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# Lagrangian in Classical Mechanics and in Special Relativity from Observer's Mathematics Point of View

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**Abstract** This work considers the Lagrangian in classical mechanics and in special relativity in a setting of arithmetic, algebra, and topology provided by observer's mathematics (see www.mathrelativity.com). Certain results and communications pertaining to solutions of these problems are provided. In particular, we show that the standard expressions for Lagrangian take place with probabilities $<1$.

**Keywords** Lagrangian · Classical mechanics · Relativity · Observer's mathematics

## 1 Introduction

It is well known that Bohr's position and the Copenhagen interpretation of quantum mechanics (QM) results in an observer-based viewpoint to physics. Another major aspect of QM is the stipulation of discretization of space-time instead of its classical continuity interpretation, which is explicitly related to the Heisenberg uncertainty principle. Heisenberg's unique contribution was not to point out that measurement affects the system being measured, but rather, it was to recognize the new fundamental limits to measurement set by the "quantum of actions". There are two such limits. First, according to classical physics we can make the disturbance as small as we wish, while according to QM, we cannot. The action of light, for instance, is quantized, so that a photon cannot avoid disturbing a particle it strikes. The second limit imposed by QM

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is that this disturbance is uncontrollable and unpredictable. This latter feature reflects the deeply statistical nature of QM. The two new features appearing in Heisenberg's analysis, therefore, are:

- The disturbance cannot be reduced in magnitude below a fundamental limit, and
- Correction for the disturbance is impossible.

There are several number-theoretic approaches to quantum physics, e.g., based on $p-$adic theoretical physics, see, for example, [1–3]. In this paper we consider another methodology - Observer's Mathematics approach, which is based on an observer's view point and discreteness of space-time. Randomness appeared in Observer's Mathematics when we considered derivatives, see [4–7]. In particular, the following theorem was proven: "From the point of view of a $W_m-$observer, a derivative calculated by a $W_n-$observer with $m > n$ is not uniquely defined, i.e., $f'(x_0)$ is a random variable for any real function $f(x)$ on a set of real numbers." In this paper we continue to consider the probability questions that appear automatically, without any additional assumptions in quantum physics, from observer's mathematics point of view.

We will also see that randomness appears here not only when we consider derivatives, but also in elementary arithmetic calculations.

## 2 The Lagrangian for a Free Particle in Classical Mechanics

The following discussion is based on [8]. Consider the simplest case, that of the free motion of a particle relative to an inertial frame of reference. The Lagrangian in this case can depend only on the square of the velocity. To discover the form of this dependence, we make use of Galileo's relativity principle. If an inertial frame $K$ is moving with an infinitesimal velocity $\varepsilon$ relative to another inertial frame $K'$, then $\mathbf{v}' = \mathbf{v} + \varepsilon$. Since the equations of motion must have the same form in every frame, the Lagrangian $L(\upsilon^2)$ must be converted by this transformation into a function $L'$ which differs from $L(\upsilon^2)$, if at all, only by the total time derivative of a function of coordinates and time.

We have $L' = L(\upsilon'^2) = L\left(\upsilon^2 + 2\mathbf{v}\cdot\varepsilon + \varepsilon^2\right)$. Expanding this expression in powers of $\varepsilon$ and neglecting terms above the first order, we obtain

$$L(\upsilon'^2) = L(\upsilon^2) + \frac{\partial L}{\partial \upsilon^2} 2\mathbf{v}\cdot\varepsilon$$

The second term on the right of this equation is a total time derivative only if it a linear function of the velocity $\mathbf{v}$. Hence $\frac{\partial L}{\partial \upsilon^2}$ is independent of the velocity, i.e., the Lagrangian is in this case proportional to the square of the velocity, and we write it as

$$L = \frac{1}{2} m \upsilon^2$$

From the fact that a Lagrangian of this form satisfies Galileo's relativity principle for an infinitesimal relative velocity, it follows at once that the Lagrangian is invariant for a finite relative velocity $\mathbf{V}$ of the frames $K$ and $K'$. For

$$L' = \frac{1}{2}m\upsilon'^2 = \frac{1}{2}m\left(\mathbf{v}+\mathbf{V}\right)^2 = \frac{1}{2}m\upsilon^2 + m\mathbf{v}\cdot\mathbf{V} + \frac{1}{2}m\mathbf{V}^2$$

or

$$L' = L + \frac{d\left(m\mathbf{v}\cdot\mathbf{V} + \frac{1}{2}\mathbf{V}^2 t\right)}{dt}$$

The second term is a total time derivative and may be omitted.

