MITTAG-LEFFLER DISTRIBUTIONS

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Abstract. The Mittag-Leffler distribution has been studied extensively in the past decade. The Mittag-Leffler distribution has been found to be useful in a variety of applications. In this paper, we review the Mittag-Leffler distribution, and study various distributional properties and characterizations related to the Mittag-Leffler distribution. We also study semi-Mittag Leffler Distribution, Generalized positive Linnik distribution and other related distributions. We derive some new results related to the distributional properties of the semi-Mittag Leffler Distribution. We also review the results related to estimation of parameters in Mittag-Leffler distribution and propose some new estimators for the periodic function in the SML distribution using Empirical Laplace transform.

Keywords: Autoregressive models, Class L distribution, Infinite divisibility, Normal Attraction, Semi-stable law, Stable law.

1 Introduction

The Mittag-Leffler distribution is the distribution F with the Laplace transform

$$\phi(\lambda) = \int_0^\infty \exp\{-\lambda x\} dF(x) = \frac{1}{1+\lambda^\alpha}, \quad 0 < \alpha \leq 1, \quad \lambda > 0.$$

It has received the attention of many researchers recently (see Christoph and Schreiber, 2000; Kozubowski, 1994, 1998, 1999, 2000 a, b, 2001; Kozubowski and Rachev, 1994, 1999; Lin, 1994, 1997, 1998, 2001; Jacques, Remillard and Theodorescu, 1999; Remillard and Theodorescu, 2002; Jayakumar and Pillai, 1993, 1996; Pillai, 1990; Sabu George, 1984; Sandhya, 1991 a, b; Jayakumar, 1992; Gnedenko and Korolev, 1996).

Even though the Laplace transform $\phi(\lambda) = \dfrac{1}{1+\lambda^\alpha}, 0 < \alpha \leq 1$ has been studied by many researchers in the past, it is Pillai (1990) who obtained the distribution function of the same and observed that the distribution function is in fact related to the Mittag- Leffler function and hence named the distribution as the Mittag-Leffler distribution. Based on Mohan et al. (1993), the Mittag-Leffler distribution can be called positive geometric right stable law. Also Pakes (1992a, 1998) called the distribution as positive Linnik distribution. Lin (1998) proved that the Mittag-Leffler distribution belong to the class of distributions with complete monotone derivative. Pillai and Anil (1996) characterized the

Mittag-Leffler distribution using integrated Cauchy functional equation. Fujita (1993) generalized some results of Pillai (1990) and obtained characterization of geometrically infinitely divisible distributions on $(0, \infty)$.

The Mittag-Leffler distribution has been found to be useful in a variety of applications. For example, Weron and Kotulski (1996) used the Mittag-Leffler distribution to describe the Cole-Cole relaxation phenomena in Physics. Jayakumar (2003) used the Mittag-Leffler distribution to model the rate of flow of water in Kallada river, Kerala, India. Kozubowski and Rachev (1994) discussed the applications of Mittag-Leffler distribution in modeling financial data.

Semi-Mittag-Leffler distribution is a generalization of the Mittag-Leffler distribution and has been used by several authors. For example, Jayakumar and Pillai (1993) used semi-Mittag-Leffler distribution to obtain the stationary solution of a first-order autoregressive equation. Bunge (1996) explained the use of semi-Mittag-Leffler distribution in the study of random stability.

The purpose of this study is to bring the works on the Mittag-Leffler distribution available in the literature together and to present some new results on this distribution. In Section 2, we describe some properties of Mittag-Leffler distribution. We review the results related to the semi-Mittag-Leffler distributions in Section 3. In Section 4, we review the results on estimation related to the Mittag-Leffler distribution. and also present some new results related to the estimation in the semi-Mittag-Leffler distribution. Other distributions such as generalized positive Linnik laws, which are related to the Mittag-Leffler distribution are considered in Section 5 along with their relationships. The paper ends with concluding remarks in Section 6, in which we outline some open problems.

2 Mittag-Leffler Distribution

In this Section we discuss the genesis of the Mittag-Leffler distribution and study the basic distributional properties and the characterization results concerning the ML distribution.

2.1 Genesis and Definition

Pillai (1990) considered the Laplace transform $\phi(\lambda) = \dfrac{1}{1+\lambda^{\alpha}}, 0 < \alpha \leq 1$ and showed that $\phi(\lambda)$ is the Laplace transform of a probability distribution. He obtained an expression for the distribution function having the Laplace transform $\phi(\lambda)$, and is given below:

THEOREM 2.1. *The function,* $\phi(\lambda) = \dfrac{1}{1+\lambda^{\alpha}}$, $0 \leq \alpha \leq 1, \lambda > 0$ *is the Laplace* *transform of a positive valued random variable with distribution function*

$$F_\alpha(x) = \sum_{k=1}^{\infty} \frac{(-1)^{(k-1}x^{k\alpha}}{\Gamma(1+k\alpha)}.$$

Now we give the Definition of the Mittag-Leffler distribution.

DEFINITION 2.1. *A random variable X on $(0,\infty)$ is said to have the Mittag-Leffler (ML) distribution and write $X \stackrel{d}{=} ML(\alpha)$ if its distribution function is*

$$F_\alpha(x) = \sum_{k=1}^{\infty} \frac{(-1)^{(k-1)}x^{k\alpha}}{\Gamma(1+k\alpha)}, \quad 0 < \alpha \leq 1.$$

Note that $F_1(x)$ is exponential distribution. Kotz and Steutel (1988) and Huang and Chen (1989) obtained some characterizations of $F_\alpha(x)$. Pillai (1990) showed that $F_\alpha(x)$ is infinitely divisible. Kozubowski (2002) observed that the ML distribution in Bondesson et al. (1996) is different from $F_\alpha(x)$ and gave an interpretation why the Bondesson *et al.* (1996) ML distribution is not infinitely divisible. Kozubowski (2000) showed that the ML random variable X having distribution function $F_\alpha(x)$ admits the representation $X = ZW_\alpha^{1/\alpha}$ where Z and W_α are independent , Z is standard exponential, while W_α is a positive random variable with density $f_\alpha(x) = \dfrac{\sin(\alpha\pi}{\alpha\pi\,(x^2+2x\cos\alpha\pi+1)}$ (see also Pakes, 1998; Kozubowski, 1998).

