the rankings in a triple of rankings that form a Latin Square, as described in Chapter 2. So, $\boldsymbol{p}_1^*$ only admits voter preference rankings that work to reinforce the PMR cycle with *AMBMCMA* and $\boldsymbol{p}_2^*$ only admits voter preference rankings that work to reinforce the reverse PMR cycle with *AMCMBMA*.

The existence of these two cyclic components is critical to the possible existence of PMR cycles. Saposnik (1975) finds the conditions on “cyclic balance” that will require transitivity of PMR voting. Cyclic balance measures the difference between these “clockwise cycles” and “counterclockwise cycles” in the preference rankings in subsets of voters. Sen (1966,1970) uses the terms “forward circle” and “backward circle” to define them, while Riker (1980) uses the terms “forward cycle” and “backward cycle”.

Zwicker (1991) examines the possibility of the existence of PMR cycles by using linear algebra to decompose the aggregation of voters' preferences into components. The notion of "spin" is developed to measure the strength of these cyclic components in a voter preference profile. The procedure used is analogous to using homology theory to decompose current flow in electrical circuits into components. Necessary and sufficient conditions are developed to determine the "spin" values on profiles that will require PMR to be transitive.

We saw in Chapter 2 that Ward's Condition, which requires that a voter preference profile does not contain any Latin Squares, necessarily results in the existence of a PMRW, but a voter preference profile that does contain Latin Squares can still have a PMRW. The impact of these two cyclic components on the creation of a PMR cycle in a voter preference profile can cancel each other out. However, an increase in the relative strength of either one of these cyclic components in voter preference profiles increases the likelihood that the associated PMR cycle exists.

Berg (1985a) shows how this phenomenon can occur by using a P-E urn model. In particular, the study starts the experiment of generating random voter preference profiles by setting $A_1 = A_4 = A_5 = \tau$ and $A_2 = A_3 = A_6 = 1$. As τ becomes large, the three preference rankings forming one of the Latin Squares will become dominant in the preferences of the population. Berg assumes that $n \to \infty$ with $\alpha = 1$ to develop a representation for the limiting Dirichlet probability that a PMR cycle exists. Computed values show that the probability that a PMR cycle exists increases rapidly as τ increases.

Saari and Valognes (1998) develop a conditional probability representation for the likelihood that a PMRW exists when voters are restricted to have preferences on three candidates that tend to reinforce one of the Latin Squares. The study assumes that all voting situations with $n_2 = n_3 = n_6 = 0$ are equally likely to be observed. The probability that a PMRW exists in this case is shown to be given by $3(n+3)\}/\,4(n+2)\}$ for odd n. This modification significantly reduces the probability that a PMRW exists from the IAC case that does not create a bias in favor of one of the Latin Squares in voters' preferences.

3.9 Other Representations

Other representations have been developed that are related to various aspects of the likelihood that a PMRW exists in three-candidate elections. Gehrlein and Fishburn (1976a) develop IAC representations for the expected proportion of times, $R^i_{PMRW}(m,n,IAC)$, that the PMRW is ranked in the i^{th} position of individual voter's linear preference rankings in preference profiles, given that a PMRW exists. Here, a rank of one refers to a voter's most preferred candidate. Representations are obtained for the case of $m = 3$ with odd n:

$$R^1_{PMRW}(3,n,IAC)=\frac{8n^2+33n+19}{15n(n+3)} \tag{3.95}$$
$$R^2_{PMRW}(3,n,IAC)=\frac{(n-1)(4n+13)}{15n(n+3)}$$
$$R^3_{PMRW}(3,n,IAC)=\frac{n^2+n-2}{5n(n+3)}.$$

In the limit as $n \to \infty$, $R^1_{PMRW}(3,\infty,IAC)=8/15$, $R^2_{PMRW}(3,\infty,IAC)=4/15$, and $R^3_{PMRW}(3,\infty,IAC)=1/5$ to indicate that the PMRW is expected to be the most preferred candidate for a majority of voters with large electorates, given that a PMRW does exist. A candidate that is ranked as most preferred by a majority of voters must be a strict PMRW. Richelson (1978), Lepelley (1992) and others have considered various aspects of this special case of a PMRW, and have given the name *Strong PMRW* to such a candidate.

Lepelley and Gehrlein (1999) develop representations for the probability that a strong PMRW exists under the assumption of IC, IAC and MC for m = 3. Let $P^{Strong}_{PMRW}(3,n,IC)$, $P^{Strong}_{PMRW}(3,n,IAC)$ and $P^{Strong}_{PMRW}(3,L,MC)$ denote these respective probabilities. The representation for $P^{Strong}_{PMRW}(3,n,IC)$ does not reduce to a simple form, but it can be used to obtain the computed values for each n = 3(6)51, along with n = 99 and the limiting probability as $n \to \infty$, in Table 3.8.

The representations for $P^{Strong}_{PMRW}(3,n,IAC)$ from Lepelley and Gehrlein (1999) are given by:

$$P^{Strong}_{PMRW}(3,n,IAC)=\frac{3(n+7)(3n+7)}{16(n+2)(n+4)}\text{, for odd } n, \tag{3.96}$$

$$P^{Strong}_{PMRW}(3,n,IAC)=\frac{3n(n+6)(3n+9)}{16(n+1)(n+3)(n+5)}\text{, for even } n.$$

The representation for odd n in Eq. 3.96 was used to obtain the computed values of $P^{Strong}_{PMRW}(3,n,IAC)$ for each n = 3(6)51 in Table 3.8. The limiting value as $n \to \infty$ of $P^{Strong}_{PMRW}(3,\infty,IAC)=9/16$, for both the odd and even n representations in Eq. 3.96, are in agreement with limiting results that are presented in Berg and Lepelley (1992).

The representation for $P^{Strong}_{PMRW}(3,L,MC)$ is given by Lepelley and Gehrlein (1999) as

$$P^{Strong}_{PMRW}(3,L,MC)=\frac{L\left(29L^4+184L^3+421L^2+434L+192\right)}{120(L+1)^5}. \tag{3.97}$$

The representation for $P_{PMRW}^{Strong}(3, L, MC)$ in Eq. 3.97 was used to obtain the computed values for each $L = 1(2)17$, along with $L = 33$ and the limiting probability as $L \to \infty$, in Table 3.8. These values of L were selected to make the value of $E(n)$ for MC as close as possible to the corresponding n values for IC and IAC in each row of Table 3.8.

Table 3.8 Computed values of $P_{PMRW}^{Strong}(3,n,IC)$, $P_{PMRW}^{Strong}(3,n,IAC)$ and $P_{PMRW}^{Strong}(3,n,MC)$ from Lepelley and Gehrlein (1999).

n	L	$P_{PMRW}^{Strong}(3,n,IC)$	$P_{PMRW}^{Strong}(3,n,IAC)$	$P_{PMRW}^{Strong}(3,n,MC)$
3	1	.7778	.8571	.3281
9	3	.4345	.7133	.3076
15	5	.2647	.6641	.2894
21	7	.1672	.6391	.2788
27	9	.1078	.6240	.2720
33	11	.0704	.6139	.2672
39	13	.0465	.6066	.2638
45	15	.0309	.6011	.2611
51	17	.0207	.5969	.2591
99	33	.0009	.5808	.2510
∞	∞	0	.5625	.2417

The results in Table. 3.8 show that $P_{PMRW}^{Strong}(3,n,IC)$, $P_{PMRW}^{Strong}(3,n,IAC)$ and $P_{PMRW}^{Strong}(3,L,MC)$ approach very different limits as $n \to \infty$, or as $E(n) \to \infty$ for MC.

Gillett (1976, 1978) develops representations for the probability that a PMRW exists for generalized $\boldsymbol{p}$ with $m = 3,4$. Recursion relations are also developed to obtain the representations for given m and n as a function of m and n for smaller values of n. The recursion relations for the case of $m = 3$ are given with the same definition that was used for the representation of $P_{PMRW}^{A\}}(3,n,\boldsymbol{p})$ in Eq. 3.60, and where $P_{PMRW}^{A*\}}(3,n,\boldsymbol{p})$ is the probability that A is the strict PMRW for even n:

For odd n:

(3.98)

$$P_{PMRW}^{A\}}(3,n,\boldsymbol{p}) = P_{PMRW}^{A*\}}(3,n-1,\boldsymbol{p}) +$$

$$\sum_{s_1=0}^{\frac{n-3}{2}} \sum_{s_2=0}^{\frac{n-3}{2}-s_1} \frac{(n-1)!(p_5+p_6)^{s_1}(p_1+p_2)^{\frac{n-1}{2}-s_2}}{s_1!s_2!\left(\frac{n-1}{2}-s_1\right)!\left(\frac{n-1}{2}-s_2\right)!} \begin{Bmatrix} (p_1+p_2+p_3)p_3^{s_2} p_4^{\frac{n-1}{2}-s_1} + \\ (p_1+p_2+p_4)p_3^{\frac{n-1}{2}-s_1} p_4^{s_2} \end{Bmatrix}$$

$$+ \sum_{s_1=0}^{\frac{n-1}{2}} \frac{(n-1)!(p_5+p_6)^{s_1} p_3^{\frac{n-1}{2}-s_1} p_4^{\frac{n-1}{2}-s_1} (p_1+p_2)^{s_1+1}}{s_1!\left(\frac{n-1}{2}-s_1\right)!\left(\frac{n-1}{2}-s_1\right)!s_1!}$$

For even n: (3.99)

$$P_{PMRW}^{A*\}}(3,n,\boldsymbol{p}) = P_{PMRW}^{A\}}(3,n-1,\boldsymbol{p}) -$$

$$\sum_{s_1=0}^{\frac{n-3}{2}} \sum_{s_2=0}^{\frac{n-3}{2}-s_1} \frac{(n-1)!(p_5+p_6)^{s_1}(p_1+p_2)^{\frac{n+1}{2}-s_2}}{s_1!s_2!\left(\frac{n-1}{2}-s_1\right)!\left(\frac{n+1}{2}-s_2\right)!} \begin{Bmatrix} (p_4+p_5+p_6)p_3{}^{s_2}p_4{}^{\frac{n-1}{2}-s_1} + \\ (p_3+p_5+p_6)p_3{}^{\frac{n-1}{2}-s_1}p_4{}^{s_2} \end{Bmatrix}$$

$$-(p_3+p_4+p_5+p_6)\sum_{s_1=0}^{\frac{n-1}{2}} \frac{(n-1)!(p_5+p_6)^{s_1}p_3{}^{\frac{n-1}{2}-s_1}p_4{}^{\frac{n-1}{2}-s_1}(p_1+p_2)^{s_1+1}}{s_1!\left(\frac{n-1}{2}-s_1\right)!\left(\frac{n-1}{2}-s_1\right)!(s_1+1)!}.$$

Similar representations can be obtained for $P_{PMRW}^{B\}}(3,n,\boldsymbol{p})$, $P_{PMRW}^{B*\}}(3,n,\boldsymbol{p})$, $P_{PMRW}^{C\}}(3,n,\boldsymbol{p})$ and $P_{PMRW}^{C*\}}(3,n,\boldsymbol{p})$ by appropriately interchanging the subscripts on the p_i 's, and the results can be accumulated to obtain the recursion relationship for $P_{PMRW}^{S}(3,n,\boldsymbol{p})$ as a function of $P_{PMRW}^{S}(3,n-1,\boldsymbol{p})$. Gillett (1978) also develops a recursion relationship for the case of $m = 4$, but the results are very cumbersome and they are not reported here. The recursion relationships in Eqs. 3.98 and 3.99 will be used in later discussion.

3.10 Conclusion

We have observed very similar behavior for the probability that a PMRW exists under the different methods of considering balanced preferences in three-candidate elections: IAC, MC, IC, DC and UC. The situation developed by Condorcet, which has a balance in social outcomes, suggests that there should be widespread occurrences of PMR cycles, if that assumption is valid. However, this assumption ignores a certain amount of coherence among voter preferences. We have typically found the greatest likelihood for PMR cycles to exist with a small number of voters. For very large electorates we expect to have a PMRW with probability approaching 15/16 = .9375 with IAC and UC, and approaching 109/120 = .9083 for MC. The results of Guilbaud (1952) show that a PMRW exists with probability approaching .9123 for large electorates with IC.

Some studies suggest that these different assumptions give extraordinarily small estimates of the probability that a PMRW exists [Stensholt (1999b), for example]. This is not a surprising observation, since none of the studies referenced above have ever suggested that IAC, IC, DC or UC reflect reality in any particular situation. As was suggested in the introduction, they have considered instead the likelihood that a PMRW exists under various interpretations of balanced preferences. If indeed balanced preferences are most likely to produce a PMR cycle, then each of these cases represent situations in which the probability that a PMRW exists would tend to be at a minimum. These situations were somewhat contrived to make PMR cycles as likely as is possible, without building in a direct bias to force the existence of PMR cycles, and we still expect to have a PMRW in over 90 percent of cases for large electorates with three candidates. That probability would certainly be significantly greater for typical situations that are not contrived to make majority rule cycles more likely to occur.

Gehrlein and Lepelley (2004) give justification for using assumptions like IC, MC and IAC to develop probability representations, despite the fact that they are generally believed to represent situations that exaggerate the probability that paradoxical events will occur:

- They are very useful when large amounts of empirical data are not available, which is typically the case with elections.
- They can show that some paradoxical events are very unlikely to be observed. If we use conditions to maximize the likelihood of observing paradoxes and find that the probability is small with such calculations, the paradox is assuredly very unlikely to be observed in reality.
- They can show the relative impact that paradoxical events can have on different types of voting situations. For example, different voting rules can be compared on the basis of their relative likelihood of electing the PMRW.
- By using probability models to obtain closed form representations, it is easy to observe the impact of varying different parameters of voting situations or voter preference profiles, which is somewhat more difficult to do with simulation studies.
- The representations that are obtained are directly reproducible and verifiable with mathematical analysis, which is not as simple to do with simulation analysis.
- It can be useful to find out if the relative probabilities of paradoxical outcomes on various voting mechanisms behave in a consistent fashion over a number of different assumptions about the likelihood that voting situations or voter preference profiles are observed.

With regard to the third item in this list, Fishburn and Gehrlein (1982) note that comparisons of differences in the likelihoods that various election outcomes might be observed could be exaggerated with balanced preference models, but there is little reason to expect that the relative likelihoods of the election outcomes would be changed with more general assumptions.

4 The Case of More than Three Candidates

4.1 Introduction

The last chapter provided an exhaustive coverage of work that has been done to obtain representations for the probability that Condorcet's Paradox is observed in one of its forms in three-candidate elections. Most of the work that has been done in the area of obtaining these probability representations has dealt with the case of three-candidate elections, and it will soon be very clear that this type of analysis becomes much more complicated when it is extended to the case of more than three candidates. However, extending this analysis to more than three candidates is important, since it is almost universally believed that the probability that a PMRW exists will decrease rapidly as the number of candidates is increased for a fixed n. For example, see the conjecture due to Black [Black (1958), page 51)]. We begin by considering the case of four-candidate elections.

4.2 Representations for Four-Candidate Elections

4.2.1 A PMRW Exists with Four Candidates

All of the assumptions that are related to the formation of random voter preference profiles that were used in the previous chapter are continued in this extension to the case of four-candidate elections. Individual voters are assumed to have linear preference rankings on the four candidates $\boldsymbol{C^4} = \{C_1, C_2, C_3, C_4\}$, and voters are assumed to have individual preferences that are independent of the preferences of other voters. The first complicating factor is that there are 24 possible linear preference rankings in a four-candidate election, as listed in Fig. 4.1. A random voter preference profile is obtained by sequentially drawing voters from a population and observing their preferences. The sampling from the population is done with replacement, and the probability that a voter with a specified preference ranking is selected on any draw is denoted by the 24-dimension vector $\boldsymbol{r}$, with components as specified in Fig. 4.1.

$$\begin{array}{lll}
C_1 \succ C_2 \succ C_3 \succ C_4 : r_1 & C_2 \succ C_3 \succ C_1 \succ C_4 : r_9 & C_3 \succ C_4 \succ C_1 \succ C_2 : r_{17} \\
C_1 \succ C_2 \succ C_4 \succ C_3 : r_2 & C_2 \succ C_3 \succ C_4 \succ C_1 : r_{10} & C_3 \succ C_4 \succ C_2 \succ C_1 : r_{18} \\
C_1 \succ C_3 \succ C_2 \succ C_4 : r_3 & C_2 \succ C_4 \succ C_1 \succ C_3 : r_{11} & C_4 \succ C_1 \succ C_2 \succ C_3 : r_{19} \\
C_1 \succ C_3 \succ C_4 \succ C_2 : r_4 & C_2 \succ C_4 \succ C_3 \succ C_1 : r_{12} & C_4 \succ C_1 \succ C_3 \succ C_2 : r_{20} \\
C_1 \succ C_4 \succ C_2 \succ C_3 : r_5 & C_3 \succ C_1 \succ C_2 \succ C_4 : r_{13} & C_4 \succ C_2 \succ C_1 \succ C_3 : r_{21} \\
C_1 \succ C_4 \succ C_3 \succ C_2 : r_6 & C_3 \succ C_1 \succ C_4 \succ C_2 : r_{14} & C_4 \succ C_2 \succ C_3 \succ C_1 : r_{22} \\
C_2 \succ C_1 \succ C_3 \succ C_4 : r_7 & C_3 \succ C_2 \succ C_1 \succ C_4 : r_{15} & C_4 \succ C_3 \succ C_1 \succ C_2 : r_{23} \\
C_2 \succ C_1 \succ C_4 \succ C_3 : r_8 & C_3 \succ C_2 \succ C_4 \succ C_1 : r_{16} & C_4 \succ C_3 \succ C_2 \succ C_1 : r_{24}
\end{array}$$

Fig. 4.1 Possible linear preference rankings for individual voters with four candidates

We begin the development of a representation for $P_{PMRW}^{S}(4,n,\boldsymbol{r})$ by obtaining a representation for the probability $P_{PMRW}^{\{C_1\}}(4,n,\boldsymbol{r})$ that C_1 will be the PMRW in a randomly generated voter preference profile, as described above. The first step is to consider the conditions that will have C_1 as the PMRW for the triple of candidates $\{C_1, C_2, C_3\}$. Using the same notation that led to Eq. 3.87, $\boldsymbol{Z^4}$ denotes the set of all possible linear preference rankings with four candidates, and let $\succ_i^4$ for $i = 1, 2, 3, ..., 24$ denote the possible linear preference rankings in $\boldsymbol{Z^4}$. We then partition $\boldsymbol{Z^4}$ into four subsets as follows:

$$\begin{aligned}
Z_1^4 &= \left\{\succ_i^4 : C_2 \succ C_1 \text{ and } C_3 \succ C_1\right\} \\
Z_2^4 &= \left\{\succ_i^4 : C_2 \succ C_1 \succ C_3\right\} \\
Z_3^4 &= \left\{\succ_i^4 : C_3 \succ C_1 \succ C_2\right\} \\
Z_4^4 &= \left\{\succ_i^4 : C_1 \succ C_2 \text{ and } C_1 \succ C_3\right\}.
\end{aligned} \tag{4.1}$$

Let s_i denote the total number of rankings in a given voter preference profile that are contained in Z_i^4, and C_1 will be the PMRW for the triple $\{C_1, C_2, C_3\}$ for odd n for any combination of s_1, s_2, s_3, s_4 with:

$$\begin{aligned}
&0 \le s_1 \le \frac{n-1}{2} \\
&0 \le s_2 \le \frac{n-1}{2} - s_1 \\
&0 \le s_3 \le \frac{n-1}{2} - s_1 \\
&s_4 = n - s_1 - s_2 - s_3 .
\end{aligned} \tag{4.2}$$

We now consider the addition of the restriction that $C_1 \boldsymbol{M} C_4$ to the conditions in Eq. 4.2. Define two additional subsets of $\boldsymbol{Z^4}$ as

$$Z_5^4 = \left\{\succ_i^4 : C_4 \succ C_1\right\}$$
$$Z_6^4 = \left\{\succ_i^4 : C_1 \succ C_4\right\}, \tag{4.3}$$

and let $s_{i,j} = \#\left\{Z_i^4 \cap Z_j^4\right\}$. The inequalities in Eq. 4.2 require that C_1 is the PMRW for the triple of candidates $\{C_1, C_2, C_3\}$, and the addition of four more inequalities in Eq. 4.4 will require that $C_1 \boldsymbol{M} C_4$ also:

$$\begin{aligned}
&0 \le s_{1,5} \le s_1 \\
&0 \le s_{2,5} \le s_2 \\
&0 \le s_{3,5} \le Min\left\{\begin{matrix} s_3 \\ \frac{n-1}{2} - s_{1,5} - s_{2,5} \end{matrix}\right\} \\
&0 \le s_{4,5} \le Min\left\{\begin{matrix} s_4 \\ \frac{n-1}{2} - s_{1,5} - s_{2,5} - s_{3,5} \end{matrix}\right\}.
\end{aligned} \tag{4.4}$$

The probability $p_{i,j}$ denotes the probability that a randomly selected voter will have a preference ranking from Fig. 4.1 that is included in $\left\{Z_i^4 \cap Z_j^4\right\}$. It follows from the definitions in Eqs. 4.1 and 4.3, along with Fig. 4.1, that

$$\begin{aligned}
&p_{1,5} = r_{10} + r_{12} + r_{16} + r_{18} + r_{22} + r_{24} \\
&p_{1,6} = r_9 + r_{15} \qquad p_{2,5} = r_{11} + r_{21} \\
&p_{2,6} = r_7 + r_8 \qquad p_{3,5} = r_{17} + r_{23} \\
&p_{3,6} = r_{13} + r_{14} \qquad p_{4,5} = r_{19} + r_{20} \\
&p_{4,6} = r_1 + r_2 + r_3 + r_4 + r_5 + r_6.
\end{aligned} \tag{4.5}$$

Using the relationship $s_{i,6} = s_i - s_{i,5}$ with previous discussion, a representation for $P_{PMRW}^{\{C_1\}}(4, n, \boldsymbol{r})$ is given by

$$P_{PMRW}^{\{C_1\}}(4, n, \boldsymbol{r}) = \Sigma_1 \Sigma_2 n! \prod_{\substack{i=1,2,3,4 \\ j=5,6}} \frac{p_{i,j}^{\ s_{i,j}}}{s_{i,j}!}. \tag{4.6}$$

Here, Σ_1 is a triple summation function with summation indexes that are consistent with Eq. 4.2 and Σ_2 is a four summation function with summation indexes that are consistent with Eq. 4.4. Representations for $P_{PMRW}^{\{C_i\}}(4, n, \boldsymbol{r})$ can be obtained in a similar fashion for each i = 2, 3, 4 and $P_{PMRW}^{S}(4, n, \boldsymbol{r})$ is obtained as the accumulation

$$P_{PMRW}^{S}(4,n,\boldsymbol{r})=\sum_{i=1}^{4}P_{PMRW}^{\{C_i\}}(4,n,\boldsymbol{r}). \tag{4.7}$$

Gehrlein (1982a) develops a closed-form representation for the limiting probability, $P_{PMRW}^{S}(4,\infty,DC)$, as $n\to\infty$ for any $\boldsymbol{r}$ meeting the condition of DC, by extending the analysis that led to the development of the representation for $P_{PMRW}^{S}(3,\infty,DC)$ in Eq. 3.69 to the four-candidate case, with

$$P_{PMRW}^{S}(4,\infty,DC)=\frac{1}{2}+\frac{\pi}{4}\sum_{i=1}^{12}Sin^{-1}(f_i), \tag{4.8}$$

where:

$$\begin{aligned}
f_1 &= 2(r_1+r_2+r_3+r_4+r_5+r_6-r_7-r_8+r_9-r_{11}-r_{13}+r_{15})\\
f_2 &= 2(r_1+r_2+r_3+r_4+r_5+r_6-r_7-r_8-r_9+r_{11}+r_{13}-r_{15})\\
f_3 &= 2(r_1+r_2+r_3+r_4+r_5+r_6+r_7+r_8-r_9-r_{11}-r_{13}-r_{15})\\
f_4 &= 2(r_4+r_6+r_7+r_8+r_9+r_{11}-r_1-r_2+r_3-r_5+r_{13}-r_{15})\\
f_5 &= 2(r_4+r_6+r_7+r_8+r_9+r_{11}-r_1-r_2-r_3+r_5-r_{13}+r_{15})\\
f_6 &= 2(r_4+r_6+r_7+r_8+r_9+r_{11}+r_1+r_2-r_3-r_5-r_{13}-r_{15})\\
f_7 &= 2(r_2+r_5+r_8+r_{11}+r_{13}+r_{15}+r_1-r_3-r_4-r_6+r_7-r_9)\\
f_8 &= 2(r_2+r_5+r_8+r_{11}+r_{13}+r_{15}-r_1-r_3-r_4+r_6-r_7+r_9)\\
f_9 &= 2(r_2+r_5+r_8+r_{11}+r_{13}+r_{15}-r_1+r_3+r_4-r_6-r_7-r_9)\\
f_{10} &= 2(r_1+r_3+r_7+r_9+r_{13}+r_{15}+r_2-r_4-r_5-r_6+r_8-r_{11})\\
f_{11} &= 2(r_1+r_3+r_7+r_9+r_{13}+r_{15}-r_2+r_4-r_5-r_6-r_8+r_{11})\\
f_{12} &= 2(r_1+r_3+r_7+r_9+r_{13}+r_{15}-r_2-r_4+r_5+r_6-r_8-r_{11}).
\end{aligned} \tag{4.9}$$

A representation for $P_{PMRW}^{S}(4,n,IC)$ could be obtained for the special case of IC directly from Eqs. 4.6 and 4.7, but a result due to May (1971) makes this an unnecessary exercise. May gave a very nice result without providing a proof, stating that the proof of the result was very complicated. Fishburn (1973c) provides a simple proof of *May's Theorem*. The basis of that proof is explained in detail here to facilitate future discussion.

Theorem 4.1. $P_{PMRW}^{S}(4,n,IC) = 2P_{PMRW}^{S}(3,n,IC)-1$, for all odd n.

Proof. Let E_i denote the event that $C_i\boldsymbol{M}C_1$ for i = 2, 3, 4, and let $P(E_i)$ denote the probability that E_i occurs with IC. The probability, P, that C_1 is not the PMRW is, by definition, the same as $P(E_2\cup E_3\cup E_4)$. As described in Hogg and Craig (1965),

$$\begin{aligned}
P = P(E_2\cup E_3\cup E_4) = P(E_2)+P(E_3)+P(E_4)-\\
P(E_2\cap E_3)-P(E_2\cap E_4)-P(E_3\cap E_4)+P(E_2\cap E_3\cap E_4).
\end{aligned} \tag{4.10}$$

It follows by definition that $P = 1 - P_{PMRW}^{\{C_1\}}(4, n, IC)$ and that $P(E_2 \cap E_3 \cap E_4)$ is equivalent to the probability that C_1 is the PMRL for odd n. Since every voter preference profile has the same probability of being drawn as its dual preference profile with IC, it therefore follows that $P(E_2 \cap E_3 \cap E_4) = P_{PMRW}^{\{C_1\}}(4, n, IC)$ and that $P(E_i) = 1/2$. Eq. 4.10 can then be algebraically reduced and rewritten as

$$P_{PMRW}^{\{C_1\}}(4, n, IC) = \frac{1}{2}\{P(E_2 \cap E_3) + P(E_2 \cap E_4) + P(E_3 \cap E_4)\} - \frac{1}{4}. \tag{4.11}$$

By definition, $P(E_i \cap E_j)$ is identical to the probability that C_1 is the PMRL on the triple of candidates in $\{C_1, C_i, C_j\}$ within a voter preference profile on four candidates with the IC assumption. Following previous discussion about the probability that any profile is observed being equal to the probability that its dual profile is observed with the IC assumption, $P(E_i \cap E_j)$ is identical to the probability, $Q_{PMRW}^{\{C_1\}}(4, n, IC)$, that C_1 is the PMRW on the triple in a voter preference profile on four candidates. Based on Eq. 3.91, we have $Q_{PMRW}^{\{C_1\}}(m, n, IC) = P_{PMRW}^{S}(3, n, IC)/3$, so that $Q_{PMRW}^{\{C_1\}}(m, n, IC)$ is not a function of m. By substituting this result into Eq. 4.11 and using the fact that the symmetry of IC requires the equality of $P_{PMRW}^{\{C_i\}}(4, n, IC)$ for all four candidates, we obtain the representation in May's Theorem. **QED**

Computed values of $P_{PMRW}^{S}(4, n, IC)$ are listed in Table 4.1 for each n = 3(2)19. The limiting value as $n \to \infty$ for $P_{PMRW}^{S}(4, \infty, IC)$ in Table 4.1 is obtained by applying May's Theorem to Guilbaud's representation for $P_{PMRW}^{S}(3, \infty, IC)$ in Eq. 3.77.

Table 4.1 Computed values of $P_{PMRW}^{S}(4, n, IC)$ and $P_{PMRW}^{S}(4, n, IAC)$

n	$P_{PMRW}^{S}(4, n, IC)$	$P_{PMRW}^{S}(4, n, IAC)$
3	.8889	.9015
5	.8611	.8730
7	.8500	.8609
9	.8440	.8545
11	.8404	.8506
13	.8379	.8481
15	.8360	.8463
17	.8347	.8450
19	.8336	.8440
∞	.8245	.8384

Gehrlein and Fishburn (1976a) incorrectly use May's Theorem to obtain a representation for $P_{PMRW}^{S}(4,n,IAC)$, and full responsibility for that error lies with the current author. The proof of May's Theorem above used the result that $Q_{PMRW}^{\{C_1\}}(m,n,IC) = P_{PMRW}^{S}(3,n,IC)/3$ from Eq. 3.91, and Gehrlein and Fishburn (1976a) applied this same result to IAC. Given the results that were observed from Eq. 3.94 and Table 3.7 with the assumption of IAC, May's Theorem clearly can not be applied to the case of IAC, due to the small amount of dependence between individual voter's preferences that is implied by IAC. Berg and Bjurulf (1983) were the first to point out this error.

Gehrlein (1990c) develops a representation for $P_{PMRW}^{S}(4,n,IAC)$ by transforming the representation for $P_{PMRW}^{S}(4,n,IC)$ in Eq. 4.6 to the case with IAC. This transformation process mimics the logic that transformed the representation for $Q_{PMRW}^{\{C_1\}}(m,n,IC)$ in Eq. 3.90 to the representation for $Q_{PMRW}^{\{C_1\}}(m,n,IAC)$ in Eq. 3.94, leading to

$$P_{PMRW}^{S}(4,n,IAC) = \tag{4.12}$$

$$\frac{n!23!}{3600(n+23)!}\Sigma_1\Sigma_2\left[\left\{\prod_{\substack{i=1,2,3,4\\ j=5,6}}\left(s_{i,j}+1\right)\right\}\left\{\prod_{j=2,3,4,5}\left(s_{1,5}+j\right)\left(s_{4,6}+j\right)\right\}\right].$$

Computed values of $P_{PMRW}^{S}(4,n,IAC)$ from Eq. 4.12 are listed in Table 4.1 for each n = 3(2)19. The computed values of $P_{PMRW}^{S}(4,n,IAC)$ in Table 4.1 precisely match results that were obtained by computer enumeration techniques in Giraud, et al. (1988).

A variation of May's Theorem is used in Gehrlein (2001) to obtain a simpler closed form representation for $P_{PMRW}^{S}(4,n,IAC)$. This is accomplished by restating Eq. 4.10, which uses probabilities that events occur in voter preference profiles with IC, to account instead for the number of voting situations in which the same events occur with IAC. Let $N(E_i)$ denote the number of voting situations in which E_i occurs with the assumption of IAC, with

$$N(E_2 \cup E_3 \cup E_4) = N(E_2) + N(E_3) + N(E_4) - N(E_2 \cap E_3) - N(E_2 \cap E_4) \\ - N(E_3 \cap E_4) + N(E_2 \cap E_3 \cap E_4). \tag{4.13}$$

The total number of possible voting situations for four-candidates is obtained from $H(v,w)$ in Eq. 3.92 with $v = n$ and w = 4! =24. Let $N_{PMRW}^{\{C_1\}}(4,n,IAC)$ denote the total number of voting situations in which C_1 is the PMRW, and our definitions lead to the identity $N(E_2 \cup E_3 \cup E_4) = H(n,24) - N_{PMRW}^{\{C_1\}}(4,n,IAC)$. $N(E_2 \cap E_3 \cap E_4)$ is equal to the total number of voting situations in which C_1 is the PMRL. There is a 1-1 mapping between each voting situation and its equally likely dual voting situation with IAC, so it follows that $N(E_2 \cap E_3 \cap E_4) = N_{PMRW}^{\{C_1\}}(4,n,IAC)$. It also follows that $N(E_i) = H(n,24)/2$ for each i = 2, 3, 4 and that $N(E_i \cap E_j) = Q_{PMRW}^{\{C_1\}}(4,n,IAC)$ from Eq. 3.94. As a result of all of this, Eq. 4.13 can be algebraically reduced to

$$N_{PMRW}^{\{C_1\}}(4,n,IAC) = \frac{1}{2}\left[3Q_{PMRW}^{\{C_1\}}(4,n,IAC) - \frac{1}{2}H(n,24)\right]. \tag{4.14}$$

The summation indexes in the definition of $Q_{PMRW}^{\{C_1\}}(4,n,IAC)$ in Eq. 3.94 do not involve *Max* or *Min* arguments, so a closed form algebraic representation can be obtained for $N_{PMRW}^{\{C_1\}}(4,n,IAC)$. Due to the symmetry of IAC with respect to candidates, we obtain a representation for $P_{PMRW}^{S}(4,n,IAC)$ from

$$P_{PMRW}^{S}(4,n,IAC) = \frac{4N_{PMRW}^{\{C_1\}}(4,n,IAC)}{H(n,24)}. \tag{4.15}$$

After algebraic reduction,

$$P_{PMRW}^{S}(4,n,IAC) = \tag{4.16}$$

$$\frac{\left[\begin{array}{c}\{\{\{\{\{27472(n+12)^2 - 5895835\}(n+12)^2 + 440508156\}(n+12)^2 - 12323060530\}(n+12)^2 \\ + 355165514572\}(n+12)^2 - 15042878194635\}\cdot(n+12)\end{array}\right]}{32768\prod_{i=1}^{11}(n+2i)}.$$

The precise limiting probability value $P_{PMRW}^{4}(4,\infty,IAC) = \frac{1717}{2048} = .8384$ that is obtained from Eq. 4.16 is only slightly different than limiting approximation result for that probability that is reported in Berg and Bjurulf (1983).

4.2.2 PMR is Transitive with Four Candidates

Representations for the probability that PMR is transitive become very complex. Gehrlein (1988, 1989) obtains some representations for these probabilities for small *m* and *n* by using by using symmetry arguments with IC. The development of these representations follows the general notions that were used to partition $\boldsymbol{Z^4}$ into the four subgroups in Eq. 4.1 that led to the restrictions on the cardinalities of these subgroups that require C_1 to be the PMRW in Eq. 4.2. For four candidates,

$\boldsymbol{Z^4}$ is partitioned into 18 subgroups, and q_i denotes the probability that a randomly selected voter will have a preference ranking in the i^{th} subgroup, for $1 \le i \le 18$. The identity of the rankings, as defined Fig. 4.1, that are included in each subgroup can be determined from the definitions of the q_i probabilities:

$$\begin{aligned} &q_1 = r_{17} + r_{18} + r_{23} + r_{24} \quad q_2 = r_{22} \quad q_3 = r_{12} \quad q_4 = r_{16} \\ &q_5 = r_{10} \quad q_6 = r_{14} \quad q_7 = r_9 \quad q_8 = r_{15} \quad q_9 = r_{13} \\ &q_{10} = r_{20} \quad q_{11} = r_{21} \quad q_{12} = r_{11} \quad q_{13} = r_{19} \quad q_{14} = r_6 \\ &q_{15} = r_4 \quad q_{16} = r_3 \quad q_{17} = r_5 \quad q_{18} = r_1 + r_2 + r_7 + r_8 . \end{aligned} \tag{4.17}$$

For any given voter preference profile, let k_i denote the cardinality of the i^{th} subgroup from the definitions in Eq. 4.17, and the outcomes $C_1\boldsymbol{MC}_3$, $C_1\boldsymbol{MC}_4$, $C_2\boldsymbol{MC}_3$ and $C_2\boldsymbol{MC}_4$ occur when:

$$\begin{aligned} &C_1\boldsymbol{MC}_3 : \ k_1 + k_2 + k_3 + k_4 + k_5 + k_6 + k_7 + k_8 + k_9 \le (n-1)/2 \\ &C_1\boldsymbol{MC}_4 : \ k_1 + k_2 + k_3 + k_4 + k_5 + k_{10} + k_{11} + k_{12} + k_{13} \le (n-1)/2 \\ &C_2\boldsymbol{MC}_3 : \ k_1 + k_4 + k_6 + k_8 + k_9 + k_{10} + k_{14} + k_{15} + k_{16} \le (n-1)/2 \\ &C_2\boldsymbol{MC}_4 : \ k_1 + k_2 + k_6 + k_{10} + k_{11} + k_{13} + k_{14} + k_{15} + k_{17} \le (n-1)/2 . \end{aligned} \tag{4.18}$$

The restrictions on the individual k_i's that lead to the simultaneous occurrence of the outcomes $C_1\boldsymbol{MC}_3$, $C_1\boldsymbol{MC}_4$, $C_2\boldsymbol{MC}_3$ and $C_2\boldsymbol{MC}_4$ are given by:

$$\begin{aligned} &0 \le k_1 \le (n-1)/2 \\ &0 \le k_2 \le z(1) \\ &0 \le k_3 \le z(2) \\ &0 \le k_4 \le z(3) \\ &0 \le k_5 \le z(4) \\ &0 \le k_6 \le z(5) \\ &0 \le k_7 \le z(6) \\ &0 \le k_8 \le z(7) \\ &0 \le k_9 \le z(8) \\ &0 \le k_{10} \le Min\{k(9), z(5)\} \\ &0 \le k_{11} \le Min\{k(10), z(5) - k_{10}\} \\ &0 \le k_{12} \le Min\{k(11), z(5) - k_{10} - k_{11}\} \\ &0 \le k_{13} \le Min\{k(12), z(5) - k_{10} - k_{11} - k_{12}\} \\ &0 \le k_{14} \le Min\{k(13), (n-1)/2 - k_1 - k_4 - k_6 - k_8 - k_9 - k_{10}\} \\ &0 \le k_{15} \le Min\{k(14), (n-1)/2 - k_1 - k_4 - k_6 - k_8 - k_9 - k_{10} - k_{14}\} \\ &0 \le k_{16} \le Min\{k(15), (n-1)/2 - k_1 - k_4 - k_6 - k_8 - k_9 - k_{10} - k_{14} - k_{15}\} \\ &0 \le k_{17} \le Min\{k(16), (n-1)/2 - k_1 - k_4 - k_6 - k_8 - k_9 - k_{10} - k_{14} - k_{15} - k_{16}\} . \end{aligned} \tag{4.19}$$

Here, we define

$$k(j) = n - \sum_{i=1}^{j} k_i \quad \text{and} \quad z(j) = \frac{n-1}{2} - \sum_{i=1}^{j} k_i \,. \tag{4.20}$$

The probability, $Q(4,n,IC)$, that all four outcomes occur simultaneously when n is odd under IC, is equivalent to the probability that we have a transitive overall PMR outcome that is given by one of the four linear rankings: $C_1\boldsymbol{M}C_2\boldsymbol{M}C_3\boldsymbol{M}C_4$, $C_1\boldsymbol{M}C_2\boldsymbol{M}C_4\boldsymbol{M}C_3$, $C_2\boldsymbol{M}C_1\boldsymbol{M}C_3\boldsymbol{M}C_4$ or $C_2\boldsymbol{M}C_1\boldsymbol{M}C_4\boldsymbol{M}C_3$. It then follows that

$$Q(4,n,IC) = \sum\nolimits_3 \left[\frac{n!}{6^{k_1+k_{18}} 24^{n-k_1-k_{18}}} \prod_{i=1}^{18} \frac{1}{k_i!} \right], \tag{4.21}$$

where $\sum_3$ is a 17-summation function with summation indexes that are consistent with Eq. 4.19 and where $k_{18} = k(18)$. Since there are 24 possible transitive PMR rankings with four candidates and $Q(4,n,IC)$ accounts for four of them, the symmetry of IC with respect to candidates allows us to obtain a representation for the probability, $P^S_{PMRT}(4,n,IC)$, that PMR is strictly transitive in this situation by using the relationship $P^S_{PMRT}(4,n,IC) = 6Q(4,n,IC)$, so that

$$P^S_{PMRT}(4,n,IC) = 6\sum\nolimits_3 \left[\frac{n!}{6^{k_1+k_{18}} 24^{n-k_1-k_{18}}} \prod_{i=1}^{18} \frac{1}{k_i!} \right]. \tag{4.22}$$

Gehrlein (1989) contains several typographical errors in the development of this representation. The representation for $P^S_{PMRT}(4,n,IC)$ in Eq. 4.22 was used to compute the associated probabilities in Table 4.2 for each n = 3(2)19.

Table 4.2 Computed values of $P^S_{PMRT}(4,n,IC)$ and $P^S_{PMRT}(4,n,IAC)$

n	$P^S_{PMRT}(4,n,IC)$	$P^S_{PMRT}(4,n,IAC)$
3	.8299	.8492
5	.7898	.8081
7	.7741	.7909
9	.7660	.7820
11	.7609	.7767
13	.7575	.7732
15	.7550	.7707
17	.7531	.7689
19	.7517	.7675
∞	.7395	?

Gehrlein (1990c) uses the logic that led to Eq. 4.12 to develop a representation for $P_{PMRT}^{S}(4,n,IAC)$ by transforming the representation for $P_{PMRT}^{S}(4,n,IC)$ in Eq. 4.22 to the case with IAC, leading to

$$P_{PMRT}^{S}(4,n,IAC) = \frac{n!23!}{6(n+23)!}\Sigma_3\left[\prod_{i=1}^{3}\{(k_1+i)(k_{18}+i)\}\right]. \tag{4.23}$$

The representation for $P_{PMRT}^{S}(4,n,IAC)$ in Eq. 4.23 is used to compute the associated probabilities in Table 4.2 for each n = 3(2)19.

4.2.3 Probabilities for Four Candidates with Large Electorates

A representation for the limiting probability $P_{PMRT}^{S}(4,\infty,IC)$ as $n \to \infty$ is obtained by Gehrlein and Fishburn (1978a). The derivation of this representation assumes that a random voter preference profile is obtained by sequentially selecting each individual voter's preference ranking from the list of possible linear rankings on four candidates in Fig. 4.1. With the assumption of IC, each of the possible linear rankings is equally likely to be selected for each voter.

We start by defining four discrete variables that are based on the linear preference ranking that is randomly selected for the i^{th} voter :

$$X_1^i = \begin{matrix} +1: \text{if } C_1 \succ C_3 \text{ for the } i^{th} \text{voter} \\ -1: \text{if } C_3 \succ C_1 \text{ for the } i^{th} \text{voter} \end{matrix} \qquad X_2^i = \begin{matrix} +1: \text{if } C_1 \succ C_4 \text{ for the } i^{th} \text{voter} \\ -1: \text{if } C_4 \succ C_1 \text{ for the } i^{th} \text{voter} \end{matrix} \tag{4.24}$$

$$X_3^i = \begin{matrix} +1: \text{if } C_2 \succ C_3 \text{ for the } i^{th} \text{voter} \\ -1: \text{if } C_3 \succ C_2 \text{ for the } i^{th} \text{voter} \end{matrix} \qquad X_4^i = \begin{matrix} +1: \text{if } C_2 \succ C_4 \text{ for the } i^{th} \text{voter} \\ -1: \text{if } C_4 \succ C_2 \text{ for the } i^{th} \text{voter} \end{matrix}$$

Following the discussion that led to the representation for $P_{PMRW}^{S}(3,\infty,DC)$ in Eq. 3.69, the joint outcome $C_1\boldsymbol{M}C_3$, $C_1\boldsymbol{M}C_4$, $C_2\boldsymbol{M}C_3$ and $C_2\boldsymbol{M}C_4$ occurs in a voter preference profile when $\overline{X}_1\sqrt{n} > 0$, $\overline{X}_2\sqrt{n} > 0$, $\overline{X}_3\sqrt{n} > 0$ and $\overline{X}_4\sqrt{n} > 0$ respectively. The symmetry of the assumption of IC with respect to candidates leads to the observation that $E(\overline{X}_j\sqrt{n}) = E(X_j^i) = 0$ for j = 1, 2, 3, 4. The Central Limit Theorem requires that the joint probability that $C_1\boldsymbol{M}C_3$, $C_1\boldsymbol{M}C_4$, $C_2\boldsymbol{M}C_3$ and $C_2\boldsymbol{M}C_4$ as $n \to \infty$ is equivalent to the four-variate normal positive orthant probability, $\Phi_4(\boldsymbol{R^1})$, that $\overline{X}_j\sqrt{n} \ge E(\overline{X}_j\sqrt{n})$ for all j = 1,2,3,4. With the assumption of IC, the correlation matrix, $\boldsymbol{R^1}$, for this joint limiting distribution follows from the variable definitions that are given in Eq. 4.24 and same logic that led to the development of Eq. 3.68, with

$$R^1 = \begin{bmatrix} 1 & \frac{1}{3} & \frac{1}{3} & 0 \\ - & 1 & 0 & \frac{1}{3} \\ - & - & 1 & \frac{1}{3} \\ - & - & - & 1 \end{bmatrix}. \tag{4.25}$$

Following previous discussion, we know that $P^S_{PMRT}(4,\infty,IC) = 6\Phi_4(\boldsymbol{R^1})$. Simple representations for such orthant probabilities were found previously for the cases of two and three variables by applying Sheppard's Theorem of Median Dichotomy. However, the situation becomes much more complicated in the case of four or more variables. The particular form of $\boldsymbol{R^1}$ in Eq. 4.25 fits a special case of a class correlation matrices for which a representation for four-variate normal positive orthant probabilities is known [David and Mallows (1961)] and

$$P^S_{PMRT}(4,\infty,IC) = \frac{3}{8} + \frac{6}{\pi^2}\int_0^{1/3} \frac{Cos^{-1}\left(-\gamma/\left(1-2\gamma^2\right)\right)}{\sqrt{1-\gamma^2}} d\gamma . \tag{4.26}$$

Gehrlein and Fishburn (1978) obtain a value of $P^S_{PMRT}(4,\infty,IC)$ by using quadrature with the representation in Eq. 4.26, as shown in Table 4.2 . The reported result is generally in agreement with a Monte-Carlo simulation estimate for $P^S_{PMRT}(4,\infty,IC)$ in Williamson and Sargent (1967). Gehrlein (1988) performs Monte-Carlo simulation analysis to obtain estimates of $P^S_{PMRT}(5,\infty,IC) \approx .529$ and $P^S_{PMRT}(6,\infty,IC) \approx .340$. No representation for $P^S_{PMRT}(4,\infty,IAC)$ has been obtained to give an estimate of its value.

4.3 More than Four Candidates

4.3.1 Complete Breakdown by PMR

Let σ define a permutation, with terms $\{\sigma(C_1),\sigma(C_2),...,\sigma(C_m)\}$, on the candidates in $\boldsymbol{C}^m = \{C_1, C_2,..., C_m\}$ in a general *m*-candidate election. A *PMR m-cycle* exists in a given voter preference profile if there is some σ on the candidates in $\boldsymbol{C}^m$ such that $\sigma(C_1)\boldsymbol{M}\sigma(C_2)\boldsymbol{M}...\boldsymbol{M}\sigma(C_m)\boldsymbol{M}\sigma(C_1)$. Marchant (2001) considers the probability that various decision rules completely breakdown with tied relationships, so that no winner can be determined. For PMR, this complete breakdown coincides with the probability that a profile is observed with an *m*-cycle.

A limited number of representations have been obtained for the limiting probability, $P^S_{PMRC}(m,\infty,IC)$, that an m-cycle will be observed for $n \to \infty$ voters with IC. The idea of approaching this problem with the graph theory concept of Hamiltonian cycles was first considered by Taylor (1968), and Bell (1981) approached this problem in the same way to show that in the limit that both $n \to \infty$ and $m \to \infty$, $P^S_{PMRC}(\infty,\infty,IC) \to 1$. McKelvey (1976) reaches the same conclusion with a different technique, and McKelvey (1979) finds the same result for a more general assumption than IC. Very little is known about the rate of convergence of $P^S_{PMRC}(m,\infty,IC)$ to its limiting value as m increases.

Results from Guilbaud (1952) in Eq. 3.77 for $m = 3$ directly lead to

$$P^S_{PMRC}(3,\infty,IC) = 1 - P^S_{PMRW}(3,\infty,IC) = \frac{1}{4} - \frac{3}{2\pi} Sin^{-1}\left(\frac{1}{3}\right) \approx .08774\,. \tag{4.27}$$

Gehrlein (2004) develops a representation for $P^S_{PMRC}(4,\infty,IC)$, and that derivation starts with the observations in Theorem 4.2.

Theorem 4.2. If $m = 4$, any voter preference profile can have only one m-cycle at a time.

Proof. Suppose, without loss of generality, that $C_1 \boldsymbol{M} C_2$, $C_2 \boldsymbol{M} C_3$, $C_3 \boldsymbol{M} C_4$ and $C_4 \boldsymbol{M} C_1$ in an existing 4-cycle in a given voter preference profile . Further, suppose that $C_1 \boldsymbol{M} C_3$ is the first link in a second 4-cycle. The second link of this second 4-cycle cannot be $C_3 \boldsymbol{M} C_4$, because the second 4-cycle must then be completed by $C_4 \boldsymbol{M} C_2$ and $C_2 \boldsymbol{M} C_1$, which is a contradiction of $C_1 \boldsymbol{M} C_2$ in the first 4-cycle. The second link of this second 4-cycle cannot be $C_3 \boldsymbol{M} C_2$, since $C_2 \boldsymbol{M} C_3$ in the first 4-cycle. Thus, we must have $C_3 \boldsymbol{M} C_1$ as the first link in a second 4-cycle. Following the same argument as before, it is easily shown that this is not possible.

QED

The development of a representation for $P^S_{PMRC}(4,\infty,IC)$, follows the logic that has been used to obtain limiting representations as $n \to \infty$ with IC and DC. We start by defining four discrete variables on the preference rankings that are randomly assigned to individual voters to form a voter preference profile:

$$Y_1^i = \begin{cases} +1 : \text{if } C_1 \succ C_2 \text{ for the } i^{th} \text{voter} \\ -1 : \text{if } C_2 \succ C_1 \text{ for the } i^{th} \text{voter} \end{cases} \quad Y_2^i = \begin{cases} +1 : \text{if } C_2 \succ C_3 \text{ for the } i^{th} \text{voter} \\ -1 : \text{if } C_3 \succ C_2 \text{ for the } i^{th} \text{voter} \end{cases} \tag{4.28}$$

$$Y_3^i = \begin{cases} +1 : \text{if } C_3 \succ C_4 \text{ for the } i^{th} \text{voter} \\ -1 : \text{if } C_4 \succ C_3 \text{ for the } i^{th} \text{voter} \end{cases} \quad Y_4^i = \begin{cases} +1 : \text{if } C_4 \succ C_1 \text{ for the } i^{th} \text{voter} \\ -1 : \text{if } C_1 \succ C_4 \text{ for the } i^{th} \text{voter.} \end{cases}$$

For any given voter preference profile with $\overline{Y}_j\sqrt{n} > 0$ for all j = 1, 2, 3, 4, an m-cycle exists with $C_1\boldsymbol{M}C_2\boldsymbol{M}C_3\boldsymbol{M}C_4\boldsymbol{M}C_1$. Using the assumption of IC with the preference rankings in Fig. 4.1, it is easily shown that $E\left(\overline{Y}_j\sqrt{n}\right) = 0$ for all j = 1, 2, 3, 4. By appealing to the Central Limit Theorem as $n \to \infty$ and earlier arguments, the joint probability that $\overline{Y}_j\sqrt{n} > 0$ for all j = 1, 2, 3, 4 is equivalent to the four-variate normal positive orthant probability $\Phi_4\left(\boldsymbol{R^2}\right)$, and Gehrlein (2004) shows that

$$\boldsymbol{R^2} = \begin{bmatrix} 1 & -\frac{1}{3} & 0 & -\frac{1}{3} \\ - & 1 & -\frac{1}{3} & 0 \\ - & - & 1 & -\frac{1}{3} \\ - & - & - & 1 \end{bmatrix}. \tag{4.29}$$

There are six possible m-cycles with m = 4. The combination of Theorem 4.2 and the symmetry of IC with respect to candidates leads to the identity $P^S_{PMRC}(4,\infty,IC) = 6\Phi_4\left(\boldsymbol{R^2}\right)$. Given the special form of $\boldsymbol{R^2}$, results from David and Mallows (1961) can be used to obtain a representation for $\Phi_4\left(\boldsymbol{R^2}\right)$, with

$$P^S_{PMRC}(4,\infty,IC) = \frac{3}{8} - \frac{3}{\pi}Sin^{-1}\left(\frac{1}{3}\right) + \frac{6}{\pi^2}\int_0^{Sin^{-1}\left(-\frac{1}{3}\right)} Sin^{-1}\left(\frac{Sin(\gamma)}{Cos(2\gamma)}\right)d\gamma\,. \tag{4.30}$$

The representation for $P^S_{PMRC}(4,\infty,IC)$ can be evaluated by quadrature to obtain the value shown in Table 4.3. This result is generally in agreement with a Monte-Carlo simulation estimate for $P^S_{PMRC}(4,\infty,IC)$ in Bell (1978).

Table 4.3 Computed values of $P^S_{PMRC}(m,\infty,IC)$ and Monte-Carlo simulation estimates (*) from Bell (1978)

m	$P^S_{PMRC}(m,\infty,IC)$
3	.08774
4	.09042
5	.099*
10	.201*
20	.451*
40	.668*
80	.801*
∞	1.00000

The same type of analysis that was just used to obtain the representation for $P^S_{PMRC}(4,\infty,IC)$ is not easy to apply to the case of $m \geq 5$ candidates, since there can be two, or more, *m*-cycles at the same time in these cases. With m = 5 we could have a first *m*-cycle in a voter preference profile with $C_1\boldsymbol{M}C_2$, $C_2\boldsymbol{M}C_3$, $C_3\boldsymbol{M}C_4$, $C_4\boldsymbol{M}C_5$ and $C_5\boldsymbol{M}C_1$. A second *m*-cycle could then exist with $C_1\boldsymbol{M}C_3$, $C_3\boldsymbol{M}C_5$, $C_5\boldsymbol{M}C_2$, $C_2\boldsymbol{M}C_4$ and $C_4\boldsymbol{M}C_1$. There are no contradictory relationships between these two *m*-cycles, and *McGarvey's Theorem* [McGarvey (1953)] requires that a voter preference profile can be constructed to obtain such an outcome, with a sufficiently large number of voters. Table 4.2 lists estimates of $P^S_{PMRC}(m,\infty,IC)$ for each $m = 5,10,20,40,80,$ that were obtained by Monte-Carlo simulation in Bell (1978). These results strongly suggest that there is a very slow rate of convergence of $P^S_{PMRC}(m,\infty,IC)$ to its limiting value as m increases.