## 3 The Lagrangian for a Free Particle in Special Relativity

The following discussion is based on [9]. The principle of least action states that a mechanical system should have a quantity called the action $S$. Such quantity is minimized (in other words, $\delta S = 0$) for the actual motion of the system. The action of a relativistic system should be

(i) a scalar, that means Lorentz transformations will not affect this quantity,
(ii) an integral of which the integrand is a first-order differential.

The only quantity that satisfies the two criteria above is the space-time interval $ds$, or a scalar multiple thereof. In short, we can conclude that the action must have the following form: $S = \kappa \int ds$. We have

$$ds = \sqrt{c^2dt - dx^2 - dy^2 - dz^2}$$

After pulling out $cdt$ from the square root and noting that $\frac{dx^2+dy^2+dz^2}{dt^2} = \upsilon^2$, we have $c^2dt^2 - dx^2 - dy^2 - dz^2 = c^2dt^2 - \upsilon^2dt^2 = \left(c^2 - \upsilon^2\right)dt$ and thus

$$ds = cdt\sqrt{1 - \frac{\upsilon^2}{c^2}}$$

Hence

$$S = c\kappa \int \sqrt{1 - \frac{\upsilon^2}{c^2}}dt$$

Now, the action integral can be expressed as a time integral of the Lagrangian between two fixed times:

$$S = \int Ldt$$

Then we can just read off the Lagrangian:

$$L = c\kappa\sqrt{1 - \frac{\upsilon^2}{c^2}}$$

What is remaining now is determining the expression for $\kappa$. At this point we should note that for low velocity $\upsilon$, this relativistic expression for the Lagrangian should resemble that of the classical free Lagrangian $L = \frac{1}{2}m\upsilon^2$. To compare the two Lagrangians, we perform a Taylor expansion on the square root:

$$L = c\kappa\left(1 - \frac{\upsilon^2}{2c^2} + O(\upsilon^4)\right)$$

The firs term, $c\kappa$, is a constant. That will not affect the equations of motion (for example, Euler–Lagrange Equation). The second term, after expanding out, is equal to $-\kappa\frac{\upsilon^2}{2c}$. To reduce to the classical limit, we can put $\kappa = -mc$. Therefore, the relativistic Lagrangian is:

$$L = -mc^2\sqrt{1 - \frac{\upsilon^2}{c^2}}$$

## 4 Lagrangian in Observer's Mathematics

Let us consider the observer's mathematics point of view. Note, that in the calculations above, we used two fundamental arithmetic formulas that use distributive property of real numbers: $(a+b)^2 = a^2 + 2ab + b^2$ and $c(a+b) = ca + cb$. In observer's mathematics, we need to re-write the first formula as follows:

$$(a +_n b) \times_n (a +_n b) = (a \times_n a +_n 2 \times_n (a \times_n b)) +_n b \times_n b$$

We now have the following

**Theorem 1** *$P\,((a +_n b) \times_n (a +_n b) = (a \times_n a +_n 2 \times_n (a \times_n b)) +_n b \times_n b) < 1$, where $P$ is the probability.*

The proof of this theorem follows from the following. Let $n = 2$. Then

(i) The left hand side is $(1.32 +_2 2.43) \times_2 (1.32 +_2 2.43) = 3.75 \times_2 3.75 = 13.99$, while the right hand side is calculated in parts. First, $1.32 \times_2 1.32 = 1.73$; second, $2 \times_2 (1.32 \times_2 2.43) = 6.38$, and third $2.43 \times_2 2.43 = 5.88$. This means that $(1.73 +_2 6.38) +_2 5.88 = 13.99$. i.e., the left hand side is indeed equal to the right hand side. However, observe the calculations in step 2.