We discuss the distributional properties and characterizations relating to the ML distributions below.

2.2 Distributional Properties

Lin (1998) obtained the following results on the ML distribution.

THEOREM 2.2. *For each $\alpha \in (0,1]$, $F_\alpha(x) \in M$, the class of distributions with complete monotone derivative.*

THEOREM 2.3. *Let $F_{\alpha,t}(x)$ be the distribution function of a random variable $X_\alpha(t)$ having Laplace transform*

$$\phi_{\alpha t}(\lambda) = \frac{1}{(1+\lambda^\alpha)^t}, \quad 0 < \alpha \leq 1, \lambda > 0.$$

Then

(i) $F_{\alpha,t}(x)$ is slowly varying at infinity.

(ii) $F_{\alpha,t}(x) \sim \dfrac{1}{\Gamma(1+\alpha t)}\left(\dfrac{x^\alpha}{1+x^\alpha}\right)^t$ as $x \to 0+$

THEOREM 2.4. *Let $\alpha \in (0,1)$ and $t > 0$. Then*

$$\text{(i)} \qquad E\left\{X_\alpha^\rho(t)\right\} = \frac{\Gamma(1-\rho/\alpha)\Gamma(1+\rho/\alpha)}{\Gamma(1-\rho)\Gamma(t)} \;\; \text{if} \rho \in (-\alpha t, \alpha t);$$

$$\text{(ii)} \qquad E\left\{X_\alpha^\rho(t)\right\} = \infty \;\; \text{if} \;\; \rho \le -\alpha t \text{or} \;\; \rho \ge \alpha.$$

Lin (1998) has used the above results to correct some inverse Laplace transforms given in the literature.

Pakes (1992a) characterized gamma mixtures of stable laws. For further characterizations of ML laws, see also Kakosyan et al. (1984) and Yeo and Milne (1989). Pillai (1990) proved that the ML distribution with parameter α is normally attracted to positive stable law. Fujita(1993) generalized the results of Pillai (1990) to obtain characterizations of geometrically infinitely divisible distributions on $[0,\infty)$. These results are summarized in Theorem 2.4 and Theorem 2.5.

If f is a non-zero Bernstein function, then there exists a unique positive measure $W[0,\infty)$ such that $\frac{1}{f(x)} = \int_0^\infty e^{-sx}W(ds)$, $x > 0$. Let $W^{n*}dx$ denote the n times convolution measure of $W(dx)$. Define

$$U_\lambda(x) = \begin{cases} -\sum_{n=1}^{\infty}(-\lambda)^n W^{n*}dx, & x \ge 0. \\ 0, & x \ge 0. \end{cases}$$

THEOREM 2.5. *For every $\lambda > 0$, U_λ is an infinitely divisible distribution with the Laplace transform $\dfrac{\lambda}{\lambda + f(\lambda)}$.*

THEOREM 2.6. *Let $\lambda > 0$. For every $0 < p < 1$, $U_\lambda(x) = \sum_{j=1}^{\infty}(p(1-p)^{j-1}U_{\lambda/p}^{j*}(x), x > 0$. Then U_λ is geometrically infinitely divisible.*

Jayakumar and Pillai (1996) considered the characterization of ML laws and showed that within the class of infinitely divisible laws with positive support, the ML distribution F_α is the unique distribution function F satisfying the relation

$$\phi(\lambda) = \exp\left\{\int_0^\infty \frac{\exp(-\lambda x) - 1}{x}\alpha(1 - F(x))dx\right\}, \;\; \lambda > 0 \qquad (2.2.1)$$

where ϕ denotes the Laplace transform of F. Lin (2001) obtained an extension of this result and is given below.

THEOREM 2.7. *Let $\alpha \in [0,1]$ be a constant and let X be a non-negative random variable with distribution function F and Laplace transform ϕ. Assume further that $P(X=0) < 1$. Then (2.2.1) holds if and only if $F = F_{\alpha,\delta}$ for some $\delta > 0$, where $F_{\alpha,\delta}$ is the distribution function having Laplace transform $\frac{1}{1+\delta\lambda^\alpha}$, $0 < \alpha \le 1$.*

Lin (2001) obtained another characterization of ML distribution in terms of the Pareto law.

THEOREM 2.8. *Let Z have the standard exponential law and for $\alpha \in (0, 1]$, let T_α have the Pareto type III law*

$$P_\alpha(t) = 1 - \frac{1}{1+t^\alpha}, \quad t > 0.$$

Further, assume that X is a positive random variable independent of Z. Then

(a) $\frac{X}{Z} \stackrel{d}{=} T_\alpha$ *if and only if X has the ML distribution F_α,*

(b) $\frac{Z}{X} \stackrel{d}{=} T_\alpha$ *if and only if X has the ML distribution F_α,*

Next, we define the class L distributions and study some results related to the class L distributions.

DEFINITION 2.2. *A random variable X with Laplace transform $\phi(\lambda)$ is in class L if for every c, $0 < c < 1$, there exists a Laplace transform $\phi_c(\lambda)$ such that $\phi(\lambda) = \phi(c\lambda)\phi_c(\lambda)$*

Sabu George and Pillai (1984) derived the following properties of ML distributions.

THEOREM 2.9. *The distribution with L.T.* $\frac{1}{1+\lambda^\alpha}$, $0 < \alpha < 1$, $\lambda > 0$ *belongs to class L.*

THEOREM 2.10. *Suppose $\phi(\lambda)$ be a completely monotone function, $\phi(0) = 1$ and $\phi(\lambda) + \phi(1/\lambda)$ and Then $\phi(\lambda) = \frac{1}{1+\lambda^\alpha L(\lambda)}$, where $L(\lambda)$ is slowly varying both at zero and infinity.*

Kotz and Steutel (1988) proved that $X_1 \stackrel{d}{=} U(X_1 + X_2)$ if only if the distribution of X_1 is exponential and generalized the result as follows:

THEOREM 2.11. *Let X_1, X_2 and U be independent, U has a uniform law on $[0, 1]$ and $X_1 \stackrel{d}{=} X_2$. Then $X_1 \stackrel{d}{=} U^{1/\alpha}(X_1 + X_2)$, if and only if X_1 is ML.*

Some further results on ML distributions are given below (see Pillai and Anil, 1996 and Jayakumar and Pillai, 1996).

