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Intuitionistic Fuzzy Information Aggregation

Theory and Applications

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With 2 figures

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Preface

Since it was introduced by Zadeh in 1965, the fuzzy set theory has been widely applied in various fields of modern society. Central to the fuzzy set is the extension of the characteristic function taking the value of 0 or 1 to the membership function which can take any value from the closed interval [0,1]. However, the membership function is only a single-valued function, which cannot be used to express the evidences of support and objection simultaneously in many practical situations. In the processes of cognition of things, people may not possess a sufficient level of knowledge of the problem domain, due to the increasing complexity of the socio-economic environments. In such cases, they usually have some uncertainty in providing their preferences over the objects considered, which makes the results of cognitive performance exhibit the characteristics of affirmation, negation, and hesitation. For example, in a voting event, in addition to support and objection, there is usually abstention which indicates hesitation and indeterminacy of the voter to the object. As the fuzzy set cannot be used to completely express all the information in such a problem, it faces a variety of limits in actual applications.

Atanassov (1983) extends the fuzzy set characterized by a membership function to the intuitionistic fuzzy set (IFS), which is characterized by a membership function, a non-membership function and a hesitancy function. As a result, the IFS can describe the fuzzy characters of things more detailedly and comprehensively, which is found to be more effective in dealing with vagueness and uncertainty. Over the last few decades, the IFS theory has been receiving more and more attention from researchers and practitioners, and has been applied to various fields, including decision making, logic programming, medical diagnosis, pattern recognition, robotic systems, fuzzy topology, machine learning and market prediction, etc.

The IFS theory is undergoing continuous in-depth study as well as continuous expansion of the scope of its applications. As such, it has been found that effective aggregation and processing of intuitionistic fuzzy information becomes increasingly important. Information processing tools, including aggregation techniques for intuitionistic fuzzy information, association measures, distance measures and similarity measures for IFSs, have broad prospects for actual applications, but pose many interesting yet challenging topics for research.

In this book, we will give a thorough and systematic introduction to the latest research results in intuitionistic fuzzy information aggregation theory and its applications to various fields such as decision making, medical diagnosis and pattern recognition, etc. The book is organized as follows:

Chapter 1 introduces the aggregation techniques for intuitionistic fuzzy information. We first define the concept of intuitionistic fuzzy number (IFN), and based on score function and accuracy function, give a ranking method for IFNs. We then define the operational laws of IFNs, and introduce a series of operators for aggregating intuitionistic fuzzy information. These include the intuitionistic fuzzy averaging operator, intuitionistic fuzzy bonferroni means, and intuitionistic fuzzy aggregation operators based on Choquet integral, to name just a few. The desirable properties of these operators are described in detail, and their applications to multi-attribute decision making are also discussed.

Chapter 2 mainly introduces the aggregation techniques for interval-valued intuitionistic fuzzy information. We first define the concept of interval-valued intuitionistic fuzzy number (IVIFN), and introduce some basic operational laws of IVIFNs. We then define the concepts of score function and accuracy function of IVIFNs, based on which a simple method for ranking IVIFNs is presented. We also introduce a number of operators for aggregating interval-valued intuitionistic fuzzy information, including the interval-valued intuitionistic fuzzy averaging operator, the interval-valued intuitionistic fuzzy geometric operator, the interval-valued intuitionistic fuzzy aggregation operators based on Choquet integral, and many others. The interval-valued intuitionistic fuzzy aggregation operators are applied to the field of decision making, and some approaches to multi-attribute decision making based on interval-valued intuitionistic fuzzy information are developed.

Chapter 3 introduces three types of measures: association measures, distance measures, and similarity measures for IFSs and interval-valued intuitionistic fuzzy sets (IVIFSs).

Chapter 4 introduces decision making approaches based on intuitionistic preference relation. We first define preference relations, then introduce the concepts of interval-valued intuitionistic fuzzy positive and negative ideal points. We also utilize aggregation tools to establish models for multi-attribute decision making. Approaches to multi-attribute decision making in interval-valued intuitionistic fuzzy environments are also developed. Finally, consistency analysis on group decision making with intuitionistic preference relations is given.

Chapter 5 introduces multi-attribute decision making with IFN/IVIFN attribute values and known or unknown attribute weights. We introduce the concepts such as relative intuitionistic fuzzy ideal solution, relative uncertain intuitionistic fuzzy ideal solution, modules of IFNs and IVIFNs, etc. We then establish projection models to measure the similarity degree between each alternative and the relative intuitionistic fuzzy ideal solution and the similarity degree between each alternative and the relative uncertain intuitionistic fuzzy ideal solution, by which the best alternative can be obtained.

Chapter 6 introduces aggregation techniques for dynamic intuitionistic fuzzy information and methods for weighting time series. We introduce the concepts of intuitionistic fuzzy variable and uncertain intuitionistic fuzzy variable. We describe dynamic intuitionistic averaging operators, based on which dynamic intuitionistic fuzzy multi-attribute decision making and uncertain dynamic intuitionistic fuzzy multi-attribute decision making problems are tackled.

Chapter 7 considers multi-attribute group decision making problems in which the attribute values provided by experts are expressed in IFNs, and the weight information about both the experts and the attributes is to be determined. We introduce two nonlinear optimization models, from which exact formulas can be obtained to derive the weights of experts. To facilitate group consensus, we introduce a nonlinear optimization model based on individual intuitionistic fuzzy decision matrices. A simple procedure is used to rank the alternatives. The results are also extended to interval-valued intuitionistic fuzzy situations.

This book is suitable for practitioners and researchers working in the fields of fuzzy mathematics, operations research, information science and management science and engineering, etc. It can also be used as a textbook for postgraduate and senior-year undergraduate students.

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Zeshui Xu, Xiaoqiang Cai
Hong Kong
September, 2011

Contents

Chapter 1

Intuitionistic Fuzzy Information Aggregation

The fuzzy set theory has been extensively applied in various fields of modern society (Chen et al., 2005) since its introduction by Zadeh (1965) in 1960s. Central to the fuzzy set is the extension from the characteristic function taking the value of 0 or 1 to the membership function which can take any value from the closed interval [0,1]. However, the membership function is only a single-valued function, which cannot be used to express the support and objection evidences simultaneously in many practical situations.

In cognition of things, people may not possess a precise or sufficient level of knowledge of the problem domain, due to the complexity of the socio-economic environment. In such cases, they usually have some uncertainty in providing their preferences over the objects considered, which makes the results of cognitive performance exhibit the characteristics of affirmation, negation and hesitation. For example, in a voting problem, in addition to "support" and "objection", there is usually "abstention" which indicates the hesitation and indeterminacy of the voter regarding the object. Because a fuzzy set cannot be used to completely express all the information in such problems, its applicability is often limited in many practical situations.

Atanassov (1986; 1983) generalizes Zadeh's fuzzy set theory with the concept of intuitionistic fuzzy set (IFS), which is characterized by a membership function, a non-member function, and a hesitancy (indeterminancy) function. It is argued that IFS can describe the fuzzy characters of things more detailedly and comprehensively, and is therefore more useful in dealing with vagueness and uncertainty than the classical fuzzy set theory. Over the last few decades, researchers have paid great attention to investigation of the IFS theory, and achieved fruitful results (Atanassov, 1999; Bustince et al., 2007). Atanassov (1986) and De et al. (2000) introduce several basic operations on IFSs, including "intersection", "union", "supplement", and "power". However, as the study of the IFS theory expands in both depth and scope, effective aggregation and handling of intuitionistic fuzzy information has become necessary and increasingly important. These basic operations on IFSs have been far from meeting the actual needs.

In recent years, Xu (2010c; 2007e), Xu and Xia (2011), Xu and Yager (2011; 2006), and Zhao et al. (2010) have focused on the subject of aggregation techniques for intuitionistic fuzzy information. They have defined the concept of intuitionistic fuzzy number and introduced, based on the score function and the accuracy function, a ranking method for intuitionistic fuzzy numbers. They have further defined operational laws of intuitionistic fuzzy numbers, and introduced a series of operators for aggregating intuitionistic fuzzy information, including the intuitionistic fuzzy averaging operator, intuitionistic fuzzy weighted averaging operator, intuitionistic fuzzy ordered weighted averaging operator, intuitionistic fuzzy hybrid averaging operator, intuitionistic fuzzy geometric operator, intuitionistic fuzzy weighted geometric operator, intuitionistic fuzzy ordered weighted geometric operator, intuitionistic fuzzy hybrid geometric operator, intuitionistic fuzzy bonferroni means, generalized intuitionistic fuzzy aggregation operators, intuitionistic fuzzy aggregation operators based on Choquet integral, induced generalized intuitionistic fuzzy aggregation operators, etc. They have also applied these operators to the field of multi-attribute decision making.

1.1 Intuitionistic Fuzzy Sets

We first introduce the concept of Zadeh's fuzzy set:

Definition 1.1.1 (Zadeh, 1965) Let X be a fixed set. Then

$$F = \{\langle x,\ \mu_F(x)\rangle \mid x \in X\ \} \tag{1.1}$$

is called a fuzzy set, where μ_F is the membership function of F, $\mu_F : X \to [0,\ 1]$, and $\mu_F(x)$ indicates the membership degree of the element x to F, which is a single value belonging to the unit closed interval [0, 1].

Atanassov (1986; 1983) generalizes Zadeh's fuzzy set with the concept of intuitionistic fuzzy set (IFS) as defined below:

Definition 1.1.2 (Atanassov, 1986; 1983) An IFS is an object having the following form:

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in X\} \tag{1.2}$$

which is characterized by a membership function:

$$\mu_A : X \to [0,1], \quad x \in X \ \to \ \mu_A(x) \in [0,1] \tag{1.3}$$

and a non-membership function:

$$\nu_A : X \to [0,1], \quad x \in X \ \to \ \nu_A(x) \in [0,1] \tag{1.4}$$

with the condition:

$$0 \leqslant \mu_A(x) + \nu_A(x) \leqslant 1, \quad \text{for all} \quad x \in X \tag{1.5}$$

where $\mu_A(x)$ and $\nu_A(x)$ represent, respectively, the membership degree and the non-membership degree of x in A.

Moreover, for each IFS A in X, if

$$\pi_A(x) = 1 - \mu_A(x) - \nu_A(x), \quad \text{for all} \quad x \in X \tag{1.6}$$

then $\pi_A(x)$ is called an indeterminancy degree of x to A.

Szmidt and Kacprzyk (2000) call $\pi_A(x)$ an intuitionistic index of x in A. It is a hesitancy degree of x to A. Obviously,

$$0 \leqslant \pi_A(x) \leqslant 1, \quad \text{for all} \quad x \in X \tag{1.7}$$

In particular, if

$$\pi_A(x) = 1 - \mu_A(x) - [1 - \mu_A(x)] = 0, \quad x \in X \tag{1.8}$$

then A reduces to Zadeh's fuzzy set. Thus, fuzzy sets are the special cases of IFSs.

For convenience, Xu (2007e) calls $\alpha = (\mu_\alpha,\ \nu_\alpha)$ an intuitionistic fuzzy number (IFN) or an intuitionistic fuzzy value (IFV), where

$$\mu_\alpha \in [0,1], \quad \nu_\alpha \in [0,1], \quad \mu_\alpha + \nu_\alpha \leqslant 1 \tag{1.9}$$

and let Θ be the set of all IFNs. Clearly, $\alpha^+ = (1,\ 0)$ is the largest IFN, and $\alpha^- = (0,\ 1)$ is the smallest IFN.

Each IFN $\alpha = (\mu_\alpha, \nu_\alpha)$ has a physical interpretation. For example, if $\alpha = (0.5,\ 0.3)$, then we can see that $\mu_\alpha = 0.5$ and $\nu_\alpha = 0.3$. It can be interpreted as "the vote for resolution is 5 in favor, 3 against, and 2 abstentions".

For any IFN $\alpha = (\mu_\alpha,\ \nu_\alpha)$, the score of α can be evaluated by the score function s (Chen and Tan, 1994) as shown below:

$$s(\alpha) = \mu_\alpha - \nu_\alpha \tag{1.10}$$

where $s(\alpha) \in [-1,\ 1]$.

From Eq.(1.10), we can see that the score $s(\alpha)$ of the IFN α is directly related to the difference between the membership degree μ_α and the non-membership degree ν_α. The greater the difference $\mu_\alpha - \nu_\alpha$, the larger the score $s(\alpha)$, and then the larger the IFN α. In particular, if $s(\alpha) = 1$, then the IFN α takes the largest value $(1,\ 0)$; If $s(\alpha) = -1$, then α takes the smallest value $(0,\ 1)$.

Example 1.1.1 Let $\alpha_1 = (0.7, 0.2)$ and $\alpha_2 = (0.5, 0.3)$ be two IFNs. Then by Eq. (1.10), we can get the scores of α_1 and α_2 respectively:

$$s(\alpha_1) = 0.7 - 0.2 = 0.5, \quad s(\alpha_2) = 0.5 - 0.3 = 0.2$$

Since $s(\alpha_1) > s(\alpha_2)$, we have $\alpha_1 > \alpha_2$.

Gau and Buehrer (1993) define the concept of vague set. Bustince and Burillo (1996) point out that vague sets are IFSs. Chen and Tan (1994) utilize the max and min operations and the score function to develop an approach to multi-attribute decision making based on vague sets. However, in some special cases, this approach cannot be used to distinguish two IFNs.

Example 1.1.2 Let $\alpha_1 = (0.6,\ 0.2)$ and $\alpha_2 = (0.7,\ 0.3)$ be two IFNs. Then by Eq. (1.10), we have

$$s(\alpha_1) = 0.6 - 0.2 = 0.4, \quad s(\alpha_2) = 0.7 - 0.3 = 0.4$$

Since $s(\alpha_1) = s(\alpha_2)$, we cannot tell the difference between α_1 and α_2 by using the score function.

Hong and Choi (2000) define an accuracy function:

$$h(\alpha) = \mu_\alpha + \nu_\alpha \tag{1.11}$$

where $\alpha = (\mu_\alpha, \nu_\alpha)$ is an IFN, h is the accuracy function of α, and $h(\alpha)$ is the accuracy degree of α. The larger $h(\alpha)$, the higher the accuracy degree of the IFN α.

From Eqs.(1.6) and (1.11), the relationship between the hesitancy degree and the accuracy degree of the IFN α can be shown as follows:

$$\pi_\alpha + h(\alpha) = 1 \tag{1.12}$$

Hence, the smaller the hesitancy degree π_α, the bigger the accuracy degree $h(\alpha)$.

By Eq.(1.11), we can calculate the accuracy degrees of the IFNs α_1 and α_2 in Example 1.1.2:

$$h(\alpha_1) = 0.6 + 0.2 = 0.8, \quad h(\alpha_2) = 0.7 + 0.3 = 1$$

Then $h(\alpha_2) > h(\alpha_1)$, i.e., the accuracy degree of the IFN α_2 is higher than that of the IFN α_1.

Hong and Choi (2000) show that the relation between the score function s and the accuracy function h is similar to the relation between the mean and the variance in statistics. It is well known that an efficient estimator is a measure of the variance of an estimate's sampling distribution in statistics, i.e., the smaller the variance, the better the performance of the estimator. Based on this idea, it is meaningful and appropriate to stipulate that the higher the accuracy degree $h(\alpha)$, the better the IFN α. Consequently, α_2 is larger than α_1.

Motivated by the above analysis, Xu and Yager (2006) develop a method for comparison and ranking of two IFNs, which is based on the score function s and the accuracy function h as defined below:

Definition 1.1.3 (Xu and Yager, 2006) Let $\alpha_1 = (\mu_{\alpha_1}, \nu_{\alpha_1})$ and $\alpha_2 = (\mu_{\alpha_2}, \nu_{\alpha_2})$ be two IFNs, $s(\alpha_1) = \mu_{\alpha_1} - \nu_{\alpha_1}$ and $s(\alpha_2) = \mu_{\alpha_2} - \nu_{\alpha_2}$ the scores of the IFNs α_1 and α_2 respectively, and $h(\alpha_1) = \mu_{\alpha_1} + \nu_{\alpha_1}$ and $h(\alpha_2) = \mu_{\alpha_2} + \nu_{\alpha_2}$ the accuracy degrees of the IFNs α_1 and α_2 respectively. Then

• If $s(\alpha_1) < s(\alpha_2)$, then the IFN α_1 is smaller than the IFN α_2, denoted by $\alpha_1 < \alpha_2$.

• If $s(\alpha_1) = s(\alpha_2)$, then

(1) If $h(\alpha_1) = h(\alpha_2)$, the IFNs α_1 and α_2 are equal, i.e., $\mu_{\alpha_1} = \mu_{\alpha_2}$ and $\nu_{\alpha_1} = \nu_{\alpha_2}$, denoted by $\alpha_1 = \alpha_2$;

(2) If $h(\alpha_1) < h(\alpha_2)$, the IFN α_1 is smaller than the IFN α_2, denoted by $\alpha_1 < \alpha_2$;

(3) If $h(\alpha_1) > h(\alpha_2)$, the IFN α_1 is larger than α_2, denoted by $\alpha_1 > \alpha_2$.

Hong and Choi (2000) further strengthen the decision making method of Chen and Tan (1994). They utilize the score function, the accuracy function, and the max and min operations to develop another technique for handling multi-attribute decision making with intuitionistic fuzzy information. However, the main problem of the aforementioned techniques using the minimum and maximum operations to carry the combination process is the loss of information, and hence a lack of precision in the final results. Therefore, "how to aggregate a collection of IFNs without any loss of information" is an interesting research topic (Xu, 2007e).

Up to now, many operators have been proposed for aggregating information in various decision making environments (Calvo et al., 2002; Xu 2007g; 2004e; Xu and Da, 2003b; Yager and Kacprzyk, 1997). Four of the most common operators for aggregating arguments are the weighted averaging operator (Harsanyi, 1955), weighted geometric operator (Saaty, 1980), ordered weighted averaging operator (Yager, 1988) and ordered weighted geometric operator (Chiclana et al., 2001b; Xu and Da, 2002a), which are defined as follows respectively:

Definition 1.1.4 (Harsanyi, 1955) Let WA : $(Re)^n \to Re$, and a_j $(j = 1, 2, \cdots, n)$ be a collection of real numbers. If

$$\mathrm{WA}_\omega(a_1, a_2, \cdots, a_n) = \sum_{j=1}^{n} \omega_j a_j \tag{1.13}$$

then the function WA is called a weighted averaging (WA) operator, where Re is the set of all real numbers, $\omega = (\omega_1, \omega_2, \cdots, \omega_n)^{\mathrm{T}}$ is the weight vector of a_j $(j =$

$1, 2, \cdots, n)$, with $\omega_j \in [0, 1]$ $(j = 1, 2, \cdots, n)$ and $\sum_{j=1}^{n} \omega_j = 1$.

Definition 1.1.5 (Saaty, 1980) Let $\mathrm{WG} : (Re)^{+n} \to (Re)^+$. If

$$\mathrm{WG}_\omega(a_1, a_2, \cdots, a_n) = \prod_{j=1}^{n} a_j^{\omega_j} \tag{1.14}$$

then the function WG is called a weighted geometric (WG) operator, where $(Re)^+$ is the set of all positive real numbers, and $\omega = (\omega_1, \omega_2, \cdots, \omega_n)^{\mathrm{T}}$ is the exponential weighting vector of a_j $(j = 1, 2, \cdots, n)$, with $\omega_j \in [0, 1]$ $(j = 1, 2, \cdots, n)$ and $\sum_{j=1}^{n} \omega_j = 1$.

Both the WA and WG operators first weight all the given arguments a_j $(j = 1, 2, \cdots, n)$, and then aggregate these weighted arguments. The difference between these two operators is that the WG operator is much more sensitive to the given arguments. Especially in the case where there is an argument taking the value of zero, the aggregated value of these arguments by using the WG operator must be zero no matter what the other given arguments are.

Definition 1.1.6 (Yager, 1988) Let $\mathrm{OWA} : (Re)^n \to Re$. If

$$\mathrm{OWA}_w(a_1, a_2, \cdots, a_n) = \sum_{j=1}^{n} w_j b_j \tag{1.15}$$

where $w = (w_1, w_2, \cdots, w_n)^{\mathrm{T}}$ is the weighting vector associated with the function OWA, with $w_j \in [0, 1]$, $j = 1, 2, \cdots, n$, $\sum_{j=1}^{n} w_j = 1$, and b_j is the j-th largest of a_j $(j = 1, 2, \cdots, n)$, then the function OWA is called an ordered weighted averaging (OWA) operator.

Definition 1.1.7 (Chiclana et al., 2001b; Xu and Da, 2002a) Let $\mathrm{OWG} : (Re)^{+n} \to (Re)^+$. If

$$\mathrm{OWG}_w(a_1, a_2, \cdots, a_n) = \prod_{j=1}^{n} b_j^{w_j} \tag{1.16}$$

where $w = (w_1, w_2, \cdots, w_n)^{\mathrm{T}}$ is the exponential weighting vector associated with the function OWG, with $w_j \in [0, 1]$, $j = 1, 2, \cdots, n$, $\sum_{j=1}^{n} w_j = 1$, and b_j is the j-th largest of a_j $(j = 1, 2, \cdots, n)$, then the function OWG is called an ordered weighted geometric (OWG) operator.

The fundamental characteristic of the OWA and OWG operators is that they first rearrange all the given arguments in descending order, and then weight the ordered positions of the arguments. These two operators aggregate the ordered arguments together with the weights of their ordered positions. Obviously, the argument a_i is not associated with the particular weight w_i. Instead, the weight w_i is associated with the particular ordered position i of the arguments. Thus, w_i is also called a position weight. The OWG operator is developed on the basis of the OWA operator and the geometric mean.

The above four operators are generally suitable to aggregate the arguments taking the values of real numbers. They have been extended to accommodate uncertain or fuzzy linguistic environments, see (Bordogna et al., 1997; Delgado et al., 1993; Herrera et al., 2005; 1996b; Herrera and Martinez, 2000a; 2000b; Xu, 2008a; 2007g; 2006a; 2006b; 2006c; 2006g; 2004a; 2004d; Xu and Da, 2002b; Yager, 2004c; 2003a; 2003b; 1996; 1995; Zhang and Xu, 2005). Xu (2010c; 2007e), Xu and Xia (2011), Xu and Yager (2011; 2006), and Zhao et al. (2010) have further generalized them to accommodate intuitionistic fuzzy environments and investigated the aggregation techniques for intuitionistic fuzzy information.

1.2 Operational Laws of Intuitionistic Fuzzy Numbers

Theorem 1.2.1 (Xu, 2007e) Let $\alpha_1 = (\mu_{\alpha_1},\ \nu_{\alpha_1})$ and $\alpha_2 = (\mu_{\alpha_2},\ \nu_{\alpha_2})$ be two IFNs. Then

$$\alpha_1 \leqslant \alpha_2 \Leftarrow \mu_{\alpha_1} \leqslant \mu_{\alpha_2} \quad \text{and} \quad \nu_{\alpha_1} \geqslant \nu_{\alpha_2} \tag{1.17}$$

Proof Since $s(\alpha_1) = \mu_{\alpha_1} - \nu_{\alpha_1}$, $s(\alpha_2) = \mu_{\alpha_2} - \nu_{\alpha_2}$, $\mu_{\alpha_1} \leqslant \mu_{\alpha_2}$ and $\nu_{\alpha_1} \geqslant \nu_{\alpha_2}$, we have

$$\begin{aligned} s(\alpha_1) - s(\alpha_2) &= (\mu_{\alpha_1} - \nu_{\alpha_1}) - (\mu_{\alpha_2} - \nu_{\alpha_2}) \\ &= (\mu_{\alpha_1} - \mu_{\alpha_2}) + (\nu_{\alpha_2} - \nu_{\alpha_1}) \end{aligned}$$

If $\mu_{\alpha_1} = \mu_{\alpha_2}$ and $\nu_{\alpha_1} = \nu_{\alpha_2}$, then $\alpha_1 = \alpha_2$; Otherwise $s(\alpha_1) - s(\alpha_2) < 0$, i.e., $s(\alpha_1) < s(\alpha_2)$. Hence, by Definition 1.1.3, we have $\alpha_1 < \alpha_2$. The proof is completed.

Goguen (1967) defines an L-fuzzy set on X as an $X \to L$ mapping, which is a generalization of the concept of fuzzy set. It covers the fuzzy set as a special case when $L = [0,\ 1]$, where L is a complete lattice equipped with an operator satisfying certain conditions. For example, Deschrijver and Kerre (2003b) define a complete lattice as a partially ordered set $(L, \leqslant_L)$ such that every nonempty subset of L has a supremum and an infimum.

A traditional relation on the lattice $(L, \leqslant_L)$, defined by

$$\alpha_1 \leqslant_L \alpha_2 \Leftrightarrow \mu_{\alpha_1} \leqslant \mu_{\alpha_2} \ \text{ and } \ \nu_{\alpha_1} \geqslant \nu_{\alpha_2} \tag{1.18}$$

is also applied to the operations of IFSs (Atanassov, 1986; Cornelis et al., 2004).

However, in some situations, Eq.(1.18) cannot be used to compare IFNs. For example, let $\alpha_1 = (\mu_{\alpha_1},\ \nu_{\alpha_1}) = (0.2,\ 0.4)$ and $\alpha_2 = (\mu_{\alpha_2}, \nu_{\alpha_2}) = (0.4,\ 0.5)$, where $\mu_{\alpha_1} = 0.2 < \mu_{\alpha_2} = 0.4$ and $\nu_{\alpha_1} = 0.4 < \nu_{\alpha_2} = 0.5$. Then it is impossible to know which one is larger by using Eq.(1.18). But in this case, we can use Definition 1.1.3 to compare them. In fact, since

$$s(\alpha_1) = 0.2 - 0.4 = -0.2, \quad s(\alpha_2) = 0.4 - 0.5 = -0.1$$

it follows from Definition 1.1.3 that $\alpha_1 < \alpha_2$.

Another well-known relation on the lattice $(L, \leqslant_L)$, defined by

$$\alpha_1 \prec_L \alpha_2 \Leftrightarrow \mu_{\alpha_1} \leqslant \mu_{\alpha_2} \quad \text{and} \quad \nu_{\alpha_1} \leqslant \nu_{\alpha_2} \tag{1.19}$$

does not conform to the implication of IFS (Atanassov, 1986).

Atanassov (1986) and De et al. (2000) introduce some basic operations on IFSs, which not only ensure that the operational results are also IFSs, but also are useful in the calculus of linguistic variables in an intuitionistic fuzzy environment:

Definition 1.2.1 (Atanassov, 1986) Let a set X be fixed, and let $A = \{\langle x,\ \mu_A(x),\ \nu_A(x)\rangle \,|\, x \in X\ \}$, $A_1 = \{\langle x,\ \mu_{A_1}(x),\ \nu_{A_1}(x)\rangle \,|\, x \in X\ \}$ and $A_2 = \{\langle x,\ \mu_{A_2}(x),\ \nu_{A_2}(x)\rangle \,|\, x \in X\ \}$ be three IFSs. Then the following operations are valid:

(1) $\bar{A} = \{\langle x,\ \nu_A(x),\ \mu_A(x)\rangle | \, x \in X\}$;

(2) $A_1 \cap A_2 = \{\langle x,\ \min\{\mu_{A_1}(x),\ \mu_{A_2}(x)\}, \max\{\nu_{A_1}(x),\ \nu_{A_2}(x)\}\rangle \,|\, x \in X\}$;

(3) $A_1 \cup A_2 = \{\langle x,\ \max\{\mu_{A_1}(x),\ \mu_{A_2}(x)\}, \min\{\nu_{A_1}(x),\ \nu_{A_2}(x)\}\rangle \,|\, x \in X\}$;

(4) $A_1 + A_2 = \{\langle x,\ \mu_{A_1}(x) + \mu_{A_2}(x) - \mu_{A_1}(x)\,\mu_{A_2}(x), \nu_{A_1}(x)\,\nu_{A_2}(x)\rangle \,|\, x \in X\}$;

(5) $A_1 \cdot A_2 = \{\langle x,\ \mu_{A_1}(x)\,\mu_{A_2}(x),\ \nu_{A_1}(x) + \nu_{A_2}(x) - \nu_{A_1}(x)\ \nu_{A_2}(x)\rangle \,|\, x \in X\}$.

De et al. (2000) further give another two operations of IFSs:

(6) $n\,A = \{\langle x,\ 1-(1-\mu_A(x))^n,\ (\nu_A(x))^n\rangle \,|\, x \in X\}$;

(7) $A^n = \{\langle x,\ (\mu_A(x))^n,\ 1-(1-\nu_A(x))^n\rangle \,|\, x \in X\}$,

where n is a positive integer.

Motivated by the above operations, Xu (2007e), Xu and Yager (2006) define some basic operational laws of IFNs, which will be used in the remainder of this book:

Definition 1.2.2 (Xu, 2007e; Xu and Yager, 2006) Let $\alpha = (\mu_\alpha,\ \nu_\alpha)$, $\alpha_1 = (\mu_{\alpha_1},\ \nu_{\alpha_1})$ and $\alpha_2 = (\mu_{\alpha_2}, \nu_{\alpha_2})$ be IFNs. Then

(1) $\bar{\alpha} = (\nu_\alpha, \mu_\alpha)$;

(2) $\alpha_1 \wedge \alpha_2 = (\min\{\mu_{\alpha_1},\ \mu_{\alpha_2}\},\ \max\{\nu_{\alpha_1},\ \nu_{\alpha_2}\})$;

(3) $\alpha_1 \vee \alpha_2 = (\max\{\mu_{\alpha_1}, \mu_{\alpha_2}\},\ \min\{\nu_{\alpha_1}, \nu_{\alpha_2}\})$;

(4) $\alpha_1 \oplus \alpha_2 = (\mu_{\alpha_1} + \mu_{\alpha_2} - \mu_{\alpha_1}\mu_{\alpha_2},\ \nu_{\alpha_1}\,\nu_{\alpha_2})$;

(5) $\alpha_1 \otimes \alpha_2 = (\mu_{\alpha_1}\mu_{\alpha_2},\ \nu_{\alpha_1} + \nu_{\alpha_2} - \nu_{\alpha_1}\,\nu_{\alpha_2})$;

(6) $\lambda\alpha = (1-(1-\mu_\alpha)^\lambda,\ \nu_\alpha^\lambda)$, $\lambda > 0$;

(7) $\alpha^\lambda = (\mu_\alpha^\lambda,\ 1-(1-\nu_\alpha)^\lambda),\ \lambda > 0$.

Let $S_P(\mu_{\alpha_1}, \mu_{\alpha_2}) = \mu_{\alpha_1} + \mu_{\alpha_2} - \mu_{\alpha_1}\mu_{\alpha_2}$ and $T_P(\nu_{\alpha_1}, \nu_{\alpha_2}) = \nu_{\alpha_1}\nu_{\alpha_2}$. Then the operational law (4) in Definition 1.2.2 can be rewritten as:

$$\alpha_1 \oplus \alpha_2 = (S_P(\mu_{\alpha_1}, \mu_{\alpha_2}),\ \ T_P(\nu_{\alpha_1}, \nu_{\alpha_2})) \tag{1.20}$$

If we let $S_P(\nu_{\alpha_1}, \nu_{\alpha_2}) = \nu_{\alpha_1} + \nu_{\alpha_2} - \nu_{\alpha_1}\nu_{\alpha_2}$ and $T_P(\mu_{\alpha_1}, \mu_{\alpha_2}) = \mu_{\alpha_1}\mu_{\alpha_2}$, then the operational law (5) in Definition 1.2.2 can be rewritten as:

$$\alpha_1 \otimes \alpha_2 = (\ T_P(\mu_{\alpha_1}, \mu_{\alpha_2}),\ S_P(\nu_{\alpha_1}, \nu_{\alpha_2})) \tag{1.21}$$

where $S_P(x_1, x_2) = x_1 + x_2 - x_1x_2$ is a well-known t-conorm satisfying (Klir and Yuan, 1995):

(1) **(Boundary):** $S_P(1,\ 1) = 1$, $S_P(x,\ 0) = S_P(0,\ x) = x$;

(2) **(Monotonicity):** If $x_1 \leqslant x_1'$ and $x_2 \leqslant x_2'$, then $S_P(x_1,\ x_2) \leqslant S_P(x_1',\ x_2')$;

(3) **(Commutativity):** $S_P(x_1,\ x_2) = S_P(x_2,\ x_1)$;

(4) **(Associativity):** $S_P(x_1,\ S_P(x_2, x_3)) = S_P(S_P(x_1, x_2),\ x_3))$,

and $T_P(y_1, y_2) = y_1y_2$ is a well-known t-norm satisfying (Klir and Yuan, 1995):

(1) **(Boundary):** $T_P(0,\ 0) = 0$, $T_P(y,\ 1) = y$;

(2) **(Monotonicity):** If $y_1 \leqslant y_1'$ and $y_2 \leqslant y_2'$, then $T_P(y_1,\ y_2) \leqslant T_P(y_1', y_2')$;

(3) **(Commutativity):** $T_P(y_1,\ y_2) = T_P(y_2,\ y_1)$;

(4) **(Associativity):** $T_P(y_1,\ T_P(y_2, y_3)) = T_P(T_P(y_1, y_2),\ y_3)$.

In particular, if $\mu_\alpha = 1 - \nu_\alpha$, $\mu_{\alpha_1} = 1 - \nu_{\alpha_1}$ and $\mu_{\alpha_2} = 1 - \nu_{\alpha_2}$, then α, α_1 and α_2 represent the traditional fuzzy numbers. In this case, the operational laws (4)–(7) in Definition 1.2.2 reduce to the following forms (Xu, 2007e):

(4′) $\alpha_1 \oplus \alpha_2 = S_P(\mu_{\alpha_1}, \mu_{\alpha_2})$;

(5′) $\alpha_1 \otimes \alpha_2 = \ T_P(\mu_{\alpha_1}, \mu_{\alpha_2})$;

(6′) $\lambda\alpha = 1 - (1-\mu_\alpha)^\lambda$, $\lambda > 0$;

(7′) $\alpha^\lambda = \mu_\alpha^\lambda$, $\lambda > 0$.

Let $\delta(\lambda,\ a) = 1 - (1-a)^\lambda$, where $a \in [0, 1]$ and $\lambda > 0$. Then the expression (6′) is a unit interval monotone increasing function on λ and a, with the following good properties (Xu, 2007e):

(1) $0 \leqslant \delta(\lambda,\ a) \leqslant 1$. Especially, $\delta(\lambda,\ 0) = 0$, $\delta(\lambda,\ 1) = 1$ and $\delta(1,\ a) = a$;

(2) If $\lambda \to 0$ and $0 < a < 1$, then $\delta(\lambda,\ a) \to 0$;

(3) If $\lambda \to +\infty$ and $0 < a < 1$, then $\delta(\lambda,\ a) \to 1$;

(4) If $\lambda_1 > \lambda_2$, then $\delta(\lambda_1,\ a) > \delta(\lambda_2,\ a)$;

(5) If $a_1 > a_2$, then $\delta(\lambda,\ a_1) > \delta(\lambda,\ a_2)$.

These desirable properties provide a theoretic basis for the application of the operational laws (4)-(7) in Definition 1.2.2 to the aggregation of IFNs.

Theorem 1.2.2 (Xu, 2007e; Xu and Yager, 2006) Let $\dot{\alpha}_1 = \alpha_1 \oplus \alpha_2$, $\dot{\alpha}_2 = \alpha_1 \otimes \alpha_2$, $\dot{\alpha}_3 = \lambda\,\alpha$, $\dot{\alpha}_4 = \alpha^{\lambda}$, and $\lambda > 0$. Then all $\dot{\alpha}_i$ $(i = 1, 2, 3, 4)$ are IFNs.

Proof Since $\alpha_1 = (\mu_{\alpha_1},\ \nu_{\alpha_1})$ and $\alpha_2 = (\mu_{\alpha_2},\ \nu_{\alpha_2})$ are IFNs,

$$\mu_{\alpha_1} \in [0,1], \quad \nu_{\alpha_1} \in [0,1], \quad \mu_{\alpha_2} \in [0,1]$$
$$\nu_{\alpha_2} \in [0,1], \quad \mu_{\alpha_1} + \nu_{\alpha_1} \leqslant 1, \quad \mu_{\alpha_2} + \nu_{\alpha_2} \leqslant 1$$

Therefore, by the operational law (4) in Definition 1.2.2, we have

$$\mu_{\alpha_1} + \mu_{\alpha_2} - \mu_{\alpha_1}\mu_{\alpha_2} = \mu_{\alpha_1}(1-\mu_{\alpha_2}) + \mu_{\alpha_2} \geqslant \mu_{\alpha_2} \geqslant 0, \quad \mu_{\alpha_1}\mu_{\alpha_2} \geqslant 0$$
$$\mu_{\alpha_1} + \mu_{\alpha_2} - \mu_{\alpha_1}\mu_{\alpha_2} + \nu_{\alpha_1}\nu_{\alpha_2} \leqslant \mu_{\alpha_1} + \mu_{\alpha_2} - \mu_{\alpha_1}\mu_{\alpha_2} + (1-\mu_{\alpha_1})(1-\mu_{\alpha_2}) = 1$$

and thus $\dot{\alpha}_1$ is an IFN.

It follows from the operational law (5) in Definition 1.2.2 that

$$0 \leqslant \mu_{\alpha_1}\mu_{\alpha_2} \leqslant 1, \quad 0 \leqslant \nu_{\alpha_1} + \nu_{\alpha_2} - \nu_{\alpha_1}\nu_{\alpha_2} \leqslant 1$$
$$\begin{aligned}\mu_{\alpha_1}\mu_{\alpha_2} + \nu_{\alpha_1} + \nu_{\alpha_2} - \nu_{\alpha_1}\nu_{\alpha_2} &\leqslant (1-\nu_{\alpha_1})(1-\nu_{\alpha_2}) + \nu_{\alpha_1} + \nu_{\alpha_2} - \nu_{\alpha_1}\nu_{\alpha_2} \\ &= (1-\nu_{\alpha_1})(1-\nu_{\alpha_2}) + 1 - (1-\nu_{\alpha_1})(1-\nu_{\alpha_2}) \\ &= 1\end{aligned}$$

hence $\dot{\alpha}_2$ is an IFN.

Since $\alpha = (\mu_\alpha,\ \nu_\alpha)$ is an IFN, by the operational law (6) in Definition 1.2.2, we can obtain

$$1-(1-\mu_\alpha)^{\lambda} \geqslant 0, \quad \nu_\alpha^{\lambda} \geqslant 0$$

and

$$1-(1-\mu_\alpha)^{\lambda} + \nu_\alpha^{\lambda} \leqslant 1-(1-\mu_\alpha)^{\lambda} + (1-\mu_\alpha)^{\lambda} = 1$$

Therefore, $\dot{\alpha}_3$ is also an IFN.

According to the operational law (7) in Definition 1.2.2, we have

$$\mu_\alpha^{\lambda} \geqslant 0, \quad 1-(1-\nu_\alpha)^{\lambda} \geqslant 0$$

and

$$\mu_\alpha^{\lambda} + 1-(1-\nu_\alpha)^{\lambda} \leqslant (1-\nu_\alpha)^{\lambda} + 1-(1-\nu_\alpha)^{\lambda} = 1$$

Thus, $\dot{\alpha}_4$ is also an IFNs. The proof is completed.

In the following, let us look at $\lambda\,\alpha$ and α^{λ} for some special cases of λ and α (Xu, 2007e):

(1) If $\alpha = (\mu_\alpha,\ \nu_\alpha) = (1,\ 0)$, then

$$\lambda\,\alpha = (1-(1-\mu_\alpha)^{\lambda},\ \nu_\alpha^{\lambda}) = (1-(1-1)^{\lambda},\ 0^{\lambda}) = (1,\ 0)$$
$$\alpha^{\lambda} = (\mu_\alpha^{\lambda},\ 1-(1-\nu_\alpha)^{\lambda}) = (1^{\lambda},\ 1-(1-0)^{\lambda}) = (1,\ 0)$$

i.e.,

$$\lambda(1,\ 0) = (1,\ 0), \quad (1,\ 0)^\lambda = (1,\ 0)$$

(2) If $\alpha = (\mu_\alpha,\ \nu_\alpha) = (0,\ 1)$, then

$$\lambda\,\alpha = (1-(1-\mu_\alpha)^\lambda,\ \ \nu_\alpha^\lambda) = (1-(1-0)^\lambda,\ \ 1^\lambda) = (0,\ 1)$$

$$\alpha^\lambda = (\mu_\alpha^\lambda,\ 1-(1-\nu_\alpha)^\lambda) = (0^\lambda,\ 1-(1-1)^\lambda) = (0,\ 1)$$

i.e.,

$$\lambda\,(0,\ 1) = (0,\ 1), \quad (0,\ 1)^\lambda = (0,\ 1)$$

(3) If $\alpha = (\mu_\alpha,\ \nu_\alpha) = (0,\ 0)$, then

$$\lambda\alpha = (1-(1-\mu_\alpha)^\lambda,\ \ \nu_\alpha^\lambda) = (1-(1-0)^\lambda,\ \ 0^\lambda) = (0,\ 0)$$

$$\alpha^\lambda = (\mu_\alpha^\lambda,\ 1-(1-\nu_\alpha)^\lambda) = (0^\lambda,\ 1-(1-0)^\lambda) = (0,\ 0)$$

i.e.,

$$\lambda\,(0,\ 0) = (0,\ 0), \quad (0,\ 0)^\lambda = (0,\ 0)$$

(4) If $\lambda \to 0$ and $0 < \mu_\alpha,\ \nu_\alpha < 1$, then

$$\lambda\,\alpha = (1-(1-\mu_\alpha)^\lambda,\ \ \nu_\alpha^\lambda) \to (0,\ 1)$$

$$\alpha^\lambda = (\mu_\alpha^\lambda,\ 1-(1-\nu_\alpha)^\lambda) \to (1,\ 0)$$

(5) If $\lambda \to +\infty$ and $0 < \mu_\alpha,\ \nu_\alpha < 1$, then

$$\lambda\,\alpha = (1-(1-\mu_\alpha)^\lambda,\ \ \nu_\alpha^\lambda) \to (1,\ 0)$$

$$\alpha^\lambda = (\mu_\alpha^\lambda,\ 1-(1-\nu_\alpha)^\lambda) \to (0,\ 1)$$

(6) If $\lambda = 1$, then

$$\lambda\alpha = (1-(1-\mu_\alpha)^\lambda,\ \ \nu_\alpha^\lambda) = (\mu_\alpha,\ \ \nu_\alpha) = \alpha$$

$$\alpha^\lambda = (\mu_\alpha^\lambda,\ 1-(1-\nu_\alpha)^\lambda) = (\mu_\alpha,\ \nu_\alpha) = \alpha$$

i.e.,

$$\lambda\alpha = \alpha, \quad \alpha^\lambda = \alpha, \quad \lambda = 1$$

Theorem 1.2.3 (Xu, 2007e; Xu and Yager, 2006) Let $\lambda,\ \lambda_1, \lambda_2 > 0$. Then

(1) $\alpha_1 \oplus \alpha_2 = \alpha_2 \oplus \alpha_1$;
(2) $\alpha_1 \otimes \alpha_2 = \alpha_2 \otimes \alpha_1$;
(3) $\lambda(\alpha_1 \oplus \alpha_2) = \lambda\alpha_1 \oplus \lambda\alpha_2$;
(4) $(\alpha_1 \otimes \alpha_2)^\lambda = \alpha_1^\lambda \otimes \alpha_2^\lambda$;

(5) $\lambda_1\alpha \oplus \lambda_2\alpha = (\lambda_1 + \lambda_2)\alpha$;
(6) $\alpha^{\lambda_1} \otimes \alpha^{\lambda_2} = \alpha^{\lambda_1+\lambda_2}$;
(7) $(\alpha_1 \oplus \alpha_2) \oplus \alpha = \alpha_1 \oplus (\alpha_2 \oplus \alpha)$;
(8) $(\alpha^{\lambda_1})^{\lambda_2} = \alpha^{\lambda_1\lambda_2}$.

Proof (1) By the operational law (4) in Definition 1.2.2, we get

$$\begin{aligned}\alpha_1 \oplus \alpha_2 &= (\mu_{\alpha_1} + \mu_{\alpha_2} - \mu_{\alpha_1}\mu_{\alpha_2},\ \ \nu_{\alpha_1}\nu_{\alpha_2})\\ &= (\mu_{\alpha_2} + \mu_{\alpha_1} - \mu_{\alpha_2}\mu_{\alpha_1},\ \ \nu_{\alpha_2}\nu_{\alpha_1})\\ &= \alpha_2 \oplus \alpha_1\end{aligned}$$

(2) From the operational law (5) in Definition 1.2.2, we obtain

$$\begin{aligned}\alpha_1 \otimes \alpha_2 &= (\mu_{\alpha_1}\mu_{\alpha_2},\ \nu_{\alpha_1} + \nu_{\alpha_2} - \nu_{\alpha_1}\nu_{\alpha_2})\\ &= (\mu_{\alpha_2}\mu_{\alpha_1},\ \nu_{\alpha_2} + \nu_{\alpha_1} - \nu_{\alpha_2}\nu_{\alpha_1})\\ &= \alpha_2 \otimes \alpha_1\end{aligned}$$

(3) Using the operational laws (4) and (6) in Definition 1.2.2, we have

$$\begin{aligned}\lambda(\alpha_1 \oplus \alpha_2) &= (1 - (1 - (\mu_{\alpha_1} + \mu_{\alpha_2} - \mu_{\alpha_1}\mu_{\alpha_2}))^{\lambda},\ \ (\nu_{\alpha_1}\nu_{\alpha_2})^{\lambda})\\ &= (1 - (1 - \mu_{\alpha_1})^{\lambda}(1 - \mu_{\alpha_2})^{\lambda},\ \ (\nu_{\alpha_1}\nu_{\alpha_2})^{\lambda})\end{aligned} \tag{1.22}$$

Also since

$$\lambda\,\alpha_1 = (1 - (1 - \mu_{\alpha_1})^{\lambda},\ \ \nu_{\alpha_1}^{\lambda})$$

$$\lambda\,\alpha_2 = (1 - (1 - \mu_{\alpha_2})^{\lambda},\ \ \nu_{\alpha_2}^{\lambda})$$

we can see that

$$\begin{aligned}\lambda\,\alpha_1 \oplus \lambda\alpha_2 &= (1 - (1 - \mu_{\alpha_1})^{\lambda} + 1 - (1 - \mu_{\alpha_2})^{\lambda}\\ &\quad - (1 - (1 - \mu_{\alpha_1})^{\lambda})(1 - (1 - \mu_{\alpha_2})^{\lambda}),\ \ (\nu_{\alpha_1}\nu_{\alpha_2})^{\lambda})\\ &= (2 - (1 - \mu_{\alpha_1})^{\lambda} - (1 - \mu_{\alpha_2})^{\lambda} - (1 - (1 - \mu_{\alpha_1})^{\lambda} - (1 - \mu_{\alpha_2})^{\lambda}\\ &\quad + (1 - \mu_{\alpha_1})^{\lambda}(1 - \mu_{\alpha_2})^{\lambda}),\ \ (\nu_{\alpha_1}\nu_{\alpha_2})^{\lambda})\\ &= (1 - (1 - \mu_{\alpha_1})^{\lambda}(1 - \mu_{\alpha_2})^{\lambda},\ \ (\nu_{\alpha_1}\nu_{\alpha_2})^{\lambda})\end{aligned} \tag{1.23}$$

Combining Eq.(1.22) with Eq.(1.23), we can get

$$\lambda(\alpha_1 \oplus \alpha_2) = \lambda\alpha_1 \oplus \lambda\alpha_2$$

(4) Using the operational laws (5) and (7) in Definition 1.2.2, we have

$$(\alpha_1 \otimes \alpha_2)^{\lambda} = ((\mu_{\alpha_1}\mu_{\alpha_2})^{\lambda},\ \ 1 - (1 - (\nu_{\alpha_1} + \nu_{\alpha_2} - \nu_{\alpha_1}\nu_{\alpha_2}))^{\lambda})$$

$$
\begin{aligned}
&= ((\mu_{\alpha_1}\mu_{\alpha_2})^{\lambda},\ \ 1-((1-\nu_{\alpha_1})(1-\nu_{\alpha_2}))^{\lambda}) \\
&= (\mu_{\alpha_1}^{\lambda}\mu_{\alpha_2}^{\lambda},\ \ 1-(1-\nu_{\alpha_1})^{\lambda}(1-\nu_{\alpha_2})^{\lambda})
\end{aligned} \tag{1.24}
$$

Also since

$$
\alpha_1^{\lambda} = (\mu_{\alpha_1}^{\lambda},\ \ 1-(1-\nu_{\alpha_1})^{\lambda}),\quad \alpha_2^{\lambda} = (\mu_{\alpha_2}^{\lambda},\ \ 1-(1-\nu_{\alpha_2})^{\lambda})
$$

we have

$$
\begin{aligned}
\alpha_1^{\lambda}\otimes\alpha_2^{\lambda} &= (\mu_{\alpha_1}^{\lambda}\mu_{\alpha_2}^{\lambda},\ 1-(1-\nu_{\alpha_1})^{\lambda}+1-(1-\nu_{\alpha_2})^{\lambda} \\
&\quad -(1-(1-(1-\nu_{\alpha_1})^{\lambda})(1-(1-\nu_{\alpha_2})^{\lambda})) \\
&= (\mu_{\alpha_1}^{\lambda}\mu_{\alpha_2}^{\lambda},\ \ 1-(1-\nu_{\alpha_1})^{\lambda}(1-\nu_{\alpha_2})^{\lambda})
\end{aligned} \tag{1.25}
$$

It thus follows from Eqs.(1.23) and (1.24) that

$$
(\alpha_1\otimes\alpha_2)^{\lambda} = \alpha_1^{\lambda}\otimes\alpha_2^{\lambda}
$$

(5) According to the operational law (6) in Definition 1.2.2, we can get

$$
\lambda_1\alpha = (1-(1-\mu_{\alpha})^{\lambda_1},\ \ \nu_{\alpha}^{\lambda_1})
$$

$$
\lambda_2\alpha = (1-(1-\mu_{\alpha})^{\lambda_2},\ \ \nu_{\alpha}^{\lambda_2})
$$

Thus

$$
\begin{aligned}
\lambda_1\alpha\oplus\lambda_2\alpha &= (2-(1-\mu_{\alpha})^{\lambda_1}-(1-\mu_{\alpha})^{\lambda_2} \\
&\quad -(1-(1-\mu_{\alpha})^{\lambda_1})(1-(1-\mu_{\alpha})^{\lambda_2}),\ \ \nu_{\alpha}^{\lambda_1}\nu_{\alpha}^{\lambda_2}) \\
&= (1-(1-\mu_{\alpha})^{\lambda_1}(1-\mu_{\alpha})^{\lambda_2},\ \ (\nu_{\alpha})^{\lambda_1+\lambda_2}) \\
&= (1-(1-\mu_{\alpha})^{\lambda_1+\lambda_2},\ \ (\nu_{\alpha})^{\lambda_1+\lambda_2}) \\
&= (\lambda_1+\lambda_2)\alpha
\end{aligned}
$$

(6) Using the operational law (7) in Definition 1.2.2, we have

$$
\alpha^{\lambda_1} = (\mu_{\alpha}^{\lambda_1},\ \ (1-\nu_{\alpha})^{\lambda_1})
$$

$$
\alpha^{\lambda_2} = (\mu_{\alpha}^{\lambda_2},\ \ (1-\nu_{\alpha})^{\lambda_2})
$$

Then

$$
\begin{aligned}
\alpha^{\lambda_1}\otimes\alpha^{\lambda_2} &= (\mu_{\alpha}^{\lambda_1}\mu_{\alpha}^{\lambda_2},\ \ (1-\nu_{\alpha})^{\lambda_1}+(1-\nu_{\alpha})^{\lambda_2}-(1-\nu_{\alpha})^{\lambda_1}(1-\nu_{\alpha})^{\lambda_2}) \\
&= (\mu_{\alpha}^{\lambda_1}\mu_{\alpha}^{\lambda_2},\ \ 1-(1-\nu_{\alpha})^{\lambda_1}(1-\nu_{\alpha})^{\lambda_2}) \\
&= ((\mu_{\alpha})^{\lambda_1+\lambda_2},\ \ (1-\nu_{\alpha})^{\lambda_1+\lambda_2})
\end{aligned}
$$

$$= \alpha^{\lambda_1+\lambda_2}$$

(7) By the operational law (4) in Definition 1.2.2, we can obtain that

$$\begin{aligned}(\alpha_1 \oplus \alpha_2) \oplus \alpha &= (\mu_{\alpha_1} + \mu_{\alpha_2} - \mu_{\alpha_1}\mu_{\alpha_2},\ \nu_{\alpha_1}\nu_{\alpha_2}) \oplus (\mu_\alpha,\ \nu_\alpha) \\ &= (\mu_{\alpha_1} + \mu_{\alpha_2} - \mu_{\alpha_1}\mu_{\alpha_2} + \mu_\alpha \\ &\quad - (\mu_{\alpha_1} + \mu_{\alpha_2} - \mu_{\alpha_1}\mu_{\alpha_2})\mu_\alpha,\ \ \nu_{\alpha_1}\nu_{\alpha_2}\nu_\alpha) \\ &= (\mu_{\alpha_1} + \mu_{\alpha_2} - \mu_{\alpha_1}\mu_{\alpha_2} + \mu_\alpha \\ &\quad - \mu_{\alpha_1}\mu_\alpha - \mu_{\alpha_2}\mu_\alpha + \mu_{\alpha_1}\mu_{\alpha_2}\mu_\alpha,\ \ \nu_{\alpha_1}\nu_{\alpha_2}\nu_\alpha) \\ \alpha_1 \oplus (\alpha_2 \oplus \alpha) &= (\mu_{\alpha_1},\ \nu_{\alpha_1}) \oplus (\mu_{\alpha_2} + \mu_\alpha - \mu_{\alpha_2}\mu_\alpha,\ \nu_{\alpha_2}\nu_\alpha) \\ &= (\mu_{\alpha_1} + \mu_{\alpha_2} + \mu_\alpha - \mu_{\alpha_2}\mu_\alpha - \mu_{\alpha_1}(\mu_{\alpha_2} \\ &\quad + \mu_\alpha - \mu_{\alpha_2}\mu_\alpha),\ \ \nu_{\alpha_1}\nu_{\alpha_2}\nu_\alpha) \\ &= (\mu_{\alpha_1} + \mu_{\alpha_2} + \mu_\alpha - \mu_{\alpha_2}\mu_\alpha - \mu_{\alpha_1}\mu_{\alpha_2} - \mu_{\alpha_1}\mu_\alpha \\ &\quad + \mu_{\alpha_1}\mu_{\alpha_2}\mu_\alpha,\ \ \nu_{\alpha_1}\nu_{\alpha_2}\nu_\alpha)\end{aligned}$$

and thus $(\alpha_1 \oplus \alpha_2) \oplus \alpha = \alpha_1 \oplus (\alpha_2 \oplus \alpha)$.

(8) According to the operational law (7) in Definition 1.2.2, we can get

$$\begin{aligned}(\alpha^{\lambda_1})^{\lambda_2} &= (\mu_\alpha^{\lambda_1},\ 1 - (1 - \nu_\alpha)^{\lambda_1})^{\lambda_2} \\ &= ((\mu_\alpha^{\lambda_1})^{\lambda_2},\ 1 - (1 - (1 - (1 - \nu_\alpha)^{\lambda_1}))^{\lambda_2}) \\ &= (\mu_\alpha^{\lambda_1\lambda_2},\ 1 - (1 - \nu_\alpha)^{\lambda_1\lambda_2}) \\ &= (\alpha)^{\lambda_1\lambda_2}\end{aligned}$$

which completes the proof.

1.3 Intuitionistic Fuzzy Aggregation Operators

In this section we will introduce, based on Definition 1.2.2, some operators for aggregating intuitionistic fuzzy information:

Definition 1.3.1 (Xu, 2007e) Let $\alpha_j = (\mu_{\alpha_j},\ \nu_{\alpha_j})$ $(j = 1, 2, \cdots, n)$ be a collection of IFNs, and let IFWA : $\Theta^n \to \Theta$. If

$$\text{IFWA}_\omega(\alpha_1, \alpha_2, \cdots, \alpha_n) = \omega_1\alpha_1 \oplus \omega_2\alpha_2 \oplus \cdots \oplus \omega_n\alpha_n \tag{1.26}$$

then the function IFWA is called an intuitionistic fuzzy weighted averaging (IFWA) operator, where $\omega = (\omega_1, \omega_2, \cdots, \omega_n)^{\mathrm{T}}$ is the weight vector of α_j $(j = 1, 2, \cdots, n)$, with $\omega_j \in [0, 1]$ $(j = 1, 2, \cdots, n)$ and $\sum_{j=1}^{n} \omega_j = 1$. In particular, if $\omega = (1/n, 1/n, \cdots, 1/n)^{\mathrm{T}}$, then the IFWA operator reduces to an intuitionistic fuzzy averaging (IFA) operator:

$$\text{IFA}\,(\alpha_1, \alpha_2, \cdots, \alpha_n) = \frac{1}{n}(\alpha_1 \oplus \alpha_2 \oplus \cdots \oplus \alpha_n) \tag{1.27}$$

Definition 1.3.2 (Xu and Yager, 2006) Let IFWG : $\Theta^n \to \Theta$. Then, if

$$\text{IFWG}_\omega(\alpha_1, \alpha_2, \cdots, \alpha_n) = \alpha_1^{\omega_1} \otimes \alpha_2^{\omega_2} \otimes \cdots \otimes \alpha_n^{\omega_n} \tag{1.28}$$

then the function IFWG is called an intuitionistic fuzzy weighted geometric (IFWG) operator, where $\omega = (\omega_1, \omega_2, \cdots, \omega_n)^{\mathrm{T}}$ is the exponential weighting vector of α_j $(j = 1, 2, \cdots, n)$, with $\omega_j \in [0, 1]\,(j = 1, 2, \cdots, n)$ and $\sum_{j=1}^{n} \omega_j = 1$. In particular, if $\omega = (1/n, 1/n, \cdots, 1/n)^{\mathrm{T}}$, then the IFWG operator reduces to an intuitionistic fuzzy geometric (IFG) operator:

$$\text{IFG}\,(\alpha_1, \alpha_2, \cdots, \alpha_n) = (\alpha_1 \otimes \alpha_2 \otimes \cdots \otimes \alpha_n)^{\frac{1}{n}} \tag{1.29}$$

Based on Definitions 1.2.2, 1.2.3, and Theorem 1.2.3, we can get the following result:

Theorem 1.3.1 (Xu, 2007e) The aggregated value by using the IFWA operator is also an IFN, where

$$\text{IFWA}_\omega(\alpha_1, \alpha_2, \cdots, \alpha_n) = \left(1 - \prod_{j=1}^{n} (1 - \mu_{\alpha_j})^{\omega_j},\ \ \prod_{j=1}^{n} \nu_{\alpha_j}^{\omega_j}\right) \tag{1.30}$$

and $\omega = (\omega_1, \omega_2, \cdots, \omega_n)^{\mathrm{T}}$ is the weight vector of α_j $(j = 1, 2, \cdots, n)$, with $\omega_j \in [0,\ 1]\,(j = 1, 2, \cdots, n)$ and $\sum_{j=1}^{n} \omega_j = 1$. In particular, if $\mu_{\alpha_j} = 1 - \nu_{\alpha_j}$ $(j = 1, 2, \cdots, n)$, then Eq.(1.30) reduces to the following form:

$$\text{IFWA}_\omega(\alpha_1, \alpha_2, \cdots, \alpha_n) = \left(1 - \prod_{j=1}^{n} (1 - \mu_{\alpha_j})^{\omega_j},\ \ \prod_{j=1}^{n} (1 - \mu_{\alpha_j})^{\omega_j}\right) \tag{1.31}$$

Proof We prove Eq.(1.30) by using mathematical induction on n.

(1) When $n = 2$, we have

$$\text{IFWA}_{\,\omega}(\alpha_1, \alpha_2) = \omega_1\alpha_1 \oplus \omega_2\alpha_2$$

By Theorem 1.2.2, we can see that both $\omega_1\alpha_1$ and $\omega_2\alpha_2$ are IFNs, and the value of $\omega_1\alpha_1 \oplus \omega_2\alpha_2$ is also an IFN. From the operational law (6) in Definition 1.2.2, we have

$$\omega_1\alpha_1 = (1 - (1 - \mu_{\alpha_1})^{\omega_1},\ \ \nu_{\alpha_1}^{\omega_1})$$

$$\omega_2\alpha_2 = (1-(1-\mu_{\alpha_2})^{\omega_2},\ \ \nu_{\alpha_2}^{\omega_2})$$

Then

$$\begin{aligned}\text{IFWA}_\omega(\alpha_1,\alpha_2) &= \omega_1\alpha_1 \oplus \omega_2\alpha_2 \\ &= (2-(1-\mu_{\alpha_1})^{\omega_1} - (1-\nu_{\alpha_2})^{\omega_2} - (1-(1-\mu_{\alpha_1})^{\omega_1}) \\ &\quad \cdot (1-(1-\mu_{\alpha_2})^{\omega_2}),\ \nu_{\alpha_1}^{\omega_1}\nu_{\alpha_2}^{\omega_2}) \\ &= (1-(1-\mu_{\alpha_1})^{\omega_1}(1-\mu_{\alpha_2})^{\omega_2},\ \ \nu_{\alpha_1}^{\omega_1}\nu_{\alpha_2}^{\omega_2})\end{aligned}$$

(2) Suppose that $n=k$, Eq.(1.30) holds, i.e.,

$$\begin{aligned}\text{IFWA}_\omega(\alpha_1,\alpha_2,\cdots,\alpha_k) &= \omega_1\alpha_1 \oplus \omega_2\alpha_2 \oplus \cdots \oplus \omega_k\alpha_k \\ &= \left(1-\prod_{j=1}^{k}(1-\mu_{\alpha_j})^{\omega_j},\ \ \prod_{j=1}^{k}\nu_{\alpha_j}^{\omega_j}\right)\end{aligned}$$

and the aggregated value is an IFN. Then when $n=k+1$, by the operational laws (4) and (6) in Definition 1.2.2, we have

$$\begin{aligned}\text{IFWA}_\omega(\alpha_1,\alpha_2,\cdots,\alpha_{k+1}) &= \omega_1\alpha_1 \oplus \omega_2\alpha_2 \oplus \cdots \oplus \omega_k\alpha_k \oplus \omega_{k+1}\alpha_{k+1} \\ &= (\omega_1\alpha_1 \oplus \omega_2\alpha_2 \oplus \cdots \oplus \omega_k\alpha_k) \oplus \omega_{k+1}\alpha_{k+1} \\ &= \left(1-\prod_{j=1}^{k}(1-\mu_{\alpha_j})^{\omega_j} + (1-(1-\mu_{\alpha_{k+1}})^{\omega_{k+1}})\right. \\ &\quad \left. -(1-\prod_{j=1}^{k}(1-\mu_{\alpha_j})^{\omega_j})(1-(1-\mu_{\alpha_{k+1}})^{\omega_{k+1}}),\ \ \prod_{j=1}^{k+1}\nu_{\alpha_j}^{\omega_j}\right) \\ &= \left(1-\prod_{j=1}^{k+1}(1-\mu_{\alpha_j})^{\omega_j},\ \ \prod_{j=1}^{k+1}\nu_{\alpha_j}^{\omega_j}\right)\end{aligned}$$

by which the aggregated value is also an IFN. Therefore, when $n=k+1$, Eq.(1.30) holds.

Thus, by (1) and (2), we know that Eq.(1.30) holds for all n. The proof is completed.

Theorem 1.3.2 (Xu and Yager, 2006) The aggregated value by using the IFWG operator is also an IFN, where

$$\text{IFWG}_\omega(\alpha_1,\alpha_2,\cdots,\alpha_n) = \left(\prod_{j=1}^{n}\mu_{\alpha_j}^{\omega_j},\ \ 1-\prod_{j=1}^{n}(1-\nu_{\alpha_j})^{\omega_j}\right) \tag{1.32}$$

and $\omega=(\omega_1,\omega_2,\cdots,\omega_n)^{\mathrm{T}}$ is the exponential weighting vector of α_j $(j=1,2,\cdots,n)$, with $\omega_j \in [0,\ 1]\,(j = 1, 2, \cdots, n)$ and $\sum_{j=1}^{n} \omega_j = 1$. In particular, if $\mu_{\alpha_j} = 1 - \nu_{\alpha_j}$ $(j = 1, 2, \cdots, n)$, then Eq.(1.32) reduces to the following form:

$$\text{IFWG}_\omega(\alpha_1, \alpha_2, \cdots, \alpha_n) = \left(\prod_{j=1}^{n} \mu_{\alpha_j}^{\omega_j},\ \ 1 - \prod_{j=1}^{n} \mu_{\alpha_j}^{\omega_j} \right) \tag{1.33}$$

Proof We prove Eq.(1.32) by using mathematical induction on n.

(1) When $n = 2$, we have

$$\text{IFWG}_\omega(\alpha_1, \alpha_2) = \alpha_1^{\omega_1} \otimes \alpha_2^{\omega_2}$$

According to Theorem 1.2.2, we can see that both $\alpha_1^{\omega_1}$ and $\alpha_2^{\omega_2}$ are IFNs, and the value of $\alpha_1^{\omega_1} \otimes \alpha_2^{\omega_2}$ is also an IFN. It follows from the operational law (7) in Definition 1.2.2 that

$$\alpha_1^{\omega_1} = (\mu_{\alpha_1}^{\omega_1},\ 1 - (1 - \nu_{\alpha_1})^{\omega_1})$$
$$\alpha_2^{\omega_2} = (\mu_{\alpha_2}^{\omega_2},\ 1 - (1 - \nu_{\alpha_2})^{\omega_2})$$

Then

$$\begin{aligned}\text{IFWG}_\omega(\alpha_1, \alpha_2) &= \alpha_1^{\omega_1} \otimes \alpha_2^{\omega_2} \\ &= (\mu_{\alpha_1}^{\omega_1}\mu_{\alpha_2}^{\omega_2},\ 1 - (1 - \nu_{\alpha_1})^{\omega_1} \\ &\quad + 1 - (1 - \nu_{\alpha_2})^{\omega_2} - (1 - (1 - \nu_{\alpha_1})^{\omega_1})(1 - (1 - \nu_{\alpha_2})^{\omega_2}) \\ &= (\mu_{\alpha_1}^{\omega_1}\mu_{\alpha_2}^{\omega_2},\ \ 1 - (1 - \nu_{\alpha_1})^{\omega_1}(1 - \nu_{\alpha_2})^{\omega_2})\end{aligned}$$

(2) Suppose that $n = k$, Eq.(1.32) holds, i.e.,

$$\begin{aligned}\text{IFWG}_\omega(\alpha_1, \alpha_2, \cdots, \alpha_k) &= \alpha_1^{\omega_1} \otimes \alpha_2^{\omega_2} \otimes \cdots \otimes \alpha_k^{\omega_k} \\ &= \left(\prod_{j=1}^{k} \mu_{\alpha_j}^{\omega_j},\ 1 - \prod_{j=1}^{k} (1 - \nu_{\alpha_j})^{\omega_j} \right)\end{aligned}$$

and the aggregated value is an IFN. Then when $n = k + 1$, by the operational laws (5) and (7) in Definition 1.2.2, we get

$$\begin{aligned}&\text{IFWG}_\omega(\alpha_1, \alpha_2, \cdots, \alpha_{k+1}) \\ &= \alpha_1^{\omega_1} \otimes \alpha_2^{\omega_2} \otimes \cdots \otimes \alpha_k^{\omega_k} \otimes \alpha_{k+1}^{\omega_{k+1}} \\ &= (\alpha_1^{\omega_1} \otimes \alpha_2^{\omega_2} \otimes \cdots \otimes \alpha_k^{\omega_k}) \otimes \alpha_{k+1}^{\omega_{k+1}} \\ &= \left(\prod_{j=1}^{k+1} \mu_{\alpha_j}^{\omega_j},\ \ 1 - \prod_{j=1}^{k} (1 - \nu_{\alpha_j})^{\omega_j} + (1 - (1 - \nu_{\alpha_{k+1}})^{\omega_{k+1}}) \right.\end{aligned}$$

$$- \left(1 - \prod_{j=1}^{k} (1-\nu_{\alpha_j})^{\omega_j}\right)(1-(1-\nu_{\alpha_{k+1}})^{\omega_{k+1}})\Bigg)$$
$$= \left(\prod_{j=1}^{k+1} \mu_{\alpha_j}^{\omega_j},\ 1 - \prod_{j=1}^{k+1} (1-\nu_{\alpha_j})^{\omega_j}\right)$$

by which the aggregated value is also IFN. Thus, when $n = k+1$, Eq.(1.32) holds.

Therefore, based on (1) and (2), we know that Eq.(1.32) holds for any n. The proof is completed.

Example 1.3.1 Let $\alpha_1 = (0.3,\ 0.5)$, $\alpha_2 = (0.2,\ 0.6)$, $\alpha_3 = (0.7,\ 0.2)$, and $\alpha_4 = (0.4,\ 0.3)$ be IFNs, and $\omega = (0.3,\ 0.4,\ 0.2,\ 0.1)^{\mathrm{T}}$ the weight vector of α_j $(j = 1, 2, 3, 4)$. Then

$$\begin{aligned}
\mathrm{IFWA}_\omega(\alpha_1,\alpha_2,\alpha_3,\alpha_4) &= \left(1 - \prod_{j=1}^{4} (1-\mu_{\alpha_j})^{\omega_j},\ \prod_{j=1}^{4} \nu_{\alpha_j}^{\omega_j}\right) \\
&= (1-(1-0.3)^{0.3} \times (1-0.2)^{0.4} \times (1-0.7)^{0.2} \times (1-0.4)^{0.1}, \\
&\quad 0.5^{0.3} \times 0.6^{0.4} \times 0.2^{0.2} \times 0.3^{0.1}) \\
&= (0.386,\ 0.425) \\
\mathrm{IFWG}_\omega(\alpha_1,\alpha_2,\alpha_3,\alpha_4) &= \left(\prod_{j=1}^{4} \mu_{\alpha_j}^{\omega_j},\ 1 - \prod_{j=1}^{4} (1-\nu_{\alpha_j})^{\omega_j}\right) \\
&= (0.3^{0.3} \times 0.2^{0.4} \times 0.7^{0.2} \times 0.4^{0.1}, \\
&\quad 1-(1-0.5)^{0.3} \times (1-0.6)^{0.4} \times (1-0.2)^{0.2} \times (1-0.3)^{0.1}) \\
&= (0.311,\ 0.480)
\end{aligned}$$

The IFWA operator has the following properties:

Theorem 1.3.3 (Xu, 2007e)(Idempotency) If all the IFNs α_j $(j = 1, 2, \cdots, n)$ are equal, i.e., $\alpha_j = \alpha$, $j = 1, 2, \cdots, n$, then

$$\mathrm{IFWA}_\omega(\alpha_1, \alpha_2, \cdots, \alpha_n) = \alpha \tag{1.34}$$

Proof Let $\alpha = (\mu_\alpha,\ \nu_\alpha)$. Then Eq.(1.30) and $\alpha_j = \alpha$ $(j = 1, 2, \cdots, n)$ yield

$$\begin{aligned}
\mathrm{IFWA}_\omega(\alpha_1,\alpha_2,\cdots,\alpha_n) &= \left(1 - \prod_{j=1}^{n} (1-\mu_{\alpha_j})^{\omega_j},\ \prod_{j=1}^{n} \nu_{\alpha_j}^{\omega_j}\right) \\
&= \left(1 - \prod_{j=1}^{n} (1-\mu_{\alpha})^{\omega_j},\ \prod_{j=1}^{n} \nu_{\alpha}^{\omega_j}\right)
\end{aligned}$$

$$= \left(1-(1-\mu_{\alpha})^{\sum\limits_{j=1}^{n}\omega_j},\ \ (\nu_{\alpha})^{\sum\limits_{j=1}^{n}\omega_j}\right)$$
$$= (\mu_{\alpha},\ \nu_{\alpha}) = \alpha$$

which completes the proof.

Theorem 1.3.4 (Xu, 2007e)(Boundedness) Let

$$\alpha^- = \left(\min_j\{\mu_{\alpha_j}\},\ \max_j\{\nu_{\alpha_j}\}\right), \quad \alpha^+ = \left(\max_j\{\mu_{\alpha_j}\},\ \min_j\{\nu_{\alpha_j}\}\right)$$

Then

$$\alpha^- \leqslant \mathrm{IFWA}_{\omega}(\alpha_1, \alpha_2, \cdots, \alpha_n) \leqslant \alpha^+ \tag{1.35}$$

Proof Since for any j, we have

$$\min_j\{\mu_{\alpha_j}\} \leqslant \mu_{\alpha_j} \leqslant \max_j\{\mu_{\alpha_j}\}, \quad \min_j\{\nu_{\alpha_j}\} \leqslant \nu_{\alpha_j} \leqslant \max_j\{\nu_{\alpha_j}\}$$

i.e.,

$$\begin{aligned} 1-\prod_{j=1}^{n}(1-\mu_{\alpha_j})^{\omega_j} &\geqslant 1-\prod_{j=1}^{n}\left(1-\min_j\{\mu_{\alpha_j}\}\right)^{\omega_j} \\ &= 1-\left(1-\min_j\{\mu_{\alpha_j}\}\right)^{\sum\limits_{j=1}^{n}\omega_j} \\ &= \min_j\{\mu_{\alpha_j}\} \end{aligned} \tag{1.36}$$

$$\prod_{j=1}^{n}\nu_{\alpha_j}^{\omega_j} \geqslant \prod_{j=1}^{n}\left(\min_j\{\nu_{\alpha_j}\}\right)^{\omega_j} = \left(\min_j\{\nu_{\alpha_j}\}\right)^{\sum\limits_{j=1}^{n}\omega_j} = \min_j\{\nu_{\alpha_j}\} \tag{1.37}$$

$$\begin{aligned} 1-\prod_{j=1}^{n}(1-\mu_{\alpha_j})^{\omega_j} &\leqslant 1-\prod_{j=1}^{n}\left(1-\max_j\{\mu_{\alpha_j}\}\right)^{\omega_j} \\ &= 1-\left(1-\max_j\{\mu_{\alpha_j}\}\right)^{\sum\limits_{j=1}^{n}\omega_j} \\ &= \max_j\{\mu_{\alpha_j}\} \end{aligned} \tag{1.38}$$

$$\prod_{j=1}^{n}\nu_{\alpha_j}^{\omega_j} \leqslant \prod_{j=1}^{n}\left(\max_j\{\nu_{\alpha_j}\}\right)^{\omega_j} = \left(\max_j\{\nu_{\alpha_j}\}\right)^{\sum\limits_{j=1}^{n}\omega_j} = \max_j\{\nu_{\alpha_j}\} \tag{1.39}$$

Let $\mathrm{IFWA}_{\omega}(\alpha_1, \alpha_2, \cdots, \alpha_n) = (\mu_{\alpha},\ \nu_{\alpha})$. Then

$$s(\alpha) = \mu_{\alpha} - \nu_{\alpha} \leqslant \max_j\{\mu_{\alpha_j}\} - \min_j\{\nu_{\alpha_j}\} = s(\alpha^+)$$

$$s(\alpha) = \mu_\alpha - \nu_\alpha \geqslant \min_j\{\mu_{\alpha_j}\} - \max_j\{\nu_{\alpha_j}\} = s(\alpha^-)$$

In what follows, we discuss three cases:

(1) $s(\alpha) < s(\alpha^+)$ and $s(\alpha) > s(\alpha^-)$: It follows from Definition 1.1.3 that Eq.(1.35) holds.

(2) $s(\alpha) = s(\alpha^+)$, i.e., $\mu_\alpha - \nu_\alpha = \max_j\{\mu_{\alpha_j}\} - \min_j\{\nu_{\alpha_j}\}$: By Eqs.(1.36) and (1.37), we have

$$\mu_\alpha = \max_j\{\mu_{\alpha_j}\}, \quad \nu_\alpha = \min_j\{\nu_{\alpha_j}\}$$

Thus

$$h(\alpha) = \mu_\alpha + \nu_\alpha = \max_j\{\mu_{\alpha_j}\} + \min_j\{\nu_{\alpha_j}\} = h(\alpha^+)$$

In this case, according to Definition 1.1.3, we have

$$\text{IFWA}_\omega(\alpha_1, \alpha_2, \cdots, \alpha_n) = \alpha^+ \tag{1.40}$$

(3) $s(\alpha) = s(\alpha^-)$, i.e., $\mu_\alpha - \nu_\alpha = \min_j\{\mu_{\alpha_j}\} - \max_j\{\nu_{\alpha_j}\}$: By Eqs.(1.36) and (1.39), one can obtain

$$\mu_\alpha = \min_j\{\mu_{\alpha_j}\}, \quad \nu_\alpha = \max_j\{\nu_{\alpha_j}\}$$

Hence

$$h(\alpha) = \mu_\alpha + \nu_\alpha = \min_j\{\mu_{\alpha_j}\} + \max_j\{\nu_{\alpha_j}\} = h(\alpha^-)$$

In this case, according to Definition 1.1.3, we have

$$\text{IFWG}_\omega(\alpha_1, \alpha_2, \cdots, \alpha_n) = \alpha^- \tag{1.41}$$

Therefore, by (1)-(3), we can see that Eq.(1.35) must hold. This completes the proof.

Theorem 1.3.5 (Xu, 2007e)(monotonicity) Let $\alpha_j = (\mu_{\alpha_j}, \nu_{\alpha_j})$ $(j = 1, 2, \cdots, n)$ and $\alpha_j^* = (\mu_{\alpha_j^*}, \nu_{\alpha_j^*})$ $(j = 1, 2, \cdots, n)$ be two collections of IFNs, if $\mu_{\alpha_j} \leqslant \mu_{\alpha_j^*}$ and $\nu_{\alpha_j} \geqslant \nu_{\alpha_j^*}$, for any j. Then

$$\text{IFWA}_\omega(\alpha_1, \alpha_2, \cdots, \alpha_n) \leqslant \text{IFWA}_\omega(\alpha_1^*, \alpha_2^*, \cdots, \alpha_n^*) \tag{1.42}$$

Proof Since $\mu_\alpha \leqslant \mu_{\alpha_j^*}$ and $\nu_{\alpha_j} \geqslant \nu_{\alpha_j^*}$ for any j, we have

$$(1 - \mu_{\alpha_j})^{\omega_j} \geqslant (1 - \mu_{\alpha_j^*})^{\omega_j}, \quad \nu_{\alpha_j}^{\omega_j} \geqslant \nu_{\alpha_j^*}^{\omega_j}$$

Moreover,

$$1 - \prod_{j=1}^{n} (1 - \mu_{\alpha_j})^{\omega_j} \leqslant 1 - \prod_{j=1}^{n} (1 - \mu_{\alpha_j^*})^{\omega_j}, \quad \prod_{j=1}^{n} \nu_{\alpha_j}^{\omega_j} \geqslant \prod_{j=1}^{n} \nu_{\alpha_j^*}^{\omega_j}$$

Hence

$$1-\prod_{j=1}^{n}(1-\mu_{\alpha_j})^{\omega_j}-\prod_{j=1}^{n}\nu_{\alpha_j}^{\omega_j}\leqslant 1-\prod_{j=1}^{n}(1-\mu_{\alpha_j^*})^{\omega_j}-\prod_{j=1}^{n}\nu_{\alpha_j^*}^{\omega_j} \tag{1.43}$$

If $\alpha=\text{IFWA}_\omega(\alpha_1,\alpha_2,\cdots,\alpha_n)$ and $\alpha^*=\text{IFWA}_\omega(\alpha_1^*,\alpha_2^*,\cdots,\alpha_n^*)$, then it follows from Eq.(1.43) that

$$s(\alpha)\leqslant s(\alpha^*)$$

If $s(\alpha)<s(\alpha^*)$, then by Definition 1.1.3, we get

$$\text{IFWA}_\omega(\alpha_1,\alpha_2,\cdots,\alpha_n)<\text{IFWA}_\omega(\alpha_1^*,\alpha_2^*,\cdots,\alpha_n^*) \tag{1.44}$$

If $s(\alpha)=s(\alpha^*)$, then

$$1-\prod_{j=1}^{n}(1-\mu_{\alpha_j})^{\omega_j}-\prod_{j=1}^{n}\nu_{\alpha_j}^{\omega_j}=1-\prod_{j=1}^{n}(1-\mu_{\alpha_j^*})^{\omega_j}-\prod_{j=1}^{n}\nu_{\alpha_j^*}^{\omega_j}$$

Therefore, by the conditions $\mu_{\alpha_j}\leqslant\mu_{\alpha_j^*}$ and $\nu_{\alpha_j}\geqslant\nu_{\alpha_j^*}$, for any j, we have

$$1-\prod_{j=1}^{n}(1-\mu_{\alpha_j})^{\omega_j}=1-\prod_{j=1}^{n}(1-\mu_{\alpha_j^*})^{\omega_j}$$

$$\prod_{j=1}^{n}\nu_{\alpha_j}^{\omega_j}=\prod_{j=1}^{n}\nu_{\alpha_j^*}^{\omega_j}$$

Hence

$$\begin{aligned}h(\alpha)&=1-\prod_{j=1}^{n}(1-\mu_{\alpha_j})^{\omega_j}+\prod_{j=1}^{n}\nu_{\alpha_j}^{\omega_j}\\&=1-\prod_{j=1}^{n}(1-\mu_{\alpha_j^*})^{\omega_j}+\prod_{j=1}^{n}\nu_{\alpha_j^*}^{\omega_j}=h(\alpha^*)\end{aligned} \tag{1.45}$$

Based on Eqs.(1.44) and (1.45), we can see that Eq.(1.42) must hold. This completes the proof of the theorem.

Similarly, the IFWG operator also has the above properties:

Theorem 1.3.6 (Xu, 2007e)

(1) **(Idempotency).** If all IFSs α_j $(j=1,2,\cdots,n)$ are equal, i.e., $\alpha_j=\alpha$ $(j=1,2,\cdots,n)$, then

$$\text{IFWG}_\omega(\alpha_1,\alpha_2,\cdots,\alpha_n)=\alpha \tag{1.46}$$

(2) **(Boundedness).** For any ω, we have

$$\alpha^-\leqslant\text{IFWG}_\omega(\alpha_1,\alpha_2,\cdots,\alpha_n)\leqslant\alpha^+ \tag{1.47}$$

where

$$\alpha^- = \left(\min_j \{\mu_{\alpha_j}\},\ \max_j \{\nu_{\alpha_j}\}\right), \quad \alpha^+ = \left(\max_j \{\mu_{\alpha_j}\},\ \min_j \{\nu_{\alpha_j}\}\right)$$

(3) **(Monotonicity).** Let $\alpha_j^* = (\mu_{\alpha_j^*},\ \nu_{\alpha_j^*})$ $(j = 1, 2, \cdots, n)$ be a collection of IFNs, if for any j, $\mu_{\alpha_j} \leqslant \mu_{\alpha_j^*}$ and $\nu_{\alpha_j} \geqslant \nu_{\alpha_j^*}$. Then

$$\text{IFWG}_\omega(\alpha_1, \alpha_2, \cdots, \alpha_n) \leqslant \text{IFWG}_\omega(\alpha_1^*, \alpha_2^*, \cdots, \alpha_n^*) \tag{1.48}$$

Based on Definitions 1.1.3 and 1.2.2, we give an intuitionistic fuzzy ordered weighted averaging operator below:

Definition 1.3.3 (Xu, 2007e) An intuitionistic fuzzy ordered weighted averaging (IFOWA) operator is a mapping IFOWA : $\Theta^n \to \Theta$, such that $w = (w_1, w_2, \cdots, w_n)^{\mathrm{T}}$, with $w_j \in [0,\ 1]$ $(j = 1, 2, \cdots, n)$ and $\sum_{j=1}^{n} w_j = 1$, and

$$\text{IFOWA}_w(\alpha_1, \alpha_2, \cdots, \alpha_n) = w_1\alpha_{\sigma(1)} \oplus w_2\alpha_{\sigma(2)} \oplus \cdots \oplus w_n\alpha_{\sigma(n)} \tag{1.49}$$

where $(\sigma(1), \sigma(2), \cdots, \sigma(n))$ is a permutation of $(1, 2, \cdots, n)$, such that $\alpha_{\sigma(j-1)} \geqslant \alpha_{\sigma(j)}$, for any j. In particular, if $w = (1/n, 1/n, \cdots, 1/n)^{\mathrm{T}}$, then the IFOWA operator reduces to an intuitionistic fuzzy averaging (IFA) operator.

Similar to Theorem 1.3.1, we can get the following result:

Theorem 1.3.7 (Xu, 2007e) The aggregated value by using the IFOWA operator is also an IFN, where

$$\text{IFOWA}_w(\alpha_1, \alpha_2, \cdots, \alpha_n) = \left(1 - \prod_{j=1}^{n}(1 - \mu_{\alpha_{\sigma(j)}})^{w_j},\ \prod_{j=1}^{n} \nu_{\alpha_{\sigma(j)}}^{w_j}\right) \tag{1.50}$$

and $w = (w_1, w_2, \cdots, w_n)^{\mathrm{T}}$ is the weighting vector associated with the IFOWA operator, with $w_j \in [0,\ 1]$, $j = 1, 2, \cdots, n$, and $\sum_{j=1}^{n} w_j = 1$. In particular, if $\mu_{\alpha_j} + \nu_{\alpha_j} = 1$, $j = 1, 2, \cdots, n$, then Eq.(1.50) reduces to the following:

$$\text{IFOWA}_w(\alpha_1, \alpha_2, \cdots, \alpha_n) = \left(1 - \prod_{j=1}^{n}(1 - \mu_{\alpha_{\sigma(j)}})^{w_j},\ \prod_{j=1}^{n}(1 - \mu_{\alpha_{\sigma(j)}})^{w_j}\right) \tag{1.51}$$

Definition 1.3.4 (Xu and Yager, 2006) An intuitionistic fuzzy ordered weighted geometric (IFOWG) operator is a mapping IFOWG : $\Theta^n \to \Theta$, such that

$$\text{IFOWG}_w(\alpha_1, \alpha_2, \cdots, \alpha_n) = \alpha_{\sigma(1)}^{w_1} \otimes \alpha_{\sigma(2)}^{w_2} \otimes \cdots \otimes \alpha_{\sigma(n)}^{w_n} \tag{1.52}$$

where $(\sigma(1), \sigma(2), \cdots, \sigma(n))$ is a permutation of $(1, 2, \cdots, n)$, such that $\alpha_{\sigma(j-1)} \geqslant \alpha_{\sigma(j)}$, for any j. In particular, if $w = (1/n, 1/n, \cdots, 1/n)^{\mathrm{T}}$, then the IFOWG operator reduces to an intuitionistic fuzzy geometric (IFG) operator.

Similar to Theorem 1.3.2, we have

Theorem 1.3.8 (Xu and Yager, 2006) The aggregated value by using the IFOWG operator is also an IFN, where

$$\mathrm{IFOWG}_w(\alpha_1, \alpha_2, \cdots, \alpha_n) = \left(\prod_{j=1}^{n} \mu_{\alpha_{\sigma(j)}}^{w_j},\ 1 - \prod_{j=1}^{n} (1 - \nu_{\alpha_{\sigma(j)}})^{w_j} \right) \tag{1.53}$$

In particular, if $\mu_{\alpha_j} + \nu_{\alpha_j} = 1$, $j = 1, 2, \cdots, n$, then Eq.(1.53) reduces to:

$$\mathrm{IFOWG}_w(\alpha_1, \alpha_2, \cdots, \alpha_n) = \left(\prod_{j=1}^{n} \mu_{\alpha_{\sigma(j)}}^{w_j},\ 1 - \prod_{j=1}^{n} \mu_{\alpha_{\sigma(j)}}^{w_j} \right) \tag{1.54}$$

The weighting vector associated with the IFOWA and IFOWG operators can be determined similar to that of the OWA operator (Xu(2005a) gives an overview of methods for determining OWA weights). For example, we can use the normal distribution based method to determine the IFOWA weights. The prominent characteristic of the method is that it can relieve the influence of unfair arguments on the decision results by assigning low weights to those "false" or "biased" ones.

Example 1.3.2 Let $\alpha_1 = (0.5,\ 0.3)$, $\alpha_2 = (0.1,\ 0.4)$, $\alpha_3 = (0.8,\ 0.1)$ and $\alpha_4 = (0.3,\ 0.4)$ be IFNs. Then by Eq.(1.10), we can calculate the scores of α_j $(j = 1, 2, 3, 4)$:

$$s(\alpha_1) = 0.5 - 0.3 = 0.2, \quad s(\alpha_2) = 0.1 - 0.4 = -0.3$$

$$s(\alpha_3) = 0.8 - 0.1 = 0.7, \quad s(\alpha_4) = 0.3 - 0.4 = -0.1$$

Since

$$s(\alpha_3) > s(\alpha_1) > s(\alpha_4) > s(\alpha_2)$$

we have

$$\alpha_{\sigma(1)} = (0.8,\ 0.1), \quad \alpha_{\sigma(2)} = (0.5,\ 0.3)$$

$$\alpha_{\sigma(3)} = (0.3,\ 0.4), \quad \alpha_{\sigma(4)} = (0.1,\ 0.4)$$

Suppose that $w = (0.155, 0.345, 0.345, 0.155)^{\mathrm{T}}$(obtained by the normal distribution based method (Xu, 2005a)) is the weighting vector associated with the IFOWA and FOWG operators. Then by Eqs.(1.50) and (1.53), we get

$$\mathrm{IFOWA}_w(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \left(1 - \prod_{j=1}^{4} (1 - \mu_{\alpha_{\sigma(j)}})^{w_j},\ \prod_{j=1}^{4} \nu_{\alpha_{\sigma(j)}}^{w_j} \right)$$

$$= (1 - 0.2^{0.155} \times 0.5^{0.345} \times 0.7^{0.345} \times 0.9^{0.155},\ 0.1^{0.155} \times 0.3^{0.345} \times 0.4^{0.345} \times 0.4^{0.155})$$
$$= (0.466,\ 0.292)$$

$$\text{IFOWG}_w(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \left(\prod_{j=1}^{4} \mu_{\alpha_{\sigma(j)}}^{w_j},\ 1 - \prod_{j=1}^{4} (1 - \nu_{\alpha_{\sigma(j)}})^{w_j} \right)$$
$$= (0.8^{0.155} \times 0.5^{0.345} \times 0.3^{0.345} \times 0.1^{0.155}, 1 - (1 - 0.1)^{0.155} \times (1 - 0.3)^{0.345} \times (1 - 0.4)^{0.345} \times (1 - 0.4)^{0.155})$$
$$= (0.351,\ 0.326)$$

In this example, if we replace $\alpha_4 = (0.3,\ 0.4)$ by $\alpha_4 = (0.3,\ 0.1)$, then

$$s(\alpha_4) = 0.3 - 0.1 = 0.2$$

which is the same as $s(\alpha_1)$. Thus, we need to calculate the accuracy degrees of α_1 and α_4:

$$h(\alpha_1) = 0.5 + 0.3 = 0.8, \quad h(\alpha_4) = 0.3 + 0.1 = 0.4$$

Since $h(\alpha_1) > h(\alpha_4)$, then by Definition 1.1.3, we have $\alpha_1 > \alpha_4$. Therefore

$$\alpha_{\sigma(1)} = (0.8,\ 0.1), \quad \alpha_{\sigma(2)} = (0.5,\ 0.3)$$

$$\alpha_{\sigma(3)} = (0.3,\ 0.1), \quad \alpha_{\sigma(4)} = (0.1,\ 0.4)$$

Then

$$\text{IFOWA}_w(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \left(1 - \prod_{j=1}^{4} (1 - \mu_{\alpha_{\sigma(j)}})^{w_j},\ \prod_{j=1}^{4} \nu_{\alpha_{\sigma(j)}}^{w_j} \right)$$
$$= (1 - 0.2^{0.155} \times 0.5^{0.345} \times 0.7^{0.345} \times 0.9^{0.155},\ 0.1^{0.155} \times 0.3^{0.345} \times 0.1^{0.345} \times 0.4^{0.155})$$
$$= (0.466,\ 0.191)$$

$$\text{IFOWG}_w(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \left(\prod_{j=1}^{4} \mu_{\alpha_{\sigma(j)}}^{w_j},\ 1 - \prod_{j=1}^{4} (1 - \nu_{\alpha_{\sigma(j)}})^{w_j} \right)$$
$$= (0.8^{0.155} \times 0.5^{0.345} \times 0.3^{0.345} \times 0.1^{0.155},\ 1 - (1 - 0.1)^{0.155} \times (1 - 0.3)^{0.345} \times (1 - 0.1)^{0.345} \times (1 - 0.4)^{0.155}$$
$$= (0.351,\ 0.225)$$

The IFOWA and IFOWG operators have some desirable properties similar to the IFWA and IFWG operators:

Theorem 1.3.9 (Xu, 2007e; Xu and Yager, 2006) If all IFNs α_j $(j = 1, 2, \cdots, n)$ are equal, i.e., $\alpha_j = \alpha$, for any j, then

$$\text{IFOWA}_w(\alpha_1, \alpha_2, \cdots, \alpha_n) = \alpha, \quad \text{IFOWG}_w(\alpha_1, \alpha_2, \cdots, \alpha_n) = \alpha$$

Theorem 1.3.10 (Xu, 2007e; Xu and Yager, 2006) Let

$$\alpha^- = \left(\min_j \{\mu_{\alpha_j}\},\ \max_j \{\nu_{\alpha_j}\}\right), \quad \alpha^+ = \left(\max_j \{\mu_{\alpha_j}\},\ \min_j \{\nu_{\alpha_j}\}\right)$$

Then

$$\alpha^- \leqslant \text{IFOWA}_w(\alpha_1, \alpha_2, \cdots, \alpha_n) \leqslant \alpha^+$$
$$\alpha^- \leqslant \text{IFOWG}_w(\alpha_1, \alpha_2, \cdots, \alpha_n) \leqslant \alpha^+$$

Theorem 1.3.11 (Xu, 2007e; Xu and Yager, 2006) Let $\alpha_j = (\mu_{\alpha_j},\ \nu_{\alpha_j})$ $(j = 1, 2, \cdots, n)$ and $\alpha_j^* = (\mu_{\alpha_j^*},\ \nu_{\alpha_j^*})$ $(j = 1, 2, \cdots, n)$ be two collections of IFNs. If $\mu_{\alpha_j} \leqslant \mu_{\alpha_j^*}$ and $\nu_{\alpha_j} \geqslant \nu_{\alpha_j^*}$, for any j, then

$$\text{IFOWA}_w(\alpha_1, \alpha_2, \cdots, \alpha_n) \leqslant \text{IFOWA}_w(\alpha_1^*, \alpha_2^*, \cdots, \alpha_n^*)$$
$$\text{IFOWG}_w(\alpha_1, \alpha_2, \cdots, \alpha_n) \leqslant \text{IFOWG}_w(\alpha_1^*, \alpha_2^*, \cdots, \alpha_n^*)$$

In addition to the properties described as above, the IFOWA and IFOWG operators also have the following desirable properties:

Theorem 1.3.12 (Xu, 2007e; Xu and Yager, 2006) (Commutativity) Let $\alpha_j = (\mu_{\alpha_j},\ \nu_{\alpha_j})$ $(j = 1, 2, \cdots, n)$ and $\alpha'_j = (\mu_{\alpha_j^*},\ \nu_{\alpha_j^*})$ $(j = 1, 2, \cdots, n)$ be two collections of IFNs. Then

$$\text{IFOWA}_w(\alpha_1, \alpha_2, \cdots, \alpha_n) = \text{IFOWA}_w(\alpha'_1, \alpha'_2, \cdots, \alpha'_n) \tag{1.55}$$
$$\text{IFOWG}_w(\alpha_1, \alpha_2, \cdots, \alpha_n) = \text{IFOWG}_w(\alpha'_1, \alpha'_2, \cdots, \alpha'_n) \tag{1.56}$$

where $(\alpha'_1, \alpha'_2, \cdots, \alpha'_n)$ is any permutation of $(\alpha_1, \alpha_2, \cdots, \alpha_n)$.

Proof Let

$$\text{IFOWA}_w(\alpha_1, \alpha_2, \cdots, \alpha_n) = w_1\alpha_{\sigma(1)} \oplus w_2\alpha_{\sigma(2)} \oplus \cdots \oplus w_n\alpha_{\sigma(n)}$$
$$\text{IFOWA}_w(\alpha'_1, \alpha'_2, \cdots, \alpha'_n) = w_1\alpha'_{\sigma(1)} \oplus w_2\alpha'_{\sigma(2)} \oplus \cdots \oplus w_n\alpha'_{\sigma(n)}$$

Since $(\alpha'_1, \alpha'_2, \cdots, \alpha'_n)$ is any permutation of $(\alpha_1, \alpha_2, \cdots, \alpha_n)$, we have

$$\alpha_{\sigma(j)} = \alpha'_{\sigma(j)}, \quad j = 1, 2, \cdots, n$$

Thus Eq.(1.55) holds. Similarly, we can prove that Eq.(1.56) also holds. The proof is completed.

From Theorem 1.3.12, both the IFOWA and IFOWG operators possess the commutativity property, but the IFWA and IFWG operators do not.

Theorem 1.3.13 (Xu, 2007e; Xu and Yager, 2006)

(1) If $w = (1, 0, \cdots, 0)^{\mathrm{T}}$, then

$$\mathrm{IFOWA}_w(\alpha_1, \alpha_2, \cdots, \alpha_n) = \max_j\{\alpha_j\}$$

$$\mathrm{IFOWG}_w(\alpha_1, \alpha_2, \cdots, \alpha_n) = \max_j\{\alpha_j\}$$

(2) If $w = (0, 0, \cdots, 1)^{\mathrm{T}}$, then

$$\mathrm{IFOWA}_w(\alpha_1, \alpha_2, \cdots, \alpha_n) = \min_j\{\alpha_j\}$$

$$\mathrm{IFOWG}_w(\alpha_1, \alpha_2, \cdots, \alpha_n) = \min_j\{\alpha_j\}$$

(3) If $w_j = 1,\ w_i = 0$ and $i \neq j$, then

$$\mathrm{IFOWA}_w(\alpha_1, \alpha_2, \cdots, \alpha_n) = \alpha_{\sigma(j)}$$

$$\mathrm{IFOWG}_w(\alpha_1, \alpha_2, \cdots, \alpha_n) = \alpha_{\sigma(j)}$$

where $\alpha_{\sigma(j)}$ is the j-th largest of α_i $(i = 1, 2, \cdots, n)$.

We know that the IFWA and IFWG operators only weight the IFNs, while the IFOWA and IFOWG operators weight the ordered positions of the IFNs instead of the IFNs themselves. Therefore, the weights represent two different aspects in these operators. However, each of the operators considers only one of the two aspects. To overcome this limitation and motivated by the idea of combining the weighted average and the OWA operator (Torra, 1997; Xu and Da, 2003b), we now introduce two intuitionistic fuzzy hybrid aggregation operators, which weight both the given IFNs and their ordered positions.

Definition 1.3.5 (Xu, 2007e) An intuitionistic fuzzy hybrid averaging (IFHA) operator is a mapping IFHA : $\Theta^n \rightarrow \Theta$, such that

$$\mathrm{IFHA}_{\omega,w}\ (\alpha_1, \alpha_2, \cdots, \alpha_n) = w_1\dot{\alpha}_{\sigma(1)} \oplus w_2\dot{\alpha}_{\sigma(2)} \oplus \cdots \oplus w_n\dot{\alpha}_{\sigma(n)} \tag{1.57}$$

where $w = (w_1, w_2, \cdots, w_n)^{\mathrm{T}}$ is the weighting vector associated with the IFHA operator, with $w_j \in [0,\ 1]\,(j = 1, 2, \cdots, n)$ and $\sum_{j=1}^{n} w_j = 1$; $\dot{\alpha}_j = n\omega_j\alpha_j$, $j = 1, 2, \cdots, n$, $(\dot{\alpha}_{\sigma(1)}, \dot{\alpha}_{\sigma(2)}, \cdots, \dot{\alpha}_{\sigma(n)})$ is any permutation of a collection of the weighted IFNs $(\dot{\alpha}_1, \dot{\alpha}_2, \cdots, \dot{\alpha}_n)$, such that $\dot{\alpha}_{\sigma(j)} \geqslant \dot{\alpha}_{\sigma(j+1)}(j = 1, 2, \cdots, n-1)$; $\omega = (\omega_1, \omega_2, \cdots, \omega_n)^{\mathrm{T}}$ is the weight vector of α_j $(j = 1, 2, \cdots, n)$, with $\omega_j \in [0,\ 1]\,(j = 1, 2, \cdots, n)$ and $\sum_{j=1}^{n} \omega_j = 1$, and n is the balancing coefficient, which plays a role of balance (in such a case, if the vector $(\omega_1, \omega_2, \cdots, \omega_n)^{\mathrm{T}}$ approaches $(1/n, 1/n, \cdots, 1/n)^{\mathrm{T}}$, then the vector $(n\omega_1\alpha_1, n\omega_2\alpha_2, \cdots, n\omega_n\alpha_n)^{\mathrm{T}}$ approaches $(\alpha_1, \alpha_2, \cdots, \alpha_n)^{\mathrm{T}}$).

Let $\dot{\alpha}_{\sigma(j)} = (\mu_{\dot{\alpha}_{\sigma(j)}},\ \nu_{\dot{\alpha}_{\sigma(j)}})\,(j = 1, 2, \cdots, n)$. Then similar to Theorem 1.3.1, we have

$$\mathrm{IFHA}_{\omega,w}(\alpha_1, \alpha_2, \cdots, \alpha_n) = \left(1 - \prod_{j=1}^{n}(1 - \mu_{\dot{\alpha}_{\sigma(j)}})^{w_j},\ \prod_{j=1}^{n}\nu_{\dot{\alpha}_{\sigma(j)}}^{w_j}\right) \tag{1.58}$$

and the aggregated value by using the IFHA operator is also an IFN. In particular, if $\mu_{\dot{\alpha}_{\sigma(j)}} = 1 - \nu_{\dot{\alpha}_{\sigma(j)}}$, for any $j = 1, 2, \cdots, n$, then Eq.(1.58) reduces to the following form:

$$\mathrm{IFHA}_{\omega,w}(\alpha_1, \alpha_2, \cdots, \alpha_n) = \left(1 - \prod_{j=1}^{n}(1 - \mu_{\dot{\alpha}_{\sigma(j)}})^{w_j},\ \prod_{j=1}^{n}(1 - \mu_{\dot{\alpha}_{\sigma(j)}})^{w_j}\right) \tag{1.59}$$

Definition 1.3.6 (Xu and Yager, 2006) An intuitionistic fuzzy hybrid geometric (IFHG) operator is a mapping IFHG : $\Theta^n \to \Theta$, such that

$$\mathrm{IFHG}_{\omega,w}(\alpha_1, \alpha_2, \cdots, \alpha_n) = \ddot{\alpha}_{\sigma(1)}^{w_1} \otimes \ddot{\alpha}_{\sigma(2)}^{w_2} \otimes \cdots \otimes \ddot{\alpha}_{\sigma(n)}^{w_n} \tag{1.60}$$

where $\ddot{\alpha}_j = \alpha_j^{n\omega_j}$ $(j = 1, 2, \cdots, n)$, $(\ddot{\alpha}_{\sigma(1)}, \ddot{\alpha}_{\sigma(2)}, \cdots, \ddot{\alpha}_{\sigma(n)})$ is any permutation of a collection of the exponential weighted IFNs $(\ddot{\alpha}_1, \ddot{\alpha}_2, \cdots, \ddot{\alpha}_n)$, such that $\ddot{\alpha}_{\sigma(j)} \geqslant \ddot{\alpha}_{\sigma(j+1)}$ $(j = 1, 2, \cdots, n-1)$.

Let $\ddot{\alpha}_{\sigma(j)} = (\mu_{\ddot{\alpha}_{\sigma(j)}},\ \nu_{\ddot{\alpha}_{\sigma(j)}})\,(j = 1, 2, \cdots, n)$. Then similar to Theorem 1.3.2, we have

$$\mathrm{IFHG}_{\omega,w}(\alpha_1, \alpha_2, \cdots, \alpha_n) = \left(\prod_{j=1}^{n}\mu_{\ddot{\alpha}_{\sigma(j)}}^{w_j},\ 1 - \prod_{j=1}^{n}(1 - \nu_{\ddot{\alpha}_{\sigma(j)}})^{w_j}\right) \tag{1.61}$$

and the aggregated value by using the IFHG operator is also an IFN. In particular, if $\mu_{\ddot{\alpha}_{\sigma(j)}} = 1 - \nu_{\ddot{\alpha}_{\sigma(j)}}$, for any $j = 1, 2, \cdots, n$, then Eq.(1.61) reduces to the following form:

$$\mathrm{IFHG}_{\omega,w}(\alpha_1, \alpha_2, \cdots, \alpha_n) = \left(\prod_{j=1}^{n}\mu_{\ddot{\alpha}_{\sigma(j)}}^{w_j},\ 1 - \prod_{j=1}^{n}\mu_{\ddot{\alpha}_{\sigma(j)}}^{w_j}\right) \tag{1.62}$$

Clearly, from Definitions 1.3.5 and 1.3.6, we know that the IFHA and IFHG operators are composed of the following three phases:

(1) They weight the IFNs α_j $(j = 1, 2, \cdots, n)$ by the associated weights ω_j $(j = 1, 2, \cdots, n)$ and get $\omega_j\alpha_j$ or $\alpha_j^{\omega_j}$ $(j = 1, 2, \cdots, n)$. Then they multiply these values by the balancing coefficient n, and then get the weighted IFNs $n\omega_j\alpha_j$ or $\alpha_j^{n\omega_j}$ $(j = 1, 2, \cdots, n)$.

(2) They reorder the weighted IFNs $\dot{\alpha} = n\omega_j\alpha_j$ or $\ddot{\alpha} = \alpha_j^{n\omega_j}$ $(j = 1, 2, \cdots, n)$ in descending order $(\dot{\alpha}_{\sigma(1)}, \dot{\alpha}_{\sigma(2)}, \cdots, \dot{\alpha}_{\sigma(n)})$ or $(\ddot{\alpha}_{\sigma(1)}, \ddot{\alpha}_{\sigma(2)}, \cdots, \ddot{\alpha}_{\sigma(n)})$, where $\dot{\alpha}_{\sigma(j)}$ is the j-th largest of $n\omega_i\alpha_i$ $(i = 1, 2, \cdots, n)$, and $\ddot{\alpha}_{\sigma(j)}$ is the j-th largest of $\alpha_i^{n\omega_i}$ $(i = 1, 2, \cdots, n)$.

(3) They weight these ordered weighted IFNs $\dot{\alpha}_{\sigma(j)}$ or $\ddot{\alpha}_{\sigma(j)}$ $(j = 1, 2, \cdots, n)$ by the IFHA or IFHG weights w_j $(j = 1, 2, \cdots, n)$ and then aggregate all the weighted IFNs $w_j\dot{\alpha}_{\sigma(j)}$ or $\ddot{\alpha}_{\sigma(j)}^{w_j}$ $(j = 1, 2, \cdots, n)$ into the collective ones.

Example 1.3.3 (Xu, 2007e) Let $\alpha_1 = (0.2,\ 0.5)$, $\alpha_2 = (0.4,\ 0.2)$, $\alpha_3 = (0.5,\ 0.4)$, $\alpha_4 =$ $(0.3,\ 0.3)$ and $\alpha_5 = (0.7,\ 0.1)$ be IFNs, and $\omega = (0.25, 0.20, 0.15, 0.18, 0.22)^{\mathrm{T}}$ the weight vector of α_j $(j = 1, 2, \cdots, 5)$.

We first utilize the operational laws (4) and (6) in Definition 1.2.2 to get the weighted IFNs:

$$\dot{\alpha}_1 = (1 - (1 - 0.2)^{5\times 0.25},\ 0.5^{5\times 0.25}) = (0.243,\ 0.420)$$

$$\dot{\alpha}_2 = (1 - (1 - 0.4)^{5\times 0.20},\ 0.2^{5\times 0.20}) = (0.4,\ 0.2)$$

$$\dot{\alpha}_3 = (1 - (1 - 0.5)^{5\times 0.15},\ 0.4^{5\times 0.15}) = (0.405,\ 0.503)$$

$$\dot{\alpha}_4 = (1 - (1 - 0.3)^{5\times 0.18},\ 0.3^{5\times 0.18}) = (0.275,\ 0.338)$$

$$\dot{\alpha}_5 = (1 - (1 - 0.7)^{5\times 0.22},\ 0.1^{5\times 0.22}) = (0.734,\ 0.079)$$

By Eq.(1.10), we calculate the scores of $\dot{\alpha}_j$ $(j = 1, 2, \cdots, 5)$:

$$s(\dot{\alpha}_1) = 0.243 - 0.420 = -0.177, \quad s(\dot{\alpha}_2) = 0.4 - 0.2 = 0.2$$

$$s(\dot{\alpha}_3) = 0.405 - 0.503 = -0.098, \quad s(\dot{\alpha}_4) = 0.275 - 0.338 = -0.063$$

$$s(\dot{\alpha}_5) = 0.734 - 0.079 = 0.655$$

Since

$$s(\dot{\alpha}_5) > s(\dot{\alpha}_2) > s(\dot{\alpha}_4) > s(\dot{\alpha}_3) > s(\dot{\alpha}_1)$$

we have

$$\dot{\alpha}_{\sigma(1)} = (0.734,\ 0.079), \quad \dot{\alpha}_{\sigma(2)} = (0.4,\ 0.2)$$

$$\dot{\alpha}_{\sigma(3)} = (0.275,\ 0.338), \quad \dot{\alpha}_{\sigma(4)} = (0.405,\ 0.503)$$

$$\dot{\alpha}_{\sigma(5)} = (0.243,\ 0.420)$$

Suppose that $w = (0.112,\ 0.236,\ 0.304,\ 0.236,\ 0.112)^{\mathrm{T}}$ (derived by the normal distribution based method (Xu, 2005a)) is the weighting vector of the IFHA operator. Then by Eq.(1.58), we get

$$\mathrm{IFHA}_{\omega,w}(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5)$$

$$= \left(1 - \prod_{j=1}^{5}(1-\mu_{\dot{\alpha}_{\sigma(j)}})^{w_j},\ \prod_{j=1}^{5}\nu_{\dot{\alpha}_{\sigma(j)}}^{w_j}\right)$$
$$= (1-(1-0.734)^{0.112}\times(1-0.4)^{0.236}\times(1-0.275)^{0.304}$$
$$\times(1-0.405)^{0.236}\times(1-0.243)^{0.112},$$
$$0.079^{0.112}\times 0.2^{0.236}\times 0.338^{0.304}\times 0.503^{0.236}\times 0.420^{0.112})$$
$$= (0.406,\ 0.286)$$

If we utilize the IFHG operator to aggregate the given IFNs, then by the operational laws (5) and (7) in Definition 1.2.2, we obtain the weighted IFNs:

$$\ddot{\alpha}_1 = (0.2^{5\times0.25},\ 1-(1-0.5)^{5\times0.25}) = (0.134,\ 0.580)$$
$$\ddot{\alpha}_2 = (0.4^{5\times0.25},\ 1-(1-0.2)^{5\times0.25}) = (0.318,\ 0.243)$$
$$\ddot{\alpha}_3 = (0.5^{5\times0.25},\ 1-(1-0.4)^{5\times0.25}) = (0.420,\ 0.472)$$
$$\ddot{\alpha}_4 = (0.3^{5\times0.25},\ 1-(1-0.3)^{5\times0.25}) = (0.222,\ 0.360)$$
$$\ddot{\alpha}_5 = (0.7^{5\times0.25},\ 1-(1-0.1)^{5\times0.25}) = (0.640,\ 0.123)$$

By Eq.(1.10), we calculate the scores of $\ddot{\alpha}_j$ $(j = 1, 2, \cdots, 5)$:

$$s(\ddot{\alpha}_1) = 0.134 - 0.580 = -0.446$$
$$s(\ddot{\alpha}_2) = 0.318 - 0.243 = 0.075$$
$$s(\ddot{\alpha}_3) = 0.420 - 0.472 = -0.052$$
$$s(\ddot{\alpha}_4) = 0.222 - 0.360 = -0.138$$
$$s(\ddot{\alpha}_5) = 0.640 - 0.123 = 0.517$$

Since

$$s(\ddot{\alpha}_5) > s(\ddot{\alpha}_2) > s(\ddot{\alpha}_3) > s(\ddot{\alpha}_4) > s(\ddot{\alpha}_1)$$

we can see that

$$\ddot{\alpha}_{\sigma(1)} = (0.640,\ 0.123), \quad \ddot{\alpha}_{\sigma(2)} = (0.318,\ 0.243)$$
$$\ddot{\alpha}_{\sigma(3)} = (0.420,\ 0.472), \quad \ddot{\alpha}_{\sigma(4)} = (0.222,\ 0.360)$$
$$\ddot{\alpha}_{\sigma(5)} = (0.134,\ 0.580)$$

Assume that $w = (0.112, 0.236, 0.304, 0.236, 0.112)^{\mathrm{T}}$ (derived by the normal distribution based method (Xu, 2005a)) is the weighting vector of the IFHG operator. Then by Eq.(1.62), we get

$$\mathrm{IFHG}_{\omega,w}(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5)$$

$$= \left(\prod_{j=1}^{5} \mu_{\ddot{\alpha}_{\sigma(j)}}^{w_j},\ 1 - \prod_{j=1}^{5} (1 - \nu_{\ddot{\alpha}_{\sigma(j)}})^{w_j} \right)$$
$$= (0.640^{0.112} \times 0.318^{0.236} \times 0.420^{0.304} \times 0.222^{0.236} \times 0.134^{0.112},$$
$$(1 - 0.123)^{0.112} \times (1 - 0.243)^{0.236} \times (1 - 0.472)^{0.304}$$
$$\times (1 - 0.360)^{0.236} \times (1 - 0.580)^{0.112})$$
$$= (0.312,\ 0.379)$$

Theorem 1.3.14 (Xu, 2007e) The IFWA and IFOWA operators are special cases of the IFHA operator.

Proof (1) Let $w = (1/n, 1/n, \cdots, 1/n)^{\mathrm{T}}$. Then

$$\begin{aligned}\mathrm{IFHA}_{\omega,w}(\alpha_1, \alpha_2, \cdots, \alpha_n) &= w_1\dot{\alpha}_{\sigma(1)} \oplus w_2\dot{\alpha}_{\sigma(2)} \oplus \cdots \oplus w_n\dot{\alpha}_{\sigma(n)} \\ &= \frac{1}{n}(\dot{\alpha}_{\sigma(1)} \oplus \dot{\alpha}_{\sigma(2)} \oplus \cdots \oplus \dot{\alpha}_{\sigma(n)}) \\ &= \omega_1\alpha_1 \oplus \omega_2\alpha_2 \oplus \cdots \oplus \omega_n\alpha_n \\ &= \mathrm{IFWA}_{\omega}(\alpha_1, \alpha_2, \cdots, \alpha_n)\end{aligned}$$

(2) Let $\omega = (1/n, 1/n, \cdots, 1/n)^{\mathrm{T}}$. Then $\dot{\alpha}_j = \alpha_j$ $(j = 1, 2, \cdots, n)$ and

$$\begin{aligned}\mathrm{IFHA}_{\omega,w}(\alpha_1, \alpha_2, \cdots, \alpha_n) &= w_1\dot{\alpha}_{\sigma(1)} \oplus w_2\dot{\alpha}_{\sigma(2)} \oplus \cdots \oplus w_n\dot{\alpha}_{\sigma(n)} \\ &= w_1\alpha_{\sigma(1)} \oplus w_2\alpha_{\sigma(2)} \oplus \cdots \oplus w_n\alpha_{\sigma(n)} \\ &= \mathrm{IFOWA}_{w}(\alpha_1, \alpha_2, \cdots, \alpha_n)\end{aligned}$$

which completes the proof.

Theorem 1.3.15 (Xu and Yager, 2006) The IFWG and IFOWG operators are special cases of the IFHG operator.

Proof (1) Let $w = (1/n, 1/n, \cdots, 1/n)^{\mathrm{T}}$. Then

$$\begin{aligned}\mathrm{IFHG}_{\omega,w}(\alpha_1, \alpha_2, \cdots, \alpha_n) &= \ddot{\alpha}_{\sigma(1)}^{w_1} \otimes \ddot{\alpha}_{\sigma(2)}^{w_2} \otimes \cdots \otimes \ddot{\alpha}_{\sigma(n)}^{w_n} \\ &= (\ddot{\alpha}_{\sigma(1)} \otimes \ddot{\alpha}_{\sigma(2)} \otimes \cdots \otimes \ddot{\alpha}_{\sigma(n)})^{\frac{1}{n}} \\ &= \alpha_1^{\omega_1} \otimes \alpha_2^{\omega_2} \otimes \cdots \otimes \alpha_n^{\omega_n} \\ &= \mathrm{IFWG}_{\omega}(\alpha_1, \alpha_2, \cdots, \alpha_n)\end{aligned}$$

(2) Let $\omega = (1/n, 1/n, \cdots, 1/n)^{\mathrm{T}}$. Then $\ddot{\alpha}_j = \alpha_j$ $(j = 1, 2, \cdots, n)$ and

$$\begin{aligned}\mathrm{IFHG}_{\omega,w}(\alpha_1, \alpha_2, \cdots, \alpha_n) &= \ddot{\alpha}_{\sigma(1)}^{w_1} \otimes \ddot{\alpha}_{\sigma(2)}^{w_2} \otimes \cdots \otimes \ddot{\alpha}_{\sigma(n)}^{w_n} \\ &= \alpha_{\sigma(1)}^{w_1} \otimes \alpha_{\sigma(2)}^{w_2} \otimes \cdots \otimes \alpha_{\sigma(n)}^{w_n} \\ &= \mathrm{IFOWG}_{w}(\alpha_1, \alpha_2, \cdots, \alpha_n)\end{aligned}$$

which completes the proof.

Obviously, from Theorems 1.3.14 and 1.3.15, we can see that the IFHA operator generalizes both the IFWA and IFOWA operators, and the IFHG operator generalizes both the IFWG and IFOWG operators. The desirable characteristic of both the IFHA and IFHG operators is that they can take into account not only the importance of the given IFNs themselves, but also that of the ordered positions of the given IFNs.

In the following, we apply the IFHA and IFHG operators to multi-attribute decision making:

For a multi-attribute decision making problem, let $Y = \{Y_1,\ Y_2,\ \cdots, Y_n\}$ be a finite set of alternatives, $G = \{G_1,\ G_2,\ \cdots,\ G_m\}$ a set of attributes, and $\omega = (\omega_1, \omega_2, \cdots, \omega_m)^{\mathrm{T}}$ the weight vector of attributes, where $\omega_j \in [0,1]$ $(j = 1,2,\cdots,m)$ and $\sum_{j=1}^{m} \omega_j = 1$. Suppose that the characteristics of the alternatives Y_i $(i = 1,2,\cdots,n)$ are represented by the IFNs:

$$Y_i = \{\langle G_j, \mu_{Y_i}(G_j),\ \nu_{Y_i}(G_j)\rangle \mid G_j \in G\}, \quad i = 1,2,\cdots,n \tag{1.63}$$

where $\mu_{Y_i}(G_j)$ indicates the degree that the alternative Y_i satisfies the attribute G_j, $\nu_{Y_i}(G_j)$ indicates the degree that the alternative Y_i does not satisfy the attribute G_j, and

$$\mu_{Y_i}(G_j) \in [0,1], \quad \nu_{Y_i}(G_j) \in [0,1], \quad \mu_{Y_i}(G_j) + \nu_{Y_i}(G_j) \leqslant 1 \tag{1.64}$$

For convenience, let $r'_{ij} = (t_{ij},\ f_{ij})$ denote the characteristic of the alternative Y_i with respect to the attribute G_j, where t_{ij} indicates the degree that the alternative Y_i satisfies the attribute G_j, and f_{ij} indicates the degree that the alternative Y_i does not satisfy the attribute G_j. Therefore, the characteristics of all the alternatives Y_i $(i = 1,2,\cdots,n)$ with respect to the attributes G_j $(j = 1,2,\cdots,m)$ can be contained in an intuitionistic fuzzy decision matrix $R' = (r'_{ij})_{n\times m}$ (Table 1.1), where $r'_{ij} = (t_{ij},\ f_{ij})$, $t_{ij} \in [0,1]$, $f_{ij} \in [0,1]$, and $t_{ij} + f_{ij} \leqslant 1$.

Table 1.1 Intuitionistic fuzzy decision matrix R'

	G_1	G_2	$\cdots$	G_m
Y_1	$(t_{11},\ f_{11})$	$(t_{12},\ f_{12})$	$\cdots$	(t_{1m}, f_{1m})
Y_2	$(t_{21},\ f_{21})$	$(t_{22},\ f_{22})$	$\cdots$	(t_{2m}, f_{2m})
$\vdots$	$\vdots$	$\vdots$		$\vdots$
Y_n	$(t_{n1},\ f_{n1})$	$(t_{n2},\ f_{n2})$	$\cdots$	$(t_{nm},\ f_{nm})$

If all the attributes G_j $(j = 1,2,\cdots,m)$ are of the same type, then the attribute values do not need normalization. However, there are generally benefit attributes (the bigger the attribute values the better) and cost attributes (the smaller the attribute values the better) in multi-attribute decision making. In such cases, we may transform the attribute values of cost type into the attribute values of benefit type, then $R' = (r'_{ij})_{n\times m}$ can be transformed into the intuitionistic fuzzy decision matrix $R = (r_{ij})_{n\times m}$, where

$$r_{ij} = (\mu_{ij}, \nu_{ij}) = \begin{cases} r'_{ij}, & \text{for benefit attribute } G_j, \\ \bar{r}'_{ij}, & \text{for cost attribute } G_j, \end{cases} \quad i = 1, 2, \cdots, n \tag{1.65}$$

where $\bar{r}'_{ij}$ is the complement of r'_{ij}, such that $\bar{r}'_{ij} = (f_{ij},\ t_{ij})$.

In what follows, we introduce an approach to multi-attribute decision making with intuitionistic fuzzy information (adapted from Xu and Yager (2006)):

Step 1 Utilize the IFHA operator:

$$\dot{r}_i = \text{IFHA}_{\omega,w}\left(r_{i1}, r_{i2}, \cdots, r_{im}\right), \quad i = 1, 2, \cdots, n \tag{1.66}$$

or the IFHG operator:

$$\ddot{r}_i = \text{IFHG}_{\omega,w}\left(r_{i1}, r_{i2}, \cdots, r_{im}\right), \quad i = 1, 2, \cdots, n \tag{1.67}$$

to aggregate the characteristics $r_{ij} = (\mu_{ij},\ \nu_{ij})$ $(j = 1, 2, \cdots, m)$ of the alternative Y_i with respect to the attributes G_j $(j = 1, 2, \cdots, m)$, where $w = (w_1, w_2, \cdots, w_m)^{\mathrm{T}}$ is the weighting vector associated with the IFHA and IFHG operators, with $w_j \in [0, 1]\,(j = 1, 2, \cdots, m)$ and $\sum_{j=1}^{m} w_j = 1$. This can be determined by the normal distribution based method (Xu, 2005a), and thus yields the overall attribute values $\dot{r}_i = (\dot{\mu}_i,\ \dot{v}_i)$ (or $\ddot{r}_i = (\ddot{\mu}_i,\ \ddot{v}_i)$) of the alternative Y_i.

Step 2 Utilize Eq.(1.10) to calculate the scores $s(\dot{r}_i)$ (or $s(\ddot{r}_i)$) of the overall attribute value $\dot{r}_i$ (or $\ddot{r}_i$) of the alternative Y_i.

Step 3 Utilize the score $s(\dot{r}_i)$ (or $s(\ddot{r}_i)$) $(i = 1, 2, \cdots, n)$ to rank and select the alternative Y_i $(i = 1, 2, \cdots, n)$ (if the two scores $s(\dot{r}_i)$ (or $s(\ddot{r}_i)$) and $s(\dot{r}_j)$ (or $s(\ddot{r}_j)$) are equal, then we need to calculate the accuracy degree $h(\dot{r}_i)$ (or $h(\ddot{r}_i)$) and $h(\dot{r}_j)$ (or $h(\ddot{r}_j)$) of the overall attribute values $\dot{r}_i$ (or $\ddot{r}_i$) and $\dot{r}_j$ (or $\ddot{r}_j$), and then utilize $h(\dot{r}_i)$ (or $h(\ddot{r}_i)$) and $h(\dot{r}_j)$ (or $h(\ddot{r}_j)$) to rank the alternatives Y_i and Y_j).

Example 1.3.4 Let us consider a customer who intends to buy a car. Five types of cars (alternatives) Y_i $(i = 1,\ 2, \cdots, 5)$ are available. The customer takes into account six attributes to decide which car to buy (Herrera and Martínez, 2000b): ① G_1: Fuel economy; ② G_2: Aerod. degree; ③ G_3: Price; ④ G_4: Comfort; ⑤ G_5: Design; and ⑥ G_6: Safety. The weight vector of the attributes G_j $(j = 1, 2, \cdots, 6)$ is $\omega = (0.15,\ 0.25,\ 0.14,\ 0.16,\ 0.20,\ 0.10)^{\mathrm{T}}$. Assume that the characteristics of the alternatives Y_i $(i = 1,\ 2, \cdots, 5)$ are represented by the IFNs, as shown in the intuitionistic fuzzy decision matrix $R' = (r'_{ij})_{5\times 6}$ (Table 1.2):

Table 1.2 Intuitionistic fuzzy decision matrix R'

	G_1	G_2	G_3	G_4	G_5	G_6
Y_1	(0.3, 0.5)	(0.6, 0.1)	(0.3, 0.4)	(0.8, 0.1)	(0.1, 0.6)	(0.5, 0.4)
Y_2	(0.6, 0.3)	(0.5, 0.2)	(0.1, 0.6)	(0.7, 0.1)	(0.3, 0.6)	(0.4, 0.3)
Y_3	(0.4, 0.4)	(0.8, 0.1)	(0.1, 0.5)	(0.6, 0.2)	(0.4, 0.5)	(0.3, 0.2)
Y_4	(0.2, 0.4)	(0.4, 0.1)	(0.0, 0.9)	(0.8, 0.1)	(0.2, 0.5)	(0.7, 0.1)
Y_5	(0.5, 0.2)	(0.3, 0.6)	(0.3, 0.6)	(0.7, 0.1)	(0.6, 0.2)	(0.5, 0.3)

Considering that the attributes have two different types, we first transform the attribute values of cost type into the attribute values of benefit type by using Eq. (1.65). Then $R' = (r'_{ij})_{5\times 6}$ is transformed into $R = (r_{ij})_{5\times 6}$ (Table 1.3):

Table 1.3 Intuitionistic fuzzy decision matrix R

	G_1	G_2	G_3	G_4	G_5	G_6
Y_1	(0.3, 0.5)	(0.6, 0.1)	(0.4, 0.3)	(0.8, 0.1)	(0.1, 0.6)	(0.5, 0.4)
Y_2	(0.6, 0.3)	(0.5, 0.2)	(0.6, 0.1)	(0.7, 0.1)	(0.3, 0.6)	(0.4, 0.3)
Y_3	(0.4, 0.4)	(0.8, 0.1)	(0.5, 0.1)	(0.6, 0.2)	(0.4, 0.5)	(0.3, 0.2)
Y_4	(0.2, 0.4)	(0.4, 0.1)	(0.9, 0.0)	(0.8, 0.1)	(0.2, 0.5)	(0.7, 0.1)
Y_5	(0.5, 0.2)	(0.3, 0.6)	(0.6, 0.3)	(0.7, 0.1)	(0.6, 0.2)	(0.5, 0.3)

To get the most desirable alternative, the following steps are involved:

We first weight all the attribute values r_{ij} $(i = 1, 2, \cdots, 5;\ j = 1, 2, \cdots, 6)$ by the weight vector $\omega = (0.15,\ 0.25,\ 0.14,\ 0.16,\ 0.20,\ 0.10)^{\mathrm{T}}$ of the attributes G_j $(j = 1, 2, \cdots, 6)$ and multiply these values by the balancing coefficient $m = 6$, and then get the weighted attribute values $6\omega_j r_{ij}$ $(i = 1, 2, \cdots, 5;\quad j = 1, 2, \cdots, 6)$, as listed in the weighted intuitionistic fuzzy decision matrix $\dot{R} = (6\omega_j r_{ij})_{5\times 6}$ (Table 1.4):

Table 1.4 The weighted intuitionistic fuzzy decisi on matrix $\dot{R}$

	G_1	G_2	G_3
Y_1	(0.275, 0.536)	(0.747, 0,032)	(0.349, 0.364)
Y_2	(0.562, 0.338)	(0.646, 0.089)	(0.537, 0.145)
Y_3	(0.369, 0.438)	(0.911, 0.032)	(0.441, 0.855)
Y_4	(0.182, 0.562)	(0.535, 0.032)	(0.855, 0.000)
Y_5	(0.464, 0.765)	(0.414, 0.465)	(0.537, 0.636)

	G_4	G_5	G_6
Y_1	(0.787, 0.110)	(0.119, 0.542)	(0.340, 0.577)
Y_2	(0.585, 0.213)	(0.348, 0.542)	(0.264, 0.586)
Y_3	(0.585, 0.213)	(0.458, 0.435)	(0.193, 0.381)
Y_4	(0.787, 0.890)	(0.235, 0.435)	(0.514, 0.251)
Y_5	(0.685, 0.251)	(0.667, 0.145)	(0.340, 0.486)

Then, by Definition 1.1.3, we reorder the weighted attribute values of each alternative in descending order, and utilize the IFHA operator (1.66) (let $w = (0.0865, 0.1716, 0.2419, 0.2419, 0.1716, 0.0865)^{\mathrm{T}}$ be its weighting vector as derived by the normal distribution based method of Xu (2005a), which can relieve the influence of unfair arguments on the decision results by assigning low weights to those unduly high or unduly low ones) to derive the overall attribute values $\dot{r}_i$ $(i = 1, 2, \cdots, 5)$ of the alternatives Y_i $(i = 1, 2, \cdots, 5)$:

$$\dot{r}_1 = \mathrm{IFHA}_{\omega,w}((0.3,\ 0.5),\ (0.6,\ 0.1),\ (0.4,\ 0.3),\ (0.8,\ 0),\ (0.1,\ 0.6),\ (0.5,\ 0.4)) = (0.4728,\ 0.2672)$$

$$\dot{r}_2 = \mathrm{IFHA}_{\omega,w}((0.6,\ 0.3),\ (0.5,\ 0.2),\ (0.6,\ 0),\ (0.7,\ 0.1),\ (0.3,\ 0.6),\ (0.4,\ 0.3)) = (0.5342,\ 0.2224)$$

$$\dot{r}_3 = \mathrm{IFHA}_{\omega,w}((0.4,\ 0.4),\ (0.8,\ 0.1),\ (0.5,\ 0.1),\ (0.6,\ 0.2),\ (0.4,\ 0.5),\ (0.3,\ 0.2)) = (0.5260,\ 0.2330)$$

$$\dot{r}_4 = \mathrm{IFHA}_{\omega,w}((0.2,\ 0.4),\ (0.4,\ 0.1),\ (0.9,\ 0),\ (0.8,\ 0.1),\ (0.2,\ 0.5),\ (0.7,\ 0.1)) = (0.5749,\ 0)$$

$$\dot{r}_5 = \mathrm{IFHA}_{\omega,w}((0.5,\ 0.2),\ (0.3,\ 0.6),\ (0.6,\ 0.3),\ (0.7,\ 0),\ (0.6,\ 0.2),\ (0.5,\ 0.3)) = (0.5315,\ 0.3034)$$

Utilizing Eq.(1.10) to calculate the scores of $\dot{r}_i$ $(i = 1, 2, \cdots, 5)$, we can get

$$s(\dot{r}_1) = 0.4728 - 0.2672 = 0.2054, \quad s(\dot{r}_2) = 0.5342 - 0.2224 = 0.3118$$

$$s(\dot{r}_3) = 0.5260 - 0.2330 = 0.2930, \quad s(\dot{r}_4) = 0.5749 - 0 = 0.5749$$

$$s(\dot{r}_5) = 0.5315 - 0.3034 = 0.2281$$

Since

$$s(\dot{r}_4) > s(\dot{r}_2) > s(\dot{r}_3) > s(\dot{r}_5) > s(\dot{r}_1)$$

we have

$$Y_4 \succ Y_2 \succ Y_3 \succ Y_5 \succ Y_1$$

Thus the best car is Y_4, where "$\succ$" denotes "be superior to".

If we utilize the IFHG operator (1.67) to aggregate the given intuitionistic fuzzy information, then by the weight vector $\omega = (0.15,\ 0.25,\ 0.14,\ 0.16,\ 0.20,\ 0.10)^{\mathrm{T}}$ of the attribute weights G_j $(j = 1, 2, \cdots, 6)$, we can first weight exponentially all the attribute values r_{ij} $(i = 1, 2, \cdots, 5;\ \ j = 1, 2, \cdots, 6)$ and multiply these values by the balancing coefficient $m = 6$, and then get the exponentially weighted attribute values $r_{ij}^{6\omega_j}$ $(i = 1, 2, \cdots, 5;\ \ j = 1, 2, \cdots, 6)$ as listed in the weighted intuitionistic fuzzy decision matrix $\ddot{R} = (r_{ij}^{6\omega_j})_{5\times 6}$ (Table 1.5):

Table 1.5 The weighted intuitionistic fuzzy decision matrix $\ddot{R}$

	G_1	G_2	G_3
Y_1	(0.338, 0.464)	(0.465, 0.146)	(0.463, 0.259)
Y_2	(0.631, 0.275)	(0.354, 0.284)	(0.651, 0.085)
Y_3	(0.438, 0.369)	(0.716, 0.146)	(0.559, 0.085)
Y_4	(0.235, 0.369)	(0.253, 0.146)	(0.915, 0.000)
Y_5	(0.536, 0.182)	(0.164, 0.747)	(0.651, 0.259)

	G_4	G_5	G_6
Y_1	(0.807, 0.096)	(0.063, 0.667)	(0.660, 0.264)
Y_2	(0.710, 0.096)	(0.236, 0.667)	(0.577, 0.193)
Y_3	(0.612, 0.193)	(0.333, 0.565)	(0.486, 0.125)
Y_4	(0.807, 0.096)	(0.145, 0.565)	(0.807, 0.061)
Y_5	(0.710, 0.096)	(0.542, 0.235)	(0.660, 0.193)

After that, we can reorder the exponentially weighted attribute values of each alternative in descending order, and utilize the IFHG operator (1.67) (let $w = (0.0865, 0.1716, 0.2419, 0.2419, 0.1716, 0.0865)^{\mathrm{T}}$ be its weighting vector) to derive the overall attribute values $\ddot{r}_i$ $(i = 1, 2, \cdots, 5)$ of the alternatives Y_i $(i = 1, 2, \cdots, 5)$:

$$\ddot{r}_1 = \mathrm{IFHG}_{\omega,w}\left((0.3,\ 0.5),\ (0.6,\ 0.1),\ (0.4,\ 0.3),\ (0.8,\ 0),\ (0.1,\ 0.6),\ (0.5,\ 0.4)\right) = (0.4138,\ 0.3110)$$

$$\ddot{r}_2 = \mathrm{IFHG}_{\omega,w}\left((0.6,\ 0.3),\ (0.5,\ 0.2),\ (0.6,\ 0),\ (0.7,\ 0.1),\ (0.3,\ 0.6),\ (0.4,\ 0.3)\right) = (0.5228,\ 0.2625)$$

$$\ddot{r}_3 = \mathrm{IFHG}_{\omega,w}\left((0.4,\ 0.4),\ (0.8,\ 0.1),\ (0.5,\ 0.1),\ (0.6,\ 0.2),\ (0.4,\ 0.5),\ (0.3,\ 0.2)\right) = (0.5183,\ 0.2311)$$

$$\ddot{r}_4 = \mathrm{IFHG}_{\omega,w}\left((0.2,\ 0.4),\ (0.4,\ 0.1),\ (0.9,\ 0),\ (0.8,\ 0.1),\ (0.2,\ 0.5),\ (0.7,\ 0.1)\right) = (0.4366,\ 0.1979)$$

$$\ddot{r}_5 = \mathrm{IFHG}_{\omega,w}\left((0.5,\ 0.2),\ (0.3,\ 0.6),\ (0.6,\ 0.3),\ (0.7,\ 0),\ (0.6,\ 0.2),\ (0.5,\ 0.3)\right) = (0.5410,\ 0.2807)$$

Utilizing Eq.(1.10) to calculate the scores of $\ddot{r}_i$ $(i = 1, 2, \cdots, 5)$, we get

$$s(\ddot{r}_1) = 0.4138 - 0.3110 = 0.1028$$

$$s(\ddot{r}_2) = 0.5228 - 0.2625 = 0.2603$$

$$s(\ddot{r}_3) = 0.5183 - 0.2311 = 0.2872$$

$$s(\ddot{r}_4) = 0.4366 - 0.1979 = 0.2387$$

$$s(\ddot{r}_5) = 0.5410 - 0.2807 = 0.2603$$

Since $s(\ddot{r}_4) \neq s(\ddot{r}_5)$, we calculate the accuracy degrees of $\ddot{r}_4$ and $\ddot{r}_5$:

$$h(\ddot{r}_4) = 0.4366 + 0.1979 = 0.6345$$

$$h(\ddot{r}_5) = 0.5410 + 0.2807 = 0.8217$$

Since $h(\ddot{r}_5) > h(\ddot{r}_4)$, we can see, from Definition 1.1.3, that $\ddot{r}_5 > \ddot{r}_4$. Thus

$$s(\ddot{r}_3) > s(\ddot{r}_2) > s(\ddot{r}_5) > s(\ddot{r}_4) > s(\ddot{r}_1)$$

and then the best alternative is Y_3.

In the above example, the ranking of the alternatives Y_4 and Y_3 changes significantly by using the IFHA and IFHG operators respectively. This is because that the IFHG operator is much more sensitive to the given arguments than the IFHA operator. When using these two operators, the two low attribute values $r_{41}{=}(0.2, 0.4)$ and $r_{45}{=}(0.2, 0.5)$ of the alternative Y_4 have great influence on their positions in the ranking of all the alternatives Y_i $(i = 1, 2, \cdots, 5)$.

In order to enable decision making to be more scientific and democratic, modern decision making problems often require multiple decision makers to participate in the decision making process. In the following, we consider a multi-attribute group decision making problem with intuitionistic fuzzy information:

For a multi-attribute group decision making problem, let $Y = \{Y_1, Y_2, \cdots, Y_n\}$ be a finite set of alternatives, $E = \{E_1, E_2, \cdots, E_l\}^{\mathrm{T}}$ the set of decision makers, and $\xi = (\xi_1, \xi_2, \cdots, \xi_l)^{\mathrm{T}}$ the weight vector of decision makers, where $\xi_k \geqslant 0$, $k = 1, 2, \cdots, l$, and $\sum_{k=1}^{l} \xi_k = 1$. Let $G = \{G_1,\ G_2,\ \cdots,\ G_m\}$ be the set of attributes, and $\omega = (\omega_1, \omega_2, \cdots, \omega_m)^{\mathrm{T}}$ the weight vector of attributes, where $\omega_j \in [0, 1]$ $(j = 1, 2, \cdots, m)$ and $\sum_{j=1}^{m} \omega_j = 1$. Λ is the set of the known information about attribute weights given by the decision makers, which can be constructed in the following forms, for $i \neq j$ (Kim and Ahn, 1999; Kim et al., 1999; Xu and Chen, 2007b; Xu, 2007a; 2006d):

(1) A weak ranking: $\{\omega_i \geqslant \omega_j\}$;

(2) A strict ranking: $\{\omega_i - \omega_j \geqslant \delta_i\ (> 0)\}$;

(3) A ranking with multiples: $\{\omega_i \geqslant \delta_i \omega_j\}$, $0 \leqslant \delta_i \leqslant 1$;

(4) An interval form: $\{\delta_i \leqslant \omega_i \leqslant \delta_i + \varepsilon_i\}$, $0 \leqslant \delta_i < \delta_i + \varepsilon_i \leqslant 1$;

(5) A ranking of differences: $\{\omega_i - \omega_j \geqslant \omega_k - \omega_l\}$, for $j \neq k \neq l$.

Let $R'_k = (r'^{(k)}_{ij})_{n \times m}$ be the intuitionistic fuzzy decision matrix given by the decision maker E_k, where $r'^{(k)}_{ij} = (t^{(k)}_{ij},\ f^{(k)}_{ij})$ is the attribute value given by E_k for the alternative Y_i with respect to the attribute $G_j \in G$. $t^{(k)}_{ij}$ indicates the degree that the alternative Y_i satisfies the attribute G_j, and $f_{ij}^{(k)}$ indicates the degree that the alternative Y_i does not satisfy the attribute G_j, where

$$t_{ij}^{(k)} \in [0,1], \quad f_{ij}^{(k)} \in [0,1], \quad t_{ij}^{(k)} + f_{ij}^{(k)} \leqslant 1,$$

$$i = 1,2,\cdots,n; \quad j = 1,2,\cdots,m; \quad k = 1,2,\cdots,l \tag{1.68}$$

By Eq.(1.65), we normalize $R'_k = ({r'}_{ij}^{(k)})_{n\times m}$ as $R_k = (r_{ij}^{(k)})_{n\times m}$, where $r_{ij}^{(k)} = (\mu_{ij}^{(k)},\ \nu_{ij}^{(k)})$ (in the case where all the attributes are of the same type, the attribute values do not need normalization). In the process of group decision making, in order to get the final decision, it is necessary to fuse all individual opinions into a collective one. Accordingly, we can utilize the IFHA operator:

$$\dot{r}_{ij} = \text{IFHA}_{\xi,w}(r_{ij}^{(1)}, r_{ij}^{(2)}, \cdots, r_{ij}^{(l)}), \quad i = 1,2,\cdots,n;\ \ j = 1,2,\cdots,m \tag{1.69}$$

or the IFHG operator

$$\ddot{r}_{ij} = \text{IFHG}_{\xi,w}(r_{ij}^{(1)}, r_{ij}^{(2)}, \cdots, r_{ij}^{(l)}), \quad i = 1,2,\cdots,n;\ \ j = 1,2,\cdots,m \tag{1.70}$$

to aggregate all the intuitionistic fuzzy decision matrices $R_k = (r_{ij}^{(k)})_{n\times m}$ $(k = 1,2,\cdots,l)$ into a collective intuitionistic fuzzy decision matrix $\dot{R} = (\dot{r}_{ij})_{n\times m}$ (or $\ddot{R} = (\ddot{r}_{ij})_{n\times m}$), where $\dot{r}_{ij} = (\dot{\mu}_{ij}, \dot{v}_{ij})$, $\ddot{r}_{ij} = (\ddot{\mu}_{ij}, \ddot{v}_{ij})$, $i = 1,2,\cdots,n;\ j = 1,2,\cdots,m$, and $w = (w_1, w_2, \cdots, w_m)^{\mathrm{T}}$ is the weighting vector associated with the IFHA and IFHG operators, with $w_j \in [0,1]\,(j = 1,2,\cdots,m)$ and $\sum_{j=1}^{m} w_j = 1$, which can be derived by the normal distribution based method (Xu, 2005a).

If the attribute weights are predefined, i.e., the weight vector $(\omega_1, \omega_2, \cdots, \omega_m)^{\mathrm{T}}$ of the attributes G_j $(j = 1,2,\cdots,m)$ is given in advance, then based on the collective intuitionistic fuzzy decision matrix $\dot{R} = (\dot{r}_{ij})_{n\times m}$ (or $\ddot{R} = (\ddot{r}_{ij})_{n\times m}$), we can utilize the IFWA operator (or the IFWG operator):

$$\dot{r}_i = \text{IFWA}_{\omega}(\dot{r}_{i1}, \dot{r}_{i2}, \cdots, \dot{r}_{im}), \quad i = 1,2,\cdots,n \tag{1.71}$$

or

$$\ddot{r}_i = \text{IFWG}_{\omega}(\ddot{r}_{i1}, \ddot{r}_{i2}, \cdots, \ddot{r}_{im}), \quad i = 1,2,\cdots,n \tag{1.72}$$

to get the overall attribute value of the alternative Y_i.

We can utilize Eq.(1.10) to calculate the scores $s(\dot{r}_i)$ (or $s(\ddot{r}_i)$) $(i = 1,2,\cdots,n)$ of the overall attribute values $\dot{r}_i$ (or $\ddot{r}_i$) $(i = 1,2,\cdots,n)$ of the alternatives Y_i $(i = 1,2,\cdots,n)$, and then utilize the scores $s(\dot{r}_i)$ (or $s(\ddot{r}_i)$) $(i = 1,2,\cdots,n)$ to rank and select the alternatives Y_i $(i = 1,2,\cdots,n)$ (if two scores $s(\dot{r}_i)$ (or $s(\ddot{r}_i)$) and $s(\dot{r}_j)$ (or $s(\ddot{r}_j)$) are equal, then we need to calculate respectively the accuracy degree $h(\dot{r}_i)$ (or

$h(\ddot{r}_i)$) and $h(\dot{r}_j)$ (or $h(\ddot{r}_j)$) of the overall attribute values $\dot{r}_i$ (or $\ddot{r}_i$) and $\dot{r}_j$ (or $\ddot{r}_j$). After that, we can utilize $h(\dot{r}_i)$ (or $h(\ddot{r}_i)$) and $h(\dot{r}_j)$ (or $h(\ddot{r}_j)$) to rank the alternatives Y_i and Y_j).

We will now discuss situations where the information about the attribute weights is incomplete (Xu, 2007i):

From Section 1.1, we can see that the score $s(\alpha)$ can be used as an index to measure the magnitude of the IFN α. This means that we can utilize the score function to define the following concept (for convenience, we denote both the collective intuitionistic fuzzy decision matrices derived by the IFHA and IFHG operators as $R = (r_{ij})_{n\times m}$):

Definition 1.3.7 (Xu, 2007i) Let $R = (r_{ij})_{n\times m}$ be the collective intuitionistic fuzzy decision matrix. Then $S = (s_{ij})_{n\times m}$ is called the score matrix of $R = (r_{ij})_{n\times m}$, where

$$s_{ij} = s(r_{ij}) = \mu_{ij} - \nu_{ij}, \quad i = 1, 2, \cdots, n; \quad j = 1, 2, \cdots, m \tag{1.73}$$

and $s(d_{ij})$ is the score of the IFN r_{ij}.

Based on the score matrix, the overall attribute value of each alternative can be expressed as:

$$s_i(\omega) = \sum_{j=1}^{m} \omega_j s_{ij}, \quad i = 1, 2, \cdots, m \tag{1.74}$$

Obviously, the larger $s_i(\omega)$, the better the alternative Y_i. If we only consider the single alternative Y_i, then a reasonable vector $\omega = (\omega_1, \omega_2, \cdots, \omega_m)^{\mathrm{T}}$ of the attribute weights should be obtained by maximizing $s_i(\omega)$. Consequently, we can establish the following optimization model:

(M-1.1)
$$\begin{aligned} \max \quad & s_i(\omega) = \sum_{j=1}^{m} \omega_j s_{ij} \\ \text{s.t.} \quad & \omega = (\omega_1, \omega_2, \cdots, \omega_m)^{\mathrm{T}} \in \Lambda \\ & \omega_j \geqslant 0,\ j = 1, 2, \cdots, m,\ \sum_{j=1}^{m} \omega_j = 1 \end{aligned}$$

By solving the model (M-1.1), we can get the optimal solution $\omega^{(i)} = (\omega_1^{(i)}, \omega_2^{(i)}, \cdots, \omega_m^{(i)})^{\mathrm{T}}$ corresponding to the alternative Y_i.