Tovey (1997) considered a variation of this problem with the assumption of IAC, while considering the probability that a given number of different PMR cycles exist in randomly generated voting situations. It is shown that the expected number of PMR cycles that are observed in any given randomly generated voting situation goes to infinity as $m \to \infty$.

Saari (2004) presents an extensive discussion of a procedure that produces "symmetric" preference patterns in profiles that lead to the existence of PMR *m*-cycles. The profiles are produced with a "stuttering process" using a "ranking wheel" to produce a profile that represents a "perfect square". The procedure is a direct extension of Ward (1965) since it can easily be seen to be generating Latin Square patterns on *m*-candidates.

4.3.2 General PMR Relationships for More than Four Candidates

Numerous early studies were performed to obtain estimates of $P^S_{PMRW}(m,n,IC)$ and $P^S_{PMRT}(m,n,IC)$ for larger m. Due to the complexity of the problem, most of these studies were performed by using Monte-Carlo simulation techniques, such as in Pomeranz and Weil (1970), Campbell and Tullock (1965) and other studies that have already been mentioned. The focus on IC was likely driven by the fact that this assumption was used in the original analytical work of Guilbaud (1952) and by some rather stunning results that are obtained if IC is assumed when m is large.

Garman and Kamien (1968) use computer enumeration techniques to obtain limited results for values of $P^S_{PMRW}(m,n,IC)$, and then use a discussion of limiting distributions to arrive at the conjecture that $P^S_{PMRW}(m,n,IC) \to 0$ as $m \to \infty$ for all n. Blin (1973) later uses an application of Stirling's Approximation to prove part of this conjecture to be true, by showing that $P^S_{PMRW}(m,n,IC) \to 0$ as

$m \to \infty$ for large *n*. This observation then leads to the obvious conclusion that $P^S_{PMRT}(m,n,IC) \to 0$ as $m \to \infty$ for large *n*. May (1971) proves that the conjecture that was posed by Garman and Kamien is true for all $n \geq 3$, but noted that the convergence to the limiting probability as $m \to \infty$ is extremely slow for small *n*. All of this suggests that the probability that a PMRW exists, or that PMR is transitive, could become quite small in elections with a large number of candidates. However, conditions like IC have already been shown to tend to exaggerate the probability that Condorcet's Paradox might occur. Tovey (1997) proves that $P^S_{PMRT}(m,n,IAC) \to 0$ as $m \to \infty$ with $n \to \infty$. This result is not surprising following the work of Blin (1973) and the fact IC and IAC converge to the same results as $m \to \infty$.

Williamson and Sargent (1967) show how strongly these rather threatening results are linked to the assumption of IC by examining the outcome of considering situations that are slightly different than IC. In particular, one possible linear preference ranking for an *m*-candidate election is given a probability $\frac{1}{m!} + \varepsilon$ of being drawn to represent the preferences of a randomly selected voter, and all other rankings are given a probability $\frac{1}{m!} - \frac{\varepsilon}{m!-1}$ of being selected. Let the *Perturbed Culture Condition* (*PC*) denote this model of assigning preference rankings to candidates. As $n \to \infty$ under PC, arguments are made to show that not only does $P^S_{PMRW}(m,\infty,PC) \to 1$, but that $P^S_{PMRT}(m,\infty,PC) \to 1$ also. Previous discussion has considered the significant impact that the assumption of unbalanced preferences, as we have with PC, can have on the probability that various voting situations are observed.

Kelly (1974) proves some general results concerning how $P^S_{PMRW}(m,n,IC)$ and $P^S_{PMRT}(m,n,IC)$ change as *m* and *n* change. The most progress was made regarding the behavior of $P^S_{PMRT}(m,n,IC)$:

Theorem 4.3. $P^S_{PMRT}(m,n,IC) > P^S_{PMRT}(m+1,n,IC)$ for $n \geq 3$ and $m \geq 2$.

Theorem 4.4. $P^S_{PMRT}(m,n,IC) > P^S_{PMRT}(m,n+1,IC)$ for odd *n* and $m \geq 3$.

Theorem 4.5. $P^S_{PMRT}(m,n+1,IC) > P^S_{PMRT}(m,n,IC)$ for even *n* and $m \geq 3$.

Theorem 4.6. $P^S_{PMRT}(m,n,IC) > P^S_{PMRT}(m,n+2,IC)$ for all *n* and $m \geq 3$.

Kelly (1974) also extended the notion of transitivity of PMR to consider the case of even *n*, so that PMR ties might exist. Let $P^{WO}_{PMRT}(m,n,IC)$ denote the probability that the PMR relationship is a weak order when *n* is even. Then

Theorem 4.7. $P^{WO}_{PMRT}(m,n,IC) > P^{WO}_{PMRT}(m+1,n,IC)$ for $n \geq 3$ and $m \geq 2$.

Theorem 4.8. $P_{PMRT}^{WO}(m,n,IC) > P_{PMRT}^{WO}(m,n+1,IC)$ for odd n and $m \geq 3$.

Theorem 4.9. $P_{PMRT}^{WO}(m,n+1,IC) > P_{PMRT}^{WO}(m,n,IC)$ for even n and $m \geq 3$.

Theorem 4.10. $P_{PMRT}^{WO}(m,n,IC) > P_{PMRT}^{WO}(m,n+2,IC)$ for all n and $m \geq 3$.

Mimiague (1973) presents a Monte-Carlo simulation based study that examines the probability that various PMR weak order structures result under IC.

Buckley and Westen (1979) prove some results that are related to the probability that a strict PMRW exists:

Theorem 4.11. $P_{PMRW}^{S}(m,n,IC) > P_{PMRW}^{S}(m,n+1,IC)$ for odd n and $m \geq 3$.

Theorem 4.12. $P_{PMRW}^{S}(m,n+1,IC) > P_{PMRW}^{S}(m,n,IC)$ for even n and $m \geq 3$.

The recursion relations for $P_{PMRW}^{S}(3,n,\boldsymbol{p})$ from Gillett (1978) in Eqs. 3.98 and 3.99 indicate that Theorems 4.11 and 4.12 are true for all $\boldsymbol{p}$ when $m = 3$, and other unreported results from Gillett further indicate that the same observation is valid when $m = 4$.

Kelly (1974) proves some results that are related to the probability that a weak PMRW exists:

Theorem 4.13. $P_{PMRW}^{W}(m,n+1,IC) > P_{PMRW}^{W}(m,n,IC)$ for odd n and $m \geq 3$.

Theorem 4.14. $P_{PMRW}^{W}(m,n,IC) > P_{PMRW}^{W}(m,n+1,IC)$ for even n and $m \geq 3$.

Kelly (1974) then leaves two conjectures that were originally suggested by Black (1948a,1958), and they remain to be proved.

Conjecture 4.1. $P_{PMRW}^{W}(m,n,IC) > P_{PMRW}^{W}(m+1,n,IC)$ for $m \geq 2$ and $n = 3$ or $n \geq 5$.

Conjecture 4.2. $P_{PMRW}^{W}(m,n,IC) > P_{PMRW}^{W}(m,n+2,IC)$ for $m \geq 3$ and $n = 1$ or $n \geq 3$.

Fishburn, et al. (1979a, b) prove Conjecture 4.1 to be true for the case special with $n = 3$. It was also proved that Conjecture 4.2 is true for all odd n with $m = 3$ [Theorem 3.1 (IC)], and for large even n with $m = 3$ [Theorem 3.5 (IC)].

Buckley and Westen (1979) give three additional conjectures that remain unproved.

Conjecture 4.3. $P_{PMRW}^{S}(m,n,IC) > P_{PMRW}^{S}(m+1,n,IC)$ for $m \geq 2$, $n \geq 3$.

Conjecture 4.4. $P_{PMRW}^{S}(m,n,IC) > P_{PMRW}^{S}(m,n+2,IC)$ for $m \geq 3$, odd $n \geq 3$.

Conjecture 4.5. $P_{PMRW}^{S}(m,n+2,IC) > P_{PMRW}^{S}(m,n,IC)$ for $m \geq 3$, even $n \geq 3$.

Krishnamoorthy and Raghavachari (2005) present a proof that Conjecture 4.3 is true for the limiting case as $n \to \infty$.

Buckley and Westen (1979) also develop another definition regarding the existence of a PMRW. Candidate $C_i \in \boldsymbol{C^m}$ is a *Semi-Strict PMRW* if C_i defeats or ties all other candidates in $\boldsymbol{C^m}$ by PMR and $C_i \boldsymbol{M} C_j$ for some $C_j \in \boldsymbol{C^m}$. Let $P_{PMRW}^{SS}(m,n,IC)$ denote the probability that a Semi-Strict PMRW exists for n voters and m candidates with the assumption of IC. It is then conjectured that:

Conjecture 4.6. $P_{PMRW}^{SS}(m,n+1,IC) > P_{PMRW}^{SS}(m,n,IC)$ for odd n and $m \geq 3$.

Conjecture 4.7. $P_{PMRW}^{SS}(m,n,IC) > P_{PMRW}^{SS}(m,n+1,IC)$ for even n and $m \geq 3$.

Conjecture 4.8. $P_{PMRW}^{SS}(m,n,IC) > P_{PMRW}^{SS}(m+1,n,IC)$ for $m \geq 2$ and $n \geq 3$.

Conjecture 4.9. $P_{PMRW}^{SS}(m,n,IC) > P_{PMRW}^{SS}(m,n+2,IC)$ for $m \geq 3$ and $n \geq 3$.

Fishburn (1976a) considers a variation of IC, which is denoted here as *Balanced Impartial Culture* (*BIC*). BIC is appropriate for the special case that $m = n$, and it is has a conditional assumption that each candidate is most preferred in the linear preference ranking of one of the voters. The conjecture is

Conjecture 4.10. $P_{PMRW}^{S}(m,n,IC) > P_{PMRW}^{S}(m,n,BIC)$ for all $m = n$.

Fishburn, et al. (1979a, b) develop some relationships that are related to Conjectures 4.3, 4.4 and 4.5, and to some interrelationships between them.

Theorem 4.15. If $P_{PMRW}^{S}(5,n,IC) > P_{PMRW}^{S}(6,n,IC)$, then $P_{PMRW}^{S}(4,n,IC) > P_{PMRW}^{S}(5,n,IC)$ for odd $n \geq 3$.

Theorem 4.16. $P_{PMRW}^{S}(3,n,IC) > P_{PMRW}^{S}(6,n,IC)$ if and only if $P_{PMRW}^{S}(4,n,IC) > P_{PMRW}^{S}(5,n,IC)$ for odd $n \geq 3$.

Theorem 4.17. If $P_{PMRW}^{S}(6,n,IC) > P_{PMRW}^{S}(6,n+2,IC)$ then $P_{PMRW}^{S}(5,n,IC) > P_{PMRW}^{S}(5,n+2,IC)$ for odd $n \geq 1$.

Theorem 4.18. If $P_{PMRW}^{S}(6,n,IC) > P_{PMRW}^{S}(6,n+2,IC)$ then $P_{PMRW}^{S}(5,n,IC) > P_{PMRW}^{S}(6,n,IC)$ for odd $n \geq 3$.

Additional relationships that are related to Conjectures 4.3, 4.4 and 4.5 come from Gehrlein (1981c)

Theorem 4.19. $P_{PMRW}^{S}(3,n,IC)^2 > P_{PMRW}^{S}(4,n,IC)$ for all odd $n \geq 1$.

Theorem 4.20. $P_{PMRW}^{S}(4,n,IC) > P_{PMRW}^{S}(3,n,IC)^3$ for all odd $n \geq 1$.

Theorem 4.21. If $P_{PMRW}^{S}(5,n,IC) \leq .780625$, then $P_{PMRW}^{S}(5,n,IC) > P_{PMRW}^{S}(6,n,IC)$ for odd $n \geq 1$.

Theorem 4.22. If $P_{PMRW}^{S}(5,n,IC) \le .799$ then $P_{PMRW}^{S}(3,n,IC) > P_{PMRW}^{S}(7,n,IC)$ for odd $n \ge 1$.

Theorem 4.23. If $P_{PMRW}^{S}(7,n,IC) \le .75$ then $P_{PMRW}^{S}(4,n,IC) > P_{PMRW}^{S}(5,n,IC)$ for odd $n \ge 1$.

Theorem 4.24. If $P_{PMRW}^{S}(6,n,IC) > P_{PMRW}^{S}(6,n+2,IC)$ for all odd $n \ge 1$ then $P_{PMRW}^{S}(3,n,IC) > P_{PMRW}^{S}(7,n,IC)$ for all odd $n \ge 1$.

Theorem 4.25. $P_{PMRW}^{S}(m,\infty,IC) > \frac{m}{2(m-1)} P_{PMRW}^{S}(m-1,\infty,IC)$ for all $m > 1$.

Gehrlein and Fishburn (1976) develop a recursion relationship concerning $P_{PMRW}^{S}(m,n,IC)$ for even *m*, generalizing the notions of May's Theorem in Theorem 4.1.

Theorem 4.26. For odd *n* and all even $m \ge 4$, there exist numbers ω_j^m such that

$$P_{PMRW}^{S}(m,n,IC) = \omega_0^m + \sum_F \omega_i^m P_{PMRW}^{S}(i,n,IC)$$

where $F = \{i\!: 3 \le i \le m \text{ and } i \text{ is odd}\}$.

Proof. The proof is a generalization for the earlier proof of Theorem 4.1. *P* is the probability that candidate $C_1 \in \boldsymbol{C^m}$ is not the PMRW in an *m*-candidate election, so that when *n* is odd, *P* is equivalent to

$$P = 1 - P_{PMRW}^{\{C_1\}}(m,n,IC). \tag{4.31}$$

Boole's Equation can be used in situations like the current case [Johnson and Kotz (1972), pg. 52] to generalize the results from Eq. 4.10 to obtain

$$P = P\left(\bigcup_{i=2}^{m} E_i\right) = \sum_{i=1}^{m-1} \binom{m-1}{i} (-1)^{i+1} P_{PMRW}^{\{C_1\}}(i+1,n,IC). \tag{4.32}$$

By equating the representations for *P* in Eqs. 4.31 and 4.32 for even *m* and reducing, we obtain

$$P_{PMRW}^{S}(m,n,IC) = \frac{m}{2} + \frac{m}{2} \sum_{i=1}^{m-2} \binom{m-1}{i} (-1)^{i} P_{PMRW}^{\{C_1\}}(i+1,n,IC). \tag{4.33}$$

It is important to note that Boole's Equation can not be used to obtain the representation in Eq. 4.33 if *m* is odd.

By starting with a representation for $P_{PMRW}^{S}(m,n,IC)$ for a desired *m*, as shown in Eq. 4.33, and sequentially replacing the $P_{PMRW}^{\{C_1\}}(i+1,n,IC)$ terms for which *i* + 1 is even and less than *m*, from largest to smallest, and expression for

$P^S_{PMRW}(m,n,IC)$ as a linear combination of $P^S_{PMRW}(j,n,IC)$ for odd $j < m$ can be obtained for all odd n. **QED**

Table 4.4 lists all of the computed values for the ω^m_j terms in Theorem 4.26 for each $m = 4(2)18$.

Table 4.4 Computed values of ω^m_j in Theorem 4.26 from Gehrlein and Fishburn (1976)

m	ω^m_0	ω^m_3	ω^m_5	ω^m_7	ω^m_9	ω^m_{11}	ω^m_{13}	ω^m_{15}	ω^m_{17}
4	-1	2							
6	3	-5	3						
8	-17	28	-14	4					
10	155	-255	126	-30	5				
12	-2073	3410	-1683	396	-55	6			
14	38227	-62881	31031	-7293	1001	-91	7		
16	-929569	1529080	-754572	177320	-24310	2184	-140	8	
18	28820619	-47408019	23394924	-5497596	753610	-67626	4284	-204	9

4.3.3 Enumerated Values of $P^S_{PMRW}(m,n,IC)$

Sevcik (1969) develops a computer enumeration procedure to count the number of possible voter preference profiles that have a PMRW with the assumption of IC to obtain exact values for $P^S_{PMRW}(m,n,IC)$ with odd $n \le 7$ and odd $m \le 7$, as shown in Table 4.5. These results were extended later by Maassen and Bezembinder (2002) for each m = 8, 9 and 10 for each n = 3, 4, 5 and 6. Computed values of the exact $P^S_{PMRW}(m,n,IC)$ forms in Table 4.5 are given in Table 4.6 for odd m to verify results in Garman and Kamien (1968) and other sources. Sevcik (1969) contains a minor typographical error for the resultant value of $P^S_{PMRW}(5,5,IC)$.

Table 4.5 Exact values of $P^S_{PMRW}(m,n,IC)$ from Sevcik (1969) and Maassen and Bezembinder (2002)

	m							
n	3	4	5	6	7	8	9	10
3	$\frac{17}{18}$	$\frac{8}{9}$	$\frac{21}{25}$	$\frac{359}{450}$	$\frac{33569}{44100}$	$\frac{536}{735}$	$\frac{13913}{19845}$	$\frac{67079}{99225}$
4	$\frac{4}{9}$	$\frac{197}{576}$	$\frac{1107}{4000}$	$\frac{1043}{4500}$	$\frac{15359}{77175}$	$\frac{91745}{526848}$	$\frac{5499323}{35562240}$	$\frac{705967}{5080320}$
5	$\frac{67}{72}$	$\frac{31}{36}$	$\frac{32019}{40000}$	$\frac{269513}{360000}$	$\frac{608721061}{864360000}$	$\frac{767419}{1152480}$	$\frac{1574336347}{2489356800}$	$\frac{37525387727}{62233920000}$
6	$\frac{989}{1944}$	$\frac{1037}{2592}$	$\frac{472549}{1440000}$	$\frac{1078499}{3888000}$	$\frac{13057391131}{54454680000}$	$\frac{1837328467}{8712748800}$	$\frac{9553049400803}{50812751001600}$	$\frac{23921196935141}{141146530560000}$
7	$\frac{10789}{11664}$	$\frac{4957}{5832}$	$\frac{15253909}{19440000}$	$\frac{2285362442560247}{3134566563824375}$	$\frac{46771945468099}{68612896800000}$			

Results for even *m* are not included in Table 4.6 since they are directly obtainable from Theorem 4.26. Gehrlein and Fishburn (1976) use the exact results from Sevcik (1969) to prove that a linear recursion relation for $P_{PMRW}^{S}(m,n,IC)$, like the one in Theorem 4.26, does not exist for odd *m*.

Table 4.6 Computed values of $P_{PMRW}^{S}(m,n,IC)$. Five digit entries are exact and three digit entries are approximations

	m					
n	3	5	7	9	11	13
3	.94444	.84000	.76120	.70108	.65356	.61484
5	.93056	.80047	.70424	.63243	.57682	.53235
7	.92498	.78467	.68168	.60551	.54703	.50063
9	.92202	.77628	.66976	.59135	.53144	.48409
11	.92019	.77108	.66238	.584	.523	.474
13	.91893	.76753	.65736	.578	.516	.467
15	.91802	.76496	.65372	.574	.511	.462
17	.91733	.76300	.65095	.571	.508	.458
19	.91678	.76146	.64879	.568	.505	.455
21	.91635	.76022	.64704	.566	.503	.453
23	.91599	.75920	.64560	.564	.501	.451
25	.91568	.75835	.64440	.563	.499	.449
27	.91543	.75763	.645	.562	.498	.448
29	.91521	.75700	.644	.561	.497	.447
31	.91501	.75646	.644	.560	.496	.446
33	.91484	.75598	.643	.559	.495	.445
35	.91470	.75556	.643	.558	.494	.444
37	.91456	.75519	.642	.557	.493	.443
39	.91444	.75485	.642	.557	.493	.442
41	.91434	.75455	.641	.556	.492	.442
43	.91424	.75427	.641	.556	.492	.441
45	.91415	.75402	.640	.555	.491	.441
47	.91407	.75379	.640	.555	.491	.440
49	.91399	.75358	.639	.555	.490	.440
∞	.91226	.74869	.63082	.54547	.48129	.43131

4.3.4 PMRW Probability Representations with Small *m*

Let $M_{PMRW}^{\{C_1\}}(m,n,IC)$ denote the number of distinct voter preference profiles for which C_1 is a strict PMRW in an *m*-candidate election for odd *n*. It follows directly from the development of Eq. 3.90 that $M_{PMRW}^{\{C_1\}}(3,n,IC)$ is given by

$$M_{PMRW}^{\{C_1\}}(3,n,IC) = n! \sum_{s_1=0}^{\frac{n-1}{2}} \sum_{s_2=0}^{\frac{n-1}{2}-s_1} \sum_{s_3=0}^{\frac{n-1}{2}-s_1} \frac{2^{n-s_2-s_3}}{s_1!s_2!s_3!(n-s_1-s_2-s_3)!}. \tag{4.34}$$

Table 4.6 (cont.)

	m					
n	15	17	19	21	23	25
3	.58249	.55495	.53111	.51021	.49168	.47511
5	.49583	.46521	.43908	.41647	.39667	.37915
7	.46280	.43128	.40455	.38154	.36150	.34385
9	.44564	.413	.386	.363	.343	.326
11	.435	.403	.375	.352	.332	.315
13	.428	.395	.368	.345	.325	.307
15	.423	.390	.363	.340	.320	.302
17	.419	.386	.359	.336	.316	.298
19	.416	.383	.356	.333	.313	.295
21	.413	.380	.353	.330	.310	.293
23	.411	.378	.351	.328	.308	.291
25	.409	.377	.349	.326	.306	.289
27	.408	.375	.348	.325	.305	.288
29	.407	.374	.347	.324	.304	.286
31	.406	.373	.346	.323	.303	.285
33	.405	.372	.345	.322	.302	.284
35	.404	.371	.344	.321	.301	.284
37	.403	.370	.343	.320	.300	.283
39	.402	.370	.342	.319	.299	.282
41	.402	.369	.342	.319	.299	.282
43	.401	.369	.341	.318	.298	.281
45	.401	.368	.341	.318	.298	.281
47	.400	.368	.340	.317	.297	.280
49	.400	.367	.340	.317	.297	.280
∞	.39127	.35844	.33100	.30771	.28768	.27025

The neutrality of IC toward candidates, coupled with the fact that each voter preference profile in an *m*-candidate election has the probability $m!^{-n}$ of being observed with the assumption of IC leads to

$$P_{PMRW}^{S}(m,n,IC) = mM_{PMRW}^{\{C_1\}}(m,n,IC)/m!^{n}. \tag{4.35}$$

Gehrlein and Fishburn (1979a) develop a recursive procedure to compute values of $M_{PMRW}^{\{C_1\}}(m,n,IC)$ from $M_{PMRW}^{\{C_1\}}(m-1,n,IC)$. The analysis starts with values of $M_{PMRW}^{\{C_1\}}(3,n,IC)$ that are obtained from Eq. 4.34 and then uses the recursive procedure to obtain $M_{PMRW}^{\{C_1\}}(m,n,IC)$ values by sequentially increasing *m*. The $M_{PMRW}^{\{C_1\}}(m,n,IC)$ values are obtained for each *m*, and the computed values of $P_{PMRW}^{S}(m,n,IC)$ then follow directly from the representation in Eq. 4.35. Gehrlein and Fishburn (1979a) contains a number of typographical errors. Gehrlein (1999b) corrects many of the typographical errors and then uses the same recursion procedure to significantly extend the list of known computed values of $P_{PMRW}^{S}(m,n,IC)$.

The recursive procedure to obtain $M_{PMRW}^{\{C_1\}}(m,n,IC)$ values starts with the representation in Eq. 4.34, which followed from Eq. 3.90. We start the description of this procedure by making some observations about the development of Eq. 3.90. $M_{PMRW}^{\{C_1\}}(3,n,IC)$ is obtained by calculating of the number of voter preference profiles on three candidates such that s_1 voters have C_1 ranked as least preferred. Similarly, s_2+s_3 voters have C_1 ranked second, and $n-s_1-s_2-s_3$ voters have C_1 ranked as most preferred. Let $f_3\left(k_1^3,k_2^3,k_3^3\right)$ denote the number of voter preference profiles that are enumerated in the computation of $M_{PMRW}^{\{C_1\}}(3,n,IC)$ that have k_i^3 voters with $(3-i)$ candidates preferred to candidate C_1 in their preference rankings, for i = 1, 2, 3. Obviously, $\sum_{i=1}^{3} k_i^3 = n$ and $k_1^3 \le (n-1)/2$. As usual, only the case of odd n is considered, to avoid complications with PMR ties.

Then, $M_{PMRW}^{\{C_1\}}(4,n,IC)$ is obtained by determining the number of unique assignments of voter preference profiles that have C_1 remaining as the PMRW when a fourth candidate, C_4 , is added to obtain linear voter preference rankings on all four candidates, as an extension of the voter preference profiles on the initial three candidates. For any combination of k_i^3's for three-candidates, let s_i^4 denote the number of voters included in k_i^3 for whom we rank C_4 ahead of C_1 in the extended rankings. There are $\binom{k_i^3}{s_i^4}$ combinations of such assignments. For any given voter of the s_i^4 in this particular assignment, C_4 can be placed in $(4-i)$ different positions above C_1 in that voter's linear preference ranking.

Similarly, any given voter among the remaining the $k_i^3 - s_i^4$ voters in this particular assignment could have C_4 placed in i different positions below C_1 in that voter's preference ranking. Given that C_1 is the PMRW for all of the three-candidate voter preference profiles that are being considered, C_1 will remain the PMRW for all of the extended four-candidate voter preference profiles whenever $\sum_{i=1}^{3} s_i^4 \le (n-1)/2$. Using all of the above, we find

$$M_{PMRW}^{\{C_1\}}(4,n,IC) = \tag{4.36}$$

$$\sum_{k_1^3=0}^{\frac{n-1}{2}} \sum_{k_2^3=0}^{n-k_1^3} f_3\left(k_1^3,k_2^3,k_3^3\right) \sum_{s_1^4=0}^{k_1^3} \binom{k_1^3}{s_1^4} 3^{s_1^4} \sum_{s_2^4=0}^{Min\left\{\frac{n-1}{2}-s_1^4,\ k_2^3\right\}} \binom{k_2^3}{s_2^4} 2^{k_2^3} \sum_{s_3^4=0}^{Min\left\{\frac{n-1}{2}-s_1^4-s_2^4,\ k_3^3\right\}} \binom{k_3^3}{s_3^4} 3^{k_3^3-s_3^4}.$$

Here, $k_3^3 = n - k_1^3 - k_2^3$.

While performing the process of computing $M^{\{C_1\}}_{PMRW}(4,n,IC)$, it is possible to accumulate values for $f_4\left(k_1^4,k_2^4,k_3^4,k_4^4\right)$, which has terms that are defined in the same fashion as for $f_3\left(k_1^3,k_2^3,k_3^3\right)$). That is, k_i^4 is the total number of individual voter preference rankings in a voter preference profile on four candidates for which $(4-i)$ of the four candidates are ranked above C_1, and $f_4\left(k_1^4,k_2^4,k_3^4,k_4^4\right)$ is the total accumulated number of extended voter preference profiles on four candidates, with the associated k_i^4 's, that have C_1 as the PMRW. In determining the contribution to $f_4\left(k_1^4,k_2^4,k_3^4,k_4^4\right)$ from a specific combination of k_i^3 's in the development of $M^{\{C_1\}}_{PMRW}(4,n,IC)$, we find that s_1^4 voters will have three candidates preferred to C_1 in the extended preference rankings on the four candidates, $k_1^3 - s_1^4 + s_2^4$ will have two candidates preferred to C_1, $k_2^3 - s_2^4 + s_3^4$ will have one candidate preferred to C_1, and $k_3^3 - s_3^4$ will have no candidates preferred to C_1.

Once the values for $f_4\left(k_1^4,k_2^4,k_3^4,k_4^4\right)$ have been obtained, we go about extending the preference ranking assignments by adding a fifth candidate, C_5, to the preference ranking assignments on four candidates. Then, for any particular set of k_i^4 's which have C_1 as the PMRW, we extend the preference rankings of s_i^5 of the voters from the number of k_i^4 by ranking C_5 ahead of C_1. Following earlier discussion, there are $\binom{k_i^4}{s_i^5}$ combinations of such assignments, and there are $(5-i)$ different positions for placing C_5 above C_1 in each individual voter's preference ranking. For voters who are counted among $k_i^4 - s_i^5$, there are *i* positions in which to place C_5 below C_1 in each voter's preference ranking. Following earlier discussion, C_1 will remain the PMRW as long as $\sum_{i=1}^4 s_i^5 \le (n-1)/2$, and

$$M^{\{C_1\}}_{PMRW}(5,n,IC) = \tag{4.37}$$

$$\sum_{k_1^4=0}^{\frac{n-1}{2}} \sum_{k_2^4=0}^{n-k_1^4} \sum_{k_3^4=0}^{n-k_1^4-k_2^4} \left[\begin{array}{l} f_4\left(k_1^4,k_2^4,k_3^4,k_4^4\right) \sum\limits_{s_1^5=0}^{k_1^4} \binom{k_1^4}{s_1^5} 4^{s_1^5} \sum\limits_{s_2^5=0}^{Min\left\{\frac{n-1}{2}-s_1^5,\ k_2^4\right\}} \binom{k_2^4}{s_2^5} 3^{s_2^5} 2^{k_2^4-s_2^5} \times \\ \sum\limits_{s_3^5=0}^{Min\left\{\frac{n-1}{2}-s_1^5-s_2^5,\ k_3^4\right\}} \binom{k_3^4}{s_3^5} 2^{s_3^5} 3^{k_3^4-s_3^5} \sum\limits_{s_4^5=0}^{Min\left\{\frac{n-1}{2}-s_1^5-s_2^5-s_3^5,\ k_4^4\right\}} \binom{k_4^4}{s_4^5} 4^{k_4^4-s_4^5} \end{array} \right],$$

where $k_4^4 = n - k_1^4 - k_2^4 - k_3^4$.

Similar representations for $M_{PMRW}^{\{C_1\}}(m,n,IC)$ with larger m become very cumbersome, but they can easily be developed in the same fashion. This was done for odd m up to seven in Gehrlein (1999b), and Table 4.6 lists the resulting computed probabilities for $P_{PMRW}^{S}(m,n,IC)$ from Eq. 4.35 for all n = 3(2)49 with m = 3, 5 and for all n = 3(2)25 for m = 7. Computed values of $P_{PMRW}^{S}(m,n,IC)$ are not reported for even m, since recursion relations are known to exist for these cases from Theorem 4.26.

4.3.5 PMRW Probability Representations with Small *n*

Other studies have developed procedures to obtain computed values of $P_{PMRW}^{S}(m,n,IC)$ for the special case of small n. May (1971) presents some results along these lines, and begins by developing a representation for $P_{PMRW}^{S}(m,3,IC)$ as

$$P_{PMRW}^{S}(m,3,IC)= \tag{4.38}$$

$$\sum_{i=0}^{m-1}\sum_{j=0}^{m-1-i}\sum_{k=0}^{m-1-i-j}\frac{(m-1)!(m-1-i)!(m-1-j)!(m-1-k)!}{[m!]^3(m-1-i-j-k)!}.$$

This representation contains a typographical error in the statement of Eq. (2) in the text in May (1971), but it is stated correctly in the development of footnote (3) of the same paper.

Gehrlein and Fishburn (1979a) develop a representation for $P_{PMRW}^{S}(m,3,IC)$ that is simpler than the one shown in Eq. 4.38. This is done by using arguments that count the possible number of voter preference profiles for which C_1 is the PMRW, following the arguments in the preceding case for small m. Since there are only three voters, it is not possible for a PMRW to be beaten by any other candidate in the preference ranking of more than one voter. Assume that candidate C_1 is the PMRW, and let m_i^3 denote the number of candidates that are ranked ahead of C_1 in the preference ranking of voter i in a given voter preference profile when there are three voters. Obviously, there are no candidates that are common among those counted in the different m_i^3's. There are $\binom{m-1}{m_1^3}$ unique combinations of m_1^3 candidates that are ranked above C_1 in the first voter's preference ranking, and there are $m_1^3!\left(m-1-m_1^3\right)!$ different complete preference rankings that the first voter might have that are consistent with a specific value of m_1^3. The second voter has $\binom{m-1-m_1^3}{m_2^3}$ unique combinations of m_2^3 candidates that have no overlap with candidates among the m_1^3 candidates that are ranked above C_1 in the first voter's preference ranking. As above, there are $m_2^3!\left(m-1-m_2^3\right)!$ different preference rankings that the second voter might have that are consistent with a specific value of m_2^3.

The third voter must have a preference ranking such that C_1 is ranked above all of the candidates that are included in m_1^3 and m_2^3. The proportion of preference rankings, of the $m!$ possible complete preference rankings on all candidates, that the third voter might have to keep C_1 as the PMRW is given by $1/\left(m_1^3+m_2^3+1\right)$. This proportion is simply obtained by noting, for any given m_1^3 and m_2^3, that there are a total of $\left(m_1^3+m_2^3+1\right)!$ permutations on the candidates included in a reduced set of those candidates counted in m_1^3 and m_2^3, along with C_1. Of these $\left(m_1^3+m_2^3+1\right)!$ permutations, $\left(m_1^3+m_2^3\right)!$ have C_1 ranked in the first position among the reduced set of candidates, regardless of the relative position of candidates in the reduced set within the third voter's complete preference ranking on all candidates. It follows that

$$M_{PMRW}^{\{C_1\}}(m,3,IC)= \tag{4.39}$$
$$\sum_{m_1^3=0}^{m-1}\sum_{m_2^3=0}^{m-1-m_1^3}\binom{m-1}{m_1^3}\binom{m-1-m_1^3}{m_2^3}\frac{m_1^3!\,m_2^3!\left(m-1-m_1^3\right)!\left(m-1-m_2^3\right)!\,m!}{\left(m_1^3+m_2^3+1\right)}.$$

Using Eqs. 4.35 and 4.39 with algebraic reduction, we obtain the representation

$$P_{PMRW}^{S}(m,3,IC)=\sum_{m_1^3=0}^{m-1}\sum_{m_2^3=0}^{m-1-m_1^3}\frac{\left(m-1-m_1^3\right)!\left(m-1-m_2^3\right)!}{m!\left(m-1-m_1^3-m_2^3\right)!\left(m_1^3+m_2^3+1\right)}. \tag{4.40}$$

Tables 4.6 and 4.7 list computed values of $P_{PMRW}^{S}(m,3,IC)$ from Eq. 4.40 for various values of odd m with $3\le m\le 449$. As noted in the observation in May (1971) that was mentioned earlier, the convergence of $P_{PMRW}^{S}(m,3,IC)$ to its limiting value of zero as $m\to\infty$ is very slow as m increases.

The development of representations for $P_{PMRW}^{S}(m,n,IC)$ becomes somewhat more complicated for n greater than three. When n = 5, we partition the voters into two sets, the first three and the last two. Consider the situation in which we wish to compute the probability that candidate C_1 is the PMRW for the five-voter case. Let a_i^3 denote the number of candidates that are ranked as preferred to C_1 in exactly i of the preference rankings of the first three voters, with $0 \le i \le 2$. Since C_1 is the PMRW, we must have $a_0^3 + a_1^3 + a_2^3 = m - 1$, and no candidate can be counted in more than one of the a_i^3 terms.

Table 4.7 Computed values of $P_{PMRW}^{S}(m,n,IC)$ from Gehrlein (1999b)

	n		
m	3	5	∞
27	.46017	.36352	.25494
29	.44660	.34947	.24138
31	.43421	.33675	.22927
33	.42283	.32517	.21840
35	.41233	.31457	.20857
37	.40260	.30482	.19964
39	.39355	.29583	.19149
41	.38150	.28749	.18402
43	.37720	.27974	.17715
45	.36977	.27251	.17080
47	.36278	.26574	.16492
49	.35618	.25940	.15945
99	.25933		
149	.21450		
199	.18720		
249	.16833		
299	.15426		
349	.14325		
399	.13433		
449	.12690		

As a first step, we compute the proportion, $Z^5\left(a_0^3, a_1^3, a_2^3\right)$, of five voter preference profiles that will have C_1 as the PMRW, given a specific combination of a_i^3 terms from the preferences of the first three voters. Once the a_i^3 terms are fixed from the first three voters, the conditions that keep C_1 as the PMRW result from placing restrictions on the preference rankings of the last two voters. Candidates counted in a_0^3 can be ranked anywhere in the preferences of the last two voters and C_1 will remain as the PMRW, so we focus our attention on the feasible placement of the reduced set of candidates included in the remaining $m - 1 - a_0^3$ candidates, relative to the position of C_1 in the preference rankings of the last two voters. There are $\left(m - a_0^3\right)!$ possible preference rankings on this reduced set of candidates for voter number four. If C_1 is the PMRW, then all candidates among those counted in a_2^3 must be ranked below C_1 in the preference ranking for voter four. Assume that exactly η candidates from a_1^3 are ranked ahead of C_1 in the preferences of voter four, with $a_1^3 - \eta$ being ranked below C_1. There are $\binom{a_1^3}{\eta}$ combinations of ways of selecting the candidates that are counted in η.

A total of η candidates from the reduced set are ranked above C_1 for voter four, with a total of $m-1-a_0^3-\eta$ from the reduced set being ranked below C_1. Then there are $\eta!\left(m-1-a_0^3-\eta\right)!$ possible rankings on the reduced set of candidates for voter four that allow C_1 as the PMRW. The preference ranking for voter five must then have candidate C_1 ranked above all candidates counted in either a_2^3 or η. Following earlier discussion from the development of the representation for $M_{PMRW}^S(m,3,IC)$, the proportion of the $m!$ possible preference rankings on all candidates for voter five that meet this criterion is given by $1/\left(\eta + a_2^3 + 1\right)$. Given this discussion, we find the representation

$$Z^5\left(a_0^3,a_1^3,a_2^3\right)=\sum_{\eta=0}^{a_1^3}\binom{a_1^3}{\eta}\frac{\eta!\left(m-1-a_0^3-\eta\right)!}{\left(m-a_0^3\right)!}\frac{1}{\eta+a_2^3+1}, \tag{4.41}$$

which can be reduce with the identity $m-1-a_0^3 = a_1^3 + a_2^3$ to

$$Z^5\left(a_0^3,a_1^3,a_2^3\right)=\sum_{\eta=0}^{a_1^3}\frac{a_1^3!\left(a_1^3+a_2^3-\eta\right)!}{\left(a_1^3-\eta\right)!\left(a_1^3+a_2^3+1\right)!\left(\eta+a_2^3+1\right)}. \tag{4.42}$$

This representation is valid, without regard to the relative placement of candidates in the reduced set within the complete preference rankings on all m candidates of the last two voters.

Attention now returns to the preference rankings of the first three voters. Let b_i denote the number of candidates that are ranked as being preferred to candidate C_1 only in the preference ranking of Voter i among the first three voters. Similarly, $b_{i,j}$ denotes the number of candidates that are ranked as being preferred to C_1 only in the preference rankings of both Voter i and Voter j among the first three voters. It then follows directly from previous definitions that $a_1^3 = b_1 + b_2 + b_3$, $a_2^3 = b_{1,2} + b_{1,3} + b_{2,3}$ and $a_0^3 = m - 1 - a_1^3 - a_2^3$. The total number of candidates that are ranked as being preferred to C_1 in the preference ranking for Voter 1 is given by $m_1 = b_1 + b_{1,2} + b_{1,3}$. There are $m!$ possible preference rankings for Voter 1, and $m_1!(m-1-m_1)!$ of them that meet this criterion. Similar arguments hold for Voter 2 and for Voter 3. A representation for $M^S_{PMRW}(m,5,IC)$ is then obtained by using a summation function to enumerate all feasible combinations of b_i 's and $b_{i,j}$'s. In this enumeration we calculate the number of combinations of ways that the $m-1$ candidates can be partitioned into the b_i 's and $b_{i,j}$'s, then compute the proportion of all possible profiles for the first three voters that are feasible with the given b_i 's and $b_{i,j}$'s, and then account for the proportion, $Z^5\left(a_0^3, a_1^3, a_2^3\right)$, of profiles for the last two voters that will have C_1 as the PMRW given the b_i 's and $b_{i,j}$'s. After using Eq. 4.35 with algebraic reduction, the resulting representation is given by

$$P^S_{PMRW}(m,5,IC) = \tag{4.43}$$

$$\Sigma_4\left[\frac{\prod_{i=1}^3 m_i!(m-1-m_i)!}{b_1!b_2!b_3!b_{1,2}!b_{1,3}!b_{2,3}!m^*!m!^2}\sum_{\eta=0}^{b_\bullet}\frac{b_\bullet!(b_\bullet+b_{\bullet\bullet}-\eta)!}{(b_\bullet-\eta)!(b_\bullet+b_{\bullet\bullet}+1)!(\eta+b_{\bullet\bullet}+1)}\right].$$

where Σ_4 is a six-summation function with has summation limits given by

$$\begin{aligned}
0 \le\ & b_1 \le m-1 \\
0 \le\ & b_2 \le m-1-b_1 \\
0 \le\ & b_3 \le m-1-b_1-b_2 \\
0 \le\ & b_{1,2} \le m-1-b_1-b_2-b_3 \\
0 \le\ & b_{1,3} \le m-1-b_1-b_2-b_3-b_{1,2} \\
0 \le\ & b_{2,3} \le m-1-b_1-b_2-b_3-b_{1,2}-b_{1,3}
\end{aligned} \tag{4.44}$$

with

$$\begin{aligned}
m_1 &= b_1+b_{1,2}+b_{1,3} \\
m_2 &= b_2+b_{1,2}+b_{2,3} \\
m_3 &= b_3+b_{1,3}+b_{2,3} \\
m^* &= m-1-b_1-b_2-b_3-b_{1,2}-b_{1,3}-b_{2,3} \\
b_\bullet &= b_1+b_2+b_3 \\
b_{\bullet\bullet} &= b_{1,2}+b_{1,3}+b_{2,3}.
\end{aligned} \tag{4.45}$$

Using similar logical arguments, a representation for $P^S_{PMRW}(m,7,IC)$ is obtained in Gehrlein and Fishburn (1979a) as

$$P^S_{PMRW}(m,7,IC) = \Sigma_5\left[\frac{\prod_{i=1}^4\left[m_i!(m-1-m_i)!\right]}{m^*!m!^3\prod_X b_X!}Z^7\left(m^*,b_\bullet,b_{\bullet\bullet},b_{\bullet\bullet\bullet}\right)\right]. \tag{4.46}$$

Here Σ_5 is a 14-summation function that has sequential indexes b_1, b_2, b_3, b_4, $b_{1,2}, b_{1,3}, b_{1,4}, b_{2,3}, b_{2,4}, b_{3,4}, b_{1,2,3}, b_{1,2,4}, b_{1,3,4}, b_{2,3,4}$ where the limits on these summation indexes keep

$$0 \leq b_1 + b_2 + b_3 + b_4 + b_{1,2} + \ldots + b_{1,3,4} + b_{2,3,4} \leq m - 1. \tag{4.47}$$

In addition,

$$\begin{aligned}
m_1 &= b_1 + b_{1,2} + b_{1,3} + b_{1,4} + b_{1,2,3} + b_{1,2,4} + b_{1,3,4} \\
m_2 &= b_2 + b_{1,2} + b_{2,3} + b_{2,4} + b_{1,2,3} + b_{1,2,4} + b_{2,3,4} \\
m_3 &= b_3 + b_{1,3} + b_{2,3} + b_{3,4} + b_{1,2,3} + b_{1,3,4} + b_{2,3,4} \\
m_4 &= b_4 + b_{1,4} + b_{2,4} + b_{3,4} + b_{1,2,4} + b_{1,3,4} + b_{2,3,4} \\
m^* &= m - 1 - b_1 - b_2 - b_3 - b_4 - b_{1,2} - \ldots - b_{1,3,4} - b_{2,3,4} \\
b_{\bullet} &= b_1 + b_2 + b_3 + b_4 \\
b_{\bullet\bullet} &= b_{1,2} + b_{1,3} + b_{1,4} + b_{2,3} + b_{2,4} + b_{3,4} \\
b_{\bullet\bullet\bullet} &= b_{1,2,3} + b_{1,2,4} + b_{1,3,4} + b_{2,3,4} \\
\prod_X b_X! &= b_1! b_2! b_3! b_4! b_{1,2}! \ldots b_{1,3,4}! b_{2,3,4}!
\end{aligned} \tag{4.48}$$

and

$$Z^7(m^*, b_{\bullet}, b_{\bullet\bullet}, b_{\bullet\bullet\bullet}) = \sum_{\beta_5=0}^{b_{\bullet}} \sum_{\beta_6=0}^{b_{\bullet}-\beta_5} \sum_{\alpha_5=0}^{b_{\bullet\bullet}} \sum_{\alpha_6=0}^{b_{\bullet\bullet}-\alpha_5} \left[\frac{\dfrac{b_{\bullet}! b_{\bullet\bullet}! (\alpha_5 + b_{\bullet} - \beta_5)! (\beta_5 + b_{\bullet\bullet} - \alpha_5 + b_{\bullet\bullet\bullet})!}{\beta_5! \beta_6! (b_{\bullet} - \beta_5 - \beta_6)! \alpha_5! \alpha_6! (b_{\bullet\bullet} - \alpha_5 - \alpha_6)!} \times (b_{\bullet} - \beta_5 - \beta_6 + \alpha_6)! (\beta_6 + b_{\bullet\bullet} + b_{\bullet\bullet\bullet} - \alpha_6)!}{(b_{\bullet} + b_{\bullet\bullet} + b_{\bullet\bullet\bullet} + 1)! (b_{\bullet} + b_{\bullet\bullet} + b_{\bullet\bullet\bullet} - \beta_5 + 1)! (b_{\bullet} + b_{\bullet\bullet\bullet} + \alpha_5 + \alpha_6 - \beta_5 - \beta_6 + 1)} \right]. \tag{4.49}$$

Tables 4.6 and 4.7 list computed values of $P^S_{PMRW}(m,5,IC)$ for each m = 3(2)49, and Table 4.6 lists computed values of $P^S_{PMRW}(m,7,IC)$ for each m = 3(2)25 from Gehrlein (1999b).

A representation for $P^S_{PMRW}(m,3,IAC)$ is developed in Gehrlein (1998), and the logic behind the development of that representation relies upon arguments for counting the number of voting situations for which a specified candidate, C_1, is the PMRW when $n = 3$. The first possibility, Case 1, occurs when all three voters have identical preference rankings on the m candidates with C_1 ranked as the most preferred candidate. There are $m!$ different linear preference rankings with m candidates, and $(m-1)!$ of them have C_1 ranked as most preferred. Case 2 deals with the situation in which two voters have identical preference rankings on the m candidates with C_1 ranked as most preferred, while the third voter has any other preference ranking. The number of voting situations of this nature is given by $(m-1)!(m!-1)$.

Case 3^i for i = 0, 1, 2, 3 considers the voting situations in which C_1 is the PMRW, when all three voters have different preference rankings, with i voters having C_1 ranked first. It follows directly that the number of voting situations in Case 3^3 is given by $\binom{(m-1)!}{3}$. Similarly, the number of voting situations in Case 3^2 is given by $\binom{(m-1)!}{2}[m!-(m-1)!]$.

In the discussion of Cases 3^0 and 3^1, r_j denotes the number of candidates that are ranked ahead of C_1 in the preference ranking of Voter j. Begin with an evaluation of Case 3^1, and assume arbitrarily that C_1 is ranked first in the preference ranking of Voter 1, so that $r_1 = 0$. This assumption can be made without any loss of generality since voters are anonymous with IAC. We can also assume without any loss of generality that $r_3 \geq r_2$. Given that there are $(m-1)!$ linear preference rankings with $r_1 = 0$, the number of voting situations in Case 3^1 is given by

$$(m-1)! \sum_{r_2=1}^{\left[\frac{m-1}{2}\right]^-} \sum_{r_3=r_2}^{m-1-r_2} \left[\binom{m-1}{r_2+r_3}\binom{r_2+r_3}{r_2} r_2!(m-1-r_2)!r_3!(m-1-r_3)!/\delta^1\right], \tag{4.50}$$

where $\delta^1 = 2$ if $r_2 = r_3$

$\delta^1 = 1$ otherwise.

The notation $[x]^-$ denotes the largest integer that is less than or equal to x. The δ^1 term prevents double counting for the case with $r_2 = r_3$, since interchanging the preferences of Voters 2 and 3 will not create a new voting situation.

To avoid the double counting of voting situations from permutations of preference assignments to voters in Case 3^0, we assume arbitrarily that $r_3 \geq r_2 \geq r_1$, with $r_1 \geq 1$. It follows directly from the logic of previous arguments that the number of voting situations for this particular case is given by

$$\sum_{r_1=1}^{\left[\frac{m-1}{3}\right]^-} \sum_{r_2=r_1}^{\left[\frac{m-1-r_1}{2}\right]^-} \sum_{r_3=r_2}^{m-1-r_1-r_2} \binom{m-1}{r_1+r_2+r_3}\binom{r_1+r_2+r_3}{r_1}\binom{r_2+r_3}{r_2}\prod_{i=1}^{3}[r_i!(m-1-r_i)!]/\delta^2, \tag{4.51}$$

where $\delta^2 = 2$ if $r_1 = r_2$ or $r_2 = r_3$, with $r_1 \neq r_3$

$\delta^2 = 6$ if $r_1 = r_3$ $\delta^2 = 1$ otherwise.

The total number of voting situations for all of these cases can be accumulated to give the number of voting situations, $N_{PMRW}^{\{C_1\}}(m,3,IAC)$, for which candidate C_1 is the PMRW with n = 3 voters for m candidates. After accumulation and algebraic reduction

$$N_{PMRW}^{\{C_1\}}(m,3,IAC) = \frac{(m-1)!\{(m-1)!+1\}}{6}\{(3m-2)(m-1)!+2\} + (m-1)!\sum_{r_1=0}^{\left[\frac{m-1}{3}\right]^-}\sum_{r_2=\delta(r_1)}^{\left[\frac{m-1-r_1}{2}\right]^-}\sum_{r_3=r_2}^{m-1-r_1-r_2}\left[\frac{\prod_{i=1}^{3}(m-1-r_i)!}{(m-1-r_1-r_2-r_3)!}\right]/\delta^2, \tag{4.52}$$

where $\delta(r_1) = r_1$ if $r_1 > 0$
$\delta(r_1) = 1$ if $r_1 = 0$.

The $\delta(r_1)$ term in Eq. 4.52 is defined to allow for the aggregation of the results from Eqs. 4.50 and 4.51.

Eq. 3.92 can be used to show that there are $H(3,m!) = m!(m!+1)(m!+2)/6$ possible voting situations for n = 3 with m candidates. By the symmetry of IAC with respect to candidates, a representation for $P_{PMRW}^{S}(m,3,IAC)$ can then be obtained from the relationship

$$P_{PMRW}^{S}(m,3,IAC) = mN_{PMRW}^{\{C_1\}}(m,n,IAC)/H(3,m!). \tag{4.53}$$

After substituting the representation for $N_{PMRW}^{\{C_1\}}(m,3,IAC)$ from Eq. 4.52 into Eq. 4.53 and performing algebraic reduction, we obtain

$$P_{PMRW}^{S}(m,3,IAC) = \frac{(m-1)!+1}{(m!+1)(m!+2)}\{(3m-2)(m-1)!+2\} + \frac{6}{(m!+1)(m!+2)}\sum_{r_1=0}^{\left[\frac{m-1}{3}\right]^-}\sum_{r_2=\delta(r_1)}^{\left[\frac{m-1-r_1}{2}\right]^-}\sum_{r_3=r_2}^{m-1-r_1-r_2}\left[\frac{\prod_{i=1}^{3}(m-1-r_i)!}{(m-1-r_1-r_2-r_3)!}\right]/\delta^2. \tag{4.54}$$

Table 4.8 contains computed values of $P_{PMRW}^{S}(m,3,IAC)$ for each m = 3(1)11 from Eq. 4.54. The associated values of $P_{PMRW}^{S}(m,3,IC)$ from Eq. 4.40 are also included in Table 4.8 for the purposes of comparison. We observe that computed values of $P_{PMRW}^{S}(m,3,IAC)$ and $P_{PMRW}^{S}(m,3,IC)$ converge to the same values very quickly as m increases, to verify earlier observations. Table 4.8 also lists values of

$P^S_{PMRW}(m,5,IAC)$ for each m = 3,4,5 that were obtained by computer enumeration in Gehrlein (1998).

Table 4.8 Calculated values of $P^S_{PMRW}(m,n,IAC)$ and $P^S_{PMRW}(m,n,IC)$ for n = 3,5 from Gehrlein (1998)

m	$P^S_{PMRW}(m,3,IAC)$	$P^S_{PMRW}(m,3,IC)$	$P^S_{PMRW}(m,5,IAC)$	$P^S_{PMRW}(m,5,IC)$
3	.94444	.96429	.93056	.95238
4	.88889	.90154	.86111	.87302
5	.84000	.84392	.80047	.80380
6	.79778	.79862		
7	.76120	.76134		
8	.72925	.72927		
9	.70108	.70109		
10	.67603	.67603		
11	.65357	.65357		

4.3.6 Limiting Probabilities with More than Four Candidates

The primary limiting probability representation that has been obtained for cases with more than four candidates considers the probability $P^S_{PMRW}(m,\infty,IC)$ that a strict PMRW exists for m candidates as $n \to \infty$ with IC. Niemi and Weisberg (1968) generalize Guilbaud's result for m equal to three from Eq. 3.77. Their results for the general case of m candidates follow directly from previous discussion.

The logic that led to Eq. 3.69 can be used to obtain Guilbaud's result for the special case of IC for three candidates. In particular, Eq. 3.69 can be reduced to $P^S_{PMRW}(3,\infty,IC) = 3\Phi_2(\rho)$, where $\Phi_2(\rho)$ is a bivariate normal positive orthant probability with correlation $\rho = 1/3$ between the two variables. The logic that led to Eq. 3.91 allows us to easily generalize Guilbaud's result to

$$P^S_{PMRW}(m,\infty,IC) = m\Phi_{m-1}\left(\boldsymbol{R^{m-1}}(1/3)\right). \tag{4.55}$$

Here, $\boldsymbol{R^{m-1}}(\rho)$ is a correlation matrix for a joint multinormal distribution on $m-1$ variables for which all correlation terms between variables are equal, with $\rho = 1/3$.