THEOREM 2.12. *The family of ML distributions coincides with the family of geometric strictly stable distributions with support from R^+.*

THEOREM 2.13. *Suppose X_1 and X_2 are i.i.d. as $F(x)$ with density function $f(x)$ and b_1 and b_2 are constants such that $0 < b_1, b_2 < 1$ and $b_1^\alpha + b_2^\alpha = 1$, $0 < \alpha \le 1$. Then $F(x)$ is ML if and only if $b_1 X_1 + b_2 X_2 \stackrel{d}{=} X_1 + X_b$, where X_b has distribution function $F(x/b_1) + F(x/b_2) - F(x/b_1)F(x/b_2)$.*

THEOREM 2.14. *Let $F(x)$ be an infinitely divisible distribution with positive support and let $P(x)$ be the spectrum function given by*

$$ln(\phi(t)) = \int_0^\infty \frac{e^{-tx} - 1}{x} P(dx).$$

Then $F(x)$ is ML if and only if

$$F(x) = 1 - \frac{1}{\alpha}\frac{dP}{dx}.$$

Kozubowski and Rachev (1999) defined a re-parameterization of the ML distribution.

DEFINITION 2.3. *For $0 < \alpha \leq 1$ and $\sigma > 0$, the probability distribution on $(0, \infty)$ with the Laplace transform,*

$$\phi_{\alpha,\sigma}(\lambda) = \frac{1}{1 + \sigma^\alpha \lambda^\alpha}, \quad \lambda \geq 0$$

is called ML and is denoted by $ML_{\alpha,\sigma}$.
Kozubowski and Rachev (1999) noted that the density functions of ML distributions are completely monotonic on $(0, \infty)$.

2.3 ML Processes

Here, we consider the Mittag Leffler process.

DEFINITION 2.4. *The stochastic process $\{X(t), t > 0\}$ having stationary independent increments with $X(0) = 0$ and $X(1)$ having the Laplace transform $\phi(\lambda) = \frac{1}{1+\lambda^\alpha}$, $0 < \alpha \leq 1$ $\lambda > 0$is called the ML stochastic process.*
Pillai (1990) gave the above Definition and obtained the following result.

THEOREM 2.15. *The ML stochastic process $X(t)$ has the distribution function, for $t > 0$,*

$$F_{\alpha,t}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(t+k) x^{\alpha(t+k)}}{\Gamma(t) k! \Gamma(1 + \alpha(t+k))}.$$

THEOREM 2.16. *$F_{\alpha,t}(x)$, has the following property, for $0 < \alpha < 1$*

$$F_{\alpha,t}(x) = \int_0^\infty S_{\alpha,t}(x) G_t(ds),$$

where $S_{\alpha,t}(x)$is the distribution function of the stable process with Laplace transform $\exp\{-tx^\alpha\}$ and $G_t(x) = \frac{1}{\Gamma(t)} \int_0^x y^{t-1} e^{-y} dy$.
Pillai and Sabu George (1984) defined a Laplace - α process given in Definition 2.5 and proved a result (given in Theorem 2.17) related to the same.

DEINITION 2.5. *A nonnegative valued stochastic process $X(t)$ is said to be Laplace -α process if its Laplace transform is given by $\frac{1}{1+t\lambda^\alpha}$.*

THEOREM 2.17. *A Laplace-α process directed by a stable β process leads to Laplace - $\alpha\beta$ process.*
For further properties of Laplace α process, see Pillai (1985).

Pillai and Anil (1996) defined a "Process" to mean a non-degenerate stochastic process $\{Y(t),\ t \geq 0\}$ with $Y(0) = 0$, which is homogeneous, continuous in probability and has independent increments, and proved the following theorems.

THEOREM 2.18. *Let $Y(t),\ t \geq 0$ be a process with $Y(t) \geq 0$. Then $\int_0^1 t dY(t)$ has the same distribution as $Y(\alpha)$ for some $\alpha > 0$ if and only if the Laplace Transform (LT) of $Y(1)$ is of the form $\exp\{-c\lambda^\alpha\},\ \lambda \geq 0$, for some $\alpha \in (0, 1]$. Let T be a random variable independent of $Y(t)$ and with standard exponential distribution function. Then, $S = \frac{\int_0^T tdY(t)}{T}$ has a ML distribution function with exponent α.*

THEOREM 2.19. *Let $Y(t),\ t > 0$ be a stable process with $E[\exp\{-\lambda Y(t)\}] = \exp\{-ct\lambda^\alpha\}$ for all $\lambda \geq 0$, for some $c > 0$, for all $t \geq 0$, for some $\alpha \in (0, 1]$ and T a positive random variable independent of $Y(t)$. Then T has an exponential distribution if and only if $S = \frac{\int_0^T tdY(t)}{T}$ has ML distribution function.*

THEOREM 2.20. *Let $\{X(t), t > 0\}$ and $\{Y(t), t > 0\}$ be two non-negative processes. Assume that the LST of $X(1)$ is given by , $\frac{1}{1+\eta(t)}$, $t \geq 0$, where $\eta(t)$ is the logarithm of the LST of $Y(1)$. Then $X(t)$ is an ML process with exponent α if and only if the stochastic integral $\int_0^\infty R(t)dY(t)$ has the same distribution as $Y(\alpha)$, where $R(t), t \geq 0$ is a strictly decreasing survival function on $[0, \infty)$ and $\alpha > 0$ is the unique solution of $\int_0^\infty R^\alpha(t)dt = \alpha$.*

Corollary 2.1. *Let T be a random variable having unit exponential density function. Then under the conditions of the above theorem, $X(t)$ is a ML process with exponent a if and only if the random variable $Y = E_T(Y(t))$ has the same distribution as $Y(1/\alpha)$.*
Jayakumar and Pillai (1993) introduced first order autoregressive ($AR(1)$) Mittag-Leffler process. Pillai and Jayakumar (1994) characterized pth autoregressive processes using specialized class L property.