(ii) The left hand side is $(1.32 +_2 2.79) \times_2 (1.32 +_2 2.79) = 4.11 \times_2 4.12 = 16.89$, while the right hand side is calculated in part as well. First, $1.32 \times_2 1.32 = 1.73$; second, $2 \times_2 (1.32 \times_2 2.79) = 7.28$, and third $2.79 \times_2 2.79 = 7.65$. This means that $(1.73 +_2 7.28) +_2 7.65 = 16.66$. i.e., the left hand side is not equal to the right hand side.

In particular, for $W_2$, direct calculation shows that $P = 0.34$. Now, consider a random variable

$$\delta_1 = (a +_n b) \times_n (a +_n b) - ((a \times_n a +_n 2 \times_n (a \times_n b)) +_n (b \times_n b))$$

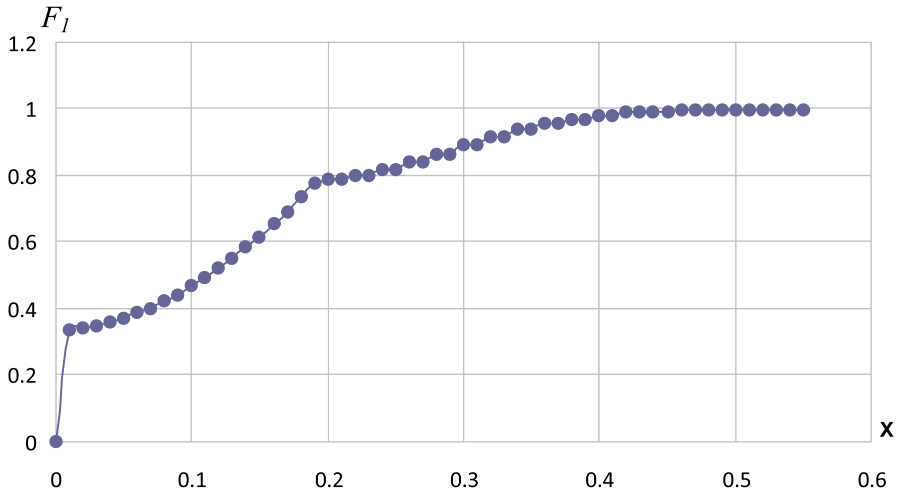


**Fig. 1** Graph of $F_1$

where $a, b \geq 0$, and $\delta_1$ and all expressions on the right hand side are in $W_n$. Now, put $n = 2$. Then using direct calculations, we can build $F_1(x)$ - distribution function of $\delta_1$, according to the following expression $F_1(x) = P(\delta_1 < x)$, where $P$ is the probability. The graph of $F_1(x)$ is given by Fig. 1.

General proof for $W_n$ follows from the information below. If $a, b$ are positive integers in $W_n$ and $(a +_n b) \times_n (a +_n b) \in W_n$, then we have $\delta_1 = 0$. Consider now $a = 0.\underbrace{9\ldots9}_{n}$ and $b = 0.\underbrace{0\ldots08}_{n}$. Then $a +_n b = 1.\underbrace{0\ldots07}_{n}$ and $(a +_n b) \times_n (a +_n b) = 1.\underbrace{0\ldots07}_{n} \times_n 1.\underbrace{0\ldots07}_{n} = 1.\underbrace{0\ldots014}_{n}$, however, $a \times_n a < 1$, $b \times_n b = 0$, and $2 \times_n (a \times_n b) = 0$. Thus, $\delta_1 \neq 0$.

We now have another theorem.