Gehrlein and Fishburn (1978a) consider the case of m = 5 and use a representation for $\Phi_4\left(\boldsymbol{R^4}(1/3)\right)$ from Posnyakov (1971) to obtain

$$P^S_{PMRW}(5,\infty,IC) = \frac{5}{16} + \frac{15}{4\pi} Sin^{-1}\left(\frac{1}{3}\right) + \frac{15}{2\pi^2} \int_0^{\frac{1}{3}} \frac{Sin^{-1}[\gamma/(1+2\gamma)]}{\sqrt{(1-\gamma^2)}} d\gamma . \tag{4.56}$$

Evaluation by quadrature in Eq. 4.56 leads to $P_{PMRW}^{S}(5,\infty,IC) \approx .74869$. The results of Theorem 4.26 can then used to extend these results a representation for $P_{PMRW}^{S}(6,\infty,IC) \approx .68476$. Niemi and Weisberg (1968) use calculated values of $\Phi_{m-1}\left(\boldsymbol{R^{m-1}}(1/3)\right)$ from Ruben (1954) to obtain values of $P_{PMRW}^{S}(m,\infty,IC)$ for each m = 3(2)49, and these values are listed in Tables 4.6 and 4.7.

Gehrlein and Fishburn (1979a) use a result from Bacon (1963) to develop an approximation for $P_{PMRW}^{S}(m,\infty,IC)$ as

$$P_{PMRW}^{S}(m,\infty,IC) \approx \frac{m}{2^{m-1}}\left[1+\sum_{k=1}^{\frac{m-1}{2}} \frac{(m-1)!\Theta^k}{(m-1-2k)!k!\prod_{i=0}^{k-1}(1-4i\Theta)}\right], \tag{4.57}$$

where $\qquad \Theta = Sin^{-1}(1/3)/\pi$.

4.4 Other Related Results

Gehrlein and Fishburn (1979a) developed an approximation to minimize the maximum absolute percentage deviation of the approximation from known values of $P_{PMRW}^{S}(m,n,IC)$ that were available at that time, including the limiting $P_{PMRW}^{S}(m,\infty,IC)$ values from Niemi and Weisberg (1968). Somewhat later, Gehrlein (1999b) extended the list of known $P_{PMRW}^{S}(m,n,IC)$ values and re-evaluated that approximation, using all of the known values that are listed in Tables 4.6 and 4.7. The resulting approximation is given by

$$P_{PMRW}^{S}(m,n,IC) \approx \frac{9.34}{m+9.71} + .18(.64)^{\frac{m-3}{2}} + \frac{\frac{82.46}{m+156.22} - Ln\left(1+.56^{\frac{m-1}{2}}\right)}{n-1-.88^{\frac{m-1}{2}}}. \tag{4.58}$$

The approximation in Eq. 4.58 was found to have an absolute percent error of 0.45% or less over the range of all of the computed values of $P_{PMRW}^{S}(m,n,IC)$ that are listed in Tables 4.6 and 4.7 with $m \le 49$. All three decimal place entries in Table 4.6 were computed with this approximation. A number of simpler functional forms were tested, but none could match the accuracy of the approximation that is shown in Eq. 4.58.

4.4.1 Spatial Models

Many studies have been performed to evaluate the probability that a PMRW exists, and to consider factors that affect the likelihood that a PMRW exists by using *spatial models*. Chamberlin and Cohen (1978) present an excellent study that is based on such a model. Each candidate is evaluated by voters on the basis of k different characteristics or issues that are of interest, and the position of any candidate with regard to any given characteristic is determined by some continuous measure of that characteristic. Each dimension in k-space then represents a characteristic, and any given point in that space represents a specific position on all of the k characteristics. Candidates are then positioned at the point in the k-space to represent their stand across all of the characteristics. Voters are then positioned at their ideal point in the space to represent their particular position on the characteristics. The ranked preferences on candidates for a given voter are then based on the relative Euclidean distance between that voter's ideal point and the position points of the various candidates in the k-space. This process follows the arguments in Chapter 3 by translating cardinal utilities to ordinal candidate rankings.

By manipulating the relative positions of candidates in the k-space to consider various situations, and then randomly generating position points for each of the voters in a Monte-Carlo simulation, Chamberlin and Cohen (1978) obtain random voter preference profiles that are used to estimate the probability that a PMRW exists in various situations. The basic conclusions of the study suggest that there are significant differences in estimates of the probability that a PMRW exists as the candidates' positions are placed in different configurations. When candidates tend to be clustered near each other, the smallest probabilities that a PMRW exists are observed. This situation would tend to result in the random generation of voters' preference rankings on candidates that are most consistent with IC.

As the positions of candidates are intentionally fixed to create significant dispersion among their relative positions, more candidates are located in fringe positions. This would suggest situations in which the fringe candidates would typically only be ranked near the top or near the bottom of voters' preference rankings, depending on the general part of the space in which the voters' ideal points were randomly positioned. This type of situation would tend to be very different than IC, and would be more in accord with the notions of Sen's class war model. It is found that as the dispersion among candidates' positions increases, there are significant increases in the probability that a PMRW exists.

Some researchers are skeptical of the results that are based on these spatial models. Skog (1994) presents the primary criticism of such studies, by stating that the requirements that such a model places on the degree of precision that is required of voters to evaluate position points of their own preferences and of candidates' positions is unrealistic. This criticism is particularly asserted for comparisons that must be made between candidates that are relatively close together in the criteria space. The process by which individuals might go about making pairwise comparisons between candidates in such situations is the topic of the final chapter of the current study.

Dutter (1982) does an empirical study of election results from Northern Ireland. The main purpose of the study is to determine if voters perform comparisons of candidates based on spatial models or based on "lexicographic models" that are very much like ordinal ranking comparison models. The results indicate patterns of voting behavior that are consistent with both models in different elections. Dutter warns that if one proceeds with the assumption that voters have preferences that are exclusively based on spatial models, when the population actually has lexicographic preferences or a mix of both, the wrong conclusions could be drawn. The same warning would obviously apply equally for the reverse situation.

Spatial models clearly serve a very useful function, and many interesting studies use them as a basis for analysis. These models are not discussed at length in the current study for two reasons. First, these models are primarily only useful for Monte-Carlo simulation based analysis. And, studies based on spatial models and their analysis of topics related to the likelihood that a PMRW exists could be the basis of a complete book on its own. See Merrill (1988) for example, which does an excellent job of summarizing and integrating a number of Monte-Carlo simulation studies that are based on spatial model analysis of election outcomes. As a result, we primarily focus our attention in the current study on the development of probability representations for the likelihood that a PMRW exists that are based on ordinal rankings.

4.4.2 Supermajority Rules

Supermajority rules have been applied in a number of different situations when the resulting decisions of an election are viewed as being of extreme importance. Consider an election between Candidates A and B with n voters where $N(A \succ B)$ denotes the number of voters who prefer A to B in a voter preference profile. Candidate A will only be viewed as superior to B under a supermajority relation $\boldsymbol{M}^{\boldsymbol{\tau}}$ if $N(A \succ B)/n \geq \tau$ where $\tau > 1/2$. Simple majority rule corresponds to the case where $\tau = 1/2$. The general underlying notion has been that τ should increase as the importance of the decision increases.

Colomer and McLean (1998) examined the history of voting procedures that were used to elect popes in the past. A 2/3 supermajority rule for voting cardinals was enacted by Pope Alexander III in 1179. The intent of imposing this rule was to require that a large coalition of voting cardinals had to be formed in order to achieve the election of a mutually agreeable candidate. The elected pope would then likely arise as a result of compromise among the supporters of other candidates, which would tend to lead to a stable situation after the election was complete. Any minority coalition that supported a losing candidate would be faced with the prospect of having to persuade a majority of the winning coalition cardinals to change their votes in favor of the losing coalition's candidate. Faced with such a formidable task, it would seem that any losing coalition would give up opposition to the final outcome. The enforcement of this 2/3 majority rule did result in a much more stable situation after elections, but problems accompanied its use.

The process of requiring a coalition of 2/3 of voting cardinals to reach a mutually agreeable decision could drag on for long periods of time. Elections in 1216, 1241, 1243, 1261 and 1265 took several months to reach a decision. An election that started in 1268 lasted for more than two years without a decision. Colomer and McLean (1998) report a number of interesting anecdotes related to drastic measures that were employed to speed up the decision processes in these cases.

Lines (1986) and Coggins and Perali (1998) discuss a prolonged process by which elections were held to elect the Doge, or Duke, of Venice. The position was a lifetime appointment, and once a doge died, the government came to a standstill until the next doge was elected. A series of nominations for electors, with lotteries and supermajority voting being used at different stages, ultimately led to the selection of a voting body of 41 members. The election of a doge finally resulted when the final committee of 41 members voted approve, uncertain or disapprove on each candidate, and at least 25 electors voted approve for a candidate, for a 61 percent supermajority approval rating. The system was used in Venice for about 500 years. While the process of electing a doge led to significant periods of time in which the government was at a standstill, it is credited with explaining the stable political climate in the area during the period in which it was used.

Nitzan and Procaccia (1986) consider voting rules as they are related to notions of corporate governance. Of particular interest is a reliance on supermajority rules at shareholder meetings, where shareholders typically have one vote for each share of stock that they own. The English Companies Act is cited as requiring the use of simple majority rule for voting at any general meting of shareholders when "ordinary resolutions" are being considered. However, "special resolutions" and "extraordinary resolutions" that can require a significant change in corporate operations require a 75 percent supermajority for passage. The California Corporation Code does not distinguish between "ordinary resolutions" and "extraordinary resolutions" and only requires simple majority voting in all cases. However, the majority reference is to a majority of shareholders of all outstanding stock, whether present at the shareholders meeting or not, which typically imposes a supermajority requirement on the shareholders who are present at any meeting.

Gehrlein and Kher (2004) consider the application of supermajority rules by the Academy of Motion Pictures Arts and Sciences in making decisions that are related to some awards that it gives out. In this particular case, a nominee is selected for an award by a committee and a supermajority vote from all Academy members is then required for the award to be given. In this way, the award has a higher prestige associated with it, and there is no disruption to the Academy if a supermajority of members does not vote in favor of granting the award to the nominee, since the award is simply not given out during the year when such an outcome occurs.

Wickström (1986) presents a survey of work that is related to the notion that the required size for the margin of victory by supermajority should increase as the relative importance of the associated decision increases. The conclusion of the study is that this notion is only valid if two assumptions are simultaneously met. In particular, voters must be risk averse, and the so-called important issues must have greater variance, or more uncertainty, associated with them, relative to their net benefit to the voters.

It is interesting to note that a *Pairwise Supermajority cycle* ($\mathrm{PM}^{\tau}\mathrm{R}$ cycle), can still exist when $\mathrm{PM}^{\tau}\mathrm{R}$ is used instead of PMR. Weber (1993) gives a simple proof of a result from Greenberg (1979) that determines the necessary conditions for a $\mathrm{PM}^{\tau}\mathrm{R}$ cycle to exist. For $n \geq 2$ and $m \geq 2$, a $\mathrm{PM}^{\tau}\mathrm{R}$ cycle can exist if and only if

$$\tau \leq \frac{m-1}{m}. \tag{4.59}$$

Coughlin (1981, 1986) considers supermajority rules in the context of "δ-relative majorities". We only have $N(A \succ B) + N(B \succ A) \leq n$ in this case since some voters might not have strict preferences on the pair. Then A beats B by a δ-relative majority if $N(A \succ B) > \delta N(B \succ A)$. It is shown that voting outcomes with δ-relative majority voting will be transitive, with the possibility of ties, if and only if

$$\delta \geq \frac{\left[\frac{mn}{(m+1)}\right]^{-}}{\left[\frac{n}{(m+1)}\right]^{+}}. \tag{4.60}$$

Here, $[x]^{-}$ denotes the largest integer that is less than or equal to x and $[x]^{+}$ denotes the smallest integer that is greater than or equal to x.

Caplin and Nalebuff (1988) also examine ranges of τ that prevent the existence of $\mathrm{PM}^{\tau}\mathrm{R}$ cycles. Linear preference rankings on candidates are assumed to exist for voters, and the preference rankings result from a spatial model. An assumption is made regarding a convexity condition on the distribution of the density of voter's most preferred points in the attribute space. This convexity condition precludes the existence of societies with some types of preference structures, such as with polarized preferences. The conditions that are assumed in the study lead to the conclusion that $\mathrm{PM}^{\tau}\mathrm{R}$ cycles are precluded with a majority rate

$$\tau = 1 - \left(\frac{n}{n+1}\right)^{n}. \tag{4.61}$$

As n increases, τ increases monotonically to a limiting value of $\tau = 1 - \frac{1}{e}$, so that a value of τ that is approximately equal to 64 percent would then always effectively preclude the possibility of a $\mathrm{PM}^{\tau}\mathrm{R}$ cycle, given the conditions specified in the study.

A number of papers have also considered the probability that a $\mathrm{PM}^{\tau}\mathrm{R}$ cycle will be observed. Buckley and Westen (1974) conjectured that the probability that a $\mathrm{PM}^{\tau}\mathrm{R}$ cycle will be observed decreases as τ increases. Monte-Carlo simulation results under IC support this conjecture and also indicate that the probability of observing a $\mathrm{PM}^{\tau}\mathrm{R}$ cycle approaches zero very quickly for large electorates with m at all large for τ only marginally greater than 1/2.

Balasko and Crès (1997) evaluate the probability that $\mathrm{PM}^{\tau}\mathrm{R}$ is transitive for m-candidate elections. The study defines the probability of observing voting situations in terms of volumes of an $m!$-dimensional simplex, which we have already seen to be equivalent to considering the limiting case of IAC as $n \to \infty$. A representation is obtained for the upper limit of the probability that there is a $\mathrm{PM}^{\tau}\mathrm{R}$ cycle as $Y(m,\tau)$, with

$$Y(m,\tau) = m!\left(\frac{1-\tau}{.4714}\right)^{m!}, \tag{4.62}$$

which becomes remarkably small. For example, with $\tau = .54$ and $m = 7$, the relative volume of the subspace containing voter preference profiles with $\mathrm{PM}^{\tau}\mathrm{R}$ cycles is less than 10^{-52}. In general, $\mathrm{PM}^{\tau}\mathrm{R}$ cycles are shown to be rare events for m at all large with $\tau \geq .53$. Black (1969) considered the case of unanimity rule, with $\tau = 1$, and found that the probability that any such $\mathrm{PM}^{\tau}\mathrm{R}$ cycle exists is so small that it can be disregarded, in agreement with the observation in Eq. 4.62.

Grofman (1972) analyzes a variation of this problem by considering the existence of a set of candidates, $C(j,i)$, who would receive at least j votes in PMR contests against each of at least $m-1-i$ of the other $m-1$ candidates with odd n voters. Obviously, if $\#C\left(\frac{n+1}{2},0\right)=1$ then the candidate in $C(j,i)$ is the strict PMRW, and if $\#C\left(\frac{n+1}{2},0\right)=0$ then there is no strict PMRW. Grofman takes the reverse option of requiring a supermajority $\mathrm{PM}^{\tau}\mathrm{R}$ and considers the submajority PMR with $j \leq \frac{n-1}{2}$ that is necessary to ensure that $\#C(j,0) > 0$. It is proved for linear voter preference rankings on candidates that

Theorem 4.27. If $\dfrac{nm(m-1)/2 - m(j-1)}{m(m-2)} > n$, then $\#C(j,0) > 0$ for $j \leq \dfrac{n-1}{2}$.

4.4.3 Condorcet Committees

All analysis to this point has been focused on elections that are trying to select a single winner. When the problem changes so that a group of voters is trying to elect a committee, a number of new paradoxes that are related to this particular problem can be observed. For example, see Staring (1986) and Mitchell and Trumbull (1992). Arguments over the way in which the notions behind PMR should be extended to the case of electing members of a committee have a long history. Dodgson (1884, 1885a, 1885b) was involved in a dispute with the "Society for Proportional Representation" regarding this very issue.

Dodgson gives an example in which a group is trying to elect a committee of three members from five candidates. In this example, the candidates are Chamberlain (A), Gladstone (B), Goschen (C), Hartington (D), and Northcote (E). The voting outcome from an election gives the voter preference ranking on candidates that are shown in Fig. 4.2.

B	D	B	A	C	E
D	B	A	B	B	-
C	C	D	D	D	-
A	A	C	C	A	-
E	E	E	E	E	-
3030	2980	2020	1100	790	2079

Fig. 4.2 Example voting situation from Dodgson (1885a)

Dodgson notes that Candidates A, B, C and D are liberal candidates, and that E is a conservative candidate. The fact that the 2079 conservative electors only rank their candidate would suggest that they are indifferent, or equally unhappy, with the possibility of any of the other candidates.

Dodgson argues that A, B and D should be elected to the committee "as a matter of justice", by comparing pairs of candidates for entry. The logic is that the pair B and D are obvious selections, since 6010 of the 11999 voters rank B and D first. Next, Dodgson argues for the inclusion of A as the third member of the committee, since "over and above these" 6010 voters, we have 3120 voters who place the pair A and B as their two most preferred candidates. Dodgson goes on to show that the system that was proposed by the Society for Proportional Representation would have elected B, C and D. Dodgson states that the election of Goschen (C) "would bring in the wrong man".

The Society for Proportional Representation responds that the election of Goschen (C) by their procedure in Dodgson's example is, in fact, the proper choice over Chamberlain (A). Their argument against Dodgson is that there are 9920 "liberal electors" in this example for whom a preference comparison between A and C are known. Of these 9920 electors, 6800 prefer C to A, with only 3120 preferring A to C. Thus, Goschen (C) should be the winner, based on a direct PMR comparison between candidates A and C. Thus, the argument of the Society for Proportional Representation is based on the notion proposed in the definition of a Condorcet committee based on individual comparisons of candidates who are in the committee versus not in the committee. Dodgson responds to the criticism of the Society with an example to show that it is possible to continue their logic and have PMR cycles in entry and removal of candidates from the elected set.

The first definition of a *Condorcet Committee* that we consider follows from the basic idea of the argument that was presented by the Society for Proportional Representation regarding the entry of the third candidate to the committee is given in Gehrlein (1985). Let $C^W \subset \boldsymbol{C^m}$ denote a possible subset of candidates to be elected to a committee. Then, C^W is a Condorcet committee if $C_i \boldsymbol{M} C_j$ for all $C_i \in C^W$ and all $C_j \in \boldsymbol{C^m} \setminus C^W$. Previous discussion makes it clear that a Condorcet committee does not necessarily exist according to this definition for a specified $\# C^W$ with a given n and m. However, since PMR is transitive when voters have single-peaked preferences or dichotomous preferences, a Condorcet committee will always exist according to this definition whenever either of these restrictions is assumed to hold.

Felsenthal and Machover (1992) develop the same definition for a Condorcet committee as in Gehrlein (1985), and they suggest that this definition is valid when the goal is to select the PMRW as a single winner, but that it might not be effective when the goal is to select a committee that reflects a "microcosm of society". Hill (1988) previously made a similar observation. Numerous studies have been conducted to develop methods to choose committees that would tend to more accurately reflect the mix of preferences of the population that the committee will represent. For example, see Good and Tideman (1976), Chamberlin and Courant (1983) and Benoit and Kornhauser (1994).

Fishburn (1981a,b) develops a second definition of a Condorcet committee that is based on the notion of PMR. In these studies, attention is moved from directly considering the relative position of individual candidates in voters' preference rankings. Instead, a determination is made of what the preference rankings of voters would be on the combinations of candidates in all possible committees with a specified number of members, given the voters' preference rankings on individual candidates. A Condorcet committee is then determined on the basis of elections that would be performed by having PMR comparisons between possible pairs of committees with the same number of members in each committee, with the Condorcet committee being defined as the possible committee of a given size that is preferred by PMR to all other committees of the same size.

Fishburn (1981a) makes some interesting observations when considering this definition of a Condorcet committee when individual voters have dichotomous preferences on candidates. For any given voter, $H(\succ)$ denotes the subset of candidates among the more preferred candidates and $L(\succ)$ denotes the subset of less preferred candidates in the voter's dichotomous preference order. Some mechanism is required to determine how each voter would then rank committees of a specified size, k, given their preferences on the candidates. Fishburn defines this mechanism as *Condition P*. Let C^X and C^Y denote two possible committees of k candidates. Then, Condition P is defined on a given voter's pairwise preference on committees such that

$$C^X \succ C^Y \Leftrightarrow \#\{C^X \cap H(\succ)\} > \#\{C^Y \cap H(\succ)\}. \tag{4.63}$$

That is, a voter will prefer committee C^X to C^Y if C^X contains more candidates in the voter's more preferred set of candidates than committee C^Y does.

Unlike the results obtained by Inada (1964) for the election of a single candidate, Fishburn (1981a) gives an example on four candidates $\{A,B,C,D\}$ in which voters with dichotomous preferences have PMR cycles on committees when Condition P determines individual voter's preferences on the committees. This PMR cycle refers to a majority of voters who actually have a preference on a given pair of committees, since Condition P allows for voter indifference between two committees. Voters who are indifferent between pairs of committees are assumed to abstain from voting in that particular PMR comparison. In this example, the individual voters' dichotomous preferences on candidates are shown in Fig 4.3.

Voter Type	$H(\succ)$	$L(\succ)$	Number of Voters
1	*AB*	*CD*	3
2	*C*	*ABD*	2
3	*D*	*ABC*	2

Fig. 4.3 Example voter preference profile with dichotomous preferences from Fishburn (1981a)

We see, for example, that committee $\{A,B\}$ has a three voters to two majority over committee $\{A,C\}$. This results under Condition P with the three voters of Type 1 preferring $\{A,B\}$ to $\{A,C\}$ and the two voters of Type 2 preferring $\{A,C\}$ to $\{A,B\}$. Voters of Type 3 are indifferent between $\{A,B\}$ and $\{A,C\}$ and do not vote for this particular PMR comparison of committees. Using the same logic, we find that $\{A,C\}$ has a three to two majority over $\{C,D\}$, with voters of Type 2 not voting. Then, the cycle is complete with $\{C,D\}$ having a four to three majority over $\{A,B\}$.

Fishburn (1981b) shows that imposing the condition of single-peaked preferences on voter's preferences on individual candidates is insufficient to ensure the existence of a majority committee, for $k > 1$. The study considers the additional restrictions that are required on individual voter's preferences on candidates to ensure the existence on a Condorcet committee. A Condorcet committee must exist when voters have single-peaked preferences on candidates; with the additional restriction that each voter must also have the same most preferred candidate in his or her preference ranking. Thus, the conditions that require the existence of a PMRW in single-candidate elections fail to be sufficient to require the existence of a Condorcet committee of more than one member, given Fishburn's definition of a Condorcet committee.

Several studies have been conducted to consider various aspects of these two definitions of a Condorcet committee. For example, see Kaymak and Sanver (2003) and Ratliff (2003). A general conclusion seems to be that the definition of a Condorcet committee from Fishburn (1981a,b) is more appropriate in situations in which committee members are expected to reflect a "microcosm" of the society that it is supposed to represent. The definition from Gehrlein (1985) is more appropriate if the elected committee represents a list of candidates that are to be passed along for further deliberation that will lead to the selection of the final winning candidate from that set. Barberà and Coelho (2004) compare the two definitions of a Condorcet committee and they formulate "Random Chooser Game" in which agents act strategically and cooperatively. Using this game as a basis, it is shown that when any procedure that meets some basic restrictions is used to select candidates to a committee, a set of candidates in a committee can be a strong Nash equilibrium outcome only if the meets the definition of a Condorcet committee from Gehrlein (1985).

A limited amount of work has been done to develop representations for the probability that a Condorcet committee exists, with either definition of the term. Gehrlein (1985) does present results for the probability, $P^S_{CC(k)}(m,n,IC)$, that a Condorcet committee with k members exists for n voters with m candidates under the assumption of IC. The results refer to the definition of a Condorcet committee as defined above in reference to that study.

Since every voter preference profile has the same probability of being observed as its dual voter preference profile

$$P^S_{CC(k)}(m,n,IC) = P^S_{CC(m-k)}(m,n,IC). \tag{4.64}$$

For the special case that $k = 1$,

$$P^S_{CC(1)}(m,n,IC) = P^S_{CC(m-1)}(m,n,IC) = P^S_{PMRW}(m,n,IC). \tag{4.65}$$

The discussion that led to the development of Eq. 4.22 for odd n leads to

$$P^S_{CC(2)}(4,n,IC) = P^S_{PMRT}(4,n,IC). \tag{4.66}$$

To develop more general relationships for $P^S_{CC(k)}(m,n,IC)$, we follow the development of Eq. 4.24 and define $k(m-k)$ discrete variables of the form $X^i_{j,\ell}$ for the i^{th} individual voter's preferences that will be used to obtain the joint probability, $Q^S_{CC(k)}(m,n,IC)$, that $C_j \boldsymbol{M} C_\ell$ for each $1 \le j \le k$ and $k+1 \le \ell \le m$ in a random voter preference profile, with

$$X^i_{j,\ell} = \begin{array}{l} +1: \text{if } C_j \succ C_\ell \text{ for the } i^{th} \text{voter} \\ -1: \text{if } C_\ell \succ C_j \text{ for the } i^{th} \text{voter.} \end{array} \tag{4.67}$$

A representation for $Q^S_{CC(k)}(m,n,IC)$ can be obtained as the joint probability that $X^i_{j,\ell} > 0$ for each $1 \le j \le k$ and $k+1 \le \ell \le m$. With the assumption of IC, it is easily shown that $E\left(X^i_{j,\ell}\right) = 0$. Previous arguments that were based on the Central Limit Theorem have shown that the limiting distribution $Q^S_{CC(k)}(m,\infty,IC)$ as $n \to \infty$ is equivalent to the multivariate-normal positive orthant probability, $\Phi_{k(m-k)}(\boldsymbol{R}(\boldsymbol{m},\boldsymbol{k}))$, with $k(m-k)$ variables that $\overline{X}_{j,\ell}\sqrt{n} \ge E\left(\overline{X}_{j,\ell}\sqrt{n}\right)$ for each $1 \le j \le k$ and $k+1 \le \ell \le m$. The correlation matrix for this multivariate normal distribution, $\boldsymbol{R}(\boldsymbol{m},\boldsymbol{k})$, can be generalized from the form of $\boldsymbol{R^1}$, which has m = 4 and k = 2, in Eq. 4.25. The correlation term between the pair of variables $X^i_{a,b}$ and $X^i_{c,d}$ is equal to 1/3 if either a = c or b = d. Otherwise, the correlation between the pair of variables is zero.

There are $\binom{m}{k}$ different combinations of candidates that could form a Condorcet committee of k candidates, and the symmetry of IC with respect to candidates leads to

$$P^S_{CC(k)}(m,\infty,IC) = \binom{m}{k} Q^S_{CC(k)}(m,\infty,IC) = \binom{m}{k} \Phi_{k(m-k)}(\boldsymbol{R}(\boldsymbol{m},\boldsymbol{k})). \tag{4.68}$$

Precise analytical representations for $P^S_{CC(k)}(m,\infty,IC)$ become intractable for $m \ge 5$, so Gehrlein (1985) obtains Monte-Carlo simulation estimates for values of $\Phi_{k(m-k)}(\boldsymbol{R}(\boldsymbol{m},\boldsymbol{k}))$, and then obtains associated estimates of $P^S_{CC(k)}(m,\infty,IC)$. The simulation estimates for $\Phi_{k(m-k)}(\boldsymbol{R}(\boldsymbol{m},\boldsymbol{k}))$ are obtained by using a process from Naylor, et al. (1966) to generate 15,000 random observations from a $k(m-k)$ variable normal distribution with correlation matrix $\boldsymbol{R}(\boldsymbol{m},\boldsymbol{k})$. Each observation was checked to determine if each of the $k(m-k)$ values were positive, which would place that random observation in the positive orthant of the distribution. Then the estimate of $\Phi_{k(m-k)}(\boldsymbol{R}(\boldsymbol{m},\boldsymbol{k}))$ is obtained as the proportion of the 15,000 observations that fell into the positive orthant. Table 4.9 lists the estimates for $P^S_{CC(k)}(m,\infty,IC)$ for each $k \le m-1$ with $m = 3,4,5,6,7$ from Gehrlein (1985). The results in Table 4.8 suggest that $P^S_{CC(k)}(m,\infty,IC)$ decreases as k increases for the range $1 \le k \le m/2$.

Table 4.9 Monte-Carlo simulation estimates for $P^S_{CC(k)}(m,\infty,IC)$ from Gehrlein (1985)

	m				
k	3	4	5	6	7
1	.916	.837	.716	.692	.641
2	.938	.736	.575	.483	.410
3	--	.824	.598	.437	.350
4	--	--	.750	.479	.312
5	--	--	--	.656	.450
6	--	--	--	--	.628

4.4.4 Linear Extension Majority Cycles

Fishburn (1974c, 1976b, 1986) develops the notion of a *Linear Extension Majority Cycle* (*LEM Cycle*). To describe the phenomenon, we start by defining a *partial order*, ***S***, in the context of the pairwise preferences of a society, where $C_i \boldsymbol{S} C_j$ denotes that the society prefers C_i to C_j. Obviously, no candidate can be preferred to itself. Suppose that not all of the societal preference relationships are known for all of the pairs of candidates in the Cartesian product $\boldsymbol{C^m} \times \boldsymbol{C^m}$ in an m-candidate election. The relationship between C_i and C_j is unknown, denoted by $C_i U C_j$, if neither $C_i \boldsymbol{S} C_j$ nor $C_j \boldsymbol{S} C_i$. The event $C_i U C_j$ does not necessarily mean that society is indifferent between C_i and C_j, but means that the societal preference is not known for that pair. Then, ***S*** is a partial order if it is transitive, but is not necessarily a weak order. That is, it might be known that $C_i \boldsymbol{S} C_j$, while $C_i U C_k$ and $C_k U C_j$ for some other candidate C_k, so that the U relationship on pairs of candidates is not transitive, as it would be if ***S*** were a weak order.

A linear extension L of ***S*** is a linear order with $\boldsymbol{S} \subseteq L$, and $\Lambda(\boldsymbol{S})$ is the set of all possible linear extensions of a given ***S***. For any two candidates $C_i, C_j \in \boldsymbol{C^m}$, $L(C_i, C_j)$ is defined as the subset of $\Lambda(\boldsymbol{S})$ such that if $L' \in L(C_i, C_j)$ then $C_i L' C_j$, and it follows from these definitions that $L(C_i, C_j) \cup L(C_j, C_i) = \Lambda(\boldsymbol{S})$ and $L(C_i, C_j) \cap L(C_j, C_i) = \phi$, since all such L' are linear orders. The LEM relation, $\boldsymbol{M}^*$, on $\boldsymbol{C^m}$ is obtained by following the same logic as PMR, with $C_i \boldsymbol{M}^* C_j$ if $\#L(C_i, C_j) > \#L(C_j, C_i)$. Fishburn (1974c, 1976b, 1986) gives examples to show that LEM cycles can exist, such that $C_i \boldsymbol{M}^* C_j \boldsymbol{M}^* ... \boldsymbol{M}^* C_k \boldsymbol{M}^* C_i$ for large numbers of candidates. Gehrlein and Fishburn (1990a) show that LEM cycles can exist for all $m \geq 9$. Ewacha, et al. (1990) and Gehrlein and Fishburn (1990b) consider the existence of LEM cycles on specializations of partial orders and conditions on partial orders that prohibit the existence of LEM cycles. Gehrlein (1991a) does a Monte-Carlo simulation analysis to support the hypothesis that the probability that a randomly generated partial order will lead to a LEM cycle increases rapidly as m increases.

4.4.5 Geometric Models

The notion of using arguments that have some form of a geometric basis to analyze elections in terms of the probabilities that various events occur has a long history. Black (1958) makes numerous references to results that are based on geometric proofs, in discussing the use of PMR in voting with complementary elections. Tullock (1967) uses geometric arguments with spatial modeling on two dimensions to consider the possibility that a PMRW exists with a large electorate.

Saari (1995b) presents a summary of much of his extensive work that is based on geometric approaches to problems that are related to voting events. That study, along with numerous related articles, provides many valuable insights regarding paradoxical voting events. Most of this work is related to the consideration of the general behavior of weighted scoring rules, with particular emphasis on the superiority of Borda Rule. Saari and Tataru (1999) extend these geometric arguments to develop probability representations for various voting events in the limiting case as $n \to \infty$. Other related studies followed in Merlin and Tataru (1997) and in Merlin, et al. (2000, 2002). These studies are based on finding probabilities as volumes of various multi-dimensional spaces, starting with first principles in sources such as Coxeter (1935) and Schläfli (1950). These studies have produced some very nice results, in using what has come to be referred to as the *geometric approach* to the problem.

The approach that has been used to obtain limiting probabilities for voting events as $n \to \infty$ in the current study is generally referred to as using the *traditional approach*. As we have seen, the traditional approach is based on defining a set of variables, and applying the Central Limit Theorem to define probability representations as multivariate normal orthant probabilities with specified correlation matrices. Known forms of representations for these orthant probabilities from numerous other well established studies are then used to obtain final representations. Two studies that are frequently used as a basis in the traditional approach to obtain limiting probability representations are due to Plackett (1954) and Slepian (1962), and both of these studies are based on first principles from Coxeter (1935) and Schläfli (1950). As a result, the traditional approach typically obtains simpler probability representations, since it is based on using well established studies that are geared toward obtaining simple representations, rather than starting from first principles to obtain representations in every case.

The use of the geometric approach to obtain probability representations for voting events has also led to the development of incorrect results in a few instances, since it is quite easy to fall into the trap of applying it to situations in which it is not applicable. In particular, the geometric approach is typically only applicable in situations in which probability distributions for observing voting situations have a spherical symmetry, as in the limiting case with IC. The approach will not always work in other cases, such as the limiting case with IAC. For example, see the work of Van Newenhizen (1992) as compared to Cervone, et al. (2005), or Saari and Valognes (1999) as compared to Bezembinder (1996).

The use of the geometric approach to consider the *relative* probabilities that various election events are observed as parameters change has produced some remarkable results. However, the traditional approach is typically more applicable to the process of obtaining simple probability representations for election outcomes in most cases.

4.5 Conclusion

There has been a significant interest over the years in developing representations both for the probability that a PMRW exists and for the probability that PMR is transitive. This interest is largely driven by the belief that the probability that a PMRW exists will go to zero as the number of candidates becomes large under the condition of IC. Given all of this effort, it is still conjecture that $P_{PMRW}^{S}(m,n,IC)$ decreases as m increases with a given n, and decreases as n increases for odd n with a given m. However, representations have been successfully obtained for $P_{PMRW}^{S}(m,n,IC)$ for small m, for small n, and for the limiting case as $n \to \infty$. It has also been shown that $P_{PMRW}^{S}(m,n,IC) \to 0$ as $m \to \infty$ for all n. Tables 4.6 and 4.7 list all of the known values for $P_{PMRW}^{S}(m,n,IC)$ for odd n, and these values all reinforce the notion that $P_{PMRW}^{S}(m,n,IC)$ decreases as m increases with a given n, and decreases as n increases for odd n with a given m.

Substantial evidence has been presented to verify the fact that values of $P_{PMRW}^{S}(m,n,IC)$ and $P_{PMRW}^{S}(m,n,IAC)$ converge to the same values very quickly as m increases. As a result of the known values of $P_{PMRW}^{S}(m,n,IC)$ in Tables 4.6 and 4.7, we can conclude that the probability that a PMRW exists for large electorates, as $n \to \infty$, is such that Condorcet's Paradox is a real threat to elections with a relatively large number of candidates if either IC or IAC is a valid assumption. We have already stressed that IC and IAC should be expected to produce exaggerated probability estimates that Condorcet's Paradox will be observed. However, the degree to which these assumptions exaggerate that probability is not known.

The next step of our analysis of this problem proceeds to consider the impact that the presence of various societal factors and the requirement of an additional degree of coherence of preference among members of the electorate will have on the probability that a PMRW exists.

5 The Impact of Societal Factors

5.1 Introduction

The previous chapter showed that the probability of observing PMR cycles could be a real threat to election processes with more than a few candidates, if conditions like IC and IAC reflect the true model that generates voter preference profiles or voting situations from a population of voters. The survey of empirical findings in Chapter 2 indicates that PMR cycles have indeed been observed in actual voting situations, but that they are not a common phenomenon. Typical explanations for this observation have hinged on the idea that assumptions like IC and IAC give too much weight to possible voting outcomes that are very unlikely to ever be observed if the population of voters has some coherent underlying structure to the process by which the preferences of individual voters are formed. We have already seen, for example, that the imposition of the condition of single-peaked preferences over the preferences of all voters in a population would be sufficient to result in a situation in which a PMRW must exist for any subset of actual voters taken from that population.

The general idea that there should be a connection between group coherence and the probability that a PMRW exists was alluded to by Tullock in some early studies. Campbell and Tullock (1965) use Monte-Carlo simulation analysis to obtain estimates of $P^S_{PMRW}(m,n,IC)$. The authors concluded that the existence of PMR cycles is clearly an important phenomenon when voters have independent preferences, as IC implies. Tullock (1967) later uses geometric arguments with spatial modeling on two dimensions to consider the possibility that PMR cycles are observed with a large electorate, and Tullock speculated that this situation should tend to maximize the probability that PMR cycles might exist. It is noted that the model that is used in the study creates a degree of interdependence between voter's preferences, and it is shown that there would be a very small probability of observing PMR cycles as a result.

Many studies have been conducted to evaluate the impact that various measures of the consistency of preference, or group coherence, of a population will have on the probability that a PMRW exists. This work has generally considered the consistency of preference of a population in the context of *social homogeneity*. In general, the preferences of a society would be totally homogeneous if every member of that society has the same preference ranking on candidates. The opposite extreme is a situation that reflects the notion of IC, in which the population has preferences that are completely dispersed over all possible preference rankings on candidates.

Berg's application of P-E models to the probability that a PMRW exists was discussed in Chapter 3, where it was shown that IC corresponds to the P-E model with $\alpha = 0$ and IAC corresponds to the P-E model with $\alpha = 1$. The observation that $P^S_{PMRW}(3,n,IAC) > P^S_{PMRW}(3,n,IC)$ has been attributed to the fact that IAC imposes a small degree of dependence among voters' preference rankings while IC does not do so. Berg (1985) generalizes this result to argue that an increase in the α parameter of P-E models that generate random voting outcomes corresponds to an increase in the probability that voting situations with greater degrees of social homogeneity will be observed. It is possible to enumerate all voting situations with a strict PMRW on three candidates, for a given n, and calculate the probability that each is observed with a P-E model for a specified parameter α. The sum of these probabilities, $P^S_{PMRW}(3,n,PE(\alpha))$, corresponds to the expected probability that a strict PMRW exists for the given value of α. Gehrlein (1995) computes values of $P^S_{PMRW}(3,n,PE(\alpha))$ for various odd n over a range of α values, to show that this expected probability does indeed increase as α increases, as shown in Table 5.1.

Table 5.1 Computed values of $P^S_{PMRW}(3,n,PE(\alpha))$ from Gehrlein (1995)

	n					
α	3	5	7	9	11	25
0	.9444	.9306	.9250	.9220	.9202	.9157
1	.9643	.9524	.9470	.9441	.9423	.9387
2	.9750	.9665	.9626	.9604	.9590	.9561
3	.9815	.9753	.9724	.9708	.9698	.9675
4	.9857	.9811	.9789	.9776	.9769	.9750
5	.9886	.9850	.9833	.9824	.9817	.9803
10	.9952	.9838	.9931	.9927	.9925	.9919
15	.9974	.9966	.9963	.9961	.9959	.9956
20	.9983	.9979	.9977	.9975	.9975	.9973
25	.9988	.9985	.9984	.9983	.9983	.9981

This general problem has also been addressed in a very interesting way with models from statistical mechanics. The general connection between voting probabilities and this topic from physics was previously addressed in Chapter 3. In particular, Galam (1997) uses the notion of a random field Ising model to consider factors that cause individuals in a group to have preferences on a pair of candidates that are "polarized" in the physical sense. When preferences are polarized in this context, they are consistently oriented, reflecting homogeneous preferences on the pair. This definition of polarization is the opposite of what is typically implied by the use of the term as related to the preferences of societies. As expected with this model, interactions between voters and small amounts of external social pressure are found to result in significant polarization, or agreement, for voters' preferences on the pair of candidates.

Raffaelli and Marsili (2004) extend this work to develop a representation for the probability that PMR is completely transitive. They describe a random field Ising model on $m(m-1)/2$ components. Each component takes a value of ± 1 to correspond with the ordering of preference that a specified voter has on a given pair of candidates, with there being $m(m-1)/2$ paired comparisons for each voter. The possible outcomes on components are then constrained to force each voter's preferences to be consistent with some linear preference ranking. A representation is then obtained for the likelihood that the PMR relations for the voters are completely transitive with this model.

The notion of interaction between voters' preferences is then introduced in the sense that voters are described as having an increased tendency towards conformity among all voters. When there is no interaction, the system is identical to IC. Results are obtained as $n \to \infty$, and the numerical equivalent of Guilbaud's number from Eq. 3.77 is obtained for the special case of $m = 3$, and the probability that PMR is completely transitive deceases rapidly as m increases, as expected. When interaction is introduced between components, there are some levels of interaction such that the probability that PMR is transitive increases rapidly toward one as m increases. This leads to the counterintuitive result that at higher levels of interaction, increasing m will increase the probability that the PMR relationship is transitive.

Attention is now turned to the consideration of measures of social homogeneity that are more directly related to the parameters of voter preference profiles and voting situations.

5.2 Population Specific Measures of Homogeneity

Population specific measures of social homogeneity are related to parameters of the population that is used to generate random voter preference profiles or voting situations. For three candidates, $A, B, C\}$, these measures are based on the p_i's from the probability vector $\boldsymbol{p}$ that describes the likelihood that a randomly selected voter will have the i^{th} possible linear preference ranking on the candidates. The possible preference rankings on the three candidates and their associated probabilities are shown in Fig. 5.1, which is repeated for convenience from Fig. 3.3.

A	A	B	C	B	C
B	C	A	A	C	B
C	B	C	B	A	A
p_1	p_2	p_3	p_4	p_5	p_6

Fig. 5.1 The probability that a randomly selected voter will have a given linear preference ranking

Population specific measures of homogeneity of this type are partitioned into two categories. *Non-Comparison Population Measures* (*NPM*) do not account for the relative positions of pairs of candidates in the preference rankings of the voters, while *Comparison Population Measures* (*CPM*) do so.

5.2.1 Non-Comparison Population Measures

NCM's are based on the concept that social homogeneity can be gauged by using some measure of the degree of dispersion among the p_i 's in a given $\boldsymbol{p}$ vector. IC has the minimum amount of dispersion among the p_i 's, and it represents one of the forms of balanced preferences, with the minimum amount of social homogeneity. The maximum amount of social homogeneity occurs when all voters have the same linear preference ranking, and this gives the maximum degree of dispersion among the p_i 's. Two studies give credence to applying this general concept.

Jamison and Luce (1972) consider the case with n voters on m alternatives, where $\boldsymbol{p}$ denotes a probability vector over the $m!$ possible linear preference rankings on candidates, with a large population. A voter preference profile is obtained when n voters from the population are sequentially assigned preference rankings according to the probabilities in $\boldsymbol{p}$. It is assumed that the specific $\boldsymbol{p}$ for the population is not known. However, $\boldsymbol{F}(\boldsymbol{p})$ denotes the probability density function over the set, Ω, of possible $\boldsymbol{p}$ vectors, following the same idea that was suggested by Buckley (1975) in the discussion of unconditional probabilities in Chapter 3. Jamison and Luce (1972) assume that $\boldsymbol{F}(\boldsymbol{p})$ has the form of a Dirichlet distribution with parameters $\nu_1, \nu_2, ..., \nu_{m!}$. The ν_i parameters are directly linked to $\boldsymbol{p}$ by the fact that $E(p_i) = \nu_i / \omega$, where $\omega = \sum_{i=1}^{m!} \upsilon_i$. When $p_i = \nu_i / \omega$, Jamison and Luce (1972) note that $P_{PMRT}^S(m, n, \boldsymbol{F}(\boldsymbol{p}))$ approaches $P_{PMRT}^S(m, n, \boldsymbol{p})$ as ω becomes large. For the IC-like situation in which $\nu_i = \nu_j$ for all i and j, the only remaining parameter is ω, so $P_{PMRT}^S(m, n, \boldsymbol{F}(\boldsymbol{p}))$ can then be specified completely in this situation as $P_{PMRT}^S(m, n, \omega)$.

Jamison and Luce argue that ω serves as a measure of social homogeneity, with increasing ω reflecting decreasing homogeneity. This follows from the fact that as ω increases, $\boldsymbol{p}$ become more IC-like, since $p_i = \nu_i / \omega$. For the special case of $m = 3$ and $n = 3$, it is shown that

$$P_{PMRT}^S(3,3,\omega) = \frac{17\omega^2 + 54\omega + 36}{18(\omega^2 + 3\omega + 2)}. \tag{5.1}$$

By taking the derivative of this representation with respect to ω, it is easily shown that there is a positive relationship between social homogeneity and the probability that a PMRW exists for this IC-like situation with $m = 3$ and $n = 3$. It should also be noted that as $\omega \to \infty$ in Eq. 5.1, the precise value from the representation in Eq. 5.1 approaches $P_{PMRT}^{S}(3,3,IC) = 17/18$ from Sevcik (1969) in Table 4.5.

Berg (1985b) considers the impact that biasing the tendency of some candidate to have a plurality majority has on the probability that a PMRW exists in three-candidate elections with a P-E model. The number of balls that are in the urn to begin the experiment to obtain random voter preference profiles with sequential draws have previously been defined with all $A_i = 1$ for each of the six rankings with three candidates with IAC. To bias the population to have preferences that favor of a specified candidate, Berg (1985b) changes the experiment so that the two rankings that have some specified candidate being ranked as most preferred will start out with $\beta + 1$ balls instead of one ball.

As β increases from zero, the two rankings with the selected candidate being listed as most preferred will have a greater likelihood for selection. The concept of IAC is maintained in this experiment since each selected ball is replaced after each draw, along with one additional ball of the same color. It is then shown that the probability that a PMRW exists under this experimental scenario as $n \to \infty$ is given by

$$1 - \frac{(\beta+1)(\beta+2)}{2^{(2\beta+5)}}. \tag{5.2}$$

The particular case in which $\beta = 0$ is equivalent to the limiting probability $P_{PMRW}^{S}(3,\infty,IAC) = 15/16$ as $n \to \infty$, from Eq. 3.30. The probability in the representation in Eq. 5.2 increases dramatically as β increases, so that the smallest probability that a PMRW exists in this case occurs with IAC, which is the least homogenous scenario that is considered under this experiment.

We clearly expect some general relationship between measures of dispersion among the p_i 's and the probability that a PMRW exists. A number of studies have been conducted to evaluate simple measures of this dispersion that act as a gauge of social homogeneity to determine if they display this general relationship to a significant degree. Abrams (1976) considers the homogeneity measure that is given by

$$NPM^1(\boldsymbol{p}) = \sum_{i=1}^{6} p_i^2 . \tag{5.3}$$

This measure is identical to the large population approximation for the "fractionalization index" in Rae (1968). It is maximized when $p_i = 1$ for some *i*, and it is minimized with the assumption of IC. Increased values of $NPM^1(\boldsymbol{p})$ reflect increased homogeneity for a population. With a large value of $NPM^1(\boldsymbol{p})$ for a population, we would expect an increased propensity to observe random voter preference profiles from such a population that have voters' preferences that are clustered around one, or a few, of the possible linear rankings on candidates. As $NPM^1(\boldsymbol{p})$ increases, $P^S_{PMRW}(m,n,\boldsymbol{p})$ should therefore be expected to increase.

Abrams (1976) gives example $\boldsymbol{p}$ vectors for which there are inconsistencies in the expected positive relationship between $NPM^1(\boldsymbol{p})$ and $P^S_{PMRW}(m,n,\boldsymbol{p})$. That is, there are $\boldsymbol{p}$ and $\boldsymbol{p}^*$ with $NPM^1(\boldsymbol{p}) > NPM^1(\boldsymbol{p}^*)$ and with $P^S_{PMRW}(m,n,\boldsymbol{p}) < P^S_{PMRW}(m,n,\boldsymbol{p}^*)$. The specific example that Abrams refers to has $\boldsymbol{p}^*$ equivalent to IC with $m = 3$ and $n = 17$. Here, $P^S_{PMRW}(3,17,IC) = .91733$ with $NPM^1(IC) = .167$. Then, $\boldsymbol{p}$ has $p_1 = 0$ and $p_i = 1/5$ for each $i = 2,3,4,5,6$. For this $\boldsymbol{p}$, we have $P^S_{PMRW}(3,17,\boldsymbol{p}) = .90907$ with $NPM^1(\boldsymbol{p}) = .200$. Thus, while we might expect there to be a general positive relationship between $NPM^1(\boldsymbol{p})$ and $P^S_{PMRW}(m,n,\boldsymbol{p})$, specific counterexamples exist to show that the relationship is not perfect. Paris (1975) indirectly makes a similar observation.

Fishburn and Gehrlein (1980) prove several results that are related to $NPM^1(\boldsymbol{p})$. First, $NPM^1(\boldsymbol{p})$ is minimized by IC over the space of $\boldsymbol{p}$ vectors in DC, and $P^S_{PMRW}(m,n,\boldsymbol{p})$ increases as $NPM^1(\boldsymbol{p})$ increases for $\boldsymbol{p}$ vectors in DC when $NPM^1(\boldsymbol{p})$ is changed by keeping one of p_1, p_2 or p_3 fixed while changing the other two. The example that Abrams (1976) presented to show a violation of the expected relationship between $NPM^1(\boldsymbol{p})$ and $P^S_{PMRW}(m,n,\boldsymbol{p})$ is generalized to show that similar examples can be observed for all odd n. Further analysis suggests that the general expected positive relationship between $NPM^1(\boldsymbol{p})$ and $P^S_{PMRW}(m,n,\boldsymbol{p})$ tends to deteriorate as the number of voters gets very large. Gehrlein (1980) considers the value of $NPM^1(\boldsymbol{p})$ that must be observed in a specific voting situation for the given observation to be categorized as having a statistically significant level of homogeneity.

Gehrlein (1981d) presents results from a Monte-Carlo simulation analysis to evaluate eight different NPM measures, denoted by $NPM^i(\boldsymbol{p})$ for $i = 1, 2, 3, \ldots 8$ with:

$$NPM^2(\boldsymbol{p}) = \sum_{i=1}^{6} p_i^3 \tag{5.4}$$

$$NPM^3(\boldsymbol{p}) = \sum_{i=1}^{6} p_i^4 \tag{5.5}$$

$$NPM^4(\boldsymbol{p}) = Maximum\ p_i\} \tag{5.6}$$

$$NPM^5(\boldsymbol{p}) = Minimum\ p_i\} \tag{5.7}$$

$$NPM^6(\boldsymbol{p}) = NPM^4(\boldsymbol{p}) - NPM^5(\boldsymbol{p}) \tag{5.8}$$

$$NPM^7(\boldsymbol{p}) = \prod_{i=1}^{6} p_i \tag{5.9}$$

$$NPM^8(\boldsymbol{p}) = \prod_{i=1}^{6} \left| p_i - \frac{1}{6} \right|. \tag{5.10}$$

Results of the study indicate that any relationship between these NPM's and the probability that a PMRW exists is very weak, as measured either by the correlation between the two, or by the proportion of sequential randomly generated $\boldsymbol{p}$ vectors that have the their respective $P^S_{PMRW}(m,n,\boldsymbol{p})$ values increasing, or decreasing, as expected by their corresponding measured values of $NPM^i(\boldsymbol{p})$. Any relationship that did exist diminished rapidly as the number of voters increased beyond a relatively small number. May (1971) presents some preliminary analysis with IC to suggest that this observation might have been expected for a very large number of voters.

5.2.2 Comparison Population Measures

CPM's are homogeneity measures that are based on $\boldsymbol{p}$, as are NPM's, but they account for the relative differences between pairs of candidates in the preference rankings of the voters. We begin by measuring the differences between candidates in rankings according to $\boldsymbol{p}$, using the $\Delta(A,B)$ definitions that were developed in conjunction with Eq. 3.62:

$$\begin{aligned} \Delta(A,B) &= p_1 + p_2 + p_4 - p_3 - p_5 - p_6 \\ \Delta(A,C) &= p_1 + p_2 + p_3 - p_4 - p_5 - p_6 \\ \Delta(B,C) &= p_1 + p_3 + p_5 - p_2 - p_4 - p_6\,. \end{aligned} \tag{5.11}$$

Gehrlein (1987) did a Monte-Carlo simulation analysis to compare four CPM's to the NPM's. To avoid confusion, it must be noted that the CPM's were referred to as "profile specific measures" in that study. The four CPM measures, denoted by $CPM^i(\boldsymbol{p})$ are:

$$CPM^1(\boldsymbol{p}) = Max\ \left|\Delta(A,B)\Delta(A,C)\right|, \left|\Delta(B,A)\Delta(B,C)\right|, \left|\Delta(C,A)\Delta(C,B)\right|\} \tag{5.12}$$

$$CPM^2(\boldsymbol{p}) = Min \ \left|\Delta(A,B)\Delta(A,C)\right|, \left|\Delta(B,A)\Delta(B,C)\right|, \left|\Delta(C,A)\Delta(C,B)\right|\} \tag{5.13}$$

$$CPM^3(\boldsymbol{p}) = CPM^1(\boldsymbol{p}) - CPM^2(\boldsymbol{p}) \tag{5.14}$$

$$CPM^4(\boldsymbol{p}) = \frac{1}{2}\begin{bmatrix} (p_5 + p_6 - p_1 - p_2)^2 + (p_2 + p_4 - p_3 - p_5)^2 + \\ (p_1 + p_3 - p_4 - p_6)^2 \end{bmatrix}. \tag{5.15}$$

Here, $CPM^4(\boldsymbol{p})$ is directly associated with Kendall's Coefficient of Concordance that Fishburn (1973) suggests as a measure of social homogeneity in a slightly different context that will be addressed later in the current study.

The results of this Monte-Carlo simulation study clearly show a stronger relationship between these CPM's and the probability that a PMRW exists than that which was observed in the same situation with NPM's. The measure $CPM^4(\boldsymbol{p})$ showed some superiority over all other homogeneity measures, except for $CPM^1(\boldsymbol{p})$. That is, the superiority of $CPM^4(\boldsymbol{p})$ over $CPM^1(\boldsymbol{p})$ is not as significant. An interesting observation is that the percentage of sequential observations of $\boldsymbol{p}$ for which $P^S_{PMRW}(3,n,\boldsymbol{p})$ changes in accordance with the change in $CPM^4(\boldsymbol{p})$ tends to increase as n increases, for the range of n considered. This percentage typically ranges above 75 percent of the observed sequential observations of $\boldsymbol{p}$ vectors.