Aly and Bouzar (2000) considered the $AR(1)$ equation

$$X_n = \rho X_{n-1} + \epsilon_n, n = 0, \pm 1, \pm 2, \ldots$$

with all the variables being $R+$ valued. They found the distribution of $\{\epsilon_n\}$ when X_n's have the Laplace transform

$$\phi(u) = \frac{1}{(1 + cu^\gamma)^r}, r > 0, c > 0, 0 < \gamma \leq 1.$$

Under stationarity assumption the innovation equation ϵ_n is given by

$$\epsilon_n = \sum_{i=1}^{N}(\alpha_i^u)W_i,$$

where $\{U_i\}$ are i.i.d. $U(0,1)$ random variables, N is Poisson with mean $-r\gamma ln(\alpha)$ and the W_i's are i.i.d. with common LT

$$\phi(u) = \frac{1}{1+cu^\gamma}, \quad c>0,\ 0<\gamma<1.$$

2.4 ML Generator

Here, we describe a method of generating samples from ML distribution.

Geometric stable (GS) distributions approximate (normalized) sums of i.i.d random variables, $S_N = X_1 + X_2 + \dots X_n$ where the number of terms has a geometric distribution with mean $1/p$ and $p \to 0$. In one dimension, GS laws form a four parameter family given by the characteristic function

$$\psi(t) = \frac{1}{1+\sigma^\alpha \|t\|^\alpha \omega_{\alpha,\beta}(t) - i\mu t} \tag{2.4.1}$$

where

$$\omega_{\alpha,\beta}(x) = \begin{cases} 1 - i\beta \text{sgn}(x)\tan(\pi\alpha/2), & \text{if } \alpha \neq 1 \\ 1 + i\beta 2/\pi \text{sgn}(x)\log\|x\|, & \text{if } \alpha = 1 \end{cases} \tag{2.4.2}$$

The parameter $\alpha \in (0,2]$ is the index of stability determining the tail of the distribution, $\beta \in (-1,1)$ is the skewness parameter and $\mu \in \Re$ and $\sigma \geq 0$ controls the location and scale, respectively. Let $GS\alpha(\sigma,\beta,\mu)$ denote the GS distribution with characteristic function (2.4.1). For properties of GS distributions, see Kozubowski and Rachev (1999), Rachev and SenGupta (1992, 1993), Ramachandran (1997). Kozubowski (2000b, 2001), Jayakumar et al. (1995) discussed the generation of GS random variables.

Let us denote $\text{ML}_{\alpha,\sigma}$as the ML random variable having Laplace transform

$$\phi_{\alpha,\sigma}(\lambda) = \frac{1}{1+\sigma^\alpha\lambda^\alpha}, \quad \lambda \geq 0, \quad 0<\alpha\leq 1, \sigma>0 \tag{2.4.3}$$

Clearly for $\alpha = 1$, (2.4.3) is the Laplace transform of an exponential distribution with mean σ, while for $0<\alpha<1$, (2.4.1) is the Laplace transform of $\text{GS}_\alpha\left(\sigma\left[\cos\left(\frac{\pi\alpha}{2}\right)\right]^{1/\alpha},1,0\right)$ $if 0<\alpha<1$ distribution. Hence, we have the following relation between $\text{ML}_{\alpha,\sigma}$ and $\text{GS}_\alpha(\sigma,\beta,\mu)$ distributions:

$$\text{ML}_{\alpha,\sigma} = \begin{cases} \text{GS}_\alpha(0,1,\sigma), & \text{if}\alpha = 1 \\ GS_\alpha\left(\sigma\left[\cos\left(\frac{\pi\alpha}{2}\right)\right]^{1/\alpha},1,0\right), & \text{if } 0<\alpha<1. \end{cases} \tag{2.4.4}$$

Note that $\sigma Y\ ML_{\alpha,\sigma}$ if $Y\ ML_{\alpha,1}$ (σ is a scale parameter).

Kozubowski and Rachev (1999) obtained the following representation of ML_r random variable. (See also Jayakumar et. al., 1995).

THEOREM 2.21. *Let $0 < \alpha < \alpha' \leq 1$ and $\rho = \frac{\alpha}{\alpha'} < 1$. Let W be a nonnegative random variable with density*

$$g_\rho(x) = \frac{sin(\pi\rho)}{\pi\rho[x^2 + 2xcos(\pi\rho) + 1]} \tag{2.4.5}$$

and let $M_{\alpha',\sigma}$ be a ML random variable (2.4.3), independent of W. Then, a ML random variable $M_{\alpha,\sigma}$ admits the representation $M_{\alpha,\sigma} \stackrel{d}{=} M_{\alpha',\sigma}W^{1/\alpha}$.

The representation given in the above theorem can be written in terms of densities. In particular, taking the largest values of α' leads to the expressions for ML densities and distribution functions that are particularly convenient for numerical approximation.

PROPOSITION 2.1. *For any $0 < \alpha < 1$, the density and distribution function of a $ML_{\alpha,1}$, distribution have the representations*

$$f_{ML}(x) = \frac{sin\pi\alpha}{\pi} \int_0^\infty \frac{y^\alpha e^{-xy}}{y^{2\alpha} + 1 + 2y^\alpha cos\pi\alpha} dy, x > 0$$

and

$$F_{ML}(x) = 1 - \frac{sin\pi\alpha}{\pi} \int_0^\infty \frac{y^{\alpha-1} e^{-xy}}{y^{2\alpha} + 1 + 2y^\alpha cos\pi\alpha} dy, x > 0$$

respectively.

Selected densities of ML distributions calculated from the above representation are presented in Figure 2.4.1.

The relation between the Linnik and ML distribution via stable law is given below.