However, in the process of determining the weight vector of attribute weights, it generally needs to consider all the alternatives Y_i $(i = 1, 2, \cdots, n)$ simultaneously. For this purpose, we consider a combination of the weight vectors:

$$\omega = v_1\omega^{(1)} + v_2\omega^{(2)} + \cdots + v_m\omega^{(m)}$$

$$= \begin{bmatrix} \omega_1^{(1)} & \omega_1^{(2)} & \cdots & \omega_1^{(m)} \\ \omega_2^{(1)} & \omega_2^{(2)} & \cdots & \omega_2^{(m)} \\ \vdots & \vdots & & \vdots \\ \omega_n^{(1)} & \omega_n^{(2)} & \cdots & \omega_n^{(m)} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix} = Wv \tag{1.75}$$

where

$$W = \begin{bmatrix} \omega_1^{(1)} & \omega_1^{(2)} & \cdots & \omega_1^{(m)} \\ \omega_2^{(1)} & \omega_2^{(2)} & \cdots & \omega_2^{(m)} \\ \vdots & \vdots & & \vdots \\ \omega_n^{(1)} & \omega_n^{(2)} & \cdots & \omega_n^{(m)} \end{bmatrix} \tag{1.76}$$

and $v = (v_1, v_2, \cdots, v_m)^{\mathrm{T}}$ is a non-negative vector to be determined, with:

$$v^{\mathrm{T}} v = 1 \tag{1.77}$$

Let $s_i = (s_{i1}, s_{i2}, \cdots, s_{im})^{\mathrm{T}}$ $(i = 1, 2, \cdots, n)$. Then the score matrix S can be expressed as $S = (s_1, s_2, \cdots, s_n)^{\mathrm{T}}$. By Eqs.(1.74) and (1.75), we get

$$s_i(\omega) = \sum_{j=1}^{m} \omega_j s_{ij} = \omega^{\mathrm{T}} s_i = (Wv)^{\mathrm{T}} s_i \tag{1.78}$$

In order to determine the combined weight vector $(\omega_1, \omega_2, \cdots, \omega_m)^{\mathrm{T}}$, we shall maximize the overall attribute values $s_i(\omega)$ $(i = 1, 2, \cdots, n)$, which is to maximize $s(\omega) = (s_1(\omega), s_2(\omega), \cdots, s_n(\omega))$ under the constraint (1.77). Accordingly, we can establish the following multi-objective optimization model:

(M-1.2) max $s(\omega) = (s_1(\omega), s_2(\omega), \cdots, s_n(\omega))$
s.t. $v^{\mathrm{T}} v = 1$

By the equal weighted summation method (French et al., 1983), the model (M-1.2) can be transformed into a single objective optimization problem:

(M-1.3) max $s(\omega)^{\mathrm{T}} s(\omega)$
s.t. $v^{\mathrm{T}} v = 1$

Let $f(v) = s(\omega)^{\mathrm{T}} s(\omega)$. Then by Eq.(1.78), we have

$$f(v) = s(\omega)^{\mathrm{T}} s(\omega) = v^{\mathrm{T}} (S^{\mathrm{T}} W)^{\mathrm{T}} (S^{\mathrm{T}} W) v \tag{1.79}$$

Let $\hat{S} = (S^{\mathrm{T}} W)^{\mathrm{T}} (S^{\mathrm{T}} W)$. Then $\hat{S}^{\mathrm{T}} = (S^{\mathrm{T}} W)^{\mathrm{T}} (S^{\mathrm{T}} W) = \hat{S}$, i.e., $\hat{S}$ is a real symmetrical matrix. Moreover, $\hat{S} \geqslant 0$. Therefore, $\hat{S}$ is a nonnegative definite matrix.

Theorem 1.3.16 (Horn and Johnson, 1990) Let U be a real symmetrical matrix, i.e., $U^{\mathrm{T}} = U$. Then

$$\max \frac{v^{\mathrm{T}} U v}{v^{\mathrm{T}} v} = \lambda_{\max} \tag{1.80}$$

where $\lambda_{\max}$ is the largest eigenvalue of U, and v is a nonzero vector.

Theorem 1.3.17 (Horn and Johnson, 1990) Let U be a real irreducible nonnegative matrix. Then

(1) U has the largest eigenvalue $\lambda_{\max}$, which is also a unique eigenvalue of U.

(2) Let $v = (v_1, v_2, \cdots, v_m)^{\mathrm{T}}$ be the eigenvector of $\lambda_{\max}$. Then $v_j > 0$ $(j = 1, 2, \cdots, m)$, i.e., v is a positive eigenvector.

By Theorems 1.3.16 and 1.3.17, we know that $f(v)$ has the largest value $\max f(v)$, which is also the largest eigenvalue $\bar{\lambda}_{\max}$ of $\hat{S}$. $v = (v_1, v_2, \cdots, v_m)^{\mathrm{T}}$ is the eigenvector of $\bar{\lambda}_{\max}$, where $\bar{\lambda}_{\max}$ is unique, and all $v_j > 0$ $(j = 1, 2, \cdots, m)$.

After normalizing the eigenvector v, we can utilize Eq.(1.75) to derive the weight vector $\omega = (\omega_1, \omega_2, \cdots, \omega_m)^{\mathrm{T}}$.

Based on the analysis above and the IFHA (or IFHG) operator, we now introduce an approach to multi-person multi-attribute intuitionistic fuzzy decision making with incomplete attribute weight information (Xu, 2007i):

Step 1 Utilize the IFHA operator (or IFHG) operator to aggregate all individual intuitionistic fuzzy decision matrices $R_k = (r_{ij}^{(k)})_{n\times m}$ $(k = 1, 2, \cdots, l)$ into the collective intuitionistic fuzzy decision matrix $R = (r_{ij})_{n\times m}$.

Step 2 Calculate the score matrix $S = (s_{ij})_{n\times m}$ of the collective intuitionistic fuzzy decision matrix D.

Step 3 Utilize the model (M-1.1) to get the optimal weight vectors $\omega^{(i)} = (\omega_1^{(i)}, \omega_2^{(i)}, \cdots, \omega_m^{(i)})^{\mathrm{T}}$ $(i = 1, 2, \cdots, n)$ corresponding to the alternatives Y_i $(i = 1, 2, \cdots, n)$, and construct the weight matrix W.

Step 4 Calculate the normalized eigenvector $v = (v_1, v_2, \cdots, v_m)^{\mathrm{T}}$ of the matrix $(S^{\mathrm{T}}W)^{\mathrm{T}}(S^{\mathrm{T}}W)$.

Step 5 Utilize Eq.(1.75) to get the weight vector $\omega = (\omega_1, \omega_2, \cdots, \omega_m)^{\mathrm{T}}$.

The other steps are the same as those with the attribute weights known completely.

Example 1.3.5 (Xu, 2007i) Consider a problem in a manufacturing company, which aims to search for the best global supplier for one of its most critical parts used in its assembling process (Chan and Kumar, 2007). Five potential global suppliers Y_i $(i = 1, 2, 3, 4, 5)$ have been identified. The attributes to be considered in the selection process are: ① G_1: Total cost of the product; ② G_2: Quality of the product; ③ G_3: Service performance of the supplier; ④ G_4: Supplier's profile; and ⑤ G_5: Risk factor. An expert group is formed which consists of four experts (decision makers): E_k $(k = 1, 2, 3, 4)$ (whose weight vector is $\xi = (0.3,\ 0.2,\ 0.3,\ 0.2)^{\mathrm{T}}$). The experts E_k $(k = 1, 2, 3, 4)$ represent, respectively, the characteristics of the potential global suppliers Y_i $(i = 1, 2, 3, 4, 5)$ by the IFNs $r_{ij}^{(k)}$ $(i, j = 1, 2, 3, 4, 5)$ with respect to the attributes G_i $(i = 1, 2, 3, 4, 5)$, as listed in Tables 1.6–1.9 (i.e., intuitionistic fuzzy decision matrices $R'_k = ({r'}_{ij}^{(k)})_{5\times 5}$ $(k = 1, 2, 3, 4)$):

Table 1.6 Intuitionistic fuzzy decision matrix R'_1 (Xu, 2007i)

	G_1	G_2	G_3	G_4	G_5
Y_1	(0.4, 0.5)	(0.5, 0.2)	(0.6, 0.2)	(0.8, 0.1)	(0.3, 0.7)
Y_2	(0.6, 0.2)	(0.7, 0.2)	(0.3, 0.4)	(0.5, 0.1)	(0.2, 0.8)
Y_3	(0.7, 0.3)	(0.8, 0.1)	(0.5, 0.5)	(0.3, 0.2)	(0.3, 0.6)
Y_4	(0.3, 0.4)	(0.7, 0.1)	(0.6, 0.1)	(0.4, 0.3)	(0.1, 0.9)
Y_5	(0.8, 0.1)	(0.3, 0.4)	(0.4, 0.5)	(0.7, 0.2)	(0.2, 0.5)

Table 1.7 Intuitionistic fuzzy decision matrix R'_2 (Xu, 2007i)

	G_1	G_2	G_3	G_4	G_5
Y_1	(0.5, 0.3)	(0.6, 0.1)	(0.7, 0.3)	(0.7, 0.1)	(0.2, 0.8)
Y_2	(0.7, 0.2)	(0.6, 0.2)	(0.4, 0.4)	(0.6, 0.2)	(0.3, 0.7)
Y_3	(0.5, 0.3)	(0.7, 0.2)	(0.6, 0.3)	(0.4, 0.2)	(0.1, 0.6)
Y_4	(0.5, 0.4)	(0.8, 0.1)	(0.4, 0.2)	(0.7, 0.2)	(0.3, 0.7)
Y_5	(0.7, 0.3)	(0.5, 0.4)	(0.6, 0.3)	(0.6, 0.2)	(0.1, 0.5)

Table 1.8 Intuitionistic fuzzy decision matrix R'_3 (Xu, 2007i)

	G_1	G_2	G_3	G_4	G_5
Y_1	(0.6, 0.3)	(0.5, 0.2)	(0.6, 0.4)	(0.8, 0.1)	(0.3, 0.7)
Y_2	(0.8, 0.2)	(0.5, 0.3)	(0.6, 0.4)	(0.5, 0.2)	(0.3, 0.6)
Y_3	(0.6, 0.1)	(0.8, 0.2)	(0.7, 0.3)	(0.4, 0.2)	(0.1, 0.8)
Y_4	(0.6, 0.3)	(0.6, 0.1)	(0.5, 0.4)	(0.9, 0.1)	(0.2, 0.5)
Y_5	(0.8, 0.1)	(0.6, 0.2)	(0.7, 0.3)	(0.5, 0.2)	(0.1, 0.7)

Table 1.9 Intuitionistic fuzzy decision matrix R'_4 (Xu, 2007i)

	G_1	G_2	G_3	G_4	G_5
Y_1	(0.3, 0.4)	(0.9, 0.1)	(0.8, 0.1)	(0.5, 0.5)	(0.6, 0.4)
Y_2	(0.7, 0.1)	(0.7, 0.3)	(0.4, 0.2)	(0.8, 0.2)	(0.1, 0.3)
Y_3	(0.4, 0.1)	(0.5, 0.2)	(0.8, 0.1)	(0.6, 0.2))	(0.3, 0.6)
Y_4	(0.8, 0.2)	(0.5, 0.1)	(0.6, 0.4)	(0.7, 0.2)	(0.2, 0.7)
Y_5	(0.6, 0.1)	(0.8, 0.2)	(0.7, 0.2)	(0.6, 0.3)	(0.1, 0.8)

Since G_5 is an attribute of cost type different from the benefit type of the other attributes, we employ Eq.(1.65) to normalize $R'_k = ({r'}_{ij}^{(k)})_{5\times5}$ $(k = 1, 2, 3, 4)$ into the intuitionistic fuzzy decision matrices $R_k = (r_{ij}^{(k)})_{5\times5}$ $(k = 1, 2, 3, 4)$(Tables 1.10–1.13) respectively:

Table 1.10 Intuitionistic fuzzy decision matrix R_1 (Xu, 2007i)

	G_1	G_2	G_3	G_4	G_5
Y_1	(0.4, 0.5)	(0.5, 0.2)	(0.6, 0.2)	(0.8, 0.1)	(0.7, 0.3)
Y_2	(0.6, 0.2)	(0.7, 0.2)	(0.3, 0.4)	(0.5, 0.1)	(0.8, 0.2)
Y_3	(0.7, 0.3)	(0.8, 0.1)	(0.5, 0.5)	(0.3, 0.2)	(0.6, 0.3)
Y_4	(0.3, 0.4)	(0.7, 0.1)	(0.6, 0.1)	(0.4, 0.3)	(0.9, 0.1)
Y_5	(0.8, 0.1)	(0.3, 0.4)	(0.4, 0.5)	(0.7, 0.2)	(0.5, 0.2)

Table 1.11 Intuitionistic fuzzy decision matrix R_2 (Xu, 2007i)

	G_1	G_2	G_3	G_4	G_5
Y_1	(0.5, 0.3)	(0.6, 0.1)	(0.7, 0.3)	(0.7, 0.1)	(0.8, 0.2)
Y_2	(0.7, 0.2)	(0.6, 0.2)	(0.4, 0.4)	(0.6, 0.2)	(0.7, 0.3)
Y_3	(0.5, 0.3)	(0.7, 0.2)	(0.6, 0.3)	(0.4, 0.2)	(0.6, 0.1)
Y_4	(0.5, 0.4)	(0.8, 0.1)	(0.4, 0.2)	(0.7, 0.2)	(0.7, 0.3)
Y_5	(0.7, 0.3)	(0.5, 0.4)	(0.6, 0.3)	(0.6, 0.2)	(0.5, 0.1)

Table 1.12 Intuitionistic fuzzy decision matrix R_3 (Xu, 2007i)

	G_1	G_2	G_3	G_4	G_5
Y_1	(0.6, 0.3)	(0.5, 0.2)	(0.6, 0.4)	(0.8, 0.1)	(0.7, 0.3)
Y_2	(0.8, 0.2)	(0.5, 0.3)	(0.6, 0.4)	(0.5, 0.2)	(0.6, 0.3)
Y_3	(0.6, 0.1)	(0.8, 0.2)	(0.7, 0.3)	(0.4, 0.2)	(0.8, 0.1)
Y_4	(0.6, 0.3)	(0.6, 0.1)	(0.5, 0.4)	(0.9, 0.1)	(0.5, 0.2)
Y_5	(0.8, 0.1)	(0.6, 0.2)	(0.7, 0.3)	(0.5, 0.2)	(0.7, 0.1)

Table 1.13 Intuitionistic fuzzy decision matrix R_4 (Xu, 2007i)

	G_1	G_2	G_3	G_4	G_5
Y_1	(0.3, 0.4)	(0.9, 0.1)	(0.8, 0.1)	(0.5, 0.5)	(0.4, 0.6)
Y_2	(0.7, 0.1)	(0.7, 0.3)	(0.4, 0.2)	(0.8, 0.2)	(0.3, 0.1)
Y_3	(0.4, 0.1)	(0.5, 0.2)	(0.8, 0.1)	(0.6, 0.2))	(0.6, 0.3)
Y_4	(0.8, 0.2)	(0.5, 0.1)	(0.6, 0.4)	(0.7, 0.2)	(0.7, 0.2)
Y_5	(0.6, 0.1)	(0.8, 0.2)	(0.7, 0.2)	(0.6, 0.3)	(0.8, 0.1)

Suppose that the weight vector $\omega = (0.2,\ 0.15,\ 0.2,\ 0.3,\ 0.15)^{\mathrm{T}}$. If the information about attribute weights is completely known in advance, then we can first utilize the score weighting function (Chen and Tan, 1994) to handle the above problem, which involves the following steps:

Step 1 Utilize Eq.(1.73) to construct the score matrices $S_k = (s_{ij}^{(k)})_{5\times 5}$ $(k = 1, 2, 3, 4)$ of $R_k = (r_{ij}^{(k)})_{5\times 5}$ $(k = 1, 2, 3, 4)$ (Tables 1.14–1.17) respectively.

Table 1.14 Score matrix S_1 (Xu, 2007i)

	G_1	G_2	G_3	G_4	G_5
Y_1	−0.1	0.3	0.4	0.7	0.4
Y_2	0.4	0.5	−0.1	0.4	0.6
Y_3	0.4	0.7	0	0.1	0.3
Y_4	−0.1	0.6	0.5	0.1	0.8
Y_5	0.7	−0.1	−0.1	0.5	0.3

Table 1.15 Score matrix S_2 (Xu, 2007i)

	G_1	G_2	G_3	G_4	G_5
Y_1	0.2	0.5	0.4	0.6	0.6
Y_2	0.5	0.4	0	0.4	0.4
Y_3	0.2	0.5	0.3	0.2	0.5
Y_4	0.1	0.7	0.2	0.5	0.4
Y_5	0.4	0.1	0.3	0.4	0.4

Table 1.16 Score matrix S_3 (Xu, 2007i)

	G_1	G_2	G_3	G_4	G_5
Y_1	0.3	0.3	0.2	0.7	0.4
Y_2	0.6	0.2	0.2	0.3	0.3
Y_3	0.5	0.6	0.4	0.2	0.7
Y_4	0.3	0.5	0.1	0.8	0.3
Y_5	0.7	0.4	0.4	0.3	0.6

Table 1.17 Score matrix S_4 (Xu, 2007i)

	G_1	G_2	G_3	G_4	G_5
Y_1	−0.1	0.8	0.7	0	−0.2
Y_2	0.6	0.4	0.2	0.6	0.2
Y_3	0.3	0.3	0.7	0.4	0.3
Y_4	0.6	0.4	0.2	0.5	0.5
Y_5	0.5	0.6	0.5	0.3	0.7

Step 2 Utilize the weight vector $\xi = (0.3,\ 0.2,\ 0.3,\ 0.2)^{\mathrm{T}}$ and the score weighting function:

$$s_{ij} = \sum_{k=1}^{4} \xi_k s_{ij}^{(k)}, \quad k = 1, 2, 3, 4$$

to aggregate the individual score matrices $S_k = (s_{ij}^{(k)})_{5\times 5}$ $(k = 1, 2, 3, 4)$ into the collective score matrix $S = (s_{ij})_{5\times 5}$ (Table 1.18):

Table 1.18 Collective score matrix S (Xu, 2007i)

	G_1	G_2	G_3	G_4	G_5
Y_1	0.08	0.52	0.37	0.20	0.60
Y_2	0.44	0.37	0.55	0.55	0.23
Y_3	0.44	0.07	0.28	0.25	0.25
Y_4	0.47	0.38	0.19	0.39	0.35
Y_5	0.28	0.36	0.39	0.48	0.43

Step 3 Utilize the weight vector $\omega = (0.2,\ 0.15,\ 0.2,\ 0.3,\ 0.15)^{\mathrm{T}}$ and the score weighting function:

$$s_i = \sum_{j=1}^{5} \omega_j s_{ij}, \quad i = 1, 2, 3, 4, 5$$

to get the overall scores s_i $(i = 1, 2, 3, 4, 5)$ of the alternatives Y_i $(i = 1, 2, 3, 4, 5)$:

$$s_1 = 0.3530, \quad s_2 = 0.3430, \quad s_3 = 0.3280, \quad s_4 = 0.3615, \quad s_5 = 0.3740$$

Thus

$$s_5 > s_4 > s_1 > s_2 > s_3$$

Step 4 Rank the alternatives Y_i $(i = 1, 2, 3, 4, 5)$ according to the overall scores s_i $(i = 1, 2, 3, 4, 5)$:

$$Y_5 \succ Y_4 \succ Y_1 \succ Y_2 \succ Y_3$$

Then, the most desirable global supplier is Y_5.

If we apply the procedure proposed above to tackle the problem, then we can first determine the associated vector $w = (0.155,\ 0.345,\ 0.345,\ 0.155)^{\mathrm{T}}$ of the IFHG operator by using the normal distribution based method (Xu, 2005a), and then use the IFHG operator (1.70) to aggregate the individual intuitionistic fuzzy decision matrices $R_k = (r_{ij}^{(k)})_{5\times 5}$ $(k = 1, 2, 3, 4)$ into the collective intuitionistic fuzzy decision matrix $R = (r_{ij})_{5\times 5}$ (Table 1.19):

Table 1.19 Collective intuitionistic fuzzy decision matrix R (Xu, 2007i)

	G_1	G_2	G_3	G_4	G_5
Y_1	(0.449, 0.370)	(0.565, 0.162)	(0.705, 0.232)	(0.730, 0.170)	(0.646, 0.354)
Y_2	(0.719, 0.188)	(0.630, 0.232)	(0.448, 0.378)	(0.557, 0.160)	(0.597, 0.192)
Y_3	(0.546, 0.192)	(0.727, 0.182)	(0.641, 0.322)	(0.399, 0.200)	(0.658, 0.192)
Y_4	(0.520, 0.337)	(0.630, 0.100)	(0.539, 0.271)	(0.679, 0.188)	(0.708, 0.198)
Y_5	(0.727, 0.128)	(0.520, 0.299)	(0.619, 0.318)	(0.618, 0.229)	(0.609, 0.120)

Based on the collective intuitionistic fuzzy decision matrix R, we can utilize the IFWG operator (1.72) to get the overall values r_i $(i = 1, 2, 3, 4, 5)$ of the alternatives Y_i $(i = 1, 2, 3, 4, 5)$:

$$r_1 = (0.6215,\ 0.2541), \quad r_2 = (0.5776,\ 0.2293), \quad r_3 = (0.5509,\ 0.2207)$$

$$r_4 = (0.6116,\ 0.2265), \quad r_5 = (0.6209,\ 0.2245)$$

Then we can utilize Eq.(1.10) to calculate the scores $s(r_i)$ $(i = 1, 2, 3, 4, 5)$ of the overall values r_i $(i = 1, 2, 3, 4, 5)$:

$$s(r_1) = 0.3674, \quad s(r_2) = 0.3483, \quad s(r_3) = 0.3302$$

$$s(r_4) = 0.3851, \quad s(r_5) = 0.3964$$

Thus

$$s(r_5) > s(r_4) > s(r_1) > s(r_2) > s(r_3)$$

which yields:

$$Y_5 \succ Y_4 \succ Y_1 \succ Y_2 \succ Y_3$$

Consequently, the most desirable global supplier is Y_5.

Compared to the former approach, the latter can not only relieve the influence of unfair arguments on the decision results, but also avoid losing or distorting the original decision information in the process of aggregation.

The decision makers may, however, only provide the information about attribute weights with value ranges or order relations, perhaps because of time pressure or lack of sufficient knowledge. For example, the information about attribute weights given by the decision makers may be described as follows, respectively:

E_1 : $\omega_1 \leqslant 0.3, \quad 0.2 \leqslant \omega_3 \leqslant 0.5$;

E_2 : $0.1 \leqslant \omega_2 \leqslant 0.2, \quad \omega_5 \leqslant 0.4$;

E_3 : $\omega_3 - \omega_2 \geqslant \omega_5 - \omega_4, \quad \omega_4 \geqslant \omega_1$;

E_4 : $\omega_3 - \omega_1 \leqslant 0.1, \quad 0.1 \leqslant \omega_4 \leqslant 0.3$.

Then

$$\Lambda = \{\omega_1 \leqslant 0.3,\ 0.2 \leqslant \omega_3 \leqslant 0.5,\ 0.1 \leqslant \omega_2 \leqslant 0.2,\ \omega_5 \leqslant 0.4,\ \omega_3 - \omega_2 \geqslant \omega_5 - \omega_4,$$
$$\omega_4 \geqslant \omega_1,\ \omega_3 - \omega_1 \leqslant 0.1,\ 0.1 \leqslant \omega_4 \leqslant 0.3\}$$

Many approaches have been developed to deal with decision making problems based on intuitionistic fuzzy set theory. However, none of them seems to be suitable to handle this case. In the following, we shall select the best global supplier following the procedure we propose above (Xu, 2007i):

Step 1 Calculate the score matrix of the collective intuitionistic fuzzy decision matrix D (Table 1.20):

Table 1.20 Score matrix S (Xu, 2007i)

	G_1	G_2	G_3	G_4	G_5
Y_1	0.079	0.403	0.473	0.560	0.292
Y_2	0.531	0.398	0.070	0.397	0.405
Y_3	0.354	0.545	0.319	0.199	0.466
Y_4	0.183	0.530	0.268	0.491	0.510
Y_5	0.599	0.221	0.301	0.389	0.489

Step 2 Utilize the model (M-1.1) to calculate the optimal vectors $\omega^{(i)} = (\omega_1^{(i)}, \omega_2^{(i)}, \omega_3^{(i)}, \omega_4^{(i)}, \omega_5^{(i)})^{\mathrm{T}}$ of attribute weights corresponding to each alternative Y_i:

$$\omega^{(1)} = (0.1,\ 0.2,\ 0.2,\ 0.3,\ 0.2)^{\mathrm{T}}$$

$$\omega^{(2)} = (0.3,\ 0.1,\ 0.2,\ 0.3,\ 0.1)^{\mathrm{T}}$$

$$\omega^{(3)} = (0.16,\ 0.2,\ 0.26,\ 0.16,\ 0.22)^{\mathrm{T}}$$

$$\omega^{(4)} = (0.1,\ 0.2,\ 0.2,\ 0.25,\ 0.25)^{\mathrm{T}}$$

$$\omega^{(5)} = (0.3,\ 0.1,\ 0.2,\ 0.3,\ 0.1)^{\mathrm{T}}$$

and construct the weight matrix:

$$W = \begin{bmatrix} 0.1 & 0.3 & 0.16 & 0.1 & 0.3 \\ 0.2 & 0.1 & 0.2 & 0.2 & 0.1 \\ 0.2 & 0.2 & 0.26 & 0.2 & 0.2 \\ 0.3 & 0.3 & 0.16 & 0.25 & 0.3 \\ 0.2 & 0.1 & 0.22 & 0.25 & 0.1 \end{bmatrix}$$

Thus,

$$(S^{\mathrm{T}}W)^{\mathrm{T}}(S^{\mathrm{T}}W) = \begin{bmatrix} 0.7443 & 0.7101 & 0.7222 & 0.7462 & 0.7101 \\ 0.7099 & 0.6871 & 0.6915 & 0.7122 & 0.6871 \\ 0.7222 & 0.6917 & 0.7033 & 0.7248 & 0.6917 \\ 0.7752 & 0.7302 & 0.7454 & 0.7685 & 0.7302 \\ 0.7204 & 0.6872 & 0.6917 & 0.7124 & 0.6872 \end{bmatrix}$$

Step 3 Calculate the normalized eignvector v of the matrix $(S^{\mathrm{T}}W)^{\mathrm{T}}(S^{\mathrm{T}}W)$:

$$v = (0.2029,\ 0.1948,\ 0.1974,\ 0.2095,\ 0.1954)^{\mathrm{T}}$$

Step 4 Utilize Eq.(1.75) to get the weight vector ω:

$$\omega = Wv = \begin{bmatrix} 0.1 & 0.3 & 0.16 & 0.1 & 0.3 \\ 0.2 & 0.1 & 0.2 & 0.2 & 0.1 \\ 0.2 & 0.2 & 0.26 & 0.2 & 0.2 \\ 0.3 & 0.3 & 0.16 & 0.25 & 0.3 \\ 0.2 & 0.1 & 0.22 & 0.25 & 0.1 \end{bmatrix} \begin{bmatrix} 0.2029 \\ 0.1948 \\ 0.1974 \\ 0.2095 \\ 0.1954 \end{bmatrix}$$
$$= (0.1899,\ 0.1610,\ 0.2118,\ 0.2619,\ 0.1754)^{\mathrm{T}}$$

Step 5 Utilize the IFWG operator (1.72) to calculate the overall attribute values r_i $(i = 1, 2, 3, 4, 5)$ of the alternatives Y_i $(i = 1, 2, 3, 4, 5)$:

$$r_1 = (0.6211,\ 0.2570), \quad r_2 = (0.5767,\ 0.2331), \quad r_3 = (0.5633,\ 0.2218)$$

$$r_4 = (0.6122,\ 0.2249), \quad r_5 = (0.6187,\ 0.2248)$$

Step 6 Utilize Eq.(1.10) to get the scores $s(r_i)$ $(i = 1, 2, 3, 4, 5)$ of the overall attribute values r_i $(i = 1, 2, 3, 4, 5)$ corresponding to the alternatives Y_i $(i = 1, 2, 3, 4, 5)$:

$$s(r_1) = 0.3641, \quad s(r_2) = 0.3436, \quad s(r_3) = 0.3415$$

$$s(r_4) = 0.3873, \quad s(r_5) = 0.3939$$

Thus

$$s(r_5) > s(r_4) > s(r_1) > s(r_2) > s(r_3)$$

Step 7 Utilize the scores $s(r_i)$ $(i = 1, 2, 3, 4, 5)$ to rank the alternatives Y_i $(i = 1, 2, 3, 4, 5)$:

$$Y_5 \succ Y_4 \succ Y_1 \succ Y_2 \succ Y_3$$

To conclude, the most desirable global supplier is Y_5.

If the information about the attribute weights in the problem considered is completely unknown, then we need to determine the attribute weights in advance. To do so, Xu (2010a) introduces the following definition:

Definition 1.3.8 (Xu, 2010a) Let $\alpha_1 = (\mu_{\alpha_1}, \nu_{\alpha_1})$ and $\alpha_2 = (\mu_{\alpha_2}, \nu_{\alpha_2})$ be two IFNs. Then we call

$$d(\alpha_1, \alpha_2) = \frac{1}{2}(|\mu_{\alpha_1} - \mu_{\alpha_2}| + |\nu_{\alpha_1} - \nu_{\alpha_2}|) \tag{1.81}$$

the distance between α_1 and α_2.

By Definition 1.3.8, we have

Theorem 1.3.18 (Xu, 2010a) Let α_i $(i = 1, 2, 3)$ be any three IFNs. Then

(1) $0 \leqslant d(\alpha_1, \alpha_2) \leqslant 1$, especially, $d(\alpha_1, \alpha_1) = 0$;

(2) $d(\alpha_1, \alpha_2) = d(\alpha_2, \alpha_1)$;

(3) $d(\alpha_1, \alpha_3) \leqslant d(\alpha_1, \alpha_2) + d(\alpha_2, \alpha_3)$.

According to the information theory, if all alternatives have similar attribute values with respect to an attribute, then a small weight should be assigned to this attribute, because such an attribute does not help in differentiating alternatives (Zeleny, 1982). As a result, based on the collective intuitionistic fuzzy decision matrix $R = (r_{ij})_{n\times m}$ and by Eq.(1.81), we introduce the deviation between the alternative Y_i and all the other alternatives with respect to the attribute G_j:

$$d_{ij}(\omega) = \sum_{k\neq i} d(r_{ij}, r_{kj})\,\omega_j, \quad i = 1, 2, \cdots, n; \quad j = 1, 2, \cdots, m \tag{1.82}$$

Let

$$d_j(\omega) = \sum_{i=1}^{n}\sum_{k\neq i} d(r_{ij}, r_{ik})\,\omega_j, \quad j = 1, 2, \cdots, m \tag{1.83}$$

denote the sum of all the deviations $d_{ij}(\omega)$ $(j = 1, 2, \cdots, m)$, and construct the deviation function:

$$d(\omega) = \sum_{j=1}^{m} d_j(\omega) = \sum_{j=1}^{m}\sum_{i=1}^{n} d_{ij}(\omega) = \sum_{j=1}^{m}\sum_{i=1}^{n}\sum_{k\neq i} d(r_{ij}, r_{kj})\,\omega_j \tag{1.84}$$

Obviously, a reasonable vector of attribute weights $\omega = (\omega_1, \omega_2, \cdots, \omega_m)^{\mathrm{T}}$ should be determined so as to maximize $d(\omega)$. For this purpose, we establish the following optimization model (Xu, 2010b):

$$\begin{aligned} \max \quad & d(\omega) = \sum_{j=1}^{m}\sum_{i=1}^{n}\sum_{k\neq i} d(r_{ij}, r_{kj})\,\omega_j \\ \text{s.t.} \quad & \sum_{j=1}^{m}\omega_j^2 = 1,\ \omega_j \in [0, 1],\ \ j = 1, 2, \cdots, m \end{aligned}$$

To solve this model, we construct the Lagrange function:

$$L(\omega,\varsigma) = d(\omega) + \frac{1}{2}\varsigma\left(\sum_{j=1}^{m}\omega_j^2 - 1\right) \tag{1.85}$$

where ς is the Lagrange multiplier.

Differentiating Eq.(1.85) with respect to ω_i $(i = 1, 2, \cdots, m)$ and ς, and setting these partial derivatives equal to zero, we can obtain the following set of equations:

$$\begin{cases} \dfrac{\partial L(\omega,\varsigma)}{\partial \omega_j} = \displaystyle\sum_{i=1}^{n}\sum_{k\neq i} d(r_{ij}, r_{kj}) + \varsigma\,\omega_j = 0 \\ \dfrac{\partial L(\omega,\varsigma)}{\partial \varsigma} = \displaystyle\sum_{j=1}^{m}\omega_j^2 - 1 = 0 \end{cases} \tag{1.86}$$

By solving Eq.(1.86), we can get the optimal solution $\omega^* = (\omega_1^*, \omega_2^*, \cdots, \omega_m^*)^{\mathrm{T}}$, where

$$\omega_j^* = \frac{\displaystyle\sum_{i=1}^{n}\sum_{k\neq i} d(r_{ij}, r_{kj})}{\sqrt{\displaystyle\sum_{j=1}^{m}\left(\sum_{i=1}^{n}\sum_{k\neq i} d(r_{ij}, r_{kj})\right)^2}}, \quad j = 1, 2, \cdots, m \tag{1.87}$$

Obviously, $\omega_j^* \in [0, 1]$, for all j. Normalizing Eq.(1.87), we can get the normalized attribute weights (Xu, 2010b):

$$\omega_j = \frac{\displaystyle\sum_{i=1}^{n}\sum_{k\neq i} d(r_{ij}, r_{kj})}{\displaystyle\sum_{j=1}^{m}\sum_{i=1}^{n}\sum_{k\neq i} d(r_{ij}, r_{kj})}, \quad j = 1, 2, \cdots, m \tag{1.88}$$

In such a case, we have $\omega_j \in [0,\ 1]$ $(j = 1, 2, \cdots, m)$, and $\sum_{j=1}^{m}\omega_j = 1$. Then, based on the attribute weights derived by Eq.(1.88), we can utilize the approach introduced previously to rank the given alternatives and then to get the most desirable one(s).

Example 1.3.6 (Xu, 2010b) Consider a multi-attribute group decision making problem involving the prioritization of a set of information technology improvement projects (adapted from Ngwenyama and Bryson(1999)): The information management steering committee of Midwest American Manufacturing Corp. must prioritize for development and implementation a set of six information technology improvement projects Y_i $(i = 1, 2, \cdots, 6)$, which have been proposed by area managers. The committee hope that the projects are prioritized from highest to lowest according to their potential contributions to the firm's strategic goal of gaining competitive advantage in the industry. In assessing the potential contribution of each project, three factors are considered: ① G_1−Productivity; ② G_2−Differentiation; and ③ G_3− Management, whose weight vector is to be determined. The productivity factor assesses the potential of a proposed project to increase effectiveness and efficiency of the firm's manufacturing and service operations. The differentiation factor assesses the potential of a proposed project to fundamentally differentiate the firm's products and services from its competitors, and to make them more desirable to its customers. The management factor assesses the potential of a proposed project to assist management in improving their planning, controlling and decision making activities. The following is the list of proposed information systems projects, ① Y_1−Quality management information; ② Y_2−Inventory control; ③ Y_3−Customer order tracking; ④ Y_4−Materials purchasing management; ⑤ Y_5−Fleet management; and ⑥ Y_6− Design change management. Suppose that there are four decision makers E_k $(k = 1, 2, 3, 4)$, whose weight vector is $\xi = (0.4,\ 0.2,\ 0.3,\ 0.1)^{\mathrm{T}}$. These four decision makers represent, respectively, the characteristics of the projects Y_i $(i = 1,\ 2,\ \cdots, 6)$ by the IFNs r'_{ij} $(i = 1, 2, \cdots, 6;\ j = 1,\ 2, 3)$ with respect to the factors G_j $(j = 1, 2, 3)$, as listed in Tables 1.21–1.24 (i.e., intuitionistic fuzzy decision matrices $R'_k = ({r'}_{ij}^{(k)})_{6\times 3}$ $(k = 1, 2, 3, 4)$).

Since all the attributes G_j $(j = 1, 2, 3)$ are of benefit type, we do not have to normalize the given intuitionistic fuzzy decision matrices $R'_k = ({r'}_{ij}^{(k)})_{6\times 3}$ $(k = 1, 2, 3, 4)$. We can use the approach introduced above to rank the projects:

Table 1.21 Intuitionisitc fuzzy decision matrix R'_1 (Xu, 2010a)

	G_1	G_2	G_3
Y_1	(0.5, 0.3)	(0.6, 0.2)	(0.3, 0.4)
Y_2	(0.4, 0.2)	(0.5, 0.4)	(0.8, 0.1)
Y_3	(0.7, 0.1)	(0.2, 0.7)	(0.5, 0.2)
Y_4	(0.2, 0.3)	(0.5, 0.3)	(0.7, 0.1)
Y_5	(0.6, 0.1)	(0.4, 0.2)	(0.1, 0.6)
Y_6	(0.3, 0.5)	(0.7, 0.1)	(0.6, 0.2)

Table 1.22 Intuitionisitc fuzzy decision matrix R'_2 (Xu, 2010a)

	G_1	G_2	G_3
Y_1	(0.6, 0.2)	(0.7, 0.1)	(0.5, 0.4)
Y_2	(0.5, 0.4)	(0.3, 0.3)	(0.8, 0.1)
Y_3	(0.7, 0.2)	(0.4, 0.4)	(0.5, 0.2)
Y_4	(0.4, 0.5)	(0.8, 0.1)	(0.5, 0.3)
Y_5	(0.5, 0.1)	(0.7, 0.2)	(0.4, 0.5)
Y_6	(0.4, 0.2)	(0.6, 0.3)	(0.5, 0.1)

Table 1.23 Intuitionisitc fuzzy decision matrix R'_3 (Xu, 2010a)

	G_1	G_2	G_3
Y_1	(0.7, 0.2)	(0.5, 0.3)	(0.5, 0.4)
Y_2	(0.6, 0.3)	(0.4, 0.4)	(0.5, 0.2)
Y_3	(0.5, 0.1)	(0.8, 0.1)	(0.3, 0.6)
Y_4	(0.4, 0.5)	(0.7, 0.2)	(0.5, 0.3)
Y_5	(0.4, 0.4)	(0.6, 0.2)	(0.5, 0.1)
Y_6	(0.7, 0.1)	(0.4, 0.5)	(0.6, 0.2)

Table 1.24 Intuitionisitc fuzzy decision matrix R'_4 (Xu, 2010a)

	G_1	G_2	G_3
Y_1	(0.6, 0.1)	(0.4, 0.5)	(0.7, 0.1)
Y_2	(0.5, 0.2)	(0.6, 0.3)	(0.7, 0.2)
Y_3	(0.4, 0.4)	(0.7, 0.1)	(0.6, 0.3)
Y_4	(0.6, 0.3)	(0.7, 0.2)	(0.5, 0.4)
Y_5	(0.8, 0.1)	(0.4, 0.4)	(0.3, 0.5)
Y_6	(0.4, 0.5)	(0.8, 0.1)	(0.5, 0.2)

Firstly, we give the associated vector $w = (0.155,\ 0.345,\ 0.345,\ 0.155)^{\mathrm{T}}$ of the IFHA operator by using the normal distribution based method (Xu, 2005a). Based on the weight vector $\xi = (0.4,\ 0.2,\ 0.3,\ 0.1)^{\mathrm{T}}$, we utilize the IFHA operator to aggregate the individual intuitionistic fuzzy decision matrices $R'_k = ({r'}_{ij}^{(k)})_{6\times 3}$ $(k = 1, 2, 3, 4)$ into the collective intuitionistic fuzzy decision matrix $R'_k = ({r'}_{ij}^{(k)})_{6\times 3}$ (Table 1.25):

Table 1.25 Collective intuitionisitc fuzzy decision matrix R' (Xu, 2010a)

	G_1	G_2	G_3
Y_1	(0.600,0.136)	(0.584,0.109)	(0.447,0.266)
Y_2	(0.523,0.222)	(0.416,0.261)	(0.561,0.073)
Y_3	(0.613,0.084)	(0.332,0.262)	(0.601,0.207)
Y_4	(0.371,0.353)	(0.671,0.135)	(0.560,0.167)
Y_5	(0.518,0.094)	(0.441,0.114)	(0.295,0.321)
Y_6	(0.447,0.175)	(0.581,0.120)	(0.569,0.087)

Secondly, we utilize Eq.(1.88) to derive the weight vector of the attributes G_j $(j = 1, 2, 3)$:

$$\omega = (0.321,\ 0.332,\ 0.347)^{\mathrm{T}}$$

and then utilize the IFWA operator to get the overall values r_i $(i = 1, 2, \cdots, 6)$ of the alternatives Y_i $(i = 1, 2, \cdots, 6)$:

$$r_1 = (0.547,\ 0.159), \quad r_2 = (0.504,\ 0.159), \quad r_3 = (0.531,\ 0.168)$$

$$r_4 = (0.552,\ 0.198), \quad r_5 = (0.422,\ 0.153), \quad r_6 = (0.537,\ 0.121)$$

Thirdly, we utilize Eq.(1.10) to calculate the scores $s(r_i)$ $(i = 1, 2, \cdots, 6)$ of the overall values r_i $(i = 1, 2, \cdots, 6)$ of the alternatives Y_i $(i = 1, 2, \cdots, 6)$:

$$s(r_1) = 0.388, \quad s(r_2) = 0.345, \quad s(r_3) = 0.363$$

$$s(r_4) = 0.354, \quad s(r_5) = 0.269, \quad s(r_6) = 0.416$$

Since

$$s(r_6) > s(r_1) > s(r_3) > s(r_4) > s(r_2) > s(r_5)$$

we have

$$Y_6 \succ Y_1 \succ Y_3 \succ Y_4 \succ Y_2 \succ Y_5$$

Therefore, the project Y_6 has the highest potential contribution that should be selected.

1.4 Intuitionistic Fuzzy Bonferroni Means

The Bonferroni mean (BM) was originally introduced by Bonferroni (1950) and then more recently generalized by Yager (2009). The desirable characteristic of the BM is its capability to capture the interrelationship between input arguments. Nevertheless, it seems that the existing literature only considers the BM for aggregating crisp numbers instead of any other types of arguments. Xu and Yager (2011) investigate the BM under intuitionistic fuzzy environments. They develop an intuitionistic fuzzy BM, discuss a variety of special cases, and apply the weighted intuitionistic fuzzy BM to multi-attribute decision making.

Definition 1.4.1 (Bonferroni, 1950) Let $p, q \geqslant 0$, and a_i $(i = 1, 2, \cdots, n)$ be a collection of non-negative numbers. If

$$B^{p,q}(a_1, a_2, \cdots, a_n) = \left(\frac{1}{n(n-1)} \sum_{\substack{i,j=1 \\ i \neq j}}^{n} a_i^p a_j^q \right)^{\frac{1}{p+q}} \tag{1.89}$$

then $B^{p,q}$ is called the Bonferroni mean (BM).

Obviously, the BM has the following properties (Yager, 2009):

(1) $B^{p,q}(0, 0, \cdots, 0) = 0$;

(2) $B^{p,q}(a, a, \cdots, a) = a$, if $a_i = a$, for all i;

(3) $B^{p,q}(a_1, a_2, \cdots, a_n) \geqslant B^{p,q}(d_1, d_2, \cdots, d_n)$, i.e., $B^{p,q}$ is monotonic, if $a_i \geqslant d_i$, for all i;

(4) $\min\limits_i\{a_i\} \leqslant B^{p,q}(a_1, a_2, \cdots, a_n) \leqslant \max\limits_i\{a_i\}$.

Furthermore, if $q = 0$, then it follows from Eq.(1.89) that

$$B^{p,0}(a_1, a_2, \cdots, a_n) = \left(\frac{1}{n}\sum_{i=1}^{n} a_i^p \left(\frac{1}{n-1}\sum_{\substack{j=1\\ j\neq i}}^{n} a_j^0\right)\right)^{\frac{1}{p+0}} = \left(\frac{1}{n}\sum_{i=1}^{n} a_i^p\right)^{\frac{1}{p}} \tag{1.90}$$

which is a generalized mean operator discussed by Dyckhoff and Pedrycz (1984). Especially, the following cases hold (Xu and Yager, 2011):

(1) If $p = 2$ and $q = 0$, then Eq.(1.90) reduces to the square mean:

$$B^{2,0}(a_1, a_2, \cdots, a_n) = \left(\frac{1}{n}\sum_{i=1}^{n} a_i^2\right)^{\frac{1}{2}} \tag{1.91}$$

(2) If $p = 1$ and $q = 0$, then Eq.(1.90) reduces to the usual average:

$$B^{1,0}(a_1, a_2, \cdots, a_n) = \frac{1}{n}\sum_{i=1}^{n} a_i \tag{1.92}$$

(3) If $p \to +\infty$ and $q = 0$, then Eq.(1.90) reduces to the max operator:

$$\lim_{p\to+\infty} B^{p,0}(a_1, a_2, \cdots, a_n) = \lim_{p\to+\infty}\left(\frac{1}{n}\sum_{i=1}^{n} a_i^p\right)^{\frac{1}{p}} = \max_i\{a_i\} \tag{1.93}$$

(4) If $p \to 0$ and $q = 0$, then Eq.(1.90) reduces to the geometric mean:

$$\lim_{p\to 0} B^{p,0}(a_1, a_2, \cdots, a_n) = \lim_{p\to 0}\left(\frac{1}{n}\sum_{i=1}^{n} a_i^p\right)^{\frac{1}{p}} = \left(\prod_{i=1}^{n} a_i\right)^{\frac{1}{n}} \tag{1.94}$$

Xu and Yager (2011) investigate the BM in intuitionistic fuzzy environments. Based on Eq.(1.89), they give the definition of intuitionistic fuzzy Bonferroni mean as follows:

Definition 1.4.2 (Xu and Yager, 2011) For any $p, q > 0$, if

$$\mathrm{IFB}^{p,q}(\alpha_1, \alpha_2, \cdots, \alpha_n) = \left(\frac{1}{n(n-1)}\left(\bigoplus_{\substack{i,j=1\\ i\neq j}}^{n} \left(\alpha_i^p \otimes \alpha_j^q\right)\right)\right)^{\frac{1}{p+q}} \tag{1.95}$$

then $\mathrm{IFB}^{p,q}$ is called the intuitionistic fuzzy Bonferroni mean (IFBM).

Based on the operational laws described in Subsection 1.1, we can derive the following result:

Theorem 1.4.1 (Xu and Yager, 2011) The aggregated value by using the IFBM is an IFN, and

$$\begin{aligned}
&\mathrm{IFB}^{p,q}(\alpha_1, \alpha_2, \cdots, \alpha_n) \\
=&\left(\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\mu_{\alpha_i}^{p}\mu_{\alpha_j}^{q}\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}},\right.\\
&\left.1-\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-(1-\nu_{\alpha_i})^{p}(1-\nu_{\alpha_j})^{q}\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}\right)
\end{aligned} \tag{1.96}$$

Proof By the operational laws (5) and (7) in Definition 1.2.2, we have

$$\alpha_i^p = (\mu_{\alpha_i}^p,\ 1-(1-\nu_{\alpha_i})^p), \quad \alpha_j^q = (\mu_{\alpha_j}^q,\ 1-(1-\nu_{\alpha_j})^q) \tag{1.97}$$

and then

$$\begin{aligned}
\alpha_i^p \otimes \alpha_j^q &= (\mu_{\alpha_i}^p\mu_{\alpha_j}^q,\ 1-(1-\nu_{\alpha_i})^p+1-(1-\nu_{\alpha_j})^q-(1-(1-\nu_{\alpha_i})^p)\left(1-(1-\nu_{\alpha_j})^q\right)) \\
&= \left(\mu_{\alpha_i}^p\mu_{\alpha_j}^q,\ 1-(1-\nu_{\alpha_i})^p(1-\nu_{\alpha_j})^q\right)
\end{aligned} \tag{1.98}$$

In what follows, we first prove

$$\left(\bigoplus_{\substack{i,j=1\\ i\neq j}}^{n}\left(\alpha_i^p\otimes\alpha_j^q\right)\right) = \left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\mu_{\alpha_i}^p\mu_{\alpha_j}^q\right),\ \prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-(1-\nu_{\alpha_i})^p(1-\nu_{\alpha_j})^q\right)\right) \tag{1.99}$$

by using mathematical induction on n:

(1) For $n=2$, we have

$$\begin{aligned}
\bigoplus_{\substack{i,j=1\\ i\neq j}}^{2}\left(\alpha_i^p\otimes\alpha_j^q\right) &= (\alpha_1^p\otimes\alpha_2^q)\oplus(\alpha_2^p\otimes\alpha_1^q) \\
&= \left(\mu_{\alpha_1}^p\mu_{\alpha_2}^q,\ 1-(1-\nu_{\alpha_1})^p(1-\nu_{\alpha_2})^q\right) \\
&\quad \oplus\left(\mu_{\alpha_2}^p\mu_{\alpha_1}^q,\ 1-(1-\nu_{\alpha_2})^p(1-\nu_{\alpha_1})^q\right) \\
&= (\mu_{\alpha_1}^p\mu_{\alpha_2}^q+\mu_{\alpha_2}^p\mu_{\alpha_1}^q-\mu_{\alpha_1}^{p+q}\mu_{\alpha_2}^{p+q},\ (1-(1-\nu_{\alpha_1})^p(1-\nu_{\alpha_2})^q) \\
&\quad (1-(1-\nu_{\alpha_2})^p(1-\nu_{\alpha_1})^q)) \\
&= (1-\left(1-\mu_{\alpha_1}^p\mu_{\alpha_2}^q\right)\left(1-\mu_{\alpha_2}^p\mu_{\alpha_1}^q\right),\ (1-(1-\nu_{\alpha_1})^p(1-\nu_{\alpha_2})^q) \\
&\quad (1-(1-\nu_{\alpha_2})^p(1-\nu_{\alpha_1})^q))
\end{aligned} \tag{1.100}$$

(2) If Eq.(1.99) holds for $n=k$, that is

$$\mathop{\oplus}_{\substack{i,j=1\\i\neq j}}^{k}\left(\alpha_i^p\otimes\alpha_j^q\right)=\left(1-\prod_{\substack{i,j=1\\i\neq j}}^{k}\left(1-\mu_{\alpha_i}^p\mu_{\alpha_j}^q\right),\ \prod_{\substack{i,j=1\\i\neq j}}^{k}\left(1-(1-\nu_{\alpha_i})^p(1-\nu_{\alpha_j})^q\right)\right) \tag{1.101}$$

then when $n=k+1$, by the operational laws (1)-(3) in Section 1.2, we have

$$\left(\mathop{\oplus}_{\substack{i,j=1\\i\neq j}}^{k+1}\left(\alpha_i^p\otimes\alpha_j^q\right)\right)=\left(\mathop{\oplus}_{\substack{i,j=1\\i\neq j}}^{k}\left(\alpha_i^p\otimes\alpha_j^q\right)\right)\oplus\left(\mathop{\oplus}_{i=1}^{k}\left(\alpha_i^p\otimes\alpha_{k+1}^q\right)\right)\oplus\left(\mathop{\oplus}_{j=1}^{k}\left(\alpha_{k+1}^p\otimes\alpha_j^q\right)\right) \tag{1.102}$$

Now we prove

$$\mathop{\oplus}_{i=1}^{k}\left(\alpha_i^p\otimes\alpha_{k+1}^q\right)=\left(1-\prod_{i=1}^{k}\left(1-\mu_{\alpha_i}^p\mu_{\alpha_{k+1}}^q\right),\ \prod_{i=1}^{k}\left(1-(1-\nu_{\alpha_i})^p(1-\nu_{\alpha_{k+1}})^q\right)\right) \tag{1.103}$$

by using mathematical induction on k:

(i) For $k=2$, then by Eq.(1.98), we have

$$\alpha_i^p\otimes\alpha_{2+1}^q=\left(\mu_{\alpha_i}^p\mu_{\alpha_{2+1}}^q,\ 1-(1-\nu_{\alpha_i})^p(1-\nu_{\alpha_{2+1}})^q\right),\quad i=1,2 \tag{1.104}$$

and thus

$$\begin{aligned}\mathop{\oplus}_{i=1}^{2}\left(\alpha_i^p\otimes\alpha_{2+1}^q\right)&=\left(\alpha_1^p\otimes\alpha_{2+1}^q\right)\oplus\left(\alpha_2^p\otimes\alpha_{2+1}^q\right)\\&=(\mu_{\alpha_1}^p\mu_{\alpha_3}^q+\mu_{\alpha_2}^p\mu_{\alpha_3}^q-\mu_{\alpha_1}^p\mu_{\alpha_2}^p\mu_{\alpha_3}^{2q},\ (1-(1-\nu_{\alpha_1})^p(1-\nu_{\alpha_3})^q)\\&\quad(1-(1-\nu_{\alpha_2})^p(1-\nu_{\alpha_3})^q))\\&=\left(1-\prod_{i=1}^{2}(1-\mu_{\alpha_i}^p\mu_{\alpha_3}^q),\ \prod_{i=1}^{2}(1-(1-\nu_{\alpha_i})^p(1-\nu_{\alpha_3})^q)\right)\end{aligned} \tag{1.105}$$

(ii) If Eq.(1.103) holds for $k=k_0$, that is

$$\mathop{\oplus}_{i=1}^{k_0}\left(\alpha_i^p\otimes\alpha_{k_0+1}^q\right)=\left(1-\prod_{i=1}^{k_0}\left(1-\mu_{\alpha_i}^p\mu_{\alpha_{k_0+1}}^q\right),\ \prod_{i=1}^{k_0}\left(1-(1-\nu_{\alpha_i})^p(1-\nu_{\alpha_{k_0+1}})^q\right)\right) \tag{1.106}$$

then when $k=k_0+1$, by Eq.(1.98) and the operational laws (5) and (7) in Definition 1.2.2, we obtain

$$\begin{aligned}\mathop{\oplus}_{i=1}^{k_0+1}\left(\alpha_i^p\otimes\alpha_{k_0+2}^q\right)&=\mathop{\oplus}_{i=1}^{k_0}\left(\alpha_i^p\otimes\alpha_{k_0+2}^q\right)\oplus\left(\alpha_{k_0+1}^p\otimes\alpha_{k_0+2}^q\right)\\&=\left(1-\prod_{i=1}^{k_0}\left(1-\mu_{\alpha_i}^p\mu_{\alpha_{k_0+2}}^q\right),\ \prod_{i=1}^{k_0}\left(1-(1-\nu_{\alpha_i})^p(1-\nu_{\alpha_{k_0+2}})^q\right)\right)\end{aligned}$$

$$\oplus\left(\mu^p_{\alpha_{k_0+1}}\mu^q_{\alpha_{k_0+2}},\ 1-(1-\nu_{\alpha_{k_0+1}})^p(1-\nu_{\alpha_{k_0+2}})^q\right)$$

$$=\left(1-\prod_{i=1}^{k_0}\left(1-\mu^p_{\alpha_i}\mu^q_{\alpha_{k_0+2}}\right)+\mu^p_{\alpha_{k_0+1}}\mu^q_{\alpha_{k_0+2}}\right.$$

$$-\left(1-\prod_{i=1}^{k_0}\left(1-\mu^p_{\alpha_i}\mu^q_{\alpha_{k_0+2}}\right)\right)\left(\mu^p_{\alpha_{k_0+1}}\mu^q_{\alpha_{k_0+2}}\right),$$

$$\left.\left(\prod_{i=1}^{k_0}\left(1-(1-\nu_{\alpha_i})^p(1-\nu_{\alpha_{k_0+2}})^q\right)\right)\left(1-(1-\nu_{\alpha_{k_0+1}})^p(1-\nu_{\alpha_{k_0+2}})^q\right)\right)$$

$$=\left(1-\prod_{i=1}^{k_0}\left(1-\mu^p_{\alpha_i}\mu^q_{\alpha_{k_0+2}}\right)+\left(\prod_{i=1}^{k_0}\left(1-\mu^p_{\alpha_i}\mu^q_{\alpha_{k_0+2}}\right)\right)\left(\mu^p_{\alpha_{k_0+1}}\mu^q_{\alpha_{k_0+2}}\right),\right.$$

$$\left.\prod_{i=1}^{k_0+1}\left(1-(1-\nu_{\alpha_i})^p(1-\nu_{\alpha_{k_0+2}})^q\right)\right)$$

$$=\left(1-\prod_{i=1}^{k_0+1}\left(1-\mu^p_{\alpha_i}\mu^q_{\alpha_{k_0+2}}\right),\ \prod_{i=1}^{k_0+1}\left(1-(1-\nu_{\alpha_i})^p(1-\nu_{\alpha_{k_0+2}})^q\right)\right)\tag{1.107}$$

i.e., Eq.(1.103) holds for $k=k_0+1$. Thus Eq.(1.103) holds for all k.

Similarly, we can prove

$$\mathop{\oplus}_{j=1}^{k}\left(\alpha^p_{k+1}\otimes\alpha^q_j\right)=\left(1-\prod_{j=1}^{k}\left(1-\mu^p_{\alpha_{k+1}}\mu^q_{\alpha_j}\right),\ \prod_{j=1}^{k}\left(1-(1-\nu_{\alpha_{k+1}})^p(1-\nu_{\alpha_j})^q\right)\right)\tag{1.108}$$

Thus, by Eqs.(1.101), (1.103) and (1.107), we further transform Eq.(1.102) to:

$$\begin{aligned}\mathop{\oplus}_{\substack{i,j=1\\ i\neq j}}^{k+1}\left(\alpha^p_i\otimes\alpha^q_j\right) &= \left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{k}\left(1-\mu^p_{\alpha_i}\mu^q_{\alpha_j}\right),\ \prod_{\substack{i,j=1\\ i\neq j}}^{k}\left(1-(1-\nu_{\alpha_i})^p(1-\nu_{\alpha_j})^q\right)\right)\\ &\oplus\left(1-\prod_{i=1}^{k}\left(1-\mu^p_{\alpha_i}\mu^q_{\alpha_{k+1}}\right),\ \prod_{i=1}^{k}\left(1-(1-\nu_{\alpha_i})^p(1-\nu_{\alpha_{k+1}})^q\right)\right)\\ &\oplus\left(1-\prod_{j=1}^{k}\left(1-\mu^p_{\alpha_{k+1}}\mu^q_{\alpha_j}\right),\ \prod_{j=1}^{k}\left(1-(1-\nu_{\alpha_{k+1}})^p(1-\nu_{\alpha_j})^q\right)\right)\end{aligned}$$

$$= \left(1 - \prod_{\substack{i,j=1 \\ i \neq j}}^{k} \left(1 - \mu_{\alpha_i}^p \mu_{\alpha_j}^q\right), \prod_{\substack{i,j=1 \\ i \neq j}}^{k} \left(1 - (1 - \nu_{\alpha_i})^p (1 - \nu_{\alpha_j})^q\right) \right)$$

$$\oplus \left(1 - \left(\prod_{i=1}^{k} \left(1 - \mu_{\alpha_i}^p \mu_{\alpha_{k+1}}^q\right) \right) \left(\prod_{j=1}^{k} \left(1 - \mu_{\alpha_{k+1}}^p \mu_{\alpha_j}^q\right) \right), \right.$$

$$\left. \left(\prod_{i=1}^{k} \left(1 - (1 - \nu_{\alpha_i})^p (1 - \nu_{\alpha_{k+1}})^q\right) \right) \left(\prod_{j=1}^{k} \left(1 - (1 - \nu_{\alpha_{k+1}})^p (1 - \nu_{\alpha_j})^q\right) \right) \right)$$

$$= \left(1 - \prod_{\substack{i,j=1 \\ i \neq j}}^{k+1} \left(1 - \mu_{\alpha_i}^p \mu_{\alpha_j}^q\right), \prod_{\substack{i,j=1 \\ i \neq j}}^{k+1} \left(1 - (1 - \nu_{\alpha_i})^p (1 - \nu_{\alpha_j})^q\right) \right) \tag{1.109}$$

i.e., Eq.(1.99) holds for $n = k + 1$. Thus Eq.(1.99) holds for all n.

Then, by Eq.(1.99) and the operational law (6) in Definition 1.2.2, we get

$$\frac{1}{n(n-1)} \left(\bigoplus_{\substack{i,j=1 \\ i \neq j}}^{n} (\alpha_i^p \otimes \alpha_j^q) \right)$$

$$= \left(1 - \left(1 - \left(1 - \prod_{\substack{i,j=1 \\ i \neq j}}^{n} \left(1 - \mu_{\alpha_i}^p \mu_{\alpha_j}^q\right) \right) \right)^{\frac{1}{n(n-1)}}, \right.$$

$$\left. \left(\prod_{\substack{i,j=1 \\ i \neq j}}^{n} \left(1 - (1 - \nu_{\alpha_i})^p (1 - \nu_{\alpha_j})^q\right) \right)^{\frac{1}{n(n-1)}} \right)$$

$$= \left(1 - \left(\prod_{\substack{i,j=1 \\ i \neq j}}^{n} \left(1 - \mu_{\alpha_i}^p \mu_{\alpha_j}^q\right) \right)^{\frac{1}{n(n-1)}}, \right.$$

$$\left. \prod_{\substack{i,j=1 \\ i \neq j}}^{n} \left(1 - (1 - \nu_{\alpha_i})^p (1 - \nu_{\alpha_j})^q\right)^{\frac{1}{n(n-1)}} \right) \tag{1.110}$$

Using Eq.(1.110) and the operational law (7) in Definition 1.2.2, it yields

$$\mathrm{IFB}^{p,q}(\alpha_1, \alpha_2, \cdots, \alpha_n)$$

$$= \left(\frac{1}{n(n-1)} \left(\bigoplus_{\substack{i,j=1 \\ i \neq j}}^{n} \left(\alpha_i^p \otimes \alpha_j^q\right) \right) \right)^{\frac{1}{p+q}}$$

$$= \left(\left(1 - \prod_{\substack{i,j=1 \\ i \neq j}}^{n} \left(1 - \mu_{\alpha_i}^{p} \mu_{\alpha_j}^{q} \right)^{\frac{1}{n(n-1)}} \right)^{\frac{1}{p+q}}, \right.$$

$$\left. 1 - \left(1 - \prod_{\substack{i,j=1 \\ i \neq j}}^{n} \left(1 - (1 - \nu_{\alpha_i})^{p} (1 - \nu_{\alpha_j})^{q} \right)^{\frac{1}{n(n-1)}} \right)^{\frac{1}{p+q}} \right) \tag{1.111}$$

i.e., Eq.(1.96) holds. In addition, since

$$0 \leqslant \left(1 - \prod_{\substack{i,j=1 \\ i \neq j}}^{n} (1 - \mu_{\alpha_i}^{p} \mu_{\alpha_j}^{q})^{\frac{1}{n(n-1)}} \right)^{\frac{1}{p+q}} \leqslant 1 \tag{1.112}$$

$$0 \leqslant 1 - \left(1 - \prod_{\substack{i,j=1 \\ i \neq j}}^{n} \left(1 - (1 - \nu_{\alpha_i})^{p} (1 - \nu_{\alpha_j})^{q} \right)^{\frac{1}{n(n-1)}} \right)^{\frac{1}{p+q}} \leqslant 1 \tag{1.113}$$

and using Eq.(1.9), we have

$$\begin{aligned}
&\left(1 - \prod_{\substack{i,j=1 \\ i \neq j}}^{n} \left(1 - \mu_{\alpha_i}^{p} \mu_{\alpha_j}^{q} \right)^{\frac{1}{n(n-1)}} \right)^{\frac{1}{p+q}} \\
&\quad + 1 - \left(1 - \prod_{\substack{i,j=1 \\ i \neq j}}^{n} \left(1 - (1 - \nu_{\alpha_i})^{p} (1 - \nu_{\alpha_j})^{q} \right)^{\frac{1}{n(n-1)}} \right)^{\frac{1}{p+q}} \\
&= 1 + \left(1 - \prod_{\substack{i,j=1 \\ i \neq j}}^{n} \left(1 - \mu_{\alpha_i}^{p} \mu_{\alpha_j}^{q} \right)^{\frac{1}{n(n-1)}} \right)^{\frac{1}{p+q}} \\
&\quad - \left(1 - \prod_{\substack{i,j=1 \\ i \neq j}}^{n} \left(1 - (1 - \nu_{\alpha_i})^{p} (1 - \nu_{\alpha_j})^{q} \right)^{\frac{1}{n(n-1)}} \right)^{\frac{1}{p+q}} \\
&\leqslant 1 + \left(1 - \prod_{\substack{i,j=1 \\ i \neq j}}^{n} \left(1 - (1 - \nu_{\alpha_i})^{p} (1 - \nu_{\alpha_j})^{q} \right)^{\frac{1}{n(n-1)}} \right)^{\frac{1}{p+q}} \\
&\quad - \left(1 - \prod_{\substack{i,j=1 \\ i \neq j}}^{n} \left(1 - (1 - \nu_{\alpha_i})^{p} (1 - \nu_{\alpha_j})^{q} \right)^{\frac{1}{n(n-1)}} \right)^{\frac{1}{p+q}} \\
&= 1
\end{aligned} \tag{1.114}$$

which completes the proof.

Now let us look at some desirable properties of the IFBM (Xu and Yager, 2011):

(1) If $\alpha_i = (\mu_{\alpha_i}, v_{\alpha_i})\,(i = 1, 2, \cdots, n)$ are a collection of the smallest IFNs, i.e., $\alpha_i = \alpha_* = (0, 1)$, for all i, then

$$\begin{aligned}
&\mathrm{IFB}^{p,q}(\alpha_1, \alpha_2, \cdots, \alpha_n)\\
&= \mathrm{IFB}^{p,q}(\alpha_*, \alpha_*, \cdots, \alpha_*)\\
&= \left(\left(1 - \prod_{\substack{i,j=1\\ i\neq j}}^{n} \left(1 - \mu_{\alpha_*}^{p}\mu_{\alpha_*}^{q}\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}},\right.\\
&\qquad \left. 1 - \left(1 - \prod_{\substack{i,j=1\\ i\neq j}}^{n} \left(1 - (1-\nu_{\alpha_*})^{p}(1-\nu_{\alpha_*})^{q}\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}\right)\\
&= \left(\left(1 - \prod_{\substack{i,j=1\\ i\neq j}}^{n} (1-0)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}, 1 - \left(1 - \prod_{\substack{i,j=1\\ i\neq j}}^{n} \left(1 - (1-1)^{p}(1-1)^{q}\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}\right)\\
&= (0, 1)
\end{aligned} \tag{1.115}$$

which is also the smallest IFN.

(2) If $\alpha_i = (\mu_{\alpha_i}, v_{\alpha_i})$ $(i = 1, 2, \cdots, n)$ are a collection of the largest IFNs, i.e., $\alpha_i = \alpha^* = (1, 0)$, for all i, then

$$\begin{aligned}
&\mathrm{IFB}^{p,q}(\alpha_1, \alpha_2, \cdots, \alpha_n)\\
&= \mathrm{IFB}^{p,q}(\alpha^*, \alpha^*, \cdots, \alpha^*)\\
&= \left(\left(1 - \prod_{\substack{i,j=1\\ i\neq j}}^{n} \left(1 - \mu_{\alpha*}^{p}\mu_{\alpha*}^{q}\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}},\right.\\
&\qquad \left. 1 - \left(1 - \prod_{\substack{i,j=1\\ i\neq j}}^{n} \left(1 - (1-\nu_{\alpha^*})^{p}(1-\nu_{\alpha^*})^{q}\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}\right)\\
&= \left(\left(1 - \prod_{\substack{i,j=1\\ i\neq j}}^{n} (1-1)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}, 1 - \left(1 - \prod_{\substack{i,j=1\\ i\neq j}}^{n} \left(1 - (1-0)^{p}(1-0)^{q}\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}\right)\\
&= (1, 0)
\end{aligned} \tag{1.116}$$

which is also the largest IFN.

(3) **(Idempotency):** If all α_i $(i = 1, 2, \cdots, n)$ are equal, i.e., $\alpha_i = \alpha = (\mu_\alpha, \nu_\alpha)$, for all i, then

$$\begin{aligned}
&\mathrm{IFB}^{p,q}(\alpha_1, \alpha_2, \cdots, \alpha_n)\\
&= \mathrm{IFB}^{p,q}(\alpha, \alpha, \cdots, \alpha)\\
&= \left(\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}(1-\mu_\alpha^p\mu_\alpha^q)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}, 1-\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}(1-(1-\nu_\alpha)^p(1-\nu_\alpha)^q)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}\right)\\
&= \left(\left(1-\left(1-\mu_\alpha^{p+q}\right)\right)^{\frac{1}{p+q}},\ 1-\left(1-\left(1-(1-\nu_\alpha)^{p+q}\right)\right)^{\frac{1}{p+q}}\right)\\
&= \left(\left(\mu_\alpha^{p+q}\right)^{\frac{1}{p+q}},\ 1-\left((1-\nu_\alpha)^{p+q}\right)^{\frac{1}{p+q}}\right)\\
&= (\mu_\alpha, \nu_\alpha)\\
&= \alpha
\end{aligned} \tag{1.117}$$

(4) **(Monotonicity):** Let $\alpha_i = (\mu_{\alpha_i}, \nu_{\alpha_i})$ $(i = 1, 2, \cdots, n)$ and $\beta_i = (\mu_{\beta_i}, \nu_{\beta_i})$ $(i = 1, 2, \cdots, n)$ be two collections of IFNs, if $\mu_{\alpha_i} \leqslant \mu_{\beta_i}$ and $\nu_{\alpha_i} \geqslant \nu_{\beta_i}$, for all i, then

$$\mathrm{IFB}^{p,q}(\alpha_1, \alpha_2, \cdots, \alpha_n) \leqslant \mathrm{IFB}^{p,q}(\beta_1, \beta_2, \cdots, \beta_n) \tag{1.118}$$

Proof Since $\mu_{\alpha_i} \leqslant \mu_{\beta_i}$ and $\nu_{\alpha_i} \geqslant \nu_{\beta_i}$, for all i, then

$$\mu_{\alpha_i}^p\mu_{\alpha_j}^q \leqslant \mu_{\beta_i}^p\mu_{\beta_j}^q, \quad \text{for all } i, j \tag{1.119}$$

$$\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\mu_{\alpha_i}^p\mu_{\alpha_j}^q\right)^{\frac{1}{n(n-1)}} \geqslant \prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\mu_{\beta_i}^p\mu_{\beta_j}^q\right)^{\frac{1}{n(n-1)}} \tag{1.120}$$

$$1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\mu_{\alpha_i}^p\mu_{\alpha_j}^q\right)^{\frac{1}{n(n-1)}} \leqslant 1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\mu_{\beta_i}^p\mu_{\beta_j}^q\right)^{\frac{1}{n(n-1)}} \tag{1.121}$$

$$\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\mu_{\alpha_i}^p\mu_{\alpha_j}^q\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}} \leqslant \left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\mu_{\beta_i}^p\mu_{\beta_j}^q\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}} \tag{1.122}$$

$$(1-\nu_{\alpha_i})^p(1-\nu_{\alpha_j})^q \leqslant (1-\nu_{\beta_i})^p(1-\nu_{\beta_j})^q, \quad \text{for all } i, j \tag{1.123}$$

$$\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-(1-\nu_{\alpha_i})^p(1-\nu_{\alpha_j})^q\right)^{\frac{1}{n(n-1)}} \geqslant \prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-(1-\nu_{\beta_i})^p(1-\nu_{\beta_j})^q\right)^{\frac{1}{n(n-1)}} \tag{1.124}$$

$$\left(1-\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-(1-\nu_{\alpha_i})^p(1-\nu_{\alpha_j})^q\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}$$

$$\leqslant\left(1-\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-(1-\nu_{\beta_i})^p(1-\nu_{\beta_j})^q\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}} \tag{1.125}$$

$$1-\left(1-\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-(1-\nu_{\alpha_i})^p(1-\nu_{\alpha_j})^q\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}$$

$$\geqslant 1-\left(1-\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-(1-\nu_{\beta_i})^p(1-\nu_{\beta_j})^q\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}} \tag{1.126}$$

$$\left(1-\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-\mu_{\alpha_i}^p\mu_{\alpha_j}^q\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}$$

$$-\left(1-\left(1-\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-(1-\nu_{\alpha_i})^p(1-\nu_{\alpha_j})^q\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}\right)$$

$$\leqslant\left(1-\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-\mu_{\beta_i}^p\mu_{\beta_j}^q\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}$$

$$-\left(1-\left(1-\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-(1-\nu_{\beta_i})^p(1-\nu_{\beta_j})^q\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}\right) \tag{1.127}$$

Let $\alpha=\mathrm{IFB}^{p,q}(\alpha_1,\alpha_2,\cdots,\alpha_n)$ and $\beta=\mathrm{IFB}^{p,q}(\beta_1,\beta_2,\cdots,\beta_n)$, and let s_α and s_β be the scores of α and β. Then Eq.(1.101) is equal to $s_\alpha\leqslant s_\beta$. Now we discuss the following cases (Xu and Yager, 2011):

Case 1 If $s_\alpha<s_\beta$, then it can be obtained from Definition 1.1.3 that

$$\mathrm{IFB}^{p,q}(\alpha_1,\alpha_2,\cdots,\alpha_n)<\mathrm{IFB}^{p,q}(\beta_1,\beta_2,\cdots,\beta_n) \tag{1.128}$$

Case 2 If $s_\alpha=s_\beta$, then

$$\left(1-\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-\mu_{\alpha_i}^p\mu_{\alpha_j}^q\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}$$

$$-\left(1-\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-(1-\nu_{\alpha_i})^p(1-\nu_{\alpha_j})^q\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}\right)$$

$$=\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\mu_{\beta_i}^p\mu_{\beta_j}^q\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}$$

$$-\left(1-\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-(1-\nu_{\beta_i})^p(1-\nu_{\beta_j})^q\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}\right) \tag{1.129}$$

Since $\mu_{\alpha_i} \leqslant \mu_{\beta_i}$ and $\nu_{\alpha_i} \geqslant \nu_{\beta_i}$, for all i, then

$$\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\mu_{\alpha_i}^p\mu_{\alpha_j}^q\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}} = \left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\mu_{\beta_i}^p\mu_{\beta_j}^q\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}} \tag{1.130}$$

$$1-\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-(1-\nu_{\alpha_i})^p(1-\nu_{\alpha_j})^q\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}$$

$$=1-\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-(1-\nu_{\beta_i})^p(1-\nu_{\beta_j})^q\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}} \tag{1.131}$$

and thus

$$h_\alpha=\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\mu_{\alpha_i}^p\mu_{\alpha_j}^q\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}$$

$$+\left(1-\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-(1-\nu_{\alpha_i})^p(1-\nu_{\alpha_j})^q\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}\right)$$

$$=\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\mu_{\beta_i}^p\mu_{\beta_j}^q\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}$$

$$+\left(1-\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-(1-\nu_{\beta_i})^p(1-\nu_{\beta_j})^q\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}\right)$$

$$= h_\beta \tag{1.132}$$

Then, it follows from Definition 1.1.3 that

$$\text{IFB}^{p,q}(\alpha_1, \alpha_2, \cdots, \alpha_n) = \text{IFB}^{p,q}(\beta_1, \beta_2, \cdots, \beta_n) \tag{1.133}$$

Thus, Eqs.(1.129) and (1.133) indicate that Eq.(1.118) holds.

(5) **(Commutativity):**

$$\text{IFB}^{p,q}(\alpha_1, \alpha_2, \cdots, \alpha_n) = \text{IFB}^{p,q}(\dot{\alpha}_1, \dot{\alpha}_2, \cdots, \dot{\alpha}_n) \tag{1.134}$$

where $(\dot{\alpha}_1, \dot{\alpha}_2, \cdots, \dot{\alpha}_n)$ is any permutation of $(\alpha_1, \alpha_2, \cdots, \alpha_n)$.

Proof Since $(\dot{\alpha}_1, \dot{\alpha}_2, \cdots, \dot{\alpha}_n)$ is any permutation of $(\alpha_1, \alpha_2, \cdots, \alpha_n)$, then

$$\begin{aligned}\text{IFB}^{p,q}(\alpha_1, \alpha_2, \cdots, \alpha_n) &= \left(\frac{1}{n(n-1)}\left(\bigoplus_{\substack{i,j=1\\ i\neq j}}^{n}(\alpha_i^p \otimes \alpha_j^q)\right)\right)^{\frac{1}{p+q}} \\ &= \left(\frac{1}{n(n-1)}\left(\bigoplus_{\substack{i,j=1\\ i\neq j}}^{n}(\dot{\alpha}_i^p \otimes \dot{\alpha}_j^q)\right)\right)^{\frac{1}{p+q}} \\ &= \text{IFB}^{p,q}(\dot{\alpha}_1, \dot{\alpha}_2, \cdots, \dot{\alpha}_n)\end{aligned} \tag{1.135}$$

(6) (**Boundedness**): Let

$$\alpha^- = \left(\min_i\{\mu_{\alpha_i}\}, \max_i\{\nu_{\alpha_i}\}\right), \quad \alpha^+ = \left(\max_i\{\mu_{\alpha_i}\}, \min_i\{\nu_{\alpha_i}\}\right) \tag{1.136}$$

Then

$$\alpha^- \leqslant \text{IFB}^{p,q}(\alpha_1, \alpha_2, \cdots, \alpha_n) \leqslant \alpha^+ \tag{1.137}$$

Proof Since $\min_i\{\mu_{\alpha_i}\} \leqslant \mu_{\alpha_i} \leqslant \max_i\{\mu_{\alpha_i}\}$ and $\min_i\{\nu_{\alpha_i}\} \leqslant \nu_{\alpha_i} \leqslant \max_i\{\nu_{\alpha_i}\}$, for all i, then

$$(\min_i\{\mu_{\alpha_i}\})^{p+q} \leqslant \mu_{\alpha_i}^p \mu_{\alpha_i}^q \leqslant (\max_i\{\mu_{\alpha_i}\})^{p+q} \tag{1.138}$$

$$\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\mu_{\alpha_i}^p\mu_{\alpha_j}^q\right)^{\frac{1}{n(n-1)}} \leqslant \prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(\min_i\{\mu_{\alpha_i}\}\right)^{p+q}\right)^{\frac{1}{n(n-1)}} = 1-\left(\min_i\{\mu_{\alpha_i}\}\right)^{p+q} \tag{1.139}$$

$$\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\mu_{\alpha_i}^p\mu_{\alpha_j}^q\right)^{\frac{1}{n(n-1)}} \geqslant \prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(\max_i\{\mu_{\alpha_i}\}\right)^{p+q}\right)^{\frac{1}{n(n-1)}} = 1-\left(\max_i\{\mu_{\alpha_i}\}\right)^{p+q} \tag{1.140}$$

Therefore,

$$\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\mu_{\alpha_i}^p\mu_{\alpha_j}^q\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}} \leqslant \left(1-\left(1-\left(\max_i\{\mu_{\alpha_i}\}\right)^{p+q}\right)\right)^{\frac{1}{p+q}} = \max_i\{\mu_{\alpha_i}\} \tag{1.141}$$

$$\left(1-\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-\mu_{\alpha_i}^{p}\mu_{\alpha_j}^{q}\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}} \geqslant \left(1-\left(1-\left(\min_i\{\mu_{\alpha_i}\}\right)^{p+q}\right)\right)^{\frac{1}{p+q}} = \min_i\{\mu_{\alpha_i}\} \tag{1.142}$$

In addition, we have

$$\left(1-\max_i\{\nu_{\alpha_i}\}\right)^{p+q} \leqslant (1-\nu_{\alpha_i})^p(1-\nu_{\alpha_j})^q \leqslant \left(1-\min_i\{\nu_{\alpha_i}\}\right)^{p+q} \tag{1.143}$$

$$\begin{aligned}\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-(1-\nu_{\alpha_i})^p(1-\nu_{\alpha_j})^q\right)^{\frac{1}{n(n-1)}} &\geqslant \prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-\left(1-\min_i\{\nu_{\alpha_i}\}\right)^{p+q}\right)^{\frac{1}{n(n-1)}}\\ &= 1-\left(1-\min_i\{\nu_{\alpha_i}\}\right)^{p+q}\end{aligned} \tag{1.144}$$

$$\begin{aligned}\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-(1-\nu_{\alpha_i})^p(1-\nu_{\alpha_j})^q\right)^{\frac{1}{n(n-1)}} &\leqslant \prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-\left(1-\max_i\{\nu_{\alpha_i}\}\right)^{p+q}\right)^{\frac{1}{n(n-1)}}\\ &= 1-\left(1-\max_i\{\nu_{\alpha_i}\}\right)^{p+q}\end{aligned} \tag{1.145}$$

and thus

$$\begin{aligned}&1-\left(1-\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-(1-\nu_{\alpha_i})^p(1-\nu_{\alpha_j})^q\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}\\ &\geqslant 1-\left(1-\left(1-\left(1-\min_i\{\nu_{\alpha_i}\}\right)^{p+q}\right)\right)^{\frac{1}{p+q}}\\ &= \min_i\{\nu_{\alpha_i}\}\end{aligned} \tag{1.146}$$

$$\begin{aligned}&1-\left(1-\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-(1-\nu_{\alpha_i})^p(1-\nu_{\alpha_j})^q\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}\\ &\leqslant 1-\left(1-\left(1-\left(1-\max_i\{\nu_{\alpha_i}\}\right)^{p+q}\right)\right)^{\frac{1}{p+q}}\\ &= \max_i\{\nu_{\alpha_i}\}\end{aligned} \tag{1.147}$$

Let $\alpha = \mathrm{IFB}^{p,q}(\alpha_1, \alpha_2, \cdots, \alpha_n) = (\mu_\alpha, \nu_\alpha)$. Then

$$s_\alpha = \mu_\alpha - \nu_\alpha \leqslant \max_i\{\mu_{\alpha_i}\} - \min_i\{\nu_{\alpha_i}\} = s_{\alpha^+} \tag{1.148}$$

$$s_\alpha = \mu_\alpha - \nu_\alpha \geqslant \min_i\{\mu_{\alpha_i}\} - \max_i\{\nu_{\alpha_i}\} = s_{\alpha^-} \tag{1.149}$$

In what follows, three cases are discussed:

Case 1 If $s_\alpha < s_{\alpha^+}$ and $s_\alpha > s_{\alpha^-}$, then it follows from Definition 1.1.3 that

$$\alpha^- < \mathrm{IFB}^{p,q}(\alpha_1, \alpha_2, \cdots, \alpha_n) < \alpha^+ \tag{1.150}$$

Case 2 If $s_\alpha = s_{\alpha^+}$, i.e., $\mu_\alpha - \nu_\alpha = \max_i\{\mu_{\alpha_i}\} - \min_i\{\nu_{\alpha_i}\}$, then by Eqs.(1.141) and (1.146), we have $\mu_\alpha = \max_i\{\mu_{\alpha_i}\}$ and $\nu_\alpha = \min_i\{\nu_{\alpha_i}\}$. Thus

$$h_\alpha = \mu_\alpha + \nu_\alpha = \max_i\{\mu_{\alpha_i}\} + \min_i\{\nu_{\alpha_i}\} = h_{\alpha^+} \tag{1.151}$$

then it follows from Definition 1.1.3 that

$$\mathrm{IFB}^{p,q}(\alpha_1, \alpha_2, \cdots, \alpha_n) = \alpha^+ \tag{1.152}$$

Case 3 If $s_\alpha = s_{\alpha^-}$, i.e., $\mu_\alpha - \nu_\alpha = \min_i\{\mu_{\alpha_i}\} - \max_i\{\nu_{\alpha_i}\}$, then from Eqs.(1.142) and (1.147), we can obtain $\mu_\alpha = \min_i\{\mu_{\alpha_i}\}$ and $\nu_\alpha = \max_i\{\nu_{\alpha_i}\}$. Consequently, we have

$$h_\alpha = \mu_\alpha + \nu_\alpha = \min_i\{\mu_{\alpha_i}\} + \max_i\{\nu_{\alpha_i}\} = h_{\alpha^-} \tag{1.153}$$

Thus, it follows from Definition 1.1.3 that

$$\mathrm{IFB}^{p,q}(\alpha_1, \alpha_2, \cdots, \alpha_n) = \alpha^- \tag{1.154}$$

Therefore, according to all the cases above, it is clear that Eq.(1.137) holds.

Let us now consider some special cases of the IFBM by taking different values of the parameters p and q (Xu and Yager, 2011):

Case 1 If $q \to 0$, then it follows from Eq.(1.95) that

$$\lim_{q\to 0} \mathrm{IFB}^{p,q}(\alpha_1, \alpha_2, \cdots, \alpha_n) = \lim_{q\to 0}\left(\frac{1}{n(n-1)}\left(\bigoplus_{\substack{i,j=1\\ i\neq j}}^{n} \left(\alpha_i^p \otimes \alpha_j^q\right)\right)\right)^{\frac{1}{p+q}}$$

$$= \lim_{q\to 0}\left(\left(1 - \prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1 - \mu_{\alpha_i}^p \mu_{\alpha_j}^q\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}},\right.$$

$$\left. 1 - \left(1 - \prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1 - (1-\nu_{\alpha_i})^p(1-\nu_{\alpha_j})^q\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}\right)$$

$$= \left(\left(1 - \prod_{i=1}^{n}\left(1 - \mu_{\alpha_i}^p\right)^{\frac{n-1}{n(n-1)}}\right)^{\frac{1}{p}},\ 1 - \left(1 - \prod_{i=1}^{n}\left(1 - (1-\nu_{\alpha_i})^p\right)^{\frac{n-1}{n(n-1)}}\right)^{\frac{1}{p}}\right)$$

$$
\begin{aligned}
&= \left(\left(1-\prod_{i=1}^{n}\left(1-\mu_{\alpha_i}^{p}\right)^{\frac{1}{n}}\right)^{\frac{1}{p}},\ 1-\left(1-\prod_{i=1}^{n}\left(1-(1-\nu_{\alpha_i})^{p}\right)^{\frac{1}{n}}\right)^{\frac{1}{p}}\right) \\
&= \left(\frac{1}{n}\left(\mathop{\oplus}_{i=1}^{n}\alpha_i^{p}\right)\right)^{\frac{1}{p}} \\
&= \mathrm{IFB}^{p,0}(\alpha_1,\alpha_2,\cdots,\alpha_n)
\end{aligned} \tag{1.155}
$$

which we call a generalized intuitionistic fuzzy mean.

Case 2 If $p=2$ and $q\to 0$, then Eq.(1.95) can be transformed to:

$$
\begin{aligned}
&\mathrm{IFB}^{2,0}(\alpha_1,\alpha_2,\cdots,\alpha_n) = \left(\frac{1}{n}\left(\mathop{\oplus}_{i=1}^{n}\alpha_i^{2}\right)\right)^{\frac{1}{2}} \\
&= \left(\left(1-\prod_{i=1}^{n}\left(1-\mu_{\alpha_i}^{2}\right)^{\frac{1}{n}}\right)^{\frac{1}{2}},\ 1-\left(1-\prod_{i=1}^{n}\left(1-(1-\nu_{\alpha_i})^{2}\right)^{\frac{1}{n}}\right)^{\frac{1}{2}}\right)
\end{aligned} \tag{1.156}
$$

which we call an intuitionistic fuzzy square mean.

Case 3 If $p=1$ and $q\to 0$, then Eq.(1.95) reduces to the IFA average:

$$
\begin{aligned}
&\mathrm{IFB}^{1,0}(\alpha_1,\alpha_2,\cdots,\alpha_n) \\
&= \left(\left(1-\prod_{i=1}^{n}(1-\mu_{\alpha_i})^{\frac{1}{n}}\right),\ 1-\left(1-\prod_{i=1}^{n}(1-(1-\nu_{\alpha_i}))^{\frac{1}{n}}\right)\right) \\
&= \left(\left(1-\prod_{i=1}^{n}(1-\mu_{\alpha_i})^{\frac{1}{n}}\right),\ \prod_{i=1}^{n}(\nu_{\alpha_i})^{\frac{1}{n}}\right) = \frac{1}{n}\left(\mathop{\oplus}_{i=1}^{n}\alpha_i\right)
\end{aligned} \tag{1.157}
$$

Case 4 If $p=q=1$, then Eq.(1.95) reduces to the following:

$$
\begin{aligned}
&\mathrm{IFB}^{1,1}(\alpha_1,\alpha_2,\cdots,\alpha_n) = \left(\frac{1}{n(n-1)}\left(\mathop{\oplus}_{\substack{i,j=1\\ i\neq j}}^{n}(\alpha_i\otimes\alpha_j)\right)\right)^{\frac{1}{2}} \\
&= \left(\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\mu_{\alpha_i}\mu_{\alpha_j}\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{2}},\right. \\
&\qquad \left. 1-\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-(1-\nu_{\alpha_i})(1-\nu_{\alpha_j})\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{2}}\right)
\end{aligned} \tag{1.158}
$$

which we call an intuitionistic fuzzy interrelated square mean.

Example 1.4.1 (Xu and Yager, 2011) Assume we have four IFNs: $\alpha_1=(0.3,\ 0.5)$, $\alpha_2=(0.6,\ 0.2)$, $\alpha_3=(0.8,\ 0.1)$, and $\alpha_4=(0.7,\ 0.2)$. Here we use the IFBM to fuse these intuitionistic fuzzy data. Without loss of generality, we let $p=q=1$, then

$\alpha_1 \otimes \alpha_2 = (0.3 \times 0.6,\ 0.5 + 0.2 - 0.5 \times 0.2) = (0.18,\ 0.60)$
$\alpha_2 \otimes \alpha_1 = \alpha_1 \otimes \alpha_2 = (0.18,\ 0.60)$
$\alpha_1 \otimes \alpha_3 = (0.3 \times 0.8,\ 0.5 + 0.1 - 0.5 \times 0.1) = (0.24,\ 0.55)$
$\alpha_3 \otimes \alpha_1 = \alpha_1 \otimes \alpha_3 = (0.24,\ 0.55)$
$\alpha_1 \otimes \alpha_4 = (0.3 \times 0.7,\ 0.5 + 0.2 - 0.5 \times 0.2) = (0.21,\ 0.60)$
$\alpha_4 \otimes \alpha_1 = \alpha_1 \otimes \alpha_4 = (0.21,\ 0.60)$
$\alpha_2 \otimes \alpha_3 = (0.6 \times 0.8,\ 0.2 + 0.1 - 0.2 \times 0.1) = (0.48,\ 0.28)$
$\alpha_3 \otimes \alpha_1 = \alpha_2 \otimes \alpha_3 = (0.48,\ 0.28)$
$\alpha_2 \otimes \alpha_4 = (0.6 \times 0.7,\ 0.2 + 0.2 - 0.2 \times 0.2) = (0.42,\ 0.36)$
$\alpha_4 \otimes \alpha_2 = \alpha_2 \otimes \alpha_4 = (0.42,\ 0.36)$
$\alpha_3 \otimes \alpha_4 = (0.8 \times 0.7,\ 0.1 + 0.2 - 0.1 \times 0.2) = (0.56,\ 0.28)$
$\alpha_4 \otimes \alpha_3 = \alpha_3 \otimes \alpha_4 = (0.56,\ 0.28)$

and thus, by Eq.(1.96), we get

$$\text{IFB}^{1,1}(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \left(\frac{1}{12} \left(\bigoplus_{\substack{i,j=1 \\ i \neq j}}^{4} (\alpha_i \otimes \alpha_j) \right) \right)^{\frac{1}{2}} = (0.60,\ 0.24)$$

In the analysis above, only the input data and their interrelationships are involved in the aggregation process. The importance of each datum is not emphasized. Nevertheless, in many practical situations, the weights of the data should be taken into account. For example, in multi-attribute decision making, the considered attributes usually have different importance, and thus need to be assigned different weights. Now we introduce a weighted intuitionistic fuzzy Bonferroni mean:

Definition 1.4.3 (Xu and Yager, 2011) Let $\alpha_i = (\mu_{\alpha_i}, \nu_{\alpha_i})\,(i = 1, 2, \cdots, n)$ be a collection of IFNs, and $\omega = (\omega_1, \omega_2, \cdots, \omega_n)^{\mathrm{T}}$ the weight vector of α_i $(i = 1, 2, \cdots, n)$, where ω_i indicates the importance degree of α_i, satisfying $\omega_i \in [0, 1]$ $(i = 1, 2, \cdots, n)$ and $\sum_{i=1}^{n} \omega_i = 1$. If

$$\text{IFB}_{\omega}(\alpha_1, \alpha_2, \cdots, \alpha_n) = \left(\frac{1}{n(n-1)} \left(\bigoplus_{\substack{i,j=1 \\ i \neq j}}^{n} ((\omega_i \alpha_i)^p \otimes (\omega_j \alpha_j)^q) \right) \right)^{\frac{1}{p+q}}, \quad p, q > 0 \tag{1.159}$$

then $\text{IFB}_{\omega}^{p,q}$ is called a weighted intuitionistic fuzzy Bonferroni mean (WIFBM).

Similar to Theorem 1.4.1, we have

Theorem 1.4.2 (Xu and Yager, 2011) The aggregated value by using the WIFBM (1.159) is an IFN, and

$$\text{IFB}_{\omega}^{p,q}(\alpha_1, \alpha_2, \cdots, \alpha_n)$$

$$= \left(\left(1 - \prod_{\substack{i,j=1 \\ i \neq j}}^{n} \left(1 - (1 - (1 - \mu_{\alpha_i})^{\omega_i})^p \left(1 - (1 - \mu_{\alpha_j})^{\omega_j}\right)^q\right)^{\frac{1}{n(n-1)}} \right)^{\frac{1}{p+q}}, \right.$$

$$\left. 1 - \left(1 - \prod_{\substack{i,j=1 \\ i \neq j}}^{n} \left(1 - \left(1 - \nu_{\alpha_i}^{\omega_i}\right)^p \left(1 - \nu_{\alpha_j}^{\omega_j}\right)^q\right)^{\frac{1}{n(n-1)}} \right)^{\frac{1}{p+q}} \right) \tag{1.160}$$

In what follows, we apply the WIFBM to multi-attribute decision making, which involves the following steps (Xu and Yager, 2011):

Step 1 For a multi-attribute decision making problem, let Y, G and ω be defined as in Section 1.3. The characteristic (attribute value) of the alternative Y_i with respect to the attribute G_j is measured by an IFN r'_{ij}. All r'_{ij} $(i = 1, 2, \cdots, n;\ j = 1, 2, \cdots, m)$ are contained in an intuitionistic fuzzy decision matrix $R' = (r'_{ij})_{n \times m}$. By using Eq.(1.65), we can transform R' into the normalized intuitionistic fuzzy decision matrix $R = (r_{ij})_{n \times m}$, where $r_{ij} = (\mu_{ij}, \nu_{ij})$, $\mu_{ij} \in [0, 1]$, $\nu_{ij} \in [0, 1]$, and $\mu_{ij} + \nu_{ij} \leqslant 1$.

Step 2 Utilize the WIFBM (in general, we can take $p = q = 1$):

$$r_i = (\mu_i, \nu_i) = \mathrm{IFB}_{\omega}^{p,q}(r_{i1}, r_{i2}, \cdots, r_{im}) \tag{1.161}$$

to aggregate all the characteristics r_{ij} $(j = 1, 2, \cdots, m)$ of the i-th line, and get the overall attribute values r_i corresponding to the alternative Y_i.

Step 3 Utilize the method in Definition 1.1.3 to rank the overall attribute values r_i $(i = 1, 2, \cdots, n)$.

Step 4 Rank all the alternatives Y_i $(i = 1, 2, \cdots, n)$ in accordance with r_i $(i = 1, 2, \cdots, n)$ in descending order, and then select the most desirable alternative with the largest overall attribute value.

In the above procedure, we have utilized the WIFBM to aggregate the characteristics of each alternative with respect to a collection of the pre-given attributes, so as to rank and select the alternatives. The desirable characteristic of the WIFBM is that it can not only consider the importance of each attribute but also the interrelationship of the individual attributes, and thus can take as much as possible the decision information into account.

Below let us give a detailed illustration of the decision making procedure above with a numerical example:

Example 1.4.2 (Xu and Yager, 2011) A city is planning to build a municipal library. One of the problems facing the city development commissioner is to determine what kind of air-conditioning system should be installed in the library (Yoon, 1989). The contractor offers five feasible alternatives Y_i $(i = 1, 2, 3, 4, 5)$, which might be adapted to the physical structure of the library. Suppose that three attributes: ① G_1:

Economic; ② G_2: Functional; and ③ G_3: Operational, are taken into consideration in the installation problem, the weight vector of the attributes G_j $(j = 1, 2, 3)$ is $\omega = (0.3,\ 0.5,\ 0.2)^{\mathrm{T}}$. Assume that the characteristics of the alternatives Y_i $(i = 1, 2, 3, 4, 5)$ with respect to the attribute G_j $(j = 1, 2, 3)$ are represented by the IFNs $r_{ij} = (\mu_{ij}, \nu_{ij})$, and all r_{ij} $(i = 1, 2, 3, 4, 5;\ j = 1, 2, 3)$ are contained in the intuitionistic fuzzy decision matrix $R' = (r'_{ij})_{5\times 3}$ (Table 1.26):

Table 1.26 Intuitionistic fuzzy decision matrix R' (Xu and Yager, 2011)

	G_1	G_2	G_3
Y_1	(0.3, 0.4)	(0.7,0.2)	(0.5, 0.3)
Y_2	(0.5, 0.2)	(0.4, 0.1)	(0.7, 0.1)
Y_3	(0.4, 0.5)	(0.7, 0.2)	(0.4, 0.4)
Y_4	(0.2, 0.6)	(0.8, 0.1)	(0.8, 0.2)
Y_5	(0.9, 0.1)	(0.6, 0.3)	(0.2, 0.5)

Considering all the attributes G_j $(j = 1, 2, 3)$ are the benefit attributes, the characteristics of the alternatives Y_i $(i = 1, 2, 3, 4, 5)$ do not need normalization.

Now we first utilize the WIFBM (here, we take $p = q = 1$):

$$r'_i = (\mu_i, \nu_i) = \mathrm{IFB}_{\omega}^{1,1}(r'_{i1}, r'_{i2}, r'_{i3}) \tag{1.162}$$

to aggregate all the characteristics r'_{ij} $(j = 1, 2, 3)$ of the i-th line, and get the overall attribute value r'_i corresponding to the alternative Y_i:

$$r'_1 = (0.198,\ 0.681),\quad r'_2 = (0.209,\ 0.531),\quad r'_3 = (0.202,\ 0.723)$$

$$r'_4 = (0.266,\ 0.668),\quad r'_5 = (0.278,\ 0.654)$$

Then we can calculate the scores of all the alternatives:

$$s(r'_1) = 0.198 - 0.681 = -0.483,\quad s(r'_2) = 0.209 - 0.531 = -0.322$$

$$s(r'_3) = 0.202 - 0.723 = -0.521,\quad s(r'_4) = 0.266 - 0.668 = -0.402$$

$$s(r'_5) = 0.278 - 0.654 = -0.376$$

Since

$$s(r'_2) > s(r'_5) > s(r'_4) > s(r'_1) > s(r'_3)$$

then we can get the ranking of the IFNs by Definition 1.1.3:

$$r'_2 > r'_5 > r'_4 > r'_1 > r'_3$$

and thus the ranking of the alternatives Y_i $(i = 1, 2, 3, 4, 5)$ is:

$$Y_2 \succ Y_5 \succ Y_4 \succ Y_1 \succ Y_3$$

Hence Y_2 is the best alternative.

If we take $p = q = 2$, then by $r'_i = (\mu_i, \nu_i) = \text{IFB}_{\omega}^{2,2}(r'_{i1}, r'_{i2}, r'_{i3})$, we get

$$r'_1 = (0.209,\ 0.673), \quad r'_2 = (0.209,\ 0.525), \quad r'_3 = (0.214,\ 0.713)$$

$$r'_4 = (0.302,\ 0.647), \quad r'_5 = (0.328,\ 0.625)$$

Then we can calculate the scores of all the alternatives:

$$s(r'_1) = -0.464, \quad s(r'_2) = -0.316, \quad s(r'_3) = -0.499$$

$$s(r'_4) = -0.345, \quad s(r'_5) = -0.297$$

Since

$$s(r'_5) > s(r'_2) > s(r'_4) > s(r'_1) > s(r'_3)$$

then by Definition 1.1.3, we can get the ranking of the IFNs:

$$r'_5 > r'_2 > r'_4 > r'_1 > r'_3$$

by which we have

$$Y_5 \succ Y_2 \succ Y_4 \succ Y_1 \succ Y_3$$

Hence, in this case, Y_5 is the best alternative.

In the above numerical results, the WIFBM $\text{IFB}_{\omega}^{1,1}$ produces the ranking of all the alternatives as $Y_2 \succ Y_5 \succ Y_4 \succ Y_1 \succ Y_3$, in which the alternative Y_2 ranks first. This ranking result is slightly different from the ranking of the alternatives: $Y_5 \succ Y_2 \succ Y_4 \succ Y_1 \succ Y_3$, derived by the WIFBM $\text{IFB}_{\omega}^{2,2}$. That is, the ranking of Y_5 and Y_2 is reversed while the ranking of the other alternatives keeps unchanged. Therefore, the decision results may be different with the change of the parameters p and q.

For intuitiveness and simplicity, we usually take the values of the two parameters as $p = q = 1$ in practical applications.

1.5 Generalized Intuitionistic Fuzzy Aggregation Operators

Yager (2004b) extends the OWA operator to provide a new class of operators called the generalized OWA (GOWA) operators. These operators add to the OWA operator an additional parameter controlling the power to which the argument values are raised. He also proves that the GOWA operators are mean operators. Yet it is worthy of pointing out that the GOWA operators have not been extended to accommodate intuitionistic fuzzy environment. Recently, based on the GOWA operators, Zhao et al. (2010) develop some intuitionistic fuzzy aggregation operators, such as the generalized intuitionistic fuzzy weighted averaging (GIFWA) operator, generalized intuitionistic fuzzy ordered weighted averaging (GIFOWA) operator, and generalized intuitionistic fuzzy hybrid averaging (GIFHA) operator, etc.

Definition 1.5.1 (Yager, 2004b) A generalized weighted averaging (GWA) operator of dimension n is a mapping GWA: $(Re)^{+n} \to (Re)^{+}$, which has the following form:

$$\mathrm{GWA}_{\omega}(a_1, a_2, \cdots, a_n) = \left(\sum_{j=1}^{n} \omega_j a_j^{\lambda}\right)^{\frac{1}{\lambda}} \tag{1.163}$$

where $\lambda > 0$, $\omega = (\omega_1, \omega_2, \cdots, \omega_n)^{\mathrm{T}}$ is the weight vector of the nonnegative real numbers a_j $(j = 1, 2, \cdots, n)$, with $\omega_j \in [0, 1]$ $(j = 1, 2, \cdots, n)$ and $\sum_{j=1}^{n} \omega_j = 1$.

Another aggregation operator called the GOWA operator (Yager, 2004b) is the generalization of the OWA operator:

Definition 1.5.2 (Yager, 2004b) A generalized ordered weighted averaging (GOWA) operator of dimension n is a mapping GOWA: $I^n \to I$, which has the following form:

$$\mathrm{GOWA}_{w}(a_1, a_2, \cdots, a_n) = \left(\sum_{j=1}^{n} w_j b_j^{\lambda}\right)^{\frac{1}{\lambda}} \tag{1.164}$$

where $\lambda > 0$, $w = (w_1, w_2, \cdots, w_n)^{\mathrm{T}}$ is the weighting vector associated with the GOWA operator, such that $w_j \in [0, 1]$, $j = 1, 2, \cdots, n$, and $\sum_{j=1}^{n} w_j = 1$, b_j is the j-th largest of a_i, $I = [0, 1]$.

Corresponding to certain choices of the parameters λ and w, we get some special cases. Some of the special operators have been used in situations where the input arguments are IFNs, such as the IFWA and IFOWA operators (Xu, 2007e), the IFWG and IFOWG operators (Xu and Yager, 2006). But there are still a large number of special operators that have not been extended to situations where the input arguments are IFNs. Zhao et al. (2010) extend the GWA and GOWA operators to accommodate situations where the input arguments are IFNs.

Definition 1.5.3 (Zhao et al., 2010) Let GIFWA: $\Theta^n \to \Theta$, if

$$\mathrm{GIFWA}_{\omega}\,(\alpha_1, \alpha_2, \cdots, \alpha_n) = (\omega_1 \alpha_1^{\lambda} \oplus \omega_2 \alpha_2^{\lambda} \oplus \cdots \oplus \omega_n \alpha_n^{\lambda})^{\frac{1}{\lambda}} \tag{1.165}$$

then the function GIFWA is called a generalized intuitionistic fuzzy weighted averaging (GIFWA) operator, where $\lambda > 0$, $\omega = (\omega_1, \omega_2, \cdots, \omega_n)^{\mathrm{T}}$ is a weight vector of α_j $(j = 1, 2, \cdots, n)$, with $\omega_j \in [0, 1]$, $j = 1, 2, \cdots, n$, and $\sum_{j=1}^{n} \omega_j = 1$.

Theorem 1.5.1 (Zhao et al., 2010) The aggregated value by using the GIFWA operator is an IFN, and

$$\begin{aligned}&\mathrm{GIFWA}_\omega(\alpha_1, \alpha_2, \cdots, \alpha_n)\\ &= \left(\left(1 - \prod_{j=1}^{n} (1 - \mu_{\alpha_j}^{\lambda})^{\omega_j} \right)^{\frac{1}{\lambda}},\ 1 - \left(1 - \prod_{j=1}^{n} (1 - (1 - \nu_{\alpha_j})^{\lambda})^{\omega_j} \right)^{\frac{1}{\lambda}} \right)\end{aligned} \tag{1.166}$$

Proof The first result follows quickly from Definition 1.2.2 and Theorem 1.2.2. In the following, we first prove

$$\omega_1 \alpha_1^{\lambda} \oplus \omega_2 \alpha_2^{\lambda} \oplus \cdots \oplus \omega_n \alpha_n^{\lambda} = \left(1 - \prod_{j=1}^{n} (1 - \mu_{\alpha_j}^{\lambda})^{\omega_j},\ \prod_{j=1}^{n} \left(1 - (1 - \nu_{\alpha_j})^{\lambda} \right)^{\omega_j} \right) \tag{1.167}$$

by using mathematical induction on n:

(1) For $n = 2$: Since

$$\alpha_1^{\lambda} = \left(\mu_{\alpha_1}^{\lambda}, 1 - (1 - \nu_{\alpha_1})^{\lambda} \right), \quad \alpha_2^{\lambda} = \left(\mu_{\alpha_2}^{\lambda}, 1 - (1 - \nu_{\alpha_2})^{\lambda} \right)$$

then

$$\omega_1 \alpha_1^{\lambda} \oplus \omega_2 \alpha_2^{\lambda} = \left(1 - \prod_{j=1}^{2} \left(1 - \mu_{\alpha_j}^{\lambda} \right)^{\omega_j}, \prod_{j=1}^{2} \left(1 - (1 - \nu_{\alpha_j})^{\lambda} \right)^{\omega_j} \right)$$

(2) If Eq.(1.167) holds for $n = k$, that is

$$\omega_1 \alpha_1^{\lambda} \oplus \omega_2 \alpha_2^{\lambda} \oplus \cdots \oplus \omega_k \alpha_k^{\lambda} = \left(1 - \prod_{j=1}^{k} \left(1 - \mu_{\alpha_j}^{\lambda} \right)^{\omega_j}, \prod_{j=1}^{k} \left(1 - (1 - \nu_{\alpha_j})^{\lambda} \right)^{\omega_j} \right)$$

then, when $n = k + 1$, by the operational laws (4), (6) and (7) in Definition 1.2.2, we have

$$\begin{aligned}\omega_1 \alpha_1^{\lambda} \oplus \omega_2 \alpha_2^{\lambda} \oplus \cdots \oplus \omega_{k+1} \alpha_{k+1}^{\lambda} &= \left(1 - \prod_{j=1}^{k} \left(1 - \mu_{\alpha_j}^{\lambda} \right)^{\omega_j}, \prod_{j=1}^{k} \left(1 - (1 - \nu_{\alpha_j})^{\lambda} \right)^{\omega_j} \right)\\ &\quad \oplus \left(1 - \left(1 - \mu_{\alpha_{k+1}}^{\lambda} \right)^{\omega_{k+1}}, \left(1 - (1 - \nu_{\alpha_{k+1}})^{\lambda} \right)^{\omega_{k+1}} \right)\\ &= \left(1 - \prod_{j=1}^{k+1} \left(1 - \mu_{\alpha_j}^{\lambda} \right)^{\omega_j}, \prod_{j=1}^{k+1} \left(1 - (1 - \nu_{\alpha_j})^{\lambda} \right)^{\omega_j} \right)\end{aligned}$$

i.e. Eq.(1.167) holds for $n = k+1$. Thus, Eq.(1.167) holds for all n. Then

$$\begin{aligned}&\text{GIFWA}_\omega(\alpha_1, \alpha_2, \cdots, \alpha_n)\\&= \left(1 - \prod_{j=1}^{n}\left(1-\mu_{\alpha_j}^{\lambda}\right)^{\omega_j}, \prod_{j=1}^{n}\left(1-(1-\nu_{\alpha_j})^{\lambda}\right)^{\omega_j}\right)^{\frac{1}{\lambda}}\\&= \left(\left(1 - \prod_{j=1}^{n}\left(1-\mu_{\alpha_j}^{\lambda}\right)^{\omega_j}\right)^{\frac{1}{\lambda}}, 1 - \left(1 - \prod_{j=1}^{n}\left(1-(1-\nu_{\alpha_j})^{\lambda}\right)^{\omega_j}\right)^{\frac{1}{\lambda}}\right)\end{aligned}$$

Example 1.5.1 (Zhao et al., 2010) Let $\alpha_1 = (0.1,\ 0.7)$, $\alpha_2 = (0.4,\ 0.3)$, $\alpha_3 = (0.6,\ 0.1)$ and $\alpha_4 = (0.2,\ 0.5)$ be four IFNs, and $\omega = (0.2,\ 0.3,\ 0.1,\ 0.4)^{\mathrm{T}}$ the weight vector of α_j $(j = 1, 2, 3, 4)$. Let $\lambda = 2$, then

$$\begin{aligned}&\text{GIFWA}_\omega(\alpha_1, \alpha_2, \alpha_3, \alpha_4)\\&= ((1-(1-0.1^2)^{0.2} \times (1-0.4^2)^{0.3} \times (1-0.6^2)^{0.1} \times (1-0.2^2)^{0.4})^{\frac{1}{2}},\\&\quad 1-(1-(1-(1-0.7)^2)^{0.2} \times (1-(1-0.3)^2)^{0.3}\\&\quad \times (1-(1-0.1)^2)^{0.1} \times (1-(1-0.5)^2)^{0.4})^{\frac{1}{2}})\\&= (0.3381, 0.3717)\end{aligned}$$

Based on Theorem 1.5.1, we have the following properties of the GIFWA operators:

Theorem 1.5.2 (Zhao et al., 2010) If all IFNs α_j $(j = 1, 2, \cdots, n)$ are equal, i.e. $\alpha_j = \alpha$, for all j, then

$$\text{GIFWA}_\omega(\alpha_1, \alpha_2, \cdots, \alpha_n) = \alpha$$

Proof By Theorem 1.5.1, we have

$$\begin{aligned}\text{GIFWA}_\omega(\alpha_1, \alpha_2, \cdots, \alpha_n) &= (\omega_1\alpha_1^{\lambda} \oplus \omega_2\alpha_2^{\lambda} \oplus \cdots \oplus \omega_n\alpha_n^{\lambda})^{\frac{1}{\lambda}}\\&= (\omega_1\alpha^{\lambda} \oplus \omega_2\alpha^{\lambda} \oplus \cdots \oplus \omega_n\alpha^{\lambda})^{\frac{1}{\lambda}}\\&= ((\omega_1+\omega_2+\cdots+\omega_n)\alpha^{\lambda})^{\frac{1}{\lambda}}\\&= (\alpha^{\lambda})^{\frac{1}{\lambda}} = \alpha\end{aligned}$$

Theorem 1.5.3 (Zhao et al., 2010) Let

$$\alpha^- = \left(\min_j(\mu_{\alpha_j}), \max_j(\nu_{\alpha_j})\right), \quad \alpha^+ = \left(\max_j(\mu_{\alpha_j}), \min_j(\nu_{\alpha_j})\right)$$

Then

$$\alpha^- \leqslant \text{GIFWA}_\omega(\alpha_1, \alpha_2, \cdots, \alpha_n) \leqslant \alpha^+ \tag{1.168}$$

Proof Since $\min_j(\mu_{\alpha_j}) \leqslant \mu_{\alpha_j} \leqslant \max_j(\mu_{\alpha_j})$ and $\min_j(\nu_{\alpha_j}) \leqslant \nu_{\alpha_j} \leqslant \max_j(\nu_{\alpha_j})$, for all j, then

$$\prod_{j=1}^{n}(1-\mu_{\alpha_j}^{\lambda})^{\omega_j} \geqslant \prod_{j=1}^{n}\left(1-\left(\max_j(\mu_{\alpha_j})\right)^{\lambda}\right)^{\omega_j} = 1-\left(\max_j(\mu_{\alpha_j})\right)^{\lambda}$$

and then

$$\left(1-\prod_{j=1}^{n}(1-\mu_{\alpha_j}^{\lambda})^{\omega_j}\right)^{\frac{1}{\lambda}} \leqslant \max_j(\mu_{\alpha_j}) \tag{1.169}$$

Similarly, we have

$$\left(1-\prod_{j=1}^{n}\left(1-\mu_{\alpha_j}^{\lambda}\right)^{\omega_j}\right)^{\frac{1}{\lambda}} \geqslant \min_j(\mu_{\alpha_j}) \tag{1.170}$$

$$\prod_{j=1}^{n}\left(1-(1-\nu_{\alpha_j})^{\lambda}\right)^{\omega_j} \leqslant \prod_{j=1}^{n}\left(1-(1-\max_j(\nu_{\alpha_j}))^{\lambda}\right)^{\omega_j} = 1-\left(1-\max_j(\nu_{\alpha_j})\right)^{\lambda}$$

$$1-\prod_{j=1}^{n}\left(1-(1-\nu_{\alpha_j})^{\lambda}\right)^{\omega_j} \geqslant \left(1-\max_j(\nu_{\alpha_j})\right)^{\lambda}$$

$$\left(1-\prod_{j=1}^{n}\left(1-(1-\nu_{\alpha_j})^{\lambda}\right)^{\omega_j}\right)^{\frac{1}{\lambda}} \geqslant 1-\max_j(\nu_{\alpha_j})$$

$$1-\left(1-\prod_{j=1}^{n}\left(1-(1-\nu_{\alpha_j})^{\lambda}\right)^{\omega_j}\right)^{\frac{1}{\lambda}} \leqslant \max_j(\nu_{\alpha_j}) \tag{1.171}$$

In a similar way, we get

$$1-\left(1-\prod_{j=1}^{n}\left(1-(1-\nu_{\alpha_j})^{\lambda}\right)^{\omega_j}\right)^{\frac{1}{\lambda}} \geqslant \min_j(\nu_{\alpha_j}) \tag{1.172}$$

Let $\text{GIFWA}_{\omega}(\alpha_1,\alpha_2,\cdots,\alpha_n)=\alpha=(\mu_\alpha,\nu_\alpha)$. Then

$$s(\alpha)=\mu_\alpha-\nu_\alpha \leqslant \max_j(\mu_{\alpha_j})-\min_j(\nu_{\alpha_j})=s(\alpha^+)$$

$$s(\alpha)=\mu_\alpha-\nu_\alpha \geqslant \min_j(\mu_{\alpha_j})-\max_j(\nu_{\alpha_j})=s(\alpha^-)$$

If $s(\alpha)<s(\alpha^+)$ and $s(\alpha)>s(\alpha^-)$, then we get by Definition 1.1.3 that

$$\alpha^- < \text{GIFWA}_{\omega}(\alpha_1,\alpha_2,\cdots,\alpha_n) < \alpha^+ \tag{1.173}$$

If $s(\alpha) = s(\alpha^+)$, i.e. $\mu_\alpha - \nu_\alpha = \max_j(\mu_{\alpha_j}) - \min_j(\nu_{\alpha_j})$, then by Eqs.(1.169) and (1.172), we have

$$\mu_\alpha = \max_j(\mu_{\alpha_j}), \quad \nu_\alpha = \min_j(\nu_{\alpha_j})$$

So

$$h(\alpha) = \mu_\alpha + \nu_\alpha = \max_j(\mu_{\alpha_j}) + \min_j(\nu_{\alpha_j}) = h(\alpha^+)$$

Then it follows from Definition 1.1.3 that

$$\text{GIFWA}_\omega(\alpha_1, \alpha_2, \cdots, \alpha_n) = \alpha^+ \tag{1.174}$$

If $s(\alpha) = s(\alpha^-)$, i.e. $\mu_\alpha - \nu_\alpha = \min_j(\mu_{\alpha_j}) - \max_j(\nu_{\alpha_j})$, then by Eqs.(1.170) and (1.171) we obtain

$$\mu_\alpha = \min_j(\mu_{\alpha_j}), \quad \nu_\alpha = \max_j(\nu_{\alpha_j})$$

So

$$h(\alpha) = \mu_\alpha + \nu_\alpha = \min_j(\mu_{\alpha_j}) + \max_j(\nu_{\alpha_j}) = h(\alpha^-)$$

Thus, it follows from Definition 1.1.3 that

$$\text{GIFWA}_\omega(\alpha_1, \alpha_2, \cdots, \alpha_n) = \alpha^- \tag{1.175}$$

and then from Eqs.(1.173)–(1.175), we know that Eq.(1.168) always holds.

Theorem 1.5.4 (Zhao et al., 2010) Let $\alpha_j = (\mu_{\alpha_j}, \nu_{\alpha_j})$ $(j = 1, 2, \cdots, n)$ and $\alpha_j^* = (\mu_{\alpha_j^*}, \nu_{\alpha_j^*})$ $(j = 1, 2, \cdots, n)$ be two collections of IFNs, $\lambda > 0$. If $\mu_{\alpha_j} \leqslant \mu_{\alpha_j^*}$ and $\nu_{\alpha_j} \geqslant \nu_{\alpha_j^*}$, for all j, then

$$\text{GIFWA}_\omega(\alpha_1, \alpha_2, \cdots, \alpha_n) \leqslant \text{GIFWA}_\omega(\alpha_1^*, \alpha_2^*, \cdots, \alpha_n^*) \tag{1.176}$$

Proof Since $\mu_{\alpha_j} \leqslant \mu_{\alpha_j^*}$ and $\nu_{\alpha_j} \geqslant \nu_{\alpha_j^*}$, for all j, then

$$\prod_{j=1}^{n}\left(1-\mu_{\alpha_j}^{\lambda}\right)^{\omega_j} \geqslant \prod_{j=1}^{n}\left(1-\mu_{\alpha_j^*}^{\lambda}\right)^{\omega_j}$$

$$1-\prod_{j=1}^{n}\left(1-\mu_{\alpha_j}^{\lambda}\right)^{\omega_j} \leqslant 1-\prod_{j=1}^{n}\left(1-\mu_{\alpha_j^*}^{\lambda}\right)^{\omega_j}$$

$$\left(1-\prod_{j=1}^{n}\left(1-\mu_{\alpha_j}^{\lambda}\right)^{\omega_j}\right)^{\frac{1}{\lambda}} \leqslant \left(1-\prod_{j=1}^{n}\left(1-\mu_{\alpha_j^*}^{\lambda}\right)^{\omega_j}\right)^{\frac{1}{\lambda}}$$

$$\prod_{j=1}^{n}\left(1-(1-\nu_{\alpha_j})^{\lambda}\right)^{\omega_j} \geqslant \prod_{j=1}^{n}\left(1-(1-\nu_{\alpha_j^*})^{\lambda}\right)^{\omega_j}$$

$$1-\prod_{j=1}^{n}\left(1-(1-\nu_{\alpha_j})^{\lambda}\right)^{\omega_j} \leqslant 1-\prod_{j=1}^{n}\left(1-(1-\nu_{\alpha_j^*})^{\lambda}\right)^{\omega_j}$$

$$\left(1-\prod_{j=1}^{n}\left(1-(1-\nu_{\alpha_j})^{\lambda}\right)^{\omega_j}\right)^{\frac{1}{\lambda}} \leqslant \left(1-\prod_{j=1}^{n}\left(1-(1-\nu_{\alpha_j^*})^{\lambda}\right)^{\omega_j}\right)^{\frac{1}{\lambda}}$$

$$1-\left(1-\prod_{j=1}^{n}\left(1-(1-\nu_{\alpha_j})^{\lambda}\right)^{\omega_j}\right)^{\frac{1}{\lambda}} \geqslant 1-\left(1-\prod_{j=1}^{n}\left(1-(1-\nu_{\alpha_j^*})^{\lambda}\right)^{\omega_j}\right)^{\frac{1}{\lambda}}$$

$$\left(1-\prod_{j=1}^{n}\left(1-\mu_{\alpha_j}^{\lambda}\right)^{\omega_j}\right)^{\frac{1}{\lambda}} - \left(1-\left(1-\prod_{j=1}^{n}\left(1-(1-\nu_{\alpha_j})^{\lambda}\right)^{\omega_j}\right)^{\frac{1}{\lambda}}\right)$$

$$\leqslant \left(1-\prod_{j=1}^{n}\left(1-\mu_{\alpha_j^*}^{\lambda}\right)^{\omega_j}\right)^{\frac{1}{\lambda}} - \left(1-\left(1-\prod_{j=1}^{n}\left(1-(1-\nu_{\alpha_j^*})^{\lambda}\right)^{\omega_j}\right)^{\frac{1}{\lambda}}\right) \tag{1.177}$$

Let $\alpha = \text{GIFWA}_{\omega}(\alpha_1, \alpha_2, \cdots, \alpha_n)$ and $\alpha^* = \text{GIFWA}_{\omega}(\alpha_1^*, \alpha_2^*, \cdots, \alpha_n^*)$. Then by Eq.(1.177), we have $s(\alpha) \leqslant s(\alpha^*)$.

If $s(\alpha) < s(\alpha^*)$, then by Definition 1.1.3 we can get

$$\text{GIFWA}_{\omega}(\alpha_1, \alpha_2, \cdots, \alpha_n) < \text{GIFWA}_{\omega}(\alpha_1^*, \alpha_2^*, \cdots, \alpha_n^*) \tag{1.178}$$

If $s(\alpha) = s(\alpha^*)$, then

$$\left(1-\prod_{j=1}^{n}\left(1-\mu_{\alpha_j}^{\lambda}\right)^{\omega_j}\right)^{\frac{1}{\lambda}} - \left(1-\left(1-\prod_{j=1}^{n}\left(1-(1-\nu_{\alpha_j})^{\lambda}\right)^{\omega_j}\right)^{\frac{1}{\lambda}}\right)$$

$$= \left(1-\prod_{j=1}^{n}\left(1-\mu_{\alpha_j^*}^{\lambda}\right)^{\omega_j}\right)^{\frac{1}{\lambda}} - \left(1-\left(1-\prod_{j=1}^{n}\left(1-(1-\nu_{\alpha_j^*})^{\lambda}\right)^{\omega_j}\right)^{\frac{1}{\lambda}}\right)$$

Since $\mu_{\alpha_j} \leqslant \mu_{\alpha_j^*}$ and $\nu_{\alpha_j} \geqslant \nu_{\alpha_j^*}$, for all j, we have

$$\left(1-\prod_{j=1}^{n}\left(1-\mu_{\alpha_j}^{\lambda}\right)^{\omega_j}\right)^{\frac{1}{\lambda}} = \left(1-\prod_{j=1}^{n}\left(1-\mu_{\alpha_j^*}^{\lambda}\right)^{\omega_j}\right)^{\frac{1}{\lambda}}$$

$$1-\left(1-\prod_{j=1}^{n}\left(1-(1-\nu_{\alpha_j})^{\lambda}\right)^{\omega_j}\right)^{\frac{1}{\lambda}} = 1-\left(1-\prod_{j=1}^{n}\left(1-(1-\nu_{\alpha_j^*})^{\lambda}\right)^{\omega_j}\right)^{\frac{1}{\lambda}}$$

Hence

$$\begin{aligned} h(\alpha) &= \left(1-\prod_{j=1}^{n}\left(1-\mu_{\alpha_j}^{\lambda}\right)^{\omega_j}\right)^{\frac{1}{\lambda}} + \left(1-\left(1-\prod_{j=1}^{n}\left(1-(1-\nu_{\alpha_j})^{\lambda}\right)^{\omega_j}\right)^{\frac{1}{\lambda}}\right) \\ &= \left(1-\prod_{j=1}^{n}\left(1-\mu_{\alpha_j^*}^{\lambda}\right)^{\omega_j}\right)^{\frac{1}{\lambda}} + \left(1-\left(1-\prod_{j=1}^{n}\left(1-(1-\nu_{\alpha_j^*})^{\lambda}\right)^{\omega_j}\right)^{\frac{1}{\lambda}}\right) \\ &= h(\alpha^*) \end{aligned}$$

thus, it follows from Definition 1.1.3 that

$$\mathrm{GIFWA}_{\omega}(\alpha_1,\alpha_2,\cdots,\alpha_n) = \mathrm{GIFWA}_{\omega}(\alpha_1^*,\alpha_2^*,\cdots,\alpha_n^*) \tag{1.179}$$

From Eqs.(1.178) and (1.179), we can see that Eq.(1.176) always holds.

We now look at some special cases corresponding to different choices of the parameters ω and λ:

Theorem 1.5.5 (Zhao et al., 2010)

(1) If λ=1, then the GIFWA operator (1.165) reduces to the IFWA operator.

(2) If $\omega = (1/n, 1/n, \cdots, 1/n)^{\mathrm{T}}$ and λ=1, then the GIFWA operator (1.165) reduces to the IFA operator.

Definition 1.5.4 (Zhao et al., 2010) Let $\alpha_j = (\mu_{\alpha_j}, \nu_{\alpha_j})$ $(j = 1, 2, \cdots, n)$ be a collection of IFNs, and let GIFOWA_w: $\Theta^n \to \Theta$. If

$$\mathrm{GIFOWA}_w(\alpha_1,\alpha_2,\cdots,\alpha_n) = \left(w_1\alpha_{\sigma(1)}^{\lambda} \oplus w_2\alpha_{\sigma(2)}^{\lambda} \oplus \cdots \oplus w_n\alpha_{\sigma(n)}^{\lambda}\right)^{\frac{1}{\lambda}} \tag{1.180}$$

where $\lambda > 0$, $w = (w_1, w_2, \cdots, w_n)^{\mathrm{T}}$ is an associated weight vector such that $w_j \in [0,1]$, $j = 1, 2, \cdots, n$, and $\sum_{j=1}^{n} w_j = 1$, $\alpha_{\sigma(j)}$ is the j-th largest of α_j $(j = 1, 2, \cdots, n)$, then the function GIFOWA is called a generalized intuitionistic fuzzy ordered weighted averaging (GIFOWA) operator.

The GIFOWA operator has some properties similar to those of the GIFWA operator:

Theorem 1.5.6 (Zhao et al., 2010) The aggregated value by using the GIFOWA operator is an IFN, and

$$\begin{aligned} &\mathrm{GIFOWA}_w(\alpha_1,\alpha_2,\cdots,\alpha_n) \\ &= \left(\left(1-\prod_{j=1}^{n}(1-\mu_{\alpha_{\sigma(j)}}^{\lambda})^{w_j}\right)^{\frac{1}{\lambda}},\ 1-\left(1-\prod_{j=1}^{n}\left(1-(1-\nu_{\alpha_{\sigma(j)}})^{\lambda}\right)^{w_j}\right)^{\frac{1}{\lambda}}\right) \end{aligned} \tag{1.181}$$

Example 1.5.2 (Zhao et al., 2010) Let $\alpha_1 = (0.3,\ 0.6)$, $\alpha_2 = (0.4,\ 0.5)$, $\alpha_3 = (0.6,\ 0.3)$, $\alpha_4 = (0.7,\ 0.1)$, and $\alpha_5 = (0.1,\ 0.6)$ be five IFNs, $w = (0.1117,\ 0.2365,\ 0.3036,\ 0.2365,\ 0.1117)^{\mathrm{T}}$ the weight vector of α_j $(j = 1, 2, 3, 4, 5)$, and λ=2. Then

$$\mu_{\alpha_1} = 0.3, \quad \mu_{\alpha_2} = 0.4, \quad \mu_{\alpha_3} = 0.6, \quad \mu_{\alpha_4} = 0.7, \quad \mu_{\alpha_5} = 0.1$$

$$\nu_{\alpha_1} = 0.6, \quad \nu_{\alpha_2} = 0.5, \quad \nu_{\alpha_3} = 0.3, \quad \nu_{\alpha_4} = 0.1, \quad \nu_{\alpha_5} = 0.6$$

Let us calculate the scores of α_j $(j = 1, 2, 3, 4, 5)$:

$$s(\alpha_1) = 0.3 - 0.6 = -0.3, \quad s(\alpha_2) = 0.4 - 0.5 = -0.1, \quad s(\alpha_3) = 0.6 - 0.3 = 0.3$$

$$s(\alpha_4) = 0.7 - 0.1 = 0.6, \quad s(\alpha_5) = 0.1 - 0.6 = -0.5$$

Since

$$s(\alpha_4) > s(\alpha_3) > s(\alpha_2) > s(\alpha_1) > s(\alpha_5)$$

we have

$$\alpha_{\sigma(1)} = (0.7,\ 0.1), \quad \alpha_{\sigma(2)} = (0.6,\ 0.3), \quad \alpha_{\sigma(3)} = (0.4,\ 0.5)$$

$$\alpha_{\sigma(4)} = (0.3,\ 0.6), \quad \alpha_{\sigma(5)} = (0.1, 0.6)$$

and thus using Eq.(1.180), we can get

$$\mathrm{GIFOWA}_w(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) = (0.4762, 0.3762)$$

Theorem 1.5.7 (Zhao et al., 2010) If all IFNs α_j $(j = 1, 2, \cdots, n)$ are equal, i.e. $\alpha_j = \alpha$, for all j, then

$$\mathrm{GIFOWA}_w(\alpha_1, \alpha_2, \cdots, \alpha_n) = \alpha$$

Theorem 1.5.8 (Zhao et al., 2010) Let

$$\alpha^- = \left(\min_j(\mu_{\alpha_j}), \max_j(\nu_{\alpha_j})\right), \quad \alpha^+ = \left(\max_j(\mu_{\alpha_j}), \min_j(\nu_{\alpha_j})\right)$$

Then

$$\alpha^- \leqslant \mathrm{GIFOWA}_w(\alpha_1, \alpha_2, \cdots, \alpha_n) \leqslant \alpha^+$$

Theorem 1.5.9 (Zhao et al., 2010) Let $\alpha_j = (\mu_{\alpha_j}, \nu_{\alpha_j})$ $(j = 1, 2, \cdots, n)$ and $\alpha_j^* = (\mu_{\alpha_j^*}, \nu_{\alpha_j^*})$ $(j = 1, 2, \cdots, n)$ be two collections of IFNs. If $\mu_{\alpha_j} \leqslant \mu_{\alpha_j^*}$ and $\nu_{\alpha_j} \geqslant \nu_{\alpha_j^*}$, for all j, then

$$\mathrm{GIFOWA}_w(\alpha_1, \alpha_2, \cdots, \alpha_n) \leqslant \mathrm{GIFOWA}_w(\alpha_1^*, \alpha_2^*, \cdots, \alpha_n^*)$$

Theorem 1.5.10 (Zhao et al., 2010) Let $\alpha_j = (\mu_{\alpha_j}, \nu_{\alpha_j})$ $(j = 1, 2, \cdots, n)$ and $\alpha_j' = (\mu_{\alpha_j'}, \nu_{\alpha_j'})$ $(j = 1, 2, \cdots, n)$ be two collections of IFNs. Then

$$\mathrm{GIFOWA}_w(\alpha_1, \alpha_2, \cdots, \alpha_n) = \mathrm{GIFOWA}_w(\alpha_1', \alpha_2', \cdots, \alpha_n') \tag{1.182}$$

where $(\alpha_1', \alpha_2', \cdots, \alpha_n')^{\mathrm{T}}$ is any permutation of $(\alpha_1, \alpha_2, \cdots, \alpha_n)^{\mathrm{T}}$.

From Eq.(1.182), we know that the GIFOWA operator possesses the commutativity property we desire to have. It is worth noting that the GIFWA operator does not have this property.

We now examine some special cases corresponding to different choices of the parameters w and λ:

Theorem 1.5.11 (Zhao et al., 2010)

(1) If λ=1, then the GIFOWA operator (1.180) reduces to the IFOWA operator.

(2) If $w = (1/n, 1/n, \cdots, 1/n)^{\mathrm{T}}$ and λ=1, then the GIFOWA operator (1.180) reduces to the IFA operator.

(3) If $w = (1, 0, \cdots, 0)^{\mathrm{T}}$, then the GIFOWA operator (1.180) reduces to the following:

$$\mathrm{IFMAX}_w(\alpha_1, \alpha_2, \cdots, \alpha_n) = \max_j(\alpha_j)$$

which is called an intuitionistic fuzzy maximum operator (Chen and Tan, 1994).

(4) If $w = (0, 0, \cdots, 1)^{\mathrm{T}}$, then the GIFOWA operator (1.180) reduces to the following:

$$\mathrm{IFMIN}_w(\alpha_1, \alpha_2, \cdots, \alpha_n) = \min_j(\alpha_j)$$

which is called an intuitionistic fuzzy minimum operator (Chen and Tan, 1994).

Note that the GIFWA operator weights only the IFNs, while the GIFOWA operator weights only the ordered positions of the IFNs instead of the IFNs themselves. To overcome this limitation, in what follows, we introduce a generalized intuitionistic fuzzy hybrid aggregation (GIFHA) operator, which weights both the given IFNs and their ordered positions:

Definition 1.5.5 (Zhao et al., 2010) A GIFHA operator of dimension n is a mapping GIFHA : $\Theta^n \to \Theta$, which has an associated vector $w = (w_1, w_2, \cdots, w_n)^{\mathrm{T}}$, with $w_j \in [0, 1]$ $(j = 1, 2, \cdots, n)$ and $\sum_{j=1}^{n} w_j = 1$, such that

$$\mathrm{GIFHA}_{w,\omega}(\alpha_1, \alpha_2, \cdots, \alpha_n) = (w_1(\dot{\alpha}_{\sigma(1)})^\lambda \oplus w_2(\dot{\alpha}_{\sigma(2)})^\lambda \oplus \cdots \oplus w_n(\dot{\alpha}_{\sigma(n)})^\lambda)^{\frac{1}{\lambda}} \quad (1.183)$$

where $\lambda > 0$, $\dot{\alpha}_{\sigma(j)}$ is the j-th largest of the weighted IFNs $\dot{\alpha}_j$ $(\dot{\alpha}_j = n\omega_j\alpha_j,\ j = 1, 2, \cdots, n)$, $\omega = (\omega_1, \omega_2, \cdots, \omega_n)^{\mathrm{T}}$ is the weight vector of α_j $(j = 1, 2, \cdots, n)$ with $\omega_j \in [0, 1]$ $j = 1, 2, \cdots, n$, $\sum_{j=1}^{n} \omega_j = 1$, and n is the balancing coefficient.

Let $\dot{\alpha}_{\sigma(j)} = (\mu_{\dot{\alpha}_{\sigma(j)}}, \nu_{\dot{\alpha}_{\sigma(j)}})$. Then, similar to Theorem 1.5.6, we have

$$\mathrm{GIFHA}_{w,\omega}(\alpha_1, \alpha_2, \cdots, \alpha_n)$$

$$= \left(\left(1 - \prod_{j=1}^{n} (1 - \mu_{\dot{\alpha}_{\sigma(j)}}^{\lambda})^{w_j} \right)^{\frac{1}{\lambda}}, 1 - \left(1 - \prod_{j=1}^{n} \left(1 - (1 - \nu_{\dot{\alpha}_{\sigma(j)}})^{\lambda}\right)^{w_j} \right)^{\frac{1}{\lambda}} \right) \quad (1.184)$$

and the aggregated value derived by using the GIFHA operator is an IFN. Especially, if $\lambda = 1$, then Eq.(1.184) reduces to the IFHA operator.

Theorem 1.5.12 (Zhao et al., 2010) The GIFOWA operator is a special case of the GIFHA operator.

Proof Let $\omega = (1/n, 1/n, \cdots, 1/n)^{\mathrm{T}}$. Then $\dot{\alpha}_j = \alpha_j$ $(j = 1, 2, \cdots, n)$, so we have

$$\begin{aligned} \mathrm{GIFHA}_{w,\omega}(\alpha_1, \alpha_2, \cdots, \alpha_n) &= (w_1(\dot{\alpha}_{\sigma(1)})^{\lambda} \oplus w_2(\dot{\alpha}_{\sigma(2)})^{\lambda} \oplus \cdots \oplus w_n(\dot{\alpha}_{\sigma(n)})^{\lambda})^{\frac{1}{\lambda}} \\ &= (w_1\alpha_{\sigma(1)}^{\lambda} \oplus w_2\alpha_{\sigma(2)}^{\lambda} \oplus \cdots \oplus w_n\alpha_{\sigma(n)}^{\lambda})^{\frac{1}{\lambda}} \\ &= \mathrm{GIFOWA}_w(\alpha_1, \alpha_2, \cdots, \alpha_n) \end{aligned}$$

This completes the proof.

Example 1.5.3 (Zhao et al., 2010) Let $\alpha_1 = (0.2, 0.5)$, $\alpha_2 = (0.4, 0.2)$, $\alpha_3 = (0.5, 0.4)$, $\alpha_4 = (0.3, 0.3)$ and $\alpha_5 = (0.7, 0.1)$ be five IFNs, and $\omega = (0.25,\ 0.20,\ 0.15,\ 0.18,\ 0.22)^{\mathrm{T}}$ the weight vector of α_j $(j = 1, 2, 3, 4, 5)$. Then, by the operational law (6) in Definition 1.2.2, the weighted IFNs can be obtained as:

$$\dot{\alpha}_1 = (0.234, 0.42), \quad \dot{\alpha}_2 = (0.4, 0.2), \quad \dot{\alpha}_3 = (0.405, 0.503)$$

$$\dot{\alpha}_4 = (0.275, 0.338), \quad \dot{\alpha}_5 = (0.734, 0.079)$$

Using Eq.(1.10), we can compute the scores of $\dot{\alpha}_j$ $(j = 1, 2, 3, 4, 5)$:

$$s(\dot{\alpha}_1) = -0.177, \quad s(\dot{\alpha}_2) = 0.2, \quad s(\dot{\alpha}_3) = -0.098$$

$$s(\dot{\alpha}_4) = -0.063, \quad s(\dot{\alpha}_5) = 0.655$$

Since

$$s(\dot{\alpha}_5) > s(\dot{\alpha}_2) > s(\dot{\alpha}_4) > s(\dot{\alpha}_3) > s(\dot{\alpha}_1)$$

we can get

$$\dot{\alpha}_{\sigma(1)} = (0.734,\ 0.079), \quad \dot{\alpha}_{\sigma(2)} = (0.4,\ 0.2), \quad \dot{\alpha}_{\sigma(3)} = (0.275,\ 0.338)$$

$$\dot{\alpha}_{\sigma(4)} = (0.405,\ 0.503), \quad \dot{\alpha}_{\sigma(5)} = (0.234,\ 0.42)$$

Suppose that $w = (0.112,\ 0.236,\ 0.304,\ 0.236,\ 0.112)^{\mathrm{T}}$ (derived by the normal distribution based method (Xu, 2005a) is the weighting vector of the GIFHA operator. Then, it follows from Eq.(1.182) that

$$\mathrm{GIFHA}_{w,\omega}(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) = (0.4295,\ 0.2767)$$

Similar to the IFWA, IFOWA and IFHA operators, the GIFWA, GIFOWA and GIFHA operators can also be applied to multi-attribute decision making based on intuitionistic fuzzy information.

1.6 Intuitionistic Fuzzy Aggregation Operators Based on Choquet Integral

The intuitionistic fuzzy aggregation operators presented previously only consider situations where all the elements in an IFS are independent, i.e., they only consider the addition of the importance of individual elements. However, in many practical situations, the elements in an IFS are usually correlative. For instance, Grabisch (1995) and Torra (2003) described such an example: "We are to evaluate a set of students in relation to three subjects: {mathematics, physics, literature}, we want to give more importance to science-related subjects than to literature, but on the other hand we want to give some advantage to students that are good both in literature and in any of the science related subjects".

As shown by the example, we need to find new ways to deal with these situations in which the considered decision data are correlative. Choquet integral (Choquet, 1953) is a very useful means of measuring the expected utility of an uncertain event, which can be used to depict the correlations of the considered data. Motivated by the correlation properties of Choquet integral, Xu (2010c) proposes some intuitionistic fuzzy aggregation operators, whose prominent characteristic is that they can address not only the importance of the elements or their ordered positions, but also the correlations of the elements or their ordered positions.

Let a finite set $X = \{x_1, x_2, \cdots, x_n\}$ be fixed. In some situations, the weight of each element $x_i \in X$ should be taken into account. For example, in multi-attribute decision making, the considered attributes usually have different importance, and thus need to be assigned different weights.

Let $\zeta(\{x_i\})$ $(i = 1, 2, \cdots, n)$ be the weights of the elements $x_i \in X$ $(i = 1, 2, \cdots, n)$, where ζ is a fuzzy measure, which can be defined as follows:

Definition 1.6.1 (Wang and Klir, 1992) A fuzzy measure ζ on the set X is a set function $\zeta: X \rightarrow [0, 1]$ satisfying the following axioms:

(1) $\zeta(\varnothing) = 0$, $\zeta(X) = 1$;

(2) $B \subseteq C$ implies $\zeta(B) \leqslant \zeta(C)$, for all $B, C \subseteq X$;

(3) $\zeta(B \cup C) = \zeta(B) + \zeta(C) + \rho\,\zeta(B)\zeta(C)$, for all $B, C \subseteq X$and $B \cap C = \varnothing$, where $\rho \in (-1, \infty)$.

Especially, if $\rho = 0$, then (3) in Definition 1.6.1 reduces to the axiom of additive measure:

$$\zeta(B \cup C) = \zeta(B) + \zeta(C), \quad \text{for all} \ \ B, C \subseteq X \quad \text{and} \quad B \cap C = \varnothing \tag{1.185}$$

In this case, all the elements in X are independent, and we have

$$\zeta(B) = \sum_{x_i \in B} \zeta(\{x_i\}), \quad \text{for all} \ \ B \subseteq X \tag{1.186}$$

Based on Definition 1.6.1, Xu (2010c) used the well-known Choquet integral to develop some operators for aggregating IFNs together with their correlative weights:

Definition 1.6.2 (Xu, 2010c) Let $\alpha(x_i) = (\mu_\alpha(x_i), \nu_\alpha(x_i))\,(i = 1, 2, \cdots, n)$ be n IFNs, and ζ a fuzzy measure on X. Then we call

$$\begin{aligned}(\mathrm{C}_1)\int \alpha \mathrm{d}\zeta =& \mathrm{IFCA}\,(\alpha(x_1), \alpha(x_2), \cdots, \alpha(x_n))\\ &= (\zeta(B_{\sigma(1)}) - \zeta(B_{\sigma(0)}))\alpha(x_{\sigma(1)}) \oplus (\zeta(B_{\sigma(2)}) - \zeta(B_{\sigma(1)}))\alpha(x_{\sigma(2)})\\ &\quad \oplus \cdots \oplus (\zeta(B_{\sigma(n)}) - \zeta(B_{\sigma(n-1)}))\alpha(x_{\sigma(n)})\end{aligned} \tag{1.187}$$

an intuitionistic fuzzy correlated averaging (IFCA) operator, where $(C_1)\int \alpha \mathrm{d}\zeta$ is a notation of Choquet integral, $(\sigma(1), \sigma(2), \cdots, \sigma(n))$ is a permutation of $(1, 2, \cdots, n)$ such that $\alpha(x_{\sigma(1)}) \geqslant \alpha(x_{\sigma(2)}) \geqslant \cdots \geqslant \alpha(x_{\sigma(n)})$, $B_{\sigma(k)} = \{x_{\sigma(j)} | j \leqslant k\}$, when $k \geqslant 1$ and $B_{\sigma(0)} = \varnothing$.

With the operations of IFNs, the IFCA operator (1.187) can be transformed into the following form by using mathematical induction on n:

$$\begin{aligned}(C_1)\int \alpha \mathrm{d}\zeta =& \mathrm{IFCA}\,(\alpha(x_1), \alpha(x_2), \cdots, \alpha(x_n))\\ &= \left(1 - \prod_{i=1}^{n} \left(1 - \mu_\alpha(x_{\sigma(i)})\right)^{\zeta(B_{\sigma(i)}) - \zeta(B_{\sigma(i-1)})},\right.\\ &\quad \left.\prod_{i=1}^{n} \left(\nu_\alpha(x_{\sigma(i)})\right)^{\zeta(B_{\sigma(i)}) - \zeta(B_{\sigma(i-1)})}\right)\end{aligned} \tag{1.188}$$

whose aggregated value is an IFN.

Now we consider three special cases of the IFCA operator (Xu, 2010c):

(1) If Eq.(1.186) holds, then

$$\zeta\left(\{x_{\sigma(i)}\}\right) = \zeta(B_{\sigma(i)}) - \zeta(B_{\sigma(i-1)}) \tag{1.189}$$

In this case, the IFCA operators (1.187) and (1.188) reduce to the IFWA operator:

$$\begin{aligned}&\mathrm{IFWA}\,(\alpha(x_1), \alpha(x_2), \cdots, \alpha(x_n))\\ &= \zeta\left(\{x_1\}\right)\alpha(x_1) \oplus \zeta\left(\{x_2\}\right)\alpha(x_2) \oplus \cdots \oplus \zeta\left(\{x_n\}\right)\alpha(x_n)\\ &= \left(1 - \prod_{i=1}^{n} (1 - \mu_\alpha(x_i))^{\zeta(\{x_i\})},\ \prod_{i=1}^{n} (\nu_\alpha(x_i))^{\zeta(\{x_i\})}\right)\end{aligned} \tag{1.190}$$

Especially, if $\zeta\left(\{x_i\}\right) = \dfrac{1}{n}$, for all $i = 1, 2, \cdots, n$, then the IFWA operator (1.190) reduces to the IFA operator:

$$\mathrm{IFWA}\,(\alpha(x_1), \alpha(x_2), \cdots, \alpha(x_n)) = \frac{1}{n}\left(\alpha(x_1) \oplus \alpha(x_2) \oplus \cdots \oplus \alpha(x_n)\right)$$

$$= \left(1 - \left(\prod_{i=1}^{n}(1-\mu_\alpha(x_i))\right)^{1/n}, \left(\prod_{i=1}^{n}\nu_\alpha(x_i)\right)^{1/n}\right) \tag{1.191}$$

(2) If

$$\zeta(B) = \sum_{i=1}^{|B|}\omega_i, \quad \text{for all} \quad B \subseteq X \tag{1.192}$$

where $|B|$ is the number of the elements in the set B, then

$$\omega_i = \zeta(B_{\sigma(i)}) - \zeta(B_{\sigma(i-1)}), \quad i = 1, 2, \cdots, n \tag{1.193}$$

where $\omega = (\omega_1, \omega_2, \cdots, \omega_n)^{\mathrm{T}}$, $\omega_i \in [0,1]$ $(i = 1, 2, \cdots, n)$ and $\sum_{i=1}^{n}\omega_i = 1$. In this case, the IFCA operators (1.187) and (1.188) reduce to the IFOWA operator.