5.3 Situation Specific Measures of Homogeneity

Situation Specific Measures of Homogeneity (*SSM*) do not measure homogeneity based on $\boldsymbol{p}$ vectors, as the population-based measures do. SSM's are based on the n_i's of particular $\boldsymbol{n}$ vectors for a given voting situation, or on the $\boldsymbol{n}$ vectors that result from accumulating individual preferences in a voter preference profile. Situation specific measures would use the observed proportions, n_i / n, as a substitute for the p_i terms in the population-specific measures. For any particular voting situation, we know with certainty whether a PMRW exists or not. One would therefore expect to have the strongest correlation between social homogeneity and the existence of a PMRW in these situation-specific studies. Any particular population-based measure of social homogeneity is fixed by $\boldsymbol{p}$. But, any given voter situation with n voters could be obtained from an experiment to create voter preference profiles from the population, with some probability for a given $\boldsymbol{p}$. As a result, the same population specific measure of social homogeneity would be associated with many different possible voting situations.

Studies have been conducted to consider the general relationship between SSM's and the probability that a PMRW exists. Lhuilier (1793) presents results in an early study that lead to such an observation. The study starts by proving that a voting procedure that was developed in Condorcet (1789) does not always select the PMRW, as Condorcet had claimed. The study continues to compute the probability of observing some election results, using a different perspective than the one that was used in Condorcet (1785d) in the analysis that led to Eq. 3.18. In particular, Lhuilier discusses the probability that election outcomes are observed with an IC-like assumption that individual voters have a conditional equally likely probability to have given linear preference rankings on three candidates, rather than using Condorcet's notion of having an equal probability of a social outcome with PMR for any particular pair of candidates.

Lhuilier discusses these conditional probabilities for possible individual voter's linear preference rankings on candidates, where these probabilities are conditioned on the fact that the most preferred candidate in each voter's preference ranking is known from the results of a plurality election. Lhuilier performs a probability analysis to examine the combinations of ways in which voters might have the remaining candidates positioned in their preference rankings, given their observed most preferred candidate. The nature of this analysis is based on combinatorial computations that make it clear that Lhuilier is assuming that voters are identifiable. This particular set of assumptions closely reflects the notions of IC, as applied in the context of the possible conditional preference rankings on candidates.

Fishburn (1982) considers the same basic type of problem that was presented in Lhuilier (1793), in which voter preference profiles, with their associated voting situations, are generated with the assumption of IC for three alternatives $A, B, C\}$. Let N_A denote the number of preference rankings in a voting situation for which A is ranked as most preferred. A given combination of N_i's is referred to as a plurality situation. There can be many different voting situations with the same plurality situation, and the minimizing voting situation for a specific plurality situation is that voting situation with the minimum associated probability that a PMRW exists with IC. As the N_i's in a plurality situation become more similar, the voters would generally tend to have preferences that are less homogeneous, which should tend to minimize the probability that a PMRW exists.

Fishburn shows that the relationship between the probability that a PMRW exists for minimizing voting situations and the similarity of the N_i's is not strict. However, the overall PMRW probability minimizing voting situation does occur for odd n when we have the least homogeneous case, with

$$|N_A - N_B| \leq 1\,,\ |N_A - N_C| \leq 1\,,\ |N_B - N_C| \leq 1\,. \tag{5.16}$$

Skog (1993) reaches the same general conclusion by employing a very different approach to the problem, using a variation of IC. It is shown that the probability that a PMR cycle exists is greatest for voting situations that are similar to the minimizing voting situation, with differences in N_i's that are relatively small.

However, the probability that a PMR cycle exists is found to become insignificant only for differences between the N_i's that are quite different than the balanced case in the overall PMRW probability minimizing voting situation. Any general relationship between the probability that a PMRW exists and the N_i's in a voting situation will therefore be relatively weak

Skog (1993) goes on in an effort to find such a general relationship by going beyond the consideration of the information that is given in plurality situations. Let N_{ABC} denote the number of voters with the linear preference ranking $A \succ B \succ C$ in a voting situation. It is first proved that a voting situation can only have a PMR cycle if every component of one of the possible PMR cycles completely dominates the respective components of the reverse cycle. That is, if the PMR cycle exists with *AMBMCMA*, then it must be true that

$$\begin{aligned} N_{ABC} &> N_{CBA} \\ N_{BCA} &> N_{ACB} \\ N_{CAB} &> N_{BAC}. \end{aligned} \tag{5.17}$$

A model is then developed in which it is assumed that the relative proportions between the N_i's can be used to determine the relative number of voter preference rankings that have specific rankings on pairs of candidates. For example, the proportion of voter preference rankings that include $A \succ B$ is assumed to be the same as the proportion $N_A/(N_A + N_B)$. This model assumes that a strict consistency exists in the voters' relative rankings on candidates at all levels of preference, according to first place rankings. The differences in the two models presented in this study are consistent with the differences between the "impartial" and "proportional" scenarios in Chamberlin, et al. (1984) in Chapter 2.

Skog's model can then be used to consider the requirements for a PMR cycle in Eq. 5.17. The first equation results in

$$\begin{gathered} N_{ABC} > N_{CBA} \\ N_A\left[\frac{N_B}{N_B + N_C}\right] > N_C\left[\frac{N_B}{N_B + N_A}\right] \\ N_A(N_A + N_B) > N_C(N_C + N_B). \end{gathered} \tag{5.18}$$

It follows directly that $N_A > N_C$. The other two equations will result in $N_B > N_A$ and $N_C > N_B$. These results are clearly inconsistent, so this model with the strict proportion breakdown of voters' preferences on pairs of candidates, according to N_i's, prohibits the existence of PMR cycles.

Monte-Carlo simulation analysis was subsequently used to consider the impact that the degree of structure of voter preferences has on the probability that a PMR cycle exists. Individual voter preference rankings on three candidates were generated randomly from a mixture of two models to obtain random voter preference profiles. The mixture used a weight, ε, with $0 \le \varepsilon \le 1$, to obtain the probability that a randomly selected voter would have a particular preference ranking on three candidates. The associated probabilities from the model described above were weighted by ε, while a uniformly random component was weighted by $(1-\varepsilon)$. Results suggest that the probability that a PMR cycle is observed is greatly reduced for ε as small as 0.5. This indicates that even a relatively low level of structure in voters' preferences will significantly reduce the probability that a PMR cycle is observed.

Kuga and Nagatani (1974) consider the impact that some societal factors have on the probability that a PMRW exists by considering a measure of the degree to which voters' preferences tended to be different in voting situations, as measured by *voter antagonism*. Two voters are antagonistic on a pair of candidates if the order on these two candidates is reversed in their respective preference rankings. When $m = 3$, $n_1 + n_2 + n_4$ voters rank A over B, while $n_3 + n_5 + n_6$ rank B over A, for a total of $(n_1 + n_2 + n_4)(n_3 + n_5 + n_6)$ antagonistic voter preferences on that pair of candidates. The total number of antagonistic pairs in a given voting situation, $\boldsymbol{n}$, is given by $AP(\boldsymbol{n})$ with

$$AP(\boldsymbol{n}) = (n_1 + n_2 + n_4)(n_3 + n_5 + n_6) + (n_1 + n_2 + n_3)(n_4 + n_5 + n_6) + (n_1 + n_3 + n_5)(n_2 + n_4 + n_6). \quad (5.19)$$

The total number of possible antagonistic pairs for n voters over three possible pairs with $m = 3$ is given by $3\binom{n}{2}$, and the intensity of antagonism in a society is measured as the ratio $AP(n)/\left\{3\binom{n}{2}\right\}$. The measure $SSM^1(\boldsymbol{n})$ of voter antagonism is then obtained by normalizing this ratio, and:

$$SSM^1(\boldsymbol{n}) = \frac{4AP(\boldsymbol{n})}{3(n-1)(n+1)}, \text{ for odd } n. \quad (5.20)$$

$$SSM^1(\boldsymbol{n}) = \frac{4AP(\boldsymbol{n})}{3n^2}, \text{ for even } n.$$

$SSM^1(\boldsymbol{n})$ is minimized at zero, when all voters have the same preference ranking, with $n_i = n$ for some i and $n_j = 0$ for all $j \neq i$. $SSM^1(\boldsymbol{n})$ is maximized at one when the n_i's are equal for even n, and when the n_i's are nearly equally balanced for odd n, making a difference of one for each set of two voters with antagonistic preferences on the pairs of candidates.

Kuga and Nagatani (1974) prove that there must be a PMRW on three candidates when the degree of antagonism is relatively small, with

$$SSM^1(\boldsymbol{n}) \le \frac{2n^2}{3\left(n^2-1\right)}, \text{ for odd } n \tag{5.21}$$

$$SSM^1(\boldsymbol{n}) \le \frac{2}{3}, \text{ for even } n.$$

Kuga and Nagatani (1974) also consider the limiting case as $n \to \infty$ with IAC to show that there is a negative correlation between the degree of voter antagonism and the probability that a PMRW exists. The correlation is negative in this case since increasing homogeneity corresponds to decreasing antagonism. In particular, the study shows that $P^S_{PMRW}(3,n,IAC)$ increases when the voting situations that were enumerated were restricted to have $SSM^1(\boldsymbol{n})$ values less than or equal to some decreasing specified value. If the preference rankings on candidates are viewed as being representative of different political parties, Berg (1985a) points out that there is a direct link between the notion of voter antagonism and the measure of "fractionalization" that was developed by Rae (1968).

Fishburn (1973a) approaches the problem of social homogeneity by using Kendall's Coefficient of Concordance to measure the degree to which voters' preferences tend to be in agreement. In the context of three candidate elections, Kendall's Coefficient is measured on a specific voting situation, $\boldsymbol{n}$, and it is denoted by $SSM^2(\boldsymbol{n})$, with

$$SSM^2(\boldsymbol{n}) = \frac{\left(n_5+n_6-n_1-n_2\right)^2+\left(n_2+n_4-n_3-n_5\right)^2+\left(n_1+n_3-n_4-n_6\right)^2}{2n^2}. \tag{5.22}$$

Kendall's Coefficient is a standard method for considering correlation between ordinal rankings. Fishburn performs a computer simulation analysis with IC to generate voter preference profiles for various m and odd n. For each profile, $SSM^2(\boldsymbol{n})$ was calculated and it was determined if a PMRW existed in the profile. The voter preference profiles were then partitioned into 20 different segments over the range of possible $SSM^2(\boldsymbol{n})$ values. It was found that the proportion of profiles that had a PMRW within each segment tended to increase as the value of $SSM^2(\boldsymbol{n})$ for the segment increased.

Studies considering SSM's typically consider all of the profiles that correspond to a particular value of the measure of homogeneity, and then measure the proportion of them which have a PMRW, and Fishburn (1973) is an example of a study of this type. That simulation study showed that the percentage of profiles that have a PMRW increased to nearly 100 percent as the associated values of Kendall's Coefficient of Concordance increased to large values. It is easy to conclude in general that the most positive relationship between measures of social homogeneity and the probability that a PMRW exists will be observed while considering situation-specific measures of homogeneity.

5.4 The Effectiveness of Measures of Social Homogeneity

Attention is now turned to the development of representations for the conditional probability that a PMRW exists, given that voting situations have specified values of some situation-specific measures of social homogeneity. These representations are based on an extension of IAC. Let X denote some measure of social homogeneity, and the *Conditional Impartial Anonymous Culture Condition* $(IAC_X(k))$ is used to develop probability representations for events with the assumption that only voting situations for which measure X has a specified value of k can be observed, and that all such voting situations are equally likely. The *Cumulative Conditional Impartial Anonymous Culture Condition* $(CIAC_X(k^+))$ is used to develop probability representations for events with the assumption that only voting situations for which measure X has a specified value of k, or more, can be observed, and that all such voting situations are equally likely. Similarly, $CIAC_X(k^-)$ is used to develop probability representations for events with the assumption that only voting situations for which measure X has a specified value of k, or less, can be observed, and that all such voting situations are equally likely.

Berg and Bjurulf (1983) consider the situation-specific measure of social homogeneity $SSM^3(\boldsymbol{n})$ that corresponds to the population-specific measure $NPM^5(\boldsymbol{p})$ in Eq. 5.7, with

$$SSM^3(\boldsymbol{n}) = Min\ n_i\}. \tag{5.23}$$

Let k denote a specified value of $SSM^3(\boldsymbol{n})$ for any given voting situation. We must have $0 \le k \le n/6$, and voters' preferences obviously become less homogeneous as k increases. Gehrlein (2004c) considers the effect that specifying a value of $SSM^3(\boldsymbol{n})$ has on the expected values of several measures of social homogeneity that were discussed above, including the situation-specific measure, $SSM^4(\boldsymbol{n})$ that corresponds to the population-specific measure $NPM^1(\boldsymbol{p})$ in Eq. 5.3, with

$$SSM^4(\boldsymbol{n}) = \sum_{i=1}^{6} n_i^2 . \tag{5.24}$$

Let φ denote that $SSM^3(\boldsymbol{n})$ is being used as a basis for measuring social homogeneity, and the number of voting situations, $K(3,n,CIAC_\varphi(k^+))$, that are restricted to have k, or more, voters associated with each preference ranking is

$$K(3,n,CIAC_\varphi(k^+)) = \sum_{n_6=k}^{n-5k} \sum_{n_5=k}^{n-4k-n_6} \sum_{n_4=k}^{n-3k-n_6-n_5} \sum_{n_3=k}^{n-2k-n_6-n_5-n_4} \sum_{n_2=k}^{n-k-n_6-n_5-n_4-n_3} 1. \tag{5.25}$$

After algebraic reduction, we find

$$K\left(3,n,CIAC_{\varphi}\left(k^{+}\right)\right)=\frac{\prod_{i=1}^{5}(n+i-6k)}{120}. \tag{5.26}$$

It follows directly from definitions that the conditional expected value, $E\,\{SSM^{i}(\boldsymbol{n})\mid CIAC_{\varphi}\left(k^{+}\right)\}$, for any $SSM^{i}(\boldsymbol{n})$, is obtained from the general representation

$$E\,\{SSM^{i}(\boldsymbol{n})\mid CIAC_{\varphi}\left(k^{+}\right)\}= \tag{5.27}$$
$$\sum_{n_6=k}^{n-5k}\ \sum_{n_5=k}^{n-4k-n_6}\ \sum_{n_4=k}^{n-3k-n_6-n_5}\ \sum_{n_3=k}^{n-2k-n_6-n_5-n_4}\ \sum_{n_2=k}^{n-k-n_6-n_5-n_4-n_3}\frac{SSM^{i}(\boldsymbol{n})}{K\left(3,n,CIAC_{\varphi}\left(k^{+}\right)\right)}.$$

Gehrlein (2004c) shows that:

$$E\,\{SSM^{1}(\boldsymbol{n})\mid CIAC_{\varphi}\left(k^{+}\right)\}=\frac{6}{7}\left[\frac{n(n-1)+2kn-6k(k-1)}{n(n-1)}\right], \tag{5.28}$$

$$E\,\{SSM^{2}(\boldsymbol{n})\mid CIAC_{\varphi}\left(k^{+}\right)\}=\frac{(n-6k)(n+6-6k)}{7n^{2}}, \tag{5.29}$$

$$E\,\{SSM^{4}(\boldsymbol{n})\mid CIAC_{\varphi}\left(k^{+}\right)\}=\frac{2n^{2}+5n-10kn-30k+30k^{2}}{7n^{2}}. \tag{5.30}$$

The representation for $E\,\{SSM^{1}(\boldsymbol{n})\mid CIAC_{\varphi}\left(k^{+}\right)\}$ in Eq. 5.28 is in disagreement with results reported in Berg (1985a), for the limiting case as $n\to\infty$ with $k=0$. The representation in Eq. 5.28 contains a typographical error in Gehrlein (2004c).

By taking derivatives with respect to *k*, we find that each of the expected values behaves as anticipated. In particular, the derivative of $E\,\{SSM^{1}(\boldsymbol{n})\mid CIAC_{\varphi}\left(k^{+}\right)\}$ is positive for $0\le k\le n/6$. This makes sense because increasing *k* will increase the level of voter antagonism that $SSM^{1}(\boldsymbol{n})$ measures. However, this corresponds to decreased homogeneity. The derivatives of both $E\,\{SSM^{2}(\boldsymbol{n})\mid CIAC_{\varphi}\left(k^{+}\right)\}$ and $E\,\{SSM^{4}(\boldsymbol{n})\mid CIAC_{\varphi}\left(k^{+}\right)\}$ are negative over the range $0\le k\le n/6$, since increasing *k* will decrease the level of homogeneity. As a result, we find that increasing values of $SSM^{3}(\boldsymbol{n})$ do indeed correspond to the notion of systematically decreasing all measures of social homogeneity that are being considered.

Gehrlein and Berg (1992) use computer enumeration techniques to obtain some conditional expected values for representations that are of the same nature as

$E\{SSM^2(\boldsymbol{n}) \mid CIAC_\varphi(0^+)\}$ in which a PMRW exists for voting situations, while using a general P-E model instead of $\mathrm{CIAC}_\varphi(k^+)$. The addition of the restriction that a PMRW exists is found to cause only a marginal change in computed values when compared to the situation in which all possible voting situations could be observed, to suggest that the general positive relationship between $SSM^2(\boldsymbol{n})$ and the probability that a PMRW exists might not be highly significant.

A very surprising result is observed when we consider the effect that increasing k has on the probability that a PMRW exists with the assumption of $\mathrm{CIAC}_\varphi(k^+)$ for a three-candidate election. A representation for the associated probability, $P^S_{PMRW}(3,n \mid CIAC_\varphi(k^+))$, follows directly from the discussion that led to the representation in Eq. 3.31 for even n, with

$$P^S_{PMRW}(3,n \mid CIAC_\varphi(k^+)) = \tag{5.31}$$

$$\sum_{n_6=k}^{\frac{n-2}{2}-2k} \sum_{n_5=k}^{\frac{n-2}{2}-n_6-k} \sum_{n_4=k}^{\frac{n-2}{2}-n_6-n_5} \sum_{n_3=k}^{\frac{n-2}{2}-n_6-n_5} \sum_{n_2=k}^{n-n_6-n_5-n_4-n_3-k} \frac{3}{K(3,n,CIAC_\varphi(k^+))}.$$

After algebraic reduction, for even $n > 6$,

$$P^S_{PMRW}(3,n|CIAC_\varphi(k^+)) = \frac{15(n-6k)(n+2-6k)(n+4-6k)}{16(n+1-6k)(n+3-6k)(n+5-6k)}. \tag{5.32}$$

By taking the derivative of this function with respect to k, we find that $P^S_{PMRW}(3,n \mid CIAC_\varphi(k^+))$ decreases as k increases for $0 \le k \le n/6$. This result is in agreement with expectations since an increase in k, which decreases expected social homogeneity, leads to a decrease in the probability that a PMRW exists.

Gehrlein (2004c) proceeds in the same fashion to find a representation for $P^S_{PMRW}(3,n \mid CIAC_\varphi(k^+))$ for odd $n > 6$, with:

$$P^S_{PMRW}(3,n \mid CIAC_\varphi(k^+)) = \frac{15(n+3-6k)^2}{16(n+2-6k)(n+4-6k)}. \tag{5.33}$$

This representation is in agreement with a similar result in Berg and Bjurulf (1983) for the special case of $k = 1$. The representations in Eqs 5.32 and 5.33 are also identical to the results in Eqs. 3.32 and 3.30 respectively when $k = 0$.

Surprisingly, we find that $P^S_{PMRW}(3,n \mid CIAC_\varphi(k^+))$ increases as k increases over the range $0 \le k \le n/6$ for odd n. Table 5.2 lists computed values of $P^S_{PMRW}(3,n \mid CIAC_\varphi(k^+))$ for each n = 48, 49 for all $0 \le k \le n/6$ from Eqs. 5.32 and 5.33.

Table 5.2 Computed values for $P^S_{PMRW}(3,48 \mid CIAC_\varphi(k^+))$ and $P^S_{PMRW}(3,49 \mid CIAC_\varphi(k^+))$ from Gehrlein (2004c)

k	$P^S_{PMRW}(3,48 \mid CIAC_\varphi(k^+))$	$P^S_{PMRW}(3,49 \mid CIAC_\varphi(k^+))$
0	.8834	.9378
1	.8763	.9379
2	.8671	.9381
3	.8546	.9383
4	.8368	.9387
5	.8091	.9394
6	.7602	.9412
7	.6494	.9470
8	.0000	1.0000

The overall expected general relationship between social homogeneity and the probability that a PMRW exists is very weak. It is so weak that a complete reversal of expected results is obtained simply by switching from an even number of voters to an odd number of voters.

Based on what we have seen so far, there seems to be little reason to pursue the notion of trying to find a strong general relationship between simple measures of social homogeneity and the probability that a PMRW exists. This observation is certainly true for the measure $SSM^3(\boldsymbol{n})$, and it would seem to be true for other simple measures of homogeneity, despite the fact that they are situation-specific measures.

5.5 Requiring More Coherence in Voters' Preferences

Voters can obviously have preferences on candidates that lead to PMR cycles, but we have observed that there is typically much more stability in most political settings than would be expected when considering computed probabilities with IC and IAC. This phenomenon has typically been explained in general by supposing the presence of some group coherence or social homogeneity in the voters' preferences. We have seen that different measures of social homogeneity have differing degrees of ability to show a general relationship with the probability that a PMRW exists, and simple measures of homogeneity perform very poorly at displaying a general relationship between social homogeneity and the probability that a PMRW exists.

List (2002) suggests that this observation should be expected since there are different levels of group coherence of preference. That is, voters might have *substantive level agreement*, to the extent that their preferences, or views, tend to have some degree of consistency. However, voters might go beyond that and have some degree of *meta-level agreement*, to the extent that they can agree on a common dimension on which issues can be conceptualized. The voters might be largely in agreement as to what this common dimension is, while being in great disagreement as to what the optimal position on the dimension is. Positioning issues along such a dimension is perfectly consistent with the notion of single-peaked preferences. List (2002) argues that agreement at the meta-level is more likely to reduce occurrences of paradoxical results like PMR cycles than is agreement on a substantive level.

Dryzek and List (2003) discuss the same notion. Two or more individuals can agree on a substantive level to the extent that their preferences are the same. However, two individuals might disagree on any common ranking of alternatives to reflect their own preferences, but still agree on some ranking of alternatives on a common dimension. This second scenario is agreement on a meta-level. As described, agreement on a meta-level might imply single-peakedness. Issue complexity might rule out any common agreement on a single dimension, but multiple relevant issue dimensions coupled with individual voter's preference rankings on candidates on the issue dimensions might lead to some "intra-dimensional single-peakedness". They also discuss the impact that deliberation and discussion might have on "preference structuration" to increase the likelihood that the resultant voters' preferences will be more like single-peaked preferences.

Grofman and Uhlaner (1985) propose a similar concept regarding the existence of "meta-preferences" that result when voters have preferences for characteristics of broadly defined processes that might be involved in determining their preferences on candidates, rather than simply having preferences for candidates. They suggest that the additional structure that results with the notion of such meta-preferences leads to more of an overall understanding of the entire decision process, and therefore to more overall stability.

The notion of associating meta-level preferences to voters' preference rankings on candidates would certainly give rise to stability, with transitive PMR, if meta-level agreement is assumed to reflect the existence of single-peaked preferences. However, it seems extremely unlikely that all voters in a population would ever have preferences on candidates that would meet the strict definition of single-peakedness. It was noted in earlier discussion that Skog(1993) found that significant stability could be found with PMR if a relatively small proportion of a population had preferences that were structured according to the definition of the model used in the study.

Niemi (1969) developed this same general notion much earlier, to consider the relationship between degrees of single-peaked preferences and the existence of a PMRW. In particular, the study considers the maximum proportion, x/n, of n voters in any voter preference profile who have preference rankings that are single-peaked along some dimension. Computer procedures were used to enumerate all possible voter preference profiles on n voters. The enumeration results were used to compute the conditional probability that PMR is transitive, given x/n. As expected, it was found that the conditional probability that PMR is transitive increases as x/n increases. Computational results surprisingly suggest that if x/n is held constant, then the probability that PMR rankings are transitive will increase as n increases. The conclusion of the study is that PMR cycles are most likely to be observed in situations where voters' preferences are unstructured, or when small numbers of voters are making decisions on alternatives with many attributes of comparison. Buckley and Westen (1974) give examples to show that this general relationship is not exact. That is, one profile with a given x/n might have a PMRW, while another profile with a greater value of x/n might not have a PMRW.

Niemi (1970) applied this notion of degree of single-peaked preferences to some empirical results. The study considers seven different three-candidate election data sets to find the maximum proportion of voters' preferences that were single-peaked on some dimension. One of the three-candidate elections resulted in a PMR cycle. In agreement with expectations, that particular election had the voter preference profile with the least maximum proportion of voters with preferences that were single-peaked on some dimension. Niemi and Wright (1987) examined survey thermometer scores for candidates in the 1980 U. S. Presidential election. Results indicate that a relatively high proportion of voters' preferences were consistent with single-peaked preferences on all subsets of three and four candidates, and that the orderings of candidates that was required to obtain this result did not generally occur in agreement with left to right political affiliation of the candidates. That is, voters seemed to have strong opinions about candidates that were unrelated to the candidates' relative positions in the political spectrum.

Radcliff (1993) does an empirical study to determine the propensity of voters to have single-peaked preferences. The study obtains weak ordered preference rankings on U. S. Presidential candidates for respondents to surveys in American National Election Studies from 1972 to 1984. The respondents did not make actual pairwise comparisons between candidates, but the respondents' thermometer ratings on candidates were used to reconstruct all paired comparisons on candidates. It was assumed that any difference in reported thermometer scores resulted in a distinct preference in pairwise comparison between candidates. The percentage of respondents with preferences that were single-peaked across a reference ranking of candidates was approximately 83 percent for three-candidate elections, with that percentage decreasing to approximately 68 percent in four and five-candidate elections.

Van Deemen and Vergunst (1998) extended their empirical work that was discussed in Chapter 2, in which no PMR cycles were found in results of national elections to find a similar result as Niemi (1970). The analysis used a reference ordering for candidates that was based on the liberal-conservative nature of the political parties that were involved in the election. Their analysis suggests that the observed transitivity of PMR comparisons in the study did not seem to be a result of single-peaked preferences in respondents' preference rankings relative to the assumed linear reference ordering. This observation is not surprising, given the observations from Niemi and Wright (1987) above.

Adams (1997) performs a Monte-Carlo simulation study of the probability that a PMRW exists using a spatial model format with k criteria, to provide additional support to the ideas from Niemi (1969). The utility, $U_i(A)$, that the i^{th} voter has for a given candidate, A, has two components:

$$U_i(A) = -b \sum_{j=1}^{k} \left(x_{ij} - A_j\right)^2 + \mu_{iA} . \tag{5.34}$$

The first term represents the Euclidean distance between the ideal point of the i^{th} voter, as represented by the x_{ij} 's, and the stated position of Candidate *A*, as represented by the A_j 's. This term has a negative coefficient, since greater Euclidean distance between a voter's ideal preference point and a candidate's position suggests less satisfaction. The *b* value represents the policy salience coefficient for the voter, and it is assumed to be the same for all voters. Increased values of *b* indicate increased concern regarding policy issues for the voter. The second term, μ_{iA} , is a uniformly random variable.

When we have $b = 0$ in this model, uniformly random utilities are given to candidates for each voter. This leads to a situation that is identical to IC. When *b* is very large, voters are driven completely by policy issues, and the random component becomes insignificant. In the special case that $k = 1$ with large *b*, candidates' positions are represented by some numerical value along a number line, so that voters' preferences will be single-peaked with this model. We can then conclude that voters' preferences will consistently tend to be more like single-peaked preferences as *b* increases. Simulation results indicate that the probability that a PMRW exists does indeed increase as *b* increases. Thus, a more structured preference format for voters does increase the probability that a PMRW will exist. A second interesting observation is that small values of *b* will effectively eliminate the possibility of a PMR cycle as *n* becomes large.

Any approach that effectively shows a strong expected positive relationship between social homogeneity and the probability that a PMRW exists will obviously have to rely heavily on some additional underlying consistency in the structure of voters' preferences, like Niemi's degree of single-peaked preferences. That issue will be directly addressed in detail in Chapter 6.

It is worth noting that Feld and Grofman (1988) present an opposing argument to the general ideas that have been presented above, in which we have assumed that if voters show some consistency of preference according to some measure of homogeneity, then the overall preference of the group of voters will also reflect a consistency of preference. They suggest that it is possible to have situations in which individual voters might have preferences that are consistent with some homogeneity measure like single-peaked preferences, while an overall voter preference profile does not really reflect consistent preferences. Moreover, they argue that individual voters might have preferences that appear to be mutually inconsistent, while the overall preference structure of the voter preference profile actually does display consistency. For now, we turn our attention to other societal factors that are related to the likelihood that a PMRW exists.

5.6 Voter Abstention

Numerous studies have been conducted to consider the simple issue of why so many people extend the effort to vote in elections when there is such a small probability that their vote might alter the outcome of an election. Empirical studies have evaluated the impact that many different factors have on voter turnout.

Settle and Abrams (1976) perform an empirical study to consider a number of factors which tend to make people choose to vote, rather than abstain. These factors include: the anticipated closeness of the election, the income level of voters, the existence of third part candidates, media usage for advertising, the level of campaign spending, per capita level of federal spending, and the impact of women's suffrage. Results indicate that closeness in an election increases voter turnout, as does increased levels of federal spending as a reflection of total payoff to be expected.

Many studies have focused on the attitude that voters have toward the candidates in an election, particularly with regard to the perceived closeness of an election and voter turnout, and very mixed results are observed. Brody and Page (1973) consider abstention rates in terms of the general attitude of voters toward the candidates. Crain, et al. (1987) consider the effect of close races on abstention rates. In this study, ballots were considered in which voters did not have to vote for all candidates in all races on a ballot sheet. Analysis was then performed on the impact of closeness of the elections for the races in which votes were cast, versus the races in which no vote was cast. Results indicate that once voters are at a polling place, they are more likely to vote in close races than in races with an expected wide margin of victory.

Kirchgässner and Schimmelpfenny (1992) analyze elections in the United Kingdom and Germany to find that a positive relationship between perceived closeness of an election and voting rate is observed only at the level of individual election districts. The result is observed in local elections that are run in conjunction with national elections, even when the likely winner at the national level is evident. Kirchgässner and Zu Himmerman (1997) found a positive relationship when stable political environments exist. However, the relationship was found to be negative for elections held during times of political instability. Grofman, et al. (1998) look at a large number of elections, and a strong relationship was found for elections for U. S. Senate in years in which there was no presidential election. Similar results were observed in elections for state representatives to the House of Representative in years when there was no election for state governor or for members of the U. S. Senate in the same state.

Silver (1973) conducted a study of survey information that was taken from voters, and it considered many different independent variables. It was concluded that the perceived closeness of an election was not a significant factor in voter participation. The individual voter's interests in politics in general, and in the campaign in particular, were found to be highly significant factors in their participation. Matsusaka (1993) reaches a similar conclusion in an empirical study of 885 ballots in California over the interval 1912-1990. The results show no consistent relationship between the perceived closeness of votes on ballot propositions and voter turnout. It is suggested that the conclusions from earlier studies that found a positive relationship between voter turnout and the perceived closeness between candidates result from an increased mobilization of party members by party organizers to get voters to the polls in close races.

Many other factors have also been examined to find relationships to explain voter turnout. Cebula (1983) argues from empirical evidence that the existence of the U. S. Electoral College has acted to reduce voter turnout, even during non-Presidential election years. Carter (1984) considers the impact that early announcement of projections of the winners of an election have on voter turnout. Capron and Kruseman (1988) consider the impact that political rivalry among candidates has on voter turnout. Empirical results indicate that the participation rate in an election can be negatively affected by having either too few or too many candidates in an election. Glazer (1987) argues that voters go through the effort to vote for particular candidates for entertainment and to project an image. Wright (1989) examines the phenomenon of reduced voter turnout for runoff elections. Heckelman (1994) conducted an empirical study suggesting that the use of secret ballots reduces voter turnout, since there is a reduced incentive for candidates to offer payoffs for votes. This results from the lack of candidate control in this situation, since the candidate is not able to observe the true behavior of the voters who are given bribes in such situations. Knack (1994) does an empirical study on voter turnout, as related to: weather conditions, homeownership status, regularity of church attendance, education, age, and other factors. The results are not consistently clear in that study.

Several studies have developed economic models to explain why people vote. Riker and Ordeshook (1968) develop such a model for two candidate elections. This model is basically driven by an attempt to determine the expected payoff from voting, and voters are assumed to vote if that expected payoff exceeds the expected payoff from abstaining. The relevant factors in this expected cost model include: the number of eligible voters in the population, the reward that the voter receives from the act of voting, the marginal reward that the voter receives if the more preferred candidate wins, the probability that the voter will be pivotal, and the cost of the act of voting. A voter will be pivotal if his or her vote causes their preferred candidate to win. Hinich (1981) also develops an economic model that is based on a voter's expected payoffs from voting for either of two candidates, but this model does not include a factor for the probability that the voter will be pivotal. Niemi (1976) presents a discussion of reasons why people might choose not to vote in an election in the context of differences in costs for voting and for not voting.

Darvish and Rosenberg (1988) test some aspects of this general model of Riker and Ordeshook (1968) by considering the impact of population size on voter turnout in elections. The size of the population of voters would also have a direct impact on the likelihood that the voter would be pivotal. Empirical results from election data indicate that population size is negatively correlated with local election turnout, but that the result is insignificant for national elections. These results support the economic model for local elections that are help separately from national elections, but the results did not hold for local elections that were held in conjunction with national elections.

Jaarsma, et al. (1986) perform an empirical test of the same model, and focused on the aspect of the size of the payoff from having a preferred candidate win. This is done by considering the abstention rates in Dutch elections for private sector workers, for local bureaucrats with government jobs in a local sector where they would vote, and for commuting bureaucrats with government jobs in a sector in which they do not vote. Local bureaucrats have the highest voter turnout rate for any of the groups in local elections. The difference in voting rates between local bureaucrats and commuting bureaucrats disappears in national elections. The results suggest that a higher turnout rate should be anticipated as the expected payoff from the election outcome increases.

Greene and Nikolaev (1999) perform a similar empirical study on voting data from the United States over the period 1972-1997, showing a positive relationship between voter turnout and income, and that public sector employees have a higher rate of turnout than private sector employees. This study was based on results that were obtained from surveys of individual voters, rather than considering relationships over aggregated totals from groups of similar voters. The results from individual voters were not found to be as strong as the results from many earlier studies in which aggregated results were used.

Variations of the model proposed by Riker and Ordeshook (1968) are proposed in several studies. Ferejohn and Fiorina (1974) evaluate the decision of individuals to vote or to abstain from voting from a utility-based decision-theoretic approach, extending notions from Downs (1957). Tideman (1985b) considers the act of voting in terms of the amount of remorse or elation that is felt by a voter as a result of the outcome of an election, in conjunction with whether or not the voter has an impact on the outcome of the election.

Game theoretic approaches are also taken to explain why people vote. Palfrey and Rosenthal (1983) use a game theoretic approach to model voter participation in terms of the voters' perceptions that they might affect the outcome of an election. Owen and Grofman (1984) develop a model of voter participation and find that a rational equilibrium condition exists if each voter adopts a small probability of actually voting. In this situation, the turnout will be sufficiently small to make it worth the time to vote, in terms of having an impact on the election, for those who actually do vote. The normal rate of turnout for elections suggests that voters must actually be enticed to vote for some reason other than an expectation that they will influence the outcome of an election.

Each of these economic and game theoretic models assumes that individual voters decide whether or not to vote based on an evaluation of how to act in order to maximize their own utility for voting. Harsanyi (1980) surveys earlier research that leads to a description of two types of decision-makers. Act-utilitarian decision makers make decisions solely to maximize social utility on the basis of the particulars of a specific problem that is facing society at any given time. Rule-utilitarian decision-makers make decisions after making a rational commitment to act as an agent would be expected to behave to maximize social welfare on an ongoing basis. It is argued that an act-utilitarian decision-maker, like those described in individual utility maximizing models, would very likely never choose to participate in an election, since their vote has an insignificant likelihood of affecting the outcome. However, rule-utilitarian decision-makers would be much more likely to participate in elections.

The recurring reference to the probability that a voter will be pivotal, or have an impact on the election outcome by voting, in the individual utility maximizing models has led to a number of studies to consider that probability. The main focus of studies in this particular area has been on the development of representations for the probability that a majority rule election on two candidates will result in a tied vote.

Beck (1975) develops a representation for the limiting probability as $n \to \infty$ that a tie exists in an election on two candidates $A, B\}$ when the probability that a randomly selected voter has the preference $A \succ B$ is p, and voters' preferences are assumed to be independent. The value of n must be even for a tie to exist. The representation is derived by applying Stirling's Approximation to the appropriate binomial probability representation. The resulting limiting probability representation is given as

$$\{4p(1-p)\}^{\frac{n}{2}}\sqrt{\frac{2}{\pi n}}\,. \tag{5.35}$$

Computational evidence shows that this probability approaches zero quite quickly for large n, except for p near 1/2 Penrose (1946) gives the same representation for the special case of IC with $p = 1/2$.

Good and Mayer (1975) and Chamberlain and Rothschild (1981) extend the notions from Beck (1975) by using Bayesian analysis with probability distributions defining the likelihood that p values are observed. Chamberlain and Rothschild (1981) consider a special case in which p has a uniform distribution over the interval $[0,1]$, and the expected probability that a tie exists for even n is then given by

$$\int_0^1 \binom{n}{\frac{n}{2}} p^{\frac{n}{2}} (1-p)^{\frac{n}{2}}\, dp = \frac{1}{n+1}\,. \tag{5.36}$$

The concept of expected probabilities with UC in Chapter 3 applies here, to require this result to be equivalent to obtaining the obvious representation for this probability with IAC, and Berg (1990) finds this to be true with a different Bayesian approach to the problem. Fischer (1999) provides an extensive analysis of the differences between the work by Beck (1975) and the Bayesian approaches.

Blais and Young (1999) did an experimental study on a group of subjects, exposing them to a film about the rationality of voting and the paradox of why so many people vote. It was found that exposure to this film significantly reduced the percentage of subjects who actually voted in a subsequent election, as compared to those who did not view the film. Framing the act of voting in terms of rational choice seemed to induce subjects to reconsider why they should feel obliged to vote. There are clearly many reasons to explain why potential voters choose to abstain from voting in elections, and simple observations show that many potential voters do indeed choose to abstain from voting. We focus our attention on the impact that voter abstention can have on the determination of a PMRW.

Gehrlein and Fishburn (1976c) consider the basic situation of an election with two candidates $A, B\}$. The set of n voters is partitioned into four categories, where υ_i^X denotes the number of voters who prefer $X \in \{A,B\}$ of the two candidates, and vote in a PMR election on the two candidates if $i = 1$, or abstain if $i = 0$. By definition, $\upsilon_0^A + \upsilon_1^A + \upsilon_0^B + \upsilon_1^B = n$, and we denote the probability that any particular combination of these terms is observed as $P\left(\upsilon_0^A, \upsilon_1^A, \upsilon_0^B, \upsilon_1^B\right)$. Candidate B is defined as the strict *Global Pairwise Majority Rule Winner* (*GPMRW*) if it is the strict PMRW for both the population of potential voters and the set of actual voters. The PMRW for the set of actual voters only requires a PMR majority that is based on the number of actual voters, not on the number of potential voters. When $\Omega(\boldsymbol{P})$ denotes some probability distribution over $P\left(\upsilon_0^A, \upsilon_1^A, \upsilon_0^B, \upsilon_1^B\right)$ for all possible combinations of υ_i^X's for a specified n, $P_{GPMRW}^{\{B\}}(m,n,\Omega(\boldsymbol{P}))$ denotes the probability that B is the strict GPMRW for m-candidate elections. It follows directly for odd n that

$$P_{GPMRW}^{\{B\}}(2,n,\Omega(\boldsymbol{P})) = \sum_{\upsilon_1^A=0}^{\frac{n-1}{2}} \sum_{\upsilon_0^A=0}^{\frac{n-1}{2}-\upsilon_1^A} \sum_{\upsilon_1^B=\upsilon_1^A+1}^{n-\upsilon_1^A-\upsilon_0^A} P\left(\upsilon_0^A, \upsilon_1^A, \upsilon_0^B, \upsilon_1^B\right)$$
$$P_{GPMRW}^{\{A\}}(2,n,\Omega(\boldsymbol{P})) = \sum_{\upsilon_1^B=0}^{\frac{n-1}{2}} \sum_{\upsilon_0^B=0}^{\frac{n-1}{2}-\upsilon_1^B} \sum_{\upsilon_1^A=\upsilon_1^B+1}^{n-\upsilon_1^B-\upsilon_0^B} P\left(\upsilon_0^A, \upsilon_1^A, \upsilon_0^B, \upsilon_1^B\right). \tag{5.37}$$

Let $P_{TPMRW}^{\{B\}}(2,n,\Omega(\boldsymbol{P}))$ denote the probability that B is the strict PMRW for the population of potential voters while A and B tie among the set of actual voters in a two-candidate election, with $\upsilon_1^A = \upsilon_1^B$, and

$$P_{TPMRW}^{\{B\}}(2,n,\Omega(\boldsymbol{P})) = \sum_{\upsilon_1^A=0}^{\frac{n-1}{2}} \sum_{\upsilon_0^A=0}^{\frac{n-1}{2}-\upsilon_1^A} P\left(\upsilon_0^A, \upsilon_1^A, \upsilon_0^B, \upsilon_1^A\right)$$
$$P_{TPMRW}^{\{A\}}(2,n,\Omega(\boldsymbol{P})) = \sum_{\upsilon_1^B=0}^{\frac{n-1}{2}} \sum_{\upsilon_0^B=0}^{\frac{n-1}{2}-\upsilon_1^B} P\left(\upsilon_0^A, \upsilon_1^B, \upsilon_0^B, \upsilon_1^B\right). \tag{5.38}$$

If a strict winner of a tied outcome for actual voters is determined by random selection with equal likelihood for either candidate being selected, the probability, $P_{GPMRW}^S(2,n,\Omega(\boldsymbol{P}))$, that a strict GPMRW exists is given by

$$P_{GPMRW}^S(2,n,\Omega(\boldsymbol{P})) = \tag{5.39}$$
$$P_{GPMRW}^{\{A\}}(2,n,\Omega(\boldsymbol{P})) + P_{GPMRW}^{\{B\}}(2,n,\Omega(\boldsymbol{P})) + \frac{1}{2}\left\{P_{TPMRW}^{\{A\}}(2,n,\Omega(\boldsymbol{P})) + P_{TPMRW}^{\{B\}}(2,n,\Omega(\boldsymbol{P}))\right\}.$$

Gehrlein and Fishburn (1976c) develop representations for the probability $P_{GPMRW}^{S}(2,n,\Omega(\boldsymbol{P}))$ with various assumptions about $\Omega(\boldsymbol{P})$. The assumption of *Impartial Anonymous Culture with Abstentions* (*IACA*) is defined as the situation in which $P(\upsilon_0^A,\upsilon_1^A,\upsilon_0^B,\upsilon_1^B)$ is identical for all combinations of υ_i^X's for a specified *n*. Based on counting arguments like those used to develop the representation for $P_{PMRW}^{S}(3,n,IAC)$ in Eq. 3.30, it is found that the representation for $P_{GPMRW}^{S}(2,n,IACA)$ from Eq. 5.39 for odd *n* is not a function of *n*, with

$$P_{GPMRW}^{S}(2,n,IACA)=3/4. \tag{5.40}$$

The case with identifiable voters is also considered, where p_A is the probability that a randomly selected potential voter prefers *A* to *B*, with $p_B=1-p_A$. Let λ_A denote the probability that a potential voter who prefers *A* to *B* will actually vote. If voters' preferences are conjoint independent, potential voters independently form their preferences on the pair *A* and *B*, and the values of λ_A and λ_B are independent of the values of p_A and p_B. In the case of conjoint independence,

$$P(\upsilon_0^A,\upsilon_1^A,\upsilon_0^B,\upsilon_1^B)= \tag{5.41}$$
$$\frac{n!}{\upsilon_0^A!\upsilon_1^A!\upsilon_0^B!\upsilon_1^B!}\{(1-\lambda_A)p_A\}^{\upsilon_0^A}(\lambda_A p_A)^{\upsilon_1^A}\{(1-\lambda_B)p_B\}^{\upsilon_0^B}(\lambda_B p_B)^{\upsilon_1^B}.$$

Gehrlein and Fishburn (1976c) consider the case in which $\lambda=\lambda_A=\lambda_B$ and prove that $P_{GPMRW}^{S}(2,n,\Omega(\boldsymbol{P}))$ increases as λ increases for all p_A when conjoint independence is assumed, with $P_{GPMRW}^{S}(2,n,\Omega(\boldsymbol{P}))$ = 1/2 when $\lambda=0$ and $P_{GPMRW}^{S}(2,n,\Omega(\boldsymbol{P}))$ = 1 when $\lambda=1$.

The conjoint independent condition of *Impartial Culture with Abstention* (*ICA*(λ)) assumes that $p_A=p_B$ = 1/2 and $\lambda=\lambda_A=\lambda_B$, and several observations are made regarding $P_{GPMRW}^{S}(2,n,ICA(\lambda))$. Theorems 5.1 and 5.2 are proved in Gehrlein and Fishburn (1976c), with

Theorem 5.1. $\{P_{GPMRW}^{S}(2,n,ICA(\lambda))+P_{GPMRW}^{S}(2,n,ICA(1-\lambda))\}/2=3/4$ for all $\lambda\in[0,1]$ and all odd $n\geq 1$.

Theorem 5.2. $\int_0^1 P_{GPMRW}^{S}(2,n,ICA(\lambda))f(\lambda)d\lambda=3/4$ for any probability density function $f(\lambda)$ over $\lambda\in[0,1]$ for which $f(\lambda)=f(1-\lambda)$.

An additional representation is also obtained for the limiting distribution of $P_{GPMRW}^{S}(2,n,ICA(\lambda))$ as $n\to\infty$, with

$$P^S_{GPMRW}(2,\infty,ICA(\lambda)) = Cos^{-1}\left(-\sqrt{\lambda}\right)/\pi \,. \qquad (5.42)$$

The conditional limiting probability, $CP^S_{GPMRW}(m,\infty,ICA(\lambda))$, that there is a GPMRW for *m* candidates, given that a PMRW exists in the population of potential voters is obtained from

$$CP^S_{GPMRW}(m,\infty,ICA(\lambda)) = P^S_{GPMRW}(m,\infty,ICA(\lambda)) / P^S_{PMRW}(m,\infty,IC) \,. \qquad (5.43)$$

Table 5.3 lists computed values of $CP^S_{GPMRW}(2,\infty,ICA(\lambda))$ for each $\lambda = 0.1(.1)1.00$ from Eqs. 5.42 and 5.43, with the fact that $P^S_{PMRW}(2,\infty,IC) \rightarrow 1$.

Table 5.3 Computed values of $CP^S_{GPMRW}(2,\infty,ICA(\lambda))$ and $CP^S_{GPMRW}(3,\infty,ICA(\lambda))$

λ	$CP^S_{GPMRW}(2,\infty,ICA(\lambda))$	$CP^S_{GPMRW}(3,\infty,ICA(\lambda))$
.00	.5000	-------
.10	.6024	.4236
.20	.6476	.4806
.30	.6845	.5290
.40	.7180	.5742
.50	.7500	.6186
.60	.7820	.6640
.70	.8155	.7126
.80	.8524	.7675
.90	.8976	.8365
1.00	1.0000	1.0000

The results of Table 5.3 indicate that there is a significant likelihood that some candidate other then the PMRW for potential voters will be elected when the abstention rate it at all large with the assumption of ICA(λ) when the potential electorate is large. Computed values in the original study indicate that these limiting values are approached quite rapidly as *n* increases, approximately for $n \geq 49$. Gehrlein and Fishburn (1976c) also consider a specialized model in which λ_A and λ_B are dependent on the magnitude of p_A and p_B.

Gehrlein and Fishburn (1978a) extend the notion of ICA(λ) to elections with three candidates $A,B,C\}$ to obtain a representation for the limiting probability $P^S_{GPMRW}(3,\infty,ICA(\lambda))$ as $n \rightarrow \infty$. Following previous arguments that have developed similar limiting probability representations, we start by defining four discrete variables to describe the preferences of a given i^{th} voter:

$$\begin{aligned}
X_1^i &= +1: \text{if } A \succ B \text{ for the } i^{th} \text{ potential voter } [p_1 + p_2 + p_4] \\
&\quad -1: \text{if } B \succ A \text{ for the } i^{th} \text{ potential voter } [p_3 + p_5 + p_6] \\
X_2^i &= +1: \text{if } A \succ C \text{ for the } i^{th} \text{ potential voter } [p_1 + p_2 + p_3] \\
&\quad -1: \text{if } C \succ A \text{ for the } i^{th} \text{ potential voter } [p_4 + p_5 + p_6] \\
X_3^i &= +1: \text{if } A \succ B \text{ for the } i^{th} \text{ potential voter, and the vote is cast } [\lambda(p_1 + p_2 + p_4)] \\
&\quad 0: \text{if the } i^{th} \text{ potential voter abstains } [1 - \lambda] \\
&\quad -1: \text{if } B \succ A \text{ for the } i^{th} \text{ potential voter, and the vote is cast } [\lambda(p_3 + p_5 + p_6)] \\
X_4^i &= +1: \text{if } A \succ C \text{ for the } i^{th} \text{ potential voter, and the vote is cast } [\lambda(p_1 + p_2 + p_3)] \\
&\quad 0: \text{if the } i^{th} \text{ potential voter abstains } [1 - \lambda] \\
&\quad -1: \text{if } C \succ A \text{ for the } i^{th} \text{ potential voter, and the vote is cast } [\lambda(p_4 + p_5 + p_6)].
\end{aligned} \tag{5.44}$$

The expressions in brackets in Eq. 5.44 denote the probability that the associated values of each variable are observed, following the definition of p_i 's from Fig. 5.1. The symmetry of ICA(λ) with respect to candidates leads to $E(\overline{X}_j) = E(X_j^i) = 0$ for j = 1, 2, 3, 4, and A will be the GPMRW with the joint probability that $\overline{X}_j\sqrt{n} \geq E(\overline{X}_j\sqrt{n})$ for j = 1,2,3,4. As $n \to \infty$ the Central Limit Theorem requires that this joint distribution is multivariate normal with correlation matrix $\boldsymbol{R^3}$ with

$$\boldsymbol{R^3} = \begin{bmatrix} 1 & 1/3 & \sqrt{\lambda} & \sqrt{\lambda}/3 \\ - & 1 & \sqrt{\lambda}/3 & \sqrt{\lambda} \\ - & - & 1 & 1/3 \\ - & - & - & 1 \end{bmatrix}. \tag{5.45}$$

Following the logic of arguments that were used previously to develop representations for limiting probabilities with IC, along with the fact that there are three possible candidates that could be the GPMRW

$$P_{GPMRW}^{S}(3, \infty, ICA(\lambda)) = 3\Phi_4(\boldsymbol{R^3}). \tag{5.46}$$

The positive multivariate normal orthant probability $\Phi_4(\boldsymbol{R^3})$ is a special case of a representation that is given in Cheng (1969), which Gehrlein and Fishburn (1978a) use with Eq. 5.46 to obtain a representation for $P_{GPMRW}^{S}(3, \infty, ICA(\lambda))$, when $0 < \lambda \leq 1$, with

$$P^S_{GPMRW}(3,\infty,ICA(\lambda)) = \frac{3}{16} + \frac{3}{4\pi}\left\{Sin^{-1}\left(\frac{1}{3}\right) + Sin^{-1}\left(\sqrt{\lambda}\right) + Sin^{-1}\left(\frac{\sqrt{\lambda}}{3}\right)\right\} \quad (5.47)$$

$$+\frac{3}{4\pi^2}\left\{\left[Sin^{-1}\left(\frac{1}{3}\right)\right]^2 + \left[Sin^{-1}\left(\sqrt{\lambda}\right)\right]^2 - \left[Sin^{-1}\left(\frac{\sqrt{\lambda}}{3}\right)\right]^2\right\}.$$

The situation with $\lambda = 0$ is problematic in this representation when $m > 2$, since there is more than one interpretation of the existence of a strict PMRW when all voters abstain.

Computed values of $CP^S_{GPMRW}(3,\infty,ICA(\lambda))$ are listed in Table 5.3 for each value of λ = .1(.1)1.0 from Eqs. 5.43 and 5.47. As $\lambda \to 1$, $P^S_{GPMRW}(3,\infty,ICA(1))$ = $P^S_{PMRW}(3,\infty,IC)$, so $CP^S_{GPMRW}(3,\infty,ICA(1)) = 1$. These computed values show that there is a significant likelihood that a GPMRW does not exist, given that a PMRW does exist on the population of all potential voters, when the rate of abstention is at all large with the assumption of ICA(λ).