Let $S_\alpha(\alpha, \beta, \mu)$ denote a stable law corresponding to characteristic function $\phi(t)$ where, $log\phi(t) = 1 - \frac{1}{\Psi(t)}$, $\Psi(t)$ as in (2.4.1). Then every Linnik distribution $L_{\alpha,\sigma}$ having characteristic function, $\Psi(t) = \frac{1}{1+\sigma^\alpha|t|^\alpha}$ is a scale mixture of $S_{\alpha'}(m^{1/\alpha}\sigma, 0, 0)$ where m has a $ML_{\alpha/\alpha',1}$ distribution.

For various representations of GS laws, see Kozubowski (1998, 2000a), Kozubowski and Podgorski (1994).

THEOREM 2.22. *Let $0 < \alpha < \alpha' \leq 2$ and $\rho = \frac{\alpha}{\alpha'} < 1$. Let $X_{\alpha',\alpha} \quad S_{\alpha'}(\sigma, 0, 0)$ and $M_{\rho,1} \quad ML_{\rho,1}$ be independent. Then $Y_{\alpha,\sigma} \quad L_{\alpha,\sigma}$ admits the representation $Y_{\alpha,\sigma} \stackrel{d}{=} ML_{\rho,1}^{1/\alpha'} X_{\alpha',\sigma}$.*

REMARK 2.1. *By taking $\alpha' = 2$, it can be seen that $L_{\alpha,\sigma}$ is a scale mixture of normal distributions, and consequently, is conditionally Gaussian.*

Now we turn to simulation of ML distributions (2.4.3). The procedure is based on representation (2.4.5) with $\alpha' = 1$, showing that a ML distribution is a scale mixture of exponential distributions. An algorithm for generating ML random variate is given below.

Algorithm: A $ML_{\alpha,\sigma}$ generator.
Generate random variate Z from $ML_{1,1}$ distribution(standard exponential) Generate uniform[0, 1] variate U, independent of Z.

Set $\rho \leftarrow \alpha$
Set $W \leftarrow sin(\pi\rho)cot(\pi\rho U) - cot(\pi\rho)$
Set $Y \leftarrow \sigma Z W^{1/\alpha}$
Return Y.

To illustrate this algorithm, Kozubowski (2001) simulated ML distributions and compared the resulting histograms with their densities.

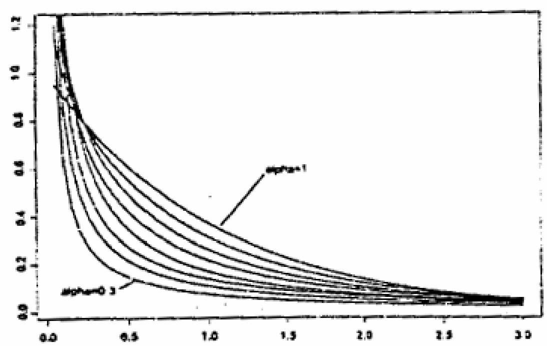

Figure 2.4.1 : Densities of Mittag-Leffler distributions with $\sigma = 1$ and $\alpha = 0.3(0.1)1$.

3 Semi-Mittag-Leffler distribution

3.1 Definition and properties

The semi-Mittag-Leffler distribution is defined as follows:

DEFINITION 3.1. *A distribution with positive support is said to be semi-Mittag-Leffler and write* $X \stackrel{d}{=} SML(\alpha, p)$ *if its Laplace transform* $\phi(\lambda)$ *is of the form*

$$\phi(\lambda) = \frac{1}{1 + \eta(\lambda)}, \tag{3.1.1}$$

where $\eta(\lambda)$ *satisfies the functional equation*

$$\eta(\lambda) = \frac{1}{p}\eta(p^{1/\alpha}\lambda), 0 < p < 1, 0 < \alpha \leq 1. \tag{3.1.2}$$

The solution of the equation (3.1.2) is derived in Kagan, Linnik and Rao (1973, p.163) and is given in the following Lemma. For the relation between Cox processes and SML distribution, see Sandhya (1991a, b).

LEMMA 3.1. *The solution of the functional equation (3.1.2) is* $\eta(\lambda) = \lambda^{\alpha} h(\lambda)$ *where* $h(\lambda)$ *is periodic in* $ln\lambda$ *with period* $\frac{-2\pi\alpha}{lnp}$.

For an example of $h(\lambda)$ function, see Jayakumar and Pillai (1993). The function $\eta(\lambda)$ of the form (3.1.2) have been studied by Dubuc (1990) and Biggins and Bingham (1991). Jayakumar and Pillai (1993) developed a first order autoregressive SML process and studied its properties. José and Pillai (1996) introduced a new autoregressive model with ML marginals as a generalization of the NEAR(1) model of Lawrance and Lewis (1981). They have further extended the same to obtain a new class called new first order autoregressive SML process.

Here, we present some new results related to the SML distribution.

THEOREM 3.2. *The function* $\phi(\lambda) = \frac{1}{1+\eta(\lambda)}$ *where* $\eta(\lambda)$ *satisfies the equation*

$$\eta(\lambda) = \frac{1}{p}\eta(p^{1/\alpha}\lambda), 0 < p < 1, 0 < \alpha \leq 1. \tag{3.1.3}$$

Proof: Consider the function $g(\lambda) = \frac{1}{1+\lambda}$, $\lambda > 0$. Since $g(\lambda)$ is the Laplace transform of unit exponential distribution, it is completely monotone. Consider the function $\eta(\lambda) = e^{-\eta(\lambda)}$, where $\eta(\lambda)$ satisfies (3.1.3). By Ramachandran and Rao (1968), $h(\lambda)$ is infinitely divisible, being the Laplace transform of a semi-stable random variable. By Feller (1966), p. 425, $\eta(\lambda)$ has complete monotone derivative. Hence, $g(\eta(\lambda)) = \frac{1}{1+\eta(\lambda)}$ is completely monotone. Also $g(\eta(0)) = 1$. Therefore, $\frac{1}{1+\eta(\lambda)}$ is the Laplace transform of a probability distribution. This completes the proof.

In Theorem 3.3 below, we decompose the SML law in terms of the semi-stable law. We omit the proof of the theorem as it follows easily.