Especially, if $\zeta(B) = \dfrac{|B|}{n}$, for all $B \subseteq X$, then the IFCA operator (1.187) reduces to the IFA operator.

(3) If

$$\zeta(B) = \psi\left(\sum_{x_i \in B}\zeta(\{x_i\})\right), \quad \text{for all} \quad B \subseteq X \tag{1.194}$$

where ψ is a basic unit-interval monotonic (BUM) function (Yager and Xu, 2006; Xu, 2005c) $\psi:\ [0,1] \to [0,1]$ and is monotonic with the properties: (i) $\psi(0) = 0$, (ii) $\psi(1) = 1$, and (iii) $\psi(x) \geqslant \psi(y)$, for $x > y$. Then we let

$$w_i = \zeta(B_{\sigma(i)}) - \zeta(B_{\sigma(i-1)}) = \psi\left(\sum_{j \leqslant i}\zeta\left(\{x_{\sigma(i)}\}\right)\right) - \psi\left(\sum_{j < i}\zeta\left(\{x_{\sigma(i)}\}\right)\right),$$

$$i = 1, 2, \cdots, n \tag{1.195}$$

where $w = (w_1, w_2, \cdots, w_n)^{\mathrm{T}}$, $w_i \in [0,1]$ $(i = 1, 2, \cdots, n)$ and $\sum_{i=1}^{n} w_i = 1$. In this case, the IFCA operators (1.187) and (1.188) reduce to the following form:

$$\text{IFWOWA}\left(\alpha(x_1), \alpha(x_2), \cdots, \alpha(x_n)\right) = \mathop{\oplus}_{i=1}^{n}\left(w_i\alpha(x_{\sigma(i)})\right)$$
$$= \left(1 - \prod_{i=1}^{n}\left(1-\mu_\alpha(x_{\sigma(i)})\right)^{w_i}, \prod_{i=1}^{n}\left(\nu_\alpha(x_{\sigma(i)})\right)^{w_i}\right) \tag{1.196}$$

which we call an intuitionistic fuzzy weighted ordered weighted averaging (IFWOWA) operator. Especially, if $\zeta(\{x_i\}) = \frac{1}{n}$, for all $i = 1, 2, \cdots, n$, then the IFWOWA operator reduces to the IFOWA operator.

Note Torra (1997) defined a weighted ordered weighted averaging (WOWA) operator which is used to aggregate numerical values, and thus the IFWOWA operator can be regarded as an extension of the WOWA operator.

Definition 1.6.3 (Xu, 2010c) An intuitionistic fuzzy correlated geometric (IFCG) operator is defined as:

$$\begin{aligned}(C_2)\int \alpha \mathrm{d}\zeta =& \mathrm{IFCG}\left(\alpha(x_1), \alpha(x_2), \cdots, \alpha(x_n)\right) \\ =& \left(\alpha(x_{\sigma(1)})\right)^{\zeta(B_{\sigma(1)})-\zeta(B_{\sigma(0)})} \otimes \left(\alpha(x_{\sigma(2)})\right)^{\zeta(B_{\sigma(2)})-\zeta(B_{\sigma(1)})} \otimes \cdots \\ & \otimes \left(\alpha(x_{\sigma(n)})\right)^{\zeta(B_{\sigma(n)})-\zeta(B_{\sigma(n-1)})} \end{aligned} \tag{1.197}$$

where $(C_2)\int \alpha \mathrm{d}\zeta$ is a notation of Choquet integral, $(\sigma(1), \sigma(2), \cdots, \sigma(n))$ is a permutation of $(1, 2, \cdots, n)$ such that $\alpha(x_{\sigma(1)}) \geqslant \alpha(x_{\sigma(2)}) \geqslant \cdots \geqslant \alpha(x_{\sigma(n)})$, $B_{\sigma(k)} = \{x_{\sigma(j)} |\ j \leqslant k\}$, for $k \geqslant 1$ and $B_{\sigma(0)} = \varnothing$.

With the operations of IFNs, the IFCG operator (1.197) can be transformed into the following form by using mathematical induction on n:

$$\begin{aligned}(C_2)\int \alpha \mathrm{d}\zeta =& \mathrm{IFCG}\left(\alpha(x_1), \alpha(x_2), \cdots, \alpha(x_n)\right) \\ =& \left(\prod_{i=1}^{n} \left(\mu_\alpha(x_{\sigma(i)})\right)^{\zeta(B_{\sigma(i)})-\zeta(B_{\sigma(i-1)})},\right. \\ & \left. 1 - \prod_{i=1}^{n} \left(1 - \nu_\alpha(x_{\sigma(i)})\right)^{\zeta(B_{\sigma(i)})-\zeta(B_{\sigma(i-1)})}\right) \end{aligned} \tag{1.198}$$

whose aggregated value is also an IFN.

Below we discuss three special cases of the IFCG operator (Xu, 2010c):

(1) If Eqs.(1.186) and (1.189) hold, then the IFCG operators (1.197) and (1.198) reduce to the IFWG operator.

Especially, if $\zeta(\{x_i\}) = \frac{1}{n}$, for all $i = 1, 2, \cdots, n$, then the IFCG operator reduces to the IFG operator.

(2) If Eqs.(1.192) and (1.193) hold, then the IFCG operators (1.197) and (1.198) reduce to the IFOWG operator.

Especially, if $\zeta(B) = \frac{|B|}{n}$, for all $B \subseteq X$, then both the IFCG operator (1.197) and the IFOWG operator reduce to the IFG operator.

(3) If Eqs.(1.194) and (1.195) hold, then the IFCG operators (1.197) and (1.198) reduce to the following:

$$\begin{aligned}
&\text{IFWOWG}\left(\alpha(x_1), \alpha(x_2), \cdots, \alpha(x_n)\right) \\
&= \left(\alpha(x_{\sigma(1)})\right)^{w_1} \otimes \left(\alpha(x_{\sigma(2)})\right)^{w_2} \otimes \cdots \otimes \left(\alpha(x_{\sigma(n)})\right)^{w_n} \\
&= \left(\prod_{i=1}^{n}\left(\mu_\alpha(x_{\sigma(i)})\right)^{w_i},\ 1-\prod_{i=1}^{n}\left(1-\nu_\alpha(x_{\sigma(i)})\right)^{w_i}\right)
\end{aligned} \tag{1.199}$$

which we call an intuitionistic fuzzy weighted ordered weighted geometric (IFWOWG) operator. Especially, if $\zeta\left(\{x_i\}\right) = \dfrac{1}{n}$, for all $i = 1, 2, \cdots, n$, then the IFWOWG operator reduces to the IFOWG operator.

In the following we apply the above aggregation operators to a practical decision making problem involving the prioritization of a set of ten information technology improvement projects (Ngwenyama and Bryson, 1999; Xu, 2010c):

Example 1.6.1 The information management steering committee of Midwest American Manufacturing Corp.(MAMC) must prioritize for development and implementation a set of ten information technology improvement projects $Y = \{Y_i \mid i = 1, 2, \cdots, 10\}$, which have been proposed by area managers, where ① Y_1−Quality management information; ② Y_2−Inventory control; ③ Y_3−Customer order tracking; ④ Y_4−Materials purchasing management; ⑤ Y_5−Fleet management; ⑥ Y_6−Design change management; ⑦ Y_7−Electronic mail; ⑧ Y_8−Customer returns and complaints; ⑨ Y_9−Employee skills tracking; and ⑩ Y_{10}−Budget analysis. The committee aims to prioritize the projects from highest to lowest potential contribution to the firm's strategic goal of gaining competitive advantage in the industry. In assessing the potential contribution of each project, a set of three factors are considered: $G = \{G_1, G_2, G_3\}$ = {Productivity, Differentiation, Management}. The committee evaluates the projects $Y_i\,(i = 1, 2, \cdots, 10)$ in relation to the factors $G_j\,(j = 1, 2, 3)$, and gives more importance to G_1 and G_2 than to G_3, but on the other hand the committee gives some advantage to the projects that are good both in G_3 and in any of G_1 and G_2. Let

$$\zeta(\varnothing) = 0, \quad \zeta(G) = \zeta\left(\{G_1,\ G_2,\ G_3\}\right) = 1, \quad \zeta\left(\{G_1\}\right) = \zeta\left(\{G_2\}\right) = 0.4$$

$$\zeta\left(\{G_3\}\right) = 0.3, \quad \zeta(\{G_1, G_2\}) = 0.6, \quad \zeta(\{G_1, G_3\}) = \zeta(\{G_2, G_3\}) = 0.8$$

The evaluation information about the projects $Y_i\,(i = 1, 2, \cdots, 10)$ under the factors $G_j\,(j = 1, 2, 3)$ is represented by the IFNs $\alpha_{Y_i}(G_j)\,(i = 1, 2, \cdots, 10;\ j = 1, 2, 3)$, as shown in Table 1.27:

Table 1.27 The evaluation information about projects (Xu, 2010c)

	G_1	G_2	G_3
Y_1	(0.7, 0.3)	(0.8, 0.1)	(0.9, 0.1)
Y_2	(0.6, 0.2)	(0.8, 0.2)	(0.8, 0.1)
Y_3	(0.4, 0.1)	(0.6, 0.1)	(0.5, 0.2)
Y_4	(0.7, 0.3)	(0.8, 0.2)	(0.6, 0.3)
Y_5	(0.5, 0.5)	(0.7, 0.3)	(0.4, 0.2)
Y_6	(0.4, 0.3)	(0.6, 0.2)	(0.8, 0.1)
Y_7	(0.3, 0.6)	(0.4, 0.3)	(0.2, 0.2)
Y_8	(0.6, 0.1)	(0.5, 0.1)	(0.8, 0.2)
Y_9	(0.4, 0.5)	(0.9, 0.1)	(0.3, 0.1)
Y_{10}	(0.3, 0.5)	(0.6, 0.4)	(0.4, 0.1)

We can rearrange the IFNs corresponding to each project in descending order by using the method presented in Section 1.2:

$$\alpha_{Y_1}(G_{\sigma(1)}) = (0.9, 0.1), \quad \alpha_{Y_1}(G_{\sigma(2)}) = (0.8, 0.1), \quad \alpha_{Y_1}(G_{\sigma(3)}) = (0.7, 0.3)$$

$$\alpha_{Y_2}(G_{\sigma(1)}) = (0.8, 0.1), \quad \alpha_{Y_2}(G_{\sigma(2)}) = (0.8, 0.2), \quad \alpha_{Y_2}(G_{\sigma(3)}) = (0.6, 0.2)$$

$$\alpha_{Y_3}(G_{\sigma(1)}) = (0.6, 0.1), \quad \alpha_{Y_3}(G_{\sigma(2)}) = (0.5, 0.2), \quad \alpha_{Y_3}(G_{\sigma(3)}) = (0.4, 0.1)$$

$$\alpha_{Y_4}(G_{\sigma(1)}) = (0.8, 0.2), \quad \alpha_{Y_4}(G_{\sigma(2)}) = (0.7, 0.3), \quad \alpha_{Y_4}(G_{\sigma(3)}) = (0.6, 0.3)$$

$$\alpha_{Y_5}(G_{\sigma(1)}) = (0.7, 0.3), \quad \alpha_{Y_5}(G_{\sigma(2)}) = (0.4, 0.2), \quad \alpha_{Y_5}(G_{\sigma(3)}) = (0.5, 0.5)$$

$$\alpha_{Y_6}(G_{\sigma(1)}) = (0.8, 0.1), \quad \alpha_{Y_6}(G_{\sigma(2)}) = (0.6, 0.2), \quad \alpha_{Y_6}(G_{\sigma(3)}) = (0.4, 0.3)$$

$$\alpha_{Y_7}(G_{\sigma(1)}) = (0.2, 0.2), \quad \alpha_{Y_7}(G_{\sigma(2)}) = (0.4, 0.3), \quad \alpha_{Y_7}(G_{\sigma(3)}) = (0.3, 0.6)$$

$$\alpha_{Y_8}(G_{\sigma(1)}) = (0.8, 0.2), \quad \alpha_{Y_8}(G_{\sigma(2)}) = (0.6, 0.1), \quad \alpha_{Y_8}(G_{\sigma(3)}) = (0.5, 0.1)$$

$$\alpha_{Y_9}(G_{\sigma(1)}) = (0.9, 0.1), \quad \alpha_{Y_9}(G_{\sigma(2)}) = (0.3, 0.1), \quad \alpha_{Y_9}(G_{\sigma(3)}) = (0.4, 0.5)$$

$$\alpha_{Y_{10}}(G_{\sigma(1)}) = (0.4, 0.1), \quad \alpha_{Y_{10}}(G_{\sigma(2)}) = (0.6, 0.4), \quad \alpha_{Y_{10}}(G_{\sigma(3)}) = (0.3, 0.5)$$

With the IFCA operator (1.188), we can calculate the overall evaluation information corresponding to each project:

$$\text{IFCA}\left(\alpha_{Y_1}(G_1), \alpha_{Y_1}(G_2), \alpha_{Y_1}(G_3)\right) = (0.82, 0.16)$$

$$\text{IFCA}\left(\alpha_{Y_2}(G_1), \alpha_{Y_2}(G_2), \alpha_{Y_2}(G_3)\right) = (0.77, 0.16)$$

$$\text{IFCA}\left(\alpha_{Y_3}(G_1), \alpha_{Y_3}(G_2), \alpha_{Y_3}(G_3)\right) = (0.53, 0.13)$$

$$\text{IFCA}\left(\alpha_{Y_4}(G_1), \alpha_{Y_4}(G_2), \alpha_{Y_4}(G_3)\right) = (0.71, 0.26)$$

$$\text{IFCA}\left(\alpha_{Y_5}(G_1), \alpha_{Y_5}(G_2), \alpha_{Y_5}(G_3)\right) = (0.56, 0.28)$$

$$\text{IFCA}\left(\alpha_{Y_6}(G_1), \alpha_{Y_6}(G_2), \alpha_{Y_6}(G_3)\right) = (0.65, 0.19)$$

$$\text{IFCA}\,(\alpha_{Y_7}(G_1), \alpha_{Y_7}(G_2), \alpha_{Y_7}(G_3)) = (0.33, 0.31)$$
$$\text{IFCA}\,(\alpha_{Y_8}(G_1), \alpha_{Y_8}(G_2), \alpha_{Y_8}(G_3)) = (0.66, 0.12)$$
$$\text{IFCA}\,(\alpha_{Y_9}(G_1), \alpha_{Y_9}(G_2), \alpha_{Y_9}(G_3)) = (0.69, 0.14)$$
$$\text{IFCA}\,(\alpha_{Y_{10}}(G_1), \alpha_{Y_{10}}(G_2), \alpha_{Y_{10}}(G_3)) = (0.49, 0.28)$$

Finally, we can rank the above IFNs by using the method presented in Definition 1.1.3:

$$\begin{aligned}(0.82, 0.12) &> (0.77, 0.16) > (0.69, 0.14) > (0.66, 0.12) > (0.65, 0.19)\\ &> (0.71, 0.26) > (0.53, 0.13) > (0.56, 0.28) > (0.49, 0.28) > (0.33, 0.31)\end{aligned}$$

Hence, the ranking of the ten projects Y_i $(i = 1, 2, \cdots, 10)$ is:

$$Y_1 \succ Y_2 \succ Y_9 \succ Y_8 \succ Y_6 \succ Y_4 \succ Y_3 \succ Y_5 \succ Y_{10} \succ Y_7$$

If we use the IFCG operator (1.196) to calculate the overall evaluation information corresponding to each project, then

$$\text{IFCG}\,(\alpha_{Y_1}(G_1), \alpha_{Y_1}(G_2), \alpha_{Y_1}(G_3)) = (0.81, 0.14)$$
$$\text{IFCG}\,(\alpha_{Y_2}(G_1), \alpha_{Y_2}(G_2), \alpha_{Y_2}(G_3)) = (0.76, 0.17)$$
$$\text{IFCG}\,(\alpha_{Y_3}(G_1), \alpha_{Y_3}(G_2), \alpha_{Y_3}(G_3)) = (0.51, 0.14)$$
$$\text{IFCG}\,(\alpha_{Y_4}(G_1), \alpha_{Y_4}(G_2), \alpha_{Y_4}(G_3)) = (0.69, 0.26)$$
$$\text{IFCG}\,(\alpha_{Y_5}(G_1), \alpha_{Y_5}(G_2), \alpha_{Y_5}(G_3)) = (0.52, 0.31)$$
$$\text{IFCG}\,(\alpha_{Y_6}(G_1), \alpha_{Y_6}(G_2), \alpha_{Y_6}(G_3)) = (0.60, 0.19)$$
$$\text{IFCG}\,(\alpha_{Y_7}(G_1), \alpha_{Y_7}(G_2), \alpha_{Y_7}(G_3)) = (0.31, 0.35)$$
$$\text{IFCG}\,(\alpha_{Y_8}(G_1), \alpha_{Y_8}(G_2), \alpha_{Y_8}(G_3)) = (0.63, 0.13)$$
$$\text{IFCG}\,(\alpha_{Y_9}(G_1), \alpha_{Y_9}(G_2), \alpha_{Y_9}(G_3)) = (0.49, 0.20)$$
$$\text{IFCG}\,(\alpha_{Y_{10}}(G_1), \alpha_{Y_{10}}(G_2), \alpha_{Y_{10}}(G_3)) = (0.46, 0.35)$$

Thus the ranking of the above IFNs is:

$$\begin{aligned}(0.81, 0.14) &> (0.76, 0.17) > (0.63, 0.13) > (0.69, 0.26) > (0.60, 0.19)\\ &> (0.51, 0.14) > (0.49, 0.20) > (0.52, 0.31) > (0.46, 0.35) > (0.31, 0.35)\end{aligned}$$

and then the ranking of the ten projects Y_i $(i = 1, 2, \cdots, 10)$ is:

$$Y_1 \succ Y_2 \succ Y_8 \succ Y_4 \succ Y_6 \succ Y_3 \succ Y_9 \succ Y_5 \succ Y_{10} \succ Y_7$$

From the above numerical results, we know that the ranking results by using the IFCA and IFCG operators are slightly different, but both the operators get the same best project Y_1.

1.7 Induced Generalized Intuitionistic Fuzzy Aggregation Operators

Based on Definition 1.6.1 and the idea of order induced aggregation (Yager, 2004a; 2003a, Yager and Filev, 1999), we can utilize the Choquet integral to develop some generalized aggregation operators for IFNs:

Definition 1.7.1 (Xu and Xia, 2011) An induced generalized intuitionistic fuzzy correlated averaging (IGIFCA) operator of dimension n is a function IGIFCA: $\Theta^n \rightarrow \Theta$, which is defined to aggregate the set of second arguments of a list of 2-tuples $(\langle \nabla_1, \alpha_1 \rangle, \langle \nabla_2, \alpha_2 \rangle, \cdots, \langle \nabla_n, \alpha_n \rangle)$ according to the following expression:

$$\begin{aligned}&\text{IGIFCA}\left(\langle \nabla_1, \alpha_1 \rangle, \langle \nabla_2, \alpha_2 \rangle, \cdots, \langle \nabla_n, \alpha_n \rangle\right)\\&=\Big((\zeta(A_{\sigma(1)}) - \zeta(A_{\sigma(0)}))\alpha_{\sigma(1)}^{\lambda} \oplus (\zeta(A_{\sigma(2)}) - \zeta(A_{\sigma(1)}))\alpha_{\sigma(2)}^{\lambda}\\&\quad\oplus \cdots \oplus (\zeta(A_{\sigma(n)}) - \zeta(A_{\sigma(n-1)}))\alpha_{\sigma(n)}^{\lambda}\Big)^{1/\lambda}\end{aligned} \tag{1.200}$$

where $\lambda > 0$, ∇_i in 2-tuples $\langle \nabla_i, \alpha_i \rangle$ is referred to as the order-inducing variable and α_i as the argument variable, $\sigma(i)$: $\{1, 2, \cdots, n\} \rightarrow \{1, 2, \cdots, n\}$ is a permutation such that $\nabla_{\sigma(1)} \geqslant \nabla_{\sigma(2)} \geqslant \cdots \geqslant \nabla_{\sigma(n)}$, $A_{\sigma(i)} = \left\{\alpha_{\sigma(1)}, \alpha_{\sigma(2)}, \cdots, \alpha_{\sigma(i)}\right\}$ when $i \geqslant 1$ and $A_{\sigma(0)} = \varnothing$.

According to Zhao et al (2010), we can get

$$\begin{aligned}&\text{IGIFCA}\left(\langle \nabla_1, \alpha_1 \rangle, \langle \nabla_2, \alpha_2 \rangle, \cdots, \langle \nabla_n, \alpha_n \rangle\right)\\&=\left(\left(1 - \prod_{i=1}^{n}(1 - \mu_{\alpha_{\sigma(i)}}^{\lambda})^{\zeta(A_{\sigma(i)}) - \zeta(A_{\sigma(i-1)})}\right)^{1/\lambda},\right.\\&\quad\left. 1 - \left(1 - \prod_{i=1}^{n}(1 - (1 - \nu_{\alpha_{\sigma(i)}})^{\lambda})^{\zeta(A_{\sigma(i)}) - \zeta(A_{\sigma(i-1)})}\right)^{1/\lambda}\right)\end{aligned} \tag{1.201}$$

Especially, if there exist two 2-tuples $\langle \nabla_i, \alpha_i \rangle$ and $\langle \nabla_j, \alpha_j \rangle$ such that $\nabla_i = \nabla_j$, then we can replace the arguments of the tied 2-tuples by the average of their arguments, i.e., replace α_i and α_j by $(\alpha_i \oplus \alpha_j)/2$. If k items are tied, then we replace these by k replica's of their average.

In the case where $\nabla_{\sigma(1)} \geqslant \nabla_{\sigma(2)} \geqslant \cdots \geqslant \nabla_{\sigma(n)}$ and $\alpha_{\sigma(1)} \geqslant \alpha_{\sigma(2)} \geqslant \cdots \geqslant \alpha_{\sigma(n)}$, the IGIFCA operator becomes the generalized intuitionistic fuzzy correlated averaging (GIFCA) operator (Xu and Xia, 2011):

$$\begin{aligned}&\text{GIFCA}\left(\alpha_1, \alpha_2, \cdots, \alpha_n\right)\\&=\Big((\zeta(A_{\sigma(1)}) - \zeta(A_{\sigma(0)}))\alpha_{\sigma(1)}^{\lambda} \oplus (\zeta(A_{\sigma(2)}) - \zeta(A_{\sigma(1)}))\alpha_{\sigma(2)}^{\lambda}\\&\quad\oplus \cdots \oplus (\zeta(A_{\sigma(n)}) - \zeta(A_{\sigma(n-1)}))\alpha_{\sigma(n)}^{\lambda}\Big)^{1/\lambda}\end{aligned}$$

$$= \left(\left(1 - \prod_{j=1}^{n} (1 - \mu_{\alpha_{\sigma(j)}}^{\lambda})^{\zeta(A_{\sigma(j)}) - \zeta(A_{\sigma(j-1)})} \right)^{1/\lambda}, \right.$$
$$\left. 1 - \left(1 - \prod_{j=1}^{n} (1 - (1 - \nu_{\alpha_{\sigma(j)}})^{\lambda})^{\zeta(A_{\sigma(j)}) - \zeta(A_{\sigma(j-1)})} \right)^{1/\lambda} \right) \tag{1.202}$$

where $\alpha_{\sigma(j)}$ is the j-th largest of α_i $(i = 1, 2, \cdots, n)$.

We can easily prove that the IGIFCA operator is commutative, monotonic, bounded, and idempotent, which are presented as follows:

Theorem 1.7.1 (Xu and Xia, 2011) (Commutativity) If $(\langle \nabla_{\sigma(1)}, \alpha_{\sigma(1)} \rangle, \langle \nabla_{\sigma(2)}, \alpha_{\sigma(2)} \rangle, \cdots, \langle \nabla_{\sigma(n)}, \alpha_{\sigma(n)} \rangle)$ is any permutation of $(\langle \nabla_1, \alpha_1 \rangle, \langle \nabla_2, \alpha_2 \rangle, \cdots, \langle \nabla_n, \alpha_n \rangle)$, then

$$\begin{aligned} &\text{IGIFCA}\left(\langle \nabla_1, \alpha_1 \rangle, \langle \nabla_2, \alpha_2 \rangle, \cdots, \langle \nabla_n, \alpha_n \rangle\right) \\ &= \text{IGIFCA}\left(\left\langle \nabla_{\sigma(1)}, \alpha_{\sigma(1)} \right\rangle, \left\langle \nabla_{\sigma(2)}, \alpha_{\sigma(2)} \right\rangle, \cdots, \left\langle \nabla_{\sigma(n)}, \alpha_{\sigma(n)} \right\rangle\right) \end{aligned} \tag{1.203}$$

Theorem 1.7.2 (Xu and Xia, 2011)(Monotonicity) Let $(\langle \nabla_1, \alpha_1 \rangle, \langle \nabla_2, \alpha_2 \rangle, \cdots, \langle \nabla_n, \alpha_n \rangle)$, $(\langle \hat{\nabla}_1, \hat{\alpha}_1 \rangle, \langle \hat{\nabla}_2, \hat{\alpha}_2 \rangle, \cdots, \langle \hat{\nabla}_n, \hat{\alpha}_n \rangle)$ be two sets of 2-tuples, such that $\alpha_i \leqslant \hat{\alpha}_i$, $i = 1, 2, \cdots, n$. Then

$$\begin{aligned} &\text{IGIFCA}\left(\langle \nabla_1, \alpha_1 \rangle, \langle \nabla_2, \alpha_2 \rangle, \cdots, \langle \nabla_n, \alpha_n \rangle\right) \\ &\leqslant \text{IGIFCA}\left(\left\langle \hat{\nabla}_1, \hat{\alpha}_1 \right\rangle, \left\langle \hat{\nabla}_2, \hat{\alpha}_2 \right\rangle, \cdots, \left\langle \hat{\nabla}_n, \hat{\alpha}_n \right\rangle\right) \end{aligned} \tag{1.204}$$

Theorem 1.7.3 (Xu and Xia, 2011)(Boundedness) Let $\alpha^- = (\min_j(\mu_{\alpha_j}), \max_j(\nu_{\alpha_j}))$, $\alpha^+ = (\max_j(\mu_{\alpha_j}), \min_j(\nu_{\alpha_j}))$. Then

$$\alpha^- \leqslant \text{IGIFCA}\left(\langle \nabla_1, \alpha_1 \rangle, \langle \nabla_2, \alpha_2 \rangle, \cdots, \langle \nabla_n, \alpha_n \rangle\right) \leqslant \alpha^+ \tag{1.205}$$

Theorem 1.7.4 (Xu and Xia, 2011) (Idempotency) If $\alpha_i = \alpha$ $(i = 1, 2, \cdots, n)$, then

$$\text{IGIFCA}\left(\langle \nabla_1, \alpha_1 \rangle, \langle \nabla_2, \alpha_2 \rangle, \cdots, \langle \nabla_n, \alpha_n \rangle\right) = \alpha \tag{1.206}$$

Especially, if $\lambda = 1$, then Eq.(1.201) reduces to

$$\begin{aligned} &\text{IIFCA}\left(\langle \nabla_1, \alpha_1 \rangle, \langle \nabla_2, \alpha_2 \rangle, \cdots, \langle \nabla_n, \alpha_n \rangle\right) \\ &= (\zeta(A_{\sigma(1)}) - \zeta(A_{\sigma(0)}))\alpha_{\sigma(1)} \oplus (\zeta(A_{\sigma(2)}) - \zeta(A_{\sigma(1)}))\alpha_{\sigma(2)} \\ &\quad \oplus \cdots \oplus (\zeta(A_{\sigma(n)}) - \zeta(A_{\sigma(n-1)}))\alpha_{\sigma(n)} \\ &= \left(1 - \prod_{i=1}^{n} \left(1 - \mu_{\alpha_{\sigma(i)}}\right)^{\zeta(A_{\sigma(i)}) - \zeta(A_{\sigma(i-1)})}, \ \prod_{i=1}^{n} \left(\nu_{\alpha_{\sigma(i)}}\right)^{\zeta(A_{\sigma(i)}) - \zeta(A_{\sigma(i-1)})} \right) \end{aligned} \tag{1.207}$$

which we call an induced intuitionistic fuzzy correlated averaging (IIFCA) operator (Xu and Xia, 2011).

Furthermore, if $\nabla_{\sigma(1)} \geqslant \nabla_{\sigma(2)} \geqslant \cdots \geqslant \nabla_{\sigma(n)}$ and $\alpha_{\sigma(1)} \geqslant \alpha_{\sigma(2)} \geqslant \cdots \geqslant \alpha_{\sigma(n)}$, then the IIFCA operator (1.207) becomes the IFCA operator (Xu, 2010c):

$$\begin{aligned}
&\text{IFCA}\left(\alpha_1, \alpha_2, \cdots, \alpha_n\right) \\
&= (\zeta(A_{\sigma(1)}) - \zeta(A_{\sigma(0)}))\alpha_{\sigma(1)} \oplus (\zeta(A_{\sigma(2)}) - \zeta(A_{\sigma(1)}))\alpha_{\sigma(2)} \\
&\quad \oplus \cdots \oplus (\zeta(A_{\sigma(n)}) - \zeta(A_{\sigma(n-1)}))\alpha_{\sigma(n)} \\
&= \left(1 - \prod_{j=1}^{n}\left(1 - \mu_{\alpha_{\sigma(j)}}\right)^{\zeta(A_{\sigma(j)}) - \zeta(A_{\sigma(i-1)})},\ \prod_{j=1}^{n}\left(\nu_{\alpha_{\sigma(j)}}\right)^{\zeta(A_{\sigma(j)}) - \zeta(A_{\sigma(j-1)})}\right) \quad (1.208)
\end{aligned}$$

where $\alpha_{(j)}$ is the j-th largest of α_i $(i = 1, 2, \cdots, n)$.

In the case where $\lambda = 0$, we have

$$\begin{aligned}
&\text{IIFCG}\left(\left\langle \nabla_1, \alpha_1 \right\rangle, \left\langle \nabla_2, \alpha_2 \right\rangle, \cdots, \left\langle \nabla_n, \alpha_n \right\rangle\right) \\
&= \left(\alpha_{\sigma(1)}\right)^{\zeta(A_{\sigma(1)}) - \zeta(A_{\sigma(0)})} \otimes \left(\alpha_{\sigma(2)}\right)^{\zeta(A_{\sigma(2)}) - \zeta(A_{\sigma(1)})} \otimes \cdots \otimes \left(\alpha_{\sigma(n)}\right)^{\zeta(A_{\sigma(n)}) - \zeta(A_{\sigma(n-1)})} \\
&= \left(\prod_{i=1}^{n}\left(\mu_{\alpha_{\sigma(i)}}\right)^{\zeta(A_{\sigma(i)}) - \zeta(A_{\sigma(i-1)})},\ 1 - \prod_{i=1}^{n}\left(1 - \nu_{\alpha_{\sigma(i)}}\right)^{\zeta(A_{\sigma(i)}) - \zeta(A_{\sigma(i-1)}}\right) \quad (1.209)
\end{aligned}$$

which is called an induced intuitionistic fuzzy correlated geometric (IIFCG) operator (Xu and Xia, 2011).

Especially, if $\nabla_{\sigma(1)} \geqslant \nabla_{\sigma(2)} \geqslant \cdots \geqslant \nabla_{\sigma(n)}$ and $\alpha_{\sigma(1)} \geqslant \alpha_{\sigma(2)} \geqslant \cdots \geqslant \alpha_{\sigma(n)}$, then the IIFCG operator (1.209) reduces to the IFCG operator (Xu, 2010c):

$$\begin{aligned}
&\text{IFCG}\left(\alpha_1, \alpha_2, \cdots, \alpha_n\right) \\
&= \left(\alpha_{\sigma(1)}\right)^{\zeta(A_{\sigma(1)}) - \zeta(A_{\sigma(0)})} \otimes \left(\alpha_{\sigma(2)}\right)^{\zeta(A_{\sigma(2)}) - \zeta(A_{\sigma(1)})} \otimes \cdots \otimes \left(\alpha_{\sigma(n)}\right)^{\zeta(A_{\sigma(n)}) - \zeta(A_{\sigma(n-1)})} \\
&= \left(\prod_{j=1}^{n}\left(\mu_{\alpha_{\sigma(j)}}\right)^{\zeta(A_{\sigma(j)}) - \zeta(A_{\sigma(j-1)})},\ 1 - \prod_{j=1}^{n}\left(1 - \nu_{\alpha_{\sigma(j)}}\right)^{\zeta(A_{\sigma(j)}) - \zeta(A_{\sigma(j-1)})}\right) \quad (1.210)
\end{aligned}$$

where $\alpha_{\sigma(j)}$ is the j-th largest of α_i $(i = 1, 2, \cdots, n)$.

With the results as discussed above, we can now introduce a method for decision making based on the IGIFCA operator (Xu and Xia, 2011):

Assume that there is a decision making problem with a collection of alternatives $Y = \{Y_1, Y_2, \cdots, Y_n\}$ and a set of the states of nature, $G = \{G_1, G_2, \cdots, G_m\}$. If the alternative Y_i is selected under the state G_j, then α_{ij} is the payoff value which is denoted by IFNs. To get the optimal alternative, we can follow the following steps:

Step 1 Calculate the correlations between the states of nature using the Choquet integral.

Step 2 Calculate the inducing variables matrix $\nabla = (\nabla_{ij})_{n\times m}$.

Step 3 Utilize Eq.(1.201) to get the expected result C_i for the alternative Y_i:

$$\begin{aligned} C_i &= \text{IGIFCA}\left(\langle\nabla_{i1},\alpha_{i2}\rangle, \langle\nabla_{i2},\alpha_{i2}\rangle, \cdots, \langle\nabla_{im},\alpha_{im}\rangle\right) \\ &= \Big((\zeta(G_{\sigma(1)}) - \zeta(G_{\sigma(0)}))\alpha_{i\sigma(1)}^{\lambda} \oplus (\zeta(G_{\sigma(2)}) - \zeta(G_{\sigma(1)}))\alpha_{i\sigma(2)}^{\lambda} \oplus \cdots \\ &\quad \oplus(\zeta(G_{\sigma(m)}) - \zeta(G_{\sigma(m-1)}))\alpha_{i\sigma(m)}^{\lambda}\Big)^{1/\lambda} \end{aligned} \tag{1.211}$$

where $\lambda > 0$, $\sigma(j)$: $\{1,2,\cdots,m\} \to \{1,2,\cdots,m\}$ is a permutation such that $\nabla_{i\sigma(1)} \geqslant \nabla_{i\sigma(2)} \geqslant \cdots \geqslant \nabla_{i\sigma(m)}$.

Step 4 Get the priority of C_i according to the ranking method of IFNs, and then generate the ranking of the alternatives Y_i $(i = 1,2,\cdots,n)$.

We now use an example given by Merigó and Casanovas (2009) to illustrate the method developed above:

Example 1.7.1 (Xu and Xia, 2011) Assume that an investor wants to invest some money in an enterprise in order to get the highest possible profit. Initially, he considers five possible alternatives: ① Y_1 is a computer company; ② Y_2 is a chemical company; ③ Y_3 is a food company; ④ Y_4 is a car company; and ⑤ Y_5 is a TV company. In order to evaluate these alternatives, the investor has brought together a group of experts. This group considers that the key factor is the economic environment in the global economy. After careful analysis, they consider four possible situations for the economic environment: ① G_1 : Negative growth rate; ② G_2 : Low growth rate; ③ G_3 : Medium growth rate; and ④ G_4 : High growth rate.

The expected results of evaluations, depending on the situation G_j that occurs and the alternative Y_i that the investor chooses, are given as IFNs $\alpha_{ij} = (\mu_{ij},\nu_{ij})$, where μ_{ij} denotes the degree that the alternative Y_i satisfies the situation G_j, and ν_{ij} denotes the degree that the alternative Y_i does not satisfy the situation G_j. The results are shown in Table 1.28:

Table 1.28 Payoff matrix (Xu and Xia, 2011)

	G_1	G_2	G_3	G_4
Y_1	(0.5,0.3)	(0.1,0.6)	(0.5,0.4)	(0.3,0.5)
Y_2	(0.6,0.1)	(0.3,0.6)	(0.4,0.3)	(0.6,0.3)
Y_3	(0.5,0.1)	(0.4,0.5)	(0.3,0.2)	(0.4,0.4)
Y_4	(0.8,0.1)	(0.2,0.5)	(0.7,0.1)	(0.2,0.4)
Y_5	(0.6,0.3)	(0.6,0.2)	(0.5,0.3)	(0.5,0.2)

Next, we use the method developed above to get the ranking of the companies:

Step 1 Assume that the weights of the situations have correlations with each other, as shown below:

$$\zeta(\varnothing) = 0, \quad \zeta(\{G_1\}) = 0.3, \quad \zeta(\{G_2\}) = 0.2, \quad \zeta(\{G_3\}) = 0.4, \quad \zeta(\{G_4\}) = 0.1$$

$$\zeta(\{G_1, G_2\}) = 0.6, \quad \zeta(\{G_1, G_3\}) = 0.5, \quad \zeta(\{G_1, G_4\}) = 0.4, \quad \zeta(\{G_2, G_3\}) = 0.5$$

$$\zeta(\{G_2, G_4\}) = 0.5, \quad \zeta(\{G_3, G_4\}) = 0.6, \quad \zeta(\{G_1, G_2, G_3\}) = 0.7$$

$$\zeta(\{G_1, G_2, G_4\}) = 0.8, \quad \zeta(\{G_1, G_3, G_4\}) = 0.7, \quad \zeta(\{G_2, G_3, G_4\}) = 0.9$$

$$\zeta(\{G_1, G_2, G_3, G_4\}) = 1.0$$

Step 2 As the attitudinal character is very complex because it involves the opinion of different members of the board of directors, the experts may use order-inducing variables to represent it. The results are shown in Table 1.29:

Table 1.29 Inducing variables (Xu and Xia, 2011)

	G_1	G_2	G_3	G_4
Y_1	17	15	22	12
Y_2	15	22	25	13
Y_3	24	20	22	15
Y_4	16	21	25	28
Y_5	18	26	23	21

Step 3 With this information, we can aggregate the expected result for each state of nature using Eq.(1.211). Table 1.30 shows the results which are different when the parameter λ changes:

Table 1.30 Aggregated results

	$\lambda = 1$	$\lambda = 2$	$\lambda = 5$	$\lambda = 10$
C_1	(0.3779,0.4507)	(0.3998,0.4457)	(0.4399,0.4295)	(0.4667,0.4047)
C_2	(0.5025,0.2581)	(0.5108,0.2517)	(0.5350,0.2330)	(0.5607,0.2053)
C_3	(0.4142,0.2402)	(0.4183,0.2313)	(0.4320,0.2064)	(0.4508,0.1772)
C_4	(0.5735,0.1862)	(0.6047,0.1777)	(0.6582,0.1573)	(0.6925,0.1376)
C_5	(0.5324,0.2352)	(0.5337,0.2344)	(0.5389,0.2316)	(0.5493,0.2266)

Step 4 According to the ranking method of IFNs in Section 1.2, we can get the rankings of the alternatives in Table 1.31:

Table 1.31 Rankings of the alternatives

λ	Rankings
1	$Y_4 \succ Y_5 \succ Y_2 \succ Y_3 \succ Y_1$
2	$Y_4 \succ Y_5 \succ Y_2 \succ Y_3 \succ Y_1$
5	$Y_4 \succ Y_5 \succ Y_2 \succ Y_3 \succ Y_1$
10	$Y_4 \succ Y_2 \succ Y_5 \succ Y_3 \succ Y_1$
0	$Y_5 \succ Y_2 \succ Y_4 \succ Y_3 \succ Y_1$

As we can see, as the parameter λ of the aggregation operator used changes, the rankings of the alternatives may be different, which reflects the indeterminacy of the final decision. The investor can decide to select the desirable alternative in accordance with his interest and actual needs.

Xu and Xia (2011) also give some induced intuitionistic fuzzy aggregation operators based on the Dempster-Shafer belief structure.

The concept of Dempster-Shafer belief structure was provided by Dempster (1968) and Shafer (1976) as follows:

Definition 1.7.2 (Dempster, 1968; Shafer, 1976) A Dempster-Shafer belief structure defined on a space X consists of a collection of n non-null subsets B_j $(j = 1, 2, \cdots, n)$ of X, called focal elements, and a mapping ζ called the probability assignment, defined as ζ: $2^X \rightarrow [0,1]$ such that

(1) $\zeta(B_j) \in [0,1]$;

(2) $\sum_{j=1}^{n} \zeta(B_j) = 1$;

(3) $\zeta(A) = 0$, $\forall A \neq B_j$.

As mentioned above, a main characteristic of the Dempster-Shafer belief structure is that it can represent some traditional cases of uncertainty. If it consists of n focal elements such that $B_j = \{x_j\}$ in which each focal elements is a singleton, then we can evidently make decision in a risk environment with $B_j = P_j = \text{prob}\{x_j\}$. Another special case is that when the belief structure consists of only one focal element B_1 which comprises all the states of nature ($B_1 = X = \{x_1, x_2, \cdots, x_n\}$), where $\zeta(X) = 1$. Then we have to make decision in ignorance environment.

Definition 1.7.3 (Xu and Xia, 2011) Let $(\langle\nabla_1, \alpha_1\rangle, \langle\nabla_2, \alpha_2\rangle, \cdots, \langle\nabla_n, \alpha_n\rangle)$ be a collection of 2-tuples on X, where ∇_i is the order-inducing variables and α_i $(i = 1, 2, \cdots, n)$ are the aggregated arguments in the form of IFNs, and

$$\begin{aligned} M &= (M_k | \, M_k = \{\langle\nabla_i, \alpha_i\rangle | \, x_i \in D_k, i = 1, 2, \cdots, n\}, k = 1, 2, \cdots, r) \\ &= (\langle\nabla_{11}, \alpha_{11}\rangle, \langle\nabla_{21}, \alpha_{21}\rangle, \cdots, \langle\nabla_{q_1 1}, \alpha_{q_1 1}\rangle, \cdots, \langle\nabla_{1r}, \alpha_{1r}\rangle, \\ &\quad \langle\nabla_{2r}, \alpha_{2r}\rangle, \cdots, \langle\nabla_{q_r r}, \alpha_{q_r r}\rangle) \end{aligned}$$

are a collection of 2-tuple arguments with r focal elements, B_k $(k = 1, 2, \cdots, r)$. A BSI-GIFOA operator of dimension r is a function $\text{BSI}-\text{GIFOA}$: $\Theta^r \rightarrow \Theta$ defined by

$$\begin{aligned} \text{BSI}-\text{GIFOA}(M) &= \left(\overset{r}{\underset{k=1}{\oplus}} \left(\zeta(B_k)\left(\left(\left(\overset{q_k}{\underset{j=1}{\oplus}} \left(\omega_{jk}\beta_{jk}^{\lambda_1}\right)\right)^{1/\lambda_1}\right)^{\lambda_2}\right)\right)\right)^{1/\lambda_2} \\ &= \left(\overset{r}{\underset{k=1}{\oplus}} \left(\zeta(B_k)\left(\overset{q_k}{\underset{j=1}{\oplus}} \left(\omega_{jk}\beta_{jk}^{\lambda_1}\right)\right)^{\lambda_2/\lambda_1}\right)\right)^{1/\lambda_2} \end{aligned} \tag{1.212}$$

where $\overset{q_k}{\underset{j=1}{\oplus}} \left(\omega_{jk}\beta_{jk}^{\lambda_1}\right) = \omega_{1k}\beta_{1k}^{\lambda_1} \oplus \omega_{2k}\beta_{2k}^{\lambda_1} \oplus \cdots \oplus \omega_{q_k k}\beta_{q_k k}^{\lambda_1}$, and $\lambda_1 > 0$, $\lambda_2 > 0$, $W_k =$

$(\omega_{1k}, \omega_{2k}, \cdots, \omega_{q_k k})^{\mathrm{T}}$ is the weighting vector for the k-th focal element B_k such that $\sum_{j=1}^{q_k} \omega_{jk} = 1$ and $\omega_{jk} \in [0,1]$, $i = 1, 2, \cdots, q_k$, where q_k is the number of elements in B_k, β_{jk} is the α_{ik} value of the pair $\langle \nabla_{ik}, \alpha_{ik} \rangle$ having the j-th largest ∇_{ik} $(i = 1, 2, \cdots, q_k)$, ∇_{ik} is the order-inducing variable, α_{ik} is the argument variable in the form of IFNs, and $\zeta(B_k)$ is the basic probability assignment.

By the operational laws of IFNs, the BSI-GIFOA operator (1.212) can be transformed into the following form (Xu and Xia, 2011):

$$\begin{aligned} &\mathrm{BSI-IFOA}\,(M) \\ &= \left(\left(\left(1 - \prod_{k=1}^{r} \left(1 - \left(1 - \prod_{j=1}^{q_k} (1 - \mu_{\beta_{jk}}^{\lambda_1})^{\omega_{jk}} \right)^{\lambda_2/\lambda_1} \right)^{\zeta(B_k)} \right)^{1/\lambda_2} \right), \right. \\ &\quad \left. 1 - \left(1 - \prod_{k=1}^{r} \left(1 - \left(1 - \prod_{j=1}^{q_k} (1 - (1 - \nu_{\beta_{jk}})^{\lambda_1})^{w_{jk}} \right)^{\lambda_2/\lambda_1} \right)^{\zeta(B_k)} \right)^{1/\lambda_2} \right), \quad \lambda_1, \lambda_2 > 0 \end{aligned} \tag{1.213}$$

Let $\mho = \{i, k \mid k = 1, 2, \cdots, r;\ i = 1, 2, \cdots, q_k\}$. Motivated by Merigó and Casanovas (2009) and Zhao et al. (2010), we can show that the BSI-GIFOA operator has some good properties, such as commutativity, monotonicity, boundedness, and idempotency.

Theorem 1.7.5 (Xu and Xia, 2011) (Commutativity) If $M^* = \{ \langle \nabla_{ik}^*, \alpha_{ik}^* \rangle | \, i, k \in \mho \}$ is any permutation of $M = \{ \langle \nabla_{ik}, \alpha_{ik} \rangle | \, i, k \in \mho \}$, then

$$\mathrm{BSI-GIFOA}\,(M) = \mathrm{BSI-GIFOA}\,(M^*) \tag{1.214}$$

Theorem 1.7.6 (Xu and Xia, 2011)(Monotonicity) Let $M = \{ \langle \nabla_{ik}, \alpha_{ik} \rangle | \, i, k \in \mho \}$ and $\hat{M} = \{ \langle \nabla_{ik}, \hat{\alpha}_{ik} \rangle | \, i, k \in \mho \}$. If $\forall i, k \in \mho$, $\alpha_{ik} \leqslant \hat{\alpha}_{ik}$. Then

$$\mathrm{BSI-GIFOA}\,(M) \leqslant \mathrm{BSI-GIFOA}\left(\hat{M}\right) \tag{1.215}$$

Theorem 1.7.7 (Xu and Xia, 2011)(Boundedness) Let $\alpha^- = (\min_{i,k \in \mho}(\mu_{\alpha_{ik}}), \max_{i,k \in \mho}(\nu_{\alpha_{ik}}))$, $\alpha^+ = (\max_{i,k \in \mho}(\mu_{\alpha_{ik}}), \min_{i,k \in \mho}(\nu_{\alpha_{ik}}))$. Then

$$\alpha^- \leqslant \mathrm{BSI-GIFOA}\,(M) \leqslant \alpha^+ \tag{1.216}$$

Theorem 1.7.8 (Xu and Xia, 2011)(Idempotency) If $\forall i, k \in \mho$, $\alpha_{ik} = \alpha$, then

$$\mathrm{BSI-GIFOA}\,(M) = \alpha \tag{1.217}$$

Next, we discuss some special cases of the BSI−GIFOA operator (Xu and Xia, 2011):

(1) If $\lambda_1 = \lambda_2 = \lambda$, then it follows from Eq.(1.210) that

$$\begin{aligned}\mathrm{BSI-GIFOA}\,(M) &= \left(\mathop{\oplus}_{k=1}^{r}\left(\zeta(B_k)\mathop{\oplus}_{j=1}^{q_k}\left(\omega_{jk}\beta_{jk}^{\lambda}\right)\right)\right)^{1/\lambda}\\ &= \left(\left(1-\prod_{k=1}^{r}\prod_{j=1}^{q_k}(1-\mu_{\beta_{jk}}^{\lambda})^{\omega_{jk}\zeta(B_k)}\right)^{1/\lambda},\right.\\ &\qquad \left.1-\left(1-\prod_{k=1}^{r}\prod_{j=1}^{q_k}(1-(1-\nu_{\beta_{jk}})^{\lambda})^{\omega_{jk}\zeta(B_k)}\right)^{1/\lambda}\right) \end{aligned} \tag{1.218}$$

(2) If $\lambda_1 = \lambda_2 = 1$, then Eq.(1.212) becomes the BSI-IFOAA operator:

$$\begin{aligned}\mathrm{BSI-IFOAA}\,(M) &= \mathop{\oplus}_{k=1}^{r}\left(\zeta(B_k)\mathop{\oplus}_{j=1}^{q_k}\left(\omega_{jk}\beta_{jk}\right)\right)\\ &= \left(1-\prod_{k=1}^{r}\prod_{j=1}^{q_k}\left(1-\mu_{\beta_{jk}}\right)^{w_{jk}\zeta(B_k)},\ \prod_{k=1}^{r}\prod_{j=1}^{q_k}\left(\nu_{\beta_{jk}}\right)^{\omega_{jk}\zeta(B_k)}\right)\end{aligned} \tag{1.219}$$

(3) If $\lambda_1 = 0$ and $\lambda_2 = 1$, then we have

$$\begin{aligned}\mathrm{BSI-IFOAG}\,(M) &= \mathop{\oplus}_{k=1}^{r}\left(\zeta(B_k)\mathop{\otimes}_{j=1}^{q_k}\left(\beta_{jk}^{\omega_{jk}}\right)\right)\\ &= \left(1-\prod_{k=1}^{r}\left(1-\prod_{j=1}^{q_k}\left(\mu_{\beta_{jk}}\right)^{\omega_{jk}}\right)^{\zeta(B_k)},\prod_{k=1}^{r}\left(1-\prod_{j=1}^{q_k}\left(1-\nu_{\beta_{jk}}\right)^{\omega_{jk}}\right)^{\zeta(B_k)}\right)\end{aligned} \tag{1.220}$$

which we call a BSI-IFOAG operator.

(4) If $\lambda_1 = 1$ and $\lambda_2 = 0$, then we have

$$\begin{aligned}\mathrm{BSI-IFOGA}\,(M) &= \mathop{\otimes}_{k=1}^{r}\left(\mathop{\oplus}_{j=1}^{q_k}\left(\omega_{jk}\beta_{jk}\right)\right)^{\zeta(B_k)}\\ &= \left(\prod_{k=1}^{r}\left(1-\prod_{j=1}^{q_k}\left(1-\mu_{\beta_{jk}}\right)^{\omega_{jk}}\right)^{\zeta(B_k)},1-\prod_{k=1}^{r}\left(1-\prod_{j=1}^{q_k}\left(\nu_{\beta_{jk}}\right)^{\omega_{jk}}\right)^{\zeta(B_k)}\right)\end{aligned} \tag{1.221}$$

which we call a BSI-IFOGA operator.

(5) If $\lambda_1 = 0$ and $\lambda_2 = 0$, then we have

$$\text{BSI} - \text{IFOGG}\,(M) = \overset{r}{\underset{k=1}{\otimes}} \left(\left(\overset{q_k}{\underset{j=1}{\otimes}} \left(\beta_{jk}^{\omega_{jk}} \right) \right)^{\zeta(B_k)} \right)$$
$$= \left(\prod_{k=1}^{r} \prod_{j=1}^{q_k} \left(\mu_{\beta_{jk}} \right)^{\omega_{jk}\zeta(B_k)},\ 1 - \prod_{k=1}^{r} \prod_{j=1}^{q_k} \left(1 - \nu_{\beta_{jk}} \right)^{\omega_{jk}\zeta(B_k)} \right) \tag{1.222}$$

which we call a BSI$-$IFOGG operator.

Xu and Xia (2011) apply the BSI$-$GIFOA operator to decision making. Assume that there is a decision making problem with a collection of alternatives $Y = \{Y_1, Y_2, \cdots, Y_n\}$ and a set of the states of nature, $G = \{G_1, G_2, \cdots, G_m\}$. If the alternative Y_i is selected under the state G_j, then α_{ij} is the payoff value which is denoted by IFNs. The knowledge of the state G_j $(j = 1, 2, \cdots, m)$ is captured in terms of a belief structure ζ with the focal elements $B_1, B_2, \cdots, B_r$, each of which is associated with a weight $\zeta(B_k)$. The problem is to select the alternative with the best result. To do so, Xu and Xia (2011) proposed a method based on the BSI-GIFOA operator, which involves the following steps:

Step 1 Calculate the attitudinal character of the decision maker to determine the inducing values matrix $\nabla = (\nabla_{ij})_{n\times m}$.

Step 2 Find $M_{ik} = \{\langle \nabla_{ij}, \alpha_{ij}\rangle |\, G_j \in B_k,\ j = 1, 2, \cdots, m\} = \{\left\langle \nabla_{ik}^1, \alpha_{ik}^1 \right\rangle, \left\langle \nabla_{ik}^2, \alpha_{ik}^2 \right\rangle, \cdots, \langle \nabla_{ik}^{q_k}, \alpha_{ik}^{q_k}\rangle\}$, the set of payoff values that are possible if we select the alternative Y_i and the focal element B_k occurs, where q_k is the number of elements in B_k.

Step 3 Utilize one of the existing methods (Xu, 2005a) to determine the weight vector $W_{q_k} = \left(\omega_{q_k}^{(1)}, \omega_{q_k}^{(2)}, \cdots, \omega_{q_k}^{(q_k)}\right)^{\mathrm{T}}$ for aggregating the q_k arguments in M_{ik}.

Step 4 Calculate the aggregated payoff value, V_{ik}, for Y_i when the focal element B_k occurs:

$$V_{ik} = \left(\omega_{q_k}^{(1)} (\beta_{ik}^{(1)})^{\lambda_1} \oplus \omega_{q_k}^{(2)} (\beta_{ik}^{(2)})^{\lambda_1} \oplus \cdots \oplus \omega_{q_k}^{(q_k)} (\beta_{ik}^{(q_k)})^{\lambda_1} \right)^{1/\lambda_1} \tag{1.223}$$

where $\lambda_1 > 0$, $\beta_{ik}^{(j)}$ is the $\alpha_{ik}^{(l)}$ value of the pair $\left\langle \nabla_{ik}^{(l)}, \alpha_{ik}^{(l)} \right\rangle$ with the j-th largest of $\nabla_{ik}^{(l)}$ $(l = 1, 2, \cdots, q_k)$.

Step 5 For each alternative Y_i, utilize the BSI-GIFOA operator (1.212) to calculate the aggregated payoff value C_i:

$$C_i = \left(\zeta(B_1)(V_{i1})^{\lambda_2} \oplus \zeta(B_2)(V_{i2})^{\lambda_2} \oplus \cdots \oplus \zeta(B_r)(V_{ir})^{\lambda_2} \right)^{1/\lambda_2}$$
$$= \left(\overset{r}{\underset{k=1}{\oplus}} \left(\zeta(B_k) \left(\overset{q_k}{\underset{j=1}{\oplus}} \left(\omega_{q_k}^{(j)} \left(\beta_{ik}^{(j)} \right)^{\lambda_1} \right) \right)^{\lambda_2/\lambda_1} \right) \right)^{1/\lambda_2}, \quad \lambda_1, \lambda_2 > 0 \tag{1.224}$$

Step 6 Select the alternative Y_i with the largest C_i as the optimal one.

Below we use the problem in Example 1.7.1 to illustrate the application of the BSI-GIFOA operator in decision making (Xu and Xia, 2011):

Example 1.7.2 In Example 1.14, assume that the states of nature representing the different economic situations are evaluated by the following world growth rates: ① G_1 : Strong recession; ② G_2 : Weak recession; ③ G_3 : Growth rate near zero; ④ G_4 :Very low growth rate; ⑤ G_5 : Low growth rate; ⑥ G_6 : Medium growth rate; ⑦ G_7 : High growth rate; and ⑧ G_8 : Very high growth. The possible results, depending on the future of nature, are represented as IFNs in Table 1.32:

Table 1.32 Payoff matrix (Xu and Xia, 2011)

	G_1	G_2	G_3	G_4	G_5	G_6	G_7	G_8
Y_1	(0.3,0.4)	(0.7,0.1)	(0.4,0.4)	(0.8,0.2)	(0.4,0.5)	(0.5,0.2)	(0.1,0.5)	(0.5,0.4)
Y_2	(0.5,0.3)	(0.5,0.2)	(0.7,0.1)	(0.6,0.1)	(0.6,0.2)	(0.7,0.2)	(0.3,0.6)	(0.4,0.3)
Y_3	(0.4,0.5)	(0.6,0.1)	(0.5,0.1)	(0.6,0.2)	(0.7,0.3)	(0.8,0.1)	(0.4,0.5)	(0.3,0.2)
Y_4	(0.3,0.4)	(0.4,0.2)	(0.8,0.1)	(0.8,0.1)	(0.3,0.4)	(0.7,0.1)	(0.2,0.5)	(0.7,0.1)
Y_5	(0.5,0.3)	(0.4,0.6)	(0.6,0.3)	(0.7,0.1)	(0.8,0.1)	(0.3,0.4)	(0.6,0.2)	(0.5,0.3)

Some probabilistic information about the states of nature is represented by the following belief function ζ:

$$\zeta(B_1) = \zeta(\{G_1, G_5, G_6, G_7\}) = 0.4, \quad \zeta(B_2) = \zeta(\{G_1, G_3, G_8\}) = 0.3$$

$$\zeta(B_3) = \zeta(\{G_2, G_3, G_4\}) = 0.3$$

Next, we use the method above to select the best alternative:

Step 1 Suppose that the induced aggregation variables are represented in Table 1.33:

Table 1.33 Inducing variables (Xu and Xia, 2011)

	G_1	G_2	G_3	G_4	G_5	G_6	G_7	G_8
Y_1	25	16	24	18	20	13	19	14
Y_2	18	34	22	12	24	16	20	26
Y_3	13	21	28	22	19	25	16	26
Y_4	20	24	14	31	27	25	19	18
Y_5	25	16	23	30	15	21	18	26

Step 2 Find $M_{ik} = \{\langle \nabla_{ij}, \alpha_{ij}\rangle | G_j \in B_k\} = \{\langle \nabla_{ik}^1, \alpha_{ik}^1 \rangle, \langle \nabla_{ik}^2, \alpha_{ik}^2 \rangle, \cdots, \langle \nabla_{ik}^{q_k}, \alpha_{ik}^{q_k} \rangle\}$

$Y_1 : M_{11} = \{\langle 25, (0.3, 0.4)\rangle, \langle 20, (0.4, 0.5)\rangle, \langle 13, (0.5, 0.2)\rangle, \langle 19, (0.1, 0.5)\rangle\}$

$M_{12} = \{\langle 25, (0.3, 0.4)\rangle, \langle 24, (0.4, 0.4)\rangle, \langle 14, (0.5, 0.4)\rangle\}$

$M_{13} = \{\langle 16, (0.7, 0.1)\rangle, \langle 24, (0.4, 0.4)\rangle, \langle 18, (0.8, 0.2)\rangle\}$

$Y_2 : M_{21} = \{\langle 18, (0.5, 0.3)\rangle, \langle 24, (0.6, 0.2)\rangle, \langle 16, (0.7, 0.2)\rangle, \langle 20, (0.3, 0.6)\rangle\}$

$M_{22} = \{\langle 18, (0.5, 0.3)\rangle, \langle 22, (0.7, 0.1)\rangle, \langle 26, (0.4, 0.3)\rangle\}$

$M_{23} = \{\langle 34, (0.5, 0.2)\rangle, \langle 22, (0.7, 0.1)\rangle, \langle 12, (0.6, 0.1)\rangle\}$

$Y_3 : M_{31} = \{\langle 20, (0.4, 0.5)\rangle, \langle 19, (0.7, 0.3)\rangle, \langle 25, (0.8, 0.1)\rangle, \langle 13, (0.4, 0.5)\rangle\}$

$M_{32} = \{\langle 13, (0.4, 0.5)\rangle, \langle 28, (0.5, 0.1)\rangle, \langle 26, (0.3, 0.2)\rangle\}$

$M_{33} = \{\langle 21, (0.6, 0.1)\rangle, \langle 28, (0.5, 0.1)\rangle, \langle 22, (0.6, 0.2)\rangle\}$

$Y_4 : M_{41} = \{\langle 20, (0.3, 0.4)\rangle, \langle 27, (0.3, 0.4)\rangle, \langle 25, (0.7, 0.1)\rangle, \langle 19, (0.2, 0.5)\rangle\}$

$M_{42} = \{\langle 20, (0.3, 0.4)\rangle, \langle 14, (0.8, 0.1)\rangle, \langle 18, (0.7, 0.1)\rangle\}$

$M_{43} = \{\langle 24, (0.4, 0.2)\rangle, \langle 14, (0.8, 0.1)\rangle, \langle 31, (0.8, 0.1)\rangle\}$

$Y_5 : M_{51} = \{\langle 25, (0.5, 0.3)\rangle, \langle 15, (0.8, 0.1)\rangle, \langle 21, (0.3, 0.4)\rangle, \langle 18, (0.6, 0.2)\rangle\}$

$M_{52} = \{\langle 25, (0.5, 0.3)\rangle, \langle 23, (0.6, 0.3)\rangle, \langle 26, (0.5, 0.3)\rangle\}$

$M_{53} = \{\langle 16, (0.4, 0.6)\rangle, \langle 23, (0.6, 0.3)\rangle, \langle 30, (0.7, 0.1)\rangle\}$

Step 3 Utilize the normal distribution based method (Xu, 2005a) to determine the weight vector W_{q_k} with q_k elements to aggregate the q_k arguments in M_{ik}:

$$W_3 = (0.4,\ 0.4,\ 0.2)^{\mathrm{T}}, \quad W_4 = (0.3,\ 0.3,\ 0.2,\ 0.2)^{\mathrm{T}}$$

Step 4 Calculate the aggregated payoff by using Eq.(1.223). The results are presented in Table 1.34 when $\lambda_1 = 0, 1, 2, 5, 10$:

Table 1.34 Aggregated payoffs for all V_{ik} $(i = 1, 2, 3, 4, 5;\ k = 1, 2, 3)$

	$\lambda_1 = 0$	$\lambda_1 = 1$	$\lambda_1 = 2$	$\lambda_1 = 5$	$\lambda_1 = 10$
V_{11}	(0.2908, 0.4198)	(0.3429, 0.3893)	(0.3616, 0.3815)	(0.3990, 0.3538)	(0.4324, 0.3093)
V_{12}	(0.3728, 0.4000)	(0.3847, 0.4000)	(0.3900, 0.4000)	(0.4085, 0.4000)	(0.4345, 0.4000)
V_{13}	(0.5903, 0.2700)	(0.6634, 0.2297)	(0.6770, 0.2246)	(0.7105, 0.2093)	(0.7410, 0.1877)
V_{21}	(0.4846, 0.3673)	(0.5329, 0.3016)	(0.5463, 0.2913)	(0.5805, 0.2666)	(0.6148, 0.2444)
V_{22}	(0.5232, 0.2260)	(0.5616, 0.1933)	(0.5722, 0.1898)	(0.6044, 0.1788)	(0.6406, 0.1614)
V_{23}	(0.5933, 0.1414)	(0.6102, 0.1320)	(0.6142, 0.1313)	(0.6277, 0.1297)	(0.6476, 0.1256)
V_{31}	(0.5825, 0.3402)	(0.6495, 0.2647)	(0.6627, 0.2533)	(0.6961, 0.2219)	(0.7280, 0.1855)
V_{32}	(0.3898, 0.2366)	(0.4067, 0.1821)	(0.4137, 0.1771)	(0.4352, 0.1646)	(0.4589, 0.1507)
V_{33}	(0.5578, 0.1414)	(0.5627, 0.1320)	(0.5640, 0.1313)	(0.5686, 0.1292)	(0.5760, 0.1256)
V_{41}	(0.3567, 0.3467)	(0.4424, 0.2759)	(0.4763, 0.2644)	(0.5597, 0.2304)	(0.6212, 0.1887)
V_{42}	(0.5123, 0.2347)	(0.6118, 0.1741)	(0.6327, 0.1686)	(0.6775, 0.1540)	(0.7123, 0.1372)
V_{43}	(0.6063, 0.1414)	(0.6896, 0.1320)	(0.7034, 0.1313)	(0.7355, 0.1292)	(0.7619, 0.1256)
V_{51}	(0.4887, 0.2782)	(0.5596, 0.2421)	(0.5789, 0.2371)	(0.6315, 0.2212)	(0.6885, 0.1966)
V_{52}	(0.5186, 0.3000)	(0.5218, 0.3000)	(0.5229, 0.3000)	(0.5272, 0.3000)	(0.5370, 0.3000)
V_{53}	(0.5885, 0.3079)	(0.6134, 0.2221)	(0.6188, 0.2124)	(0.6340, 0.1888)	(0.6517, 0.1638)

Step 5 Let $\lambda_2 = 1$ and $\lambda_2 = 0$. Calculate the expected value of each alternative by using Eq.(1.224). The results are given in Tables 1.35–1.36, respectively:

Table 1.35 Aggregated payoffs for all Y_i $(i = 1, 2, 3, 4, 5)$ with $\lambda_2 = 1$

λ_1	C_1	C_2	C_3	C_4	C_5
0	(0.4202, 0.3624)	(0.5310, 0.2384)	(0.5240, 0.2344)	(0.4891, 0.2357)	(0.5295, 0.2934)
1	(0.4729, 0.3350)	(0.5659, 0.2060)	(0.5614, 0.1920)	(0.5804, 0.1926)	(0.5659, 0.2516)
2	(0.4867, 0.3301)	(0.5754, 0.2017)	(0.5700, 0.1868)	(0.6030, 0.1873)	(0.5757, 0.2462)
5	(0.5196, 0.3136)	(0.6023, 0.1903)	(0.5934, 0.1725)	(0.6558, 0.1716)	(0.6037, 0.2311)
10	(0.5519, 0.2876)	(0.6327, 0.1767)	(0.6180, 0.1550)	(0.6966, 0.1518)	(0.6372, 0.2113)

Table 1.36 Aggregated payoffs for all Y_i $(i = 1, 2, 3, 4, 5)$ with $\lambda_2 = 0$

λ_1	C_1	C_2	C_3	C_4	C_5
0	(0.3874, 0.3721)	(0.5269, 0.2634)	(0.5097, 0.2540)	(0.4662, 0.2564)	(0.5260, 0.2938)
1	(0.4326, 0.3487)	(0.5638, 0.2216)	(0.5406, 0.2021)	(0.5570, 0.2047)	(0.5633, 0.2542)
2	(0.4465, 0.3441)	(0.5738, 0.2158)	(0.5482, 0.1955)	(0.5830, 0.1978)	(0.5728, 0.2494)
5	(0.4778, 0.3286)	(0.6015, 0.2012)	(0.5690, 0.1778)	(0.6433, 0.1783)	(0.5989, 0.2364)
10	(0.5090, 0.3049)	(0.6322, 0.1855)	(0.5909, 0.1575)	(0.6881, 0.1548)	(0.6286, 0.2198)

Step 6 Select the best alternative with different values λ_1 and λ_2, as listed in Table 1.37:

Table 1.37 Rankings of the alternatives

λ_1	$\lambda_2 = 0$	$\lambda_2 = 1$
0	$Y_2 \succ Y_3 \succ Y_5 \succ Y_4 \succ Y_1$	$Y_3 \succ Y_2 \succ Y_4 \succ Y_5 \succ Y_1$
1	$Y_4 \succ Y_2 \succ Y_3 \succ Y_5 \succ Y_1$	$Y_3 \succ Y_4 \succ Y_2 \succ Y_5 \succ Y_1$
2	$Y_4 \succ Y_2 \succ Y_3 \succ Y_5 \succ Y_1$	$Y_4 \succ Y_3 \succ Y_2 \succ Y_5 \succ Y_1$
5	$Y_4 \succ Y_2 \succ Y_3 \succ Y_5 \succ Y_1$	$Y_4 \succ Y_3 \succ Y_2 \succ Y_5 \succ Y_1$
10	$Y_4 \succ Y_2 \succ Y_3 \succ Y_5 \succ Y_1$	$Y_4 \succ Y_3 \succ Y_2 \succ Y_5 \succ Y_1$

As we can see from Table 1.37, with the change of the parameters λ_1 and λ_2, the rankings of the alternatives may be different. Thus, the investor can select the desirable alternative according to his interest and actual needs.

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Chapter 2

Interval-Valued Intuitionistic Fuzzy Information Aggregation

In many real-world decision problems the values of the membership function and the non-membership function in an IFS are difficult to be expressed as exact numbers. Instead, the ranges of their values can usually be specified. In such cases, Atanassov and Gargov (1989) generalized the concept of IFS to interval-valued intuitionistic fuzzy set (IVIFS), and define some basic operational laws of IVIFSs. Xu (2007h) defined the concept of interval-valued intuitionistic fuzzy number (IVIFN), and gave some basic operational laws of IVIFNs. He put forward an interval-valued intuitionistic fuzzy weighted averaging operator and an interval-valued intuitionistic fuzzy weighted geometric operator, and defines the score function and the accuracy function of IVIFNs. He further presents a simple ranking method for IVIFNs, based on which an approach is proposed for multi-attribute decision making with intuitionistic fuzzy information. Xu and Chen (2007a) define an interval-valued intuitionistic fuzzy ordered weighted averaging operator and an interval-valued intuitionistic fuzzy hybrid averaging operator. Xu and Chen (2007c) investigate an interval-valued intuitionistic fuzzy ordered weighted geometric operator and an interval-valued intuitionistic fuzzy hybrid geometric operator. Xu and Yager (2011) extended the IFBMs to accommodate interva-valued intuitionistic fuzzy environments. Zhao et al.(2010) developed a series of generalized aggregation operators for IVIFNs. Xu (2010c) used Choquet integral to propose some operators for aggregating IVIFNs together with their correlative weights. All these aggregation techniques for interval-valued intuitionistic fuzzy information are generalizations of the intuitionistic fuzzy aggregation techniques introduced in Chapter 1.

2.1 Interval-Valued Intuitionistic Fuzzy Sets

Atanassov and Gargov (1989) defined the concept of interval-valued intuitionistic fuzzy set:

Definition 2.1.1 (Atanassov and Gargov, 1989) Let X be a fixed set. Then

$$\tilde{A} = \{\langle x, \tilde{\mu}_{\tilde{A}}(x), \tilde{\nu}_{\tilde{A}}(x)\rangle | \, x \in X\} \tag{2.1}$$

is called an interval-valued intuitionistic fuzzy set (IVIFS), where $\tilde{\mu}_{\tilde{A}}(x) \subset [0,1]$ and $\tilde{\nu}_{\tilde{A}}(x) \subset [0,1]$, $x \in X$, with the condition:

$$\sup \tilde{\mu}_{\tilde{A}}(x) + \sup \tilde{\nu}_{\tilde{A}}(x) \leqslant 1, \quad x \in X \tag{2.2}$$

Clearly, if $\inf \tilde{\mu}_{\tilde{A}}(x) = \sup \tilde{\mu}_{\tilde{A}}(x)$ and $\inf \tilde{\nu}_{\tilde{A}}(x) = \sup \tilde{\nu}_{\tilde{A}}(x)$, then the IVIFS $\tilde{A}$ reduces to a traditional IFS.

Atanassov and Gargov (1989) further gave some basic operational laws of IVIFSs:

Definition 2.1.2 (Atanassov and Gargov, 1989) Let $\tilde{A} = \{\langle x, \tilde{\mu}_{\tilde{A}}(x), \tilde{\nu}_{\tilde{A}}(x)\rangle | \, x \in X\}$, $\tilde{A}_1 = \{\langle x, \tilde{\mu}_{\tilde{A}_1}(x), \tilde{\nu}_{\tilde{A}_1}(x)\rangle | \, x \in X\}$ and $\tilde{A}_2 = \{\langle x, \tilde{\mu}_{\tilde{A}_2}(x), \tilde{\nu}_{\tilde{A}_2}(x)\rangle | \, x \in X\}$ be three IVIFSs. Then

(1) $\bar{\tilde{A}} = \{\langle x, \tilde{\nu}_{\tilde{A}}(x), \tilde{\mu}_{\tilde{A}}(x)\rangle | \, x \in X\}$;

(2) $\tilde{A}_1 \cap \tilde{A}_2 = \{\langle x, [\min\{\inf \tilde{\mu}_{\tilde{A}_1}(x), \inf \tilde{\mu}_{\tilde{A}_2}(x)\}, \min\{\sup \tilde{\mu}_{\tilde{A}_1}(x), \sup \tilde{\mu}_{\tilde{A}_2}(x)\}],$
$[\max\{\inf \tilde{\nu}_{\tilde{A}_1}(x), \inf \tilde{\nu}_{\tilde{A}_2}(x)\}, \max\{\sup \tilde{\nu}_{\tilde{A}_1}(x), \sup \tilde{\nu}_{\tilde{A}_2}(x)\}]\rangle | \, x \in X\}$;

(3) $\tilde{A}_1 \cup \tilde{A}_2 = \{\langle x, [\max\{\inf \tilde{\mu}_{\tilde{A}_1}(x), \inf \tilde{\mu}_{\tilde{A}_2}(x)\}, \max\{\sup \tilde{\mu}_{\tilde{A}_1}(x), \sup \tilde{\mu}_{\tilde{A}_2}(x)\}],$
$[\min\{\inf \tilde{\nu}_{\tilde{A}_1}(x), \inf \tilde{\nu}_{\tilde{A}_2}(x)\}, \min\{\sup \tilde{\nu}_{\tilde{A}_1}(x)), \sup \tilde{\nu}_{\tilde{A}_2}(x)\}]\rangle | \, x \in X\}$,

(4) $\tilde{A}_1 + \tilde{A}_2 = \{\langle x, [\inf \tilde{\mu}_{\tilde{A}_1}(x) + \inf \tilde{\mu}_{\tilde{A}_2}(x) - \inf \tilde{\mu}_{\tilde{A}_1}(x) \cdot \inf \tilde{\mu}_{\tilde{A}_2}(x),$
$\sup \tilde{\mu}_{\tilde{A}_1}(x) + \sup \tilde{\mu}_{\tilde{A}_2}(x) - \sup \tilde{\mu}_{\tilde{A}_1}(x) \cdot \sup \tilde{\mu}_{\tilde{A}_2}(x)],$
$[\inf \tilde{\nu}_{\tilde{A}_1}(x) \cdot \inf \tilde{\nu}_{\tilde{A}_2}(x), \sup \tilde{\nu}_{\tilde{A}_1}(x) \cdot \sup \tilde{\nu}_{\tilde{A}_2}(x)]\rangle | \, x \in X\}$;

(5) $\tilde{A}_1 \cdot \tilde{A}_2 = \{\langle x, [\inf \tilde{\mu}_{\tilde{A}_1}(x) \cdot \inf \tilde{\mu}_{\tilde{A}_2}(x), \sup \tilde{\mu}_{\tilde{A}_1}(x) \cdot \sup \tilde{\mu}_{\tilde{A}_2}(x)],$
$[\inf \tilde{\nu}_{\tilde{A}_1}(x) + \inf \tilde{\nu}_{\tilde{A}_2}(x) - \inf \tilde{\nu}_{\tilde{A}_1}(x). \inf \tilde{\nu}_{\tilde{A}_2}(x),$
$\sup \tilde{\nu}_{\tilde{A}_1}(x) + \sup \tilde{\nu}_{\tilde{A}_2}(x) - \sup \tilde{\nu}_{\tilde{A}_1}(x) \cdot \sup \tilde{\nu}_{\tilde{A}_2}(x)]\rangle | \, x \in X\}$.

Considering the needs in applications, Xu and Chen (2007c) introduced other two operational laws:

(6) $\lambda \tilde{A} = \{\langle x, [1 - (1 - \inf \tilde{\mu}_{\tilde{A}}(x))^{\lambda}, 1 - (1 - \sup \tilde{\mu}_{\tilde{A}}(x))^{\lambda}],$
$[(\inf \tilde{\nu}_{\tilde{A}}(x))^{\lambda}, (\sup \tilde{\nu}_{\tilde{A}}(x))^{\lambda}] > | \, x \in X\}, \quad \lambda > 0$;

(7) $\tilde{A}^{\lambda} = \{\langle x, [(\inf \tilde{\mu}_{\tilde{A}}(x))^{\lambda}, (\sup \tilde{\mu}_{\tilde{A}}(x))^{\lambda}],$
$[1 - (1 - \inf \tilde{\nu}_{\tilde{A}}(x))^{\lambda}, 1 - (1 - \sup \tilde{\mu}_{\tilde{A}}(x))^{\lambda}] > | \, x \in X\}, \quad \lambda > 0$.

2.2 Operational Laws of Interval-Valued Intuitionistic Fuzzy Numbers

According to Definition 2.1.1, the basic component of an IVIFS is an ordered pair, characterized by an interval-valued membership degree and an interval-valued non-membership degree of x in $\tilde{A}$. This ordered pair is called an interval-valued intuitionistic fuzzy number (IVIFN) (Xu, 2007h).

For convenience, an IVIFN is generally simplified as $([a, b], [c, d])$ (Xu, 2007h), where

$$[a, b] \subset [0, 1], [c, d] \subset [0, 1], \quad b + d \leqslant 1 \tag{2.3}$$

and $\tilde{\Theta}$ is the set of all IVIFNs. Obviously, $\tilde{\alpha}^+ = ([1, 1], [0, 0])$ is the largest IVIFN, and $\tilde{\alpha}^- = ([0, 0], [1, 1])$ is the smallest IVIFN.

In particular, if $\tilde{\alpha}_1 = ([a_1, b_1], [c_1, d_1])$ and $\tilde{\alpha}_2 = ([a_2, b_2], [c_2, d_2])$ are IVIFNs, then $\tilde{\alpha}_1 = \tilde{\alpha}_2$ if and only if $a_1 = a_2$, $b_1 = b_2$, $c_1 = c_2$ and $d_1 = d_2$.

Similar to Definition 2.1.2, we introduce some operational laws of IVIFNs as follows:

Definition 2.2.1 (Xu, 2007h) Let $\tilde{\alpha} = ([a, b], [c, d])$, $\tilde{\alpha}_1 = ([a_1, b_1], [c_1, d_1])$ and $\tilde{\alpha}_2 = ([a_2, b_2], [c_2, d_2])$ be IVIFNs. Then

(1) $\bar{\tilde{\alpha}} = ([c, d], [a, b])$;

(2) $\tilde{\alpha}_1 \wedge \tilde{\alpha}_2 = ([\min\{a_1, a_2\}, \min\{b_1, b_2\}],\ [\max\{c_1, c_2\}, \max\{d_1, d_2\}])$;

(3) $\tilde{\alpha}_1 \vee \tilde{\alpha}_2 = ([\max\{a_1, a_2\}, \max\{b_1, b_2\}],\ [\min\{c_1, c_2\}, \min\{d_1, d_2\}])$;

(4) $\tilde{\alpha}_1 \oplus \tilde{\alpha}_2 = ([a_1 + a_2 - a_1 a_2, b_1 + b_2 - b_1 b_2],\ [c_1 c_2, d_1 d_2])$;

(5) $\tilde{\alpha}_1 \otimes \tilde{\alpha}_2 = ([a_1 a_2, b_1 b_2],\ [c_1 + c_2 - c_1 c_2, d_1 + d_2 - d_1 d_2])$;

(6) $\lambda\tilde{\alpha} = ([1 - (1 - a)^\lambda, 1 - (1 - b)^\lambda],\ [c^\lambda, d^\lambda]),\ \lambda > 0$;

(7) $\tilde{\alpha}^\lambda = ([a^\lambda, b^\lambda], [1 - (1 - c)^\lambda,\ 1 - (1 - d)^\lambda]),\ \lambda > 0$.

Theorem 2.2.1 (Xu, 2007h) All the operational results in Definition 2.2.1 are IVIFNs.

Proof (1) Since $\tilde{\alpha} = ([a, b], [c, d])$ is an IVIFN, $\bar{\tilde{\alpha}} = ([c, d], [a, b])$ satisfies the condition (2.3). Thus, $\bar{\tilde{\alpha}}$ is an IVIFN.

(2) Since both $\tilde{\alpha}_1 = ([a_1, b_1], [c_1, d_1])$ and $\tilde{\alpha}_2 = ([a_2, b_2], [c_2, d_2])$ are IVIFNs, $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ satisfy the condition (2.3), we have

$$[\min\{a_1, a_2\}, \min\{b_1, b_2\}] \subset [0, 1]$$

$$[\max\{c_1, c_2\}, \max\{d_1, d_2\}] \subset [0, 1]$$

$$\min\{b_1, b_2\} + \max\{d_1, d_2\} \leqslant 1$$

Then $\tilde{\alpha}_1 \wedge \tilde{\alpha}_2$ satisfies the condition (6), i.e., $\tilde{\alpha}_1 \wedge \tilde{\alpha}_2$ is an IVIFN.

(3) Similar to (2), we can prove that $\tilde{\alpha}_1 \vee \tilde{\alpha}_2$ is also an IVIFN.

(4) Since both $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ satisfy the condition (2.3), it follows that

$$a_1 + a_2 - a_1 a_2 = a_1(1 - a_2) + a_2 \geqslant a_2 \geqslant 0, \quad c_1 c_2 \geqslant 0$$

$$b_1 + b_2 - b_1 b_2 + d_1 d_2 \leqslant b_1 + b_2 - b_1 b_2 + (1 - b_1)(1 - b_2) = 1$$

Therefore, the value of $\tilde{\alpha}_1 \oplus \tilde{\alpha}_2$ is an IVIFN. In a similar way, (5) can be proven.

(6) Since $1 - (1 - a)^\lambda \geqslant 0$, $c^\lambda \geqslant 0$ and $1 - (1 - b)^\lambda + d^\lambda \leqslant 1 - (1 - b)^\lambda + (1 - b)^\lambda = 1$, the value of $\lambda\tilde{\alpha}$ is an IVIFN.

(7) can be proven similarly. This completes the proof.

Theorem 2.2.2 (Xu, 2007h) Let $\lambda, \lambda_1, \lambda_2 \geqslant 0$. Then

(1) $\tilde{\alpha}_1 \oplus \tilde{\alpha}_2 = \tilde{\alpha}_2 \oplus \tilde{\alpha}_1$;

(2) $\tilde{\alpha}_1 \otimes \tilde{\alpha}_2 = \tilde{\alpha}_2 \otimes \tilde{\alpha}_1$;

(3) $\lambda(\tilde{\alpha}_1 \oplus \tilde{\alpha}_2) = \lambda\tilde{\alpha}_1 \oplus \lambda\tilde{\alpha}_2$;

(4) $(\tilde{\alpha}_1 \otimes \tilde{\alpha}_2)^\lambda = \tilde{\alpha}_1^\lambda \otimes \tilde{\alpha}_2^\lambda$;

(5) $\lambda_1\tilde{\alpha} \oplus \lambda_2\tilde{\alpha} = (\lambda_1 + \lambda_2)\tilde{\alpha}$;

(6) $\tilde{\alpha}^{\lambda_1} \otimes \tilde{\alpha}^{\lambda_2} = \tilde{\alpha}^{\lambda_1+\lambda_2}$.

Proof (1) By (4) in Definition 2.2.1, we have

$$\begin{aligned}\tilde{\alpha}_1 \oplus \tilde{\alpha}_2 &= ([a_1 + a_2 - a_1a_2, b_1 + b_2 - b_1b_2],\ [c_1c_2, d_1d_2]) \\ &= ([a_2 + a_1 - a_2a_1, b_2 + b_1 - b_2b_1],\ [c_2c_1, d_2d_1]) \\ &= \tilde{\alpha}_2 \oplus \tilde{\alpha}_1\end{aligned}$$

(2) According to (5) in Definition 2.2.1, we can get

$$\begin{aligned}\tilde{\alpha}_1 \otimes \tilde{\alpha}_2 &= ([a_1a_2, b_1b_2],\ [c_1 + c_2 - c_1c_2, d_1 + d_2 - d_1d_2]) \\ &= ([a_2a_1, b_2b_1],\ [c_2 + c_1 - c_2c_1, d_2 + d_1 - d_2d_1]) \\ &= \tilde{\alpha}_2 \otimes \tilde{\alpha}_1\end{aligned}$$

(3) It follows from (4) in Definition 2.2.1 that

$$\tilde{\alpha}_1 \oplus \tilde{\alpha}_2 = ([a_1 + a_2 - a_1a_2, b_1 + b_2 - b_1b_2],\ [c_1c_2, d_1d_2])$$

According to (6) in Definition 2.2.1, we get

$$\begin{aligned}\lambda(\tilde{\alpha}_1 \oplus \tilde{\alpha}_2) &= ([1-(1-(a_1+a_2-a_1a_2)^\lambda, 1-(1-(b_1+b_2-b_1b_2)^\lambda],\ [(c_1c_2)^\lambda, (d_1d_2)^\lambda]) \\ &= ([1 - (1 - a_1)^\lambda(1 - a_2)^\lambda, 1 - (1 - b_1)^\lambda(1 - b_2)^\lambda],\ [(c_1c_2)^\lambda, (d_1d_2)^\lambda])\end{aligned}$$

Also since

$$\lambda\tilde{\alpha}_1 = ([1 - (1 - a_1)^\lambda, 1 - (1 - b_1)^\lambda],\ [c_1^\lambda, d_1^\lambda])$$

$$\lambda\tilde{\alpha}_2 = ([1 - (1 - a_2)^\lambda, 1 - (1 - b_2)^\lambda],\ [c_2^\lambda, d_2^\lambda])$$

we have

$$\begin{aligned}\lambda\tilde{\alpha}_1 \oplus \lambda\tilde{\alpha}_2 =&([1 - (1 - a_1)^\lambda + (1 - a_2)^\lambda - (1 - (1 - a_1)^\lambda)(1 - a_2)^\lambda, \\ &1 - (1 - b_1)^\lambda + 1 - (1 - b_2)^\lambda - (1 - (1 - b_1)^\lambda)(1 - (1 - b_2)^\lambda], \\ &[(c_1c_2)^\lambda, (d_1d_2)^\lambda]) \\ =&([1 - (1 - a_1)^\lambda(1 - a_2)^\lambda, 1 - (1 - b_1)^\lambda(1 - b_2)^\lambda],\ [(c_1c_2)^\lambda, (d_1d_2)^\lambda])\end{aligned}$$

Thus

$$\lambda(\tilde{\alpha}_1 \oplus \tilde{\alpha}_2) = \lambda\tilde{\alpha}_1 \oplus \lambda\tilde{\alpha}_2$$

Similarly, we can prove (4).

(5) Since

$$\lambda_1\tilde{\alpha}_1 = ([1-(1-a_1)^{\lambda_1}, 1-(1-b_1)^{\lambda_1}],\ [c_1^{\lambda_1}, d_1^{\lambda_1}])$$
$$\lambda_2\tilde{\alpha}_1 = ([1-(1-a_1)^{\lambda_2}, 1-(1-b_1)^{\lambda_2}],\ [c_1^{\lambda_2}, d_1^{\lambda_2}])$$

we can obtain

$$\begin{aligned}\lambda_1\tilde{\alpha}_1 \oplus \lambda_2\tilde{\alpha}_1 &= ([2-(1-a_1)^{\lambda_1}-(1-a_1)^{\lambda_2}-(1-(1-a_1)^{\lambda_1})(1-(1-a_1)^{\lambda_2}),\\ &\quad 2-(1-b_1)^{\lambda_1}-(1-b_1)^{\lambda_2}-(1-(1-b_1)^{\lambda_1})(1-(1-b_1)^{\lambda_2})],\ [c_1^{\lambda_1}c_1^{\lambda_2}, d_1^{\lambda_1}d_1^{\lambda_2}])\\ &= ([1-(1-a_1)^{\lambda_1}(1-a_1)^{\lambda_2}, [1-(1-b_1)^{\lambda_1}(1-b_1)^{\lambda_2}],\ [(c_1)^{\lambda_1+\lambda_2}, (d_1)^{\lambda_1+\lambda_2}])\\ &= ([1-(1-a_1)^{\lambda_1+\lambda_2}, [1-(1-b_1)^{\lambda_1+\lambda_2}],\ [(c_1)^{\lambda_1+\lambda_2}, (d_1)^{\lambda_1+\lambda_2}])\\ &= (\lambda_1+\lambda_2)\tilde{\alpha}_1\end{aligned}$$

In a similar way, (6) can be proven. The proof is completed.

2.3 Interval-Valued Intuitionistic Fuzzy Aggregation Operators

We now introduce, based on Definition 2.2.1, some operators for aggregating IVIFNs:

Definition 2.3.1 (Xu, 2007h) Let $\tilde{\alpha}_j$ $(j=1,2,\cdots,n)$ be a collection of IVIFNs, and let IIFWA : $\tilde{\Theta}^n \to \tilde{\Theta}$. If

$$\text{IIFWA}_\omega(\tilde{\alpha}_1, \tilde{\alpha}_2, \cdots, \tilde{\alpha}_n) = \omega_1\tilde{\alpha}_1 \oplus \omega_2\tilde{\alpha}_2 \oplus \cdots \oplus \omega_n\tilde{\alpha}_n \tag{2.4}$$

where $\omega = (\omega_1, \omega_2, \cdots, \omega_n)^{\mathrm{T}}$ is the weight vector of $\tilde{\alpha}_j$ $(j=1,2,\cdots,n)$, with $\omega_j \in [0,1]$, $j=1,2,\cdots,n$, and $\sum_{j=1}^{n}\omega_j = 1$, then the function IIFWA is called an interval-valued intuitionistic fuzzy weighted averaging (IIFWA) operator. In particular, if $\omega = (1/n, 1/n, \cdots, 1/n)^{\mathrm{T}}$, then the IIFWA operator reduces to an interval-valued intuitionistic fuzzy averaging (IIFA) operator:

$$\text{IIFA}(\tilde{\alpha}_1, \tilde{\alpha}_2, \cdots, \tilde{\alpha}_n) = \frac{1}{n}(\tilde{\alpha}_1 \oplus \tilde{\alpha}_2 \oplus \cdots \oplus \tilde{\alpha}_n) \tag{2.5}$$

Definition 2.3.2 (Xu, 2007h) Let IIFWG : $\tilde{\Theta}^n \to \tilde{\Theta}$. If

$$\text{IIFWG}_\omega(\tilde{\alpha}_1, \tilde{\alpha}_2, \cdots, \tilde{\alpha}_n) = \tilde{\alpha}_1^{\omega_1} \otimes \tilde{\alpha}_2^{\omega_2} \otimes \cdots \otimes \tilde{\alpha}_n^{\omega_n} \tag{2.6}$$

then the function IIFWG is called an interval-valued intuitionistic fuzzy weighted geometric (IIFWG) operator. In particular, if $\omega = (1/n, 1/n, \cdots, 1/n)^{\mathrm{T}}$, then the

IIFWG operator reduces to an interval-valued intuitionistic fuzzy geometric (IIFG) operator:

$$\text{IIFG}(\tilde{\alpha}_1, \tilde{\alpha}_2, \cdots, \tilde{\alpha}_n) = (\tilde{\alpha}_1 \otimes \tilde{\alpha}_2 \otimes \cdots \otimes \tilde{\alpha}_n)^{\frac{1}{n}} \tag{2.7}$$

Theorem 2.3.1 (Xu, 2007h) Let $\tilde{\alpha}_j = ([a_j, b_j], [c_j, d_j])$ $(j = 1, 2, \cdots, n)$ be a collection of IVIFNs. Then the aggregated value by using Eq.(2.4) is an IVIFN, and

$$\text{IIFWA}_\omega(\tilde{\alpha}_1, \tilde{\alpha}_2, \cdots, \tilde{\alpha}_n) = \left(\left[1 - \prod_{j=1}^{n}(1 - a_j)^{\omega_j}, 1 - \prod_{j=1}^{n}(1 - b_j)^{\omega_j}\right], \left[\prod_{j=1}^{n} c_j^{\omega_j}, \prod_{j=1}^{n} d_j^{\omega_j}\right]\right) \tag{2.8}$$

where $\omega = (\omega_1, \omega_2, \cdots, \omega_n)^{\text{T}}$ is the weight vector of $\tilde{\alpha}_j$ $(j = 1, 2, \cdots, n)$, with $\omega_j \in [0, 1]$ $(j = 1, 2, \cdots, n)$ and $\sum_{j=1}^{n} \omega_j = 1$.

Proof We prove Eq.(2.8) by using mathematical induction on n:

(1) When $n = 2$,

$$\text{IIFWA}_\omega(\alpha_1, \alpha_2) = \omega_1 \tilde{\alpha}_1 \oplus \omega_2 \tilde{\alpha}_2$$

According to Theorem 2.2.2, we can see that both $\omega_1 \tilde{\alpha}_1$ and $\omega_2 \tilde{\alpha}_2$ are IVIFNs, and the value of $\omega_1 \tilde{\alpha}_1 \oplus \omega_2 \tilde{\alpha}_2$ is an IVIFN. By the operational law (6) in Definition 2.3.1, we have

$$\omega_1 \tilde{\alpha}_1 = ([1 - (1 - a_1)^{\omega_1}, 1 - (1 - b_1)^{\omega_1}],\ [c_1^{\omega_1}, d_1^{\omega_1}])$$

$$\omega_2 \tilde{\alpha}_2 = ([1 - (1 - a_2)^{\omega_2}, 1 - (1 - b_2)^{\omega_2}],\ [c_2^{\omega_2}, d_2^{\omega_2}])$$

Then

$$\begin{aligned}
\text{IIFWA}_\omega(\tilde{\alpha}_1, \tilde{\alpha}_2) =& \omega_1 \tilde{\alpha}_1 \oplus \omega_2 \tilde{\alpha}_2 \\
=& ([2 - (1 - a_1)^{\omega_1} - (1 - a_2)^{\omega_2} - (1 - (1 - a_1)^{\omega_1})(1 - (1 - a_2)^{\omega_2}), \\
& 2 - (1 - b_1)^{\omega_1} - (1 - b_2)^{\omega_2} - (1 - (1 - b_1)^{\omega_1})(1 - (1 - b_2)^{\omega_2})], \\
& [c_1^{\omega_1} c_2^{\omega_2}, d_1^{\omega_1} d_2^{\omega_2}]) \\
=& ([1 - (1 - a_1)^{\omega_1}(1 - a_2)^{\omega_2}, 1 - (1 - b_1)^{\omega_1}(1 - b_2)^{\omega_2}],\ [c_1^{\omega_1} c_2^{\omega_2}, d_1^{\omega_1} d_2^{\omega_2}])
\end{aligned}$$

(2) Suppose that $n = k$, Eq.(2.8) holds, i.e.,

$$\text{IIFWA}_\omega(\tilde{\alpha}_1, \tilde{\alpha}_2, \cdots, \tilde{\alpha}_k) = \left(\left[1 - \prod_{j=1}^{k}(1 - a_j)^{\omega_j}, 1 - \prod_{j=1}^{k}(1 - b_j)^{\omega_j}\right], \left[\prod_{j=1}^{k} c_j^{\omega_j}, \prod_{j=1}^{k} d_j^{\omega_j}\right]\right)$$

Then when $n = k + 1$, by (4) and (6) in Definition 2.3.1, we get

$$\text{IIFWA}_\omega(\tilde{\alpha}_1, \tilde{\alpha}_2, \cdots, \tilde{\alpha}_{k+1})$$

$$= \left(\left[1 - \prod_{j=1}^{k} (1-a_j)^{\omega_j} + (1-(1-a_{k+1})^{\omega_{k+1}}) - \left(1 - \prod_{j=1}^{k} (1-a_j)^{\omega_j}\right)(1-(1-a_{k+1})^{\omega_{k+1}}), \right.\right.$$

$$\left. 1 - \prod_{j=1}^{k} (1-b_j)^{\omega_j} + (1-(1-b_{k+1})^{\omega_{k+1}}) - \left(1 - \prod_{j=1}^{k} (1-b_j)^{\omega_j}\right)(1-(1-b_{k+1})^{\omega_{k+1}}) \right],$$

$$\left.\left[\prod_{j=1}^{k+1} c_j^{\omega_j}, \prod_{j=1}^{k+1} d_j^{\omega_j} \right]\right)$$

$$= \left(\left[1 - \prod_{j=1}^{k+1} (1-a_j)^{\omega_j}, 1 - \prod_{j=1}^{k+1} (1-b_j)^{\omega_j} \right], \left[\prod_{j=1}^{k+1} c_j^{\omega_j}, \prod_{j=1}^{k+1} d_j^{\omega_j} \right] \right)$$

i.e., when $n = k + 1$, Eq.(2.8) also holds.

From (1) and (2), Eq.(2.8) holds for any n. The proof is completed.

In particular, if $a_j = b_j$ and $c_j = d_j$ for all $j = 1, 2, \cdots, n$, i.e., all $\tilde{\alpha}_j$ $(j = 1, 2, \cdots, n)$ reduce to IFNs, then the IIFWA operator reduces to an IFWA operator.

Similarly, we can prove the following theorem:

Theorem 2.3.2 (Xu, 2007h) The aggregated value by using Eq.(2.6) is also an IVIFN, and

$$\text{IIFWG}_\omega(\tilde{\alpha}_1, \tilde{\alpha}_2, \cdots, \tilde{\alpha}_n) = \left(\left[\prod_{j=1}^{n} a_j^{\omega_j}, \prod_{j=1}^{n} b_j^{\omega_j} \right], \left[1 - \prod_{j=1}^{n} (1-c_j)^{\omega_j}, 1 - \prod_{j=1}^{n} (1-d_j)^{\omega_j} \right] \right) \tag{2.9}$$

In particular, if all $\tilde{\alpha}_j$ $(j = 1, 2, \cdots, n)$ reduce to IFNs, then the IIFWG operator reduces to the IFWG operator.

Exampe 2.3.1 Suppose that $\tilde{\alpha}_1 = ([0.3, 0.5], [0.2, 0.3])$, $\tilde{\alpha}_2 = ([0.4, 0.7], [0.1, 0.2])$, $\tilde{\alpha}_3 = ([0.1, 0.2], [0.7, 0.8])$, $\tilde{\alpha}_4 = ([0.5, 0.7], [0.1, 0.3])$, and $\omega = (0.2, 0.3, 0.1, 0.4)^{\mathrm{T}}$ is the weight vector of $\tilde{\alpha}_j$ $(j = 1, 2, 3, 4)$. Then

$$\begin{aligned} \text{IIFWA}_\omega(\tilde{\alpha}_1, \tilde{\alpha}_2, \tilde{\alpha}_3, \tilde{\alpha}_4) =& ([1 - (1-0.3)^{0.2}(1-0.4)^{0.3}(1-0.1)^{0.1}(1-0.5)^{0.4}, \\ & 1 - (1-0.5)^{0.2}(1-0.7)^{0.3}(1-0.2)^{0.1}(1-0.7)^{0.4}], \\ & [0.2^{0.2} \times 0.1^{0.3} \times 0.7^{0.1} \times 0.1^{0.4}, 0.3^{0.2} \times 0.2^{0.3} \times 0.8^{0.1} \times 0.3^{0.4}]) \\ =& ([0.4009, 0.6335], [0.1395, 0.2930]) \end{aligned}$$

$$\begin{aligned} \text{IIFWG}_\omega(\tilde{\alpha}_1, \tilde{\alpha}_2, \tilde{\alpha}_3, \tilde{\alpha}_4) =& ([0.3^{0.2} \times 0.4^{0.3} \times 0.1^{0.1} \times 0.5^{0.4}, 0.5^{0.2} \times 0.7^{0.3} \times 0.2^{0.1} \times 0.7^{0.4}], \\ & [1 - (1-0.2)^{0.2}(1-0.1)^{0.3}(1-0.7)^{0.1}(1-0.1)^{0.4}, \\ & 1 - (1-0.3)^{0.2}(1-0.2)^{0.3}(1-0.8)^{0.1}(1-0.3)^{0.4}]) \end{aligned}$$

$$=([0.3594, 0.5774],\ [0.2124, 0.3572])$$

In order to rank the IVIFNs, we now introduce the score function and the accuracy function of IVIFNs:

Definition 2.3.3 (Xu, 2007h; Xu, 2010b) Let $\tilde{\alpha} = ([a, b],\ [c, d])$ be an IVIFN. Then we call

$$s(\tilde{\alpha}) = \frac{1}{2}(a - c + b - d) \tag{2.10}$$

the score of $\tilde{\alpha}$, where s is the score function of $\tilde{\alpha}$, $s(\tilde{\alpha}) \in [-1, 1]$.

Clearly, the greater the $s(\tilde{\alpha})$, the larger the $\tilde{\alpha}$. In particular, if $s(\tilde{\alpha}) = 1$, then $\tilde{\alpha}$ is the largest IVIFN: $([1, 1], [0, 0])$; If $s(\tilde{\alpha}) = -1$, then $\tilde{\alpha}$ is the smallest IVIFN: $([0, 0],\ [1, 1])$.

However, if we take $\tilde{\alpha}_1 = ([0.4, 0.5],\ [0.4, 0.5])$ and $\tilde{\alpha}_2 = ([0.2, 0.3],\ [0.2, 0.3])$, then $s(\tilde{\alpha}_1) = s(\tilde{\alpha}_2) = 0$. In this case, the score function cannot distinguish between the IVIFNs $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$. To solve this issue, Xu (2007h) defined an accuracy function:

Definition 2.3.4 (Xu, 2007h) The accuracy function of an IVIFN $\tilde{\alpha}$ is defined as:

$$h(\tilde{\alpha}) = \frac{1}{2}(a + b + c + d) \tag{2.11}$$

where $h(\tilde{\alpha}) \in [0, 1]$.

For the above IVIFNs: $\tilde{\alpha}_1 = ([0.4, 0.5],\ [0.4, 0.5])$ and $\tilde{\alpha}_2 = ([0.2, 0.3],\ [0.2, 0.3])$, by using Eq.(2.11) we can get

$$h(\tilde{\alpha}_1) = 0.9, \quad h(\tilde{\alpha}_2) = 0.5$$

Based on the above analysis, we introduce an approach to ranking the IVIFNs as follows:

Definition 2.3.5 (Xu, 2007h) Let $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ be any two IVIFNs. Then

- If $s(\tilde{\alpha}_1) < s(\tilde{\alpha}_2)$, then $\tilde{\alpha}_1 < \tilde{\alpha}_2$;
- If $s(\tilde{\alpha}_1) = s(\tilde{\alpha}_2)$, then

(1) If $h(\tilde{\alpha}_1) = h(\tilde{\alpha}_2)$, then $\tilde{\alpha}_1 \sim \tilde{\alpha}_2$;
(2) If $h(\tilde{\alpha}_1) < h(\tilde{\alpha}_2)$, then $\tilde{\alpha}_1 < \tilde{\alpha}_2$;
(3) If $h(\tilde{\alpha}_1) > h(\tilde{\alpha}_2)$, then $\tilde{\alpha}_1 > \tilde{\alpha}_2$.

Recently, Wang et al.(2009) gave another two indices called the membership uncertainty index:

$$q(\tilde{\alpha}) = b + c - a - d \tag{2.12}$$

and the hesitation uncertainty index:

$$g(\tilde{\alpha}) = b + d - a - c \tag{2.13}$$

respectively, to supplement the above procedure. In the case where $s(\tilde{\alpha}_1) = s(\tilde{\alpha}_2)$ and $h(\tilde{\alpha}_1) = h(\tilde{\alpha}_2)$, one can further consider these two indices, i.e.,

(a) If $q(\tilde{\alpha}_1) < q(\tilde{\alpha}_2)$, then $\tilde{\alpha}_1$ is larger than $\tilde{\alpha}_2$, denoted by $\tilde{\alpha}_1 > \tilde{\alpha}_2$.

(b) If $q(\tilde{\alpha}_1) = q(\tilde{\alpha}_2)$, then

(i) If $g(\tilde{\alpha}_1) < g(\tilde{\alpha}_2)$, then $\tilde{\alpha}_1$ is larger than $\tilde{\alpha}_2$, denoted by $\tilde{\alpha}_1 > \tilde{\alpha}_2$;

(ii) If $g(\tilde{\alpha}_1) = g(\tilde{\alpha}_2)$, then $\tilde{\alpha}_1$ is equal to $\tilde{\alpha}_2$, denoted by $\tilde{\alpha}_1 = \tilde{\alpha}_2$.

Based on Definitions 2.2.1 and 2.3.5, in what follows, we introduce some ordered weighted aggregation operators for IVIFNs:

Definition 2.3.6 (Xu and Chen, 2007a) Let IIFOWA : $\tilde{\Theta}^n \to \tilde{\Theta}$. If

$$\text{IIFOWA}_w(\tilde{\alpha}_1, \tilde{\alpha}_2, \cdots, \tilde{\alpha}_n) = w_1\tilde{\alpha}_{\sigma(1)} \oplus w_2\tilde{\alpha}_{\sigma(2)} \oplus \cdots \oplus w_n\tilde{\alpha}_{\sigma(n)} \tag{2.14}$$

where $w = (w_1, w_2, \cdots, w_n)^{\mathrm{T}}$ is the weighting vector associated with the function IIFOWA, with $w_j \in [0,1]$ $(j = 1, 2, \cdots, n)$ and $\sum_{j=1}^{n} w_j = 1$, $(\sigma(1), \sigma(2), \cdots, \sigma(n))$ is any permutation of $(1, 2, \cdots, n)$, such that $\tilde{\alpha}_{\sigma(j-1)} > \tilde{\alpha}_{\sigma(j)}$, for any j, then the function IIFOWA is called an interval-valued intuitionistic fuzzy ordered weighted averaging (IIFOWA) operator.

In particular, if $w = (1/n, 1/n, \cdots, 1/n)^{\mathrm{T}}$, then the function IIFOWA reduces to the IIFA operator.

Definition 2.3.7 (Xu and Chen, 2007c) Let IIFOWG : $\tilde{\Theta}^n \to \tilde{\Theta}$. If

$$\text{IIFOWG}_w(\tilde{\alpha}_1, \tilde{\alpha}_2, \cdots, \tilde{\alpha}_n) = \tilde{\alpha}_{\sigma(1)}^{w_1} \otimes \tilde{\alpha}_{\sigma(2)}^{w_2} \otimes \cdots \otimes \tilde{\alpha}_{\sigma(n)}^{w_n} \tag{2.15}$$

where $(\sigma(1), \sigma(2), \cdots, \sigma(n))$ is any permutation of $(1, 2, \cdots, n)$, such that $\tilde{\alpha}_{\sigma(j-1)} > \tilde{\alpha}_{\sigma(j)}$, for any j, then the function IIFOWG is called an interval-valued intuitionistic fuzzy ordered weighted geometric (IIFOWG) operator.

In particular, if $w = (1/n, 1/n, \cdots, 1/n)^{\mathrm{T}}$, then the function IIFOWG reduces to an interval-valued intuitionistic fuzzy geometric (IIFG) operator.

Similar to Theorems 2.3.1 and 2.3.2, we can show the following result:

Theorem 2.3.3 (Xu and Chen, 2007c) Let $(\tilde{\alpha}_{\sigma(1)}, \tilde{\alpha}_{\sigma(2)}, \cdots, \tilde{\alpha}_{\sigma(n)})$ be any permutation of $(\tilde{\alpha}_1, \tilde{\alpha}_2, \cdots, \tilde{\alpha}_n)$, such that $\tilde{\alpha}_{\sigma(j-1)} > \tilde{\alpha}_{\sigma(j)}$, for any j, and $\tilde{\alpha}_{\sigma(j)} = ([a_{\sigma(j)}, b_{\sigma(j)}], [c_{\sigma(j)}, d_{\sigma(j)}])$. Then the aggregated value by using Eq.(2.14) is an IVIFN, and

$$\text{IIFOWA}_w(\tilde{\alpha}_1, \tilde{\alpha}_2, \cdots, \tilde{\alpha}_n) = \left(\left[1 - \prod_{j=1}^{n} (1 - a_{\sigma(j)})^{w_j}, 1 - \prod_{j=1}^{n} (1 - b_{\sigma(j)})^{w_j} \right], \left[\prod_{j=1}^{n} c_{\sigma(j)}^{w_j}, \prod_{j=1}^{n} d_{\sigma(j)}^{w_j} \right] \right) \tag{2.16}$$

In particular, if all $\tilde{\alpha}_j$ $(j = 1, 2, \cdots, n)$ reduce to the IFNs, then the IIFOWA operator reduces to the IFOWA operator.

Theorem 2.3.4 (Xu and Chen, 2007c) The aggregated value by using Eq.(2.15) is also an IVIFN, and

$$\text{IIFOWG}_w(\tilde{\alpha}_1, \tilde{\alpha}_2, \cdots, \tilde{\alpha}_n) = \left(\left[\prod_{j=1}^{n} a_{\sigma(j)}^{w_j}, \prod_{j=1}^{n} b_{\sigma(j)}^{w_j} \right], \left[1 - \prod_{j=1}^{n} (1 - c_{\sigma(j)})^{w_j}, 1 - \prod_{j=1}^{n} (1 - d_{\sigma(j)})^{w_j} \right] \right) \quad (2.17)$$

In particular, if all $\tilde{\alpha}_j$ $(j = 1, 2, \cdots, n)$ reduce to the IFNs, then the IIFOWG operator reduces to the IFOWG operator.

Note The weighting vector associated with the IIFOWA and IIFOWG operators can be obtained by using one of the methods (Xu, 2005a) for determining the OWA weights.

Example 2.3.2 Suppose that $\tilde{\alpha}_1 = ([0.5, 0.7], [0.1, 0.2])$, $\tilde{\alpha}_2 = ([0.1, 0.6], [0.2, 0.4])$, $\tilde{\alpha}_3 = ([0.2, 0.3], [0.4, 0.5])$ and $\tilde{\alpha}_4 = ([0.3, 0.5], [0.2, 0.5])$. Then we can utilize Eq.(2.10) to calculate the score of $\tilde{\alpha}_i$ $(i = 1, 2, 3, 4)$:

$$s(\tilde{\alpha}_1) = \frac{1}{2}(0.5 - 0.1 + 0.7 - 0.2) = 0.45$$

$$s(\tilde{\alpha}_2) = \frac{1}{2}(0.1 - 0.2 + 0.6 - 0.4) = 0.05$$

$$s(\tilde{\alpha}_3) = \frac{1}{2}(0.2 - 0.4 + 0.3 - 0.5) = -0.2$$

$$s(\tilde{\alpha}_4) = \frac{1}{2}(0.3 - 0.2 + 0.5 - 0.5) = 0.05$$

Therefore

$$s(\tilde{\alpha}_1) > s(\tilde{\alpha}_2) = s(\tilde{\alpha}_4) > s(\tilde{\alpha}_3)$$

Since $s(\tilde{\alpha}_2)$ and $s(\tilde{\alpha}_4)$ are equal, we need to calculate the accuracy degree of $\tilde{\alpha}_2$ and $\tilde{\alpha}_4$ by using Eq.(2.11):

$$h(\tilde{\alpha}_2) = \frac{1}{2}(0.1 + 0.6 + 0.2 + 0.4) = 0.65$$

$$h(\tilde{\alpha}_4) = \frac{1}{2}(0.3 + 0.5 + 0.2 + 0.5) = 0.75$$

Then

$$\tilde{\alpha}_1 > \tilde{\alpha}_4 > \tilde{\alpha}_2 > \tilde{\alpha}_3$$

and hence

$$\tilde{a}_{\sigma(1)} = ([0.5, 0.7], [0.1, 0.2]), \quad \tilde{a}_{\sigma(2)} = ([0.3, 0.5], [0.2, 0.5])$$

$$\tilde{a}_{\sigma(3)} = ([0.1,\ 0.6],\ [0.2,\ 0.4]), \quad \tilde{a}_{\sigma(4)} = ([0.2,\ 0.3],\ [0.4,\ 0.5])$$

If we use the normal distribution based method (Xu, 2005a) to determine the weighting vector associated with the IIFOWA and IIFOWG operators, then we get $w = (0.155,\ 0.345,\ 0.345,\ 0.155)^{\mathrm{T}}$, and thus

$$\begin{aligned}\text{IIFOWA}_w(\tilde{\alpha}_1, \tilde{\alpha}_2, \tilde{\alpha}_3, \tilde{\alpha}_4) =& ([1-(1-0.5)^{0.155} \times (1-0.3)^{0.345} \times (1-0.1)^{0.345} \\ & \times (1-0.2)^{0.155}, 1-(1-0.7)^{0.155} \times (1-0.5)^{0.345} \\ & \times (1-0.6)^{0.345} \times (1-0.3)^{0.155}], \\ & [0.1^{0.155} \times 0.2^{0.345} \times 0.2^{0.345} \times 0.4^{0.155}, \\ & 0.2^{0.155} \times 0.5^{0.345} \times 0.4^{0.345} \times 0.5^{0.155}]) \\ =& ([0.2602,\ 0.5494],\ [0.2000,\ 0.4017])\end{aligned}$$

$$\begin{aligned}\text{IIFOWG}_w(\tilde{\alpha}_1, \tilde{\alpha}_2, \tilde{\alpha}_3, \tilde{\alpha}_4) =& ([0.5^{0.155} \times 0.3^{0.345} \times 0.1^{0.345} \times 0.2^{0.155}, \\ & 0.7^{0.155} \times 0.5^{0.345} \times 0.6^{0.345} \times 0.3^{0.155}], \\ & [1-(1-0.1)^{0.155} \times (1-0.2)^{0.345} \times (1-0.2)^{0.345} \\ & \times (1-0.4)^{0.155}, 1-(1-0.2)^{0.155} \times (1-0.5)^{0.345} \\ & \times (1-0.4)^{0.345} \times (1-0.5)^{0.155}]) \\ =& ([0.2087,\ 0.5183],\ [0.2208,\ 0.4273])\end{aligned}$$

If we revise the IVIFNs $\tilde{\alpha}_2 = ([0.1,\ 0.6],\ [0.2,\ 0.4])$ and $\tilde{\alpha}_4 = ([0.3,\ 0.5],\ [0.2,\ 0.5])$ in Example 2.3.1 as $\tilde{\alpha}_2' = ([0.2,\ 0.6],\ [0.2,\ 0.4])$ and $\tilde{\alpha}_4' = ([0.3,\ 0.5],\ [0.1,\ 0.5])$, then we can utilize Eq.(2.10) to calculate the scores of $\tilde{\alpha}_i'(i = 2, 4)$:

$$s(\tilde{\alpha}_2') = \frac{1}{2}(0.2 - 0.2 + 0.6 - 0.4) = 0.1$$

$$s(\tilde{\alpha}_4') = \frac{1}{2}(0.3 - 0.1 + 0.5 - 0.5) = 0.1$$

Thus $s(\tilde{\alpha}_2) = s(\tilde{\alpha}_4)$. In such a case, we need to calculate the accuracy degree of $\tilde{\alpha}_2$ and $\tilde{\alpha}_4$ by using Eq.(2.11):

$$h(\tilde{\alpha}_2) = \frac{1}{2}(0.2 + 0.6 + 0.2 + 0.4) = 0.7$$

$$h(\tilde{\alpha}_4) = \frac{1}{2}(0.3 + 0.5 + 0.1 + 0.5) = 0.7$$

i.e., $h(\tilde{\alpha}_2) = h(\tilde{\alpha}_4)$. Then we calculate the membership uncertainty index by using Eq.(2.12):

$$q(\tilde{\alpha}_2) = 0.6 + 0.2 - 0.2 - 0.4 = 0.2$$

$$q(\tilde{\alpha}_4) = 0.5 + 0.1 - 0.3 - 0.5 = -0.2$$

To conclude, $q(\tilde{\alpha}_4) < q(\tilde{\alpha}_2)$, i.e., $\tilde{\alpha}_4 > \tilde{\alpha}_2$. Accordingly,

$$\begin{aligned}\text{IIFOWA}_w(\tilde{\alpha}_1, \tilde{\alpha}'_2, \tilde{\alpha}_3, \tilde{\alpha}'_4) =&([1 - (1-0.5)^{0.155} \times (1-0.3)^{0.345} \times (1-0.1)^{0.345} \\ &\times (1-0.2)^{0.155}, 1 - (1-0.7)^{0.155} \times (1-0.5)^{0.345} \\ &\times (1-0.6)^{0.345} \times (1-0.3)^{0.155}], \\ &[0.1^{0.155} \times 0.2^{0.345} \times 0.2^{0.345} \times 0.4^{0.155}, \\ &0.2^{0.155} \times 0.5^{0.345} \times 0.4^{0.345} \times 0.5^{0.155}]) \\ =&([0.26,\ 0.62],\ [0.20,\ 0.40])\end{aligned}$$

If we use the IIFOWG operator to aggregate the IVIFNs $\tilde{\alpha}_i$ $(i = 1, 2, 3, 4)$, then

$$\begin{aligned}\text{IIFOWG}_w(\tilde{\alpha}_1, \tilde{\alpha}'_2, \tilde{\alpha}_3, \tilde{\alpha}'_4) =&([0.5^{0.155} \times 0.3^{0.345} \times 0.1^{0.345} \times 0.2^{0.155}, \\ &0.7^{0.155} \times 0.5^{0.345} \times 0.6^{0.345} \times 0.3^{0.155}], \\ &[1 - (1-0.1)^{0.155} \times (1-0.2)^{0.345} \times (1-0.2)^{0.345} \\ &\times (1-0.4)^{0.155}, 1 - (1-0.2)^{0.155} \times (1-0.5)^{0.345} \\ &\times (1-0.4)^{0.345} \times (1-0.5)^{0.155}]) \\ =&([0.21,\ 0.50],\ [0.22,\ 0.43])\end{aligned}$$

We can see that the IIFWA and IIFWG operators only consider the importance degree of each given IVIFN, while the IIFOWA and IIFOWG operators only weight the ordered position of each IVIFN instead of the IVIFN itself. Thus, these operators consider only one of the two different aspects. To overcome this limitation, Xu and Chen (2007a, 2007c) investigated the hybrid aggregation techniques for IVIFNs:

Definition 2.3.8 (Xu and Chen, 2007a) Let IIFHA : $\tilde{\Theta}^n \to \tilde{\Theta}$. Suppose

$$\text{IIFHA}_{\omega,w}(\tilde{\alpha}_1, \tilde{\alpha}_2, \cdots, \tilde{\alpha}_n) = \omega_1 \dot{\tilde{\alpha}}_{\sigma(1)} \oplus \omega_2 \dot{\tilde{\alpha}}_{\sigma(2)} \oplus \cdots \oplus \omega_n \dot{\tilde{\alpha}}_{\sigma(n)} \tag{2.18}$$

where $w = (w_1, w_2, \cdots, w_n)^{\mathrm{T}}$ is the weighting vector associated with the function IIFHA, with $w_j \in [0, 1]$, $j = 1, 2, \cdots, n$, and $\sum_{j=1}^{n} w_j = 1$. $\dot{\tilde{\alpha}}_{\sigma(j)}$ is the j-th largest of the weighted IVIFNs $\dot{\tilde{\alpha}}_i$ $(i = 1, 2, \cdots, n)$, here $\dot{\tilde{\alpha}}_i = n\omega_i\tilde{\alpha}_i$ $(i = 1, 2, \cdots, n)$, $\omega = (\omega_1, \omega_2, \cdots, \omega_n)^{\mathrm{T}}$ is the weight vector of a collection of the IVIFNs $\tilde{\alpha}_i$ $(i = 1, 2, \cdots, n)$, with $\omega_j \in [0, 1]$ $(j = 1, 2, \cdots, n)$ and $\sum_{j=1}^{n} \omega_j = 1$, and n is the balancing coefficient. Then the function IIFHA is called an interval-valued intuitionistic fuzzy hybrid averaging (IIFHA) operator. In particular, if $w = (1/n, 1/n, \cdots, 1/n)^{\mathrm{T}}$, then the function IIFHA reduces to an IIFWA operator; if $\omega = (1/n, 1/n, \cdots, 1/n)^{\mathrm{T}}$, then the function IIFHA reduces to the IIFOWA operator.

Definition 2.3.9 (Xu and Chen, 2007c) Let IIFHG : $\tilde{\Theta}^n \to \tilde{\Theta}$. Suppose

$$\text{IIFHG}_{\omega,w}(\tilde{\alpha}_1, \tilde{\alpha}_2, \cdots, \tilde{\alpha}_n) = \ddot{\tilde{\alpha}}_{\sigma(1)}^{\omega_1} \otimes \ddot{\tilde{\alpha}}_{\sigma(2)}^{\omega_2} \otimes \cdots \otimes \ddot{\tilde{\alpha}}_{\sigma(n)}^{\omega_n} \tag{2.19}$$

where $\ddot{\tilde{\alpha}}_{\sigma(j)}$ is the j-th largest of the exponential weighted IVIFNs $\ddot{\tilde{\alpha}}_i$ $(i = 1, 2, \cdots, n)$, here $\ddot{\tilde{\alpha}}_i = \tilde{\alpha}_i^{n\omega_i}$ $(i = 1, 2, \cdots, n)$, and n is the balancing coefficient. Then the function IIFHG is called an interval-valued intuitionistic fuzzy hybrid geometric (IIFHG) operator. In particular, if $w = (1/n, 1/n, \cdots, 1/n)^{\mathrm{T}}$, then the function IIFHG reduces to the IIFWG operator; if $\omega = (1/n, 1/n, \cdots, 1/n)^{\mathrm{T}}$, then the function IIFHG reduces to the IIFOWG operator.

Clearly, the IIFHA operator generalizes both the IIFWA and IIFOWA operators, and the IIFHG operator generalizes both the IIFWG and IIFOWG operators. They can consider not only the importance of each given IVIFN itself, but also the importance of the ordered position of the IVIFN.

Similar to Theorems 2.3.3 and 2.3.4, we can get the following results:

Theorem 2.3.5 (Xu and Chen, 2007a) Assume that $\dot{\tilde{\alpha}}_i = ([\dot{a}_i, \dot{b}_i],\ [\dot{c}_i, \dot{d}_i])$ $(i = 1, 2, \cdots, n)$ and $\dot{\tilde{\alpha}}_{\sigma(j)} = ([\dot{a}_{\sigma(j)}, \dot{b}_{\sigma(j)}],\ [\dot{c}_{\sigma(j)}, \dot{d}_{\sigma(j)}])$ $(j = 1, 2, \cdots, n)$. Then the aggregated value by using Eq.(2.18) is an IVIFN, and

$$\text{IIFHA}_{\omega,w}(\tilde{\alpha}_1, \tilde{\alpha}_2, \cdots, \tilde{\alpha}_n) = \left(\left[1 - \prod_{j=1}^{n}(1 - \dot{a}_{\sigma(j)})^{w_j}, 1 - \prod_{j=1}^{n}(1 - \dot{b}_{\sigma(j)})^{w_j}\right],\right.$$
$$\left.\left[\prod_{j=1}^{n}\dot{c}_{\sigma(j)}^{w_j}, \prod_{j=1}^{n}\dot{d}_{\sigma(j)}^{w_j}\right]\right) \tag{2.20}$$

In particular, if all $\tilde{\alpha}_j$ $(j = 1, 2, \cdots, n)$ reduce to the IFNs, then Eq.(2.20) reduces to Eq.(1.58).

Theorem 2.3.6 (Xu and Chen, 2007c) Suppose that $\ddot{\tilde{\alpha}}_i = ([\ddot{a}_i, \ddot{b}_i],\ [\ddot{c}_i, \ddot{d}_i])$ $(i = 1, 2, \cdots, n)$ and $\ddot{\tilde{\alpha}}_{\sigma(j)} = ([\ddot{a}_{\sigma(j)}, \ddot{b}_{\sigma(j)}],\ [\ddot{c}_{\sigma(j)}, \ddot{d}_{\sigma(j)}])$ $(j = 1, 2, \cdots, n)$. Then the aggregated value by using Eq.(2.19) is an IVIFN, and

$$\text{IIFHG}_{\omega,w}(\tilde{\alpha}_1, \tilde{\alpha}_2, \cdots, \tilde{\alpha}_n) = \left(\left[\prod_{j=1}^{n}\ddot{a}_{\sigma(j)}^{w_j}, \prod_{j=1}^{n}\ddot{b}_{\sigma(j)}^{w_j}\right],\right.$$
$$\left.\left[1 - \prod_{j=1}^{n}(1 - \ddot{c}_{\sigma(j)})^{w_j}, 1 - \prod_{j=1}^{n}(1 - \ddot{d}_{\sigma(j)})^{w_j}\right]\right) \tag{2.21}$$

In particular, if all $\tilde{\alpha}_j$ $(j = 1, 2, \cdots, n)$ reduce to the IFNs, then Eq.(2.21) reduces to Eq.(1.61).

Example 2.3.3 Suppose that $\tilde{\alpha}_1 = ([0.2, 0.3],\ [0.5, 0.6])$, $\tilde{\alpha}_2 = ([0.7, 0.8],\ [0.1, 0.2])$, $\tilde{\alpha}_3 = ([0.5, 0.6],\ [0.2, 0.4])$, $\tilde{\alpha}_4 = ([0.3, 0.4],\ [0.4, 0.6])$ and $\tilde{\alpha}_5 = ([0.6, 0.7],\ [0.2, 0.3])$, and let $\omega = (0.25, 0.20, 0.15, 0.18, 0.22)^{\mathrm{T}}$ be the weight vector of $\tilde{\alpha}_j$ $(j = 1, 2, \cdots, 5)$. If we utilize the IIFHA operator to aggregate the given data, then we first determine the weighting vector $w = (0.112, 0.236, 0.304, 0.236, 0.112)^{\mathrm{T}}$ associated with the IIFHA operator by using the normal distribution based method (Xu, 2005a). After that, by the operational law (6) in Definition 2.2.1, we have

$$\begin{aligned}\dot{\tilde{\alpha}}_1 &= ([1-(1-0.2)^{5\times 0.25}, 1-(1-0.3)^{5\times 0.25}],\ [0.5^{5\times 0.25}, 0.6^{5\times 0.25}]) \\ &= ([0.2434, 0.3597],\ [0.4204, 0.5281])\end{aligned}$$

Similarly, we get

$$\begin{aligned}\dot{\tilde{\alpha}}_2 &= ([0.7, 0.8],\ [0.1, 0.2]) \\ \dot{\tilde{\alpha}}_3 &= ([0.4054, 0.4970],\ [0.2991, 0.5030]) \\ \dot{\tilde{\alpha}}_4 &= ([0.2746, 0.3686],\ [0.4384, 0.6314]) \\ \dot{\tilde{\alpha}}_5 &= ([0.6350, 0.7340],\ [0.1703, 0.2660])\end{aligned}$$

Then we can utilize Eq.(2.10) to calculate the score of $\dot{\tilde{\alpha}}_i$ $(i = 1, 2, \cdots, 5)$:

$$\begin{aligned}s(\dot{\tilde{\alpha}}_1) &= \frac{1}{2}(0.2434 - 0.4204 + 0.3597 - 0.5281) = -0.3450 \\ s(\dot{\tilde{\alpha}}_2) &= \frac{1}{2}(0.7 - 0.1 + 0.8 - 0.2) = 0.6000 \\ s(\dot{\tilde{\alpha}}_3) &= \frac{1}{2}(0.4054 - 0.2991 + 0.4970 - 0.5030) = 0.0502 \\ s(\dot{\tilde{\alpha}}_4) &= \frac{1}{2}(0.2746 - 0.4384 + 0.3686 - 0.6314) = -0.2133 \\ s(\dot{\tilde{\alpha}}_5) &= \frac{1}{2}(0.6350 - 0.1703 + 0.7340 - 0.2660) = 0.4664\end{aligned}$$

Since

$$s(\dot{\tilde{\alpha}}_2) > s(\dot{\tilde{\alpha}}_5) > s(\dot{\tilde{\alpha}}_3) > s(\dot{\tilde{\alpha}}_4) > s(\dot{\tilde{\alpha}}_1)$$

we have

$$\begin{aligned}\dot{\tilde{\alpha}}_{\sigma(1)} &= ([0.7, 0.8]\ , [0.1, 0.2]) \\ \dot{\tilde{\alpha}}_{\sigma(2)} &= ([0.6350, 0.7340],\ [0.1703, 0.2660]) \\ \dot{\tilde{\alpha}}_{\sigma(3)} &= ([0.4054, 0.4970],\ [0.2991, 0.5030]) \\ \dot{\tilde{\alpha}}_{\sigma(4)} &= ([0.2746, 0.3686],\ [0.4384, 0.6314]) \\ \dot{\tilde{\alpha}}_{\sigma(5)} &= ([0.2434, 0.3597],\ [0.4204, 0.5281])\end{aligned}$$

and by Eq.(2.20), we can obtain

$$\mathrm{IIFHA}_{\omega,w}(\tilde{\alpha}_1, \tilde{\alpha}_2, \tilde{\alpha}_3, \tilde{\alpha}_4, \tilde{\alpha}_5)$$

$$
\begin{aligned}
=&([1-(1-0.7)^{0.112}\times(1-0.6350)^{0.236}\times(1-0.4054)^{0.304}\\
&\times(1-0.2746)^{0.236}\times(1-0.2434)^{0.112},\ 1-(1-0.8000)^{0.112}\times(1-0.7340)^{0.236}\\
&\times(1-0.4970)^{0.304}\times(1-0.3686)^{0.236}\times(1-0.3597)^{0.112}],\\
&[0.1^{0.112}\times 0.1703^{0.236}\times 0.2991^{0.304}\times 0.4384^{0.236}\times 0.4204^{0.112},\\
&0.2^{0.112}\times 0.2660^{0.236}\times 0.5030^{0.304}\times 0.6314^{0.236}\times 0.5281^{0.112}])\\
=&([0.4715, 0.5769],\ [0.2634, 0.4141])
\end{aligned}
$$

If we utilize the IIFHG operator to aggregate the IVIFNs $\tilde{\alpha}_j$ $(j=1,2,\cdots,5)$, then similar to the IIFHA operator, we first determine the exponential weighted vector $w=(0.112,\ 0.236,\ 0.304,\ 0.236,\ 0.112)^{\mathrm{T}}$ associated with the IIFHG operator by using the normal distribution based method (Xu, 2005a). After that, by the operational law (7) in Definition 2.2.1, we have

$$
\begin{aligned}
\ddot{\tilde{\alpha}}_1 &= ([0.2^{5\times 0.25}, 0.3^{5\times 0.25}],\ [1-(1-0.5)^{5\times 0.25}, 1-(1-0.6)^{5\times 0.25}])\\
&= ([0.1337, 0.2220],\ [0.5796, 0.6819])
\end{aligned}
$$

In a similar way, we get

$$
\begin{aligned}
\ddot{\tilde{\alpha}}_2 &= ([0.7, 0.8],\ [0.1, 0.2])\\
\ddot{\tilde{\alpha}}_3 &= ([0.5946, 0.6817],\ [0.1541, 0.3183])\\
\ddot{\tilde{\alpha}}_4 &= ([0.3384, 0.4384],\ [0.3686, 0.5616])\\
\ddot{\tilde{\alpha}}_5 &= ([0.6314, 0.7254],\ [0.2177, 0.3245])
\end{aligned}
$$

Then we can utilize Eq.(2.10) to calculate the scores of $\ddot{\tilde{\alpha}}_i$ $(i=1,2,\cdots,5)$:

$$
\begin{aligned}
s(\ddot{\tilde{\alpha}}_1) &= \frac{1}{2}(0.1337-0.5796+0.2220-0.6819) = -0.4529\\
s(\ddot{\tilde{\alpha}}_2) &= \frac{1}{2}(0.7-0.1+0.8-0.2) = 0.6000\\
s(\ddot{\tilde{\alpha}}_3) &= \frac{1}{2}(0.5946-0.15410+0.6817-0.3182) = 0.4020\\
s(\ddot{\tilde{\alpha}}_4) &= \frac{1}{2}(0.3384-0.3686+0.4384-0.5616) = -0.0767\\
s(\ddot{\tilde{\alpha}}_5) &= \frac{1}{2}(0.6314-0.2177+0.7254-0.3245) = 0.4073
\end{aligned}
$$

Since

$$
s(\ddot{\tilde{\alpha}}_2) > s(\ddot{\tilde{\alpha}}_5) > s(\ddot{\tilde{\alpha}}_3) > s(\ddot{\tilde{\alpha}}_4) > s(\ddot{\tilde{\alpha}}_1)
$$

we have

$$
\ddot{\tilde{\alpha}}_{\sigma(1)} = ([0.7, 0.8],\ [0.1, 0.2])
$$

$$\ddot{\tilde{\alpha}}_{\sigma(2)} = ([0.6314, 0.7254],\ [0.2177, 0.3245])$$
$$\ddot{\tilde{\alpha}}_{\sigma(3)} = ([0.5946, 0.6817],\ [0.1541, 0.3183])$$
$$\ddot{\tilde{\alpha}}_{\sigma(4)} = ([0.3384, 0.4384],\ [0.3686, 0.5616])$$
$$\ddot{\tilde{\alpha}}_{\sigma(5)} = ([0.1337, 0.2220],\ [0.5796, 0.6819])$$

Accordingly, by Eq.(2.20) we get

$$\begin{aligned}\text{IIFHG}_{\omega,w}(\tilde{a}_1, \tilde{a}_2, \tilde{a}_3, \tilde{a}_4, \tilde{a}_5) =&([0.7^{0.112} \times 0.6314^{0.236} \times 0.5964^{0.304} \\ &\times 0.3384^{0.236} \times 0.1337^{0.112},\ 0.8^{0.112} \times 0.7254^{0.236} \\ &\times 0.6817^{0.304} \times 0.4384^{0.236} \times 0.2220^{0.112}], \\ &[1-(1-0.1)^{0.112} \times (1-0.2177)^{0.236} \times (1-0.1541)^{0.304} \\ &\times (1-0.3686)^{0.236} \times (1-0.5796)^{0.112}, 1-(1-0.2)^{0.112} \\ &\times (1-0.3245)^{0.236} \times (1-0.3183)^{0.304} \\ &\times (1-0.5616)^{0.236} \times (1-0.6819)^{0.112}]) \\ =&([0.4554, 0.5597],\ [0.2783, 0.4270])\end{aligned}$$

We now apply the IIFWA and IIFWG operators to multi-attribute decision making problems in interval-valued intuitionistic fuzzy environments:

For a multi-attribute decision making problem, let Y, G and ω be defined as in Section 1.3. Suppose that the characteristic information on the alternative Y_i is expressed in IVIFSs:

$$Y_i = \{\langle G_j, \tilde{\mu}_{Y_i}(G_j), \tilde{\nu}_{Y_i}(G_j)\rangle |\, G_j \in G\}, \quad i = 1, 2, \cdots, n$$

where $\tilde{\mu}_{Y_i}(G_j)$ indicates the degree that the alternative Y_i satisfies the attribute G_j, and $\tilde{\nu}_{Y_i}(G_j)$ indicates the degree that the alternative Y_i does not satisfy the attribute G_j. Here, $\tilde{\mu}_{Y_i}(G_j)$ and $\tilde{\nu}_{Y_i}(G_j)$ are given in the value ranges, i.e., interval numbers, and

$$\tilde{\mu}_{Y_i}(G_j) \subset [0, 1], \quad \tilde{\nu}_{Y_i}(G_j) \subset [0, 1], \quad \sup \tilde{\mu}_{Y_i}(G_j) + \sup \tilde{\nu}_{Y_i}(x) \leqslant 1$$

For convenience, let $\tilde{\mu}_{Y_i}(G_j) = \tilde{\mu}_{ij} = [\tilde{\mu}_{ij}^{\rm L}, \tilde{\mu}_{ij}^{\rm U}]$ and $\tilde{\nu}_{Y_i}(G_j) = \tilde{\nu}_{ij} = [\tilde{\nu}_{ij}^{\rm L}, \tilde{\nu}_{ij}^{\rm U}]$, $i = 1, 2, \cdots, n$; $j = 1, 2, \cdots, m$. Consequently, we can get the interval-valued intuitionistic fuzzy decision matrix $\tilde{R}' = (\tilde{r}'_{ij})_{n\times m}$, where $\tilde{r}'_{ij} = (\tilde{\mu}'_{ij}, \tilde{\nu}'_{ij}) = ([\tilde{\mu}'^{\rm L}_{ij}, \tilde{\mu}'^{\rm U}_{ij}],\ [\tilde{\nu}'^{\rm L}_{ij}, \tilde{\nu}'^{\rm U}_{ij}])$.

In general, there are two types of attributes, i.e., benefit attributes and cost attributes. Similar to Section 1.3, we can normalize $\tilde{R}' = (\tilde{r}'_{ij})_{n\times m}$ into the interval-valued intuitionistic fuzzy decision matrix $\tilde{R} = (\tilde{r}_{ij})_{n\times m}$, where

$$\tilde{r}_{ij} = (\tilde{\mu}_{ij}, \tilde{v}_{ij}) = ([\tilde{\mu}_{ij}^{\rm L}, \tilde{\mu}_{ij}^{\rm U}],\ [\tilde{v}_{ij}^{\rm L}, \tilde{v}_{ij}^{\rm U}]) = \begin{cases} \tilde{r}'_{ij}, & \text{for benefit attribute } G_j, \\ \bar{\tilde{r}}'_{ij}, & \text{for cos}\,t \text{ attribute } G_j, \end{cases}$$

$$i=1,2,\cdots,n;\ j=1,2,\cdots,m \tag{2.22}$$

and $\bar{\tilde{r}}'_{ij}$ is the complement of $\tilde{r}'_{ij}$, i.e., $\bar{\tilde{r}}'_{ij}=(\tilde{\nu}'_{ij},\tilde{\mu}'^{(k)}_{ij})=([\tilde{\nu}'^{\rm L}_{ij},\tilde{\nu}'^{\rm U}_{ij}],\ [\tilde{\mu}'^{\rm L}_{ij},\tilde{\mu}'^{\rm U}_{ij}])$.

Xu (2007h) utilized the IIFWA (or IIFWG) operator to develop an approach to multi-attribute decision making with interval-valued intuitionistic fuzzy information, which involves the following steps:

Step 1 Utilize the IIFWA operator:

$$\dot{\tilde{r}}_i=\mathrm{IIFWA}_\omega(\tilde{r}_{i1},\tilde{r}_{i2},\cdots,\tilde{r}_{im}),\quad i=1,2,\cdots,n \tag{2.23}$$

or the IIFWG operator:

$$\ddot{\tilde{r}}_i=\mathrm{IIFWG}_\omega(\tilde{r}_{i1},\tilde{r}_{i2},\cdots,\tilde{r}_{im}),\quad i=1,2,\cdots,n \tag{2.24}$$

to aggregate all the elements $\tilde{r}_{ij}$ $(j=1,2,\cdots,m)$ in the i-th line of the interval-valued intuitionistic fuzzy decision matrix $\tilde{R}$, and get the overall values $\dot{\tilde{r}}_i$ (or $\ddot{\tilde{r}}_i$) $(i=1,2,\cdots,n)$ corresponding to the alternatives Y_i $(i=1,2,\cdots,n)$.

Step 2 Utilize the score function (2.10) and the accuracy function (2.11) to calculate the scores $s(\dot{\tilde{r}}_i)$ (or $s(\dot{\tilde{r}}_i)$) $(i=1,2,\cdots,n)$ and the accuracy degrees $h(\dot{\tilde{r}}_i)$ (or $h(\ddot{\tilde{r}}_i)$) $(i=1,2,\cdots,n)$ of $\dot{\tilde{r}}_i$ (or $\ddot{\tilde{r}}_i$) $(i=1,2,\cdots,n)$.

Step 3 By Definition 2.3.5, rank the alternatives Y_i $(i=1,2,\cdots,n)$, and then derive the most desirable alternative.

Example 2.3.4 (Xu, 2007h) A practical use of the developed approach involves the evaluation of staff for tenure and promotion in a unit. The attributes which are considered here in evaluation of five candidates Y_i $(i=1,2,\cdots,5)$ are: ① G_1: Moral level; ② G_2: Work attitude; ③ G_3: Working style; ④ G_4: Literacy level and knowledge structure; ⑤ G_5: Leadership ability; and ⑥ G_6: Exploration capacity. The weight vector of the attributes G_j $(j=1,2,\cdots,6)$ is $\omega=(0.20,\ 0.10,\ 0.25,\ 0.10,\ 0.15,\ 0.20)^{\rm T}$. The evaluation information on the candidates Y_i $(i=1,2,\cdots,5)$ with respect to the attributes G_j $(i=1,2,\cdots,6)$ is characterized by IVIFNs, which are contained in the interval-valued intuitionistic fuzzy decision matrix $\tilde{R}$, as shown in Table 2.1:

Table 2.1 Interval-valued intuitionistic fuzzy decision matrix $\tilde{R}$

	G_1	G_2	G_3
Y_1	([0.2,0.3],[0.4,0.5])	([0.5,0.6],[0.1,0.3])	([0.4,0.5], [0.2,0.4])
Y_2	([0.6,0.7], [0.2,0.3])	([0.5,0.6], [0.1,0.3])	([0.6,0.7], [0.2,0.3])
Y_3	([0.4,0.5], [0.3,0.4])	([0.7,0.8], [0.1,0.3])	([0.5,0.6], [0.3,0.4])
Y_4	([0.6,0.7], [0.2,0.3])	([0.5,0.7], [0.1,0.3])	([0.7,0.8], [0.1,0.2])
Y_5	([0.5,0.6],[0.3,0.5])	([0.3,0.4],[0.3,0.5])	([0.6,0.7],[0.1,0.3])

	G_4	G_5	G_6
Y_1	([0.7,0.8],[0.1,0.2])	([0.1,0.3], [0.5,0.6])	([0.5,0.7], [0.2,0.3])
Y_2	([0.6,0.7], [0.1,0.2])	([0.3,0.4], [0.5,0.6])	([0.4,0.7], [0.1,0.2])
Y_3	([0.6,0.7], [0.1,0.3])	([0.4,0.5], [0.3,0.4])	([0.3,0.5], [0.1,0.3])
Y_4	([0.3,0.4], [0.1,0.2])	([0.5,0.6], [0.1,0.3])	([0.7,0.8], [0.1,0.2])
Y_5	([0.6,0.8], [0.1,0.2])	([0.6,0.7], [0.2,0.3])	([0.5,0.6], [0.2,0.4])

In what follows, we use the approach developed to rank and select the candidates Y_i $(i = 1, 2, \cdots, 5)$:

Step 1 Utilize the IIFWA operator (2.23) to aggregate all the elements $\tilde{r}_{ij}$ $(j = 1, 2, \cdots, 6)$ in the i-th line of the interval-valued intuitionistic fuzzy decision matrix $\tilde{R}$, and get the overall values $\dot{\tilde{r}}_i$ $(i = 1, 2, \cdots, 5)$ corresponding to the candidates Y_i $(i = 1, 2, \cdots, 5)$:

$$\dot{\tilde{r}}_1 = ([0.4165, 0.5597], [0.2459, 0.3804])$$

$$\dot{\tilde{r}}_2 = ([0.5176, 0.6574], [0.1739, 0.2947])$$

$$\dot{\tilde{r}}_3 = ([0.4703, 0.5900], [0.1933, 0.3424])$$

$$\dot{\tilde{r}}_4 = ([0.5407, 0.6702], [0.1149, 0.2400])$$

$$\dot{\tilde{r}}_5 = ([0.5375, 0.6536], [0.1772, 0.3557])$$

Step 2 Utilize the score function (2.10) to calculate the scores $s(\dot{\tilde{r}}_i)$ $(i = 1, 2, \cdots, 5)$ of $\dot{\tilde{r}}_i$ $(i = 1, 2, \cdots, 5)$:

$$s(\dot{\tilde{r}}_1) = 0.1749, \quad s(\dot{\tilde{r}}_2) = 0.3532, \quad s(\dot{\tilde{r}}_3) = 0.2623$$

$$s(\dot{\tilde{r}}_4) = 0.4280, \quad s(\dot{\tilde{r}}_5) = 0.3291$$

Step 3 According to $s(\dot{\tilde{r}}_i)$ $(i = 1, 2, \cdots, 5)$, rank the candidates Y_i $(i = 1, 2, \cdots, 5)$:

$$Y_4 \succ Y_2 \succ Y_5 \succ Y_3 \succ Y_1$$

Therefore, the best alternative is Y_4.

We can utilize the IIFWG operator (2.24) to aggregate all the elements $\tilde{r}_{ij}$ $(j = 1, 2, \cdots, 6)$ in the i-th line of the interval-valued intuitionistic fuzzy decision matrix $\tilde{R}$, and get the overall values $\ddot{\tilde{r}}_i$ $(i = 1, 2, \cdots, 5)$ corresponding to the candidates Y_i $(i = 1, 2, \cdots, 5)$:

$$\ddot{\tilde{r}}_1 = ([0.3257, 0.4848],\ [0.2878, 0.4132])$$

$$\ddot{\tilde{r}}_2 = ([0.4896, 0.6338],\ [0.2185, 0.3301])$$

$$\ddot{\tilde{r}}_3 = ([0.4398, 0.5673],\ [0.2260, 0.3533])$$

$$\ddot{\tilde{r}}_4 = ([0.4972, 0.6190],\ [0.1210, 0.2467])$$

$$\ddot{\tilde{r}}_5 = ([0.5204, 0.6307],\ [0.1991, 0.3782])$$

Then we can utilize the score function (2.10) to calculate the scores $s(\ddot{\tilde{r}}_i)$ $(i = 1, 2, \cdots, 5)$ of $\ddot{\tilde{r}}_i$ $(i = 1, 2, \cdots, 5)$:

$$s(\ddot{\tilde{r}}_1) = 0.0547, \quad s(\ddot{\tilde{r}}_2) = 0.2874, \quad s(\ddot{\tilde{r}}_3) = 0.2139$$

$$s(\ddot{\tilde{r}}_4) = 0.3742, \quad s(\ddot{\tilde{r}}_5) = 0.2869$$

According to $s(\ddot{\tilde{r}}_i)$ $(i = 1, 2, \cdots, 5)$, we can rank the candidates Y_i $(i = 1, 2, \cdots, 5)$:

$$Y_4 \succ Y_2 \succ Y_5 \succ Y_3 \succ Y_1$$

To conclude, Y_4 is also the best alternative.