Gehrlein and Fishburn (1979b) consider the probability that a *Local Pairwise Majority Rule Winner* (*LPMRW*) exists. Candidate *B* is a LPMRW if it is the PMRW for the set of actual voters, when either *A* or *C* is the PMRW among the population of all potential voters. The determination of the PMRW for the set of actual voters only requires a PMR majority that is based on the number of actual voters, not on the number of potential voters. To develop a representation for this probability, we define four discrete variables to describe the preferences of a given i^{th} voter:

$$\begin{aligned}
Y_1^i &= +1: \text{if } A \succ B \text{ for the } i^{th} \text{ potential voter } [p_1 + p_2 + p_4] \quad (5.48)\\
&\quad -1: \text{if } B \succ A \text{ for the } i^{th} \text{ potential voter } [p_3 + p_5 + p_6]\\
Y_2^i &= +1: \text{if } A \succ C \text{ for the } i^{th} \text{ potential voter } [p_1 + p_2 + p_3]\\
&\quad -1: \text{if } C \succ A \text{ for the } i^{th} \text{ potential voter } [p_4 + p_5 + p_6]\\
Y_3^i &= +1: \text{if } B \succ A \text{ for the } i^{th} \text{ potential voter, and the vote is cast } [\lambda(p_3 + p_5 + p_6)]\\
&\quad 0: \text{if the } i^{th} \text{ potential voter abstains } [1 - \lambda]\\
&\quad -1: \text{if } A \succ B \text{ for the } i^{th} \text{ potential voter, and the vote is cast } [\lambda(p_1 + p_2 + p_4)]\\
Y_4^i &= +1: \text{if } B \succ C \text{ for the } i^{th} \text{ potential voter, and the vote is cast } [\lambda(p_1 + p_3 + p_5)]\\
&\quad 0: \text{if the } i^{th} \text{ potential voter abstains } [1 - \lambda]\\
&\quad -1: \text{if } C \succ B \text{ for the } i^{th} \text{ potential voter, and the vote is cast } [\lambda(p_2 + p_4 + p_6)].
\end{aligned}$$

The symmetry of ICA(λ) with respect to candidates leads to $E(\overline{Y}_j) = E(Y_j^i) = 0$ for *j* = 1, 2, 3, 4, and *A* will be the PMRW for the set of all potential voters while *B* is the PMRW for the set of actual voters with the joint probability that

$\overline{Y}_j\sqrt{n} \geq E\left(\overline{Y}_j\sqrt{n}\right)$ for j = 1, 2, 3, 4. As $n \to \infty$ the Central Limit Theorem requires that this joint distribution is multivariate normal with correlation matrix $\boldsymbol{R^4}$ with

$$\boldsymbol{R^4} = \begin{bmatrix} 1 & 1/3 & -\sqrt{\lambda} & -\sqrt{\lambda}/3 \\ - & 1 & -\sqrt{\lambda}/3 & \sqrt{\lambda}/3 \\ - & - & 1 & 1/3 \\ - & - & - & 1 \end{bmatrix}. \tag{5.49}$$

Following the logic of arguments that were used previously to develop representations for limiting probabilities with IC, along with the fact that there are six possible combinations of candidates that could create the existence of a LPMRW, the limiting probability, $P^S_{LPMRW}(3,\infty,ICA(\lambda))$ that a LPMRW exists with the assumption of ICA(λ) is given by

$$P^S_{LPMRW}(3,\infty,ICA(\lambda)) = 6\Phi_4\left(\boldsymbol{R^4}\right). \tag{5.50}$$

The form of the correlation matrix $\boldsymbol{R^4}$ does not lead to a simple representation for $\Phi_4\left(\boldsymbol{R^4}\right)$, so Gehrlein and Fishburn (1979b) obtain values for $\Phi_4\left(\boldsymbol{R^4}\right)$ by using quadrature with a representation from Gehrlein (1979). These computed values are then used with Eq. 5.50 and the logic of Eq. 5.43 to obtain computed values for the conditional probability, $CP^S_{LPMRW}(3,\infty,ICA(\lambda))$, that a LPMRW exists, given that a PMRW exists for the set of all potential voters. The computed values of $CP^S_{LPMRW}(3,\infty,ICA(\lambda))$ for each $\lambda = .1(.1)1.0$ are given in Table 5.4

Table 5.4 Computed values of $CP^S_{LPMRW}(3,\infty,ICA(\lambda))$ and $CP^S_{RPMRW}(3,\infty,ICA(\lambda))$

λ	$CP^S_{LPMRW}(3,\infty,ICA(\lambda))$	$CP^S_{RPMRW}(3,\infty,ICA(\lambda))$
.10	.4906	.1999
.20	.4359	.1596
.30	.3901	.1295
.40	.3480	.1046
.50	.3073	.0829
.60	.2663	.0636
.70	.2236	.0459
.80	.1767	.0296
.90	.1202	.0143
1.00	.0000	.0000

A *Reverse Pairwise Majority Rule Winner* (*RPMRW*) exists when some candidate is the both the PMRL for the population of all potential voters and the PMRW for the set of actual voters. The determination of the PMRW for the set of actual voters only requires a PMR majority that is based on the number of actual voters, not on the number of potential voters. The derivation of a representation for the limiting probability, $P^S_{RPMRW}(3,\infty,ICA(\lambda))$, that a RPMRW exists can be obtained as a simple extension of the development that led to the representation for $P^S_{GPMRW}(3,\infty,ICA(\lambda))$ in Eq. 5.47. The modification that must be made is the reversal of signs in the definitions of variables X_1^i and X_2^i in Eq. 5.44. The result that is obtained is given by

$$P^S_{RPMRW}(3,\infty,ICA(\lambda)) = \frac{3}{16} + \frac{3}{4\pi}\left\{Sin^{-1}\left(\frac{1}{3}\right) - Sin^{-1}\left(\sqrt{\lambda}\right) - Sin^{-1}\left(\frac{\sqrt{\lambda}}{3}\right)\right\} \tag{5.51}$$
$$+\frac{3}{4\pi^2}\left\{\left[Sin^{-1}\left(\frac{1}{3}\right)\right]^2 + \left[Sin^{-1}\left(\sqrt{\lambda}\right)\right]^2 - \left[Sin^{-1}\left(\frac{\sqrt{\lambda}}{3}\right)\right]^2\right\}.$$

Computed values for the conditional probability, $CP^S_{RPMRW}(3,\infty,ICA(\lambda))$, that a RPMRW exists, given that a PMRL exists for the population of potential voters with the assumption of ICA(λ) are obtained by following the logic of Eq. 5.43, and the corresponding values are listed in Table 5.4 for each $\lambda = .1(.1)1.0$.

The calculated values in Table 5.4 show that $CP^S_{RPMRW}(3,\infty,ICA(\lambda))$ is significantly smaller than corresponding values for $CP^S_{LPMRW}(3,\infty,ICA(\lambda))$. However, $CP^S_{RPMRW}(3,\infty,ICA(\lambda))$ is still quite significant for low rates of voter participation. These observations are completely consistent with some empirical observations that are related to such outcomes.

In particular, Regenwetter, et al. (2002) examine survey results from eight different three-candidate elections in three different countries. Results show that no PMR cycles were found in any of the survey results. However, it was discovered that using samples of survey respondents could frequently misrepresent the true PMR ranking of the entire population of survey respondents, unless the size of the sample was relatively large. In this situation the entire population of the data set would reflect the preferences of all potential voters, and the sample set would represent the preferences of the set of actual voters. In some cases the true PMR rankings of the population would only have a high probability of being represented by the sample if the sample size was in the hundreds or thousands, to reflect a relatively low abstention rate. A key factor for sample convergence to the population PMR ranking was shown to be linked to the minimum marginal PMR comparison on a pair in the population preferences.

Tsetlin and Regenwetter (2003) consider the sample size requirements that are associated with various confidence levels that the PMRW in a sample is the same as the PMRW in a population. As in the previous study, the size of the sample reflects the abstention rate. Relatively small sample sizes are shown to be sufficient to have a high confidence that the sample PMRW is the same as the population

PMRW for samples from populations that do not have a strict balance in pairwise preferences on candidates like that suggested by ICA(λ). As in our previous findings, the conclusions that are reached with the probability representations that have been obtained are strongly linked to the balanced preference basis of the assumption of ICA(λ).

5.7 Degrees of Voter Indifference between Candidates

The work of Inada (1964) was mentioned in Chapter 2, in considering conditions that require that a PMRW must exist. In particular, Inada shows that PMR must be transitive for an odd number of voters with dichotomous preferences. As in our previous discussion about single-peaked preferences, we should expect there to be a positive general relationship between the proportion of voters who have dichotomous preferences and the probability that a PMRW exists.

Radcliff (1993) does an empirical study to determine the propensity of voters to have dichotomous preferences. The study obtains individual voter's weak ordered preference rankings on U. S. Presidential candidates from reported thermometer scores in surveys from American National Election Studies from 1972 to 1984. Results suggest that approximately 30 percent of respondents had dichotomous preferences in three-candidate elections, with that percentage decreasing dramatically as the number of candidates increased to four or five candidates.

Gehrlein and Valognes (2001) develop a probability representation to consider the impact that voter indifference between candidates can have on the probability that a PMRW exists in three-candidate elections. There is a direct link between indifference and dichotomous preferences in three-candidate elections. Let $A \sim B$ denote the situation of voter indifference between candidates *A* and *B*. If individual indifference is allowed, while individual transitivity of preference is still required, individual voter's preferences must be weak orders. There are 13 possible weak ordered preference rankings on three candidates that might represent the preferences of a given voter:

A	A	B	C	B	C
B	C	A	A	C	B
C	B	C	B	A	A
p_1	p_2	p_3	p_4	p_5	p_6

$A \sim B$	$A \sim C$	$B \sim C$	A	B	C
C	B	A	$B \sim C$	$A \sim C$	$A \sim B$
p_7	p_8	p_9	p_{10}	p_{11}	p_{12}

$$A \sim B,\ A \sim C, B \sim C$$

$$p_{13}$$

Fig. 5.2 Possible weak ordered preference types for individual voters with three candidates

Here, p_{13} denotes the probability that a randomly selected voter has preferences that are represented by complete indifference among the candidates.

Let k_1 denote the probability that a randomly selected voter is in the class of voters with linear preference rankings. Similarly, k_2 denotes the probability that voters have weak ordered preferences that do not reflect complete indifference between candidates, which must be dichotomous preferences with three candidates. Then, k_3 denotes the probability that voters have complete indifference on candidates, with $k_1 + k_2 + k_3 = 1$. Let the vector $\boldsymbol{k} = (k_1, k_2, k_3)$ define the probabilities that voter's preferences fall into the associated preference classes.

The *Impartial Weak Order Culture Condition* (IWOC$(\boldsymbol{k})$) assumes that all preference structures within a class of voter preference types are equally likely to be observed for a specified $\boldsymbol{k}$ with independent voters. With the voter preferences as defined in Fig. 5.2, IWOC$(\boldsymbol{k})$ requires that $p_i = k_1/6$ for i = 1, 2, 3, 4, 5, 6; that $p_j = k_2/6$ for j = 7, 8, 9, 10, 11, 12; and that $p_{13} = k_3$. Gehrlein and Valognes (2001) apply the same basic techniques that have been presented earlier to IWOC$(\boldsymbol{k})$ in order to develop a representation for the limiting probability, $P^S_{PMRW}(3, \infty, IWOC(\boldsymbol{k}))$, that a PMRW exists as $n \to \infty$, and

$$P^S_{PMRW}(3, \infty, IWOC(\boldsymbol{k})) = \frac{3}{4} + \frac{3}{2\pi} Sin^{-1}(\rho_3), \tag{5.52}$$

where

$$\rho_3 = \frac{k_1 + k_2}{3k_1 + 2k_2}. \tag{5.53}$$

The determination of the PMRW in this case only requires a PMR majority on pairs of candidates that is based on the number of voters who have a preference on that pair of candidates, not on the total number of voters. The result in Eq. 5.52 is a direct extension of work in Fishburn and Gehrlein (1980b), where the identical result was obtained for the special case with $k_3 = 0$. Obviously, $k_1 + k_2 > 0$ since no PMRW can exists if all voters have complete indifference between all candidates. Table 5.5 lists computed values of $P^S_{PMRW}(3, \infty, IWOC(\boldsymbol{k}))$ for $\rho_3 = 1/3$, and for each value of $\rho_3 = .350(.025).500$.

Two results are obtained by Gehrlein and Valognes (2001) by simply taking derivatives of the representation in Eq. 5.52:

Theorem 5.3 For any fixed $k_1 > 0$, $P^S_{PMRW}(3, \infty, IWOC(\boldsymbol{k}))$ increases as k_2 increases. For $k_1 = 0$, $P^S_{PMRW}(3, \infty, IWOC(\boldsymbol{k})) = 1$ with any $k_2 > 0$.

Theorem 5.4 For any fixed $k_2 > 0$, $P^S_{PMRW}(3, \infty, IWOC(\boldsymbol{k}))$ decreases as k_1 increases.

Table 5.5 Computed values of $P_{PMRW}^{S}(3,\infty,IWOC(\boldsymbol{k}))$, $P_{PMRW}^{S}(4,\infty,IWOC(\boldsymbol{k}))$ and $P_{PMRW}^{S}(5,\infty,IWOC(\boldsymbol{k}))$

ρ	$P_{PMRW}^{S}(3,\infty,IWOC(\boldsymbol{k}))$	$P_{PMRW}^{S}(4,\infty,IWOC(\boldsymbol{k}))$	$P_{PMRW}^{S}(5,\infty,IWOC(\boldsymbol{k}))$
1/3	.9123	.8245	.7487
.350	.9207	.8415	.7726
.375	.9335	.8671	.8089
.400	.9465	.8930	.8459
.425	.9596	.9192	.8834
.450	.9729	.9457	.9215
.475	.9863	.9727	.9604
.500	1.0000	1.0000	1.0000

Theorem 5.3 generalizes the result of Inada (1964) for the case of three candidate elections by showing that when the preferences of a population of voters shifts, so that fewer individual voters will have linear preferences and more will have dichotomous preferences, then there is an increased likelihood that a PMRW will exist. And, when all voters who hold any preferences at all have dichotomous preferences a PMRW must exist. Theorem 5.4 shows that the same result is true when individual voter's preferences shift from having linear rankings to having complete indifference between candidates. For the special case with $k_2 = 0$, Eq. 5.52 reduces to the result of Guilbaud (1952) for $P_{PMRW}^{S}(3,\infty,IC)$ in Eq. 3.77 for all $k_1 > 0$.

For the case of more than three candidates, the relationship between voter indifference and dichotomous preferences is not as direct, since voters can have transitive preferences with some indifference, without having dichotomous preferences in such cases. A number of studies have been conducted to establish a general link between voter indifference between candidates and the probability that a PMRW exists for more than three candidates. Voter's preferences are assumed to be weak ordered preferences in these studies, so that transitivity of preference is still a necessary condition. Bjurulf (1972) performs a Monte-Carlo simulation analysis to reach the conclusion that an increased propensity for voters to have weak ordered preferences leads to an expected increase in the probability that a PMRW exists.

Analysis in Fishburn and Gehrlein (1980b) can easily be extended to obtain representations for the limiting probabilities $P_{PMRW}^{S}(4,\infty,IWOC(\boldsymbol{k}))$ and $P_{PMRW}^{S}(5,\infty,IWOC(\boldsymbol{k}))$, since IWOC$(\boldsymbol{k})$ is a simple extension of the assumption of "Permutation Invariance" that was used as a basis in that study. We develop the representation for $P_{PMRW}^{S}(4,\infty,IWOC(\boldsymbol{k}))$ to show how the procedure works. Let $\boldsymbol{K}$ denote the set of all possible weak orders on a set of four candidates $\boldsymbol{C^4} = C_1,C_2,C_3,C_4\}$, and let K_i denote the subset of $\boldsymbol{K}$ with $m+1-i$ indifference classes. It follows from simple counting arguments that $\#K_1 = 4! = 24$, $\#K_2 = 6\binom{4}{2} = 36$, $\#K_3 = 2\binom{4}{3} + \binom{4}{2} = 14$, and $\#K_4 = 1$. For example, the calculation of $\#K_2$ results from the fact that there are $\binom{4}{2}$ different pairs of candidates that could be placed in the single indifference class in a weak order in this particular class of weak orders. Then, the weak order could be any of the six possible linear rankings on the two remaining candidates and the pair in the single indifference class. Following earlier discussion, IWOC$(\boldsymbol{k})$ assigns a probability k_i to the likelihood that a randomly selected voter will have preferences that match a weak order in K_i. And, each weak order in a specified K_i is equally likely to be observed with probability equal to $k_i / \#K_i$.

The development of a closed-form representation for the limiting probability $P_{PMRW}^S(4, \infty, IWOC(\boldsymbol{k}))$ follows earlier arguments. We start by defining three discrete variables X_j^i on the preferences of the i^{th} voter in a randomly generated voter preference profile, for j = 1, 2, 3, with

$$
\begin{aligned}
X_j^i = 1 \quad & \text{for the 12 weak orders in } K_1 \text{ with } C_1 \succ C_{i+1} && (5.54)\\
& \text{for the 15 weak orders in } K_2 \text{ with } C_1 \succ C_{i+1}\\
& \text{for the 4 weak orders in } K_3 \text{ with } C_1 \succ C_{i+1}\\
X_j^i = 0 \quad & \text{for none of the weak orders in } K_1\\
& \text{for the 6 weak orders in } K_2 \text{ with } C_1 \sim C_{i+1}\\
& \text{for the 4 weak orders in } K_3 \text{ with } C_1 \sim C_{i+1}\\
& \text{for the 1 weak order in } K_4 \text{ with } C_1 \sim C_{i+1}\\
X_j^i = \text{-}1 \quad & \text{for the 12 weak orders in } K_1 \text{ with } C_{i+1} \succ C_1\\
& \text{for the 15 weak orders in } K_2 \text{ with } C_{i+1} \succ C_1\\
& \text{for the 4 weak orders in } K_3 \text{ with } C_{i+1} \succ C_1 .
\end{aligned}
$$

Given the definitions in Eq. 5.54, Candidate C_1 will be the strict PMRW for a specified random voter preference profile if $\overline{X}_j > 0$ for all j = 1, 2, 3. By the symmetry of IWOC$(\boldsymbol{k})$ with respect to candidates, $E(\overline{X}_j) = E(X_j^i) = 0$, so the probability that C_1 will be the strict PMRW is the same as the positive orthant probability that $\overline{X}_j\sqrt{n} \geq E(\overline{X}_j\sqrt{n})$ for all j = 1, 2, 3. As $n \to \infty$, this joint distribution of the $\overline{X}_j\sqrt{n}$ variables is trivariate normal.

The correlations between the $\overline{X}_j\sqrt{n}$ variables in the joint distribution are the same as the correlations between the original X_j^i variables, as a result of the Central Limit Theorem. Since $E\left(X_j^i\right)=0$, the variance of X_j^i is simply obtained as $E\left({X_j^i}^2\right)$ for each $j = 1, 2, 3$. The symmetry of IWOC$(\boldsymbol{k})$ leads to

$$E\left({X_j^i}^2\right)=24\frac{k_1}{24}+30\frac{k_2}{36}+8\frac{k_3}{14}=k_1+\frac{5}{6}k_2+\frac{4}{7}k_3\,, \tag{5.55}$$

for each $j = 1, 2, 3$. A representation of the covariance term between the variables X_j^i and X_ℓ^i is equivalent to $E\left(X_j^i X_\ell^i\right)$ since $E\left(X_j^i\right)=E\left(X_\ell^i\right)=0$. Moreover, the definitions in Eq. 5.54 lead to

$X_j^i X_\ell^i = +1$ when either $C_1 \succ C_{j+1}$ and $C_1 \succ C_{\ell+1}$ or $C_{j+1} \succ C_1$ and $C_{\ell+1} \succ C_1$ (5.56)

$X_j^i X_\ell^i = 0$ when either $C_1 \sim C_{j+1}$ or $C_1 \sim C_{\ell+1}$

$X_j^i X_\ell^i = -1$ when either $C_{\ell+1} \succ C_1 \succ C_{j+1}$ or $C_{j+1} \succ C_1 \succ C_{\ell+1}$.

It then follows directly that

$X_j^i X_\ell^i = 1$ for the 16 weak orders in K_1 (5.57)
for the 18 weak orders in K_2
for the 4 weak orders in K_3

$X_j^i X_\ell^i = 0$ for none of the weak orders in K_1
for the 12 weak orders in K_2
for the 10 weak orders in K_3
for the 1 weak order in K_4

$X_j^i X_\ell^i = -1$ for the 8 weak orders in K_1
for the 6 weak orders in K_2
for none of the weak orders in K_3.

After algebraic reduction,

$$E\left(X_j^i X_\ell^i\right)=\frac{1}{3}k_1+\frac{1}{3}k_2+\frac{2}{7}k_3\,. \tag{5.58}$$

The correlation term between all pairs of variables X_j^i and X_ℓ^i, ρ_4, can then be obtained for $k_1+k_2+k_3>0$, so that $E\left({X_j^i}^2\right)>0$, with

$$\rho_4 = \frac{E\left(X_j^i X_\ell^i\right)}{E\left({X_j^i}^2\right)} = \frac{14k_1 + 14k_2 + 12k_3}{42k_1 + 35k_2 + 24k_3}. \tag{5.59}$$

The symmetry of IWOC$(\boldsymbol{k})$ with respect to candidates then leads to the representation

$$P_{PMRW}^S(4,\infty,IWOC(\boldsymbol{k})) = 4\Phi_3\left(\boldsymbol{R^3}(\rho_4)\right). \tag{5.60}$$

Following earlier arguments, $\boldsymbol{R^3}(\rho_4)$ is a correlation matrix on three variables in which all correlations are equal to ρ_4, and the representation in Eq. 5.60 can be reduced by using the trivariate extension of Sheppard's Theorem of Median Dichotomy to obtain

$$P_{PMRW}^S(4,\infty,IWOC(\boldsymbol{k})) = \frac{1}{2} + \frac{3}{\pi} Sin^{-1}(\rho_4), \tag{5.61}$$

for $k_1 + k_2 + k_3 > 0$.

Following the same procedure with m = 5, it is possible to develop a representation for $P_{PMRW}^S(5,\infty,IWOC(\boldsymbol{k}))$, with

$$P_{PMRW}^S(5,\infty,IWOC(\boldsymbol{k})) = 5\Phi_4\left(\boldsymbol{R^4}(\rho_5)\right), \tag{5.62}$$

where

$$\rho_5 = \frac{50k_1 + 50k_2 + 48k_3 + 40k_4}{150k_1 + 135k_2 + 114k_3 + 80k_4}, \tag{5.63}$$

for $k_1 + k_2 + k_3 + k_4 > 0$.

The representations in Eqs. 5.59 and 5.63 indicate that ρ_4 and ρ_5 are minimized at 1/3, with $k_1 > 0$ and $k_2 + k_3 = 0$ for $m = 4$, or $k_2 + k_3 + k_4 = 0$ for $m = 5$ They are maximized at 1/2, when $k_3 > 0$ and $k_1 + k_2 = 0$ for $m = 4$, or when $k_4 > 0$ and $k_1 + k_2 + k_3 = 0$ for $m = 5$ to require all voters who are not completely indifferent between candidates to have dichotomous preferences.

Table 5.5 lists computed vales for each of $P_{PMRW}^S(4,\infty,IWOC(\boldsymbol{k}))$ and $P_{PMRW}^S(5,\infty,IWOC(\boldsymbol{k}))$ for ρ_4 and ρ_5 equal to 1/3 and for each ρ_4 and ρ_5 equal to .350(.025).500. The values for $P_{PMRW}^S(4,\infty,IWOC(\boldsymbol{k}))$ are obtained with Eq. 5.61. The values of $P_{PMRW}^S(5,\infty,IWOC(\boldsymbol{k}))$ are obtained from tabular values from Gehrlein and Saniga (1975) where the representation that led to Eq. 4.56 was used to obtain extensive listings of values of $\Phi_4\left(\boldsymbol{R^4}(\rho_5)\right)$.

Lepelley and Martin (2001) consider a special case of $\text{IWOC}(\boldsymbol{k})$, denoted as IWOC*, that is very similar to the notion of IC. For three-candidate elections, IWOC* assumes that each of the thirteen possible weak ordered preference structures in Fig. 5.2 is equally likely to represent the individual preferences of a randomly selected voter. Thus, IWOC* is equivalent to the special case of $\text{IWOC}(\boldsymbol{k})$ with $k_1 = k_2 = 6/13$ and $k_3 = 1/13$ with m = 3. When m = 4, IWOC* is the special case of $\text{IWOC}(\boldsymbol{k})$ with $k_1 = 24/75$, $k_2 = 36/75$, $k_3 = 14/75$ and $k_4 = 1/75$. Representations are obtained for the limiting probability as $n \to \infty$ for $P_{PMRW}^{S}(3,\infty,IWOC*)$ and $P_{PMRW}^{S}(4,\infty,IWOC*)$, with

$$P_{PMRW}^{S}(3,\infty,IWOC*) = \frac{3}{2} - \frac{3}{2\pi}Cos^{-1}\left(\frac{2}{5}\right) = .9465. \tag{5.64}$$

$$P_{PMRW}^{S}(4,\infty,IWOC*) = 2 - \frac{3}{\pi}Cos^{-1}\left(\frac{12}{31}\right) = .8792. \tag{5.65}$$

The representations for $P_{PMRW}^{S}(3,\infty,IWOC*)$ and $P_{PMRW}^{S}(4,\infty,IWOC*)$ in Eqs. 5.64 and 5.65 are respectively equivalent to the representation in Eqs. 5.52 and 5.61 for this special case. Van Deemen (1999) presents some limited computed values of $P_{PMRW}^{S}(3,3,IWOC*)$ and $P_{PMRC}^{S}(3,3,IWOC*)$ that are obtained by enumeration.

Jones, et al. (1995) do a Monte-Carlo Simulation study to obtain estimates of $P_{PMRW}^{S}(m,n,IWOC*)$. Results lead to the conjecture that $P_{PMRW}^{S}(m,n,IWOC*)$ is minimized, for a given m, when $n = m$. In addition, $P_{PMRW}^{S}(m,n,IWOC*)$ is conjectured to decrease monotonically, for a given m, as n increases for $n \le m$, while it increases monotonically as n increases for $n \ge m$.

Lepelley and Martin (2001) also consider the probability that a PMRW exists under the assumption of the *Impartial Anonymous Weak Ordered Culture Condition* (*IAWOC*). Following the notion of IAC, the use of IAWOC assumes that each possible voting situation with a specified number of voters is equally likely to be observed, given that individual voters have weak ordered preferences on candidates, as listed in Fig. 5.2. Representations for both $P_{PMRW}^{S}(3,n,IAWOC)$ and $P_{PMRC}^{S}(3,n,IAWOC)$ are obtained for all odd n, for which n + 1 is a multiple of four, with

$$P_{PMRW}^{S}(3,n,IAWOC) = \frac{3\left(37n^6 + 1968n^5 + 41191n^4 + 429864n^3 + 2341099n^2 + 6337896n + 6814665\right)}{512(n+12)(n+10)(n+8)(n+6)(n+4)(n+2)}. \tag{5.66}$$

$$P_{PMRC}^{S}(3,n,IAWOC)= \tag{5.67}$$
$$\frac{(n+13)(n+17)(n+21)\begin{bmatrix} n^6+66n^5+1603n^4+17292n^3+ \\ 75967n^2+63426n-156915 \end{bmatrix}}{4096(n+12)(n+11)(n+10)(n+8)(n+7)(n+6)(n+4)(n+3)(n+2)}.$$

The probability representations in Eqs. 5.66 and 5.67 are based on having *AMB* for candidates *A* and *B* if $A \succ B$ for $(n+1)/2$ or more voters. The PMR relationship on any pair of candidates must therefore hold on the basis of a majority of all voters in these two representations, not just for a majority of voters who have a preference on the pair.

Crès (2001) is an extension of earlier work that related to the possible existence of supermajority cycles. The primary result in the study is that the use of $\mathrm{PM}^{\tau}\mathrm{R}$ with rate $\tau \geq 1-\frac{1}{\hbar}$ is necessary and sufficient to prohibit the existence of $\mathrm{PM}^{\tau}\mathrm{R}$ cycles of any length, where $\hbar$ is the maximum number of allowable equivalence sets in any voter's weak ordered preference rankings on candidates. Increased levels of indifference, as reflected by smaller values of $\hbar$, therefore reduces the value of τ that is required to prevent the existence of $\mathrm{PM}^{\tau}\mathrm{R}$ cycles.

5.8 The Impact of Intransitive Voter Preferences

It was stressed in Chapter 1 that individual voters who displayed intransitive preferences were viewed as acting irrationally. Rose (1957) presents an interesting early study that evaluates the presence of intransitive responses in paired comparison responses of individual subjects in repeated experiments. Three types of intransitive responses were observed. Intransitivity due to random error occurs when a subject is indifferent between all alternatives in some subset and intransitive responses result because the subject makes random preference selections on pairs from that indifference subset when forced to give a preference response. Intransitivity due to carelessness occurs when a subject makes clerical errors in reporting pairwise preferences, or simply does not give adequate thought to determining true pairwise preferences before responding. True intransitivity refers to actual cyclic preferences that a subject would consistently report after giving adequate thought to determining the true preferences on alternatives. Most reported intransitivities in this empirical study are explained as being due to either random error or carelessness. It is concluded that the evidence shows that examples of true intransitivity appear are a very rare phenomenon, if they exist at all.

Van Acker (1990) surveys much of the work that is related to finding various types of intransitivities in individual preferences, referring to the three types of intransitivities above, as representing “inconsistent behavior”, “mistakes” and “genotypic intransitivity”. Van Acker also surveys much of the work that attempts to explain the occasional presence of various forms of individual intransitivity of preference in empirical studies. This particular topic will be addressed in detail in Chapter 7.

Fishburn and Gehrlein (1980b) develop a representation for the probability that a PMRW exists when voters can have intransitive preferences. Begin by defining a partition of all possible individual preference structures on three candidates $A, B, C\}$ into six subsets, denoted as R_i for 1 = 1, 2, …, 6. The various types of individual voter preference structures are displayed in Fig. 5.3.

Preference Subset	Prototype Structure	$\# R_i$
R_1	$A \sim B, A \succ C, B \succ C$	6
R_2	$A \succ B, A \succ C, B \succ C$	6
R_3	$A \sim B, A \succ C, B \sim C$	6
R_4	$A \succ B, A \sim C, B \succ C$	6
R_5	$A \succ B, C \succ A, B \succ C$	2
R_6	$A \sim B, A \sim C, B \sim C$	1

Fig. 5.3 Six types on individual preference structures on thee candidates

Preference subset R_1 corresponds to all dichotomous preferences, while subset R_2 corresponds to all linear preference rankings and R_6 corresponds to the situation of complete indifference between rankings. Preference subset R_5 contains individual voter's preferences that are intransitive. Voter preference structures in subsets R_3 and R_4 are *quasi-transitive*. Let $C_i \succ' C_j$ denote that a voter either prefers C_i to C_j or is indifferent between the two candidates. A voter's preference ranking on candidates is quasi-transitive if $C_i \succ' C_j$ and $C_j \succ' C_k$ require that $C_i \succ' C_k$, while maintaining transitivity on pairwise preference comparisons. The analysis in Fishburn and Gehrlein (1980b) did not consider the possibility of complete indifference between candidates, but the results are easily extended to include R_6 as a possibility.

Let $\boldsymbol{r}$ denote a six-dimensional vector such that r_i is the probability that a randomly selected voter has a preference structure in R_i. The assumption of *Permutation Invariance* ($PI(\boldsymbol{r})$) assumes that all preferences within any R_i are equally likely to be observed, with probability equal to $r_i / \# R_i$. Fishburn and Gehrlein (1980b) develop a representation for the limiting probability, $P_{PMRW}^S(3, \infty, PI(\boldsymbol{r}))$, that a PMRW exists as $n \to \infty$, that can be applied to this case, when $r_6 < 1$, to lead to

$$P_{PMRW}^{S}(3,\infty,PI(\boldsymbol{r}))=\frac{3}{4}+\frac{3}{2\pi}Sin^{-1}(\rho_3'), \tag{5.68}$$

where

$$\rho_3'=\frac{r_1+r_2-r_4-3r_5}{2r_1+3r_2+r_3+2r_4+3r_5}. \tag{5.69}$$

The minimum correlation coefficient value for Eq. 5.69 has $\rho_3'=-1$ when $r_5>0$ with $r_1+r_2+r_3+r_4=0$, and the maximum value is $\rho_3'=1/2$ when $r_1>0$ with $r_2+r_3+r_4+r_5=0$. Simple analysis will show that $P_{PMRW}^{S}(3,\infty,PI(\boldsymbol{r}))$ decreases if r_5 increases, while any other r_i decreases accordingly.

Table 5.6 lists calculated values of $P_{PMRW}^{S}(3,\infty,PI(\boldsymbol{r}))$ for each value of $\rho_3'=-1.0(.25).30$ from Eq. 5.68. The identical form of Eqs. 5.52 and 5.68 allows the determination of $P_{PMRW}^{S}(3,\infty,PI(\boldsymbol{r}))$ for $\rho_3'>1/3$ from values that are associated with $P_{PMRW}^{S}(3,\infty,IWOC(\boldsymbol{k}))$ in Table 5.5.

Table 5.6 Computed values of $P_{PMRW}^{S}(3,\infty,PI(\boldsymbol{r}))$

ρ	$P_{PMRW}^{S}(3,\infty,PI(\boldsymbol{r}))$
-1.00	.0000
-.75	.3451
-.50	.5000
-.25	.6294
.00	.7500
.25	.8706
.30	.8955

Roff (1977) considers the impact that irrational voters have on the probability that a PMRW exists in three-candidate elections. The basis of the study is a variation of IC, and is connected to the notion of Permutation Invariance. Let *r* denote the proportion of voters with rational preferences that are complete and transitive. Each of these six possible linear rankings has a probability of selection for a voter's preference equal to *r*/6. There are two possible cycles on three alternatives, and each cycle represents the preferences of a voter with probability $(1-r)/2$. This is equivalent to the notion of Permutation Invariance with all cases of voter indifference being disallowed, and the probability that a PMRW exists is calculated for small *n*.

Jain (1986) considers necessary and sufficient conditions for PM^{τ}R to be quasi-transitive when voters' preferences on candidates are quasi-transitive. The conditions that are necessary and sufficient to require transitivity of PM^{τ}R are found to be linked to the degree of Latin Square agreement in voter preference profiles.

5.9 The Impact of Uncertainty

Shepsle (1970,1972) considers PMR cycles in the context of positions or platforms that candidates might take in an election. Voters will form preference rankings on candidates based on the positions that the candidates adopt. For three possible positions $P_1, P_2, P_3\}$, we consider three equally sized blocs of voters with preference rankings on the positions as shown in Fig. 5.4

Bloc 1	Bloc 2	Bloc 3
P_1	P_2	P_3
P_2	P_3	P_1
P_3	P_1	P_2

Fig. 5.4 An example voting situation of bloc preferences on three possible candidate positions

Thus, a candidate can not choose any position to adopt from $P_1, P_2, P_3\}$ such that the electorate would select that candidate as a PMRW, given the voter blocs' preferences on these possible platforms. However, a candidate might introduce "uncertainty" into the process by not taking a precise stand on the issues. As a result, the selection of the candidate would amount to selecting a "lottery" on the positions. It is shown that such a "lottery position" might be the PMRW in comparison to the three specific positions. Necessary and sufficient conditions are given so that a lottery position must exist as a PMRW in such situations. Dacey (1979) addresses the notion of how such a "lottery" might be created through the use of ambiguous candidate statements about their true positions on issues.

Holler (1980,1982) addresses this same problem of PMR cycles in the context of candidates selecting positions or platforms, given the known preference rankings of voters over the positions. The preference rankings that the candidates themselves have for adopting the possible positions are brought in as an additional issue. The candidates are then concerned not only with adopting a position to maximize the likelihood that they will be a PMRW with the voters, but also with the utility that they themselves have for adopting, and supposedly implementing, that particular position. Analysis suggests that the addition of this additional information to the candidate's problem of selecting a position, or taking the same position as another candidate should tend to lead to a stable, or non-cyclical, selection process. Petry (1982) surveys earlier work, based on spatial voting models, regarding candidates who select positions to either maximize the number of votes that they receive or to maximize their utility from adopting positions.

Shepsle (1970,1972) is an extension of the work of Zeckhauser (1969), who showed that the reverse situation could exist. That is, there can be a PMRW on pure candidate positions, and the introduction of a lottery on possible candidate positions could create an unstable situation in which there is no equilibrium position for candidates to select. Fishburn (1972), McKelvey and Richelson (1974) and Flood (1980) develop similar observations as those in Zeckhauser's analysis.

5.10 Conclusion

A number of societal factors have been found to have an impact on the probability that a PMRW exists. The factor that has been most thoroughly examined in this context is the degree of coherence or consistency of voters' preferences, as measured according to some definition of social homogeneity. Numerous studies have sought to find general relationships between these measures of social homogeneity and the probability that a PMRW exists. Some theoretical studies have successfully shown that such relationships can exist on an expected value basis, without measuring the overall strength of the relationships. Strong evidence has also been provided to indicate that simple measures of social homogeneity are very ineffective at showing such a relationship at all. In particular, the connection between some of these measures of social homogeneity and the probability that a PMRW exists is found to be reversed, based on whether the number of voters is restricted to be odd or even. If such a general relationship is to be found with a significant degree of strength, the measure of homogeneity has to be based on an understanding of the mechanism by which voters preferences are formed. The proximity of the voters' preferences to single-peaked preferences is an example of such a measure of social homogeneity.

Other significant societal factors that have an impact on the probability that a PMRW exists include: the propensity of voters to abstain from the election, the degree of voter indifference between candidates, the propensity of voters to have intransitive preferences, and the presence of uncertainty regarding the exact position that candidates represent on issues.

6 The Impact of Coherent Preferences

6.1 Introduction

The possibility that a PMRW does not exist, to result in an occurrence of Condorcet's Paradox, has been seen to be a potentially significant threat to the stability of election processes, if the conditions of assumptions like IC and IAC are valid in a given situation. Moreover, we have found that when simple measures of social homogeneity are used to evaluate the level of coherence of voters' preferences, it is difficult to observe a strong general relationship between homogeneity and the probability that a PMRW exists. When voters' preferences are formed by a process that imposes some internal structural consistency or coherence to voter preference profiles or voting situations, much stronger relationships can be found between measures of homogeneity and the probability that a PMRW exists. One such assumption is met if voters' preferences are consistent with the condition of single-peaked preferences, which assures the existence of transitive PMR relationships for odd n. However, this is generally a very restrictive assumption, which led Niemi (1969) to propose a measure of proximity to single-peaked preferences to gauge social homogeneity. The goal of this chapter is to develop representations to evaluate the impact that several similar types of internal structural coherence or consistency will have on the probability that a PMRW exists.

6.2 Methods for Obtaining Representations

Methods that can be used to obtain simple closed-form representations for the probability that election outcomes are observed with the assumption of IAC or MC were developed in Chapter 3. With these assumptions, the representations were obtained by using counting arguments to determine the number of voting situations that result in a given outcome.

Figure 6.1 shows the six possible linear preference rankings that voters might have on three candidates, $A,B,C\}$, following the development of Figure 3.2. Consistent with previous discussion n_i denotes the number of voters with the associated linear preference ranking, and $n = \sum_{i=1}^{6} n_i$.

A	A	B	C	B	C
B	C	A	A	C	B
C	B	C	B	A	A
n_1	n_2	n_3	n_4	n_5	n_6

Fig. 6.1 Voting situations with individual linear preference rankings for three-candidate elections

A representation was developed for $P^S_{PMRW}(3,n,IAC)$ with odd n in Chapter 3 by starting with a definition of the conditions on n_i's that require Candidate A to be the PMRW in a voting situation. Following the logic that led to Eq 3.21, this event is observed for any voting situation in which

$$\begin{aligned} n_3 + n_5 + n_6 &\le \frac{n-1}{2} \Rightarrow A\boldsymbol{M}B \\ n_4 + n_5 + n_6 &\le \frac{n-1}{2} \Rightarrow A\boldsymbol{M}C. \end{aligned} \tag{6.1}$$

The specific restrictions on the individual n_i's that are necessary to lead to the outcome in Eq. 6.1 are developed in the discussion that led to Eq. 3.22, with

$$\begin{aligned} &0 \le n_6 \le \frac{n-1}{2} \\ &0 \le n_5 \le \frac{n-1}{2} - n_6 \\ &0 \le n_4 \le \frac{n-1}{2} - n_6 - n_5 \\ &0 \le n_3 \le \frac{n-1}{2} - n_6 - n_5 \\ &0 \le n_2 \le n - n_6 - n_5 - n_4 - n_3 \\ &n_1 = n - n_6 - n_5 - n_4 - n_3 - n_2. \end{aligned} \tag{6.2}$$

A representation for the total number of voting situations, $N^{A\}}_{PMRW}(3,n,IAC)$, for which A is the strict PMRW, with the restrictions on n_i's that are defined in Eq. 6.2, is developed in Eq 3.23 with

$$N^{A\}}_{PMRW}(3,n,IAC) = \sum_{n_6=0}^{\frac{n-1}{2}} \sum_{n_5=0}^{\frac{n-1}{2}-n_6} \sum_{n_4=0}^{\frac{n-1}{2}-n_6-n_5} \sum_{n_3=0}^{\frac{n-1}{2}-n_6-n_5} \sum_{n_2=0}^{n-n_6-n_5-n_4-n_3} 1, \quad \text{for odd } n. \tag{6.3}$$

The symmetry of IAC with respect to candidates leads to the identity

$$P^S_{PMRW}(3,n,IAC) = \frac{3N^{A\}}_{PMRW}(3,n,IAC)}{K(3,n,IAC)}. \tag{6.4}$$

Here, $K(3,n,IAC)$ is the total number of possible voting situations with three candidates, following the notion of IAC with fixed n, and

$$K(3,n,IAC) = \frac{\prod_{i=1}^{5}(n+i)}{120}. \tag{6.5}$$

A simple closed-form representation for $N_{PMRW}^{A\}}(3,n,IAC)$ was obtained by algebraic methods in Chapter 3 by using known relations for sums of powers of integers to sequentially make simplifying reductions to the representation in Eq. 6.3. This sequential reduction process was found to be cumbersome, but it follows a very simple and direct logic.

Simple closed form equations for representations of the type shown in Eq. 6.3, are currently very easy to obtain with standard software packages, as long as the necessary conditions for an event to occur result in bounds on the upper and lower summation indexes are like those specified above in Eq 6.3. That is, where each upper and lower summation bound is expressed as a simple linear function of n and of n_i's that are defined earlier in the sequence of summation indexes.

We define this as the *simple linear form restriction*, which also requires that each of the coefficients in the linear equations that bound the summation indexes are rational numbers that can be expressed as ratios of integer numbers. Huang and Chua (2000) note that a generalization can be made when a representation is being developed for the count of the number of voting situations that meet conditions that have a simple linear form restriction. In particular, the general form of the identities for sums of powers of integers requires that the resulting representation for the count of voting situations must be expressible as a polynomial in n. With five summation signs in the function, the degree of the polynomial must be five or less. Moreover, the constants in the polynomial must also be rational numbers. Huang and Chua (2000) then suggest that this could lead to an easier way to obtain representations for IAC probabilities than using the cumbersome process of sequential algebraic reduction. These arguments can easily be extended to representations with MC, by replacing n with L in the discussion above.

6.2.1 EUPIA

Gehrlein (2002b) develops a computer algorithm, *EUPIA* (*Effectively Unlimited Precision Integer Arithmetic*), to efficiently implement the basic notions from Huang and Chua to obtain closed form probability representations for election outcomes with IAC and MC. To describe how this procedure works, let $E^A(n)$ denote the number of voting situations for which Candidate A meets the conditions of voting Event F with n voters. Based on the preceding discussion:

Axiom 6.1 If the restrictions on n_i's that are necessary for Event F to be observed in a voting situation for a three-candidate election meet the simple linear form restriction, then

$$E^A(n) = \sum_{i=0}^{5} \tau_i n^i \,, \tag{6.6}$$

for some integer sequence $n = \psi + pj$, with $j = 0, 1, 2, \ldots$.

Here, each τ_i coefficient is a rational number that is expressible as the ratio of two integers. The term p is the periodicity of the representation, and ψ is the starting point of the integer sequence for which the given representation is valid. As observed in Chapter 3, the representation for $N_{PMRW}^{A\}}(3,n,IAC)$ in Eq. 3.26 has $p = 2$ and $\psi = 3$, since it is only valid for odd $n \geq 3$, and the representation for $K(3,n,IAC)$ in Eq. 6.5 has $p = 1$ and $\psi = 1$, since it is valid for all positive integers.

The periodicity of the series of n values for which a given representation is valid is driven by restrictions that are needed to keep all summation limits at integer values. For example, suppose that a summation limit contains the term $\frac{n+x}{y}$ for integer constants x and y. To keep this ratio integer valued, it can only hold for a series of n values with periodicity y. The specific values of n that are used in a sequence with a specified periodicity must also be such that the ratios are integer valued, so that $(n + x)$ must be an integer multiple of y. This has a direct impact on the starting point, ψ, which can be used for the series.

Suppose that we arbitrarily fix ψ and p, and use computer enumeration techniques to evaluate the exact integer values for the number of voting situations, $NVS^A(\psi + pj)$, for which Candidate A meets the conditions of Event F with $\psi + pj$ voters, for each $j = 0(1)5$. We then use the computed values of $NVS^A(\psi + pj)$ to establish six simultaneous equations of the form

$$E^A(\psi + pj) = NVS^A(\psi + pj). \tag{6.7}$$

The τ_i terms in the $E^A(\psi + pj)$ functions from Eq. 6.6 are identical in all six of these equations for each given i, and they can then be found by using precise algebraic methods to solve these six simultaneous equations with six unknowns.

For example, suppose that we wish to determine the coefficients of $E^A(n)$ for the event that A is the PMRW for odd n, so that $\psi = 3$ and $p = 2$. As a first step, computer enumeration is used to obtain the values $NVS^A(3) = 18$, $NVS^A(5) = 80$, $NVS^A(7) = 250$, $NVS^A(9) = 630$, $NVS^A(11) = 1372$ and $NVS^A(13) = 2688$.

Using these computed values with Eqs. 6.6 and 6.7, we then set up six simultaneous equations with six unknowns $\tau_0, \tau_1, \tau_2, \tau_3, \tau_4, \tau_5,\}$ that correspond to the six rational coefficients in $E^A(n)$:

$$\begin{aligned}
&\tau_0 + \tau_1 3 + \tau_2 3^2 + \tau_3 3^3 + \tau_4 3^4 + \tau_5 3^5 = 18 \\
&\tau_0 + \tau_1 5 + \tau_2 5^2 + \tau_3 5^3 + \tau_4 5^4 + \tau_5 5^5 = 80 \\
&\tau_0 + \tau_1 7 + \tau_2 7^2 + \tau_3 7^3 + \tau_4 7^4 + \tau_5 7^5 = 250 \\
&\tau_0 + \tau_1 9 + \tau_2 9^2 + \tau_3 9^3 + \tau_4 9^4 + \tau_5 9^5 = 630 \\
&\tau_0 + \tau_1 11 + \tau_2 11^2 + \tau_3 11^3 + \tau_4 11^4 + \tau_5 11^5 = 1372 \\
&\tau_0 + \tau_1 13 + \tau_2 13^2 + \tau_3 13^3 + \tau_4 13^4 + \tau_5 13^5 = 2688.
\end{aligned} \tag{6.8}$$

Algebraic reduction is then used in Eq. 6.8 to solve for the six unknown variables, to find:

$$\begin{aligned}
&\tau_0 = \frac{45}{128} \quad \tau_1 = \frac{99}{128} \quad \tau_2 = \frac{39}{64} \\
&\tau_3 = \frac{43}{192} \quad \tau_4 = \frac{5}{128} \quad \tau_5 = \frac{1}{384}.
\end{aligned} \tag{6.9}$$

The resulting representation for $E^A(n)$ from Eq. 6.9 is therefore identical to the representation for $N_{PMRW}^{A\}}(3,n,IAC)$ in Eq. 3.26. The procedure is obviously very simple to implement when ψ and p are known in advance.

When ψ and p are not known in advance EUPIA performs an additional search in order to determine them. Suppose that we arbitrarily fix ψ at a relatively large number, and start the process with $p = 1$. Computer enumeration is then used to evaluate the exact integer values of, $NVS^A(\psi + pj)$ such that A meets the conditions of Event F for each j = 0(1)7. The first six computed values of $NVS^A(\psi + pj)$, with j = 0(1)5, are then used to establish the six simultaneous equations of the form in Eq. 6.7, and the resulting functional form of $E^A(n)$ is obtained. A functional form must always exist to fit the six equations with six unknowns. However, if the true periodicity for the representation does not actually have $p = 1$, the functional form that has just been obtained for $E^A(n)$ will not accurately give values of $E^A(n)$ for $n > \psi + 5p$.

EUPIA therefore determines if the $E^A(n)$ function that has just been obtained by the procedure will correctly determine the computer enumeration values of $NVS^A(\psi + pj)$ for each j = 6, 7. If the numerical values from the computer enumeration and the derived $E^A(n)$ are identical for each j = 6, 7, it is concluded that the correct $E^A(n)$ representation and p have been found, for the given ψ. If these results do not match, then the correct periodicity is not being used to obtain the $E^A(n)$ representation. In this case, EUPIA iterates through this process and sequentially increases p, while keeping ψ fixed, until the computer enumeration results and the derived $E^A(n)$ that has been obtained for the iteration are identical with j = 6, 7. The minimum value of ψ for which the representation $E^A(n)$ is correct can easily be determined by finding the smallest value of n for which the obtained representation matches computer enumeration results; given that the determined periodicity is maintained as the number of voters is reduced from the ψ that was arbitrarily set to use the EUPIA procedure.

The necessary conditions that are given in Eq. 6 .2 to identify voting situations that are included in $N_{PMRW}^{A\}}(3,n,IAC)$ clearly result in summation limits in Eq. 6.3 that meet the definition of the simple linear form restriction. However, if we consider the restrictions in Eq. 3.41 that identify voting situations that are included in $N_{PMRW}^{A*\}}(3,n,MC)$, a much more complicated situation results, due to the presence of *Max* and *Min* arguments on sets of linear functions in the summation bounds. This complication was dealt with in Chapter 3 by partitioning the set of all voting situations that are included in $N_{PMRW}^{A*\}}(3,n,MC)$ into thirteen subspaces, such that each of these subspaces has summation bounds that meet the simple linear form restriction. As a result, each of the thirteen subspaces has a representation for the number of voting situations that it contains that is of the form of Eq. 6.6. It follows directly that the ultimate representation for $N_{PMRW}^{A*\}}(3,n,MC)$ that is obtained by accumulating the associated representations for the thirteen subspaces must also have a form like that shown in Eq. 6.6. There must be some periodicity for the accumulated representation that is consistent with the periodicities of all of the individual subspace representations.

Following this logic, it is easy to generalize the earlier definition the simple linear form restriction to include situations in which each upper and lower summation bound is expressed as the *Max* or *Min* of some set of simple linear functions of n and of n_i's that are previously defined in the series of summation indexes. As with the original definition, the coefficients in the simple linear functions must be rational numbers. As discussed above, these arguments can easily be extended to representations with either IAC or MC

A number of probability representations for voting outcomes are obtained with the use of the EUPIA procedure in Gehrlein (2002b, 2003, 2006a). Representations are obtained in these studies with the assumptions of IAC, MC, and the condition in which all single-peaked voting situations are assumed to be equally likely to be observed.

6.2.2 EUPIA2

Gehrlein (2004c, 2006b) develops an extension of EUPIA that obtains representations for the conditional probability that voting outcomes are observed, given that voting situations are constrained to have some specified value of a measurable parameter. The representation for $P_{PMRW}^{S}\left(3,n \mid CIAC_{\varphi}\left(k^{+}\right)\right)$ in Eq. 5.33 is an example of such a result. The measurable parameter of that is of primary interest at this point of our analysis is the proximity of voting situations to the condition of perfectly single-peaked preferences.

As discussed in Chapter 5, Niemi (1969) considers an extension of the strict definition of single-peaked preferences. In particular, the maximum proportion of n voters in any voting situation who have preference rankings that are single-peaked along some common dimension is used as a rough measure for proximity of the voting situation to being perfectly single peaked. It was also noted in Chapter 2 that Arrow (1963) proves that a voting situation on three candidates represents perfectly single-peaked preferences if some candidate is never ranked as least preferred by any voter. Gehrlein (2004c) adopts a hybrid of these ideas by using the number, b, that measures the minimum number of times that some candidate is bottom ranked, or least preferred, in the preferences of n voters to serve as a simple measure of the proximity of a voting situation to being perfectly single-peaked in three-candidate elections, where

$$b = Min\{n_1 + n_3, n_2 + n_4, n_5 + n_6\}. \tag{6.10}$$

If b is equal to zero for a voting situation with three candidates, it represents perfectly single-peaked preferences, and when b is maximized at $n/3$, a voting situation reflects very disperse preferences of voters over candidates to reflect a situation that is very far removed from perfect single-peakedness. Another perspective on this issue is that a voting situation with a small parameter b reflects a situation in which there is some candidate that very few voters think is the worst of the three candidates. In that sense, this candidate can be viewed as a *positively unifying candidate* that voters would not generally think of as reflecting the worst outcome if that candidate were to be elected.

Following the definitions of Chapter 5, let $IAC_b(k)$ denote an extension of IAC that is conditional on the statement that attention is restricted only to voting situations with a specified value of $b = k$. In particular, all such voting situations are assumed to be equally likely to be observed. The conditional probability that a strict PMRW exists for n voters with three candidates, given the assumption of $IAC_b(k)$, is denoted by $P_{PMRW}^{S}(3,n \mid IAC_b(k))$. Following the logic that led to Eq. 6.4,

$$P_{PMRW}^{S}(3,n \mid IAC_b(k)) = \frac{3N_{PMRW}^{A\}}(3,n,IAC_b(k))}{K(3,n,IAC_b(k))}. \tag{6.11}$$

Here, $N_{PMRW}^{A\}}(3, n, IAC_b(k))$ and $K(3, n, IAC_b(k))$ are defined in the obvious fashion, following the development of Eq. 6.4.

Gehrlein (2006b) develops *EUPIA2* to obtain representations for functions like $N_{PMRW}^{A\}}(3, n, IAC_b(k))$ and $K(3, n, IAC_b(k))$. Under the assumption of IAC, EUPIA obtains a representation for the number of voting situations, $E^A(n)$, such that the n_i's meet the necessary conditions for Candidate *A* to meet the requirements of Event *F*. With the assumption of $IAC_b(k)$, EUPIA2 obtains a representation for the number of voting situations, $E^A(n,k)$, such that the n_i's both meet the necessary conditions for Candidate *A* to meet the requirements of Event *F* and meet the necessary conditions to match a specified integer value *k* for some defined parameter of the voting situation, like *b*.

The *extended linear form restriction* requires that each upper and lower summation bound on the representation to obtain $E^A(n,k)$ is expressed as the *Max* or *Min* of some set of simple linear functions of *n*, a specified *k* for some defined parameter and n_i's that are previously defined in the series of summation indexes. As with the definition of a simple linear form restriction, the coefficients in the simple linear functions must be rational numbers. Given the nature of identities for sums of powers of integers, it obvious that:

Axiom 6.2 If the restrictions on n_i's in a three-candidate voting situation that are necessary for Event *F* to be observed and also meet the necessary conditions to match a specified integer value *k* for some defined parameter meet the extended linear form restriction, then

$$E^A(n,k) = \sum_{i=0}^{5} \sum_{j=0}^{5-i} \tau_{ij} n^i k^j \tag{6.12}$$

for some integer sequence $n = \psi + p\upsilon$, with $\upsilon = 0, 1, 2, \ldots$.

As in Axiom 6.1, the τ_{ij} coefficients are rational numbers, and these arguments can easily be extended to representations with MC by replacing *n* with *L* in the definition of the extended linear form restriction.