THEOREM 3.3. *If* X *and* Y *are independent random variables such that* X *has exponential distribution with unit mean and* Y *has semi-stable distribution having Laplace transform* $e^{-\eta(\lambda)}$, *where* $\eta(\lambda)$ *satisfies (3.1.3), then* $X^{1/\alpha}Y \stackrel{d}{=} Z$, *where* $Z \stackrel{d}{=} SML(\alpha, p)$.

REMARK 3.1. *Since* $\exp\{\eta(\lambda)\}$ *is the Laplace transform of an infinitely divisible distribution, the canonical representation of* $\eta(\lambda)$ *in terms of the Levy spectrum* M *is*

$$\eta(\lambda) = \int_0^{\infty} \frac{1 - e^{-\lambda x}}{x} dM(x).$$

But $\eta(\lambda)$ *satisfies (3.1.3). Hence, on simplification we get*

$$\eta(\lambda) = \sum_{n=-\infty}^{\infty} \frac{1}{p^n} \eta_0(p^{1/\alpha}\lambda),$$

where

$$\eta_0(x) = \int_{p^{1/\alpha}}^{1} \frac{1-e^{-xu}}{u} dM(u).$$

Now we consider the renewal process $\{S_n\}$ with semi-Mittag-Leffler waiting time distribution defined by

$$S_0 = 0, \quad S_n = \sum_{i=1}^{n} X_i,$$

where X_i's are i.i.d. SML(α, p).

THEOREM 3.4. *The expected number of epochs of the renewal process defined by (3.1.4) is given by $U(x)$ having the Laplace transform $\frac{1}{\eta(\lambda)}$.*

Proof : The expected number of renewals is given by

$$U(x) = \sum_{n=1}^{\infty} F^{n*}(x),$$

where $F(x)$ has Laplace transform $\frac{1}{\eta(\lambda)}$ Taking Laplace transform on both sides of (3.1.4), we get

$$\begin{aligned} \phi(\lambda) &= \sum_{n=1}^{\infty} \left[\frac{1}{1+\eta(\lambda)}\right]^n \\ &= \frac{1}{\eta(\lambda)}. \end{aligned}$$

This completes the proof.

REMARK 3.2. *If $\eta(\lambda) = \lambda^\alpha$, then $U(x) = \frac{x^\alpha}{\Gamma(1+\alpha)}$.*

4 Estimation

4.1 Estimation in ML Distribution

Using the theory of geometric stable laws, Kozubowski (2001) developed procedures for estimating the parameters of ML distributions with the approach based on fractional moments, used by Nikias and Shao (1995) for estimating stable parameters. Let $e(p) = E\,||Y||^p$, where $0 < p < \alpha$ denote the pth absolute moment of Y ML(α, σ). By Pillai (1990), we have

$$e(p) = \frac{p\sigma^p \pi}{\alpha\Gamma(1-p)\sin(\pi p.\alpha)}. \tag{4.1.1}$$

Consider a random sample $Y_1, Y_2, \ldots, Y_n$ from a $ML_{\alpha,\sigma}$ distribution. Formula (4.1.1) leads to the method of moment estimators for α and σ. Choose two values of p, say p_1 and p_2 , replace $e(p_k)$ in (4.1.1) with its sample counterpart

$$\hat{e}(p_k) - \frac{1}{n}\sum \|Y_i\|^{p_k}, \quad k = 1, 2, .$$

and solve the resulting equation for α and σ. For an illustration of this procedure, see Kuzubowski (2001). Kozubowski (2001) also verfied the procedure using simulated results and found them to be quite satisfactory even for small sample sizes. For other methods of estimation, see Kozubowski (1999), Jacques et al. (1999).

4.2 Estimation of $\eta(\lambda)$in Semi Mittag-Leffler distribution

Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from SML distribution with Laplace transform given in (3.1.1)-(3.1.2). The function

$$\phi_n(\lambda) = \frac{1}{n}\sum_{i=1}^{n} \mathrm{e}^{-\lambda X_i} \qquad (4.2.1)$$

is called the empirical Laplace transform. We derive an estimator for $\eta(\lambda)$using the method of moments and is given by

$$\hat{\eta}(\lambda) = \frac{1}{\phi_n(\lambda)} - 1. \qquad (4.2.2)$$

To study the asymptotic properties of the estimator , we first study the asymptotic properties of $\phi_n(\lambda)$. Note that $\phi_n(\lambda) = \frac{1}{n}\sum_{i=1}^{n} \mathrm{e}^{-\lambda X_i} = \frac{1}{n}\sum_{i=1}^{n} Y_i$, where Y_i's are i.i.d. with $Y = e^{-\lambda X}$. Since $E(\phi_n(\lambda) = \phi(\lambda)$ for all $\lambda > 0$, we have by strong law of large numbers, $\phi_n(\lambda) \stackrel{\text{a.s.}}{\to} \phi(\lambda)$. It can be easily shown that the convergence is uniform. This follows on the same lines as the uniform convergence of empirical characteristic function. (see Laha and Rohatgi, 1979, p.156.). Since $\hat{\eta}(\lambda) = \dfrac{1}{\phi_n(\lambda)} - 1$ is a continuous function for all $\lambda > 0$, it follows, using the invariance principle, that $\hat{(\eta(\lambda))} \to \eta(\lambda)$ uniformly in λ. Note that $E[\mathrm{e}^{-\lambda X}] = \phi(\lambda)$ and $\mathrm{Var}[\mathrm{e}^{-\lambda X}] = \phi(2\lambda) - \phi^2(\lambda) = M(\lambda)$ say, $0 < M(\lambda) < \infty$. Since $\phi_n(\lambda) = \sum_{i=1}^{n} Y_i$ using Central Limit Theorem, it follows that

$$\sqrt{n}(\phi_n(\lambda) - \phi(\lambda)) \stackrel{d}{\to} N(0, M(\lambda)).$$

Using the invariance property of CAN estimators (see Kale, 1999, p.116), we get

$$\sqrt{n}(\hat{(\eta(\lambda))} - \eta(\lambda)) \to aN\left(0, (\eta'(\lambda))^2 \left\{\frac{1}{1+\eta(2\lambda)} - \left[\frac{1}{1+\eta(\lambda)}\right]^2\right\}\right).$$

That is $\hat{\eta}(\lambda)$, is a CAN estimator with variance given by

$$(\eta'(\lambda))^2 \left\{ \frac{1}{1+\eta(\lambda)} - \left[\frac{1}{1+\eta(\lambda)} \right]^2 \right\}.$$

In the next Section, we will study the small sample properties of $\hat{\eta}(\lambda)$.