Below we consider the multi-attribute group decision making problem with interval-valued intuitionistic fuzzy information:

Let Y, E, ξ and ω be defined as in Section 1.3, and let $\tilde{R}_k = (\tilde{r}_{ij}^{(k)})_{n\times m}$ be the interval-valued intuitionistic fuzzy decision matrix given by the decision maker E_k, where $\tilde{r}_{ij}^{(k)} = (\tilde{\mu}_{ij}^{(k)}, \tilde{\nu}_{ij}^{(k)})$ is the attribute value given by the decision maker E_k for the alternative Y_i with respect to the attribute $G_j \in G$. $\tilde{\mu}_{ij}^{(k)}$ indicates the degree that the alternative Y_i satisfies the attribute G_j, and $\tilde{\nu}_{ij}^{(k)}$ indicates the degree that the alternative Y_i does not satisfy the attribute G_j. Here $\tilde{\mu}_{ij}^{(k)}$ and $\tilde{\nu}_{ij}^{(k)}$ are given in the value ranges, denoted by $\tilde{\mu}_{ij}^{(k)} = [\tilde{\mu}_{ij}^{\mathrm{L}(k)}, \tilde{\mu}_{ij}^{\mathrm{U}(k)}]$, $\tilde{\nu}_{ij}^{(k)} = [\tilde{\nu}_{ij}^{\mathrm{L}(k)}, \tilde{\nu}_{ij}^{\mathrm{U}(k)}]$, and

$$\tilde{\mu}_{ij}^{(k)} \subset [0,1], \quad \tilde{\nu}_{ij}^{(k)} \subset [0,1], \quad \tilde{\mu}_{ij}^{\mathrm{U}(k)} + \tilde{\nu}_{ij}^{\mathrm{U}(k)} \leqslant 1, \quad i = 1, 2, \cdots, n; \; j = 1, 2, \cdots, m;$$
$$k = 1, 2, \cdots, l \tag{2.25}$$

In order to fuse the individual opinions into the group one, we utilize the IIFHA operator:

$$\dot{\tilde{r}}_{ij} = \mathrm{IIFHA}_{\xi,w}(\tilde{r}_{ij}^{(1)}, \tilde{r}_{ij}^{(2)}, \cdots, \tilde{r}_{ij}^{(l)}), \quad i = 1, 2, \cdots, n; \; j = 1, 2, \cdots, m \tag{2.26}$$

or the IIFHG operator:

$$\ddot{\tilde{r}}_{ij} = \mathrm{IIFHG}_{\xi,w}(\tilde{r}_{ij}^{(1)}, \tilde{r}_{ij}^{(2)}, \cdots, \tilde{r}_{ij}^{(l)}), \quad i = 1, 2, \cdots, n; \; j = 1, 2, \cdots, m \tag{2.27}$$

to aggregate all the interval-valued intuitionistic fuzzy decision matrices $\tilde{R}_k = (\tilde{R}_{ij}^{(k)})_{n\times m}$ $(k = 1, 2, \cdots, l)$ into the collective interval-valued intuitionistic fuzzy decision matrix $\dot{\tilde{R}} = (\dot{\tilde{r}}_{ij})_{n\times m}$ (or $\ddot{\tilde{R}} = (\ddot{\tilde{r}}_{ij})_{n\times m}$), where $\dot{\tilde{r}}_{ij} = (\dot{\tilde{\mu}}_{ij}, \dot{\tilde{\nu}}_{ij})$, $\ddot{\tilde{r}}_{ij} = (\ddot{\tilde{\mu}}_{ij}, \ddot{\tilde{\nu}}_{ij})$ $(i = 1, 2, \cdots, n; \; j = 1, 2, \cdots, m)$, $w = (w_1, w_2, \cdots, w_m)^{\mathrm{T}}$ is the weighting vector associated with the IIFHA and IIFHG operators, with $w_j \in [0,1]$, $j = 1, 2, \cdots, m$, and $\sum_{j=1}^{m} w_j = 1$, which can be determined by the normal distribution based method or the others (Xu, 2005a).

Based on the collective interval-valued intuitionistic fuzzy decision matrix $\dot{\tilde{R}} = (\dot{\tilde{r}}_{ij})_{n\times m}$ (or $\ddot{\tilde{R}} = (\ddot{\tilde{r}}_{ij})_{n\times m}$), we utilize the IIFWA (or IIFWG) operator:

$$\dot{\tilde{r}}_i = \mathrm{IIFWA}_{\omega}(\dot{\tilde{r}}_{i1}, \dot{\tilde{r}}_{i2}, \cdots, \dot{\tilde{r}}_{im}), \quad i = 1, 2, \cdots, n \tag{2.28}$$

or

$$\ddot{\tilde{r}}_i = \mathrm{IIFWG}_{\omega}(\ddot{\tilde{r}}_{i1}, \ddot{\tilde{r}}_{i2}, \cdots, \ddot{\tilde{r}}_{im}), \quad i = 1, 2, \cdots, n \tag{2.29}$$

to get the overall attribute value of the alternative Y_i.

After that, we utilize Eq.(2.10) to calculate the scores $s(\dot{\tilde{r}}_i)$ (or $s(\ddot{\tilde{r}}_i)$) $(i = 1, 2, \cdots, n)$ of the overall attribute values $\dot{\tilde{r}}_i$ (or $\ddot{\tilde{r}}_i$) $(i = 1, 2, \cdots, n)$ of the alternatives Y_i $(i = 1, 2, \cdots, n)$, and then utilize the scores $s(\dot{\tilde{r}}_i)$ (or $s(\ddot{\tilde{r}}_i)$) $(i = 1, 2, \cdots, n)$ to rank the alternatives Y_i $(i = 1, 2, \cdots, n)$. If two scores $s(\dot{\tilde{r}}_i)$ (or $s(\ddot{\tilde{r}}_i)$) and $s(\dot{\tilde{r}}_j)$ (or $s(\ddot{\tilde{r}}_j)$) are equal, then we need to calculate respectively the accuracy degree $h(\dot{\tilde{r}}_i)$ (or $h(\ddot{\tilde{r}}_i)$) and $h(\dot{\tilde{r}}_j)$ (or $h(\ddot{\tilde{r}}_j)$) of the overall attribute values $\dot{\tilde{r}}_i$ (or $\ddot{\tilde{r}}_i$) and $\dot{\tilde{r}}_j$ (or $\ddot{\tilde{r}}_j$). After that, we can utilize $h(\dot{\tilde{r}}_i)$ (or $h(\ddot{\tilde{r}}_i)$) and $h(\dot{\tilde{r}}_j)$ (or $h(\ddot{\tilde{r}}_j)$) to rank the alternatives Y_i and Y_j. In the case where both the score function and the accuracy function cannot be used to distinguish between the overall attribute values $\dot{\tilde{r}}_i$ (or $\ddot{\tilde{r}}_i$) and $\dot{\tilde{r}}_j$ (or $\ddot{\tilde{r}}_j$)), Wang et al.(2009)' s method can be applied (see Section 2.3).

Example 2.3.5 We use Example 1.3.5 to illustrate the above approach. Suppose that the experts E_k $(k = 1, 2, 3, 4)$ (whose weight vector is $\xi = (0.3, 0.2, 0.3, 0.2)^{\mathrm{T}}$) utilize the IVIFNs $r_{ij}^{\prime(k)}$ $(i, j = 1, 2, 3, 4, 5)$ to describe the characteristics of the potential global suppliers Y_i $(i = 1, 2, 3, 4, 5)$ with respect to the attributes G_j $(j = 1, 2, 3, 4, 5)$; see Tables 2.2-2.5 (i.e., the interval-valued intuitionistic fuzzy decision matrices $\tilde{R}'_k = (\tilde{r}_{ij}^{\prime(k)})_{5\times 5}$ $(k = 1, 2, 3, 4)$), and $\omega = (0.2, 0.15, 0.2, 0.3, 0.15)^{\mathrm{T}}$ is the weight vector of the attributes G_j $(j = 1, 2, 3, 4, 5)$:

Table 2.2 Interval-valued intuitionistic fuzzy decision matrix $\widetilde{R'_1}$

	G_1	G_2	G_3	G_4	G_5
Y_1	([0.3,0.4],[0.4,0.6])	([0.5,0.6],[0.1,0.2])	([0.6,0.7],[0.2,0.3])	([0.7,0.8],[0.0,0.1])	([0.2,0.3],[0.6,0.7])
Y_2	([0.6,0.8],[0.1,0.2])	([0.6,0.7],[0.2,0.3])	([0.2,0.3],[0.4,0.6])	([0.5,0.6],[0.1,0.3])	([0.0,0.2],[0.7,0.8])
Y_3	([0.5,0.8],[0.1,0.2])	([0.7,0.8],[0.0,0.1])	([0.5,0.5],[0.4,0.5])	([0.2,0.3],[0.2,0.4])	([0.2,0.3],[0.4,0.6])
Y_4	([0.2,0.3],[0.4,0.5])	([0.5,0.7],[0.1,0.3])	([0.6,0.7],[0.1,0.2])	([0.4,0.5],[0.1,0.3])	([0.0,0.1],[0.6,0.9])
Y_5	([0.6,0.8],[0.1,0.2])	([0.3,0.5],[0.4,0.5])	([0.4,0.6],[0.3,0.4])	([0.6,0.8],[0.1,0.2])	([0.2,0.3],[0.5,0.6])

Table 2.3 Interval-valued intuitionistic fuzzy decision matrix $\widetilde{R'_2}$

	G_1	G_2	G_3	G_4	G_5
Y_1	([0.4,0.5],[0.3,0.4])	([0.5,0.6],[0.1,0.2])	([0.6,0.7],[0.2,0.3])	([0.7,0.8],[0.1,0.2])	([0.0,0.2],[0.7,0.8])
Y_2	([0.6,0.8],[0.1,0.2])	([0.5,0.6],[0.3,0.4])	([0.4,0.5],[0.3,0.4])	([0.4,0.6],[0.3,0.4])	([0.1,0.3],[0.4,0.7])
Y_3	([0.5,0.6],[0.3,0.4])	([0.5,0.7],[0.1,0.2])	([0.5,0.6],[0.3,0.4])	([0.3,0.4],[0.2,0.5])	([0.2,0.3],[0.6,0.7])
Y_4	([0.5,0.6],[0.3,0.4])	([0.7,0.8],[0.0,0.1])	([0.4,0.5][0.2,0.4])	([0.5,0.7],[0.1,0.2])	([0.2,0.3],[0.5,0.7])
Y_5	([0.4,0.7],[0.2,0.3])	([0.5,0.6][0.2,0.4])	([0.3,0.6],[0.3,0.4])	([0.6,0.8],[0.1,0.2])	([0.2,0.3],[0.4,0.5])

Table 2.4 Interval-valued intuitionistic fuzzy decision matrix $\widetilde{R'_3}$

	G_1	G_2	G_3	G_4	G_5
Y_1	([0.4,0.6],[0.3,0.4])	([0.5,0.7],[0.0,0.2])	([0.5,0.6],[0.2,0.4])	([0.6,0.8],[0.1,0.2])	([0.2,0.3],[0.4,0.7])
Y_2	([0.5,0.8],[0.1,0.2])	([0.3,0.5],[0.2,0.3])	([0.3,0.6],[0.2,0.4])	([0.4,0.5],[0.2,0.4])	([0.2,0.3],[0.3,0.6])
Y_3	([0.5,0.6],[0.0,0.1])	([0.5,0.8],[0.1,0.2])	([0.4,0.7],[0.2,0.3])	([0.2,0.4],[0.2,0.3])	([0.0,0.2],[0.5,0.8])
Y_4	([0.5,0.7],[0.1,0.3])	([0.4,0.6],[0.0,0.1])	([0.3,0.5],[0.2,0.4])	([0.7,0.9],[0.0,0.1])	([0.2,0.2],[0.3,0.5])
Y_5	([0.7,0.8],[0.0,0.1])	([0.4,0.6],[0.0,0.2])	([0.4,0.7],[0.2,0.3])	([0.3,0.5],[0.1,0.3])	([0.1,0.2],[0.6,0.7])

Table 2.5 Interval-valued intuitionistic fuzzy decision matrix $\widetilde{R'_4}$

	G_1	G_2	G_3	G_4	G_5
Y_1	([0.3,0.4],[0.4,0.5])	([0.8,0.9],[0.1,0.1])	([0.7,0.8],[0.1,0.2])	([0.4,0.5],[0.3,0.5])	([0.3,0.6],[0.2,0.4])
Y_2	([0.5,0.7],[0.1,0.3])	([0.4,0.7],[0.2,0.3])	([0.4,0.5],[0.2,0.2])	([0.6,0.8],[0.1,0.2])	([0.0,0.1],[0.2,0.3])
Y_3	([0.2,0.4],[0.1,0.2])	([0.4,0.5],[0.2,0.4])	([0.5,0.8],[0.0,0.1])	([0.4,0.6],[0.2,0.3])	([0.2,0.3],[0.5,0.6])
Y_4	([0.7,0.8],[0.0,0.2])	([0.5,0.7],[0.1,0.2])	([0.6,0.7],[0.1,0.3])	([0.4,0.5],[0.1,0.2])	([0.1,0.2],[0.7,0.8])
Y_5	([0.5,0.6],[0.2,0.4])	([0.5,0.8],[0.0,0.2])	([0.4,0.7],[0.2,0.3])	([0.3,0.6],[0.2,0.3])	([0.0,0.1],[0.7,0.8])

Since G_5 is an attribute of cost type and the other attributes are benefit type, we employ Eq.(2.22) to normalize $\tilde{R'}_k = (\tilde{r}'^{(k)}_{ij})_{5\times 5}$ $(k = 1, 2, 3, 4)$ into the interval-valued intuitionistic fuzzy decision matrices $\tilde{R}_k = (\tilde{r}^{(k)}_{ij})_{5\times 5}$ $(k = 1, 2, 3, 4)$ (Tables 2.6-2.9) respectively:

Table 2.6 Interval-valued intuitionistic fuzzy decision matrix $\widetilde{R_1}$

	G_1	G_2	G_3	G_4	G_5
Y_1	([0.3,0.4],[0.4,0.6])	([0.5,0.6],[0.1,0.2])	([0.6,0.7],[0.2,0.3])	([0.7,0.8],[0.0,0.1])	([0.6,0.7],[0.2,0.3])
Y_2	([0.6,0.8],[0.1,0.2])	([0.6,0.7],[0.2,0.3])	([0.2,0.3],[0.4,0.6])	([0.5,0.6],[0.1,0.3])	([0.7,0.8],[0.0,0.2])
Y_3	([0.5,0.8],[0.1,0.2])	([0.7,0.8],[0.0,0.1])	([0.5,0.5],[0.4,0.5])	([0.2,0.3],[0.2,0.4])	([0.4,0.6],[0.2,0.3])
Y_4	([0.2,0.3],[0.4,0.5])	([0.5,0.7],[0.1,0.3])	([0.6,0.7],[0.1,0.2])	([0.4,0.5],[0.1,0.3])	([0.6,0.9],[0.0,0.1])
Y_5	([0.6,0.8],[0.1,0.2])	([0.3,0.5],[0.4,0.5])	([0.4,0.6],[0.3,0.4])	([0.6,0.8],[0.1,0.2])	([0.5,0.6],[0.2,0.3])

Table 2.7 Interval-valued intuitionistic fuzzy decision matrix $\widetilde{R_2}$

	G_1	G_2	G_3	G_4	G_5
Y_1	([0.4,0.5],[0.3,0.4])	([0.5,0.6],[0.1,0.2])	([0.6,0.7],[0.2,0.3])	([0.7,0.8],[0.1,0.2])	([0.7,0.8],[0.0,0.2])
Y_2	([0.6,0.8],[0.1,0.2])	([0.5,0.6],[0.3,0.4])	([0.4,0.5],[0.3,0.4])	([0.4,0.6],[0.3,0.4])	([0.4,0.7],[0.1,0.3])
Y_3	([0.5,0.6],[0.3,0.4])	([0.5,0.7],[0.1,0.2])	([0.5,0.6],[0.3,0.4])	([0.3,0.4],[0.2,0.5])	([0.6,0.7],[0.2,0.3])
Y_4	([0.5,0.6],[0.3,0.4])	([0.7,0.8],[0.0,0.1])	([0.4,0.5][0.2,0.4])	([0.5,0.7],[0.1,0.2])	([0.5,0.7],[0.2,0.3])
Y_5	([0.4,0.7],[0.2,0.3])	([0.5,0.6][0.2,0.4])	([0.3,0.6],[0.3,0.4])	([0.6,0.8],[0.1,0.2])	([0.4,0.5],[0.2,0.3])

Table 2.8 Interval-valued intuitionistic fuzzy decision matrix $\widetilde{R_3}$

	G_1	G_2	G_3	G_4	G_5
Y_1	([0.4,0.6],[0.3,0.4])	([0.5,0.7],[0.0,0.2])	([0.5,0.6],[0.2,0.4])	([0.6,0.8],[0.1,0.2])	([0.4,0.7],[0.2,0.3])
Y_2	([0.5,0.8],[0.1,0.2])	([0.3,0.5],[0.2,0.3])	([0.3,0.6],[0.2,0.4])	([0.4,0.5],[0.2,0.4])	([0.3,0.6],[0.2,0.3])
Y_3	([0.5,0.6],[0.0,0.1])	([0.5,0.8],[0.1,0.2])	([0.4,0.7],[0.2,0.3])	([0.2,0.4],[0.2,0.3])	([0.5,0.8],[0.0,0.2])
Y_4	([0.5,0.7],[0.1,0.3])	([0.4,0.6],[0.0,0.1])	([0.3,0.5],[0.2,0.4])	([0.7,0.9],[0.0,0.1])	([0.3,0.5],[0.2,0.2])
Y_5	([0.7,0.8],[0.0,0.1])	([0.4,0.6],[0.0,0.2])	([0.4,0.7],[0.2,0.3])	([0.3,0.5],[0.1,0.3])	([0.6,0.7],[0.1,0.2])

Table 2.9 Interval-valued intuitionistic fuzzy decision matrix $\widetilde{R_4}$

	G_1	G_2	G_3	G_4	G_5
Y_1	([0.3,0.4],[0.4,0.5])	([0.8,0.9],[0.1,0.1])	([0.7,0.8],[0.1,0.2])	([0.4,0.5],[0.3,0.5])	([0.2,0.4],[0.3,0.6])
Y_2	([0.5,0.7],[0.1,0.3])	([0.4,0.7],[0.2,0.3])	([0.4,0.5],[0.2,0.2])	([0.6,0.8],[0.1,0.2])	([0.2,0.3],[0.0,0.1])
Y_3	([0.2,0.4],[0.1,0.2])	([0.4,0.5],[0.2,0.4])	([0.5,0.8],[0.0,0.1])	([0.4,0.6],[0.2,0.3])	([0.5,0.6],[0.2,0.3])
Y_4	([0.7,0.8],[0.0,0.2])	([0.5,0.7],[0.1,0.2])	([0.6,0.7],[0.1,0.3])	([0.4,0.5],[0.1,0.2])	([0.7,0.8],[0.1,0.2])
Y_5	([0.5,0.6],[0.2,0.4])	([0.5,0.8],[0.0,0.2])	([0.4,0.7],[0.2,0.3])	([0.3,0.6],[0.2,0.3])	([0.7,0.8],[0.0,0.1])

Without loss of generality, here we utilize the IIFHA and IIFWA operators to aggregate the given data: We first utilize the normal distribution based method (Xu, 2005a) to determine the weighting vector $w = (0.155,\ 0.345,\ 0.345,\ 0.155)^{\mathrm{T}}$ associated with the IIFHA operator, and then utilize the IIFHA operator (2.26) to aggregate the individual interval-valued intuitionistic fuzzy decision matrices $\tilde{R}_k = (\tilde{r}_{ij}^{(k)})_{5\times 5}$ $(k = 1, 2, 3, 4)$ into the collective interval-valued intuitionistic fuzzy decision matrix $\dot{\tilde{R}} = (\dot{\tilde{r}}_{ij})_{5\times 5}$:

Table 2.10 Collective interval-valued intuitionistic fuzzy decision matrix $\dot{\tilde{R}}$

	G_1	G_2	G_3	G_4	G_5
Y_1	([0.348,0.471], [0.338,0.473])	([0.577,0.741], [0.000,0.162])	([0.595,0.698], [0.165,0.296])	([0.564,0.735], [0.000,0.254])	([0.536,0.716], [0.000,0.284])
Y_2	([0.549,0.790], [0.100,0.210])	([0.420,0.616], [0.210,0.311])	([0.322,0.462], [0.288,0.382])	([0.447,0.670], [0.153,0.278])	([0.417,0.652], [0.000,0.243])
Y_3	([0.470,0.630], [0.000,0.120])	([0.535,0.749], [0.000,0.192])	([0.489,0.667], [0.000,0.306])	([0.273,0.428], [0.2000,0.360])	([0.493,0.675], [0.000,0.278])
Y_4	([0.488,0.638], [0.000,0.340])	([0.520,0.707], [0.000,0.144])	([0.470,0.605], [0.145,0.325])	([0.498,0.678], [0.000,0.208])	([0.521,0.730], [0.000,0.185])
Y_5	([0.564,0.756], [0.000,0.214])	([0.420,0.631], [0.000,0.324])	([0.388,0.650], [0.249,0.350])	([0.460,0.682], [0.109,0.249])	([0.574,0.678], [0.000,0.205])

Then, we utilize the IIFWA operator (2.28) to aggregate the elements in each line of the collective interval-valued intuitionistic fuzzy decision matrix $\dot{\tilde{R}}$, and get the overall attribute values $\dot{\tilde{r}}_i$ $(i = 1, 2, 3, 4, 5)$ of the alternatives Y_i $(i = 1, 2, 3, 4, 5)$:

$$\dot{\tilde{r}}_1 = ([0.5321, 0.6855], [0.0000, 0.2819])$$

$$\dot{\tilde{r}}_2 = ([0.4386, 0.6572], [0.0000, 0.2791])$$

$$\dot{\tilde{r}}_3 = ([0.4365, 0.5694], [0.0000, 0.2449])$$

$$\dot{\tilde{r}}_4 = ([0.4975, 0.6703], [0.0000, 0.2333])$$

$$\dot{\tilde{r}}_5 = ([0.4825, 0.6850], [0.0000, 0.2613])$$

After that, we utilize Eq.(2.10) to calculate the scores $s(\dot{\tilde{r}}_i)$ $(i = 1, 2, 3, 4, 5)$ of the overall attribute values $\dot{\tilde{r}}_i$ $(i = 1, 2, 3, 4, 5)$ of the alternatives Y_i $(i = 1, 2, 3, 4, 5)$:

$$s(\dot{\tilde{r}}_1) = 0.4678, \quad s(\dot{\tilde{r}}_2) = 0.4084, \quad s(\dot{\tilde{r}}_3) = 0.3805$$

$$s(\dot{\tilde{r}}_4) = 0.4672, \quad s(\dot{\tilde{r}}_5) = 0.4531$$

Then

$$s(\dot{\tilde{r}}_1) > s(\dot{\tilde{r}}_4) > s(\dot{\tilde{r}}_5) > s(\dot{\tilde{r}}_2) > s(\dot{\tilde{r}}_3)$$

Consequently,

$$Y_1 \succ Y_4 \succ Y_5 \succ Y_2 \succ Y_3$$

To conclude, the best potential global supplier is Y_1.

To tackle the problem where the information about the attribute weights in the problem considered is completely unknown, Xu (2010a) introduces the following definition:

Definition 2.3.10 (Xu, 2010a) Let $\tilde{\alpha}_1=([a_1,b_1],\ [c_1,d_1])$ and $\tilde{\alpha}_2=([a_2,b_2],\ [c_2,d_2])$ be two IVIFNs. Then we call

$$d(\tilde{\alpha}_1,\tilde{\alpha}_2)=\frac{1}{4}(|a_1-a_2|+|b_1-b_2|+|c_1-c_2|+|d_1-d_2|) \tag{2.30}$$

the distance between $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$.

By Definition 2.3.10, we have

Theorem 2.3.7 (Xu, 2010a) Let $\tilde{\alpha}_i$ $(i=1,2,3)$ be any three IVIFNs. Then

(1) $0\leqslant d(\tilde{\alpha}_1,\tilde{\alpha}_2)\leqslant 1$, especially, $d(\tilde{\alpha}_1,\tilde{\alpha}_1)=0$;

(2) $d(\tilde{\alpha}_1,\tilde{\alpha}_2)=d(\tilde{\alpha}_2,\tilde{\alpha}_1)$;

(3) $d(\tilde{\alpha}_1,\tilde{\alpha}_3)\leqslant d(\tilde{\alpha}_1,\tilde{\alpha}_2)+d(\tilde{\alpha}_2,\tilde{\alpha}_3)$.

Similar to Section 1.3, we can obtain the following formula to derive the attribute weights:

$$\omega_j=\frac{\sum_{i=1}^{n}\sum_{k\neq i}d(\tilde{r}_{ij},\tilde{r}_{kj})}{\sum_{j=1}^{m}\sum_{i=1}^{n}\sum_{k\neq i}d(\tilde{r}_{ij},\tilde{r}_{kj})},\quad j=1,2,\cdots,m \tag{2.31}$$

Then, based on the attribute weights derived by Eq.(2.31), we can utilize the approach introduced previously to rank the given alternatives and then to get the most desirable one(s).

2.4 Interval-Valued Intuitionistic Fuzzy Bonferroni Means

Xu and Chen (2011) extended the IFBMs to accommodate interval-valued intuitionistic fuzzy environments. They defined the interval-valued intuitionistic fuzzy Bonferroni mean as follows, which is to be used to aggregate interval-valued intuitionistic fuzzy information:

Definition 2.4.1 (Xu and Chen, 2011) Let $\tilde{\alpha}_i=(\tilde{\mu}_{\tilde{\alpha}_i},\tilde{\nu}_{\tilde{\alpha}_i})=([\tilde{\mu}^{\mathrm{L}}_{\tilde{\alpha}_i},\tilde{\mu}^{\mathrm{U}}_{\tilde{\alpha}_i}],\ [\tilde{\nu}^{\mathrm{L}}_{\tilde{\alpha}_i},\tilde{\nu}^{\mathrm{U}}_{\tilde{\alpha}_i}])$ $(i=1,2,\cdots,n)$ be a collection of IVIFNs, and $p,q>0$. If

$$\mathrm{IIFB}^{p,q}(\tilde{\alpha}_1,\tilde{\alpha}_2,\cdots,\tilde{\alpha}_n)=\left(\frac{1}{n(n-1)}\left(\mathop{\oplus}_{\substack{i,j=1\\ i\neq j}}^{n}\left(\tilde{\alpha}_i^p\otimes\tilde{\alpha}_j^q\right)\right)\right)^{\frac{1}{p+q}} \tag{2.32}$$

then $\mathrm{IIFB}^{p,q}$ is called an interval-valued intuitionistic fuzzy Bonferroni mean (IIFBM).

Based on the operational laws described in Section 2.2, and similar to Eq.(1.96), it can be proven that the aggregated value by using the IIFBM (2.32) is an IVIFN,

and

$$\begin{aligned}
&\text{IIFB}^{p,q}(\tilde{\alpha}_1, \tilde{\alpha}_2, \cdots, \tilde{\alpha}_n) \\
&= \left(\left[\left(1 - \prod_{\substack{i,j=1 \\ i \neq j}}^{n} \left(1 - (\tilde{\mu}_{\tilde{\alpha}_i}^{\text{L}})^p (\tilde{\mu}_{\tilde{\alpha}_j}^{\text{L}})^q \right)^{\frac{1}{n(n-1)}} \right)^{\frac{1}{p+q}}, \left(1 - \prod_{\substack{i,j=1 \\ i \neq j}}^{n} \left(1 - (\tilde{\mu}_{\tilde{\alpha}_i}^{\text{U}})^p (\tilde{\mu}_{\tilde{\alpha}_j}^{\text{U}})^q \right)^{\frac{1}{n(n-1)}} \right)^{\frac{1}{p+q}} \right], \right. \\
&\quad \left[1 - \left(1 - \prod_{\substack{i,j=1 \\ i \neq j}}^{n} \left(1 - (1 - \tilde{\nu}_{\tilde{\alpha}_i}^{\text{L}})^p (1 - \tilde{\nu}_{\tilde{\alpha}_j}^{\text{L}})^q \right)^{\frac{1}{n(n-1)}} \right)^{\frac{1}{p+q}}, \right. \\
&\quad \left. \left. 1 - \left(1 - \prod_{\substack{i,j=1 \\ i \neq j}}^{n} \left(1 - (1 - \tilde{\nu}_{\tilde{\alpha}_i}^{\text{U}})^p (1 - \tilde{\nu}_{\tilde{\alpha}_j}^{\text{U}})^q \right)^{\frac{1}{n(n-1)}} \right)^{\frac{1}{p+q}} \right] \right) \qquad (2.33)
\end{aligned}$$

The IIFBM has the following properties (Xu and Chen, 2011):

(1) If $\tilde{\alpha}_i = \tilde{\alpha}_* = ([0,0],[1,1])$ $(i = 1, 2, \cdots, n)$, then

$$\text{IIFB}^{p,q}(\tilde{\alpha}_*, \tilde{\alpha}_*, \cdots, \tilde{\alpha}_*) = ([0,0],\ [1,1]) \qquad (2.34)$$

(2) If $\tilde{\alpha}_i = \tilde{\alpha}^* = ([1,1],[0,0])$ $(i = 1, 2, \cdots, n)$, then

$$\text{IIFB}^{p,q}(\tilde{\alpha}^*, \tilde{\alpha}^*, \cdots, \tilde{\alpha}^*) = ([1,1],\ [0,0]) \qquad (2.35)$$

(3) **(Idempotency)**: If all the IVIFNs $\tilde{\alpha}_i$ $(i = 1, 2, \cdots, n)$ are equal, i.e., $\tilde{\alpha}_i = \tilde{\alpha}$, for all i, then

$$\text{IIFB}^{p,q}(\tilde{\alpha}, \tilde{\alpha}, \cdots, \tilde{\alpha}) = \tilde{\alpha} \qquad (2.36)$$

(4) **(Monotonicity)**: Let $\tilde{\alpha}_i = (\tilde{\mu}_{\tilde{\alpha}_i}, \tilde{\nu}_{\tilde{\alpha}_i}) = ([\tilde{\mu}_{\tilde{\alpha}_i}^{\text{L}}, \tilde{\mu}_{\tilde{\alpha}_i}^{\text{U}}], [\tilde{\nu}_{\tilde{\alpha}_i}^{\text{L}}, \tilde{\nu}_{\tilde{\alpha}_i}^{\text{U}}])$ $(i = 1, 2, \cdots, n)$ and $\tilde{\beta}_i = (\tilde{\mu}_{\tilde{\beta}_i}, \tilde{\nu}_{\tilde{\beta}_i}) = ([\tilde{\mu}_{\tilde{\beta}_i}^{\text{L}}, \tilde{\mu}_{\tilde{\beta}_i}^{\text{U}}],\ [\tilde{\nu}_{\tilde{\beta}_i}^{\text{L}}, \tilde{\nu}_{\tilde{\beta}_i}^{\text{U}}])$ $(i = 1, 2, \cdots, n)$ be two collections of IVIFNs. If $\tilde{\mu}_{\tilde{\alpha}_i}^{\text{L}} \leqslant \tilde{\mu}_{\tilde{\beta}_i}^{\text{L}}$, $\tilde{\mu}_{\tilde{\alpha}_i}^{\text{U}} \leqslant \tilde{\mu}_{\tilde{\beta}_i}^{\text{U}}$, $\tilde{\nu}_{\tilde{\alpha}_i}^{\text{L}} \geqslant \tilde{\nu}_{\tilde{\beta}_i}^{\text{L}}$ and $\tilde{\nu}_{\tilde{\alpha}_i}^{\text{U}} \geqslant \tilde{\nu}_{\tilde{\beta}_i}^{\text{U}}$, for all i, then

$$\text{IIFB}^{p,q}(\tilde{\alpha}_1, \tilde{\alpha}_2, \cdots, \tilde{\alpha}_n) \leqslant \text{IIFB}^{p,q}(\tilde{\beta}_1, \tilde{\beta}_2, \cdots, \tilde{\beta}_n) \qquad (2.37)$$

(5) **(Commutativity)**: Let $\tilde{\alpha}_i$ $(i = 1, 2, \cdots, n)$ be a collection of IVIFNs. Then

$$\text{IIFB}^{p,q}(\tilde{\alpha}_1, \tilde{\alpha}_2, \cdots, \tilde{\alpha}_n) = \text{IIFB}^{p,q}(\dot{\tilde{\alpha}}_1, \dot{\tilde{\alpha}}_2, \cdots, \dot{\tilde{\alpha}}_n) \qquad (2.38)$$

where $(\dot{\tilde{\alpha}}_1, \dot{\tilde{\alpha}}_2, \cdots, \dot{\tilde{\alpha}}_n)$ is any permutation of $(\tilde{\alpha}_1, \tilde{\alpha}_2, \cdots, \tilde{\alpha}_n)$.

(6) **(Boundedness)**: Let

$$\tilde{\alpha}^- = \left(\left[\min_i \{\tilde{\mu}_{\tilde{\alpha}_i}^{\text{L}}\}, \min_i \{\tilde{\mu}_{\tilde{\alpha}_i}^{\text{U}}\} \right],\ \left[\max_i \{\tilde{\nu}_{\tilde{\alpha}_i}^{\text{L}}\}, \max_i \{\tilde{\nu}_{\tilde{\alpha}_i}^{\text{U}}\} \right] \right) \qquad (2.39)$$

$$\tilde{\alpha}^+ = \left(\left[\max_i\{\tilde{\mu}^{\mathrm{L}}_{\tilde{\alpha}_i}\}, \max_i\{\tilde{\mu}^{\mathrm{U}}_{\tilde{\alpha}_i}\}\right],\ \left[\min_i\{\tilde{\nu}^{\mathrm{L}}_{\tilde{\alpha}_i}\}, \min_i\{\tilde{\nu}^{\mathrm{U}}_{\tilde{\alpha}_i}\}\right]\right) \tag{2.40}$$

Then

$$\tilde{\alpha}^- \leqslant \mathrm{IIFB}^{p,q}(\tilde{\alpha}_1, \tilde{\alpha}_2, \cdots, \tilde{\alpha}_n) \leqslant \tilde{\alpha}^+ \tag{2.41}$$

Now we discuss some special cases of the IIFBM with respect to the parameters p and q (Xu and Chen, 2011).

Case 1 If $q \to 0$, then it follows from Eq.(2.32) that

$$\begin{aligned}
\lim_{q\to 0}\mathrm{IIFB}^{p,q}(\tilde{\alpha}_1, \tilde{\alpha}_2, \cdots, \tilde{\alpha}_n) &= \lim_{q\to 0}\left(\frac{1}{n(n-1)}\left(\bigoplus_{\substack{i,j=1\\ i\neq j}}^{n}\left(\tilde{\alpha}_i^p \otimes \tilde{\alpha}_j^q\right)\right)\right)^{\frac{1}{p+q}} \\
&= \left(\frac{1}{n}\left(\bigoplus_{i=1}^{n}\tilde{\alpha}_i^p\right)\right)^{\frac{1}{p}} \\
&= \left(\left[\left(1-\prod_{i=1}^{n}\left(1-(\tilde{\mu}^{\mathrm{L}}_{\tilde{\alpha}_i})^p\right)^{\frac{1}{n}}\right)^{\frac{1}{p}}, \left(1-\prod_{i=1}^{n}\left(1-(\tilde{\mu}^{\mathrm{U}}_{\tilde{\alpha}_i})^p\right)^{\frac{1}{n}}\right)^{\frac{1}{p}}\right],\right. \\
&\quad \left[1-\left(1-\prod_{i=1}^{n}\left(1-(1-\tilde{\nu}^{\mathrm{L}}_{\tilde{\alpha}_i})^p\right)^{\frac{1}{n}}\right)^{\frac{1}{p}},\right. \\
&\quad \left.\left. 1-\left(1-\prod_{i=1}^{n}\left(1-(1-\tilde{\nu}^{\mathrm{U}}_{\tilde{\alpha}_i})^p\right)^{\frac{1}{n}}\right)^{\frac{1}{p}}\right]\right) \\
&= \mathrm{IIFB}^{p,0}(\tilde{\alpha}_1, \tilde{\alpha}_2, \cdots, \tilde{\alpha}_n)
\end{aligned} \tag{2.42}$$

which we call a generalized interval-valued intuitionistic fuzzy mean.

Case 2 If $p = 2$ and $q \to 0$, then by Eq.(2.32), we have

$$\begin{aligned}
&\mathrm{IIFB}^{2,0}(\tilde{\alpha}_1, \tilde{\alpha}_2, \cdots, \tilde{\alpha}_n) \\
&= \left(\left[\left(1-\prod_{i=1}^{n}\left(1-(\tilde{\mu}^{\mathrm{L}}_{\tilde{\alpha}_i})^2\right)^{\frac{1}{n}}\right)^{\frac{1}{2}}, \left(1-\prod_{i=1}^{n}\left(1-(\tilde{\mu}^{\mathrm{U}}_{\tilde{\alpha}_i})^2\right)^{\frac{1}{n}}\right)^{\frac{1}{2}}\right],\right. \\
&\quad \left.\left[1-\left(1-\prod_{i=1}^{n}\left(1-(1-\tilde{\nu}^{\mathrm{L}}_{\tilde{\alpha}_i})^2\right)^{\frac{1}{n}}\right)^{\frac{1}{2}}, 1-\left(1-\prod_{i=1}^{n}\left(1-(1-\tilde{\nu}^{\mathrm{U}}_{\tilde{\alpha}_i})^2\right)^{\frac{1}{n}}\right)^{\frac{1}{2}}\right]\right) \\
&= \left(\frac{1}{n}\left(\bigoplus_{i=1}^{n}\tilde{\alpha}_i^2\right)\right)^{\frac{1}{2}}
\end{aligned} \tag{2.43}$$

which we call an interval-valued intuitionistic fuzzy square mean.

Case 3 If $p = 1$ and $q \to 0$, then Eq.(2.32) reduces to the IIFA operator:

$$\begin{aligned}&\text{IIFB}^{1,0}(\tilde{\alpha}_1,\tilde{\alpha}_2,\cdots,\tilde{\alpha}_n)\\&=\left(\left[\left(1-\prod_{i=1}^{n}\left(1-\tilde{\mu}_{\tilde{\alpha}_i}^{\mathrm{L}}\right)^{\frac{1}{n}}\right),\left(1-\prod_{i=1}^{n}\left(1-\tilde{\mu}_{\tilde{\alpha}_i}^{\mathrm{U}}\right)^{\frac{1}{n}}\right)\right],\ \left[\prod_{i=1}^{n}\left(\tilde{\nu}_{\tilde{\alpha}_i}^{\mathrm{L}}\right)^{\frac{1}{n}},\prod_{i=1}^{n}\left(\tilde{\nu}_{\tilde{\alpha}_i}^{\mathrm{U}}\right)^{\frac{1}{n}}\right]\right)\\&=\frac{1}{n}\left(\overset{n}{\underset{i=1}{\oplus}}\tilde{\alpha}_i\right)\end{aligned}\tag{2.44}$$

Case 4 If $p=q=1$, then we get from Eq.(2.32):

$$\begin{aligned}&\text{IIFB}^{1,1}(\tilde{\alpha}_1,\tilde{\alpha}_2,\cdots,\tilde{\alpha}_n)=\left(\frac{1}{n(n-1)}\left(\overset{n}{\underset{\substack{i,j=1\\i\neq j}}{\oplus}}(\tilde{\alpha}_i\otimes\tilde{\alpha}_j)\right)\right)^{\frac{1}{2}}\\&=\left(\left[\left(1-\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-\tilde{\mu}_{\tilde{\alpha}_i}^{\mathrm{L}}\tilde{\mu}_{\tilde{\alpha}_j}^{\mathrm{L}}\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{2}},\left(1-\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-\tilde{\mu}_{\tilde{\alpha}_i}^{\mathrm{U}}\tilde{\mu}_{\tilde{\alpha}_j}^{\mathrm{U}}\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{2}}\right],\right.\\&\left[1-\left(1-\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-(1-\tilde{\nu}_{\tilde{\alpha}_i}^{\mathrm{L}})(1-\tilde{\nu}_{\tilde{\alpha}_j}^{\mathrm{L}})\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{2}},\right.\\&\left.\left.1-\left(1-\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-(1-\tilde{\nu}_{\tilde{\alpha}_i}^{\mathrm{U}})(1-\tilde{\nu}_{\tilde{\alpha}_j}^{\mathrm{U}})\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{2}}\right]\right)\end{aligned}\tag{2.45}$$

which we call an interval-valued intuitionistic fuzzy interrelated square mean.

Below we give an example to illustrate the IIFBM:

Example 2.4.1 Suppose that there are four IVIFNs: $\tilde{\alpha}_1=([0.1,0.3],\ [0.5,0.6])$, $\tilde{\alpha}_2=([0.4,0.5],\ [0.3,0.4])$, and $\tilde{\alpha}_3=([0.5,0.7],\ [0.2,0.3])$. Then we can utilize the IIFBM to aggregate these IVIFNs. Assume that $p=q=2$, and since

$$\begin{aligned}\tilde{\alpha}_i^2\otimes\tilde{\alpha}_j^2=&\left(\left[\left(\tilde{\mu}_{\tilde{\alpha}_i}^{\mathrm{L}}\tilde{\mu}_{\tilde{\alpha}_j}^{\mathrm{L}}\right)^2,\left(\tilde{\mu}_{\tilde{\alpha}_i}^{\mathrm{U}}\tilde{\mu}_{\tilde{\alpha}_j}^{\mathrm{U}}\right)^2\right],\right.\\&\left.\left[1-\left((1-\tilde{\nu}_{\tilde{\alpha}_i}^{\mathrm{L}})(1-\tilde{\nu}_{\tilde{\alpha}_j}^{\mathrm{L}})\right)^2,1-\left((1-\tilde{\nu}_{\tilde{\alpha}_i}^{\mathrm{U}})(1-\tilde{\nu}_{\tilde{\alpha}_j}^{\mathrm{U}})\right)^2\right]\right)\end{aligned}\tag{2.46}$$

we have

$$\begin{aligned}\tilde{\alpha}_1^2\otimes\tilde{\alpha}_2^2=&\left(\left[(0.1\times 0.4)^2,(0.3\times 0.5)^2\right],\right.\\&\left.\left[1-((1-0.5)\times(1-0.3))^2,1-((1-0.6)\times(1-0.4))^2\right]\right)\\=&([0.0016,0.0225],\ [0.8775,0.9424])=\tilde{\alpha}_2^2\otimes\tilde{\alpha}_1^2\\\tilde{\alpha}_1^2\otimes\tilde{\alpha}_3^2=&\left(\left[(0.1\times 0.5)^2,(0.3\times 0.7)^2\right],\right.\\&\left.\left[1-((1-0.5)\times(1-0.2))^2,1-((1-0.6)\times(1-0.3))^2\right]\right)\end{aligned}$$

$$= ([0.0025, 0.0441],\ [0.8400, 0.9216]) = \tilde{\alpha}_3^2 \otimes \tilde{\alpha}_1^2$$

$$\tilde{\alpha}_2^2 \otimes \tilde{\alpha}_3^2 = \left(\left[(0.4 \times 0.5)^2, (0.5 \times 0.7)^2\right],\right.$$
$$\left.\left[1-((1-0.3)\times(1-0.2))^2, 1-((1-0.4)\times(1-0.3))^2\right]\right)$$
$$= ([0.0400, 0.1225],\ [0.6864, 0.8236]) = \tilde{\alpha}_3^2 \otimes \tilde{\alpha}_2^2$$

Then, it follows from Eq.(2.32) that

$$\mathrm{IIFB}^{2,2}(\tilde{\alpha}_1, \tilde{\alpha}_2, \tilde{\alpha}_3) = \left(\frac{1}{6}\left(\bigoplus_{\substack{i,j=1\\ i\neq j}}^{3} \left(\tilde{\alpha}_i^2 \otimes \tilde{\alpha}_j^2\right)\right)\right)^{\frac{1}{4}}$$
$$= ([0.349, 0.503],\ [0.329, 0.430])$$

Considering that the input data may have different importance degrees, we now introduce the concept of weighted interval-valued intuitionistic fuzzy Bonferroni mean:

Definition 2.4.2 (Xu and Chen, 2011) If

$$\mathrm{IIFB}_{\omega}^{p,q}(\tilde{\alpha}_1, \tilde{\alpha}_2, \cdots, \tilde{\alpha}_n) = \left(\frac{1}{n(n-1)}\left(\bigoplus_{\substack{i,j=1\\ i\neq j}}^{n} \left((\omega_i\tilde{\alpha}_i)^p \otimes (\omega_j\tilde{\alpha}_j)^q\right)\right)\right)^{\frac{1}{p+q}} \tag{2.47}$$

then $\mathrm{IIFB}_{\omega}^{p,q}$ is called a weighted interval-valued intuitionistic fuzzy Bonferroni mean (WIIFBM), where $\omega = (\omega_1, \omega_2, \cdots, \omega_n)^{\mathrm{T}}$ is the weight vector of $\tilde{\alpha}_i$ $(i = 1, 2, \cdots, n)$, and ω_i indicates the importance degree of $\tilde{\alpha}_i$, satisfying $\omega_i \in [0, 1]$ $(i = 1, 2, \cdots, n)$, $\sum_{i=1}^{n} \omega_i = 1$.

Similar to Eq.(2.32), Eq.(2.47) can also be transformed into the following form:

$$\mathrm{IIFB}_{\omega}^{p,q}(\tilde{\alpha}_1, \tilde{\alpha}_2, \cdots, \tilde{\alpha}_n)$$
$$= \left(\left[\left(1 - \prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(1-(1-\tilde{\mu}_{\tilde{\alpha}_i}^{\mathrm{L}})^{\omega_i}\right)^p\left(1-(1-\tilde{\mu}_{\tilde{\alpha}_j}^{\mathrm{L}})^{\omega_j}\right)^q\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}},\right.\right.$$
$$\left.\left(1 - \prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(1-(1-\tilde{\mu}_{\tilde{\alpha}_i}^{\mathrm{U}})^{\omega_i}\right)^p\left(1-(1-\tilde{\mu}_{\tilde{\alpha}_j}^{\mathrm{U}})^{\omega_j}\right)^q\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}\right],$$
$$\left[1-\left(1 - \prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(1-(\tilde{\nu}_{\tilde{\alpha}_i}^{\mathrm{L}})^{\omega_i}\right)^p\left(1-(\tilde{\nu}_{\tilde{\alpha}_j}^{\mathrm{L}})^{\omega_j}\right)^q\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}},\right.$$
$$\left.\left.1-\left(1 - \prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-\left(1-(\tilde{\nu}_{\tilde{\alpha}_i}^{\mathrm{U}})^{\omega_i}\right)^p\left(1-(\tilde{\nu}_{\tilde{\alpha}_j}^{\mathrm{U}})^{\omega_j}\right)^q\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{p+q}}\right]\right)$$

whose aggregated value is an IVIFN.

In the above we have developed an approach to multi-attribute decision making in intuitionistic fuzzy environments, where the performance values (attribute values) are measured in IFNs. We now apply the WIIFBM to multi-attribute decisionmaking with interval-valued intuitionistic fuzzy information, which involves the following steps (Xu and Chen, 2011):

Step 1 For a multi-attribute decision making problem, let Y,G and ω be defined as in Subsection 1.3. The performance of the alternative $Y_i \in Y$ with respect to the attribute $G_j \in G$ is measured by an IVIFN $\tilde{r}'_{ij} = (\tilde{\mu}'_{ij}, \tilde{\nu}'_{ij})$, where $\tilde{\mu}'_{ij} = [\mu'^{\rm L}_{ij}, \mu'^{\rm U}_{ij}]$ indicates the degree range that the alternative Y_i satisfies the attribute G_j, $\tilde{\nu}'_{ij} = [\tilde{\nu}'^{\rm L}_{ij}, \tilde{\nu}'^{\rm U}_{ij}]$ indicates the degree range that the alternative Y_i does not satisfy the attribute G_j, such that $\tilde{\mu}'_{ij} \subseteq [0,1]$, $\tilde{\nu}'_{ij} \subseteq [0,1]$, $\tilde{\mu}'^{\rm U}_{ij} + \tilde{\nu}'^{\rm U}_{ij} \leqslant 1$. All $\tilde{r}'_{ij} = (\tilde{\mu}'_{ij}, \tilde{\nu}'_{ij})$ $(i = 1, 2, \cdots, n;\ j = 1, 2, \cdots, m)$ are contained in an interval-valued intuitionistic fuzzy decision matrix $\tilde{R}' = (\tilde{r}'_{ij})_{n\times m}$.

If all the attributes G_j $(j = 1, 2, \cdots, m)$ are of the same type, then the performance values do not need normalization; Otherwise, using Eq.(2.22), we normalize the matrix $\tilde{R}' = (\tilde{r}'_{ij})_{n\times m}$ into the matrix $\tilde{R} = (\tilde{r}_{ij})_{n\times m}$, where

$$\tilde{r}_{ij} = (\tilde{\mu}_{ij}, \tilde{\nu}_{ij}), \quad \tilde{\mu}_{ij} = [\tilde{\mu}^{\rm L}_{ij}, \tilde{\mu}^{\rm U}_{ij}], \quad \tilde{\nu}_{ij} = [\tilde{\nu}^{\rm L}_{ij}, \tilde{\nu}^{\rm U}_{ij}], \quad \tilde{\mu}^{\rm U}_{ij} + \tilde{\nu}^{\rm U}_{ij} \leqslant 1$$
$$i = 1, 2, \cdots, n; \quad j = 1, 2, \cdots, m \tag{2.48}$$

Step 2 Utilize the WIIFBM (in general, we can take $p = q = 1$):

$$\tilde{r}_i = {\rm IIFB}^{p,q}_{w}(\tilde{r}_{i1}, \tilde{r}_{i2}, \cdots, \tilde{r}_{im}) \tag{2.49}$$

to aggregate all the performance values $\tilde{r}_{ij}$ $(i = 1, 2, \cdots, m)$ of the j-th column, and get the overall performance value $\tilde{r}_i = (\tilde{\mu}_i, \tilde{\nu}_i) = ([\tilde{\mu}^{\rm L}_i, \tilde{\mu}^{\rm U}_i],\ [\tilde{\nu}^{\rm L}_i, \tilde{\nu}^{\rm U}_i])$ corresponding to the alternative Y_i.

Step 3 Utilize the method presented in Definition 2.3.5 to rank the overall performance values $\tilde{r}_i$ $(i = 1, 2, \cdots, n)$.

Step 4 Rank all the alternatives Y_i $(i = 1, 2, \cdots, n)$ in accordance with $\tilde{r}_i$ $(i = 1, 2, \cdots, n)$ in descending order, and then select the best one.

Example 2.4.2 Let us continue Example 1.4.2. If the characteristics of the alternatives Y_i $(i = 1, 2, 3, 4, 5)$ with respect to the attributes G_j $(j = 1, 2, 3)$ are represented by the IVIFNs $\tilde{r'}_{ij} = (\tilde{\mu}'_{ij}, \tilde{\nu}'_{ij})$ $(i = 1, 2, 3, 4, 5;\ j = 1, 2, 3)$ as shown in the matrix $\tilde{R}' = (\tilde{r}'_{ij})_{5\times 3}$ (Table 2.11):

Table 2.11 Interval-valued intuitionistic fuzzy decision matrix $\widetilde{R'}$

	G_1	G_2	G_3
Y_1	([0.2,0.4],[0.4,0.5])	([0.6,0.8],[0.1,0.2])	([0.4,0.5],[0.2,0.4])
Y_2	([0.5,0.7],[0.1,0.2]	([0.4,0.7],[0,0.1])	([0.7,0.8],[0.1,0.2])
Y_3	([0.2,0.4],[0.4,0.5])	([0.6,0.7],[0.2,0.3])	([0.3,0.4],[0.4,0.6])
Y_4	([0.2,0.3],[0.4,0.6])	([0.7,0.8],[0.1,0.2])	([0.6,0.8],[0,0.2])
Y_5	([0.,7,0.9],[0,0.1])	([0.5,0.6],[0.3,0.4])	([0.1,0.3],[0.4,0.6])

Since all the attributes are benefit attributes, the characteristics of the alternatives do not need normalization. To get the decision results, we first utilize the WIIFBM (without loss of generality, we take $p = q = 1$):

$$\tilde{r}'_i = (\tilde{\mu}'_i, \tilde{\nu}'_i) = \mathrm{IIFB}^{1,1}_{\omega}(\tilde{r}'_{i1}, \tilde{r}'_{i2}, \tilde{r}'_{i3}) \tag{2.50}$$

to aggregate all the performance values $\tilde{r}'_{ij}$ $(j = 1, 2, 3)$ of the i-th line, and get the overall performance value $\tilde{r}'_i$ corresponding to the alternative Y_i:

$$\tilde{r}'_1 = ([0.148, 0.238],\ [0.624, 0.723])$$

$$\tilde{r}'_2 = ([0.209, 0.340],\ [0.397, 0.567])$$

$$\tilde{r}'_3 = ([0.134, 0.202],\ [0.701, 0.778])$$

$$\tilde{r}'_4 = ([0.198, 0.283],\ [0.356, 0.698])$$

$$\tilde{r}'_5 = ([0.186, 0.290],\ [0.500, 0.697])$$

Then we calculate the scores of all the alternatives:

$$s(\tilde{r}'_1) = \frac{1}{2}(0.148 - 0.624 + 0.238 - 0.723) = -0.480$$

$$s(\tilde{r}'_2) = \frac{1}{2}(0.209 - 0.397 + 0.340 - 0.567) = -0.207$$

$$s(\tilde{r}'_3) = \frac{1}{2}(0.134 - 0.701 + 0.202 - 0.778) = -0.572$$

$$s(\tilde{r}'_4) = \frac{1}{2}([0.198 - 0.356 + 0.283 - 0.698) = -0.286$$

$$s(\tilde{r}'_5) = \frac{1}{2}(0.186 - 0.500 + 0.290 - 0.697) = -0.360$$

by which we have $s(\tilde{r}'_2) > s(\tilde{r}'_4) > s(\tilde{r}'_5) > s(\tilde{r}'_1) > s(\tilde{r}'_3)$. Therefore, from the ranking procedure of IVIFNs introduced in Definition 2.3.5, we get $\tilde{r}'_2 > \tilde{r}'_4 > \tilde{r}'_5 > \tilde{r}'_1 > \tilde{r}'_3$, and thus $Y_2 \succ Y_4 \succ Y_5 \succ Y_1 \succ Y_3$. To conclude, the optimal alternative is Y_2.

2.5 Generalized Interval-Valued Intuitionistic Fuzzy Aggregation Operators

Based on the aggregation techniques introduced in Section 1.5, Zhao et al. (2010) developed a series of generalized aggregation operators for IVIFNs:

Definition 2.5.1 (Zhao et al, 2010) Let GIIFWA : $\tilde{\Theta}^n \to \tilde{\Theta}$. If

$$\mathrm{GIIFWA}_{\omega}(\tilde{\alpha}_1, \tilde{\alpha}_2, \cdots, \tilde{\alpha}_n) = (\omega_1 \tilde{\alpha}_1^{\lambda} \oplus \omega_2 \tilde{\alpha}_2^{\lambda} \oplus \cdots \oplus \omega_n \tilde{\alpha}_n^{\lambda})^{\frac{1}{\lambda}} \tag{2.51}$$

then the function GIIFWA is called a generalized interval-valued intuitionistic fuzzy weighted averaging (GIIFWA) operator, where $\lambda > 0$, $\omega = (\omega_1, \omega_2, \cdots, \omega_n)^{\mathrm{T}}$ is the weight vector of $\tilde{\alpha}_j = ([a_j, b_j],\ [c_j, d_j])$ $(j = 1, 2, \cdots, n)$, with $\omega_j \in [0, 1]$ $(j = 1, 2, \cdots, n)$ and $\sum_{j=1}^{n} \omega_j = 1$.

Similar to the discussion of Section 1.5, we have

Theorem 2.5.1 (Zhao et al., 2010) The aggregated value by using the GIIFWA operator is an IVIFN, and

$$\begin{aligned} \mathrm{GIIFWA}_\omega(\tilde{\alpha}_1, \tilde{\alpha}_2, \cdots, \tilde{\alpha}_n) = & \left(\left[\left(1 - \prod_{j=1}^{n} (1 - a_j^{\lambda})^{\omega_j}\right)^{\frac{1}{\lambda}}, \left(1 - \prod_{j=1}^{n} (1 - b_j^{\lambda})^{\omega_j}\right)^{\frac{1}{\lambda}} \right], \right. \\ & \left[1 - \left(1 - \prod_{j=1}^{n} \left(1 - (1 - c_j)^{\lambda}\right)^{\omega_j}\right)^{\frac{1}{\lambda}}, \right. \\ & \left. \left. 1 - \left(1 - \prod_{j=1}^{n} \left(1 - (1 - d_j)^{\lambda}\right)^{\omega_j}\right)^{\frac{1}{\lambda}} \right] \right) \end{aligned} \tag{2.52}$$

Especially, we have

(1) If $a_j = b_j$, and $c_j = d_j$, for all j, i.e., all $\tilde{\alpha}_j$ $(j = 1, 2, \cdots, n)$ reduce to the IFNs, then the GIIFWA operator reduces to the GIFWA operator, which has the following form:

$$\mathrm{GIFWA}_\omega(\tilde{\alpha}_1, \tilde{\alpha}_2, \cdots, \tilde{\alpha}_n) = \left(\left(1 - \prod_{j=1}^{n} \left(1 - a_j^{\lambda}\right)^{\omega_j}\right)^{\frac{1}{\lambda}}, 1 - \left(1 - \prod_{j=1}^{n} (1 - (1 - c_j)^{\lambda})^{\omega_j}\right)^{\frac{1}{\lambda}} \right)$$

(2) If $a_j = b_j, c_j = d_j$ and $a_j + c_j = 1$, for all j, i.e., all $\tilde{\alpha}_j$ $(j = 1, 2, \cdots, n)$ reduce to the traditional fuzzy numbers, then the GIIFWA operator reduces to the GFWA operator, which has the following form:

$$\mathrm{GFWA}_\omega(\tilde{\alpha}_1, \tilde{\alpha}_2, \cdots, \tilde{\alpha}_n) = \left(\left(1 - \prod_{j=1}^{n} (1 - a_j^{\lambda})^{\omega_j}\right)^{\frac{1}{\lambda}}, 1 - \left(1 - \prod_{j=1}^{n} (1 - a_j^{\lambda})^{\omega_j}\right)^{\frac{1}{\lambda}} \right)$$

We now examine some special cases obtained by using different choices of the parameters ω and λ (Zhao et al., 2010):

(1) If λ=1, then the GIIFWA operator (2.52) reduces to the IIFWA operator.

(2) If $\omega = (1/n, 1/n, \cdots, 1/n)^{\mathrm{T}}$ and λ=1, then the GIIFWA operator (2.52) reduces to the IIFA operator.

Definition 2.5.2 (Zhao et al., 2010) Let GIIFOWA : $\tilde{\Theta}^n \rightarrow \tilde{\Theta}$. If

$$\text{GIIFOWA}_w(\tilde{\alpha}_1, \tilde{\alpha}_2, \cdots, \tilde{\alpha}_n) = (w_1(\tilde{\alpha}_{\sigma(1)})^\lambda \oplus w_2(\tilde{\alpha}_{\sigma(2)})^\lambda \oplus \cdots \oplus w_n(\tilde{\alpha}_{\sigma(n)})^\lambda)^{\frac{1}{\lambda}} \quad (2.53)$$

then the function GIIFOWA is called a generalized interval-valued intuitionistic fuzzy ordered weighted averaging (GIIFOWA) operator, where $\lambda > 0$, $w = (w_1, w_2, \cdots, w_n)^{\mathrm{T}}$ is the weight vector associated with the GIIFOWA operator, with $w_j \in [0,1]$ $(j = 1, 2, \cdots, n)$ and $\sum_{j=1}^{n} w_j = 1$, $\tilde{\alpha}_{\sigma(j)}$ is the j-th largest of $\tilde{\alpha}_j$ $(j = 1, 2, \cdots, n)$.

Similar to Theorem 2.5.1, we have

Theorem 2.5.2 (Zhao et al., 2010) The aggregated value by using the GIIFOWA operator is also an IVIFN, and

$$\begin{aligned}\text{GIIFOWA}_w(\tilde{\alpha}_1, \tilde{\alpha}_2, \cdots, \tilde{\alpha}_n) = \Bigg(& \left[\left(1-\prod_{j=1}^{n}(1-a_{\sigma(j)}^\lambda)^{w_j}\right)^{\frac{1}{\lambda}}, \left(1-\prod_{j=1}^{n}(1-b_{\sigma(j)}^\lambda)^{w_j}\right)^{\frac{1}{\lambda}}\right], \\ & \left[1-\left(1-\prod_{j=1}^{n}\left(1-(1-c_{\sigma(j)})^\lambda\right)^{w_j}\right)^{\frac{1}{\lambda}}, \right. \\ & \left. 1-\left(1-\prod_{j=1}^{n}\left(1-(1-d_{\sigma(j)})^\lambda\right)^{w_j}\right)^{\frac{1}{\lambda}}\right]\Bigg) \end{aligned} \quad (2.54)$$

where $\lambda > 0$ and $\tilde{\alpha}_{\sigma(j)} = ([a_{\sigma(j)}, b_{\sigma(j)}], [c_{\sigma(j)}, d_{\sigma(j)}])$ is the j-th largest of $\tilde{\alpha}_j$ $(j = 1, 2, \cdots, n)$.

Especially, we have

(1) If $a_j = b_j, c_j = d_j$ for all j, i.e., all $\tilde{\alpha}_j$ $(j = 1, 2, \cdots, n)$ reduce to the IFNs, then the GIIFOWA operator reduces to the GIFOWA operator.

(2) If $a_j = b_j, c_j = d_j$ and $a_j + c_j = 1$ for all j, i.e., all $\tilde{\alpha}_j$ $(j = 1, 2, \cdots, n)$ reduce to the ordinary fuzzy numbers, then the GIIFOWA operator reduces to the GFOWA operator, which has the following form:

$$\text{GFOWA}_w(\tilde{\alpha}_1, \tilde{\alpha}_2, \cdots, \tilde{\alpha}_n) = \left(\left(1-\prod_{j=1}^{n}(1-a_{\sigma(j)}^\lambda)^{w_j}\right)^{\frac{1}{\lambda}}, 1-\left(1-\prod_{j=1}^{n}\left(1-a_{\sigma(j)}^\lambda\right)^{w_j}\right)^{\frac{1}{\lambda}}\right)$$

We now study some special cases obtained by using different choices of the parameters w and λ (Zhao et al., 2010):

(1) If λ=1, then the GIIFOWA operator (2.54) reduces to the IIFOWA operator.

(2) If $w = (1/n, 1/n, 1/n, \cdots, 1/n)^{\mathrm{T}}$ and λ=1, then the GIIFOWA operator (2.54) reduces to the IIFA operator.

(3) If $w=(1,0,\cdots,0)^{\mathrm{T}}$, then the GIIFOWA operator (2.54) reduces to the following:

$$\mathrm{IIFMAX}_w(\tilde{\alpha}_1,\tilde{\alpha}_2,\cdots,\tilde{\alpha}_n)=\max_j(\tilde{\alpha}_j) \tag{2.55}$$

(4) If $w=(0,0,\cdots,1)^{\mathrm{T}}$, then the GIIFOWA operator (2.54) reduces to the following:

$$\mathrm{IIFMIN}_w(\tilde{\alpha}_1,\tilde{\alpha}_2,\cdots,\tilde{\alpha}_n)=\min_j(\tilde{\alpha}_j) \tag{2.56}$$

The GIIFWA operator weights only the IVIFNs, while the GIIFOWA operator weights only the ordered positions of the IVIFNs instead of the IVIFNs themselves. To overcome this limitation, we now introduce a generalized interval-valued intuitionistic fuzzy hybrid aggregation (GIIFHA) operator, which weights both the given IVIFNs and their ordered positions:

Definition 2.5.3 (Zhao et al., 2010) A generalized interval-valued intuitionistic fuzzy hybrid averaging (GIIFHA) operator of dimension n is a mapping GIIFHA : $\tilde{\Theta}^n \to \tilde{\Theta}$, which has an associated vector $w=(w_1,w_2,\cdots,w_n)^{\mathrm{T}}$, with $w_j\in[0,1]$ $(j=1,2,\cdots,n)$ and $\sum_{j=1}^{n} w_j=1$, $\lambda>0$, such that

$$\mathrm{GIIFHA}_{w,\omega}(\tilde{\alpha}_1,\tilde{\alpha}_2,\cdots,\tilde{\alpha}_n)=(w_1(\dot{\tilde{\alpha}}_{\sigma(1)})^{\lambda}\oplus w_2(\dot{\tilde{\alpha}}_{\sigma(2)})^{\lambda}\oplus\cdots\oplus w_n(\dot{\tilde{\alpha}}_{\sigma(n)})^{\lambda})^{\frac{1}{\lambda}} \tag{2.57}$$

where $\dot{\tilde{\alpha}}_{\sigma(j)}$ is the j-th largest of the weighted IVIFNs $\dot{\tilde{\alpha}}_j$ ($\dot{\tilde{\alpha}}_j=n\omega_j\tilde{\alpha}_j,\ j=1,2,\cdots,n$), $\omega=(\omega_1,\omega_2,\cdots,\omega_n)^{\mathrm{T}}$ is the weight vector of $\tilde{\alpha}_j$ $(j=1,2,\cdots,n)$ with $\omega_j\in[0,1]$, $\sum_{j=1}^{n}\omega_j=1$, and n is the balancing coefficient, which plays a role of balance.

Let $\dot{\tilde{\alpha}}_{\sigma(j)}=\left([\dot{a}_{\sigma(j)},\dot{b}_{\sigma(j)}],\ [\dot{c}_{\sigma(j)},\dot{d}_{\sigma(j)}]\right)$. Then, similar to Theorem 2.5.1, we have

$$\begin{aligned}\mathrm{GIIFHA}_{w,\omega}(\tilde{\alpha}_1,\tilde{\alpha}_2,\cdots,\tilde{\alpha}_n)=&\left(\left[\left(1-\prod_{j=1}^{n}(1-\dot{a}_{\sigma(j)}^{\lambda})^{w_j}\right)^{\frac{1}{\lambda}},\left(1-\prod_{j=1}^{n}(1-\dot{b}_{\sigma(j)}^{\lambda})^{w_j}\right)^{\frac{1}{\lambda}}\right],\right.\\&\left[1-\left(1-\prod_{j=1}^{n}\left(1-(1-\dot{c}_{\sigma(j)})^{\lambda}\right)^{w_j}\right)^{\frac{1}{\lambda}},\right.\\&\left.\left.1-\left(1-\prod_{j=1}^{n}\left(1-(1-\dot{d}_{\sigma(j)})^{\lambda}\right)^{w_j}\right)^{\frac{1}{\lambda}}\right]\right)\end{aligned} \tag{2.58}$$

and the aggregated value derived by using the GIIFHA operator is an IVIFN.

Especially, if $\lambda=1$, then Eq.(2.58) reduces to the IIFHA operator.

Theorem 2.5.3 (Zhao et al, 2010) The GIIFOWA operator is a special case of the GIIFHA operator.

Proof Let $\omega = (1/n, 1/n, 1/n, \cdots, 1/n)^{\mathrm{T}}$. Then $\dot{\tilde{\alpha}}_j = \tilde{\alpha}_j$ $(j = 1, 2, \cdots, n)$, so we have

$$\begin{aligned}\mathrm{GIIFHA}_{w,\omega}(\tilde{\alpha}_1, \tilde{\alpha}_2, \cdots, \tilde{\alpha}_n) &= (w_1(\dot{\tilde{\alpha}}_{\sigma(1)})^\lambda \oplus w_2(\dot{\tilde{\alpha}}_{\sigma(2)})^\lambda \oplus \cdots \oplus w_n(\dot{\tilde{\alpha}}_{\sigma(n)})^\lambda)^{\frac{1}{\lambda}} \\ &= (w_1(\tilde{\alpha}_{\sigma(1)})^\lambda \oplus w_2(\tilde{\alpha}_{\sigma(2)})^\lambda \oplus \cdots \oplus w_n(\tilde{\alpha}_{\sigma(n)})^\lambda)^{\frac{1}{\lambda}} \\ &= \mathrm{GIIFOWA}_w(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n)\end{aligned}$$

which completes the proof.

Similar to the IIFWA, IIFOWA and IFHA operators, the GIIFWA, GIIFOWA and GIIFHA operators can also be applied to multi-attribute decision making based on interval-valued intuitionistic fuzzy information.

2.6 Interval-Valued Intuitionistic Fuzzy Aggregation Operators Based on Choquet Integral

Based on Definition 1.5.1, Xu (2010c) uses Choquet integral to propose some operators for aggregating IVIFNs together with their correlative weights:

Definition 2.6.1 (Xu, 2010c) Let $\tilde{\alpha}(x_i) = ([a(x_i), b(x_i)],\ [c(x_i), d(x_i)])$ $(i = 1, 2, \cdots, n)$ be n IVIFNs, and ζ a fuzzy measure on $X = \{x_1, x_2, \cdots, x_n\}$. Then we call

$$\begin{aligned}(C_3)\int \tilde{\alpha}\mathrm{d}\zeta &= \mathrm{IIFCA}\,(\tilde{\alpha}(x_1), \tilde{\alpha}(x_2), \cdots, \tilde{\alpha}(x_n)) \\ &= (\zeta(B_{\sigma(1)}) - \zeta(B_{\sigma(0)}))\tilde{\alpha}(x_{\sigma(1)}) \oplus (\zeta(B_{\sigma(2)}) - \zeta(B_{\sigma(1)}))\tilde{\alpha}(x_{\sigma(2)}) \\ &\quad \oplus \cdots \oplus (\zeta(B_{\sigma(n)}) - \zeta(B_{\sigma(n-1)}))\tilde{\alpha}(x_{\dot{\sigma}(n)}) \end{aligned} \tag{2.59}$$

an interval-valued intuitionistic fuzzy correlated averaging (IIFCA) operator, where $(C_3)\int \tilde{\alpha}\mathrm{d}\zeta$ is a notation of Choquet integral, $\tilde{\alpha}(x_{\sigma(i)})$ indicates that the indices have been permuted so that $\tilde{\alpha}(x_{\sigma(1)}) \geqslant \tilde{\alpha}(x_{\sigma(2)}) \geqslant \cdots \geqslant \tilde{\alpha}(x_{\sigma(n)})$, $B_{\sigma(k)} = \{x_{\sigma(j)} \,|\, j \leqslant k\}$, when $k \geqslant 1$ and $B_{\sigma(0)} = \varnothing$.

With the operations of IVIFNs, the IIFCA operator (2.59) can be transformed into the following form by using mathematical induction on n:

$$\begin{aligned}(C_3)\int \tilde{\alpha}\mathrm{d}\zeta &= \mathrm{IIFCA}\,(\tilde{\alpha}(x_1), \tilde{\alpha}(x_2), \cdots, \tilde{\alpha}(x_n)) \\ &= \left(\left[1 - \prod_{i=1}^{n}\left(1 - a(x_{\sigma(i)})\right)^{\zeta(B_{\sigma(i)}) - \zeta(B_{\sigma(i-1)})}, \right.\right. \\ &\quad \left. 1 - \prod_{i=1}^{n}\left(1 - b(x_{\sigma(i)})\right)^{\zeta(B_{\sigma(i)}) - \zeta(B_{\sigma(i-1)})}\right],\end{aligned}$$

$$\left[\prod_{i=1}^{n}\left(c(x_{\sigma(i)})\right)^{\zeta(B_{\sigma(i)})-\zeta(B_{\sigma(i-1)})},\ \prod_{i=1}^{n}\left(d(x_{\sigma(i)})\right)^{\zeta(B_{\sigma(i)})-\zeta(B_{\sigma(i-1)})}\right]\right) \tag{2.60}$$

whose aggregated value is an IVIFN.

In what follows we discuss some special cases of the IIFCA operator (Xu, 2010c):

(1) If Eqs.(1.186) and (1.189) hold, then the IIFCA operators (2.59) and (2.60) reduce to the IIFWA operator

$$\begin{aligned}
&\text{IIFWA}\left(\tilde{\alpha}(x_1),\tilde{\alpha}(x_2),\cdots,\tilde{\alpha}(x_n)\right)\\
=&\zeta\left(\{x_1\}\right)\tilde{\alpha}(x_1)\oplus\zeta\left(\{x_2\}\right)\tilde{\alpha}(x_2)\oplus\cdots\oplus\zeta\left(\{x_n\}\right)\tilde{\alpha}(x_n)\\
=&\left(\left[1-\prod_{i=1}^{n}(1-a(x_i))^{\zeta(\{x_i\})},1-\prod_{i=1}^{n}(1-b(x_i))^{\zeta(\{x_i\})}\right],\right.\\
&\left.\left[\prod_{i=1}^{n}(c(x_i))^{\zeta(\{x_i\})},\prod_{i=1}^{n}(d(x_i))^{\zeta(\{x_i\})}\right]\right)
\end{aligned} \tag{2.61}$$

Especially, if $\zeta\left(\{x_i\}\right)=\dfrac{1}{n}$, for all $i=1,2,\cdots,n$, then the IIFWA operator (2.60) reduces to the IIFA operator:

$$\begin{aligned}
&\text{IIFA}\left(\tilde{\alpha}(x_1),\tilde{\alpha}(x_2),\cdots,\tilde{\alpha}(x_n)\right)\\
=&\left(\left[1-\left(\prod_{i=1}^{n}(1-a(x_i))\right)^{1/n},1-\left(\prod_{i=1}^{n}(1-b(x_i))\right)^{1/n}\right],\right.\\
&\left.\left[\left(\prod_{i=1}^{n}c(x_i)\right)^{1/n},\left(\prod_{i=1}^{n}d(x_i)\right)^{1/n}\right]\right)
\end{aligned} \tag{2.62}$$

(2) If Eqs.(1.192) and (1.193) hold, then the IIFCA operators (2.59) and (2.60) reduce to the IIFOWA operator:

$$\begin{aligned}
&\text{IIFOWA}\left(\tilde{\alpha}(x_1),\tilde{\alpha}(x_2),\cdots,\tilde{\alpha}(x_n)\right)\\
=&\omega_1\tilde{\alpha}(x_{\sigma(1)})\oplus\omega_2\tilde{\alpha}(x_{\sigma(2)})\oplus\cdots\oplus\omega_n\tilde{\alpha}(x_{\sigma(n)})\\
=&\left(\left[1-\prod_{i=1}^{n}\left(1-a(x_{\sigma(i)})\right)^{\omega_i},1-\prod_{i=1}^{n}\left(1-b(x_{\sigma(i)})\right)^{\omega_i}\right],\right.\\
&\left.\left[\prod_{i=1}^{n}\left(c(x_{\sigma(i)})\right)^{\omega_i},\prod_{i=1}^{n}\left(d(x_{\sigma(i)})\right)^{\omega_i}\right]\right)
\end{aligned} \tag{2.63}$$

Especially, if $\zeta(B)=\dfrac{|B|}{n}$, for all $B\subseteq X$, then both the IIFCA operator (2.59) and the IIFOWA operator (2.63) reduce to the IIFA operator.

(3) If Eqs.(1.194) and (1.195) hold, then the IIFCA operators (2.59) and (2.60) reduce to the following form:

$$\begin{aligned}&\text{IIFWOWA}\left(\tilde{\alpha}(x_1),\tilde{\alpha}(x_2),\cdots,\tilde{\alpha}(x_n)\right)\\ =&w_1\tilde{\alpha}(x_{\sigma(1)})\oplus w_2\tilde{\alpha}(x_{\sigma(2)})\oplus\cdots\oplus w_n\tilde{\alpha}(x_{\sigma(n)})\\ =&\left(\left[1-\prod_{i=1}^{n}\left(1-a(x_{\sigma(i)})\right)^{w_i},1-\prod_{i=1}^{n}\left(1-b(x_{\sigma(i)})\right)^{w_i}\right],\right.\\ &\left.\left[\prod_{i=1}^{n}\left(c(x_{\sigma(i)})\right)^{w_i},\prod_{i=1}^{n}\left(d(x_{\sigma(i)})\right)^{w_i}\right]\right)\end{aligned} \tag{2.64}$$

which we call an interval-valued intuitionistic fuzzy weighted ordered weighted averaging (IIFWOWA) operator. Especially, if $\zeta\left(\{x_i\}\right)=\dfrac{1}{n}$, for all $i=1,2,\cdots,n$, then the IIFWOWA operator reduces to the IIFOWA operator.

Definition 2.6.2 (Xu, 2010c) An interval-valued intuitionistic fuzzy correlated geometric (IIFCG) operator is defind as:

$$\begin{aligned}(C_4)\int\tilde{\alpha}\mathrm{d}\zeta&=\text{IIFCG}\left(\tilde{\alpha}(x_1),\tilde{\alpha}(x_2),\cdots,\tilde{\alpha}(x_n)\right)\\ &=\left(\tilde{\alpha}(x_{\sigma(1)})\right)^{\zeta(B_{\sigma(1)})-\zeta(B_{\sigma(0)})}\otimes\left(\tilde{\alpha}(x_{\sigma(2)})\right)^{\zeta(B_{\sigma(2)})-\zeta(B_{\dot{\sigma}(1)})}\\ &\quad\otimes\cdots\otimes\left(\tilde{\alpha}(x_{\sigma(n)})\right)^{\zeta(B_{\sigma(n)})-\zeta(B_{\sigma(n-1)})}\end{aligned} \tag{2.65}$$

where $(C_4)\int\tilde{\alpha}\mathrm{d}\zeta$ is a notation of Choquet integral, $\tilde{\alpha}(x_{s(i)})$ indicates that the indices have been permuted so that $\tilde{\alpha}(x_{\sigma(1)})\geqslant\tilde{\alpha}(x_{\sigma(2)})\geqslant\cdots\geqslant\tilde{\alpha}(x_{\sigma(n)})$, $B_{\sigma(k)}=\{x_{\sigma(j)}|\,j\leqslant k\}$, when $k\geqslant 1$ and $B_{\sigma(0)}=\varnothing$.

With the operations of IVIFNs, the IIFCG operator (2.65) can be transformed into the following form by applying mathematical induction on n:

$$\begin{aligned}(C_4)\int\tilde{\alpha}\mathrm{d}\zeta=&\text{IIFCG}\left(\tilde{\alpha}(x_1),\tilde{\alpha}(x_2),\cdots,\tilde{\alpha}(x_n)\right)\\ =&\left(\left[\prod_{i=1}^{n}\left(a(x_{\sigma(i)})\right)^{(\zeta(B_{\sigma(i)})-\zeta(B_{\sigma(i-1)})},\prod_{i=1}^{n}\left(b(x_{\sigma(i)})\right)^{\zeta(B_{\sigma(i)})-\zeta(B_{\sigma(i-1)})}\right],\right.\\ &\left[1-\prod_{i=1}^{n}\left(1-c(x_{\sigma(i)})\right)^{\zeta(B_{\sigma(i)})-\zeta(B_{\sigma(i-1)})},\right.\\ &\left.\left.1-\prod_{i=1}^{n}\left(1-d(x_{\sigma(i)})\right)^{\zeta(B_{\sigma(i)})-\zeta(B_{\sigma(i-1)})}\right]\right)\end{aligned} \tag{2.66}$$

whose aggregated value is an IVIFN.

Below we discuss some special cases of the IIFCG operator (Xu, 2010c):

(1) If Eqs.(1.184) and (1.187) hold, then the IIFCG operators (2.65) and (2.66) reduce to the IIFWG operator:

$$\begin{aligned}
&\text{IIFWG}\left(\tilde{\alpha}(x_1), \tilde{\alpha}(x_2), \cdots, \tilde{\alpha}(x_n)\right) \\
&= (\tilde{\alpha}(x_1))^{\zeta(\{x_1\})} \otimes (\tilde{\alpha}(x_2))^{\zeta(\{x_2\})} \otimes \cdots \otimes (\tilde{\alpha}(x_n))^{\zeta(\{x_n\})} \\
&= \left(\left[\prod_{i=1}^{n} (a(x_i))^{\zeta(\{x_i\})}, \prod_{i=1}^{n} (b(x_i))^{\zeta(\{x_i\})}\right],\right. \\
&\quad \left.\left[1-\prod_{i=1}^{n} (1-c(x_i))^{\zeta(\{x_i\})}, 1-\prod_{i=1}^{n} (1-d(x_i))^{\zeta(\{x_i\})}\right]\right)
\end{aligned} \tag{2.67}$$

Especially, if $\zeta\left(\{x_i\}\right) = \dfrac{1}{n}$, for all $i = 1, 2, \cdots, n$, then the IIFWG operator (2.67) reduces to the IIFG operator:

$$\begin{aligned}
&\text{IIFG}\left(\tilde{\alpha}(x_1), \tilde{\alpha}(x_2), \cdots, \tilde{\alpha}(x_n)\right) \\
&= \left(\left[\left(\prod_{i=1}^{n} a(x_i)\right)^{1/n}, \left(\prod_{i=1}^{n} b(x_i)\right)^{1/n}\right],\right. \\
&\quad \left.\left[1-\left(\prod_{i=1}^{n} (1-c(x_i))\right)^{1/n}, \left(1-\prod_{i=1}^{n} (1-d(x_i))\right)^{1/n}\right]\right)
\end{aligned} \tag{2.68}$$

(2) If Eqs.(1.192) and (1.193) hold, then the IIFCG operators (2.65) and (2.66) reduce to the IIFOWG operator:

$$\begin{aligned}
&\text{IIFOWG}\left(\tilde{\alpha}(x_1), \tilde{\alpha}(x_2), \cdots, \tilde{\alpha}(x_n)\right) \\
&= \left(\tilde{\alpha}(x_{\sigma(1)})\right)^{\omega_1} \otimes \left(\tilde{\alpha}(x_{\sigma(2)})\right)^{\omega_2} \otimes \cdots \otimes \left(\tilde{\alpha}(x_{\sigma(n)})\right)^{\omega_n} \\
&= \left(\left[\prod_{i=1}^{n} \left(a(x_{\sigma(i)})\right)^{\omega_i}, \prod_{i=1}^{n} \left(b(x_{\sigma(i)})\right)^{\omega_i}\right],\right. \\
&\quad \left.\left[1-\prod_{i=1}^{n} \left(1-c(x_{\sigma(i)})\right)^{\omega_i}, 1-\prod_{i=1}^{n} \left(1-d(x_{\sigma(i)})\right)^{\omega_i}\right]\right)
\end{aligned} \tag{2.69}$$

Especially, if $\mu(B) = \dfrac{|B|}{n}$, for all $B \subseteq X$, then both the IIFCG operator (2.65) and the IIFOWG operator (2.69) reduce to the IIFG operator.

(3) If Eqs.(1.194) and (1.195) hold, then the IIFCG operators (2.65) and (2.69) reduce to the following form:

$$\text{IIFWOWG}\left(\tilde{\alpha}(x_1), \tilde{\alpha}(x_2), \cdots, \tilde{\alpha}(x_n)\right)$$

$$= \left(\tilde{\alpha}(x_{\sigma(1)})\right)^{w_1} \otimes \left(\tilde{\alpha}(x_{\sigma(2)})\right)^{w_2} \otimes \cdots \otimes \left(\tilde{\alpha}(x_{\sigma(n)})\right)^{w_n}$$
$$= \left(\left[\prod_{i=1}^{n}\left(a(x_{\sigma(i)})\right)^{w_i}, \prod_{i=1}^{n}\left(b(x_{\sigma(i)})\right)^{w_i}\right], \right.$$
$$\left.\left[1-\prod_{i=1}^{n}\left(1-c(x_{\sigma(i)})\right)^{\omega_i}, 1-\prod_{i=1}^{n}\left(1-d(x_{\sigma(i)})\right)^{\omega_i}\right]\right) \tag{2.70}$$

which we call an interval-valued intuitionistic fuzzy weighted ordered weighted geometric (IIFWOWG) operator. Especially, if $\zeta(\{x_i\}) = \dfrac{1}{n}$, for all $i = 1, 2, \cdots, n$, then the IIFWOWG operator reduces to the IIFOWG operator.

Example 2.6.1 (Xu, 2010c) If the evaluation information in Example 1.6.1 is represented by the IVIFNs $\tilde{\alpha}_{Y_i}$ (G_j) $(i = 1, 2, \cdots, 10;\ j = 1, 2, 3)$, as shown in Table 2.12:

Table 2.12 The evaluation information about projects

	G_1	G_2	G_3
Y_1	([0.7,0.8],[0.1,0.2])	([0.8,0.9],[0.0,0.1])	([0.6,0.9],[0.0,0.1])
Y_2	([0.6,0.7],[0.1,0.3])	([0.7,0.8],[0.1,0.2])	([0.5,0.6],[0.2,0.3])
Y_3	([0.4,0.5],[0.3,0.4])	([0.6,0.7],[0.2,0.3])	([0.4,0.6],[0.1,0.2])
Y_4	([0.5,0.7],[0.1,0.2])	([0.6,0.7],[0.1,0.2])	([0.5,0.6],[0.3,0.4])
Y_5	([0.3,0.4],[0.4,0.5])	([0.4,0.6],[0.2,0.3])	([0.3,0.4],[0.3,0.5])
Y_6	([0.4,0.5],[0.2,0.3])	([0.3,0.6],[0.1,0.2])	([0.5,0.8],[0.1,0.2])
Y_7	([0.2,0.3],[0.4,0.6])	([0.3,0.5],[0.4,0.5])	([0.2,0.4],[0.4,0.5])
Y_8	([0.4,0.6],[0.1,0.3])	([0.5,0.7],[0.1,0.3])	([0.7,0.8],[0.1,0.2])
Y_9	([0.4,0.5],[0.3,0.4])	([0.7,0.8],[0.1,0.2])	([0.2,0.3],[0.1,0.3])
Y_{10}	([0.1,0.3],[0.5,0.7])	([0.6,0.7],[0.2,0.3])	([0.3,0.5],[0.3,0.4])

then we can rearrange the IVIFNs corresponding to each project in descending order by using the method presented in Section 2.2:

$$\tilde{\alpha}_{Y_1}(G_{\sigma(1)}) = ([0.8, 0.9],\ [0.0, 0.1]), \quad \tilde{\alpha}_{Y_1}(G_{\sigma(2)}) = ([0.6, 0.9],\ [0.0, 0.1])$$
$$\tilde{\alpha}_{Y_1}(G_{\sigma(3)}) = ([0.7, 0.8],\ [0.1, 0.2])$$
$$\tilde{\alpha}_{Y_2}(G_{\sigma(1)}) = ([0.7, 0.8],\ [0.1, 0.2]), \quad \tilde{\alpha}_{Y_2}(G_{\sigma(2)}) = ([0.6, 0.7],\ [0.1, 0.3])$$
$$\tilde{\alpha}_{Y_2}(G_{\sigma(3)}) = ([0.5, 0.6],\ [0.2, 0.3])$$
$$\tilde{\alpha}_{Y_3}(G_{\sigma(1)}) = ([0.6, 0.7],\ [0.2, 0.3]), \quad \tilde{\alpha}_{Y_3}(G_{\sigma(2)}) = ([0.4, 0.6],\ [0.1, 0.2])$$
$$\tilde{\alpha}_{Y_3}(G_{\sigma(3)}) = ([0.4, 0.5],\ [0.3, 0.4])$$
$$\tilde{\alpha}_{Y_4}(G_{\sigma(1)}) = ([0.6, 0.7],\ [0.1, 0.2]), \quad \tilde{\alpha}_{Y_4}(G_{\sigma(2)}) = ([0.5, 0.7],\ [0.1, 0.2])$$
$$\tilde{\alpha}_{Y_4}(G_{\sigma(3)}) = ([0.5, 0.6],\ [0.3, 0.4])$$

$$\tilde{\alpha}_{Y_5}(G_{\sigma(1)}) = ([0.4, 0.6],\ [0.2, 0.3]), \quad \tilde{\alpha}_{Y_5}(G_{\sigma(2)}) = ([0.3, 0.4],\ [0.3, 0.5])$$

$$\tilde{\alpha}_{Y_5}(G_{\sigma(3)}) = ([0.3, 0.4],\ [0.4, 0.5])$$

$$\tilde{\alpha}_{Y_6}(G_{\sigma(1)}) = ([0.5, 0.8],\ [0.1, 0.2]), \quad \tilde{\alpha}_{Y_6}(G_{\sigma(2)}) = ([0.3, 0.6],\ [0.1, 0.2])$$

$$\tilde{\alpha}_{Y_6}(G_{\sigma(3)}) = ([0.4, 0.5],\ [0.2, 0.3])$$

$$\tilde{\alpha}_{Y_7}(G_{\sigma(1)}) = ([0.3, 0.5],\ [0.4, 0.5]), \quad \tilde{\alpha}_{Y_7}(G_{\sigma(2)}) = ([0.2, 0.4],\ [0.4, 0.5])$$

$$\tilde{\alpha}_{Y_7}(G_{\sigma(3)}) = ([0.2, 0.3],\ [0.4, 0.6])$$

$$\tilde{\alpha}_{Y_8}(G_{\sigma(1)}) = ([0.7, 0.8],\ [0.1, 0.2]), \quad \tilde{\alpha}_{Y_8}(G_{\sigma(2)}) = ([0.5, 0.7],\ [0.1, 0.3])$$

$$\tilde{\alpha}_{Y_8}(G_{\sigma(3)}) = ([0.4, 0.6],\ [0.1, 0.3])$$

$$\tilde{\alpha}_{Y_9}(G_{\sigma(1)}) = ([0.7, 0.8],\ [0.1, 0.2]), \quad \tilde{\alpha}_{Y_9}(G_{\sigma(2)}) = ([0.4, 0.5],\ [0.3, 0.4])$$

$$\tilde{\alpha}_{Y_9}(G_{\sigma(3)}) = ([0.2, 0.3],\ [0.1, 0.3])$$

$$\tilde{\alpha}_{Y_{10}}(G_{\sigma(1)}) = ([0.6, 0.7],\ [0.2, 0.3]), \quad \tilde{\alpha}_{Y_{10}}(G_{\sigma(2)}) = ([0.3, 0.5],\ [0.3, 0.4])$$

$$\tilde{\alpha}_{Y_{10}}(G_{\sigma(3)}) = ([0.1, 0.3],\ [0.5, 0.7])$$

If we use the IIFCA operator (2.60) to calculate the overall evaluation information corresponding to each project, then

$$\text{IIFCA}\,(\tilde{\alpha}_{Y_1}(G_1), \tilde{\alpha}_{Y_1}(G_2), \tilde{\alpha}_{Y_1}(G_3)) = ([0.71, 0.89],\ [0.00, 0.11])$$

$$\text{IIFCA}\,(\tilde{\alpha}_{Y_2}(G_1), \tilde{\alpha}_{Y_2}(G_2), \tilde{\alpha}_{Y_2}(G_3)) = ([0.61, 0.71],\ [0.13, 0.26])$$

$$\text{IIFCA}\,(\tilde{\alpha}_{Y_3}(G_1), \tilde{\alpha}_{Y_3}(G_2), \tilde{\alpha}_{Y_3}(G_3)) = ([0.49, 0.63],\ [0.16, 0.27])$$

$$\text{IIFCA}\,(\tilde{\alpha}_{Y_4}(G_1), \tilde{\alpha}_{Y_4}(G_2), \tilde{\alpha}_{Y_4}(G_3)) = ([0.54, 0.66],\ [0.16, 0.26])$$

$$\text{IIFCA}\,(\tilde{\alpha}_{Y_5}(G_1), \tilde{\alpha}_{Y_5}(G_2), \tilde{\alpha}_{Y_5}(G_3)) = ([0.34, 0.50],\ [0.29, 0.41])$$

$$\text{IIFCA}\,(\tilde{\alpha}_{Y_6}(G_1), \tilde{\alpha}_{Y_6}(G_2), \tilde{\alpha}_{Y_6}(G_3)) = ([0.39, 0.66],\ [0.11, 0.22])$$

$$\text{IIFCA}\,(\tilde{\alpha}_{Y_7}(G_1), \tilde{\alpha}_{Y_7}(G_2), \tilde{\alpha}_{Y_7}(G_3)) = ([0.24, 0.42],\ [0.40, 0.52])$$

$$\text{IIFCA}\,(\tilde{\alpha}_{Y_8}(G_1), \tilde{\alpha}_{Y_2}(G_2), \tilde{\alpha}_{Y_8}(G_3)) = ([0.56, 0.72],\ [0.10, 0.27])$$

$$\text{IIFCA}\,(\tilde{\alpha}_{Y_9}(G_1), \tilde{\alpha}_{Y_9}(G_2), \tilde{\alpha}_{Y_9}(G_3)) = ([0.49, 0.60],\ [0.12, 0.27])$$

$$\text{IIFCA}\,(\tilde{\alpha}_{Y_{10}}(G_1), \tilde{\alpha}_{Y_{10}}(G_2), \tilde{\alpha}_{Y_{10}}(G_3)) = ([0.41, 0.56],\ [0.25, 0.37])$$

After that, we can rank the above IVIFNs by using the method presented in Section 2.2:

$$\begin{aligned}([0.71, 0.89],\ [0.00, 0.11]) &> ([0.61, 0.71],\ [0.13, 0.26]) > ([0.56, 0.72],\ [0.10, 0.27]) \\ &> ([0.54, 0.66],\ [0.16, 0.26]) > ([0.39, 0.66],\ [0.11, 0.22])\end{aligned}$$

$$> ([0.49, 0.60],\ [0.12, 0.27]) > ([0.49, 0.63],\ [0.16, 0.27])$$
$$> ([0.41, 0.56],\ [0.25, 0.37]) > ([0.34, 0.50],\ [0.29, 0.41])$$
$$> ([0.24, 0.42],\ [0.40, 0.52])$$

The ranking of the four projects Y_i $(i = 1, 2, \cdots, 10)$ can now be obtained:

$$Y_1 \succ Y_2 \succ Y_8 \succ Y_4 \succ Y_6 \succ Y_9 \succ Y_3 \succ Y_{10} \succ Y_5 \succ Y_7$$

If we use the IIFCG operator (2.66) to calculate the overall evaluation information corresponding to each project, then

$$\text{IIFCG}\,(\tilde{\alpha}_{Y_1}(G_1), \tilde{\alpha}_{Y_1}(G_2), \tilde{\alpha}_{Y_1}(G_3)) = ([0.69, 0.88],\ [0.02, 0.12])$$

$$\text{IIFCG}\,(\tilde{\alpha}_{Y_2}(G_1), \tilde{\alpha}_{Y_2}(G_2), \tilde{\alpha}_{Y_2}(G_3)) = ([0.59, 0.69],\ [0.14, 0.26])$$

$$\text{IIFCG}\,(\tilde{\alpha}_{Y_3}(G_1), \tilde{\alpha}_{Y_3}(G_2), \tilde{\alpha}_{Y_3}(G_3)) = ([0.47, 0.62],\ [0.18, 0.28])$$

$$\text{IIFCG}\,(\tilde{\alpha}_{Y_4}(G_1), \tilde{\alpha}_{Y_4}(G_2), \tilde{\alpha}_{Y_4}(G_3)) = ([0.54, 0.66],\ [0.19, 0.29])$$

$$\text{IIFCG}\,(\tilde{\alpha}_{Y_5}(G_1), \tilde{\alpha}_{Y_5}(G_2), \tilde{\alpha}_{Y_5}(G_3)) = ([0.34, 0.47],\ [0.31, 0.43])$$

$$\text{IIFCG}\,(\tilde{\alpha}_{Y_6}(G_1), \tilde{\alpha}_{Y_6}(G_2), \tilde{\alpha}_{Y_6}(G_3)) = ([0.37, 0.63],\ [0.14, 0.24])$$

$$\text{IIFCG}\,(\tilde{\alpha}_{Y_7}(G_1), \tilde{\alpha}_{Y_7}(G_2), \tilde{\alpha}_{Y_7}(G_3)) = ([0.24, 0.41],\ [0.40, 0.52])$$

$$\text{IIFCG}\,(\tilde{\alpha}_{Y_8}(G_1), \tilde{\alpha}_{Y_8}(G_2), \tilde{\alpha}_{Y_8}(G_3)) = ([0.53, 0.71],\ [0.10, 0.27])$$

$$\text{IIFCG}\,(\tilde{\alpha}_{Y_9}(G_1), \tilde{\alpha}_{Y_9}(G_2), \tilde{\alpha}_{Y_9}(G_3)) = ([0.38, 0.49],\ [0.14, 0.28])$$

$$\text{IIFCG}\,(\tilde{\alpha}_{Y_{10}}(G_1), \tilde{\alpha}_{Y_{10}}(G_2), \tilde{\alpha}_{Y_{10}}(G_3)) = ([0.32, 0.52],\ [0.31, 0.44])$$

We can rank the above IVIFNs by using the method presented in Section 2.2:

$$([0.69, 0.88],\ [0.02, 0.12]) > ([0.59, 0.69],\ [0.14, 0.26]) > ([0.53, 0.71],\ [0.10, 0.27])$$
$$> ([0.54, 0.66],\ [0.19, 0.29]) > ([0.47, 0.62],\ [0.18, 0.28])$$
$$> ([0.37, 0.63],\ [0.14, 0.24]) > ([0.38, 0.49],\ [0.14, 0.28])$$
$$> ([0.32, 0.52],\ [0.31, 0.44]) > ([0.34, 0.47],\ [0.31, 0.43])$$
$$> ([0.24, 0.41],\ [0.40, 0.52])$$

Thus the ranking of the ten projects Y_i $(i = 1, 2, \cdots, 10)$ is:

$$Y_1 \succ Y_2 \succ Y_8 \succ Y_4 \succ Y_3 \succ Y_6 \succ Y_9 \succ Y_{10} \succ Y_5 \succ Y_7$$

which indicates that the IIFCA and IIFCG operators produce slightly different ranking results, but the best project is Y_1 in both the cases.

2.7 Induced Generalized Interval-Valued Intuitionistic Fuzzy Aggregation Operators

Xu and Xia (2011) extend the generalized intuitionistic fuzzy correlated averaging operators to aggregate interval-valued intuitionistic fuzzy information:

Definition 2.7.1 (Xu and Xia, 2011) An induced generalized interval-valued intuitionistic fuzzy correlated averaging IGIIFCA operator of dimension n is a function IGIIFCA : $\tilde{\Theta}^n \rightarrow \tilde{\Theta}$, which is defined to aggregate the set of second arguments of a list of 2-tuples $\{\langle\nabla_1,\tilde{\alpha}_1\rangle,\langle\nabla_2,\tilde{\alpha}_2\rangle,\cdots,\langle\nabla_n,\tilde{\alpha}_n\rangle\}$ according to the following expression:

$$\begin{aligned}&\text{IGIIFCA}\left(\langle\nabla_1,\tilde{\alpha}_1\rangle,\langle\nabla_2,\tilde{\alpha}_2\rangle,\cdots,\langle\nabla_n,\tilde{\alpha}_n\rangle\right)\\ =&\Big((\zeta(A_{\sigma(1)})-\zeta(A_{\sigma(0)}))\tilde{\alpha}^{\lambda}_{\sigma(1)}\oplus(\zeta(A_{\sigma(2)})-\zeta(A_{\sigma(1)}))\tilde{\alpha}^{\lambda}_{\sigma(2)}\oplus\cdots\\ &\oplus(\zeta(A_{\sigma(n)})-\zeta(A_{\sigma(n-1)}))\tilde{\alpha}^{\lambda}_{\sigma(n)}\Big)^{1/\lambda}\end{aligned}\tag{2.71}$$

where $\lambda>0$, ∇_i in 2-tuples $\langle\nabla_i,\tilde{\alpha}_i\rangle$ is referred to as the order-inducing variable and $\tilde{\alpha}_i$ as the argument variable, $\sigma(i)$: $\{1,2,\cdots,n\}\rightarrow\{1,2,\cdots,n\}$ is a permutation such that $\nabla_{\sigma(1)}\geqslant\nabla_{\sigma(2)}\geqslant\cdots\geqslant\nabla_{\sigma(n)}$, $A_{\sigma(i)}=\left\{\tilde{\alpha}_{\sigma(1)},\tilde{\alpha}_{\sigma(2)},\cdots,\tilde{\alpha}_{\sigma(i)}\right\}$ when $i\geqslant 1$ and $A_{\sigma(0)}=\varnothing$. Using the operations for IVIFNs, we can get

$$\begin{aligned}&\text{IGIIFCA}\left(\langle\nabla_1,\tilde{\alpha}_1\rangle,\langle\nabla_2,\tilde{\alpha}_2\rangle,\cdots,\langle\nabla_n,\tilde{\alpha}_n\rangle\right)\\ =&\left(\left[\left(1-\prod_{i=1}^{n}(1-(\tilde{\mu}^{\mathrm{L}}_{\tilde{\alpha}_{\sigma(i)}})^{\lambda})^{\zeta(A_{\sigma(i)})-\zeta(A_{\sigma(i-1)})}\right)^{1/\lambda},\right.\right.\\ &\left.\left(1-\prod_{i=1}^{n}(1-(\tilde{\mu}^{\mathrm{U}}_{\tilde{\alpha}_{\sigma(i)}})^{\lambda})^{\zeta(A_{\sigma(i)})-\zeta(A_{\sigma(i-1)})}\right)^{1/\lambda}\right],\\ &\left[1-\left(1-\prod_{i=1}^{n}(1-(1-\tilde{\nu}^{\mathrm{L}}_{\tilde{\alpha}_{\sigma(i)}})^{\lambda})^{\zeta(A_{\sigma(i)})-\zeta(A_{\sigma(i-1)})}\right)^{1/\lambda},\right.\\ &\left.\left.1-\left(1-\prod_{i=1}^{n}(1-(1-\tilde{\nu}^{\mathrm{U}}_{\tilde{\alpha}_{\sigma(i)}})^{\lambda})^{\zeta(A_{\sigma(i)})-\zeta(A_{\sigma(i-1)})}\right)^{1/\lambda}\right]\right)\end{aligned}\tag{2.72}$$

In particular, if $\nabla_i=\nabla_j$ in two 2-tuples $\langle\nabla_i,\tilde{\alpha}_i\rangle$ and $\langle\nabla_j,\tilde{\alpha}_j\rangle$, then we can replace $\tilde{\alpha}_i$ and $\tilde{\alpha}_j$ by their average, i.e., $(\tilde{\alpha}_i\oplus\tilde{\alpha}_j)/2$. If k items are tied, we can replace these by k replica's of their average.

In the case where $\nabla_{\sigma(1)}\geqslant\nabla_{\sigma(2)}\geqslant\cdots\geqslant\nabla_{\sigma(n)}$ and $\tilde{\alpha}_{\sigma(1)}\geqslant\tilde{\alpha}_{\sigma(2)}\geqslant\cdots\geqslant\tilde{\alpha}_{\sigma(n)}$, the IGIIFCA operator (2.72) becomes

$$\begin{aligned}&\text{GIIFCA}(\tilde{\alpha}_1,\tilde{\alpha}_2,\cdots,\tilde{\alpha}_n)\\ =&(\zeta(A_{\sigma(1)})-\zeta(A_{\sigma(0)}))\tilde{\alpha}_{\sigma(1)}\oplus(\zeta(A_{\sigma(2)})-\zeta(A_{\sigma(1)}))\tilde{\alpha}_{\sigma(2)}\oplus\cdots\end{aligned}$$

$$
\begin{aligned}
&\oplus (\zeta(A_{\sigma(n)}) - \zeta(A_{\sigma(n-1)}))\tilde{\alpha}_{\sigma(n)} \\
=&\left(\left[\left(1-\prod_{j=1}^{n}(1-(\tilde{\mu}^{\mathrm{L}}_{\tilde{\alpha}_{\sigma(j)}})^{\lambda})^{\zeta(A_{\sigma(j)})-\zeta(A_{\sigma(j-1)})}\right)^{1/\lambda},\right.\right. \\
&\left.\left(1-\prod_{j=1}^{n}(1-(\tilde{\mu}^{\mathrm{U}}_{\tilde{\alpha}_{\sigma(j)}})^{\lambda})^{\zeta(A_{\sigma(j)})-\zeta(A_{\sigma(j-1)})}\right)^{1/\lambda}\right], \\
&\left[1-\left(1-\prod_{j=1}^{n}(1-(1-\tilde{\nu}^{\mathrm{L}}_{\tilde{\alpha}_{\sigma(j)}})^{\lambda})^{\zeta(A_{\sigma(j)})-\zeta(A_{\sigma(j-1)})}\right)^{1/\lambda},\right. \\
&\left.\left.1-\left(1-\prod_{j=1}^{n}(1-(1-\tilde{\nu}^{\mathrm{U}}_{\tilde{\alpha}_{\sigma(j)}})^{\lambda})^{\zeta(A_{\sigma(j)})-\zeta(A_{\sigma(j-1)})}\right)^{1/\lambda}\right]\right) \qquad (2.73)
\end{aligned}
$$

which we call a generalized interval-valued intuitionistic fuzzy correlated averaging (GIIFCA) operator, where $\tilde{\alpha}_{\sigma(j)}$ is the j-th largest of $\tilde{\alpha}_i$ $(i = 1, 2, \cdots, n)$ (Xu and Xia, 2011).

Especially, if $\lambda = 1$, then Eq.(2.73) reduces to the induced interval-valued intuitionistic fuzzy correlated averaging (IIIFCA) operator (Xu and Xia, 2011):

$$
\begin{aligned}
&\mathrm{IIIFCA}\left(\langle \nabla_1, \tilde{\alpha}_1\rangle, \langle \nabla_2, \tilde{\alpha}_2\rangle, \cdots, \langle \nabla_n, \tilde{\alpha}_n\rangle\right) \\
=&(\zeta(A_{\sigma(1)}) - \zeta(A_{\sigma(0)}))\tilde{\alpha}_{\sigma(1)} \oplus (\zeta(A_{\sigma(2)}) - \zeta(A_{\sigma(1)}))\tilde{\alpha}_{\sigma(2)} \oplus \cdots \\
&\oplus (\zeta(A_{\sigma(n)}) - \zeta(A_{\sigma(n-1)}))\tilde{\alpha}_{\sigma(n)} \\
=&\left(\left[1-\prod_{i=1}^{n}\left(1-\tilde{\mu}^{\mathrm{L}}_{\tilde{\alpha}_{\sigma(i)}}\right)^{\zeta(A_{\sigma(i)})-\zeta(A_{\sigma(i-1)})}, 1-\prod_{i=1}^{n}\left(1-\tilde{\mu}^{\mathrm{U}}_{\tilde{\alpha}_{\sigma(i)}}\right)^{\zeta(A_{\sigma(i)})-\zeta(A_{\sigma(i-1)})}\right],\right. \\
&\left.\left[\prod_{i=1}^{n}\left(\tilde{\nu}^{\mathrm{L}}_{\tilde{\alpha}_{\sigma(i)}}\right)^{\zeta(A_{\sigma(i)})-\zeta(A_{\sigma(i-1)})}, \prod_{i=1}^{n}\left(\tilde{\nu}^{\mathrm{U}}_{\tilde{\alpha}_{\sigma(i)}}\right)^{\zeta(A_{\sigma(i)})-\zeta(A_{\sigma(i-1)})}\right]\right) \qquad (2.74)
\end{aligned}
$$

Furthermore, if $u_{\sigma(1)} \geqslant u_{\sigma(2)} \geqslant \cdots \geqslant u_{\sigma(n)}$ and $\tilde{\alpha}_{\sigma(1)} \geqslant \tilde{\alpha}_{\sigma(2)} \geqslant \cdots \geqslant \tilde{\alpha}_{\sigma(n)}$, then the IIIFCA operator (2.74) becomes the IIFCA operator (Xu, 2010c):

$$
\begin{aligned}
&\mathrm{IIFCA}(\tilde{\alpha}_1, \tilde{\alpha}_2, \cdots, \tilde{\alpha}_n) \\
=&(\zeta(A_{\sigma(1)}) - \zeta(A_{\sigma(0)}))\tilde{\alpha}_{\sigma(1)} \oplus (\zeta(A_{\sigma(2)}) - \zeta(A_{\sigma(1)}))\tilde{\alpha}_{\sigma(2)} \oplus \cdots \\
&\oplus (\zeta(A_{\sigma(n)}) - \zeta(A_{\sigma(n-1)}))\tilde{\alpha}_{\sigma(n)} \\
=&\left(\left[1-\prod_{j=1}^{n}\left(1-\tilde{\mu}^{\mathrm{L}}_{\tilde{\alpha}_{\sigma(j)}}\right)^{\zeta(A_{\sigma(j)})-\zeta(A_{\sigma(j-1)})}, 1-\prod_{j=1}^{n}\left(1-\tilde{\mu}^{\mathrm{U}}_{\tilde{\alpha}_{\sigma(j)}}\right)^{\zeta(A_{\sigma(j)})-\zeta(A_{\sigma(j-1)})}\right],\right.
\end{aligned}
$$

$$\left[\prod_{j=1}^{n}\left(\tilde{\nu}_{\tilde{\alpha}_{\sigma(j)}}^{\mathrm{L}}\right)^{\zeta(A_{\sigma(j)})-\zeta(A_{\sigma(j-1)})},\prod_{j=1}^{n}\left(\tilde{\nu}_{\tilde{\alpha}_{\sigma(j)}}^{\mathrm{U}}\right)^{\zeta(A_{\sigma(j)})-\zeta(A_{\sigma(j-1)})}\right]\Bigg) \tag{2.75}$$

where $\tilde{\alpha}_{\sigma(j)}$ is the j-th largest of $\tilde{\alpha}_i$ $(i=1,2,\cdots,n)$.

If $\lambda=0$, then we have

$$\begin{aligned}&\mathrm{IIIFCG}\left(\langle\nabla_1,\tilde{\alpha}_1\rangle,\langle\nabla_2,\tilde{\alpha}_2\rangle,\cdots,\langle\nabla_n,\tilde{\alpha}_n\rangle\right)\\ =&(\tilde{\alpha}_{\sigma(1)})^{\zeta(A_{\sigma(1)})-\zeta(A_{\sigma(0)})}\otimes(\tilde{\alpha}_{\sigma(2)})^{\zeta(A_{\sigma(2)})-\zeta(A_{\sigma(1)})}\otimes\cdots\\ &\otimes(\tilde{\alpha}_{\sigma(n)})^{\zeta(A_{\sigma(n)})-\zeta(A_{\sigma(n-1)})}\\ =&\Bigg(\left[\prod_{i=1}^{n}\left(\tilde{\mu}_{\tilde{\alpha}_{\sigma(i)}}^{\mathrm{L}}\right)^{\zeta(A_{\sigma(i)})-\zeta(A_{\sigma(i-1)})},\prod_{i=1}^{n}\left(\tilde{\mu}_{\tilde{\alpha}_{\sigma(i)}}^{\mathrm{U}}\right)^{\zeta(A_{\sigma(i)})-\zeta(A_{\sigma(i-1)})}\right],\\ &\left[1-\prod_{i=1}^{n}\left(1-\tilde{\nu}_{\tilde{\alpha}_{\sigma(i)}}^{\mathrm{L}}\right)^{\zeta(A_{\sigma(i)})-\zeta(A_{\sigma(i-1)})},1-\prod_{i=1}^{n}\left(1-\tilde{\nu}_{\tilde{\alpha}_{\sigma(i)}}^{\mathrm{U}}\right)^{\zeta(A_{\sigma(i)})-\zeta(A_{\sigma(i-1)})}\right]\Bigg)\end{aligned} \tag{2.76}$$

which we call an induced interval-valued intuitionistic fuzzy correlated geometric (IIFCG) operator (Xu and Xia, 2011).

Especially, if $\nabla_{\sigma(1)}\geqslant\nabla_{\sigma(2)}\geqslant\cdots\geqslant\nabla_{\sigma(n)}$ and $\tilde{\alpha}_{\sigma(1)}\geqslant\tilde{\alpha}_{\sigma(2)}\geqslant\cdots\geqslant\tilde{\alpha}_{\sigma(n)}$, then the IIFCG operator (2.76) reduces to the interval-valued intuitionistic fuzzy correlated geometric (IIFCG) operator (Xu, 2010c):

$$\begin{aligned}&\mathrm{IIFCG}(\tilde{\alpha}_1,\tilde{\alpha}_2,\cdots,\tilde{\alpha}_n)\\ =&(\tilde{\alpha}_{\sigma(1)})^{\zeta(A_{\sigma(1)})-\zeta(A_{\sigma(0)})}\otimes(\tilde{\alpha}_{\sigma(2)})^{\zeta(A_{\sigma(2)})-\zeta(A_{\sigma(1)})}\otimes\cdots\\ &\otimes(\tilde{\alpha}_{\sigma(n)})^{\zeta(A_{\sigma(n)})-\zeta(A_{\sigma(n-1)})}\\ =&\Bigg(\left[\prod_{j=1}^{n}\left(\tilde{\mu}_{\tilde{\alpha}_{\sigma(j)}}^{\mathrm{L}}\right)^{\zeta(A_{\sigma(j)})-\zeta(A_{\sigma(j-1)})},\prod_{j=1}^{n}\left(\tilde{\mu}_{\tilde{\alpha}_{\sigma(j)}}^{\mathrm{U}}\right)^{\zeta(A_{\sigma(j)})-\zeta(A_{\sigma(j-1)})}\right],\\ &\left[1-\prod_{j=1}^{n}\left(1-\tilde{\nu}_{\tilde{\alpha}_{\sigma(j)}}^{\mathrm{L}}\right)^{\zeta(A_{\sigma(j)})-\zeta(A_{\sigma(j-1)})},1-\prod_{j=1}^{n}\left(1-\tilde{\nu}_{\tilde{\alpha}_{\sigma(j)}}^{\mathrm{U}}\right)^{\zeta(A_{\sigma(j)})-\zeta(A_{\sigma(j-1)})}\right]\Bigg)\end{aligned} \tag{2.77}$$

where $\tilde{\alpha}_{\sigma(j)}$ is the j-th largest of $\tilde{\alpha}_i$ $(i=1,2,\cdots,n)$.

If each payoff value of the alternative Y_i under the state G_j in Section 1.7 is given by an IVIFN $\tilde{\alpha}_{ij}$, then we can develop a method for decision making based on the IGIIFCA operator (Xu and Xia, 2011).

Step 1 Calculate the correlations between the states of nature using the Choquet integral.

Step 2 Calculate the inducing variables matrix $\nabla=(\nabla_{ij})_{n\times m}$.

Step 3 Utilize Eq.(2.72) to get the expected result $\tilde{C}_i$ for the alternative Y_i:

$$\tilde{C}_i=\mathrm{IGIIFCA}\left(\langle\nabla_{i1},\tilde{\alpha}_{i1}\rangle,\langle\nabla_{i2},\tilde{\alpha}_{i2}\rangle,\cdots,\langle\nabla_{im},\tilde{\alpha}_{im}\rangle\right)$$

$$= \Big((\zeta(A_{\sigma(1)}) - \zeta(A_{\sigma(0)}))\tilde{\alpha}_{i\sigma(1)}^{\lambda} \oplus (\zeta(A_{\sigma(2)}) - \zeta(A_{\sigma(1)}))\tilde{\alpha}_{i\sigma(2)}^{\lambda} \oplus \cdots \oplus(\zeta(A_{\sigma(m)}) - \zeta(A_{\sigma(m-1)}))\tilde{\alpha}_{i\sigma(m)}^{\lambda}\Big)^{1/\lambda} \tag{2.78}$$

Step 4 Derive the priority of $\tilde{C}_i$ according to the comparison method of IVIFNs, and generate the ranking of the alternatives Y_i $(i = 1, 2, \cdots, n)$.

Xu and Xia (2011) further extend the intuitionistic fuzzy Dempster-Shafer operators to interval-valued intuitionistic fuzzy environments:

Definition 2.7.2 Let

$$\tilde{M} = (\langle \nabla_{11}, \tilde{\alpha}_{11}\rangle, \langle \nabla_{21}, \tilde{\alpha}_{21}\rangle, \cdots, \langle \nabla_{q_1 1}, \tilde{\alpha}_{q_1 1}\rangle, \cdots, \langle \nabla_{1r}, \tilde{\alpha}_{1r}\rangle, \langle \nabla_{2r}, \tilde{\alpha}_{2r}\rangle, \cdots, \langle \nabla_{q_r r}, \tilde{\alpha}_{q_r r}\rangle)$$

be a collection of 2-tuple arguments with r focal elements, B_k $(k = 1, 2, \cdots, r)$. A BSI-GIIFOA operator of dimension r is a function BSI-GIVIFCOA : $\tilde{\Theta}^r \to \tilde{\Theta}$ defined by

$$\text{BSI-GIIFOA}\left(\tilde{M}\right) = \left(\mathop{\oplus}_{k=1}^{r}\left(\zeta(B_k)\left(\mathop{\oplus}_{j=1}^{q_k}\left(\omega_{jk}\tilde{\beta}_{jk}^{\lambda_1}\right)\right)^{\lambda_2/\lambda_1}\right)\right)^{1/\lambda_2} \tag{2.79}$$

where $\mathop{\oplus}_{j=1}^{q_k}\left(\omega_{jk}\tilde{\beta}_{jk}^{\lambda_1}\right) = \omega_{1k}\tilde{\beta}_{1k}^{\lambda_1} \oplus \omega_{2k}\tilde{\beta}_{2k}^{\lambda_1} \oplus \cdots \oplus \omega_{q_k k}\tilde{\beta}_{q_k k}^{\lambda_1}$, and $\lambda_1 > 0$, $\lambda_2 > 0$, $W_k = (w_{1k}, w_{2k}, \cdots, w_{q_k k})^{\mathrm{T}}$ is the weighting vector of the k-th focal element B_k such that $\sum_{j=1}^{q_k}\omega_{jk} = 1$ and $\omega_{jk} \in [0, 1]$, q_k is the number of elements in B_k, $\tilde{\beta}_{jk}$ is the $\tilde{\alpha}_{jk}$ value of the pair $\langle \nabla_{ik}, \tilde{\alpha}_{ik}\rangle$ having the j-th largest of $\nabla_{ik}(i = 1, 2, \cdots, q_k)$, u_{ik} is the order-inducing variable, $\tilde{\alpha}_{ik}$ is the argument variable, and $\zeta(B_k)$ is the basic probability assignment.

By the operational laws of IVIFN, the BSI-GIIFOA operator (2.79) can be transformed into the following form by using mathematical induction on n:

$$\text{BSI-GIIFOA}\left(\tilde{M}\right) = \left(\left[\left(1 - \prod_{k=1}^{r}\left(1 - \left(1 - \prod_{j=1}^{q_k}(1 - (\tilde{\mu}_{\tilde{\beta}_{jk}}^{\mathrm{L}})^{\lambda_1})^{w_{jk}}\right)^{\lambda_2/\lambda_1}\right)^{\zeta(B_k)}\right)^{1/\lambda_2},\right.\right.$$
$$\left.\left(1 - \prod_{k=1}^{r}\left(1 - \left(1 - \prod_{j=1}^{q_k}(1 - (\tilde{\mu}_{\tilde{\beta}_{jk}}^{\mathrm{U}})^{\lambda_1})^{\omega_{jk}}\right)^{\lambda_2/\lambda_1}\right)^{\zeta(B_k)}\right)^{1/\lambda_2}\right],$$

$$\left[1-\left(1-\prod_{k=1}^{r}\left(1-\left(1-\prod_{j=1}^{q_k}(1-(1-\tilde{\nu}_{\tilde{\beta}_{jk}}^{\mathrm{L}})^{\lambda_1})^{\omega_{jk}}\right)^{\lambda_2/\lambda_1}\right)^{\zeta(B_k)}\right)^{1/\lambda_2},\right.$$
$$\left.1-\left(1-\prod_{k=1}^{r}\left(1-\left(1-\prod_{j=1}^{q_k}(1-(1-\tilde{\nu}_{\tilde{\beta}_{jk}}^{\mathrm{U}})^{\lambda_1})^{\omega_{jk}}\right)^{\lambda_2/\lambda_1}\right)^{\zeta(B_k)}\right)^{1/\lambda_2}\right]\Bigg) \tag{2.80}$$

Some special cases of the BSI-GIIFOA can be given as follows (Xu and Xia, 2011):

(1) If $\lambda_1 = \lambda_2 = \lambda$, then Eq.(2.80) reduces to

$$\begin{aligned}\text{BSI-GIIFOA}\left(\tilde{M}\right) &= \left(\underset{k=1}{\overset{r}{\oplus}}\left(\zeta(B_k)\underset{j=1}{\overset{q_k}{\oplus}}\left(\omega_{jk}\tilde{\beta}_{jk}^{\lambda}\right)\right)\right)^{1/\lambda}\\ &= \left(\left[\left(1-\prod_{k=1}^{r}\prod_{j=1}^{q_k}(1-(\tilde{\mu}_{\tilde{\beta}_{jk}}^{\mathrm{L}})^{\lambda})^{\omega_{jk}\zeta(B_k)}\right)^{1/\lambda},\right.\right.\\ &\quad \left.\left(1-\prod_{k=1}^{r}\prod_{j=1}^{q_k}(1-(\tilde{\mu}_{\tilde{\beta}_{jk}}^{\mathrm{U}})^{\lambda})^{\omega_{jk}\zeta(B_k)}\right)^{1/\lambda}\right],\\ &\quad \left[1-\left(1-\prod_{k=1}^{r}\prod_{j=1}^{q_k}(1-(1-\tilde{\nu}_{\tilde{\beta}_{jk}}^{\mathrm{L}})^{\lambda})^{\omega_{jk}\zeta(B_k)}\right)^{1/\lambda},\right.\\ &\quad \left.\left.1-\left(1-\prod_{k=1}^{r}\prod_{j=1}^{q_k}(1-(1-\tilde{\nu}_{\tilde{\beta}_{jk}}^{\mathrm{U}})^{\lambda})^{\omega_{jk}\zeta(B_k)}\right)^{1/\lambda}\right]\right)\end{aligned} \tag{2.81}$$

(2) If $\lambda_1 = \lambda_2 = 1$, then Eq.(2.80) becomes:

$$\begin{aligned}\text{BSI-IIFOAA}\left(\tilde{M}\right) &= \underset{k=1}{\overset{r}{\oplus}}\left(\zeta(B_k)\underset{j=1}{\overset{q_k}{\oplus}}\left(\omega_{jk}\tilde{\beta}_{jk}\right)\right)\\ &= \left(\left[1-\prod_{k=1}^{r}\prod_{j=1}^{q_k}\left(1-\tilde{\mu}_{\tilde{\beta}_{jk}}^{\mathrm{L}}\right)^{\omega_{jk}\zeta(B_k)}, 1-\prod_{k=1}^{r}\prod_{jk=1}^{q_k}\left(1-\tilde{\mu}_{\tilde{\beta}_{jk}}^{\mathrm{U}}\right)^{\omega_{jk}\zeta(B_k)}\right],\right.\\ &\quad \left.\left[\prod_{k=1}^{r}\prod_{j=1}^{q_k}\left(\tilde{\nu}_{\tilde{\beta}_{jk}}^{\mathrm{L}}\right)^{\omega_{jk}\zeta(B_k)}, \prod_{k=1}^{r}\prod_{j=1}^{q_k}\left(\tilde{\nu}_{\tilde{\beta}_{jk}}^{\mathrm{U}}\right)^{\omega_{jk}\zeta(B_k)}\right]\right)\end{aligned} \tag{2.82}$$

which we call a BSI-IIFOAA operator.

(3) If $\lambda_1 = 0$ and $\lambda_2 = 1$, then we have

$$\begin{aligned}
\text{BSI-IIFOAG}\left(\tilde{M}\right) &= \underset{k=1}{\overset{r}{\oplus}} \left(\zeta(B_k) \underset{j=1}{\overset{q_k}{\otimes}} \left(\tilde{\beta}_{jk}^{\omega_{jk}}\right)\right) \\
&= \left(\left[1 - \prod_{k=1}^{r}\left(1 - \prod_{j=1}^{q_k}\left(\tilde{\mu}_{\tilde{\beta}_{jk}}^{\mathrm{L}}\right)^{\omega_{jk}}\right)^{\zeta(B_k)}, \right.\right. \\
&\quad \left. 1 - \prod_{k=1}^{r}\left(1 - \prod_{j=1}^{q_k}\left(\tilde{\mu}_{\tilde{\beta}_{jk}}^{\mathrm{U}}\right)^{\omega_{jk}}\right)^{\zeta(B_k)}\right], \\
&\quad \left[\prod_{k=1}^{r}\left(1 - \prod_{j_k=1}^{q_k}\left(1 - \tilde{\nu}_{\tilde{\beta}_{j_k}}^{\mathrm{L}}\right)^{\omega_{j_k}}\right)^{\zeta(B_k)}, \right. \\
&\quad \left.\left. \prod_{k=1}^{r}\left(1 - \prod_{j_k=1}^{q_k}\left(1 - \tilde{\nu}_{\tilde{\beta}_{j_k}}^{\mathrm{U}}\right)^{\omega_{j_k}}\right)^{\zeta(B_k)}\right]\right) \qquad (2.83)
\end{aligned}$$

which we call a BSI-IIFOAG operator.

(4) If $\lambda_1 = 1$ and $\lambda_2 = 0$, then we obtain the BSI-IIFOGA operator:

$$\begin{aligned}
\text{BSI-IIFOGA}\left(\tilde{M}\right) &= \underset{k=1}{\overset{r}{\otimes}} \left(\underset{j=1}{\overset{q_k}{\oplus}} \left(\omega_{jk}\tilde{\beta}_{jk}\right)\right)^{\zeta(B_k)} \\
&= \left(\left[\prod_{k=1}^{r}\left(1 - \prod_{j=1}^{q_k}\left(1 - \tilde{\mu}_{\tilde{\beta}_{jk}}^{\mathrm{L}}\right)^{\omega_{jk}}\right)^{\zeta(B_k)}, \right.\right. \\
&\quad \left. \prod_{k=1}^{r}\left(1 - \prod_{j=1}^{q_k}\left(1 - \tilde{\mu}_{\tilde{\beta}_{jk}}^{\mathrm{U}}\right)^{\omega_{jk}}\right)^{\zeta(B_k)}\right], \\
&\quad \left[1 - \prod_{k=1}^{r}\left(1 - \prod_{j=1}^{q_k}\left(\tilde{\nu}_{\tilde{\beta}_{jk}}^{\mathrm{L}}\right)^{\omega_{jk}}\right)^{\zeta(B_k)}, \right. \\
&\quad \left.\left. 1 - \prod_{k=1}^{r}\left(1 - \prod_{j=1}^{q_k}\left(\tilde{\nu}_{\tilde{\beta}_{jk}}^{\mathrm{U}}\right)^{\omega_{jk}}\right)^{\zeta(B_k)}\right]\right) \qquad (2.84)
\end{aligned}$$

which we call a BSI-IIFOGA operator.

(5) If $\lambda_1 = 0$ and $\lambda_2 = 0$, then we have

$$\text{BSI-IIFOGG}\left(\tilde{M}\right) = \underset{k=1}{\overset{r}{\otimes}} \left(\left(\underset{j=1}{\overset{q_k}{\otimes}} \left(\tilde{\beta}_{jk}^{\omega_{jk}}\right)\right)^{\zeta(B_k)}\right)$$

$$= \left(\left[\prod_{k=1}^{r} \prod_{j=1}^{q_k} \left(\tilde{\mu}^{\mathrm{L}}_{\tilde{\beta}_{jk}} \right)^{\omega_{jk}\zeta(B_k)}, \prod_{k=1}^{r} \prod_{j=1}^{q_k} \left(\tilde{\mu}^{\mathrm{U}}_{\tilde{\beta}_{jk}} \right)^{\omega_{jk}\zeta(B_k)} \right], \right.$$
$$\left. \left[1 - \prod_{k=1}^{r} \prod_{j=1}^{q_k} \left(1 - \tilde{\nu}^{\mathrm{L}}_{\tilde{\beta}_{jk}} \right)^{\omega_{jk}\zeta(B_k)}, 1 - \prod_{k=1}^{r} \prod_{j=1}^{q_k} \left(1 - \tilde{\nu}^{\mathrm{U}}_{\tilde{\beta}_{jk}} \right)^{\omega_{jk}\zeta(B_k)} \right] \right) \tag{2.85}$$

which we call a BSI-IIFOGG operator.

If each payoff value of the alternative Y_i under the state G_j in Section 1.7 is given by an IVIFN $\tilde{\alpha}_{ij}$, then we can develop a method for decision making based on the BSI-GIIFOA operator (Xu and Xia, 2011) as follows:

Step 1 Calculate the attitudinal character of the decision maker to determine the inducing values matrix $\nabla = (\nabla_{ij})_{n \times m}$.

Step 2 Find the set of payoff values:

$$\tilde{M}_{ik} = \{ \langle \nabla_{ij}, \tilde{\alpha}_{ij} \rangle | \, G_j \in B_k \} = \left\{ \left\langle \nabla^1_{ik}, \tilde{\alpha}^1_{ik} \right\rangle, \left\langle \nabla^2_{ik}, \tilde{\alpha}^2_{ik} \right\rangle, \cdots, \langle \nabla^{q_k}_{ik}, \tilde{\alpha}^{q_k}_{ik} \rangle \right\} \tag{2.86}$$

Step 3 Determine the weight vector $W_{q_k} = \left(\omega^{(1)}_{q_k}, \omega^{(2)}_{q_k}, \cdots, \omega^{(q_k)}_{q_k} \right)^{\mathrm{T}}$, and calculate the aggregated payoff $\tilde{V}_{ik}$:

$$\tilde{V}_{ik} = \left(\omega^{(1)}_{q_k} (\tilde{\beta}^{(1)}_{ik})^{\lambda_1} \oplus \omega^{(2)}_{q_k} (\tilde{\beta}^{(2)}_{ik})^{\lambda_1} \oplus \cdots \oplus \omega^{(q_k)}_{q_k} (\tilde{\beta}^{(q_k)}_{ik})^{\lambda_1} \right)^{1/\lambda_1} \tag{2.87}$$

where $\tilde{\beta}^{(j)}_{ik}$ is the $\tilde{\alpha}^{(l)}_{ik}$ value of the pair $\left\langle \nabla^{(l)}_{ik}, \tilde{\alpha}^{(l)}_{ik} \right\rangle$ with the j-th largest of $\nabla^{(l)}_{ik} (l = 1, 2, \cdots, q_k)$.

Step 4 Utilize the BSI-GIIFOA operator (2.80) to calculate the aggregated payoff $\tilde{C}_i$ for each alternative Y_i:

$$\tilde{C}_i = \left(\mathop{\oplus}_{k=1}^{r} \left(\zeta(B_k) \left(\tilde{V}_{ik} \right)^{\lambda_2} \right) \right)^{1/\lambda_2} = \left(\mathop{\oplus}_{k=1}^{r} \left(\zeta(B_k) \left(\mathop{\oplus}_{j=1}^{q_k} \left(\omega^{(j)}_{q_k} \left(\tilde{\beta}^{(j)}_{ik} \right)^{\lambda_1} \right) \right)^{\lambda_2/\lambda_1} \right) \right)^{1/\lambda_2} \tag{2.88}$$

Step 5 Select the best alternative Y_i with the largest $\tilde{C}_i$.

One can apply the above method to Example 1.7.2 with the evaluation information expressed in IVIFNs.

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Chapter 3

Correlation, Distance and Similarity Measures of Intuitionistic Fuzzy Sets

Correlation, distance and similarity measures are an important research topic in the IFS theory, which has received great attention in recent years. In this chapter, we shall give a thorough and systematic introduction to the existing research results on this topic.

3.1 Correlation Measures of Intuitionistic Fuzzy Sets

Gerstenkorn and Mafiko (1991) suggest a correlation measure function of IFSs, and define the concept of correlation coefficient. Bustince and Burillo (1995) define the correlation degree of IVIFSs, and also introduce two decomposition theorems, one on the correlation of interval-valued fuzzy sets and the entropy of IFSs, and the other on the correlation of IFSs. Hong and Hwang (1995) study the concepts of correlation and correlation coefficient of IFSs in probability spaces. Hung and Wu (2002) propose a method to calculate the correlation coefficient of IFSs by means of "centroid". The method reflects not only the strength of relationship between IFSs, but also their positive or negative correlation. Furthermore, they extend the "centroid" method to IVIFSs. Hung (2001) investigates the correlation measure of IFSs from the viewpoint of statistics. Hong (1998) generalizes the concepts of correlation and correlation coefficient of IVIFSs to a general probability space and extends the results of Bustince and Burillo (1995). Xu et al. (2008) define a correlation coefficient of IFSs from the set-theoretic viewpoint. Xu (2006h) provides a survey on correlation analysis of IFSs, and proposes a new method for deriving the correlation coefficients of IFSs, which has some advantages over the existing methods. Furthermore, the developed method is extended to the IVIFS theory, and an application in medical diagnosis is illustrated. These results are described in more details below:

Let $X = \{x_1, x_2, \cdots, x_n\}$ be a fixed finite set, $A_1 = \{\langle x_i, \mu_{A_1}(x_i), \nu_{A_1}(x_i)\rangle \mid x_i \in X\}$ and $A_2 = \{\langle x_i, \mu_{A_2}(x_i), \nu_{A_2}(x_i)\rangle \mid x_i \in X\}$ two IFSs. If

$$\rho_1(A_1, A_2) = \frac{c_1(A_1, A_2)}{(c_1(A_1, A_1) \cdot c_1(A_2, A_2))^{1/2}} \tag{3.1}$$

where

$$c_1(A_1, A_2) = \sum_{i=1}^{n} (\mu_1(x_i) \cdot \mu_{A_2}(x_i) + \nu_1(x_i) \cdot \nu_{A_2}(x_i)) \tag{3.2}$$

then $\rho_1(A_1, A_2)$ is called the correlation coefficient of the IFSs A_1 and A_2 (Gerstenkorn andMafiko, 1991).

Definition 3.1.1 (Gerstenkorn and Manko, 1991) A correlation coefficient $\rho_1(A_1, A_2)$ should satisfy the following properties:

(1) $0 \leqslant \rho_1(A_1, A_2) \leqslant 1$;

(2) $A_1 = A_2 \Rightarrow \rho_1(A_1, A_2) = 1$;

(3) $\rho_1(A_1, A_2) = \rho_1(A_2, A_1)$.

Hong and Hwang (1995) consider situations where the set X is an infinite set, and define

$$\rho_2(A_1, A_2) = \frac{c_2(A_1, A_2)}{(c_2(A_1, A_1) \cdot c_2(A_2, A_2))^{1/2}} \tag{3.3}$$

as a correlation coefficient of the IFSs A_1 and A_2, where

$$c_2(A_1, A_2) = \int_X (\mu_{A_1}(x)\mu_{A_2}(x) + \nu_{A_1}(x)\nu_{A_2}(x))\ \mathrm{d}x \tag{3.4}$$

Clearly, $\rho_2(A_1, A_2)$ also has all the properties in Definition 3.1.1.

Hung (2001) defines the correlation coefficient of IFSs A_1 and A_2 from the viewpoint of statistics:

$$\rho_3(A_1, A_2) = \frac{1}{2}(\rho_{1,1} + \rho_{2,2}) \tag{3.5}$$

where

$$\rho_{1,1} = \frac{\displaystyle\int_X (\mu_{A_1}(x) - \bar{\mu}_{A_1}) \times (\mu_{A_2}(x) - \bar{\mu}_{A_2})\mathrm{d}x}{\left(\displaystyle\int_X (\mu_{A_1}(x) - \bar{\mu}_{A_1})^2\mathrm{d}x \int_X (\mu_{A_2}(x) - \bar{\mu}_{A_2})^2\mathrm{d}x\right)^{1/2}} \tag{3.6}$$

is the correlation coefficient of $\mu_{A_1}(x)$ and $\mu_{A_2}(x)$, and

$$\rho_{2,2} = \frac{\displaystyle\int_{X_1} (\nu_{A_1}(x) - \bar{\nu}_{A_1}) \times (\nu_{A_2}(x) - \bar{\nu}_{A_2})\mathrm{d}x}{\left(\displaystyle\int_{X_1} (\nu_{A_1}(x) - \bar{\nu}_{A_1})^2\mathrm{d}x \int_{X_1} (\nu_{A_2}(x) - \bar{\nu}_{A_2})^2\mathrm{d}x\right)^{1/2}} \tag{3.7}$$

is the correlation coefficient of $\nu_{A_1}(x)$ and $\nu_{A_2}(x)$, $\bar{\mu}_{A_1}$ and $\bar{\mu}_{A_2}$ are, respectively, the sample means of the membership functions $\mu_{A_1}(x)$ and $\mu_{A_2}(x)$; $\bar{\nu}_{A_1}$ and $\bar{\nu}_{A_2}$ are the sample means of the non-membership functions $\nu_{A_1}(x)$ and $\nu_{A_2}(x)$.

Mitchell (2004) gives an improved version of Hung's results. He interprets the IFSs A_1 and A_2 as the ensembles of the ordinary membership functions $\varphi_{A_1}^{(s)}(x)$ and $\varphi_{A_2}^{(k)}(x)$ $(s, k = 1, 2, \cdots, n)$:

$$\varphi_{A_1}^{(s)}(x) = \mu_{A_1}(x) + \pi_{A_1}(x) \times p_s(x), \quad \varphi_{A_2}^{(k)}(x) = \mu_{A_2}(x) + \pi_{A_2}(x) \times p_k(x) \tag{3.8}$$

where for each $x \in X$, $p_s(x)$ and $p_k(x)$ are two uniform random numbers chosen from the interval [0, 1]. The correlation coefficient $\rho_{s,k}$ of each pair of membership functions $\varphi_{A_1}^{(s)}(x)$ and $\varphi_{A_2}^{(k)}(x)$ can be calculated as follows:

$$\rho_{s,k} = \frac{\int_X (\varphi_{A_1}^{(s)}(x) - \bar{\varphi}_{A_1}^{(s)}) \times (\varphi_{A_2}^{(k)}(x) - \bar{\varphi}_{A_2}^{(k)})\mathrm{d}x}{\left(\int_X (\varphi_{A_1}^{(s)}(x) - \bar{\varphi}_{A_1}^{(s)})^2\mathrm{d}x \int_X (\varphi_{A_2}^{(k)}(x) - \bar{\varphi}_{A_2}^{(k)})\mathrm{d}x\right)^{1/2}} \tag{3.9}$$

where $\rho_{s,k} \in [-1, 1]$, $\bar{\varphi}_{A_1}^{(s)}$ and $\bar{\varphi}_{A_2}^{(k)}$ are, respectively, the sample means of the ordinary membership functions $\varphi_{A_1}^{(s)}(x)$ and $\varphi_{A_2}^{(k)}(x)$ $(s, k = 1, 2, \cdots, n)$. Then he defines the correlation coefficient of the IFSs A_1 and A_2 as:

$$\rho_3(A_1, A_2) = f(\rho_{s,k} \mid s, k = 1, 2, \cdots, n) \tag{3.10}$$

where f is a mean aggregation function.

Huang and Wu (2002) propose a method to calculate the correlation coefficient of the IFSs A_1 and A_2 by means of "centroid", as follows:

$$\rho_4(A_1, A_2) = \frac{c_4(A_1, A_2)}{(c_4(A_1, A_1) \cdot c_4(A_2, A_2))^{1/2}} \tag{3.11}$$

where

$$c_4(A_1, A_2) = m(\mu_{A_1})m(\mu_{A_2}) + m(v_{A_1})m(v_{A_2})$$

is the correlation of A_1 and A_2, and

$$m(\mu_{A_1}) = \frac{\int_X x\mu_{A_1}(x)\mathrm{d}x}{\int_X \mu_{A_1}(x)\mathrm{d}x}, \quad m(\nu_{A_1}) = \frac{\int_X x\nu_{A_1}(x)\mathrm{d}x}{\int_X \nu_{A_1}(x)\mathrm{d}x} \tag{3.12}$$

$$m(\mu_{A_2}) = \frac{\int_X x\mu_{A_2}(x)\mathrm{d}x}{\int_X \mu_{A_2}(x)\mathrm{d}x}, \quad m(\nu_{A_2}) = \frac{\int_X x\nu_{A_2}(x)\mathrm{d}x}{\int_X \nu_{A_2}(x)\mathrm{d}x} \tag{3.13}$$

are, respectively, the centroids of μ_{A_1}, ν_{A_1}, μ_{A_2} and ν_{A_2}.

Huang and Wu (2002) further extend the "centroid" method to the IVIFS theory:

Let $\tilde{A}_1 = \{\langle x, \tilde{\mu}_{\tilde{A}_1}(x), \tilde{\nu}_{\tilde{A}_1}(x)\rangle \mid x \in X\}$ and $\tilde{A}_2 = \{\langle x, \tilde{\mu}_{\tilde{A}_2}(x), \tilde{\nu}_{\tilde{A}_2}(x)\rangle \mid x \in X\}$ be two IVIFSs, where

$$\tilde{\mu}_{\tilde{A}_1}(x) = [\tilde{\mu}_{\tilde{A}_1}^{\mathrm{L}}(x), \tilde{\mu}_{\tilde{A}_1}^{\mathrm{U}}(x)], \quad \tilde{\mu}_{\tilde{A}_2}(x) = [\tilde{\mu}_{\tilde{A}_2}^{\mathrm{L}}(x), \tilde{\mu}_{\tilde{A}_2}^{\mathrm{U}}(x)]$$

$$\tilde{\nu}_{\tilde{A}_1}(x) = [\tilde{\nu}^{\mathrm{L}}_{\tilde{A}_1}(x), \tilde{\nu}^{\mathrm{U}}_{\tilde{A}_1}(x)], \quad \tilde{\nu}_{\tilde{A}_2}(x) = [\tilde{\nu}^{\mathrm{L}}_{\tilde{A}_2}(x), \tilde{\nu}^{\mathrm{U}}_{\tilde{A}_2}(x)]$$

$$\tilde{\mu}^{\mathrm{L}}_{\tilde{A}_1}(x) = \inf \tilde{\mu}_{\tilde{A}_1}(x), \quad \tilde{\mu}^{\mathrm{U}}_{\tilde{A}_1}(x) = \sup \tilde{\mu}_{\tilde{A}_1}(x), \quad \tilde{\nu}^{\mathrm{L}}_{\tilde{A}_1}(x) = \inf \tilde{\nu}_{\tilde{A}_1}(x)$$

$$\tilde{\nu}^{\mathrm{U}}_{\tilde{A}_1}(x) = \sup \tilde{\nu}_{\tilde{A}_1}(x), \quad \tilde{\mu}^{\mathrm{L}}_{\tilde{A}_2}(x) = \inf \tilde{\mu}_{\tilde{A}_2}(x), \quad \tilde{\mu}^{\mathrm{U}}_{\tilde{A}_2}(x) = \sup \tilde{\mu}_{\tilde{A}_2}(x)$$

$$\tilde{\nu}^{\mathrm{L}}_{\tilde{A}_2}(x) = \inf \tilde{\nu}_{\tilde{A}_2}(x), \quad \tilde{\nu}^{\mathrm{U}}_{\tilde{A}_2}(x) = \sup \tilde{\nu}_{\tilde{A}_2}(x)$$

and let the centroids of $\tilde{\mu}^{\mathrm{L}}_{\tilde{A}_1}$, $\tilde{\mu}^{\mathrm{U}}_{\tilde{A}_1}$, $\tilde{\nu}^{\mathrm{L}}_{\tilde{A}_1}$, $\tilde{\nu}^{\mathrm{U}}_{\tilde{A}_1}$, $\tilde{\mu}^{\mathrm{L}}_{\tilde{A}_2}$, $\tilde{\mu}^{\mathrm{U}}_{\tilde{A}_2}$, $\tilde{\nu}^{\mathrm{L}}_{\tilde{A}_2}$ and $\tilde{\nu}^{\mathrm{U}}_{\tilde{A}_2}$ be $m(\tilde{\mu}^{\mathrm{L}}_{\tilde{A}_1})$, $m(\tilde{\mu}^{\mathrm{U}}_{\tilde{A}_1})$, $m(\tilde{\nu}^{\mathrm{L}}_{\tilde{A}_1})$, $m(\tilde{\nu}^{\mathrm{U}}_{\tilde{A}_1})$, $m(\tilde{\mu}^{\mathrm{L}}_{\tilde{A}_2})$, $m(\tilde{\mu}^{\mathrm{U}}_{\tilde{A}_2})$, $m(\tilde{\nu}^{\mathrm{L}}_{\tilde{A}_2})$ and $m(\tilde{\nu}^{\mathrm{U}}_{\tilde{A}_2})$ respectively. Then the correlation coefficient of the IVIFSs $\tilde{A}_1$ and $\tilde{A}_2$ is defined as:

$$\rho_5(\tilde{A}_1, \tilde{A}_2) = \frac{c_5(\tilde{A}_1, \tilde{A}_2)}{\left(c_5(\tilde{A}_1, \tilde{A}_1) \cdot c_5(\tilde{A}_2, \tilde{A}_2)\right)^{1/2}} \tag{3.14}$$

where

$$\begin{aligned} c_5(\tilde{A}_1, \tilde{A}_2) = & m(\tilde{\mu}^{\mathrm{L}}_{\tilde{A}_1})m(\tilde{\mu}^{\mathrm{L}}_{\tilde{A}_2}) + m(\tilde{\mu}^{\mathrm{U}}_{\tilde{A}_1})m(\tilde{\mu}^{\mathrm{U}}_{\tilde{A}_2}) \\ & + m(\tilde{\nu}^{\mathrm{L}}_{\tilde{A}_1})m(\tilde{\nu}^{\mathrm{L}}_{\tilde{A}_2}) + m(\tilde{\nu}^{\mathrm{U}}_{\tilde{A}_1})m(\tilde{\nu}^{\mathrm{U}}_{\tilde{A}_2}) \end{aligned} \tag{3.15}$$

is the correlation measure of the IVIFSs $\tilde{A}_1$ and $\tilde{A}_2$.

Bustince and Burillo (1995) also investigate the correlation coefficient of the IVIFSs $\tilde{A}_1$ and $\tilde{A}_2$, and define it as:

$$\rho_6(\tilde{A}_1, \tilde{A}_2) = \frac{c_6(\tilde{A}_1, \tilde{A}_2)}{\left(c_6(\tilde{A}_1, \tilde{A}_1) \cdot c_6(\tilde{A}_2, \tilde{A}_2)\right)^{1/2}} \tag{3.16}$$

where

$$\begin{aligned} c_6(\tilde{A}_1, \tilde{A}_2) = & \frac{1}{2}\sum_{i=1}^{n} \Big(\tilde{\mu}^{\mathrm{L}}_{\tilde{A}_1}(x_i)\tilde{\mu}^{\mathrm{L}}_{\tilde{A}_2}(x_i) + \tilde{\mu}^{\mathrm{U}}_{\tilde{A}_1}(x_i)\tilde{\mu}^{\mathrm{U}}_{\tilde{A}_2}(x_i) \\ & + \tilde{\nu}^{\mathrm{L}}_{\tilde{A}_1}(x_i)\tilde{\nu}^{\mathrm{L}}_{\tilde{A}_2}(x_i) + \tilde{\nu}^{\mathrm{U}}_{\tilde{A}_1}(x_i)\tilde{\nu}^{\mathrm{U}}_{\tilde{A}_2}(x_i)\Big), \quad x_i \in X \end{aligned} \tag{3.17}$$

is the correlation measure of $\tilde{A}_1$ and $\tilde{A}_2$.