**Theorem 2** *$P(c \times_n (a +_n b) = c \times_n a +_n c \times_n b) < 1$, where $P$ is the probability.*

The proof of this theorem follows from the following. Let $n = 2$. Then

(i) The left hand side is $2 \times_2 (3 +_2 6) = 2 \times_2 9 = 18$, and the right hand side is calculated in parts. First, $2 \times_2 3 = 6$, then $2 \times_2 6 = 12$ and $6 +_2 12 = 18$ i.e., the left hand side is indeed equal to the right hand side. However, observe the calculations in step 2.

(ii) The left hand side is $2.41 \times_2 (3.14 +_2 0.58) = 2.41 \times_2 3.72 = 8.95$, and the right hand side is calculated in parts. First, $2.41 \times_2 3.14 = 7.55$, then $2.41 \times_2 0.58 = 1.36$ and $7.55 +_2 1.36 = 8.91$ i.e., the left hand side is not equal to the right hand side.

In particular, for $W_2$, direct calculation shows that $P = 0.34$. Now, consider a random variable

$$\delta_2 = c \times_n (a +_n b) -_n (c \times_n a +_n c \times_n b)$$

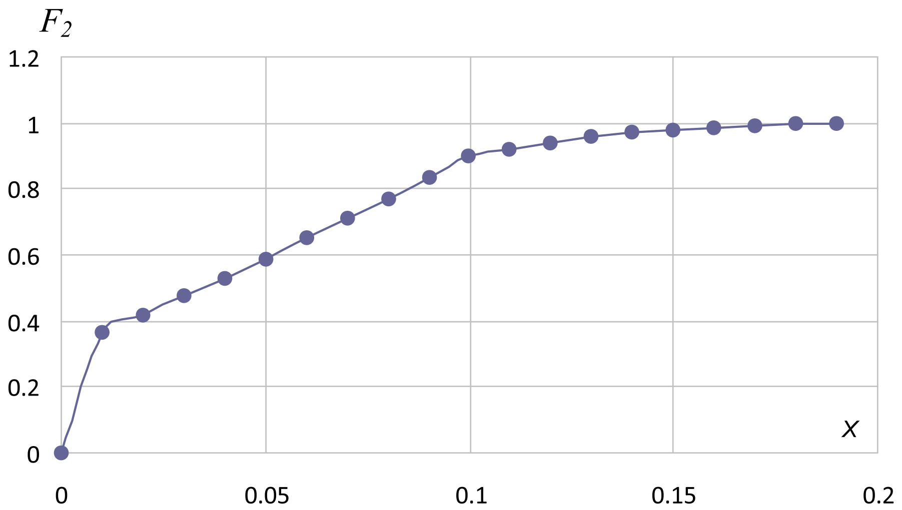


**Fig. 2** Graph of $F_2$

where $a, b, c \geq 0$, and $\delta_2$ and all expressions on the right hand side are in $W_n$. Now, put $n = 2$. Then using direct calculations, we can build $F_2(x)$ - distribution function of $\delta_2$, according to the following expression $F_2(x) = P(\delta_2 < x)$, where $P$ is the probability. The graph of $F_2(x)$ is given by Fig. 2.

General proof for $W_n$ follows from the information below. If $a, b, c$ are positive integers in $W_n$ and $a \times_n (b \times_n c) \in W_n$, then we have $\delta_2 = 0$. Consider now $a = 2$, $b = 0.\underbrace{9\ldots9}_{n}$ and $c = 0.\underbrace{0\ldots01}_{n}$. Then $b \times_n c = 0$, $a \times_n (b \times_n c) = 0$, $a \times_n b = 1.\underbrace{9\ldots98}_{n}$, and $(a \times_n b) \times_n c = 0.\underbrace{0\ldots01}_{n}$. Thus, $\delta_2 \neq 0$.

Therefore, we have proved the following theorems.

**Theorem 3** *In classical mechanics,* $P\left(L = \frac{mv^2}{2}\right) < 1$, *where* $P$ *is the probability.*

**Theorem 4** *In special relativity,* $P\left(L = -mc^2\sqrt{1 - \frac{v^2}{c^2}}\right) < 1$, *where* $P$ *is the probability.*

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