It follows easily from results in Gehrlein (2004c) that:

Axiom 6.3 If that the necessary conditions to obtain $E^A(n)$ for some Event *F* in a three-candidate election meet the simple linear form restriction, then $E^A(n,k)$ must result in a functional form as specified in Eq. 6.12, if *k* is a specified value for parameter *b*.

Proof. This result is easily proved by noting that $E^A(n,k)$ can be obtained as the sum of three individual functions in this case. Each function is obtained by starting with the restrictions on the n_i's that require Candidate A to meet the conditions of Event F, and an additional set of restrictions is added in each case. Following the definition of b in Eq. 6.10, the first function adds restrictions such that $n_1 + n_3 = k$, $n_2 + n_4 \ge k$ and $n_5 + n_6 \ge k$. The second function adds restrictions such that $n_1 + n_3 \ge k+1$, $n_2 + n_4 = k$ and $n_5 + n_6 \ge k$. The third function adds restrictions such that $n_1 + n_3 \ge k+1$, $n_2 + n_4 \ge k+1$ and $n_5 + n_6 = k$. Assuming that the necessary conditions that define $E^A(n)$ meet the simple linear form restriction, the addition of the restrictions for each of these three functions will obviously meet the extended linear form restriction for each function. Based on Axiom 6.2, each of the three individual functions must then have the general form of Eq. 6.12, and so must their sum. **QED**

The first step in developing a representation for $P^S_{PMRW}(3,n \mid IAC_b(k))$ with EUPIA2 is to obtain a representation for the number of voting situations, $K(3,n,IAC_b(k))$, with n voters that have a specified value, k, for parameter b, as defined in Eq. 6.10. The representation for $K(3,n,IAC)$ in Eq. 3.28 is clearly consistent with the simple linear form restriction, so Axiom 6.3 requires that the representation for $K(3,n,IAC_b(k))$ must have the general form of Eq. 6.12.

The EUPIA2 process starts by fixing k at a specified numerical value and using computer enumeration procedures to obtain values of $NVS^A(\psi + pj \mid k)$ for each value of $j = 0(1)7$. In this case, $NVS^A(\psi + pj \mid k)$ is a count of the number of voting situations with $\psi + pj$ voters for which parameter b is equal to the specified value of k. In addition, k is treated as a constant in Eq. 6.12, so that the general form can be reduced to a linear function with a single variable, n, as in Eq. 6.6, for that specified k, with the k^j term being absorbed into the τ_{ij} term in Eq. 6.12.

EUPIA is then used directly to find the conditional representation for $K(3,n,IAC_b(k))$, denoted as $K(3,n,IAC_b(k) \mid k)$, when the k value has been specified at the fixed value, and

$$K(3,n,IAC_b(k) \mid k) = \sum_{i=0}^{5} C_i^k n^i \,, \tag{6.13}$$

for some integer sequence $n = \psi + pj$, with $j = 0, 1, 2, \ldots$

The process is repeated for each integer k value with $0 \le k < n/3$, and the C_i^k terms that are obtained for these $K(3,n,IAC_b(k) \mid k)$ representations will typically be different for each given k. For the process to work effectively, we need to start the search process in EUPIA2 with a relatively large value of ψ. Table 6.1 summarizes the C_i^k values that were obtained for $0 \le i \le 3$ for each $0 \le k \le 11$ when

EUPIA2 was run while arbitrarily setting $\psi = 35$ in all cases. The results give $C_i^k = 0$, for all $i \geq 4$, and the periodicity for all cases is found to have $p = 1$. Furthermore, additional EUPIA2 runs were performed to verify that the relevant entries in Table 6.1 remain valid for all integer values of $\psi \geq 1$.

A representation for $K(3, n, IAC_b(k) \mid k)$ can then be obtained very easily for any specified k in the range $0 \leq k \leq (n-2)/3$ by using the known form of the representation in Eq. 6.13 along with the C_i^k entries in Table 6.1.

Table 6.1 Computed C_i^k values with the specified k for ψ = 35 and $p = 1$

k	C_0^k	C_1^k	C_2^k	C_3^k
0	0	5/2	3	1/2
1	12	-22	3	1
2	171	-165/2	0	3/2
3	720	-188	-6	2
4	2010	-695/2	-15	5/2
5	4500	-570	-27	3
6	8757	-1729/2	-42	7/2
7	15456	-1240	-60	4
8	25380	-3411/2	-81	9/2
9	39420	-2270	-105	5
10	58575	-5885/2	-132	11/2
11	83952	-3732	-162	6

The general form of the representations that are given in Eqs. 6.6 and 6.12, along with the specific representation for $K(3, n, IAC_b(k) \mid k)$ that is given in Eq. 6.13, lead directly to the conclusion that each C_i^k coefficient must be obtainable from a function of k, with

$$C_i^k = \sum_{j=0}^{5-i} \partial_{ij} k^j , \tag{6.14}$$

for some rational ∂_{ij} coefficients for a specified i.

Following the earlier logic of EUPIA with a specified i, the known values of C_i^k that are given in Table 6.1 can be used for k = 0, 1, 2,…, 5-i to establish a set of 6-i simultaneous equations, following the format of Eq. 6.14, with $6-i$ unknowns. The solution of the $6-i$ simultaneous equations will give the $6-i$ values of the ∂_{ij} coefficients in the general representation for C_i^k. When the particular case with $i = 0$ is considered, six variables $\partial_{00}, \partial_{01}, \partial_{02}, \partial_{03}, \partial_{04}, \partial_{05}, \}$ are defined. Using the associated entries from Table 6.1, the six simultaneous equations are given by:

$$\partial_{00} + \partial_{01}0 + \partial_{02}0^2 + \partial_{03}0^3 + \partial_{04}0^4 + \partial_{05}0^5 = 0 \tag{6.15}$$
$$\partial_{00} + \partial_{01}1 + \partial_{02}1^2 + \partial_{03}1^3 + \partial_{04}1^4 + \partial_{05}1^5 = 12$$
$$\partial_{00} + \partial_{01}2 + \partial_{02}2^2 + \partial_{03}2^3 + \partial_{04}2^4 + \partial_{05}2^5 = 171$$
$$\partial_{00} + \partial_{01}3 + \partial_{02}3^2 + \partial_{03}3^3 + \partial_{04}3^4 + \partial_{05}3^5 = 720$$
$$\partial_{00} + \partial_{01}4 + \partial_{02}4^2 + \partial_{03}4^3 + \partial_{04}4^4 + \partial_{05}4^5 = 2010$$
$$\partial_{00} + \partial_{01}5 + \partial_{02}5^2 + \partial_{03}5^3 + \partial_{04}5^4 + \partial_{05}5^5 = 4500.$$

Algebraic techniques are then used to solve the six simultaneous equations in Eq. 6.15 for the six unknown variables, with:

$$\partial_{00} = 0 \quad \partial_{01} = \frac{-15}{2} \quad \partial_{02} = \frac{3}{2} \tag{6.16}$$
$$\partial_{03} = \frac{27}{2} \quad \partial_{04} = \frac{9}{2} \quad \partial_{05} = 0.$$

Given these results,

$$C_0^k = \frac{-15}{2}k + \frac{3}{2}k^2 + \frac{27}{2}k^3 + \frac{9}{2}k^4 = \frac{3k(k+1)(3k^2+6k-5)}{2}. \tag{6.17}$$

Similar analysis leads to:

$$C_1^k = -\frac{1}{2}(k+1)(3k^2+24k-5) \tag{6.18}$$

$$C_2^k = -\frac{3}{2}(k+1)(k-2) \qquad C_3^k = \frac{(k+1)}{2}$$

It is easily verified that these functional forms generate the values that appear in the associated columns of Table 6.1 for any specified k.

After substitution into Eq. 6.13 and algebraic reduction, we obtain

$$K(3,n,IAC_b(k)) = \frac{(k+1)(n-3k)}{2}\ (n+1)(n+5) - 3k(2+k)\}, \tag{6.19}$$

$$\text{for } n \geq 1 \text{ and } k \leq (n-2)/3.$$

This result is exactly the same as the representation for $K(3,n,IAC_b(k))$ in Gehrlein (2004c) that was obtained by using algebraic reduction of functions involving sums of powers of integers.

For the special case that $k = n/3$ when n is a multiple of three, it is easily shown that

$$K\left(3,n,IAC_b\left(\frac{n}{3}\right)\right) = \left(\frac{n+3}{3}\right)^3. \tag{6.20}$$

The proportion of all possible IAC voting situations that are single-peaked for *n* voters with three candidates is obtained from Eqs. 6.19 and 3.29 as

$$\frac{K(3,n,IAC_b(0))}{K(3,n,IAC)} = \frac{60n}{(n+2)(n+3)(n+4)}, \text{ for } n \geq 1. \tag{6.21}$$

When $n = 91$, this proportion is given by 0.0066, to provide firm evidence that the assumption of single-peaked preferences is extremely restrictive, to the point that this assumption excludes more than 99 percent of all possible voting situations with $n = 91$.

A representation for $N_{PMRW}^{A\}}(3,n,IAC_b(k))$ is obtained in the same general fashion that was used to obtain the representation for $K(3,n,IAC_b(k))$ in Eq. 6.19. The conditions on n_i's that result in Candidate *A* being the strict PMRW for odd *n* in Eq. 3.22 clearly meet the simple linear form restriction. Axiom 6.3 therefore requires that the representation for $N_{PMRW}^{A\}}(3,n,IAC_b(k))$ must have the form of Eq. 6.12.

Following the development of Table 6.1 that led to representations for $K(3,n,IAC_b(k)\,|\,k)$ with specified values of *k*, we use EUPIA to find coefficients D_i^k for specified *k* that give representations for $N_{PMRW}^{A\}}(3,n,IAC_b(k)\,|\,k)$. The computations were performed with $\psi = 91$, and attempts were made to obtain D_i^k coefficients for all *k* with $0 \leq k \leq 30$, where

$$N_{PMRW}^{A\}}(3,n,IAC_b(k)\,|\,k) = \sum_{i=0}^{3} D_i^k n^i. \tag{6.22}$$

The results are summarized in Table 6.2 for all $0 \leq k \leq 22$ and the periodicity was found to be $p = 2$ for all *k* entries. Before we proceed with further analysis, it is necessary to consider why Table 6.2 terminates at $k = 22$.

EUPIA2 consistently obtains representations for $N_{PMRW}^{A\}}(3,n,IAC_b(k)\,|\,k)$ with $p = 2$ and $\psi = 91$, for all $0 \leq k \leq 22$ in Table 6.2. However, no such representation was found with $k = 23$. The reason for this is that representations to obtain $N_{PMRW}^{A\}}(3,n,IAC_b(k))$ have one functional form for $k \leq \frac{n-3}{4}$ and a second functional form for $k \geq \frac{n+1}{4}$.

EUPIA2 began this process by using computer enumeration techniques to count the number of voting situations, $NVS_{PMRW}^{A}(n\,|\,k)$, for which Candidate *A* is the PMRW with a specified value of *k* for parameter *b*, for a series of *n* values with $n = \psi + jp$ for $j = 0(1)7$. The first term in the series is $n = \psi = 91$. With $k = 23$, $k \geq \frac{n+1}{4}$ so the second functional form should be used to obtain the observed value of $NVS^{A}_{PMRW}(91\,|\,23)$. The third enumerated value that is listed in the series is $n = \psi + 2p = 95$. With k = 23, $k \le \frac{n-3}{4}$ so the first functional form should be used to obtain the observed value of $NVS^{A}_{PMRW}(95\,|\,23)$.

Table 6.2 Computed D_i^k values with the specified k for ψ = 91 and p = 2

k	D_0^k	D_1^k	D_2^k	D_3^k
0	0	5/6	1	1/6
1	5	-25/3	1	1/3
2	69	-63/2	0	1/2
3	290	-218/3	-2	2/3
4	810	-815/6	-5	5/6
5	1815	-225	-9	1
6	3535	-2065/6	-14	7/6
7	6244	-1492/3	-20	4/3
8	10260	-1377/2	-27	3/2
9	15945	-2765/3	-35	5/3
10	23705	-7205/6	-44	11/6
11	33990	-1530	-54	2
12	47294	-11479/6	-65	13/6
13	64155	-7063/3	-77	7/3
14	85155	-5715/2	-90	5/2
15	110920	-10280/3	-104	8/3
16	142120	-24395/6	-119	17/6
17	179469	-4779	-135	3
18	223725	-33421/6	-152	19/6
19	275690	-19330/3	-170	10/3
20	336210	-14805/2	-189	7/2
21	406175	-25355/3	-209	11/3
22	486519	-57569/6	-230	23/6

This conflict explains why a single functional form is not obtained as a representation for $N^{A\}}_{PMRW}(3, n, IAC_b(23)\,|\,23)$ when $\psi = 91$ is used to start the series of n values to get the values in Table 6.2. The exact break point of this type in such series can be precisely determined as a function of n by running EUPIA2 with a number of ψ values, to look for consistency in terms of the value of ψ where the first functional form stops working for each ψ. We find that the first functional form for $N^{A\}}_{PMRW}(3, n, IAC_b(k))$ holds over the range of k values with $0 \le k \le [(n-1)/4]^{-}$, where $[z]^{-}$ denotes the greatest integer value that is less than or equal to z.

A representation for $N^{A\}}_{PMRW}(3, n, IAC_b(k))$ is obtained for this range of k values in the same fashion that was used to developed the representation for

$K(3,n,IAC_b(k))$ in Eq. 6.19. Using the data from Table 6.2, with the necessary functional form like that in Eq. 6.14, we obtain

$$D_0^k = \frac{k(k+1)}{6}\left(11k^2+21k-17\right) \quad D_1^k = -\frac{(k+1)}{6}\left(4k^2+26k-5\right) \tag{6.23}$$

$$D_2^k = -\frac{(k+1)(k-2)}{2} \quad D_3^k = \frac{(k+1)}{6}.$$

Using the identity that is given in Eq. 6.11 with the representation for $N_{PMRW}^{A\}}(3,n,IAC_b(k)\,|\,k)$ that follows from Eqs. 6.22 and 6.23, substitution and algebraic reduction lead to

$$P_{PMRW}^S(3,n\,|\,IAC_b(k)) = \frac{-k\left(17-21k-11k^2\right)+\left(5-26k-4k^2\right)n+3(2-k)n^2+n^3}{(n-3k)\left[(n+1)(n+5)-3k(2+k)\right]} \tag{6.24}$$

$$\text{for odd } n \text{ with } 0 \le k \le \left[(n-1)/4\right]^-.$$

This representation for $P_{PMRW}^S(3,n,IAC_b(k))$ in Eq. 6.24 is equivalent to the one that was obtained by algebraic methods in Gehrlein (2004c).

Table 6.3 Computed $F_i^{k^*}$ values with the specified k^* for ψ = 91 and p = 4

k^*	$F_0^{k^*}$	$F_1^{k^*}$	$F_2^{k^*}$	$F_3^{k^*}$	$F_4^{k^*}$
0	-231/512	-59/128	17/768	5/128	11/1536
1	5385/512	-751/128	-343/768	-7/128	11/1536
2	60345/512	-2883/128	-415/768	-19/128	11/1536
3	261417/512	-7607/128	-199/768	-31/128	11/1536
4	760665/512	-16075/128	305/768	-43/128	11/1536
5	1765449/512	-29439/128	1097/768	-55/128	11/1536
6	3538425/512	-48851/128	2177/768	-67/128	11/1536
7	6397545/512	-75463/128	3545/768	-79/128	11/1536

The determination of a representation for $P_{PMRW}^S(3,n,IAC_b(k))$ with $k \ge \frac{n+1}{4}$ requires some manipulation of EUPIA2. We obtained computer enumeration values for $NVS_{PMRW}^A(n\,|\,k)$ in the last phase for each $n = \psi + pj$ with j = 0(1)7 for each k = 0(1)22 to obtain the entries in Table 6.2. To obtain the associated representation for $N_{PMRW}^{A\}}(3,n,IAC_b(k))$ over the range of k values with $\frac{n+1}{4} \le k \le \frac{n}{3}$, we start with computer enumeration values for $NVS_{PMRW}^A\left(n\,|\,\frac{n+1}{4}+k^*\right)$ for each

$n = \psi + pj$ with j = 0(1)7, for each k *= 0(1)7, with $\psi = 91$. Table 6.3 summarizes the resulting F_i^{k*} values such that

$$N_{PMRW}^{A\}}\left(3, n, IAC_b\left(\frac{n+1}{4} + k*\right) \mid \frac{n+1}{4} + k*\right) = \sum_{i=0}^{4} F_i^{k*} n^i. \tag{6.25}$$

The entries in Table 6.3 all have p = 4.

A representation for $N_{PMRW}^{A\}}\left(3, n, IAC_b\left(\frac{n+1}{4} + k*\right)\right)$ is then obtained for this range of k values with $\frac{n+1}{4} \le k < \frac{n}{3}$ in the same fashion that was used to developed the representation for the range of k values $0 \le k \le [(n-1)/4]^-$ in Eq. 6.24. Using the data from Table 6.3, with the necessary functional form like that in Eq. 6.14, we obtain

$$F_0^{k*} = \frac{3}{512}(4k*+1)(192k*^3 + 144k*^2 + 100k* - 77) \tag{6.26}$$

$$F_1^{k*} = \frac{-1}{128}(59 + 356k* + 144k*^2 + 192k*^3)$$

$$F_2^{k*} = \frac{1}{768}(17 - 504k* + 144k*^2)$$

$$F_3^{k*} = \frac{5 - 12k*}{128} \qquad F_4^{k*} = \frac{11}{1536}.$$

By substituting $k - \frac{n+1}{4}$ for k* in the representations for F_i^{k*} in Eq. 6.26, and in Eq. 6.25, a representation for $N_{PMRW}^{A\}}(3, n, IAC_b(k))$ can be obtained for the range of k values with $\frac{n+1}{4} \le k < \frac{n}{3}$, with

$$N_{PMRW}^{A\}}(3, n, IAC_b(k)) = \frac{(n-3k)\,3(3 - 2k - 6k^3) + (11 + 18k^2)n + 3(1-2k)n^2 + n^3\}}{12}. \tag{6.27}$$

Additional runs with $p = 4$ verify that this representation is valid for all $n = 7, 11, 15, 19, \ldots$.

By repeating this procedure with $\psi = 93$, this representation is found to be valid for all odd $n \ge 7$ with $[(n+1)/4]^+ \le k \le [(n-1)/3]^-$, where $[z]^+$ denotes the smallest integer value that is greater than or equal to z.

Using the identity in Eq. 6.11 with algebraic reduction leads to

$$P^S_{PMRW}(3,n \mid IAC_b(k)) = \frac{3(3-2k-6k^3)+(11+18k^2)n+3(1-2k)n^2+n^3}{2(k+1)[(n+1)(n+5)-3k(2+k)]} \tag{6.28}$$

for odd $n \geq 7$, with $\left[\frac{n+1}{4}\right]^+ \leq k \leq \left[\frac{n-1}{3}\right]^-$.

The case of $k = n/3$ when n is an odd multiple of three must be handled as a special case, to obtain $P^S_{PMRW}(3,n,IAC_b(n/3)) = 3/4$.

Similar analysis for even n results in the following representations for even $n \geq 8$:

$$P^S_{PMRW}(3,n \mid IAC_b(k)) \tag{6.29}$$

$$= \frac{2k(6+31k+11k^2)-4(2+13k+2k^2)n+3(3-2k)n^2+2n^3}{2(n-3k)[(n+1)(n+5)-3k(2+k)]},$$

$$\text{for } 0 \leq k \leq \left[\frac{n-4}{4}\right]^-$$

$$= \frac{2(2-3k+18k^2-9k^3)+2(1-12k+9k^2)n+(5-6k)n^2+n^3}{2(k+1)[(n+1)(n+5)-3k(2+k)]},$$

$$\text{for } \left[\frac{n}{4}\right]^+ \leq k \leq \left[\frac{n-1}{3}\right]^-$$

$$= \frac{3n^2}{4(n+3)^2}, \text{ for } k = \frac{n}{3}. \qquad \text{For even } n \geq 8.$$

Table 6.4 lists values of $P^S_{PMRW}(3,91 \mid IAC_b(k))$ and $P^S_{PMRW}(3,92 \mid IAC_b(k))$ that were obtained from Eqs. 6.24, 6.28 and 6.29 for each $0 \leq k \leq 30$. One very important observation from Table 6.4 is that $P^S_{PMRW}(3,n \mid IAC_b(k))$ increases as k increases for both odd and even n. The use of proximity to perfectly single peaked preferences as a situation-specific measure of social homogeneity adds enough internal consistency or coherence to the voters' preferences to avoid the very disappointing results that were observed Chapter 5 when $SSM^3(\boldsymbol{n})$ was used to measure social homogeneity.

The computed valued in Table 6.4 also indicate that the presence of a strong positively unifying candidate significantly increases the probability that a PMRW exists. For example, $P^S_{PMRW}(3,91 \mid IAC_b(k)) > .99$ for all values of $k \leq 7$ and $P^S_{PMRW}(3,91 \mid IAC_b(k)) < .80$ for all $k \geq 25$. As we have observed in earlier analyses, the rate of convergence of $P^S_{PMRW}(3,n \mid IAC_b(k))$ to its limiting value of 3/4 is much faster for odd n than it is for even n.

Table 6.4 Computed values of $P_{PMRW}^{S}\left(3{,}91 \mid IAC_b(k)\right)$, $P_{PMRW}^{S}\left(3{,}92 \mid IAC_b(k)\right)$ and $P_{PMRW}^{S}\left(3{,}91 \mid IAC_c(k)\right)$

k	$P_{PMRW}^{S}\left(3{,}91 \mid IAC_b(k)\right)$	$P_{PMRW}^{S}\left(3{,}92 \mid IAC_b(k)\right)$	$P_{PMRW}^{S}\left(3{,}91 \mid IAC_c(k)\right)$
0	1.0000	.9837	1.0000
1	.9997	.9828	.9920
2	.9991	.9817	.9894
3	.9982	.9803	.9841
4	.9971	.9786	.9810
5	.9957	.9766	.9762
6	.9939	.9743	.9729
7	.9919	.9715	.9683
8	.9894	.9684	.9648
9	.9866	.9649	.9602
10	.9833	.9608	.9565
11	.9795	.9562	.9520
12	.9751	.9509	.9481
13	.9700	.9450	.9435
14	.9641	.9382	.9394
15	.9574	.9304	.9347
16	.9496	.9215	.9304
17	.9404	.9112	.9255
18	.9297	.8993	.9211
19	.9170	.8853	.9160
20	.9017	.8686	.9115
21	.8832	.8485	.9063
22	.8601	.8239	.9016
23	.8325	.7947	.8965
24	.8088	.7693	.8921
25	.7900	.7490	.8875
26	.7754	.7331	.8839
27	.7645	.7211	.8803
28	.7569	.7125	.8779
29	.7523	.7069	.8758
30	.7503	.7040	.8751

6.3 Proximity to Single-Troughed Preferences

It was noted in Chapter 2 that Vickery (1960) observed that PMR must be transitive for odd n if voters' preferences are restricted to be single-troughed. For three-candidate elections, this condition is equivalent to requiring that some candidate is never top-ranked, or most preferred, by any voter. Such a candidate can be thought of as being a *negatively unifying candidate*, in the sense that the voters' are unified in their general opposition to having such a candidate selected as the winner of an election.

Following the logic of the discussion of proximity to perfectly single-peaked preferences in the immediately preceding section, we measure the proximity of a voting situation to perfectly single-troughed preferences by t, where

$$t = Min\ n_1 + n_2, n_3 + n_5, n_4 + n_6\}. \tag{6.30}$$

Vickery (1960) points out that the condition of single-troughed preferences is equivalent to the assumption of single-peaked preferences, when all of the voter preference rankings are inverted. As a result, we expect to find a very similar impact of proximity to perfectly single-troughed preferences on the likelihood that a PMRW exists when compared to that which was observed with proximity to perfectly single-peaked preferences. Intuition turns out to be correct in this case.

Lemma 6.1. $K(3,n,IAC_b(k)) = K(3,n,IAC_t(k))$ for all n.

Proof: Consider any voting situation $\boldsymbol{n}$ such that b has any given value k. For any such voting situation there is a unique voting situation, $\boldsymbol{n}'$ that is obtained from $\boldsymbol{n}$ by taking the dual of the preferences of voters, with:

$$n_1 \leftrightarrow n_6, n_2 \leftrightarrow n_5, n_3 \leftrightarrow n_4. \tag{6.31}$$

If $b = k$ in $\boldsymbol{n}$, the nature of the mapping in Eq. 6.31 requires that $t = k$ in $\boldsymbol{n}'$. Since this constitutes a 1-1 mapping between all possible voting situations with n voters, the result follows directly. **QED**

Lemma 6.2. $N_{PMRW}^{A\}}(3,n,IAC_b(k)) = N_{PMRW}^{A\}}(3,n,IAC_t(k))$ for odd $n \geq 3$.

Proof. Consider any voting situation $\boldsymbol{n}$ such that b has any given value k, and suppose that a strict PMRW exists for this voting situation. Each such voting situation can be mapped to its unique dual voting situation, $\boldsymbol{n}'$ according to Eq. 6.31. Based on Lemma 6.1, $t = k$ in $\boldsymbol{n}'$. Given that all preferences are reversed in obtaining $\boldsymbol{n}'$ from $\boldsymbol{n}$, the PMRW in $\boldsymbol{n}$ must be the PMRL in $\boldsymbol{n}'$. When n is odd, the existence of a PMRL (PMRW) requires that a PMRW (PMRL) must also exist for three candidates. The same statement is not necessarily true when n is even. The mapping in Eq. 6.31 is 1-1, so that when n is odd, there must be the same number of voting situations with a strict PMRW and associated PMRL when $b = k$ as there are with a strict PMRL and associated PMRW when $t = k$. The symmetry of conditions $IAC_b(k)$ and $IAC_t(k)$ with respect to candidates leads directly to the stated result. **QED**

Lemma 6.3. $P^S_{PMRW}\left(3,n \mid IAC_b(k)\right) = P^S_{PMRW}\left(3,n \mid IAC_t(k)\right)$ for odd $n \geq 3$.

Proof. Following the development of Eq. 6.11,

$$P^S_{PMRW}\left(3,n \mid IAC_t(k)\right) = \frac{3N^{A\}}_{PMRW}\left(3,n,IAC_t(k)\right)}{K\left(3,n,IAC_t(k)\right)}. \quad (6.32)$$

The result follows directly from Eq. 6.32 with Lemmas 6.1 and 6.2. **QED**

Thus, the impact of having voters' preferences reflect some degree of proximity to perfectly single-troughed preferences is identical to the impact of having the same degree of proximity to perfectly single-peaked preferences. At least this is true with regard to the relationship of these two measures of social homogeneity to the probability that a PMRW exists.

6.4 The Impact of Polarizing Candidates

It was mentioned in Chapter 2 that Ward (1965) developed another restriction on voting situations that will assure transitivity of PMR for odd *n*. In particular, Ward noted that PMR must be transitive if some candidate in every triple of candidates is never the middle-ranked candidate among the three candidates for any voter. Let *c* define the minimum number of voters who rank some candidate at the center of their preference ranking in a three-candidate election, with

$$c = Min\ n_3 + n_4, n_1 + n_6, n_2 + n_5\}. \quad (6.33)$$

If *c* is equal to zero, every voter ranks some candidate as being either most preferred or least preferred in a three-candidate election. This candidate can therefore be viewed as a *polarizing candidate* that tends to split voters among the electorate into antagonistic groups, rather than unify them.. Following previous discussion about measures *b* and *t*, *c* can be viewed as a measure of the proximity of voting situations to perfectly polarized preferences. One observation follows quite easily.

Lemma 6.4. $K\left(3,n,IAC_b(k)\right) = K\left(3,n,IAC_c(k)\right)$ for all *n*.

Proof. Consider any voting situation ***n*** such that *b* has any given value *k*. For any such voting situation there is a unique voting situation, ***n***' that is mapped to each ***n*** by reversing the ranking of the two less preferred candidates in the preferences of voters, with:

$$n_1 \leftrightarrow n_2, n_3 \leftrightarrow n_5, n_4 \leftrightarrow n_6. \quad (6.34)$$

If $b = k$ in ***n***, the nature of the mapping in Eq. 6.34 requires that $c = k$ in ***n***'. Since this constitutes a 1-1 mapping between all possible voting situations with *n* voters, the result follows directly. **QED**

The result of Lemma 6.2 does not apply to the comparison of *b* and *c* with the mapping in Eq. 6.34, and $P^S_{PMRW}\left(3,n \mid IAC_b(k)\right) \neq P^S_{PMRW}\left(3,n \mid IAC_c(k)\right)$.

In order to develop a representation for $P_{PMRW}^{S}(3,n \mid IAC_c(k))$, the next step is to obtain a representation for $N_{PMRW}^{A\}}(3,n,IAC_c(k))$. Given the definition of c in Eq. 6.33, the logic of Axiom 6.3 can clearly be applied in this case and the resulting development of Eq. 6.22 leads to

$$N_{PMRW}^{A\}}(3,n,IAC_c(k) \mid k) = \sum_{i=0}^{3} G_i^k n^i. \tag{6.35}$$

Gehrlein (2006b) uses EUPIA2 to obtain the G_i^k coefficients with $\psi = 91$. The results are summarized in Table 6.5 for all $0 \le k \le 23$, and the periodicity was found to be $p = 4$ for all k entries.

Table 6.5 Computed G_i^k values with specified k values with $\psi = 91$ and $p = 4$

k	G_0^k	G_1^k	G_2^k	G_3^k
0	0	5/6	1	1/6
1	25/4	-22/3	3/4	1/3
2	127/2	-51/2	-1/2	1/2
3	260	-167/3	-3	2/3
4	1399/2	-599/6	-13/2	5/6
5	6213/4	-159	-45/4	1
6	2978	-1417/6	-17	7/6
7	5234	-994/3	-24	4/3
8	8525	-897/2	-32	3/2
9	52825/4	-1760/3	-165/4	5/3
10	39045/2	-4505/6	-103/2	11/6
11	27930	-939	-63	2
12	77417/2	-6943/6	-151/2	13/6
13	209685/4	-4207/3	-357/4	7/3
14	69377	-3363/2	-104	5/2
15	90252	-5972/3	-120	8/3
16	115378	-14027/6	-137	17/6
17	582201/4	-2718	-621/4	3
18	362235/2	-18841/6	-349/2	19/6
19	223000	-10795/3	-195	10/3
20	543115/2	-8205/2	-433/2	7/2
21	1311365/4	-13937/3	-957/4	11/3
22	392216	-31433/6	-263	23/6
23	465870	-5874	-288	4

The fact that Table 6.5 terminates at k = 23 while $\psi = 91$ results from the fact that there are different functional forms for $N_{PMRW}^{A\}}(3,n,IAC_c(k))$, depending upon the value of k. Following the development of $N_{PMRW}^{A\}}(3,n,IAC_b(k) \mid k)$, the first functional form for $N_{PMRW}^{A\}}(3,n,IAC_c(k))$ is valid for $0 \le k \le \frac{n+1}{4}$ and the second is valid for $\frac{n+5}{4} \le k \le \frac{n}{3}$. The results in Table 6.5 can therefore be used to determine the functional form of $N_{PMRW}^{A\}}(3,n,IAC_c(k))$ for $0 \le k \le \frac{n+1}{4}$.

This representation is not directly obtainable from the data that is given in Table 6.5, since this series presents a new problem in the process of generating representations with EUPIA2. In particular, there are two different representations for $N_{PMRW}^{A\}}(3,n,IAC_c(k))$ in the range $0 \le k \le \frac{n+1}{4}$, depending upon whether k has an odd or even value

We consider the case of odd values of k first, to obtain

$$N_{PMRW}^{A\}}(3,n,IAC_c(k) \mid k) = \sum_{i=0}^{3} GO_i^k n^i, \text{ for } n = 7,11,15... \quad \text{with odd } k \text{ and } 0 \le k \le \frac{n+1}{4}. \tag{6.36}$$

Using the data from Table 6.5 for odd k

$$GO_0^k = \frac{1}{96}(k+1)(139k^3 + 333k^2 - 169k - 3)$$
$$GO_1^k = \frac{-1}{24}(k+1)(7k^2 + 95k - 14)$$
$$GO_2^k = \frac{-3}{16}(k+1)(3k-5) \quad GO_3^k = \frac{(k+1)}{6}. \tag{6.37}$$

Following the development of earlier representations,

$$P_{PMRW}^{S}(3,n \mid IAC_c(k)) = \frac{(139k^3 + 333k^2 - 169k - 3) - 4(7k^2 + 95k - 14)n - 18(3k-5)n^2 + 16n^3}{16(n-3k)[(n+1)(n+5) - 3k(2+k)]}, \tag{6.38}$$

$$\text{for } n = 7,11,15 \text{ and odd } k \text{ with } 0 \le k \le \frac{n+1}{4}.$$

The same procedure is then used for the sequence of even values of k to obtain

$$P_{PMRW}^{S}(3,n \mid IAC_c(k)) = \frac{\left[\begin{array}{c}(139k^3 + 472k^2 + 146k - 244)k - 4(7k^3 + 102k^2 + 84k - 20)n \\ - 6(9k^2 - 6k - 16)n^2 + 16(k+1)n^3\end{array}\right]}{16(k+1)(n-3k)[(n+1)(n+5) - 3k(2+k)]}, \tag{6.39}$$

$$\text{for } n = 7,11,15 \text{ and even } k \text{ with } 0 \le k \le \frac{n+1}{4}.$$

Similar analysis can be performed with ψ = 93 to find that the representations in Eqs. 6.38 and 6.39 are valid over the range $0 \le k \le \frac{n-1}{4}$ for all odd $n \ge 3$. These results can be combined to obtain

$$P^S_{PMRW}(3,n \mid IAC_c(k)) = \tag{6.40}$$
$$\frac{\left[\begin{array}{c}(139k^3+472k^2+146k-244)k-4(7k^3+102k^2+84k-20)n \\ -6(9k^2-6k-16)n^2+16(k+1)n^3+3\delta^2_{k+1}\left[(6k^2+24k-1)+4(k-2)n-2n^2\right]\end{array}\right]}{16(k+1)(n-3k)[(n+1)(n+5)-3k(2+k)]}$$
$$\text{for all odd } n \ge 3, \text{with } 0 \le k \le \frac{n-1}{4}.$$

Here $\delta^y_x = 1$ if x is an integer multiple of y. Otherwise, $\delta^y_x = 0$. The representation in Eq. 6.40 is used to compute the $P^S_{PMRW}(3{,}91 \mid IAC_c(k))$ entries that are shown in Table 6.4 for $0 \le k \le 22$.

The values in Table 6.4 show some very interesting results, with $P^S_{PMRW}(3{,}91 \mid IAC_b(k)) > P^S_{PMRW}(3{,}91 \mid IAC_c(k))$ for $0 \le k \le 19$ and with $P^S_{PMRW}(3{,}91 \mid IAC_c(k)) > P^S_{PMRW}(3{,}91 \mid IAC_b(k))$ for $20 \le k \le 22$. This suggests that proximity to perfectly single-peaked preferences has more of an impact on the probability that a PMRW exists than does proximity to perfectly polarized preferences for small values of k. However, as k increases the reverse situation exists. In order to investigate this phenomenon further, the representation for $P^S_{PMRW}(3,n \mid IAC_c(k))$ must be obtained for $k > \frac{n+1}{4}$.

The development of the representation for $P^S_{PMRW}(3{,}91 \mid IAC_c(k))$ over the range $\frac{n+1}{4} \le k < \frac{n}{3}$ follows previous analysis, and

$$P^S_{PMRW}(3,n \mid IAC_c(k)) = \tag{6.41}$$
$$\frac{\left[\begin{array}{c}3(-39k^4+72k^3+38k^2-76k+1)+4(57k^3-54k^2-80k+19)n \\ -2(75k^2+6k-47)n^2+4(8k+5)n^3-n^4+3\delta^2_{k+1}\left[(6k^2+24k-1)+4(k-2)n-2n^2\right]\end{array}\right]}{16(k+1)(n-3k)[(n+1)(n+5)-3r(2+k)]},$$
$$\text{for all odd } n \ge 3 \text{ with } \left[\frac{n+1}{4}\right]^+ \le k \le \left[\frac{n-1}{3}\right]^-.$$

The case with $k = n/3$ when n is an odd multiple of three is a special case, with

$$P^S_{PMRW}(3,n \mid IAC_c(n/3)) = \frac{7n^2+42n+27}{8(n+3)^2}. \tag{6.42}$$

Computed values of $P_{PMRW}^{S}(3{,}91 \mid IAC_c(k))$ from Eq. 6.41 are shown in Table 6.4 for each $23 \le k \le 30$. The pattern continues with $P_{PMRW}^{S}(3{,}91 \mid IAC_c(k)) >$ $P_{PMRW}^{S}(3{,}91 \mid IAC_b(k))$ for all k in this range. Moreover, $P_{PMRW}^{S}(3{,}91 \mid IAC_c(k))$ and $P_{PMRW}^{S}(3{,}91 \mid IAC_b(k))$ do not seem to be approaching the same limiting value as $k \to n/3$. In order to examine these observations further, we consider these representations in the limiting case $n \to \infty$.

6.5 Limiting Distributions for Large Electorates

The representations for $P_{PMRW}^{S}(3, n \mid IAC_b(k))$ in Eqs. 6.24 and 6.28, and for $P_{PMRW}^{S}(3, n \mid IAC_c(k))$ in Eq. 6.40 and 6.41 can easily be modified to account for the limiting case as $n \to \infty$. To do this, *k* is replaced with $\alpha_k n$, so that *k* is expressed as a proportion, α_k, of *n*, rather than as an integer value. Then, the limiting representation as $n \to \infty$ is determined. The resulting representations for the limiting distributions are:

$$P_{PMRW}^{S}(3, \infty \mid IAC_b(\alpha_k)) = \frac{11\alpha_k^3 - 4\alpha_k^2 - 3\alpha_k + 1}{(1 - 3\alpha_k)(1 - 3\alpha_k^2)} \text{ for } 0 \le \alpha_k \le 1/4, \tag{6.43}$$

$$= \frac{-18\alpha_k^3 + 18\alpha_k^2 - 6\alpha_k + 1}{2a_k(1 - 3\alpha_k^2)} \text{ for } 1/4 \le \alpha_k < 1/3.$$

$$P_{PMRW}^{S}(3, \infty \mid IAC_c(\alpha_k)) = \frac{139\alpha_k^3 - 28\alpha_k^2 - 54\alpha_k + 16}{16(1 - 3\alpha_k)(1 - 3\alpha_k^2)} \text{ for } 0 \le \alpha_k \le 1/4,$$

$$= \frac{39\alpha_k^3 - 63\alpha_k^2 + 29\alpha_k - 1}{16\alpha_k(1 - 3\alpha_k^2)} \text{ for } 1/4 \le \alpha_k < 1/3.$$

Table 6.6 lists computed values of limiting probabilities as $n \to \infty$ for both $P_{PMRW}^{S}(3, \infty \mid IAC_b(\alpha_k))$ and $P_{PMRW}^{S}(3, \infty \mid IAC_c(\alpha_k))$ from Eq. 6.43 for each value of $\alpha_k = .01(.02).33$. Table 6.6 also includes the limiting value of $P_{PMRW}^{S}(3, \infty \mid IAC_b(1/3)) = .75$ that follows from previous discussion and from Eq. 6.29. The value of the limiting probability $P_{PMRW}^{S}(3, \infty \mid IAC_c(1/3)) = 7/8$ is obtained from Eq. 6.42. The same representation is obtained for the limiting probability $P_{PMRW}^{S}(3, \infty \mid IAC_b(\alpha_k))$ regardless of whether the starting representation for $P_{PMRW}^{S}(3, n \mid IAC_b(k))$ is for odd or even *n*.

Table 6.6 Computed values of the limiting probabilities $P^S_{PMRW}(3,\infty \mid IAC_b(\alpha_k))$ and $P^S_{PMRW}(3,\infty \mid IAC_c(\alpha_k))$

α_k	$P^S_{PMRW}(3,\infty \mid IAC_b(\alpha_k))$	$P^S_{PMRW}(3,\infty \mid IAC_c(\alpha_k))$
0	1.0000	1.0000
.01	.9999	.9963
.03	.9991	.9888
.05	.9973	.9814
.07	.9946	.9740
,09	.9907	.9665
,11	.9854	.9589
.13	.9784	.9511
.15	.9693	.9431
.17	.9574	.9348
.19	.9416	.9263
.21	.9203	.9174
.23	.8905	.9083
.25	.8462	.8990
.27	.8009	.8903
.29	.7720	.8828
.31	.7559	.8775
.33	.7501	.8751
1/3	.7500	.8750

By comparing relevant values in Table 6.6, in which $n \to \infty$, to corresponding values in Table 6.2, in which $n = 91$, it is evident that the rate of convergence to the limiting distribution values is very rapid for odd *n*. The relevant root of the equation $P^S_{PMRW}(3,\infty \mid IAC_b(\alpha_k *)) = P^S_{PMRW}(3,\infty \mid IAC_c(\alpha_k *))$ is found at $\alpha_k * = \left(18 - \sqrt{102}\right)/37 \approx .2135$. As a result, the previously observed pattern holds up, with $P^S_{PMRW}(3,\infty \mid IAC_b(\alpha_k)) < P^S_{PMRW}(3,\infty \mid IAC_c(\alpha_k))$ for all $\alpha_k < \alpha_k *$ and $P^S_{PMRW}(3,\infty \mid IAC_c(\alpha_k)) > P^S_{PMRW}(3,\infty \mid IAC_b(\alpha_k))$ for all $\alpha_k > \alpha_k *$. Moreover, the limiting values of these functions as $k \to n/3$ are not the same.

It was noted in Chapter 5 that Radcliff (1993) does an empirical study to determine the propensity of voters to have single-peaked preferences. The percentage of respondents with preferences that were single-peaked across a reference ranking of candidates was found to be approximately 83 percent for three-candidate elections. This corresponds to a measure of $\alpha_k \approx .17$ according to parameter *b*. Table 6.6 entries suggest that a PMRW should exist with probability of approximately .96 for large electorates in this case. As a result, it is not surprising that Radcliff found that a PMRW existed in each study that was considered.

6.6 Cumulative Probabilities that a PMRW Exists

Representations for the cumulative probabilities $P^S_{PMRW}\left(3,n \mid CIAC_b\left(k^-\right)\right)$, $P^S_{PMRW}\left(3,n \mid CIAC_t\left(k^-\right)\right)$ and $P^S_{PMRW}\left(3,n \mid CIAC_c\left(k^-\right)\right)$ follow earlier definitions. That is, all voting situations with a specified parameter value k^* with the range $0 \le k^* \le k$ are assumed to be equally likely to be observed. For example, it follows Eq. 6.11 and definitions:

$$P^S_{PMRW}\left(3,n \mid CIAC_b\left(k^-\right)\right) = \frac{3\sum_{k^*=0}^{k} N^{A\}}_{PMRW}\left(3,n,IAC_b(k^*)\right)}{\sum_{k^*=0}^{k} K\left(3,n,IAC_b(k^*)\right)}. \tag{6.44}$$

Gehrlein (2006c) performs the algebraic manipulations to obtain these representations. With parameters b and t for odd n:

$$P^S_{PMRW}\left(3,n \mid CIAC_b\left(k^-\right)\right) = P^S_{PMRW}\left(3,n \mid CIAC_t\left(k^-\right)\right) \tag{6.45}$$

$$= \frac{2\left\{\left(-41+69k+22k^2\right)k+5\left(5-18k-2k^2\right)n+10(3-k)n^2+5n^3\right\}}{\left\{\left(-73+117k+36k^2\right)k+5\left(10-33k-3k^2\right)n+20(3-k)n^2+10n^3\right\}},$$
$$\text{for } 0 \le k \le (n-1)/4.$$

$$= \frac{\left[\begin{array}{c} 195-1968k-720k^2+3840k^3+4320k^4+1728k^5 \\ +\left(1661-1680k-6000k^2-5760k^3-2880k^4\right)n+10\left(165+200k+216k^2+192k^3\right)n^2 \\ +30\left(9-8k-24k^2\right)n^3+5(15+32k)n^4-11n^5 \end{array}\right]}{16(k+1)(k+2)\left\{\left(-73+117k+36k^2\right)k+5\left(10-33k-3k^2\right)n+20(3-k)n^2+10n^3\right\}},$$
$$\text{for } (n+1)/4 \le k \le (n-1)/3.$$

$$\frac{15(n+3)^2}{16(n+2)(n+4)}, \quad \text{for } k = n/3.$$

This representation is quite unwieldy to serve as the basis of any useful analysis, so attention will be focused on the limiting probability as $n \to \infty$. As in previous analysis, this is done by first substituting $\alpha_k n$ for k in Eq. 6.45, and then letting $n \to \infty$.

$$P^S_{PMRW}\left(3,\infty \mid CIAC_b\left(\alpha_k^-\right)\right) = P^S_{PMRW}\left(3,\infty \mid CIAC_t\left(\alpha_k^-\right)\right) \tag{6.46}$$

$$= \frac{10-20\alpha_k-20\alpha_k^2+44\alpha_k^3}{10-20\alpha_k-15\alpha_k^2+36\alpha_k^3} \quad \text{for } 0 \le \alpha_k \le 1/4.$$

$$= \frac{-11+160\alpha_k-720\alpha_k^2+1920\alpha_k^3-2880\alpha_k^4+1728\alpha_k^5}{16\alpha_k^2\left(10-20\alpha_k-15\alpha_k^2+36\alpha_k^3\right)}$$
$$\text{for } 1/4 \le \alpha_k \le 1/3.$$

Here, $CIAC_b\left(\alpha_k^{\,-}\right)$ assumes that all voting situations with $0 \le \alpha_k{}^* \le \alpha_k$ are equally likely to be observed.

For parameter c with odd n:

$$P^S_{PMRW}\left(3, n \mid CIAC_c\left(k^-\right)\right) \tag{6.47}$$

$$= \frac{\left[\begin{array}{c}(k+1)\left[\begin{array}{c}165-783k+1743k^2+1597k^3+278k^4+10\left(71-233k-143k^2-7k^3\right)n \\ +30\left(31+3k-6k^2\right)n^2+80(k+2)n^3\end{array}\right] \\ -15\delta_k^2\ 11+30k+6k^2-2(3-2k)n-2n^2\}\end{array}\right]}{8(k+1)(k+2)\left(-73+117k+36k^2\right)k+5\left(10-33k-3k^2\right)n+20(3-k)n^2+10n^3\}},$$

for $0 \le k \le (n-1)/4$.

$$= \frac{\left[\begin{array}{c}435-952k+480k^2+2200k^3-90k^4-468k^5 \\ +\left(1349-2520k-4160k^2+840k^3+1140k^4\right)n+10\left(177+120k-162k^2-100k^3\right)n^2 \\ +10\left(39+72k+32k^2\right)n^3-5(3+4k)n^4+n^5-30\delta_k^2\ 11+30k+6k^2-2(3-2k)n-2n^2\}\end{array}\right]}{16(k+1)(k+2)\left(-73+117k+36k^2\right)k+5\left(10-33k-3k^2\right)n+20(3-k)n^2+10n^3\}},$$

for $(n+1)/4 \le k \le (n-1)/3$.

$$\frac{15(n+3)^2}{16(n+2)(n+4)}, \text{ for } k = n/3.$$

As defined previously, $\delta_x^y = 1$ if x is an integer multiple of y. Otherwise, $\delta_x^y = 0$.

The associated limiting representations for parameter c are given by:

$$P^S_{PMRW}\left(3, \infty, \mid CIAC_c\left(\alpha_k^{\,-}\right)\right) \tag{6.48}$$

$$= \frac{40-90\alpha_k-35\alpha_k{}^2+139\alpha_k{}^3}{40-80\alpha_k-60\alpha_k{}^2+144\alpha_k{}^3} \text{ for } 0 \le \alpha_k \le 1/4,$$

$$= \frac{1-20\alpha_k+320\alpha_k{}^2-1000\alpha_k{}^3+1140\alpha_k{}^4-468\alpha_k{}^5}{16\alpha_k{}^2\left(10-20\alpha_k-15\alpha_k{}^2+36\alpha_k{}^3\right)}$$

for $1/4 \le \alpha_k \le 1/3$.

These limiting representations are much more tractable, and they also represent the potentially most interesting case of large electorates. Following earlier discussion, these limiting representations result in specific values such that $P^S_{PMRW}\left(3, \infty \mid CIAC_b\left(0^-\right)\right) = P^S_{PMRW}\left(3, \infty \mid CIAC_t\left(0^-\right)\right) = P^S_{PMRW}\left(3, \infty \mid CIAC_c\left(0^-\right)\right) = 1$ and such that $P^S_{PMRW}\left(3, \infty \mid CIAC_b\left(1/3^-\right)\right) = P^S_{PMRW}\left(3, \infty \mid CIAC_t\left(1/3^-\right)\right) = P^S_{PMRW}\left(3, \infty \mid CIAC_c\left(1/3^-\right)\right) = 15/16$. This follows from the fact that $CIAC_b\left(0^-\right)$ is the same as the limiting case as $n \to \infty$ when voters must have preferences that are perfectly single-peaked, and $CIAC_b\left(1/3^-\right)$ is equivalent to the limiting case of IAC as $n \to \infty$.

6.7 Proportions of Profiles with Specified Parameters

Previous analyses of computed values for the probabilities $P^S_{PMRW}(3,n \mid IAC_b(k))$, $P^S_{PMRW}(3,n \mid IAC_t(k))$ and $P^S_{PMRW}(3,n \mid IAC_c(k))$ have produced some very interesting observations. In particular, voting situations with small values of *k*, to reflect close proximity to perfect single-peakedness, perfect single-troughedness or perfect polarization, have a high probability that a PMRW exists. This observation can however be misleading. We find that these probabilities remain quite large for relatively large range of *k* values. But, this does not account for the proportions of all possible voting situations that this range of *k* values represents. That is, $P^S_{PMRW}(3,n \mid IAC_b(k))$ can be quite large for a relatively wide range of *k* values, but the results are meaningless if this range of *k* only accounts for a very small proportion of all possible voting situations. It was seen earlier in this chapter that the proportion of all possible voting situations for which *k* is equal to zero in Eq. 6.21 is very small, which would tend to make results that focus only on profiles that are perfectly single-peaked to be of very limited interest.

To address this issue, it is necessary to develop representations for the proportion of all possible voting situations that have a specified parameter k^* in some given range $0 \le k^* \le k$. The logic behind the development of Eq. 6.21 leads to a representation for this proportion, $P_{VS}\left(3, n \mid CIAC_b\left(k^-\right)\right)$, when the specified parameter that is being measured is *b*.

$$P_{VS}\left(3, n \mid CIAC_b\left(k^-\right)\right) = \frac{\sum_{k^*=0}^{k} K\left(3, n, IAC_b\left(k^*\right)\right)}{K\left(3, n, IAC\right)}. \tag{6.49}$$

Gehrlein (2006c) performs the algebraic reduction of Eq. 6.49 to obtain

$$\begin{aligned} &P_{VS}\left(3, n \mid CIAC_b\left(k^-\right)\right) \\ &= \frac{\left[3(k+1)(k+2)\left\{\begin{matrix}\left(-73+117k+36k^2\right)k + \\ 5\left(10-33k-3k^2\right)n + 20(3-k)n^2 + 10n^3\end{matrix}\right\}\right]}{(n+1)(n+2)(n+3)(n+4)(n+5)}, \\ &\quad \text{for } 0 \le k \le (n-1)/3. \\ &\quad = 1 \text{ for } k = n/3. \end{aligned} \tag{6.50}$$

Based on Lemmas 6.1 and 6.4, $P_{VS}\left(3, n \mid CIAC_b\left(k^-\right)\right) = P_{VS}\left(3, n \mid CIAC_t\left(k^-\right)\right) = P_{VS}\left(3, n \mid CIAC_{c.}\left(k^-\right)\right)$.

Attention will be focused on the limiting distribution, $P_{VS}\left(3, \infty \mid CIAC_b\left(\alpha_k^{\ -}\right)\right)$, as $n \to \infty$, and following earlier analyses,

$$P_{VS}\left(3,\infty \mid CIAC_b\left(\alpha_k^-\right)\right) = P_{VS}\left(3,\infty \mid CIAC_t\left(\alpha_k^-\right)\right) = P_{VS}\left(3,\infty \mid CIAC_c\left(\alpha_k^-\right)\right) = 3\alpha_k^2\left(10 - 20\alpha_k - 15\alpha_k^2 + 36\alpha_k^3\right), \text{ for } 0 \le \alpha_k \le 1/3. \tag{6.51}$$

The representation in Eq. 6.51 can be used to find values of β_b^p such that $P_{VS}\left(n,\infty \mid CIAC_b\left(\beta_b^{p-}\right)\right) = p$ for each proportion p = 0.00(.05)1.00, and the results are listed in Table 6.7. Based on previous discussion, $\beta_b^p = \beta_t^p = \beta_c^p$ for all p. The results in Table 6.7 indicate for example that 65 percent of all possible voting situations are included in the range of α_k parameter values with $0 \le \alpha_k \le .1924$ for parameter b, t, or c.

Table 6.7 Computed values of $\beta_b^p, \beta_t^p, \beta_c^p, \beta_u^p$, and β_ℓ^p for each proportion p = 0.00(.05)1.00

p	$\beta_b^p = \beta_t^p = \beta_c^p$	β_u^p	β_ℓ^p
.00	.0000	.0000	.0000
.05	.0428	.0308	.0256
.10	.0619	.0449	.0375
.15	.0772	.0564	.0473
.20	.0908	.0667	.0562
.25	.1033	.0763	.0646
.30	.1150	.0854	.0727
.35	.1264	.0943	.0806
.40	.1374	.1031	.0885
.45	.1483	.1118	.0965
.50	.1591	.1206	.1046
.55	.1700	.1296	.1130
.60	.1811	.1388	.1217
.65	.1924	.1484	.1308
.70	.2042	.1585	.1407
.75	.2166	.1695	.1514
.80	.2298	.1815	.1634
.85	.2445	.1951	.1774
.90	.2614	.2117	.1946
.95	.2829	.2344	.2191
1.00	.3333	.3333	.3333

The results of Table 6.7 can be used in conjunction with the limiting representations from Eq. 6.45 to compute the limiting conditional cumulative probability

$P^S_{PMRW}\left(n,\infty \mid CIAC_b\left(\beta_b^{p^-}\right)\right)$ that a PMRW exists for the p percent of all voting situations that are closest to being perfectly single-peaked. In the same fashion, it is also possible to obtain similar values for $P^S_{PMRW}\left(n,\infty \mid CIAC_t\left(\beta_t^{p^-}\right)\right)$ and for $P^S_{PMRW}\left(n,\infty \mid CIAC_c\left(\beta_c^{p^-}\right)\right)$. Computed results from the associated representations for all three of these probabilities are summarized in Table 6.8 for each proportion $p = 0.00(.05)1.00$.