Hong (1998) introduces the concept of correlation coefficient of the IVIFSs $\tilde{A}_1$ and $\tilde{A}_2$ in a general probability space. This is given as:

$$\rho_7(\tilde{A}_1, \tilde{A}_2) = \frac{c_7(\tilde{A}_1, \tilde{A}_2)}{\left(c_7(\tilde{A}_1, \tilde{A}_1) \cdot c_7(\tilde{A}_2, \tilde{A}_2)\right)^{1/2}} \tag{3.18}$$

where

$$c_7(\tilde{A}_1, \tilde{A}_2) = \frac{1}{2}\int_X \Big(\tilde{\mu}^{\mathrm{L}}_{\tilde{A}_1}(x)\tilde{\mu}^{\mathrm{L}}_{\tilde{A}_2}(x) + \tilde{\mu}^{\mathrm{U}}_{\tilde{A}_1}(x)\tilde{\mu}^{\mathrm{U}}_{\tilde{A}_2}(x)$$

$$+\tilde{\nu}^{\mathrm{L}}_{\tilde{A}_1}(x)\tilde{\nu}^{\mathrm{L}}_{\tilde{A}_2}(x)+\tilde{\nu}^{\mathrm{U}}_{\tilde{A}_1}(x)\tilde{\nu}^{\mathrm{U}}_{\tilde{A}_2}(x)\Big)\,\mathrm{d}x \tag{3.19}$$

is the correlation measure of $\tilde{A}_1$ and $\tilde{A}_2$.

All the correlation coefficients $\rho_i(A_1, A_2)$ $(i = 3, 4)$ and $\rho_i(\tilde{A}_1, \tilde{A}_2)$ $(i = 5, 6, 7)$ have the properties (2) and (3) in Definition 3.1.1, and satisfy the following:

(1) $|\rho_i(A_1, A_2)| \leqslant 1$, $i = 3, 4$;

(2) $|\rho_5(\tilde{A}_1, \tilde{A}_2)| \leqslant 1$;

(3) $0 \leqslant \rho_i(\tilde{A}_1, \tilde{A}_2) \leqslant 1$, $i = 6, 7$.

Xu (2006h) develops a new method to calculate the correlation coefficient of the IFSs A_1 and A_2:

Definition 3.1.2 (Xu, 2006h) Let $A_1 = \{\langle x_i, \mu_{A_1}(x_i), \nu_{A_1}(x_i)\rangle \,|\, x_i \in X\}$ and $A_2 = \{\langle x_i, \mu_{A_2}(x_i), \nu_{A_2}(x_i)\rangle \,|\, x_i \in X\}$ be two IFSs. Then

$$\rho_8(A_1, A_2) = \frac{1}{2n}\sum_{i=1}^{n}\left(\frac{\Delta\mu_{\min}+\Delta\mu_{\max}}{\Delta\mu_i+\Delta\mu_{\max}}+\frac{\Delta\nu_{\min}+\Delta\nu_{\max}}{\Delta\nu_i+\Delta\nu_{\max}}\right) \tag{3.20}$$

is called the correlation coefficient of A_1 and A_2, where

$$\Delta\mu_i = |\mu_{A_1}(x_i)-\mu_{A_2}(x_i)|, \quad \Delta v_i = |\nu_{A_1}(x_i)-\nu_{A_2}(x_i)|$$

$$\Delta\mu_{\min} = \min_i\{|\mu_{A_1}(x_i)-\mu_{A_2}(x_i)|\}, \quad \Delta v_{\min} = \min_i\{|\nu_{A_1}(x_i)-\nu_{A_2}(x_i)|\}$$

$$\Delta\mu_{\max} = \max_i\{|\mu_{A_1}(x_i)-\mu_{A_2}(x_i)|\}, \quad \Delta v_{\max} = \max_i\{|\nu_{A_1}(x_i)-\nu_{A_2}(x_i)|\}$$

In many situations, the weight of the element $x_j \in X$ should be taken into account. For example, in multi-attribute decision making, the considered attributes usually have different importance, and thus need to be assigned different weights. As a result, Xu (2006h) further extends Eq.(3.20) as:

$$\rho_9(A_1, A_2) = \frac{1}{2}\sum_{i=1}^{n}\omega_i\left(\frac{\Delta\mu_{\min}+\Delta\mu_{\max}}{\Delta\mu_i+\Delta\mu_{\max}}+\frac{\Delta\nu_{\min}+\Delta\nu_{\max}}{\Delta\nu_i+\Delta\nu_{\max}}\right) \tag{3.21}$$

where $\omega = (\omega_1, \omega_2, \cdots, \omega_n)^{\mathrm{T}}$ is the weight vector of $x_i\,(i = 1, 2, \cdots, n)$, with $\omega_i \in [0, 1]$, $i = 1, 2, \cdots, n$, and $\sum\limits_{i=1}^{n}\omega_i = 1$. In particular, if $\omega = (1/n, 1/n, \cdots, 1/n)^{\mathrm{T}}$, then Eq.(3.21) reduces to Eq.(3.20).

Xu (2006h) also generalizes Definition 3.1.2 to the IVIFS theory:

Definition 3.1.3 (Xu, 2006h) The correlation coefficient of two IVIFSs $\tilde{A}_1$ and $\tilde{A}_2$ is defined as:

$$\rho_{10}(\tilde{A}_1, \tilde{A}_2) = \frac{1}{4n}\sum_{i=1}^{n}\left(\frac{\Delta\tilde{\mu}^{\mathrm{L}}_{\min}+\Delta\tilde{\mu}^{\mathrm{L}}_{\max}}{\Delta\tilde{\mu}^{\mathrm{L}}_i+\Delta\tilde{\mu}^{\mathrm{L}}_{\max}}+\frac{\Delta\tilde{\mu}^{\mathrm{U}}_{\min}+\Delta\tilde{\mu}^{\mathrm{U}}_{\max}}{\Delta\tilde{\mu}^{\mathrm{U}}_i+\Delta\tilde{\mu}^{\mathrm{U}}_{\max}}\right.$$

$$+\frac{\Delta\tilde{\nu}^{\mathrm{L}}_{\min}+\Delta\tilde{\nu}^{\mathrm{L}}_{\max}}{\Delta\tilde{\nu}^{\mathrm{L}}_{i}+\Delta\tilde{\nu}^{\mathrm{L}}_{\max}}+\frac{\Delta\tilde{\nu}^{\mathrm{U}}_{\min}+\Delta\tilde{\nu}^{\mathrm{U}}_{\max}}{\Delta\tilde{\nu}^{\mathrm{U}}_{i}+\Delta\tilde{\nu}^{\mathrm{U}}_{\max}}\Bigg) \tag{3.22}$$

where

$$\Delta\tilde{\mu}^{\mathrm{L}}_i=|\tilde{\mu}^{\mathrm{L}}_{\tilde{A}_1}(x_i)-\tilde{\mu}^{\mathrm{L}}_{\tilde{A}_2}(x_i)|,\quad \Delta\tilde{\mu}^{\mathrm{U}}_i=|\tilde{\mu}^{\mathrm{U}}_{\tilde{A}_1}(x_i)-\tilde{\mu}^{\mathrm{U}}_{\tilde{A}_2}(x_i)|$$

$$\Delta\tilde{\nu}^{\mathrm{L}}_i=|\tilde{\nu}^{\mathrm{L}}_{\tilde{A}_1}(x_i)-\tilde{\nu}^{\mathrm{L}}_{\tilde{A}_2}(x_i)|,\quad \Delta\tilde{\nu}^{\mathrm{U}}_i=|\tilde{\nu}^{\mathrm{U}}_{\tilde{A}_1}(x_i)-\tilde{\nu}^{\mathrm{U}}_{\tilde{A}_2}(x_i)|$$

$$\Delta\tilde{\mu}^{\mathrm{L}}_{\min}=\min_i\{|\tilde{\mu}^{\mathrm{L}}_{\tilde{A}_1}(x_i)-\tilde{\mu}^{\mathrm{L}}_{\tilde{A}_2}(x_i)|\},\quad \Delta\tilde{\mu}^{\mathrm{U}}_{\min}=\min_i\{|\tilde{\mu}^{\mathrm{U}}_{\tilde{A}_1}(x_i)-\tilde{\mu}^{\mathrm{U}}_{\tilde{A}_2}(x_i)|\}$$

$$\Delta\tilde{\nu}^{\mathrm{L}}_{\min}=\min_i\{|\tilde{\nu}^{\mathrm{L}}_{\tilde{A}_1}(x_i)-\tilde{\nu}^{\mathrm{L}}_{\tilde{A}_2}(x_i)|\},\quad \Delta\tilde{\nu}^{\mathrm{U}}_{\min}=\min_i\{|\tilde{\nu}^{\mathrm{U}}_{\tilde{A}_1}(x_i)-\tilde{\nu}^{\mathrm{U}}_{\tilde{A}_2}(x_i)|\}$$

$$\Delta\tilde{\mu}^{\mathrm{L}}_{\max}=\max_i\{|\tilde{\mu}^{\mathrm{L}}_{\tilde{A}_1}(x_i)-\tilde{\mu}^{\mathrm{L}}_{\tilde{A}_2}(x_i)|\},\quad \Delta\tilde{\mu}^{\mathrm{U}}_{\max}=\max_i\{|\tilde{\mu}^{\mathrm{U}}_{\tilde{A}_1}(x_i)-\tilde{\mu}^{\mathrm{U}}_{\tilde{A}_2}(x_i)|\}$$

$$\Delta\tilde{\nu}^{\mathrm{L}}_{\max}=\max_i\{|\tilde{\nu}^{\mathrm{L}}_{\tilde{A}_1}(x_i)-\tilde{\nu}^{\mathrm{L}}_{\tilde{A}_2}(x_i)|\},\quad \Delta\tilde{\nu}^{\mathrm{U}}_{\max}=\max_i\{|\tilde{\nu}^{\mathrm{U}}_{\tilde{A}_1}(x_i)-\tilde{\nu}^{\mathrm{U}}_{\tilde{A}_2}(x_i)|\}$$

Eq.(3.22) can be generalized to a more general form:

$$\rho_{11}(\tilde{A}_1,\tilde{A}_2)=\frac{1}{4}\sum_{i=1}^{n}\omega_i\Bigg(\frac{\Delta\tilde{\mu}^{\mathrm{L}}_{\min}+\Delta\tilde{\mu}^{\mathrm{L}}_{\max}}{\Delta\tilde{\mu}^{\mathrm{L}}_{i}+\Delta\tilde{\mu}^{\mathrm{L}}_{\max}}+\frac{\Delta\tilde{\mu}^{\mathrm{U}}_{\min}+\Delta\tilde{\mu}^{\mathrm{U}}_{\max}}{\Delta\tilde{\mu}^{\mathrm{U}}_{i}+\Delta\tilde{\mu}^{\mathrm{U}}_{\max}}$$
$$+\frac{\Delta\tilde{\nu}^{\mathrm{L}}_{\min}+\Delta\tilde{\nu}^{\mathrm{L}}_{\max}}{\Delta\tilde{\nu}^{\mathrm{L}}_{i}+\Delta\tilde{\nu}^{\mathrm{L}}_{\max}}+\frac{\Delta\tilde{\nu}^{\mathrm{U}}_{\min}+\Delta\tilde{\nu}^{\mathrm{U}}_{\max}}{\Delta\tilde{\nu}^{\mathrm{U}}_{i}+\Delta\tilde{\nu}^{\mathrm{U}}_{\max}}\Bigg) \tag{3.23}$$

where $\omega=(\omega_1,\omega_2,\cdots,\omega_n)^{\mathrm{T}}$ is the weight vector of $x_i\,(i=1,2,\cdots,n)$, with $\omega_i\in[0,1]$, $i=1,2,\cdots,n$, and $\sum_{i=1}^{n}\omega_i=1$. In particular, when $\omega=(1/n,1/n,\cdots,1/n)^{\mathrm{T}}$, then Eq.(3.23) reduces to Eq.(3.22).

However, from Definition 3.1.1 we notice that all the correlation coefficients above cannot guarantee that the correlation coefficient of any two IFSs (or IVIFSs) equals one when these two IFSs (or IVIFSs) are the same. Thus, how to derive the correlation coefficients of the IFSs (or IVIFSs) that satisfy this desirable property is an interesting research topic. To resolve this issue, below we improve Eq.(3.20) as:

$$\rho'_8(A_1,A_2)=\frac{1}{3n}\sum_{i=1}^{n}\Bigg(\frac{\Delta\mu_{\min}+\Delta\mu_{\max}}{\Delta\mu_i+\Delta\mu_{\max}}+\frac{\Delta\nu_{\min}+\Delta\nu_{\max}}{\Delta\nu_i+\Delta\nu_{\max}}$$
$$+\frac{\Delta\pi_{\min}+\Delta\pi_{\max}}{\Delta\pi_i+\Delta\pi_{\max}}\Bigg) \tag{3.24}$$

where

$$\pi_{A_1}(x_i)=1-\mu_{A_1}(x_i)-\nu_{A_1}(x_i),\quad \pi_{A_2}(x_i)=1-\mu_{A_2}(x_i)-\nu_{A_2}(x_i)$$

$$\Delta\pi_i=|\pi_{A_1}(x_i)-\pi_{A_2}(x_i)|,\quad \Delta\pi_{\min}=\min_i\{|\pi_{A_1}(x_i)-\pi_{A_2}(x_i)|\}$$

$$\Delta\pi_{\max}=\max_i\{|\pi_{A_1}(x_i)-\pi_{A_2}(x_i)|\}$$

If we take the weights $\omega_i\ (i=1,2,\cdots,n)$ of the elements $x_i\ (i=1,2,\cdots,n)$ into account, then Eq.(3.24) can be extended as:

$$\rho_9'(A_1,A_2)=\frac{1}{3}\sum_{i=1}^{n}\omega_i\left(\frac{\Delta\mu_{\min}+\Delta\mu_{\max}}{\Delta\mu_i+\Delta\mu_{\max}}+\frac{\Delta\nu_{\min}+\Delta\nu_{\max}}{\Delta\nu_i+\Delta\nu_{\max}}\right.$$
$$\left.+\frac{\Delta\pi_{\min}+\Delta\pi_{\max}}{\Delta\pi_i+\Delta\pi_{\max}}\right) \tag{3.25}$$

In particular, if $\omega=(1/n,1/n,\cdots,1/n)^{\mathrm{T}}$, then Eq.(3.25) reduces to Eq.(3.24).

Obviously, the correlation coefficient derived by Eqs.(3.24) and (3.25) not only involve the first two parameters (the membership degree and the non-membership degree) describing IFSs, but also contain the third parameter (the indeterminacy degree), and thus can take all the information into account.

By Eq.(3.25), we have

Theorem 3.1.1 The correlation coefficient $\rho_9'(A_1,A_2)$ of two IFSs A_1 and A_2 satisfies the following properties:

(1) $0\leqslant\rho_9'(A_1,A_2)\leqslant 1$;

(2) $A_1=A_2\Leftrightarrow\rho_9'(A_1,A_2)=1$;

(3) $\rho_9'(A_1,A_2)=\rho_9'(A_2,A_1)$.

Similar to Eq.(3.25), we can improve Eq.(3.23) as:

$$\rho_{11}'(\tilde{A}_1,\tilde{A}_2)=\frac{1}{6}\sum_{i=1}^{n}\omega_i\left(\frac{\Delta\tilde{\mu}_{\min}^{\mathrm{L}}+\Delta\tilde{\mu}_{\max}^{\mathrm{L}}}{\Delta\tilde{\mu}_i^{\mathrm{L}}+\Delta\tilde{\mu}_{\max}^{\mathrm{L}}}+\frac{\Delta\tilde{\mu}_{\min}^{\mathrm{U}}+\Delta\tilde{\mu}_{\max}^{\mathrm{U}}}{\Delta\tilde{\mu}_i^{\mathrm{U}}+\Delta\tilde{\mu}_{\max}^{\mathrm{U}}}+\frac{\Delta\tilde{\nu}_{\min}^{\mathrm{L}}+\Delta\tilde{\nu}_{\max}^{\mathrm{L}}}{\Delta\tilde{\nu}_i^{\mathrm{L}}+\Delta\tilde{\nu}_{\max}^{\mathrm{L}}}\right.$$
$$\left.+\frac{\Delta\tilde{\nu}_{\min}^{\mathrm{U}}+\Delta\tilde{\nu}_{\max}^{\mathrm{U}}}{\Delta\tilde{\nu}_i^{\mathrm{U}}+\Delta\tilde{\nu}_{\max}^{\mathrm{U}}}+\frac{\Delta\tilde{\pi}_{\min}^{\mathrm{L}}+\Delta\tilde{\pi}_{\max}^{\mathrm{L}}}{\Delta\tilde{\pi}_i^{\mathrm{L}}+\Delta\tilde{\pi}_{\max}^{\mathrm{L}}}+\frac{\Delta\tilde{\pi}_{\min}^{\mathrm{U}}+\Delta\tilde{\pi}_{\max}^{\mathrm{U}}}{\Delta\tilde{\pi}_i^{\mathrm{U}}+\Delta\tilde{\pi}_{\max}^{\mathrm{U}}}\right) \tag{3.26}$$

where

$$\tilde{\pi}_{A_1}^{\mathrm{L}}(x_i)=1-\tilde{\mu}_{A_1}^{\mathrm{U}}(x_i)-\tilde{\nu}_{A_1}^{\mathrm{U}}(x_i),\quad \tilde{\pi}_{A_1}^{\mathrm{U}}(x_i)=1-\tilde{\mu}_{A_1}^{\mathrm{L}}(x_i)-\tilde{\nu}_{A_1}^{\mathrm{L}}(x_i)$$

$$\tilde{\pi}_{A_2}^{\mathrm{L}}(x_i)=1-\tilde{\mu}_{A_2}^{\mathrm{U}}(x_i)-\tilde{\nu}_{A_2}^{\mathrm{U}}(x_i),\quad \tilde{\pi}_{A_2}^{\mathrm{U}}(x_i)=1-\tilde{\mu}_{A_2}^{\mathrm{L}}(x_i)-\tilde{\nu}_{A_2}^{\mathrm{L}}(x_i)$$

$$\Delta\tilde{\pi}_i^{\mathrm{L}}=|\tilde{\pi}_{\tilde{A}_1}^{\mathrm{L}}(x_i)-\tilde{\pi}_{\tilde{A}_2}^{\mathrm{L}}(x_i)|,\quad \Delta\tilde{\pi}_i^{\mathrm{U}}=|\tilde{\pi}_{\tilde{A}_1}^{\mathrm{U}}(x_i)-\tilde{\pi}_{\tilde{A}_2}^{\mathrm{U}}(x_i)|$$

$$\Delta\tilde{\pi}_{\min}^{\mathrm{L}}=\min_i\{|\tilde{\pi}_{\tilde{A}_1}^{\mathrm{L}}(x_i)-\tilde{\pi}_{\tilde{A}_2}^{\mathrm{L}}(x_i)|\},\quad \Delta\tilde{\pi}_{\min}^{\mathrm{U}}=\min_i\{|\tilde{\pi}_{\tilde{A}_1}^{\mathrm{U}}(x_i)-\tilde{\pi}_{\tilde{A}_2}^{\mathrm{U}}(x_i)|\}$$

$$\Delta\tilde{\pi}_{\max}^{\mathrm{L}}=\max_i\{|\tilde{\pi}_{\tilde{A}_1}^{\mathrm{L}}(x_i)-\tilde{\pi}_{\tilde{A}_2}^{\mathrm{L}}(x_i)|\},\quad \Delta\tilde{\pi}_{\max}^{\mathrm{U}}=\max_i\{|\tilde{\pi}_{\tilde{A}_1}^{\mathrm{U}}(x_i)-\tilde{\pi}_{\tilde{A}_2}^{\mathrm{U}}(x_i)|\}$$

In particular, if $\omega=(1/n,1/n,\cdots,1/n)^{\mathrm{T}}$, then Eq.(3.26) reduces to the counterpart of Eq.(3.22):

$$\rho_{10}'(\tilde{A}_1,\tilde{A}_2)=\frac{1}{6n}\sum_{i=1}^{n}\left(\frac{\Delta\tilde{\mu}_{\min}^{\mathrm{L}}+\Delta\tilde{\mu}_{\max}^{\mathrm{L}}}{\Delta\tilde{\mu}_i^{\mathrm{L}}+\Delta\tilde{\mu}_{\max}^{\mathrm{L}}}+\frac{\Delta\tilde{\mu}_{\min}^{\mathrm{U}}+\Delta\tilde{\mu}_{\max}^{\mathrm{U}}}{\Delta\tilde{\mu}_i^{\mathrm{U}}+\Delta\tilde{\mu}_{\max}^{\mathrm{U}}}+\frac{\Delta\tilde{\nu}_{\min}^{\mathrm{L}}+\Delta\tilde{\nu}_{\max}^{\mathrm{L}}}{\Delta\tilde{\nu}_i^{\mathrm{L}}+\Delta\tilde{\nu}_{\max}^{\mathrm{L}}}\right.$$

$$+\frac{\Delta\tilde{\nu}_{\min}^{\mathrm{U}}+\Delta\tilde{\nu}_{\max}^{\mathrm{U}}}{\Delta\tilde{\nu}_{i}^{\mathrm{U}}+\Delta\tilde{\nu}_{\max}^{\mathrm{U}}}+\frac{\Delta\tilde{\pi}_{\min}^{\mathrm{L}}+\Delta\tilde{\pi}_{\max}^{\mathrm{L}}}{\Delta\tilde{\pi}_{i}^{\mathrm{L}}+\Delta\tilde{\pi}_{\max}^{\mathrm{L}}}+\frac{\Delta\tilde{\pi}_{\min}^{\mathrm{U}}+\Delta\tilde{\pi}_{\max}^{\mathrm{U}}}{\Delta\tilde{\pi}_{i}^{\mathrm{U}}+\Delta\tilde{\pi}_{\max}^{\mathrm{U}}}\Bigg) \quad (3.27)$$

Similar to Theorem 3.1.1, we have

Theorem 3.1.2 The correlation coefficient $\rho'_{11}(\tilde{A}_1,\tilde{A}_2)$ (derived by Eq.(3.26)) of two IVIFSs $\tilde{A}_1$ and $\tilde{A}_2$ satisfies the following properties:

(1) $0\leqslant\rho'_{11}(\tilde{A}_1,\tilde{A}_2)\leqslant 1$;

(2) $\tilde{A}_1=\tilde{A}_2\Leftrightarrow\rho'_{11}(\tilde{A}_1,\tilde{A}_2)=1$;

(3) $\rho'_{11}(\tilde{A}_1,\tilde{A}_2)=\rho'_{11}(\tilde{A}_2,\tilde{A}_1)$.

Furthermore, Xu et al. (2008) define a correlation coefficient of IFSs from the set-theoretic viewpoint:

Definition 3.1.4 (Xu et al, 2008) The correlation coefficient of two IFSs A_1 and A_2 can be defined as:

$$\rho_{12}(A_1,A_2)$$
$$=\frac{\sum\limits_{i=1}^{n}(\mu_{A_1}(x_i)\cdot\mu_{A_2}(x_i)+\nu_{A_1}(x_i)\cdot\nu_{A_2}(x_i)+\pi_{A_1}(x_i)\cdot\pi_{A_2}(x_i))}{\max\left(\sum\limits_{i=1}^{n}(\mu_{A_1}^2(x_i)+\nu_{A_1}^2(x_i)+\pi_{A_1}^2(x_i)),\sum\limits_{i=1}^{n}(\mu_{A_2}^2(x_i)+\nu_{A_2}^2(x_i)+\pi_{A_2}^2(x_i))\right)} \quad (3.28)$$

Obviously, Eq.(3.28) satisfies the three properties in Theorem 3.1.1.

If we take the weights of the elements $x_i\,(i=1,2,\cdots,n)$ into account, then Eq.(3.28) can be extended as:

$$\rho_{13}(A_1,A_2)$$
$$=\frac{\sum\limits_{i=1}^{n}\omega_i(\mu_{A_1}(x_i)\cdot\mu_{A_2}(x_i)+\nu_{A_1}(x_i)\cdot\nu_{A_2}(x_i)+\pi_{A_1}(x_i)\cdot\pi_{A_2}(x_i))}{\max\left(\sum\limits_{i=1}^{n}\omega_i(\mu_{A_1}^2(x_i)+\nu_{A_1}^2(x_i)+\pi_{A_1}^2(x_i)),\sum\limits_{i=1}^{n}\omega_i(\mu_{A_2}^2(x_i)+\nu_{A_2}^2(x_i)+\pi_{A_2}^2(x_i))\right)} \quad (3.29)$$

where $\omega=(\omega_1,\omega_2,\cdots,\omega_n)^{\mathrm{T}}$ is the weight vector of $x_i\,(i=1,2,\cdots,n)$, with $\omega_i\in[0,1]$, $i=1,2,\cdots,n$, and $\sum\limits_{i=1}^{n}\omega_i=1$. In particular, if $\omega=(1/n,1/n,\cdots,1/n)^{\mathrm{T}}$, then Eq.(3.29) reduces to Eq.(3.28).

If the universe of discourse X and the weights of its elements are continuous, and the weight of $x\in X=[a,b]$ is $\omega(x)$, where $\omega(x)\in[0,\,1]$ and $\int_a^b\omega(x)\mathrm{d}x=1$, then Eq.(3.29) can be transformed into its continuous form:

$$\rho_{14}(A_1,A_2)$$

$$= \frac{\displaystyle\int_a^b \omega(x)(\mu_A(x)\cdot\mu_{A_2}(x)+\nu_{A_1}(x)\cdot\nu_{A_2}(x)+\pi_{A_1}(x)\cdot\pi_{A_2}(x))\mathrm{d}x}{\max\left(\displaystyle\int_a^b \omega(x)(\mu_{A_1}^2(x)+\nu_{A_1}^2(x)+\pi_{A_1}^2(x))\mathrm{d}x, \int_a^b \omega(x)(\mu_{A_2}^2(x)+\nu_{A_2}^2(x)+\pi_{A_2}^2(x))\mathrm{d}x\right)} \tag{3.30}$$

If $\omega(x) = 1/(b-a)$, for any $x \in [a,b]$, then Eq.(3.30) reduces to the following:

$\rho_{15}(A_1, A_2)$

$$= \frac{\displaystyle\int_a^b (\mu_A(x)\cdot\mu_{A_2}(x)+\nu_{A_1}(x)\cdot\nu_{A_2}(x)+\pi_{A_1}(x)\cdot\pi_{A_2}(x))\mathrm{d}x}{\max\left(\displaystyle\int_a^b (\mu_{A_1}^2(x)+\nu_{A_1}^2(x)+\pi_{A_1}^2(x))\mathrm{d}x, \int_a^b \omega(x)(\mu_{A_2}^2(x)+\nu_{A_2}^2(x)+\pi_{A_2}^2(x))\mathrm{d}x\right)} \tag{3.31}$$

In what follows, we generalize Definition 3.1.4 to the IVIFS theory:

Definition 3.1.5 (Xu et al, 2008) Let $X = \{x_1, x_2, \cdots, x_n\}$ be a finite universe of discourse, $\tilde{A}_1 = \{\langle x_i, \tilde{\mu}_{\tilde{A}_1}(x_i), \tilde{\nu}_{\tilde{A}_1}(x_i)\rangle \mid x_i \in X\}$ and $\tilde{A}_2 = \{\langle x_i, \tilde{\mu}_{\tilde{A}_2}(x_i), \tilde{\nu}_{\tilde{A}_2}(x_i)\rangle \mid x_i \in X\}$ two IVIFSs, where $\tilde{\mu}_{\tilde{A}_1}(x_i) = [\tilde{\mu}^{\mathrm{L}}_{\tilde{A}_1}(x_i), \tilde{\mu}^{\mathrm{U}}_{\tilde{A}_1}(x_i)]$, $\tilde{\mu}_{\tilde{A}_2}(x_i) = [\tilde{\mu}^{\mathrm{L}}_{\tilde{A}_2}(x_i), \tilde{\mu}^{\mathrm{U}}_{\tilde{A}_2}(x_i)]$, $\tilde{\nu}_{\tilde{A}_1}(x_i) = [\tilde{\nu}^{\mathrm{L}}_{\tilde{A}_1}(x_i), \tilde{\nu}^{\mathrm{U}}_{\tilde{A}_1}(x_i)]$, $\tilde{\nu}_{\tilde{A}_2}(x_i) = [\tilde{\nu}^{\mathrm{L}}_{\tilde{A}_2}(x_i), \tilde{\nu}^{\mathrm{U}}_{\tilde{A}_2}(x_i)]$. Then

$$\rho_{16}(\tilde{A}_1, \tilde{A}_2) = \frac{\displaystyle\sum_{i=1}^n \varphi_{\tilde{A}_1,\tilde{A}_2}(x_i)}{\max\left(\displaystyle\sum_{i=1}^n \varphi_{\tilde{A}_1}(x_i), \sum_{i=1}^n \varphi_{\tilde{A}_2}(x_i)\right)} \tag{3.32}$$

is called the correlation coefficient of $\tilde{A}_1$ and $\tilde{A}_2$, where

$$\begin{aligned}
\varphi_{\tilde{A}_1}(x_i) &= (\tilde{\mu}^{\mathrm{L}}_{\tilde{A}_1}(x_i))^2 + (\tilde{\nu}^{\mathrm{L}}_{\tilde{A}_1}(x_i))^2 + (\tilde{\pi}^{\mathrm{L}}_{\tilde{A}_1}(x_i))^2 + (\tilde{\mu}^{\mathrm{U}}_{\tilde{A}_1}(x_i))^2 \\
&\quad + (\tilde{\nu}^{\mathrm{U}}_{\tilde{A}_1}(x_i))^2 + (\tilde{\pi}^{\mathrm{U}}_{\tilde{A}_1}(x_i))^2 \\
\varphi_{\tilde{A}_2}(x_i) &= (\tilde{\mu}^{\mathrm{L}}_{\tilde{A}_2}(x_i))^2 + (\tilde{\nu}^{\mathrm{L}}_{\tilde{A}_2}(x_i))^2 + (\tilde{\pi}^{\mathrm{L}}_{\tilde{A}_2}(x_i))^2 + (\tilde{\mu}^{\mathrm{U}}_{\tilde{A}_2}(x_i))^2 \\
&\quad + (\tilde{\nu}^{\mathrm{U}}_{\tilde{A}_2}(x_i))^2 + (\tilde{\pi}^{\mathrm{U}}_{\tilde{A}_2}(x_i))^2 \\
\varphi_{\tilde{A}_1,\tilde{A}_2}(x_i) &= \tilde{\mu}^{\mathrm{L}}_{\tilde{A}_1}(x_i)\cdot\tilde{\mu}^{\mathrm{L}}_{\tilde{A}_2}(x_i) + \tilde{\mu}^{\mathrm{U}}_{\tilde{A}_1}(x_i)\cdot\tilde{\mu}^{\mathrm{U}}_{\tilde{A}_2}(x_i) + \tilde{\nu}^{\mathrm{L}}_{\tilde{A}_1}(x_i)\cdot\tilde{\nu}^{\mathrm{L}}_{\tilde{A}_2}(x_i) \\
&\quad + \tilde{\nu}^{\mathrm{U}}_{\tilde{A}_1}(x_i)\cdot\tilde{\nu}^{\mathrm{U}}_{\tilde{A}_2}(x_i) + \tilde{\pi}^{\mathrm{L}}_{\tilde{A}_1}(x_i)\cdot\tilde{\pi}^{\mathrm{L}}_{\tilde{A}_2}(x_i) + \tilde{\pi}^{\mathrm{U}}_{\tilde{A}_1}(x_i)\cdot\tilde{\pi}^{\mathrm{U}}_{\tilde{A}_2}(x_i)
\end{aligned}$$

If we take the weights of the elements $x_i\,(i = 1, 2, \cdots, n)$ into account, then Eq.(3.32) can be extended as the weighted form:

$$\rho_{17}(\tilde{A}_1, \tilde{A}_2) = \frac{\displaystyle\sum_{i=1}^n \omega_i\varphi_{\tilde{A}_1,\tilde{A}_2}(x_i)}{\max\left(\displaystyle\sum_{i=1}^n \omega_i\varphi_{\tilde{A}_1}(x_i), \sum_{i=1}^n \omega_i\varphi_{\tilde{A}_2}(x_i)\right)} \tag{3.33}$$

where $\omega = (\omega_1, \omega_2, \cdots, \omega_n)^{\mathrm{T}}$ is the weight vector of $x_i\,(i = 1, 2, \cdots, n)$, $\omega_i \in [0, 1]$, $i = 1, 2, \cdots, n$, and $\sum_{i=1}^{n} \omega_i = 1$. In particular, if $\omega = (1/n, 1/n, \cdots, 1/n)^{\mathrm{T}}$, then Eq.(3.33) reduces to Eq.(3.32).

If the universe of discourse X and the weights of its elements are continuous, and the weight of $x \in X = [a, b]$ is $\omega(x)$, where $\omega(x) \in [0,\ 1]$ and $\int_a^b \omega(x)\mathrm{d}x = 1$, then Eq.(3.33) can be transformed into its continuous form:

$$\rho_{18}(\tilde{A}_1, \tilde{A}_2) = \frac{\displaystyle\int_a^b \omega(x)\varphi_{\tilde{A}_1,\tilde{A}_2}(x)\mathrm{d}x}{\max\left(\displaystyle\int_a^b \omega(x)\varphi_{\tilde{A}_1}(x)\mathrm{d}x, \int_a^b \omega(x)\varphi_{\tilde{A}_2}(x)\mathrm{d}x\right)} \tag{3.34}$$

where

$$\begin{aligned}
\varphi_{\tilde{A}_1}(x) &= (\tilde{\mu}^{\mathrm{L}}_{\tilde{A}_1}(x))^2 + (\tilde{\nu}^{\mathrm{L}}_{\tilde{A}_1}(x))^2 + (\tilde{\pi}^{\mathrm{L}}_{\tilde{A}_1}(x))^2 + (\tilde{\mu}^{\mathrm{U}}_{\tilde{A}_1}(x))^2 + (\tilde{\nu}^{\mathrm{U}}_{\tilde{A}_1}(x))^2 + (\tilde{\pi}^{\mathrm{U}}_{\tilde{A}_1}(x))^2 \\
\varphi_{\tilde{A}_2}(x) &= (\tilde{\mu}^{\mathrm{L}}_{\tilde{A}_2}(x))^2 + (\tilde{\nu}^{\mathrm{L}}_{\tilde{A}_2}(x))^2 + (\tilde{\pi}^{\mathrm{L}}_{\tilde{A}_2}(x))^2 + (\tilde{\mu}^{\mathrm{U}}_{\tilde{A}_2}(x))^2 + (\tilde{\nu}^{\mathrm{U}}_{\tilde{A}_2}(x))^2 + (\tilde{\pi}^{\mathrm{U}}_{\tilde{A}_2}(x))^2 \\
\varphi_{\tilde{A}_1,\tilde{A}_2}(x) &= \tilde{\mu}^{\mathrm{L}}_{\tilde{A}_1}(x) \cdot \tilde{\mu}^{\mathrm{L}}_{\tilde{A}_2}(x) + \tilde{\mu}^{\mathrm{U}}_{\tilde{A}_1}(x) \cdot \tilde{\mu}^{\mathrm{U}}_{\tilde{A}_2}(x) + \tilde{\nu}^{\mathrm{L}}_{\tilde{A}_1}(x) \cdot \tilde{\nu}^{\mathrm{L}}_{\tilde{A}_2}(x) + \tilde{\nu}^{\mathrm{U}}_{\tilde{A}_1}(x) \cdot \tilde{\nu}^{\mathrm{U}}_{\tilde{A}_2}(x) \\
&\quad + \tilde{\pi}^{\mathrm{L}}_{\tilde{A}_1}(x) \cdot \tilde{\pi}^{\mathrm{L}}_{\tilde{A}_2}(x) + \tilde{\pi}^{\mathrm{U}}_{\tilde{A}_1}(x) \cdot \tilde{\pi}^{\mathrm{U}}_{\tilde{A}_2}(x)
\end{aligned}$$

Especially, if the elements $x_i\,(i = 1, 2, \cdots, n)$ have the same importance, i.e., $\omega = (1/n, 1/n, \cdots, 1/n)^{\mathrm{T}}$, then Eq.(3.34) reduces to:

$$\rho_{19}(\tilde{A}_1, \tilde{A}_2) = \frac{\displaystyle\int_a^b \varphi_{\tilde{A}_1,\tilde{A}_2}(x)\mathrm{d}x}{\max\left(\displaystyle\int_a^b \varphi_{\tilde{A}_1}(x)\mathrm{d}x, \int_a^b \varphi_{\tilde{A}_2}(x)\mathrm{d}x\right)} \tag{3.35}$$

We now apply Eqs.(3.20) and (3.28) to an example in medical diagnosis (adapted from Szmidt and Kacprzyk (2004))

Example 3.1.1 To make a proper diagnosis $D =$ {Viral fever, Malaria, Typhoid, Stomach problem, Chest problem} for a patient with the given values of the symptoms: $S =$ {Temperature, Headache, Stomach pain, Cough, Chest pain}, a medical knowledge base is necessary that involves elements described in terms of IFSs. The data are given in Table 3.1—each symptom is described by a pair of parameters (μ, ν), i.e., the membership μ and the non-membership ν. The set of patients is $Y =$ {Al, Bob, Joe, Ted}. The symptoms are given in Table 3.2. We need to seek a diagnosis for each patient Y_i $(i = 1, 2, 3, 4)$.

We utilize the correlation measure (3.20) to derive a diagnosis for each patient. All the results for the considered patients are listed in Table 3.3:

Table 3.1 Symptoms characteristic for the considered diagnoses (Szmidt and Kacprzyk, 2004)

	Viral fever	Malaria	Typhoid	Stomach problem	Chest problem
Temperature	(0.4, 0.0)	(0.7, 0.0)	(0.3, 0.3)	(0.1, 0.7)	(0.1, 0.8)
Headache	(0.3, 0.5)	(0.2, 0.6)	(0.6, 0.1)	(0.2, 0.4)	(0.0, 0.8)
Stomach pain	(0.1, 0.7)	(0.0, 0.9)	(0.2, 0.7)	(0.8, 0.0)	(0.2, 0.8)
Cough	(0.4, 0.3)	(0.7, 0.0)	(0.2, 0.6)	(0.2, 0.7)	(0.2, 0.8)
Chest pain	(0.1, 0.7)	(0.1, 0.8)	(0.1, 0.9)	(0.2, 0.7)	(0.8, 0.1)

Table 3.2 Symptoms characteristic for the considered patients (Szmidt and Kacprzyk, 2004)

	Temperature	Headache	Stomach pain	Cough	Chest pain
Al	(0.8, 0.1)	(0.6, 0.1)	(0.2, 0.8)	(0.6, 0.1)	(0.1, 0.6)
Bob	(0.0, 0.8)	(0.4, 0.4)	(0.6, 0.1)	(0.1, 0.7)	(0.1, 0.8)
Joe	(0.8, 0.1)	(0.8, 0.1)	(0.0, 0.6)	(0.2, 0.7)	(0.0, 0.5)
Ted	(0.6, 0.1)	(0.5, 0.4)	(0.3, 0.4)	(0.7, 0.2)	(0.3, 0.4)

Table 3.3 Correlation coefficients of symptoms for each patient to the considered set of possible diagnoses (Xu, 2006h)

	Viral fever	Malaria	Typhoid	Stomach problem	Chest problem
Al	0.7918	0.7705	0.7485	0.7308	0.6258
Bob	0.7332	0.6688	0.7536	0.8000	0.7381
Joe	0.8207	0.7459	0.7529	0.7121	0.7092
Ted	0.9167	0.6977	0.8435	0.6862	0.7702

From the arguments in Table 3.3, we can derive a proper diagnosis as follows: Al suffers from malaria, Bob from a stomach problem, and both Joe and Ted from viral fever.

If we utilize the correlation formulas (3.1) and (3.2) to derive a diagnosis, then we can get the results as shown in Table 3.4:

Table 3.4 Correlation coefficients of symptoms for each patient to the considered set of possible diagnoses (Xu, 2006h)

	Viral fever	Malaria	Typhoid	Stomach problem	Chest problem
Al	0.8856	0.9003	0.8316	0.4546	0.4194
Bob	0.6096	0.4258	0.7872	0.9714	0.6642
Joe	0.8082	0.7066	0.8822	0.5083	0.4828
Ted	0.8709	0.8645	0.7548	0.5997	0.5810

If we utilize the correlation formula (3.28) to derive a diagnosis, then we can get the results as shown in Table 3.5.

The results in Tables 3.4 and 3.5 show that Al suffers from malaria, Bob from a stomach problem, Joe from typhoid, and Ted from viral fever. The difference between the results derived by the above three methods is only the diagnosis for Joe.

Table 3.5 Correlation coefficients of symptoms for each patient to the considered set of possible diagnoses

	Viral fever	Malaria	Typhoid	Stomach problem	Chest problem
Al	0.7721	0.8377	0.7941	0.4779	0.4048
Bob	0.5815	0.4318	0.7741	0.9556	0.5863
Joe	0.6806	0.6591	0.7708	0.4965	0.4315
Ted	0.8190	0.7013	0.6692	0.5451	0.4464

From the data in Table 3.3, we know that for Joe the correlation coefficient of his symptoms and the symptoms characteristic for viral fever is the largest one, while the correlation coefficient of his symptoms and the symptoms characteristic for typhoid ranks second. But in Tables 3.4 and 3.5, the ranking is just reversed. The difference is because the results derived by using Eqs.(3.1), (3.2) and (3.24) are prone to the influence of unfair arguments with too high or too low values, while Eq.(3.21) can relieve the influence of these unfair arguments by emphasizing the role of the considered arguments as a whole.

3.2 Distance and Similarity Measures of Intuitionistic Fuzzy Sets

The IFS was originally introduced by Atanassov (1986; 1983). Since then, many different distance and similarity measures of IFSs have been proposed. Xu and Chen (2008) give a comprehensive overview of distance and similarity measures of IFSs, and propose some new distance and similarity measures of IFSs. They also extend these distance and similarity measures for IVIFSs.

Let X be a finite universe of discourse, and $\Phi(X)$ the set of all IFSs on X. Li and Cheng (2002) define the concept of the similarity measure of IFSs:

Definition 3.2.1 (Li and Cheng, 2002) Let $\vartheta : (\Phi(X))^2 \to [0, 1]$ be a mapping, and let $A_j \in \Phi(X)$ $(j = 1, 2, 3)$. Then $\vartheta(A_1, A_2)$ is called the similarity degree of A_1 and A_2, if it satisfies the following conditions:

(1) $0 \leqslant \vartheta(A_1, A_2) \leqslant 1$;

(2) If $A_1 = A_2$, then $\vartheta(A_1, A_2) = 1$;

(3) $\vartheta(A_1, A_2) = \vartheta(A_2, A_1)$;

(4) If $A_1 \subseteq A_2 \subseteq A_3$, then $\vartheta(A_1, A_3) \leqslant \vartheta(A_1, A_2)$ and $\vartheta(A_1, A_3) \leqslant \vartheta(A_2, A_3)$.

The condition (2) in Definition 3.2.1 only considers the sufficiency "if", and hence has some limitations. Mitchell (2003) improves it by this condition with the following:

(2′) $\vartheta(A_1, A_2) = 1$ if and only if $A_1 = A_2$.

Distance measure is another important measure in the IFS theory. In what follows, we introduce, based on the similarity measures of IFSs, the concept of distance measure of IFSs:

Let d be a mapping $d : (\Phi(X))^2 \rightarrow [0, 1]$. Then the distance between A_1 and A_2 can be defined as:

$$d(A_1, A_2) = 1 - \vartheta(A_1, A_2) \tag{3.36}$$

where $d(A_1, A_2)$ satisfies the following conditions:

(1) $0 \leqslant d(A_1, A_2) \leqslant 1$;

(2) $d(A_1, A_2) = 0$ if and only if $A_1 = A_2$;

(3) $d(A_1, A_2) = d(A_2, A_1)$;

(4) If $A_1 \subseteq A_2 \subseteq A_3$, $A_1, A_2, A_3 \in \Phi(X)$, then $d(A_1, A_3) \geqslant d(A_1, A_2)$ and $d(A_1, A_3) \geqslant d(A_2, A_3)$.

Let $X = \{x_1, x_2, \cdots, x_n\}$ be a finite universe of discourse Bustince and Burillo (1995) define two distance measures for IFNs:

(1) The normalized Hamming distance:

$$d_1(A_1, A_2) = \frac{1}{2n}\sum_{j=1}^{n}(|\mu_{A_1}(x_j) - \mu_{A_2}(x_j)| + |\nu_{A_1}(x_j) - \nu_{A_2}(x_j)|) \tag{3.37}$$

(2) The normalized Euclidean distance:

$$d_2(A_1, A_2) = \sqrt{\frac{1}{2n}\sum_{j=1}^{n}((\mu_{A_1}(x_j) - \mu_{A_2}(x_j))^2 + (\nu_{A_1}(x_j) - \nu_{A_2}(x_j))^2)} \tag{3.38}$$

Obviously, the distance measures (3.37) and (3.38) only involve the first two parameters describing IFSs. Szmidt and Kacprzyk (2000) indicate that distance measures should take into account all three parameters of IFSs, that is, the third parameter should not be omitted when calculating distance between IFSs. As a result, they propose the following distances to improve Eqs.(3.34) and (3.38):

(1) The normalized Hamming distance:

$$\begin{aligned} d_3(A_1, A_2) = \frac{1}{2n}\sum_{j=1}^{n}(&|\mu_{A_1}(x_j) - \mu_{A_2}(x_j)| + |\nu_{A_1}(x_j) - \nu_{A_2}(x_j)| \\ &+ |\pi_{A_1}(x_j) - \pi_{A_2}(x_j)|) \end{aligned} \tag{3.39}$$

(2) The normalized Euclidean distance:

$$\begin{aligned} &d_4(A_1, A_2) \\ &= \sqrt{\frac{1}{2n}\sum_{j=1}^{n}((\mu_{A_1}(x_j)-\mu_{A_2}(x_j))^2+(\nu_{A_1}(x_j)-\nu_{A_2}(x_j))^2+(\pi_{A_1}(x_j)-\pi_{A_2}(x_j))^2)} \end{aligned} \tag{3.40}$$

Szmidt and Kacprzyk (2004) define a similarity measure combining the distance measure of A_1 and A_2, and the distance measure of A_1 and $\bar{A}_2$:

$$\vartheta_1(A_1, A_2) = \frac{d(A_1, A_2)}{d(A_1, \bar{A}_2)} \tag{3.41}$$

where $d(A_1, A_2)$ and $d(A_1, \bar{A}_2)$ can be calculated by Eqs.(3.39) and (3.40), $\bar{A}_2$ is the complement of A_2 (see Definition 1.2.1).

The similarity measure (3.41) contains both the similarity and dissimilarity between two IFSs, which has the following properties:

(1) $\vartheta_1(A_1, A_2) = 0$ indicates that the IFSs A_1 and A_2 are equal, i.e., $A_1 = A_2$;

(2) $\vartheta_1(A_1, A_2) = 1$ indicates that the similarity degree between the IFSs A_1 and A_2 is the same as the similarity degree between the IFSs A_1 and $\bar{A}_2$;

(3) $\vartheta_1(A_1, A_2) \to +\infty$, i.e., $A_1 = \bar{A}_2$ (or $\bar{A}_1 = A_2$) indicates that the IFSs A_1 and A_2 are completely dissimilar (or $A_1 = \bar{A}_2$);

(4) $A_1 = A_2 = \bar{A}_2$ indicates that the entropy between the IFSs A_1 and A_2 is the highest (Szmidt and Kacprzyk, 2004).

Note that $0 \leqslant \vartheta_1(A_1, A_2) \leqslant +\infty$ does not satisfy the condition (1) in Definition 3.2.1. Motivated by the idea of the TOPSIS of Hwang and Yoon (1981), Xu and Chen (2008) adjust Eq.(3.41) as:

$$\vartheta_2(A_1, A_2) = 1 - \frac{d(A_1, A_2)}{d(A_1, A_2) + d(A_1, \bar{A}_2)} = \frac{d(A_1, \bar{A}_2)}{d(A_1, A_2) + d(A_1, \bar{A}_2)} \tag{3.42}$$

where $0 \leqslant \vartheta_2(A_1, A_2) \leqslant 1$. In particular, if $A_1 = A_2$, then $\vartheta_2(A_1, A_2) = 1$.

The prominent characteristic of the similarity measures (3.41) and (3.42) is that they not only take into account the pure distance between IFSs, but also examine whether the IFSs compared are more similar, or more dissimilar, to each other so as to avoid drawing conclusions on the basis of small distances.

Based on the geometric distance model, Xu (2007k) generalizes the distance measures (3.27) and (3.28):

$$\begin{aligned} d_5(A_1, A_2) = \Bigg[\frac{1}{2n}\sum_{j=1}^{n}(&|\mu_{A_1}(x_j) - \mu_{A_2}(x_j)|^\lambda + |\nu_{A_1}(x_j) - \nu_{A_2}(x_j)|^\lambda \\ &+ |\pi_{A_1}(x_j) - \pi_{A_2}(x_j)|^\lambda)\Bigg]^{1/\lambda} \end{aligned} \tag{3.43}$$

where $\lambda \geqslant 1$.

In particular, if $\lambda = 1$, then Eq.(3.43) reduces to Eq.(3.43); If $\lambda = 2$, then Eq.(3.43) reduces to Eq.(3.37); if $\lambda \to +\infty$, then Eq.(3.43) reduces to:

$$\begin{aligned} \lim_{\lambda \to +\infty} d_5(A_1, A_2) = \max_j\{&|\mu_{A_1}(x_j) - \mu_{A_2}(x_j)|,\ |\nu_{A_1}(x_j) \\ &- \nu_{A_2}(x_j)|,\ |\pi_{A_1}(x_j) - \pi_{A_2}(x_j)|\} \end{aligned} \tag{3.44}$$

In fact, we can let

$$\ell = \max_j\{|\mu_{A_1}(x_j) - \mu_{A_2}(x_j)|,\ |\nu_{A_1}(x_j) - \nu_{A_2}(x_j)|,\ |\pi_{A_1}(x_j) - \pi_{A_2}(x_j)|\} \tag{3.45}$$

Then by Eq.(3.43), we have

$$\lim_{\lambda\to+\infty} d_5(A_1,A_2) = \lim_{\lambda\to+\infty}\left[\frac{1}{2n}\sum_{j=1}^{n}(|\mu_{A_1}(x_j)-\mu_{A_2}(x_j)|^\lambda + |\nu_{A_1}(x_j)-\nu_{A_2}(x_j)|^\lambda + |\pi_{A_1}(x_j)-\pi_{A_2}(x_j)|^\lambda)\right]^{1/\lambda}$$

$$= \lim_{\lambda\to+\infty} \ell\left[\frac{1}{2n}\sum_{j=1}^{n}\left(\frac{|\mu_{A_1}(x_j)-\mu_{A_2}(x_j)|}{\ell}\right)^\lambda + \left(\frac{|\nu_{A_1}(x_j)-\nu_{A_2}(x_j)|}{\ell}\right)^\lambda + \left(\frac{|\pi_{A_1}(x_j)-\pi_{A_2}(x_j)|}{\ell}\right)^\lambda\right]^{1/\lambda}$$

$$= \ell \lim_{\lambda\to+\infty}\left(\frac{1}{2n}\right)^{1/\lambda}\cdot \lim_{\lambda\to+\infty}\left[\sum_{j=1}^{n}\left(\frac{|\mu_{A_1}(x_j)-\mu_{A_2}(x_j)|}{\ell}\right)^\lambda + \left(\frac{|\nu_{A_1}(x_j)-\nu_{A_2}(x_j)|}{\ell}\right)^\lambda + \left(\frac{|\pi_{A_1}(x_j)-\pi_{A_2}(x_j)|}{\ell}\right)^\lambda\right]^{1/\lambda}$$

$$= \ell \tag{3.46}$$

i.e., Eq.(3.44) holds.

Xu (2007k) takes into account the weight of the element $x_j \in X$, and generalizes Eq.(3.43) to the weighted form:

$$d_6(A_1,A_2) = \left[\frac{1}{2}\sum_{j=1}^{n}\omega_j(|\mu_{A_1}(x_j)-\mu_{A_2}(x_j)|^\lambda + |\nu_{A_1}(x_j)-\nu_{A_2}(x_j)|^\lambda + |\pi_{A_1}(x_j)-\pi_{A_2}(x_j)|^\lambda)\right]^{1/\lambda} \tag{3.47}$$

where $\lambda \geqslant 1$, $\omega = (\omega_1, \omega_2, \cdots, \omega_n)^{\mathrm{T}}$ is the weight vector of x_j ($j = 1, 2, \cdots, n$), with $\omega_j \in [0, 1]$, $j = 1, 2, \cdots, n$, and $\sum_{j=1}^{n}\omega_j = 1$.

In particular, if $\lambda = 1$, then Eq.(3.47) reduces to the weighted Hamming distance:

$$d_7(A_1,A_2) = \frac{1}{2}\sum_{j=1}^{n}\omega_j(|\mu_{A_1}(x_j)-\mu_{A_2}(x_j)| + |\nu_{A_1}(x_j)-\nu_{A_2}(x_j)| + |\pi_{A_1}(x_j)-\pi_{A_2}(x_j)|) \tag{3.48}$$

If $\omega = (1/n, 1/n, \cdots, 1/n)^{\mathrm{T}}$, then Eq.(3.47) reduces to Eq.(3.43), and Eq.(3.48) reduces to Eq.(3.39).

If $\lambda = 2$, then Eq.(3.47) reduces to the weighted Euclidean distance:

$$d_8\,(A_1,A_2) = \left[\frac{1}{2}\sum_{j=1}^{n}\omega_j((\mu_{A_1}(x_j)-\mu_{A_2}(x_j))^2 + (\nu_{A_1}(x_j)-\nu_{A_2}(x_j))^2\right.$$

$$+ \left(\pi_{A_1}(x_j) - \pi_{A_2}(x_j))^2\right)\Bigg]^{1/2} \tag{3.49}$$

If $\omega = (1/n, 1/n, \cdots, 1/n)^{\mathrm{T}}$, then Eq.(3.49) reduces to Eq.(3.40).

If $\lambda \to +\infty$, then Eq.(3.47) reduces to Eq.(3.44).

Based on Eq.(3.47), Xu (2007k) defines a similarity measure between the IFSs A_1 and A_2:

$$\vartheta_3(A_1, A_2) = 1 - \left[\frac{1}{2}\sum_{j=1}^{n} \omega_j (|\mu_{A_1}(x_j) - \mu_{A_2}(x_j)|^\lambda + |\nu_{A_1}(x_j) - \nu_{A_2}(x_j)|^\lambda + |\pi_{A_1}(x_j) - \pi_{A_2}(x_j)|^\lambda)\right]^{1/\lambda} \tag{3.50}$$

where $\lambda \geqslant 1$, $\vartheta_3(A_1, A_2)$ is a similarity measure between the IFSs A_1 and A_2.

In particular, if all the elements have the same importance, i.e., $\omega = (1/n, 1/n, \cdots, 1/n)^{\mathrm{T}}$, then Eq.(3.50) reduces to:

$$\vartheta_4(A_1, A_2) = 1 - \left[\frac{1}{2n}\sum_{j=1}^{n} (|\mu_{A_1}(x_j) - \mu_{A_2}(x_j)|^\lambda + |\nu_{A_1}(x_j) - \nu_{A_2}(x_j)|^\lambda + |\pi_{A_1}(x_j) - \pi_{A_2}(x_j)|^\lambda)\right]^{1/\lambda} \tag{3.51}$$

where $\lambda \geqslant 1$.

Assuming that the universe of discourse X and the weights of its elements are continuous, and the weight of $x \in X = [a, b]$ is $\omega(x)$, where $\omega(x) \in [0, 1]$ and $\int_a^b \omega(x)\mathrm{d}x = 1$, Xu (2007k) defines the following distance and similarity measures based on Eqs.(3.47) and (3.50):

$$d_9\,(A_1, A_2) = \left[\frac{1}{2}\int_a^b \omega(x)(|\mu_{A_1}(x) - \mu_{A_2}(x)|^\lambda + |\nu_{A_1}(x) - \nu_{A_2}(x)|^\lambda + |\pi_{A_1}(x) - \pi_{A_2}(x)|^\lambda)\mathrm{d}x\right]^{1/\lambda} \tag{3.52}$$

$$\vartheta_5(A_1, A_2) = 1 - \left[\frac{1}{2}\int_a^b \omega(x)(|\mu_{A_1}(x) - \mu_{A_2}(x)|^\lambda + |\nu_{A_1}(x) - \nu_{A_2}(x)|^\lambda + |\pi_{A_1}(x) - \pi_{A_2}(x)|^\lambda)\mathrm{d}x\right]^{1/\lambda} \tag{3.53}$$

where $\lambda \geqslant 1$.

If $\omega(x) = 1/(b-a)$, for any $x \in [a, b]$, then Eq.(3.52) reduces to Eq.(3.54), and Eq.(3.53) reduces to Eq.(3.55):

$$d_{10}(A_1, A_2) = \frac{1}{(2b-2a)^{1/\lambda}} \left[\int_a^b (|\mu_{A_1}(x) - \mu_{A_2}(x)|^\lambda + |\nu_{A_1}(x) - \nu_{A_2}(x)|^\lambda + |\pi_{A_1}(x) - \pi_{A_2}(x)|^\lambda) \mathrm{d}x \right]^{1/\lambda} \quad (3.54)$$

$$\vartheta_6(A_1, A_2) = 1 - \frac{1}{(2b-2a)^{1/\lambda}} \left[\int_a^b (|\mu_{A_1}(x) - \mu_{A_2}(x)|^\lambda + |\nu_{A_1}(x) - \nu_{A_2}(x)|^\lambda + |\pi_{A_1}(x) - \pi_{A_2}(x)|^\lambda) \mathrm{d}x \right]^{1/\lambda} \quad (3.55)$$

where $\lambda \geqslant 1$.

For any $A \in \Phi(X)$, let

$$\varphi_A(x_j) = \frac{\mu_A(x_j) + 1 - \nu_A(x_j)}{2} \quad (3.56)$$

Li and Cheng (2002) define a similarity measure based on the membership degrees and the non-membership degrees of IFSs:

$$\vartheta_7(A_1, A_2) = 1 - \left[\sum_{j=1}^n \omega_j |\varphi_{A_1}(x_j) - \varphi_{A_2}(x_j)|^\lambda \right]^{1/\lambda}, \quad \lambda \geqslant 1 \quad (3.57)$$

In particular, if all the elements have the same importance, i.e., $\omega = (1/n, 1/n, \cdots, 1/n)^{\mathrm{T}}$, then Eq.(3.57) reduces to:

$$\vartheta_8(A_1, A_2) = 1 - \left[\frac{1}{n} \sum_{j=1}^n |\varphi_{A_1}(x_j) - \varphi_{A_2}(x_j)|^\lambda \right]^{1/\lambda}, \quad \lambda \geqslant 1 \quad (3.58)$$

If the universe of discourse X and the weights of its elements are continuous, and the weight of $x \in X = [a, b]$ is $\omega(x)$, as defined before, then Eq.(3.57) can be transformed into the following form:

$$\vartheta_9(A_1, A_2) = 1 - \left[\int_a^b \omega(x) |\varphi_{A_1}(x) - \varphi_{A_2}(x)|^\lambda \mathrm{d}x \right]^{1/\lambda}, \quad \lambda \geqslant 1 \quad (3.59)$$

If $\omega(x) = 1/(b-a)$, for any $x \in [a, b]$, then Eq.(3.59) reduces to:

$$\vartheta_{10}(A_1, A_2) = 1 - \left[\frac{1}{b-a} \int_a^b |\varphi_{A_1}(x) - \varphi_{A_2}(x)|^\lambda \mathrm{d}x \right]^{1/\lambda}, \quad \lambda \geqslant 1 \quad (3.60)$$

Liang and Shi (2003) use several examples to show that Eqs.(3.57) and (3.59) are not always reasonable in some cases (i.e., the modified condition $2'$) in Definition 3.2.1 sometimes does not hold, and propose to improve (3.57) as:

$$\vartheta_{11}(A_1,A_2)=1-\left[\sum_{j=1}^{n}\omega_j\left(\sum_{i=1}^{3}\beta_i\varphi_i(x_j)\right)^{\lambda}\right]^{1/\lambda},\quad \lambda\geqslant 1 \tag{3.61}$$

where $\beta_i\in[0,\,1]\,(i=1,2,3)$, $\sum\limits_{i=1}^{3}\beta_i=1$, and

$$\varphi_1(x_j)=\varphi_{\mu_{A_1A_2}}(x_j)+\varphi_{\nu_{A_1A_2}}(x_j)$$

$$\varphi_2(x_j)=|\varphi_{A_1}(x_j)-\varphi_{A_2}(x_j)|$$

$$\varphi_3(x_j)=\max\{l_{A_1}(x_j),l_{A_2}(x_j)\}-\min\{l_{A_1}(x_j),l_{A_2}(x_j)\}$$

$$l_{A_1}(x_j)=\frac{1-\nu_{A_1}(x_j)-\mu_{A_1}(x_j)}{2}$$

$$l_{A_2}(x_j)=\frac{1-\nu_{A_2}(x_j)-\mu_{A_2}(x_j)}{2}$$

$$\varphi_{\mu_{A_1A_2}}(x_j)=\frac{|\mu_{A_1}(x_j)-\mu_{A_2}(x_j)|}{2}$$

$$\varphi_{\nu_{A_1A_2}}(x_j)=\frac{|\nu_{A_1}(x_j)-\nu_{A_2}(x_j)|}{2}$$

They also propose to improve (3.59) as:

$$\vartheta_{12}(A_1,A_2)=1-\left[\int_a^b\omega(x)\left(\sum_{i=1}^{3}\beta_i\varphi_i(x)\right)^{\lambda}\mathrm{d}x\right]^{1/\lambda},\quad \lambda\geqslant 1 \tag{3.62}$$

If $\omega=(1/n,1/n,\cdots,1/n)^{\mathrm{T}}$, and for any $x\in[a,b]$, $\omega(x)=1/(b-a)$, then Eqs.(3.61) and (3.62) reduce to Eqs.(3.63) and Eq.(3.64) respectively:

$$\vartheta_{13}(A_1,A_2)=1-\left[\frac{1}{n}\sum_{j=1}^{n}\left(\sum_{i=1}^{3}\beta_i\varphi_i(x_j)\right)^{\lambda}\right]^{1/\lambda},\quad \lambda\geqslant 1 \tag{3.63}$$

$$\vartheta_{14}(A_1,A_2)=1-\left[\frac{1}{n}\int_a^b\left(\sum_{i=1}^{3}\beta_i\varphi_i(x)\right)^{\lambda}\mathrm{d}x\right]^{1/\lambda},\quad \lambda\geqslant 1 \tag{3.64}$$

Compared to Eqs.(3.57) and (3.59), Eqs.(3.61) and (3.62) contain more intuitionistic fuzzy information. Consequently, in general, the latter can distinguish more effectively IFSs.

Mitchell (2004) modifies the similarity measure (3.57) by adopting a statistical approach:

$$\vartheta_{15}(A_1,A_2)=\frac{\vartheta_{\mu}(A_1,A_2)+\vartheta_{\nu}(A_1,A_2)}{2} \tag{3.65}$$

where

$$\vartheta_\mu(A_1, A_2) = \vartheta(\mu_{A_1}, \mu_{A_2}) = 1 - \left[\frac{1}{n}\sum_{j=1}^{n}|\mu_{A_1}(x_j) - \mu_{A_2}(x_j)|^\lambda\right]^{1/\lambda}, \quad \lambda \geqslant 1 \tag{3.66}$$

$$\vartheta_\nu(A_1, A_2) = \vartheta(1 - \nu_{A_1}, 1 - \nu_{A_2}) = 1 - \left[\frac{1}{n}\sum_{j=1}^{n}|\nu_{A_1}(x_j) - \nu_{A_2}(x_j)|^\lambda\right]^{1/\lambda}, \quad \lambda \geqslant 1 \tag{3.67}$$

$\vartheta_\mu(A_1, A_2)$ and $\vartheta_\nu(A_1, A_2)$ denote the similarity measures of the "low" membership functions μ_{A_1} and μ_{A_2}, and the "high" membership functions $1 - \nu_{A_1}$ and $1 - \nu_{A_2}$ respectively.

The modified similarity measures (3.65)–(3.67) satisfy the conditions (1), (3) and (4) in Definition 3.2.1, and the revised condition (2′).

Considering that the elements in the universe may have different importance, here we define the weighted forms of Eqs.(3.62) and (3.63) respectively:

$$\vartheta'_\mu(A_1, A_2) = \vartheta'(\mu_{A_1}, \mu_{A_2}) = 1 - \left[\sum_{j=1}^{n}\omega_j|\mu_{A_1}(x_j) - \mu_{A_2}(x_j)|^\lambda\right]^{1/\lambda}, \quad \lambda \geqslant 1 \tag{3.68}$$

$$\vartheta'_\nu(A_1, A_2) = \vartheta'(1 - \nu_{A_1}, 1 - \nu_{A_2}) = 1 - \left[\sum_{j=1}^{n}\omega_j|\nu_{A_1}(x_j) - \nu_{A_2}(x_j)|^\lambda\right]^{1/\lambda}, \quad \lambda \geqslant 1 \tag{3.69}$$

If $\omega = (1/n, 1/n, \cdots, 1/n)^{\mathrm{T}}$, then Eqs.(3.68) and (3.69) reduce to Eqs.(3.66) and (3.67) respectively.

Assuming that the universe of discourse X and the weights of its elements are continuous, Xu and Chen (2008) define the continuous forms of Eqs.(3.68) and (3.69) as follows:

$$\vartheta''_\mu(A_1, A_2) = \vartheta''(\mu_{A_1}, \mu_{A_2}) = 1 - \left[\int_a^b \omega(x)|\mu_{A_1}(x) - \mu_{A_2}(x)|^\lambda \mathrm{d}x\right]^{1/\lambda}, \quad \lambda \geqslant 1 \tag{3.70}$$

$$\vartheta''_\nu(A_1, A_2) = \vartheta''(1 - \nu_{A_1}, 1 - \nu_{A_2}) = 1 - \left[\int_a^b \omega(x)|\nu_{A_1}(x) - \nu_{A_2}(x)|^\lambda \mathrm{d}x\right]^{1/\lambda}, \quad \lambda \geqslant 1 \tag{3.71}$$

In particular, if $\omega(x) = 1/(b-a)$, for any $x \in [a,b]$, then Eqs.(3.70) and (3.71) reduce to Eqs.(3.72) and (3.73) respectively:

$$\vartheta''_{\mu}(A_1,A_2) = \vartheta''(\mu_{A_1},\mu_{A_2}) = 1 - \frac{1}{(b-a)^{1/\lambda}}\left[\int_a^b |\mu_{A_1}(x)-\mu_{A_2}(x)|^{\lambda}\mathrm{d}x\right]^{1/\lambda}, \quad \lambda \geqslant 1 \tag{3.72}$$

$$\vartheta''_{\nu}(A_1,A_2) = \vartheta''(1-\nu_{A_1},1-\nu_{A_2}) = 1 - \frac{1}{(b-a)^{1/\lambda}}\left[\int_a^b |\nu_{A_1}(x)-\nu_{A_2}(x)|^{\lambda}\mathrm{d}x\right]^{1/\lambda}, \quad \lambda \geqslant 1 \tag{3.73}$$

The Hausdorff distance (Nadler 1978) is a measure on how much two non-empty compact (closed and bounded) sets differ and resemble each other with respect to their positions in a metric space, it has the properties: homogeneity, symmetry and triangular inequality.

Let $\tilde{a} = [a^{\mathrm{L}}, a^{\mathrm{U}}]$ and $\tilde{b} = [b^{\mathrm{L}}, b^{\mathrm{U}}]$ be two interval numbers. Then the Hausdorff distance $\widehat{d}(\tilde{a},\tilde{b})$ is defined as:

$$\widehat{d}(\tilde{a},\tilde{b}) = \max\{|a^{\mathrm{L}} - b^{\mathrm{L}}|, |a^{\mathrm{U}} - b^{\mathrm{U}}|\} \tag{3.74}$$

Hung and Yang (2004) present a method to calculate the distance between IFSs on the basis of the Hausdorff distance (3.74), which is described as follows:

Let $\tilde{I}_{A_1}(x_j)$ and $\tilde{I}_{A_2}(x_j)$ be the subintervals on $[0,1]$, denoted by

$$\tilde{I}_{A_1}(x_j) = [\mu_{A_1}(x_j), 1-\nu_{A_1}(x_j)] \tag{3.75}$$

$$\tilde{I}_{A_2}(x_j) = [\mu_{A_2}(x_j), 1-\nu_{A_2}(x_j)] \tag{3.76}$$

Then, the distance between the IFSs A_1 and A_2 is defined as:

$$d_{11}(A_1,A_2) = \sum_{j=1}^{n} \omega_j \widehat{d}(\tilde{I}_{A_1}(x_j), \tilde{I}_{A_2}(x_j)) \tag{3.77}$$

If the universe of discourse X and the weights of its elements are continuous, then Eq.(3.77) can be transformed into the following form:

$$d_{12}(A_1,A_2) = \int_a^b \omega(x)\widehat{d}(\tilde{I}_{A_1}(x), \tilde{I}_{A_2}(x))\mathrm{d}x \tag{3.78}$$

Based on Eqs.(3.77) and (3.78), Xu and Chen (2008) define the similarity measure between the IFSs A_1 and A_2 as:

$$\vartheta_{16}(A_1,A_2) = 1 - \sum_{j=1}^{n} \omega_j \widehat{d}(\tilde{I}_{A_1}(x_j), \tilde{I}_{A_2}(x_j)) \tag{3.79}$$

$$\vartheta_{17}(A_1, A_2) = 1 - \int_a^b \omega(x)\widehat{d}(\tilde{I}_{A_1}(x), \tilde{I}_{A_2}(x))\mathrm{d}x \tag{3.80}$$

If $\omega = (1/n, 1/n, \cdots, 1/n)^{\mathrm{T}}$, and $\omega(x) = 1/(b-a)$, for any $x \in [a, b]$, then Eqs.(3.79) and (3.80) reduces to:

$$\vartheta_{18}(A_1, A_2) = 1 - \frac{1}{n}\sum_{j=1}^{n} \widehat{d}(\tilde{I}_{A_1}(x_j), \tilde{I}_{A_2}(x_j)) \tag{3.81}$$

$$\vartheta_{19}(A_1, A_2) = 1 - \frac{1}{b-a}\int_a^b \widehat{d}(\tilde{I}_{A_1}(x), \tilde{I}_{A_2}(x))\mathrm{d}x \tag{3.82}$$

The similarity measures (3.79)–(3.82) are not only simple, but also suitable for measuring linguistic variables.

Grzegorzewski (2004) also proposes some distance measures based on Hausdorff metric, which are the generalizations of the normalized Hamming and Euclidean distances:

(1) The normalized Hamming distance based on Hausdorff metric:

$$d_{13}(A_1, A_2) = \frac{1}{n}\sum_{j=1}^{n} \max\{|\mu_{A_1}(x_j) - \mu_{A_2}(x_j)|, |\nu_{A_1}(x_j) - \nu_{A_2}(x_j)|\} \tag{3.83}$$

(2) The normalized Euclidean distance based on Hausdorff metric:

$$d_{14}(A_1, A_2) = \frac{1}{n}\sum_{j=1}^{n} \max\{(\mu_{A_1}(x_j) - \mu_{A_2}(x_j))^2, (\nu_{A_1}(x_j) - \nu_{A_2}(x_j))^2\} \tag{3.84}$$

Obviously, the distance between A_1 and A_2 derived by Eq.(3.83) is actually compressed after squaring each difference value. Here, we modify Eq.(3.84) as follows:

$$d_{15}(A_1, A_2) = \sqrt{\frac{1}{n}\sum_{j=1}^{n} \max\{(\mu_{A_1}(x_j) - \mu_{A_2}(x_j))^2, (\nu_{A_1}(x_j) - \nu_{A_2}(x_j))^2\}} \tag{3.85}$$

and generalize Eqs.(3.83) and (3.85) to their weighted forms respectively:

(1) The weighted Hamming distance based on Hausdorff metric:

$$d_{16}(A_1, A_2) = \sum_{j=1}^{n} \omega_j \max\{|\mu_{A_1}(x_j) - \mu_{A_2}(x_j)|, |\nu_{A_1}(x_j) - \nu_{A_2}(x_j)|\} \tag{3.86}$$

which is equivalent to Eq.(3.77).

(2) The weighted Euclidean distance based on Hausdorff metric:

$$d_{17}(A_1, A_2) = \sqrt{\sum_{j=1}^{n} \omega_j \max\{(\mu_{A_1}(x_j) - \mu_{A_2}(x_j))^2, (\nu_{A_1}(x_j) - \nu_{A_2}(x_j))^2\}} \tag{3.87}$$

If the universe of discourse X and the weights of its elements are continuous, then we can define the continuous weighted Euclidean distance:

$$d_{18}(A_1, A_2) = \sqrt{\int_a^b \omega(x) \max\{(\mu_{A_1}(x) - \mu_{A_2}(x))^2, (\nu_{A_1}(x) - \nu_{A_2}(x))^2\} \,\mathrm{d}x} \quad (3.88)$$

Based on the geometric distance model, we can further generalize the distance measures (3.77) and (3.78):

$$d_{19}(A_1, A_2) = \left[\sum_{j=1}^{n} \omega_j \max\{(\mu_{A_1}(x_j) - \mu_{A_2}(x_j))^\lambda, (\nu_{A_1}(x_j) - \nu_{A_2}(x_j))^\lambda\}\right]^{1/\lambda}, \quad \lambda \geqslant 1 \quad (3.89)$$

$$d_{20}(A_1, A_2) = \left[\int_a^b \omega(x) \max\{(\mu_{A_1}(x) - \mu_{A_2}(x))^\lambda, (\nu_{A_1}(x) - \nu_{A_2}(x))^\lambda\}\mathrm{d}x\right]^{1/\lambda}, \quad \lambda \geqslant 1 \quad (3.90)$$

Based on Eqs.(3.89) and (3.90), Xu and Chen (2008) define two similarity measures between the IFSs A_1 and A_2:

$$\vartheta_{20}(A_1, A_2) = 1 - \left[\sum_{j=1}^{n} \omega_j \max\{(\mu_{A_1}(x_j) - \mu_{A_2}(x_j))^\lambda, (\nu_{A_1}(x_j) - \nu_{A_2}(x_j))^\lambda\}\right]^{1/\lambda}, \quad \lambda \geqslant 1 \quad (3.91)$$

$$\vartheta_{21}(A_1, A_2) = 1 - \left[\int_a^b \omega(x) \max\{(\mu_{A_1}(x) - \mu_{A_2}(x))^\lambda, (\nu_{A_1}(x) - \nu_{A_2}(x))^\lambda\}\mathrm{d}x\right]^{1/\lambda}, \quad \lambda \geqslant 1 \quad (3.92)$$

If $\omega = (1/n, 1/n, \cdots, 1/n)^{\mathrm{T}}$ and $\omega(x) = 1/(b-a)$, for any $x \in [a, b]$, then Eqs.(3.91) and (3.92) reduce to the following forms:

$$\vartheta_{22}(A_1, A_2) = 1 - \left[\frac{1}{n}\sum_{j=1}^{n} \max\{(\mu_{A_1}(x_j) - \mu_{A_2}(x_j))^\lambda, (\nu_{A_1}(x_j) - \nu_{A_2}(x_j))^\lambda\}\right]^{1/\lambda}, \quad \lambda \geqslant 1 \quad (3.93)$$

$$\vartheta_{23}(A_1, A_2) = 1 - \frac{1}{(b-a)^{1/\lambda}}\left[\int_a^b \max\{(\mu_{A_1}(x) - \mu_{A_2}(x))^\lambda, (\nu_{A_1}(x) - \nu_{A_2}(x))^\lambda\}\mathrm{d}x\right]^{1/\lambda}, \quad \lambda \geqslant 1 \quad (3.94)$$

The similarity measures (3.91)–(3.94) are not only simple but also convenient for practical applications.

Wang and Xin (2005a) show that the normalized Hamming distance measure satisfies the conditions of the metric and has some good geometric properties, but it may not fit quite well with the reality. To overcome this drawback, they define the following distance:

$$d_{21}(A_1, A_2) = \sum_{j=1}^{n} \omega_j \left[\frac{|\mu_{A_1}(x_j) - \mu_{A_2}(x_j)| + |\nu_{A_1}(x_j) - \nu_{A_2}(x_j)|}{4} + \frac{\max\{|\mu_{A_1}(x_j) - \mu_{A_2}(x_j)|, |\nu_{A_1}(x_j) - \nu_{A_2}(x_j)|\}}{2} \right] \quad (3.95)$$

Clearly, the distance measure (3.95) is the combination of the normalized Hamming distance and Hausdorff distance. It can not only satisfy all the conditions of the metric, but also avoid the unreasonable results produced by the normalized Hamming distance in practical applications.

If the universe of discourse X and the weights of its elements are continuous, then the continuous counterpart of Eq.(3.95) is as follows:

$$d_{22}(A_1, A_2) = \int_a^b \omega(x) \left[\frac{|\mu_{A_1}(x) - \mu_{A_2}(x)| + |\nu_{A_1}(x) - \nu_{A_2}(x)|}{4} + \frac{\max\{|\mu_{A_1}(x) - \mu_{A_2}(x)|, |\nu_{A_1}(x) - \nu_{A_2}(x)|\}}{2} \right] \mathrm{d}x \quad (3.96)$$

Based on Eqs.(3.95) and (3.96), Xu and Chen (2008) define the following two similarity measures of A_1 and A_2 respectively:

$$\vartheta_{24}(A_1, A_2) = 1 - \sum_{j=1}^{n} \omega_j \left[\frac{|\mu_{A_1}(x_j) - \mu_{A_2}(x_j)| + |\nu_{A_1}(x_j) - \nu_{A_2}(x_j)|}{4} + \frac{\max\{| \mu_{A_1}(x_j) - \mu_{A_2}(x_j)|, |\nu_{A_1}(x_j) - \nu_{A_2}(x_j)|\}}{2} \right] \quad (3.97)$$

$$\vartheta_{25}(A_1, A_2) = 1 - \int_a^b \omega(x) \left[\frac{|\mu_{A_1}(x) - \mu_{A_2}(x)| + |\nu_{A_1}(x) - \nu_{A_2}(x)|}{4} + \frac{\max\{|\mu_{A_1}(x) - \mu_{A_2}(x)|, |\nu_{A_1}(x) - \nu_{A_2}(x)|\}}{2} \right] \mathrm{d}x \quad (3.98)$$

Eqs.(3.97) and (3.98) can be further generalized:

$$\vartheta_{26}(A_1, A_2) = 1 - \left[\sum_{j=1}^{n} \omega_j \left[\frac{|\mu_{A_1}(x_j) - \mu_{A_2}(x_j)|^\lambda + |\nu_{A_1}(x_j) - \nu_{A_2}(x_j)|^\lambda}{4} + \frac{\max\{| \mu_{A_1}(x_j) - \mu_{A_2}(x_j)|^\lambda, |\nu_{A_1}(x_j) - \nu_{A_2}(x_j)|^\lambda\}}{2} \right] \right]^{1/\lambda}, \quad \lambda \geqslant 1 \quad (3.99)$$

$$\vartheta_{27}(A_1, A_2) = 1 - \left[\int_a^b \omega(x) \left[\frac{|\mu_{A_1}(x) - \mu_{A_2}(x)|^\lambda + |\nu_{A_1}(x) - \nu_{A_2}(x)|^\lambda}{4} + \frac{\max\{|\mu_{A_1}(x) - \mu_{A_2}(x)|^\lambda, |\nu_{A_1}(x) - \nu_{A_2}(x)|^\lambda\}}{2} \right] \mathrm{d}x \right]^{1/\lambda}, \quad \lambda \geqslant 1 \tag{3.100}$$

If $\omega = (1/n, 1/n, \cdots, 1/n)^{\mathrm{T}}$ and $\omega(x) = 1/(b-a)$, for any $x \in [a, b]$, then Eqs.(3.99) and (3.100) reduce to Eqs.(3.101) and (3.102) respectively:

$$\vartheta_{28}(A_1, A_2) = 1 - \left[\frac{1}{n} \sum_{j=1}^n \left[\frac{|\mu_{A_1}(x_j) - \mu_{A_2}(x_j)|^\lambda + |\nu_{A_1}(x_j) - \nu_{A_2}(x_j)|^\lambda}{4} + \frac{\max\{|\mu_{A_1}(x_j) - \mu_{A_2}(x_j)|^\lambda, |\nu_{A_1}(x_j) - \nu_{A_2}(x_j)|^\lambda\}}{2} \right] \right]^{1/\lambda}, \quad \lambda \geqslant 1 \tag{3.101}$$

$$\vartheta_{29}(A_1, A_2) = 1 - \frac{1}{(b-a)^{1/\lambda}} \left[\int_a^b \left[\frac{|\mu_{A_1}(x) - \mu_{A_2}(x)|^\lambda + |\nu_{A_1}(x) - \nu_{A_2}(x)|^\lambda}{4} + \frac{\max\{|\mu_{A_1}(x) - \mu_{A_2}(x)|^\lambda, |\nu_{A_1}(x) - \nu_{A_2}(x)|^\lambda\}}{2} \right] \mathrm{d}x \right]^{1/\lambda}, \quad \lambda \geqslant 1 \tag{3.102}$$

If $\lambda = 1$, then Eqs.(3.99) and (3.100) reduce to Eqs.(3.97) and (3.98) respectively. Thus, Eqs.(3.97) and (3.98) are the special cases of Eqs.(3.99) and (3.100) respectively.

If $\lambda = 2$, then Eqs.(3.99) and (3.100) reduce to the following similarity measures respectively:

$$\vartheta_{30}(A_1, A_2) = 1 - \left[\sum_{j=1}^n \omega_j \left[\frac{|\mu_{A_1}(x_j) - \mu_{A_2}(x_j)|^2 + |\nu_{A_1}(x_j) - \nu_{A_2}(x_j)|^2}{4} + \frac{\max\{|\mu_{A_1}(x_j) - \mu_{A_2}(x_j)|^2, |\nu_{A_1}(x_j) - \nu_{A_2}(x_j)|^2\}}{2} \right] \right]^{1/2} \tag{3.103}$$

$$\vartheta_{31}(A_1, A_2) = 1 - \left[\int_a^b \omega(x) \left[\frac{|\mu_{A_1}(x) - \mu_{A_2}(x)|^2 + |\nu_{A_1}(x) - \nu_{A_2}(x)|^2}{4} + \frac{\max\{|\mu_{A_1}(x) - \mu_{A_2}(x)|^2, |\nu_{A_1}(x) - \nu_{A_2}(x)|^2\}}{2} \right] \mathrm{d}x \right]^{1/2} \tag{3.104}$$

Obviously, the similarity measure (3.103) is based on the weighted Euclidean distance and Hausdorff metric, and the similarity measure (3.104) is the continuous counterpart of the similarity measure (3.103).

In the next section, we present the generalized results of this section to the IVIFS theory (Xu and Chen, 2008).

3.3 Distance and Similarity Measures of Interval-Valued Intuitionistic Fuzzy Sets

Let $\tilde{A}_1 \in \tilde{\Phi}(X)$ and $\tilde{A}_2 \in \tilde{\Phi}(X)$ be two IVIFSs. Atanassov and Gargov (1989) define two operational laws of the IVIFSs $\tilde{A}_1$ and $\tilde{A}_2$:

(1) $\tilde{A}_1 \subseteq \tilde{A}_2$ if and only if $\tilde{\mu}^{\mathrm{U}}_{\tilde{A}_1}(x) \leqslant \tilde{\mu}^{\mathrm{U}}_{\tilde{A}_2}(x)$, $\mu^{\mathrm{L}}_{\tilde{A}_1}(x) \leqslant \mu^{\mathrm{L}}_{\tilde{A}_2}(x)$, $\tilde{\nu}^{\mathrm{U}}_{\tilde{A}_1}(x) \geqslant \tilde{\nu}^{\mathrm{U}}_{\tilde{A}_2}(x)$ and $\tilde{\nu}^{\mathrm{L}}_{\tilde{A}_1}(x) \geqslant \tilde{\nu}^{\mathrm{L}}_{\tilde{A}_2}(x)$, for any $x \in X$;

(2) $\tilde{A}_1 = \tilde{A}_2$ if and only if $\tilde{A}_1 \subseteq \tilde{A}_2$ and $\tilde{A}_1 \supseteq \tilde{A}_2$.

In what follows, we introduce the concept of similarity measure between two IVIFSs:

Definition 3.3.1 (Xu and Chen, 2008) Let ϑ be a mapping: $\vartheta : (\tilde{\Phi}(X))^2 \to [0,1]$. Then $\vartheta(\tilde{A}_1, \tilde{A}_2)$ is called the similarity degree between $\tilde{A}_1$ and $\tilde{A}_2$, if it satisfies the following conditions:

(1) $0 \leqslant \vartheta(\tilde{A}_1, \tilde{A}_2) \leqslant 1$;

(2) $\vartheta(\tilde{A}_1, \tilde{A}_2) = 1$ if and only if $\tilde{A}_1 = \tilde{A}_2$;

(3) $\vartheta(\tilde{A}_1, \tilde{A}_2) = \vartheta(\tilde{A}_2, \tilde{A}_1)$;

(4) If $\tilde{A}_1 \subseteq \tilde{A}_2 \subseteq \tilde{A}_3$, $\tilde{A}_1, \tilde{A}_2, \tilde{A}_3 \in \tilde{\Phi}(X)$, then $\vartheta(\tilde{A}_1, \tilde{A}_3) \leqslant \vartheta(\tilde{A}_1, \tilde{A}_2)$ and $\vartheta(\tilde{A}_1, \tilde{A}_3) \leqslant \vartheta(\tilde{A}_2, \tilde{A}_3)$.

Definition 3.3.2 (Xu and Chen, 2008) Let d be a mapping: $d : (\tilde{\Phi}(X))^2 \to [0,1]$. Then the distance between $\tilde{A}_1$ and $\tilde{A}_2$ is defined as:

$$d(\tilde{A}_1, \tilde{A}_2) = 1 - \vartheta(\tilde{A}_1, \tilde{A}_2) \tag{3.105}$$

which satisfies the properties:

(1) $0 \leqslant d(\tilde{A}_1, \tilde{A}_2) \leqslant 1$;

(2) $d(\tilde{A}_1, \tilde{A}_2) = 0$ if and only if $\tilde{A}_1 = \tilde{A}_2$;

(3) $d(\tilde{A}_1, \tilde{A}_2) = d(\tilde{A}_2, \tilde{A}_1)$;

(4) If $\tilde{A}_1 \subseteq \tilde{A}_2 \subseteq \tilde{A}_3$, $\tilde{A}_1, \tilde{A}_2, \tilde{A}_3 \in \tilde{\Phi}(X)$, then $d(\tilde{A}_1, \tilde{A}_3) \geqslant d(\tilde{A}_1, \tilde{A}_2)$ and $d(\tilde{A}_1, \tilde{A}_3) \geqslant d(\tilde{A}_2, \tilde{A}_3)$.

Let $X = \{x_1, x_2, \cdots, x_n\}$ be a finite universe of discourse. Then similar to those in Section 3.1, Xu and Chen (2008) propose some distance and similarity measures based on geometric distance model and set-theoretic approach.

3.3.1 Distance and Similarity Measures Based on Geometric Distance Models

We first define the following distance measure of the IVIFSs $\tilde{A}_1$ and $\tilde{A}_2$:

$$d_{23}(\tilde{A}_1, \tilde{A}_2) = \left[\frac{1}{4} \sum_{j=1}^{n} \omega_j (|\tilde{\mu}^{\mathrm{L}}_{\tilde{A}_1}(x_j) - \tilde{\mu}^{\mathrm{L}}_{\tilde{A}_2}(x_j)|^{\lambda} + |\tilde{\mu}^{\mathrm{U}}_{\tilde{A}_1}(x_j) - \tilde{\mu}^{\mathrm{U}}_{\tilde{A}_2}(x_j)|^{\lambda} \right.$$

$$+|\tilde{\nu}^{\rm L}_{\tilde{A}_1}(x_j)-\tilde{\nu}^{\rm L}_{\tilde{A}_2}(x_j)|^\lambda+|\tilde{\nu}^{\rm U}_{\tilde{A}_1}(x_j)-\tilde{\nu}^{\rm U}_{\tilde{A}_2}(x_j)|^\lambda)\Bigg]^{1/\lambda},\quad \lambda\geqslant 1 \tag{3.106}$$

where $\omega=(\omega_1,\omega_2,\cdots,\omega_n)^{\rm T}$ is the weight vector of $x_j\,(j=1,2,\cdots,n)$, $\omega_j\in[0,1]$, $j=1,2,\cdots,n$, and $\sum_{j=1}^{n}\omega_j=1$.

The distance measure (3.106) involves not only the lower and upper limits of the membership degree and the non-membership degree describing IVIFSs, but also the positive parameter λ. Especially, if $\lambda=1$, then Eq.(3.106) reduces to the weighted Hamming distance:

$$\begin{aligned}d_{24}(\tilde{A}_1,\tilde{A}_2)=&\frac{1}{4}\sum_{j=1}^{n}\omega_j(|\tilde{\mu}^{\rm L}_{A_1}(x_j)-\tilde{\mu}^{\rm L}_{A_2}(x_j)|+|\tilde{\mu}^{\rm U}_{A_1}(x_j)-\tilde{\mu}^{\rm U}_{A_2}(x_j)|\\&+|\tilde{\nu}^{\rm L}_{A_1}(x_j)-\tilde{\nu}^{\rm L}_{A_2}(x_j)|+|\tilde{\nu}^{\rm U}_{A_1}(x_j)-\tilde{\nu}^{\rm U}_{A_2}(x_j)|)\end{aligned} \tag{3.107}$$

If $\omega=(1/n,1/n,\cdots,1/n)^{\rm T}$, then Eq.(106) reduces to the normalized Hamming distance:

$$\begin{aligned}d_{25}(\tilde{A}_1,\tilde{A}_2)=&\frac{1}{4n}\sum_{j=1}^{n}(|\tilde{\mu}^{\rm L}_{\tilde{A}_1}(x_j)-\tilde{\mu}^{\rm L}_{\tilde{A}_2}(x_j)|+|\tilde{\mu}^{\rm U}_{\tilde{A}_1}(x_j)-\tilde{\mu}^{\rm U}_{\tilde{A}_2}(x_j)|\\&+|\tilde{\nu}^{\rm L}_{\tilde{A}_1}(x_j)-\tilde{\nu}^{\rm L}_{\tilde{A}_2}(x_j)|+|\tilde{\nu}^{\rm U}_{\tilde{A}_1}(x_j)-\tilde{\nu}^{\rm U}_{\tilde{A}_2}(x_j)|)\end{aligned} \tag{3.108}$$

If $\lambda=2$, then Eq.(3.106) reduces the weighted Euclidean distance:

$$\begin{aligned}d_{26}(\tilde{A}_1,\tilde{A}_2)=&\Bigg[\frac{1}{4}\sum_{j=1}^{n}\omega_j((\tilde{\mu}^{\rm L}_{\tilde{A}_1}(x_j)-\tilde{\mu}^{\rm L}_{\tilde{A}_2}(x_j))^2+(\tilde{\mu}^{\rm U}_{\tilde{A}_1}(x_j)-\tilde{\mu}^{\rm U}_{\tilde{A}_2}(x_j))^2\\&+(\tilde{\nu}^{\rm L}_{\tilde{A}_1}(x_j)-\tilde{\nu}^{\rm L}_{\tilde{A}_2}(x_j))^2+(\tilde{\nu}^{\rm U}_{\tilde{A}_1}(x_j)-\tilde{\nu}^{\rm U}_{\tilde{A}_2}(x_j))^2)\Bigg]^{1/2}\end{aligned} \tag{3.109}$$

If $\omega=(1/n,1/n,\cdots,1/n)^{\rm T}$, then Eq.(109) reduces to the normalized Euclidean distance:

$$\begin{aligned}d_{27}(\tilde{A}_1,\tilde{A}_2)=&\Bigg[\frac{1}{4n}\sum_{j=1}^{n}((\tilde{\mu}^{\rm L}_{\tilde{A}_1}(x_j)-\tilde{\mu}^{\rm L}_{\tilde{A}_2}(x_j))^2+(\tilde{\mu}^{\rm U}_{\tilde{A}_1}(x_j)-\tilde{\mu}^{\rm U}_{\tilde{A}_2}(x_j))^2\\&+(\tilde{\nu}^{\rm L}_{\tilde{A}_1}(x_j)-\tilde{\nu}^{\rm L}_{\tilde{A}_2}(x_j))^2+(\tilde{\nu}^{\rm U}_{\tilde{A}_1}(x_j)-\tilde{\nu}^{\rm U}_{\tilde{A}_2}(x_j))^2)\Bigg]^{1/2}\end{aligned} \tag{3.110}$$

If $\lambda\to+\infty$, then Eq.(3.106) reduces to:

$$\lim_{\lambda\to+\infty}d_{23}(\tilde{A}_1,\tilde{A}_2)=\max_j\{|\tilde{\mu}^{\rm L}_{\tilde{A}_1}(x_j)-\tilde{\mu}^{\rm L}_{\tilde{A}_2}(x_j)|,|\tilde{\mu}^{\rm U}_{\tilde{A}_1}(x_j)-\tilde{\mu}^{\rm U}_{\tilde{A}_2}(x_j)|,$$

$$|\tilde{\nu}^{\rm L}_{\tilde{A}_1}(x_j) - \tilde{\nu}^{\rm L}_{\tilde{A}_2}(x_j)|, |\tilde{\nu}^{\rm U}_{\tilde{A}_1}(x_j) - \tilde{\nu}^{\rm U}_{\tilde{A}_2}(x_j)|\}$$

Based on Eq.(3.106), we introduce a similarity measure between the IVIFSs $\tilde{A}_1$ and $\tilde{A}_2$:

$$\vartheta_{32}(\tilde{A}_1, \tilde{A}_2) = 1 - \left[\frac{1}{4}\sum_{j=1}^{n}\omega_j(|\tilde{\mu}^{\rm L}_{\tilde{A}_1}(x_j) - \tilde{\mu}^{\rm L}_{\tilde{A}_2}(x_j)|^\lambda + |\tilde{\mu}^{\rm U}_{\tilde{A}_1}(x_j) - \tilde{\mu}^{\rm U}_{\tilde{A}_2}(x_j)|^\lambda + |\tilde{\nu}^{\rm L}_{\tilde{A}_1}(x_j) - \tilde{\nu}^{\rm L}_{\tilde{A}_2}(x_j)|^\lambda + |\tilde{\nu}^{\rm U}_{\tilde{A}_1}(x_j) - \tilde{\nu}^{\rm U}_{\tilde{A}_2}(x_j)|^\lambda)\right]^{1/\lambda}, \quad \lambda \geqslant 1 \quad (3.111)$$

If the universe of discourse X and the weights of its elements are continuous, then Eqs.(3.106) and (3.111) can be transformed into Eqs.(3.112) and (3.113) respectively:

$$d_{28}(\tilde{A}_1, \tilde{A}_2) = \left[\frac{1}{4}\int_a^b \omega(x)(|\tilde{\mu}^{\rm L}_{\tilde{A}_1}(x) - \tilde{\mu}^{\rm L}_{\tilde{A}_2}(x)|^\lambda + |\tilde{\mu}^{\rm U}_{\tilde{A}_1}(x) - \tilde{\mu}^{\rm U}_{\tilde{A}_2}(x)|^\lambda + |\tilde{\nu}^{\rm L}_{\tilde{A}_1}(x) - \tilde{\nu}^{\rm L}_{\tilde{A}_2}(x)|^\lambda + |\tilde{\nu}^{\rm U}_{\tilde{A}_1}(x) - \tilde{\nu}^{\rm U}_{\tilde{A}_2}(x)|^\lambda){\rm d}x\right]^{1/\lambda}, \quad \lambda \geqslant 1 \quad (3.112)$$

$$\vartheta_{33}(\tilde{A}_1, \tilde{A}_2) = 1 - \left[\frac{1}{4}\int_a^b \omega(x)(|\tilde{\mu}^{\rm L}_{\tilde{A}_1}(x) - \tilde{\mu}^{\rm L}_{\tilde{A}_2}(x)|^\lambda + |\tilde{\mu}^{\rm U}_{\tilde{A}_1}(x) - \tilde{\mu}^{\rm U}_{\tilde{A}_2}(x)|^\lambda + |\tilde{\nu}^{\rm L}_{\tilde{A}_1}(x) - \tilde{\nu}^{\rm L}_{\tilde{A}_2}(x)|^\lambda + |\tilde{\nu}^{\rm U}_{\tilde{A}_1}(x) - \tilde{\nu}^{\rm U}_{\tilde{A}_2}(x)|^\lambda){\rm d}x\right]^{1/\lambda}, \quad \lambda \geqslant 1 \quad (3.113)$$

In particular, if $\omega(x) = 1/(b-a)$, for any $x \in [a, b]$, then Eqs.(3.112) and (3.113) reduce to Eqs.(3.114) and (3.115) respectively:

$$d_{29}(\tilde{A}_1, \tilde{A}_2) = \frac{1}{(4b-4a)^{1/\lambda}}\left[\int_a^b (|\tilde{\mu}^{\rm L}_{\tilde{A}_1}(x) - \tilde{\mu}^{\rm L}_{\tilde{A}_2}(x)|^\lambda + |\tilde{\mu}^{\rm U}_{\tilde{A}_1}(x) - \tilde{\mu}^{\rm U}_{\tilde{A}_2}(x)|^\lambda + |\tilde{\nu}^{\rm L}_{\tilde{A}_1}(x) - \tilde{\nu}^{\rm L}_{\tilde{A}_2}(x)|^\lambda + |\tilde{\nu}^{\rm U}_{\tilde{A}_1}(x) - \tilde{\nu}^{\rm U}_{\tilde{A}_2}(x)|^\lambda){\rm d}x\right]^{1/\lambda}, \quad \lambda \geqslant 1 \quad (3.114)$$

$$\vartheta_{34}(\tilde{A}_1, \tilde{A}_2) = 1 - \frac{1}{(4b-4a)^{1/\lambda}}\left[\int_a^b (|\tilde{\mu}^{\rm L}_{\tilde{A}_1}(x) - \tilde{\mu}^{\rm L}_{\tilde{A}_2}(x)|^\lambda + |\tilde{\mu}^{\rm U}_{\tilde{A}_1}(x) - \tilde{\mu}^{\rm U}_{\tilde{A}_2}(x)|^\lambda + |\tilde{\nu}^{\rm L}_{\tilde{A}_1}(x) - \tilde{\nu}^{\rm L}_{\tilde{A}_2}(x)|^\lambda + |\tilde{\nu}^{\rm U}_{\tilde{A}_1}(x) - \tilde{\nu}^{\rm U}_{\tilde{A}_2}(x)|^\lambda){\rm d}x\right]^{1/\lambda}, \quad \lambda \geqslant 1 \quad (3.115)$$

Similar to Eqs.(3.65), (3.68) and (3.69), we introduce a similarity measure between the IVIFSs $\tilde{A}_1$ and $\tilde{A}_2$:

$$\vartheta_{35}(\tilde{A}_1, \tilde{A}_2) = \frac{\vartheta_{\tilde{\mu}}(\tilde{A}_1, \tilde{A}_2) + \vartheta_{\tilde{v}}(\tilde{A}_1, \tilde{A}_2)}{2} \quad (3.116)$$

where

$$\vartheta_{\tilde{\mu}}(\tilde{A}_1, \tilde{A}_2) = 1 - \left[\frac{1}{2}\sum_{j=1}^{n}\omega_j(|\tilde{\mu}^{\mathrm{L}}_{\tilde{A}_1}(x_j) - \tilde{\mu}^{\mathrm{L}}_{\tilde{A}_2}(x_j)|^\lambda + |\tilde{\mu}^{\mathrm{U}}_{\tilde{A}_1}(x_j) - \tilde{\mu}^{\mathrm{U}}_{\tilde{A}_2}(x_j)|^\lambda)\right]^{1/\lambda}, \quad \lambda \geqslant 1 \tag{3.117}$$

$$\vartheta_{\tilde{\nu}}(\tilde{A}_1, \tilde{A}_2) = 1 - \left[\frac{1}{2}\sum_{j=1}^{n}\omega_j(|\tilde{\nu}^{\mathrm{L}}_{\tilde{A}_1}(x_j) - \tilde{\nu}^{\mathrm{L}}_{\tilde{A}_2}(x_j)|^\lambda + |\tilde{\nu}^{\mathrm{U}}_{\tilde{A}_1}(x_j) - \tilde{\nu}^{\mathrm{U}}_{\tilde{A}_2}(x_j)|^\lambda)\right]^{1/\lambda}, \quad \lambda \geqslant 1 \tag{3.118}$$

Obviously, the similarity measure (3.116) is very similar to (3.111).

Assuming that the universe of discourse X and the weights of its elements are continuous, and the weight of $x \in X = [a, b]$ is $\omega(x)$, where $\omega(x) \in [0, 1]$ and $\int_a^b \omega(x)\mathrm{d}x = 1$, Xu and Chen (2008) define the following similarity measures:

$$\vartheta'_{\tilde{\mu}}(\tilde{A}_1, \tilde{A}_2) = 1 - \left[\frac{1}{2}\int_a^b \omega(x)(|\tilde{\mu}^{\mathrm{L}}_{\tilde{A}_1}(x) - \tilde{\mu}^{\mathrm{L}}_{\tilde{A}_2}(x)|^\lambda + |\tilde{\mu}^{\mathrm{U}}_{\tilde{A}_1}(x) - \tilde{\mu}^{\mathrm{U}}_{\tilde{A}_2}(x)|^\lambda)\mathrm{d}x\right]^{1/\lambda}, \quad \lambda \geqslant 1 \tag{3.119}$$

$$\vartheta'_{\tilde{\nu}}(\tilde{A}_1, \tilde{A}_2) = 1 - \left[\frac{1}{2}\int_a^b \omega(x)(|\tilde{\nu}^{\mathrm{L}}_{\tilde{A}_1}(x) - \tilde{\nu}^{\mathrm{L}}_{\tilde{A}_2}(x)|^\lambda + |\tilde{\nu}^{\mathrm{U}}_{\tilde{A}_1}(x) - \tilde{\nu}^{\mathrm{U}}_{\tilde{A}_2}(x)|^\lambda)\mathrm{d}x\right]^{1/\lambda}, \quad \lambda \geqslant 1 \tag{3.120}$$

In particular, if $\omega(x) = 1/(b-a)$, for any $x \in [a, b]$, then Eqs.(3.119) and (3.120) reduce to:

$$\vartheta'_{\tilde{\mu}}(\tilde{A}_1, \tilde{A}_2) = 1 - \frac{1}{(2b-2a)^{1/\lambda}}\left[\int_a^b (|\tilde{\mu}^{\mathrm{L}}_{\tilde{A}_1}(x) - \tilde{\mu}^{\mathrm{L}}_{\tilde{A}_2}(x)|^\lambda + |\tilde{\mu}^{\mathrm{U}}_{\tilde{A}_1}(x) - \tilde{\mu}^{\mathrm{U}}_{\tilde{A}_2}(x)|^\lambda)\mathrm{d}x\right]^{1/\lambda}, \quad \lambda \geqslant 1 \tag{3.121}$$

and

$$\vartheta'_{\tilde{\nu}}(\tilde{A}_1, \tilde{A}_2) = 1 - \frac{1}{(2b-2a)^{1/\lambda}}\left[\int_a^b (|\tilde{\nu}^{\mathrm{L}}_{\tilde{A}_1}(x) - \tilde{\nu}^{\mathrm{L}}_{\tilde{A}_2}(x)|^\lambda + |\tilde{\nu}^{\mathrm{U}}_{\tilde{A}_1}(x) - \tilde{\nu}^{\mathrm{U}}_{\tilde{A}_2}(x)|^\lambda)\mathrm{d}x\right]^{1/\lambda}, \quad \lambda \geqslant 1 \tag{3.122}$$

respectively.

3.3.2 Distance and Similarity Measures Based on Set-Theoretic Approaches

Based on Hausdorff metric, Xu and Chen (2008) first define some distance measures between the IVIFSs $\tilde{A}_1$ and $\tilde{A}_2$, which are generalizations of the normalized Hamming and Euclidean distances:

(1) The normalized Hamming distance based on Hausdorff metric:

$$d_{30}(\tilde{A}_1, \tilde{A}_2) = \frac{1}{n}\sum_{j=1}^{n}\max\{|\tilde{\mu}^{\mathrm{L}}_{\tilde{A}_1}(x_j) - \tilde{\mu}^{\mathrm{L}}_{\tilde{A}_2}(x_j)|,\ |\tilde{\mu}^{\mathrm{U}}_{\tilde{A}_1}(x_j) - \tilde{\mu}^{\mathrm{U}}_{\tilde{A}_2}(x_j)|,$$

$$|\tilde{\nu}^{\mathrm{L}}_{\tilde{A}_1}(x_j) - \tilde{\nu}^{\mathrm{L}}_{\tilde{A}_2}(x_j)|,\ |\tilde{\nu}^{\mathrm{U}}_{\tilde{A}_1}(x_j) - \tilde{\nu}^{\mathrm{U}}_{\tilde{A}_2}(x_j)|\} \tag{3.123}$$

(2) The normalized Euclidean distance based on Hausdorff metric:

$$d_{31}(\tilde{A}_1, \tilde{A}_2) = \left[\frac{1}{n}\sum_{j=1}^{n} \max\{(\tilde{\mu}^{\mathrm{L}}_{\tilde{A}_1}(x_j) - \tilde{\mu}^{\mathrm{L}}_{\tilde{A}_2}(x_j))^2,\ (\tilde{\mu}^{\mathrm{U}}_{\tilde{A}_1}(x_j) - \tilde{\mu}^{\mathrm{U}}_{\tilde{A}_2}(x_j))^2,\right.$$
$$\left.(\tilde{\nu}^{\mathrm{L}}_{\tilde{A}_1}(x_j) - \tilde{\nu}^{\mathrm{L}}_{\tilde{A}_2}(x_j))^2,\ (\tilde{\nu}^{\mathrm{U}}_{\tilde{A}_1}(x_j) - \tilde{\nu}^{\mathrm{U}}_{\tilde{A}_2}(x_j))^2\}\right]^{1/2} \tag{3.124}$$

In many practical situations, the elements $x_j \in X$ $(j = 1, 2, \cdots, n)$ usually have different importance, and thus need to be assigned different weights. This motivates us to generalize Eqs.(3.123) and (3.124) to their weighted forms respectively (Xu and Chen, 2008):

(1) The weighted Hamming distance based on Hausdorff metric:

$$d_{32}(\tilde{A}_1, \tilde{A}_2) = \sum_{j=1}^{n} \omega_j \max\{|\tilde{\mu}^{\mathrm{L}}_{\tilde{A}_1}(x_j) - \tilde{\mu}^{\mathrm{L}}_{\tilde{A}_2}(x_j)|,\ |\tilde{\mu}^{\mathrm{U}}_{\tilde{A}_1}(x_j) - \tilde{\mu}^{\mathrm{U}}_{\tilde{A}_2}(x_j)|,$$
$$|\tilde{\nu}^{\mathrm{L}}_{\tilde{A}_1}(x_j) - \tilde{\nu}^{\mathrm{L}}_{\tilde{A}_2}(x_j)|,\ |\tilde{\nu}^{\mathrm{U}}_{\tilde{A}_1}(x_j) - \tilde{\nu}^{\mathrm{U}}_{\tilde{A}_2}(x_j)|\} \tag{3.125}$$

(2) The weighted Euclidean distance based on Hausdorff metric:

$$d_{33}(\tilde{A}_1, \tilde{A}_2) = \left[\sum_{j=1}^{n} \omega_j \max\{(\tilde{\mu}^{\mathrm{L}}_{\tilde{A}_1}(x_j) - \tilde{\mu}^{\mathrm{L}}_{\tilde{A}_2}(x_j))^2,\ (\tilde{\mu}^{\mathrm{U}}_{\tilde{A}_1}(x_j) - \tilde{\mu}^{\mathrm{U}}_{\tilde{A}_2}(x_j))^2,\right.$$
$$\left.(\tilde{\nu}^{\mathrm{L}}_{\tilde{A}_1}(x_j) - \tilde{\nu}^{\mathrm{L}}_{\tilde{A}_2}(x_j))^2,\ (\tilde{\nu}^{\mathrm{U}}_{\tilde{A}_1}(x_j) - \tilde{\nu}^{\mathrm{U}}_{\tilde{A}_2}(x_j))^2\}\right]^{1/2} \tag{3.126}$$

Obviously, if $\omega = (1/n, 1/n, \cdots, 1/n)^{\mathrm{T}}$, then Eqs.(3.125) and (3.126) reduce to Eqs.(3.123) and (3.124) respectively.

Xu and Chen (2008) further generalize Eqs.(3.125) and (3.126) to the following form:

$$d_{34}(\tilde{A}_1, \tilde{A}_2) = \left[\sum_{j=1}^{n} \omega_j \max\{|\tilde{\mu}^{\mathrm{L}}_{\tilde{A}_1}(x_j) - \tilde{\mu}^{\mathrm{L}}_{\tilde{A}_2}(x_j)|^{\lambda},\ |\tilde{\mu}^{\mathrm{U}}_{\tilde{A}_1}(x_j) - \tilde{\mu}^{\mathrm{U}}_{\tilde{A}_2}(x_j)|^{\lambda},\right.$$
$$\left.|\tilde{\nu}^{\mathrm{L}}_{\tilde{A}_1}(x_j) - \tilde{\nu}^{\mathrm{L}}_{\tilde{A}_2}(x_j)|^{\lambda},\ |\tilde{\nu}^{\mathrm{U}}_{\tilde{A}_1}(x_j) - \tilde{\nu}^{\mathrm{U}}_{\tilde{A}_2}(x_j)|^{\lambda}\}\right]^{1/\lambda}, \quad \lambda \geqslant 1 \tag{3.127}$$

In particular, if $\lambda = 1$, then Eq.(3.127) reduces to Eq.(3.123); If $\lambda = 2$, then Eq.(3.127) reduces to Eq.(3.124); If $\lambda \to +\infty$, then Eq.(3.127) reduces to the following form:

$$\lim_{\lambda \to +\infty} d_{34}(\tilde{A}_1, \tilde{A}_2) = \max_j\{|\tilde{\mu}^{\mathrm{L}}_{\tilde{A}_1}(x_j) - \tilde{\mu}^{\mathrm{L}}_{\tilde{A}_2}(x_j)|,\ |\tilde{\mu}^{\mathrm{U}}_{\tilde{A}_1}(x_j) - \tilde{\mu}^{\mathrm{U}}_{\tilde{A}_2}(x_j)|,$$

$$|\tilde{\nu}^{\mathrm{L}}_{\tilde{A}_1}(x_j) - \tilde{\nu}^{\mathrm{L}}_{\tilde{A}_2}(x_j)|,\ |\tilde{\nu}^{\mathrm{U}}_{\tilde{A}_1}(x_j) - \tilde{\nu}^{\mathrm{U}}_{\tilde{A}_2}(x_j)|\} \tag{3.128}$$

If the universe of discourse X and the weights of its elements are continuous, then Eq.(3.127) can be transformed to:

$$d_{35}(\tilde{A}_1, \tilde{A}_2) = \left[\int_a^b \omega(x) \max\{|\tilde{\mu}^{\mathrm{L}}_{\tilde{A}_1}(x) - \tilde{\mu}^{\mathrm{L}}_{\tilde{A}_2}(x)|^\lambda,\ |\tilde{\mu}^{\mathrm{U}}_{\tilde{A}_1}(x) - \tilde{\mu}^{\mathrm{U}}_{\tilde{A}_2}(x)|^\lambda,\right.$$
$$\left.|\tilde{\nu}^{\mathrm{L}}_{\tilde{A}_1}(x) - \tilde{\nu}^{\mathrm{L}}_{\tilde{A}_2}(x)|^\lambda,\ |\tilde{\nu}^{\mathrm{U}}_{\tilde{A}_1}(x) - \tilde{\nu}^{\mathrm{U}}_{\tilde{A}_2}(x)|^\lambda\}\mathrm{d}x\right]^{1/\lambda}, \quad \lambda \geqslant 1 \tag{3.129}$$

In particular, if $\lambda = 1, 2$, then Eq.(3.127) reduces to the following distance measures:

(1) The continuous weighted Hamming distance based on Hausdorff metric:

$$d_{36}(\tilde{A}_1, \tilde{A}_2) = \int_a^b \omega(x) \max\{|\tilde{\mu}^{\mathrm{L}}_{\tilde{A}_1}(x) - \tilde{\mu}^{\mathrm{L}}_{\tilde{A}_2}(x)|,\ |\tilde{\mu}^{\mathrm{U}}_{\tilde{A}_1}(x) - \tilde{\mu}^{\mathrm{U}}_{\tilde{A}_2}x)|,$$
$$|\tilde{\nu}^{\mathrm{L}}_{\tilde{A}_1}(x) - \tilde{\nu}^{\mathrm{L}}_{\tilde{A}_2}(x)|,\ |\tilde{\nu}^{\mathrm{U}}_{\tilde{A}_1}(x) - \tilde{\nu}^{\mathrm{U}}_{\tilde{A}_2}(x)|\}\mathrm{d}x \tag{3.130}$$

(2) The continuous weighted Euclidean distance based on Hausdorff metric:

$$d_{37}(\tilde{A}_1, \tilde{A}_2) = \left[\int_a^b \omega(x) \max\{(\tilde{\mu}^{\mathrm{L}}_{\tilde{A}_1}(x) - \tilde{\mu}^{\mathrm{L}}_{\tilde{A}_2}(x))^2,\ (\tilde{\mu}^{\mathrm{U}}_{\tilde{A}_1}(x) - \tilde{\mu}^{\mathrm{U}}_{\tilde{A}_2}(x))^2,\right.$$
$$\left.(\tilde{\nu}^{\mathrm{L}}_{\tilde{A}_1}(x) - \tilde{\nu}^{\mathrm{L}}_{\tilde{A}_2}(x))^2,\ (\tilde{\nu}^{\mathrm{U}}_{\tilde{A}_1}(x) - \tilde{\nu}^{\mathrm{U}}_{\tilde{A}_2}(x))^2\}\,\mathrm{d}x\right]^{1/2} \tag{3.131}$$

Based on Eqs.(3.127) and (3.129), Xu and Chen (2008) define two similarity measures of $\tilde{A}_1$ and $\tilde{A}_2$ as follows:

$$\vartheta_{36}(\tilde{A}_1, \tilde{A}_2) = 1 - \left[\sum_{j=1}^n \omega_j \max\{|\tilde{\mu}^{\mathrm{L}}_{\tilde{A}_1}(x_j) - \tilde{\mu}^{\mathrm{L}}_{\tilde{A}_2}(x_j)|^\lambda,\ |\tilde{\mu}^{\mathrm{U}}_{\tilde{A}_1}(x_j) - \tilde{\mu}^{\mathrm{U}}_{\tilde{A}_2}(x_j)|^\lambda,\right.$$
$$\left.|\tilde{\nu}^{\mathrm{L}}_{\tilde{A}_1}(x_j) - \tilde{\nu}^{\mathrm{L}}_{\tilde{A}_2}(x_j)|^\lambda,\ |\tilde{\nu}^{\mathrm{U}}_{\tilde{A}_1}(x_j) - \tilde{\nu}^{\mathrm{U}}_{\tilde{A}_2}(x_j)|^\lambda\}\right]^{1/\lambda}, \quad \lambda \geqslant 1 \tag{3.132}$$

$$\vartheta_{37}(\tilde{A}_1, \tilde{A}_2) = 1 - \left[\int_a^b \omega(x) \max\{|\tilde{\mu}^{\mathrm{L}}_{\tilde{A}_1}(x) - \tilde{\mu}^{\mathrm{L}}_{\tilde{A}_2}(x)|^\lambda,\ |\tilde{\mu}^{\mathrm{U}}_{\tilde{A}_1}(x) - \tilde{\mu}^{\mathrm{U}}_{\tilde{A}_2}(x)|^\lambda,\right.$$
$$\left.|\tilde{\nu}^{\mathrm{L}}_{\tilde{A}_1}(x) - \tilde{\nu}^{\mathrm{L}}_{\tilde{A}_2}(x)|^\lambda,\ |\tilde{\nu}^{\mathrm{U}}_{\tilde{A}_1}(x) - \tilde{\nu}^{\mathrm{U}}_{\tilde{A}_2}(x)|^\lambda\}\mathrm{d}x\right]^{1/\lambda}, \quad \lambda \geqslant 1 \tag{3.133}$$

In particular, if $\omega(x) = 1/(b-a)$, for any $x \in [a, b]$, then Eqs.(3.129) and (3.133) reduce to Eqs.(3.134) and (3.135) respectively:

$$d_{38}(\tilde{A}_1, \tilde{A}_2) = \frac{1}{(b-a)^{1/\lambda}}\left[\int_a^b \max\{|\tilde{\mu}^{\rm L}_{\tilde{A}_1}(x) - \tilde{\mu}^{\rm L}_{\tilde{A}_2}(x)|^\lambda,\ |\tilde{\mu}^{\rm U}_{\tilde{A}_1}(x) - \tilde{\mu}^{\rm U}_{\tilde{A}_2}(x)|^\lambda,\right.$$
$$\left.|\tilde{\nu}^{\rm L}_{\tilde{A}_1}(x) - \tilde{\nu}^{\rm L}_{\tilde{A}_2}(x)|^\lambda,\ |\tilde{\nu}^{\rm U}_{\tilde{A}_1}(x) - \tilde{\nu}^{\rm U}_{\tilde{A}_2}(x)|^\lambda\}{\rm d}x\right]^{1/\lambda}, \quad \lambda \geqslant 1 \qquad (3.134)$$

$$\vartheta_{38}(\tilde{A}_1, \tilde{A}_2) = 1 - \frac{1}{(b-a)^{1/\lambda}}\left[\int_a^b \max\{|\tilde{\mu}^{\rm L}_{\tilde{A}_1}(x) - \tilde{\mu}^{\rm L}_{\tilde{A}_2}(x)|^\lambda,\ |\tilde{\mu}^{\rm U}_{\tilde{A}_1}(x) - \tilde{\mu}^{\rm U}_{\tilde{A}_2}(x)|^\lambda,\right.$$
$$\left.|\tilde{\nu}^{\rm L}_{\tilde{A}_1}(x) - \tilde{\nu}^{\rm L}_{\tilde{A}_2}(x)|^\lambda,\ |\tilde{\nu}^{\rm U}_{\tilde{A}_1}(x) - \tilde{\nu}^{\rm U}_{\tilde{A}_2}(x)|^\lambda\}{\rm d}x\right]^{1/\lambda}, \quad \lambda \geqslant 1 \qquad (3.135)$$

Although the weighted Hamming distance measure (3.107) satisfies the conditions of the metric and has some good geometric properties, it may not fit the reality quite well. As such, let us now combine its generalized form Eq.(3.106) and the distance measure (3.127) to get the following distance formula (Xu and Chen, 2008):

$$d_{39}(\tilde{A}_1, \tilde{A}_2) = \left[\sum_{j=1}^n \omega_j \left[\frac{1}{8}(|\tilde{\mu}^{\rm L}_{\tilde{A}_1}(x_j) - \tilde{\mu}^{\rm L}_{\tilde{A}_2}(x_j)|^\lambda + |\tilde{\mu}^{\rm U}_{\tilde{A}_1}(x_j) - \tilde{\mu}^{\rm U}_{\tilde{A}_2}(x_j)|^\lambda\right.\right.$$
$$+ |\tilde{\nu}^{\rm L}_{\tilde{A}_1}(x_j) - \tilde{\nu}^{\rm L}_{\tilde{A}_2}(x_j)|^\lambda + |\tilde{\nu}^{\rm U}_{\tilde{A}_1}(x_j) - \tilde{\nu}^{\rm U}_{\tilde{A}_2}(x_j)|^\lambda)$$
$$+ \frac{1}{2}(\max\{|\tilde{\mu}^{\rm L}_{\tilde{A}_1}(x_j) - \tilde{\mu}^{\rm L}_{\tilde{A}_2}(x_j)|^\lambda,\ |\tilde{\mu}^{\rm U}_{\tilde{A}_1}(x_j) - \tilde{\mu}^{\rm U}_{\tilde{A}_2}(x_j)|^\lambda,$$
$$\left.\left.|\tilde{\nu}^{\rm L}_{\tilde{A}_1}(x_j) - \tilde{\nu}^{\rm L}_{\tilde{A}_2}(x_j)|^\lambda,\ |\tilde{\nu}^{\rm U}_{\tilde{A}_1}(x_j) - \tilde{\nu}^{\rm U}_{\tilde{A}_2}(x_j)|^\lambda\})\right]\right]^{1/\lambda} \qquad (3.136)$$

where $\lambda \geqslant 1$.

If the universe of discourse X and the weights of its elements are continuous, then Eq.(3.136) can be transformed as:

$$d_{40}(\tilde{A}_1, \tilde{A}_2) = \left[\int_a^b \omega(x) \left[\frac{1}{8}(|\tilde{\mu}^{\rm L}_{\tilde{A}_1}(x) - \tilde{\mu}^{\rm L}_{\tilde{A}_2}(x)|^\lambda + |\tilde{\mu}^{\rm U}_{\tilde{A}_1}(x) - \tilde{\mu}^{\rm U}_{\tilde{A}_2}(x)|^\lambda\right.\right.$$
$$+ |\tilde{\nu}^{\rm L}_{\tilde{A}_1}(x) - \tilde{\nu}^{\rm L}_{\tilde{A}_2}(x)|^\lambda + |\tilde{\nu}^{\rm U}_{\tilde{A}_1}(x) - \tilde{\nu}^{\rm U}_{\tilde{A}_2}(x)|^\lambda)$$
$$+ \frac{1}{2}(\max\{|\tilde{\mu}^{\rm L}_{\tilde{A}_1}(x) - \tilde{\mu}^{\rm L}_{\tilde{A}_2}(x)|^\lambda,\ |\tilde{\mu}^{\rm U}_{\tilde{A}_1}(x) - \tilde{\mu}^{\rm U}_{\tilde{A}_2}(x)|^\lambda,$$
$$\left.\left.|\tilde{\nu}^{\rm L}_{\tilde{A}_1}(x) - \tilde{\nu}^{\rm L}_{\tilde{A}_2}(x)|^\lambda,\ |\tilde{\nu}^{\rm U}_{\tilde{A}_1}(x) - \tilde{\nu}^{\rm U}_{\tilde{A}_2}(x)|^\lambda\})\right]{\rm d}x\right]^{1/\lambda} \qquad (3.137)$$

where $\lambda \geqslant 1$.

Based on Eqs.(3.136) and (3.137), we can define the following two similarity measures of $\tilde{A}_1$ and $\tilde{A}_2$ respectively (Xu and Chen, 2008):

$$\vartheta_{39}(\tilde{A}_1, \tilde{A}_2) = 1 - \left[\sum_{j=1}^n \omega_j \left[\frac{1}{8}(|\tilde{\mu}^{\rm L}_{\tilde{A}_1}(x_j) - \tilde{\mu}^{\rm L}_{\tilde{A}_2}(x_j)|^\lambda + |\tilde{\mu}^{\rm U}_{\tilde{A}_1}(x_j) - \tilde{\mu}^{\rm U}_{\tilde{A}_2}(x_j)|^\lambda\right.\right.$$

$$+|\tilde{\nu}^{\mathrm{L}}_{\tilde{A}_1}(x_j)-\tilde{\nu}^{\mathrm{L}}_{\tilde{A}_2}(x_j)|^\lambda+|\tilde{\nu}^{\mathrm{U}}_{\tilde{A}_1}(x_j)-\tilde{\nu}^{\mathrm{U}}_{\tilde{A}_2}(x_j)|^\lambda)$$
$$+\frac{1}{2}(\max\{|\tilde{\mu}^{\mathrm{L}}_{\tilde{A}_1}(x_j)-\tilde{\mu}^{\mathrm{L}}_{\tilde{A}_2}(x_j)|^\lambda,\ |\tilde{\mu}^{\mathrm{U}}_{\tilde{A}_1}(x_j)-\tilde{\mu}^{\mathrm{U}}_{\tilde{A}_2}(x_j)|^\lambda,$$
$$|\tilde{\nu}^{\mathrm{L}}_{\tilde{A}_1}(x_j)-\tilde{\nu}^{\mathrm{L}}_{\tilde{A}_2}(x_j)|^\lambda,\ |\tilde{\nu}^{\mathrm{U}}_{\tilde{A}_1}(x_j)-\tilde{\nu}^{\mathrm{U}}_{\tilde{A}_2}(x_j)|^\lambda\})\Bigg]\Bigg]^{1/\lambda} \tag{3.138}$$

$$\vartheta_{40}(\tilde{A}_1,\tilde{A}_2)=1-\Bigg[\int_a^b\omega(x)\Bigg[\frac{1}{8}(|\tilde{\mu}^{\mathrm{L}}_{\tilde{A}_1}(x)-\tilde{\mu}^{\mathrm{L}}_{\tilde{A}_2}(x)|^\lambda+|\tilde{\mu}^{\mathrm{U}}_{\tilde{A}_1}(x)-\tilde{\mu}^{\mathrm{U}}_{\tilde{A}_2}(x)|^\lambda$$
$$+|\tilde{\nu}^{\mathrm{L}}_{\tilde{A}_1}(x)-\tilde{\nu}^{\mathrm{L}}_{\tilde{A}_2}(x)|^\lambda+|\tilde{\nu}^{\mathrm{U}}_{\tilde{A}_1}(x)-\tilde{\nu}^{\mathrm{U}}_{\tilde{A}_2}(x)|^\lambda)$$
$$+\frac{1}{2}(\max\{|\tilde{\mu}^{\mathrm{L}}_{\tilde{A}_1}(x)-\tilde{\mu}^{\mathrm{L}}_{\tilde{A}_2}(x)|^\lambda,\ |\tilde{\mu}^{\mathrm{U}}_{\tilde{A}_1}(x)-\tilde{\mu}^{\mathrm{U}}_{\tilde{A}_2}(x)|^\lambda,$$
$$|\tilde{\nu}^{\mathrm{L}}_{\tilde{A}_1}(x)-\tilde{\nu}^{\mathrm{L}}_{\tilde{A}_2}(x)|^\lambda,\ |\tilde{\nu}^{\mathrm{U}}_{\tilde{A}_1}(x)-\tilde{\nu}^{\mathrm{U}}_{\tilde{A}_2}(x)|^\lambda\})\Bigg]\mathrm{d}x\Bigg]^{1/\lambda} \tag{3.139}$$

where $\lambda \geqslant 1$.

If $\omega=(1/n,1/n,\cdots,1/n)^{\mathrm{T}}$ and $\omega(x)=1/(b-a)$, for any $x\in[a,b]$, then Eqs.(3.138) and (3.139) reduce to Eqs.(4.140) and (4.141) respectively:

$$\vartheta_{41}(\tilde{A}_1,\tilde{A}_2)=1-\frac{1}{n^{1/\lambda}}\Bigg[\sum_{j=1}^n\Bigg[\frac{1}{8}(|\tilde{\mu}^{\mathrm{L}}_{\tilde{A}_1}(x_j)-\tilde{\mu}^{\mathrm{L}}_{\tilde{A}_2}(x_j)|^\lambda+|\tilde{\mu}^{\mathrm{U}}_{\tilde{A}_1}(x_j)-\tilde{\mu}^{\mathrm{U}}_{\tilde{A}_2}(x_j)|^\lambda$$
$$+|\tilde{\nu}^{\mathrm{L}}_{\tilde{A}_1}(x_j)-\tilde{\nu}^{\mathrm{L}}_{\tilde{A}_2}(x_j)|^\lambda+|\tilde{\nu}^{\mathrm{U}}_{\tilde{A}_1}(x_j)-\tilde{\nu}^{\mathrm{U}}_{\tilde{A}_2}(x_j)|^\lambda)$$
$$+\frac{1}{2}(\max\{|\tilde{\mu}^{\mathrm{L}}_{\tilde{A}_1}(x_j)-\tilde{\mu}^{\mathrm{L}}_{\tilde{A}_2}(x_j)|^\lambda,\ |\tilde{\mu}^{\mathrm{U}}_{\tilde{A}_1}(x_j)-\tilde{\mu}^{\mathrm{U}}_{\tilde{A}_2}(x_j)|^\lambda,$$
$$|\tilde{\nu}^{\mathrm{L}}_{\tilde{A}_1}(x_j)-\tilde{\nu}^{\mathrm{L}}_{\tilde{A}_2}(x_j)|^\lambda,\ |\tilde{\nu}^{\mathrm{U}}_{\tilde{A}_1}(x_j)-\tilde{\nu}^{\mathrm{U}}_{\tilde{A}_2}(x_j)|^\lambda\})\Bigg]\Bigg]^{1/\lambda} \tag{3.140}$$

$$\vartheta_{42}(\tilde{A}_1,\tilde{A}_2)=1-\frac{1}{(b-a)^{1/\lambda}}\Bigg[\int_a^b\Bigg[\frac{1}{8}(|\tilde{\mu}^{\mathrm{L}}_{\tilde{A}_1}(x)-\tilde{\mu}^{\mathrm{L}}_{\tilde{A}_2}(x)|^\lambda+|\tilde{\mu}^{\mathrm{U}}_{\tilde{A}_1}(x)-\tilde{\mu}^{\mathrm{U}}_{\tilde{A}_2}(x)|^\lambda$$
$$+|\tilde{\nu}^{\mathrm{L}}_{\tilde{A}_1}(x)-\tilde{\nu}^{\mathrm{L}}_{\tilde{A}_2}(x)|^\lambda+|\tilde{\nu}^{\mathrm{U}}_{\tilde{A}_1}(x)-\tilde{\nu}^{\mathrm{U}}_{\tilde{A}_2}(x)|^\lambda)$$
$$+\frac{1}{2}(\max\{|\tilde{\mu}^{\mathrm{L}}_{\tilde{A}_1}(x)-\tilde{\mu}^{\mathrm{L}}_{\tilde{A}_2}(x)|^\lambda,\ |\tilde{\mu}^{\mathrm{U}}_{\tilde{A}_1}(x)-\tilde{\mu}^{\mathrm{U}}_{\tilde{A}_2}(x)|^\lambda,$$
$$|\tilde{\nu}^{\mathrm{L}}_{\tilde{A}_1}(x)-\tilde{\nu}^{\mathrm{L}}_{\tilde{A}_2}(x)|^\lambda,\ |\tilde{\nu}^{\mathrm{U}}_{\tilde{A}_1}(x)-\tilde{\nu}^{\mathrm{U}}_{\tilde{A}_2}(x)|^\lambda\})\Bigg]\mathrm{d}x\Bigg]^{1/\lambda} \tag{3.141}$$

where $\lambda \geqslant 1$.

If $\lambda=1$, then Eqs.(3.138) and (3.139) reduce to Eqs.(3.142) and (3.143) respectively:

$$\vartheta_{43}(\tilde{A}_1,\tilde{A}_2)=1-\sum_{j=1}^n\omega_j\Bigg[\frac{1}{8}(|\tilde{\mu}^{\mathrm{L}}_{\tilde{A}_1}(x_j)-\tilde{\mu}^{\mathrm{L}}_{\tilde{A}_2}(x_j)|+|\tilde{\mu}^{\mathrm{U}}_{\tilde{A}_1}(x_j)-\tilde{\mu}^{\mathrm{U}}_{\tilde{A}_2}(x_j)|$$

$$+ |\tilde{\nu}^{\rm L}_{\tilde{A}_1}(x_j) - \tilde{\nu}^{\rm L}_{\tilde{A}_2}(x_j)| + |\tilde{\nu}^{\rm U}_{\tilde{A}_1}(x_j) - \tilde{\nu}^{\rm U}_{\tilde{A}_2}(x_j)|)$$
$$+ \frac{1}{2}(\max\{|\tilde{\mu}^{\rm L}_{\tilde{A}_1}(x_j) - \tilde{\mu}^{\rm L}_{\tilde{A}_2}(x_j)|,\ |\tilde{\mu}^{\rm U}_{\tilde{A}_1}(x_j) - \tilde{\mu}^{\rm U}_{\tilde{A}_2}(x_j)|,$$
$$|\tilde{\nu}^{\rm L}_{\tilde{A}_1}(x_j) - \tilde{\nu}^{\rm L}_{\tilde{A}_2}(x_j)|,\ |\tilde{\nu}^{\rm U}_{\tilde{A}_1}(x_j) - \tilde{\nu}^{\rm U}_{\tilde{A}_2}(x_j)|\})\Bigg] \tag{3.142}$$

$$\vartheta_{44}(\tilde{A}_1, \tilde{A}_2) = 1 - \int_a^b \omega(x)\left[\frac{1}{8}(|\tilde{\mu}^{\rm L}_{\tilde{A}_1}(x) - \tilde{\mu}^{\rm L}_{\tilde{A}_2}(x)| + |\tilde{\mu}^{\rm U}_{\tilde{A}_1}(x) - \tilde{\mu}^{\rm U}_{\tilde{A}_2}(x)|\right.$$
$$+ |\tilde{\nu}^{\rm L}_{\tilde{A}_1}(x) - \tilde{\nu}^{\rm L}_{\tilde{A}_2}(x)| + |\tilde{\nu}^{\rm U}_{\tilde{A}_1}(x) - \tilde{\nu}^{\rm U}_{\tilde{A}_2}(x)|)$$
$$+ \frac{1}{2}(\max\{|\tilde{\mu}^{\rm L}_{\tilde{A}_1}(x) - \tilde{\mu}^{\rm L}_{\tilde{A}_2}(x)|, |\tilde{\mu}^{\rm U}_{\tilde{A}_1}(x) - \tilde{\mu}^{\rm U}_{\tilde{A}_2}(x)|,$$
$$\left.|\tilde{\nu}^{\rm L}_{\tilde{A}_1}(x) - \tilde{\nu}^{\rm L}_{\tilde{A}_2}(x)|,\ |\tilde{\nu}^{\rm U}_{\tilde{A}_1}(x) - \tilde{\nu}^{\rm U}_{\tilde{A}_2}(x)|\})\right]{\rm d}x \tag{3.143}$$

where $\lambda \geqslant 1$.

If $\lambda = 2$, then Eqs.(3.138) and (3.139) reduce to Eqs.(3.144) and (3.145) respectively:

$$\vartheta_{45}(\tilde{A}_1, \tilde{A}_2) = 1 - \left[\sum_{j=1}^{n} \omega_j \left[\frac{1}{8}(|\tilde{\mu}^{\rm L}_{\tilde{A}_1}(x_j) - \tilde{\mu}^{\rm L}_{\tilde{A}_2}(x_j)|^2 + |\tilde{\mu}^{\rm U}_{\tilde{A}_1}(x_j) - \tilde{\mu}^{\rm U}_{\tilde{A}_2}(x_j)|^2\right.\right.$$
$$+ |\tilde{\nu}^{\rm L}_{\tilde{A}_1}(x_j) - \tilde{\nu}^{\rm L}_{\tilde{A}_2}(x_j)|^2 + |\tilde{\nu}^{\rm U}_{\tilde{A}_1}(x_j) - \tilde{\nu}^{\rm U}_{\tilde{A}_2}(x_j)|^2)$$
$$+ \frac{1}{2}(\max\{|\tilde{\mu}^{\rm L}_{\tilde{A}_1}(x_j) - \tilde{\mu}^{\rm L}_{\tilde{A}_2}(x_j)|^2,\ |\tilde{\mu}^{\rm U}_{\tilde{A}_1}(x_j) - \tilde{\mu}^{\rm U}_{\tilde{A}_2}(x_j)|^2,$$
$$\left.\left.|\tilde{\nu}^{\rm L}_{\tilde{A}_1}(x_j) - \tilde{\nu}^{\rm L}_{\tilde{A}_2}(x_j)|^2,\ |\tilde{\nu}^{\rm U}_{\tilde{A}_1}(x_j) - \tilde{\nu}^{\rm U}_{\tilde{A}_2}(x_j)|^2\})\right]\right]^{1/2} \tag{3.144}$$

$$\vartheta_{46}(\tilde{A}_1, \tilde{A}_2) = 1 - \left[\int_a^b \omega(x)\left[\frac{1}{8}(|\tilde{\mu}^{\rm L}_{\tilde{A}_1}(x) - \tilde{\mu}^{\rm L}_{\tilde{A}_2}(x)|^2 + |\tilde{\mu}^{\rm U}_{\tilde{A}_1}(x) - \tilde{\mu}^{\rm U}_{\tilde{A}_2}(x)|^2\right.\right.$$
$$+ |\tilde{\nu}^{\rm L}_{\tilde{A}_1}(x) - \tilde{\nu}^{\rm L}_{\tilde{A}_2}(x)|^2 + |\tilde{\nu}^{\rm U}_{\tilde{A}_1}(x) - \tilde{\nu}^{\rm U}_{\tilde{A}_2}(x)|^2)$$
$$+ \frac{1}{2}(\max\{|\tilde{\mu}^{\rm L}_{\tilde{A}_1}(x) - \tilde{\mu}^{\rm L}_{\tilde{A}_2}(x)|^2,\ |\tilde{\mu}^{\rm U}_{\tilde{A}_1}(x) - \tilde{\mu}^{\rm U}_{\tilde{A}_2}(x)|^2,$$
$$\left.\left.|\tilde{\nu}^{\rm L}_{\tilde{A}_1}(x) - \tilde{\nu}^{\rm L}_{\tilde{A}_2}(x)|^2,\ |\tilde{\nu}^{\rm U}_{\tilde{A}_1}(x) - \tilde{\nu}^{\rm U}_{\tilde{A}_2}(x)|^\lambda\})\right]{\rm d}x\right]^{1/2} \tag{3.145}$$

where $\lambda \geqslant 1$.

Clearly, the similarity measure (3.142) is based on the weighted Hamming distance and Hausdorff metric, and the similarity measure (3.144) is based on the weighted Euclidean distance and Hausdorff metric, while the similarity measures (3.143) and (3.145) are the continuous counterparts of the similarity measures (3.142) and (3.144) respectively.

Let us now utilize two practical examples to illustrate the above distance and similarity measures:

Example 3.3.1 Consider four building materials: sealant, floor varnish, wall paint, carpet, and polyvinyl chloride flooring, which are represented by the IFSs A_j ($j = 1, 2, 3, 4$) in the feature space $X = \{x_1, x_2, \cdots, x_{12}\}$. The weight vector of x_i ($i = 1, 2, \cdots, 12$) is:

$$\omega = (0.12,\ 0.10,\ 0.08,\ 0.05,\ 0.10,\ 0.11,\ 0.09,\ 0.06,\ 0.12,\ 0.10,\ 0.07)^{\mathrm{T}}$$

Now we consider another kind of unknown building material A, with data as listed in Table 3.6. Based on the weight vector ω and the data in Table 3.6, we can use the above distance and similarity measures to identify to which type the unknown material A belongs:

(1) By Eqs.(3.39) and (3.42), we have

$$\vartheta_2(A_1, A) = 0.480, \quad \vartheta_2(A_2, A) = 0.476$$

$$\vartheta_2(A_3, A) = 0.714, \quad \vartheta_2(A_4, A) = 0.941$$

Table 3.6 The data on building materials

	A_1	A_2	A_3	A_4	A
x_1	(0.173, 0.524)	(0.510, 0.365)	(0.495, 0.387)	(1.000, 0.000)	(0.978, 0.003)
x_2	(0.102, 0.818)	(0.627, 0.125)	(0.603, 0.298)	(1.000, 0.000)	(0.980, 0.012)
x_3	(0.530, 0.326)	(1.000, 0.000)	(0.987, 0.006)	(0.857, 0.123)	(0.987, 0.132)
x_4	(0.965, 0.008)	(0.125, 0.648)	(0.073, 0.849)	(0.734, 0.158)	(0.693, 0.0876)
x_5	(0.420, 0.351)	(0.026, 0.823)	(0.037, 0.923)	(0.021, 0.896)	(0.051, 0.876)
x_6	(0.008, 0.956)	(0.732, 0.153)	(0.690, 0.268)	(0.076, 0.912)	(0.123, 0.756)
x_7	(0.331, 0.512)	(0.556, 0.303)	(0.147, 0.812)	(0.152, 0.712)	(0.152, 0.732)
x_8	(1.000, 0.000)	(0.650, 0.267)	(0.213, 0.653)	(0.113, 0.756)	(0.113, 0.732)
x_9	(0.215, 0.625)	(1.000, 0.000)	(0.501, 0.284)	(0.489, 0.389)	(0.494, 0.368)
x_{10}	(0.432, 0.534)	(0.145, 0.762)	(1.000, 0.000)	(1.000, 0.000)	(0.987, 0.000)
x_{11}	(0.750, 0.126)	(0.047, 0.923)	(0.324, 0.483)	(0.386, 0.485)	(0.376, 0.423)
x_{12}	(0.432, 0.432)	(0.760, 0.231)	(0.045, 0.912)	(0.028, 0.912)	(0.012, 0.897)

Then

$$\vartheta_2(A_4, A) > \vartheta_2(A_3, A) > \vartheta_2(A_1, A) > \vartheta_2(A_2, A)$$

(2) By Eq.(3.50) (without loss of generality, let $\lambda = 1$), we get

$$\vartheta_3(A_1, A) = 0.511, \quad \vartheta_3(A_2, A) = 0.527$$

$$\vartheta_3(A_3, A) = 0.737, \quad \upsilon_3(A_4, A) = 0.874$$

Then

$$\vartheta_3(A_4, A) > \vartheta_3(A_3, A) > \vartheta_3(A_2, A) > \vartheta_3(A_1, A)$$

(3) By Eq.(3.57) (let $\lambda = 1$), we have

$$\vartheta_7(A_1, A) = 0.685, \quad \vartheta_7(A_2, A) = 0.578$$

$$\vartheta_7(A_3, A) = 0.796, \quad \vartheta_7(A_4, A) = 0.974$$

Then

$$\vartheta_7(A_4, A) > \vartheta_7(A_3, A) > \vartheta_7(A_1, A) > \vartheta_7(A_2, A)$$

(4) According to Eq.(3.61) (let $\lambda = 1$ and $\beta_1 = \beta_2 = \beta_3 = 1/3$), we have

$$\vartheta_{11}(A_1, A) = 0.693, \quad \vartheta_{11}(A_2, A) = 0.705$$

$$\vartheta_{11}(A_3, A) = 0.854, \quad \vartheta_{11}(A_4, A) = 0.976$$

Then

$$\vartheta_{11}(A_4, A) > \vartheta_{11}(A_3, A) > \vartheta_{11}(A_2, A) > \vartheta_{11}(A_1, A)$$

(5) It follows from Eqs.(3.65), (3.68) and (3.69) (let $\lambda = 1$) that

$$\vartheta_{15}(A_1, A) = 0.563, \quad \vartheta_{15}(A_2, A) = 0.575$$

$$\vartheta_{15}(A_3, A) = 0.795, \quad \vartheta_{15}(A_4, A) = 0.972$$

Then

$$\vartheta_{15}(A_4, A) > \vartheta_{15}(A_3, A) > \vartheta_{15}(A_2, A) > \vartheta_{15}(A_1, A)$$

(6) By Eq.(3.81), we can obtain

$$\vartheta_{18}(A_1, A) = 0.522, \quad \vartheta_{18}(A_2, A) = 0.529$$

$$\vartheta_{18}(A_3, A) = 0.767, \quad \vartheta_{18}(A_4, A) = 0.958$$

Then

$$\vartheta_{18}(A_4, A) > \vartheta_{18}(A_3, A) > \vartheta_{18}(A_2, A) > \vartheta_{18}(A_1, A)$$

Obviously, the ranking result derived by Eq.(3.89) (let $\lambda = 1$) is the same as that derived by Eq.(3.81).

(7) Using Eq.(3.99) (let $\lambda = 1$), we get

$$\vartheta_{26}(A_1, A) = 0.542, \quad \vartheta_{26}(A_2, A) = 0.552$$

$$\vartheta_{26}(A_3, A) = 0.781, \quad \vartheta_{26}(A_4, A) = 0.966$$

To conclude

$$\vartheta_{26}(A_4, A) > \vartheta_{26}(A_3, A) > \vartheta_{26}(A_2, A) > \vartheta_{26}(A_1, A)$$

From the above numerical results, we know that the degree of similarity between A_4 and A is the largest one as derived by each similarity measure. That is, all the similarity measures assign the unknown building material A to the class of building material A_4 according to the principle of the maximum degree of similarity between IFSs. Yet, there exist two slightly different ranking results: the similarity measures (3.42) and (3.57) derive the same ranking of the building materials, in which the degree of similarity between A_3 and A ranks the second, the degree of similarity between A_1 and A ranks the third, and the degree of similarity between A_2 and A is the smallest one. While the similarity measures (3.50), (3.61), (3.65), (3.81) and (3.99) derive the same ranking of the building materials, the ranking of the degree of similarity between A_1 and A, and the degree of similarity between A_2 and A, is reversed.

Example 3.3.2 Consider three kinds of mineral fields, represented by the IVIFSs $\tilde{A}_j$ $(j = 1, 2, 3)$, each of which is featured by the content of six minerals in the feature space $X = \{x_1, x_2, \cdots, x_6\}$. The weight vector of x_i $(i = 1, 2, \cdots, 6)$ is:

$$\omega = (0.20,\ 0.10,\ 0.15,\ 0.25,\ 0.10,\ 0.20)^{\mathrm{T}}$$

Cosndier another kind of mineral $\tilde{A}$ (the data are listed in Table 3.7). Our aim is to justify to which field the mineral $\tilde{A}$ should belong.