4.3 Numerical Results

Here we compare the estimator with the actual value using a simulation study when the samples are drawn from the semi-Mittag-Leffler distribution with $\eta(\lambda) = \lambda^\alpha$, for specific values of $a = 0.8$ and 0.9 with sample sizes 30 and 50. In this study, we have taken simulation size to be 1000. The simulation results are presented in Figures 4.3.1-4.3.4. In these figures, we have used the notation $g(s)$ to denote the periodic function and we have plotted the function $g(s)$ and the average of the estimates of $g(s)$ in the simulated samples, in the interval $(0, 1)$.

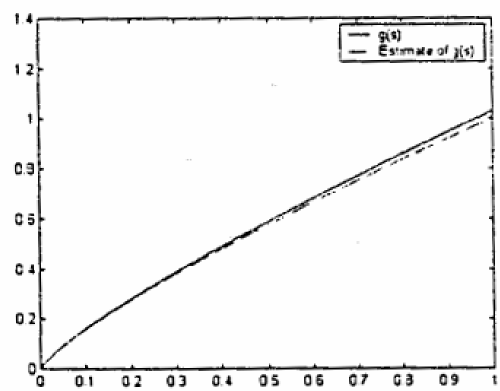

Figure 4.3.1. Periodic Function and its estimate ($\alpha = 0.8, n = 30$)

As is clear from the figures, the estimator performs very well for even sample size of 30, and the performance improves as n increases. It can be noted that for n=50, the average of the estimates becomes almost indistinguishable as compared to the function itself.

5 Other related distributions

5.1 Generalized Linnik distribution

Christoph and Schreiber (2000) have defined the generalized positive Linnik distribution as follows:

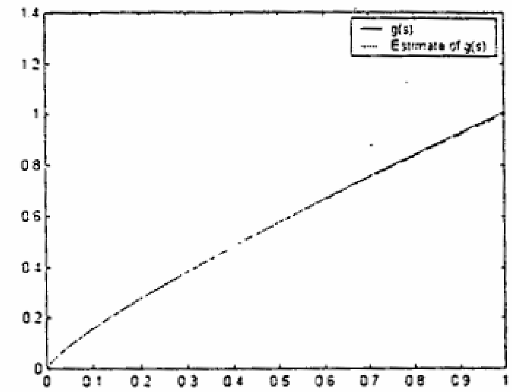

Figure 4.3.2 Periodic Function and its estimate ($\alpha = 0.8, n = 50$)

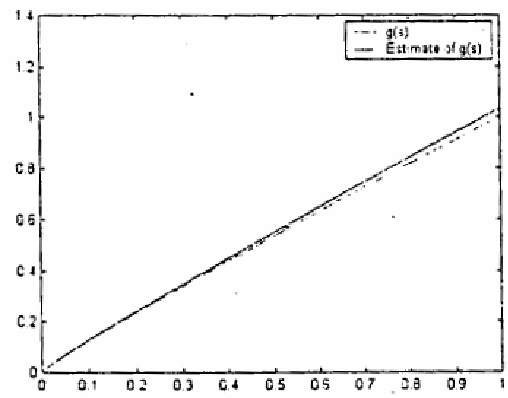

Figure 4.3.3 Periodic Function and its estimate ($\alpha = 0.9, n = 30$)

DEFINITION 5.1. *A random random variable X on $(0, \infty)$ is said to have generalized positive Linnik distribution with characteristic exponent $\gamma \in (0, 1]$, scale parameter $\delta > 0$ and shape parameter $\beta > 0$ if it has the Laplace transform*

$$\phi_x(\lambda) = \begin{cases} \left(1 + \delta \frac{\lambda^r}{\beta}\right)^{-\beta}, & \text{for } 0 < \beta < \infty \\ \exp\{-\delta\lambda^r\}, & \text{for } 0 < \beta = \infty. \end{cases} \quad (5.1.1)$$

If $\beta = 1$, (5.1.1) defines the Laplace transform of the ML distribution with corresponding distribution function $1 - E_r(-x^r)$, where $E_r(x)$ is the ML function.

In contrast, Bingham, Goldie and Teugels (1987, p.329 and p.392) defined ML distribution as limit laws for occupation times of Markov process when the corresponding Laplace transform $E_r(u)$ being ML function (see also Pakes, 1992b). For $r = 1$ well known distribution occurs; degenerate distribution at the point δ if $\beta = \infty$, gamma distribution with density $\frac{1}{\Gamma(\beta)}(\frac{\beta}{\delta})^\beta e^{-x\beta/\delta} x^{\beta-1}$, $x > 0$, if $\beta < \infty$ with the special case of exponential distribution with parameter

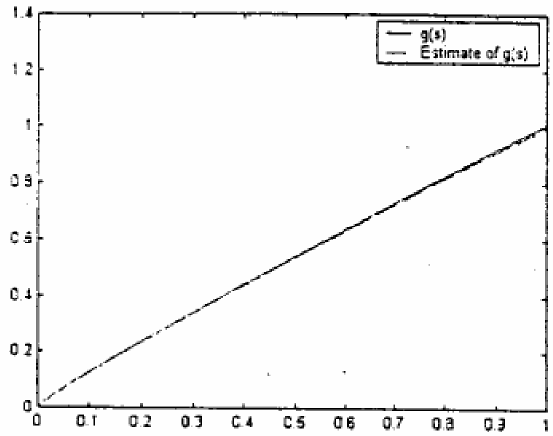

Figure 4.3.4 Periodic Function and its estimate ($\alpha = 0.9, n = 50$)

$\delta^{-1} > 0$ if $\beta = 1$. Lin (1994) characterized Linnik distribution through closure under geometric compounding. For properties of Linnik distribution, see also Kotz *et. al.* (2001).