Table 6.8 Computed Values of $P^S_{PMRW}\left(n,\infty \mid CIAC_X\left(\beta_X^{p^-}\right)\right)$, for $X = b, t, c, u, \ell$ for each proportion $p = 0.00(.05)1.00$

p	b, t	c	u	ℓ
.00	1.0000	1.0000	1.0000	1.0000
.05	.9991	.9895	.9995	.9976
.10	.9980	.9850	.9989	.9963
.15	.9969	.9814	.9983	.9951
.20	.9956	.9782	.9975	.9940
.25	.9943	.9753	.9967	.9928
.30	.9929	.9726	.9958	.9916
.35	.9913	.9701	.9948	.9903
.40	.9896	.9676	.9936	.9890
.45	.9877	.9652	.9924	.9876
.50	.9857	.9628	.9910	.9860
.55	.9834	.9605	.9894	.9843
.60	.9809	.9582	.9876	.9825
.65	.9781	.9558	.9856	.9804
.70	.9749	.9535	.9832	.9781
.75	.9712	.9510	.9804	.9753
.80	.9669	.9486	.9770	.9721
.85	.9616	.9460	.9728	.9680
.90	.9548	.9433	.9671	.9628
.95	.9466	.9405	.9583	.9550
1.00	.9375	.9375	.9375	.9375

The values in Table 6.8 show some very interesting results. For example, the 50 percent of all possible voting situations that are closest to being perfectly single-peaked have a PMRW with probability of .9857 for large electorates. And, the 35 percent of all possible voting situations that are closest to being perfectly single-peaked have a PMRW with probability of .9913 for large electorates. Clearly, any significant degree of internal consistency of voters' preferences that approaches perfectly single-peaked preferences leads to a very high probability that a PMRW exists. The impact of having voters' preferences that suggest the presence of a candidate approaching a perfectly polarizing candidate in voting situations is also quite strong, but it is not as significant as the presence of the same degree of proximity to perfect single-peakedness or single-troughedness in large electorates, assuming equivalence of these factors as measured by α_k, since $P_{PMRW}^{S}\left(n,\infty \mid CIAC_b\left(\beta_b^{p^-}\right)\right) > P_{PMRW}^{S}\left(n,\infty \mid CIAC_c\left(\beta_c^{p^-}\right)\right)$ for $0 < p < 1$.

6.8 The Impact of an Overall Unifying Candidate

Parameters *b* and *t* have been shown to have a significant impact on the probability that a PMRW exists. While *b* measures the proximity of a voting situation to perfect single-peakedness, it has also been described as reflecting the existence of a positively unifying candidate. Similarly, *t* measures proximity of a voting situation to perfect single-troughedness, but it has also been described as reflecting the existence of a negatively unifying candidate.

Both of these parameters can be combined if the presence of an *overall unifying candidate* is considered. By ignoring the distinction between positively unifying and negatively unifying, parameter, *u*, measures the presence of an overall unifying candidate in a voting situation by

$$u = Min\ b,t\}. \tag{6.52}$$

Gehrlein (2006c) develops representations with parameter *u* for odd *n*:

$$\begin{aligned}
&P_{VS}\left(3,n \mid CIAC_u\left(k^-\right)\right) \qquad (6.53)\\
&= \frac{6(k+1)(k+2)\,2\left(15+56k+111k^2+13k^3\right)-5\left(2+27k-7k^2\right)n+10(3-4k)n^2+10n^3\}}{(n+1)(n+2)(n+3)(n+4)(n+5)}\\
&\text{for } 0 \le k \le (n-1)/4.\\
&= \frac{3(n-2k)\begin{bmatrix}18(k+1)\left(13+42k+63k^2+27k^3\right)-3\left(35+250k+360k^2+144k^3\right)n\\ +(25+24k)(5+6k)n^2-3(5+6k)n^3+n^4\end{bmatrix}}{(n+1)(n+2)(n+3)(n+4)(n+5)}\\
&\text{for } (n+1)/4 \le k \le (n-1)/3,\\
&= 1 \text{ for } k = n/3.
\end{aligned}$$

The same logic that was used in previous discussion is then used to obtain the limiting representation for $P_{VS}\left(3,\infty \mid CIAC_u\left(\alpha_k^-\right)\right)$ as $n \to \infty$, with

$$\begin{aligned}
&P_{VS}\left(3,\infty \mid CIAC_u\left(\alpha_k^-\right)\right) \qquad (6.54)\\
&= 6\alpha_k^2\left(10-40\alpha_k+35\alpha_k^2+26\alpha_k^3\right) \text{for } 0 \le \alpha_k \le 1/4.\\
&= 3(1-2\alpha_k)\left(1-18\alpha_k+144\alpha_k^2-432\alpha_k^3+486\alpha_k^4\right)\\
&\text{for } 1/4 \le \alpha_k \le 1/3.
\end{aligned}$$

Values of β_u^p for each proportion p = 0.00(.05)1.00, are obtained from manipulation of Eq. 6.54, and the results are listed in Table 6.7. While 65 percent of all possible voting situations are included in the range of $0 \le \alpha_k \le .1924$ for parameter b, 65 percent of all possible voting situations are included in a much smaller range for parameter u, with $0 \le \alpha_k \le .1484$. This results from the existence of voting situations with a large b, but a small t, or the converse situation.

Representations for the cumulative probability $P_{PMRW}^S\left(3,n \mid CIAC_u\left(k^-\right)\right)$ for odd n are:

$$P_{PMRW}^S\left(3,n \mid CIAC_u\left(k^-\right)\right) \tag{6.55}$$

$$= \frac{30+121k+261k^2+38k^3-10\left(1+15k-3k^2\right)n+10(3-4k)n^2+10n^3}{2\left(15+56k+111k^2+13k^3\right)-5\left(2+27k-7k^2\right)n+10(3-4k)n^2+10n^3}$$

$$\text{for } 0 \le k \le (n-1)/4.$$

$$= \frac{\left[\begin{array}{c} 27\left(25+64k+480k^2+1280k^3+1440k^4+576k^5\right) \\ +9\left(101-960k-3840k^2-5760k^3-2880k^4\right)n+90\left(29+128k+288k^2+192k^3\right)n^2 \\ -10\left(85+576k+576k^2\right)n^3+15(37+64k)n^4-59n^5 \end{array}\right]}{16(n-2u)\left[\begin{array}{c} 18(k+1)\left(13+42k+63k^2+27k^3\right)-3\left(35+250k+360k^2+144k^3\right)n \\ +(25+24k)(5+6k)n^2-3(5+6k)n^3+n^4 \end{array}\right]}$$

$$\text{for } (n+1)/4 \le k \le (n-1)/3.$$

$$= \frac{15(n+3)^2}{16(n+2)(n+4)} \text{ for } k = n/3.$$

The limiting probability representations for $P_{PMRW}^S\left(3,\infty \mid CIAC_u\left(\alpha_k{}^-\right)\right)$ are:

$$P_{PMRW}^S\left(3,\infty \mid CIAC_u\left(\alpha_k{}^-\right)\right) \tag{6.56}$$

$$= \frac{10-40\alpha_k+30\alpha_k{}^2+38\alpha_k{}^3}{10-40\alpha_k+35\alpha_k{}^2+26\alpha_k{}^3} \text{ for } 0 \le \alpha_k \le 1/4,$$

$$= \frac{-59+960\alpha_k-5760\alpha_k{}^2+17280\alpha_k{}^3-25920\alpha_k{}^4+15552\alpha_k{}^5}{16\left(1-2\alpha_k\right)\left(1-18\alpha_k+144\alpha_k{}^2-432\alpha_k{}^3+486\alpha_k{}^4\right)}$$

$$\text{for } 1/4 \le \alpha_k \le 1/3.$$

The representation in Eq. 6.56 is used with values of β_u^p from Table 6.7 to compute numerical values of $P_{PMRW}^S\left(n,\infty \mid CIAC_u\left(\beta_u^{p\,-}\right)\right)$ for each p = 0.00(.05)1.00, and these values are shown in Table 6.8. The use of the joint measure of voter preference unification, u, has a significantly greater impact on the probability that a PMRW exists than the use of the individual measures b and t. The results from Table 6.8 show that the 50 percent of voting situations that are most closely related to perfect overall voter unification have a probability .9910 of having a PMRW, and that the 65 percent of voting situations that are most closely related to perfect voter unification have a probability .9856 of having a PMRW. It is remarkable that any voting situation that is remotely close to representing perfectly unified preferences, as measure by *u*, will have a very high probability of yielding a PMRW with large electorates.

6.9 The Impact of Ward's Condition

It was noted in Chapter 2 that Ward (1965) defines another condition on voting situations that requires the existence of a PMRW for three candidates. This condition requires that voters' preferences do not contain any Latin Squares, which is equivalent to the requirement that there is some candidate that is never ranked first, is never ranked last, or is never ranked in the middle by any voter. Parameter ℓ measures the proximity of a voting situation to meeting Ward's Condition, with

$$\ell = Min\ b,t,c\} . \tag{6.57}$$

If ℓ is equal to zero for a voting situation, then that voting situation does not contain any Latin Squares, and it perfectly meets Ward's Condition. Parameter ℓ is therefore used as a measure of the proximity of a voting situation to perfectly meeting Ward's Condition. Gehrlein (2006c) obtains a representation for $P_{VS}\left(3, n \mid CIAC_\ell\left(k^-\right)\right)$, with

$$P_{VS}\left(3, n \mid CIAC_\ell\left(k^-\right)\right) \tag{6.58}$$
$$= \frac{9(k+1)(k+2) - 3k\left(17 + 27k + 36k^2\right) + 15\left(2 + 3k + 9k^2\right)n - 60kn^2 + 10n^3\}}{(n+1)(n+2)(n+3)(n+4)(n+5)},$$
$$\text{for } 0 \le k \le (n-1)/3.$$
$$= 1 \text{ for } k = n/3.$$

A representation for the limiting distribution $P_{VS}\left(3, \infty \mid CIAC_b\left(\alpha_k^-\right)\right)$ as $n \to \infty$ is then given by

$$P_{VS}\left(3, \infty \mid CIAC_\ell\left(\alpha_k^-\right)\right) = \tag{6.59}$$
$$9\alpha_k^2\left(10 - 60\alpha_k + 135\alpha_k^2 - 108\alpha_k^3\right), \text{ for } 0 \le \alpha_k \le 1/3.$$

Eq. 6.59 is used to obtain the values of β_ℓ^p for each proportion p = 0.00(.05)1.00, and the results are listed in Table 6.7. It was noted above the 65 percent of all possible voting situations are included in the range $0 \le \alpha_k \le .1484$ for parameter *u*. Here, 65 percent of all possible voting situations are contained in the smaller range $0 \le \alpha_k \le .1308$ for parameter ℓ .

A representation for $P^S_{PMRW}\left(3,n \mid CIAC_\ell\left(k^-\right)\right)$ for odd n is obtained following the logic of previous discussion, with

$$P^S_{PMRW}\left(3,n \mid CIAC_\ell\left(k^-\right)\right) \tag{6.60}$$

$$= \frac{\left[\begin{array}{c}(k+1)\left[\begin{array}{c}-135-2547k-4293k^2-6687k^3-2538k^4+10\left(153+273k+759k^2+327k^3\right)n \\ -10\left(3+295k+146k^2\right)n^2+240(2+k)n^3\end{array}\right] \\ +15\delta_k^2\ 9\left(1+2k+2k^2\right)-6(1+2k)n+2n^2\}\end{array}\right]}{24(k+1)(k+2)-3k\left(17+27k+36k^2\right)+15\left(2+3k+9k^2\right)n-60kn^2+10n^3\}},$$

$$\text{for } 0 \le k \le (n-1)/4.$$

$$= \frac{\left[\begin{array}{c}27\left(25+96k+440k^2+840k^3+810k^4+324k^5\right)+9\left(69-880k-2520k^2-3240k^3-1620k^4\right)n \\ +30\left(83+252k+486k^2+324k^3\right)n^2-10\left(41+324k+324k^2\right)n^3+15(23+36k)n^4-31n^5 \\ +30\delta_k^2\ 9\left(1+2k+2k^2\right)-6(1+2k)n+2n^2\}\end{array}\right]}{48(k+1)(k+2)-3k\left(17+27k+36k^2\right)+15\left(2+3k+9k^2\right)n-60kn^2+10n^3\}},$$

$$\text{for } (n+1)/4 \le k \le (n-1)/3.$$

$$= \frac{15(n+3)^2}{16(n+2)(n+4)}, \text{ for } k = n/3.$$

The limiting distribution $P^S_{PMRW}\left(3,\infty \mid CIAC_\ell\left(\alpha_k{}^-\right)\right)$ is given by

$$P^S_{PMRW}\left(3,\infty \mid CIAC_\ell\left(\alpha_k{}^-\right)\right) \tag{6.61}$$

$$= \frac{120-730\alpha_k+1635\alpha_k{}^2-1269\alpha_k{}^3}{12\left(10-60\alpha_k+135\alpha_k{}^2-108\alpha_k{}^3\right)}, \text{ for } 0 \le \alpha_k \le 1/4.$$

$$= \frac{-31+540\alpha_k-3240\alpha_k{}^2+9720\alpha_k{}^3-14580\alpha_k{}^4+8748\alpha_k{}^5}{48\alpha_k{}^2\left(10-60\alpha_k+135\alpha_k{}^2-108\alpha_k{}^3\right)},$$

$$\text{for } 1/4 \le \alpha_k \le 1/3.$$

Eq. 6.61 is used with entries from Table 6.7 to compute numerical values of $P^S_{PMRW}\left(n,\infty \mid CIAC_\ell\left(\beta_\ell^{p^-}\right)\right)$ for each p = 0.00(.05)1.00, and these resulting values are given in Table 6.8.

Some interesting observations can be made about the values of $P^S_{PMRW}\left(n,\infty \mid CIAC_\ell\left(\beta_\ell^{p^-}\right)\right)$ in Table 6.8. It was noted previously that

$$P^S_{PMRW}\left(n,\infty \mid CIAC_b\left(\beta_b^{p^-}\right)\right) > P^S_{PMRW}\left(n,\infty \mid CIAC_c\left(\beta_c^{p^-}\right)\right) \text{ for all } 0 < p < 1.$$

The impact that a polarizing candidate has on the probability that a PMRW exists is not as strong as the impact having a positively-unifying candidate or a negatively-unifying candidate. As a result, despite the fact that $\beta_\ell^p \le \beta_u^p$ for all p, we find $P_{PMRW}^{S}\left(n,\infty \mid CIAC_{\ell}\left(\beta_{\ell}^{p^{-}}\right)\right) < P_{PMRW}^{S}\left(n,\infty \mid CIAC_{u}\left(\beta_{u}^{p^{-}}\right)\right)$ for all $0 < p < 1$.

6.10 Ehrhart Polynomials

All of the representations that have been obtained in the current study could not conceivably have been developed without the EUPIA and EUPIA2 procedures. These procedures work because of the known polynomial form that representations must have for counting the number of voting situations with specified characteristics for three-candidate elections, from Axioms 6.1, 6.2, 6.3. It has recently been pointed out that these procedures are based on much more general principles that have been developed under the topic of *Ehrhart Polynomials*. For example, see Ehrhart (1967a, 1967b), where the general problem is developed in the context of counting the number lattice points in a polyhedron.

Lepelley, et al. (2006) develop an algorithm that is based on the notions of Ehrhart Polynomials to produce representations of the type that have been obtained in the current study. Preliminary results indicate that their procedure is very efficient, to open the door to many possible investigations into the probability that voting events can occur. Mbih, et al. (2006) formally develop many of the links between the type of work that has been done in the current study and the notions of Ehrhart Polynomials.

6.11 Conclusion

When voters' preferences in a three-candidate voting situation reflect any significant degree of proximity to perfect single-peakedness, perfect single-troughedness, or perfect polarization, the probability that a PMRW exists is quite high. When voters' preferences are at all close to reflecting a situation in which a unifying candidate exists, the probability that a PMRW exists is very high. It is very important to note that the associated underlying models that lead to single-peaked, single-troughed, or polarized preferences do not actually have to be the basis of the mechanism by which the voters' preference rankings on candidates are actually formed. It is only required that the preferences in a given voting situation could have been obtained by one of these models. As a result, Condorcet's Paradox should rarely be observed in any real elections on a small number of candidates with large electorates, as long as voters' preferences reflect any significant degree of group coherence or consistency.

7 Individual Intransitivity

7.1 Introduction

The requirement that individual voters must have transitive preferences on candidates has been assumed as a basis of rational individual behavior from the start of this study. However, it was mentioned that models exist to explain situations in which individuals might have intransitive preferences, and that is the topic of the current chapter.

Consider a subject who is making pairwise preference comparisons on elements from a set of three alternatives $\{A,B,C\}$. The pairwise comparisons of alternatives are made on the basis of some common set of attributes. The subject has some perceived ranking on alternatives for each of the particular attributes. These rankings will quite likely be different for different attributes. For example, the subject might perceive *A* as being superior to both *B* and *C* on the basis of one attribute, while being inferior to both *B* and *C* on the basis of some other attribute. The subject then makes a pairwise comparison between alternatives on the basis of these perceived attribute rankings on the alternatives. Using the notation from Chapter 1, $A \succ B$ denotes the outcome that the subject responds with an overall pairwise preference for *A* over *B*, after considering the relative attribute rankings of these two alternatives.

Suppose that there are *n* different common attributes of comparison that are used to make comparisons between the three alternatives. It follows that there are six possible complete rankings on the alternatives that a subject might have for any particular attribute, as noted in Fig. 7.1.

A	A	B	C	B	C
B	C	A	A	C	B
C	B	C	B	A	A
n_1	n_2	n_3	n_4	n_5	n_6

Fig. 7.1 Possible rankings on attribute values for three alternatives

The alternative rankings in Fig. 7.1 are ordered so that the top ranked item is perceived as being the best alternative according to any particular attribute, and n_i denotes the number of times that the i^{th} ranking represents the subject's perceived ranking on alternatives for attributes in a given *attribute ranking situation*, and ***n*** denotes a specified combination of n_i 's with $\sum_{i=1}^{6} n_i = n$.

This scenario is completely analogous to earlier developments with regard to the formation of voter preference profiles and voting situations. The three alternatives in this case correspond to the three candidates in earlier discussion, and the *n* attribute rankings on alternatives correspond to the *n* voter preference rankings on candidates.

7.2 Algebraic and Probabilistic Choosers

Luce and Suppes (1965) note two different general types of models to describe how a subject might go about selecting his or her more-preferred alternative from a pair of available alternatives that are under consideration. An *algebraic chooser* approaches this process by performing a precise evaluation of all of the information that is contained in the rankings of alternatives by the attributes, to reach an absolute and precise determination of the more preferred alternative. A *probabilistic chooser* is less precise in the process of performing preference comparisons. As a result, a probabilistic chooser will only have some associated probability of selecting the same more-preferred alternative that an algebraic chooser would select in an identical situation.

DeSoete, et al. (1989) present a survey of different models that have been developed to explain the basis of preference comparisons that are made by probabilistic choosers. By resorting to the use of probabilistic models, it is inherently assumed that a subject is unwilling to take the time and effort to precisely evaluate the perceived alternative rankings on each of the attributes to definitively determine the more preferred alternative from a pair. A typical example would involve decisions involving the selection of a low impact item when there are many available alternatives to pick from with many different attributes of comparison. There are many types of situations in which it might be expected that subjects will act as probabilistic choosers, rather than act as a “utility maximizing, omniscient, indefatigable consumer” [Swait and Adamowicz (2001), pg 135]. A probabilistic chooser is not expected necessarily to select the same more preferred alternative from a pair as an algebraic chooser picks, but it is expected that there should be a positive general relationship between a probabilistic chooser’s ultimate selection of a more preferred alternative from a pair and the choice that would be made if the subject would undergo a thorough consideration of the decision, given the perceived attribute rankings on alternatives.

A number of empirical studies have been performed to find that subjects will often resort to the use of simple decision-making heuristic processes when they are making preference choices, rather than act as precise algebraic choosers. Subjects are typically found to resort to the use of such simple heuristic decision processes when the decision task becomes more difficult and complicated. See for example: Mazzotta and Opaluch (1995), Stone and Kadous (1997) and Bettman, et al. (1998).

7.3 May's Model

Several studies have identified one particular simplifying heuristic that is used in making pairwise preference comparisons. This specific heuristic operates by making preference comparisons between alternatives with the use of a PMR-like relationship on pairs of alternatives. That is, the subject would respond that $A \succ B$ if the alternative rankings on attributes are such that $n_1 + n_2 + n_4 > n_3 + n_5 + n_6$, so that A is viewed as being superior to B on more attributes than B is viewed as being superior to A.

Bettman (1979) summarizes a number of empirical studies that have given support to the notion that this model accurately reflects the process by which some subjects make pairwise preference comparisons in various situations. Wright and Barbour (1977) refer to this model as the "attribute dominance model" and it was found that approximately 20 percent of the subjects in their experiment were using it to make pairwise preference comparisons. They also note that this model should be more widely recognized in studies of behavioral decision making. Russo and Dosher (1983) refer to this model as the "multiple confirming dimensions heuristic" and found that approximately 50 percent of the subjects in their study were using it to make pairwise preference comparisons. They also noted that the propensity of subjects to use this model increases as the number of attributes of comparison increases. Arrow and Raynaud (1986) refer to this model as the "outranking problem". The term *May's Model* is used in the current study, since this model was first proposed in May (1954).

A subject who makes choices according to May's Model is acting as an algebraic chooser, since the n attribute rankings in a given situation will be precisely evaluated by the subject to determine the results of a particular pairwise preference comparison. The model sounds plausible from its definition, and empirical studies indicate that it is often used by subjects to make pairwise comparisons. The major problem that develops is that May's Model can lead to intransitive preferences for individuals. Given the analogy between individual preference comparisons and voting procedures that was outlined above, along with the basic definition of May's Model, it is easily seen that the existence of intransitive preferences for individuals in this case is directly linked to the existence of Condorcet's Paradox in the context of voting.

May (1954) conducts an experiment in which 62 undergraduate students perform all pairwise comparisons on three hypothetical marriage partners. Each hypothetical marriage partner was evaluated on the basis of three specified characteristics, and the given rankings of the three possible choices on the three characteristics formed a perfect Latin Square. Some subjects had individual intransitivity in their reported pairwise preference responses that was consistent with using May's Model to make pairwise preference comparisons.

Lansdowne (1996) considers an example of the same type of problem as applied to the evaluation of five possible light helicopter systems by the U. S. Department of Defense. Each system was initially given a numerical score for each of seven different criteria. An ordinal ranking of systems was then obtained for each criterion, according to the relative numerical values that each system had for the criteria measurements. May's Model did not result in any intransitivity in this case. Obviously, May's Model will not always result in an intransitive result.

As in the previous analysis of Condorcet's Paradox, the focus here turns to considering the probability that May's Model will result in transitive preference responses from a subject who is performing pairwise preference comparisons on three alternatives. By the definition of May's Model, Alternative A will be the strictly most preferred alternative, or *Strict Maximal Alternative* (*SMA*), for a subject with a specified attribute ranking situation $\boldsymbol{n}$ when:

$$n_3 + n_5 + n_6 \le \frac{n-1}{2} \Rightarrow A \succ B$$
$$n_4 + n_5 + n_6 \le \frac{n-1}{2} \Rightarrow A \succ C. \tag{7.1}$$

The logic behind Eq. 7.1 directly follows the notions behind Eq. 3.21, except that each of the conditions on the n_i's in Eq. 7.1 lead to a pairwise preference outcomes for the subject, rather than lead to a PMR preference on a pair of candidates for a group of voters. If a SMA exists on three candidates with May's Model, then the subject's pairwise preferences must be transitive.

Social homogeneity was seen to have an impact on the probability that Condorcet's Paradox is observed. In the context of individual preferences, this corresponds to consideration of the possibility that a correlation exists between the relative rankings of alternatives in attribute rankings. The existence of a "halo effect" occurs if a subject tends to perceive some alternative as rating very well on most attributes. Increasing the likelihood that such an alternative exists would seem to tend to increase the likelihood that a SMA exists with May's Model.

Representations for the probability that a SMA exists with May's Model are obtained in the context of determining if a SMA exists when a profile of n different criteria rankings on the alternatives is randomly generated in an urn experiment, following the discussion in Section 3.7. In order to determine the impact of increasing the degree of dependence among the criteria rankings, the urn experiments are conducted with a P-E model with parameter α, and the experiments start with all $A_i = 1$ for i = 1, 2, 3, 4, 5, 6. The symmetry of P-E models with respect to candidates makes it equally likely that A, B or C is the SMA in a randomly generated profile of criteria rankings. Given all of these arguments, along with the logic that led to Eq. 3.60, a representation for the probability, $P_{SMA}(3, n, PE(\alpha))$, that a SMA exists with May's Model for n attributes of comparison under the assumption of a P-E model with parameter α is given by

$$P_{SMA}(3, n, PE(\alpha)) = \tag{7.2}$$
$$3 \sum_{n_6=0}^{\frac{n-1}{2}} \sum_{n_5=0}^{\frac{n-1}{2}-n_6} \sum_{n_4=0}^{\frac{n-1}{2}-n_6-n_5} \sum_{n_3=0}^{\frac{n-1}{2}-n_6-n_5} \sum_{n_2=0}^{n-n_6-n_5-n_4-n_3} P^1(\boldsymbol{n}, \alpha).$$

Here, the definition of $P^1(\boldsymbol{n},\alpha)$ follows from Eqs. 3.84 and 3.86, and n_1 is obtained from $n_1 = n - n_2 - n_3 - n_4 - n_5 - n_6$.

Table 7.1 lists computed values of $P_{SMA}(3,n,PE(\alpha))$ for each value of n = 3, 5, 9, 15, 25, 45 with each α =0, 1, 2, 3, 4, 5, 10, 15, 20, 25. The computed values in Table 7.1 show two distinct trends. The probability of that a SMA exists with May's Model decreases as the number of attributes of comparison increases. And, the probability increases as α increases, to indicate that an increase in the degree of dependence among attribute rankings will increase the probability that a SMA will exist with May's Model. The probability that a SMA exists exceeds .91 in all cases with three alternatives.

Table 7.1 Computed values of $P_{SMA}(3,n,PE(\alpha))$

	Number of Attributes (n)					
α	3	5	9	15	25	45
0	.9444	.9306	.9220	.9180	.9157	.9142
1	.9643	.9524	.9441	.9404	.9387	.9379
2	.9750	.9665	.9604	.9575	.9561	.9554
3	.9815	.9753	.9708	.9686	.9675	.9668
4	.9857	.9811	.9776	.9760	.9750	.9745
5	.9886	.9850	.9824	.9810	.9803	.9798
10	.9952	.9938	.9927	.9922	.9919	.9917
15	.9974	.9966	.9961	.9958	.9956	.9955
20	.9983	.9979	.9975	.9974	.9973	.9972
25	.9988	.9985	.9983	.9982	.9981	.9981

It is interesting to note the impact that increasing the number of attributes of comparison has on the probability that a SMA exists for an algebraic chooser using May's Model. An analogous situation was found in the context of voting situations where Tullock and Campbell (1970) use Monte-Carlo simulation with a spatial model for voter preferences to estimate the probability that there are intransitives in PMR voting. Attention is restricted to small numbers of voters and a small number of candidates, since it is speculated that this situation is most likely to give rise to the existence of PMR cycles. Of particular interest is the effect that varying the number of dimensions in the preference space has on the propensity of PMR to exist. With random placement of both the ideal voter preference positions and the positions of candidates in the feasible space, the addition of dimensions to the space does tend to increase the probability that PMR cycles exist. However, in changing from two to three dimensions, and from three to five dimensions, the increase in the probability is not particularly significant.

7.4 Probabilistic Chooser Models

Gehrlein (1990b, 1990d, 2006d) considers variations of May's Model in which individual subjects are probabilistic choosers. Using May's Model as a basis for modeling a probabilistic chooser, a subject should logically be expected to become more likely to respond $A \succ B$ as *A* is ranked above *B* in an increasing number of attribute rankings. However, since the subject is not a precise algebraic chooser, there will typically always be some probability that the subject responds $B \succ A$. A *pairwise preference model* (*PPM*) defines the probabilities that a subject will respond with $A \succ B$ or $B \succ A$ for an attribute ranking situation on alternatives in a given $\boldsymbol{n}$.

Let $P_{A \succ B}^{PPM}(\boldsymbol{n})$ denote the probability that a subject will respond with $A \succ B$ for a given attribute ranking situation, as specified by $\boldsymbol{n}$. It is assumed throughout that PPM's do not permit a subject response of indifference in pairwise comparison, so $P_{A \succ B}^{PPM}(\boldsymbol{n}) + P_{B \succ A}^{PPM}(\boldsymbol{n}) = 1$ for any $\boldsymbol{n}$. Given that $A \succ B$ for an algebraic chooser with May's Model for a specified $\boldsymbol{n}$, PPM models with greater values of $P_{A \succ B}^{PPM}(\boldsymbol{n})$ then represent probabilistic choosers with greater discriminatory power in making pairwise preference comparisons.

Gehrlein (1990b) describes a particular PPM called *Model L* for a probabilistic chooser who will respond $A \succ B$ for an attribute ranking situation with a given $\boldsymbol{n}$ with a probability $P_{A \succ B}^{L}(\boldsymbol{n})$ that is obtained as the proportion of rankings in which *A* is ranked ahead of *B*, with

$$P_{A \succ B}^{L}(\boldsymbol{n}) = \frac{n_1 + n_2 + n_4}{n} \qquad P_{B \succ A}^{L}(\boldsymbol{n}) = \frac{n_3 + n_5 + n_6}{n}. \tag{7.3}$$

The term Model L is used in this case since $P_{A \succ B}^{L}(\boldsymbol{n})$ increases linearly as more attribute rankings have *A* ranked ahead of *B*. This model is consistent with some standards that are required of all PPM's.

First, Model L is *unbiased* towards candidates in any pairwise preference comparison, since there is an equal probability of alternative selection when there is a tie between candidates in their relative position in attribute rankings. That is, $P_{A \succ B}^{PPM}(\boldsymbol{n}) = P_{B \succ A}^{PPM}(\boldsymbol{n}) = 1/2$ when $n_1 + n_2 + n_4 = n_3 + n_5 + n_6$. *Decisiveness* is also required in all PPM's, so that the subject must respond $A \succ B$ if *A* is ranked ahead of *B* in every attribute ranking, so that $P_{A \succ B}^{PPM}(\boldsymbol{n}) = 1$ and $P_{B \succ A}^{PPM}(\boldsymbol{n}) = 0$ if $n_1 + n_2 + n_4 = n$.

Even if the algebraic form of May's Model does have a SMA, Model L does not necessarily have one. Model L might also produce a different SMA than the one that is obtained by May's Model. It is of interest to develop representations for the probability that Model L and other PPM's will result in the same SMA as the algebraic form of May's Model, so that the probabilistic chooser will effectively be behaving as an algebraic chooser with May's Model. The number of attributes is assumed to be odd throughout, to avoid complications with ties with May's Model.

7.5 Algebraic and Probabilistic Chooser Coincidence

Let $P_{SMA}^{PPM}(3,n,PE(\alpha))$ denote the conditional probability that a given PPM and May's Model will both have the same SMA for n attributes of comparison with three alternatives, when attribute ranking situations are being randomly generated by to a P-E model with parameter α, given that May's Model has a SMA. The starting point for developing a general representation for $P_{SMA}^{PPM}(3,n,PE(\alpha))$, is a representation for the joint probability, $JP_{SMA}^{PPM}(3,n,PE(\alpha))$, that a given PPM and May's Model both have the same SMA for n attributes for a given α. Then

$$P_{SMA}^{PPM}(3,n,PE(\alpha)) = \frac{JP_{SMA}^{PPM}(3,n,PE(\alpha))}{P_{SMA}(3,n,PE(\alpha))}. \tag{7.4}$$

The symmetry of P-E Models with respect to alternatives requires that $JP_{SMA}^{PPM}(3,n,PE(\alpha))$ can be obtained as three times the probability that A is the SMA both with May's Models and with Model L. Sequential pairwise preference responses from subjects are assumed to be obtained independently, and it follows from definitions that

$$JP_{SMA}^{L}(3,n,PE(\alpha)) = \tag{7.5}$$

$$3\sum_{n_6=0}^{\frac{n-1}{2}}\sum_{n_5=0}^{\frac{n-1}{2}-n_6}\sum_{n_4=0}^{\frac{n-1}{2}-n_6-n_5}\sum_{n_3=0}^{\frac{n-1}{2}-n_6-n_5}\sum_{n_2=0}^{n-n_6-n_5-n_4-n_3}\left(\frac{n-n_3-n_5-n_6}{n}\right)\left(\frac{n-n_4-n_5-n_6}{n}\right)P^1(\boldsymbol{n},\alpha).$$

The summation indexes in Eq. 7.5 require that A is the SMA with May's Model, since they enumerate all possible alternative ranking outcomes with $n_4+n_5+n_6 \le \frac{n-1}{2}$ and $n_3+n_5+n_6 \le \frac{n-1}{2}$. The product terms that are being summed over give the probabilities $P_{A\succ B}^{L}(\boldsymbol{n})$ and $P_{A\succ C}^{L}(\boldsymbol{n})$ with Model L for each $\boldsymbol{n}$ that is defined by the summation indexes.

Computed values of $P_{SMA}^{L}(3,n,PE(\alpha))$ that result from Eq. 7.4, after obtaining values with the representation for in Eq. 7.5 are listed in Table 7.2 for each n = 3, 5, 9, 15, 25, 45 for each α = 0, 1, 2, 3, 4, 5, 10, 15, 20, 25. The values that are listed in Table 7.2 indicate that $P_{SMA}^{L}(3,n,PE(\alpha))$ increases as α increases, and that $P_{SMA}^{L}(3,n,PE(\alpha))$ decreases as n increases. Values of $P_{SMA}^{L}(3,n,PE(\alpha))$ are less than .50 when n = 45 for both α equal to zero and one.

Table 7.2 Computed values of $P^{L}_{SMA}(3, n, PE(\alpha))$

	Number of Attributes (n)					
α	3	5	9	15	25	45
0	.5882	.4945	.4218	.3781	.3463	.3200
1	.6420	.5690	.5179	.4910	.4741	.4626
2	.6838	.6224	.5811	.5601	.5474	.5387
3	.7170	.6635	.6282	.6106	.6001	.5930
4	.7440	.6963	.6653	.6500	.6409	.6348
5	.7663	.7233	.6955	.6819	.6738	.6684
10	.8374	.8082	.7896	.7806	.7752	.7717
15	.8753	.8531	.8391	.8323	.8283	.8257
20	.8989	.8810	.8697	.8642	.8610	.8589
25	.9150	.9000	.8905	.8860	.8833	.8815

The remarkably small conditional probabilities that May's Model and Model L select the same SMA, given that a SMA exists with May's Model, for larger values of n for small α raises interest in a determination of limiting values of these probabilities as $n \to \infty$. Following previous discussion, the case with α equal to zero is IC-like in the context of the current problem, with complete independence between attribute rankings on alternatives. Similarly, the case with α equal to one is the same as IAC-like, suggesting the presence of a small degree of dependence between attribute rankings on alternatives.

To develop a limiting representation for $P^{L}_{SMA}(3, \infty, PE(0))$, we begin by considering the limiting probability, $P^{A\}}_{SMA}(3, \infty, PE(0))$, that A is the SMA with May's Model for three alternatives, following the development of the representation for $P^{S}_{PMRW}(3, \infty, DC)$ in Eq. 3.69. The process is replicated here in the context of the current problem to facilitate an extension of this analysis in later discussion.

A random profile of n attribute rankings on alternatives is obtained by sequentially drawing balls from an urn in a P-E experiment. When the i^{th} ball is drawn during the experiment, the color of the selected ball determines which of the six possible complete rankings from Fig. 7.1 represents the i^{th} attribute ranking. Here, p_j denotes the probability that the j^{th} colored ball is drawn. With α equal to zero, these p_j probabilities do not change from draw to draw. Two discrete variables X_1^i and X_2^i describe two joint events that can result as each ball is drawn in the experiment, with:

$$X_1^i = \begin{matrix} +1 : p_1 + p_2 + p_4 \\ -1 : p_3 + p_5 + p_6 \end{matrix} \qquad (7.6)$$

$$X_2^i = \begin{matrix} +1 : p_1 + p_2 + p_3 \\ -1 : p_4 + p_5 + p_6. \end{matrix}$$

The definitions of these variables are such that, $X_1^i = +1$ if the subject perceives A as being superior to B in the basis of the i^{th} attribute, and $X_1^i = -1$ if the subject perceives B as being superior to A on that attribute. Then, according to May's Model the subject will respond with $A \succ B$ if $\sum_{i=1}^{n} X_1^i > 0$. Similarly, the subject will respond $A \succ C$ if $\sum_{i=1}^{n} X_2^i > 0$. Let $\overline{X}_1$ denote the average value of X_1^i, with $\overline{X}_1 = \left[\sum_{i=1}^{n} X_1^i\right]/n$. Then, A will be the SMA with May's Model with the joint probability that $\overline{X}_1 > 0$ and $\overline{X}_2 > 0$. This can be stated in the alternative form that A will be the SMA with May's Model in a randomly drawn profile of attribute rankings on alternatives with the joint probability that $\overline{X}_1\sqrt{n} > 0$ and $\overline{X}_2\sqrt{n} > 0$. The argument that is used here is the same as the one that is used to develop Eq. 3.69, with $X_1^i = X_B^i$ and $X_2^i = X_C^i$.

The Central Limit Theorem applies as the number of voters gets large, with $n \to \infty$ and the limiting joint distribution of $\overline{X}_1\sqrt{n}$ and $\overline{X}_2\sqrt{n}$ will have a bivariate normal distribution. The probability that each of $\overline{X}_1\sqrt{n}$ and $\overline{X}_2\sqrt{n}$ take on any particular value, including zero, in a bivariate normal distribution is zero, so the probability that A is the SMA with May's Model can be restated as the joint probability that $\overline{X}_1\sqrt{n} \geq 0$ and $\overline{X}_2\sqrt{n} \geq 0$. The Central Limit Theorem also states that the correlation between $\overline{X}_1\sqrt{n}$ and $\overline{X}_2\sqrt{n}$ in this bivariate normal distribution is identical to the correlation, $Cor\left(X_1^i, X_2^i\right)$, between the original variables X_1^i and X_2^i.

Given the assumption above that the P-E urn experiment starts with all $A_i = 1$ for i = 1, 2, 3, 4, 5, 6, $p_i = 1/6$ for i = 1, 2, 3, 4, 5, 6, and these probabilities are constant on each draw since $\alpha = 0$. Following the development of Eq. 3.64,

$$E\left(X_1^i\right) = E\left(X_2^i\right) = 0. \tag{7.7}$$

The logic of the development of Eq. 3.68 leads to

$$Cor\left(X_1^i, X_2^i\right) = 1/3. \tag{7.8}$$

The limiting probability $P_{SMA}^{A\}}(3, \infty, PE(0))$ is therefore given as the joint probability that $\overline{X}_1\sqrt{n} \geq 0$ and $\overline{X}_2\sqrt{n} \geq 0$, in a bivariate normal distribution with a coefficient of correlation that is equal to 1/3. Since $E\left(X_1^i\right) = E\left(X_2^i\right) = 0$, it follows that $E\left(X_1^i\sqrt{n}\right)=E\left(X_2^i\sqrt{n}\right)=0$, so $P_{SMA}^{A\}}(3,\infty,PE(0))$ is the same as the joint probability that $\overline{X}_1\sqrt{n}\geq E\left(\overline{X}_1\sqrt{n}\right)$ and $\overline{X}_2\sqrt{n}\geq E\left(\overline{X}_2\sqrt{n}\right)$, in a bivariate normal distribution with a coefficient of correlation equal to 1/3. Using the notation of Chapter 4, this limiting joint probability is the bivariate normal positive orthant probability $\Phi_2(1/3)$.

Sheppard's 1898 Theorem of Median Dichotomy applies to bivariate normal positive orthant probabilities, and $P_{SMA}^{A\}}(3,\infty,PE(0))=\frac{1}{4}+\frac{1}{2\pi}Sin^{-1}\left(\frac{1}{3}\right)$. The symmetry of PE models with respect to candidates leads to $P_{SMA}^{A\}}(3,\infty,PE(0))=$ $P_{SMA}^{B\}}(3,\infty,PE(0))=P_{SMA}^{C\}}(3,\infty,PE(0))$, and it directly follows that

$$P_{SMA}(3,\infty,PE(0))=\frac{3}{4}+\frac{3}{2\pi}Sin^{-1}\left(\frac{1}{3}\right). \tag{7.9}$$

The representation for $P_{SMA}(3,\infty,PE(0))$ in Eq. 7.9 is identical to the representation for Guilbaud's result in Eq. 3.77.

A representation for the limiting probability $JP_{SMA}^{L}(3,\infty,PE(0))$ is obtained by using the same notions from expected value theory that led to the representation for $JP_{SMA}^{L}(3,n,PE(\alpha))$ in Eq. 7.5. To start, consider the limiting joint probability, $JP_{A\}}^{L}(3,\infty,PE(0))$, that A is the SMA with May's Model and with Model L.

$$JP_{A\}}^{L}(3,\infty,PE(0))=E^2\left[\frac{(n_1+n_2+n_4)(n_1+n_2+n_3)}{n^2}\right]. \tag{7.10}$$

The expectation, E^2, in Eq. 7.10 is taken over the two dimensions of the positive orthant of the bivariate normal distribution. Let, $Z_j=\overline{X}_j\sqrt{n}$, for j = 1, 2 and then define $JP_{A\}}^{L}(3,\infty,PE(0))$ in terms of Z_1 and Z_2. Using the fact that $n_1+n_2+n_4=\frac{n+\sum_{i=1}^{n}X_1^i}{2}$ and $n_1+n_2+n_3=\frac{n+\sum_{i=1}^{n}X_2^i}{2}$, $JP_{A\}}^{L}(3,\infty,PE(0))$ can be rewritten as

$$\begin{aligned}JP_{A\}}^{L}(3,\infty,PE(0))&=E^2\left[\left(\frac{1}{2}+\frac{Z_1}{2\sqrt{n}}\right)\left(\frac{1}{2}+\frac{Z_2}{2\sqrt{n}}\right)\right]\\&=\frac{1}{4}\Phi_2(1/3)+\frac{1}{4\sqrt{n}}E^2[Z_1+Z_2]+\frac{1}{4n}E^2[Z_1Z_2].\end{aligned} \tag{7.11}$$

Johnson and Kotz (1972) give general representations of the form of $E^2[Z_1 + Z_2]$ and $E^2[Z_1 Z_2]$ for expectation over the positive orthant of bivariate normal probability distributions with correlation equal to ρ as functions of ρ. For the special case with ρ = 1/3,

$$E^2[Z_1] = E^2[Z_2] = \frac{1}{4}\sqrt{\frac{2}{\pi}}(1+\rho) = \frac{1}{3}\sqrt{\frac{2}{\pi}} \tag{7.12}$$

$$E^2[Z_1 Z_2] = \frac{1}{2\pi}\left[\rho\left(\frac{\pi}{2} + Sin^{-1}(\rho)\right) + \sqrt{1-\rho^2}\right] = \frac{1}{6\pi}\left[\frac{\pi}{2} + Sin^{-1}\left(\frac{1}{3}\right) + 2\sqrt{2}\right].$$

In the limit as $n \to \infty$, after substituting the appropriate terms from Eq. 7.12 into Eq. 7.11, we find

$$JP^L_{A\}}(3,\infty,PE(0)) = \frac{1}{4}\Phi_2\left(\frac{1}{3}\right) = \tfrac{1}{12}P_{SMA}(3,\infty,PE(0)). \tag{7.13}$$

Using the symmetry of P-E models with respect to Alternatives *A*, *B* and *C*, along with Eq. 7.4,

$$P^L_{SMA}(3,n,PE(0)) = \frac{3JP^L_{A\}}(3,\infty,PE(0))}{P_{SMA}(3,\infty,PE(0))} = \frac{1}{4}. \tag{7.14}$$

A Model L chooser is quite likely either to select some SMA, or to have pairwise preferences on alternatives that cycle, when May's Model selects a different SMA. It is also noted that the rate of convergence of this probability to the limiting value as $n \to \infty$ is very slow, since $P^L_{SMA}(3,45,PE(0))$ = .3200 in Table 4.2.

The given representation for $P^1(\boldsymbol{n},\alpha)$ for the special case in which α is equal to one in Eq. 3.86 allows for a direct algebraic reduction of Eq. 7.5 for this special case. Gehrlein (1990b) performs the necessary algebraic reduction to obtain a representation for $P^L_{SMA}(3,n,PE(1))$ with odd *n*, and

$$P^L_{SMA}(3,n,PE(1)) = \frac{47n^4 + 354n^3 + 787n^2 + 450n + 42}{105n^2(n+3)^2}. \tag{7.15}$$

The limiting value of $P^L_{SMA}(3,n,PE(1))$ as $n \to \infty$ is given by $P^L_{SMA}(3,\infty,PE(1))$ = 47/105 $\approx$ .4476, which is a very significant improvement over the limiting result from Eq. 7.14, which has $P^L_{SMA}(3,\infty,PE(0))$ = .2500. The introduction of some small degree of dependence among attribute rankings by using α equal to one clearly increases the propensity of Model L choosers to behave like algebraic choosers with May's Model, when compared to the case with α equal to zero, with complete independence between attribute rankings.

7.6 Weak Maximal Alternatives

The representation for $P^{L}_{SMA}(3,n,PE(\alpha))$ that follows from Eq. 7.4 is a strict conditional probability since it requires the same SMA with both May's Model and Model L. However, there could be a SMA with May's Model, while there is a pairwise preference cycle with Model L. If a subject is unable to make a direct determination of a SMA, due to the presence of a pairwise preference cycle with any PPM, it is assumed that the subject selects a maximal alternative at random, with an equal likelihood of selecting any of the alternatives. A *Weak Maximal Alternative* (WMA) is an alternative that a subject either directly selects as a SMA, or indirectly selects as a result of a random determination when a pairwise preference cycle exists. Let $P^{PPM}_{WMA}(3,n,PE(\alpha))$ denote the conditional probability that a subject selects a WMA that is the same as the SMA with May's Model, given that May's Model has a SMA.

In order to develop a representation for the probability $P^{PPM}_{WMA}(3,n,PE(\alpha))$ from $P^{PPM}_{SMA}(3,n,PE(\alpha))$, we start with a representation for the joint probability, $T^{PPM}_{A\}}(3,n,PE(\alpha))$ that a subject has a pairwise preference cycle with a PPM when A is the SMA with May's Model. There are two possible cycles that must be accounted for when A is the SMA with May's Model. For Model L:

$$T^{L}_{A\}}(3,n,PE(\alpha)) = \tag{7.16}$$

$$\sum_{n_6=0}^{\frac{n-1}{2}} \sum_{n_5=0}^{\frac{n-1}{2}-n_6} \sum_{n_4=0}^{\frac{n-1}{2}-n_6-n_5} \sum_{n_3=0}^{\frac{n-1}{2}-n_6-n_5} \sum_{n_2=0}^{n-n_6-n_5-n_4-n_3} \left[\begin{array}{l} \left(\frac{n_1+n_2+n_4}{n}\right)\left(\frac{n_1+n_3+n_5}{n}\right)\left(\frac{n_4+n_5+n_6}{n}\right)+ \\ \left(\frac{n_3+n_5+n_6}{n}\right)\left(\frac{n_2+n_4+n_6}{n}\right)\left(\frac{n_1+n_2+n_3}{n}\right) \end{array} \right] P^1(\boldsymbol{n},\alpha)$$

The summation indexes in Eq. 7.16 require that A is the SMA for an algebraic chooser with May's Model, and the two triple product terms that are being summed account for the two possible pairwise preference cycles on the three alternatives for a probabilistic chooser who is using Model L. The first product is $P^{L}_{A\succ B}(\boldsymbol{n})\, P^{L}_{B\succ C}(\boldsymbol{n})\, P^{L}_{C\succ A}(\boldsymbol{n})$ for a given $\boldsymbol{n}$ that is based on the summation indexes, and the second product is $P^{L}_{B\succ A}(\boldsymbol{n})\, P^{L}_{C\succ B}(\boldsymbol{n})\, P^{L}_{A\succ C}(\boldsymbol{n})$.

A general representation for $P^{PPM}_{WMA}(3,n,PE(\alpha))$ is obtained by appealing to both the symmetry of P-E models with respect to alternatives and the use of a random selection procedure to determine the most preferred alternative in the presence of cycles in pairwise preferences, with

$$P^{PPM}_{WMA}(3,n,PE(\alpha)) = \frac{JP^{PPM}_{SMA}(3,n,PE(\alpha)) + \frac{1}{3}\left\{3T^{PPM}_{A\}}(3,n,PE(\alpha))\right\}}{P_{SMA}(3,n,PE(\alpha))}. \tag{7.17}$$

Computed values of $P_{WMA}^{L}(3,n,PE(\alpha))$ from Eq. 7.17 that use calculations from the representation for $T_{A\}}^{L}(3,n,PE(\alpha))$ in Eq. 7.16 are listed in Table 7.3 for each n = 3, 5, 9, 15, 25, 45 for each α =0, 1, 2, 3, 4, 5, 10, 15, 20, 25. The results in Table 7.3 indicate both that $P_{WMA}^{L}(3,n,PE(\alpha))$ increases as α increases, and that $P_{WMA}^{L}(3,n,PE(\alpha))$ decreases as n increases.

Table 7.3 Computed values of $P_{WMA}^{L}(3,n,PE(\alpha))$

	Number of Attributes (n)					
α	3	5	9	15	25	45
0	.6405	.5589	.4944	.4550	.4258	.4012
1	.6872	.6240	.5796	.5560	.5412	.5311
2	.7236	.6706	.6349	.6169	.6058	.5984
3	.7526	.7064	.6761	.6611	.6520	.6459
4	.7762	.7351	.7085	.6954	.6877	.6825
5	.7957	.7586	.7348	.7232	.7163	.7118
10	.8578	.8327	.8168	.8092	.8047	.8017
15	.8910	.8719	.8599	.8542	.8508	.8486
20	.9116	.8962	.8866	.8820	.8793	.8775
25	.9257	.9128	.9047	.9009	.8986	.8971

By definition, $P_{WMA}^{PPM}(3,n,PE(\alpha)) > P_{SMA}^{PPM}(3,n,PE(\alpha))$ for all PPM, but with the assumption that α is equal to zero, $P_{WMA}^{L}(3,45,PE(0))$ = .4012, which is still a relatively small probability. It is therefore of interest to develop a representation for the limiting value as $n \to \infty$ of $P_{WMA}^{L}(3,\infty,PE(0))$.

The development a representation for $P_{WMA}^{L}(3,\infty,PE(0))$ starts with the limiting joint limiting probability $JP_{ABC}^{L}(n,\infty,PE(0))$ that both May's Model and Model L will have the transitive preference ranking $A \succ B \succ C$. Variables X_1^i and X_2^i were defined in Eq. 7.6 to maintain $A \succ B$ and $A \succ C$ respectively with May's Model. We now define a third variable X_3^i to require $B \succ C$ with May's Model.

$$X_3^i = +1: \ p_1 + p_3 + p_5 \qquad (7.18)$$
$$-1: \ p_2 + p_4 + p_6 .$$

Following earlier discussion, this definition leads to $B \succ C$ in Model L when $Z_3 > 0$, with $Z_3 = \overline{X}_3\sqrt{n}$. The probability that May's Model will have the transitive ranking $A \succ B \succ C$ is obtained as the joint distribution that $Z_1 \geq 0$, $Z_2 \geq 0$ and $Z_3 \geq 0$. This joint distribution is trivariate normal, with correlation matrix $\boldsymbol{\rho}^*$, which is given by

$$\boldsymbol{\rho}^* = \begin{bmatrix} 1 & \frac{1}{3} & -\frac{1}{3} \\ \frac{1}{3} & 1 & \frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} & 1 \end{bmatrix}. \tag{7.19}$$

A representation for $JP_{ABC}^L(n,\infty,PE(0))$ is then obtained as an expected value over the three dimensions covering the positive orthant of the trivariate normal distribution for Z_1, Z_2, Z_3:

$$JP_{ABC}^L(n,\infty,PE(0)) = E^3\left[\left(\frac{1}{2}+\frac{Z_1}{2\sqrt{n}}\right)\left(\frac{1}{2}+\frac{Z_2}{2\sqrt{n}}\right)\left(\frac{1}{2}+\frac{Z_3}{2\sqrt{n}}\right)\right], \tag{7.20}$$

which reduces to

$$JP_{ABC}^L(n,\infty,PE(0)) = \frac{1}{8}\left[\begin{array}{c}\Phi_3(\boldsymbol{\rho}^*) + n^{-\frac{1}{2}}E^3[Z_1+Z_2+Z_3] + n^{-1}E^3[Z_1Z_2+Z_1Z_3+Z_2Z_3] + \\ n^{-\frac{3}{2}}E^3[Z_1Z_2Z_3]\end{array}\right]. \tag{7.21}$$

Here, $\Phi_3(\boldsymbol{\rho}^*)$ is the trivariate orthant probability that $Z_1 \geq 0$, $Z_2 \geq 0$ and $Z_3 \geq 0$. A representation for $\Phi_3(\boldsymbol{\rho}^*)$ is obtained, from the trivariate extension of Sheppard's (1898) Theorem of Median Dichotomy.

$$\Phi_3(\boldsymbol{\rho}^*) = \frac{1}{8} + \frac{1}{4\pi}\left\{Sin^{-1}(\rho_{12}^*) + Sin^{-1}(\rho_{13}^*) + Sin^{-1}(\rho_{23}^*)\right\} = \frac{1}{8} + \frac{1}{4\pi}Sin^{-1}\left(\frac{1}{3}\right). \tag{7.22}$$

Eq. 7.21 can be algebraically reduced by using representations for the expected value terms $E^3(Z_i)$, $E^3(Z_iZ_j)$ and $E^3(Z_1Z_2Z_3)$ from Kamat (1956). Each of these terms is a function of the ρ_{ij}^* entries from $\boldsymbol{\rho}^*$. The representation for $E^3(Z_1Z_2Z_3)$ includes the determinant of $\boldsymbol{\rho}^*$ matrix. Each of these representations reduces to a simple finite value, as in the analysis of $E^2(Z_i)$ and $E^2(Z_1Z_2)$ above. In the limit as $n \to \infty$,

$$JP_{ABC}^L(n,\infty,PE(0)) = \frac{1}{8}\left[\frac{1}{8} + \frac{1}{4\pi}Sin^{-1}\left(\frac{1}{3}\right)\right] = \frac{1}{48}P_{SMA}(3,\infty,PE(0)). \tag{7.23}$$

There are eight possible Model L pairwise preference rankings on the alternatives that could result when the transitive ranking $A \succ B \succ C$ is obtained from May's Model. These are the six possible transitive rankings and the two pairwise preference cycles that can be obtained on three alternatives. A representation for each of these rankings is obtained from one of the eight combinations of

$$E^3\left[\left(\frac{1}{2}\pm\frac{Z_1}{2\sqrt{n}}\right)\left(\frac{1}{2}\pm\frac{Z_2}{2\sqrt{n}}\right)\left(\frac{1}{2}\pm\frac{Z_3}{2\sqrt{n}}\right)\right]. \tag{7.24}$$

A direct consequence of Eq. 7.24 and the arguments that lead to develop the representation for $JP^L_{ABC}(n,\infty,PE(0))$ in Eq. 7.23 is that each of these eight possible pairwise preference rankings is equally likely to be observed. The result that A is the SMA with May's Model occurs because one of the two transitive preference rankings $A \succ B \succ C$ or $A \succ C \succ B$ occurs, and there are two Model L pairwise preference cycles for each of these two underlying transitive rankings. The symmetry of P-E models with respect to alternatives then leads to

$$T^L_{A\}}(3,\infty,PE(0)) = 4JP^L_{ABC}(3,\infty,PE(0)) = \frac{1}{12}P_{SMA}(3,\infty,PE(0)). \tag{7.25}$$

A representation for $P^L_{WMA}(3,\infty,PE(0))$ follows from Eq 7.17 as

$$P^L_{WMA}(3,\infty,PE(0)) = \frac{JP^L_{SMA}(3,\infty,PE(0)) + 3\,\frac{1}{3}T^L_{A\}}(3,\infty,PE(0))\}}{P_{SMA}(3,\infty,PE(0))} = \frac{1}{3}. \tag{7.26}$$

A probabilistic chooser with Model L is therefore equivalent to a subject who makes random preference selections on three candidates as $n \to \infty$ with α equal to zero, when pairwise preference cycles are broken at random. Since $P^L_{WMA}(3,45,PE(0))$ = .4012, the convergence of $P^L_{WMA}(3,n,PE(0))$ to its limiting value as $n \to \infty$ is very slow.