Table 3.7 The data on minerals

	$\tilde{A}_1$	$\tilde{A}_2$
x_1	([0.72,0.74], [0.10,0.12])	([0.42,0.45], [0.38,0.40])
x_2	([0.00,0.05], [0.80,0.82])	([0.65,0.67], [0.28,0.30])
x_3	([0.18,0.20], [0.62,0.63])	([1.00,1.00], [0.00,0.00])
x_4	([0.49,0.50], [0.35,0.37])	([0.70,0.90], [0.00,0.10])
x_5	([0.01,0.02], [0.60,0.63])	([0.80,1.00], [0.00,0.00])
x_6	([0.72,0.74], [0.12,0.13])	([0.90,1.00], [0.00,0.00])
	$\tilde{A}_3$	$\tilde{A}$
x_1	([0.30,0.32], [0.45,0.47])	([0.60,0.63], [0.30,0.35])
x_2	([0.90,1.00], [0.00,0.00])	([0.50,0.53], [0.34,0.36])
x_3	([0.18,0.20], [0.70,0.73])	([0.20,0.21], [0.68,0.70])
x_4	([0.15,0.16], [0.75,0.78])	([0.20,0.22], [0.75,0.77])
x_5	([0.00,0.05], [0.88,0.90])	([0.05,0.07], [0.87,0.90])
x_6	([0.65,0.68], [0.25,0.30])	([0.65,0.70], [0.25,0.30])

(1) According to Eq.(3.111) (without loss of generality, let $\lambda = 2$), we have

$$\vartheta_{32}(\tilde{A}_1, \tilde{A}) = 0.744, \quad \vartheta_{32}(\tilde{A}_2, \tilde{A}) = 0.446, \quad \vartheta_{32}(\tilde{A}_3, \tilde{A}) = 0.806$$

Then

$$\vartheta_{32}(\tilde{A}_3, \tilde{A}) > \vartheta_{32}(\tilde{A}_1, \tilde{A}) > \vartheta_{32}(\tilde{A}_2, \tilde{A})$$

(2) By Eqs.(3.116)–(3.118) (let $\lambda = 2$), we get

$$\vartheta_{35}(\tilde{A}_1, \tilde{A}) = 0.747, \quad \vartheta_{35}(\tilde{A}_2, \tilde{A}) = 0.465, \quad \vartheta_{35}(\tilde{A}_3, \tilde{A}) = 0.838$$

Thus

$$\vartheta_{32}(\tilde{A}_3, \tilde{A}) > \vartheta_{32}(\tilde{A}_1, \tilde{A}) > \vartheta_{32}(\tilde{A}_2, \tilde{A})$$

(3) It follows from Eq.(3.132) (let $\lambda = 2$) that

$$\vartheta_{36}(\tilde{A}_1, \tilde{A}) = 0.729, \quad \vartheta_{36}(\tilde{A}_2, \tilde{A}) = 0.458, \quad \vartheta_{36}(\tilde{A}_3, \tilde{A}) = 0.794$$

To conclude

$$\vartheta_{36}(\tilde{A}_3, \tilde{A}) > \vartheta_{36}(\tilde{A}_1, \tilde{A}) > \vartheta_{36}(\tilde{A}_2, \tilde{A})$$

(4) Using Eq.(3.144) (let $\lambda = 2$), we have

$$\vartheta_{45}(\tilde{A}_1, \tilde{A}) = 0.722, \quad \vartheta_{45}(\tilde{A}_2, \tilde{A}) = 0.436, \quad \vartheta_{45}(\tilde{A}_3, \tilde{A}) = 0.813$$

Then

$$\vartheta_{45}(\tilde{A}_3, \tilde{A}) > \vartheta_{45}(\tilde{A}_1, \tilde{A}) > \vartheta_{45}(\tilde{A}_2, \tilde{A})$$

In the above numerical results, all the similarity measures derive the same ranking, in which the degree of similarity between $\tilde{A}_3$ and $\tilde{A}$ is the largest one, the degree of similarity between $\tilde{A}_1$ and $\tilde{A}$ ranks the second, and the degree of similarity between $\tilde{A}_2$ and $\tilde{A}$ is the smallest one. Therefore, the mineral $\tilde{A}$ should belong to the kind of mineral field $\tilde{A}_3$ according to the principle of the maximum degree of similarity between IVIFSs.

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Chapter 4

Decision Making Models and Approaches Based on Intuitionistic Preference Relations

In real-life situations, such as partner selection in supply chain management, and performance assessment of military systems, a decision maker may be unable to express accurately his/her preferences for alternatives, because ① the decision maker may not possess a precise or sufficient level of knowledge (i.e., lack of knowledge to a certain degree (Mitchell, 2004), and ② he/she is unable to discriminate explicitly the degree to which one alternative is better than the others (Herrera-Viedma et al., 2007), and so there is a certain degree of hesitation (Szmidt and Kacprzyk, 2000). The decision maker may express, to a certain degree, his/her preferences for alternatives, but it is possible that he/she is not so sure about it (Deschrijver and Kerre, 2003a). In these problems, it is very suitable to study the decision maker's preferences using IFNs rather than exact numerical values or linguistic variables (Dai et al., 2007; Herrera et al., 2005; Szmidt and Kacprzyk, 2003; 2002; Xu, 2007c; 2007f; Xu and Chen, 2007a).

We examine this issue in this chapter. We will present approaches to group decision making based on intuitionistic preference relations or incomplete intuitionistic preference relations. We will first introduce concepts such as intuitionistic preference relation, consistent intuitionistic preference relation, incomplete intuitionistic preference relation, etc. We will then introduce their properties, and give the relationships among the interval-valued intuitionistic preference relation, intuitionistic preference and fuzzy preference relation. After that, we will utilize some aggregation tools such as the IFWA operator, etc., to establish the model for multi-attribute decision making based on intuitionistic preference relation or incomplete intuitionistic preference relation. We will also develop a model for multi-attribute group decision making based on intuitionistic preference relation and incomplete intuitionistic preference relation. We will introduce a number of approaches to multi-attribute decision making with distinct preference structures. Moreover, we will describe some decision making approaches under different interval-valued intuitionistic fuzzy environments by using aggregation tools including the IIFA and IIFG operators.

4.1 Intuitionistic Preference Relations

Preference relation is a common and effective way for a decision maker to express his/her preferences over objects (or alternatives).

For a decision making problem, let $Y = \{Y_1, Y_2, \cdots, Y_n\}$ be a discrete set of alternatives. In the process of decision making, a decision maker generally needs to provide his/her preferences for each pair of alternatives, and then constructs a preference relation, which can be defined as follows:

Definition 4.1.1 (Herrera-Viedma et al., 2007) A preference relation P on the set Y is characterized by a function $\mu_P : Y \times Y \rightarrow \Omega$, where Ω is the domain of representation of preference degrees.

A number of studies have been conducted on decision making problems with preference relations (Chen and Fan, 2005; Chiclana et al., 2002; 2001a; 2001b; 1998; Fan et al., 2002; Herrera and Herrera-Viedma, 2000a; 2000b; Herrera et al., 2005; 2000; 1997; 1996a; 1996b; Herrera and Martinez, 2000a; 2000b; Ma et al., 2006; Orlovsky, 1978; Saaty, 1980; Tanino, 1984; Wang and Xu, 2005; 1990; Xu, 2008b; 2007b; 2007f; 2007j; 2006a; 2006c; 2006e; 2006f; 2005b; 2005d; 2004a; 2004b; 2004c; 2004e; 2001a; 2001b; 1999; Xu and Da, 2005). These preference relations can be grouped into the following three main categories:

(1) Multiplicative preference relation (Saaty, 1980). A multiplicative preference relation P on the set Y is represented by a reciprocal matrix $P = (p_{ij})_{n\times n} \subset Y \times Y$ with:

$$p_{ij} > 0, \quad p_{ij} \cdot p_{ji} = 1, \quad p_{ii} = 1, \quad \text{for all } i, j = 1, 2, \cdots, n \tag{4.1}$$

where p_{ij} is interpreted as the ratio of the preference intensity of the alternative Y_i to that of Y_j. In particular, $p_{ij} = 1$ indicates indifference between Y_i and Y_j, $p_{ij} > 1$ indicates that Y_i is preferred to Y_j, and $p_{ij} < 1$ indicates that Y_j is preferred to Y_i.

(2) Fuzzy preference relation (Orlovsky, 1978). A fuzzy preference relation B on the set Y is represented by a complementary matrix $B = (b_{ij})_{n\times n} \subset Y \times Y$ with:

$$b_{ij} \geqslant 0, \quad b_{ij} + b_{ji} = 1, \quad b_{ii} = 0.5, \quad i, j = 1, 2, \cdots, n \tag{4.2}$$

where b_{ij} denotes the preference degree of the alternative Y_i over Y_j. In particular, $b_{ij} = 0.5$ indicates indifference between Y_i and Y_j, $b_{ij} > 0.5$ indicates that Y_i is preferred to Y_j, and $b_{ij} < 0.5$ indicates that Y_j is preferred to Y_i.

(3) Linguistic preference relation (Herrera and Herrera-Viedma, 2000b). Consider a finite and totally ordered discrete linguistic label set $\Gamma = \{\tau_i |\ i = -t, \cdots, t\}$, where τ_i represents a linguistic variable (Zadeh, 2005) and satisfies the following characteristics:

(1) The set is ordered: $\tau_i > \tau_j$ if $i > j$;

(2) There is a negation operator: $\mathrm{neg}(\tau_i) = \tau_{-i}$. The cardinality of Γ must be small enough so as not to impose useless precision on the decision maker and it must be rich enough in order to allow a discrimination of the performances of each alternative in a limited number of grades (Bordogna et al., 1997). To preserve all the given information, the discrete label set Γ should be extended to a continuous label set $\overline{\Gamma} = \{\tau_a |\ a \in [-q, q]\}$, where $q(q > t)$ is a sufficiently large positive integer. If $\tau_a \in \Gamma$, then τ_a is termed an original linguistic label; Otherwise, τ_a is termed a virtual linguistic label (Xu, 2005b).

Note In general, a decision maker uses the original linguistic labels to evaluate alternatives, and the virtual linguistic labels can only appear in operation.

Considering any two linguistic labels $\tau_a, \tau_b \in \overline{\Gamma}$, we define their operational laws as follows (Xu, 2005b):

(1) $\tau_a \oplus \tau_b = \tau_{a+b}$;

(2) $\lambda \tau_a = \tau_{\lambda a}$, $\lambda \in [0, 1]$.

A linguistic preference relation L on the set Y is represented by a linguistic decision matrix $L = (l_{ij})_{n\times n} \subset Y \times Y$ with:

$$l_{ij} \in \overline{\Gamma},\ \ l_{ij} \oplus l_{ji} = \tau_0, \quad l_{ii} = \tau_0, \quad i, j = 1, 2, \cdots, n \tag{4.3}$$

where l_{ij} denotes the preference degree of the alternative Y_i over Y_j. In particular, $l_{ij} = \tau_0$ indicates indifference between Y_i and Y_j, $l_{ij} > \tau_0$ indicates that Y_i is preferred to Y_j, and $l_{ij} < \tau_0$ indicates that Y_j is preferred to Y_i.

In some real-life situations, a decision maker may provide his/her preferences for alternatives to a certain degree, but it is possible that he/she is not so sure about it (Deschrijver and Kerre, 2003a). Thus, it can be more comprehensive, detailed and visually effective for decision makers to describe and characterize their preferences over the alternatives by means of IFNs rather than exact numerical values or linguistic variables. Szmidt and Kacprzyk (2003) generalize the fuzzy preference relation to the intuitionistic preference relation, and define the concepts of intuitionistic fuzzy core and consensus winner. They aggregate the individual intuitionistic preference relations into a social fuzzy preference relation on the basis of fuzzy majority equated with a fuzzy linguistic quantifier. Xu (2007f) introduces the concept of intuitionistic preference relation as follows:

Definition 4.1.2 (Xu, 2007f) An intuitionistic preference relation Q on the set Y is represented by a matrix $Q = (q_{ij})_{n\times n} \subset Y \times Y$ with $q_{ij} = \langle (Y_i, Y_j),\ \mu(Y_i, Y_j), \nu(Y_i, Y_j)\rangle$, for all $i, j = 1, 2, \cdots, n$. For convenience, we let $q_{ij} = (\mu_{ij}, \nu_{ij})$, for all $i, j = 1, 2, \cdots, n$, where q_{ij} is an IFN, composed by the certainty degree μ_{ij} to which Y_i is preferred to Y_j and the certainty degree ν_{ij} to which Y_i is non-preferred to Y_j, and $\pi_{ij} = 1 - \mu_{ij} - \nu_{ij}$ is interpreted as the indeterminancy degree to which Y_i is preferred to Y_j. Furthermore, μ_{ij} and ν_{ij} satisfy the following characteristics: $0 \leqslant \mu_{ij}+\nu_{ij} \leqslant 1$, $\mu_{ji} = \nu_{ij}$, $\nu_{ji} = \mu_{ij}$, $\mu_{ii} = \nu_{ii} = 0.5$, for all $i, j = 1, 2, \cdots, n$.

In particular, if $1 - \mu_{ij} - \nu_{ij} = 0$, for any i, j, then the intuitionistic preference relation Q can be decomposed into two fuzzy preference relations (Xu, 2007b): $Q_1 = (\mu_{ij})_{n\times n}$ and $Q_2 = (\nu_{ij})_{n\times n}$, where $\mu_{ij}, \nu_{ij} \in [0, 1]$, $\mu_{ij} + \mu_{ji} = 1$, $\nu_{ij} + \nu_{ji} = 1$, $\mu_{ii} = 0.5$, $\nu_{ii} = 0.5$, $i, j = 1, 2, \cdots, n$.

According to Definition 4.1.2, we can see that each element q_{ij} of the intuitionistic preference relation Q is an ordered pair (μ_{ij}, ν_{ij}), which satisfies the condition: $\mu_{ij} + \nu_{ij} \leqslant 1$, i.e., $\mu_{ij} \leqslant 1 - \nu_{ij}$, from which Xu (2007b) transforms the element $q_{ij} = (\mu_{ij}, \nu_{ij})$ into an interval value $\dot{q}_{ij} = [\mu_{ij}, 1 - \nu_{ij}]$. Consequently, the intuitionistic preference relation $Q = (q_{ij})_{n\times n}$ is in fact equivalent, mathematically, to an interval-valued fuzzy preference relation $\dot{Q} = (\dot{q}_{ij})_{n\times n}$, where $\dot{q}_{ij} = [\dot{q}_{ij}^{\mathrm{L}}, \dot{q}_{ij}^{\mathrm{U}}] = [\mu_{ij}, 1 - \nu_{ij}]$, $i, j = 1, 2, \cdots, n$, and

$$\dot{q}_{ij}^{\mathrm{L}} + \dot{q}_{ji}^{\mathrm{U}} = \dot{q}_{ij}^{\mathrm{U}} + \dot{q}_{ji}^{\mathrm{L}} = 1, \quad \dot{q}_{ij}^{\mathrm{U}} \geqslant \dot{q}_{ij}^{\mathrm{L}} \geqslant 0, \quad \dot{q}_{ii}^{\mathrm{U}} = \dot{q}_{ii}^{\mathrm{L}} = 0.5, \quad i, j = 1, 2, \cdots, n \quad (4.4)$$

Definition 4.1.3 (Xu, 2007f) Let $Q = (q_{ij})_{n\times n}$ be an intuitionistic preference relation. If it satisfies

$$q_{ij} = q_{ik} \otimes q_{kj}, \quad \text{for all } i < k < j \quad (4.5)$$

then Q is called a consistent intuitionistic preference relation.

A consistent intuitionistic preference relation can be interpreted as follows: For all $i < k < j$, the alternative Y_i is preferred to Y_j with an IFN q_{ik} that should be equal to the product of the intensities of preferences when using an intermediate alternative Y_k.

From Definition 4.1.2, we can derive directly the following theorem:

Theorem 4.1.1 (Xu, 2007f) If we remove the i-th row and i-th column from an intuitionistic preference relation Q, then the preference relation composed by the remainder $(n-1)$ rows and $(n-1)$ columns of Q is also an intuitionistic preference relation.

Proof It follows immediately from Definition 4.1.2.

Based on Definitions 1.1.3 and 4.1.2, we can introduce the following properties of an intuitionistic preference relation (Xu, 2007f) $Q = (q_{ij})_{n\times n}$:

(1) If $q_{ik} \oplus q_{kj} \geqslant q_{ij}$, for all $i, j, k = 1, 2, \cdots, n$, then we say Q satisfies the triangle condition.

This condition can be explained geometrically: if we regard the alternatives Y_i, Y_k and Y_j as the vertices of a triangle with length sides q_{ik}, q_{kj} and q_{ij}, then the length corresponding to the vertices Y_i, Y_j should not exceed the sum of the lengths corresponding to the vertices Y_i, Y_k, and Y_k, Y_j.

(2) If $q_{ik} \geqslant (0.5, 0.5)$, $q_{kj} \geqslant (0.5, 0.5) \Rightarrow q_{ij} \geqslant (0.5, 0.5)$, for all $i, j, k = 1, 2, \cdots, n$, then we say Q satisfies the weak transitivity property.

This property can be interpreted as follows: If the alternative Y_i is preferred to Y_k, and Y_k is preferred to Y_j, then Y_i should be preferred to Y_j.

(3) If $q_{ij} \geqslant \min\{q_{ik}, q_{kj}\}$, for all $i, j, k = 1, 2, \cdots, n$, then we say Q satisfies the max-min transitivity property.

The max-min transitivity property is that the IFN derived from a direct comparison between two alternatives should be equal to or greater than the minimum partial values derived from comparing both alternatives with an intermediate one.

(4) If $q_{ij} \geqslant \max\{q_{ik}, q_{kj}\}$, for all $i, j, k = 1, 2, \cdots, n$, then we say Q satisfies the max-max transitivity property.

The max-max transitivity property can be described as follows: The IFN derived from a direct comparison between two alternatives should be equal to or greater than the maximum partial values derived from comparing both alternatives with an intermediate one.

(5) If $q_{ik} \geqslant (0.5, 0.5)$, $q_{kj} \geqslant (0.5, 0.5) \Rightarrow q_{ij} \geqslant \min\{q_{ik}, q_{kj}\}$, for all $i, j, k = 1, 2, \cdots, n$, then we say Q satisfies the restricted max-min transitivity property.

The restricted max-min transitivity property can be interpreted in the following way: When the alternative Y_i is preferred to Y_k with an IFN q_{ik}, and Y_k is preferred to Y_j with a value q_{kj}, then Y_i should be preferred to Y_j with at least an IFN q_{ij} equal to the minimum of the above values. The equality holds only when there is indifference between at least two of the three alternatives.

(6) If $q_{ik} \geqslant (0.5, 0.5)$, $q_{kj} \geqslant (0.5, 0.5) \Rightarrow q_{ij} \geqslant \max\{q_{ik}, q_{kj}\}$, for all $i, j, k = 1, 2, \cdots, n$, then we say Q satisfies the restricted max-max transitivity property.

The restricted max-max transitivity property implies that when the alternative Y_i is preferred to Y_k with an IFN q_{ik}, and Y_k is preferred to Y_j with an IFN q_{kj}, then Y_i should be preferred to Y_j with at least an IFN q_{ij} equal to the maximum of the above values. The equality holds only when there is indifference between at least two of the three alternatives.

4.2 Group Decision Making Based on Intuitionistic Preference Relations

In this section, we introduce an approach to group decision making based on intuitionistic preference relations, which can be described as follows (Xu, 2007f):

Step 1 Consider a group decision making problem. Let Y, E and ξ be defined as in Section 1.3. The decision maker $E_k \in E$ provides his/her intuitionistic fuzzy preference for each pair of alternatives, and constructs an intuitionistic preference relation $Q_k = (q_{ij}^{(k)})_{n \times n}$, where

$$q_{ij}^{(k)} = (\mu_{ij}^{(k)}, \nu_{ij}^{(k)}), \quad 0 \leqslant \mu_{ij}^{(k)} + \nu_{ij}^{(k)} \leqslant 1, \quad \mu_{ji}^{(k)} = \nu_{ij}^{(k)}, \quad \nu_{ji}^{(k)} = \mu_{ij}^{(k)}$$

$$\mu_{ii}^{(k)} = \nu_{ii}^{(k)} = 0.5, \quad \text{for all } i, j = 1, 2, \cdots, n$$

Step 2 Utilize the IFA operator:

$$q_i^{(k)} = \frac{1}{n}(q_{i1}^{(k)} \oplus q_{i2}^{(k)} \oplus \cdots \oplus q_{in}^{(k)}), \quad i = 1, 2, \cdots, n \tag{4.6}$$

to aggregate all $q_{ij}^{(k)}$ $(j = 1, 2, \cdots, n)$ corresponding to the alternative Y_i, and then get the averaged IFN $q_i^{(k)}$ of the alternative Y_i over all the other alternatives.

Step 3 Utilize the IFWA operator:

$$q_i = \xi_1 q_i^{(1)} \oplus \xi_2 q_i^{(2)} \oplus \cdots \oplus \xi_l q_i^{(l)}, \quad i = 1, 2, \cdots, n \tag{4.7}$$

to aggregate all $q_i^{(k)}$ $(k = 1, 2, \cdots, l)$ corresponding to l decision makers into a collective IFN q_i of the alternative Y_i over all the other alternatives.

Step 4 Rank all q_i $(i = 1, 2, \cdots, n)$ by means of the score function (1.10) and the accuracy function (1.11), and then rank all the alternatives Y_i $(i = 1, 2, \cdots, n)$ and select the best one in accordance with the values of q_i $(i = 1, 2, \cdots, n)$.

4.3 Incomplete Intuitionistic Preference Relations

A complete preference relation needs all $n(n-1)/2$ judgments in its entire top triangular portion. Sometimes, however, it is difficult to obtain such a preference relation, especially for the preference relation with high order, because of time pressure, lack of knowledge, and the decision maker's limited expertise related to the problem domain. An incomplete preference relation may be developed, in which some of the elements cannot be provided. Xu (2007f) investigates the decision making problem with incomplete intuitionistic preference relations. He first defined the concept of incomplete intuitionistic fuzzy relation:

Definition 4.3.1 Let $Q = (q_{ij})_{n \times n}$ be an intuitionistic preference relation, where $q_{ij} = (\mu_{ij}, \nu_{ij})$, for all $i, j = 1, 2, \cdots, n$. Then Q is called an incomplete intuitionistic preference relation, if some of its elements cannot be given by the decision maker, which we denote by the unknown variable "x", and the others can be provided by the decision maker, which satisfy

$$0 \leqslant \mu_{ij} + \nu_{ij} \leqslant 1, \quad \mu_{ji} = \nu_{ij}, \quad \nu_{ji} = \mu_{ij}, \quad \mu_{ii} = \nu_{ii} = 0.5$$

Similar to the properties of an intuitionistic preference relation, Xu (2007f) introduces the properties of an incomplete intuitionistic preference relation:

Let $Q = (q_{ij})_{n \times n}$ be an incomplete intuitionistic preference relation, where $q_{ij} =$

(μ_{ij}, ν_{ij}), $i, j = 1, 2, \cdots, n$, and let Δ be the set of all the known elements. Then

(1) If $q_{ik} \oplus q_{kj} \geqslant q_{ij}$, for all $q_{ik}, q_{kj}, q_{ij} \in \Delta$, we say Q satisfies the triangle condition.

(2) If $q_{ik} \geqslant (0.5, 0.5)$, $q_{kj} \geqslant (0.5, 0.5) \Rightarrow q_{ij} \geqslant (0.5, 0.5)$, for all $q_{ik}, q_{kj}, q_{ij} \in \Delta$, we say Q satisfies the weak transitivity property.

(3) If $q_{ij} \geqslant \min\{q_{ik}, q_{kj}\}$, for all $q_{ik}, q_{kj}, q_{ij} \in \Delta$, we say Q satisfies the max-min transitivity property.

(4) If $q_{ij} \geqslant \max\{q_{ik}, q_{kj}\}$, for all $q_{ik}, q_{kj}, q_{ij} \in \Delta$, we say Q satisfies the max-max transitivity property.

(5) If $q_{ik} \geqslant (0.5, 0.5)$, $q_{kj} \geqslant (0.5, 0.5) \Rightarrow q_{ij} \geqslant \min\{ q_{ik}, q_{kj}\}$, for all $q_{ik}, q_{kj}, q_{ij} \in \Delta$, we say Q satisfies the restricted max-min transitivity property.

(6) If $q_{ik} \geqslant (0.5, 0.5)$, $q_{kj} \geqslant (0.5, 0.5) \Rightarrow q_{ij} \geqslant \max\{q_{ik}, q_{kj}\}$, for all $q_{ik}, q_{kj}, q_{ij} \in \Delta$, we say Q satisfies the restricted max-max transitivity property.

Definition 4.3.2 (Xu, 2007f) Let $Q = (q_{ij})_{n \times n}$ be an incomplete intuitionistic preference relation. If it satisfies

$$q_{ij} = q_{ik} \otimes q_{kj}, \quad \text{for all } q_{ik}, q_{kj}, q_{ij} \in \Delta, \quad i < k < j \tag{4.8}$$

then Q is called a consistent incomplete intuitionistic preference relation.

Definition 4.3.3 (Xu, 2007f) Let $Q = (q_{ij})_{n \times n}$ be an incomplete intuitionistic preference relation. If $(i, j) \cap (k, s) \neq \varnothing$, then the elements q_{ij} and q_{ks} are said to be adjoining. For the unknown element q_{ij}, if there exist two adjoining known elements q_{ik} and q_{kj}, then q_{ij} is called available. The element q_{ij} can be obtained indirectly by using $q_{ij} = q_{ik} \otimes q_{kj}$, which means that the estimated element q_{ij} should be taken according to the known elements q_{ik} and q_{kj}.

Definition 4.3.4 (Xu, 2007f) Let $Q = (q_{ij})_{n \times n}$ be an incomplete intuitionistic preference relation. If each unknown element can be derived from its adjoining known elements, then Q is called acceptable; Otherwise, Q is called unacceptable.

Obviously, for an incomplete intuitionistic preference relation $Q = (q_{ij})_{n \times n}$, if Q is acceptable, then there exists at least one known element (except diagonal elements) in each line or each column of Q, i.e., there exist at least $(n - 1)$ judgments provided by the decision maker (that is to say, each one of the alternatives is compared at least once).

Let $Q = (q_{ij})_{n \times n}$ be an acceptable incomplete intuitionistic preference relation. Then based on Eq.(4.8), each unknown element q_{ij} can be obtained indirectly from:

$$\dot{q}_{ij} = \left(\underset{k \in N_{ij}}{\otimes} (q_{ik} \otimes q_{kj}) \right)^{\frac{1}{n_{ij}}} \tag{4.9}$$

where $N_{ij} = \{k | q_{ik}, q_{kj} \in \Delta,\ i < k < j\}$, n_{ij} is the number of the elements in N_{ij}. Therefore, we get an improved intuitionistic preference relation $\dot{Q} = (\dot{q}_{ij})_{n \times n}$, where

$$\dot{q}_{ij} = \begin{cases} q_{ij}, & q_{ij} \in \varDelta \\ \dot{q}_{ij}, & q_{ij} \notin \varDelta \end{cases} \tag{4.10}$$

Clearly, an unknown element q_{ij} can be estimated if there exists at least one k so that the elements q_{ik} and q_{kj} are known. The improved intuitionistic preference relation $\dot{Q}$ contains both the direct intuitionistic preference information given by the decision maker and the indirect intuitionistic preference information derived from the known intuitionistic preference information.

4.4 Group Decision Making Based on Incomplete Intuitionistic Preference Relations

We now introduce an approach to group decision making based on incomplete intuitionistic preference relations (Xu, 2007f):

Step 1 Consider a group decision making problem. Let Y, E and ξ be defined as in Section 1.3. The decision maker $E_k \in E$ provides his/her intuitionistic preferences by comparing at least $n-1$ pairs of the alternatives (Y_i, Y_j) $(i = 1, 2, \cdots, n-1;\ j = i+1)$, and constructs an incomplete intuitionistic preference relations $Q_k = (q_{ij}^{(k)})_{n\times n}$, where

$$q_{ij}^{(k)} = (\mu_{ij}^{(k)}, \nu_{ij}^{(k)}), \quad 0 \leqslant \mu_{ij}^{(k)} + \nu_{ij}^{(k)} \leqslant 1, \quad \mu_{ji}^{(k)} = \nu_{ij}^{(k)}, \quad \mu_{ii}^{(k)} = \nu_{ii}^{(k)} = 0.5, \quad i, j \in \varDelta$$

Step 2 Utilize Eq.(4.9) to construct the improved intuitionistic preference relations $\dot{Q}_k = (\dot{q}_{ij}^{(k)})_{n\times n}$ $(k = 1, 2, \cdots, l)$ of $Q_k = (q_{ij}^{(k)})_{n\times n}$ $(k = 1, 2, \cdots, l)$.

Step 3 Utilize the IFA operator:

$$\dot{q}_i^{(k)} = \frac{1}{n}(\dot{q}_{i1}^{(k)} \oplus \dot{q}_{i2}^{(k)} \oplus \cdots \oplus \dot{q}_{in}^{(k)}), \quad i = 1, 2, \cdots, n \tag{4.11}$$

to aggregate all $q_{ij}^{(k)}$ $(j = 1, 2, \cdots, n)$ corresponding to the alternative Y_i, and then get the averaged IFN $q_i^{(k)}$ of the alternative Y_i over all the other alternatives.

Step 4 Utilize the IFWA operator:

$$\dot{q}_i = \xi_1 \dot{q}_i^{(1)} \oplus \xi_2 \dot{q}_i^{(2)} \oplus \cdots \oplus \xi_l \dot{q}_i^{(l)}, \quad i = 1, 2, \cdots, n \tag{4.12}$$

to aggregate all $\dot{q}_i^{(k)}$ $(k = 1, 2, \cdots, l)$ corresponding to l decision makers into a collective intuitionistic fuzzy preference value q_i of the alternative Y_i over all the other alternatives.

Step 5 Rank all $\dot{q}_i$ $(i = 1, 2, \cdots, n)$ by means of the score function (1.10) and the accuracy function (1.11), and then rank all the alternatives Y_i $(i = 1, 2, \cdots, n)$ and select the best one in accordance with the values of $\dot{q}_i$ $(i = 1, 2, \cdots, n)$.

Example 4.4.1 (Xu, 2007f) We now utilize a practical example (adapted from Li et al.(2005)) involving the assessment of a set of agroecological regions in Hubei Province, China, to illustrate the developed approaches.

Locating in Central China and the middle reaches of the Changjiang (Yangtze) River, Hubei Province is distributed in a transitional belt where physical conditions and landscapes are on the transition from north to south and from east to west. Thus, Hubei Province is well known as "a land of rice and fish" since the region enjoys some of the favorable physical conditions, with a diversity of natural resources and the suitability for growing various crops. At the same time, however, there are also some restrictive factors for developing agriculture such as a tight man–land relation between a constant degradation of natural resources and a growing population pressure on land resource reserve. Despite cherishing a burning desire to promote their standard of living, people living in the area are frustrated because they have no ability to enhance their power to accelerate economic development because of a dramatic decline in quantity and quality of natural resources and a deteriorating environment. Based on the distinctness and differences in environment and natural resources, Hubei Province can be roughly divided into seven agroecological regions: ① Y_1−Wuhan-Ezhou Huanggang; ② Y_2−Northeast of Hubei; ③ Y_3−Southeast of Hubei; ④ Y_4− Jianghan region; ⑤ Y_5−North of Hubei; ⑥ Y_6−Northwest of Hubei; and ⑦ Y_7− Southwest of Hubei. In order to prioritize these agroecological regions Y_i $(i=1,2,\cdots,7)$ with respect to their comprehensive functions, a committee comprised of three decision makers E_k $(k=1,2,3)$ (whose weight vector is $\xi=(0.5,0.2,0.3)^{\mathrm{T}}$) has been set up to provide assessment information on Y_i $(i=1,2,\cdots,7)$. The decision makers E_k $(k=1,2,3)$ provide intuitionistic preferences for each pair of agroecological regions with respect to their comprehensive functions and construct the intuitionistic preference relations $Q_k=(q_{ij}^{(k)})_{7\times 7}$ $(q_{ij}^{(k)}=(\mu_{ij}^{(k)},\nu_{ij}^{(k)})$, $i,j=1,2,\cdots,7;\ k=1,2,3)$ as follows respectively:

$$Q_1=\begin{bmatrix}(0.5,0.5)&(0.5,0.2)&(0.7,0.1)&(0.5,0.3)&(0.6,0.4)&(0.9,0.1)&(0.8,0.1)\\(0.2,0.5)&(0.5,0.5)&(0.6,0.2)&(0.3,0.6)&(0.7,0.1)&(0.8,0.2)&(0.6,0.3)\\(0.1,0.7)&(0.2,0.6)&(0.5,0.5)&(0.3,0.6)&(0.4,0.5)&(0.7,0.1)&(0.7,0.2)\\(0.3,0.5)&(0.6,0.3)&(0.6,0.3)&(0.5,0.5)&(0.6,0.1)&(0.8,0.1)&(0.7,0.3)\\(0.4,0.6)&(0.1,0.7)&(0.5,0.4)&(0.1,0.6)&(0.5,0.5)&(0.5,0.2)&(0.4,0.1)\\(0.1,0.9)&(0.2,0.8)&(0.1,0.7)&(0.1,0.8)&(0.2,0.5)&(0.5,0.5)&(0.3,0.7)\\(0.1,0.8)&(0.3,0.6)&(0.2,0.7)&(0.3,0.7)&(0.1,0.4)&(0.7,0.3)&(0.5,0.5)\end{bmatrix}$$

$$Q_2=\begin{bmatrix}(0.5,0.5)&(0.6,0.1)&(0.8,0.2)&(0.6,0.3)&(0.7,0.2)&(0.8,0.1)&(0.8,0.2)\\(0.1,0.6)&(0.5,0.5)&(0.5,0.1)&(0.3,0.7)&(0.6,0.1)&(0.7,0.2)&(0.6,0.2)\\(0.2,0.8)&(0.1,0.5)&(0.5,0.5)&(0.4,0.6)&(0.3,0.5)&(0.6,0.2)&(0.5,0.1)\\(0.3,0.6)&(0.7,0.3)&(0.6,0.4)&(0.5,0.5)&(0.7,0.3)&(0.8,0.2)&(0.6,0.2)\\(0.2,0.7)&(0.1,0.6)&(0.5,0.3)&(0.3,0.7)&(0.5,0.5)&(0.6,0.2)&(0.4,0.3)\\(0.1,0.8)&(0.2,0.7)&(0.2,0.6)&(0.2,0.8)&(0.2,0.6)&(0.5,0.5)&(0.3,0.6)\\(0.2,0.8)&(0.2,0.6)&(0.1,0.5)&(0.2,0.6)&(0.3,0.4)&(0.6,0.3)&(0.5,0.5)\end{bmatrix}$$

$$Q_3 = \begin{bmatrix} (0.5,0.5) & (0.6,0.2) & (0.8,0.1) & (0.7,0.2) & (0.8,0.2) & (0.9,0.1) & (0.7,0.1) \\ (0.2,0.6) & (0.5,0.5) & (0.6,0.1) & (0.2,0.7) & (0.6,0.2) & (0.8,0.1) & (0.8,0.2) \\ (0.1,0.8) & (0.1,0.6) & (0.5,0.5) & (0.2,0.3) & (0.3,0.4) & (0.9,0.1) & (0.6,0.1) \\ (0.2,0.7) & (0.7,0.2) & (0.3,0.2) & (0.5,0.5) & (0.6,0.2) & (0.8,0.1) & (0.7,0.2) \\ (0.2,0.8) & (0.2,0.6) & (0.4,0.3) & (0.2,0.6) & (0.5,0.5) & (0.7,0.2) & (0.7,0.3) \\ (0.1,0.9) & (0.1,0.8) & (0.1,0.9) & (0.1,0.8) & (0.2,0.7) & (0.5,0.5) & (0.2,0.8) \\ (0.1,0.7) & (0.2,0.8) & (0.1,0.6) & (0.2,0.7) & (0.3,0.7) & (0.8,0.2) & (0.5,0.5) \end{bmatrix}$$

We first use Szmidt and Kacprzyk's approach (2002) to derive the decision result, which involves the following steps:

Step 1 Based on Q_k $(k = 1,2,3)$, we construct the following matrices respectively:

$$T_1 = (t_{ij}^{(1)})_{7\times 7} = \begin{bmatrix} — & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & — & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & — & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & — & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & — & 0 & [0] \\ 1 & 1 & 1 & 1 & 1 & — & 1 \\ 1 & 1 & 1 & 1 & [0] & 0 & — \end{bmatrix}$$

$$T_2 = (t_{ij}^{(2)})_{7\times 7} = \begin{bmatrix} — & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & — & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & — & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & — & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & — & 0 & [0] \\ 1 & 1 & 1 & 1 & 1 & — & 1 \\ 1 & 1 & 1 & 1 & [0] & 0 & — \end{bmatrix}$$

$$T_3 = (t_{ij}^{(3)})_{7\times 7} = \begin{bmatrix} — & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & — & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & — & [0] & [0] & 0 & 0 \\ 1 & 0 & [0] & — & 0 & 0 & 0 \\ 1 & 1 & [0] & 1 & — & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & — & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & — \end{bmatrix}$$

$$\Pi_1 = (\pi_{ij}^{(1)})_{7\times 7} = \begin{bmatrix} — & 0.3 & 0.2 & 0.2 & 0 & 0 & 0.1 \\ 0.3 & — & 0.2 & 0.1 & 0.2 & 0 & 0.1 \\ 0.2 & 0.2 & — & 0.1 & 0.1 & 0.2 & 0.1 \\ 0.2 & 0.1 & 0.1 & — & 0.3 & 0.1 & 0 \\ 0 & 0.2 & 0.1 & 0.3 & — & 0.3 & 0.5 \\ 0 & 0 & 0.2 & 0.1 & 0.3 & — & 0 \\ 0.1 & 0.1 & 0.1 & 0 & 0.5 & 0 & — \end{bmatrix}$$

$$
\Pi_2 = (\pi_{ij}^{(2)})_{7\times 7} = \begin{bmatrix} — & 0.3 & 0 & 0.1 & 0.1 & 0.1 & 0 \\ 0.3 & — & 0.4 & 0 & 0.3 & 0.1 & 0.2 \\ 0 & 0.4 & — & 0 & 0.2 & 0.2 & 0.4 \\ 0.1 & 0 & 0 & — & 0 & 0 & 0.2 \\ 0.1 & 0.3 & 0.2 & 0 & — & 0.2 & 0.2 \\ 0.1 & 0.1 & 0.2 & 0 & 0.2 & — & 0.1 \\ 0 & 0.2 & 0.4 & 0.2 & 0.3 & 0.1 & — \end{bmatrix}
$$

$$
\Pi_3 = (\pi_{ij}^{(3)})_{7\times 7} = \begin{bmatrix} — & 0.2 & 0.1 & 0.1 & 0 & 0 & 0.2 \\ 0.2 & — & 0.3 & 0.1 & 0.2 & 0.1 & 0 \\ 0.1 & 0.3 & — & 0.5 & 0.3 & 0 & 0.3 \\ 0.1 & 0.1 & 0.5 & — & 0.2 & 0.1 & 0.1 \\ 0 & 0.2 & 0.3 & 0.2 & — & 0.1 & 0 \\ 0 & 0.1 & 0 & 0.1 & 0.1 & — & 0 \\ 0.2 & 0 & 0.3 & 0.1 & 0 & 0 & — \end{bmatrix}
$$

where

$$
t_{ij}^{(k)} = \begin{cases} —, & b_{ii}^{(k)} = (\mu_{ii}^{(k)}, v_{ii}^{(k)}), \\ 1, & \mu_{ij}^{(k)} < 0.5 \text{ and } v_{ij}^{(k)} < 0.5, \\ 0, & \mu_{ij}^{(k)} \geqslant 0.5, \\ [0], & \text{others} \end{cases}
$$

and $\pi_{ij}^{(k)} = 1 - \mu_{ij}^{(k)} - \nu_{ij}^{(k)}$, $i, j = 1, 2, \cdots, 7$; $k = 1, 2, 3$. $t_{ij}^{(k)} =$ "—" means $q_{ii}^{(k)}$ does not matter, $t_{ij}^{(k)} = 1$ means that the decision maker E_k prefers Y_j over Y_i, $t_{ij}^{(k)} = 0$ means that the decision maker E_k prefers Y_i over Y_j, and $t_{ij}^{(k)} = [0]$ means no option is preferred.

Step 2 Use the formula:

$$
t_j = \frac{1}{l(n-1)} \sum_{k=1}^{l} \sum_{i=1,\, i\neq j}^{n} t_{ij}^{(k)}, \quad l = 3; \quad n = 7; \quad j = 1, 2, \cdots, 7
$$

to calculate the extent to which all the decision makers E_k $(k = 1, 2, 3)$ are not against Y_j:

$$
t_1 = 1, \quad t_2 = \frac{12}{18}, \quad t_3 = \frac{6}{18}, \quad t_4 = \frac{14}{18}, \quad t_5 = \frac{6}{18}, \quad t_6 = 0, \quad t_7 = \frac{3}{18}
$$

Step 3 Use the formula:

$$
\pi_j = \frac{1}{l(n-1)} \sum_{k=1}^{l} \sum_{i=1,\, i\neq j}^{n} \pi_{ij}^{(k)}, \quad l = 3; \quad n = 7; \quad j = 1, 2, \cdots, 7
$$

to calculate the hesitation margin related to Y_j:

$$
\pi_1 = \frac{2}{18}, \quad \pi_2 = \frac{31}{180}, \quad \pi_3 = \frac{36}{180}, \quad \pi_4 = \frac{22}{180}
$$

$$\pi_5 = \frac{33}{180}, \quad \pi_6 = \frac{16}{180}, \quad \pi_7 = \frac{25}{180}$$

Step 4 Add the value π_j to t_j, which gives the upper bound of the interval $t'_j = [t_j,\ t_j + \pi_j]$, and thus get the following ranges:

$$t'_1 = \left[1,\ \frac{20}{18}\right], \quad t'_2 = \left[\frac{12}{18},\ \frac{151}{180}\right], \quad t'_3 = \left[\frac{6}{18},\ \frac{96}{180}\right], \quad t'_4 = \left[\frac{14}{18},\ \frac{162}{180}\right]$$

$$t'_5 = \left[\frac{6}{18},\ \frac{93}{180}\right], \quad t'_6 = \left[0,\ \frac{16}{180}\right], \quad t'_7 = \left[\frac{3}{18},\ \frac{55}{180}\right]$$

Step 5 Assume a fuzzy majority given as a fuzzy linguistic quantifier "most":

$$\delta_{\text{“most”}} = \begin{cases} 1, & x \geqslant 0.8, \\ 2x - 0.6, & 0.3 < x < 0.8, \\ 0, & x \leqslant 0.3 \end{cases}$$

and use $\lambda^{(j)}_{\text{“most”}} = \delta_{\text{“most”}}(t'_j)$ to calculate the extent to which most decision makers are not against Y_j:

$$\lambda^{(1)}_{\text{“most”}} = 1, \quad \lambda^{(2)}_{\text{“most”}} = \left[\frac{66}{90},\ 1\right], \quad \lambda^{(3)}_{\text{“most”}} = \left[\frac{6}{90},\ \frac{42}{90}\right], \quad \lambda^{(4)}_{\text{“most”}} = \left[\frac{86}{90},\ 1\right]$$

$$\lambda^{(5)}_{\text{“most”}} = \left[\frac{6}{90},\ \frac{39}{90}\right], \quad \lambda^{(6)}_{\text{“most”}} = 0, \quad \lambda^{(7)}_{\text{“most”}} = \left[0,\ \frac{1}{90}\right]$$

Then, by the definition of the intuitionistic fuzzy core (Szmidt and Kacprzyk, 2000), we have

$$\begin{aligned} C_{\text{“most”}} =& 1/Y_1 + \left[\frac{66}{90},\ 1\right]/Y_2 + \left[\frac{6}{90},\ \frac{42}{90}\right]/Y_3 + \left[\frac{86}{90},\ 1\right]/Y_4 \\ &+ \left[\frac{6}{90},\ \frac{39}{90}\right]/Y_5 + \left[0,\ \frac{1}{90}\right]/Y_7 \end{aligned}$$

which means that Y_1 is certainly an element of the intuitionistic fuzzy "most"-core, Y_6 is certainly not, and Y_2, Y_3, Y_4, Y_5, and Y_7 belong to this core to the extent as measured by values from the intervals $\left[\frac{66}{90},\ 1\right]$, $\left[\frac{6}{90},\ \frac{42}{90}\right]$, $\left[\frac{86}{90},\ 1\right]$, $\left[\frac{6}{90},\ \frac{39}{90}\right]$, and $\left[0,\ \frac{1}{90}\right]$ respectively.

Clearly, Szmidt and Kacprzyk's approach aggregates the individual intuitionistic preference relations into the group opinion on the basis of fuzzy majority equated with a fuzzy linguistic quantifier. A main limitation to this approach appears because it not only loses some original decision information in the process of information aggregation, but also is unable to prioritize the given alternatives. To resolve this issue, we now apply the algorithm introduced in Subsection 4.2 to the ranking and selection of the agroecological regions:

Step 1 Use Eq.(4.6) to aggregate all $q_{ij}^{(k)}$ $(j = 1, 2, \cdots, 7)$ corresponding to the agroecological region Y_i, and then get the averaged IFN $q_i^{(k)}$ of the agroecological region Y_i over all the other agroecological regions:

$$q_1^{(1)} = (0.6861, 0.1982), \quad q_2^{(1)} = (0.5707, 0.2918)$$

$$q_3^{(1)} = (0.4587, 0.3853), \quad q_4^{(1)} = (0.6112, 0.2536)$$

$$q_5^{(1)} = (0.3769, 0.3732), \quad q_6^{(1)} = (0.2281, 0.6847)$$

$$q_7^{(1)} = (0.3527, 0.5441), \quad q_1^{(2)} = (0.7055, 0.1982)$$

$$q_2^{(2)} = (0.5023, 0.2617), \quad q_3^{(2)} = (0.3933, 0.3826)$$

$$q_4^{(2)} = (0.6268, 0.3306), \quad q_5^{(2)} = (0.3933, 0.4283)$$

$$q_6^{(2)} = (0.2536, 0.6488), \quad q_7^{(2)} = (0.3240, 0.5072)$$

$$q_1^{(3)} = (0.7440, 0.1694), \quad q_2^{(3)} = (0.5870, 0.2617)$$

$$q_3^{(3)} = (0.4892, 0.3120), \quad q_4^{(3)} = (0.5880, 0.2469)$$

$$q_5^{(3)} = (0.4575, 0.4271), \quad q_6^{(3)} = (0.1999, 0.7590)$$

$$q_7^{(3)} = (0.3773, 0.5562)$$

Step 2 Utilize Eq.(4.7) to aggregate all $q_i^{(k)}$ $(k = 1, 2, 3)$ into a collective IFN q_i of the agroecological region Y_i over all the other agroecological regions:

$$q_1 = (0.7085, 0.1891), \quad q_2 = (0.5629, 0.2763)$$

$$q_3 = (0.4558, 0.3612), \quad q_4 = (0.6076, 0.2653)$$

$$q_5 = (0.4054, 0.3995), \quad q_6 = (0.2250, 0.6986)$$

$$q_7 = (0.3546, 0.5401)$$

Step 3 By Eq.(1.10), calculate the scores $s(q_i)$ $(i = 1, 2, \cdots, 7)$ of the IFN q_i:

$$s(q_1) = 0.5194, \quad s(q_2) = 0.2866, \quad s(q_3) = 0.0946, \quad s(q_4) = 0.3423$$

$$s(q_5) = 0.0059, \quad s(q_6) = -0.4736, \quad s(q_7) = -0.1855$$

Then

$$q_1 > q_4 > q_2 > q_3 > q_5 > q_7 > q_6$$

Hence

$$Y_1 \succ Y_4 \succ Y_2 \succ Y_3 \succ Y_5 \succ Y_7 \succ Y_6$$

Therefore, the agroecological region with the most comprehensive functions is Wuhan-Ezhou-Huanggang.

From the above example, we know that the approach developed uses the IFA and IFWA operators, rather than fuzzy majority, to aggregate the intuitionistic preference information, and applies the order relation between any pair of IFNs to prioritize the agroecological regions. The main advantage over Szmidt and Kacprzyk's approach is that the approach developed here can not only prioritize the agroecological regions, but also preserve the information in the process of aggregation. Note that the aggregated preference values are also expressed in IFNs.

If the decision makers E_k $(k=1,2,3)$ provide with their preference information over the alternatives Y_i $(i=1,2,\cdots,7)$ by using incomplete intuitionistic preference relations $Q_k=(q_{ij}^{(k)})_{4\times 4}$ $(k=1,2,3)$ as follows:

$$Q_1=\begin{bmatrix} (0.5,0.5) & (0.5,0.2) & (0.7,0.1) & (0.5,0.3) & (0.6,0.4) & (0.9,0.1) & x \\ (0.2,0.5) & (0.5,0.5) & (0.6,0.2) & x & (0.7,0.1) & (0.8,0.2) & (0.6,0.3) \\ (0.1,0.7) & (0.2,0.6) & (0.5,0.5) & (0.3,0.6) & (0.4,0.5) & (0.7,0.1) & (0.7,0.2) \\ (0.3,0.5) & x & (0.6,0.3) & (0.5,0.5) & (0.6,0.1) & (0.8,0.1) & (0.7,0.3) \\ (0.4,0.6) & (0.1,0.7) & (0.5,0.4) & (0.1,0.6) & (0.5,0.5) & (0.5,0.2) & (0.4,0.1) \\ (0.1,0.9) & (0.2,0.8) & (0.1,0.7) & (0.1,0.8) & (0.2,0.5) & (0.5,0.5) & (0.3,0.7) \\ x & (0.3,0.6) & (0.2,0.7) & (0.3,0.7) & (0.1,0.4) & (0.7,0.3) & (0.5,0.5) \end{bmatrix}$$

$$Q_2=\begin{bmatrix} (0.5,0.5) & (0.6,0.1) & x & (0.6,0.3) & (0.7,0.2) & (0.8,0.1) & (0.8,0.2) \\ (0.1,0.6) & (0.5,0.5) & (0.5,0.1) & (0.3,0.7) & (0.6,0.1) & (0.7,0.2) & x \\ x & (0.1,0.5) & (0.5,0.5) & (0.4,0.6) & (0.3,0.5) & (0.6,0.2) & (0.5,0.1) \\ (0.3,0.6) & (0.7,0.3) & (0.6,0.4) & (0.5,0.5) & (0.7,0.3) & x & (0.6,0.2) \\ (0.2,0.7) & (0.1,0.6) & (0.5,0.3) & (0.3,0.7) & (0.5,0.5) & (0.6,0.2) & (0.4,0.3) \\ (0.1,0.8) & (0.2,0.7) & (0.2,0.6) & x & (0.2,0.6) & (0.5,0.5) & (0.3,0.6) \\ (0.2,0.8) & x & (0.1,0.5) & (0.2,0.6) & (0.3,0.4) & (0.6,0.3) & (0.5,0.5) \end{bmatrix}$$

$$Q_3=\begin{bmatrix} (0.5,0.5) & (0.6,0.2) & (0.8,0.1) & (0.7,0.2) & (0.8,0.2) & (0.9,0.1) & (0.7,0.1) \\ (0.2,0.6) & (0.5,0.5) & (0.6,0.1) & (0.2,0.7) & (0.6,0.2) & (0.8,0.1) & (0.8,0.2) \\ (0.1,0.8) & (0.1,0.6) & (0.5,0.5) & (0.2,0.3) & (0.3,0.4) & (0.9,0.1) & (0.6,0.1) \\ (0.2,0.7) & (0.7,0.2) & (0.3,0.2) & (0.5,0.5) & (0.6,0.2) & (0.8,0.1) & (0.7,0.2) \\ (0.2,0.8) & (0.2,0.6) & (0.4,0.3) & (0.2,0.6) & (0.5,0.5) & (0.7,0.2) & x \\ (0.1,0.9) & (0.1,0.8) & (0.1,0.9) & (0.1,0.8) & (0.2,0.7) & (0.5,0.5) & (0.2,0.8) \\ (0.1,0.7) & (0.2,0.8) & (0.1,0.6) & (0.2,0.7) & x & (0.8,0.2) & (0.5,0.5) \end{bmatrix}$$

where "x" denotes the unknown variable, then we can use the algorithm introduced in Section 4.4 to prioritize the agroecological regions, which involves the following steps:

Step 1 Use Eq.(4.9) to construct the improved intuitionistic preference relations $\dot{Q}_k=(\dot{q}_{ij}^{(k)})_{7\times 7}$ $(k=1,2,3)$ of $Q_k=(q_{ij}^{(k)})_{7\times 7}$ $(k=1,2,3)$:

$$\dot{Q}_1=\begin{bmatrix}(0.5,0.5)&(0.5,0.2)&(0.7,0.1)&(0.5,0.3)&(0.6,0.4)&(0.9,0.1)&(0.32,0.51)\\(0.2,0.5)&(0.5,0.5)&(0.6,0.2)&(0.18,0.68)&(0.7,0.1)&(0.8,0.2)&(0.6,0.3)\\(0.1,0.7)&(0.2,0.6)&(0.5,0.5)&(0.3,0.6)&(0.4,0.5)&(0.7,0.1)&(0.7,0.2)\\(0.3,0.5)&(0.68,0.18)&(0.6,0.3)&(0.5,0.5)&(0.6,0.1)&(0.8,0.1)&(0.7,0.3)\\(0.4,0.6)&(0.1,0.7)&(0.5,0.4)&(0.1,0.6)&(0.5,0.5)&(0.5,0.2)&(0.4,0.1)\\(0.1,0.9)&(0.2,0.8)&(0.1,0.7)&(0.1,0.8)&(0.2,0.5)&(0.5,0.5)&(0.3,0.7)\\(0.51,0.32)&(0.3,0.6)&(0.2,0.7)&(0.3,0.7)&(0.1,0.4)&(0.7,0.3)&(0.5,0.5)\end{bmatrix}$$

$$\dot{Q}_2=\begin{bmatrix}(0.5,0.5)&(0.6,0.1)&(0.30,0.19)&(0.6,0.3)&(0.7,0.2)&(0.8,0.1)&(0.8,0.2)\\(0.1,0.6)&(0.5,0.5)&(0.5,0.1)&(0.3,0.7)&(0.6,0.1)&(0.7,0.2)&(0.22,0.56)\\(0.19,0.30)&(0.1,0.5)&(0.5,0.5)&(0.4,0.6)&(0.3,0.5)&(0.6,0.2)&(0.5,0.1)\\(0.3,0.6)&(0.7,0.3)&(0.6,0.4)&(0.5,0.5)&(0.7,0.3)&(0.42,0.44)&(0.6,0.2)\\(0.2,0.7)&(0.1,0.6)&(0.5,0.3)&(0.3,0.7)&(0.5,0.5)&(0.6,0.2)&(0.4,0.3)\\(0.1,0.8)&(0.2,0.7)&(0.2,0.6)&(0.44,0.42)&(0.2,0.6)&(0.5,0.5)&(0.3,0.6)\\(0.2,0.8)&(0.56,0.22)&(0.1,0.5)&(0.2,0.6)&(0.3,0.4)&(0.6,0.3)&(0.5,0.5)\end{bmatrix}$$

$$\dot{Q}_3=\begin{bmatrix}(0.5,0.5)&(0.6,0.2)&(0.8,0.1)&(0.7,0.2)&(0.8,0.2)&(0.9,0.1)&(0.7,0.1)\\(0.2,0.6)&(0.5,0.5)&(0.6,0.1)&(0.2,0.7)&(0.6,0.2)&(0.8,0.1)&(0.8,0.2)\\(0.1,0.8)&(0.1,0.6)&(0.5,0.5)&(0.2,0.3)&(0.3,0.4)&(0.9,0.1)&(0.6,0.1)\\(0.2,0.7)&(0.7,0.2)&(0.3,0.2)&(0.5,0.5)&(0.6,0.2)&(0.8,0.1)&(0.7,0.2)\\(0.2,0.8)&(0.2,0.6)&(0.4,0.3)&(0.2,0.6)&(0.5,0.5)&(0.7,0.2)&(0.14,0.84)\\(0.1,0.9)&(0.1,0.8)&(0.1,0.9)&(0.1,0.8)&(0.2,0.7)&(0.5,0.5)&(0.2,0.8)\\(0.1,0.7)&(0.2,0.8)&(0.1,0.6)&(0.2,0.7)&(0.84,0.14)&(0.8,0.2)&(0.5,0.5)\end{bmatrix}$$

Step 2 Use Eq.(4.11) to aggregate all $\dot{q}_{ij}^{(k)}$ $(j=1,2,\cdots,7)$ corresponding to the agroecological region Y_i, and then get the averaged IFN $\dot{q}_i^{(k)}$ of the agroecological region Y_i over all the other agroecological regions:

$$\dot{q}_1^{(1)}=(0.6262,\ 0.2501),\quad \dot{q}_2^{(1)}=(0.5609,\ 0.2970)$$

$$\dot{q}_3^{(1)}=(0.4587,\ 0.3853),\quad \dot{q}_4^{(1)}=(0.6234,\ 0.2358)$$

$$\dot{q}_5^{(1)}=(0.3769,\ 0.3732),\quad \dot{q}_6^{(1)}=(0.2281,\ 0.6847)$$

$$\dot{q}_7^{(1)}=(0.4065,\ 0.4773),\quad \dot{q}_1^{(2)}=(0.6478,\ 0.1967)$$

$$\dot{q}_2^{(2)}=(0.4525,\ 0.3031),\quad \dot{q}_3^{(2)}=(0.3923,\ 0.3326)$$

$$\dot{q}_4^{(2)}=(0.5655,\ 0.3701),\quad \dot{q}_5^{(2)}=(0.3933,\ 0.4283)$$

$$\dot{q}_6^{(2)}=(0.2907,\ 0.5917),\quad \dot{q}_7^{(2)}=(0.3793,\ 0.4395)$$

$$\dot{q}_1^{(3)}=(0.7440,\ 0.1694),\quad \dot{q}_2^{(3)}=(0.5870,\ 0.2617)$$

$$\dot{q}_3^{(3)}=(0.4892,\ 0.3120),\quad \dot{q}_4^{(3)}=(0.5880,\ 0.2469)$$

$$\dot{q}_5^{(3)}=(0.3695,\ 0.4948),\quad \dot{q}_6^{(3)}=(0.1999,\ 0.7590)$$

$$\dot{q}_7^{(3)}=(0.4957,\ 0.4419)$$

Step 3 Use Eq (4.12) to aggregate all $\dot{q}_i^{(k)}$ $(k=1,2,3)$ into a collective IFN $\dot{q}_i$ of the agroecological region Y_i over all the other agroecological regions:

$$\dot{q}_1=(0.6703,\ 0.2121),\quad \dot{q}_2=(0.5494,\ 0.2871)$$

$$\dot{q}_3=(0.4556,\ 0.3512),\quad \dot{q}_4=(0.6019,\ 0.2616)$$

$$\dot{q}_5=(0.3780,\ 0.4175),\quad \dot{q}_6=(0.2328,\ 0.6859)$$

$$\dot{q}_7=(0.4297,\ 0.4588)$$

Step 4 By Eq.(1.9), calculate the scores $s(\dot{q}_i)$ $(i=1,2,\cdots,7)$ of $\dot{q}_i$ $(i=1,2,\cdots,7)$:

$$s(\dot{q}_1)=0.4582,\quad s(\dot{q}_2)=0.2623,\quad s(\dot{q}_3)=0.1044,\quad s(\dot{q}_4)=0.3403$$

$$s(\dot{q}_5)=-0.0395,\quad s(\dot{q}_6)=-0.4531,\quad s(\dot{q}_7)=-0.0291$$

Then

$$\dot{q}_1>\dot{q}_4>\dot{q}_2>\dot{q}_3>\dot{q}_7>\dot{q}_5>\dot{q}_6$$

and hence

$$Y_1\succ Y_4\succ Y_2\succ Y_3\succ Y_7\succ Y_5\succ Y_6$$

Therefore, the agroecological region with the most comprehensive functions is also Wuhan-Ezhou-Huanggang.

4.5 Interval-Valued Intuitionistic Preference Relations

Due to the complexity of objective things and the ambiguity of human thought, in the process of decision making, the membership degree and non-membership degree in the preference information provided by a decision maker are sometimes difficult to be expressed in exact real values. Instead, it is very convenient and proper to express them with interval numbers. In what follows, we introduce the concept of interval-valued intuitionistic preference relation (Xu and Chen, 2007a):

Definition 4.5.1 (Xu and Chen, 2007a) Let $\tilde{Q}=(\tilde{q}_{ij})_{n\times n}$ be a preference relation, where $\tilde{q}_{ij}=(\tilde{\mu}_{ij},\tilde{\nu}_{ij})$ is an IVIFN, $\tilde{\mu}_{ij}$ indicates the degree range that the decision maker prefer the alternative Y_i to the alternative Y_j, while $\tilde{\nu}_{ij}$ means the degree range that the decision maker prefer the alternative Y_j to the alternative Y_i. $\tilde{\mu}_{ij}$ and $\tilde{\nu}_{ij}$ satisfy the following conditions:

$$\tilde{\mu}_{ij}=[\tilde{\mu}_{ij}^{\mathrm{L}},\tilde{\mu}_{ij}^{\mathrm{U}}]\subset[0,1],\quad \tilde{\nu}_{ij}=[\tilde{\nu}_{ij}^{\mathrm{L}},\tilde{\nu}_{ij}^{\mathrm{U}}]\subset[0,1],\quad \tilde{\mu}_{ji}=\tilde{\nu}_{ij},\quad \tilde{\nu}_{ji}=\tilde{\mu}_{ij}$$

$$\tilde{\mu}_{ii}=\tilde{\nu}_{ii}=[0.5,0.5],\quad \tilde{\mu}_{ij}^{\mathrm{U}}+\tilde{\nu}_{ij}^{\mathrm{U}}\leqslant 1,\quad i,j=1,2,\cdots,n$$

where $\tilde{\mu}_{ij}^{\mathrm{L}}$ and $\tilde{\mu}_{ij}^{\mathrm{U}}$ indicate, respectively, the lower and upper limits of $\tilde{\mu}_{ij}$, $\tilde{\nu}_{ij}^{\mathrm{L}}$ and $\tilde{\nu}_{ij}^{\mathrm{U}}$ indicate, respectively, the lower and upper limits of $\tilde{\nu}_{ij}$. Then $\tilde{Q}$ is called an interval-valued intuitionistic preference relation.

In particular, if $\tilde{\mu}_{ij}^{\mathrm{L}} = \tilde{\mu}_{ij}^{\mathrm{U}}$ and $\tilde{\nu}_{ij}^{\mathrm{L}} = \tilde{\nu}_{ij}^{\mathrm{U}}$, for any i, j, then the interval-valued intuitionistic preference relation $\tilde{Q}$ reduces to an intuitionistic preference relation. Therefore, the intuitionistic preference relation is a special case of the interval-valued intuitionistic preference relation.

Definition 4.5.2 (Xu and Chen, 2007a) If $\tilde{q}_{ij} = \tilde{q}_{ik} \otimes \tilde{q}_{kj}$, for all $i < k < j$, then $\tilde{Q} = (\tilde{q}_{ij})_{n\times n}$ is called a consistent interval-valued intuitionistic preference relation.

Definition 4.5.3 (Xu and Chen, 2007a) Let $\tilde{Q} = (\tilde{q}_{ij})_{n\times n}$ be an interval-valued intuitionistic preference relation, where

$$\tilde{q}_{ij} = (\tilde{\mu}_{ij}, \tilde{\nu}_{ij}), \quad \tilde{\mu}_{ij} = [\tilde{\mu}_{ij}^{\mathrm{L}}, \tilde{\mu}_{ij}^{\mathrm{U}}] \subset [0,1], \quad \tilde{\nu}_{ij} = [\tilde{\nu}_{ij}^{\mathrm{L}}, \tilde{\nu}_{ij}^{\mathrm{U}}] \subset [0,1], \quad i,j = 1,2,\cdots,n$$

Then $S = (s_{ij})_{n\times n}$ is called the score matrix of $\tilde{Q}$, where s_{ij} can be determined by the score function, i.e.,

$$s_{ij} = \frac{1}{2}(\tilde{\mu}_{ij}^{\mathrm{L}} - \tilde{\nu}_{ij}^{\mathrm{L}} + \tilde{\mu}_{ij}^{\mathrm{U}} - \tilde{\nu}_{ij}^{\mathrm{U}}), \quad i,j = 1,2,\cdots,n$$

Definition 4.5.4 (Xu and Chen, 2007a) $H = (h_{ij})_{n\times n}$ is called the accuracy matrix of $\tilde{Q}$, where h_{ij} can be determined by the accuracy function, i.e.,

$$h_{ij} = \frac{1}{2}(\tilde{\mu}_{ij}^{\mathrm{L}} + \tilde{\nu}_{ij}^{\mathrm{L}} + \tilde{\mu}_{ij}^{\mathrm{U}} + \tilde{\nu}_{ij}^{\mathrm{U}}), \quad i,j = 1,2,\cdots,n$$

Theorem 4.5.1 (Xu and Chen, 2007a) Let S be the score matrix of $\tilde{Q}$, and $\bar{S}$ the transpose of S. Then $\bar{S} = -S$, i.e., S is an anti-symmetrical matrix.

Proof By Definitions 4.5.1 and 4.5.3, we have

$$s_{ij} = \frac{1}{2}(\tilde{\mu}_{ij}^{\mathrm{L}} - \tilde{\nu}_{ij}^{\mathrm{L}} + \tilde{\mu}_{ij}^{\mathrm{U}} - \tilde{\nu}_{ij}^{\mathrm{U}}), \quad s_{ji} = \frac{1}{2}(\tilde{\nu}_{ij}^{\mathrm{L}} - \tilde{\mu}_{ij}^{\mathrm{L}} + \tilde{\nu}_{ij}^{\mathrm{U}} - \tilde{\mu}_{ij}^{\mathrm{U}}), \quad i,j = 1,2,\cdots,n$$

Thus

$$s_{ij} + s_{ji} = 0, \quad i,j = 1,2,\cdots,n$$

Then $\bar{S} = -S$.

Theorem 4.5.2 (Xu and Chen, 2007a) Let H be the accuracy matrix of $\tilde{Q}$, and $\bar{H}$ the transpose of H. Then $\bar{H} = H$, i.e., H is a symmetrical matrix.

Proof By Definitions 4.5.1 and 4.5.4, we have

$$h_{ij} = \frac{1}{2}(\tilde{\mu}_{ij}^{\mathrm{L}} + \tilde{\nu}_{ij}^{\mathrm{L}} + \tilde{\mu}_{ij}^{\mathrm{U}} + \tilde{\nu}_{ij}^{\mathrm{U}}), \quad h_{ji} = \frac{1}{2}(\tilde{\nu}_{ij}^{\mathrm{L}} + \tilde{\mu}_{ij}^{\mathrm{L}} + \tilde{\nu}_{ij}^{\mathrm{U}} + \tilde{\mu}_{ij}^{\mathrm{U}}), \quad i,j = 1,2,\cdots,n$$

Then, $h_{ij} = h_{ji}$, $i, j = 1, 2, \cdots, n$, and hence $\bar{H} = H$.

The interval-valued intuitionistic preference relation has the following properties:

Theorem 4.5.3 (Xu and Chen, 2007a) If we remove the between row and between column from the interval-valued intuitionistic preference relation $\tilde{Q}$, then the preference relation composed by the remainder $(n-1)$ rows and $(n-1)$ columns of $\tilde{Q}$ is also an interval-valued intuitionistic preference relation.

Theorem 4.5.4 (Xu and Chen, 2007a) Let $\tilde{Q} = (\tilde{q}_{ij})_{n\times n}$ be an interval-valued intuitionistic preference relation, where $\tilde{q}_{ij} = (\tilde{\mu}_{ij}, \tilde{\nu}_{ij})$, $i, j = 1, 2, \cdots, n$. Then

(1) If $\tilde{q}_{ik} \oplus \tilde{q}_{kj} \geqslant \tilde{q}_{ij}$, for all $i, j, k = 1, 2, \cdots, n$, then $\tilde{Q}$ satisfies the triangle condition.

(2) If $\tilde{q}_{ik} \geqslant ([0.5, 0.5], [0.5, 0.5])$, $\tilde{q}_{kj} \geqslant ([0.5, 0.5], [0.5, 0.5]) \Rightarrow \tilde{q}_{ij} \geqslant ([0.5, 0.5], [0.5, 0.5])$, for all $i, j, k = 1, 2, \cdots, n$, then $\tilde{Q}$ satisfies the weak transitivity property.

(3) If $\tilde{q}_{ij} \geqslant \min\{\tilde{q}_{ik}, \tilde{q}_{kj}\}$, for all $i, j, k = 1, 2, \cdots, n$, then $\tilde{Q}$ satisfies the max-min transitivity property.

(4) If $\tilde{q}_{ij} \geqslant \max\{\tilde{q}_{ik}, \tilde{q}_{kj}\}$, for all $i, j, k = 1, 2, \cdots, n$, then $\tilde{Q}$ satisfies the max-max transitivity property.

(5) If $\tilde{q}_{ik} \geqslant ([0.5, 0.5],\ [0.5, 0.5])$, $\tilde{q}_{kj} \geqslant ([0.5, 0.5],\ [0.5, 0.5]) \Rightarrow \tilde{q}_{ij} \geqslant \min\{\tilde{q}_{ik}, \tilde{q}_{kj}\}$, for all $i, j, k = 1, 2, \cdots, n$, then $\tilde{Q}$ satisfies the restricted max-min transitivity property.

(6) If $\tilde{q}_{ik} \geqslant ([0.5, 0.5],\ [0.5, 0.5])$, $\tilde{q}_{kj} \geqslant ([0.5, 0.5],\ [0.5, 0.5]) \Rightarrow \tilde{q}_{ij} \geqslant \max\{\tilde{q}_{ik}, \tilde{q}_{kj}\}$, for all $i, j, k = 1, 2, \cdots, n$, then $\tilde{Q}$ satisfies the restricted max-max transitivity property.

(7) If $\min\{\tilde{q}_{ik}, \tilde{q}_{kj}\} \geqslant ([0.5, 0.5], [0.5, 0.5]) \Rightarrow \tilde{q}_{ij} \geqslant \max\{\tilde{q}_{ik}, \tilde{q}_{kj}\}$, for all $i, j, k = 1, 2, \cdots, n$, then $\tilde{Q}$ satisfies the strong stochastic transitivity.

(8) If $\min\{\tilde{q}_{ik}, \tilde{q}_{kj}\} \geqslant ([0.5, 0.5], [0.5, 0.5]) \Rightarrow \tilde{q}_{ij} \geqslant \min\{\tilde{q}_{ik}, \tilde{q}_{kj}\}$, for all $i, j, k = 1, 2, \cdots, n$, then $\tilde{Q}$ satisfies the moderate stochastic transitivity.

(9) If $\min\{\tilde{q}_{ik}, \tilde{q}_{kj}\} \geqslant ([0.5, 0.5], [0.5, 0.5]) \Rightarrow \tilde{q}_{ij} \geqslant ([0.5, 0.5], [0.5, 0.5])$, for all $i, j, k = 1, 2, \cdots, n$, then $\tilde{Q}$ satisfies the weak stochastic transitivity.

4.6 Group Decision Making Based on Interval-Valued Intuitionistic Preference Relations

We now describe an approach to group decision making with interval-valued intuitionistic preference relation, which involves the following steps (Xu and Chen, 2007a):

Step 1 For a group decision making problem, let Y, E and ξ be defined as in Section 1.3. The decision maker E_k compares each pair of n alternatives, and constructs an interval-valued intuitionistic preference relation $\tilde{Q}_k = (\tilde{q}_{ij}^{(k)})_{n\times n}$, where

$$\tilde{q}_{ij}^{(k)} = (\tilde{\mu}_{ij}^{(k)}, \tilde{\nu}_{ij}^{(k)}), \quad \tilde{\mu}_{ij}^{(k)} \subset [0,1], \quad \tilde{\nu}_{ij}^{(k)} \subset [0,1], \quad \tilde{\mu}_{ji}^{(k)} = \tilde{\nu}_{ij}^{(k)}, \quad \tilde{\nu}_{ji}^{(k)} = \tilde{\mu}_{ij}^{(k)}$$

$$\tilde{\mu}_{ii}^{(k)} = \tilde{\nu}_{ii}^{(k)} = [0.5, 0.5], \quad \sup \tilde{\mu}_{ij}^{(k)} + \sup \tilde{\nu}_{ij}^{(k)} \leqslant 1, \quad i, j = 1, 2, \cdots, n; \quad k = 1, 2, \cdots, l$$

Step 2 Utilize the IIFA operator:

$$\tilde{q}_i^{(k)} = \text{IIFA}(\tilde{q}_{i1}^{(k)}, \tilde{q}_{i2}^{(k)}, \cdots, \tilde{q}_{in}^{(k)}), \quad i = 1, 2, \cdots, n;\ k = 1, 2, \cdots, l \tag{4.13}$$

to aggregate all the elements in each line of the interval-valued intuitionistic preference relation $\tilde{Q}_k$, and get the overall IVIFNs of each alternative corresponding to each decision maker E_k.

Step 3 Utilize the IIFWA operator:

$$\tilde{q}_i = \text{IIFWA}_\xi(\tilde{q}_i^{(1)}, \tilde{q}_i^{(2)}, \cdots, \tilde{q}_i^{(l)}), \quad i = 1, 2, \cdots, n \tag{4.14}$$

to aggregate $\tilde{q}_i^{(k)}$ $(k = 1, 2, \cdots, l)$, and get the collective overall IVIFNs $\tilde{q}_i$ $(i = 1, 2, \cdots, n)$ of the alternatives Y_i $(i = 1, 2, \cdots, n)$, where ξ is the weight vector of the decision makers.

Step 4 Calculate the scores $s(\tilde{q}_i)$ $(i = 1, 2, \cdots, n)$ and the accuracy degrees $h(\tilde{q}_i)$ $(i = 1, 2, \cdots, n)$ of $\tilde{q}_i$ $(i = 1, 2, \cdots, n)$.

Step 5 Utilize the scores $s(\tilde{q}_i)$ $(i = 1, 2, \cdots, n)$ and the accuracy degrees $h(\tilde{q}_i)$ $(i = 1, 2, \cdots, n)$ to rank and select the alternatives Y_i $(i = 1, 2, \cdots, n)$.

Example 4.6.1 (Xu and Chen, 2007a) Let us consider a problem concerning the selection of critical factors used to assess the potential partners of a company. Supply chain management focuses on strategic relationships between companies involved in a supply chain. By effective coordination, companies benefit from lower cost, lower inventory levels, information sharing and thus stronger competitive edge. Many factors may impact the coordination of companies. Among them, the following is the list of four critical factors (Chen and Xu, 2001): ① G_1 : Response time and supply capacity; ② G_2 : Quality and technical skills; ③ G_3 : Price and cost; and ④ G_4 : Service level. In order to prioritize these five critical factors G_j $(j = 1, 2, 3, 4)$, three decision makers E_k $(k = 1, 2, 3)$ (whose weight vector is $\xi = (0.35,\ 0.35,\ 0.30)^{\mathrm{T}}$) are invited to assess them. The decision makers compare each pair of these factors and provide intuitionistic preferences contained in the intuitionistic preference relations $\tilde{Q}_k = (q_{ij}^{(k)})_{4\times 4}$ $(k = 1, 2, 3)$ respectively.

$$\tilde{Q}_1 = \begin{bmatrix} ([0.5,0.5],[0.5,0.5]) & ([0.6,0.7],[0.1,0.2]) & ([0.5,0.6],[0.2,0.3]) & ([0.3,0.5],[0.2,0.4]) \\ ([0.1,0.2],[0.6,0.7]) & ([0.5,0.5],[0.5,0.5]) & ([0.4,0.6],[0.1,0.2]) & ([0.6,0.7],[0.1,0.3]) \\ ([0.2,0.3],[0.5,0.6]) & ([0.1,0.2],[0.4,0.6]) & ([0.5,0.5],[0.5,0.5]) & ([0.3,0.4],[0.5,0.6]) \\ ([0.2,0.4],[0.3,0.5]) & ([0.1,0.3],[0.6,0.7]) & ([0.5,0.6],[0.3,0.4]) & ([0.5,0.5],[0.5,0.5]) \end{bmatrix}$$

$$\tilde{Q}_2 = \begin{bmatrix} ([0.5,0.5],[0.5,0.5]) & ([0.2,0.3],[0.5,0.6]) & ([0.5,0.7],[0.1,0.2]) & ([0.2,0.4],[0.1,0.3]) \\ ([0.5,0.6],[0.2,0.3]) & ([0.5,0.5],[0.5,0.5]) & ([0.5,0.8],[0.1,0.2]) & ([0.3,0.6],[0.2,0.3]) \\ ([0.1,0.2],[0.5,0.7]) & ([0.1,0.2],[0.5,0.8]) & ([0.5,0.5],[0.5,0.5]) & ([0.4,0.6],[0.1,0.4]) \\ ([0.1,0.3],[0.2,0.4]) & ([0.2,0.3],[0.3,0.6]) & ([0.1,0.4],[0.4,0.6]) & ([0.5,0.5],[0.5,0.5]) \end{bmatrix}$$

$$\tilde{Q}_3 = \begin{bmatrix} ([0.5,0.5],[0.5,0.5]) & ([0.4,0.5],[0.2,0.3]) & ([0.6,0.7],[0.1,0.2]) & ([0.5,0.7],[0.2,0.3]) \\ ([0.2,0.3],[0.4,0.5]) & ([0.5,0.5],[0.5,0.5]) & ([0.5,0.6],[0.2,0.4]) & ([0.7,0.8],[0.1,0.2]) \\ ([0.1,0.2],[0.6,0.7]) & ([0.2,0.4],[0.5,0.6]) & ([0.5,0.5],[0.5,0.5]) & ([0.6,0.7],[0.1,0.3]) \\ ([0.2,0.3],[0.5,0.7]) & ([0.1,0.2],[0.7,0.8]) & ([0.1,0.3],[0.6,0.7]) & ([0.5,0.5],[0.5,0.5]) \end{bmatrix}$$

In order to select the most influential factor, we first utilize Eq.(4.13) to aggregate all the elements in each line of the interval-valued intuitionistic preference relation $\tilde{Q}_k$, and then get the overall IVIFNs of each alternative corresponding to the decision maker E_k:

$$\tilde{q}_1^{(1)} = ([0.4856, 0.5838],\ [0.2115, 0.3310])$$
$$\tilde{q}_2^{(1)} = ([0.4267, 0.5319],\ [0.2340, 0.3807])$$
$$\tilde{q}_3^{(1)} = ([0.2915, 0.3598],\ [0.4729, 0.5733])$$
$$\tilde{q}_4^{(1)} = ([0.3486, 0.4616],\ [0.4054, 0.5144])$$
$$\tilde{q}_1^{(2)} = ([0.3675, 0.4990],\ [0.2236, 0.3663])$$
$$\tilde{q}_2^{(2)} = ([0.4561, 0.6443],\ [0.2115, 0.3080])$$
$$\tilde{q}_3^{(2)} = ([0.2979, 0.4019],\ [0.2455, 0.3808])$$
$$\tilde{q}_4^{(2)} = ([0.2455, 0.3808],\ [0.3310, 0.5180])$$
$$\tilde{q}_1^{(3)} = ([0.5051, 0.6127],\ [0.2115, 0.3080])$$
$$\tilde{q}_2^{(3)} = ([0.5051, 0.5909],\ [0.2515, 0.3761])$$
$$\tilde{q}_3^{(3)} = ([0.3840, 0.4820],\ [0.3500, 0.5010])$$
$$\tilde{q}_4^{(4)} = ([0.2455, 0.3346],\ [0.5692, 0.6654])$$

Then we utilize the IIFWA operator (4.14) to aggregate $\tilde{q}_i^{(k)}$ $(k = 1, 2, 3)$, and get the collective overall IVIFNs of each alternative:

$$\tilde{q}_1 = ([0.4534, 0.5654],\ [0.2157, 0.3356]),\ \tilde{q}_2 = ([0.4615, 0.5916],\ [0.2308, 0.3522])$$

$$\tilde{q}_3 = ([0.3228, 0.4133],\ [0.3435, 0.4771]),\ \tilde{q}_4 = ([0.2833, 0.3975],\ [0.4181, 0.5571])$$

Finally, we calculate the scores of $\tilde{q}_i$ $(i = 1, 2, 3, 4)$:

$$s(\tilde{q}_1) = 0.2338, \quad s(\tilde{q}_2) = 0.2351$$

$$s(\tilde{q}_3) = -0.0423, \quad s(\tilde{q}_4) = -0.1472$$

Since

$$s(\tilde{q}_2) > s(\tilde{q}_1) > s(\tilde{q}_3) > s(\tilde{q}_4)$$

the ranking of the factors Y_i $(i = 1, 2, 3, 4)$ is as follows:

$$Y_2 \succ Y_1 \succ Y_3 \succ Y_4$$

To conclude, the most influential factor is Y_2.

4.7 Group Decision Making Based on Incomplete Interval-Valued Intuitionistic Preference Relations

Sometimes, a decision maker may be unable or unwilling to provide their preference over some of the given alternatives, because of time pressure, lack of knowledge, individual emotions, or limited expertise in the problem domain. In such cases, he/she may construct an incomplete interval-valued intuitionistic preference relation, where some of the elements in an interval-valued intuitionistic preference relation are missing, which we denote by the unknown variable "x". Xu and Cai (2009) investigate decision making problems with incomplete interval-valued intuitionistic preference relations:

Let $\tilde{Q} = (\tilde{q}_{ij})_{n\times n}$ be an incomplete interval-valued intuitionistic preference relation, where $\tilde{q}_{ij} = (\tilde{\mu}_{ij}, \tilde{\nu}_{ij})$, $i, j = 1, 2, \cdots, n$, and let $\tilde{\Delta}$ be the set of all the known elements in $\tilde{Q}$. Then $\tilde{Q}$ has the following properties:

Theorem 4.7.1 (Xu and Cai, 2009)

(1) If $(\tilde{q}_{ik} \oplus \tilde{q}_{kj}) \geqslant \tilde{q}_{ij}$, for all $\tilde{q}_{ik}, \tilde{q}_{kj}, \tilde{q}_{ij} \in \tilde{\Delta}$, $\tilde{Q}$ satisfies the triangle condition.

(2) If $\tilde{q}_{ik} \geqslant ([0.5, 0.5], [0.5, 0.5])$, $\tilde{q}_{kj} \geqslant ([0.5, 0.5], [0.5, 0.5]) \Rightarrow \tilde{r}_{ij} \geqslant ([0.5, 0.5], [0.5, 0.5])$, for all $\tilde{q}_{ik}, \tilde{q}_{kj}, \tilde{q}_{ij} \in \tilde{\Delta}$, $\tilde{Q}$ satisfies the weak transitivity property.

(3) If $\tilde{q}_{ij} \geqslant \min\{\tilde{q}_{ik}, \tilde{q}_{kj}\}$, for all $\tilde{q}_{ik}, \tilde{q}_{kj}, \tilde{q}_{ij} \in \tilde{\Delta}$, $\tilde{Q}$ satisfies the max-min transitivity property.

(4) If $\tilde{q}_{ij} \geqslant \max\{\tilde{q}_{ik}, \tilde{q}_{kj}\}$, for all $\tilde{q}_{ik}, \tilde{q}_{kj}, \tilde{q}_{ij} \in \tilde{\Delta}$, $\tilde{Q}$ satisfies the max-max transitivity property.

(5) If $\tilde{q}_{ik} \geqslant ([0.5, 0.5], [0.5, 0.5])$, $\tilde{q}_{kj} \geqslant ([0.5, 0.5], [0.5, 0.5]) \Rightarrow \tilde{q}_{ij} \geqslant \min\{\tilde{q}_{ik}, \tilde{q}_{kj}\}$, for all $\tilde{q}_{ik}, \tilde{q}_{kj}, \tilde{q}_{ij} \in \tilde{\Delta}$, $\tilde{Q}$ satisfies the restricted max-min transitivity property.

(6) If $\tilde{q}_{ik} \geqslant ([0.5, 0.5], [0.5, 0.5])$, $\tilde{q}_{kj} \geqslant ([0.5, 0.5], [0.5, 0.5]) \Rightarrow \tilde{q}_{ij} \geqslant \max\{\tilde{q}_{ik}, \tilde{q}_{kj}\}$, for all $\tilde{q}_{ik}, \tilde{q}_{kj}, \tilde{q}_{ij} \in \tilde{\Delta}$, $\tilde{Q}$ satisfies the restricted max-max transitivity property.

Xu and Cai (2009) further define some preference relations, including additive consistent incomplete interval-valued intuitionistic preference relation, multiplicative consistent incomplete interval-valued intuitionistic preference relation, and acceptable incomplete interval-valued intuitionistic preference relation:

Definition 4.7.1 (Xu and Cai, 2009) $\tilde{Q} = (\tilde{q}_{ij})_{n\times n}$ is called an additive consistent incomplete interval-valued intuitionistic preference relation, if it satisfies the arithmetic average:

$$\tilde{q}_{ij} = \frac{1}{2}(\tilde{q}_{ik} \oplus \tilde{q}_{kj}), \quad \text{for all } \tilde{q}_{ik}, \tilde{q}_{kj}, \tilde{q}_{ij} \in \tilde{\Delta} \tag{4.15}$$

Example 4.7.1 (Xu and Cai, 2009) Suppose that an incomplete interval-valued intuitionistic fuzzy preference relation is given as follows:

$$\tilde{R} = \begin{bmatrix} ([0.5, 0.5], [0.5, 0.5]) & ([0.5, 0.5], [0.5, 0.5]) & ([0.5, 0.5], [0.5, 0.5]) \\ ([0.5, 0.5], [0.5, 0.5]) & ([0.5, 0.5], [0.5, 0.5]) & x \\ ([0.5, 0.5], [0.5, 0.5]) & x & ([0.5, 0.5], [0.5, 0.5]) \end{bmatrix}$$

Then Eq.(4.15) holds for all $\tilde{q}_{ik}, \tilde{q}_{kj}, \tilde{q}_{ij} \in \tilde{\Delta}$, where $\tilde{\Delta}$ is the set of all the known elements in $\tilde{Q}$, i.e., $\tilde{\Delta} = \{\tilde{q}_{11}, \tilde{q}_{12}, \tilde{q}_{13}, \tilde{q}_{21}, \tilde{q}_{22}, \tilde{q}_{31}, \tilde{q}_{33}\}$. Accordingly, $\tilde{Q}$ is an additive consistent incomplete interval-valued intuitionistic preference relation.

Definition 4.7.2 (Xu and Cai, 2009) $\tilde{Q} = (\tilde{q}_{ij})_{n\times n}$ is called a multiplicative consistent incomplete interval-valued intuitionistic preference relation, if it satisfies the geometric mean:

$$\tilde{q}_{ij} = (\tilde{q}_{ik} \otimes \tilde{q}_{kj})^{1/2}, \quad \text{for all } \tilde{q}_{ik}, \tilde{q}_{kj}, \tilde{q}_{ij} \in \tilde{\Delta} \tag{4.16}$$

In Example 4.7.1, Eq.(4.16) holds for all $\tilde{q}_{ik}, \tilde{q}_{kj}, \tilde{q}_{ij} \in \tilde{\Delta}$. Thus, $\tilde{Q}$ is also a multiplicative consistent incomplete interval-valued intuitionistic preference relation.

Note We can also consider to add the more strict condition $i < k < j$ to Eqs.(4.15) and (4.16) just as in Definition 4.5.2.

Definition 4.7.3 (Xu and Cai, 2009) Let $\tilde{Q} = (\tilde{q}_{ij})_{n\times n}$ be an incomplete interval-valued intuitionistic preference relation. If $(i, j) \cap (k, s) \neq \varnothing$, then the elements $\tilde{q}_{ij}$ and $\tilde{q}_{ks}$ are called adjoining. For the unknown element $\tilde{q}_{ij}$, if there exist two adjoining known elements $\tilde{q}_{ik}$ and $\tilde{q}_{kj}$, then $\tilde{q}_{ij}$ is called available. In this case, we can use some known function to calculate $\tilde{q}_{ij}$ indirectly with the values of the known elements $\tilde{q}_{ik}$ and $\tilde{q}_{kj}$. For example, we can estimate the element $\tilde{q}_{ij}$ indirectly by using Eq.(4.15) or (4.16).

Definition 4.7.4 (Xu and Cai, 2009) If each unknown element can be estimated indirectly by using some known function (such as Eq.(4.15) or (4.16)) according to its adjoining known elements, then the incomplete interval-valued intuitionistic preference relation $\tilde{Q}$ is called acceptable; Otherwise, $\tilde{Q}$ is called unacceptable.

Based on Definition 4.7.4, the following result can be easily obtained:

Theorem 4.7.2 (Xu and Cai, 2009) If an incomplete interval-valued intuitionistic preference relation $\tilde{Q}$ is acceptable, then there exists at least one known element (except diagonal elements) in each line or each column of $\tilde{Q}$, i.e., there exist at least $n-1$ judgments provided by the decision maker (that is to say, each one of the alternatives is compared at least once).

Proof Let $\tilde{q}_{ij}$ be an arbitrary unknown element in the interval-valued intuitionistic preference relation $\tilde{Q}$. If $\tilde{Q}$ is acceptable, then by Definition 4.7.4, we know that there must exist at least two adjoining known elements $\tilde{q}_{ik}$ and $\tilde{q}_{kj}$. Therefore, there is at least an element $\tilde{q}_{ik}$ $(i \neq k)$ in the line i, and there is at least an element $\tilde{q}_{kj}$ $(k \neq j)$ in the column j. By the arbitrariness property of i and j, there exist at least $n-1$ judgments provided by the decision maker. This completes the proof of Theorem 4.7.2.

In order to extend an acceptable incomplete interval-valued intuitionistic preference relation to a complete interval-valued intuitionistic preference relation, the following procedure is given based on the acceptable incomplete interval-valued intuitionistic preference relation with the least judgments (i.e., $n-1$ judgments):

(Procedure 4.1) (Xu and Cai, 2009)

Step 1 For a decision making problem with a collection of alternatives Y_i $(i = 1, 2, \cdots, n)$, the decision maker compares one alternative with each other alternative, and constructs an acceptable incomplete interval-valued intuitionistic preference relation $\tilde{Q} = (\tilde{q}_{ij})_{n\times n}$, with only the $n-1$ judgments $\tilde{q}_{i_0j_1},\ \tilde{q}_{i_0j_2},\cdots,\ \tilde{q}_{i_0i_0-1}$, $\tilde{q}_{i_0i_0+1},\cdots,\ \tilde{q}_{i_0j_n}$. By Definition 4.5.1, we know that the $n-1$ elements $\tilde{q}_{j_1i_0},\ \tilde{q}_{j_2i_0},\cdots$, $\tilde{q}_{i_0-1\,i_0},\ \tilde{q}_{i_0+1\,i_0},\cdots,\ \tilde{q}_{j_ni_0}$ can be determined directly from the $n-1$ judgments $\tilde{q}_{i_0j_1},\tilde{q}_{i_0j_2},\cdots,\tilde{q}_{i_0i_0-1},\ \tilde{q}_{i_0i_0+1},\cdots,\ \tilde{q}_{i_0j_n}$, where $\tilde{\varDelta} = \{\tilde{q}_{i_0j_1},\ \tilde{q}_{i_0j_2},\cdots,\ \tilde{q}_{i_0i_0-1}$, $\tilde{q}_{i_0i_0+1},\cdots,\ \tilde{q}_{j_1i_0},\tilde{q}_{j_2i_0},\cdots,\tilde{q}_{i_0-1\,i_0},\ \tilde{q}_{i_0+1\,i_0},\cdots,\ \tilde{q}_{j_ni_0}\}$.

Step 2 Utilize the known elements in $\tilde{Q}$ and a known function (here we use the arithmetic average $\tilde{q}_{ij} = (\tilde{q}_{ik}\oplus\tilde{q}_{kj})/2$ or the geometric mean $\tilde{q}_{ij} = (\tilde{q}_{ik}\otimes\tilde{q}_{kj})^{1/2}$) to determine all the unknown elements in $\tilde{Q}$, and thus get a complete interval-valued intuitionistic preference relation $\dot{\tilde{Q}} = (\dot{\tilde{q}}_{ij})_{n\times n}$, where

$$\dot{\tilde{q}}_{ij} = \begin{cases} \dfrac{1}{2}(\tilde{q}_{ik}\oplus\tilde{q}_{kj}), & \tilde{q}_{ij}\notin\tilde{\varDelta},\ \ \tilde{q}_{ik},\tilde{q}_{kj}\in\tilde{\varDelta} \\ \tilde{q}_{ij}, & \tilde{q}_{ij}\in\tilde{\varDelta} \end{cases} \tag{4.17}$$

or

$$\dot{\tilde{q}}_{ij} = \begin{cases} (\tilde{q}_{ik}\otimes\tilde{q}_{kj})^{1/2}, & \tilde{q}_{ij}\notin\tilde{\varDelta},\ \ \tilde{q}_{ik},\tilde{q}_{kj}\in\tilde{\varDelta} \\ \tilde{q}_{ij}, & \tilde{q}_{ij}\in\tilde{\varDelta} \end{cases} \tag{4.18}$$

Based on Procedure 4.1, Xu and Cai (2009) develop a simple approach to the ranking and selection of the alternatives:

(Approach 4.1)

Step 1 Utilize the IIFA operator:

$$\dot{\tilde{q}}_i = \text{IIFA}\,(\dot{\tilde{q}}_{i1},\dot{\tilde{q}}_{i2},\cdots,\dot{\tilde{q}}_{in}) = \frac{1}{n}(\dot{\tilde{q}}_{i1}\oplus\dot{\tilde{q}}_{i2}\oplus\cdots\oplus\dot{\tilde{q}}_{in}),\quad i=1,2,\cdots,n \tag{4.19}$$

or the IIFG operator:

$$\dot{\tilde{q}}_i = \text{IIFG}\,(\dot{\tilde{q}}_{i1},\dot{\tilde{q}}_{i2},\cdots,\dot{\tilde{q}}_{in}) = (\dot{\tilde{q}}_{i1}\otimes\dot{\tilde{q}}_{i2}\otimes\cdots\otimes\dot{\tilde{q}}_{in})^{1/n},\quad i=1,2,\cdots,n \tag{4.20}$$

to aggregate all $\dot{\tilde{q}}_{ij}$ $(j=1,2,\cdots,n)$ corresponding to the alternative Y_i, and then get the complex IVIFN $\dot{\tilde{q}}_i$ of the alternative Y_i over all the other alternatives:

Step 2 Rank all $\dot{\tilde{q}}_i$ $(i=1,2,\cdots,n)$ by means of the score function (1.10) and the accuracy function (1.11).

Step 3 Rank all the alternatives $Y_i\,(i=1,2,\cdots,n)$ in accordance with the values of $\dot{\tilde{q}}_i$ $(i=1,2,\cdots,n)$, and then select the best one(s).

Example 4.7.2 (Xu and Cai, 2009) Consider a decision making problem involving the evaluation of five schools Y_i $(i=1,2,3,4,5)$ of a university. Suppose that the decision maker compares the school Y_1 with all other schools Y_i $(i=2,3,4,5)$ under the criterion "research", and gives the judgments by using IVIFNs: $\tilde{q}_{12} = ([0.2,0.3],[0.4,0.6])$, $\tilde{q}_{13} = ([0.4,0.5],[0.2,0.4])$, $\tilde{q}_{14} = ([0.5,0.7],[0.1,0.2])$, and $\tilde{q}_{15} = ([0.4,0.6],[0.1,0.3])$, respectively. Thus, based on these judgments and Definition 4.5.1, we can construct the following acceptable incomplete interval-valued intuitionistic preference relation $\tilde{Q} = (\tilde{q}_{ij})_{5\times 5}$:

$$\tilde{Q} = \begin{bmatrix} ([0.5,0.5],[0.5,0.5]) & ([0.2,0.3],[0.4,0.6]) & ([0.4,0.5],[0.2,0.4]) & ([0.5,0.7],[0.1,0.2]) & ([0.4,0.6],[0.1,0.3]) \\ ([0.4,0.6],[0.2,0.3]) & ([0.5,0.5],[0.5,0.5]) & x & x & x \\ ([0.2,0.4],[0.4,0.5]) & x & ([0.5,0.5],[0.5,0.5]) & x & x \\ ([0.1,0.2],[0.5,0.7]) & x & x & ([0.5,0.5],[0.5,0.5]) & x \\ ([0.1,0.3],[0.4,0.6]) & x & x & x & ([0.5,0.5],[0.5,0.5]) \end{bmatrix}$$

By Eq.(4.17), we have

$$\begin{aligned}\dot{\tilde{q}}_{23} &= \frac{1}{2}(\tilde{q}_{21} \oplus \tilde{q}_{13}) = \frac{1}{2}\left(([0.4,0.6],[0.2,0.3]) \oplus ([0.4,0.5],[0.2,0.4])\right) \\ &= ([0.40,0.55],[0.20,0.35])\end{aligned}$$

$$\begin{aligned}\dot{\tilde{q}}_{24} &= \frac{1}{2}(\tilde{q}_{21} \oplus \tilde{q}_{14}) = \frac{1}{2}\left(([0.4,0.6],[0.2,0.3]) \oplus ([0.5,0.7],[0.1,0.2])\right) \\ &= ([0.45,0.65],[0.14,0.24])\end{aligned}$$

$$\begin{aligned}\dot{\tilde{q}}_{25} &= \frac{1}{2}(\tilde{q}_{21} \oplus \tilde{q}_{15}) = \frac{1}{2}\left(([0.4,0.6],[0.2,0.3]) \oplus ([0.4,0.6],[0.1,0.3])\right) \\ &= ([0.40,0.60],[0.14,0.30])\end{aligned}$$

$$\begin{aligned}\dot{\tilde{q}}_{34} &= \frac{1}{2}(\tilde{q}_{31} \oplus \tilde{q}_{14}) = \frac{1}{2}\left(([0.2,0.4],[0.4,0.5]) \oplus ([0.5,0.7],[0.1,0.2])\right) \\ &= ([0.37,0.58],[0.20,0.32])\end{aligned}$$

$$\begin{aligned}\dot{\tilde{q}}_{35} &= \frac{1}{2}(\tilde{q}_{31} \oplus \tilde{q}_{15}) = \frac{1}{2}\left(([0.2,0.4],[0.4,0.5]) \oplus ([0.4,0.6],[0.1,0.3])\right) \\ &= ([0.31,0.51],[0.20,0.39])\end{aligned}$$

$$\begin{aligned}\dot{\tilde{q}}_{45} &= \frac{1}{2}(\tilde{q}_{41} \oplus \tilde{q}_{15}) = \frac{1}{2}\left(([0.1,0.2],[0.5,0.7]) \oplus ([0.4,0.6],[0.1,0.3])\right) \\ &= ([0.27,0.43],[0.22,0.46])\end{aligned}$$

Then, based on the above judgments and Definition 4.5.1, we can get the following complete interval-valued intuitionistic preference relation:

$$\dot{\tilde{Q}} = \begin{bmatrix} ([0.5,0.5],[0.5,0.5]) & ([0.2,0.3],[0.4,0.6]) & ([0.4,0.5],[0.2,0.4]) \\ ([0.4,0.6],[0.2,0.3]) & ([0.5,0.5],[0.5,0.5]) & ([0.40,0.55],[0.20,0.35]) \\ ([0.2,0.4],[0.4,0.5]) & ([0.20,0.35],[0.40,0.55]) & ([0.5,0.5],[0.5,0.5]) \\ ([0.1,0.2],[0.5,0.7]) & ([0.14,0.24],[0.45,0.65]) & ([0.20,0.32],[0.37,0.58]) \\ ([0.1,0.3],[0.4,0.6]) & ([0.14,0.30],[0.40,0.60]) & ([0.20,0.39],[0.31,0.51]) \end{bmatrix}$$

$$\begin{matrix} ([0.5,0.7],[0.1,0.2]) & ([0.4,0.6],[0.1,0.3]) \\ ([0.45,0.65],[0.14,0.24]) & ([0.40,0.60],[0.14,0.30]) \\ ([0.37,0.58],[0.20,0.32]) & ([0.31,0.51],[0.20,0.39]) \\ ([0.5,0.5],[0.5,0.5]) & ([0.27,0.43],[0.22,0.46]) \\ ([0.22,0.46],[0.27,0.43]) & ([0.5,0.5],[0.5,0.5]) \end{matrix}\Bigg]$$

By the IIFA operator (2.5), we can aggregate all $\dot{\tilde{q}}_{ij}$ $(j = 1, 2, \cdots, 5)$ corresponding to the school Y_i, and then get the complex IVIFN $\dot{\tilde{q}}_i$ of the school Y_i over all the other schools:

$$\dot{\tilde{q}}_1 = ([0.409, 0.538],\ [0.209, 0.373])$$

$$\dot{\tilde{q}}_2 = ([0.431, 0.583],\ [0.208, 0.328])$$

$$\dot{\tilde{q}}_3 = ([0.326, 0.474],\ [0.317, 0.444])$$

$$\dot{\tilde{q}}_4 = ([0.257, 0.348],\ [0.391, 0.571])$$

$$\dot{\tilde{q}}_5 = ([0.247, 0.396],\ [0.367, 0.524])$$

Hence, by Eq.(2.10), we have

$$s(\dot{\tilde{q}}_1) = 0.183, \quad s(\dot{\tilde{q}}_2) = 0.239, \quad s(\dot{\tilde{q}}_3) = 0.020$$

$$s(\dot{\tilde{q}}_4) = -0.178, \quad s(\dot{\tilde{q}}_5) = -0.124$$

Consequently,

$$s(\dot{\tilde{q}}_2) > s(\dot{\tilde{q}}_1) > s(\dot{\tilde{q}}_3) > s(\dot{\tilde{q}}_5) > s(\dot{\tilde{q}}_4)$$

by which we get

$$\dot{\tilde{q}}_2 > \dot{\tilde{q}}_1 > \dot{\tilde{q}}_3 > \dot{\tilde{q}}_5 > \dot{\tilde{q}}_4$$

Therefore, the ranking of the schools Y_i $(i = 1, 2, 3, 4, 5)$ is

$$Y_2 \succ Y_1 \succ Y_3 \succ Y_5 \succ Y_4$$

i.e, Y_2 is the best school.

If we utilize Eq.(4.18) to extend $\tilde{Q} = (\tilde{q}_{ij})_{5\times5}$, then we have

$$\begin{aligned} \dot{\tilde{q}}_{23} &= (\tilde{q}_{21} \otimes \tilde{q}_{13})^{\frac{1}{2}} = (([0.4, 0.6], [0.2, 0.3]) \otimes ([0.4, 0.5], [0.2, 0.4]))^{\frac{1}{2}} \\ &= ([0.40, 0.55], [0.20, 0.35]) \\ \dot{\tilde{q}}_{24} &= (\tilde{q}_{21} \otimes \tilde{q}_{14})^{\frac{1}{2}} = (([0.4, 0.6], [0.2, 0.3]) \otimes ([0.5, 0.7], [0.1, 0.2]))^{\frac{1}{2}} \\ &= ([0.45, 0.65], [0.15, 0.25]) \\ \dot{\tilde{q}}_{25} &= (\tilde{q}_{21} \otimes \tilde{q}_{15})^{\frac{1}{2}} = (([0.4, 0.6], [0.2, 0.3]) \otimes ([0.4, 0.6], [0.1, 0.3]))^{\frac{1}{2}} \\ &= ([0.40, 0.60], [0.15, 0.30]) \\ \dot{\tilde{q}}_{34} &= (\tilde{q}_{31} \otimes \tilde{q}_{14})^{\frac{1}{2}} = (([0.2, 0.4], [0.4, 0.5]) \otimes ([0.5, 0.7], [0.1, 0.2]))^{\frac{1}{2}} \\ &= ([0.32, 0.53], [0.27, 0.37]) \end{aligned}$$

$$\begin{aligned}\dot{\tilde{q}}_{35} &= (\tilde{q}_{31} \otimes \tilde{q}_{15})^{\frac{1}{2}} = (([0.2, 0.4], [0.4, 0.5]) \otimes ([0.4, 0.6], [0.1, 0.3]))^{\frac{1}{2}} \\ &= ([0.28, 0.49], [0.27, 0.41]) \\ \dot{\tilde{q}}_{45} &= (\tilde{q}_{41} \otimes \tilde{q}_{15})^{\frac{1}{2}} = (([0.1, 0.2], [0.5, 0.7]) \otimes ([0.4, 0.6], [0.1, 0.3]))^{\frac{1}{2}} \\ &= ([0.20, 0.35], [0.33, 0.54])\end{aligned}$$

Then, based on the above judgments and Definition 4.5.1, we can get the following complete interval-valued intuitionistic preference relation:

$$\dot{\tilde{Q}} = \left[\begin{array}{ccc} ([0.5, 0.5], [0.5, 0.5]) & ([0.2, 0.3], [0.4, 0.6]) & ([0.4, 0.5], [0.2, 0.4]) \\ ([0.4, 0.6], [0.2, 0.3]) & ([0.5, 0.5], [0.5, 0.5]) & ([0.40, 0.55], [0.20, 0.35]) \\ ([0.2, 0.4], [0.4, 0.5]) & ([0.20, 0.35], [0.40, 0.55]) & ([0.5, 0.5], [0.5, 0.5]) \\ ([0.1, 0.2], [0.5, 0.7]) & ([0.15, 0.25], [0.45, 0.65]) & ([0.27, 0.37], [0.32, 0.53]) \\ ([0.1, 0.3], [0.4, 0.6]) & ([0.15, 0.30], [0.40, 0.60]) & ([0.27, 0.41], [0.28, 0.49]) \end{array}\right.$$

$$\left.\begin{array}{cc} ([0.5, 0.7], [0.1, 0.2]) & ([0.4, 0.6], [0.1, 0.3]) \\ ([0.45, 0.65], [0.15, 0.25]) & ([0.40, 0.60], [0.15, 0.30]) \\ ([0.32, 0.53], [0.27, 0.37]) & ([0.28, 0.49], [0.27, 0.41]) \\ ([0.5, 0.5], [0.5, 0.5]) & ([0.20, 0.35], [0.33, 0.54]) \\ ([0.33, 0.54], [0.20, 0.35]) & ([0.5, 0.5], [0.5, 0.5]) \end{array}\right]$$

By the IIFG operator (2.7), we can aggregate all $\dot{\tilde{q}}_{ij}$ $(j = 1, 2, \cdots, 5)$ corresponding to the school Y_i, and then get the complex IVIFN $\dot{\tilde{q}}_i$ of the school Y_i over all the other schools:

$$\dot{\tilde{q}}_1 = ([0.381, 0.501],\ [0.279, 0.417])$$

$$\dot{\tilde{q}}_2 = ([0.428, 0.578],\ [0.254, 0.346])$$

$$\dot{\tilde{q}}_3 = ([0.282, 0.449],\ [0.374, 0.470])$$

$$\dot{\tilde{q}}_4 = ([0.210, 0.318],\ [0.425, 0.592])$$

$$\dot{\tilde{q}}_5 = ([0.232, 0.398],\ [0.364, 0.516])$$

Thus, by Eq.(2.10), we have

$$s(\dot{\tilde{q}}_1) = 0.093, \quad s(\dot{\tilde{q}}_2) = 0.203, \quad s(\dot{\tilde{q}}_3) = -0.056$$

$$s(\dot{\tilde{q}}_4) = -0.244, \quad s(\dot{\tilde{q}}_5) = -0.125$$

Accordingly,

$$s(\dot{\tilde{q}}_2) > s(\dot{\tilde{q}}_1) > s(\dot{\tilde{q}}_3) > s(\dot{\tilde{q}}_5) > s(\dot{\tilde{q}}_4)$$

by which we have $\dot{\tilde{q}}_2 > \dot{\tilde{q}}_1 > \dot{\tilde{q}}_3 > \dot{\tilde{q}}_5 > \dot{\tilde{q}}_4$. Therefore, we get the ranking of the schools Y_i $(i = 1, 2, 3, 4, 5)$ as:

$$Y_2 \succ Y_1 \succ Y_3 \succ Y_5 \succ Y_4$$

i.e, the best school is Y_2.

In what follows, we further consider the acceptable incomplete interval-valued intuitionistic preference relation with more than $n-1$ judgments:

(Procedure 4.2) (Xu and Cai, 2009)

Step 1 For a decision making problem, the decision maker compares the given alternatives and constructs an acceptable incomplete interval-valued intuitionistic preference relation $\tilde{Q} = (\tilde{q}_{ij})_{n\times n}$, with more than $n-1$ judgments.

Step 2 Based on the known elements in $\tilde{Q}$, each unknown element $\tilde{q}_{ij}$ can be estimated indirectly by using a known function, such as:

$$\dot{\tilde{q}}_{ij} = \frac{1}{n_{ij}}\left(\underset{k\in\tilde{N}_{ij}}{\oplus}\left(\frac{1}{2}(\tilde{q}_{ik}\oplus\tilde{q}_{kj})\right)\right) = \frac{1}{2n_{ij}}\left(\underset{k\in\tilde{N}_{ij}}{\oplus}(\tilde{q}_{ik}\oplus\tilde{q}_{kj})\right) \tag{4.21}$$

or

$$\dot{\tilde{q}}_{ij} = \left(\underset{k\in\tilde{N}_{ij}}{\otimes}(\tilde{q}_{ik}\otimes\tilde{q}_{kj})^{1/2}\right)^{1/n_{ij}} = \left(\underset{k\in\tilde{N}_{ij}}{\otimes}(\tilde{q}_{ik}\otimes\tilde{q}_{kj})\right)^{1/(2n_{ij})} \tag{4.22}$$

where $\tilde{N}_{ij} = \{k\,|\,\tilde{q}_{ik},\tilde{q}_{kj}\in\tilde{\Delta}\}$, n_{ij} is the number of the elements in $\tilde{N}_{ij}$. Then we can get a complete interval-valued intuitionistic preference relation $\ddot{\tilde{Q}} = (\ddot{\tilde{q}}_{ij})_{n\times n}$, where

$$\ddot{\tilde{q}}_{ij} = \begin{cases} \dot{\tilde{q}}_{ij}, & \tilde{q}_{ij}\notin\tilde{\Delta} \\ \tilde{q}_{ij}, & \tilde{q}_{ij}\in\tilde{\Delta} \end{cases} \tag{4.23}$$

Obviously, the complete interval-valued intuitionistic preference relation $\ddot{\tilde{Q}}$ contains both the direct interval-valued intuitionistic preference information given by the decision maker and the indirect interval-valued intuitionistic preference information derived from the known interval-valued intuitionistic preference information.

Based on Procedure 4.2, Approach 4.1 developed previously can be used to the ranking and selection of the alternatives (Xu and Cai, 2009).

Consider Example 4.7.2 again. Suppose that the decision maker can provide more than $n-1$ judgments, and constructs the following acceptable incomplete interval-valued intuitionistic preference relation $\tilde{Q} = (\tilde{q}_{ij})_{5\times 5}$:

$$\tilde{Q} = \begin{bmatrix} ([0.5,0.5],[0.5,0.5]) & ([0.2,0.3],[0.4,0.6]) & ([0.4,0.5],[0.2,0.4]) & ([0.5,0.7],[0.1,0.2]) & ([0.4,0.6],[0.1,0.3]) \\ ([0.4,0.6],[0.2,0.3]) & ([0.5,0.5],[0.5,0.5]) & x & x & ([0.4,0.6],[0.2,0.3]) \\ ([0.2,0.4],[0.4,0.5]) & x & ([0.5,0.5],[0.5,0.5]) & ([0.4,0.6],[0.2,0.3]) & x \\ ([0.1,0.2],[0.5,0.7]) & x & ([0.2,0.3],[0.4,0.6]) & ([0.5,0.5],[0.5,0.5]) & ([0.3,0.4],[0.2,0.5]) \\ ([0.1,0.3],[0.4,0.6]) & ([0.2,0.3],[0.4,0.6]) & x & ([0.2,0.5],[0.3,0.4]) & ([0.5,0.5],[0.5,0.5]) \end{bmatrix}$$

By Eq.(4.19), we have

$$\begin{aligned}
\dot{\tilde{q}}_{23} &= \frac{1}{2}(\tilde{q}_{21} \oplus \tilde{q}_{13}) \\
&= \frac{1}{2}\left(([0.4, 0.6], [0.2, 0.3]) \oplus ([0.4, 0.5], [0.2, 0.4])\right) \\
&= ([0.40, 0.55], [0.20, 0.35]) \\
\dot{\tilde{q}}_{24} &= \frac{1}{4}((\tilde{q}_{21} \oplus \tilde{q}_{14}) \oplus (\tilde{q}_{25} \oplus \tilde{q}_{54})) \\
&= \frac{1}{4}\left(([0.4, 0.6], [0.2, 0.3]) \oplus ([0.5, 0.7], [0.1, 0.2])\right. \\
&\quad \left.\oplus([0.4, 0.6], [0.2, 0.3]) \oplus ([0.2, 0.5], [0.3, 0.4])\right) \\
&= ([0.38, 0.61], [0.19, 0.29]) \\
\dot{\tilde{q}}_{35} &= \frac{1}{4}((\tilde{q}_{31} \oplus \tilde{q}_{15}) \oplus (\tilde{q}_{34} \oplus \tilde{q}_{45})) \\
&= \frac{1}{4}\left(([0.2, 0.4], [0.4, 0.5]) \oplus ([0.4, 0.6], [0.1, 0.3])\right. \\
&\quad \left.\oplus\ ([0.4, 0.6], [0.2, 0.3]) \oplus ([0.3, 0.4], [0.2, 0.5])\right) \\
&= ([0.33, 0.51], [0.20, 0.39])
\end{aligned}$$

Then, based on the above judgments and Definition 4.5.1, we can get the following complete interval-valued intuitionistic preference relation $\ddot{\tilde{Q}} = (\ddot{\tilde{q}}_{ij})_{5\times5}$:

$$\ddot{\tilde{Q}} = \begin{bmatrix}
([0.5, 0.5], [0.5, 0.5]) & ([0.2, 0.3], [0.4, 0.6]) & ([0.4, 0.5], [0.2, 0.4]) & ([0.5, 0.7], [0.1, 0.2]) & ([0.4, 0.6], [0.1, 0.3]) \\
([0.4, 0.6], [0.2, 0.3]) & ([0.5, 0.5], [0.5, 0.5]) & ([0.40, 0.55], [0.20, 0.35]) & ([0.38, 0.61], [0.19, 0.29]) & ([0.4, 0.6], [0.2, 0.3]) \\
([0.2, 0.4], [0.4, 0.5]) & ([0.20, 0.35], [0.40, 0.55]) & ([0.5, 0.5], [0.5, 0.5]) & ([0.4, 0.6], [0.2, 0.3]) & ([0.33, 0.51], [0.20, 0.39]) \\
([0.1, 0.2], [0.5, 0.7]) & ([0.19, 0.29], [0.38, 0.61]) & ([0.2, 0.3], [0.4, 0.6]) & ([0.5, 0.5], [0.5, 0.5]) & ([0.3, 0.4], [0.2, 0.5]) \\
([0.1, 0.3], [0.4, 0.6]) & ([0.2, 0.3], [0.4, 0.6]) & ([0.20, 0.39], [0.33, 0.51]) & ([0.2, 0.5], [0.3, 0.4]) & ([0.5, 0.5], [0.5, 0.5])
\end{bmatrix}$$

By the IIFA operator (2.5), we can aggregate all $\ddot{\tilde{q}}_{ij}$ $(j = 1, 2, \cdots, 5)$ corresponding to the school Y_i, and then get the complex IVIFN $\ddot{\tilde{q}}_i$ of the school Y_i over all the other schools:

$$\begin{aligned}
\ddot{\tilde{q}}_1 &= ([0.409, 0.538],\ [0.209, 0.373]) \\
\ddot{\tilde{q}}_2 &= ([0.418, 0.574],\ [0.238, 0.340]) \\
\ddot{\tilde{q}}_3 &= ([0.336, 0.479],\ [0.317, 0.438]) \\
\ddot{\tilde{q}}_4 &= ([0.272, 0.346],\ [0.377, 0.577]) \\
\ddot{\tilde{q}}_5 &= ([0.254, 0.405],\ [0.438, 0.516])
\end{aligned}$$

Thus, by Eq.(2.10), we have

$$s(\ddot{\tilde{q}}_1) = 0.183, \quad s(\ddot{\tilde{q}}_2) = 0.207, \quad s(\ddot{\tilde{q}}_3) = 0.030$$

$$s(\ddot{\tilde{q}}_4) = -0.168, \quad s(\ddot{\tilde{q}}_5) = -0.147$$

from which we get

$$s(\ddot{\tilde{q}}_2) > s(\ddot{\tilde{q}}_1) > s(\ddot{\tilde{q}}_3) > s(\ddot{\tilde{q}}) > s(\ddot{\tilde{q}}_4)$$

Hence,

$$\ddot{\tilde{q}}_2 > \ddot{\tilde{q}}_1 > \ddot{\tilde{q}}_3 > \ddot{\tilde{q}}_5 > \ddot{\tilde{q}}_4$$

To conclude, we get the ranking of the schools Y_i $(i = 1, 2, \cdots, 5)$ as:

$$Y_2 \succ Y_1 \succ Y_3 \succ Y_5 \succ Y_4$$

i.e., the best school is Y_2.

If we utilize Eq.(4.20) to extend $\tilde{Q} = (\tilde{q}_{ij})_{5\times 5}$, then we have

$$\begin{aligned}
\ddot{\tilde{q}}_{23} &= (\tilde{q}_{21} \otimes \tilde{q}_{13})^{1/2} \\
&= (([0.4, 0.6], [0.2, 0.3]) \otimes ([0.4, 0.5], [0.2, 0.4]))^{1/2} \\
&= ([0.40, 0.55], [0.20, 0.35]) \\
\ddot{\tilde{q}}_{24} &= ((\tilde{q}_{21} \otimes \tilde{q}_{14}) \otimes (\tilde{q}_{25} \otimes \tilde{q}_{54}))^{1/4} \\
&= (([0.4, 0.6], [0.2, 0.3]) \otimes ([0.5, 0.7], [0.1, 0.2]) \\
&\quad \otimes([0.4, 0.6], [0.2, 0.3]) \otimes ([0.2, 0.5], [0.3, 0.4]))^{1/4} \\
&= ([0.36, 0.60], [0.20, 0.30]) \\
\ddot{\tilde{q}}_{35} &= ((\tilde{q}_{31} \otimes \tilde{q}_{15}) \otimes (\tilde{q}_{34} \otimes \tilde{q}_{45}))^{1/4} \\
&= (([0.2, 0.4], [0.4, 0.5]) \otimes ([0.4, 0.6], [0.1, 0.3]) \\
&\quad \otimes([0.4, 0.6], [0.2, 0.3]) \otimes ([0.3, 0.4], [0.2, 0.5]))^{1/4} \\
&= ([0.31, 0.49], [0.23, 0.41])
\end{aligned}$$

Then, based on the above judgments and Definition 4.5.1, we can get the following complete interval-valued intuitionistic preference relation $\ddot{\tilde{Q}} = (\ddot{\tilde{q}}_{ij})_{5\times 5}$:

$$\ddot{\tilde{Q}} = \left[\begin{array}{ccccc}
([0.5, 0.5], [0.5, 0.5]) & ([0.2, 0.3], [0.4, 0.6]) & ([0.4, 0.5], [0.2, 0.4]) & ([0.5, 0.7], [0.1, 0.2]) & ([0.4, 0.6], [0.1, 0.3]) \\
([0.4, 0.6], [0.2, 0.3]) & ([0.5, 0.5], [0.5, 0.5]) & ([0.40, 0.55], [0.20, 0.35]) & ([0.36, 0.60], [0.20, 0.30]) & ([0.4, 0.6], [0.2, 0.3]) \\
([0.2, 0.4], [0.4, 0.5]) & ([0.20, 0.35], [0.40, 0.55]) & ([0.5, 0.5], [0.5, 0.5]) & ([0.4, 0.6], [0.2, 0.3]) & ([0.31, 0.49], [0.23, 0.41]) \\
([0.1, 0.2], [0.5, 0.7]) & ([0.20, 0.30], [0.36, 0.60]) & ([0.2, 0.3], [0.4, 0.6]) & ([0.5, 0.5], [0.5, 0.5]) & ([0.3, 0.4], [0.2, 0.5]) \\
([0.1, 0.3], [0.4, 0.6]) & ([0.2, 0.3], [0.4, 0.6]) & ([0.23, 0.41], [0.31, 0.49]) & ([0.2, 0.5], [0.3, 0.4]) & ([0.5, 0.5], [0.5, 0.5])
\end{array}\right]$$

By the IIFG operator (2.7), we can aggregate all $\ddot{\tilde{q}}_{ij}$ $(j = 1, 2, \cdots, 5)$ corresponding to the school Y_i, and then get the complex IVIFN $\ddot{\tilde{q}}_i$ of the school Y_i over all the other schools:

$$\ddot{\tilde{q}}_1 = ([0.381, 0.501],\ [0.279, 0.417])$$
$$\ddot{\tilde{q}}_2 = ([0.410, 0.569],\ [0.272, 0.355])$$
$$\ddot{\tilde{q}}_3 = ([0.301, 0.460],\ [0.356, 0.459])$$
$$\ddot{\tilde{q}}_4 = ([0.227, 0.325],\ [0.401, 0.587])$$
$$\ddot{\tilde{q}}_5 = ([0.215, 0.392],\ [0.386, 0.524])$$

Thus, by Eq.(2.10), we have

$$s(\ddot{\tilde{q}}_1) = 0.093, \quad s(\ddot{\tilde{q}}_2) = 0.176, \quad s(\ddot{\tilde{q}}_3) = -0.027$$
$$s(\ddot{\tilde{q}}_4) = -0.218, \quad s(\ddot{\tilde{q}}_5) = -0.152$$

by which, we get

$$s(\ddot{\tilde{q}}_2) > s(\ddot{\tilde{q}}_1) > s(\ddot{\tilde{q}}_3) > s(\ddot{\tilde{q}}_5) > s(\ddot{\tilde{q}}_4)$$

and thus

$$\ddot{\tilde{q}}_2 > \ddot{\tilde{q}}_1 > \ddot{\tilde{q}}_3 > \ddot{\tilde{q}}_5 > \ddot{\tilde{q}}_4$$

Therefore, we get the ranking of the schools Y_i $(i = 1, 2, 3, 4, 5)$ as:

$$Y_2 \succ Y_1 \succ Y_3 \succ Y_5 \succ Y_4$$

i.e, the best school is Y_2.

In the above example, we have utilized the approach developed based on the arithmetic average or the geometric mean respectively to derive the ranking of the given alternatives in two different cases. The numerical results have shown that the approach is easy to apply and practical, which is suitable for decision making in fuzzy or uncertain environments.

4.8 Multi-Attribute Decision Making with Intuitionistic Fuzzy Preference Information on Alternatives

In order to enable the decision maker to exert his/her initiative and make the decision results more scientific and reasonable, it is necessary for the decision maker to actively participate in the process of decision making. Xu (2007c) investigates multi-attribute decision making problems in which the attribute values are given as IFNs and the preference information on alternatives are provided by the decision maker. In situations where the preference information on alternatives is expressed in intuitionistic preference relation and the information on attribute weights is incomplete, Xu (2007c) defines the concepts of additive consistent intuitionistic preference relation, multiplicative consistent intuitionistic preference relation and score matrix of intuitionistic fuzzy decision matrix. On the basis of the score matrix and intuitionistic preference relation, he establishes some simple linear programming models by using two transformation functions, from which the attribute weights can be derived. He further proposes two approaches to multiple attribute decision making with intuitionistic fuzzy preference information on alternatives.

4.8.1 Consistent Intuitionistic Preference Relations

In what follows, we first introduce the concept of consistent fuzzy preference relation:

Definition 4.8.1 (Xu, 2004e) Let $B = (b_{ij})_{n\times n}$ be a fuzzy preference relation. If $b_{ij} = b_{ik} - b_{jk} + 0.5$ $(i, j, k = 1, 2, \cdots, n)$, then B is called an additive consistent fuzzy preference relation.

Let $\omega = (\omega_1, \omega_2, \cdots, \omega_n)^{\mathrm{T}}$ be the weight vector of the additive consistent fuzzy preference relation$B$, where $\omega_j \in [0, 1]$ $(j = 1, 2, \cdots, n)$, and $\sum_{j=1}^{n} \omega_j = 1$. Then the element b_{ij} in B can be expressed as:

$$b_{ij} = 0.5(\omega_i - \omega_j + 1), \quad i, j = 1, 2, \cdots, n$$

Definition 4.8.2 (Xu, 2004e) If $b_{ik}b_{kj}b_{ji} = b_{ki}b_{jk}b_{ij}$ $(i, j, k = 1, 2, \cdots, n)$, then $B = (b_{ij})_{n\times n}$ is called a multiplicative consistent fuzzy preference relation.

If $\omega = (\omega_1, \omega_2, \cdots, \omega_n)^T$ is the weight vector of the multiplicative consistent fuzzy preference relation B, then the element b_{ij} in B can be expressed as:

$$b_{ij} = \frac{\omega_i}{\omega_i + \omega_j}, \quad i, j = 1, 2, \cdots, n$$

In particular, if $\omega_i = \omega_j = 0$, then we stipulate that $b_{ij} = 0.5$.

Mikhailov (2002), and Wang and Xu (2005) investigate the interval-valued multiplicative preference relation (Arbel, 1989; Haines, 1998; Islam et al., 1997; Mikhailov, 2003; Saaty and Vargas, 1987; Salo and Hämäläinen, 1995; Xu, 2007j; 2005d; 2004e; Xu and Da, 2003c; Yager and Xu, 2006), and define the concept of consistent interval-valued multiplicative preference relation.

Based on Definitions 4.8.1 and 4.8.2, we now introduce two intuitionistic preference relations called additive consistent intuitionistic preference relation and multiplicative consistent intuitionistic preference relation (Xu, 2007c):

Definition 4.8.3 (Xu, 2007c) Let $Q = (q_{ij})_{n\times n}$ be an intuitionistic preference relation, where $q_{ij} = (\mu_{ij},\ \nu_{ij})$ $(i, j = 1, 2, \cdots, n)$. If there exists a vector $\omega = (\omega_1, \omega_2, \cdots, \omega_n)^{\mathrm{T}}$, such that

$$\mu_{ij} \leqslant 0.5(\omega_i - \omega_j + 1) \leqslant 1 - \nu_{ij}, \quad i, j = 1, 2, \cdots, n \tag{4.24}$$

where $\omega_j \in [0,1]$, $j = 1,2,\cdots,n$, and $\sum_{j=1}^{n}\omega_j = 1$, then Q is called an additive consistent intuitionistic preference relation.

By Eq.(4.4), it can be seen that Eq.(4.24) is equivalent to:

$$\mu_{ij} \leqslant 0.5(\omega_i - \omega_j + 1) \leqslant 1 - \nu_{ij}, \quad i = 1,2,\cdots,n-1;\ j = i+1,\cdots,n \tag{4.25}$$

Definition 4.8.4 (Xu, 2007c) If there exists a vector $\omega = (\omega_1, \omega_2, \cdots, \omega_n)^{\mathrm{T}}$, such that

$$\mu_{ij} \leqslant \frac{\omega_i}{\omega_i + \omega_j} \leqslant 1 - \nu_{ij}, \quad i,j = 1,2,\cdots,n \tag{4.26}$$

where $\omega_j \in [0,1]$ $(j = 1,2,\cdots,n)$ and $\sum_{j=1}^{n}\omega_j = 1$, then $Q = (q_{ij})_{n\times n}$ is called a multiplicative consistent intuitionistic preference relation, where $q_{ij} = (\mu_{ij},\ \nu_{ij})$, $i,j = 1,2,\cdots,n$.

By Eq.(4.4), we know that Eq.(4.26) is equivalent to:

$$\mu_{ij} \leqslant \frac{\omega_i}{\omega_i + \omega_j} \leqslant 1 - \nu_{ij}, \quad i = 1,2,\cdots,n-1;\ j = i+1,\cdots,n \tag{4.27}$$

i.e.,

$$\mu_{ij}(\omega_i + \omega_j) \leqslant \omega_i \leqslant (1 - \nu_{ij})(\omega_i + \omega_j), \quad i = 1,2,\cdots,n-1;\ j = i+1,\cdots,n \tag{4.28}$$

4.8.2 Linear Programming Models with Intuitionistic Fuzzy Information

For a multi-attribute decision making problem, let Y, G, R, ω and Λ be defined as Section 1.3, and $\omega \in \Lambda$. According to Xu (2007b), we know that the interval form of the element $r_{ij} = (\mu_{ij}, \nu_{ij})$ in the intuitionistic fuzzy decision matrix is equivalent to $r_{ij} = [\mu_{ij}, 1-\nu_{ij}]$ $(i = 1,2,\cdots,n;\ j = 1,2,\cdots,m)$. To exert his/her initiative sufficiently, a decision maker can compare each pair of the alternatives Y_i $(i = 1,2,\cdots,n)$, and construct an intuitionistic preference relation $Q = (q_{ij})_{n\times n}$, where $q_{ij} = (t_{ij}, f_{ij})$, $0 \leqslant t_{ij} + f_{ij} \leqslant 1$, $t_{ji} = f_{ij}$, $f_{ji} = t_{ij}$, $t_{ii} = f_{ii} = 0.5$, $i,j = 1,2,\cdots,n$, t_{ij} indicates the degree that the decision maker prefers the alternative Y_i to the alternative Y_j, and f_{ij} indicates the degree that the decision maker prefers the alternative Y_j to the alternative Y_i, $1 - t_{ij} - f_{ij}$ is interpreted as the uncertainty degree to which Y_i is preferred to Y_j.

By Definition 1.3.7, the score matrix of the intuitionistic fuzzy decision matrix D can be denoted as $S = (s(r_{ij}))_{n\times m}$, where $s(r_{ij}) \in [-1,\ 1]$, $i = 1,2,\cdots,n$; $j = 1,2,\cdots,m$. We use the following formula to normalize the score matrix $S = (s(r_{ij}))_{n\times m}$ into $\bar{S} = (\bar{s}(r_{ij}))_{n\times m}$:

$$\bar{s}(r_{ij}) = \frac{s(r_{ij}) - \min\limits_i\{s(r_{ij})\}}{\max\limits_i\{s(r_{ij})\} - \min\limits_i\{s(r_{ij})\}}, \quad i = 1,2,\cdots,n;\ j = 1,2,\cdots,m \tag{4.29}$$

where $\bar{s}(r_{ij}) \in [0,1]$, $i = 1,2,\cdots,n;\ j = 1,2,\cdots,m$.

According to the normalized score matrix $\bar{S} = (\bar{s}(r_{ij}))_{n\times m}$, the overall score of each alternative can be denoted as:

$$\bar{s}(r_i) = \sum_{j=1}^{m} w_j \bar{s}(r_{ij}), \quad i = 1,2,\cdots,n \tag{4.30}$$

In order to derive the attribute weights, in what follows, we establish the decision models from the viewpoints of the additive transitivity and the multiplicative transitivity:

1. Linear Programming Models Based on Additive Transitivity

To make the decision information uniform, we utilize the linear transformation function and the overall scores of all the alternatives Y_i $(i = 1,2,\cdots,n)$ to construct the additive consistent fuzzy preference relation $\bar{Q} = (\bar{q}_{ij})_{n\times n}$, where

$$\bar{q}_{ij} = 0.5(\bar{s}(r_i) - \bar{s}(r_j) + 1), \quad i,j = 1,2,\cdots,n$$

(1) If the additive consistent fuzzy preference relation $\bar{Q}$ is the same as the decision maker's intuitionistic preference relation Q, then the following inequality holds:

$$t_{ij} \leqslant 0.5\left(\bar{s}(r_i) - \bar{s}(r_j) + 1\right) \leqslant 1 - f_{ij}, \quad i = 1,2,\cdots,n-1;\ j = i+1,\cdots,n$$

i.e.,

$$t_{ij} \leqslant 0.5\left(\sum_{k=1}^{m} \omega_k\left(\bar{s}(r_{ik}) - \bar{s}(r_{jk})\right) + 1\right) \leqslant 1-f_{ij}, \quad i = 1,2,\cdots,n-1;\ j = i+1,\cdots,n \tag{4.31}$$

In general, there exist at least two weight vectors $\omega = (\omega_1,\omega_2,\cdots,\omega_m)^{\mathrm{T}}$ satisfying the condition (4.31), with each weight ω_k belonging to an interval. Therefore, based on the inequality (4.31) and the known weight information $\varLambda$, Xu (2007c) establishes the following linear programming model:

(M-4.1) $\quad \omega_k^- = \min \omega_k$

$$\text{s. t. } 0.5\left(\sum_{k=1}^{m} \omega_k\left(\bar{s}(r_{ik}) - \bar{s}(r_{jk})\right) + 1\right) \geqslant t_{ij}, \quad i=1,2,\cdots,n-1;\ j=i+1,\cdots,n$$

$$0.5\left(\sum_{k=1}^{m} \omega_k\left(\bar{s}(r_{ik}) - \bar{s}(r_{jk})\right) + 1\right) \leqslant 1-f_{ij}, \quad i=1,2,\cdots,n-1;\ j=i+1,\cdots,n$$

$$\omega = (\omega_1,\omega_2,\cdots,\omega_m)^{\mathrm{T}} \in \varLambda, \quad \omega_k \in [0,1], \ \ k = 1,2,\cdots,m, \ \ \sum_{k=1}^{m} \omega_k = 1$$

and

(M-4.2) $\omega_k^+ = \max \omega_k$

$$\text{s. t. } 0.5\left(\sum_{k=1}^{m} \omega_k \left(\bar{s}(r_{ik}) - \bar{s}(r_{jk})\right) + 1\right) \geqslant t_{ij}, \quad i=1,2,\cdots,n-1;\ j=i+1,\cdots,n$$

$$0.5\left(\sum_{k=1}^{m} \omega_k \left(\bar{s}(r_{ik}) - \bar{s}(r_{jk})\right) + 1\right) \leqslant 1-f_{ij}, \quad i=1,2,\cdots,n-1;\ j=i+1,\cdots,n$$

$$\omega = (\omega_1, \omega_2, \cdots, \omega_m)^{\mathrm{T}} \in \Lambda, \quad \omega_k \in [0,1],\ \ k=1,2,\cdots,m,\ \ \sum_{k=1}^{m} \omega_k = 1$$

Solving the models (M-4.1) and (M-4.2), we can get the set of the weight vectors of attributes:

$$\Upsilon_1 = \left\{ \omega = (\omega_1, \omega_2, \cdots, \omega_m)^{\mathrm{T}} \,|\, \omega_k \in [\omega_k^{\mathrm{L}}, \omega_k^{\mathrm{U}}],\ \omega_k \in [0,1],\ k=1,2,\cdots,m,\ \sum_{k=1}^{m} \omega_k = 1 \right\} \tag{4.32}$$

(2) If there is a difference between the additive consistent fuzzy preference relation $\bar{Q}$ and the decision maker's intuitionistic preference relation Q, i.e., the inequality (4.31) does not hold, then we cannot use the models (M-4.1) and (M-4.2) to derive the weight vectors. To resolve this issue, Xu (2007c) extends the models (M-4.1) and (M-4.2) by introducing the deviation variables ε_{ij}^- and ε_{ij}^+ to relax the inequality (4.31) and get

$$t_{ij} - \varepsilon_{ij}^- \leqslant 0.5\left(\sum_{k=1}^{m} \omega_k \left(\bar{s}(r_{ik}) - \bar{s}(r_{jk})\right) + 1\right) \leqslant 1 - f_{ij} + \varepsilon_{ij}^+,$$
$$i = 1,2,\cdots,n-1;\ j = i+1,\cdots,n \tag{4.33}$$

where both ε_{ij}^- and ε_{ij}^+ are non-negative real numbers. In particular, if both ε_{ij}^- and ε_{ij}^+ are zero, then the inequality (4.33) reduces to the inequality (4.31).

Clearly, the smaller the values of the deviation variables ε_{ij}^- and ε_{ij}^+, the closer the additive consistent fuzzy preference relation $\bar{Q}$ and the decision maker's intuitionistic preference relation Q. Therefore, we can establish the following optimization model (Xu, 2007c):

(M-4.3) $\varphi_1^* = \min \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} (\varepsilon_{ij}^- + \varepsilon_{ij}^+)$

$$\text{s. t. } 0.5\left(\sum_{k=1}^{m} \omega_k \left(\bar{s}(r_{ik}) - \bar{s}(r_{jk})\right) + 1\right) + \varepsilon_{ij}^- \geqslant t_{ij}, \quad i=1,2,\cdots,n-1; \quad j=i+1,\cdots,n$$

$$0.5\left(\sum_{k=1}^{m} \omega_k \left(\bar{s}(r_{ik}) - \bar{s}(r_{jk})\right) + 1\right) - \varepsilon_{ij}^+ \leqslant 1 - f_{ij}, \quad i=1,2,\cdots,n-1; \quad j=i+1,\cdots,n$$

$$\omega = (\omega_1, \omega_2, \cdots, \omega_m)^{\mathrm{T}} \in \Lambda, \ \ \omega_k \in [0,1], \ \ k = 1, 2, \cdots, m, \ \ \sum_{k=1}^{m} \omega_k = 1$$

$$\varepsilon_{ij}^{-}, \ \varepsilon_{ij}^{+} \geqslant 0, \ \ i = 1, 2, \cdots, n-1; \quad j = i+1, \cdots, n$$

Solving the model, we can get the optimal deviation variables $\dot{\varepsilon}_{ij}^{-}$ and $\dot{\varepsilon}_{ij}^{+}$, $i = 1, 2, \cdots, n-1$; $j = i+1, \cdots, n$.

From the model (M-4.3), we can derive the following theorem:

Theorem 4.8.1 (Xu, 2007c) The additive consistent fuzzy preference relation $\bar{Q}$ is the same as the intuitionistic preference relation Q if and only if $\varphi_1^* = 0$.

If $\varphi_1^* \neq 0$, then based on the optimal deviation variables (Xu, 2007c) $\dot{\varepsilon}_{ij}^{-}$ and $\dot{\varepsilon}_{ij}^{+}$ $(i=1, 2, \cdots, n-1;\ j = i+1, \cdots, n)$, and similar to the models (M-4.1) and (M-4.2), we can further establish the following two linear programming models:

(M-4.4) $\omega_k^{-} = \min \omega_k$

$$\text{s. t. } 0.5\left(\sum_{k=1}^{m} \omega_k (\bar{s}(r_{ik}) - \bar{s}(r_{jk})) + 1\right) + \dot{\varepsilon}_{ij}^{-} \geqslant t_{ij}, \quad i=1, 2, \cdots, n-1; \quad j=i+1, \cdots, n$$

$$0.5\left(\sum_{k=1}^{m} \omega_k (\bar{s}(r_{ik}) - \bar{s}(r_{jk})) + 1\right) - \dot{\varepsilon}_{ij}^{+} \leqslant 1 - f_{ij}, \quad i=1, 2, \cdots, n-1; \ \ j=i+1, \cdots, n$$

$$\omega = (\omega_1, \omega_2, \cdots, \omega_m)^{\mathrm{T}} \in \Lambda, \quad \omega_k \in [0,1], \ \ k = 1, 2, \cdots, m, \ \ \sum_{k=1}^{m} \omega_k = 1$$

(M-4.5) $\omega_k^{+} = \max \omega_k$

$$\text{s. t. } 0.5\left(\sum_{k=1}^{m} \omega_k \left(\bar{s}(r_{ik}) - \bar{s}(r_{jk})\right) + 1\right) + \dot{\varepsilon}_{ij}^{-} \geqslant t_{ij}, \quad i=1, 2, \cdots, n-1; \ \ j=i+1, \cdots, n$$

$$0.5\left(\sum_{k=1}^{m} \omega_k (\bar{s}(r_{ik}) - \bar{s}(r_{jk})) + 1\right) - \dot{\varepsilon}_{ij}^{+} \leqslant 1 - f_{ij}, \quad i=1, 2, \cdots, n-1; \ \ j=i+1, \cdots, n$$

$$\omega = (\omega_1, \omega_2, \cdots, \omega_m)^{\mathrm{T}} \in \Lambda, \quad \omega_k \in [0,1], \ \ k = 1, 2, \cdots, m, \ \ \sum_{k=1}^{m} \omega_k = 1$$

Solving the models (M-4.4) and (M-4.5), we can get the set of the weight vectors of attributes:

$$\Upsilon_2 = \left\{ \omega = (\omega_1, \omega_2, \cdots, \omega_m)^{\mathrm{T}} \,|\, \omega_k \in [\omega_k^{\mathrm{L}}, \omega_k^{\mathrm{U}}], \ \omega_k \in [0,1], \ k=1, 2, \cdots, m, \ \sum_{k=1}^{m} \omega_k = 1 \right\} \tag{4.34}$$

2. Linear Programming Models Based on Multiplicative Transitivity

Similar to the subsection 4.8.2'1, we can utilize the overall scores of all the alternatives Y_i $(i = 1, 2, \cdots, n)$ to construct the multiplicative consistent fuzzy preference relation

$\dot{Q} = (\dot{q}_{ij})_{n\times n}$, where

$$\dot{q}_{ij} = \frac{\bar{s}(r_i)}{\bar{s}(r_i) + \bar{s}(r_j)}, \quad i, j = 1, 2, \cdots, n \tag{4.35}$$

(1) If the multiplicative consistent fuzzy preference relation $\dot{Q}$ is the same as the decision maker's intuitionistic preference relation Q, then the following inequality holds:

$$t_{ij} \leqslant \frac{\bar{s}(r_i)}{\bar{s}(r_i) + \bar{s}(r_j)} \leqslant 1 - f_{ij}, \quad i = 1, 2, \cdots, n-1;\ j = i+1, \cdots, n \tag{4.36}$$

which is equivalent to the following form:

$$t_{ij}\left(\bar{s}(r_i) + \bar{s}(r_j)\right) \leqslant \bar{s}(r_i) \leqslant (1 - f_{ij})\left(\bar{s}(r_i) + \bar{s}(r_j)\right), \quad i = 1, 2, \cdots, n-1;\ j = i+1, \cdots, n \tag{4.37}$$

i.e.,

$$t_{ij}\left(\sum_{k=1}^{m} \omega_k \left(\bar{s}(r_{ik}) + \bar{s}(r_{jk})\right)\right) \leqslant \sum_{k=1}^{m} \omega_k \bar{s}(r_{ik}) \leqslant (1 - f_{ij})\left(\sum_{k=1}^{m} \omega_k \left(\bar{s}(r_{ik}) + \bar{s}(r_{jk})\right)\right), \quad i = 1, 2, \cdots, n-1;\ j = i+1, \cdots, n \tag{4.38}$$

Generally, the attribute weights ω_k $(k = 1, 2, \cdots, m)$ satisfying the condition (4.38) should belong to an interval. Thus, based on the inequality (4.38) and the known weight information Λ, Xu (2007c) establishes the following linear programming model:

(M-4.6) $\omega_k^- = \min \omega_k$

$$\text{s. t.} \ \sum_{k=1}^{m} \omega_k \left((1 - \mu_{ij})\bar{s}(r_{ik}) - \bar{s}(r_{jk})\right) \geqslant 0, \quad i = 1, 2, \cdots, n-1;\ j = i+1, \cdots, n$$

$$\sum_{k=1}^{m} \omega_k \left(f_{ij}\bar{s}(r_{ik}) - (1 - f_{ij})\bar{s}(r_{jk})\right) \leqslant 0, \quad i = 1, 2, \cdots, n-1;\ j = i+1, \cdots, n$$

$$\omega = (\omega_1, \omega_2, \cdots, \omega_m)^T \in \Lambda, \ \ \omega_i \in [0, 1], \ \ k = 1, 2, \cdots, m, \ \ \sum_{k=1}^{m} \omega_k = 1$$

and

(M-4.7) $\omega_k^+ = \max \omega_k$

$$\text{s. t.} \ \sum_{k=1}^{m} \omega_k \left((1 - t_{ij})\bar{s}(r_{ik}) - \bar{s}(r_{jk})\right) \geqslant 0, \quad i = 1, 2, \cdots, n-1;\ j = i+1, \cdots, n$$

$$\sum_{k=1}^{m} \omega_k \left(f_{ij}\bar{s}(r_{ik} - (1 - f_{ij})\bar{s}(r_{jk})\right) \leqslant 0, \quad i = 1, 2, \cdots, n-1;\ j = i+1, \cdots, n$$

$$\omega = (\omega_1, \omega_2, \cdots, \omega_m)^{\mathrm{T}} \in \Lambda,\ \omega_k \in [0,1],\ \ k = 1,2,\cdots,m,\ \ \sum_{k=1}^{m} \omega_k = 1$$

Solving the models (M-4.6) and (M-4.7), we can get the set of the weight vector of attributes:

$$\Upsilon_3 = \left\{ \omega = (\omega_1, \omega_2, \cdots, \omega_m)^{\mathrm{T}} \,|\, \omega_k \in [\omega_k^{\mathrm{L}}, \omega_k^{\mathrm{U}}],\ \omega_k \in [0,1],\ k=1,2,\cdots,m,\ \sum_{k=1}^{m} \omega_k = 1 \right\} \tag{4.39}$$

(2) If there is a difference between the multiplicative consistent fuzzy preference relation $\dot{Q}$ and the decision maker's intuitionistic preference relation Q, i.e., the inequality (4.38) does not hold, then we cannot use the models (M-4.6) and (M-4.7) to derive the attribute weights. To resolve this issue, Xu (2007c) extends these two models by introducing the deviation variables ε_{ij}^- and ε_{ij}^+ to relax the inequality (4.38):

$$t_{ij}\left(\sum_{k=1}^{m} \omega_k \left(\bar{s}(r_{ik}) + \bar{s}(r_{jk})\right)\right) - \varepsilon_{ij}^- \leqslant \sum_{k=1}^{m} \omega_k \bar{s}(r_{ik}) \leqslant (1 - f_{ij})\left(\sum_{k=1}^{m} \omega_k \left(\bar{s}(r_{ik}) + \bar{s}(r_{jk})\right)\right) + \varepsilon_{ij}^+$$
$$i = 1,2,\cdots,n-1;\ j = i+1,\cdots,n \tag{4.40}$$

where both ε_{ij}^- and ε_{ij}^+ are non-negative real numbers. In particular, if both ε_{ij}^- and ε_{ij}^+ are zero, then the inequality (4.40) reduces to the inequality (4.38).

Note again that the smaller the values of the deviation variables ε_{ij}^- and ε_{ij}^+, the closer the multiplicative consistent fuzzy preference relation $\dot{Q}$ to the decision maker's intuitionistic preference relation Q. Consequently, we can establish the following optimization model (Xu, 2007c):

(M-4.8) $\quad \varphi_2^* = \min \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} (\varepsilon_{ij}^- + \varepsilon_{ij}^+)$

$$\text{s. t. } \sum_{k=1}^{m} \omega_k \left((1 - t_{ij})\bar{s}(r_{ik}) - \bar{s}(r_{jk})\right) + \varepsilon_{ij}^- \geqslant 0, \quad i = 1,2,\cdots,n-1;\ j = i+1,\cdots,n$$
$$\sum_{k=1}^{m} \omega_k \left(f_{ij}\bar{s}(r_{ik}) - (1 - f_{ij})\bar{s}(r_{jk})\right) - \varepsilon_{ij}^+ \leqslant 0, \quad i = 1,2,\cdots,n-1;\ j = i+1,\cdots,n$$
$$\omega = (\omega_1, \omega_2, \cdots, \omega_m)^{\mathrm{T}} \in \Lambda,\ \ \omega_k \in [0,1],\ \ k = 1,2,\cdots,m,\ \ \sum_{k=1}^{m} \omega_k = 1$$
$$\varepsilon_{ij}^-,\ \varepsilon_{ij}^+ \geqslant 0,\ \ i = 1,2,\cdots,n-1;\ j = i+1,\cdots,n$$

Solving the model, we can get the optimal deviation variables $\dot{\varepsilon}_{ij}^-$ and $\dot{\varepsilon}_{ij}^+$, $i = 1,2,\cdots,n-1;\ j = i+1,\cdots,n$.

By the model (M-4.8), we can establish the following theorem:

Theorem 4.8.2 (Xu, 2007c) The multiplicative consistent fuzzy preference relation $\dot{Q}$ is the same as the intuitionistic preference relation Q if and only if $\varphi_2^* = 0$.