The relation between the generalized Linnik distributions with other distributions is given in Figure 5.1.1.

Pakes(1992 a) obtained a characterization of generalized Positive Linnik law. For distributions related to Linnik laws, see also Remillard and Theodorescu (2002).

Aly and Bouzar (2000) obtained the following results.

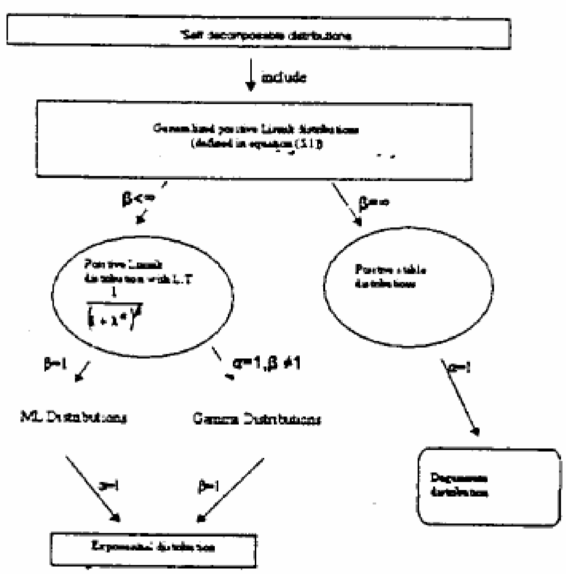

Figure 5.1.1 Relations between various classes of distributions

THEOREM 5.1 *An R+-valued random variable X is said to be compound gamma if $X \stackrel{d}{=} Y(T)$ for some R+-valued Levy process $\{Y(.)\}$ and some gamma distributed random variable T, independent of $\{Y(.)\}$. The LST of a compound gamma distribution is given by $\phi(u) = \frac{1}{[1 - c\log\phi_1(u)]^r}$, for some $c > 0$ and $r > 0$ and some infinitely divisible LST ϕ_1. That is, X is compound-gamma if and only if its LST is given by*

$$\varphi(u) = \frac{1}{[1 - \psi_1(u)]^r}, \quad u \geq 0 \tag{5.1.2}$$

where ψ has completely monotone derivative with $\psi(0) = 0$ and $r > 0$.
The compound exponential distribution arises as a special case of compound gamma and corresponds to $r = 1$, in (5.1.2). Aly and Bouzar (2000) proved the following results.

PROPOSITION 5.1. *Let X be an R+ -valued random variable. The following assertions are equivalent:*

(i) X is g.i.d.
(ii) $N_\lambda(X)$ is compound geometric for all $\lambda > 0$, where $N_\lambda(.)$ is a Poisson process of intensity λ.
(iii) X is compound exponential.

Moreover, if the distribution of X has an atom at 0 , then the above assertions are equivalent to
(iv) X satisfies the stability equation

$$X \stackrel{d}{=} B(X + S) \tag{5.1.3}$$

for some Z+ valued random variable S and some mixed Bernoulli variable B with mixing variable W taking values in $(0,1)$ and with mean $E(W) = \frac{c}{1+c}$, $c > 0$, where the random variables X, B and S are independent.

THEOREM 5.2 *Let $X(.)$ be a R+- valued Levy process. Let $\psi(u)$ be a function on R+ with a completely monotone derivative such that $\psi(0) = 0$ and let $a > 0$.*
(i) $X(t)$ has LST

$$\phi_t(u) = (1 + \psi(u))^{-t/\alpha}$$

for all $t > 0$ if and only if $X(.)$ can be represented as subordinated to a gamma process $T(.)$, in the sense that $X(t)$ can be written in the form $X(t) \stackrel{d}{=} Y[T(t)]$, where for all $t > o$, $T(t)$ has $\Gamma(t/a, 1)$ distribution and $Y(t)$ is a Levy process with LST $\exp\{-t\psi(u)\}$.

(ii) Moreover, if $\lim_{u\to\infty} \psi(u) < \infty$, then $X(.)$ will satisfy the stability equation (5.1.3) and the distribution of $X(a)$, and hence of $X(t)$ for each $t > 0$, will have an atom at 0.

THEOREM 5.3 *An $R+$ valued random variable X is geometrically strictly stable if and only if its Laplace transform is given by*

$$\phi(u) = \frac{1}{1+cu^{r}}$$

for some $0 < \gamma < 1$ *and* $c > 0$.

6 Concluding Remarks

In this paper, we have reviewed the ML distributions, and studied various distributional properties and characterizations related to the ML distributions. We also studied Semi-Mittag Leffler Distributions, Generalized positive Linnik distributions and other related distributions. We also reviewed the results related to estimation of parameters in ML distributions and proposed some new estimators for the function $\eta(\lambda)$ in the SML distribution using Empirical Laplace Transform. Here, we present some of the open problems related to the Mittag Leffler distribution.

a. General inference procedures need further study. Some estimators for parameters related to the ML distribution are available. Other general inference procedures such as testing of hypothesis, goodness of fit etc., need to be developed.

b. As has been noted in the paper, the ML distributions are generalizations of Exponential distributions. The ML distribution, therefore, can be studied as an alternative model for analyzing lifetime data. Reliability measures such as Failure Rate, Mean Residual Life, etc., need to be studied. Through such a study one can establish bounds for the distribution function of the ML distribution, which will be quite useful, as the distribution function of the ML distribution is an infinite series. We conjecture, here, that the ML distribution represents the negative ageing phenomenon.

c. In Theorem 3.1.2, we have established the relation between Semi-Stable distributions and SML distributions. If we can generate random numbers from semi-stable distributions, we can use this result to generate random numbers from SML distributions, which will be quite useful in simulation and other numerical studies. However, to the best of our knowledge, generation of random numbers from semi-stable distribution having Laplace transform $e^{-\eta(\lambda)}$, where $\eta(\lambda)$ satisfies (3.1.3), is not available.

Acknowledgements. The authors wish to thank the referee for valuable comments which have resulted in an improved presentation of the paper. The second author wishes to thank Indian Institute of Management, Kozhikode, for supporting this work through a small grant research project.

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Received : March, 2003
Revised : October, 2003