Gehrlein (2006d) develops representation for $T^L_{A\}}(3,n,PE(1))$ by using algebraic reduction on Eq. 7.16 to obtain

$$T^L_{A\}}(3,n,PE(1)) = \frac{3(n-1)(59n^3+347n^2+493n-35)}{896n^2(n+2)(n+4)}. \tag{7.27}$$

After substitution into Eq. 7.17 and algebraic reduction, a representation for $P^L_{WMA}(3,n,PE(1))$ with odd n is given by

$$P^L_{WMA}(3,n,PE(1)) = \frac{(435n^4+3120n^3+6442n^2+3072n+371)}{840n^2(n+3)^2}. \tag{7.28}$$

As $n \to \infty$ with α equal to one, $P^L_{WMA}(3,\infty,PE(1)) = 435/840 = .5179$, so that $P^L_{WMA}(3,\infty,PE(\alpha)) >$ .5179 for all $\alpha > 0$, while $P^L_{WMA}(3,\infty,PE(0)) = .333$. There is a very significant increase in the probability $P^L_{WMA}(3,n,PE(\alpha))$ in changing α from zero to one, which implies that some degree of dependence has been imposed on attribute rankings. However, a probabilistic chooser who behaves like a Model L chooser still does not display particularly good performance at selecting the SMA with May's Model with either case as $n \to \infty$, and the SMA selection is effectively random when α is equal zero as $n \to \infty$. This poor performance could result from two different sources. The first possibility is that Model L choosers are simply too weak in their ability to discriminate their true preferences on pairs of alternatives to avoid making random selections. The second possibility is that all probabilistic choosers might be reduced to making random pairwise preference in the presence of independence of attribute rankings as $n \to \infty$.

7.7 Attribute Independence - Discriminatory Power

Several PPM models that reflect probabilistic choosers with different levels of discriminatory power than a Model L chooser are developed in Gehrlein (1990b). One set of these PPM's for probabilistic choosers shows greater discriminatory power in making pairwise comparisons than Model L choosers display, in terms of the probability that the subject tends to select the more preferred alternative in pairwise comparisons that is more in agreement with the pairwise choice that would be selected by an algebraic chooser with May's Model.

Let $R(A,B)$ denote the proportion of the *n* attribute rankings in which the subject perceives *A* as being superior to *B*. Given the linear ranking definitions from Figure 7.1, $R(A,B)=\frac{n_1+n_2+n_4}{n}$ and $R(B,A)=\frac{n_3+n_5+n_6}{n}$. The PPM's for probabilistic choosers who have greater discriminatory power than Model L choosers are based on these $R(A,B)$ measures and an integer parameter, κ, with $\kappa \geq 1$, that reflects the level of discriminatory power of a probabilistic chooser. We denote these PPM models as $BTL(\kappa)$ choosers, since they are better discriminators than Model L choosers. For any pair of alternatives *A* and *B*, we define $P^{BTL(\kappa)}_{A\succ B}(\boldsymbol{n})$ according to whether $A \succ B$ or $B \succ A$ by May's Model. If $A \succ B$ with May's Model, it must be true that $R(A,B) > R(B,A)$ and the definitions of $P^{BTL(\kappa)}_{A\succ B}(\boldsymbol{n})$ and $P^{BTL(\kappa)}_{B\succ A}(\boldsymbol{n})$ are given by

$$P^{BTL(\kappa)}_{A\succ B}(\boldsymbol{n}) = 1 - 2^{\kappa-1} R(B,A)^{\kappa} \tag{7.29}$$

$$P^{BTL(\kappa)}_{B\succ A}(\boldsymbol{n}) = 2^{\kappa-1} R(B,A)^{\kappa}.$$

All $BTL(\kappa)$ choosers are decisive, since $P^{BTL(\kappa)}_{A\succ B}(\boldsymbol{n})=1$ when $R(B,A)=0$, and they are unbiased since $P^{BTL(\kappa)}_{A\succ B}(\boldsymbol{n}) = P^{BTL(\kappa)}_{B\succ A}(\boldsymbol{n}) = 1/2$ when $R(B,A)=1/2$. By definition, $P_{A\succ B}^{BTL(1)}(\boldsymbol{n}) = P_{A\succ B}^{L}(\boldsymbol{n})$. For all $\kappa > 1$, $P_{A\succ B}^{BTL(\kappa)}(\boldsymbol{n}) > P_{A\succ B}^{L}(\boldsymbol{n})$ when $A \succ B$ for an algebraic chooser according to May's Model, and $P_{A\succ B}^{BTL(\kappa)}(\boldsymbol{n})$ increases as κ increases. Since $R(B,A) < 1/2$ if $A \succ B$ according to May's Model, it is easily shown that $BTL(\kappa)$ choosers match the behavior of a precise algebraic chooser with May's Model as $\kappa \to \infty$.

Fig. 7.2 shows graphed values of the $P_{A\succ B}^{BTL(\kappa)}(\boldsymbol{n})$ pairwise comparison probabilities for κ =1, 2, 3, 4 with n = 25 attributes of comparison. The values of $P_{A\succ B}^{BTL(\kappa)}(\boldsymbol{n})$ are a function of $R(B,A)$ in the definitions in Eq. 7.29. However, Fig. 7.2 shows $P_{A\succ B}^{BTL(\kappa)}(\boldsymbol{n})$ as a function of "attribute dominance", rather than $R(B,A)$. Attribute dominance measures the number of attribute rankings for which a subject perceives *A* as being ranked as superior to *B*, or $nR(A,B)$, given that $A \succ B$ according to May's Model. The plots in Fig. 7.2 give strong evidence to indicate that $BTL(\kappa)$ choosers tend to have significantly increasing discriminatory power as κ increases.

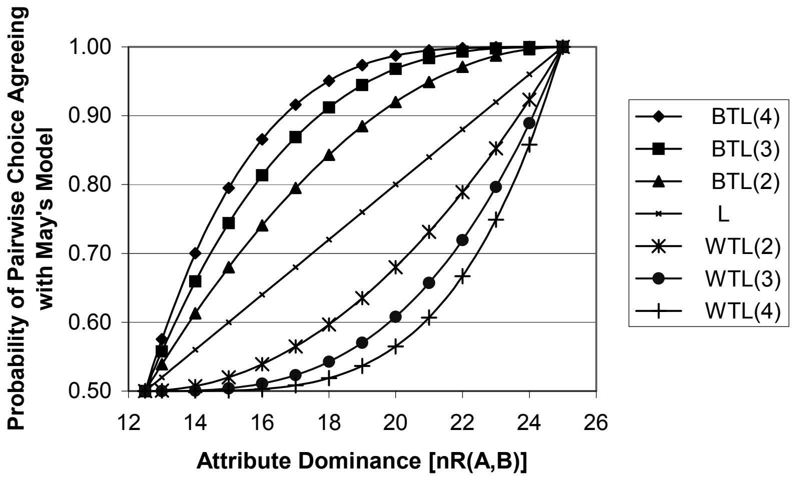

Fig. 7.2 Calculated values of $P_{A\succ B}^{BTL(\kappa)}(\boldsymbol{n})$ and $P_{A\succ B}^{WTL(\kappa)}(\boldsymbol{n})$ with $n = 25$

The same basic concepts that led to the development of PPM models for $BTL(\kappa)$ choosers, lead to the notion of $WTL(\kappa)$ choosers, who are worse than Model L choosers at making pairwise preference comparisons that are in agreement with an algebraic chooser who is using May's Model. These $WTL(\kappa)$ choosers therefore have a weaker ability to discriminate their true pairwise preferences than Model L choosers have. The PPM probabilities are defined as above for the pair *A* and *B*, with $A \succ B$ by May's Model as

$$P_{A\succ B}^{WTL(\kappa)}(\boldsymbol{n}) = \frac{1}{2}\left[1 + \left[R(A,B) - R(B,A)\right]^{\kappa}\right] \tag{7.30}$$

$$P_{B\succ A}^{WTL(\kappa)}(\boldsymbol{n}) = \frac{1}{2}\left[1 - \left[R(A,B) - R(B,A)\right]^{\kappa}\right].$$

The definitions in Eq. 7.30 lead to the observations that $P_{A\succ B}^{WTL(1)}(\boldsymbol{n})$ = $P_{A\succ B}^{L}(\boldsymbol{n})$, and for all $\kappa > 1$ $P_{A\succ B}^{WTL(\kappa)}(\boldsymbol{n}) < P_{A\succ B}^{L}(\boldsymbol{n})$ when $A \succ B$ according to May's Model. In addition, $P_{A\succ B}^{WTL(\kappa)}(\boldsymbol{n})$ decreases as κ increases. Probabilistic choosers who behave like $WTL(\kappa)$ choosers become equivalent to totally random pairwise preference choosers, with $P_{A\succ B}^{WTL(\infty)}(\boldsymbol{n})$ = 1/2 as $\kappa \to \infty$, except for the special case with $R(A,B) = 1$.

Fig. 7.2 shows plots of computed values of $P_{A\succ B}^{WTL(\kappa)}(\boldsymbol{n})$ pairwise preference comparison probabilities for each κ =1, 2, 3, 4 with *n* = 25 attributes of comparison. The numerical evidence in Fig. 7.2 clearly indicates that $WTL(\kappa)$ choosers have significantly weaker discriminatory power as κ increases.

Following the logic of the general discussion that led to Eq. 7.5, a general representation for $JP_{SMA}^{PPM}(3, n, PE(\alpha))$ is obtained as

$$JP_{SMA}^{PPM}(3, n, PE(\alpha)) = \tag{7.31}$$

$$3\sum_{n_6=0}^{\frac{n-1}{2}}\sum_{n_5=0}^{\frac{n-1}{2}-n_6}\sum_{n_4=0}^{\frac{n-1}{2}-n_6-n_5}\sum_{n_3=0}^{\frac{n-1}{2}-n_6-n_5}\sum_{n_2=0}^{n-n_6-n_5-n_4-n_3} P_{A\succ B}^{PPM}(\boldsymbol{n})P_{A\succ C}^{PPM}(\boldsymbol{n})P^{1}(\boldsymbol{n},\alpha).$$

Computed values of $P_{SMA}^{PPM}(3, n, PE(\alpha))$ can be obtained by using Eq. 7.31 in conjunction with Eq. 7.4. Tables 7.4, 7.5 and 7.6 list the associated values of $P_{SMA}^{BTL(\kappa)}(3, n, PE(\alpha))$ for the respective values of κ = 2, 3 and 4. These tables have entries for each *n* = 3, 5, 9, 15, 25, 45 with each value of α =0, 1, 2, 3, 4, 5, 10, 15, 20, 25. Tables 4.7, 4.8 and 4.9 list similar computed values of $P_{SMA}^{WTL(\kappa)}(3, n, PE(\alpha))$. An analysis of the computed probabilities in Tables 7.4 through 7.9 show that $P_{SMA}^{PPM}(3, n, PE(\alpha))$ values increase significantly for all given *n* and α as probabilistic choosers have PPM's that tend to reflect increased levels of discriminatory power.

Table 7.4 Computed values of $P_{SMA}^{BTL(2)}(3, n, PE(\alpha))$

	Number of Attributes (n)					
α	3	5	9	15	25	45
0	.7124	.6248	.5389	.4774	.4275	.3827
1	.7503	.6938	.6485	.6220	.6043	.5915
2	.7797	.7377	.7077	.6915	.6813	.6742
3	.8029	.7696	.7474	.7361	.7292	.7245
4	.8218	.7941	.7767	.7682	.7631	.7597
5	.8374	.8137	.7994	.7927	.7888	.7862
10	.8869	.8732	.8659	.8629	.8612	.8602
15	.9134	.9037	.8989	.8970	.8961	.8955
20	.9298	.9224	.9188	.9175	.9168	.9164
25	.9410	.9349	.9321	.9311	.9306	.9303

Table 7.5 Computed values of $P_{SMA}^{BTL(3)}(3, n, PE(\alpha))$

	Number of Attributes (n)					
α	3	5	9	15	25	45
0	.8025	.7113	.6220	.5540	.4948	.4380
1	.8287	.7708	.7275	.7026	.6860	.6740
2	.8489	.8066	.7788	.7647	.7561	.7501
3	.8649	.8317	.8115	.8021	.7965	.7929
4	.8779	.8506	.8349	.8279	.8240	.8215
5	.8886	.8655	.8527	.8473	.8443	.8425
10	.9225	.9096	.9032	.9007	.8995	.8989
15	.9407	.9317	.9275	.9260	.9253	.9249
20	.9519	.9451	.9420	.9409	.9405	.9402
25	.9596	.9541	.9516	.9508	.9505	.9503

Table 7.6 Computed values of $P_{SMA}^{BTL(4)}(3, n, PE(\alpha))$

	Number of Attributes (n)					
α	3	5	9	15	25	45
0	.8657	.7727	.6836	.6140	.5505	.4863
1	.8836	.8224	.7799	.7566	.7413	.7303
2	.8974	.8514	.8239	.8109	.8033	.7981
3	.9083	.8714	.8511	.8424	.8376	.8345
4	.9171	.8862	.8702	.8637	.8603	.8582
5	.9244	.8979	.8846	.8794	.8769	.8754
10	.9474	.9318	.9247	.9223	.9212	.9206
15	.9597	.9487	.9438	.9423	.9416	.9413
20	.9674	.9588	.9551	.9540	.9535	.9533
25	.9726	.9656	.9626	.9617	.9613	.9612

Table 7.7 Computed values of $P_{SMA}^{WTL(2)}(3, n, PE(\alpha))$

	Number of Attributes (n)					
α	3	5	9	15	25	45
0	.4771	.3796	.3190	.2904	.2739	.2631
1	.5446	.4578	.4024	.3757	.3600	.3496
2	.5973	.5188	.4682	.4435	.4288	.4191
3	.6394	.5679	.5213	.4984	.4848	.4758
4	.6736	.6080	.5650	.5438	.5311	.5227
5	.7020	.6414	.6015	.5818	.5700	.5622
10	.7924	.7490	.7199	.7054	.6967	.6909
15	.8408	.8070	.7843	.7729	.7660	.7614
20	.8710	.8433	.8247	.8153	.8096	.8058
25	.8915	.8681	.8523	.8443	.8395	.8363

Table 7.8 Computed values of $P_{SMA}^{WTL(3)}(3, n, PE(\alpha))$

	Number of Attributes (n)					
α	3	5	9	15	25	45
0	.4430	.3384	.2861	.2667	.2577	.2532
1	.5146	.4139	.3557	.3301	.3161	.3072
2	.5706	.4763	.4185	.3916	.3763	.3665
3	.6154	.5277	.4721	.4455	.4301	.4201
4	.6519	.5704	.5175	.4919	.4768	.4669
5	.6821	.6062	.5562	.5316	.5171	.5075
10	.7785	.7229	.6847	.6655	.6539	.6461
15	.8301	.7865	.7561	.7406	.7312	.7249
20	.8623	.8265	.8013	.7883	.7804	.7752
25	.8842	.8538	.8324	.8213	.8145	.8100

Table 7.9 Computed values of $P_{SMA}^{WTL(4)}(3, n, PE(\alpha))$

	Number of Attributes (n)					
α	3	5	9	15	25	45
0	.4319	.3189	.2719	.2580	.2529	.2509
1	.5048	.3920	.3316	.3071	.2945	.2869
2	.5620	.4548	.3912	.3630	.3474	.3376
3	.6076	.5072	.4442	.4149	.3982	.3875
4	.6448	.5511	.4902	.4610	.4440	.4330
5	.6756	.5881	.5298	.5013	.4845	.4736
10	.7740	.7094	.6636	.6403	.6261	.6167
15	.8267	.7758	.7390	.7199	.7082	.7003
20	.8594	.8177	.7870	.7709	.7610	.7543
25	.8818	.8464	.8202	.8063	.7977	.7920

Table 7.10 summarizes the computed values of $P_{SMA}^{PPM}(3,45,PE(\alpha))$ for α equal to zero and one with all PPM's that have been considered so far.

Table 7.10 Summary of $P_{SMA}^{PPM}(3,45,PE(\alpha))$ and $P_{SMA}^{PPM}(3,\infty,PE(\alpha))$ values for $\alpha = 0,1$

	$P_{SMA}^{PPM}(3,n,PE(0))$		$P_{SMA}^{PPM}(3,n,PE(1))$	
PPM	$n = 45$	$n = \infty$	$n = 45$	$n = \infty$
$BTL(4)$	.4863	.2500	.7303	.7154
$BTL(3)$	.4380	.2500	.6740	.6576
$BTL(2)$	.3827	.2500	.5915	.5743
Model L	.3200	.2500	.4626	.4476
$WTL(2)$	.2631	.2500	.3496	.3368
$WTL(3)$	.2532	.2500	.3072	.2969
$WTL(4)$	.2509	.2500	.2869	.2784

The results in Table 7.10 show that a probabilistic chooser has $P_{SMA}^{PPM}(3,45,PE(0))$ increase from .2509 for a $WTL(4)$ chooser to .4683 for a $BTL(4)$ chooser, with α equal to zero. The increase in the associated probability values is even more dramatic with α equal to one, with an increase from .2869 for a $WTL(4)$ chooser to .7303 for a $BTL(4)$ chooser. The value of $P_{SMA}^{BTL(4)}(3,45,PE(0)) = .4683$ also suggests the possibility that a significant improvement in discriminatory power beyond that of a Model L chooser might prevent a probabilistic chooser from becoming equivalent to a random chooser as $n \to \infty$.

The same procedure that was used above to obtain a representation for the limiting probability $P_{SMA}^{L}(3,\infty,PE(0))$, can be used to find some results for $BTL(2)$ and $WTL(2)$ probabilistic choosers. By using identities of the form

$$R(B,A) = \frac{1}{2} - \frac{1}{2}[R(A,B) - R(B,A)] = \frac{1}{2} - \frac{1}{2}\overline{X}_1, \tag{7.32}$$

it can be shown that

$$JP_{A\}}^{BTL(2)}(3,\infty,PE(0)) = E^2\left[\left(\frac{1}{2} + \frac{Z_1}{\sqrt{n}} - \frac{Z_1^2}{2n}\right)\left(\frac{1}{2} + \frac{Z_2}{\sqrt{n}} - \frac{Z_2^2}{2n}\right)\right], \tag{7.33}$$

and that

$$JP_{A\}}^{WTL(2)}(3,\infty,PE(0)) = E^2\left[\left(\frac{1}{2} + \frac{Z_1^2}{2n}\right)\left(\frac{1}{2} + \frac{Z_2^2}{2n}\right)\right]. \tag{7.34}$$

These expected values are being taken over both dimensions of the positive orthant of a bivariate normal distribution for variables $\overline{X}_1\sqrt{n}$ and $\overline{X}_2\sqrt{n}$. Kamat (1956) develops representations of these expected values, and by using algebraic reduction after the substitution of the resulting representations into Eqs. 7.33 and 7.34, it is found that $JP_{A\}}^{BTL(2)}(3,\infty,PE(0))$ = $JP_{A\}}^{WTL(2)}(3,\infty,PE(0))$ = $\frac{1}{4}\Phi_2\left(\frac{1}{3}\right)$ = $\frac{1}{12}P_{SMA}(3,\infty,PE(0))$ in the limit as $n\to\infty$. It then follows that $P_{SMA}^{BTL(2)}(3,\infty,PE(0))$= $P_{SMA}^{WTL(2)}(3,\infty,PE(0))$ = $P_{SMA}^{L}(3,\infty,PE(0))$, so that probabilistic choosers with PPM's of the type we are considering will all turn out to be equivalent to random choosers as $n\to\infty$ when α is equal to zero, with its associated assumption that attribute rankings on alternatives are completely independent.

The rate of convergence to this limiting probability is quite different for the PPM's. The values in Table 7.10 show that $P_{SMA}^{WTL(3)}(3,45,PE(0))$ = .2532 and $P_{SMA}^{WTL(4)}(3,45,PE(0))$ = .2509 for 45 attributes, so that both are very near their limiting value. However, the probabilistic choosers with the greatest discriminatory power have very different convergence results, with $P_{SMA}^{BTL(4)}(3,45,PE(0))$ = .4863 which is nearly double the limiting value for n = 45.

All of this creates an interest in the impact that the small amount of dependence between attribute rankings that is suggested with α equal to one might have on these coincidence probabilities. Gehrlein (2006d) uses the same process that was employed to develop the representation for $P_{SMA}^{L}(3,n,PE(1))$ in Eq. 7.15 to obtain closed form representations for both $P_{SMA}^{BTL(\kappa)}(3,n,PE(1))$ and $P_{SMA}^{WTL(\kappa)}(3,n,PE(1))$ for each κ = 2, 3 and 4 with odd n. The resulting representations are given by:

$$P_{SMA}^{BTL(2)}(3,n,PE(1))= \tag{7.35}$$
$$\frac{8684n^6+64323n^5+126089n^4+31794n^3+2594n^2+9003n-567}{15120n^4(n+3)^2}$$

$$P_{SMA}^{BTL(3)}(3,n,PE(1))= \tag{7.36}$$
$$\frac{\begin{bmatrix}182295n^8+1308753n^7+2401958n^6+752994n^5+670175n^4\\ -801903n^3+653372n^2-760164n+27720\end{bmatrix}}{277200n^6(n+3)^2}$$

$$P_{SMA}^{BTL(4)}(3,n,PE(1)) = \tag{7.37}$$

$$\frac{\left[\begin{array}{c} 3866775n^{10} + 27027675n^{9} + 48264234n^{8} + 19640268n^{7} \\ + 15165796n^{6} - 14924448n^{5} + 29326876n^{4} - 32470368n^{3} \\ - 113499671n^{2} + 104900073n - 810810 \end{array}\right]}{5405400n^{8}(n+3)^{2}}$$

$$P_{SMA}^{WTL(2)}(3,n,PE(1)) = \tag{7.38}$$

$$\frac{5093n^{6} + 39186n^{5} + 100511n^{4} + 92652n^{3} + 19667n^{2} - 14622n - 567}{15120n^{4}(n+3)^{2}}$$

$$P_{SMA}^{WTL(3)}(3,n,PE(1)) = \tag{7.39}$$

$$\frac{\left[\begin{array}{c} 164577n^{8} + 1236753n^{7} + 3409072n^{6} + 4492818n^{5} + 2499013n^{4} \\ - 2217903n^{3} - 2533862n^{2} + 1764492n + 55440 \end{array}\right]}{554400n^{6}(n+3)^{2}}$$

$$P_{SMA}^{WTL(4)}(3,n,PE(1)) = \tag{7.40}$$

$$\frac{\left[\begin{array}{c} 752448n^{10} + 5482080n^{9} + 15569784n^{8} + 26352936n^{7} \\ + 23861303n^{6} - 18274194n^{5} - 48939439n^{4} + 22080444n^{3} \\ + 83434709n^{2} - 66671466n - 405405 \end{array}\right]}{2702700n^{8}(n+3)^{2}}.$$

The limiting values, as $n \to \infty$, for the probabilities that are given in the representations that are shown in Eqs. 7.35 through 7.40 are summarized in Table 7.10, and two distinct observations stand out. First, the limiting $P_{SMA}^{PPM}(3,\infty,PE(1))$ values are significantly greater than the limiting probabilities of the associated $P_{SMA}^{PPM}(3,\infty,PE(0))$ for probabilistic choosers with any PPM model that has greater discriminatory power than a Model L chooser. The limiting probability values as $n \to \infty$ are approached much faster with $P_{SMA}^{PPM}(3,n,PE(1))$ than with $P_{SMA}^{PPM}(3,n,PE(0))$. The listed values of $P_{SMA}^{PPM}(3,45,PE(1))$ are very near their limiting probability values of $P_{SMA}^{PPM}(3,\infty,PE(1))$ for each of the PPM's that are considered in Table 7.10.

The development of representations for the probability that the WMA with $BTL(\kappa)$ and $WTL(\kappa)$ choosers coincides with the SMA for an algebraic chooser with May's Model becomes substantially more complicated than in the earlier situation of the Model L case. In order to obtain a representation for the probability that there is a pairwise preference cycle with Model L, a representation for $T^{L}_{A\}}(3,n,PE(\alpha))$ was developed in Eq. 7.16 with summation indexes that enumerate all possible attribute ranking situations for which both $A \succ B$ and $A \succ C$ with May's Model. The primary source of the additional difficulty with the cases of $BTL(\kappa)$ and $WTL(\kappa)$ probabilistic choosers results from the fact that the basic definitions of both of the $P^{BTL(\kappa)}_{A\succ B}(\boldsymbol{n})$ and $P^{WTL(\kappa)}_{A\succ B}(\boldsymbol{n})$ pairwise comparison probabilities are based on whether $A \succ B$ or $B \succ A$ for an algebraic chooser with May's Model.

In order to develop a representation for $T^{BTL(\kappa)}_{A\}}(3,n,PE(\alpha))$, it is necessary to consider summation indexes that enumerate all of the possible attribute ranking outcomes in which $A \succ B$, $A \succ C$ and $B \succ C$ in May's Model. The additional restriction that $B \succ C$ requires that the summation indexes result in the outcome that $n_2 + n_4 + n_6 \le \frac{n-1}{2}$. We compute the probability that either of the two possible pairwise preference cycles on alternatives exist when the transitive ranking $A \succ B \succ C$ results from May's Model. It then follows from symmetry arguments of P-E models with respect to alternatives that $T^{BTL(\kappa)}_{A\}}(3,n,PE(\alpha))$ is obtained as twice this probability, since both of the possible pairwise preference cycles exist with the same likelihood when the transitive ranking $A \succ C \succ B$ results from May's Model. As a result

$$T^{BTL(\kappa)}_{A\}}(3,n,PE(\alpha)) = \tag{7.41}$$

$$2 * \sum_{n_6=0}^{\frac{n-1}{2}} \sum_{n_5=0}^{\frac{n-1}{2}-n_6} \sum_{n_4=0}^{\frac{n-1}{2}-n_6-n_5} \sum_{n_3=0}^{\frac{n-1}{2}-n_6-n_5} \sum_{n_2=0}^{\frac{n-1}{2}-n_6-n_4} \begin{bmatrix} P^{BTL(\kappa)}_{A\succ B}(\boldsymbol{n}) P^{BTL(\kappa)}_{B\succ C}(\boldsymbol{n}) P^{BTL(\kappa)}_{A\succ C}(\boldsymbol{n}) + \\ P^{BTL(\kappa)}_{A\succ B}(\boldsymbol{n}) P^{BTL(\kappa)}_{B\succ C}(\boldsymbol{n}) P^{BTL(\kappa)}_{A\succ C}(\boldsymbol{n}) \end{bmatrix} P^{1}(\boldsymbol{n},\alpha).$$

A general representation for the probability $P^{BTL(\kappa)}_{WMA}(3,n,PE(\alpha))$ then follows directly from the definition that is given in Eq. 7.17, and computed values of $P^{BTL(\kappa)}_{WMA}(3,n,PE(\alpha))$ are listed in Tables 7.11, 7.12 and 7.13 for respective values of $\kappa = 2$, 3 and 4. These tables have entries for each value of n = 3, 5, 9, 15, 25, 45 with each α =0, 1, 2, 3, 4, 5, 10, 15, 20, 25. Tables 7.14, 7.15 and 7.16 list computed values of $P^{WTL(\kappa)}_{WMA}(3,n,PE(\alpha))$ respectively for each $\kappa = 2$, 3 and 4.

These computed values show a marked increase in $P^{PPM}_{WMA}(3,n,PE(\alpha))$ for all n and α as probabilistic choosers have PPM's with increased levels of discriminatory power. Table 7.17 summarizes $P^{PPM}_{WMA}(3,45,PE(\alpha))$ values for each $\alpha = 0,1$.

Table 7.11. Computed values of $P_{WMA}^{BTL(2)}(3, n, PE(\alpha))$

	Number of Attributes (n)					
α	3	5	9	15	25	45
0	.7509	.6734	.5982	.5441	.4998	.4593
1	.7842	.7337	.6939	.6708	.6554	.6444
2	.8097	.7722	.7456	.7315	.7227	.7165
3	.8300	.8000	.7803	.7705	.7645	.7604
4	.8463	.8214	.8059	.7984	.7940	.7911
5	.8598	.8385	.8257	.8198	.8164	.8142
10	.9026	.8902	.8836	.8809	.8795	.8786
15	.9254	.9167	.9123	.9106	.9098	.9093
20	.9396	.9328	.9296	.9284	.9278	.9275
25.	.9492	.9437	.9412	.9402	.9398	.9396

Table 7.12 Computed values of $P_{WMA}^{BTL(3)}(3, n, PE(\alpha))$

	Number of Attributes (n)					
α	3	5	9	15	25	45
0	.8297	.7489	.6707	.6114	.5595	.5091
1	.8529	.8010	.7625	.7408	.7263	.7159
2	.8705	.8324	.8075	.7951	.7875	.7823
3	.8844	.8543	.8361	.8277	.8229	.8197
4	.8955	.8708	.8566	.8503	.8469	.8447
5	.9048	.8837	.8721	.8672	.8646	.8630
10	.9339	.9220	.9161	.9139	.9128	.9122
15	.9494	.9411	.9372	.9358	.9352	.9349
20	.9590	.9527	.9498	.9488	.9484	.9481
25	.9656	.9605	.9581	.9574	.9570	.9569

Table 7.13 Computed values of $P_{WMA}^{BTL(4)}(3, n, PE(\alpha))$

	Number of Attributes (n)					
α	3	5	9	15	25	45
0	.8846	.8025	.7242	.6636	.6083	.5520
1	.9005	.8461	.8081	.7876	.7742	.7646
2	.9125	.8715	.8467	.8352	.8284	.8239
3	.9219	.8890	.8706	.8627	.8585	.8558
4	.9295	.9019	.8873	.8814	.8784	.8766
5	.9357	.9120	.8998	.8952	.8929	.8915
10	.9554	.9414	.9348	.9326	.9316	.9311
15	.9659	.9560	.9514	.9500	.9493	.9490
20	.9724	.9647	.9612	.9601	.9597	.9595
25	.9768	.9705	.9677	.9668	.9665	.9663

Table 7.14 Computed values of $P_{WMA}^{WTL(2)}(3, n, PE(\alpha))$

	Number of Attributes (n)					
α	3	5	9	15	25	45
0	.5396	.4546	.3996	.3727	.3568	.3463
1	.5976	.5233	.4751	.4513	.4372	.4278
2	.6436	.5767	.5332	.5117	.4990	.4904
3	.6804	.6196	.5799	.5602	.5485	.5407
4	.7105	.6547	.6182	.6001	.5893	.5821
5	.7355	.6840	.6502	.6334	.6234	.6167
10	.8155	.7785	.7539	.7417	.7344	.7295
15	.8584	.8296	.8104	.8008	.7950	.7912
20	.8851	.8616	.8458	.8379	.8332	.8300
25	.9034	.8835	.8701	.8634	.8594	.8567

Table 7.15 Computed values of $P_{WMA}^{WTL(3)}(3, n, PE(\alpha))$

	Number of Attributes (n)					
α	3	5	9	15	25	45
0	.5080	.4167	.3684	.3498	.3410	.3365
1	.5694	.4831	.4321	.4092	.3963	.3882
2	.6182	.5377	.4879	.4645	.4510	.4423
3	.6575	.5827	.5351	.5123	.4989	.4902
4	.6896	.6200	.5750	.5531	.5402	.5317
5	.7163	.6515	.6089	.5881	.5757	.5675
10	.8019	.7542	.7218	.7055	.6958	.6893
15	.8480	.8105	.7846	.7715	.7636	.7583
20	.8767	.8458	.8244	.8135	.8069	.8025
25	.8963	.8701	.8518	.8425	.8368	.8330

Table 7.16 Computed values of $P_{WMA}^{WTL(4)}(3, n, PE(\alpha))$

	Number of Attributes (n)					
α	3	5	9	15	25	45
0	.4977	.3985	.3548	.3413	.3363	.3343
1	.5601	.4630	.4098	.3877	.3760	.3690
2	.6099	.5178	.4629	.4382	.4244	.4157
3	.6500	.5636	.5096	.4844	.4699	.4605
4	.6828	.6020	.5499	.5250	.5105	.5010
5	.7100	.6345	.5847	.5605	.5462	.5368
10	.7975	.7414	.7024	.6826	.6707	.6628
15	.8445	.8003	.7689	.7527	.7428	.7362
20	.8739	.8374	.8112	.7976	.7893	.7837
25	.8939	.8629	.8405	.8288	.8216	.8168

Table 7.17 Summary of $P_{WMA}^{PPM}(3,45,PE(\alpha))$ and $P_{WMA}^{PPM}(3,\infty,PE(\alpha))$ values for $\alpha = 0,1$

	$P_{WMA}^{PPM}(3,n,PE(0))$		$P_{WMA}^{PPM}(3,n,PE(1))$	
PPM	$n = 45$	$n = \infty$	$n = 45$	$n = \infty$
$BTL(4)$	.5520	.3333	.7646	.7517
$BTL(3)$	.5091	.3333	.7159	.7018
$BTL(2)$	.4593	.3333	.6444	.6295
Model L	.4012	.3333	.5311	.5179
$WTL(2)$	.3463	.3333	.4278	.4162
$WTL(3)$	.3365	.3333	.3882	.3786
$WTL(4)$	.3343	.3333	.3690	.3610

The results in Table 7.17 indicate that $P_{WMA}^{PPM}(3,\infty,PE(0))$ values converge to the limiting result as $n \to \infty$ at different rates for different PPM's. This result is very similar to the behavior that was observed with $P_{SMA}^{PPM}(3,\infty,PE(0))$ computations.

Gehrlein (2006d) obtains representations for both $P_{WMA}^{BTL(\kappa)}(3,n,PE(1))$ and $P_{WMA}^{WTL(\kappa)}(3,n,PE(1))$ for κ = 2, 3 and 4 with odd n.

$$P_{WMA}^{BTL(2)}(3,n,PE(1)) = \frac{28553n^6 + 203049n^5 + 381812n^4 + 92238n^3 + 15821n^2 + 1641n + 2646}{45360n^4(n+3)^2} \tag{7.42}$$

$$P_{WMA}^{BTL(3)}(3,n,PE(1)) = \frac{\left[\begin{array}{c} 35371n^8 + 245886n^7 + 440418n^6 + 130770n^5 + 114444n^4 \\ -151326n^3 + 49142n^2 - 72690n + 14385 \end{array}\right]}{50400n^6(n+3)^2} \tag{7.43}$$

$$P_{WMA}^{BTL(4)}(3,n,PE(1)) = \frac{\left[\begin{array}{c} 12190428n^{10} + 83025981n^9 + 145899915n^8 + 54785664n^7 \\ +45386818n^6 - 64664190n^5 + 27910240n^4 - 41232264n^3 \\ -101245286n^2 + 81861369n + 15540525 \end{array}\right]}{16216200n^8(n+3)^2} \tag{7.44}$$

$$P_{WMA}^{WTL(2)}(3,n,PE(1)) = \frac{9439n^6 + 68490n^5 + 157597n^4 + 114564n^3 + 17545n^2 - 6078n + 1323}{22680n^4(n+3)^2} \tag{7.45}$$

$$P_{WMA}^{WTL(3)}(3,n,PE(1)) = \frac{\begin{bmatrix} 3816n^8 + 27135n^7 + 66052n^6 + 68166n^5 + 28549n^4 \\ - 23505n^3 - 20078n^2 + 8268n + 2877 \end{bmatrix}}{10080n^6(n+3)^2} \tag{7.46}$$

$$P_{WMA}^{WTL(4)}(3,n,PE(1)) = \frac{\begin{bmatrix} 5853435n^{10} + 40622490n^9 + 101190939n^8 + 134427096n^7 \\ + 93363604n^6 - 79134984n^5 - 104690384n^4 + 29494704n^3 \\ + 182686921n^2 - 159895146n + 15540525 \end{bmatrix}}{16216200n^8(n+3)^2} \tag{7.47}$$

Limiting probability values as $n \to \infty$ for these representations are summarized in Table 7.17. Following the general observations from $P_{SMA}^{BTL(\kappa)}(3,\infty,PE(1))$ and $P_{SMA}^{WTL(\kappa)}(3,\infty,PE(1))$ computations, there is an extreme improvement in the limiting probabilities with α equal to one compared to α equal to zero for any probabilistic chooser that has greater discriminatory power than a Model L chooser. The limiting probability values are approached much faster with α equal to one than with α equal to zero for all of the PPM's that are considered.

7.8 The Impact of Single-Peakedness

The imposition of the assumption of single-peaked preferences has been seen to be sufficient to force PMR relationships to be transitive for odd numbers of voters. Gehrlein (1993) adapts this notion to the current problem. The application of the structural consistency of single-peakedness is applied instead to the attribute rankings on alternatives in this case. The idea of single-peakedness does not have the same intuitive appeal in the context of attribute ranking situations, as it does in the context of voter preference rankings. But, the assumption does suggest that there will be more internal consistency in attribute ranking situations when the condition of single-peakedness is applied to the individual preference scenario.

Consistent with earlier notation, $P_{SMA}^{PPM}(3,n,IAC_b(0))$ denotes the probability that a PPM probabilistic chooser will have the same SMA as an algebraic chooser with May's Model when all single-peaked attribute ranking situations are equally likely to me observed. Using the definition of Mays Model and previous arguments, it is easily shown that $P_{SMA}(3,n,IAC_b(0))=1$, since May's Model must have transitive preferences if single-peakedness is required.

Gehrlein (1993) develops representations for $P_{SMA}^{PPM}(3,n,IAC_b(0))$ with three types of PPM's for odd n:

$$P_{SMA}^{L}(3,n,IAC_b(0))=\frac{94n^4+511n^3+109n^2+201n+45}{160(n+5)n^3} \tag{7.48}$$

$$P_{SMA}^{BTL(2)}(3,n,IAC_b(0))= \frac{1216n^6+6274n^5+684n^4+1661n^3-457n^2+1017n-315}{1680(n+5)n^5} \tag{7.49}$$

$$P_{SMA}^{WTL(2)}(3,n,IAC_b(0))= \frac{782n^6+4524n^5+1559n^4+2466n^3-1622n^2+558n-315}{1680(n+5)n^5}. \tag{7.50}$$

Eqs. 7.48. 7.49 and 7.50 are used respectively to obtain computed values of $P_{SMA}^{L}(3,n,IAC_b(0))$, $P_{SMA}^{BTL(2)}(3,n,IAC_b(0))$ and $P_{SMA}^{WTL(2)}(3,n,IAC_b(0))$ for each n = 1(2)15 and for their associated limiting values as $n\to\infty$. The resulting values are listed Table 7.18, to clearly show that the imposition of the condition of single-peakedness does indeed increase the probability that PPM choosers are in agreement with an algebraic chooser with May's Model.

Table 7.18 Computed values of $P_{SMA}^{PPM}(3,n,IAC_b(0))$

n	$P_{SMA}^{L}(3,n,IAC_b(0))$	$P_{SMA}^{BTL(2)}(3,n,IAC_b(0))$	$P_{SMA}^{WTL(2)}(3,n,IAC_b(0))$
1	1.0000	1.0000	1.0000
3	.6667	.7685	.5741
5	.6320	.7473	.5272
7	.6192	.7399	.5097
9	.6123	.7361	.5003
11	.6080	.7338	.4944
13	.6051	.7323	.4903
15	.6029	.7312	.4872
∞	.5875	.7238	.4655

7.9 Strict Maximal-Minimal Reversal

Results have indicated that there can be a significant probability that a probabilistic chooser might select a SMA that is not the same as the SMA that an algebraic chooser with May's Model would pick. This leads to the consideration of the conditional probability that a probabilistic chooser might have responses that are in drastic disagreement with a chooser using May's Model, with the SMA from May's Model being selected as the strictly minimal, or least preferred, alternative (SLA) by the probabilistic chooser, given that May's Model has a SMA.

Gehrlein (1990d) develops representations, denoted as $P_{SLA}^{PPM}(3,n,PE(1))$, for this conditional probability for some PPM's for odd n, assuming a P-E model with α equal to one as the basis for generating random attribute ranking situations:

$$P_{SLA}^{L}(3,n,PE(1)) = \frac{3(n-1)(n+2)(4n^2+9n-7)}{105n^2(n+3)^2} \tag{7.51}$$

$$P_{SLA}^{BTL(2)}(3,n,PE(1)) = \frac{980n^6+2115n^5-1207n^4-5358n^3-4966n^2+9003n-567}{15120n^4(n+3)^2} \tag{7.52}$$

$$P_{SLA}^{WTL(2)}(3,n,PE(1)) = \frac{2717n^6+10674n^5-4033n^4-21396n^3+27227n^2-14622n-567}{15120n^4(n+3)^2}. \tag{7.53}$$

Gehrlein (1994) contains a typographical error in restating a representation that is associated with Eq. 7.53.

Table 7.19 summarizes computed values of $P_{SLA}^{PPM}(3,n,PE(1))$ from Eqs. 7.51, 7.52 and 7.53 for each n = 1(2)15 and their associated limiting values as $n \to \infty$.

Table 7.19 Computed values of $P_{SLA}^{PPM}(3,n,PE(1))$

n	$P_{SLA}^{L}(3,n,PE(1))$	$P_{SLA}^{BTL(2)}(3,n,PE(1))$	$P_{SLA}^{WTL(2)}(3,n,PE(1))$
1	.0000	.0000	.0000
3	.0494	.0219	.0878
5	.0690	.0338	.1178
7	.0793	.0402	.1331
9	.0858	.0444	.1424
11	.0902	.0473	.1486
13	.0934	.0495	.1530
15	.0959	.0512	.1564
∞	.1143	.0648	.1797

The results in Table 7.19 give strong evidence to support the idea that probabilistic choosers have a relatively high chance of giving pairwise preference responses that are in drastic disagreement with an algebraic chooser using May's Model. This is particularly true for $WTL(\kappa)$ choosers. Gehrlein (1990d) extends this analysis by developing representations for $P_{SLA}^{PPM}(3,n,IAC_b(0))$, to determine the impact that the increased coherence of imposing the condition single-peakedness on attribute ranking situations will have on these probabilities. The representations are given as :

$$P_{SLA}^{L}(3,n,IAC_b(0)) = \frac{(n-1)(n+3)(9n^2+18n-15)}{160n^3(n+5)} \tag{7.54}$$

$$P_{SLA}^{BTL(2)}(3,n,IAC_b(0)) = \frac{(n-1)(n+3)(40n^4+125n^3-97n^2-269n+105)}{1680n^5(n+5)} \tag{7.55}$$

$$P_{SLA}^{WTL(2)}(3,n,IAC_b(0)) = \frac{(n-1)(n+3)(173n^4+272n^3-230n^2+256n+105)}{1680n^5(n+5)}. \tag{7.56}$$

Table 7.20 summarizes computed values of $P_{SLA}^{PPM}(3,n,IAC_b(0))$ from Eqs. 7.54, 7.55 and 7.56 for each n = 1(2)15, along with their associated limiting values as $n \to \infty$.

Table 7.20 Computed values of $P_{SLA}^{PPM}(3,n,IAC_b(0))$

n	$P_{SLA}^{L}(3,n,IAC_b(0))$	$P_{SLA}^{BTL(2)}(3,n,IAC_b(0))$	$P_{SLA}^{WTL(2)}(3,n,IAC_b(0))$
1	.0000	.0000	.0000
3	.0417	.0185	.0741
5	.0480	.0225	.0840
7	.0503	.0234	.0884
9	.0515	.0237	.0910
11	.0523	.0238	.0928
13	.0528	.0239	.0941
15	.0532	.0239	.0951
∞	.0563	.0238	.1030

The additional internal consistency that the imposition of single-peakedness produces in attribute ranking situations does significantly reduce the probability that a probabilistic chooser will select a SLA that a chooser with May's Model will select as a SMA. However, a relatively high risk still exists that a probabilistic chooser will have pairwise preferences that are in drastic disagreement with an algebraic chooser using May's Model.

7.10 Other Related Representations

A representation for the probability that a Model L chooser has transitive preferences on three alternatives with odd n, regardless of the outcomes of preferences by an algebraic chooser with May's model, is considered in Gehrlein (1990b). This probability is denoted by $P_{Tran}^{L}(3,n,PE(1))$ when a P-E model with α equal to one is applied to generating random attribute ranking situations, and

$$P_{Tran}^{L}(3,n,PE(1)) = \frac{11n+3}{14n}\,. \tag{7.57}$$

Gehrlein (1994) develops the notion of a different type algebraic chooser model in which the subject is only able to accurately determine the highest rated alternative on each of the attributes. A *Model P chooser* will respond with $A \succ B$ in a pairwise preference comparison if A is rated as the highest ranked alternative on more attributes of comparison than B is. A Model P chooser therefore acts as an algebraic chooser. In the context of earlier discussion that was related to three-candidate elections, a Model P chooser is operating in the same fashion as when elections are based on plurality rule.

Results from Gehrlein (1982b) can be modified to obtain a representation for the conditional probability, $P_{SMA}^{P}(3,n,PE(1))$, that the SMA for a Model P chooser and a chooser with May's Model are the same, given that a SMA exists with May's Model, with

$$P_{SMA}^{P}(3,n,PE(1)) = \frac{119n^4+1110n^3+3980n^2+7050n+5021}{135(n+1)(n+3)^3}, \quad \text{for } n = 1,7,13,19,\ldots \tag{7.58}$$

Computed values of $P_{SMA}^{P}(3,n,PE(1))$ from Eq. 7.58 are listed in Table 7. 21 for each n = 1(6)25 and for the limiting probability as $n \to \infty$.

Other results from Gehrlein (2003) can be extended to obtain a representation for $P_{SMA}^{P}(3,n,IAC_b(0))$ to determine the impact that the imposition of the assumption of single-peakedness on attribute ranking situations has on the probability that the SMA's by Model P choosers and choosers by May's Model coincide.

$$P_{SMA}^{P}(3,n,IAC_b(0)) = \frac{31n^2+28n+13}{36n(n+1)}, \quad \text{for } n = 1,7,13,19,\ldots \tag{7.59}$$

Table 7.21 Computed values of $P_{SMA}^{P}(3,n,PE(1))$ and $P_{SMA}^{P}(3,n,IAC_b(0))$

n	$P_{SMA}^{P}(3,n,PE(1))$	$P_{SMA}^{P}(3,n,IAC_b(0))$
1	1.0000	1.0000
7	.8480	.8571
13	.8534	.8571
19	.8591	.8579
25	.8630	.8585
∞	.8815	.8611

Computed values of $P_{SMA}^{P}(3,n,IAC_b(0))$ from Eq. 7.59 are listed in Table 7.21 for each n = 1(6)25 and for the limiting probability as $n \to \infty$. The very surprising result from the values in Table 7.21 is that the imposition of the assumption of single-peakedness does almost nothing to improve the likelihood that the same SMA results for Model P choosers and choosers who use May's Model.

Results from Gehrlein and Fishburn (1978b) can be extended to obtain a representation for $P_{SMA}^{P}(3,\infty,PE(0))$, with

$$P_{SMA}^{P}(3,\infty,PE(0)) = \frac{\left[\begin{array}{c} \frac{1}{4}+\frac{3}{4\pi}\left\{Sin^{-1}\left(\sqrt{\frac{2}{3}}\right)+Sin^{-1}\left(\sqrt{\frac{1}{6}}\right)+\frac{1}{2}Sin^{-1}\left(\frac{1}{3}\right)\right\}+ \\ \frac{3}{4\pi^2}\left\{\begin{array}{c}\left(Sin^{-1}\left(\sqrt{\frac{2}{3}}\right)\right)^2-\frac{1}{2}\left(Sin^{-1}\left(\frac{1}{3}\right)\right)^2+2Li_2\left(f,Cos^{-1}\left(\frac{1}{3}\right)\right) \\ -Li_2\left(f^2,Cos^{-1}\left(\frac{1}{3}\right)\right)+\frac{1}{4}Li_2\left(f^4\right)-\frac{1}{2}Li_2\left(f^2\right)\end{array}\right\}\end{array}\right]}{P_{SMA}(3,\infty,PE(0))}, \tag{7.60}$$

where, $f = 2-\sqrt{3}$ and

$$Li_2(a,\theta) = -\frac{1}{2}\int_0^a\left\{\frac{\log\left(1-2\nu Cos(\theta)+\nu^2\right)}{\nu}\right\}d\nu. \tag{7.61}$$

$Li_2(a)$ is the dilogarithmic function and $Li_2(a,\theta)$ is the real part of the dilogarithmic function with a complex argument. Lewin (1958) describes the dilogarithmic function in detail and describes methods for calculating values of $Li_2(a)$ with an infinite series. The $Li_2(a,\theta)$ terms are evaluated by quadrature. After performing all of the necessary calculations, we find $P_{SMA}^{P}(3,\infty,PE(0)) \approx .7572$.

By comparing this finding to the limiting value of $P_{SMA}^{P}(3,\infty,PE(1))$ from Table 7.21, it can be seen that changing the basic assumption for generating attribute ranking situations from a P-E model with α equal to zero to a P-E model with α equal to one does have a significant impact on the probability that a Model P chooser and a chooser with May's Model will have the same SMA, despite the earlier observation that imposing the single-peakedness condition has no significant impact on this probability.

Following the logic of earlier discussion, attention is now turned to the consideration of evidence to support the idea that a Model P chooser might have a relatively high chance of giving pairwise preference responses that are in drastic disagreement with an algebraic chooser using May's Model. Analysis that is performed in Gehrlein (2002b) can be extended to develop a representation for the probability that a Model P chooser selects an alternative as being the SLA, when that same alternative is the SMA for a May's Model chooser. When alternative ranking situations are generated on the basis of a P-E model with α equal to one

$$P_{SLA}^{P}(3,n,PE(1)) = \frac{2(n-1)(2n^2+n+39)}{135(n+1)(n+3)^2}, \quad \text{for } n = 1, 7, 13, 19, \ldots \tag{7.62}$$

If the condition of single-peakedness is imposed on attribute ranking situations, work in Lepelley (1986) can be extended for P-E model with α equal to one, to obtain the representation

$$P_{SLA}^{P}(3,n,IAC_b(0)) = \frac{(n-1)^2}{36n(n+1)}, \quad \text{for } n = 1, 7, 13, 19, \ldots \tag{7.63}$$

Table 7.22 lists computed values of $P_{SLA}^{P}(3,n,PE(1))$ and $P_{SLA}^{P}(3,n,IAC_b(0))$ from Eqs. 7.62 and 7.63 respectively for each n = 1(6)25, along with the limiting probability as $n \to \infty$.

Table 7.22 Computed values of $P_{SLA}^{P}(3,n,PE(1))$ and $P_{SLA}^{P}(3,n,IAC_b(0))$

n	$P_{SLA}^{P}(3,n,PE(1))$	$P_{SLA}^{P}(3,n,IAC_b(0))$
1	.0000	.0000
7	.0160	.0179
13	.0193	.0220
19	.0215	.0237
25	.0229	.0246
∞	.0296	.0278

While the probabilities in Table 7.22 remain relatively small, the imposition of the condition of single-peakedness has almost no impact on changing these probabilities. This result is not surprising at this point, since the imposition of single-peakedness had almost no impact on the probability that Model P choosers and choosers with May's Model have the same SMA.

7.11 Conclusion

Three factors have been found to be of importance in determining the probability that a probabilistic chooser will select the same SMA or WMA as an algebraic chooser who is using May's Model. An increase in the number of attributes of comparison decreases the probability that a probabilistic chooser matches the responses of an algebraic chooser in all cases. This result would be expected intuitively, since having a subject react to pairwise preferences with a large number of attributes of comparison would typically confound the subject with information overload. An increase in the discriminatory power of a probabilistic chooser leads to an increase in the probability of matching the results of an algebraic chooser, as one would expect.

The third factor concerns the impact of the degree of dependence among attribute rankings for alternatives. When there is complete independence between the subject's perceived attribute rankings, all probabilistic chooser models that were considered were found to become equivalent to random choosers as the number of attributes increased to its limiting value, with $n \to \infty$. However, the rate of convergence to this limiting value is extremely slow for probabilistic choosers who have discriminatory power that is better than that of a Model L chooser. Moreover, scenarios that reflect only a relatively low level of dependence between attribute rankings, can lead to dramatic increases in the probability that a probabilistic chooser will match the results of an algebraic chooser, even for $n \to \infty$.

Model P choosers are found to display unusual behavior. As expected, increasing the dependence among attribute rankings increases the probability that Model P choosers will be in agreement with choosers using May's Model. However, the imposition of the assumption of single peakedness on attribute ranking situations is found to have almost no impact on this relationship.

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