If $\varphi_2^* \neq 0$, then based on the optimal deviation variables $\dot{\varepsilon}_{ij}^-$ and $\dot{\varepsilon}_{ij}^+$, $i = 1, 2, \cdots, n-1$; $j = i+1, \cdots, n$, and similar to the models (M-4.6) and (M-4.7), we can further establish the following two linear programming models (Xu, 2007c):

(M-4.9) $\omega_k^- = \min \omega_k$

$$\text{s. t.} \sum_{k=1}^m \omega_k \left((1-t_{ij})\bar{s}(r_{ik})-\bar{s}(r_{jk})\right)+\dot{\varepsilon}_{ij}^- \geqslant 0, \quad i=1,2,\cdots,n-1;\ j=i+1,\cdots,n$$

$$\sum_{k=1}^m \omega_k (f_{ij}\bar{s}(r_{ik})-(1-f_{ij})\bar{s}(r_{jk}))-\dot{\varepsilon}_{ij}^+ \leqslant 0, \quad i=1,2,\cdots,n-1;\ j=i+1,\cdots,n$$

$$\omega = (\omega_1, \omega_2, \cdots, \omega_m)^{\mathrm{T}} \in \Lambda, \quad \omega_k \in [0,1],\ \ k = 1, 2, \cdots, m,\ \ \sum_{k=1}^m \omega_k = 1$$

and

(M-4.10) $\omega_k^+ = \max \omega_k$

$$\text{s. t.} \sum_{k=1}^m \omega_k \left((1-t_{ij})\bar{s}(r_{ik})-\bar{s}(r_{jk})\right)+\dot{\varepsilon}_{ij}^- \geqslant 0, \quad i=1,2,\cdots,n-1;\ j=i+1,\cdots,n$$

$$\sum_{k=1}^m \omega_k \left(f_{ij}\bar{s}(r_{ik})-(1-f_{ij})\bar{s}(r_{jk})\right)-\dot{\varepsilon}_{ij}^+ \leqslant 0, \quad i=1,2,\cdots,n-1;\ j=i+1,\cdots,n$$

$$\omega = (\omega_1, \omega_2, \cdots, \omega_m)^{\mathrm{T}} \in \Lambda,\ \ \omega_k \in [0,1],\ \ k = 1, 2, \cdots, m,\ \ \sum_{k=1}^m \omega_k = 1$$

Solving the models (M-4.9) and (M-4.10), we can get the set of the weight vector of attributes:

$$\Upsilon_4 = \left\{\omega=(\omega_1,\omega_2,\cdots,\omega_m)^{\mathrm{T}} \,|\, \omega_k \in [\omega_k^{\mathrm{L}}, \omega_k^{\mathrm{U}}],\ \omega_k \in [0,1],\ k=1,2,\cdots,m,\ \sum_{k=1}^m \omega_k = 1\right\} \tag{4.41}$$

4.8.3 Intuitionistic Fuzzy Decision Making Based on Linear Programming Models

In Subsection 4.8.2'2, we have utilized the known objective decision information and the decision maker's subjective preferences to establish the corresponding linear programming models so as to derive the weight intervals of attributes. Without loss of generality, we can uniformly denote them as:

$$\bar{\Upsilon} = \left\{\omega=(\omega_1,\omega_2,\cdots,\omega_m)^{\mathrm{T}} \,|\, \omega_k \in [\omega_k^{\mathrm{L}}, \omega_k^{\mathrm{U}}],\ \omega_k \in [0,1],\ k=1,2,\cdots,m,\ \sum_{k=1}^m \omega_k = 1\right\} \tag{4.42}$$

In this section, we utilize, based on the intuitionistic fuzzy decision matrix and the derived weight intervals of attributes, the decision models above to derive the weight vector of attributes, and then determine the best alternative, which involves the following steps (Xu, 2007c):

Step 1 Based on the intuitionistic fuzzy decision matrix and the weight intervals of attributes derived in Subsection 4.8.2'1, we establish the following linear programming models:

$$\varphi_3^* = \max \sum_{i=1}^{n} \sum_{j=1}^{m} (1 - \nu_{ij} - \mu_{ij})\omega_j$$

$$\text{s. t.} \quad \omega = (\omega_1, \omega_2, \cdots, \omega_m)^{\mathrm{T}} \in \bar{\Upsilon}$$

Solving the above model, we can determine the optimal weight vector of attributes $\omega^* = (\omega_1^*, \omega_2^*, \cdots, \omega_m^*)^{\mathrm{T}}$.

Step 2 Calculate the overall attribute values of all the alternatives Y_i $(i = 1, 2, \cdots, n)$:

$$z_i(\omega^*) = [z_i^-(\omega^*),\ z_i^+(\omega^*)], \quad i = 1, 2, \cdots, n$$

where

$$z_i^-(\omega^*) = \sum_{j=1}^{m} \omega_j^* \mu_{ij}, \quad z_i^+(\omega^*) = \sum_{j=1}^{m} \omega_j^* (1 - \nu_{ij}), \quad i = 1, 2, \cdots, n \tag{4.43}$$

Step 3 Utilize the possibility degree formula for the comparison between interval numbers (Xu and Da, 2003a; 2003c):

$$p(z_i(\omega^*) \geqslant z_j(\omega^*)) = \max\left\{1 - \max\left(\frac{z_j^+(\omega^*) - z_i^-(\omega^*)}{z_i^+(\omega^*) - z_i^-(\omega^*) + z_j^+(\omega^*) - z_j^-(\omega^*)},\ 0\right),\ 0\right\} \tag{4.44}$$

to get the possibility degrees of comparing each pair of the overall attribute values $z_i(\omega^*)$ $(i = 1, 2, \cdots, n)$:

$$b_{ij} = p(z_i(\omega^*) \geqslant z_j(\omega^*)), \quad i, j = 1, 2, \cdots, n$$

and then establish the possibility degree matrix $B = (b_{ij})_{n \times n}$, where

$$b_{ij} \geqslant 0, \quad b_{ij} + b_{ji} = 1, \quad b_{ii} = 0.5, \quad i, j = 1, 2, \cdots, n$$

Thus, the possibility degree matrix $B = (b_{ij})_{n \times n}$ is a fuzzy preference relation.

Step 4 Utilize the formula for the priority of fuzzy preference relation (Xu, 2001a):

$$w_i = \frac{1}{n(n-1)} \left(\sum_{j=1}^{n} b_{ij} + \frac{n}{2} - 1 \right), \quad i = 1, 2, \cdots, n \tag{4.45}$$

to derive the priority vector $w = (w_1, w_2, \cdots, w_n)^{\mathrm{T}}$ of P.

Step 5 Rank and select the alternatives Y_i $(i = 1, 2, \cdots, n)$ according to the elements of w.

Example 4.8.1 (Xu, 2007c) A family has a plan to buy a refrigerator. There are five kinds of brand refrigerators Y_i $(i = 1, 2, \cdots, 5)$ to be chosen. The evaluation indices are: ① G_1 : Safety; ② G_2 : Refrigeration performance; ③ G_3 : Structure; ④ G_4 : Reliability; ⑤ G_5 : Economics; and ⑥ G_6 : Appearance. By the statistical analysis, the characteristics of the alternatives Y_i $(i = 1, 2, \cdots, 5)$ with respect to the attribute G_j $(j = 1, 2, \cdots, 6)$ are expressed as the IFNs $r_{ij} = (\mu_{ij}, \nu_{ij})$ $(i = 1, 2, \cdots, 5;\ j = 1, 2, \cdots, 6)$. All these IFNs are contained in the intuitionistic fuzzy decision matrix R, as listed in Table 4.1:

Table 4.1 Intuitionistic fuzzy decision matrix R (Xu, 2007c)

	G_1	G_2	G_3	G_4	G_5	G_6
Y_1	(0.3, 0.5)	(0.6, 0.3)	(0.6, 0.4)	(0.8, 0.2)	(0.4, 0.5)	(0.5, 0.3)
Y_2	(0.7, 0.3)	(0.5, 0.3)	(0.7, 0.2)	(0.7, 0.1)	(0.5, 0.4)	(0.4, 0.1)
Y_3	(0.4, 0.3)	(0.7, 0.2)	(0.5, 0.4)	(0.6, 0.3)	(0.4, 0.3)	(0.3, 0.2)
Y_4	(0.6, 0.2)	(0.5, 0.4)	(0.7, 0.2)	(0.3, 0.2)	(0.5, 0.4)	(0.7, 0.3)
Y_5	(0.5, 0.3)	(0.3, 0.5)	(0.6, 0.3)	(0.6, 0.2)	(0.6, 0.2)	(0.5, 0.2)

The known weight information of attributes is as follows:

$$\Lambda = \{\omega_1 \leqslant 0.3\ ,\quad 0.2 \leqslant \omega_3 \leqslant 0.5,\quad \omega_2 \leqslant 0.2,\quad \omega_3 - \omega_2 \geqslant \omega_5 - \omega_4,\quad 0.1 \leqslant \omega_5 \leqslant 0.4,$$
$$\omega_4 \leqslant \omega_1,\quad \omega_4 \leqslant 0.1,\quad \omega_6 \geqslant 0.2\}$$

and the evaluation information on comparing each pair of five kinds of brand refrigerators Y_i $(i = 1, 2, \cdots, 5)$ is given in the intuitionistic preference relation Q:

$$Q = \begin{bmatrix} (0.5, 0.5) & (0.2, 0.7) & (0.4, 0.6) & (0.2, 0.3) & (0.3, 0.4) \\ (0.7, 0.2) & (0.5, 0.5) & (0.6, 0.2) & (0.5, 0.4) & (0.6, 0.4) \\ (0.6, 0.4) & (0.2, 0.6) & (0.5, 0.5) & (0.3, 0.6) & (0.4, 0.6) \\ (0.3, 0.2) & (0.4, 0.5) & (0.6, 0.3) & (0.5, 0.5) & (0.6, 0.3) \\ (0.4, 0.3) & (0.4, 0.6) & (0.6, 0.4) & (0.3, 0.6) & (0.5, 0.5) \end{bmatrix}$$

Since all the evaluation indices G_j $(j = 1, 2, \cdots, 5)$ are of benefit type, we do not need to normalize the evaluation values r_{ij} $(i = 1, 2, \cdots, 5;\ j = 1, 2, \cdots, 6)$.

We first calculate the score matrix S of the intuitionistic fuzzy decision matrix R, as shown in Table 4.2:

Table 4.2 Score matrix S (Xu, 2007c)

	G_1	G_2	G_3	G_4	G_5	G_6
Y_1	−0.2	0.3	0.2	0.6	−0.1	0.2
Y_2	0.4	0.2	0.5	0.6	0.1	0.3
Y_3	0.1	0.5	0.1	0.3	0.1	0.1
Y_4	0.4	0.1	0.5	0.1	0.1	0.4
Y_5	0.2	−0.2	0.3	0.4	0.4	0.3

By Eq.(4.29), we normalize the score matrix S into $\bar{S}$ (Table 4.3):

Table 4.3 Normalized score matrix $\bar{S}$ (Xu, 2007c)

	G_1	G_2	G_3	G_4	G_5	G_6
Y_1	0	0.714	0.25	1	0	0.333
Y_2	1	0.571	1	1	0.4	0.667
Y_3	0.5	1	0	0.4	0.4	0
Y_4	1	0.429	1	0	0.4	1
Y_5	0.667	0	0.5	0.6	1	0.667

By the model (M-4.3), we get $\varphi_1^* = 0.0907$, and the optimal deviation variables:

$$\dot{\varepsilon}_{12}^- = \dot{\varepsilon}_{12}^+ = 0, \quad \dot{\varepsilon}_{13}^- = 0, \quad \dot{\varepsilon}_{13}^+ = 0.0739, \quad \dot{\varepsilon}_{14}^- = \dot{\varepsilon}_{14}^+ = 0$$

$$\dot{\varepsilon}_{15}^- = \dot{\varepsilon}_{15}^+ = 0, \quad \dot{\varepsilon}_{23}^- = \dot{\varepsilon}_{23}^+ = 0, \quad \dot{\varepsilon}_{24}^- = \dot{\varepsilon}_{24}^+ = 0, \quad \dot{\varepsilon}_{25}^- = 0$$

$$\dot{\varepsilon}_{25}^+ = 0.0168, \quad \dot{\varepsilon}_{34}^- = \dot{\varepsilon}_{34}^+ = 0, \quad \dot{\varepsilon}_{35}^- = \dot{\varepsilon}_{35}^+ = 0, \quad \dot{\varepsilon}_{45}^- = \dot{\varepsilon}_{45}^+ = 0$$

Since $\varphi_1^* \neq 0$, then based on the above optimal deviation variables $\dot{\varepsilon}_{ij}^-$ and $\dot{\varepsilon}_{ij}^+$, $i = 1, 2, \cdots, 4$; $j = i+1, \cdots, 5$, we further solve the models (M-4.4) and (M-4.5), and get

$$\omega_1^{\mathrm{L}} = 0.1197, \quad \omega_1^{\mathrm{U}} = 0.2440, \quad \omega_2^{\mathrm{L}} = 0.1605, \quad \omega_2^{\mathrm{U}} = 0.1998$$

$$\omega_3^{\mathrm{L}} = 0.2000, \quad \omega_3^{\mathrm{U}} = 0.2788, \quad \omega_4^{\mathrm{L}} = 0.0614, \quad \omega_4^{\mathrm{U}} = 0.0899$$

$$\omega_5^{\mathrm{L}} = 0.1062, \quad \omega_5^{\mathrm{U}} = 0.1474, \quad \omega_6^{\mathrm{L}} = 0.2000, \quad \omega_6^{\mathrm{U}} = 0.2472$$

i.e.,

$$\omega_1 \in [0.1197,\ 0.2440], \quad \omega_2 \in [0.1605,\ 0.1998]$$

$$\omega_3 \in [0.2000,\ 0.2788], \quad \omega_4 \in [0.0614,\ 0.0899]$$

$$\omega_5 \in [0.1062,\ 0.1474], \quad \omega_6 \in [0.2000,\ 0.2472]$$

Based on the intuitionistic fuzzy decision matrix R and the weight intervals of attributes above, we solve the model (M-4.11), and get the optimal weight vector of attributes:

$$\omega^* = (0.1962, 0.1605, 0.2000, 0.0899, 0.1062, 0.2472)^{\mathrm{T}}$$

Then according to Eq.(4.43), we calculate the overall evaluation values:

$$z_1(\omega^*) = [0.5132, 0.6285], \quad z_2(\omega^*) = [0.5725, 0.7768]$$

$$z_3(\omega^*) = [0.4614, 0.7208], \quad z_4(\omega^*) = [0.5911, 0.7219]$$

$$z_5(\omega^*) = [0.5075, 0.7122]$$

After that, we utilize Eq.(4.44) to get the possibility degrees of comparing each pair of the overall attribute values $z_i(\omega^*)$ $(i = 1, 2, \cdots, 5)$, and then establish the possibility degree matrix:

$$B = \begin{bmatrix} 0.5000 & 0.1752 & 0.4460 & 0.1520 & 0.3781 \\ 0.8248 & 0.5000 & 0.6802 & 0.5422 & 0.6584 \\ 0.5540 & 0.3198 & 0.5000 & 0.3324 & 0.4596 \\ 0.8480 & 0.4458 & 0.6676 & 0.5000 & 0.6390 \\ 0.6219 & 0.3416 & 0.5404 & 0.3610 & 0.5000 \end{bmatrix}$$

By using Eq.(4.45), we derive the priority vector of the fuzzy preference relation B:

$$w = (0.1576,\ 0.2359,\ 0.1833,\ 0.2300,\ 0.1932)^{\mathrm{T}}$$

by which we rank the five kinds of brand refrigerators Y_i $(i = 1, 2, \cdots, 5)$:

$$Y_2 \succ Y_4 \succ Y_5 \succ Y_3 \succ Y_1$$

Therefore, the best brand refrigerator is Y_2.

If we utilize the model (M-4.8) to derive the weights of attributes, then we get $\varphi_2^* = 3.117$, and the optimal deviation variables:

$$\dot{\varepsilon}_{12}^- = 0.481, \quad \dot{\varepsilon}_{12}^+ = 0.033, \quad \dot{\varepsilon}_{13}^- = 0.010, \quad \dot{\varepsilon}_{13}^+ = 0.099$$

$$\dot{\varepsilon}_{14}^- = 0.579, \quad \dot{\varepsilon}_{14}^+ = 0, \quad \dot{\varepsilon}_{15}^- = 0.242, \quad \dot{\varepsilon}_{15}^+ = 0$$

$$\dot{\varepsilon}_{23}^- = 0.016, \quad \dot{\varepsilon}_{23}^+ = 0, \quad \dot{\varepsilon}_{24}^- = 0.333, \quad \dot{\varepsilon}_{24}^+ = 0$$

$$\dot{\varepsilon}_{25}^- = 0.196, \quad \dot{\varepsilon}_{25}^+ = 0.010, \quad \dot{\varepsilon}_{34}^- = 0.495, \quad \dot{\varepsilon}_{34}^+ = 0$$

$$\dot{\varepsilon}_{35}^- = 0.313, \quad \dot{\varepsilon}_{35}^+ = 0, \quad \dot{\varepsilon}_{45}^- = 0.220, \quad \dot{\varepsilon}_{45}^+ = 0$$

Considering $\varphi_2^* \neq 0$, we further solve the models (M-4.9) and (M-4.10), and get

$$\omega_1 \in [0.0994,\ 0.1013], \quad \omega_2 \in [0.1996,\ 0.2000]$$

$$\omega_3 \in [0.2978,\ 0.3003], \quad \omega_4 \in [0.0993,\ 0.1000]$$

$$\omega_5 \in [0.1000,\ 0.1008], \quad \omega_6 \in [0.2000,\ 0.2018]$$

Solving the model (M-4.11), we get the optimal weight vector of attributes:

$$\omega^* = (0.1013, 0.1996, 0.2978, 0.1000, 0.0995, 0.2018)^{\mathrm{T}}$$

Then by Eq.(4.43), we calculate the overall attribute values:

$$z_1(\omega^*) = [0.5495,\ 0.6401], \quad z_2(\omega^*) = [0.5796,\ 0.7802]$$

$$z_3(\omega^*) = [0.4895,\ 0.7104], \quad z_4(\omega^*) = [0.5901,\ 0.7200]$$

$$z_5(\omega^*) = [0.5098,\ 0.7002]$$

Using Eq.(4.44), we get the possibility degree matrix:

$$B = \begin{bmatrix} 0.5000 & 0.2078 & 0.4835 & 0.2268 & 0.4637 \\ 0.7922 & 0.5000 & 0.6897 & 0.5752 & 0.6916 \\ 0.5165 & 0.3103 & 0.5000 & 0.3429 & 0.4877 \\ 0.7732 & 0.4248 & 0.6571 & 0.5000 & 0.6563 \\ 0.5363 & 0.3084 & 0.5123 & 0.3437 & 0.5000 \end{bmatrix}$$

Then we utilize Eq.(4.45) to derive the priority vector of the fuzzy preference relation B:

$$w = (0.1691,\ 0.2374,\ 0.1829,\ 0.2256,\ 0.1850)^{\mathrm{T}}$$

by which we rank the five kinds of brand refrigerators Y_i $(i = 1, 2, \cdots, 5)$:

$$Y_2 \succ Y_4 \succ Y_5 \succ Y_3 \succ Y_1$$

Hence, the best brand refrigerator is Y_2.

From the above analysis, it can be seen that the calculation process by using the two approaches introduced in this section are very similar, and the ranking results derived by these two approaches are also the same.

4.9 Multi-Attribute Decision Making Based on Various Intuitionistic Preference Structures

Dai et al (2007) investigate intuitionistic fuzzy multi-attribute decision making problems, where the attribute values are given as real numbers and the decision makers have preference information on attributes. The preference information is expressed in the form of intuitionistic preference relation or incomplete intuitionistic preference relation. They establish two models for multi-attribute decision making based on intuitionistic preference relation and incomplete intuitionistic preference relation respectively, and a model for multi-attribute group decision making based on intuitionistic preference relations and incomplete intuitionistic preference relations. Furthermore, they develop an approach to multi-attribute decision making based on various intuitionistic preference structures. The method does not need to unify different preference structures and can derive the optimal weight vector from the established model directly, which can avoid losing or distorting the original preference information in the process of unifying the structures. They also apply the developed approach to the evaluation of the competence of enterprise technology innovation in Jiangsu province.

4.9.1 Multi-Attribute Decision Making Models Based on Intuitionistic Preference Relations

For a multi-attribute decision making problem, let Y, G, R and ω be as defined in Subsection 4.8.2. Let $D = (d_{ij})_{n\times m}$ be the decision matrix , where d_{ij} is the attribute value, which is expressed in real number, given by the decision maker for the alternative Y_i with respect to the attribute G_j. If all the attributes G_j $(j = 1, 2, \cdots, m)$ are of the same type, then the attribute values do not need normalization. When there are benefit and cost attributes, we may transform the attribute values of cost type into the attribute values of benefit type. Let I_1 and I_2 be the subscript sets of the benefit attributes and cost attributes respectively. In order to eliminate the impact of the different physical dimension on the decision results, we can normalize the decision matrix D by the following formula:

$$\bar{d}_{ij} = \frac{d_{ij}}{\max\limits_i(d_{ij})}, \quad j \in I_1, \ \ i = 1, 2, \cdots, n \tag{4.46}$$

$$\bar{d}_{ij} = \frac{\min\limits_i(d_{ij})}{d_{ij}}, \quad j \in I_2, \ \ i = 1, 2, \cdots, n \tag{4.47}$$

and get the normalized decision matrix $\bar{D} = (\bar{d}_{ij})_{n\times m}$. Based on the decision information in $\bar{D}$, the overall attribute values of all the alternatives can be defined as follows:

$$z_i(\omega) = \sum_{j=1}^{m} \bar{d}_{ij}\omega_j, \quad i = 1, 2, \cdots, n \tag{4.48}$$

Note that the greater the value $z_i(\omega)$, the better the alternative Y_i.

Furthermore, in order to exert the decision maker's initiative, the decision maker is asked to compare each pair of the attributes G_j $(j = 1, 2, \cdots, m)$, and construct the intuitionistic preference relation $Q = (q_{ij})_{m\times m}$, where $q_{ij} = (\mu_{ij}, \nu_{ij})$, $0 \leqslant \mu_{ij} + \nu_{ij} \leqslant 1$, $\mu_{ji} = \nu_{ij}$, $\nu_{ji} = \mu_{ij}$, $\mu_{ii} = \nu_{ii} = 0.5$, $i, j = 1, 2, \cdots, m$.

Definition 4.8.4 gives the concept of multiplicative consistent intuitionistic preference relation, i.e., when the inequality (4.26) holds, Q is a multiplicative consistent intuitionistic preference relation. However, in general, the inequality (4.26) does not hold, i.e., Q is not generally a multiplicative consistent intuitionistic preference relation. In this case, we need to extend the inequality (4.26) by introducing the deviation variables ε_{ij}^- and ε_{ij}^+ to relax the inequality (4.26):

$$\mu_{ij}(\omega_i + \omega_j) - \varepsilon_{ij}^- \leqslant \omega_i \leqslant (1 - \nu_{ij})(\omega_i + \omega_j) + \varepsilon_{ij}^+, \quad i, j = 1, 2, \cdots, m \tag{4.49}$$

where both ε_{ij}^- and ε_{ij}^+ are non-negative real numbers. If both ε_{ij}^- and ε_{ij}^+ are zero, then the inequality (4.49) reduces to the inequality (4.26).

Note that the smaller the values of the deviation variables ε_{ij}^- and ε_{ij}^+, the higher the consistency degree of the decision maker's intuitionistic preference relation Q. In order to determine the weight vector of attributes, we establish, by Eqs.(4.48) and (4.49), the following multi-objective optimization model (Dai et al. 2007):

(M-4.11) $$\max\ (z_1(\omega),\ z_2(\omega), \cdots, z_m(\omega))$$

$$\min \sum_{i,j=1}^{m} (\varepsilon_{ij}^- + \varepsilon_{ij}^+)$$

$$\text{s. t. } \mu_{ij}(\omega_i+\omega_j)-\varepsilon_{ij}^- \leqslant \omega_i \leqslant (1-\nu_{ij})(\omega_i+\omega_j)+\varepsilon_{ij}^+, \quad i,j=1,2,\cdots,m$$
$$\omega_j \in [0,1], \quad j=1,2,\cdots,m, \ \sum_{j=1}^m \omega_j = 1$$
$$\varepsilon_{ij}^-, \ \varepsilon_{ij}^+ \geqslant 0, \quad i,j=1,2,\cdots,m$$

By the linear equal-weighted summation method (Igizio, 1976), the model (M-4.11) can be transformed into the following single-objective programming model (Dai et al. 2007):

(M-4.12)
$$\max \left(\sum_{i=1}^m \varepsilon_i - \sum_{i,j=1}^n (\varepsilon_{ij}^- + \varepsilon_{ij}^+) \right)$$
$$\text{s. t. } z_i(\omega) \geqslant \varepsilon_i, \quad i=1,2,\cdots,n$$
$$\mu_{ij}(\omega_i+\omega_j)-\varepsilon_{ij}^- \leqslant \omega_i \leqslant (1-\nu_{ij})(\omega_i+\omega_j)+\varepsilon_{ij}^+, \quad i,j=1,2,\cdots,m$$
$$\omega_j \in [0,1], \quad j=1,2,\cdots,m, \ \sum_{j=1}^m \omega_j = 1$$
$$\varepsilon_{ij}^-, \ \varepsilon_{ij}^+ \geqslant 0, \quad i,j=1,2,\cdots,m$$

Solving the model, we can get the optimal weight vector $\omega=(\omega_1,\omega_2,\cdots,\omega_m)^{\mathrm{T}}$, and the deviation variables ε_{ij}^- and ε_{ij}^+, $i,j=1,2,\cdots,m$. Then by Eq.(4.48), we can get the overall attribute values $z_i(\omega)$ $(i=1,2,\cdots,n)$, by which the alternatives can be ranked and selected.

4.9.2 Multi-Attribute Decision Making Models Based on Incomplete Intuitionistic Preference Relations

Based on Definition 4.8.3, in what follows, we introduce the concept of consistent incomplete intuitionistic preference relation:

Definition 4.9.1 (Dai et al., 2007) Let $Q=(q_{ij})_{n\times n}$ be an incomplete intuitionistic preference relation, $q_{ij}=(\mu_{ij},\nu_{ij})\in \varDelta$, and let $O=\{(i,j)\,|\,q_{ij}=(\mu_{ij},\nu_{ij})\in \varDelta\}$. If there exists a vector $\omega=(\omega_1,\omega_2,\cdots,\omega_m)^{\mathrm{T}}$ such that

$$\mu_{ij}(\omega_i+\omega_j) \leqslant \omega_i \leqslant (1-\nu_{ij})(\omega_i+\omega_j), \quad (i,j)\in O \tag{4.50}$$

where $\omega_j \in [0,1]$, $j=1,2,\cdots,m$, and $\sum_{j=1}^m \omega_j = 1$, then Q is called a multiplicative consistent incomplete intuitionistic preference relation.

If Q is not a multiplicative consistent incomplete intuitionistic preference relation, then the inequality (4.50) does not hold. Consequently, we extend the inequality (4.50) to the following form:

$$\mu_{ij}(\omega_i+\omega_j)-\varepsilon_{ij}^- \leqslant \omega_i \leqslant (1-\nu_{ij})(\omega_i+\omega_j)+\varepsilon_{ij}^+, \quad (i,j)\in \varDelta \tag{4.51}$$

where both ε_{ij}^- and ε_{ij}^+ are deviation variables, taking non-negative real values.

In order to determine the weight vector of attributes, we can establish, by Eq.(4.51) and similar to the model (M-4.11), the following single-objective programming model (Dai et al. 2007):

(M-4.13)
$$\max \left(\sum_{i=1}^{n} \varepsilon_i - \sum_{(i,j)\in O} (\varepsilon_{ij}^- + \varepsilon_{ij}^+)\right)$$

$$\begin{aligned} \text{s. t. } & z_i(\omega) \geqslant \varepsilon_i, \quad i = 1, 2, \cdots, n \\ & \mu_{ij}(\omega_i + \omega_j) - \varepsilon_{ij}^- \leqslant \omega_i \leqslant (1 - \nu_{ij})(\omega_i + \omega_j) + \varepsilon_{ij}^+, \quad (i, j) \in O \\ & \omega_j \in [0, 1], \quad j = 1, 2, \cdots, m, \ \sum_{j=1}^{m} \omega_j = 1 \\ & \varepsilon_{ij}^-, \ \varepsilon_{ij}^+ \geqslant 0, \ \ (i, j) \in O \end{aligned}$$

Solving the model, we can get the optimal weight vector $\omega = (\omega_1, \omega_2, \cdots, \omega_m)^{\mathrm{T}}$, and the deviation variables ε_{ij}^- and ε_{ij}^+, $(i, j) \in$ O. Then by Eq.(4.48), we can get the overall attribute values $z_i(\omega)$ $(i = 1, 2, \cdots, n)$, by which the alternatives can be ranked and selected.

4.9.3 Multi-Attribute Decision Making Models Based on Different Types of Intuitionistic Preference Relations

In the above, we have discussed situations where a single decision maker provides his/her preferences over the attributes by means of intuitionistic preference relation or incomplete intuitionistic preference relation, and established the corresponding multi-attribute decision models. In some large or important decision making problems, it is necessary for multiple decision makers to participate in the process of decision making. Since the decision makers may come from different professional fields and usually have different knowledge backgrounds, they may have some differences in the understanding the assessed objects. Thus, in the process of decision making, the decision makers may use different preference structures to express their preferences over the objects.

Based on the results presented in Subsections 4.9.1 and 4.9.2, we now consider multi-attribute group decision making problems in which the decision makers express their preferences over the attributes by means of intuitionistic preference relations and incomplete intuitionistic preference relations (Dai et al., 2007):

Suppose that there are l decision makers E_k $(k = 1, 2, \cdots, l_1)$, who are asked to compare each pair of m attributes G_j $(j = 1, 2, \cdots, m)$, and then construct intuitionistic preference relations. Let the decision makers E_k $(k = 1, 2, \cdots, l_1)$ construct the intuitionistic preference relations $Q_k = (q_{ij}^{(k)})_{m\times m}$ $(k = 1, 2, \cdots, l_1)$, where $q_{ij}^{(k)} =$

$(\mu_{ij}^{(k)}, \nu_{ij}^{(k)})$, $i,j=1,2,\cdots,m$, and the decision makers E_k $(k=l_1+1,\cdots,l)$ construct the incomplete intuitionistic preference relations $Q_k=(q_{ij}^{(k)})_{m\times m}$ $(k=l_1+1,\cdots,l)$, where $q_{ij}^{(k)}=(\mu_{ij}^{(k)}, v_{ij}^{(k)})\in\Delta_k$, Δ_k is the set of all the known elements in Q_k.

In order to determine the weight vector of attributes, we establish, based on the models (M-4.12) and (M-4.13), the following single-objective programming model (Dai et al. 2007):

$$\textbf{(M-4.14)}\quad \max\left(\sum_{i=1}^{n}\varepsilon_i-\sum_{k=1}^{t_1}\sum_{i,j=1}^{m}(\varepsilon_{ij}^{-(k)}+\varepsilon_{ij}^{+(k)})-\sum_{k=t_1+1}^{t}\sum_{(i,j)\in\Delta_k}(\varepsilon_{ij}^{-(k)}+\varepsilon_{ij}^{+(k)})\right)$$

$$\begin{aligned}\text{s.t. } & z_i(\omega)\geqslant\varepsilon_i,\quad i=1,2,\cdots,n\\ & \mu_{ij}^{(k)}(\omega_i+\omega_j)-\varepsilon_{ij}^{-(k)}\leqslant\omega_i\leqslant(1-\nu_{ij}^{(k)})(\omega_i+\omega_j)+\varepsilon_{ij}^{+(k)},\\ & \qquad\qquad i,j=1,2,\cdots,m;\quad k=1,2,\cdots,l_1\\ & \mu_{ij}^{(k)}(\omega_i+\omega_j)-\varepsilon_{ij}^{-(k)}\leqslant\omega_i\leqslant(1-\nu_{ij}^{(k)})(\omega_i+\omega_j)+\varepsilon_{ij}^{+(k)},\\ & \qquad\qquad (i,j)\in\Delta_k,\ k=l_1+1,\cdots,l\\ & \omega_j\in[0,1],\quad j=1,2,\cdots,m,\ \sum_{j=1}^{m}\omega_j=1\\ & \varepsilon_{ij}^{-(k)},\varepsilon_{ij}^{+(k)}\geqslant 0,\quad i,j=1,2,\cdots,m;\ k=1,2,\cdots,l_1\\ & \varepsilon_{ij}^{-(k)},\ \varepsilon_{ij}^{+(k)}\geqslant 0,\quad (i,j)\in\Delta_k,\ k=l_1+1,\cdots,l\end{aligned}$$

where

$$\Delta_k=\left\{(i,j)\,|\,q_{ij}^{(k)}=(\mu_{ij}^{(k)},\nu_{ij}^{(k)})\in\Delta_k\right\},\quad k=l_1+1,\cdots,l$$

Solving the model, we can get the optimal weight vector $\omega=(\omega_1,\omega_2,\cdots,\omega_m)^{\mathrm{T}}$, and the deviation variables $\varepsilon_{ij}^{-(k)}$, $\varepsilon_{ij}^{+(k)}$ $(i,j=1,2,\cdots,n;\ k=1,2,\cdots,l_1)$, $\varepsilon_{ij}^{-(k)}$, $\varepsilon_{ij}^{+(k)}$ $((i,j)\in\Delta_k,\ k=t_1+1,\cdots,t)$. Then by Eq.(4.48), we can get the overall attribute values $z_i(\omega)$ $(i=1,2,\cdots,n)$, by which the alternatives Y_i $(i=1,2,\cdots,n)$ can be ranked and selected.

Example 4.9.1 (Dai et al., 2007) Technological innovation affects not only an enterprise's survival and development, but also impacts a region as well as the country's economic development. Let us now consider a problem of evaluating the technological innovation ability of six large enterprises Y_i $(i=1,2,\cdots,6)$ in Jiangsu province by means of the following five evaluation indices:

(1) Strategic management innovation (G_1): Primarily refers to the formulation and implementation of an enterprise's technological innovation strategy; the completeness degree of an enterprise's monitoring and evaluation system, etc.

(2) Innovation incentive management (G_2): Mainly refers to the ratio of the average revenue of R & D personnel to the average income of all employees in an enterprise;

the degree of an enterprise's distribution system stimulating innovation, etc.

(3) Innovation system management (G_3): Mainly refers to the number of cooperation organizations of production, learning, research or other forms that an enterprise participates in; the number of the certified laboratories; and the number of bases for post-doctoral researches, etc.

(4) Production strength (G_4): Primarily refers to the number of computers owned by 100 people in an enterprise; the original value of the per capita (production and R & D) equipment; the advanced degree of the key production technologies of the main products, etc.

(5) Marketing strength (G_5): Mainly refers to the role of market research sectors; the proportion of marketing costs account to the total sales revenue, etc.

The decision matrix after assessing the six large enterprises Y_i $(i = 1, 2, \cdots, 6)$ with respect to the above indices G_j $(j = 1, 2, \cdots, 5)$ by using centesimal grade is shown in Table 4.4. Two decision makers E_k $(k = 1, 2)$ provide their preferences over the five indices above, and construct the intuitionistic preference relation Q_1 and the incomplete intuitionistic preference relation Q_2 respectively:

Table 4.4 Decision matrix D (Dai et al., 2007)

	G_1	G_2	G_3	G_4	G_5
Y_1	80	75	90	85	90
Y_2	95	85	70	90	95
Y_3	90	80	75	60	95
Y_4	70	90	95	65	85
Y_5	85	75	90	80	90
Y_6	90	85	80	90	80

$$Q_1 = \begin{bmatrix} (0.5,0.5) & (0.7,0.2) & (0.8,0.2) & (0.6,0.3) & (0.5,0.4) \\ (0.2,0.7) & (0.5,0.5) & (0.6,0.3) & (0.4,0.5) & (0.3,0.6) \\ (0.2,0.8) & (0.3,0.6) & (0.5,0.5) & (0.4,0.6) & (0.3,0.4) \\ (0.3,0.6) & (0.5,0.4) & (0.6,0.4) & (0.5,0.5) & (0.4,0.6) \\ (0.4,0.5) & (0.6,0.3) & (0.4,0.3) & (0.6,0.4) & (0.5,0.5) \end{bmatrix}$$

$$Q_2 = \begin{bmatrix} (0.5,0.5) & (0.6,0.2) & x & (0.5,0.3) & x \\ (0.2,0.6) & (0.5,0.5) & x & x & (0.4,0.5) \\ x & x & (0.5,0.5) & (0.3,0.5) & x \\ (0.3,0.5) & x & (0.5,0.3) & (0.5,0.5) & (0.3,0.6) \\ x & (0.5,0.4) & x & (0.6,0.3) & (0.5,0.5) \end{bmatrix}$$

Considering that all the attributes are benefit attributes, we can utilize Eq.(4.46) to normalize the decision matrix D, and get the normalized decision matrix $\bar{D}$, as shown in Table 4.5.

By the model (M-4.15), we get the optimal weight vector:

$$\omega = (0.3559,\ 0.1525,\ 0.1017,\ 0.1525,\ 0.2374)^{\mathrm{T}}$$

Table 4.5 Normalized decision matrix $\overline{D}$ (Dai et al., 2007)

	G_1	G_2	G_3	G_4	G_5
Y_1	0.842	0.833	0.947	0.944	0.947
Y_2	1.000	0.944	0.737	1.000	1.000
Y_3	0.947	0.889	0.789	0.667	1.000
Y_4	0.737	1.000	1.000	0.722	0.895
Y_5	0.895	0.833	0.947	0.889	0.947
Y_6	0.947	0.944	0.842	1.000	0.842

and the corresponding deviation variables:

$$\varepsilon_1 = 0.8918, \quad \varepsilon_2 = 0.9647, \quad \varepsilon_3 = 0.8919, \quad \varepsilon_4 = 0.8391$$

$$\varepsilon_5 = 0.9023, \quad \varepsilon_6 = 0.9190, \quad \varepsilon_{12}^{-(1)} = \varepsilon_{12}^{+(1)} = \varepsilon_{21}^{-(1)} = \varepsilon_{21}^{+(1)} = 0$$

$$\varepsilon_{13}^{-(1)} = \varepsilon_{31}^{-(1)} = 0.0102, \quad \varepsilon_{13}^{+(1)} = \varepsilon_{31}^{+(1)} = 0$$

$$\varepsilon_{14}^{-(1)} = \varepsilon_{14}^{+(1)} = \varepsilon_{41}^{-(1)} = \varepsilon_{41}^{+(1)} = 0, \quad \varepsilon_{15}^{-(1)} = \varepsilon_{15}^{+(1)} = \varepsilon_{51}^{-(1)} = \varepsilon_{51}^{+(1)} = 0$$

$$\varepsilon_{23}^{-(1)} = \varepsilon_{23}^{+(1)} = \varepsilon_{32}^{-(1)} = \varepsilon_{32}^{+(1)} = 0, \quad \varepsilon_{24}^{-(1)} = \varepsilon_{24}^{+(1)} = \varepsilon_{42}^{-(1)} = \varepsilon_{42}^{+(1)} = 0$$

$$\varepsilon_{25}^{-(1)} = \varepsilon_{25}^{+(1)} = \varepsilon_{52}^{-(1)} = \varepsilon_{52}^{+(1)} = 0, \quad \varepsilon_{34}^{-(1)} = \varepsilon_{34}^{+(1)} = \varepsilon_{43}^{-(1)} = \varepsilon_{43}^{+(1)} = 0$$

$$\varepsilon_{35}^{-(1)} = \varepsilon_{35}^{+(1)} = \varepsilon_{53}^{-(1)} = \varepsilon_{53}^{+(1)} = 0, \quad \varepsilon_{45}^{-(1)} = \varepsilon_{54}^{-(1)} = 0.0034$$

$$\varepsilon_{12}^{-(2)} = \varepsilon_{12}^{+(2)} = \varepsilon_{21}^{-(2)} = \varepsilon_{21}^{+(2)} = 0, \quad \varepsilon_{14}^{-(2)} = \varepsilon_{14}^{+(2)} = \varepsilon_{41}^{-(2)} = \varepsilon_{41}^{+(2)} = 0$$

$$\varepsilon_{25}^{-(2)} = \varepsilon_{52}^{-(2)} = 0.0034, \quad \varepsilon_{25}^{+(2)} = \varepsilon_{52}^{+(2)} = 0$$

$$\varepsilon_{34}^{-(2)} = \varepsilon_{34}^{+(2)} = \varepsilon_{43}^{-(2)} = \varepsilon_{43}^{+(2)} = 0, \quad \varepsilon_{45}^{-(2)} = \varepsilon_{45}^{+(2)} = \varepsilon_{54}^{-(2)} = \varepsilon_{54}^{+(2)} = 0$$

Then by Eq.(4.48), we get the overall attribute values of all the enterprises Y_i $(i = 1, 2, \cdots, 6)$:

$$z_1(\omega) = 0.8918, \quad z_2(\omega) = 0.9647, \quad z_3(\omega) = 0.8920$$

$$z_4(\omega) = 0.8391, \quad z_5(\omega) = 0.9023, \quad z_6(\omega) = 0.9190$$

by which we get the ranking of the enterprises Y_i $(i = 1, 2, \cdots, 6)$ as follows:

$$Y_2 \succ Y_6 \succ Y_5 \succ Y_3 \succ Y_1 \succ Y_4$$

Accordingly, the best enterprise is Y_2.

4.10 Consistency Analysis on Group Decision Making with Intuitionistic Preference Relations

Definition 4.10.1 (Xu and Yager, 2009) Let $\alpha_1 = (\mu_{\alpha_1}, \nu_{\alpha_1})$ and $\alpha_2 = (\mu_{\alpha_2}, \nu_{\alpha_2})$ be two IFNs. Then

$$d(\alpha_1, \alpha_2) = \frac{1}{2}(|\mu_{\alpha_1} - \mu_{\alpha_2}| + |\nu_{\alpha_1} - \nu_{\alpha_2}|) \tag{4.52}$$

is called the normalized Hamming distance between α_1 and α_2.

Szmidt and Kacprzyk (2004) define a similarity measure between α_1 and α_2 as

$$\vartheta(\alpha_1, \alpha_2) = \frac{d(\alpha_1, \alpha_2)}{d(\alpha_1, \bar{\alpha}_2)} \tag{4.53}$$

where $\bar{\alpha}_2 = (\nu_{\alpha_2}, \mu_{\alpha_2})$ is the complement of $\alpha_2 = (\mu_{\alpha_2}, \nu_{\alpha_2})$.

The prominent characteristic of the similarity measure (4.53) is that: It takes into account not only a pure distance between two IFNs but also examines if the compared IFNs are more similar, or more dissimilar to each other. However, in practical applications, it is generally expected that the degree of similarity would describe to what extent the IFNs are similar, so the most similar (identical) IFNs should have the largest degree of similarity, which cannot be reflected by Eq.(4.53). Moreover, we find that if $d(\alpha_1, \bar{\alpha}_2) \to 0$, then $\vartheta(\alpha_1, \alpha_2) \to \infty$, which is inconvenient in practical applications because it is difficult to express these enormous numbers. To resolve this issue, and motivated by the idea of the TOPSIS of Hwang and Yoon (1981), Xu and Yager (2009) revise Eq.(4.53) as:

Definition 4.10.2 (Xu and Yager, 2009) Let α_1 and α_2 be two IFNs, $\bar{\alpha}_2$ is the complement of α_2. Then

$$\dot{\vartheta}(\alpha_1, \alpha_2) = \begin{cases} 1, & \alpha_1 = \alpha_2 = \bar{\alpha}_2, \\ \dfrac{d(\alpha_1, \bar{\alpha}_2)}{d(\alpha_1, \alpha_2) + d(\alpha_1, \bar{\alpha}_2)}, & \text{others} \end{cases} \tag{4.54}$$

is called the similarity degree between α_1 and α_2.

From Eq.(4.54), we know that $\dot{\vartheta}(\alpha_1, \alpha_2)$ has the following desirable properties:

(1) $0 \leqslant \dot{\vartheta}(\alpha_1, \alpha_2) \leqslant 1$;

(2) $\dot{\vartheta}(\alpha_1, \alpha_2) = \dot{\vartheta}(\alpha_2, \alpha_1) = \dot{\vartheta}(\bar{\alpha}_1, \bar{\alpha}_2)$;

(3) $\dot{\vartheta}(\alpha_1, \bar{\alpha}_2) = \dot{\vartheta}(\bar{\alpha}_1, \alpha_2)$;

(4) $\dot{\vartheta}(\alpha_1, \alpha_2) = 1$ if and only if $\alpha_1 = \alpha_2$, which means the identity of α_1 and α_2;

(5) $\dot{\vartheta}(\alpha_1, \alpha_2) = 0.5$ if and only if $d(\alpha_1, \alpha_2) = d(\alpha_1, \bar{\alpha}_2) \neq 0$, which means that α_1 is to the same extent similar to α_2 and $\bar{\alpha}_2$;

(6) $\dot{\vartheta}(\alpha_1, \alpha_2) > 0.5$ if and only if $d(\alpha_1, \alpha_2) < d(\alpha_1, \bar{\alpha}_2)$, which means that α_1 is more similar to α_2 than $\bar{\alpha}_2$;

(7) $\dot{\vartheta}(\alpha_1, \alpha_2) < 0.5$ if and only if $d(\alpha_1, \alpha_2) > d(\alpha_1, \bar{\alpha}_2)$, which means that α_1 is more similar to $\bar{\alpha}_2$ than α_2;

(8) $\dot{\vartheta}(\alpha_1, \alpha_2) = 0$ if and only if $\alpha_1 = \bar{\alpha}_2$, which means the complete dissimilarity of α_1 and α_2.

We now apply Eq.(4.54) to the consensus analysis in group decision making based on intuitionistic preference relations (Xu and Yager, 2009):

Let Y, E and ω be defined as in Section 1.3. Suppose that the decision makers $E_k \in E$ $(k = 1, 2, \cdots, l)$ compare each pair of the alternatives in Y, and construct the intuitionistic preference relations $Q_k = (q_{ij}^{(k)})_{n\times n}$ $(k = 1, 2, \cdots, l)$. Then the following theorem can be easily established:

Theorem 4.10.1 (Xu and Yager, 2009) Let $Q_k = (q_{ij}^{(k)})_{n\times n}$ $(k = 1, 2, \cdots, l)$ be the intuitionistic preference relations given by the l decision makers e_k $(k = 1, 2, \cdots, l)$, where $q_{ij}^{(k)} = (\mu_{ij}^{(k)}, v_{ij}^{(k)})$ $(i, j = 1, 2, \cdots, n;\ \ k = 1, 2, \cdots, l)$. Then the aggregation $Q = (q_{ij})_{n\times n}$ of $Q_k = (q_{ij}^{(k)})_{n\times n}$ $(k = 1, 2, \cdots, l)$ is also an intuitionistic preference relation, where

$$q_{ij} = (\mu_{ij}, \nu_{ij}),\quad \mu_{ij} = \sum_{k=1}^{l} \omega_k \mu_{ij}^{(k)},\quad v_{ij} = \sum_{k=1}^{l} \omega_k \nu_{ij}^{(k)},\quad \mu_{ii} = \nu_{ii} = 0.5,\quad i, j = 1, 2, \cdots, n \tag{4.55}$$

Definition 4.10.3 (Xu and Yager, 2009) The similarity degree between the individual intuitionistic preference relation Q_k and the aggregated intuitionistic preference relation Q is defined as:

$$\dot{\vartheta}(Q_k, Q) = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \dot{\vartheta}(q_{ij}^{(k)}, q_{ij}) \tag{4.56}$$

where $\dot{\vartheta}(q_{ij}^{(k)}, q_{ij})$ is the similarity degree between $q_{ij}^{(k)}$ and q_{ij} (which can be derived by Eq.(4.54)).

$\dot{\vartheta}(Q_k, Q)$ has the following properties:

(1) $0 \leqslant \dot{\vartheta}(Q_k, Q) \leqslant 1$;

(2) $\dot{\vartheta}(Q_k, Q) = \dot{\vartheta}(Q, Q_k) = \dot{\vartheta}(\bar{Q}, \bar{Q}_k)$;

(3) $\dot{\vartheta}(Q_k, \bar{Q}) = \dot{\vartheta}(\bar{Q}_k, Q)$;

(4) $\dot{\vartheta}(Q_k, Q) = 1$ if and only if Q_k and Q are equal;

(5) $\dot{\vartheta}(Q_k, Q) = 0$ if and only if Q_k and Q are completely dissimilar.

Definition 4.10.4 (Xu and Yager, 2009) If

$$\dot{\vartheta}(Q_k, Q) > \lambda_0 \tag{4.57}$$

then the individual intuitionistic preference relation Q_k and the aggregated intuitionistic preference relation Q are called of acceptable consensus, where λ_0 is the threshold of acceptable consensus, which can be determined by the experts in advance in practical applications. We take $\lambda_0 >= 0.5$.

In the process of group decision making, if $\dot{\vartheta}(Q_k, Q) \leqslant \lambda_0$, then we shall return Q_k together with Q to the expert E_k, and inform him/her of some elements of Q_k with small degrees of similarity, which are needed to be revaluated. We repeat this procedure until Q_k and Q are of acceptable similarity.

Example 4.10.1 (Xu and Yager, 2009) Consider a group decision making problem, where there are four alternatives Y_i $(i = 1, 2, 3, 4)$ to be selected, and there are three decision makers E_k $(k = 1, 2, 3)$(whose weight vector is $\xi = (0.5,\ 0.3,\ 0.2)^{\mathrm{T}}$). The decision makers E_k $(k = 1, 2, 3)$ compare each pair of the alternatives Y_i $(i = 1, 2, 3, 4)$, and construct the intuitionistic preference relations $Q_k = (q_{ij}^{(k)})_{4\times 4}$ $(k = 1, 2, 3)$ respectively:

$$Q_1 = \begin{bmatrix} (0.5, 0.5) & (0.2, 0.4) & (0.5, 0.4) & (0.7, 0.1) \\ (0.4, 0.2) & (0.5, 0.5) & (0.3, 0.5) & (0.4, 0.5) \\ (0.4, 0.5) & (0.5, 0.3) & (0.5, 0.5) & (0.8, 0.2) \\ (0.1, 0.7) & (0.5, 0.4) & (0.2, 0.8) & (0.5, 0.5) \end{bmatrix}$$

$$Q_2 = \begin{bmatrix} (0.5, 0.5) & (0.3, 0.4) & (0.4, 0.5) & (0.6, 0.3) \\ (0.4, 0.3) & (0.5, 0.5) & (0.4, 0.4) & (0.5, 0.3) \\ (0.5, 0.4) & (0.4, 0.4) & (0.5, 0.5) & (0.7, 0.2) \\ (0.3, 0.6) & (0.3, 0.5) & (0.2, 0.7) & (0.5, 0.5) \end{bmatrix}$$

$$Q_3 = \begin{bmatrix} (0.5, 0.5) & (0.8, 0.1) & (0.3, 0.4) & (0.6, 0.4) \\ (0.1, 0.8) & (0.5, 0.5) & (0.5, 0.3) & (0.4, 0.5) \\ (0.4, 0.3) & (0.3, 0.5) & (0.5, 0.5) & (0.3, 0.7) \\ (0.4, 0.6) & (0.5, 0.4) & (0.7, 0.3) & (0.5, 0.5) \end{bmatrix}$$

By Eq.(4.55), we can get the overall intuitionistic preference relation $Q = (q_{ij})_{4\times 4}$:

$$Q = \begin{bmatrix} (0.50, 0.50) & (0.25, 0.34) & (0.43, 0.43) & (0.65, 0.22) \\ (0.34, 0.25) & (0.50, 0.50) & (0.37, 0.43) & (0.43, 0.44) \\ (0.43, 0.43) & (0.43, 0.37) & (0.50, 0.50) & (0.67, 0.30) \\ (0.22, 0.65) & (0.44, 0.43) & (0.30, 0.67) & (0.50, 0.50) \end{bmatrix}$$

Using Eq.(4.54), we get

$$\dot{\vartheta}(q_{11}^{(1)}, q_{11}) = \dot{\vartheta}(q_{22}^{(1)}, q_{22}) = \dot{\vartheta}(q_{33}^{(1)}, q_{33}) = \dot{\vartheta}(q_{44}^{(1)}, q_{44}) = 1$$

$$\dot{\vartheta}(q_{12}^{(1)}, q_{12}) = \dot{\vartheta}(q_{21}^{(1)}, q_{21}) = 0.73, \quad \dot{\vartheta}(q_{13}^{(1)}, q_{13}) = \dot{\vartheta}(q_{31}^{(1)}, q_{31}) = 0.50$$

$$\dot{\vartheta}(q_{14}^{(1)}, q_{14}) = \dot{\vartheta}(q_{41}^{(1)}, q_{41}) = 0.86, \quad \dot{\vartheta}(q_{23}^{(1)}, q_{23}) = \dot{\vartheta}(q_{32}^{(1)}, q_{32}) = 0.65$$

$$\dot{\vartheta}(q_{24}^{(1)}, q_{24}) = \dot{\vartheta}(q_{42}^{(1)}, q_{42}) = 0.55, \quad \dot{\vartheta}(q_{34}^{(1)}, q_{34}) = \dot{\vartheta}(q_{43}^{(1)}, q_{43}) = 0.81$$

$$\dot{\vartheta}(q_{11}^{(2)}, q_{11}) = \dot{\vartheta}(q_{22}^{(2)}, q_{22}) = \dot{\vartheta}(q_{33}^{(2)}, q_{33}) = \dot{\vartheta}(q_{44}^{(2)}, q_{44}) = 1$$

$$\dot{\vartheta}(q_{12}^{(2)}, q_{12}) = \dot{\vartheta}(q_{21}^{(2)}, q_{21}) = 0.63, \quad \dot{\vartheta}(q_{13}^{(2)}, q_{13}) = \dot{\vartheta}(q_{31}^{(2)}, q_{31}) = 0.50$$

$$\dot{\vartheta}(q_{14}^{(2)}, q_{14}) = \dot{\vartheta}(q_{41}^{(2)}, q_{41}) = 0.85, \quad \dot{\vartheta}(q_{23}^{(2)}, q_{23}) = \dot{\vartheta}(q_{32}^{(2)}, q_{32}) = 0.05$$

$$\dot{\vartheta}(q_{24}^{(2)}, q_{24}) = \dot{\vartheta}(q_{42}^{(2)}, q_{42}) = 0.48, \quad \dot{\vartheta}(q_{34}^{(2)}, q_{34}) = \dot{\vartheta}(q_{43}^{(2)}, q_{43}) = 0.87$$

$$\dot{\vartheta}(q_{11}^{(3)}, q_{11}) = \dot{\vartheta}(q_{22}^{(3)}, q_{22}) = \dot{\vartheta}(q_{33}^{(3)}, q_{33}) = \dot{\vartheta}(q_{44}^{(3)}, q_{44}) = 1$$

$$\dot{\vartheta}(q_{12}^{(3)}, q_{12}) = \dot{\vartheta}(q_{21}^{(3)}, q_{21}) = 0.44, \quad \dot{\vartheta}(q_{13}^{(3)}, q_{13}) = \dot{\vartheta}(q_{31}^{(3)}, q_{31}) = 0.50$$

$$\dot{\vartheta}(q_{14}^{(3)}, q_{14}) = \dot{\vartheta}(q_{41}^{(3)}, q_{41}) = 0.73, \quad \dot{\vartheta}(q_{23}^{(3)}, q_{23}) = \dot{\vartheta}(q_{32}^{(3)}, q_{32}) = 0.35$$

$$\dot{\vartheta}(q_{24}^{(3)}, q_{24}) = \dot{\vartheta}(q_{42}^{(3)}, q_{42}) = 0.55, \quad \dot{\vartheta}(q_{34}^{(3)}, q_{34}) = \dot{\vartheta}(q_{43}^{(3)}, q_{43}) = 0.04$$

Thus, it follows from Eq.(4.56) that

$$\dot{\vartheta}(Q_1, Q) = 0.76, \quad \dot{\vartheta}(Q_2, Q) = 0.73, \quad \dot{\vartheta}(Q_3, Q) = 0.58$$

Without loss of generality, suppose that $\lambda_0 = 0.7$. Since $\dot{\vartheta}(Q_1, Q) > 0.7$, $\dot{\vartheta}(Q_2, Q) > 0.7$ and $\dot{\vartheta}(Q_3, Q) < 0.7$, Q_1 and Q, Q_2 and Q are of acceptable consensus, but the consensus degree between Q_3 and Q is unacceptable.

Note that the similarity degrees $\dot{\vartheta}(q_{24}^{(2)}, q_{24})$, $\dot{\vartheta}(q_{42}^{(2)}, q_{42})$, $\dot{\vartheta}(q_{12}^{(3)}, q_{12})$, $\dot{\vartheta}(q_{21}^{(3)}, q_{21})$, $\dot{\vartheta}(q_{23}^{(3)}, q_{23})$, $\dot{\vartheta}(q_{32}^{(3)}, q_{32})$, $\dot{\vartheta}(q_{34}^{(3)}, q_{34})$ and $\dot{\vartheta}(q_{43}^{(3)}, q_{43})$ are less than somewhat small. In particular, $\dot{\vartheta}(q_{34}^{(3)}, q_{34})$ and $\dot{\vartheta}(q_{43}^{(3)}, q_{43})$ have the smallest similarity degrees. Thus, we need to return Q_2 together with R to the decision maker E_2, and return Q_3 together with Q to the decision maker E_3, and suggest them to revaluate the elements $q_{24}^{(2)}$, $q_{12}^{(3)}$, $q_{23}^{(3)}$ and $q_{34}^{(3)}$.

Suppose that the reevaluated intuitionistic preference relations are $Q'_2 = (q_{ij}^{\prime(2)})_{4\times4}$ and $Q'_3 = (q_{ij}^{\prime(3)})_{4\times4}$, i.e.,

$$Q'_2 = \begin{bmatrix} (0.5, 0.5) & (0.3, 0.4) & (0.4, 0.5) & (0.6, 0.3) \\ (0.4, 0.3) & (0.5, 0.5) & (0.4, 0.4) & (0.4, 0.5) \\ (0.5, 0.4) & (0.4, 0.4) & (0.5, 0.5) & (0.7, 0.2) \\ (0.3, 0.6) & (0.5, 0.4) & (0.2, 0.7) & (0.5, 0.5) \end{bmatrix}$$

$$Q'_3 = \begin{bmatrix} (0.5, 0.5) & (0.4, 0.5) & (0.3, 0.4) & (0.6, 0.4) \\ (0.5, 0.4) & (0.5, 0.5) & (0.3, 0.5) & (0.4, 0.5) \\ (0.4, 0.3) & (0.5, 0.3) & (0.5, 0.5) & (0.6, 0.3) \\ (0.4, 0.6) & (0.5, 0.4) & (0.3, 0.6) & (0.5, 0.5) \end{bmatrix}$$

Then by Eq.(4.55), we aggregate the individual intuitionistic preference relations Q_1, Q'_2 and Q'_3 into the collective intuitionistic preference relation $Q' = (q'_{ij})_{4\times4}$, i.e.,

$$Q' = \begin{bmatrix} (0.50, 0.50) & (0.27, 0.42) & (0.43, 0.43) & (0.65, 0.22) \\ (0.42, 0.27) & (0.50, 0.50) & (0.33, 0.47) & (0.40, 0.50) \\ (0.43, 0.43) & (0.47, 0.33) & (0.50, 0.50) & (0.73, 0.22) \\ (0.22, 0.65) & (0.40, 0.45) & (0.22, 0.73) & (0.50, 0.50) \end{bmatrix}$$

Accordingly, Eq.(4.54) yields

$$\dot{\vartheta}(q_{11}^{(1)}, q'_{11}) = \dot{\vartheta}(q_{22}^{(1)}, q'_{22}) = \dot{\vartheta}(q_{33}^{(1)}, q'_{33}) = \dot{\vartheta}(q_{44}^{(1)}, q'_{44}) = 1$$

$$\dot{\vartheta}(q_{12}^{(1)}, q'_{12}) = \dot{\vartheta}(q_{21}^{(1)}, q'_{21}) = 0.80, \quad \dot{\vartheta}(q_{13}^{(1)}, q'_{13}) = \dot{\vartheta}(q_{31}^{(1)}, q'_{31}) = 0.50$$

$$\dot{\vartheta}(q_{14}^{(1)}, q'_{14}) = \dot{\vartheta}(q_{41}^{(1)}, q'_{41}) = 0.86, \quad \dot{\vartheta}(q_{23}^{(1)}, q'_{23}) = \dot{\vartheta}(q_{32}^{(1)}, q'_{32}) = 0.85$$

$$\dot{\vartheta}(q_{24}^{(1)}, q'_{24}) = \dot{\vartheta}(q_{42}^{(1)}, q'_{42}) = 1.00, \quad \dot{\vartheta}(q_{34}^{(1)}, q'_{34}) = \dot{\vartheta}(q_{43}^{(1)}, q'_{43}) = 0.92$$

$$\dot{\vartheta}(q_{11}^{(2)}, q'_{11}) = \dot{\vartheta}(q_{22}^{(2)}, q'_{22}) = \dot{\vartheta}(q_{33}^{(2)}, q'_{33}) = \dot{\vartheta}(q_{44}^{(2)}, q'_{44}) = 1$$

$$\dot{\vartheta}(q_{12}^{(2)}, q'_{12}) = \dot{\vartheta}(q_{21}^{(2)}, q'_{21}) = 0.83, \quad \dot{\vartheta}(q_{13}^{(2)}, q'_{13}) = \dot{\vartheta}(q_{31}^{(2)}, q'_{31}) = 0.50$$

$$\dot{\vartheta}(q_{14}^{(2)}, q'_{14}) = \dot{\vartheta}(q_{41}^{(2)}, q'_{41}) = 0.85, \quad \dot{\vartheta}(q_{23}^{(2)}, q'_{23}) = \dot{\vartheta}(q_{32}^{(2)}, q'_{32}) = 0.50$$

$$\dot{\vartheta}(q_{24}^{(2)}, q'_{24}) = \dot{\vartheta}(q_{42}^{(2)}, q'_{42}) = 1.00, \quad \dot{\vartheta}(q_{34}^{(2)}, q'_{34}) = \dot{\vartheta}(q_{43}^{(2)}, q'_{43}) = 0.95$$

$$\dot{\vartheta}(q_{11}^{(3)}, q'_{11}) = \dot{\vartheta}(q_{22}^{(3)}, q'_{22}) = \dot{\vartheta}(q_{33}^{(3)}, q'_{33}) = \dot{\vartheta}(q_{44}^{(3)}, q'_{44}) = 1$$

$$\dot{\vartheta}(q_{12}^{(3)}, q'_{12}) = \dot{\vartheta}(q_{21}^{(3)}, q'_{21}) = 0.54, \quad \dot{\vartheta}(q_{13}^{(3)}, q'_{13}) = \dot{\vartheta}(q_{31}^{(3)}, q'_{31}) = 0.50$$

$$\dot{\vartheta}(q_{14}^{(3)}, q'_{14}) = \dot{\vartheta}(q_{41}^{(3)}, q'_{41}) = 0.73, \quad \dot{\vartheta}(q_{23}^{(3)}, q'_{23}) = \dot{\vartheta}(q_{32}^{(3)}, q'_{32}) = 0.85$$

$$\dot{\vartheta}(q_{24}^{(3)}, q'_{24}) = \dot{\vartheta}(q_{42}^{(3)}, q'_{42}) = 1.00, \quad \dot{\vartheta}(q_{34}^{(3)}, q'_{34}) = \dot{\vartheta}(q_{43}^{(3)}, q'_{43}) = 0.79$$

Therefore, using Eq.(4.56), we get

$$\dot{\vartheta}(Q_1, Q') = 0.87, \quad \dot{\vartheta}(Q'_2, Q') = 0.83, \quad \dot{\vartheta}(Q'_3, Q') = 0.80$$

As a result, each individual intuitionistic preference relation and the collective intuitionistic preference relation are of acceptable consensus.

In the next section, we small extend the similarity measure to the interval-valued intuitionistic fuzzy set theory.

4.11 Consistency Analysis on Group Decision Making with Interval-Valued Intuitionistic Preference Relations

Definition 4.11.1 (Xu and Yager, 2009) Let $\tilde{\alpha}_i = ([a_i, b_i], [c_i, d_i])(i = 1, 2)$ be two IVIFNs. Then

$$d(\tilde{\alpha}_1, \tilde{\alpha}_2) = \frac{1}{4}(|a_1 - a_2| + |b_1 - b_2| + |c_1 - c_2| + |d_1 - d_2|) \tag{4.58}$$

is called the normalized Hamming distance between $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$.

Definition 4.11.2 (Xu and Yager, 2009) Let $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ be two IVIFNs, and $\bar{\tilde{\alpha}}_2$ the complement of $\tilde{\alpha}_2$. Then

$$\vartheta(\tilde{\alpha}_1, \tilde{\alpha}_2) = \begin{cases} 1, & \tilde{\alpha}_1 = \tilde{\alpha}_2 = \bar{\tilde{\alpha}}_2, \\ \dfrac{d(\tilde{\alpha}_1, \bar{\tilde{\alpha}}_2)}{d(\tilde{\alpha}_1, \tilde{\alpha}_2) + d(\tilde{\alpha}_1, \bar{\tilde{\alpha}}_2)}, & \text{others} \end{cases} \tag{4.59}$$

is called the similarity degree between $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$.

$\vartheta(\tilde{\alpha}_1, \tilde{\alpha}_2)$ has the following properties:

(1) $0 \leqslant \vartheta(\tilde{\alpha}_1, \tilde{\alpha}_2) \leqslant 1$;

(2) $\vartheta(\tilde{\alpha}_1, \tilde{\alpha}_2) = \vartheta(\tilde{\alpha}_2, \tilde{\alpha}_1) = \vartheta(\bar{\tilde{\alpha}}_1, \bar{\tilde{\alpha}}_2)$;

(3) $\vartheta(\tilde{\alpha}_1, \bar{\tilde{\alpha}}_2) = \vartheta(\bar{\tilde{\alpha}}_1, \tilde{\alpha}_2)$;

(4) $\vartheta(\tilde{\alpha}_1, \tilde{\alpha}_2) = 1$ if and only if $\tilde{\alpha}_1 = \tilde{\alpha}_2$;

(5) $\vartheta(\tilde{\alpha}_1, \tilde{\alpha}_2) > 0.5$ if and only if $d(\tilde{\alpha}_1, \tilde{\alpha}_2) < d(\tilde{\alpha}_1, \bar{\tilde{\alpha}}_2)$, which means that $\tilde{\alpha}_1$ is more similar to $\tilde{\alpha}_2$ than $\bar{\tilde{\alpha}}_2$;

(6) $\vartheta(\tilde{\alpha}_1, \alpha_2) = 0.5$ if and only if $d(\tilde{\alpha}_1, \tilde{\alpha}_2) = d(\tilde{\alpha}_1, \bar{\tilde{\alpha}}_2) \neq 0$, which means that $\tilde{\alpha}_1$ is to the same extent similar to $\tilde{\alpha}_2$ and $\bar{\tilde{\alpha}}_2$;

(7) $\vartheta(\tilde{\alpha}_1, \tilde{\alpha}_2) < 0.5$ if and only if $d(\tilde{\alpha}_1, \tilde{\alpha}_2) > d(\tilde{\alpha}_1, \bar{\tilde{\alpha}}_2)$, which means that $\tilde{\alpha}_1$ is more similar to $\bar{\tilde{\alpha}}_2$ than $\tilde{\alpha}_2$;

(8) $\vartheta(\tilde{\alpha}_1, \tilde{\alpha}_2) = 0$ if and only if $\tilde{\alpha}_1 = \bar{\tilde{\alpha}}_2$, which means the complete dissimilarity of $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$.

We now apply Eq.(4.59) to the consensus analysis in group decision making with interval-valued intuitionistic preference relations:

Let Y, E and ξ be defined as in Section 1.3. The decision makers $E_k \in E$ $(k = 1, 2, \cdots, l)$ compare each pair of the alternatives in Y, and construct the interval-valued intuitionistic preference relations $\tilde{Q}_k = (\tilde{q}_{ij}^{(k)})_{n\times n}$ $(k = 1, 2, \cdots, l)$. Then we have the following result:

Theorem 4.11.1 (Xu and Yager, 2009) Let $\tilde{Q}_k = (\tilde{q}_{ij}^{(k)})_{n\times n}$ $(k = 1, 2, \cdots, l)$ be the interval-valued intuitionistic preference relations given by the l decision makers E_k $(k = 1, 2, \cdots, l)$, and let $\xi = (\xi_1, \xi_2, \cdots, \xi_l)^{\mathrm{T}}$ be the weight vector of the decision makers, where $\tilde{q}_{ij}^{(k)} = (\tilde{\mu}_{ij}^{(k)}, \tilde{v}_{ij}^{(k)})$, $\tilde{\mu}_{ij}^{(k)} = [(\tilde{\mu}_{ij}^{(k)})^{\mathrm{L}}, (\tilde{\mu}_{ij}^{(k)})^{\mathrm{U}}]$, $\tilde{\nu}_{ij}^{(k)} = [(\tilde{\nu}_{ij}^{(k)})^{\mathrm{L}}, (\tilde{\nu}_{ij}^{(k)})^{\mathrm{U}}]$, $\xi_k \in [0, 1]$ $(k = 1, 2, \cdots, l)$ and $\sum_{k=1}^{l} \xi_k = 1$. Then the aggregation $\tilde{Q} = (\tilde{q}_{ij})_{n\times n}$ of the individual interval-valued intuitionistic preference relations $\tilde{Q}_k = (\tilde{q}_{ij}^{(k)})_{n\times n}$ $(k = 1, 2, \cdots, l)$ is also the interval-valued intuitionistic preference relation, where

$$\tilde{q}_{ij} = (\tilde{\mu}_{ij}, \tilde{\nu}_{ij}), \quad \tilde{\mu}_{ij} = [\tilde{\mu}_{ij}^{\mathrm{L}}, \tilde{\mu}_{ij}^{\mathrm{U}}], \quad \tilde{v}_{ij} = [\tilde{\nu}_{ij}^{\mathrm{L}}, \tilde{\nu}_{ij}^{\mathrm{U}}]$$

$$\tilde{\mu}_{ij}^{\mathrm{L}} = \sum_{k=1}^{m} \omega_k (\tilde{\mu}_{ij}^{(k)})^{\mathrm{L}}, \quad \tilde{\mu}_{ij}^{\mathrm{U}} = \sum_{k=1}^{m} \omega_k (\tilde{\mu}_{ij}^{(k)})^{\mathrm{U}}$$

$$\tilde{v}_{ij}^{\mathrm{L}} = \sum_{k=1}^{m} \omega_k (\tilde{\nu}_{ij}^{(k)})^{\mathrm{L}}, \quad \tilde{v}_{ij}^{\mathrm{U}} = \sum_{k=1}^{m} \omega_k (\tilde{\nu}_{ij}^{(k)})^{\mathrm{U}}, \quad i, j = 1, 2, \cdots, n \tag{4.60}$$

Definition 4.11.3 (Xu and Yager, 2009) The similarity degree between the individual interval-valued intuitionistic preference relation $\tilde{Q}_k$ and the collective interval-valued intuitionistic preference relation $\tilde{Q}$ is defined as:

$$\vartheta(\tilde{Q}_k, \tilde{Q}) = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \vartheta(\tilde{q}_{ij}^{(k)}, \tilde{q}_{ij}) \tag{4.61}$$

where $\vartheta(\tilde{q}_{ij}^{(k)}, \tilde{q}_{ij})$ is the similarity degree between $\tilde{q}_{ij}^{(k)}$ and $\tilde{q}_{ij}$ (which can be derived by Eq.(4.59)).

$\vartheta(\tilde{Q}_k, \tilde{Q})$ has the following properties:

(1) $0 \leqslant \vartheta(\tilde{Q}_k, \tilde{Q}) \leqslant 1$;

(2) $\vartheta(\tilde{Q}_k, \tilde{Q}) = \vartheta(\tilde{Q}, \tilde{Q}_k) = \vartheta(\bar{\tilde{Q}}, \bar{\tilde{Q}}_k)$;

(3) $\vartheta(\tilde{Q}_k, \bar{\tilde{Q}}) = \vartheta(\bar{\tilde{Q}}_k, \tilde{Q})$;

(4) $\vartheta(\tilde{Q}_k, \tilde{Q}) = 1$ if and only if $\tilde{Q}_k$ and $\tilde{Q}$ are equal (or completely similar);

(5) $\vartheta(\tilde{Q}_k, \tilde{Q}) = 0$ if and only if $\tilde{Q}_k$ and $\tilde{Q}$ are completely dissimilar.

Definition 4.11.4 (Xu and Yager, 2009) If

$$\vartheta(\tilde{Q}_k, \tilde{Q}) > \eta_0 \tag{4.62}$$

then the individual interval-valued intuitionistic preference relation $\tilde{Q}_k$ and the collective interval-valued intuitionistic preference relation $\tilde{Q}$ are called of acceptable consensus, where η_0 is the threshold of acceptable consensus, which can be determined by the experts in advance in practical applications. In general, we take $\eta_0 >= 0.5$.

In the process of group decision making, if $\vartheta(\tilde{Q}_k, \tilde{Q}) \leqslant \eta_0$, then we shall return $\tilde{Q}_k$ together with $\tilde{Q}$ to the decision maker E_k, and inform him/her of some elements of $\tilde{Q}_k$ with small degrees of similarity, which need to be revaluated. We repeat this procedure until $\tilde{Q}_k$ and $\tilde{Q}$ are of acceptable consensus.

Example 4.11.1 (Xu and Yager, 2009) Suppose that three decision makers E_k $(k = 1, 2, 3)$ (whose weight vector is $\xi = (0.4,\ 0.3,\ 0.3)^{\mathrm{T}}$) compare three alternatives Y_i $(i = 1, 2, 3)$ and construct the following interval-valued intuitionistic preference relations $\tilde{Q}_k = (\tilde{q}_{ij}^{(k)})_{3\times 3}$ $(k = 1, 2, 3)$, respectively:

$$\tilde{Q}_1 = \begin{bmatrix} ([0.5,0.5],\ [0.5,0.5]) & ([0.1,0.2],\ [0.7,0.8]) & ([0.6,0.9],\ [0,0.1]) \\ ([0.7,0.8],\ [0.1,0.2]) & ([0.5,0.5],\ [0.5,0.5]) & ([0,0.1],\ [0.8,0.9]) \\ ([0,0.1],\ [0.6,0.9]) & ([0.8,0.9],\ [0,0.1]) & ([0.5,0.5],\ [0.5,0.5]) \end{bmatrix}$$

$$\tilde{Q}_2 = \begin{bmatrix} ([0.5,0.5],\ [0.5,0.5]) & ([0,0.1],\ [0.7,0.9]) & ([0.7,0.8],\ [0.1,0.2]) \\ ([0.7,0.9],\ [0,0.1]) & ([0.5,0.5],\ [0.5,0.5]) & ([0.2,0.3],\ [0.5,0.7]) \\ ([0.1,0.2],\ [0.7,0.8]) & ([0.5,0.7],\ [0.2,0.3]) & ([0.5,0.5],\ [0.5,0.5]) \end{bmatrix}$$

$$\tilde{Q}_3 = \begin{bmatrix} ([0.5,0.5],\ [0.5,0.5]) & ([0.2,0.3],\ [0.4,0.6]) & ([0.6,0.8],\ [0,0.1]) \\ ([0.4,0.6],\ [0.2,0.3]) & ([0.5,0.5],\ [0.5,0.5]) & ([0.1,0.2],\ [0.7,0.8]) \\ ([0,0.1],\ [0.6,0.8]) & ([0.7,0.8],\ [0.1,0.2]) & ([0.5,0.5],\ [0.5,0.5]) \end{bmatrix}$$

By Eq.(4.60), we get the collective interval-valued intuitionistic preference relation $\tilde{Q} = (\tilde{q}_{ij})_{4\times 4}$:

$$\tilde{Q} = \begin{bmatrix} ([0.5,0.5],\ [0.5,0.5]) & ([0.10,0.20],\ [0.61,0.77]) & ([0.63,0.84],\ [0.03,0.13]) \\ ([0.61,0.77],\ [0.10,0.20]) & ([0.5,0.5],\ [0.5,0.5]) & ([0.09,0.19],\ [0.68,0.81]) \\ ([0.03,0.13],\ [0.63,0.84]) & ([0.68,0.81],\ [0.09,0.19]) & ([0.5,0.5],\ [0.5,0.5]) \end{bmatrix}$$

Then by Eq.(4.59), we have

$$\vartheta(\tilde{q}_{11}^{(1)}, \tilde{q}_{11}) = \vartheta(\tilde{q}_{22}^{(1)}, \tilde{q}_{22}) = \vartheta(\tilde{q}_{33}^{(1)}, \tilde{q}_{33}) = 1, \quad \vartheta(\tilde{q}_{12}^{(1)}, \tilde{q}_{12}) = \vartheta(\tilde{q}_{21}^{(1)}, \tilde{q}_{21}) = 0.95$$

$$\vartheta(\tilde{q}_{13}^{(1)}, \tilde{q}_{13}) = \vartheta(\tilde{q}_{31}^{(1)}, \tilde{q}_{31}) = 0.95, \quad \vartheta(\tilde{q}_{23}^{(1)}, \tilde{q}_{23}) = \vartheta(\tilde{q}_{32}^{(1)}, \tilde{q}_{32}) = 0.88$$

$$\vartheta(\tilde{q}_{11}^{(2)}, \tilde{q}_{11}) = \vartheta(\tilde{q}_{22}^{(2)}, \tilde{q}_{22}) = \vartheta(\tilde{q}_{33}^{(2)}, \tilde{q}_{33}) = 1, \quad \vartheta(\tilde{q}_{12}^{(2)}, \tilde{q}_{12}) = \vartheta(\tilde{q}_{21}^{(2)}, \tilde{q}_{21}) = 0.86$$

$$\vartheta(\tilde{q}_{13}^{(2)}, \tilde{q}_{13}) = \vartheta(\tilde{q}_{31}^{(2)}, \tilde{q}_{31}) = 0.91, \quad \vartheta(\tilde{q}_{23}^{(2)}, \tilde{q}_{23}) = \vartheta(\tilde{q}_{32}^{(2)}, \tilde{q}_{32}) = 0.79$$

$$\vartheta(\tilde{q}_{11}^{(3)}, \tilde{q}_{11}) = \vartheta(\tilde{q}_{22}^{(3)}, \tilde{q}_{22}) = \vartheta(\tilde{q}_{33}^{(3)}, \tilde{q}_{33}) = 1, \quad \vartheta(\tilde{q}_{12}^{(3)}, \tilde{q}_{12}) = \vartheta(\tilde{q}_{21}^{(3)}, \tilde{q}_{21}) = 0.73$$
$$\vartheta(\tilde{q}_{13}^{(3)}, \tilde{q}_{13}) = \vartheta(\tilde{q}_{31}^{(3)}, \tilde{q}_{31}) = 0.95, \quad \vartheta(\tilde{q}_{23}^{(3)}, \tilde{q}_{23}) = \vartheta(\tilde{q}_{32}^{(3)}, \tilde{q}_{32}) = 0.98$$

Therefore, it follows from Eq.(4.61) that

$$\vartheta(\tilde{Q}_1, \tilde{Q}) = 0.95, \quad \vartheta(\tilde{Q}^{(2)}, \tilde{Q}) = 0.90, \quad \vartheta(\tilde{Q}^{(3)}, \tilde{Q}) = 0.92$$

If we take $\eta_0 = 0.7$, then each individual interval-valued intuitionistic preference relation and the collective interval-valued intuitionistic preference relation are of acceptable consensus. Consequently, we do not need to return them to the decision makers for revaluation.

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Chapter 5

Projection Model-Based Approaches to Intuitionistic Fuzzy Multi-Attribute Decision Making

Xu and Hu (2010) investigate intuitionistic fuzzy multi-attribute decision making problems where the attribute values are expressed in IFNs or IVIFNs. They introduce some concepts, such as the relative intuitionistic fuzzy ideal solution, the relative uncertain intuitionistic fuzzy ideal solution, the modules of IFNs and IVIFNs, etc. They also introduce the cosine of the included angle between the attribute value vectors of each alternative and the relative intuitionistic fuzzy ideal solution, and the cosine of the included angle between the attribute value vectors of each alternative and the relative uncertain intuitionistic fuzzy ideal solution. They further establish two projection models to measure the similarity degrees between each alternative and the relative intuitionistic fuzzy ideal solution, and between each alternative and the relative uncertain intuitionistic fuzzy ideal solution. Based on the projection models, the given alternatives can be ranked and then the most desirable one can be selected.

5.1 Multi-Attribute Decision Making with Intuitionistic Fuzzy Information

The intuitionistic fuzzy multi-attribute decision making problem considered in this Chapter is represented as follows (Xu and Hu, 2010):

Let Y, G and ω be defined as in Section 1.3. Let $R' = (r'_{ij})_{n\times m}$ be an intuitionistic fuzzy decision matrix, with $r'_{ij} = (t_{ij}, f_{ij}, \pi_{ij})$ being an attribute value, donated by an IFN, where t_{ij} indicates the degree that the alternative Y_i satisfies the attribute G_j, f_{ij} indicates the degree that the alternative Y_i does not satisfy the attribute G_j, and π_{ij} indicates the indeterminacy degree such that

$$t_{ij} \in [0,1], \quad f_{ij} \in [0,1], \quad t_{ij} + f_{ij} \leqslant 1, \quad \pi_{ij} = 1 - t_{ij} - f_{ij},$$
$$i = 1, 2, \cdots, n; \quad j = 1, 2, \cdots, m \tag{5.1}$$

There are generally benefit attributes and cost attributes in multi-attribute decision making. In such cases, we may transform the attribute values of cost type into the attribute values of benefit type. Then $R' = (r'_{ij})_{n\times m}$ can be transformed into the intuitionistic fuzzy decision matrix $R = (r_{ij})_{n\times m}$, where

$$r_{ij} = (\mu_{ij}, \nu_{ij}, \pi_{ij}) = \begin{cases} r'_{ij}, & \text{for benefit attribute } G_j, \\ \bar{r}'_{ij}, & \text{for cos t attribute } G_j, \end{cases} \quad j = 1, 2, \cdots, n \tag{5.2}$$

where $\bar{r}'_{ij}$ is the complement of r'_{ij}, such that $\bar{r}'_{ij} = (f_{ij}, t_{ij}, \pi_{ij})$, $\pi_{ij} = 1 - t_{ij} - f_{ij} = 1 - \mu_{ij} - \nu_{ij}$.

Let us consider first situations where the information about the attribute weights is completely unknown:

For convenience of depiction, we denote the alternatives Y_i $(i = 1, 2, \cdots, n)$, based on $R = (r_{ij})_{n\times m}$, as:

$$Y_i = (r_{i1}, r_{i2}, \cdots, r_{im}), \quad i = 1, 2, \cdots, n \tag{5.3}$$

and introduce the module of Y_i as:

$$|Y_i| = \sqrt{\sum_{j=1}^{m} |r_{ij}|^2} \tag{5.4}$$

where $|r_{ij}|$ is the module of the attribute value r_{ij}, calculated as follows:

$$|r_{ij}| = \sqrt{\mu_{ij}^2 + \nu_{ij}^2 + \pi_{ij}^2} \tag{5.5}$$

By Eq.(5.4), we have $0 \leqslant |Y_i| \leqslant \sqrt{m}$.

Let $\alpha_j^* = (\mu_j^*, \nu_j^*, \pi_j^*)$ $(j = 1, 2, \cdots, m)$, where

$$\mu_j^* = \max_i\{\mu_{ij}\}, \quad \nu_j^* = \min_i\{\nu_{ij}\}, \quad \pi_j^* = 1 - \mu_j^* - \nu_j^* = 1 - \max_i\{\mu_{ij}\} - \min_i\{\nu_{ij}\}, \quad j = 1, 2, \cdots, m \tag{5.6}$$

Then we call

$$Y^* = (\alpha_1^*, \alpha_2^*, \cdots, \alpha_m^*) \tag{5.7}$$

an relative intuitionistic fuzzy ideal solution (RIFIS), whose module is denoted as:

$$|Y^*| = \sqrt{\sum_{j=1}^{m} |\alpha_j^*|^2} \tag{5.8}$$

where $|\alpha_j^*| = \sqrt{(\mu_j^*)^2 + (\nu_j^*)^2 + (\pi_j^*)^2}$ is the module of α_j^*.

Based on Eqs.(5.5)-(5.8), we introduce the following concept:

Definition 5.1.1 (Xu and Hu, 2010) Let $Y_i = (r_{i1}, r_{i2}, \cdots, r_{im})$ be the between alternative, and $Y^* = (\alpha_1^*, \alpha_2^*, \cdots, \alpha_m^*)$ the RIFIS, where $r_{ij} = (\mu_{ij}, \nu_{ij}, \pi_{ij})$, $i = 1, 2, \cdots, n$, and $\alpha_j^* = (\mu_j^*, \nu_j^*, \pi_j^*)$, $j = 1, 2, \cdots, m$. Then we call

$$\cos(Y_i,\ Y^*) = \frac{\sum\limits_{j=1}^{m} \left(\mu_{ij}\mu_j^* + \nu_{ij}\nu_j^* + \pi_{ij}\pi_j^*\right)}{|Y_i|\ |Y^*|} \tag{5.9}$$

the cosine of the included angle between Y_j and Y^*.

Obviously, the following theorem holds according to Eq.(5.9):

Theorem 5.1.1 (Xu and Hu, 2010)

(1) $\cos(Y_i,\ Y^*) = \cos(Y^*, Y_i)$;

(2) $0 \leqslant \cos(Y_i,\ Y^*) \leqslant 1$;

(3) $\cos(Y_i,\ Y^*) = 1$ if $Y_i = Y^*$.

Note that a vector is composed of direction and module. $\cos(Y_i,\ Y^*)$, however, only reflects the similarity measure of the direction of Y_j and Y^*. In order to measure the similarity degree between Y_i and Y^*, we introduce a formula of projection of Y_i on Y^*as follows:

$$\begin{aligned} \mathrm{Pr\,j}_{Y^*} Y_i &= |Y_i| \cos(Y_i, Y^*) = |Y_i| \frac{\sum\limits_{j=1}^{m} \left(\mu_{ij}\mu_j^* + \nu_{ij}\nu_j^* + \pi_{ij}\pi_j^*\right)}{|Y_i|\ |Y^*|} \\ &= \frac{1}{|Y^*|} \sum_{j=1}^{m} \left(\mu_{ij}\mu_j^* + \nu_{ij}\nu_i^* + \pi_{ij}\pi_j^*\right) \end{aligned} \tag{5.10}$$

Obviously, the greater the value $\mathrm{Pr\,j}_{Y^*} Y_i$, the closer Y_i to Y^*, and thus the closer the alternative Y_i to the IFIS Y^* (i.e., the better the alternative Y_i).

If the weight vector $\omega = (\omega_1, \omega_2, \cdots, \omega_m)^{\mathrm{T}}$ of the attributes G_j $(j = 1, 2, \cdots, m)$ is known, then we denote

$$|Y_i|_\omega = \sqrt{\sum_{j=1}^{m} (\omega_j |r_{ij}|)^2} \tag{5.11}$$

as the weighted module of the alternative $Y_i = (r_{i1}, r_{i2}, \cdots, r_{im})$, where $\omega = (\omega_1, \omega_2, \cdots, \omega_m)^{\mathrm{T}}$ is the weight vector of the attributes G_j $(j = 1, 2, \cdots, m)$, with $\omega_j \in [0, 1]$, $j = 1, 2, \cdots, n$, and $\sum\limits_{j=1}^{m} \omega_j = 1$. Furthermore, we denote

$$|Y^*|_\omega = \sqrt{\sum_{j=1}^{m} \left(\omega_j |\alpha_j^*|\right)^2} \tag{5.12}$$

as the weighted module of the RIFIS $Y^* = (\alpha_1^*, \alpha_2^*, \cdots, \alpha_m^*)$.

Similar to Eq.(5.9), we introduce the weighted cosine of the included angle between the alternative $Y_i = (r_{i1}, r_{i2}, \cdots, r_{im})$ and the RIFIS $Y^* = (\alpha_1^*, \alpha_2^*, \cdots, \alpha_m^*)$ as:

$$\cos(Y_i,\ Y^+)_\omega = \frac{\sum_{j=1}^{m} \omega_j^2 \left(\mu_{ij}\mu_j^* + \nu_{ij}\nu_j^* + \pi_{ij}\pi_j^*\right)}{|Y_i|_\omega\ |Y^*|_\omega} \tag{5.13}$$

Then we can introduce a formula of projection of Y_i on Y^*:

$$\begin{aligned}\mathrm{Prj}_{Y^*} Y_i &= |Y_i|_\omega \cos(Y_i, Y^*)_\omega = |Y_i|_\omega \frac{\sum_{j=1}^{m} \omega_j^2 \left(\mu_{ij}\mu_j^* + \nu_{ij}\nu_j^* + \pi_{ij}\pi_j^*\right)}{|Y_i|_\omega\ |Y^*|_\omega} \\ &= \frac{1}{|Y^*|_\omega} \sum_{j=1}^{m} \omega_j^2 \left(\mu_{ij}\mu_j^* + \nu_{ij}\nu_j^* + \pi_{ij}\pi_j^*\right)\end{aligned} \tag{5.14}$$

In the next section, we shall extend the above results to the interval-valued intuitionistic fuzzy set theory.

5.2 Multi-Attribute Decision Making with Interval-Valued Intuitionistic Fuzzy Information

Let $\tilde{R}' = (\tilde{r}'_{ij})_{n\times m}$ be an interval-valued intuitionistic fuzzy decision matrix, where $\tilde{r}'_{ij} = (\tilde{t}_{ij},\ \tilde{f}_{ij},\ \tilde{\pi}_{ij})$ is an attribute value, denoted by IVIFN, $\tilde{t}_{ij}$ indicates the degree range that the alternative Y_i satisfies the attribute G_j, and $\tilde{f}_{ij}$ indicates the degree range that the alternative Y_i does not satisfy the attribute G_j, and $\tilde{\pi}_{ij}$ indicates the indeterminacy degree range. Let $\tilde{t}_{ij} = [\tilde{t}_{ij}^{\mathrm{L}},\ \tilde{t}_{ij}^{\mathrm{U}}]$, $\tilde{f}_{ij} = [\tilde{f}_{ij}^{\mathrm{L}},\ \tilde{f}_{ij}^{\mathrm{U}}]$, $\tilde{\pi}_{ij} = [\tilde{\pi}_{ij}^{\mathrm{L}},\ \tilde{\pi}_{ij}^{\mathrm{U}}]$, and

$$\begin{gathered}[\tilde{t}_{ij}^{\mathrm{L}},\ \tilde{t}_{ij}^{\mathrm{U}}] \subset [0,1], \quad [\tilde{f}_{ij}^{\mathrm{L}},\ \tilde{f}_{ij}^{\mathrm{U}}] \subset [0,1], \quad \tilde{t}_{ij}^{\mathrm{U}} + \tilde{f}_{ij}^{\mathrm{U}} \leqslant 1, \quad \tilde{\pi}_{ij}^{\mathrm{L}} = 1 - \tilde{t}_{ij}^{\mathrm{U}} - \tilde{f}_{ij}^{\mathrm{U}}, \\ \tilde{\pi}_{ij}^{\mathrm{U}} = 1 - \tilde{t}_{ij}^{\mathrm{L}} - \tilde{f}_{ij}^{\mathrm{L}}, \quad i = 1, 2, \cdots, n; \quad j = 1, 2, \cdots, m\end{gathered} \tag{5.15}$$

In cases where the attributes are of benefit and cost types, we can normalize $\tilde{R}' = (\tilde{r}'_{ij})_{n\times m}$ into the interval-valued intuitionistic fuzzy decision matrix $\tilde{R} = (\tilde{r}_{ij})_{n\times m}$, where

$$\tilde{r}_{ij} = (\tilde{\mu}_{ij}, \tilde{\nu}_{ij}, \tilde{\pi}_{ij}) = ([\tilde{\mu}_{ij}^{\mathrm{L}}, \tilde{\mu}_{ij}^{\mathrm{U}}], [\tilde{\nu}_{ij}^{\mathrm{L}}, \tilde{\nu}_{ij}^{\mathrm{U}}], [\tilde{\pi}_{ij}^{\mathrm{L}}, \tilde{\pi}_{ij}^{\mathrm{U}}]) = \begin{cases} \tilde{d}_{ij}, & \text{for benefit attribute } G_j, \\ \bar{\tilde{d}}_{ij}, & \text{for cost attribute } G_j, \end{cases} \\ i = 1, 2, \cdots, n; \quad j = 1, 2, \cdots, m \tag{5.16}$$

and $\bar{\tilde{r}}_{ij}$ is the complement of $\tilde{r}_{ij}$, i.e.,

$$\bar{\tilde{r}}_{ij} = (\tilde{f}_{ij}, \tilde{t}_{ij}, \tilde{\pi}_{ij}), \quad \tilde{\pi}_{ij} = [\tilde{\pi}_{ij}^{\mathrm{L}},\ \tilde{\pi}_{ij}^{\mathrm{U}}]$$

$$\tilde{\pi}_{ij}^{\mathrm{L}} = 1 - \tilde{t}_{ij}^{\mathrm{U}} - \tilde{f}_{ij}^{\mathrm{U}} = 1 - \tilde{\mu}_{ij}^{\mathrm{U}} - \tilde{\nu}_{ij}^{U}, \quad \tilde{\pi}_{ij}^{\mathrm{U}} = 1 - \tilde{t}_{ij}^{\mathrm{L}} - \tilde{f}_{ij}^{\mathrm{L}} = 1 - \tilde{\mu}_{ij}^{\mathrm{L}} - \tilde{\nu}_{ij}^{\mathrm{L}}$$

Based on $\tilde{R} = (\tilde{r}_{ij})_{n\times m}$, we denote the alternatives Y_i $(i = 1, 2, \cdots, n)$ by $\tilde{Y}_i$ $(i = 1, 2, \cdots, n)$, where

$$\tilde{Y}_i = (\tilde{r}_{i1}, \tilde{r}_{i2}, \cdots, \tilde{r}_{im}), \quad i = 1, 2, \cdots, n \tag{5.17}$$

If the information about attribute weights is completely unknown, then we denote

$$|\tilde{Y}_i| = \sqrt{\sum_{j=1}^{m} |\tilde{r}_{ij}|^2} \tag{5.18}$$

as the module of the alternative $\tilde{Y}_i = (\tilde{r}_{i1}, \tilde{r}_{i2}, \cdots, \tilde{r}_{im})$, where $|\tilde{r}_{ij}|$ is the module of $\tilde{r}_{ij}$, calculated by

$$|\tilde{r}_{ij}| = \sqrt{(\tilde{\mu}_{ij}^{\mathrm{L}})^2 + (\tilde{\mu}_{ij}^{\mathrm{U}})^2 + (\tilde{\nu}_{ij}^{\mathrm{L}})^2 + (\tilde{\nu}_{ij}^{\mathrm{U}})^2 + (\tilde{\pi}_{ij}^{\mathrm{L}})^2 + (\tilde{\pi}_{ij}^{\mathrm{U}})^2} \tag{5.19}$$

Let $\tilde{\alpha}_j^* = (\tilde{\mu}_j^*, \tilde{\nu}_j^*, \tilde{\pi}_j^*)$ $(j = 1, 2, \cdots, m)$, where

$$\tilde{\mu}_j^* = [\tilde{\mu}_j^{*\mathrm{L}}, \tilde{\mu}_j^{*\mathrm{U}}] = [\max_i\{\tilde{\mu}_{ij}^{\mathrm{L}}\}, \max_i\{\tilde{\mu}_{ij}^{\mathrm{U}}\}], \quad \tilde{\nu}_j^* = [\tilde{\nu}_j^{*\mathrm{L}}, \tilde{\nu}_j^{*\mathrm{U}}] = [\min_i\{\tilde{\nu}_{ij}^{\mathrm{L}}\}, \min_i\{\tilde{\nu}_{ij}^{\mathrm{U}}\}]$$

$$\tilde{\pi}_j^* = [\tilde{\pi}_j^{*\mathrm{L}}, \tilde{\pi}_j^{*\mathrm{U}}], \quad \tilde{\pi}_j^{*\mathrm{L}} = 1 - \tilde{\mu}_j^{*\mathrm{U}} - \nu_j^{*\mathrm{U}} = 1 - \max_i\{\tilde{\mu}_{ij}^{\mathrm{U}}\} - \min_i\{\tilde{\nu}_{ij}^{\mathrm{U}}\}$$

$$\tilde{\pi}_j^{*\mathrm{U}} = 1 - \tilde{\mu}_j^{*\mathrm{L}} - \tilde{\nu}_j^{*\mathrm{L}} = 1 - \max_i\{\tilde{\mu}_{ij}^{\mathrm{L}}\} - \min_i\{\tilde{\nu}_{ij}^{\mathrm{L}}\}, \quad j = 1, 2, \cdots, m \tag{5.20}$$

Then we call

$$\tilde{Y}^* = (\tilde{\alpha}_1^*, \tilde{\alpha}_2^*, \cdots, \tilde{\alpha}_m^*) \tag{5.21}$$

an relative uncertain intuitionistic fuzzy ideal solution (RUIFIS), whose module is denoted as:

$$|\tilde{Y}^*| = \sqrt{\sum_{j=1}^{m} |\tilde{\alpha}_j^*|^2} \tag{5.22}$$

where $|\tilde{\alpha}_j^*|$ is the module of $\tilde{\alpha}_j^*$, calculated by

$$|\tilde{\alpha}_j^*| = \sqrt{(\tilde{\mu}_j^{\mathrm{L}})^2 + (\tilde{\mu}_j^{\mathrm{U}})^2 + (\tilde{\nu}_j^{\mathrm{L}})^2 + (\tilde{\nu}_j^{\mathrm{U}})^2 + (\tilde{\pi}_j^{\mathrm{L}})^2 + (\tilde{\pi}_j^{\mathrm{U}})^2} \tag{5.23}$$

Based on Eqs.(5.20)-(5.23), we have

Definition 5.2.1 (Xu and Hu, 2010) Let $\tilde{Y}_i = (\tilde{r}_{i1}, \tilde{r}_{i2}, \cdots, \tilde{r}_{im})$ be the between alternative, and $\tilde{Y}^* = (\tilde{\alpha}_1^*, \tilde{\alpha}_2^*, \cdots, \tilde{\alpha}_m^*)$ the RUIFIS, where $\tilde{r}_{ij} = (\tilde{\mu}_{ij},\ \tilde{\nu}_{ij},\ \tilde{\pi}_{ij}) = ([\tilde{\mu}_{ij}^{\mathrm{L}}, \tilde{\mu}_{ij}^{\mathrm{U}}], [\tilde{\nu}_{ij}^{\mathrm{L}}, \tilde{\nu}_{ij}^{\mathrm{U}}], [\tilde{\pi}_{ij}^{\mathrm{L}}, \tilde{\pi}_{ij}^{\mathrm{U}}])$ $(j = 1, 2, \cdots, m)$. Then we call

$$\cos(\tilde{Y}_i,\ \tilde{Y}^*) = \frac{\sum_{j=1}^{m} \left(\tilde{\mu}_{ij}^{\mathrm{L}}\tilde{\mu}_j^{*\mathrm{L}} + \tilde{\mu}_{ij}^{\mathrm{U}}\tilde{\mu}_j^{*\mathrm{U}} + \tilde{\nu}_{ij}^{\mathrm{L}}\tilde{\nu}_j^{*\mathrm{L}} + \tilde{\nu}_{ij}^{\mathrm{U}}\tilde{\nu}_j^{*\mathrm{U}} + \tilde{\pi}_{ij}^{\mathrm{L}}\tilde{\pi}_j^{*\mathrm{L}} + \tilde{\pi}_{ij}^{\mathrm{U}}\tilde{\pi}_j^{*\mathrm{U}}\right)}{|\tilde{Y}_i|\ |\tilde{Y}^*|} \tag{5.24}$$

the cosine of the included angle between $\tilde{Y}_i$ and $\tilde{Y}^*$.

Obviously, the following theorem holds according to Eq.(5.24):

Theorem 5.2.1 (Xu and Hu, 2010)

(1) $\cos(\tilde{Y}_i,\ \tilde{Y}^*) = \cos(\tilde{Y}^*, \tilde{Y}_i)$;

(2) $0 \leqslant \cos(\tilde{Y}_i,\ \tilde{Y}^*) \leqslant 1$;

(3) $\cos(\tilde{Y}_i,\ \tilde{Y}^*) = 1$ if $\tilde{Y}_i = \tilde{Y}^*$.

Based on Definition 5.2.1, we give a formula about the projection of $\tilde{Y}_i$ on $\tilde{Y}^*$:

$$\begin{aligned}&\mathrm{Prj}_{\tilde{Y}^*}\tilde{Y}_i = |\tilde{Y}_i|\cos(\tilde{Y}_i, \tilde{Y}^*)\\ =&\frac{1}{|\tilde{Y}^*|}\sum_{j=1}^{m}\left(\tilde{\mu}_{ij}^{\mathrm{L}}\tilde{\mu}_j^{*\mathrm{L}} + \tilde{\mu}_{ij}^{\mathrm{U}}\tilde{\mu}_j^{*\mathrm{U}} + \tilde{\nu}_{ij}^{\mathrm{L}}\tilde{\nu}_j^{*\mathrm{L}} + \tilde{\nu}_{ij}^{\mathrm{U}}\tilde{\nu}_j^{*\mathrm{U}} + \tilde{\pi}_{ij}^{\mathrm{L}}\tilde{\pi}_j^{*\mathrm{L}} + \tilde{\pi}_{ij}^{\mathrm{U}}\tilde{\pi}_j^{*\mathrm{U}}\right)\end{aligned} \tag{5.25}$$

It is clear that the greater the value $\mathrm{Prj}_{\tilde{Y}^*}\tilde{Y}_i$, the closer $\tilde{Y}_i$ to $\tilde{Y}^*$, and thus the better the alternative $\tilde{Y}_i$.

If the weight vector $\omega = (\omega_1, \omega_2, \cdots, \omega_m)^{\mathrm{T}}$ of the attributes G_j $(j = 1, 2, \cdots, m)$ is known, then we denote

$$|\tilde{Y}_i|_\omega = \sqrt{\sum_{j=1}^{m}(\omega_j|\tilde{r}_{ij}|)^2} \tag{5.26}$$

as the weighted module of the alternative $\tilde{Y}_i = (\tilde{r}_{i1}, \tilde{r}_{i2}, \cdots, \tilde{r}_{im})$. Furthermore, we call

$$|\tilde{Y}^*|_\omega = \sqrt{\sum_{j=1}^{m}(\omega_j|\tilde{\alpha}_j^*|)^2} \tag{5.27}$$

the weighted module of the RUIFIS $\tilde{Y}^* = (\tilde{\alpha}_1^*, \tilde{\alpha}_2^*, \cdots, \tilde{\alpha}_m^*)$.

Based on Eqs.(5-25)-(5.27), we can introduce the weighted cosine of the included angle between the alternative $\tilde{Y}_i = (\tilde{r}_{i1}, \tilde{r}_{i2}, \cdots, \tilde{r}_{im})$ and the RUIFIS $\tilde{Y}^* = (\tilde{\alpha}_1^*, \tilde{\alpha}_2^*, \cdots, \tilde{\alpha}_m^*)$:

$$\cos(\tilde{Y}_i,\ \tilde{Y}^*)_\omega = \frac{\sum\limits_{j=1}^{m}\omega_j^2\left(\tilde{\mu}_{ij}^{\mathrm{L}}\tilde{\mu}_j^{*\mathrm{L}} + \tilde{\mu}_{ij}^{\mathrm{U}}\tilde{\mu}_j^{*\mathrm{U}} + \tilde{\nu}_{ij}^{\mathrm{L}}\tilde{\nu}_j^{*\mathrm{L}} + \tilde{\nu}_{ij}^{\mathrm{U}}\tilde{\nu}_j^{*\mathrm{U}} + \tilde{\pi}_{ij}^{\mathrm{L}}\tilde{\pi}_j^{*\mathrm{L}} + \tilde{\pi}_{ij}^{\mathrm{U}}\tilde{\pi}_j^{*\mathrm{U}}\right)}{|\tilde{Y}_i|_\omega\ |\tilde{Y}^*|_\omega} \tag{5.28}$$

where $\tilde{r}_{ij} = (\tilde{\mu}_{ij},\ \tilde{v}_{ij},\ \tilde{\pi}_{ij}) = ([\tilde{\mu}_{ij}^{\mathrm{L}}, \tilde{\mu}_{ij}^{\mathrm{U}}], [\tilde{\nu}_{ij}^{\mathrm{L}}, \tilde{\nu}_{ij}^{\mathrm{U}}], [\tilde{\pi}_{ij}^{\mathrm{L}}, \tilde{\pi}_{ij}^{\mathrm{U}}])$, $i = 1, 2, \cdots, m$.

Similar to Eq.(5.25), we give a formula about the projection of $\tilde{Y}_i$ on $\tilde{Y}^*$:

$$\begin{aligned}&\mathrm{Prj}_{\tilde{Y}^*}\tilde{Y}_i = |\tilde{Y}_i|_\omega\cos(\tilde{Y}_i, \tilde{Y}^*)_\omega\\ =&\frac{1}{|\tilde{Y}^*|_\omega}\sum_{j=1}^{m}\omega_j^2\left(\tilde{\mu}_{ij}^{\mathrm{L}}\tilde{\mu}_j^{*\mathrm{L}} + \tilde{\mu}_{ij}^{\mathrm{U}}\tilde{\mu}_j^{*\mathrm{U}} + \tilde{\nu}_{ij}^{\mathrm{L}}\tilde{\nu}_j^{*\mathrm{L}} + \tilde{\nu}_{ij}^{\mathrm{U}}\tilde{\nu}_j^{*\mathrm{U}} + \tilde{\pi}_{ij}^{\mathrm{L}}\tilde{\pi}_j^{*\mathrm{L}} + \tilde{\pi}_{ij}^{\mathrm{U}}\tilde{\pi}_j^{*\mathrm{U}}\right)\end{aligned} \tag{5.29}$$

Clearly, the greater the value $\mathrm{Prj}_{\tilde{Y}^*}\tilde{Y}_i$, the closer $\tilde{Y}_i$ to $\tilde{Y}^*$, and thus the better the alternative $\tilde{Y}_i$.

Example 5.2.1 (Xu and Hu, 2010) Let us consider a customer who intends to buy a car. Five types of cars (alternatives) Y_i $(i = 1, 2, 3, 4, 5)$ are available. The customer takes into account six attributes to decide which car to buy (Hung, 2001): ① G_1: Fuel economy; ② G_2: Aerod. degree; ③ G_3: Price; ④ G_4: Comfort; ⑤ G_5: Design; and ⑥ G_6: Safety, where the attribute G_3 is a cost attribute, and the other five attributes are benefit attributes. Assume that the characteristics of the alternatives Y_i $(i = 1, 2, 3, 4, 5)$ are represented by the intuitionistic fuzzy decision matrix $R' = (r'_{ij})_{6\times 5}$ as shown in Table 5.1:

Table 5.1 Intuitionistic fuzzy decision matrix R' (Xu and Hu, 2010)

	G_1	G_2	G_3	G_4	G_5	G_6
Y_1	(0.5,0.4,0.1)	(0.7,0.2,0.1)	(0.4,0.3,0.3)	(0.6, 0.2, 0.2	(0.4,0.5,0.1)	(0.3,0.1,0.6)
Y_2	(0.4,0.3,0.3)	(0.8,0.2,0.0)	(0.5,0.2,0.3)	(0.6,0.3,0.1)	(0.6,0.4,0.0)	(0.7,0.1,0.2)
Y_3	(0.5,0.2,0.3)	(0.9,0.1,0.0)	(0.6,0.1,0.3)	(0.8,0.1,0.1)	(0.3,0.5,0.2)	(0.6,0.2,0.2)
Y_4	(0.4,0.2,0.4)	(0.8,0.0,0.0)	(0.7,0.3,0.0)	(0.9,0.1,0.0)	(0.5,0.3,0.2)	(0.6,0.1,0.3
Y_5	(0.6,0.4,0.0)	(0.5,0.2,0.3)	(0.8,0.1,0.1)	(0.4,0.2,0.4)	(0.9,0.0,0.1)	(0.4,0.3,0.3)

Considering that the attributes have two different types, we first transform the attribute values of cost type into attribute values of benefit type by using Eq.(5.2). Consequently, $R' = (r'_{ij})_{5\times 6}$ is transformed into $R = (r_{ij})_{5\times 6}$ (Table 5.2):

Table 5.2 Intuitionistic fuzzy decision matrix R (Xu and Hu, 2010)

	G_1	G_2	G_3	G_4	G_5	G_6
Y_1	(0.5,0.4,0.1)	(0.7,0.2,0.1)	(0.3,0.4,0.3)	(0.6,0.2,0.2)	(0.4,0.5,0.1)	(0.3,0.1,0.6)
Y_2	(0.4,0.3,0.3)	(0.8,0.2,0.0)	(0.2,0.5,0.3)	(0.6,0.3,0.1)	(0.6,0.4,0.0)	(0.7,0.1,0.2)
Y_3	(0.5,0.2,0.3)	(0.9,0.1,0.0)	(0.1,0.6,0.3)	(0.8,0.1,0.1)	(0.3,0.5,0.2)	(0.6,0.2,0.2)
Y_4	(0.4,0.2,0.4)	(0.8,0.0,0.0)	(0.3,0.7,0.0)	(0.9,0.1,0.0)	(0.5,0.3,0.2)	(0.6,0.1,0.3)
Y_5	(0.6,0.4,0.0)	(0.5,0.2,0.3)	(0.1,0.8,0.1)	(0.4,0.2,0.4)	(0.9,0.0,0.1)	(0.4,0.3,0.3)

To get the most desirable car, the following steps are followed:

Step 1 Based on $R = (r_{ij})_{5\times 6}$, we denote the alternatives Y_i $(i = 1, 2, 3, 4, 5)$ by

$$Y_1 =((0.5, 0.4, 0.1), (0.7, 0.2, 0.1), (0.3, 0.4, 0.3),\\ (0.6, 0.2, 0.2), (0.4, 0.5, 0.1), (0.3, 0.1, 0.6))$$

$$Y_2 =((0.4, 0.3, 0.3), (0.8, 0.2, 0.0), (0.2, 0.5, 0.3),\\ (0.6, 0.3, 0.1), (0.6, 0.4, 0.0), (0.7, 0.1, 0.2))$$

$$Y_3 =((0.5, 0.2, 0.3), (0.9, 0.1, 0.0), (0.1, 0.6, 0.3),\\ (0.8, 0.1, 0.1), (0.3, 0.5, 0.2), (0.6, 0.2, 0.2))$$

$$Y_4 = ((0.4, 0.2, 0.4), (0.8, 0.0, 0.2), (0.3, 0.7, 0.0), (0.9, 0.1, 0.0), (0.5, 0.3, 0.2), (0.6, 0.1, 0.3))$$

$$Y_5 = ((0.6, 0.4, 0.0), (0.5, 0.2, 0.3), (0.1, 0.8, 0.1), (0.4, 0.2, 0.4), (0.9, 0.0, 0.1), (0.4, 0.3, 0.3))$$

$$Y^* = ((0.6, 0.2, 0.2), (0.9, 0.0, 0.1), (0.3, 0.4, 0.3), (0.9, 0.1, 0.0), (0.9, 0.0, 0.1), (0.7, 0.1, 0.2))$$

Then using Eqs.(5.6) and (5.7), we get the RIFIS Y^*:

$$Y^* = ((0.6, 0.2, 0.2), (0.9, 0.0, 0.1), (0.3, 0.4, 0.3), (0.9, 0.1, 0.0), (0.9, 0.0, 0.1), (0.7, 0.1, 0.2))$$

Step 2 Calculate the projection of Y_i on Y^* by using Eq.(5.10):

$$\mathrm{Prj}_{Y^*} Y_1 = 1.363, \quad \mathrm{Prj}_{Y^*} Y_2 = 1.584, \quad \mathrm{Prj}_{Y^*} Y_3 = 1.579$$

$$\mathrm{Prj}_{Y^*} Y_4 = 1.661, \quad \mathrm{Prj}_{Y^*} Y_5 = 1.476$$

Step 3 Rank the alternatives Y_i $(i = 1, 2, 3, 4, 5)$ in accordance with the values $\mathrm{Prj}_{Y^*} Y_i$ $(i = 1, 2, 3, 4, 5)$:

$$Y_4 \succ Y_2 \succ Y_3 \succ Y_5 \succ Y_1$$

Thus, the most desirable car Y_4 is identified.

If the characteristics of the alternatives Y_i $(i = 1, 2, 3, 4, 5)$ are represented by the interval-valued intuitionistic fuzzy decision matrix $\tilde{R}' = (\tilde{r}'_{ij})_{5\times 6}$ as shown in Table 5.3:

Table 5.3 Interval-valued intuitionistic fuzzy decision matrix $\tilde{R}'$ (Xu and Hu, 2010)

	G_1	G_2	G_3
Y_1	([0.2,0.5],[0.4,0.5],[0.0,0.4])	([0.3,0.4],[0.3,0.4],[0.2,0.4])	([0.3,0.4],[0.4,0.6],[0.0,0.3])
Y_2	([0.3,0.4],[0.3,0.5],[0.1,0.4])	([0.7,0.8],[0.0,0.2],[0.0,0.3])	([0.5,0.6],[0.2,0.4],[0.0,0.3])
Y_3	([0.4,0.5],[0.2,0.3],[0.2,0.4])	([0.8,0.9],[0.0,0.1],[0.0,0.2])	([0.6,0.7],[0.1,0.3],[0.0,0.3])
Y_4	([0.4,0.6],[0.2,0.4],[0.0,0.4])	([0.8,0.9],[0.0,0.1],[0.0,0.2])	([0.5,0.6],[0.2,0.3],[0.1,0.3])
Y_5	([0.5,0.6],[0.3,0.4],[0.0,0.2])	([0.4,0.6],[0.1, 0.2],0.2,0.5])	([0.8,0.9],[0.0,0.1],[0.0,0.2])
	G_4	G_5	G_6
Y_1	([0.4,0.5],[0.2,0.3],[0.2,0.4])	([0.2,0.4],[0.5,0.6],[0.0,0.3])	([0.3,0.4],[0.2,0.5],[0.1,0.5])
Y_2	([0.5,0.6],[0.2,0.3],[0.1,0.3])	([0.4,0.6],[0.2,0.4],[0.0,0.4])	([0.6,0.7],[0.1,0.3],[0.0,0.3])
Y_3	([0.7,0.8],[0.1,0.2],[0.0,0.2])	([0.3,0.4],[0.4,0.5],[0.1,0.3])	([0.5,0.6],[0.1,0.2],[0.2,0.4])
Y_4	([0.8,0.9],[0.0,0.1],[0.0,0.2])	([0.5,0.7],[0.2,0.3],[0.0,0.3])	([0.6,0.8],[0.1,0.2],[0.0,0.3])
Y_5	([0.3,0.4],[0.2,0.3],[0.3,0.5])	([0.8,0.9],[0.0,0.1],[0.0,0.2])	([0.4,0.5],[0.2,0.3],[0.2,0.4])

then, we can first transform the attribute values of cost type into attribute values of benefit type by using Eq.(5.16). As a result, $\tilde{R}' = (\tilde{r}'_{ij})_{5\times 6}$ is transformed into $\tilde{R} = (\tilde{r}_{ij})_{5\times 6}$ (Table 5.4):

Table 5.4 Interval-valued intuitionistic fuzzy decision matrix $\tilde{R}$ (Xu and Hu, 2010)

	G_1	G_2	G_3
Y_1	([0.2,0.5],[0.4,0.5],[0.0,0.4])	([0.3,0.4],[0.3,0.4],[0.2,0.4])	([0.4,0.6],[0.3,0.4],[0.0,0.3])
Y_2	([0.3,0.4],[0.3,0.5],[0.1,0.4])	([0.7,0.8],[0.0,0.2],[0.0,0.3])	([0.2,0.4],[0.5,0.6],[0.0,0.3])
Y_3	([0.4,0.5],[0.2,0.3],[0.2,0.4])	([0.8,0.9],[0.0,0.1],[0.0,0.2])	([0.1,0.3],[0.6,0.7],[0.0,0.3])
Y_4	([0.4,0.6],[0.2,0.4],[0.0,0.4])	([0.8,0.9],[0.0,0.1],[0.0,0.2])	([0.2,0.3],[0.5,0.6],[0.1,0.3])
Y_5	([0.5,0.6],[0.3,0.4],[0.0,0.2])	([0.4,0.6],[0.1,0.2],[0.2,0.5])	([0.0,0.1],[0.8,0.9],[0.0,0.2])
	G_4	G_5	G_6
Y_1	([0.4,0.5],[0.2,0.3],[0.2,0.4])	([0.2,0.4],[0.5,0.6],[0.0,0.3])	([0.3,0.4],[0.2,0.5],[0.1,0.5])
Y_2	([0.5,0.6],[0.2,0.3],[0.1,0.3])	([0.4,0.6],[0.2,0.4],[0.0,0.4])	([0.6,0.7],[0.1,0.3],[0.0,0.3])
Y_3	([0.7,0.8],[0.1,0.2],[0.0,0.2])	([0.3,0.4],[0.4,0.5],[0.1,0.3])	([0.5,0.6],[0.1,0.2],[0.2,0.4])
Y_4	([0.8,0.9],[0.0,0.1],[0.0,0.2])	([0.5,0.7],[0.2,0.3],[0.0,0.3])	([0.6,0.8],[0.1,0.2],[0.0,0.3])
Y_5	([0.3,0.4],[0.2,0.3],[0.3,0.5])	([0.8,0.9],[0.0,0.1],[0.0,0.2])	([0.4,0.5],[0.2,0.3],[0.2,0.4])

Step 1 Based on $\tilde{R}$, we denote the alternatives $\tilde{Y}_i$ $(i = 1, 2, \cdots, 5)$ by

$$\begin{aligned}\tilde{Y}_1 =&(([0.2, 0.5], [0.4, 0.5], [0.0, 0.4]),\ ([0.3, 0.4], [0.3, 0.4], [0.2, 0.4]),\\ &([0.4, 0.6], [0.3, 0.4], [0.0, 0.3]), ([0.4, 0.5], [0.2, 0.3], [0.2, 0.4]),\\ &([0.2, 0.4], [0.5, 0.6], [0.0, 0.3]),\ ([0.3, 0.4], [0.2, 0.5], [0.1, 0.5]))\\ \tilde{Y}_2 =&(([0.3, 0.4], [0.3, 0.5], [0.1, 0.4]),\ ([0.7, 0.8], [0.0, 0.2], [0.0, 0.3]),\\ &([0.2, 0.4], [0.5, 0.6], [0.0, 0.3]), ([0.5, 0.6], [0.2, 0.3], [0.1, 0.3]),\\ &([0.4, 0.6], [0.2, 0.4], [0.0, 0.4]),\ ([0.6, 0.7], [0.1, 0.3], [0.0, 0.3]))\\ \tilde{Y}_3 =&(([0.4, 0.5], [0.2, 0.3], [0.2, 0.4]),\ ([0.8, 0.9], [0.0, 0.1], [0.0, 0.23]),\\ &([0.1, 0.3], [0.6, 0.7], [0.0, 0.3]), ([0.7, 0.8], [0.1, 0.2], [0.0, 0.2]),\\ &([0.3, 0.4], [0.4, 0.5], [0.1, 0.3]),\ ([0.5, 0.6], [0.1, 0.2], [0.2, 0.4]))\\ \tilde{Y}_4 =&(([0.4, 0.6], [0.2, 0.4], [0.0, 0.4]),\ ([0.8, 0.9], [0.0, 0.1], [0.0, 0.2]),\\ &([0.2, 0.3], [0.5, 0.6], [0.1, 0.3]), ([0.8, 0.9], [0.0, 0.1], [0.0, 0.2]),\\ &([0.5, 0.7], [0.2, 0.3], [0.0, 0.3]),\ ([0.6, 0.8], [0.1, 0.2], [0.0, 0.3]))\\ \tilde{Y}_5 =&(([0.5, 0.6], [0.3, 0.4], [0.0, 0.2]),\ ([0.4, 0.6], [0.1, 0.2], [0.2, 0.5]),\\ &([0.0, 0.1], [0.8, 0.9], [0.0, 0.2]), ([0.3, 0.4], [0.2, 0.3], [0.3, 0.5]),\\ &([0.8, 0.9], [0.0, 0.1], [0.0, 0.2]),\ ([0.4, 0.5], [0.2, 0.3], [0.2, 0.4]))\end{aligned}$$

Consequently, by Eqs.(5.20) and (5.21), we get the RUIFIS $\tilde{Y}^*$:

$$\begin{aligned}\tilde{Y}^* =&(([0.5, 0.6], [0.2, 0.3], [0.1, 0.3]), ([0.8, 0.9], [0.0, 0.1], [0.0, 0.2]),\\ &([0.4, 0.6], [0.3, 0.4], [0.0, 0.3]), ([0.8, 0.9], [0.0, 0.1], [0.0, 0.2]),\\ &([0.8, 0.9], [0.0, 0.1], [0.0, 0.2]), ([0.6, 0.8], [0.1, 0.2], [0.0, 0.3]))\end{aligned}$$

Step 2 Calculate the projection of $\tilde{Y}_i$ on $\tilde{Y}^*$ by using Eq.(5.29):

$$\mathrm{Pr\,j}_{\tilde{Y}^*}\tilde{Y}_1 = 1.694, \quad \mathrm{Pr\,j}_{\tilde{Y}^*}\tilde{Y}_2 = 2.207, \quad \mathrm{Pr\,j}_{\tilde{Y}^*}\tilde{Y}_3 = 2.229$$

$$\mathrm{Pr\,j}_{\tilde{Y}^*}\tilde{Y}_4 = 2.525, \quad \mathrm{Pr\,j}_{\tilde{Y}^*}\tilde{Y}_5 = 2.074$$

Step 3 Rank the alternatives Y_i $(i = 1, 2, \cdots, 5)$ in accordance with the values $\mathrm{Pr\,j}_{\tilde{Y}^*}\tilde{Y}_i$ $(i = 1, 2, \cdots, 5)$:

$$Y_4 \succ Y_3 \succ Y_2 \succ Y_5 \succ Y_1$$

and thus, the most desirable car is also Y_4.

References

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Chapter 6

Dynamic Intuitionistic Fuzzy Multi-Attribute Decision Making

In the previous chapters, we have discussed the problems where all the intuitionstic fuzzy data are collected in the same time period or at the same stage. However, in many practical problems, such as multi-period investment decision making, medical diagnosis, personnel dynamic examination, and military system efficiency dynamic evaluation, the decision information is usually collected at different periods of time. Recently, Xu and Yager (2008) investigate aggregation techniques for dynamic intuitionistic fuzzy information, develop methods for weighting time series, and propose an approach to dynamic intuitionistic fuzzy multi-attribute decision making. They also extend the derived results to uncertain dynamic intuitionistic fuzzy environments.

6.1 Dynamic Intuitionistic Fuzzy Weighted Averaging Operators

Based on Eq.(1.9), we first introduce the concept of intuitionistic fuzzy variable:

Definition 6.1.1 (Xu and Yager, 2008) Let t be a time variable. Then $\alpha(t) = (\mu_{\alpha(t)}, \nu_{\alpha(t)}, \pi_{\alpha(t)})$ is called an intuitionistic fuzzy variable, where

$$\mu_{\alpha(t)} \in [0,1], \quad \nu_{\alpha(t)} \in [0,1], \quad \mu_{\alpha(t)} + \nu_{\alpha(t)} \leqslant 1, \quad \pi_{\alpha(t)} = 1 - \mu_{\alpha(t)} - \nu_{\alpha(t)} \tag{6.1}$$

For an intuitionistic fuzzy variable $\alpha(t) = (\mu_{\alpha(t)}, \nu_{\alpha(t)}, \pi_{\alpha(t)})$, if $t = t_1, t_2, \cdots, t_p$, then $\alpha(t_1), \alpha(t_2), \cdots, \alpha(t_n)$ indicate p IFNs collected at p different periods.

Below we slightly improve two operational laws given in Definition 1.2.2:

Definition 6.1.2 (Xu and Yager, 2008) Let $\alpha(t) = (\mu_{\alpha(t)}, \nu_{\alpha(t)}, \pi_{\alpha(t)})$ be an intuitionistic fuzzy variable, $\alpha(t_1) = (\mu_{\alpha_1(t_1)}, \nu_{\alpha_1(t_1)}, \pi_{\alpha_1(t_1)})$ and $\alpha(t_2) = (\mu_{\alpha_2(t_2)}, \nu_{\alpha_2(t_2)}, \pi_{\alpha_2(t_2)})$ the values of the intuitionistic fuzzy variable $\alpha(t)$ taking $t = t_1, t_2$. Then

(1) $\alpha(t_1) \oplus \alpha(t_2) = (\mu_{\alpha(t_1)} + \mu_{\alpha(t_2)} - \mu_{\alpha(t_1)}\mu_{\alpha(t_2)}, \nu_{\alpha(t_1)}\nu_{\alpha(t_2)},$

$$(1 - \mu_{\alpha(t_1)})(1 - \mu_{\alpha(t_2)}) - \nu_{\alpha(t_1)}\nu_{\alpha(t_2)});$$

(2) $\lambda\,\alpha(t_1) = (1 - (1 - \mu_{\alpha(t_1)})^{\lambda}, \nu_{\alpha(t_1)}^{\lambda}, (1 - \mu_{\alpha(t_1)})^{\lambda} - \nu_{\alpha(t_1)}^{\lambda})$, $\lambda > 0$.

Definition 6.1.3 (Xu and Yager, 2008) Let $\alpha(t_1), \alpha(t_2), \cdots, \alpha(t_p)$ be a collection of IFNs collected at p different periods $t_k\,(k=1,2,\cdots,p)$, and $\omega(t)=(\omega(t_1),\omega(t_2),\cdots,\omega(t_p))^{\mathrm{T}}$ the weight vector of the periods $t_k\,(k=1,2,\cdots,p)$, with $\omega(t_k)\in[0,1]$, $k=1,2,\cdots,p$, and $\sum_{k=1}^{p}\omega(t_k)=1$. Then we call

$$\mathrm{DIFWA}_{\omega(t)}(\alpha(t_1),\alpha(t_2),\cdots,\alpha(t_p))=\omega(t_1)\alpha(t_1)\oplus\omega(t_2)\alpha(t_2)\oplus\cdots\oplus\omega(t_p)\alpha(t_p) \quad (6.2)$$

a dynamic intuitionistic fuzzy weighted averaging (DIFWA) operator.

By Definition 6.1.2, Eq.(6.2) can be rewritten as follows:

$$\begin{aligned}&\mathrm{DIFWA}_{\omega(t)}(\alpha(t_1),\alpha(t_2),\cdots,\alpha(t_p))\\&=\left(1-\prod_{k=1}^{p}(1-\mu_{\alpha(t_k)})^{\omega(t_k)},\ \prod_{k=1}^{p}\nu_{\alpha(t_k)}^{\omega(t_k)},\ \prod_{k=1}^{p}(1-\mu_{\alpha(t_k)})^{\omega(t_k)}-\prod_{k=1}^{p}\nu_{\alpha(t_k)}^{\omega(t_k)}\right)\end{aligned} \quad (6.3)$$

where

$$\omega(t_k)\in[0,1],\quad k=1,2,\cdots,p,\quad \sum_{k=1}^{p}\omega(t_k)=1 \quad (6.4)$$

In what follows, we introduce some methods to determine the weight vector $\omega(t)=(\omega(t_1),\omega(t_2),\cdots,\omega(t_p))^{\mathrm{T}}$ of the periods $t_k\,(k=1,2,\cdots,p)$:

(1) Arithmetic series based method (Xu, 2008c): Suppose that the difference value between the weight $\omega(t_{k+1})$ and its adjacent weight $\omega(t_k)$ is a constant β, i.e.,

$$\omega(t_{k+1})-\omega(t_k)=\beta,\quad k=1,2,\cdots,p-1 \quad (6.5)$$

In this case, we have

$$\omega(t_k)=\eta+(k-1)\,\beta,\quad \eta+(k-1)\,\beta\geqslant 0 \quad (6.6)$$

under the condition (6.4).

From Eq.(6.4), we have

(i) If $\beta=0$, then $\eta=1/n$, i.e., $\omega(t_k)=1/n\,(k=1,2,\cdots,p)$, which indicates that all the weights $\omega(t_k)$ $(k=1,2,\cdots,p)$ are equal.

(ii) If $\beta>0$, then $\omega(t_k)<\omega(t_{k+1})\,(k=1,2,\cdots,p-1)$, i.e., $\{\omega(t_k)\}$ is a strictly monotonically increasing sequence;

(iii) If $\beta<0$, then $\omega(t_k)>\omega(t_{k+1})\,(k=1,2,\cdots,p-1)$, i.e., $\{\omega(t_k)\}$ is a strictly monotonically decreasing sequence.

(2) Geometric series based method (Xu, 2008c): Suppose that the weight $\omega(t_{k+1})$ is β times as good as its adjacent weight $\omega(t_k)$, i.e.,

$$\omega(t_{k+1})=\beta\,\omega(t_k),\quad \beta>0,\quad k=1,2,\cdots,p-1 \quad (6.7)$$

In this case, we have

$$\omega(t_k) = \eta\,\beta^{k-1}, \quad \eta, \quad \beta > 0, \quad k = 1, 2, \cdots, p-1 \tag{6.8}$$

Using Eq.(6.4), we have

$$\eta = \frac{1}{\sum_{k=1}^{p} \beta^{k-1}}, \quad \beta > 0 \tag{6.9}$$

Hence

$$\omega(t_k) = \frac{\beta^{k-1}}{\sum_{j=1}^{p} \beta^{j-1}}, \quad \beta > 0, \quad k = 1, 2, \cdots, p \tag{6.10}$$

From Eq.(6.10), we have

(i) If $\beta = 1$, then $\eta = 1/n$, i.e., $\omega(t_k) = 1/n\,(k = 1, 2, \cdots, p)$. In this case, all the weights $\omega(t_k)$ $(k = 1, 2, \cdots, p)$ are equal.

(ii) If $\beta > 1$, then $\omega(t_k) < \omega(t_{k+1})\,(k = 1, 2, \cdots, p-1)$. In this case, $\{\omega(t_k)\}$ is a strictly monotonically increasing sequence;

(iii) If $\beta < 1$, then $\omega(t_k) > \omega(t_{k+1})\,(k = 1, 2, \cdots, p-1)$. In this case, $\{\omega(t_k)\}$ is a strictly monotonically decreasing sequence.

(3) BUM function based method (Xu and Yager, 2008): Let f be a BUM function. Then we can obtain the weight vector $\omega(t_k)$ as follows:

$$\omega(t_k) = f\left(\frac{k}{p}\right) - f\left(\frac{k-1}{p}\right), \quad k = 1, 2, \cdots, p \tag{6.11}$$

under the condition (6.4). For example, if $f(x) = x^r$, $r > 0$, then

$$\omega(t_k) = \left(\frac{k}{p}\right)^r - \left(\frac{k-1}{p}\right)^r = \left(\frac{k}{p}\right)^r - \left(\frac{k}{p} - \frac{1}{p}\right)^r, \quad k = 1, 2, \cdots, p \tag{6.12}$$

Let

$$g(x) = x^r - \left(x - \frac{1}{p}\right)^r, \quad x \geqslant \frac{1}{p} \tag{6.13}$$

Then

$$g'(x) = rx^{r-1} - r\left(x - \frac{1}{p}\right)^{r-1} = r\left(x^{r-1} - \left(x - \frac{1}{p}\right)^{r-1}\right) \tag{6.14}$$

Thus

(i) If $r > 1$, then $g'(x) > 0$, i.e., $g(x)$ is a strictly monotonically increasing function;

(ii) If $r = 1$, then $g'(x) = 0$, i.e., $g(x)$ is a constant;

(iii) If $r < 1$, then $g'(x) < 0$, i.e., $g(x)$ is a strictly monotonically decreasing function.

Consequently, from Eq.(6.11), we have

(i) If $r > 1$, then $\omega(t_{k+1}) > \omega(t_k)$, $k = 1, 2, \cdots, p-1$, i.e., $\{\omega(t_k)\}$ is a strictly monotonically increasing sequence. In particular, if $r = 2$, then

$$\omega(t_{k+1}) - \omega(t_k) = \left(\frac{k+1}{p}\right)^2 - \left(\frac{k}{p}\right)^2 - \left(\frac{k}{p}\right)^2 + \left(\frac{k-1}{p}\right)^2 = \frac{2}{p^2}, \quad k = 1, 2, \cdots, p-1 \tag{6.15}$$

i.e., $\{\omega(t_k)\}$ is a strictly monotonically increasing arithmetic sequence;

(ii) If $r = 1$, then

$$\omega(t_k) = \frac{k}{p} - \frac{k-1}{p} = \frac{1}{p}, \quad k = 1, 2, \cdots, p \tag{6.16}$$

hence $\omega(t) = (1/p, 1/p, \cdots, 1/p)^{\mathrm{T}}$;

(iii) If $r < 1$, then $\omega(t_{k+1}) < \omega(t_k)$, $k = 1, 2, \cdots, p-1$, i.e., $\{\omega(t_k)\}$ is a strictly monotonically decreasing sequence.

(4) Normal distribution based method (Xu and Yager, 2008): The normal distribution is one of the most commonly observed and is the starting point for modeling many natural processes. It is usually found in events that are aggregation of many smaller, but independent random events.

The probability density function of a normally distributed variable x is defined as follows:

$$g(x) = \frac{1}{\sqrt{2\pi}\sigma}\mathrm{e}^{-\frac{(x-u)^2}{2\sigma^2}}, \quad -\infty < x < \infty \tag{6.17}$$

where u is the mean and $\sigma(\sigma > 0)$ is the standard deviation.

We can utilize the normal distribution based method to determine the weight vector of the time series $\{t_k\}$ $(k = 1, 2, \cdots, p)$ (Xu, 2005a):

$$\omega(t_k) = \frac{1}{\sqrt{2\pi}\sigma_p}\mathrm{e}^{-\frac{(k-\bar{\mu}_p)^2}{2\sigma_p^2}}, \quad k = 1, 2, \cdots, p \tag{6.18}$$

where $\bar{\mu}_p$ is the mean of the collection of $1, 2, \cdots, p$, and σ_p $(\sigma_p > 0)$ is the standard deviation of the collection of $1, 2, \cdots, p$. $\bar{\mu}_p$ and σ_p can be obtained by using the following formulas respectively:

$$\bar{\mu}_p = \frac{1}{p}\frac{p(1+p)}{2} = \frac{1+p}{2} \tag{6.19}$$

$$\sigma_p = \sqrt{\frac{1}{p}\sum_{k=1}^{p}(k - \bar{\mu}_p)^2} \tag{6.20}$$

It follows from Eqs.(6.4) and (6.18) that

$$\omega(t_k) = \frac{\mathrm{e}^{-\frac{(k-\bar{\mu}_p)^2}{2\sigma_p^2}}}{\sum_{k=1}^{p} \mathrm{e}^{-\frac{(j-\bar{\mu}_p)^2}{2\sigma_p^2}}}, \quad k = 1, 2, \cdots, p \tag{6.21}$$

According to Eq.(6.20), we have

(i) The weights $\omega(t_k)(k = 1, 2, \cdots, p)$ are symmetrical, i.e.,

$$\omega(t_k) = \omega(t_{p+1-k}), \quad k = 1, 2, \cdots, p \tag{6.22}$$

(ii) (a) $\omega(t_k) < \omega(t_{k+1})$, $k = 1, \cdots, \mathrm{round}\left(\frac{1+p}{2}\right)$;

(b) If p is odd, then

$$\omega(t_k) > \omega(t_{k+1}), \quad k = \mathrm{round}\left(\frac{1+p}{2}\right), \cdots, p \tag{6.23}$$

(c) If p is even, then

$$\omega(t_k) > \omega(t_{k+1}), \quad k = \mathrm{round}\left(\frac{1+p}{2}\right) + 1, \cdots, p \tag{6.24}$$

where "round" is the usual round operation. In particular, if p is odd, and $k = \mathrm{round}\left(\frac{1+p}{2}\right)$, then the weight $\omega(t_k)$ reaches its maximum; if p is even, and $k = \mathrm{round}\left(\frac{1+p}{2}\right)$ or $1 + \mathrm{round}\left(\frac{1+p}{2}\right)$, then the weight $\omega(t_k)$ reaches its minimum.

Clearly, the normal distribution based method assigns the intermediate period the maximal weight. Then the farther the period deviated from the intermediate period, the smaller the weight assigned.

(5) Exponential distribution based method (Sadiq and Tesfamariam, 2007): The exponential distribution is a memoryless continuous distribution. The exponential distribution is often used to model the time between random arrivals of events that occur at a constant average rate. The probability density function of an exponential variable x is defined as follows:

$$q(x) = \frac{1}{\bar{\mu}} \mathrm{e}^{-\frac{x}{\bar{\mu}}}, \quad x > 0 \tag{6.25}$$

where $\bar{\mu}$ is the mean time between failures.

To generate the weight vector $\omega(t)$ using the probability density function of an exponential distribution, Eq.(6.25) can be rewritten as follows:

$$\omega(t_k) = \frac{1}{\bar{\mu}_p} \mathrm{e}^{-\frac{k}{\bar{\mu}_p}}, \quad k = 1, 2, \cdots, p \tag{6.26}$$

where $\bar{\mu}_p$ can be determined by Eq.(6.19).

According to Eqs.(6.4) and (6.25), we have

$$\omega(t_k) = \frac{\mathrm{e}^{-\frac{k}{\bar{\mu}_p}}}{\sum_{j=1}^{p} \mathrm{e}^{-\frac{j}{\bar{\mu}_p}}}, \quad k = 1, 2, \cdots, p \tag{6.27}$$

From Eq.(6.27), we know that $\{\omega(t_k)\}$ is a strictly monotonically decreasing sequence, i.e., the larger k, the smaller the weight assigned to the period t_k.

If we use the inverse form of exponential distribution to determine the weight vector $\omega(t)$, then

$$\omega(t_k) = \frac{1}{\bar{\mu}_p}\mathrm{e}^{\frac{k}{\bar{\mu}_p}}, \quad k = 1, 2, \cdots, p \tag{6.28}$$

By Eqs.(6.4) and (6.28), we have

$$\omega(t_k) = \frac{\mathrm{e}^{\frac{k}{\bar{\mu}_p}}}{\sum_{j=1}^{p} \mathrm{e}^{\frac{j}{\bar{\mu}_p}}}, \quad k = 1, 2, \cdots, p \tag{6.29}$$

where $\{\omega(t_k)\}$ is a strictly monotonically increasing sequence, i.e., the larger k, the greater the weight assigned to the period t_k.

Clearly, the weights generated by the exponential distribution based method are similar to those generated by the BUM function based method.

(6) Poisson distribution based method (Xu, 2011): Poisson distribution, developed by Poisson in 1837, is a discrete probability distribution for the counts of events that occur randomly in a given interval of time (or space), which satisfies:

(i) The number of successes in two disjoint time intervals (or regions of space) is independent;

(ii) The probability of a success during a small time interval or region of space is proportional to the entire length of time interval or region of space.

Poisson distribution is most commonly used to model the number of random occurrences of some phenomenon in a specified unit of space or time. The formula for Poisson probability density function is given by

$$P(z = k) = \frac{\lambda^k}{k!}\mathrm{e}^{-\lambda}, \quad \lambda > 0, \; k = 0, 1, 2, \cdots \tag{6.30}$$

where z denotes the number of events in a given time interval, λ is the shape parameter, equal to the mean number of events in the given time interval, and e is the base of the natural logarithm ($\mathrm{e} = 2.71828\cdots$). Fig. 6.1 gives a graph of Poisson distribution taking some special values of the parameter λ:

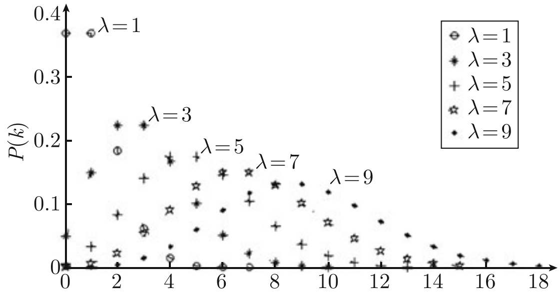

Fig. 6.1 A graph of Poisson distribution taking $\lambda = 1, 3, 5, 7, 9$ (Xu, 2011)

The characteristics of the Poisson distribution (6.30) make it more suitable than the other probability distributions to derive the weights of the time series (different stages, or disjoint time intervals). As a result, by Eq.(6.30), we can develop a Poisson distribution based method for determining the weight vector $\omega(t) = (\omega(t_1), \omega(t_2), \cdots, \omega(t_p))^{\mathrm{T}}$ as follows:

$$\omega(t_k) = \frac{\lambda^k}{k!}\mathrm{e}^{-\lambda}, \quad \lambda > 0, \quad k = 1, 2, \cdots, p \tag{6.31}$$

Considering that the weight vector needs to satisfy the condition (6.4), we can obtain, by Eq.(6.31):

$$\omega(t_k) = \frac{\dfrac{\lambda^k}{k!}\mathrm{e}^{-\lambda}}{\sum\limits_{k=1}^{p}\dfrac{\lambda^k}{k!}\mathrm{e}^{-\lambda}} = \frac{\dfrac{\lambda^k}{k!}}{\sum\limits_{k=1}^{p}\dfrac{\lambda^k}{k!}}, \quad \lambda > 0,\ k = 1, 2, \cdots, p \tag{6.32}$$

Clearly, Eq.(6.32) has the following properties (Xu, 2011):

① If $0 < \lambda < 2$, then $\omega(t_{k+1}) < \omega(t_k)$, $k = 1, 2, \cdots, p-1$, i.e., $\{\omega(t_k)\}$ is a strictly monotonically decreasing sequence;

② If $\lambda \geqslant 2$, then

(i) If λ is a non-negative integer, then there exists an integer $k_0 = \mathrm{int}\,(\lambda)$(here, "int" indicates the integral part of λ), such that

(a) $\omega(t_{k+1}) > \omega(t_k)\,(k = 1, \cdots, k_0)$, i.e., $\{\omega(t_k)\}$ $(k = 1, \cdots, k_0)$ is a strictly monotonically increasing sequence;

(b) $\omega(t_{k+1}) < \omega(t_k)\,(k = k_0, \cdots, p-1)$, i.e., $\{\omega(s_k)\}$ $(k = k_0, \cdots, p-1)$ is a strictly monotonically decreasing sequence;

(ii) If λ is a non-negative integer, there exist two integers $k_0 = \mathrm{int}\,(\lambda) - 1$ and $k_0 + 1$, such that

(a) $\omega(t_{k+1}) > \omega(t_k)\,(k = 1, \cdots, k_0)$, i.e., $\{\omega(t_k)\}$ $(k = 1, \cdots, k_0)$ is a strictly monotonically increasing sequence;

(b) $\omega(t_{k_0+1}) = \omega(t_{k_0})$;

(c) $\omega(t_{k+1}) < \omega(t_k)\,(k = k_0+1, \cdots, p-1)$, i.e., $\{\omega(t_k)\}$ $(k = k_0, \cdots, p-1)$ is a strictly monotonically decreasing sequence.

If we use the inverse form of Poisson distribution to determine the weight vector, then

$$\omega(t_k) = \frac{1}{\dfrac{\lambda^k}{k!}\mathrm{e}^{-\lambda}} = \frac{k!}{\lambda^k}\mathrm{e}^{\lambda}, \quad \lambda > 0, \quad k = 1, 2, \cdots, p \tag{6.33}$$

From Eqs.(6.4) and (6.33), we have

$$\omega(t_k) = \frac{\dfrac{k!}{\lambda^k}\mathrm{e}^{\lambda}}{\sum\limits_{k=1}^{p}\dfrac{k!}{\lambda^k}\mathrm{e}^{\lambda}} = \frac{\dfrac{k!}{\lambda^k}}{\sum\limits_{k=1}^{p}\dfrac{k!}{\lambda^k}}, \quad \lambda > 0, \quad k = 1, 2, \cdots, p \tag{6.34}$$

Eq.(6.34) has the following properties (Xu, 2011):

① If $0 < \lambda < 2$, then $\omega(t_{k+1}) > \omega(t_k)$, $k = 1, 2, \cdots, p-1$, i.e., $\{\omega(t_k)\}$ is a strictly monotonically increasing sequence;

② If $\lambda \geqslant 2$, then

(i) If λ is a non-negative integer, then there exists an integer $k_0 = \mathrm{int}\,(\lambda)$, such that

(a) $\omega(t_{k+1}) < \omega(t_k)\,(k = 1, \cdots, k_0)$, i.e., $\{\omega(t_k)\}$ $(k = 1, \cdots, k_0)$ is a strictly monotonically decreasing sequence;

(b) $\omega(t_{k+1}) > \omega(t_k)\,(k = k_0, \cdots, p-1)$, i.e., $\{\omega(t_k)\}$ $(k = k_0, \cdots, p-1)$ is a strictly monotonically increasing sequence;

(ii) If λ is a non-negative integer, then there exist two integers $k_0 = \mathrm{int}\,(\lambda) - 1$ and $k_0 + 1$, such that

(a) $\omega(t_{k+1}) < \omega(t_k)\,(k = 1, \cdots, k_0)$, i.e., $\{\omega(t_k)\}$ $(k = 1, \cdots, k_0)$ is a strictly monotonically decreasing sequence;

(b) $\omega(t_{k_0+1}) = \omega(t_{k_0})$;

(c) $\omega(t_{k+1}) > \omega(t_k)\,(k = k_0+1, \cdots, p-1)$, i.e., $\{\omega(t_k)\}$ $(k = k_0, \cdots, p-1)$ is a strictly monotonically increasing sequence.

(7) Binomial distribution based method (Xu, 2007d): The binomial distribution is one of the most commonly used probability distributions, which arises in many practical situations, such as quality control, public opinion surveys, medical research, and insurance problems. In statistics the binomial distribution describes the possible number of times that a particular observation will succeed in a sequence of observations. The observation is binary, which may succeed or fail. The binomial distribution is specified by the number of observations, $p-1$, and the probability of occurrence,

denoted by u. It is a discrete probability distribution:

$$P(z=k)=\mathrm{C}_{p-1}^{k}u^{k}(1-u)^{p-1-k},\quad k=0,1,2,\cdots,p-1,\quad u\in(0,1) \tag{6.35}$$

of obtaining exactly k successes in $p-1$ observations, where $P(z=k)$ is the probability of exactly k successes, and C_{p-1}^{k} is a binomial coefficient with:

$$\mathrm{C}_{p-1}^{k}=\frac{(p-1)!}{k!(p-1-k)!},\quad k=0,1,2,\cdots,p-1 \tag{6.36}$$

The binomial distribution assumes that

(i) The number of observations $p-1$ is fixed;

(ii) Each observation is independent;

(iii) Each observation represents one of two outcomes ("success" or "failure");

(iv) The probability of "success" u is the same in each observation.

It follows from Eq.(6.36) that

$$\sum_{k=0}^{p-1}P(z=k)=\sum_{k=0}^{p-1}\mathrm{C}_{p-1}^{k}\,u^{k}(1-u)^{p-1-k}=(u+1-u)^{p-1}=1 \tag{6.37}$$

By Eq.(6.37), we can develop a binomial distribution based method to determine the time series weights (Xu, 2007d):

$$\omega(t_k)=\mathrm{C}_{p-1}^{k}\,u^{k}(1-u)^{p-1-k},\quad k=0,1,2,\cdots,p-1,\quad u\in(0,1) \tag{6.38}$$

which satisfies the condition (6.4).

In Fig. 6.2, we give a graph to show the time series weights derived by the binomial distribution based method (6.38) taking $p=10$ and some special values of the parameter u:

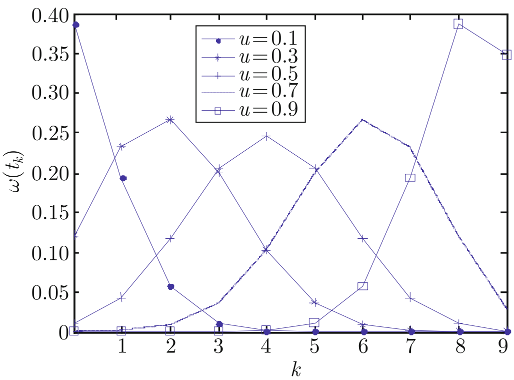

Fig. 6.2 Some special time series weights derived by the binomial distribution based method (Xu, 2007d)

In particular, if we take $u = 1/2$, then Eq.(6.38) can be transformed into the following form:

$$\omega(t_k) = \frac{C_{p-1}^k}{2^{p-1}}, \quad k = 0, 1, 2, \cdots, p-1 \tag{6.39}$$

The time series weights $\omega(t_k)$ $(k = 0, 1, 2, \cdots, p-1)$ derived from Eq.(6.39) has the similar properties as those of the normal distribution based method:

(i) $\omega(t_k)$ $(k = 0, 1, 2, \cdots, p-1)$ are symmetry, i.e.,

$$\omega(t_k) = \omega(t_{p-1-k}), \quad k = 0, 1, 2, \cdots, p-1 \tag{6.40}$$

(ii) If p is odd, then

(a) $\omega(t_k) < \omega(t_{k+1})$ $\left(k = 0, 1, \cdots, \dfrac{p-1}{2} - 1\right)$;

(b) $\omega(t_k) > \omega(t_{k+1})$ $\left(k = \dfrac{p-1}{2}, \cdots, p-2\right)$;

(c) $\omega(t_{(p-1)/2}) = \max\limits_k\{\omega(t_k)\}$;

(iii) If p is even, then

(a) $\omega(t_k) < \omega(t_{k+1})$ $\left(k = 0, 1, \cdots, \dfrac{p}{2} - 2\right)$;

(b) $\omega(t_k) > \omega(t_{k+1})$ $\left(k = \dfrac{p}{2}, \cdots, p-2\right)$;

(c) $\omega(t_{p/2-1}) = \omega(t_{p/2}) = \max\limits_k\{\omega(t_k)\}$.

(8) Average age method (Xu and Yager, 2008): We can associate with a set of weights $\omega(t_k)$ $(k = 1, 2, \cdots, p)$ a concept of the average age of the data (Yager, 2008). Assume that $\omega(t_1), \omega(t_2), \cdots, \omega(t_p)$ are the weights with t_p being the most recent and t_1 being the earliest. We can then calculate

$$\bar{t} = \sum_{k=1}^{p} (p-k)\omega(t_k) \tag{6.41}$$

where $\bar{t}$ indicates the average age of the data. Note that for the BUM approach the area under $\mathbb{Z}$ can be used to approximate $\bar{t}$:

$$\bar{t} \approx (p-1)\int_0^1 \mathbb{Z}(x)\mathrm{d}x \tag{6.42}$$

More generally, we can obtain the weights by specifying a value for $\bar{t}$ and then find a set of weights that satisfies the following mathematical programming model for the $\omega(t_k)$ (Xu and Yager, 2008):

$$\begin{aligned}
&\max \sum_{k=1}^{p} (\omega(t_k))^2 \\
&\text{s.t.} \quad \sum_{k=1}^{p} (p-k)\omega(t_k) = \bar{t} \\
&\qquad \sum_{k=1}^{p} \omega(t_k) = 1, \quad \omega(t_k) \geqslant 0, \quad k = 1, 2, \cdots, p
\end{aligned}$$

To solve this model, we can construct the Lagrange function:

$$L(\omega(t), \varsigma_1, \varsigma_2) = \sum_{k=1}^{p} (\omega(t_k))^2 - 2\varsigma_1 \left(\sum_{k=1}^{p} (p-k)\omega(t_k) - \bar{t} \right) - 2\varsigma_2 \left(\sum_{k=1}^{p} \omega(t_k) - 1 \right) \tag{6.43}$$

where $\omega(t) = (\omega(t_1), \omega(t_2), \cdots, \omega(t_p))^{\mathrm{T}}$, ς_1 and ς_2 are the Lagrange multipliers.

Differentiating Eq.(6.43) with respect to $\omega(t_k)\,(k = 1, 2, \cdots, p)$, ς_1 and ς_2, and setting these partial derivatives equal to zero, the following set of equations are obtained:

$$\begin{cases}
\dfrac{\partial L(\omega(t), \varsigma_1, \varsigma_2)}{\partial \lambda(t_k)} = 2\omega(t_k) - 2\varsigma_1(p-k) - 2\varsigma_2 = 0 & (6.44) \\
\dfrac{\partial L(\omega(t), \varsigma_1, \varsigma_2)}{\partial \eta_1} = -2 \left(\displaystyle\sum_{k=1}^{p} (p-k)\omega(t_k) - \bar{t} \right) = 0 & (6.45) \\
\dfrac{\partial L(\omega(t), \varsigma_1, \varsigma_2)}{\partial \varsigma_2} = -2 \left(\sum_{k=1}^{p} \omega(t_k) - 1 \right) = 0 & (6.46)
\end{cases}$$

Simplifying Eqs.(6.44)–(6.46), we have

$$\begin{cases}
\omega(t_k) = (p-k)\varsigma_1 + \varsigma_2 & (6.47) \\
\displaystyle\sum_{k=1}^{p} (p-k)\omega(t_k) = \bar{t} & (6.48) \\
\displaystyle\sum_{k=1}^{p} \omega(t_k) = 1 & (6.49)
\end{cases}$$

Combining Eqs.(6.47) and (6.48), (6.47) and (6.49), we obtain

$$\begin{cases}
\varsigma_1 \displaystyle\sum_{k=1}^{p} (p-k)^2 + \varsigma_2 \sum_{k=1}^{p} (p-k) = \bar{t} & (6.50) \\
\varsigma_1 \displaystyle\sum_{k=1}^{p} (p-k) + \varsigma_2 p = 1 & (6.51)
\end{cases}$$

Since

$$\sum_{k=1}^{p} (p-k)^2 = \frac{1}{6}p(p+1)(2p+1) - p^2 \tag{6.52}$$

and

$$\sum_{k=1}^{p}(p-k)=\frac{1}{2}p(p-1) \tag{6.53}$$

We get by solving Eqs.(6.50) and (6.51):

$$\begin{cases} \varsigma_1=\dfrac{12\bar{t}-6(p-1)}{p(p-1)(p+1)} & (6.54)\\ \varsigma_2=\dfrac{4p-2-6\bar{t}}{p(p+1)} & (6.55)\end{cases}$$

Thus, it follows from Eq.(6.47) that

$$\omega(t_k)=\frac{(12\bar{t}-6p+6)(p-k)+(4p-2-6\bar{t})(p-1)}{p(p-1)(p+1)},\quad k=1,2,\cdots,p \tag{6.56}$$

Since $\omega(t_k)\geqslant 0$, for all k, we have

$$\frac{(12\bar{t}-6p+6)(p-k)+(4p-2-6\bar{t})(p-1)}{p(p-1)(p+1)}\geqslant 0,\quad k=1,2,\cdots,p \tag{6.57}$$

i.e,

$$(3p-6k+3)\bar{t}\geqslant(p-1)(p-3k+1),\quad k=1,2,\cdots,p \tag{6.58}$$

Thus,

(i) If $3p-6k+3=0$, i.e., $k=\dfrac{p+1}{2}$, then Eq.(6.58) holds, for all $\bar{t}$;

(ii) If $3p-6k+3>0$, i.e., $k<\dfrac{p+1}{2}$, then Eq.(6.58) holds, for $\bar{t}\geqslant\dfrac{p-2}{3}$;

(iii) If $3p-6k+3<0$, i.e., $k>\dfrac{p+1}{2}$, then Eq.(6.58) holds, for $\bar{t}\leqslant\dfrac{2p-1}{3}$.

Therefore, we can obtain the weights $\omega(t_k)$ $(k=1,2,\cdots,p)$ by using Eq.(6.56) with the following condition (Xu and Yager, 2008):

$$\frac{p-2}{3}\leqslant\bar{t}\leqslant\frac{2p-1}{3} \tag{6.59}$$

If we let

$$g(x)=\frac{(12\bar{t}-6p+6)(p-x)+(4p-2-6\bar{t})(p-1)}{p(p-1)(p+1)} \tag{6.60}$$

then

$$g'(x)=-\frac{12\bar{t}-6p+6}{p(p-1)(p+1)} \tag{6.61}$$

As a result,

(i) If $\dfrac{p-2}{3}\leqslant\bar{t}<\dfrac{p-1}{2}$, then $g'(x)>0$, i.e., $g(x)$ is a strictly monotonic increasing function.

(ii) If $\bar{t} = \dfrac{p-1}{2}$, then $g'(x) = 0$, i.e., $g(x)$ is a constant function.

(iii) If $\dfrac{p-1}{2} < \bar{t} \leqslant \dfrac{2p-1}{3}$, then $g'(x) < 0$, i.e., $g(x)$ is a strictly monotonic decreasing function.

Consequently, by Eq.(6.61), Xu and Yager (2008) get that

(i) If $\dfrac{p-2}{3} \leqslant \bar{t} < \dfrac{p-1}{2}$, then $\omega(t_{k+1}) > \omega(t_k)\,(k = 1, 2, \cdots, p-1)$, i.e., $\{\omega(t_k)\}$ is a monotonic increasing sequence. Also since

$$\begin{aligned}\omega(t_{k+1}) - \omega(t_k) =& \frac{(12\bar{t} - 6p + 6)(p - (k+1) + (4p - 2 - 6\bar{t})(p-1)}{p(p-1)(p+1)} \\ &- \frac{(12\bar{t} - 6p + 6)(p - k) + (4p - 2 - 6\bar{t})(p-1)}{p(p-1)(p+1)} \\ =& -\frac{(12\bar{t} - 6p + 6)}{p(p-1)(p+1)} > 0, \quad k = 1, 2, \cdots, p-1 \end{aligned} \tag{6.62}$$

then $\{\omega(t_k)\}$ is an increasing arithmetic sequence.

(ii) If $\bar{t} = \dfrac{p-1}{2}$, then

$$\omega(t_k) = \frac{(12\bar{t} - 6p + 6)(p - k) + (4p - 2 - 6\bar{t})(p-1)}{p(p-1)(p+1)} = \frac{1}{p}, \quad k = 1, 2, \cdots, p \tag{6.63}$$

Hence, $\omega(t) = (1/p, 1/p, \cdots, 1/p)^{\mathrm{T}}$.

(iii) If $\dfrac{p-1}{2} < \bar{t} \leqslant \dfrac{2p-1}{3}$, then $\omega(t_{k+1}) < \omega(t_k)\,(k = 1, 2, \cdots, p-1)$, i.e., $\{\omega(t_k)\}$ is a monotonic decreasing sequence. Similar to Eq.(6.62), we have $\omega(t_{k+1}) - \omega(t_k) = -\dfrac{(12\bar{t} - 6p + 6)}{p(p-1)(p+1)} < 0$ $(k = 1, 2, \cdots, p-1)$. Thus, $\{\omega(t_k)\}$ is a decreasing arithmetic sequence.

6.2 Dynamic Intuitionistic Fuzzy Multi-Attribute Decision Making

In this section, we consider dynamic intuitionistic fuzzy multi-attribute decision making problems where all the attribute values are expressed in IFNs, which are collected at different periods of time. The following notations are used to depict the problems (Xu and Yager, 2008):

- Y, ω and G are defined as in Section 1.3.
- There are p periods t_k $(k = 1, 2, \cdots, p)$, whose weight vector is $\omega(t) = (\omega(t_1), \omega(t_2), \cdots, \omega(t_p))^{\mathrm{T}}$, where $\omega(t_k) \geqslant 0\,(k = 1, 2, \cdots, p)$ and $\sum_{k=1}^{p} \omega(t_k) = 1$.

• $R'(t_k) = (r'_{ij}(t_k))_{n\times m}$: An intuitionistic fuzzy decision matrix of the period t_k, where $r'_{ij}(t_k) = (\mu'_{ij}(t_k), \nu'_{ij}(t_k), \pi'_{ij}(t_k))$ is an attribute value, denoted by an IFN, $\mu'_{ij}(t_k)$ indicates the degree that the alternative Y_i should satisfy the attribute G_j at the period t_k, $\nu'_{ij}(t_k)$ indicates the degree that the alternative Y_i should not satisfy the attribute G_j at the period t_k, and $\pi'_{ij}(t_k)$ indicates the degree of indeterminacy of the alternative Y_i to the attribute G_j, such that

$$\begin{gathered}\mu'_{ij}(t_k) \in [0,1], \quad \nu'_{ij}(t_k) \in [0,1], \quad \mu'_{ij}(t_k) + \nu'_{ij}(t_k) \leqslant 1, \\ \pi'_{ij}(t_k) = 1 - \mu'_{ij}(t_k) - \nu'_{ij}(t_k), \quad i = 1,2,\cdots,n, \quad j = 1,2,\cdots,m\end{gathered} \tag{6.64}$$

In cases where the attributes are of benefit and cost types, we can normalize $R'(t_k) = (r'_{ij}(t_k))_{n\times m}$ into the intuitionistic fuzzy decision matrix $R(t_k) = (r_{ij}(t_k))_{n\times m}$, where

$$r_{ij}(t_k) = (\mu_{ij}(t_k), \nu_{ij}(t_k), \pi_{ij}(t_k)) = \begin{cases} r'_{ij}(t_k), & \text{for benefit attribute } G_j, \\ \bar{r}'_{ij}(t_k), & \text{for } \cos t \text{ attribute } G_j, \end{cases}$$

$$i = 1,2,\cdots,n; \quad j = 1,2,\cdots,m \tag{6.65}$$

and $\bar{r}'_{ij}(t_k)$ is the complement of $r'_{ij}(t_k)$, i.e., $\bar{r}'_{ij}(t_k) = (\nu'_{ij}(t_k), \mu'_{ij}(t_k), \pi'_{ij}(t_k))$.

Based on the above decision information, we now introduce a practical procedure to rank and select the most desirable alternative(s) (Xu and Yager, 2008):

Step 1 Utilize the DIFWA operator:

$$\begin{aligned} r_{ij} =& \text{DIFWA}_{\omega(t)}(r_{ij}(t_1), r_{ij}(t_2), \cdots, r_{ij}(t_p)) \\ =& \left(1 - \prod_{k=1}^{p} (1-\mu_{ij}(t_k))^{\omega(t_k)}, \prod_{k=1}^{p} (\nu_{ij}(t_k))^{\omega(t_k)}, \right. \\ & \left. \prod_{k=1}^{p} (1-\mu_{ij}(t_k))^{\omega(t_k)} - \prod_{k=1}^{p} (\nu_{ij}(t_k))^{\omega(t_k)} \right) \end{aligned} \tag{6.66}$$

to aggregate all the intuitionistic fuzzy decision matrices $R(t_k) = (r_{ij}(t_k))_{n\times m}$ $(k = 1,2,\cdots,p)$ into a complex intuitionistic fuzzy decision matrix $R = (r_{ij})_{n\times m}$, where

$$r_{ij} = (\mu_{ij}, \nu_{ij}, \pi_{ij}), \quad \mu_{ij} = 1 - \prod_{k=1}^{p} (1-\mu_{ij}(t_k))^{\omega(t_k)}, \quad \nu_{ij} = \prod_{k=1}^{p} (\nu_{ij}(t_k))^{\omega(t_k)}$$

$$\pi_{ij} = \prod_{k=1}^{p} (1-\mu_{ij}(t_k))^{\omega(t_k)} - \prod_{k=1}^{p} (\nu_{ij}(t_k))^{\omega(t_k)}, \quad i = 1,2,\cdots,n;\ j = 1,2,\cdots,m$$

Step 2 Define $Y^+ = (\alpha_1^+, \alpha_2^+, \cdots, \alpha_m^+)^{\mathrm{T}}$ and $\alpha^- = (\alpha_1^-, \alpha_2^-, \cdots, \alpha_m^-)^{\mathrm{T}}$ as the intuitionistic fuzzy ideal solution (IFIS) and the intuitionistic fuzzy negative ideal solution (IFNIS) respectively, where $\alpha_i^+ = (1,0,0)$ $(i = 1,2,\cdots,m)$ are the m largest

IFNs, and $\alpha_i^- = (0,1,0)$ $(i = 1,2,\cdots,m)$ are the m smallest IFNs. Furthermore, we denote the alternatives Y_i $(i = 1,2,\cdots,n)$ by $Y_i = (r_{i1}, r_{i2}, \cdots, r_{im})^{\mathrm{T}}$ $(i = 1,2,\cdots,n)$.

Step 3 Calculate the distance between the alternative Y_i and the IFIS Y^+ and the distance between the alternative Y_i and the IFNIS Y^- respectively:

$$\begin{aligned} d(Y_i, Y^+) &= \sum_{j=1}^{m} \omega_j d(r_{ij}, Y_j^+) = \frac{1}{2}\sum_{j=1}^{m} \omega_j(|\mu_{ij} - 1| + |\nu_{ij} - 0| + |\pi_{ij} - 0|) \\ &= \frac{1}{2}\sum_{j=1}^{m} \omega_j(1 - \mu_{ij} + \nu_{ij} + \pi_{ij}) \\ &= \frac{1}{2}\sum_{j=1}^{m} \omega_j(1 - \mu_{ij} + \nu_{ij} + 1 - \mu_{ij} - \nu_{ij}) \\ &= \sum_{j=1}^{m} \omega_j(1 - \mu_{ij}) \end{aligned} \tag{6.67}$$

$$\begin{aligned} d(Y_i, Y^-) &= \sum_{j=1}^{m} \omega_j d(r_{ij}, \alpha_j^-) = \frac{1}{2}\sum_{j=1}^{m} \omega_j(|\mu_{ij} - 0| + |\nu_{ij} - 1| + |\pi_{ij} - 0|) \\ &= \frac{1}{2}\sum_{j=1}^{m} \omega_j(1 + \mu_{ij} - \nu_{ij} + \pi_{ij}) \\ &= \frac{1}{2}\sum_{j=1}^{m} \omega_j(1 + \mu_{ij} - \nu_{ij} + 1 - \mu_{ij} - \nu_{ij}) \\ &= \frac{1}{2}\sum_{j=1}^{m} \omega_j(1 - \nu_{ij}) \end{aligned} \tag{6.68}$$

where $r_{ij} = (\mu_{ij}, \nu_{ij}, \pi_{ij})$ $(i = 1,2,\cdots,n;\ j = 1,2,\cdots,m)$.

Step 4 Calculate the closeness coefficient of each alternative:

$$c(Y_i) = \frac{d(Y_i, Y^-)}{d(Y_i, Y^+) + d(Y_i, Y^-)}, \quad i = 1,2,\cdots,n \tag{6.69}$$

Since

$$\begin{aligned} d(Y_i, Y^+) + d(Y_i, Y^-) &= \sum_{j=1}^{m} \omega_j(1 - \mu_{ij}) + \sum_{j=1}^{m} \omega_j(1 - \nu_{ij}) \\ &= \sum_{j=1}^{m} \omega_j(2 - \mu_{ij} - \nu_{ij}) \\ &= \sum_{j=1}^{m} \omega_j(1 + \pi_{ij}) \end{aligned} \tag{6.70}$$

Eq.(6.69) can be transformed as:

$$c(Y_i) = \frac{\sum_{j=1}^{m} \omega_j (1 - \nu_{ij})}{\sum_{j=1}^{m} \omega_j (1 + \pi_{ij})}, \quad i = 1, 2, \cdots, n \tag{6.71}$$

Step 5 Rank all the alternatives Y_i $(i = 1, 2, \cdots, n)$ according to the closeness coefficients $c(Y_i)$ $(i = 1, 2, \cdots, n)$: the greater the value $c(Y_i)$, the better the alternative Y_i.

6.3 Uncertain Dynamic Intuitionistic Fuzzy Multi-Attribute Decision Making

Similar to Definition 6.1.1, we introduce the concept of uncertain intuitionistic fuzzy variable (or called interval-valued intuitionistic fuzzy variable):

Definition 6.3.1 (Xu and Yager, 2008) Let t be a time variable. Then $\tilde{\alpha}(t) = ([\tilde{\mu}^{\mathrm{L}}_{\tilde{\alpha}(t)}, \tilde{\mu}^{\mathrm{U}}_{\tilde{\alpha}(t)}], [\tilde{\nu}^{\mathrm{L}}_{\tilde{\alpha}(t)}, \tilde{\nu}^{\mathrm{U}}_{\tilde{\alpha}(t)}], [\tilde{\pi}^{\mathrm{L}}_{\tilde{\alpha}(t)}, \tilde{\pi}^{\mathrm{U}}_{\tilde{\alpha}(t)}])$ is called an uncertain intuitionistic fuzzy variable, where

$$[\tilde{\mu}^{\mathrm{L}}_{\tilde{\alpha}(t)}, \tilde{\mu}^{\mathrm{U}}_{\tilde{\alpha}(t)}] \subset [0, 1], \quad [\tilde{\nu}^{\mathrm{L}}_{\tilde{\alpha}(t)}, \tilde{\nu}^{\mathrm{U}}_{\tilde{\alpha}(t)}] \subset [0, 1], \quad \tilde{\mu}^{\mathrm{U}}_{\tilde{\alpha}(t)} + \tilde{\nu}^{\mathrm{U}}_{\tilde{\alpha}(t)} \leqslant 1$$

$$[\tilde{\pi}^{\mathrm{L}}_{\tilde{\alpha}(t)}, \tilde{\pi}^{\mathrm{U}}_{\tilde{\alpha}(t)}] = [1 - \tilde{\mu}^{\mathrm{U}}_{\tilde{\alpha}(t)} - \tilde{\nu}^{\mathrm{U}}_{\tilde{\alpha}(t)}, 1 - \tilde{\mu}^{\mathrm{L}}_{\tilde{\alpha}(t)} - \tilde{\nu}^{\mathrm{L}}_{\tilde{\alpha}(t)}] \tag{6.72}$$

Let $\tilde{\alpha}(t) = ([\tilde{\mu}^{\mathrm{L}}_{\tilde{\alpha}(t)}, \tilde{\mu}^{\mathrm{U}}_{\tilde{\alpha}(t)}], [\tilde{\nu}^{\mathrm{L}}_{\tilde{\alpha}(t)}, \tilde{\nu}^{\mathrm{U}}_{\tilde{\alpha}(t)}], [\tilde{\pi}^{\mathrm{L}}_{\tilde{\alpha}(t)}, \tilde{\pi}^{\mathrm{U}}_{\tilde{\alpha}(t)}])$ be an uncertain intuitionistic fuzzy variable. If $t = t_1, t_2, \cdots, t_p$, then $\tilde{\alpha}(t_1), \tilde{\alpha}(t_2), \cdots, \tilde{\alpha}(t_n)$ denote p IVIFNs collected at p different periods.

Based on Eq.(6.72), Xu and Yager (2008) modify two operational laws of IVIFNs defined in Section 2.2 as follows:

Definition 6.3.2 (Xu and Yager, 2008) Let $\tilde{\alpha}(t_k) = ([\tilde{\mu}^{\mathrm{L}}_{\tilde{\alpha}(t_k)}, \tilde{\mu}^{\mathrm{U}}_{\tilde{\alpha}(t_k)}], [\tilde{\nu}^{\mathrm{L}}_{\tilde{\alpha}(t_k)}, \tilde{\nu}^{\mathrm{U}}_{\tilde{\alpha}(t_k)}], [\tilde{\pi}^{\mathrm{L}}_{\tilde{\alpha}(t_k)}, \tilde{\pi}^{\mathrm{U}}_{\tilde{\alpha}(t_k)}])$ $(k = 1, 2)$ be two IVIFNs. Then

(1)
$$\begin{aligned}\tilde{\alpha}(t_1) \oplus \tilde{\alpha}(t_2) = {} & ([\tilde{\mu}^{\mathrm{L}}_{\tilde{\alpha}(t_1)} + \tilde{\mu}^{\mathrm{L}}_{\tilde{\alpha}(t_2)} - \tilde{\mu}^{\mathrm{L}}_{\tilde{\alpha}(t_1)}\tilde{\mu}^{\mathrm{L}}_{\tilde{\alpha}(t_2)}, \tilde{\mu}^{\mathrm{U}}_{\tilde{\alpha}(t_1)} + \tilde{\mu}^{\mathrm{U}}_{\tilde{\alpha}(t_2)} - \tilde{\mu}^{\mathrm{U}}_{\tilde{\alpha}(t_1)}\tilde{\mu}^{\mathrm{U}}_{\tilde{\alpha}(t_2)}], \\ & [\tilde{\nu}^{\mathrm{L}}_{\tilde{\alpha}(t_1)}\tilde{\nu}^{\mathrm{L}}_{\tilde{\alpha}(t_2)}, \tilde{\nu}^{\mathrm{U}}_{\tilde{\alpha}(t_1)}\tilde{\nu}^{\mathrm{U}}_{\tilde{\alpha}(t_2)}], [(1 - \tilde{\mu}^{\mathrm{U}}_{\tilde{\alpha}(t_1)})(1 - \tilde{\mu}^{\mathrm{U}}_{\tilde{\alpha}(t_2)}) - \tilde{\nu}^{\mathrm{U}}_{\tilde{\alpha}(t_1)}\tilde{\nu}^{\mathrm{U}}_{\tilde{\alpha}(t_2)}, \\ & (1 - \tilde{\mu}^{\mathrm{L}}_{\tilde{\alpha}(t_1)})(1 - \tilde{\mu}^{\mathrm{L}}_{\tilde{\alpha}(t_2)}) - \tilde{\nu}^{\mathrm{L}}_{\tilde{\alpha}(t_1)}\tilde{\nu}^{\mathrm{L}}_{\tilde{\alpha}(t_2)}]);\end{aligned} \tag{6.73}$$

(2)
$$\begin{aligned}\lambda\,\tilde{\alpha}(t_1) = {} & ([1 - (1 - \tilde{\mu}^{\mathrm{L}}_{\tilde{\alpha}(t_1)})^{\lambda}, 1 - (1 - \tilde{\mu}^{\mathrm{U}}_{\tilde{\alpha}(t_1)})^{\lambda}], [(\tilde{\nu}^{\mathrm{L}}_{\tilde{\alpha}(t_1)})^{\lambda}, (\tilde{\nu}^{\mathrm{U}}_{\tilde{\alpha}(t_1)})^{\lambda}], \\ & [(1 - \tilde{\mu}^{\mathrm{U}}_{\tilde{\alpha}(t_1)})^{\lambda} - (\tilde{\nu}^{\mathrm{U}}_{\tilde{\alpha}(t_1)})^{\lambda}, (1 - \tilde{\mu}^{\mathrm{L}}_{\tilde{\alpha}(t_1)})^{\lambda} - (\tilde{\nu}^{\mathrm{L}}_{\tilde{\alpha}(t_1)})^{\lambda}]), \quad \lambda > 0.\end{aligned} \tag{6.74}$$

Definition 6.3.3 (Xu and Yager, 2008) Let $\tilde{\alpha}(t_1), \tilde{\alpha}(t_2), \cdots, \tilde{\alpha}(t_p)$ be a collection of IVIFNs collected at p different periods $t_k\,(k = 1, 2, \cdots, p)$, and $\omega(t) =$

$(\omega(t_1), \omega(t_2), \cdots, \omega(t_p))^{\mathrm{T}}$ the weight vector of the time series $\{t_k\}$ $(k = 1, 2, \cdots, p)$, which can be obtained by the methods proposed in Section 6.1, with $\omega(t_k) \in [0, 1]$ $(k = 1, 2, \cdots, p)$ and $\sum_{k=1}^{p} \omega(t_k) = 1$. Then we call

$$\mathrm{UDIFWA}_{\omega(t)}(\tilde{\alpha}(t_1), \tilde{\alpha}(t_2), \cdots, \tilde{\alpha}(t_p)) = \omega(t_1)\tilde{\alpha}(t_1) \oplus \omega(t_2)\tilde{\alpha}(t_2) \oplus \cdots \oplus \omega(t_p)\tilde{\alpha}(t_p) \tag{6.75}$$

an uncertain dynamic intuitionistic fuzzy weighted averaging (UDIFWA) operator, which can be rewritten as follows:

$$\begin{aligned}
&\mathrm{UDIFWA}_{\omega(t)}(\tilde{\alpha}(t_1), \tilde{\alpha}(t_2), \cdots, \tilde{\alpha}(t_p)) \\
=&\left(\left[1 - \prod_{k=1}^{p} (1 - \tilde{\mu}^{\mathrm{L}}_{\tilde{\alpha}(t_k)})^{\omega(t_k)}, 1 - \prod_{k=1}^{p} (1 - \tilde{\mu}^{\mathrm{U}}_{\tilde{\alpha}(t_k)})^{\omega(t_k)} \right], \right. \\
&\left[\prod_{k=1}^{p} (\tilde{\nu}^{\mathrm{L}}_{\tilde{\alpha}(t_k)})^{\omega(t_k)}, \prod_{k=1}^{p} (\tilde{\nu}^{\mathrm{U}}_{\tilde{\alpha}(t_k)})^{\omega(t_k)} \right], \\
&\left[\prod_{k=1}^{p} (1 - \tilde{\mu}^{\mathrm{U}}_{\tilde{\alpha}(t_k)})^{\omega(t_k)} - \prod_{k=1}^{p} (\tilde{\nu}^{\mathrm{U}}_{\tilde{\alpha}(t_k)})^{\omega(t_k)}, \right. \\
&\left. \left. \prod_{k=1}^{p} (1 - \tilde{\mu}^{\mathrm{L}}_{\tilde{\alpha}(t_k)})^{\omega(t_k)} - \prod_{k=1}^{p} (\tilde{\nu}^{\mathrm{L}}_{\tilde{\alpha}(t_k)})^{\omega(t_k)} \right] \right)
\end{aligned} \tag{6.76}$$

with the condition (6.4).

Below we consider dynamic intuitionistic fuzzy multi-attribute decision making problems under interval uncertainty where all the attribute values are expressed in IVIFNs, which are collected at different periods. The following notations are used to depict the problems:

Let Y, G, w and $\omega(t)$ be defined as in Section 6.2, and let $\tilde{R}'(t_k) = (\tilde{r}'_{ij}(t_k))_{n \times m}$ be an interval-valued intuitionistic fuzzy decision matrix of the period t_k, where

$$\begin{aligned}
\tilde{r}'_{ij}(t_k) =& (\tilde{\mu}'_{ij}(t_k), \tilde{\nu}'_{ij}(t_k), \tilde{\pi}'_{ij}(t_k)) \\
=& ([\tilde{\mu}'^{\mathrm{L}}_{ij}(t_k), \tilde{\mu}'^{\mathrm{U}}_{ij}(t_k)], [\tilde{\nu}'^{\mathrm{L}}_{ij}(t_k), \tilde{\nu}'^{\mathrm{U}}_{ij}(t_k)], \\
& [\tilde{\pi}'^{\mathrm{L}}_{ij}(t_k), \tilde{\pi}'^{\mathrm{U}}_{ij}(t_k)])
\end{aligned} \tag{6.77}$$

is an attribute value, denoted by an IVIFN, and $[\tilde{\mu}'^{\mathrm{L}}_{ij}(t_k), \tilde{\mu}'^{\mathrm{U}}_{ij}(t_k)]$ indicates the uncertain degree that the alternative Y_i should satisfy the attribute G_j at the period t_k, $[\tilde{\nu}'^{\mathrm{L}}_{ij}(t_k), \tilde{\nu}'^{\mathrm{U}}_{ij}(t_k)]$ indicates the uncertain degree that the alternative Y_i should not satisfy the attribute G_j at the period t_k, and $[\tilde{\pi}'^{\mathrm{L}}_{ij}(t_k), \tilde{\pi}'^{\mathrm{U}}_{ij}(t_k)]$ indicates the range of indeterminacy of the alternative Y_i to the attribute G_j, such that

$$[\tilde{\mu}'^{\mathrm{L}}_{ij}(t_k), \tilde{\mu}'^{\mathrm{U}}_{ij}(t_k)] \in [0, 1], \quad [\tilde{\nu}'^{\mathrm{L}}_{ij}(t_k), \tilde{\nu}'^{\mathrm{U}}_{ij}(t_k)] \in [0, 1], \quad \tilde{\mu}'^{\mathrm{U}}_{ij}(t_k) + \tilde{\nu}'^{\mathrm{U}}_{ij}(t_k) \leqslant 1,$$

$$[\tilde{\pi}_{ij}^{\prime\mathrm{L}}(t_k), \tilde{\pi}_{ij}^{\prime\mathrm{U}}(t_k)] = [1 - \tilde{\mu}_{ij}^{\prime\mathrm{U}}(t_k) - \tilde{\nu}_{ij}^{\prime\mathrm{U}}(t_k), 1 - \tilde{\mu}_{ij}^{\prime\mathrm{L}}(t_k) - \tilde{\nu}_{ij}^{\mathrm{L}}(t_k)], \quad i = 1, 2, \cdots, n; \quad j = 1, 2, \cdots, m \tag{6.78}$$

In cases where the attributes are of benefit and cost types, we can normalize $\tilde{R}'(t_k) = (\tilde{r}'_{ij}(t_k))_{n\times m}$ into the interval-valued intuitionistic fuzzy decision matrix $\tilde{R}(t_k) = (\tilde{r}_{ij}(t_k))_{n\times m}$, where

$$\tilde{r}_{ij}(t_k) = (\tilde{\mu}_{ij}(t_k), \tilde{\nu}_{ij}(t_k), \tilde{\pi}_{ij}(t_k)) = \begin{cases} \tilde{r}'_{ij}(t_k), & \text{for benefit attribute } G_j \\ \bar{\tilde{r}}'_{ij}(t_k), & \text{for cos}\, t \text{ attribute } G_j, \end{cases} \quad i = 1, 2, \cdots, n; \quad j = 1, 2, \cdots, m \tag{6.79}$$

and $\bar{\tilde{r}}'_{ij}(t_k)$ is the complement of $\tilde{r}'_{ij}(t_k)$, i.e., $\bar{\tilde{r}}'_{ij}(t_k) = (\tilde{\nu}'_{ij}(t_k), \tilde{\mu}'_{ij}(t_k), \tilde{\pi}'_{ij}(t_k))$.

Similar to Section 6.2, a procedure for solving the above problems can be described as follows (Xu and Yager, 2008):

Step 1 Utilize the UDIFWA operator:

$$\begin{aligned}\tilde{r}_{ij} =& \mathrm{UDIFWA}_{\omega(t)}(\tilde{r}_{ij}(t_1), \tilde{r}_{ij}(t_2), \cdots, \tilde{r}_{ij}(t_p)) \\ =& \left(\left[1 - \prod_{k=1}^{p} (1 - \tilde{\mu}_{ij}^{\mathrm{L}}(t_k))^{\omega(t_k)}, 1 - \prod_{k=1}^{p} (1 - \tilde{\mu}_{ij}^{\mathrm{U}}(t_k))^{\omega(t_k)}\right], \right. \\ & \left[\prod_{k=1}^{p} (\tilde{\nu}_{ij}^{\mathrm{L}}(t_k))^{\omega(t_k)}, \prod_{k=1}^{p} (\tilde{\nu}_{ij}^{\mathrm{U}}(t_k))^{\omega(t_k)}\right], \\ & \left[\prod_{k=1}^{p} (1 - \tilde{\mu}_{ij}^{\mathrm{U}}(t_k))^{\omega(t_k)} - \prod_{k=1}^{p} (\tilde{\nu}_{ij}^{\mathrm{U}}(t_k))^{\omega(t_k)}, \right. \\ & \left.\left. \prod_{k=1}^{p} (1 - \tilde{\mu}_{ij}^{\mathrm{L}}(t_k))^{\omega(t_k)} - \prod_{k=1}^{p} (\tilde{\nu}_{ij}^{\mathrm{L}}(t_k))^{\omega(t_k)}\right] \right)\end{aligned} \tag{6.80}$$

to aggregate all the interval-valued intuitionistic fuzzy decision matrices $\tilde{R}(t_k) = (\tilde{r}_{ij}(t_k))_{n\times m}$ $(k = 1, 2, \cdots, p)$ into a complex interval-valued intuitionistic fuzzy decision matrix $\tilde{R}(t_k) = (\tilde{r}_{ij}(t_k))_{n\times m}$, where $\tilde{r}_{ij} = ([\tilde{\mu}_{ij}^{\mathrm{L}}, \tilde{\mu}_{ij}^{\mathrm{U}}], [\tilde{\nu}_{ij}^{\mathrm{L}}, \tilde{\nu}_{ij}^{\mathrm{U}}], [\tilde{\pi}_{ij}^{\mathrm{L}}, \tilde{\pi}_{ij}^{\mathrm{U}}])$, $i = 1, 2, \cdots, n$; $j = 1, 2, \cdots, m$.

Step 2 Define $Y^+ = (\tilde{\alpha}_1^+, \tilde{\alpha}_2^+, \cdots, \tilde{\alpha}_m^+)^{\mathrm{T}}$ and $Y^- = (\tilde{\alpha}_1^-, \tilde{\alpha}_2^-, \cdots, \tilde{\alpha}_m^-)^{\mathrm{T}}$ as the uncertain intuitionistic fuzzy ideal solution (UIFIS) and the uncertain intuitionistic fuzzy negative ideal solution (UIFNIS) respectively, where $\tilde{\alpha}_i^+ = ([1, 1], [0, 0], [0, 0])$ $(i = 1, 2, \cdots, m)$ are the m largest IVIFNs, and $\tilde{\alpha}_i^- = ([0, 0], [1, 1], [0, 0])$ $(i = 1, 2, \cdots, m)$ are the m smallest IVIFNs. Moreover, denote the alternatives Y_i $(i = 1, 2, \cdots, n)$ as $Y_i = (\tilde{r}_{i1}, \tilde{r}_{i2}, \cdots, \tilde{r}_{im})^{\mathrm{T}}$, $i = 1, 2, \cdots, n$.

Step 3 Calculate the distance between the alternative Y_i and the UIFIS Y^+ and the distance between the alternative Y_i and the UIFNIS Y^- respectively:

$$\begin{aligned}d(Y_i, Y^+) =&\sum_{j=1}^{m} \omega_j d(\tilde{r}_{ij}, \tilde{\alpha}_j^+) = \frac{1}{4}\sum_{j=1}^{m} \omega_j(|\tilde{\mu}_{ij}^{\mathrm{L}} - 1| + |\tilde{\mu}_{ij}^{\mathrm{U}} - 1| + |\tilde{\nu}_{ij}^{\mathrm{L}} - 0| + |\tilde{\nu}_{ij}^{\mathrm{U}} - 0| \\ &+ |\tilde{\pi}_{ij}^{\mathrm{L}} - 0| + |\tilde{\pi}_{ij}^{\mathrm{U}} - 0|) \\ =&\frac{1}{4}\sum_{j=1}^{m} \omega_j[2 - (\tilde{\mu}_{ij}^{\mathrm{L}} + \tilde{\mu}_{ij}^{\mathrm{U}}) + \tilde{\nu}_{ij}^{\mathrm{L}} + \tilde{\nu}_{ij}^{\mathrm{U}} + \tilde{\pi}_{ij}^{\mathrm{L}} + \tilde{\pi}_{ij}^{\mathrm{U}}] \\ =&\frac{1}{4}\sum_{j=1}^{m} \omega_j[2 - (\tilde{\mu}_{ij}^{\mathrm{L}} + \tilde{\mu}_{ij}^{\mathrm{U}}) + \tilde{\nu}_{ij}^{\mathrm{L}} + \tilde{\nu}_{ij}^{\mathrm{U}} + 1 - \tilde{\mu}_{ij}^{\mathrm{U}} - \tilde{\nu}_{ij}^{\mathrm{U}} + 1 - \tilde{\mu}_{ij}^{\mathrm{L}} - \tilde{\nu}_{ij}^{\mathrm{L}}] \\ =&\frac{1}{4}\sum_{j=1}^{m} \omega_j[4 - 2(\tilde{\mu}_{ij}^{\mathrm{L}} + \tilde{\mu}_{ij}^{\mathrm{U}})] \\ =&\frac{1}{2}\sum_{j=1}^{m} \omega_j[2 - (\tilde{\mu}_{ij}^{\mathrm{L}} + \tilde{\mu}_{ij}^{\mathrm{U}})]\end{aligned} \tag{6.81}$$

$$\begin{aligned}d(Y_i, Y^-) =&\sum_{j=1}^{m} \omega_j d(\tilde{r}_{ij}, \tilde{\alpha}_j^-) \\ =&\frac{1}{4}\sum_{j=1}^{m} \omega_j(|\tilde{\mu}_{ij}^{\mathrm{L}} - 0| + |\tilde{\mu}_{ij}^{\mathrm{U}} - 0| + |\tilde{\nu}_{ij}^{\mathrm{L}} - 1| + |\tilde{\nu}_{ij}^{\mathrm{U}} - 1| \\ &+ |\tilde{\pi}_{ij}^{\mathrm{L}} - 0| + |\tilde{\pi}_{ij}^{\mathrm{U}} - 0|) \\ =&\frac{1}{4}\sum_{j=1}^{m} \omega_j[2 + \tilde{\mu}_{ij}^{\mathrm{L}} + \tilde{\mu}_{ij}^{\mathrm{U}} - (\tilde{\nu}_{ij}^{\mathrm{L}} + \tilde{\nu}_{ij}^{\mathrm{U}}) + 1 - \tilde{\mu}_{ij}^{\mathrm{U}} - \tilde{\nu}_{ij}^{\mathrm{U}} + 1 - \tilde{\mu}_{ij}^{\mathrm{L}} - \tilde{\nu}_{ij}^{\mathrm{L}}] \\ =&\frac{1}{4}\sum_{j=1}^{m} \omega_j[4 - 2(\tilde{\nu}_{ij}^{\mathrm{L}} + \tilde{\nu}_{ij}^{\mathrm{U}})] = \frac{1}{2}\sum_{j=1}^{m} \omega_j[2 - (\tilde{\nu}_{ij}^{\mathrm{L}} + \tilde{\nu}_{ij}^{\mathrm{U}})] \\ =&\frac{1}{2}\sum_{j=1}^{m} \omega_j[2 - (\tilde{\nu}_{ij}^{\mathrm{L}} + \tilde{\nu}_{ij}^{\mathrm{U}})]\end{aligned} \tag{6.82}$$

where

$$\tilde{r}_{ij} = ([\tilde{\mu}_{ij}^{\mathrm{L}}, \tilde{\mu}_{ij}^{\mathrm{U}}], [\tilde{\nu}_{ij}^{\mathrm{L}}, \tilde{\nu}_{ij}^{\mathrm{U}}], [\tilde{\pi}_{ij}^{\mathrm{L}}, \tilde{\pi}_{ij}^{\mathrm{U}}]), \quad i = 1, 2, \cdots, n; \quad j = 1, 2, \cdots, m$$

Step 4 Calculate the closeness coefficient of each alternative:

$$\tilde{c}(Y_i) = \frac{d(Y_i, Y^-)}{d(Y_i, Y^+) + d(Y_i, Y^-)}, \quad i = 1, 2, \cdots, n \tag{6.83}$$

Since

$$d(Y_i, Y^+) + d(Y_i, Y^-)$$

$$=\frac{1}{2}\sum_{j=1}^{m}\omega_j[2-(\tilde{\mu}_{ij}^{\mathrm{L}}+\tilde{\mu}_{ij}^{\mathrm{U}})]+\frac{1}{2}\sum_{j=1}^{m}\omega_j[2-(\tilde{\nu}_{ij}^{\mathrm{L}}+\tilde{\nu}_{ij}^{\mathrm{U}})]$$
$$=\frac{1}{2}\sum_{j=1}^{m}\omega_j[2-(\tilde{\mu}_{ij}^{\mathrm{L}}+\tilde{\mu}_{ij}^{\mathrm{U}})-(\tilde{\nu}_{ij}^{\mathrm{L}}+\tilde{\nu}_{ij}^{\mathrm{U}})]$$
$$=\frac{1}{2}\sum_{j=1}^{m}\omega_j[4-(\tilde{\mu}_{ij}^{\mathrm{L}}+\tilde{\mu}_{ij}^{\mathrm{U}})-(\tilde{\nu}_{ij}^{\mathrm{L}}+\tilde{\nu}_{ij}^{\mathrm{U}})]$$
$$=\frac{1}{2}\sum_{j=1}^{m}\omega_j[2+(\tilde{\pi}_{ij}^{\mathrm{L}}+\tilde{\pi}_{ij}^{\mathrm{U}})] \tag{6.84}$$

Eq.(6.83) can be transformed as:

$$\tilde{c}(Y_i)=\frac{\sum_{j=1}^{m}\omega_j[2-(\tilde{\nu}_{ij}^{\mathrm{L}}+\tilde{\nu}_{ij}^{\mathrm{U}})]}{\sum_{j=1}^{m}\omega_j[2+(\tilde{\pi}_{ij}^{\mathrm{L}}+\tilde{\pi}_{ij}^{\mathrm{U}})]},\quad i=1,2,\cdots,n \tag{6.85}$$

Step 5 Rank all the alternatives Y_i $(i=1,2,\cdots,n)$ according to the closeness coefficients $\tilde{c}(Y_i)$ $(i=1,2,\cdots,n)$: the greater the value $\tilde{c}(Y_i)$, the better the alternative Y_i.

Example 6.3.1 (Xu and Yager, 2008) Here we use a slightly revised version of Example 4.4.1 to illustrate the decision making approaches developed above:

In order to prioritize these agroecological regions Y_i $(i=1,2,\cdots,7)$ with respect to their comprehensive functions, a committee has been set up to provide assessment information on the agroecological regions Y_i $(i=1,2,\cdots,7)$. The attributes which are considered here in assessment of Y_i $(i=1,2,\cdots,7)$ are: ① G_1 is ecological benefit; ② G_2 is economic benefit; and ③ G_3 is social benefit. The committee evaluates the performance of agroecological regions x_i $(i=1,2,\cdots,7)$ in the years 2004-2006 according to the attributes G_j $(j=1,2,3)$ and constructs, respectively, the intuitionistic fuzzy decision matrices $R(t_k)$ ($k=1,2,3$, and t_1 denotes the year "2004", t_2 the year "2005", and t_3 the year "2006"), see Tables 6.1~6.3. Let $\omega(t)=(1/6,\ 2/6,\ 3/6)^{\mathrm{T}}$ be

Table 6.1 Intuitionistic fuzzy decision matrix $R(t_1)$ (Xu and Yager, 2008)

	G_1	G_2	G_3
Y_1	(0.8,0.1,0.1)	(0.9,0.1,0.0)	(0.7,0.2,0.1)
Y_2	(0.7,0.3,0.0)	(0.6,0.2,0.2)	(0.6,0.1,0.3)
Y_3	(0.5,0.4,0.1)	(0.7,0.3,0.0)	(0.6,0.1,0.3)
Y_4	(0.9,0.1,0.0)	(0.7,0.1,0.2)	(0.8,0.2,0.0)
Y_5	(0.6,0.1,0.3)	(0.8,0.2,0.0)	(0.5,0.1,0.4)
Y_6	(0.3,0.6,0.1)	(0.5,0.4,0.0)	(0.4,0.5,0.1)
Y_7	(0.5,0.2,0.3)	(0.4,0.6,0.0)	(0.5,0.5,0.0)

Table 6.2 Intuitionistic fuzzy decision matrix $R(t_2)$ (Xu and Yager, 2008)

	G_1	G_2	G_3
Y_1	(0.9,0.1,0.0)	(0.8,0.2,0.0)	(0.8,0.1,0.1)
Y_2	(0.8,0.2,0.0)	(0.5,0.1,0.4)	(0.7,0.2,0.1)
Y_3	(0.5,0.5,0.0)	(0.7,0.2,0.1)	(0.8,0.2,0.0)
Y_4	(0.9,0.1,0.0)	(0.9,0.1,0.0)	(0.7,0.3,0.0)
Y_5	(0.5,0.2,0.3)	(0.6,0.3,0.1)	(0.6,0.2,0.2)
Y_6	(0.4,0.6,0.0)	(0.3,0.4,0.3)	(0.5,0.5,0.0)
Y_7	(0.3,0.5,0.2)	(0.5,0.3,0.2)	(0.6,0.4,0.0)

Table 6.3 Intuitionistic fuzzy decision matrix $R(t_3)$ (Xu and Yager, 2008)

	G_1	G_2	G_3
Y_1	(0.7,0.1,0.2)	(0.9,0.1,0.0)	(0.9,0.1,0.0)
Y_2	(0.9,0.1,0.0)	(0.6,0.2,0.2)	(0.5,0.2,0.3)
Y_3	(0.4,0.5,0.1)	(0.8,0.1,0.1)	(0.7,0.1,0.2)
Y_4	(0.8,0.1,0.1)	(0.7,0.2,0.1)	(0.9,0.1,0.0)
Y_5	(0.6,0.3,0.1)	(0.8,0.2,0.0)	(0.7,0.2,0.1)
Y_6	(0.2,0.7,0.1)	(0.5,0.1,0.4)	(0.3,0.1,0.6)
Y_7	(0.4,0.6,0.0)	(0.7,0.3,0.0)	(0.5,0.5,0.0)

the weight vector of the years t_k $(k = 1, 2, 3)$, and $\omega = (0.3,\ 0.4,\ 0.3)^{\mathrm{T}}$ the weight vector of the attributes G_j $(j = 1, 2, 3)$.

Since all the attributes G_j $(j = 1, 2, 3)$ are of benefit type, normalization is not needed. Now we utilize the approach introduced in Section 6.3 to prioritize these agroecological regions:

Step 1 Utilize the DIFWA operator (6.2) to aggregate all the intuitionistic fuzzy decision matrices $R(t_k)$ $(k = 1, 2, 3)$ into a complex intuitionistic fuzzy decision matrix R (Table 6.4):

Table 6.4 Complex intuitionistic fuzzy decision matrix R (Xu and Yager, 2008)

	G_1	G_2	G_3
Y_1	(0.806,0.100,0.094)	(0.874,0.126,0.000)	(0.849,0.112,0.039)
Y_2	(0.849,0.151,0.066)	(0.569,0.159,0.272)	(0.594,0.214,0.192)
Y_3	(0.452,0.482,0.066)	(0.755,0.151,0.094)	(0.725,0.126,0.149)
Y_4	(0.859,0.100,0.041)	(0.792,0.141,0.067)	(0.838,0.162,0.000)
Y_5	(0.569,0.218,0.213)	(0.748,0.229,0.023)	(0.640,0.178,0.182)
Y_6	(0.289,0.648,0.063)	(0.441,0.200,0.359)	(0.390,0.224,0.383)
Y_7	(0.387,0.470,0.143)	(0.601,0.337,0.062)	(0.536,0.464,0.000)

Step 2 Denote the IFIS Y^+, IFNIS Y^-, and the alternatives Y_i $(i=1,2,\cdots,7)$ by

$$Y^+ = ((1,0,0),(1,0,0),(1,0,0))^{\mathrm{T}}, \quad Y^- = ((0,1,0),(0,1,0),(0,1,0))^{\mathrm{T}}$$

$$Y_1 = ((0.806,0.100,0.094),(0.874,0.126,0.000),(0.849,0.112,0.039))^{\mathrm{T}}$$

$$Y_2 = ((0.849,0.151,0.000),(0.569,0.159,0.272),(0.594,0.214,0.192))^{\mathrm{T}}$$

$$Y_3 = ((0.452,0.482,0.066),(0.755,0.151,0.094),(0.725,0.126,0.149))^{\mathrm{T}}$$

$$Y_4 = ((0.859,0.100,0.041),(0.792,0.141,0.067),(0.838,0.162,0.000))^{\mathrm{T}}$$

$$Y_5 = ((0.569,0.218,0.213),(0.748,0.229,0.023),(0.640,0.178,0.182))^{\mathrm{T}}$$

$$Y_6 = ((0.289,0.648,0.063),(0.441,0.200,0.359),(0.390,0.224,0.386))^{\mathrm{T}}$$

$$Y_7 = ((0.387,0.470,0.143),(0.601,0.337,0.062),(0.536,0.464,0.000))^{\mathrm{T}}$$

and utilize Eq.(6.71) to calculate the closeness coefficient of each alternative:

$$c(Y_1)=0.852, \quad c(Y_2)=0.709, \quad c(Y_3)=0.687, \quad c(Y_4)=0.833$$

$$c(Y_5)=0.700, \quad c(Y_6)=0.515, \quad c(Y_7)=0.548$$

Step 3 Rank all the alternatives Y_i $(i=1,2,\cdots,7)$ according to the closeness coefficients $c(Y_i)$ $(i=1,2,\cdots,7)$:

$$Y_1 \succ Y_4 \succ Y_2 \succ Y_5 \succ Y_3 \succ Y_7 \succ Y_6$$

Thus the agroecological region with the most comprehensive functions is Wuhan-Ezhou-Huanggang.

The committee can also evaluate the performance of agroecological regions Y_i $(i=1,2,\cdots,7)$ in the years 2004-2006 according to the attributes G_j $(j=1,2,3)$ and construct, respectively, the interval-valued intuitionistic fuzzy decision matrices $\tilde{R}(t_k)$ $(k=1,2,3)$ as listed in Tables 6.5–6.7:

Table 6.5 Interval-valued intuitionistic fuzzy decision matrix $\tilde{R}(t_1)$ (Xu and Yager, 2008)

	G_1	G_2	G_3
Y_1	([0.8, 0.9], [0.0, 0.1], [0.0, 0.2])	([0.7, 0.8], [0.1, 0.2], [0.0, 0.2])	([0.6, 0.8], [0.0, 0.2], [0.0, 0.4])
Y_2	([0.6, 0.7], [0.2, 0.3], [0.0, 0.2])	([0.5, 0.7], [0.2, 0.3], [0.0, 0.3])	([0.5, 0.6], [0.2, 0.3], [0.1, 0.3])
Y_3	([0.4, 0.5], [0.2, 0.4], [0.1, 0.4])	([0.5, 0.6], [0.2, 0.3], [0.1, 0.3])	([0.4, 0.6], [0.1, 0.2], [0.2, 0.5])
Y_4	([0.7, 0.8], [0.1, 0.2], [0.0, 0.2])	([0.6, 0.8], [0.0, 0.1], [0.1, 0.4])	([0.6, 0.7], [0.1, 0.2], [0.1, 0.3])
Y_5	([0.5, 0.7], [0.1, 0.3], [0.0, 0.4])	([0.7, 0.8], [0.1, 0.2], [0.0, 0.2])	([0.4, 0.5], [0.2, 0.4], [0.1, 0.4])
Y_6	([0.2, 0.3], [0.5, 0.6], [0.1, 0.3])	([0.3, 0.5], [0.4, 0.5], [0.0, 0.3])	([0.4, 0.6], [0.3, 0.4], [0.0, 0.3])
Y_7	([0.4, 0.5], [0.3, 0.4], [0.1, 0.3])	([0.2, 0.5], [0.3, 0.5], [0.0, 0.5])	([0.4, 0.7], [0.2, 0.3], [0.0, 0.4])

Table 6.6 Interval-valued intuitionistic fuzzy decision matrix $\tilde{R}(t_2)$ (Xu and Yager, 2008)

	G_1	G_2	G_3
Y_1	([0.7, 0.8], [0.1, 0.2], [0.0, 0.2])	([0.8, 0.9], [0.0, 0.1], [0.0, 0.2])	([0.7, 0.9], [0.0, 0.1], [0.0, 0.3])
Y_2	([0.5, 0.7], [0.1, 0.2], [0.1, 0.4])	([0.6, 0.7], [0.1, 0.3], [0.0, 0.3])	([0.4, 0.5], [0.2, 0.4], [0.1, 0.4])
Y_3	([0.3, 0.5], [0.1, 0.3], [0.2, 0.6])	([0.4, 0.5], [0.1, 0.3], [0.2, 0.5])	([0.3, 0.6], [0.3, 0.4], [0.0, 0.4])
Y_4	([0.6, 0.7], [0.1, 0.2], [0.1, 0.3])	([0.7, 0.8], [0.1, 0.2], [0.0, 0.2])	([0.5, 0.7], [0.1, 0.3], [0.0, 0.4])
Y_5	([0.5, 0.7], [0.2, 0.3], [0.0, 0.3])	([0.5, 0.7], [0.1, 0.3], [0.0, 0.4])	([0.4, 0.6], [0.2, 0.3], [0.1, 0.4])
Y_6	([0.3, 0.4], [0.4, 0.6], [0.0, 0.3])	([0.2, 0.4], [0.5, 0.6], [0.0, 0.3])	([0.4, 0.5], [0.4, 0.5], [0.0, 0.2])
Y_7	([0.3, 0.5], [0.3, 0.5], [0.0, 0.4])	([0.4, 0.6], [0.3, 0.4], [0.0, 0.3])	([0.4, 0.5], [0.2, 0.4], [0.1, 0.4])

Table 6.7 Interval-valued intuitionistic fuzzy decision matrix $\tilde{R}(t_3)$ (Xu and Yager, 2008)

	G_1	G_2	G_3
Y_1	([0.6, 0.7], [0.1, 0.3], [0.0, 0.3])	([0.7, 0.9], [0.0, 0.1], [0.0, 0.3])	([0.8, 0.9], [0.0, 0.1], [0.0, 0.2])
Y_2	([0.4, 0.6], [0.1, 0.2], [0.2, 0.5])	([0.5, 0.7], [0.1, 0.2], [0.1, 0.4])	([0.6, 0.7], [0.1, 0.3], [0.0, 0.3])
Y_3	([0.2, 0.4], [0.2, 0.3], [0.3, 0.6])	([0.3, 0.6], [0.2, 0.3], [0.1, 0.5])	([0.4, 0.6], [0.2, 0.4], [0.0, 0.4])
Y_4	([0.7, 0.8], [0.0, 0.1], [0.1, 0.3])	([0.8, 0.9], [0.0, 0.1], [0.0, 0.2])	([0.4, 0.7], [0.2, 0.3], [0.0, 0.4])
Y_5	([0.5, 0.6], [0.2, 0.3], [0.1, 0.3])	([0.4, 0.5], [0.1, 0.2], [0.3, 0.5])	([0.6, 0.7], [0.2, 0.3], [0.0, 0.2])
Y_6	([0.2, 0.3], [0.5, 0.6], [0.1, 0.3])	([0.3, 0.5], [0.3, 0.4], [0.1, 0.4])	([0.3, 0.6], [0.2, 0.4], [0.0, 0.5])
Y_7	([0.5, 0.6], [0.3, 0.4], [0.0, 0.2])	([0.2, 0.3], [0.4, 0.5], [0.2, 0.4])	([0.7, 0.8], [0.1, 0.2], [0.0, 0.2])

In such a case, we can utilize the approach introduced in Section 6.3 to prioritize these agroecological regions.

To do so, we first utilize the UDIFWA operator (6.80) to aggregate all the interval-valued intuitionistic fuzzy decision matrices $\tilde{R}(t_k)$ $(k = 1, 2, 3)$ into a complex interval-valued intuitionistic fuzzy decision matrix $\tilde{R}$:

Table 6.8 Complex interval-valued intuitionistic fuzzy decision matrix $\tilde{R}$ (Xu and Yager, 2008)

	G_1	G_2	G_3
Y_1	([0.676, 0.782], [0, 0.218], [0.000, 0.324])	([0.738, 0.888], [0, 0.112], [0.000, 0.262])	([0.743, 0.888], [0, 0.112], [0.000, 0.257])
Y_2	([0.472, 0.654], [0.112, 0.214], [0.132, 0.416])	([0.536, 0.700], [0.112, 0.245], [0.055, 0352])	([0.525, 0.627], [0.141, 0.330], [0.043, 0.334])
Y_3	([0.271, 0.452], [0.159, 0.315], [0.233, 0.570])	([0.371, 0.569], [0.159, 0.300], [0.131, 0.470])	([0.368, 0.600], [0.204, 0.356], [0.044, 0.428])
Y_4	([0.670, 0.771], [0, 0.141], [0.088, 0.330])	([0.743, 0.859], [0, 0.126], [0.015, 0.257])	([0.472, 0.700], [0.141, 0.280], [0.020, 0.387])
Y_5	([0.500, 0.654], [0.178, 0.300], [0.046, 0.322])	([0.497, 0.638], [0.100, 0.229], [0.333, 0.403])	([0.510, 0.640], [0.200, 0.315], [0.045, 0.290])
Y_6	([0.235, 0.335], [0.464, 0.600], [0.065, 0.301])	([0.268, 0.469], [0.373, 0.475], [0.056, 0.359])	([0.352, 0.569], [0.270, 0.431], [0.000, 0.378])
Y_7	([0.423, 0.553], [0.300, 0.431], [0.016, 0.277])	([0.273, 0.450], [0.346, 0.464], [0.086, 0.381])	([0.576, 0.710], [0.141, 0.270], [0.020, 0.283])

and then denote the UIFIS Y^+, UIFNIS Y^-, and the alternatives Y_i $(i = 1, 2, \cdots, 7)$ by

$$
\begin{aligned}
Y^+ =&(([1,1],[0,0],[0,0]),([1,1],[0,0],[0,0]),([1,1],[0,0],[0,0]))^{\mathrm{T}}\\
Y^- =&(([0,0],[1,1],[0,0]),([0,0],[1,1],[0,0]),([0,0],[1,1],[0,0]))^{\mathrm{T}}\\
Y_1 =&(([0.676,0.782],[0.000,0.218],[0.000,0.324]),([0.738,0.888],\\
&[0.000,0.112],[0.000,0.262]),([0.743,0.888],[0.000,0.112],[0.000,0.257]))^{\mathrm{T}}\\
Y_2 =&(([0.472,0.654],[0.112,0.214],[0.132,0.416]),([0.536,0.700],[0.112,0.245],\\
&[0.055,0.352]),([0.525,0.627],[0.141,0.330],[0.043,0.334]))^{\mathrm{T}}\\
Y_3 =&(([0.271,0.452],[0.159,0.315],[0.233,0.570]),\\
&([0.371,0.569],[0.159,0.300],[0.131,0.470]),\\
&([0.368,0.600],[0.204,0.356],[0.044,0.428]))^{\mathrm{T}}\\
Y_4 =&(([0.670,0.771],[0.000,0.141],[0.088,0.330]),([0.743,0.859],[0.000,0.126],\\
&[0.015,0.257]),([0.472,0.700],[0.141,0.280],[0.020,0.387]))^{\mathrm{T}}\\
Y_5 =&(([0.500,0.654],[0.178,0.300],[0.046,0.322]),([0.497,0.638],[0.100,0.229],\\
&[0.333,0.403]),([0.510,0.640],[0.200,0.315],[0.045,0.290]))^{\mathrm{T}}\\
Y_6 =&(([0.235,0.335],[0.464,0.600],[0.065,0.301]),([0.268,0.469],[0.373,0.475],\\
&[0.056,0.359]),([0.352,0.569],[0.270,0.431],[0.000,0.378]))^{\mathrm{T}}\\
Y_7 =&(([0.423,0.553],[0.300,0.431],[0.016,0.277]),([0.273,0.450],[0.346,0.464],\\
&[0.086,0.381]),([0.576,0.710],[0.141,0.270],[0.020,0.283]))^{\mathrm{T}}
\end{aligned}
$$

By Eq.(6.85), we can calculate the closeness coefficient of each alternative as follows:

$$\tilde{c}(Y_1) = 0.814, \quad \tilde{c}(Y_2) = 0.663, \quad \tilde{c}(Y_3) = 0.574, \quad \tilde{c}(Y_4) = 0.794$$

$$\tilde{c}(Y_5) = 0.627, \quad \tilde{c}(Y_6) = 0.474, \quad \tilde{c}(Y_7) = 0.564$$

and rank all the alternatives Y_i $(i = 1, 2, \cdots, 7)$ according to the values $\tilde{c}(Y_i)$ $(i = 1, 2, \cdots, 7)$:

$$Y_1 \succ Y_4 \succ Y_2 \succ Y_5 \succ Y_3 \succ Y_7 \succ Y_6$$

Thereby, the best alternative is also Y_1 (Wuhan-Ezhou-Huanggang).

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Chapter 7

Nonlinear Optimization Models for Multi-Attribute Group Decision Making with Intuitionistic Fuzzy Information

Xu and Cai (2010) investigate multi-attribute group decision making problems in which the attribute values provided by experts are expressed in IFNs. Each of the attributes is composed of a membership degree, a non-membership degree, and an indeterminancy degree, and the weight information about both the experts and the attributes is to be determined. They first unify different types of attribute values so as to facilitate inter-attribute comparisons, and employ the simple additive weighting method to fuse all the individual opinions into the group one. They then develop two nonlinear optimization models, one minimizing the divergence between each individual opinion and the group one, and the other minimizing the divergence among the individual opinions, from which two exact formulas can be obtained to derive the weights of experts. To maximize group consensus, they establish a nonlinear optimization model based on all the individual intuitionistic fuzzy decision matrices to determine the weights of attributes. The simple additive weighting method is used to aggregate all the intuitionistic fuzzy attribute values corresponding to each alternative, and then the score function and the accuracy function are employed to rank and select the given alternatives. Moreover, they extend all the above results to interval intuitionistic fuzzy situations, and apply the models developed to an air-condition system selection problem.

7.1 Nonlinear Optimization Models for Determining Decision Makers' Weights

Definition 7.1.1 (Xu and Cai, 2010) Let $\alpha_1 = (\mu_{\alpha_1}, \nu_{\alpha_1}, \pi_{\alpha_1})$ and $\alpha_2 = (\mu_{\alpha_2}, \nu_{\alpha_2}, \pi_{\alpha_2})$ be two IFNs. Then

$$d(\alpha_1, \alpha_2) = (\mu_{\alpha_1} - \mu_{\alpha_2})^2 + (\nu_{\alpha_1} - \nu_{\alpha_2})^2 + (\pi_{\alpha_1} - \pi_{\alpha_2})^2 \tag{7.1}$$

is called the square deviation between α_1 and α_2:

If we take the weight of each IFN into account, then it follows from Eq.(7.1) that

$$d(\omega_1\alpha_1,\omega_2\alpha_2)=(\omega_1\mu_{\alpha_1}-\omega_2\mu_{\alpha_2})^2+(\omega_1\nu_{\alpha_1}-\omega_2\nu_{\alpha_2})^2+(\omega_1\pi_{\alpha_1}-\omega_2\pi_{\alpha_2})^2 \quad (7.2)$$

which is called the weighted square deviation between α_1 and α_2, where ω_1 and ω_2 are the weights of α_1 and α_2, respectively, $\omega=(\omega_1,\omega_2)^{\mathrm{T}}$, $\omega_i\in[0,1]$, $i=1,2$, and $\omega_1+\omega_2=1$.

In the decision making field, IFNs are a very useful tool used by experts to depict their fuzzy preference information over objects. Now, we consider the group decision making problem with intuitionistic fuzzy information, which can be briefly described as follows (Xu and Cai, 2010):

Let Y,G,ω,E and ξ be defined as in Section 1.3, where ω and ξ are to be determined. Experts (decision makers) $E_k\in E$ $(k=1,2,\cdots,l)$ are invited to provide their assessment information on the alternatives $Y_i\in Y$ $(i=1,2,\cdots,n)$ with respect to the attributes $G_j\in G$ $(j=1,2,\cdots,m)$, and construct l intuitionistic fuzzy decision matrices $R'_k=(r_{ij}^{'(k)})_{n\times m}$ $(k=1,2,\cdots,l)$, where $r_{ij}^{(k)}=(t_{ij}^{(k)},f_{ij}^{(k)},\pi_{ij}^{(k)})$ is an attribute value, donated by an IFN, $t_{ij}^{(k)}$ indicates the degree that the alternative Y_i should satisfy the attribute G_j expressed by the expert E_k, $f_{ij}^{(k)}$ indicates the degree that the alternative Y_i should not satisfy the attribute G_j expressed by the expert E_k, and $\pi_{ij}^{(k)}$ indicates the indeterminacy degree of the alternative Y_i to the attribute G_j, such that

$$\begin{aligned}t_{ij}^{(k)}\in[0,1],\quad f_{ij}^{(k)}\in[0,1],\quad t_{ij}^{(k)}+f_{ij}^{(k)}\leqslant 1,\quad \pi_{ij}^{(k)}=1-t_{ij}^{(k)}-f_{ij}^{(k)}\\ i=1,2,\cdots,n;\quad j=1,2,\cdots,n\end{aligned} \quad (7.3)$$

If all the attributes $G_j\,(j=1,2,\cdots,m)$ are of the same type, then the attribute values do not need normalization. In the case where there are benefit attributes and cost attributes in multi-attribute decision making, we may transform the attribute values of cost type into the attribute values of benefit type. Then $R'_k=(r_{ij}^{'(k)})_{n\times m}$ $(k=1,2,\cdots,l)$ are transformed into the intuitionistic fuzzy decision matrices $R_k=(r_{ij}^{(k)})_{n\times m}$ $(k=1,2,\cdots,l)$, where

$$r_{ij}^{(k)}=(\mu_{ij}^{(k)},\nu_{ij}^{(k)},\pi_{ij}^{(k)})=\begin{cases} r_{ij}^{'(k)}, & \text{for benefit attribute } G_j,\\ \bar{r}_{ij}^{'(k)}, & \text{for } \cos t \text{ attribute } G_j,\end{cases}$$

$$i=1,2,\cdots,n;\quad k=1,2,\cdots,l \quad (7.4)$$

and $\bar{r}_{ij}^{'(k)}$ is the complement of $r_{ij}^{'(k)}$, i.e., $\bar{r}_{ij}^{'(k)}=(t_{ij}^{(k)},f_{ij}^{(k)},\pi_{ij}^{(k)})$, $\pi_{ij}^{(k)}=1-\mu_{ij}^{(k)}-\nu_{ij}^{(k)}=1-t_{ij}^{(k)}-f_{ij}^{(k)}$.

Based on Eq.(7.4), Xu and Cai (2010) introduce the weighted square deviation between each pair of the individual intuitionistic fuzzy decision matrices (R_k, R_s) as:

$$d(\xi_k R_k, \xi_s R_s) = \sum_{i=1}^{m}\sum_{j=1}^{n}\left((\xi_k \mu_{ij}^{(k)} - \xi_s \mu_{ij}^{(s)})^2 + (\xi_k \nu_{ij}^{(k)} - \xi_s \nu_{ij}^{(s)})^2 + (\xi_k \pi_{ij}^{(k)} - \xi_s \pi_{ij}^{(s)})^2\right) \tag{7.5}$$

If we consider the weighted square deviations among all pairs of the individual intuitionistic fuzzy decision matrices, then from Eq.(7.5), we can define

$$\begin{aligned} f(\xi) =& \sum_{k=1}^{l}\sum_{s=1,s\neq k}^{l} d(\xi_k R_k, \xi_s R_s) \\ =& \sum_{k=1}^{l}\sum_{l=1,k\neq s}^{l}\sum_{i=1}^{n}\sum_{j=1}^{m} \\ & \left((\xi_k \mu_{ij}^{(k)} - \xi_s \mu_{ij}^{(s)})^2 + (\xi_k \nu_{ij}^{(k)} - \xi_s \nu_{ij}^{(s)})^2 + (\xi_k \pi_{ij}^{(k)} - \xi_s \pi_{ij}^{(s)})^2\right) \end{aligned} \tag{7.6}$$

In the case where all the individual weighted intuitionistic fuzzy decision matrices $R_k = (r_{ij}^{(k)})_{n\times m}$ $(k = 1, 2, \cdots, s)$ are the same, it is obvious that the group is of high consensus. Nevertheless, in actual applications this case generally does not occur since the experts may have different experiences and specialties. Consequently, we construct the following nonlinear optimization model so as to make the group consensus as high as possible (Xu and Cai, 2010):

(M-7.1)

$$\begin{aligned} \min f(\xi) =& \min \sum_{k=1}^{l}\sum_{s=1,k\neq s}^{l}\sum_{i=1}^{n}\sum_{j=1}^{m} \\ & \left((\xi_k \mu_{ij}^{(k)} - \xi_s \mu_{ij}^{(s)})^2 + (\xi_k \nu_{ij}^{(k)} - \xi_s \nu_{ij}^{(s)})^2 + (\xi_k \pi_{ij}^{(k)} - \xi_s \pi_{ij}^{(s)})^2\right) \\ \text{s.t.} \quad & \xi_k \in [0,1], \quad k = 1, 2, \cdots, l, \quad \sum_{k=1}^{l} \xi_k = 1 \end{aligned}$$

Below we employ the Lagrange multiplier technique to solve the model (M-7.1). The Lagrange function can be written as:

$$\begin{aligned} L(\xi, \varsigma) =& \sum_{k=1}^{l}\sum_{s=1,k\neq s}^{l}\sum_{i=1}^{n}\sum_{j=1}^{m}\left((\xi_k \mu_{ij}^{(k)} - \xi_s \mu_{ij}^{(s)})^2 + (\xi_k \nu_{ij}^{(k)}\right. \\ & \left. - \xi_s \nu_{ij}^{(s)})^2 + (\xi_k \pi_{ij}^{(k)} - \xi_s \pi_{ij}^{(s)})^2\right) - 2\varsigma\left(\sum_{k=1}^{l}\xi_k - 1\right) \end{aligned} \tag{7.7}$$

where ς is the Lagrange multiplier.

Differentiating Eq.(7.7) with respect to ξ_k $(k = 1, 2, \cdots, l)$ and setting these partial derivatives equal to zero, the following set of equations are obtained:

$$\begin{aligned}\frac{\partial L(\xi,\varsigma)}{\partial \xi_k} =& 2\sum_{s=1,k\neq s}^{l}\sum_{i=1}^{n}\sum_{j=1}^{m}\Bigg((\xi_k\mu_{ij}^{(k)}-\xi_s\mu_{ij}^{(s)})\mu_{ij}^{(k)}+(\xi_k\nu_{ij}^{(k)}\\ &-\xi_s\nu_{ij}^{(s)})\nu_{ij}^{(k)}+(\xi_k\pi_{ij}^{(k)}-\xi_s\pi_{ij}^{(s)})\pi_{ij}^{(k)}\Bigg)-2\varsigma=0,\quad k=1,2,\cdots,l \end{aligned}\tag{7.8}$$

which can be simplified as:

$$\begin{aligned}&(l-1)\left(\sum_{i=1}^{n}\sum_{j=1}^{m}\left((\mu_{ij}^{(k)})^2+(\nu_{ij}^{(k)})^2+(\pi_{ij}^{(k)})^2\right)\right)\xi_k\\ &-\sum_{s=1,k\neq s}^{l}\left(\sum_{i=1}^{n}\sum_{j=1}^{m}\left(\xi_{ij}^{(s)}\xi_{ij}^{(k)}+\nu_{ij}^{(s)}\nu_{ij}^{(k)}+\pi_{ij}^{(s)}\pi_{ij}^{(k)}\right)\right)\xi_l-\varsigma=0,\\ &\qquad k=1,2,\cdots,l\end{aligned}\tag{7.9}$$

Eq.(7.9) can be rewritten in the matrix form as follows:

$$D\xi-\varsigma e=0\tag{7.10}$$

where $e = (1, 1, \cdots, 1)^{\mathrm{T}}$ and

$$D=\begin{pmatrix} (l-1)\left(\sum_{i=1}^{n}\sum_{j=1}^{m}\left((\mu_{ij}^{(1)})^2+(\nu_{ij}^{(1)})^2+(\pi_{ij}^{(1)})^2\right)\right) \\ -\sum_{i=1}^{n}\sum_{j=1}^{m}\left(\mu_{ij}^{(2)}\mu_{ij}^{(1)}+\nu_{ij}^{(2)}\nu_{ij}^{(1)}+\pi_{ij}^{(2)}\pi_{ij}^{(1)}\right) \\ \vdots \\ -\sum_{i=1}^{n}\sum_{j=1}^{m}\left(\mu_{ij}^{(l)}\mu_{ij}^{(1)}+\nu_{ij}^{(l)}\nu_{ij}^{(1)}+\pi_{ij}^{(l)}\pi_{ij}^{(1)}\right) \end{pmatrix}$$

$$\begin{matrix} -\sum_{i=1}^{n}\sum_{j=1}^{m}\left(\mu_{ij}^{(1)}\mu_{ij}^{(2)}+\nu_{ij}^{(1)}\nu_{ij}^{(2)}+\pi_{ij}^{(1)}\pi_{ij}^{(2)}\right) & \cdots \\ (l-1)\left(\sum_{i=1}^{n}\sum_{j=1}^{m}\left((\mu_{ij}^{(2)})^2+(\nu_{ij}^{(2)})^2+(\pi_{ij}^{(2)})^2\right)\right) & \cdots \\ \vdots & \\ -\sum_{i=1}^{n}\sum_{j=1}^{m}\left(\mu_{ij}^{(l)}\mu_{ij}^{(2)}+\nu_{ij}^{(l)}\nu_{ij}^{(2)}+\pi_{ij}^{(l)}\pi_{ij}^{(2)}\right) & \cdots \end{matrix}$$

$$
\left.\begin{array}{c}
-\sum_{i=1}^{n}\sum_{j=1}^{m}\left(\mu_{ij}^{(1)}\mu_{ij}^{(l)}+\nu_{ij}^{(1)}\nu_{ij}^{(l)}+\pi_{ij}^{(1)}\pi_{ij}^{(l)}\right)\\
-\sum_{i=1}^{n}\sum_{j=1}^{m}\left(\mu_{ij}^{(2)}\mu_{ij}^{(l)}+\nu_{ij}^{(2)}\nu_{ij}^{(l)}+\pi_{ij}^{(2)}\pi_{ij}^{(l)}\right)\\
\vdots\\
(l-1)\left(\sum_{i=1}^{n}\sum_{j=1}^{m}\left((\mu_{ij}^{(l)})^2+(\nu_{ij}^{(l)})^2+(\pi_{ij}^{(l)})^2\right)\right)
\end{array}\right) \tag{7.11}
$$

Obviously, from Eq.(7.11), the determinant of the matrix D is zero if and only if $R_k = (r_{ij}^{(k)})_{n\times m}$ $(k = 1, 2, \cdots, l)$ are proportional to one another. In this case, it follows from Eq.(7.3) that all $R_k = (r_{ij}^{(k)})_{n\times m}$ $(k = 1, 2, \cdots, l)$ are the same. Thus, it is reasonable to assign the experts E_k $(k = 1, 2, \cdots, l)$ with the same weights, i.e., $\xi = (1/l, 1/l, \cdots, 1/l)^{\mathrm{T}}$. In real situations it is very unlikely that all the different individuals' intuitionistic fuzzy decision matrices would be the same. Consequently, D is positive definite and invertible. Also, noting $\sum_{k=1}^{l}\xi_k = 1$ can be rewritten as:

$$
e^{\mathrm{T}}\xi = 1 \tag{7.12}
$$

Then, solving Eqs.(7.10) and (7.12), we get

$$
\varsigma = \frac{1}{e^{\mathrm{T}}D^{-1}e} \tag{7.13}
$$

and

$$
\xi = \frac{D^{-1}e}{e^{\mathrm{T}}D^{-1}e} \tag{7.14}
$$

Since D is a positive definite matrix, $e^{\mathrm{T}}D^{-1}e > 0$ and D^{-1} is nonnegative. Thus, $\xi \geqslant 0$, i.e., $\xi_k \in [0, 1]$ $(k = 1, 2, \cdots, l)$.

In what follows, we give another similar method for deriving the weights of the experts E_k $(k = 1, 2, \cdots, l)$ from the angle of minimizing the divergence between each individual opinion and the group one:

In order to get the group opinion, we aggregate all the individual intuitionistic fuzzy decision matrices $R_k = (r_{ij}^{(k)})_{n\times m}$ $(k = 1, 2, \cdots, l)$ into the collective decision matrix $R = (r_{ij})_{n\times m}$, where $r_{ij} = (\mu_{ij}, \nu_{ij}, \pi_{ij})$, $i = 1, 2, \cdots, n$; $j = 1, 2, \cdots, m$, and

$$
\mu_{ij} = \sum_{k=1}^{l}\xi_k\mu_{ij}^{(k)}, \quad \nu_{ij} = \sum_{k=1}^{l}\xi_k\nu_{ij}^{(k)}, \quad \pi_{ij} = \sum_{k=1}^{l}\xi_k\pi_{ij}^{(k)}
$$
$$
\text{for all } i = 1, 2, \cdots, n;\ j = 1, 2, \cdots, m \tag{7.15}
$$

Clearly, all r_{ij} $(i = 1,2,\cdots,n;\ j = 1,2,\cdots,m)$ are IFNs, and thus R is an intuitionistic fuzzy decision matrix.

Based on Eq.(7.2), we define the square deviation between each individual intuitionistic fuzzy decision matrix R_k and the collective intuitionistic fuzzy decision matrix R as:

$$d(R_k, R) = \sum_{i=1}^{n}\sum_{j=1}^{m}\left(\left(\mu_{ij}^{(k)} - \sum_{s=1}^{l}\xi_s\mu_{ij}^{(s)}\right)^2 + \left(\nu_{ij}^{(k)} - \sum_{s=1}^{l}\xi_s\nu_{ij}^{(s)}\right)^2 + \left(\pi_{ij}^{(k)} - \sum_{s=1}^{l}\xi_s\pi_{ij}^{(s)}\right)^2\right) \tag{7.16}$$

from which we can define the square deviations among all the individual intuitionistic fuzzy decision matrices $\tilde{R}_k$ $(k = 1,2,\cdots,l)$ and the collective intuitionistic fuzzy decision matrix R as:

$$\dot{f}(\xi) = \sum_{k=1}^{l} d(R_k, R) = \sum_{k=1}^{l}\sum_{i=1}^{n}\sum_{j=1}^{m}\left(\left(\mu_{ij}^{(k)} - \sum_{s=1}^{l}\xi_s\mu_{ij}^{(s)}\right)^2 + \left(\nu_{ij}^{(k)} - \sum_{s=1}^{l}\xi_s\nu_{ij}^{(s)}\right)^2 + \left(\pi_{ij}^{(k)} - \sum_{s=1}^{l}\xi_s\pi_{ij}^{(s)}\right)^2\right) \tag{7.17}$$

In group decision making, a desirable decision result should be reached with a high group consensus. Consequently, we establish the following nonlinear optimization model (Xu and Cai, 2010):

(M-7.2)

$$\min \dot{f}(\xi) = \min \sum_{k=1}^{l}\sum_{i=1}^{n}\sum_{j=1}^{m}\left(\left(\mu_{ij}^{(k)} - \sum_{s=1}^{l}\xi_s\mu_{ij}^{(s)}\right)^2 + \left(\nu_{ij}^{(k)} - \sum_{s=1}^{l}\xi_s\nu_{ij}^{(s)}\right)^2 + \left(\pi_{ij}^{(k)} - \sum_{s=1}^{l}\xi_s\pi_{ij}^{(s)}\right)^2\right)$$

$$\text{s.t.}\quad \xi_k \in [0,1],\ k = 1,2,\cdots,l,\quad \sum_{k=1}^{l}\xi_k = 1$$

The solution to the this model can be derived as follows:

$$\xi = \frac{\dot{\Omega}^{-1}e\left(1 - e^{\mathrm{T}}\dot{\Omega}^{-1}r\right)}{e^{\mathrm{T}}\dot{\Omega}^{-1}e} + \dot{\Omega}^{-1}r \tag{7.18}$$

which is the weight vector of the experts E_k $(k = 1,2,\cdots,l)$, where

$$r = \left(\sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{l}\left(\xi_{ij}^{(k)}\xi_{ij}^{(1)} + \nu_{ij}^{(k)}\nu_{ij}^{(1)} + \pi_{ij}^{(k)}\pi_{ij}^{(1)}\right),\right.$$

$$\sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{l}\left(\mu_{ij}^{(k)}\mu_{ij}^{(2)}+\nu_{ij}^{(k)}\nu_{ij}^{(2)}+\pi_{ij}^{(k)}\pi_{ij}^{(2)}\right),\cdots,$$

$$\left.\sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{l}\left(\mu_{ij}^{(k)}\mu_{ij}^{(l)}+\nu_{ij}^{(k)}\nu_{ij}^{(l)}+\pi_{ij}^{(k)}\pi_{ij}^{(l)}\right)\right)^{\mathrm{T}},\quad e=(1,1,\cdots,1)^{\mathrm{T}} \tag{7.19}$$

and

$$\dot{\Omega}=\begin{pmatrix}
\sum_{i=1}^{n}\sum_{j=1}^{m} l\left(\left(\mu_{ij}^{(1)}\right)^2+\left(\nu_{ij}^{(1)}\right)^2+\left(\pi_{ij}^{(1)}\right)^2\right) & \sum_{i=1}^{n}\sum_{j=1}^{m} l\left(\mu_{ij}^{(1)}\mu_{ij}^{(2)}+\nu_{ij}^{(1)}\nu_{ij}^{(2)}+\pi_{ij}^{(1)}\pi_{ij}^{(2)}\right) & \cdots & \sum_{i=1}^{n}\sum_{j=1}^{m} l\left(\mu_{ij}^{(1)}\mu_{ij}^{(l)}+\nu_{ij}^{(1)}\nu_{ij}^{(l)}+\pi_{ij}^{(1)}\pi_{ij}^{(l)}\right) \\
\sum_{i=1}^{n}\sum_{j=1}^{m} l\left(\mu_{ij}^{(1)}\mu_{ij}^{(2)}+\nu_{ij}^{(1)}\nu_{ij}^{(2)}+\pi_{ij}^{(1)}\pi_{ij}^{(2)}\right) & \sum_{j=1}^{m} l\left(\left(\mu_{ij}^{(2)}\right)^2+\left(\nu_{ij}^{(2)}\right)^2+\left(\pi_{ij}^{(2)}\right)^2\right) & \cdots & \sum_{i=1}^{n}\sum_{j=1}^{m} l\left(\mu_{ij}^{(2)}\mu_{ij}^{(l)}+\nu_{ij}^{(2)}\nu_{ij}^{(l)}+\pi_{ij}^{(2)}\pi_{ij}^{(l)}\right) \\
\vdots & \vdots & & \\
\sum_{i=1}^{n}\sum_{j=1}^{m} l\left(\mu_{ij}^{(1)}\mu_{ij}^{(s)}+\nu_{ij}^{(1)}\nu_{ij}^{(s)}+\pi_{ij}^{(1)}\pi_{ij}^{(s)}\right) & \sum_{i=1}^{n}\sum_{j=1}^{m} l\left(\mu_{ij}^{(2)}\mu_{ij}^{(s)}+\nu_{ij}^{(2)}\nu_{ij}^{(s)}+\pi_{ij}^{(2)}\pi_{ij}^{(s)}\right) & \cdots & \sum_{i=1}^{n}\sum_{j=1}^{m} l\left(\left(\mu_{ij}^{(l)}\right)^2+\left(\nu_{ij}^{(l)}\right)^2+\left(\pi_{ij}^{(l)}\right)^2\right)
\end{pmatrix}_{l\times l} \tag{7.20}$$

where $\dot{\Omega}$ is a positive definite matrix, and $e^{\mathrm{T}}\dot{\Omega}^{-1}e>0$.

In what follows, we further investigate the approach to determining the weight vector $\omega=(\omega_1,\omega_2,\cdots,\omega_m)^{\mathrm{T}}$ of the attributes G_j $(j=1,2,\cdots,m)$;

In cases where the expert E_k's opinion is consistent with the group opinion, the individual intuitionistic fuzzy decision matrix R_k should be equal to the collective intuitionistic fuzzy decision matrix R, i.e, $r_{ij}^{(k)} = r_{ij}$, for all $i = 1, 2, \cdots, n$; $j = 1, 2, \cdots, m$. By Eq.(7.15), we have

$$\mu_{ij}^{(k)} = \sum_{s=1}^{l} \xi_s \mu_{ij}^{(s)}, \quad \nu_{ij}^{(k)} = \sum_{s=1}^{l} \xi_s \nu_{ij}^{(s)}, \quad \pi_{ij}^{(k)} = \sum_{s=1}^{l} \xi_s \pi_{ij}^{(s)}$$

$$\text{for all } i = 1, 2, \cdots, n;\ j = 1, 2, \cdots, m \tag{7.21}$$

Noting that each attribute G_j has its own importance weight ω_j, we can express the weighted form of Eq.(7.21) as:

$$\omega_j \mu_{ij}^{(k)} = \sum_{s=1}^{l} \omega_j \xi_s \mu_{ij}^{(s)}, \quad \omega_j \nu_{ij}^{(k)} = \sum_{s=1}^{l} \omega_s \xi_s \nu_{ij}^{(s)}, \quad \omega_j \pi_{ij}^{(k)} = \sum_{s=1}^{l} \omega_j \xi_s \pi_{ij}^{(s)},$$

$$\text{for all } i = 1, 2, \cdots, n;\ j = 1, 2, \cdots, m \tag{7.22}$$

However, Eq.(7.22) generally does not hold because the experts may have different experiences and specialties. Consequently, we introduce a deviation variable $e_{ij}^{(k)}$ as:

$$\begin{aligned} e_{ij}^{(k)} &= \left(\omega_j \mu_{ij}^{(k)} - \sum_{s=1}^{l} \omega_j \xi_s \mu_{ij}^{(s)}\right)^2 + \left(\omega_j \nu_{ij}^{(k)} - \sum_{s=1}^{l} \omega_j \xi_s \nu_{ij}^{(s)}\right)^2 + \left(\omega_j \pi_{ij}^{(k)} - \sum_{s=1}^{l} \omega_j \xi_s \pi_{ij}^{(s)}\right)^2 \\ &= \left(\left(\mu_{ij}^{(k)} - \sum_{s=1}^{l} \xi_s \mu_{ij}^{(s)}\right)^2 + \left(\nu_{ij}^{(k)} - \sum_{s=1}^{l} \xi_s \nu_{ij}^{(s)}\right)^2 + \left(\pi_{ij}^{(k)} - \sum_{s=1}^{l} \xi_s \pi_{ij}^{(s)}\right)^2\right) \omega_j^2 \\ &\quad \text{for all } k = 1, 2, \cdots, l;\ i = 1, 2, \cdots, n;\ j = 1, 2, \cdots, m \end{aligned} \tag{7.23}$$

and construct a deviation function:

$$\begin{aligned} e(\omega) &= \sum_{k=1}^{l} \sum_{i=1}^{n} \sum_{j=1}^{m} e_{ij}^{(k)} \\ &= \sum_{k=1}^{l} \sum_{i=1}^{n} \sum_{j=1}^{m} \left(\left(\mu_{ij}^{(k)} - \sum_{s=1}^{l} \xi_s \mu_{ij}^{(s)}\right)^2 \right. \\ &\quad \left. + \left(\nu_{ij}^{(k)} - \sum_{s=1}^{l} \xi_s \nu_{ij}^{(s)}\right)^2 + \left(\pi_{ij}^{(k)} - \sum_{s=1}^{l} \xi_s \pi_{ij}^{(s)}\right)^2 \right) \omega_j^2 \end{aligned} \tag{7.24}$$

In order to reach a desirable decision result with as high group consensus as possible, we establish the following nonlinear optimization model (Xu and Cai, 2010):

(M-7.3)

$$\min e(\omega) = \min \sum_{k=1}^{l}\sum_{i=1}^{n}\sum_{j=1}^{m}\left(\left(\mu_{ij}^{(k)} - \sum_{s=1}^{l}\xi_s\mu_{ij}^{(s)}\right)^2 + \left(\nu_{ij}^{(k)} - \sum_{s=1}^{l}\xi_s\nu_{ij}^{(s)}\right)^2 + \left(\pi_{ij}^{(k)} - \sum_{s=1}^{l}\xi_s\pi_{ij}^{(s)}\right)^2\right)\omega_j^2$$

$$\text{s.t.} \quad \omega_j \in [0,1], \quad j = 1,2,\cdots,m, \quad \sum_{j=1}^{m}\omega_j = 1$$

In the following, we utilize the Lagrange multiplier technique to solve the model (M-7.3). We first introduce the Lagrange function:

$$L(\omega,\varsigma) = \sum_{k=1}^{l}\sum_{i=1}^{n}\sum_{j=1}^{m}\left(\left(\mu_{ij}^{(k)} - \sum_{s=1}^{l}\xi_s\mu_{ij}^{(s)}\right)^2 + \left(\nu_{ij}^{(k)} - \sum_{s=1}^{l}\xi_s\nu_{ij}^{(s)}\right)^2 + \left(\pi_{ij}^{(k)} - \sum_{s=1}^{l}\xi_s\pi_{ij}^{(s)}\right)^2\right)\omega_j^2 - 2\varsigma\left(\sum_{j=1}^{m}\omega_j - 1\right) \tag{7.25}$$

where ς is the Lagrange multiplier.

Letting $\dfrac{\partial L(\omega,\varsigma)}{\partial \omega_j} = 0$, for all $j = 1,2,\cdots,m$, we get the following set of equations:

$$\frac{\partial L(\omega,\varsigma)}{\partial \omega_j} = 2\sum_{k=1}^{l}\sum_{i=1}^{n}\left(\left(\mu_{ij}^{(k)} - \sum_{s=1}^{l}\xi_s\mu_{ij}^{(s)}\right)^2 + \left(\nu_{ij}^{(k)} - \sum_{s=1}^{l}\xi_s\nu_{ij}^{(s)}\right)^2 + \left(\pi_{ij}^{(k)} - \sum_{s=1}^{l}\xi_s\pi_{ij}^{(s)}\right)^2\right)\omega_i - 2\varsigma = 0, \quad j = 1,2,\cdots,m \tag{7.26}$$

which can be simplified as:

$$\sum_{k=1}^{l}\sum_{i=1}^{n}\left(\left(\mu_{ij}^{(k)} - \sum_{s=1}^{l}\xi_s\mu_{ij}^{(s)}\right)^2 + \left(\nu_{ij}^{(k)} - \sum_{s=1}^{l}\xi_s\nu_{ij}^{(s)}\right)^2 + \left(\pi_{ij}^{(k)} - \sum_{s=1}^{l}\xi_s\pi_{ij}^{(s)}\right)^2\right)\omega_j - \varsigma = 0, \quad j = 1,2,\cdots,m \tag{7.27}$$

i.e.,

$$\omega_j = \frac{\varsigma}{\sum_{k=1}^{l}\sum_{i=1}^{n}\left(\left(\mu_{ij}^{(k)} - \sum_{s=1}^{l}\xi_s\mu_{ij}^{(s)}\right)^2 + \left(\nu_{ij}^{(k)} - \sum_{l=1}^{s}\xi_l\nu_{ij}^{(s)}\right)^2 + \left(\pi_{ij}^{(k)} - \sum_{s=1}^{l}\xi_s\pi_{ij}^{(s)}\right)^2\right)},$$

$$j = 1, 2, \cdots, m \tag{7.28}$$

Since $\sum_{j=1}^{m}\omega_j = 1$, from Eq.(7.28), we can obtain

$$\varsigma = \frac{1}{\sum_{j=1}^{m}\frac{1}{\sum_{k=1}^{l}\sum_{i=1}^{n}\left(\left(\mu_{ij}^{(k)} - \sum_{s=1}^{l}\xi_s\mu_{ij}^{(s)}\right)^2 + \left(\nu_{ij}^{(k)} - \sum_{s=1}^{l}\xi_s\nu_{ij}^{(s)}\right)^2 + \left(\pi_{ij}^{(k)} - \sum_{s=1}^{l}\xi_s\pi_{ij}^{(s)}\right)^2\right)}} \tag{7.29}$$

Thus, by Eqs.(7.28) and (7.29), we have

$$\omega_j = \frac{\frac{1}{\sum_{j=1}^{m}\frac{1}{\sum_{k=1}^{l}\sum_{i=1}^{n}\left(\left(\mu_{ij}^{(k)} - \sum_{s=1}^{l}\xi_s\mu_{ij}^{(s)}\right)^2 + \left(\nu_{ij}^{(k)} - \sum_{s=1}^{l}\xi_s\nu_{ij}^{(s)}\right)^2 + \left(\pi_{ij}^{(k)} - \sum_{s=1}^{l}\xi_s\pi_{ij}^{(s)}\right)^2\right)}}}{\sum_{k=1}^{s}\sum_{j=1}^{n}\left(\left(\mu_{ij}^{(k)} - \sum_{s=1}^{l}\xi_s\mu_{ij}^{(s)}\right)^2 + \left(\nu_{ij}^{(k)} - \sum_{s=1}^{l}\xi_s\nu_{ij}^{(s)}\right)^2 + \left(\pi_{ij}^{(k)} - \sum_{s=1}^{l}\xi_s\pi_{ij}^{(s)}\right)^2\right)},$$

$$j = 1, 2, \cdots, m \tag{7.30}$$

Based on the collective decision matrix $R = (r_{ij})_{n\times m}$ and the weight vector ω of the attributes G_j $(j = 1, 2, \cdots, m)$, we can utilize the simple additive weighting method to get the overall IFN $r_i = (\mu_i, \nu_i, \pi_i)$ corresponding to each alternative Y_i, where

$$\mu_i = \sum_{j=1}^{m}\omega_j\mu_{ij} = \sum_{j=1}^{m}\sum_{k=1}^{l}\omega_j\xi_k\mu_{ij}^{(k)}, \quad \nu_i = \sum_{j=1}^{m}\omega_j\nu_{ij} = \sum_{j=1}^{m}\sum_{k=1}^{l}\omega_j\xi_k\nu_{ij}^{(k)}$$

$$\pi_i = \sum_{j=1}^{m}\omega_j\pi_{ij} = \sum_{j=1}^{m}\sum_{k=1}^{l}\omega_j\xi_k\pi_{ij}^{(k)}, \quad i = 1, 2, \cdots, n \tag{7.31}$$

Then, we need to rank the IFNs $r_i = (\mu_i, \nu_i, \pi_i)$ $(i = 1, 2, \cdots, n)$. This is described as follows:

We first calculate the scores $s(r_i)$ $(i = 1, 2, \cdots, n)$ and the accuracy degrees $h(r_i)$ $(i = 1, 2, \cdots, n)$ of the IFNs $r_i = (\mu_i, \nu_i, \pi_i)$ $(i = 1, 2, \cdots, n)$, where

$$s(r_i) = \mu_i - \nu_i, \quad h(r_i) = \mu_i + \nu_i, \quad i = 1, 2, \cdots, n \tag{7.32}$$

We then rank the IFNs according to Definition 1.1.3, and finally, we rank the alternatives Y_i $(i = 1, 2, \cdots, n)$ in accordance with the ranking of the IFNs r_i $(i = 1, 2, \cdots, n)$, and then select the optimal one.

7.2 Extended Nonlinear Optimization Models in Interval-Valued Intuitionistic Fuzzy Situations

Definition 7.2.1 (Xu and Cai, 2010) Let $\tilde{\alpha}_1 = ([\tilde{\mu}^{\mathrm{L}}_{\tilde{\alpha}_1}, \tilde{\mu}^{\mathrm{U}}_{\tilde{\alpha}_1}], [\tilde{\nu}^{\mathrm{L}}_{\tilde{\alpha}_1}, \tilde{\nu}^{\mathrm{U}}_{\tilde{\alpha}_1}])$ and $\tilde{\alpha}_2 = ([\tilde{\mu}^{\mathrm{L}}_{\tilde{\alpha}_2}, \tilde{\mu}^{\mathrm{U}}_{\tilde{\alpha}_2}], [\tilde{\nu}^{\mathrm{L}}_{\tilde{\alpha}_2}, \tilde{\nu}^{\mathrm{U}}_{\tilde{\alpha}_2}])$ be two IVIFNs. Then

$$d(\tilde{\alpha}_1, \tilde{\alpha}_2) = (\tilde{\mu}^{\mathrm{L}}_{\tilde{\alpha}_1} - \tilde{\mu}^{\mathrm{L}}_{\tilde{\alpha}_2})^2 + (\tilde{\mu}^{\mathrm{U}}_{\tilde{\alpha}_1} - \tilde{\mu}^{\mathrm{U}}_{\tilde{\alpha}_2})^2 + (\tilde{\nu}^{\mathrm{L}}_{\tilde{\alpha}_1} - \tilde{\nu}^{\mathrm{L}}_{\tilde{\alpha}_2})^2 + (\tilde{\nu}^{\mathrm{U}}_{\tilde{\alpha}_1} - \tilde{\nu}^{\mathrm{U}}_{\tilde{\alpha}_2})^2 \tag{7.33}$$

is called the square deviation between $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$, whose weighted form is:

$$\begin{aligned} d(\omega_1\tilde{\alpha}_1, \omega_2\tilde{\alpha}_2) =& (\omega_1\tilde{\mu}^{\mathrm{L}}_{\tilde{\alpha}_1} - \omega_2\tilde{\mu}^{\mathrm{L}}_{\tilde{\alpha}_2})^2 + (\omega_1\tilde{\mu}^{\mathrm{U}}_{\tilde{\alpha}_1} - \omega_2\tilde{\mu}^{\mathrm{U}}_{\tilde{\alpha}_2})^2 \\ &+ (\omega_1\tilde{\nu}^{\mathrm{L}}_{\tilde{\alpha}_1} - \omega_2\tilde{\nu}^{\mathrm{L}}_{\tilde{\alpha}_2})^2 + (\omega_1\tilde{\nu}^{\mathrm{U}}_{\tilde{\alpha}_1} - \omega_2\tilde{\nu}^{\mathrm{U}}_{\tilde{\alpha}_2})^2 \end{aligned} \tag{7.34}$$

where ω_1 and ω_2 are the weights of $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$, respectively, $\omega = (\omega_1, \omega_2)^{\mathrm{T}}$, $\omega_i \in [0, 1]$, $i = 1, 2$, and $\omega_1 + \omega_2 = 1$.

Below we consider the group decision making problem with interval-valued intuitionistic fuzzy information (Xu and Cai, 2010):

Let Y, G, ω and ξ be defined as in Section 1.3. The decision makers E_k $(k = 1, 2, \cdots, l)$ provide their assessment information on the alternatives Y_i $(i = 1, 2, \cdots, n)$ with respect to the attributes G_j $(j = 1, 2, \cdots, m)$, and construct the interval-valued intuitionistic fuzzy decision matrices $\tilde{R}'_k = (\tilde{r}^{'(k)}_{ij})_{n\times m}$ $(k = 1, 2, \cdots, l)$, where $\tilde{r}^{'(k)}_{ij} = (\tilde{t}^{(k)}_{ij}, \tilde{f}^{(k)}_{ij}) = ([\tilde{t}^{\mathrm{L}(k)}_{ij}, \tilde{t}^{\mathrm{U}(k)}_{ij}], [\tilde{f}^{\mathrm{L}(k)}_{ij}, \tilde{f}^{\mathrm{U}(k)}_{ij}])$ is an attribute value, donated by an IVIFN, and

$$\tilde{t}^{(k)}_{ij} = [\tilde{t}^{\mathrm{L}(k)}_{ij}, \tilde{t}^{\mathrm{U}(k)}_{ij}] \subseteq [0, 1], \quad \tilde{f}^{(k)}_{ij} = [\tilde{f}^{\mathrm{L}(k)}_{ij}, \tilde{f}^{\mathrm{U}(k)}_{ij}] \subseteq [0, 1], \quad \tilde{t}^{\mathrm{U}(k)}_{ij} + \tilde{f}^{\mathrm{U}(k)}_{ij} \leqslant 1,$$
$$i = 1, 2, \cdots, n; \quad j = 1, 2, \cdots, m \tag{7.35}$$

In cases where the attributes are of benefit and cost types, we can normalize $\tilde{R}'_k = (\tilde{r}^{'(k)}_{ij})_{n\times m}$ $(k = 1, 2, \cdots, l)$ into the interval-valued intuitionistic fuzzy decision matrices $\tilde{R}_k = (\tilde{r}^{(k)}_{ij})_{n\times m}$ $(k = 1, 2, \cdots, l)$, where

$$\tilde{r}^{(k)}_{ij} = ([\mu^{\mathrm{L}(k)}_{ij}, \mu^{\mathrm{U}(k)}_{ij}], [\nu^{\mathrm{L}(k)}_{ij}, \nu^{\mathrm{U}(k)}_{ij}]) = \begin{cases} \tilde{r}^{'(k)}_{ij}, & \text{for benefit attribute } G_j, \\ \bar{\tilde{r}}^{'(k)}_{ij}, & \text{for } \cos t \text{ attribute } G_j, \end{cases}$$
$$i = 1, 2, \cdots, n; \quad k = 1, 2, \cdots, n \tag{7.36}$$

and $\bar{\tilde{r}}_{ij}^{\prime(k)}$ is the complement of $\tilde{r}_{ij}^{\prime(k)}$, i.e., $\bar{\tilde{r}}_{ij}^{\prime(k)} = (\tilde{f}_{ij}^{(k)}, \tilde{t}_{ij}^{(k)})$.

Based on Eq.(7.35), we can define the weighted square deviation between each pair of the individual interval-valued intuitionistic fuzzy decision matrices $(\tilde{R}_k, \tilde{R}_s)$ as:

$$d(\xi_k \tilde{R}_k, \lambda_s \tilde{R}_s) = \sum_{i=1}^{n}\sum_{j=1}^{m}\Big((\xi_k \mu_{ij}^{\mathrm{L}(k)} - \xi_s \mu_{ij}^{\mathrm{L}(s)})^2 + (\xi_k \mu_{ij}^{\mathrm{U}(k)} - \xi_s \mu_{ij}^{\mathrm{U}(s)})^2 + (\xi_k \nu_{ij}^{\mathrm{L}(k)} - \xi_s \nu_{ij}^{\mathrm{L}(s)})^2 + (\xi_k \nu_{ij}^{\mathrm{U}(k)} - \xi_s \nu_{ij}^{\mathrm{U}(s)})^2\Big) \tag{7.37}$$

and define the weighted square deviation among all the pairs of the individual interval-valued intuitionistic fuzzy decision matrices as:

$$\begin{aligned}\tilde{f}(\xi) =& \sum_{k=1}^{l}\sum_{s=1,s\neq k}^{l} d(\xi_k \tilde{R}_k, \xi_s \tilde{R}_s)\\ =& \sum_{k=1}^{l}\sum_{s=1,k\neq s}^{l}\sum_{i=1}^{n}\sum_{j=1}^{m}\Big((\xi_k \mu_{ij}^{\mathrm{L}(k)} - \xi_s \mu_{ij}^{\mathrm{L}(s)})^2 + (\xi_k \mu_{ij}^{\mathrm{U}(k)} - \xi_s \mu_{ij}^{\mathrm{U}(s)})^2\\ &+ (\xi_k \nu_{ij}^{\mathrm{L}(k)} - \xi_s \nu_{ij}^{\mathrm{L}(s)})^2 + (\xi_k \nu_{ij}^{\mathrm{U}(k)} - \xi_s \nu_{ij}^{\mathrm{U}(s)})^2\Big)\end{aligned} \tag{7.38}$$

Then, similar to (M-7.1), we can construct the following nonlinear optimization model (Xu and Cai, 2010):

(M-7.4)

$$\begin{aligned}\min \tilde{f}(\xi) =& \min \sum_{k=1}^{l}\sum_{s=1,k\neq s}^{l}\sum_{i=1}^{n}\sum_{j=1}^{m}((\xi_k \mu_{ij}^{\mathrm{L}(k)} - \xi_s \mu_{ij}^{\mathrm{L}(s)})^2 + (\xi_k \mu_{ij}^{\mathrm{U}(k)} - \xi_s \mu_{ij}^{\mathrm{U}(s)})^2\\ &+(\xi_k \nu_{ij}^{\mathrm{L}(k)} - \xi_s \nu_{ij}^{\mathrm{L}(s)})^2 + (\xi_k \nu_{ij}^{\mathrm{U}(k)} - \xi_s \nu_{ij}^{\mathrm{U}(s)})^2)\\ \text{s.t.}\quad & \xi_k \geqslant 0,\ k = 1, 2, \cdots, l, \quad \sum_{k=1}^{l} \xi_k = 1\end{aligned}$$

If $\tilde{R}_k = (\tilde{r}_{ij}^{(k)})_{n\times m}$ $(k = 1, 2, \cdots, l)$ are the same, then it is reasonable to assign the decision makers E_k $(k = 1, 2, \cdots, l)$ with the same weights, i.e., $\xi = (1/l, 1/l, \cdots, 1/l)^{\mathrm{T}}$; Otherwise, solving the model (M-7.4), we have

$$\xi = \frac{\tilde{D}^{-1}e}{e^{\mathrm{T}}\tilde{D}^{-1}e} \tag{7.39}$$

where $e = (1, 1, \cdots, 1)^{\mathrm{T}}$, and

$$
\tilde{D} = \begin{pmatrix}
(l-1)\left(\sum_{i=1}^{n}\sum_{j=1}^{m}\left((\mu_{ij}^{\mathrm{L}(1)})^2 + (\mu_{ij}^{\mathrm{U}(1)})^2 + (\nu_{ij}^{\mathrm{L}(1)})^2 + (\nu_{ij}^{\mathrm{U}(1)})^2\right)\right) \\
-\sum_{i=1}^{n}\sum_{j=1}^{m}\left(\mu_{ij}^{\mathrm{L}(2)}\mu_{ij}^{\mathrm{L}(1)} + \mu_{ij}^{\mathrm{U}(2)}\mu_{ij}^{\mathrm{U}(1)} + \nu_{ij}^{\mathrm{L}(2)}\nu_{ij}^{\mathrm{L}(1)} + \nu_{ij}^{\mathrm{U}(2)}\nu_{ij}^{\mathrm{U}(1)}\right) \\
\vdots \\
-\sum_{i=1}^{n}\sum_{j=1}^{m}\left(\mu_{ij}^{\mathrm{L}(l)}\mu_{ij}^{\mathrm{L}(1)} + \mu_{ij}^{\mathrm{U}(l)}\mu_{ij}^{\mathrm{U}(1)} + \nu_{ij}^{\mathrm{L}(l)}\nu_{ij}^{\mathrm{L}(1)} + \nu_{ij}^{\mathrm{U}(l)}\nu_{ij}^{\mathrm{U}(1)}\right) \\
-\sum_{i=1}^{n}\sum_{j=1}^{m}\left(\mu_{ij}^{\mathrm{L}(1)}\mu_{ij}^{\mathrm{L}(2)} + \mu_{ij}^{\mathrm{U}(1)}\mu_{ij}^{\mathrm{U}(2)} + \nu_{ij}^{\mathrm{L}(1)}\nu_{ij}^{\mathrm{L}(2)} + \nu_{ij}^{\mathrm{U}(1)}\nu_{ij}^{\mathrm{U}(2)}\right) \quad \cdots \\
(l-1)\left((\mu_{ij}^{\mathrm{L}(2)})^2 + (\mu_{ij}^{\mathrm{U}(2)})^2 + (\nu_{ij}^{\mathrm{L}(2)})^2 + (\nu_{ij}^{\mathrm{U}(2)})^2\right) \quad \cdots \\
\vdots \\
-\sum_{i=1}^{n}\sum_{j=1}^{m}\left(\mu_{ij}^{\mathrm{L}(l)}\mu_{ij}^{\mathrm{L}(2)} + \mu_{ij}^{\mathrm{U}(l)}\mu_{ij}^{\mathrm{U}(2)} + \nu_{ij}^{\mathrm{L}(l)}\nu_{ij}^{\mathrm{L}(2)} + \nu_{ij}^{\mathrm{U}(l)}\nu_{ij}^{\mathrm{U}(2)}\right) \quad \cdots \\
-\sum_{i=1}^{n}\sum_{j=1}^{m}\left(\mu_{ij}^{\mathrm{L}(1)}\mu_{ij}^{\mathrm{L}(l)} + \mu_{ij}^{\mathrm{U}(1)}\mu_{ij}^{\mathrm{U}(l)} + \nu_{ij}^{\mathrm{L}(1)}\nu_{ij}^{\mathrm{L}(l)} + \nu_{ij}^{\mathrm{U}(1)}\nu_{ij}^{\mathrm{U}(l)}\right) \\
-\sum_{i=1}^{n}\sum_{j=1}^{m}\left(\mu_{ij}^{\mathrm{L}(2)}\mu_{ij}^{\mathrm{L}(l)} + \mu_{ij}^{\mathrm{U}(2)}\mu_{ij}^{\mathrm{U}(l)} + \nu_{ij}^{\mathrm{L}(2)}\nu_{ij}^{\mathrm{L}(l)} + \nu_{ij}^{\mathrm{U}(2)}\nu_{ij}^{\mathrm{U}(l)}\right) \\
\vdots \\
(l-1)\left(\sum_{i=1}^{n}\sum_{j=1}^{m}\left((\mu_{ij}^{\mathrm{L}(l)})^2 + (\mu_{ij}^{\mathrm{U}(l)})^2 + (\nu_{ij}^{\mathrm{L}(l)})^2 + (\nu_{ij}^{\mathrm{U}(l)})^2\right)\right)
\end{pmatrix}_{l\times l} \tag{7.40}
$$

Since $\tilde{D}$ is a positive definite matrix, $e^{\mathrm{T}}\tilde{D}^{-1}e > 0$, and $\tilde{D}^{-1}$ is nonnegative. Thus, $\xi \geqslant 0$, i.e., $\xi_k \geqslant 0$ $(k = 1, 2, \cdots, l)$.

Analogous to Section 7.1, we now propose another method for deriving the weights of the decision makers E_k $(k = 1, 2, \cdots, l)$:

We first aggregate all the individual interval-valued intuitionistic fuzzy decision matrices $\tilde{R}_k = (\tilde{r}_{ij}^{(k)})_{n\times m}$ $(k = 1, 2, \cdots, l)$ into the collective interval-valued intuitionistic fuzzy decision matrix $\tilde{R} = (\tilde{r}_{ij})_{m\times n}$, where $\tilde{r}_{ij} = (\tilde{\mu}_{ij}, \tilde{\nu}_{ij})$, $i = 1, 2, \cdots, n;\ j = 1, 2, \cdots, m$, and

$$
\tilde{\mu}_{ij} = [\tilde{\mu}_{ij}^{\mathrm{L}}, \tilde{\mu}_{ij}^{\mathrm{U}}] = \left[\sum_{k=1}^{l}\xi_k\mu_{ij}^{\mathrm{L}(k)}, \sum_{k=1}^{l}\xi_k\mu_{ij}^{\mathrm{U}(k)}\right]
$$

$$\tilde{\nu}_{ij} = [\tilde{\nu}_{ij}^{\mathrm{L}}, \tilde{\nu}_{ij}^{\mathrm{U}}] = \left[\sum_{k=1}^{l} \xi_k \tilde{\nu}_{ij}^{\mathrm{L}(k)}, \sum_{k=1}^{l} \xi_k \tilde{\nu}_{ij}^{\mathrm{U}(k)}\right]$$
$$\text{for all } i = 1, 2, \cdots, n;\ j = 1, 2, \cdots, m \tag{7.41}$$

On the basis of Eq.(7.33), we define the square deviation between each individual interval-valued intuitionistic fuzzy decision matrix $\tilde{R}_k$ and the collective interval-valued intuitionistic fuzzy decision matrix $\tilde{R}$ as:

$$d(\tilde{R}_k, \tilde{R}) = \sum_{i=1}^{n}\sum_{j=1}^{m}\left(\left(\tilde{\mu}_{ij}^{\mathrm{L}(k)} - \sum_{s=1}^{l}\xi_s \tilde{\mu}_{ij}^{\mathrm{L}(s)}\right)^2 + \left(\tilde{\mu}_{ij}^{\mathrm{U}(k)} - \sum_{s=1}^{l}\xi_s \tilde{\mu}_{ij}^{\mathrm{U}}(s)\right)^2 + \left(\tilde{\nu}_{ij}^{\mathrm{L}(k)} - \sum_{s=1}^{l}\xi_s \tilde{\nu}_{ij}^{\mathrm{L}(s)}\right)^2 + \left(\tilde{\nu}_{ij}^{\mathrm{U}(k)} - \sum_{s=1}^{l}\xi_s \tilde{\nu}_{ij}^{\mathrm{U}(s)}\right)^2\right) \tag{7.42}$$

and define the square deviation among all the individual interval-valued intuitionistic fuzzy decision matrices $\tilde{R}_k$ $(k = 1, 2, \cdots, l)$ and the collective interval-valued intuitionistic fuzzy decision matrix $\tilde{R}$ as:

$$\begin{aligned}\dot{\tilde{f}}(\xi) &= \sum_{k=1}^{l} d(\tilde{R}_k, \tilde{R}) \\ &= \sum_{k=1}^{l}\sum_{i=1}^{n}\sum_{j=1}^{m}\left(\left(\tilde{\mu}_{ij}^{\mathrm{L}(k)} - \sum_{s=1}^{l}\xi_s \tilde{\mu}_{ij}^{\mathrm{L}(s)}\right)^2 + \left(\tilde{\mu}_{ij}^{\mathrm{U}(k)} - \sum_{s=1}^{l}\xi_s \tilde{\mu}_{ij}^{\mathrm{U}(l)}\right)^2 \right.\\ &\quad \left. + \left(\tilde{\nu}_{ij}^{\mathrm{L}(k)} - \sum_{s=1}^{l}\xi_s \tilde{\nu}_{ij}^{\mathrm{L}(s)}\right)^2 + \left(\tilde{\nu}_{ij}^{\mathrm{U}(k)} - \sum_{s=1}^{l}\xi_s \tilde{\nu}_{ij}^{\mathrm{U}(s)}\right)^2\right)\end{aligned} \tag{7.43}$$

Then, similar to the model (M-7.2), we establish the following nonlinear optimization model (Xu and Cai, 2010):

(M-7.5)

$$\begin{aligned}\min \dot{\tilde{f}}(\xi) = \min \sum_{k=1}^{l}\sum_{i=1}^{m}\sum_{j=1}^{n}&\left(\left(\tilde{\mu}_{ij}^{\mathrm{L}(k)} - \sum_{s=1}^{l}\xi_s \tilde{\mu}_{ij}^{\mathrm{L}(s)}\right)^2 + \left(\tilde{\mu}_{ij}^{\mathrm{U}(k)} - \sum_{s=1}^{l}\xi_s \tilde{\mu}_{ij}^{\mathrm{U}(s)}\right)^2 \right.\\ &\left. + \left(\tilde{\nu}_{ij}^{\mathrm{L}(k)} - \sum_{s=1}^{l}\xi_s \tilde{\nu}_{ij}^{\mathrm{L}(s)}\right)^2 + \left(\tilde{\nu}_{ij}^{\mathrm{U}(k)} - \sum_{s=1}^{l}\xi_s \tilde{\nu}_{ij}^{\mathrm{U}(s)}\right)^2\right)\end{aligned}$$

$$\text{s.t.} \quad \xi_k \in [0, 1], \quad k = 1, 2, \cdots, l, \quad \sum_{k=1}^{l}\xi_k = 1$$

The solution to the model (M-7.5) can be derived as:

$$\xi = \frac{\dot{\tilde{\Omega}}^{-1} e \left(1 - e^{\mathrm{T}} \dot{\tilde{\Omega}}^{-1} \tilde{r}\right)}{e^{\mathrm{T}} \dot{\tilde{\Omega}}^{-1} e} + \dot{\tilde{\Omega}}^{-1} \tilde{r} \tag{7.44}$$

which is the weight vector of the decision makers E_k $(k = 1, 2, \cdots, l)$, where

$$\begin{aligned}
\tilde{r} =& \Bigg(\sum_{i=1}^{n} \sum_{j=1}^{m} \sum_{k=1}^{l} \left(\tilde{\mu}_{ij}^{\mathrm{L}}(k) \tilde{\mu}_{ij}^{\mathrm{L}(1)} + \tilde{\mu}_{ij}^{\mathrm{U}(k)} \tilde{\mu}_{ij}^{\mathrm{U}(1)} + \tilde{\nu}_{ij}^{\mathrm{L}(k)} \tilde{\nu}_{ij}^{\mathrm{L}(1)} + \tilde{\nu}_{ij}^{\mathrm{U}(k)} \tilde{\nu}_{ij}^{\mathrm{U}(1)} \right), \\
& \sum_{i=1}^{n} \sum_{j=1}^{m} \sum_{k=1}^{l} \left(\tilde{\mu}_{ij}^{\mathrm{L}(k)} \tilde{\mu}_{ij}^{\mathrm{L}(2)} + \tilde{\mu}_{ij}^{\mathrm{U}(k)} \tilde{\mu}_{ij}^{\mathrm{U}(2)} + \tilde{\nu}_{ij}^{\mathrm{L}(k)} \tilde{\nu}_{ij}^{\mathrm{L}(2)} + \tilde{\nu}_{ij}^{\mathrm{U}(k)} \tilde{\nu}_{ij}^{\mathrm{U}(2)} \right), \cdots, \\
& \sum_{i=1}^{n} \sum_{j=1}^{m} \sum_{k=1}^{l} \left(\tilde{\mu}_{ij}^{\mathrm{L}(k)} \tilde{\mu}_{ij}^{\mathrm{L}(l)} + \tilde{\mu}_{ij}^{\mathrm{U}(k)} \tilde{\mu}_{ij}^{\mathrm{U}(l)} + \tilde{\nu}_{ij}^{\mathrm{L}(k)} \tilde{\nu}_{ij}^{\mathrm{L}(l)} + \tilde{\nu}_{ij}^{\mathrm{U}(k)} \tilde{\nu}_{ij}^{\mathrm{U}(l)} \right) \Bigg)^{\mathrm{T}}, \\
& e = (1, 1, \cdots, 1)^{\mathrm{T}}
\end{aligned} \tag{7.45}$$

and

$$\tilde{\Omega} = \begin{pmatrix}
\sum_{i=1}^{n} \sum_{j=1}^{m} l \left(\left(\tilde{\mu}_{ij}^{\mathrm{L}(1)}\right)^2 + \left(\tilde{\mu}_{ij}^{\mathrm{U}(1)}\right)^2 + \left(\tilde{\nu}_{ij}^{\mathrm{L}(1)}\right)^2 + \left(\tilde{\nu}_{ij}^{\mathrm{U}(1)}\right)^2 \right) & \sum_{i=1}^{n} \sum_{j=1}^{m} l \left(\tilde{\mu}_{ij}^{\mathrm{L}(1)} \tilde{\mu}_{ij}^{\mathrm{L}(2)} + \tilde{\mu}_{ij}^{\mathrm{U}(1)} \tilde{\mu}_{ij}^{\mathrm{U}(2)} + \tilde{\nu}_{ij}^{\mathrm{L}(1)} \tilde{\nu}_{ij}^{\mathrm{L}(2)} + \tilde{\nu}_{ij}^{\mathrm{U}(1)} \tilde{\nu}_{ij}^{\mathrm{U}(2)} \right) & \cdots & \sum_{i=1}^{n} \sum_{j=1}^{m} l \left(\tilde{\mu}_{ij}^{\mathrm{L}(1)} \tilde{\mu}_{ij}^{\mathrm{L}(l)} + \tilde{\mu}_{ij}^{\mathrm{U}(1)} \tilde{\mu}_{ij}^{\mathrm{U}(l)} + \tilde{\nu}_{ij}^{\mathrm{L}(1)} \tilde{\nu}_{ij}^{\mathrm{L}(l)} + \tilde{\nu}_{ij}^{\mathrm{U}(1)} \tilde{\nu}_{ij}^{\mathrm{U}(l)} \right) \\
\sum_{i=1}^{n} \sum_{j=1}^{m} l \left(\tilde{\mu}_{ij}^{\mathrm{L}(1)} \tilde{\mu}_{ij}^{\mathrm{L}(2)} + \tilde{\mu}_{ij}^{\mathrm{U}(1)} \tilde{\mu}_{ij}^{\mathrm{U}(2)} + \tilde{\nu}_{ij}^{\mathrm{L}(1)} \tilde{\nu}_{ij}^{\mathrm{L}(2)} + \tilde{\nu}_{ij}^{\mathrm{U}(1)} \tilde{\nu}_{ij}^{\mathrm{U}(2)} \right) & \sum_{i=1}^{n} \sum_{j=1}^{m} l \left(\left(\tilde{\mu}_{ij}^{\mathrm{L}(2)}\right)^2 + \left(\tilde{\mu}_{ij}^{\mathrm{U}(2)}\right)^2 + \left(\tilde{\nu}_{ij}^{\mathrm{L}(2)}\right)^2 + \left(\tilde{\nu}_{ij}^{\mathrm{U}(2)}\right)^2 \right) & \cdots & \sum_{i=1}^{n} \sum_{j=1}^{m} l \left(\tilde{\mu}_{ij}^{\mathrm{L}(2)} \tilde{\mu}_{ij}^{\mathrm{L}(l)} + \tilde{\mu}_{ij}^{\mathrm{U}(2)} \tilde{\mu}_{ij}^{\mathrm{U}(l)} + \tilde{\nu}_{ij}^{\mathrm{L}(2)} \tilde{\nu}_{ij}^{\mathrm{L}(l)} + \tilde{\nu}_{ij}^{\mathrm{U}(2)} \tilde{\nu}_{ij}^{\mathrm{U}(l)} \right) \\
\vdots & & & \vdots \\
\sum_{i=1}^{n} \sum_{j=1}^{m} l \left(\tilde{\mu}_{ij}^{\mathrm{L}(1)} \tilde{\mu}_{ij}^{\mathrm{L}(l)} + \tilde{\mu}_{ij}^{\mathrm{U}(1)} \tilde{\mu}_{ij}^{\mathrm{U}(l)} + \tilde{\nu}_{ij}^{\mathrm{L}(1)} \tilde{\nu}_{ij}^{\mathrm{L}(l)} + \tilde{\nu}_{ij}^{\mathrm{U}(1)} \tilde{\nu}_{ij}^{\mathrm{U}(l)} \right) & \sum_{i=1}^{n} \sum_{j=1}^{m} l \left(\tilde{\mu}_{ij}^{\mathrm{L}(2)} \tilde{\mu}_{ij}^{\mathrm{L}(l)} + \tilde{\mu}_{ij}^{\mathrm{U}(2)} \tilde{\mu}_{ij}^{\mathrm{U}(l)} + \tilde{\nu}_{ij}^{\mathrm{L}(2)} \tilde{\nu}_{ij}^{\mathrm{L}(l)} + \tilde{\nu}_{ij}^{\mathrm{U}(2)} \tilde{\nu}_{ij}^{\mathrm{U}(l)} \right) & \cdots & \sum_{i=1}^{n} \sum_{j=1}^{m} l \left(\left(\tilde{\mu}_{ij}^{\mathrm{L}(l)}\right)^2 + \left(\tilde{\mu}_{ij}^{\mathrm{U}(l)}\right)^2 + \left(\tilde{\nu}_{ij}^{\mathrm{L}(l)}\right)^2 + \left(\tilde{\nu}_{ij}^{\mathrm{U}(l)}\right)^2 \right)
\end{pmatrix}_{l \times l} \tag{7.46}$$

In the following, we investigate the approach to deriving the attribute weights from the angle of maximizing the group consensus:

We first introduce a weighted deviation variable $\tilde{e}_{ij}^{(k)}$:

$$\begin{aligned}\tilde{e}_{ij}^{(k)} &= \left(\omega_j\tilde{\mu}_{ij}^{\mathrm{L}(k)} - \sum_{s=1}^{l}\omega_j\xi_s\tilde{\mu}_{ij}^{\mathrm{L}(s)}\right)^2 + \left(\omega_j\tilde{\mu}_{ij}^{\mathrm{U}(k)} - \sum_{s=1}^{l}\omega_j\xi_s\tilde{\mu}_{ij}^{\mathrm{U}(s)}\right)^2 \\ &\quad + \left(\omega_j\tilde{\nu}_{ij}^{\mathrm{L}(k)} - \sum_{s=1}^{l}\omega_j\xi_s\tilde{\nu}_{ij}^{\mathrm{L}(s)}\right)^2 + \left(\omega_j\tilde{\nu}_{ij}^{\mathrm{U}(k)} - \sum_{s=1}^{l}\omega_j\xi_s\tilde{\nu}_{ij}^{\mathrm{U}(s)}\right)^2 \\ &= \left(\left(\tilde{\mu}_{ij}^{\mathrm{L}(k)} - \sum_{s=1}^{l}\xi_s\tilde{\mu}_{ij}^{\mathrm{L}(s)}\right)^2 + \left(\tilde{\mu}_{ij}^{\mathrm{U}(k)} - \sum_{s=1}^{l}\xi_s\tilde{\mu}_{ij}^{\mathrm{U}(s)}\right)^2 \right. \\ &\quad \left. + \left(\tilde{\nu}_{ij}^{\mathrm{L}(k)} - \sum_{s=1}^{l}\xi_s\tilde{\nu}_{ij}^{\mathrm{L}(s)}\right)^2 + \left(\tilde{\nu}_{ij}^{\mathrm{U}(k)} - \sum_{s=1}^{l}\xi_s\tilde{\nu}_{ij}^{\mathrm{U}(s)}\right)^2\right)\omega_j^2 \\ &\quad \text{for all } k=1,2,\cdots,l;\ i=1,2,\cdots,n;\ j=1,2,\cdots,m \end{aligned} \tag{7.47}$$

and construct a deviation function:

$$\begin{aligned}\tilde{e}_k(\omega) &= \sum_{i=1}^{n}\sum_{j=1}^{m}\tilde{e}_{ij}^{(k)} \\ &= \sum_{i=1}^{n}\sum_{j=1}^{m}\left(\left(\tilde{\mu}_{ij}^{\mathrm{L}(k)} - \sum_{s=1}^{l}\xi_s\tilde{\mu}_{ij}^{\mathrm{L}(s)}\right)^2 + \left(\tilde{\mu}_{ij}^{\mathrm{U}(k)} - \sum_{s=1}^{l}\xi_s\tilde{\mu}_{ij}^{\mathrm{U}(s)}\right)^2 \right. \\ &\quad \left. + \left(\tilde{\nu}_{ij}^{\mathrm{L}(k)} - \sum_{s=1}^{l}\xi_s\tilde{\nu}_{ij}^{\mathrm{L}(s)}\right)^2 + \left(\tilde{\nu}_{ij}^{\mathrm{U}(k)} - \sum_{s=1}^{l}\xi_s\tilde{\nu}_{ij}^{\mathrm{U}(s)}\right)^2\right)\omega_j^2 \end{aligned} \tag{7.48}$$

to measure the deviation between each individual interval-valued intuitionistic fuzzy decision matrix $\tilde{R}_k$ and the collective interval-valuedintuitionistic fuzzy decision matrix $\tilde{R}$. Then, based on Eq.(7.48), we define

$$\begin{aligned}\tilde{e}(\omega) &= \sum_{k=1}^{l}\tilde{e}_k(\omega) \\ &= \sum_{k=1}^{l}\sum_{i=1}^{n}\sum_{j=1}^{m}\left(\left(\tilde{\mu}_{ij}^{\mathrm{L}(k)} - \sum_{s=1}^{l}\xi_s\tilde{\mu}_{ij}^{\mathrm{L}(s)}\right)^2 + \left(\tilde{\mu}_{ij}^{\mathrm{U}(k)} - \sum_{s=1}^{l}\xi_s\tilde{\mu}_{ij}^{\mathrm{U}(s)}\right)^2 \right. \\ &\quad \left. + \left(\tilde{\nu}_{ij}^{\mathrm{L}(k)} - \sum_{s=1}^{l}\xi_s\tilde{\nu}_{ij}^{\mathrm{L}(s)}\right)^2 + \left(\tilde{\nu}_{ij}^{\mathrm{U}(k)} - \sum_{s=1}^{l}\xi_s\tilde{\nu}_{ij}^{\mathrm{U}(s)}\right)^2\right)\omega_j^2 \end{aligned} \tag{7.49}$$

to measure all the deviation among all the individual interval-valued intuitionistic fuzzy decision matrices $\tilde{R}_k$ $(k = 1, 2, \cdots, l)$ and the collective interval-valued intuitionistic fuzzy decision matrix $\tilde{R}$.

In order to make the divergence among the individual opinions and the group opinion as small as possible, we establish the following nonlinear optimization model (Xu and Cai, 2010):

(M-7.6)

$$\min \tilde{e}(\omega) = \min \sum_{k=1}^{l}\sum_{i=1}^{n}\sum_{j=1}^{m}\left(\left(\tilde{\mu}_{ij}^{\mathrm{L}(k)} - \sum_{s=1}^{l}\xi_s\tilde{\mu}_{ij}^{\mathrm{L}(s)}\right)^2 + \left(\tilde{\mu}_{ij}^{\mathrm{U}(k)} - \sum_{s=1}^{l}\xi_s\tilde{\mu}_{ij}^{\mathrm{U}(s)}\right)^2\right.$$
$$\left. + \left(\tilde{\nu}_{ij}^{\mathrm{L}(k)} - \sum_{s=1}^{l}\xi_s\tilde{\nu}_{ij}^{\mathrm{L}(s)}\right)^2 + \left(\tilde{\nu}_{ij}^{\mathrm{U}(k)} - \sum_{s=1}^{l}\xi_s\tilde{\nu}_{ij}^{\mathrm{U}(s)}\right)^2\right)\omega_j^2$$

$$\text{s.t.} \quad \omega_j \in [0,1], \quad j = 1,2,\cdots,m, \quad \sum_{j=1}^{m}\omega_j = 1$$

By using the Lagrange multiplier technique, the solution to the model (M-7.6) can be derived as:

$$\omega_j = \frac{\dfrac{1}{\sum\limits_{j=1}^{m}\dfrac{1}{\sum\limits_{k=1}^{l}\sum\limits_{i=1}^{n}\left(\left(\tilde{\mu}_{ij}^{\mathrm{L}(k)} - \sum\limits_{s=1}^{l}\xi_s\tilde{\mu}_{ij}^{\mathrm{L}(s)}\right)^2 + \left(\tilde{\mu}_{ij}^{\mathrm{U}(k)} - \sum\limits_{s=1}^{l}\xi_s\tilde{\mu}_{ij}^{\mathrm{U}(s)}\right)^2 + \left(\tilde{\nu}_{ij}^{\mathrm{L}(k)} - \sum\limits_{s=1}^{l}\xi_s\tilde{\nu}_{ij}^{\mathrm{L}(s)}\right)^2 + \left(\tilde{\nu}_{ij}^{\mathrm{U}(k)} - \sum\limits_{s=1}^{l}\xi_s\tilde{\nu}_{ij}^{\mathrm{U}(s)}\right)^2\right)}}}{\sum\limits_{k=1}^{l}\sum\limits_{i=1}^{n}\left(\left(\tilde{\mu}_{ij}^{\mathrm{L}(k)} - \sum\limits_{s=1}^{l}\xi_s\tilde{\mu}_{ij}^{\mathrm{L}(s)}\right)^2 + \left(\tilde{\mu}_{ij}^{\mathrm{U}(k)} - \sum\limits_{s=1}^{l}\xi_s\tilde{\mu}_{ij}^{\mathrm{U}(s)}\right)^2 + \left(\tilde{\nu}_{ij}^{\mathrm{L}(k)} - \sum\limits_{s=1}^{l}\xi_s\tilde{\nu}_{ij}^{\mathrm{L}(s)}\right)^2 + \left(\tilde{\nu}_{ij}^{\mathrm{U}(k)} - \sum\limits_{s=1}^{l}\xi_s\tilde{\nu}_{ij}^{\mathrm{U}(s)}\right)^2\right)},$$

$$j = 1,2,\cdots,m \tag{7.50}$$

On the basis of the derived weight vector ω and the collective interval-valued intuitionistic fuzzy decision matrix $\tilde{R} = (\tilde{r}_{ij})_{n\times m}$, we can utilize the simple additive weighting method to get the overall IVIFN $\tilde{r}_i = (\tilde{\mu}_i, \tilde{\nu}_i)$ corresponding to each alternative Y_i, where

$$\tilde{\mu}_i = [\tilde{\mu}_i^{\mathrm{L}}, \tilde{\mu}_i^{\mathrm{U}}] = \left[\sum_{j=1}^{m}\omega_j\tilde{\mu}_{ij}^{\mathrm{L}}, \sum_{j=1}^{m}\omega_j\tilde{\mu}_{ij}^{\mathrm{U}}\right]$$
$$= \left[\sum_{j=1}^{m}\sum_{k=1}^{l}\omega_j\xi_k\tilde{\mu}_{ij}^{\mathrm{L}(k)}, \sum_{j=1}^{m}\sum_{k=1}^{l}\omega_j\xi_k\tilde{\mu}_{ij}^{\mathrm{U}(k)}\right]$$
$$\tilde{\nu}_i = [\tilde{\nu}_i^{\mathrm{L}}, \tilde{\nu}_i^{\mathrm{U}}] = \left[\sum_{j=1}^{m}\omega_j\tilde{\nu}_{ij}^{\mathrm{L}}, \sum_{j=1}^{m}\omega_j\tilde{\nu}_{ij}^{\mathrm{U}}\right]$$
$$= \left[\sum_{j=1}^{m}\sum_{k=1}^{l}\omega_j\xi_k\tilde{\nu}_{ij}^{\mathrm{L}(k)}, \sum_{j=1}^{m}\sum_{k=1}^{l}\omega_j\xi_k\tilde{\nu}_{ij}^{\mathrm{U}(k)}\right] \quad i = 1,2,\cdots,n \tag{7.51}$$

Then, we can rank the IVIFNs $\tilde{r}_i = (\tilde{\mu}_i, \tilde{\nu}_i)$ $(i = 1, 2, \cdots, n)$ according to Definition (2.3.5).

After that, we can rank the alternatives Y_i $(i = 1, 2, \cdots, n)$ in accordance with the ranking of the IVIFNs $\tilde{r}_i$ $(i = 1, 2, \cdots, n)$, and then select the optimal one.

7.3 Numerical Analysis

Consider an air-condition system selection problem (adapted from Yoon (1989), Xu and Cai (2010)). Suppose that there are three air-condition systems Y_i $(i = 1, 2, 3)$ to be selected, and the following is the list of five attributes G_j $(j = 1, 2, 3, 4, 5)$ to consider (whose weight vector $\omega = (\omega_1, \omega_2, \omega_3, \omega_4, \omega_5)^{\mathrm{T}}$ is to be determined): ① G_1: Good quality; ② G_2: Easy to operate; ③ G_3: Economical; ④ G_4: Good service after selling; and ⑤ G_5 : Cost. Among these attributes, G_j $(j = 1, 2, 3, 4)$ are of benefit type, and G_5 is of cost type. An expert group which consists of three experts (decision makers) E_k $(k = 1, 2, 3)$ (whose weight vector $\xi = (\xi_1, \xi_2, \xi_3)^{\mathrm{T}}$ is to be determined) has been set up. These experts evaluate the air-condition systems $Y_i\,(i = 1, 2, 3)$ with respect to the attributes $G_j\,(j = 1, 2, 3, 4, 5)$, and construct the following three intuitionistic fuzzy decision matrices $R'_k = (r_{ij}^{'(k)})_{3\times 5}$ (Tables 7.1–7.3):

Table 7.1 Intuitionistic fuzzy decision matrix R'_1 (Xu and Cai, 2010)

	G_1	G_2	G_3	G_4	G_5
Y_1	(0.8, 0.1, 0.1)	(0.7, 0.1, 0.2)	(0.7, 0.2, 0.1)	(0.9, 0.0, 0.1)	(0.4, 0.5, 0.1)
Y_2	(0.7, 0.1, 0.2)	(0.8, 0.2, 0.0)	(0.6, 0.4, 0.0)	(0.7, 0.1, 0.2)	(0.6, 0.4, 0.0)
Y_3	(0.8, 0.2, 0.0)	(0.9, 0.1, 0.0)	(0.7, 0.0, 0.3)	(0.7, 0.2, 0.1)	(0.5, 0.5, 0.0)

Table 7.2 Intuitionistic fuzzy decision matrix R'_2 (Xu and Cai, 2010)

	G_1	G_2	G_3	G_4	G_5
Y_1	(0.9, 0.1, 0.0)	(0.8, 0.1, 0.1)	(0.7, 0.0, 0.3)	(0.9, 0.1, 0.0)	(0.3, 0.7, 0.0)
Y_2	(0.7, 0.2, 0.1)	(0.8, 0.1, 0.1)	(0.9, 0.1, 0.0)	(0.7, 0.3, 0.0)	(0.7, 0.2, 0.1)
Y_3	(0.7, 0.1, 0.2)	(0.9, 0.0, 0.1)	(0.8, 0.0, 0.2)	(0.8, 0.2, 0.0)	(0.6, 0.3, 0.1)

Table 7.3 Intuitionistic fuzzy decision matrix R'_3 (Xu and Cai, 2010)

	G_1	G_2	G_3	G_4	G_5
Y_1	(0.8, 0.0, 0.2)	(0.7, 0.1, 0.2)	(0.9, 0.0, 0.1)	(0.8, 0.1, 0.1)	(0.4, 0.6, 0.0)
Y_2	(0.8, 0.2, 0.0)	(0.7, 0.3, 0.0)	(0.8, 0.1, 0.1)	(0.9, 0.1, 0.0)	(0.6, 0.3, 0.1)
Y_3	(0.9, 0.1, 0.0)	(0.8, 0.0, 0.2)	(0.8, 0.1, 0.1)	(0.9, 0.0, 0.1)	(0.5, 0.4, 0.1)

Considering that the attributes have two different types (benefit and cost types), we first transform the attribute values of cost type into the attribute values of benefit type by using Eq.(74), then $R'_k = (r'^{(k)}_{ij})_{3\times 5}$ $(k = 1, 2, 3)$ are transformed into $R_k = (r^{(k)}_{ij})_{3\times 5}$ $(k = 1, 2, 3)$ (Tables 7.4–7.6):

Table 7.4 Intuitionistic fuzzy decision matrix R_1 (Xu and Cai, 2010)

	G_1	G_2	G_3	G_4	G_5
Y_1	(0.8, 0.1, 0.1)	(0.7, 0.1, 0.2)	(0.7, 0.2, 0.1)	(0.9, 0.0, 0.1)	(0.5, 0.4, 0.1)
Y_2	(0.7, 0.1, 0.2)	(0.8, 0.2, 0.0)	(0.6, 0.4, 0.0)	(0.7, 0.1, 0.2)	(0.4, 0.6, 0.0)
Y_3	(0.8, 0.2, 0.0)	(0.9, 0.1, 0.0)	(0.7, 0.0, 0.3)	(0.7, 0.2, 0.1)	(0.5, 0.5, 0.0)

Table 7.5 Intuitionistic fuzzy decision matrix R_2 (Xu and Cai, 2010)

	G_1	G_2	G_3	G_4	G_5
Y_1	(0.9, 0.1, 0.0)	(0.8, 0.1, 0.1)	(0.7, 0.0, 0.3)	(0.9, 0.1, 0.0)	(0.7, 0.3, 0.0)
Y_2	(0.7, 0.2, 0.1)	(0.8, 0.1, 0.1)	(0.9, 0.1, 0.0)	(0.7, 0.3, 0.0)	(0.2, 0.7, 0.1)
Y_3	(0.7, 0.1, 0.2)	(0.9, 0.0, 0.1)	(0.8, 0.0, 0.2)	(0.8, 0.2, 0.0)	(0.3, 0.6, 0.1)

Table 7.6 Intuitionistic fuzzy decision matrix R_3 (Xu and Cai, 2010)

	G_1	G_2	G_3	G_4	G_5
Y_1	(0.8, 0.0, 0.2)	(0.7, 0.1, 0.2)	(0.9, 0.0, 0.1)	(0.8, 0.1, 0.1)	(0.6, 0.4, 0.0)
Y_2	(0.8, 0.2, 0.0)	(0.7, 0.3, 0.0)	(0.8, 0.1, 0.1)	(0.9, 0.1, 0.0)	(0.3, 0.6, 0.1)
Y_3	(0.9, 0.1, 0.0)	(0.8, 0.0, 0.2)	(0.8, 0.1, 0.1)	(0.9, 0.0, 0.1)	(0.4, 0.5, 0.1)

Then, by Eq.(7.14) (or the model (M-7.1)), we can derive the weight vector of the experts E_k $(k = 1, 2, 3)$:

$$\xi = (0.344, 0.328, 0.328)^{\mathrm{T}} \tag{7.52}$$

By Eqs.(7.15) and (7.54), we aggregate all the individual intuitionistic fuzzy decision matrices $R_k = (r^{(k)}_{ij})_{3\times 5}$ $(k = 1, 2, 3)$ into the collective intuitionistic fuzzy decision matrix $R = (r_{ij})_{3\times 5}$ (Table 7.7):

Table 7.7 Collective intuitionistic fuzzy decision matrix R (Xu and Cai, 2010)

	G_1	G_2	G_3	G_4	G_5
Y_1	(0.83, 0.07, 0.10)	(0.73, 0.10, 0.17)	(0.77, 0.07, 0.16)	(0.87, 0.07, 0.06)	(0.60, 0.37, 0.03)
Y_2	(0.73, 0.17, 0.10)	(0.77, 0.20, 0.03)	(0.74, 0.20, 0.06)	(0.77, 0.17, 0.06)	(0.30, 0.63, 0.07)
Y_3	(0.80, 0.13, 0.07)	(0.87,0.03, 0.10)	(0.77, 0.03, 0.20)	(0.80, 0.13, 0.07)	(0.40, 0.53, 0.07)

After that, we employ Eq.(7.30) (or the model (M-7.3)) to derive the weight vector of the attributes G_j $(j = 1, 2, 3, 4, 5)$:

$$\omega = (0.200,\ 0.299,\ 0.106,\ 0.156,\ 0.239)^{\mathrm{T}} \tag{7.53}$$

Then, based on the collective decision matrix $R = (r_{ij})_{3\times 5}$ and the weight vector (7.53), we utilize the simple additive weighting method (7.31) to get the overall IFNs r_i $(i = 1, 2, 3)$ corresponding to the air-condition systems Y_i $(i = 1, 2, 3)$:

$$r_1 = (0.745,\ 0.151,\ 0.104), \quad r_2 = (0.646,\ 0.292,\ 0.062), \quad r_3 = (0.722,\ 0.185,\ 0.093)$$

In order to rank the IFNs r_i $(i = 1, 2, 3)$, we first calculate the scores $s(r_i)$ $(i = 1, 2, 3)$:

$$s(r_1) = 0.594, \quad s(r_2) = 0.354, \quad s(r_3) = 0.537$$

and thus $r_1 > r_3 > r_2$, by which we get the ranking of the alternatives: $x_1 \succ x_3 \succ x_2$. Therefore, Y_1 is the optimal air-condition system.

If we utilize Eq.(7.18) (or the model (M-7.2)) to determine the expert weights, then

$$\xi = (1/3,\ 1/3,\ 1/3)^{\mathrm{T}} \tag{7.54}$$

Clearly, the weight vectors of the experts E_k $(k = 1, 2, 3)$ derived by Eqs.(7.14) and (7.18) are almost the same, which lead to the same ranking of the air-condition systems Y_i $(i = 1, 2, 3)$. Similarly, if the preferences given by the experts are expressed in interval-valued intuitionistic fuzzy decision matrices, then we can utilize the models (M-7.4), (M-7.5) and (M-7.6) to derive the weights of both the experts and the attributes.

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